

MAPPING CONES IN THE BOUNDED DERIVED CATEGORY OF A GENTLE ALGEBRA

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ABSTRACT. Gentle algebras are a class of algebras that are derived tame. They therefore provide a concrete setting to study the structure of the (bounded) derived category in detail. In this article we explicitly describe the triangulated structure of the bounded derived category of a gentle algebra by describing its triangles. In particular, we develop a graphical calculus which gives the indecomposable summands of the mapping cones of morphisms in a canonical basis of the Hom-space between any two indecomposable complexes.

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INTRODUCTION

Derived categories provide a common framework for homological algebra in subjects such as algebra, geometry and mathematical physics. For example, in mathematical physics, in the context of homological mirror symmetry, they are the natural setting for Bridgeland's stability conditions [12]. In algebraic geometry, they arise in the study of non-commutative crepant resolutions which are often studied via an algebra whose derived category is equivalent to the derived category of the smooth variety resolving the singularity [46]. In representation theory, derived categories are the natural setting for tilting theory of finite dimensional algebras, see for example [33]. Thus understanding the structure of derived categories and their properties is an important problem. However, owing to their complexity, in general, this is difficult to achieve. Therefore in the cases, where this is achievable, it is of great value to obtain as much detailed knowledge of the derived category as possible.

In the context of the representation theory of finite dimensional algebras, we will now describe a situation where it is possible to gain insight into the structure of derived categories. According to the tame-wild dichotomy [27], algebras are either of tame representation type, that is, all indecomposable finite-dimensional modules can be classified or they are of wild representation type and it is considered that a complete classification is impossible. Therefore much of the work in representation theory has been focused on tame algebras and for particular classes of tame algebras we have a good understanding of their (classical) representation theory, that is of

2010 *Mathematics Subject Classification.* 18E30, 16G10, 05E10.

Key words and phrases. bounded derived category, gentle algebra, homotopy string and band, string combinatorics, mapping cone

This work was supported through EPSRC grants EP/K026364/1, EP/K022490/1 and EP/N005457/1.

their module categories. A good example of this are special biserial algebras [47], a prominent class of tame algebras, that has been widely studied, with many exciting recent developments, see for example [1, 2, 3, 22, 24, 25, 28, 30, 45, 48]. One of the reasons that special biserial algebras are so well-understood is that their indecomposable representations are classified in terms of strings and bands [32, 42, 47].

The notion of derived-tameness was introduced in [31]. In the case of derived-tame algebras we can gain concrete insight into the structure of the derived category. Gentle algebras are derived tame [8]. They form a subclass of special biserial algebras which is closed under derived equivalence [44]. They are, therefore, a natural class of algebras whose derived category is the object of intensive study. As a result, our understanding of the structure of derived categories of gentle algebras parallels that of our understanding of their module categories.

Gentle algebras first arose in the setting of tilting theory in the classification of iterated tilted algebras of type A and type \hat{A} in [5] and [7] respectively. They now play an important role in many areas of mathematics: in algebra, they occur in cluster theory as Jacobian algebras associated to surface triangulations [6, 29, 38], and in recent advances in invariant theory [23]. In addition to their widespread appearance in representation theory and algebra, gentle algebras also are instrumental in geometric contexts. For example, their singularity category [15] – which measures how far an algebra or variety is away from being nonsingular – was described in [36]. They feature prominently in the programme to understand singularities of nodal curves [16, 17, 18] and similar combinatorics occur in an algebraic approach to mirror symmetry based on dimer models [10].

From now on let Λ be a gentle algebra and $D^b(\Lambda)$ be its bounded derived category with shift functor Σ . The indecomposable objects in $D^b(\Lambda)$ have been classified [8] in terms of string combinatorics: namely they are given in terms of *homotopy strings* and *homotopy bands*; see also [16] for a similar approach in the context of nodal algebras. Note that the terminology originates in [9].

Using the Happel functor [33], the Auslander–Reiten (AR) structure of the perfect category $K^b(\text{proj}(\Lambda))$ was determined in [9]. See also [4, §6] for similar results without the use of the Happel functor. A canonical basis for the morphisms between indecomposable objects in $D^b(\Lambda)$ was given in [4] in terms of three types of morphism: *graph maps*, *single maps* and *double maps*. Note that the last two classes may have nontrivial homotopies between them which can be detected by so-called *quasi-graph maps*. We will call this the *ALP basis*.

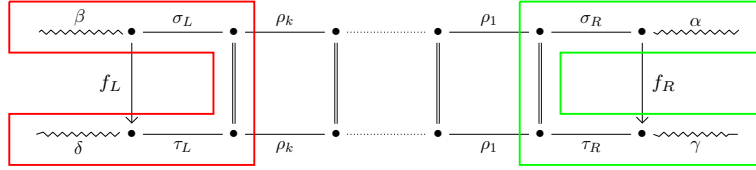
However, while the AR triangles are well understood [9], a complete description of the triangulated structure of $D^b(\Lambda)$ is not known. In particular, given two indecomposable complexes P^\bullet and Q^\bullet in $D^b(\Lambda)$ the middle terms of extensions $Q^\bullet \rightarrow E^\bullet \rightarrow P^\bullet \rightarrow \Sigma Q^\bullet$ were, up to now, not well understood. In this paper, based on (homotopy) string combinatorics, we give a complete description of the middle terms of such extensions. Note that a complete understanding of middle terms of extensions and the triangulated structure is important for many applications, for example:

- in cluster theory, these form the basis of the so-called exchange triangles and exchange relations [14, 19, 21, 34, 35];
- understanding middle terms of extensions is a key ingredient in classifying torsion pairs in triangulated categories [26, 39, 41, 49];
- the description of middle terms of extensions in the derived category of derived-discrete algebras is instrumental in the classification of thick subcategories of discrete derived categories [13];
- for a gentle algebra Λ and extending the geometric model in [13], Oppermann recently announced [40] a geometric model of $D^b(\Lambda)$ which uses the results of this paper;
- in an upcoming sequel to this paper [20], the results and techniques developed in this article will be used to show that the short exact sequences given in [43] form a basis of the Ext^1 -space between any two indecomposable Λ -modules, thus answering this longstanding open question.

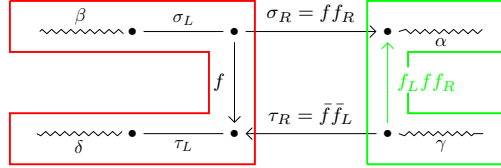
In $D^b(\Lambda)$ the middle term E^\bullet of an extension $Q^\bullet \rightarrow E^\bullet \rightarrow P^\bullet \xrightarrow{f^\bullet} \Sigma Q^\bullet$ is given by the inverse shift of the mapping cone $M_{f^\bullet}^\bullet$ of the map f^\bullet . In this paper, we describe the indecomposable summands of the mapping cones of the ALP basis elements, that is of maps f^\bullet in the canonical basis of $\text{Hom}_{D^b(\Lambda)}(P^\bullet, Q^\bullet)$, where P^\bullet and Q^\bullet are indecomposable complexes. Our description is general in that our results cover both string and band complexes. Our main results can be summarised in the following combinatorial description of the mapping cones of ALP basis elements:

Theorem. *Let Λ be a gentle algebra. Suppose σ and τ are homotopy strings or bands and that Q_σ^\bullet and Q_τ^\bullet are the corresponding indecomposable string or band complexes. Let $f^\bullet \in \text{Hom}_{D^b(\Lambda)}(Q_\sigma^\bullet, Q_\tau^\bullet)$ be an ALP basis element. Then $M_{f^\bullet}^\bullet$ can be described by the following graphical calculus.*

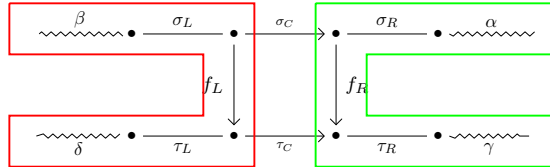
- (1) *Let $\sigma = \beta\sigma_L\rho\sigma_R\alpha$ and $\tau = \delta\tau_L\rho\tau_R\gamma$ and suppose f^\bullet is a graph map corresponding to the overlap ρ . Then $M_{f^\bullet}^\bullet$ is given by:*



- (2) *Let $\sigma = \beta\sigma_L\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_R\gamma$ and suppose f^\bullet is a single map. Then $M_{f^\bullet}^\bullet$ is given by:*



- (3) *Let $\sigma = \beta\sigma_L\sigma_C\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_C\tau_R\gamma$ and suppose f^\bullet is a double map. Then $M_{f^\bullet}^\bullet$ is given by:*



We refer the reader to Sections 4, 5 and 6 for precise statements and details.

Convention. Throughout this article, all modules will be left modules and all maps will be composed from left to right.

1. BACKGROUND

1.1. The homotopy category and mapping cones. The required background on derived and triangulated categories in the setting of representation theory can be found in [33].

Let Λ be a finite-dimensional algebra over an algebraically closed field \mathbf{k} . The category of interest in this article will be $K^{b,-}(\text{proj}(\Lambda))$, the homotopy category of right bounded complexes of projective left Λ -modules with bounded cohomology. The morphisms of $K^{b,-}(\text{proj}(\Lambda))$ are cochain maps of complexes up to homotopy. Let (P^\bullet, d_P) and (Q^\bullet, d_Q) be complexes in $K^{b,-}(\text{proj}(\Lambda))$. Two maps $f^\bullet, g^\bullet: P^\bullet \rightarrow Q^\bullet$ are said to be *homotopic*, written $f^\bullet \simeq g^\bullet$, if there exists a family of maps $\{h^n: P^n \rightarrow Q^{n-1}\}_{n \in \mathbb{Z}}$ such that $f^n - g^n = d_P^n h^{n+1} + h^n d_Q^{n-1}$. Below is

a schematic of the situation:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_P^{n-2}} & P^{n-1} & \xrightarrow{d_P^{n-1}} & P^n & \xrightarrow{d_P^n} & P^{n+1} \xrightarrow{d_P^{n+1}} \cdots \\
& & \downarrow f^{n-1}-g^{n-1} & \nearrow h^n & \downarrow h^{n+1} & \nearrow f^n-g^n & \downarrow f^{n+1}-g^{n+1} \\
\cdots & \xrightarrow{d_Q^{n-2}} & Q^{n-1} & \xrightarrow{d_Q^{n-1}} & Q^n & \xrightarrow{d_Q^n} & Q^{n+1} \xrightarrow{d_Q^{n+1}} \cdots
\end{array}$$

If $f^\bullet \simeq 0^\bullet$ then f^\bullet is called *null-homotopic*. The family $\{h^n\}_{n \in \mathbb{Z}}$ is called a *homotopy*. Homotopy forms an equivalence relation on the set of cochain maps $P^\bullet \rightarrow Q^\bullet$.

Let \mathbf{K} be a triangulated category. Throughout this article we shall denote the shift (or suspension) functor by $\Sigma: \mathbf{K} \rightarrow \mathbf{K}$. The first axiom of triangulated categories asserts that for any morphism $f: X \rightarrow Y$ in \mathbf{K} there exists a completion to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

The third object Z in the triangle above is often called the *cone of f* owing to the fact that the triangulated structure put on the homotopy category $\mathbf{K}^{b,-}(\text{proj}(\Lambda))$ comes from the mapping cone construction.

Definition 1.1. Let Λ be a finite-dimensional algebra and consider the homotopy category $\mathbf{K}^{b,-}(\text{proj}(\Lambda))$. Let (P^\bullet, d_P) and (Q^\bullet, d_Q) be complexes in $\mathbf{K}^{b,-}(\text{proj}(\Lambda))$ and suppose $f^\bullet: P^\bullet \rightarrow Q^\bullet$ is a cochain map. The *mapping cone of f^\bullet* is the complex $(M_{f^\bullet}^\bullet, d_{M_{f^\bullet}})$ given by

$$M_f^n = P^{n+1} \oplus Q^n \quad \text{and} \quad d_{M_f}^n = \begin{bmatrix} -d_P^{n+1} & f^{n+1} \\ 0 & d_Q^n \end{bmatrix}$$

It is well known that $\mathbf{K}^{b,-}(\text{proj}(\Lambda)) \simeq \mathbf{D}^b(\Lambda)$, the bounded derived category of finitely generated left Λ -modules.

1.2. Gentle algebras. The definition of a gentle algebra goes back to [7], where they first occurred as iterated tilted algebras of type A .

Definition 1.2. A finite dimensional k -algebra is called *gentle* if it is Morita equivalent to an algebra kQ/I where Q is a quiver and I an admissible ideal in kQ such that

- (1) For each vertex $i \in Q_0$ there are at most two arrows starting at i and at most two arrows ending at i ;
- (2) For each arrow $a \in Q_1$ there is at most one arrow b such that $ba \notin I$ and at most one arrow c such that $ac \notin I$;
- (3) For each arrow $a \in Q_1$ there is at most one arrow b such that $ba \in I$ and at most one arrow c such that $ac \in I$;
- (4) The ideal I is generated by length two monomial relations.

Note that if one removes condition (3) and relaxes condition (4) so that the monomial ideal is generated simply by monomial relations one obtains a so-called *string algebra*.

Throughout, Λ will be a gentle algebra over an algebraically closed field k . For convenience we shall always identify $\mathbf{D}^b(\Lambda) \simeq \mathbf{K}^{b,-}(\text{proj}(\Lambda))$.

2. STRING AND BAND COMPLEXES

The indecomposable complexes in $\mathbf{K}^{b,-}(\text{proj}(\Lambda))$ were classified by Bekkert and Merklen in [8]. They are classified in terms of *homotopy strings* and *homotopy bands*, where we use the terminology of Bobiński [9]. In this article we shall use the basis of the morphism space between indecomposable complexes given in [4]. As such, we provide a brief summary of the description of homotopy strings and bands using the notation of [4].

Let $\Lambda = \mathbf{k}Q/I$ be a gentle algebra. Since we are reading paths from right to left, all modules in this article will be left Λ -modules. Recall that for $i, j \in Q_0$ there is a bijection:

$$\begin{aligned} \{\text{paths } p: j \rightsquigarrow i \text{ in } (Q, I)\} &\xrightarrow{1-1} \{\text{basis elements of } \text{Hom}_\Lambda(P(i), P(j))\} \\ p &\longmapsto (u \mapsto up). \end{aligned}$$

Bekkert and Merklen's key insight was that one could use this correspondence together with the length two monomial relations for a gentle algebra to produce indecomposable complexes. They were then able to use Bondarenko's [11] technology concerning matrix problems to show that these were all indecomposable complexes.

2.1. Homotopy strings. A *homotopy letter* is a triple (p, i, j) where p is a path in Q with no subpath in I and $i, j \in \mathbb{Z}$ are such that $|i - j| \leq 1$. The homotopy letter (p, i, j) is called *direct* if $i < j$ and *inverse* if $i > j$. We shall see below that i and j are the cohomological degrees of the homotopy letter. If $i = j$ then p is a trivial path and is called a *trivial* homotopy letter. The inverse of (p, i, j) is $(p, i, j)^{-1} = (\bar{p}, j, i)$, where \bar{p} denotes the inverse path of p . The starting and ending vertices of a homotopy letter are defined by

$$s(p, i, j) = \begin{cases} s(p) & \text{if } (p, i, j) \text{ is direct;} \\ e(p) & \text{if } (p, i, j) \text{ is inverse;} \end{cases} \quad \text{and} \quad e(p, i, j) = \begin{cases} e(p) & \text{if } (p, i, j) \text{ is direct;} \\ s(p) & \text{if } (p, i, j) \text{ is inverse.} \end{cases}$$

The composition $(p, i, j)(p', i', j')$ is defined if $j = i'$ and $s(p, i, j) = e(p', i', j')$.

A *homotopy string* is a sequence of pairwise composable homotopy letters

$$\sigma = \prod_{r=n}^1 (\sigma_r, i_r, j_r) = (\sigma_n, i_n, j_n) \cdots (\sigma_2, i_2, j_2)(\sigma_1, i_1, j_1)$$

such that

- (1) whenever $(\sigma_r, i, i+1)(\sigma_{r-1}, i+1, i+2)$ occurs $\sigma_r \sigma_{r-1}$ has a subpath in I ;
- (2) dually, whenever $(\bar{\sigma}_r, i, i-1)(\bar{\sigma}_{r-1}, i-1, i-2)$ occurs $\sigma_{r-1} \sigma_r$ has a subpath in I ;
- (3) whenever $(\sigma_r, i, i+1)(\bar{\sigma}_{r-1}, i+1, i)$ occurs σ_r and σ_{r-1} do not start with the same arrow;
- (4) dually, whenever $(\bar{\sigma}_r, i, i-1)(\sigma_{r-1}, i-1, i)$ occurs σ_r and σ_{r-1} do not end with the same arrow.

The starting and ending vertices of a homotopy string σ are defined in the obvious fashion. Similarly, the inverse of a homotopy string is defined in the obvious way. Often we shall not require to record the degrees explicitly and may write the homotopy string as $\sigma = \sigma_n \cdots \sigma_2 \sigma_1$ for short.

Remark 2.1. In light of the correspondence between paths and between indecomposable projective modules and elements of a basis of the Hom-space between them, we observe that for a homotopy string $\sigma = (\sigma_r, i, i+1)(\sigma_{r-1}, i+1, i+2)$ condition (1) above induces a composition of maps

$$P(e(\sigma_r)) \xrightarrow{\sigma_r} P(s(\sigma_r)) = P(e(\sigma_{r-1})) \xrightarrow{\sigma_{r-1}} P(s(\sigma_{r-1})).$$

Since $\sigma_r \sigma_{r-1}$ has a subpath in I , i.e. the relation induces the $\partial^2 = 0$ property of the differential in the corresponding homotopy string complex. Similarly for homotopy strings satisfying condition (2).

We recall the following more intuitive way of expressing a homotopy string from [4].

Definition 2.2. Let $\sigma = \prod_{r=n}^1 (\sigma_r, i_r, j_r)$ be a homotopy string. The *unfolded diagram* of σ is a diagram where each indecomposable projective module is represented by a dot and each homotopy letter is represented by either a left-pointing or right-pointing arrow. Left-pointing arrows correspond to inverse homotopy letters and right-pointing arrows correspond to direct homotopy letters. When the direction of the arrow is undetermined, we simply draw it as a

line.

$$\begin{array}{ccccccccc}
\text{degree:} & i_n & j_n = i_{n-1} & j_{n-1} = i_{n-2} & j_3 = i_2 & j_2 = i_1 & j_1 & & \\
\sigma: & \bullet & \xrightarrow{\sigma_n} & \bullet & \xrightarrow{\sigma_{n-1}} & \bullet & \cdots & \bullet & \xrightarrow{\sigma_2} & \bullet & \xrightarrow{\sigma_1} & \bullet
\end{array}$$

Note that the arrows in an unfolded diagram point in the direction of the (differential) map, i.e. in the opposite direction to the path in the quiver that constitutes the homotopy letter.

Definition 2.3. For a homotopy string σ there is an indecomposable complex $P_\sigma^\bullet \in \mathcal{K}^b(\text{proj}(\Lambda))$ called the corresponding *string complex* of σ , whose construction is described explicitly in [8, Def 2]. Since $P_\sigma^\bullet \cong P_{\sigma^{-1}}^\bullet$, as such there is an equivalence relation \sim^{-1} on homotopy strings given by identifying a homotopy string with its inverse. We write HSt for the set of all homotopy strings under \sim^{-1} .

The passage from a homotopy string σ to the corresponding *string complex* P_σ^\bullet is described in the notation of the present article in [4, §2]. However, the passage is easily understood via an example, which also motivates the unfolded diagram notation.

Example 2.4. Consider the algebra $\Lambda = \mathbf{k}Q/I$ given by the quiver:

$$1 \xrightarrow[a]{a} 2 \xrightarrow[c]{c} 3 \xrightarrow[d]{d} 4$$

subject to the relations $ca = dc = 0$. Consider the homotopy string

$$\sigma = (\bar{d}, 0, -1)(ec, -1, 0)(a, 0, 1)(\bar{b}\bar{c}\bar{e}, 1, 0)(d, 0, 1)(c, 1, 2)(a, 2, 3)$$

which corresponds to the unfolded diagram:

$$\begin{array}{ccccccccccc}
\text{degree:} & 0 & -1 & 0 & 1 & 0 & 1 & 2 & 3 & & \\
\sigma: & \bullet & \xleftarrow{\bar{d}} & \bullet & \xrightarrow{ec} & \bullet & \xrightarrow{a} & \bullet & \xleftarrow{\bar{b}\bar{c}\bar{e}} & \bullet & \xrightarrow{d} & \bullet & \xrightarrow{c} & \bullet & \xrightarrow{a} & \bullet
\end{array}$$

This corresponds to the following unconventional way of writing a complex of projective modules,

$$\begin{array}{ccccccccccc}
0 & -1 & 0 & 1 & 0 & 1 & 2 & 3 & & & \\
P(3) & \xleftarrow{\bar{d}} & P(4) & \xrightarrow{ec} & P(2) & \xrightarrow{a} & P(1) & \xleftarrow{\bar{b}\bar{c}\bar{e}} & P(4) & \xrightarrow{d} & P(3) & \xrightarrow{c} & P(2) & \xrightarrow{a} & P(1),
\end{array}$$

which can be re-arranged into the following, more traditional, diagram of a complex of projective modules,

$$P_\sigma^\bullet: \quad P(4) \xrightarrow{\begin{bmatrix} d & ec & 0 \end{bmatrix}} P(3) \oplus P(2) \oplus P(4) \xrightarrow{\begin{bmatrix} 0 & 0 \\ a & 0 \\ ec & d \end{bmatrix}} P(1) \oplus P(3) \xrightarrow{\begin{bmatrix} 0 \\ c \end{bmatrix}} P(2) \xrightarrow{[a]} P(1).$$

We make the following brief note on our convention for reading homotopy strings (and therefore paths) from right to left, and note that as a consequence, in this article we compose maps from left to right.

Remark 2.5. Remark 2.1 explains our convention to read paths in the quiver from right to left: this corresponds to reading maps between indecomposable projective modules from left to right. Consider the following quiver:

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

with a relation at 3. Let us now consider the two conventions in turn:

- *Reading strings from left to right.* Consider the homotopy string $\sigma = abcd$ which decomposes into homotopy letters $\sigma = \sigma_1\sigma_2$ with $\sigma_1 = ab$ and $\sigma_2 = cd$. The relation says $bc = 0$. The corresponding homotopy string complex is

$$P(5) \xrightarrow{cd} P(3) \xrightarrow{ab} P(1),$$

where we compose maps in the usual way from right to left. Notice that this re-orders the homotopy letters in the string complex in comparison with those in the homotopy string as a word. In particular, the relation $bc = 0$ giving rise to $\partial^2 = 0$ is ‘separated’ when written as a complex.

- *Reading strings from right to left.* Consider the same homotopy string $\sigma = dcba$, again decomposing into homotopy letters $\sigma = \sigma_2\sigma_1$ with $\sigma_1 = ba$ and $\sigma_2 = dc$. The relation says $cb = 0$. The corresponding homotopy string complex is

$$P(5) \xrightarrow{dc} P(3) \xrightarrow{ba} P(1),$$

where we now compose maps from left to right. Notice that the string complex and homotopy string written as a word now coincide, and the relation is no longer separated.

2.2. Homotopy bands. A non-trivial homotopy string $\sigma = \prod_{r=n}^1 (\sigma_r, i_r, j_r)$ is called a *homotopy band* if $s(\sigma) = e(\sigma)$, $i_n = j_1$, one of $\{\sigma_1, \sigma_n\}$ is direct and the other is inverse, and σ is primitive, i.e. not a proper power of another homotopy string. Note that the condition $i_n = j_1$ implies that there must be as many inverse homotopy letters as there are direct homotopy letters, so in particular, any homotopy band contains an even number of homotopy letters.

Definition 2.6. The *unfolded diagram* for a homotopy band $\sigma = \prod_{r=n}^1 (\sigma_r, i_r, j_r)$ is an infinite repeating diagram using the same conventions as the unfolded diagram for a homotopy string:

$$\begin{array}{ccccccccccc} i_n & & j_n & & j_{n-1} & & j_3 & & j_2 & & j_1 = i_n \\ \dots & \xleftarrow{\lambda\sigma_1} & \bullet & \xrightarrow{\sigma_n} & \bullet & \xrightarrow{\sigma_{n-1}} & \bullet & \dots & \bullet & \xleftarrow{\sigma_2} & \bullet & \xrightarrow{\lambda\sigma_1} & \bullet & \xrightarrow{\sigma_n} & \dots \end{array},$$

where the scalar $\lambda \in \mathbf{k}^*$ is added to the component of the differential corresponding to the ‘first’ homotopy letter of σ in much the same way that a ‘twist’ is added to the algebra action on a band module.

Definition 2.7. For a homotopy band σ and $\lambda \in \mathbf{k}$ there is an indecomposable complex $B_{\sigma, \lambda}^\bullet \in \mathcal{K}^b(\text{proj}(\Lambda))$ called the corresponding *band complex* of σ , whose construction is described explicitly in [8, Def 3]. Define an equivalence relation \sim^r on the set of homotopy bands by identifying a homotopy band with its cyclic rotations and their inverses. For homotopy bands σ and τ , $\sigma \sim^r \tau$ if and only if $B_{\sigma, \lambda}^\bullet \cong B_{\tau, \lambda^\pm}^\bullet$. We write HBa for the set of all homotopy bands under \sim^r .

Example 2.8. Consider the algebra of Example 2.4 above. The homotopy string

$$\sigma = (d, 0, 1)(cb, 1, 2)(\bar{a}, 2, 1)(\bar{c}\bar{e}, 1, 0)$$

defines a homotopy band whose unfolded diagram is

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & 1 & & 0 \\ \dots & \xleftarrow{\lambda\bar{c}\bar{e}} & \bullet & \xrightarrow{d} & \bullet & \xrightarrow{cb} & \bullet & \xleftarrow{\bar{a}} & \bullet & \xleftarrow{\lambda\bar{c}\bar{e}} & \bullet & \xrightarrow{d} & \dots \end{array},$$

which corresponds to the following complex of projective modules:

$$P(4) \xrightarrow{[d \ \lambda\bar{e}c]} P(3) \oplus P(2) \xrightarrow{\begin{bmatrix} cb \\ \bar{a} \end{bmatrix}} P(1).$$

We note that w also determines a homotopy string complex, whose unfolded diagram is

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & 1 & & 0 \\ \bullet & \xrightarrow{d} & \bullet & \xrightarrow{cb} & \bullet & \xleftarrow{a} & \bullet & \xleftarrow{ec} & \bullet \end{array},$$

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whose corresponding homotopy string complex is

$$P(4) \oplus P(4) \xrightarrow{\begin{bmatrix} d & 0 \\ 0 & ec \end{bmatrix}} P(3) \oplus P(2) \xrightarrow{\begin{bmatrix} cb \\ a \end{bmatrix}} P(1).$$

2.3. Infinite homotopy strings. The final class of indecomposable complex are those corresponding to infinite homotopy strings, which, as will become clear later, only occur when Λ has infinite global dimension. They arise from taking projective resolutions of certain string complexes. For details on how to take projective resolutions of complexes we refer the reader to [37, Ch. 6]. The original reference for the following is [8], but we shall employ the notation and terminology of [4]. First, we recall a definition from [9].

Definition 2.9. By a *direct (resp. inverse) antipath* we mean a homotopy string in which each homotopy letter is direct (resp. inverse) of length one, i.e. is a direct or inverse arrow in the quiver.

Suppose $\Lambda = \mathbf{k}Q/I$ is a gentle algebra of infinite global dimension. Then Q contains oriented cycles with ‘full’ relations. Let $\mathcal{C}(\Lambda)$ denote the set of arrows $a \in Q_1$ such that there is a cyclic path $a_n \cdots a_2 a_1$ in Q that is repetition-free and satisfies $a_{i+1} a_i \in I$ for $0 \leq i \leq n$, where we take $a_1 = a$ and $a_0 = a_n$.

Definition 2.10. Let $\sigma = \prod_{k=n}^1 (\sigma_k, i_k, j_k)$ be a homotopy string. Following [4, Def. 2.6] we say that σ is

- (1) *right resolvable* if (σ_1, i_1, j_1) is inverse, $j_1 \leq i_k, j_k$ for all $1 < k \leq n$, and there exists $a \in \mathcal{C}(\Lambda)$ such that $\sigma(a, j_1, j_1 - 1)$ is a homotopy string; in this case we say that σ is *right resolvable by a* ,
- (2) *primitive right resolvable* if there exists no inverse antipath $\prod_{k=r}^1 (\sigma_k, i_k, j_k)$ such that $\prod_{k=n}^{r+1} (\sigma_k, i_k, j_k)$ is right resolvable.

There are dual notions of *(primitive) left resolvable*. A homotopy string is *(primitive) two-sided resolvable* if it is both (primitive) right resolvable and (primitive) left resolvable.

Note that if σ is a (primitive) right resolvable homotopy string then σ^{-1} is a (primitive) left resolvable homotopy string. As such, a homotopy string that is not (primitive) two-sided resolvable but is (primitive) right resolvable or (primitive) left resolvable is called *(primitive) one-sided resolvable*.

Remark 2.11. The point of the second part of Definition 2.10 is that a right resolvable homotopy string admits a projective resolution of infinite length, but without the primitive assumption, we would have multiple finite and right resolvable homotopy strings giving rise to the same (right) infinite homotopy string.

We next explain how to form the corresponding infinite homotopy strings from a right resolvable homotopy string.

Definition 2.12. Suppose $\sigma = \prod_{k=n}^1 (\sigma_k, i_k, j_k)$ is a homotopy string that is right resolvable by $a_1 \in Q_1$ with $a_l \cdots a_1$ a repetition-free cyclic path in (Q, I) with full relations. The *right infinite homotopy string* defined by σ is

$$\sigma^\infty := \sigma(a_1, j_1, j_1 - 1)(a_2, j_1 - 1, j_1 - 2) \cdots (a_l, j_1 + 1 - l, j_1 - l)(a_1, j_1 - l, j_1 - l - 1) \cdots$$

The unfolded diagram of σ^∞ is

$$\bullet \xrightarrow{\sigma_n} \bullet \cdots \bullet \xrightarrow{\sigma_2} \bullet \xleftarrow{\sigma_1} \bullet \xleftarrow{a_1} \bullet \xleftarrow{a_2} \bullet \cdots \bullet \xleftarrow{a_l} \bullet \xleftarrow{a_1} \bullet \cdots$$

Left infinite homotopy strings and *two-sided infinite homotopy strings* are defined analogously. A left or right infinite homotopy string that is not two-sided infinite is called a *one-sided infinite homotopy string*. We denote by \mathbf{HSt}_1 the set of all one-sided infinite homotopy strings under \sim^{-1} ; cf. Definition 2.3. Similarly we denote the set of all two-sided infinite homotopy strings under \sim^{-1} by \mathbf{HSt}_2 .

string complexes arising from infinite homotopy strings. In the unfolded diagram notation we shall use \cdots to indicate that the homotopy string is infinite. The notation \sim can denote both a finite or an infinite homotopy substring.

Definition 3.3. The following unfolded diagrams define the *graph map right endpoint conditions* of [4, §3.2],

$$\begin{array}{ll}
\text{(RG1)} \quad \begin{array}{ccc} \bullet & \xrightarrow{\sigma_R} & \bullet \sim \alpha \\ \parallel & & \downarrow f_R \\ \bullet & \xrightarrow{\tau_R = \sigma_R f_R} & \bullet \sim \gamma \end{array} & \text{(RG2)} \quad \begin{array}{ccc} \bullet & \xleftarrow{\sigma_R = \bar{\omega} f_R} & \bullet \sim \alpha \\ \parallel & & \downarrow f_R \\ \bullet & \xleftarrow{\tau_R = \bar{\omega}} & \bullet \sim \gamma \end{array} \\
\text{(RG3)} \quad \begin{array}{ccc} \bullet & \xrightarrow{\sigma_R} & \bullet \sim \alpha \\ \parallel & & \downarrow f_R \\ \bullet & \xleftarrow{\tau_R} & \bullet \sim \gamma \end{array} & \text{(RG}\infty\text{)} \quad \begin{array}{ccccccc} \bullet & \xrightarrow{\sigma_L} & \bullet & \xrightarrow{\rho-1} & \bullet & \xrightarrow{\rho-2} & \bullet & \xrightarrow{\rho-3} & \bullet \cdots \\ \downarrow f_L & (*) & \parallel & & \parallel & & \parallel & & \\ \bullet & \xleftarrow{\tau_L} & \bullet & \xleftarrow{\rho-1} & \bullet & \xleftarrow{\rho-2} & \bullet & \xleftarrow{\rho-3} & \bullet \cdots \end{array}
\end{array}$$

where $(*)$ satisfies a graph map left endpoint condition, and where ω is a direct homotopy letter. The *graph map left endpoint conditions* are defined dually.

Definition 3.4. If, in Setup 3.2, one of the graph map right endpoint conditions holds and one of the graph map left endpoint conditions holds then the data in the unfolded diagrams (1) and (2) determines a map of complexes $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$, which is called a *graph map*.

The situation of (2) will additionally be called a *one-sided infinite graph map*. Note that when (LG ∞) and (RG ∞) both hold, we get a *two-sided infinite graph map*, which is necessarily the identity map $\text{id}^\bullet: P_\sigma^\bullet \rightarrow P_\sigma^\bullet$.

3.2. Single maps. Consider the following setup at the level of unfolded diagrams for Q_σ^\bullet and Q_τ^\bullet :

$$\begin{array}{ccc}
(3) & Q_\sigma^\bullet: & \begin{array}{ccccc} \sim \beta & \bullet & \xrightarrow{\sigma_L} & \bullet & \xrightarrow{\sigma_R} & \bullet \sim \alpha \\ & & & \downarrow f & & \\ Q_\tau^\bullet: & \begin{array}{ccccc} \sim \delta & \bullet & \xrightarrow{\tau_L} & \bullet & \xrightarrow{\tau_R} & \bullet \sim \gamma \end{array} \end{array}
\end{array}$$

where f is a nontrivial path in (Q, I) .

In the following, let $\mathbf{C} := \mathbf{C}^{b,-}(\text{proj}(\Lambda))$ be the category of right bounded complexes of finitely generated projective Λ -modules whose cohomology is bounded.

Definition 3.5. A map in $\text{Hom}_{\mathbf{C}}(Q_\sigma^\bullet, Q_\tau^\bullet)$ is a *single map* if it has only one nonzero component which occurs in an unfolded diagram as in (3) above and which satisfies the following conditions:

- (L1) The homotopy letter σ_L is either inverse or is direct and $\sigma_L f$ has a subpath in I .
- (L2) The homotopy letter τ_L is either direct or is inverse and $f \tau_L$ has a subpath in I .
- (R1) The homotopy letter σ_R is either direct or is inverse and $\sigma_R f$ has a subpath in I .
- (R2) The homotopy letter τ_R is either inverse or is direct and $f \tau_R$ has a subpath in I .

It turns out that not all single maps in \mathbf{C} actually give rise to maps in $\mathbf{K}^{b,-}(\text{proj}(\Lambda))$, since some may be null-homotopic. Furthermore, a single map may not be a unique representative of its homotopy class, even in the case that it is not null-homotopic. It is useful to highlight the following instances of single maps that are unique representatives of their homotopy classes, and in particular are not null-homotopic.

Definition 3.6. A single map $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$, with single component f , is called a *singleton single map* if its unfolded diagram is one of the following, up to inversion of one of the homotopy strings.

$$\begin{array}{ccc}
\text{(i)} & \begin{array}{ccc} \sim \beta & \bullet & \xrightarrow{\sigma_L} & \bullet \\ & & & \downarrow f \\ \sim \delta & \bullet & \xrightarrow{\tau_L} & \bullet \end{array} & \text{with } f \text{ not a subletter of } \sigma_L \text{ or } \tau_L;
\end{array}$$

- (ii) $\begin{array}{c} \beta \text{---} \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{\sigma_R = f f_R} \bullet \text{---} \alpha \\ \downarrow f \\ \delta \text{---} \bullet \xrightarrow{\tau_L} \bullet \end{array}$ with f not a subletter of τ_L and σ_L possibly zero;
- (iii) $\begin{array}{c} \beta \text{---} \bullet \xrightarrow{\sigma_L} \bullet \\ \downarrow f \\ \delta \text{---} \bullet \xrightarrow{\tau_L} \bullet \xleftarrow{\tau_R = f f_L} \bullet \text{---} \gamma \end{array}$ with f not a subletter of σ_L and τ_L possibly zero;
- (iv) $\begin{array}{c} \beta \text{---} \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{\sigma_R = f f_R} \bullet \text{---} \alpha \\ \downarrow f \\ \delta \text{---} \bullet \xrightarrow{\tau_L} \bullet \xleftarrow{\tau_R = f f_L} \bullet \text{---} \gamma \end{array}$ with σ_L or τ_L possibly zero.

3.3. Double maps. Now consider the following setup at the level of unfolded diagrams for Q_σ^\bullet and Q_τ^\bullet :

$$(4) \quad \begin{array}{c} Q_\sigma^\bullet : \quad \beta \text{---} \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{\sigma_C} \bullet \xrightarrow{\sigma_R} \bullet \text{---} \alpha \\ \quad \quad \quad \downarrow f_L \quad \quad \downarrow f_R \\ Q_\tau^\bullet : \quad \delta \text{---} \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{\tau_C} \bullet \xrightarrow{\tau_R} \bullet \text{---} \gamma \end{array}$$

where f_L and f_R are nontrivial paths in (Q, I) such that $f_L \tau_j = \sigma_i f_R$ has no subpath in I .

Definition 3.7. A map $f^\bullet \in \text{Hom}_{\mathbb{C}}(Q_\sigma^\bullet, Q_\tau^\bullet)$ is *double map* if it admits an unfolded diagram with components f_L and f_R as in (4) in which (L1) and (L2) of Definition 3.5 hold for f_L and (R1) and (R2) of Definition 3.5 hold for f_R .

As is the case with single maps, double maps in \mathbb{C} may not give rise to maps in $\mathbb{K}^{b,-}(\text{proj}(\Lambda))$. They may even occur in the same homotopy class as single maps. Again, it is useful to highlight the class of double maps which are unique representatives of their homotopy class, which means that they are therefore not null-homotopic.

Definition 3.8. A double map $f^\bullet : Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ as in diagram (4) above is a *singleton double map* if there exists a nontrivial path f' in (Q, I) such that $\sigma_C = f_L f'$ and $\tau_C = f' f_R$.

3.4. Quasi-graph maps. We finally turn to the definition of quasi-graph maps. Suppose we are in Setup 3.2. The next definition makes explicit the situations in which the squares (*) and (**) of the unfolded diagram (1) do not commute.

Definition 3.9. The following unfolded diagrams define the *quasi-graph map right endpoint conditions*,

$$(RQ1) \quad \begin{array}{c} \bullet \xrightarrow{\sigma_R = \tau_R \tau'_R} \bullet \text{---} \alpha \\ \parallel \\ \bullet \xrightarrow{\tau_R} \bullet \text{---} \gamma \end{array} \quad (RQ2) \quad \begin{array}{c} \bullet \xleftarrow{\sigma_R = \bar{\omega}} \bullet \text{---} \alpha \\ \parallel \\ \bullet \xleftarrow{\tau_R = \bar{\omega} \bar{\omega}'} \bullet \text{---} \gamma \end{array} \quad (RQ3) \quad \begin{array}{c} \bullet \xleftarrow{\sigma_R} \bullet \text{---} \alpha \\ \parallel \\ \bullet \xleftarrow{\tau_R} \bullet \text{---} \gamma \end{array}$$

where ω and ω' are direct homotopy letters, and all primed homotopy (sub)letters are non-trivial. In (RQ3) we also permit σ_R or τ_R (but not both) to be zero. The *quasi-graph map left endpoint conditions* are defined dually.

Definition 3.10. If, in Setup 3.2, one of the quasi-graph map right endpoint conditions holds and one of the quasi-graph map left endpoint conditions holds, then we say there is a *quasi-graph map* $Q_\sigma^\bullet \rightsquigarrow Q_\tau^\bullet$.

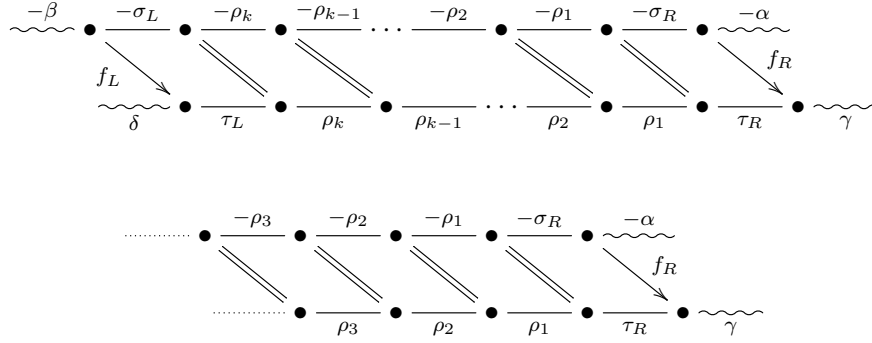
Remark 3.11. Note that a quasi-graph map $Q_\sigma^\bullet \rightsquigarrow Q_\tau^\bullet$ does **not** define a map. However, a quasi-graph map $Q_\sigma^\bullet \rightsquigarrow \Sigma^{-1} Q_\tau^\bullet$ determines a family of homotopy equivalent single and/or double maps $Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$; see [4, Prop. 4.8].

Remark 3.12. A graph map $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ determines a quasi-graph map $Q_\tau^\bullet \rightsquigarrow Q_\sigma^\bullet$ simply by reading the components of the graph maps that are isomorphisms in the opposite direction. This, therefore determines a homotopy class of single and double maps $Q_\tau^\bullet \rightarrow \Sigma Q_\sigma^\bullet$. In the case that $\sigma = \tau \in \text{HBa}$, this can be thought of as a concrete combinatorial manifestation of (Auslander-Reiten-)Serre duality.

4. MAPPING CONES OF GRAPH MAPS

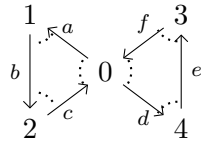
In this section we compute the mapping cones of graph maps. We first start by describing the unfolded diagram of the mapping cone of a graph map. We then tackle the problem in the case of a graph map between two string complexes and then deal with the cases that one of the indecomposable complexes is a band complex.

4.1. The unfolded diagram of the mapping cone of a graph map. Let $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ be a graph map. The unfolded diagram of $M_{f^\bullet}^\bullet$ is the following, depending on which case of Setup 3.2 we are in.

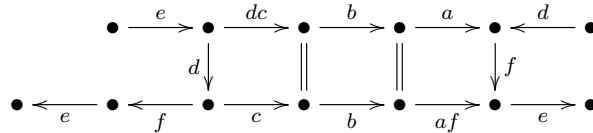


We illustrate this in an example.

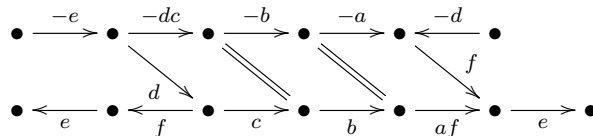
Example 4.1. Let Λ be the algebra given by the following quiver with relations.



Let $\sigma = edcbad$ and $\tau = \bar{e}\bar{f}\bar{c}\bar{b}\bar{a}\bar{f}\bar{e}$ and consider the graph map $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ whose unfolded diagram is shown below.



The unfolded diagram of $M_{f^\bullet}^\bullet$ is thus



Encoding this information back into complexes gives rise to the following standard triangle in $K^{b,-}(\text{proj}(\Lambda))$.

$$\begin{array}{c}
P_\sigma^\bullet: \\
f^\bullet \downarrow \\
P_\tau^\bullet: \\
g^\bullet \downarrow \\
M_{f^\bullet}^\bullet: \\
h^\bullet \downarrow \\
\Sigma P_\sigma^\bullet
\end{array}
\begin{array}{ccccccc}
P(3) & \xrightarrow{[e]} & P(4) & \xrightarrow{[dc]} & P(2) & \xrightarrow{[b \ 0]} & P(1) \oplus P(4) \xrightarrow{[d]} P(0) \\
& & \downarrow [d] & & \downarrow [1 \ 0] & & \downarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
& & P(0) & \xrightarrow{[c \ f]} & P(2) \oplus P(3) & \xrightarrow{\begin{bmatrix} b & 0 \\ 0 & e \end{bmatrix}} & P(1) \oplus P(4) \xrightarrow{\begin{bmatrix} af \\ 0 \end{bmatrix}} P(3) \xrightarrow{[e]} P(4) \\
& & \downarrow & & \downarrow & & \downarrow \\
P(3) & \xrightarrow{[-e]} & P(4) & \xrightarrow{[-dc]} & P(2) \oplus P(0) & \xrightarrow{\begin{bmatrix} -b & 0 & 1 & 0 \\ 0 & 0 & c & f \end{bmatrix}} & P(1) \oplus P(4) \oplus P(2) \oplus P(3) \xrightarrow{\begin{bmatrix} -a & 1 & 0 \\ -d & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & e \end{bmatrix}} P(0) \oplus P(1) \oplus P(4) \xrightarrow{\begin{bmatrix} f \\ af \\ 0 \end{bmatrix}} P(3) \xrightarrow{[e]} P(4) \\
& & \downarrow & & \downarrow & & \downarrow \\
P(3) & \xrightarrow{[-e]} & P(4) & \xrightarrow{[-dc]} & P(2) & \xrightarrow{[-b \ 0]} & P(1) \oplus P(4) \xrightarrow{[-a]} P(0)
\end{array}$$

where g^\bullet and h^\bullet are the obvious degreewise split maps.

4.2. Mapping cones of graph maps between string complexes. Let σ and τ be homotopy strings and $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ be a graph map between the corresponding string complex. In this case the generic situation is for $M_{f^\bullet}^\bullet$ to have two indecomposable summands, which are again string complexes and whose strings can be read off from σ , τ and f^\bullet in a natural way. We start with a technical definition which ensures that the unfolded diagram of the graph map is properly oriented for the mapping cone calculus in the case that f^\bullet is a graph map supported in precisely one degree.

Definition 4.2. Let σ and τ be homotopy strings and suppose $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ is a graph map concentrated in precisely one degree, i.e. the maximal common homotopy substring ρ is trivial and satisfies the graph map endpoint conditions (RG3) and (LG3):

$$\begin{array}{ccccc}
\beta & & \xleftarrow{\sigma_L} & \bullet & \xrightarrow{\sigma_R} & \bullet & & \alpha \\
& & & \parallel & & & & \\
& & & \bullet & & \bullet & & \\
& & \xrightarrow{\tau_L} & \bullet & \xleftarrow{\tau_R} & \bullet & & \gamma
\end{array}$$

We say that the homotopy strings σ and τ are *compatibly oriented* if $\sigma_L \tau_L \neq 0$ and $\sigma_R \tau_R \neq 0$.

We observe that if a graph map is supported in more than one degree, either by having precisely one isomorphism and satisfying endpoint conditions so that one of f_L or f_R is nonzero, or by having isomorphisms in at least two cohomological degrees (making ρ of length at least 1), then the homotopy strings and graph map between them is forced to be ‘compatibly oriented’.

Note that in the following if a homotopy string $\sigma = \emptyset$ then the corresponding string complex $P_\sigma^\bullet \cong 0^\bullet$.

Theorem 4.3. Let σ and τ be (possibly infinite) homotopy strings and suppose $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ is a graph map. Suppose that σ and τ are compatibly oriented. Then we have the following cases.

- (1) If $\sigma = \beta \sigma_L \rho \sigma_R \alpha$ and $\tau = \delta \tau_L \rho \tau_R \gamma$ then the mapping cone $M_{f^\bullet}^\bullet$ is isomorphic in $K^{b,-}(\text{proj}(\Lambda))$ to the direct sum of $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where

$$c_1 = \begin{cases} \bar{\gamma} \bar{\tau}_R \sigma_R \alpha & \text{if } \sigma_R \neq 0 \text{ and } \tau_R \neq 0; \\ \bar{\gamma} & \text{if } \sigma_R = 0; \\ \alpha & \text{if } \tau_R = 0; \\ \emptyset & \text{if } \sigma_R = 0 \text{ and } \tau_R = 0, \end{cases} \quad \text{and } c_2 = \begin{cases} \beta \sigma_L \bar{\tau}_L \bar{\delta} & \text{if } \sigma_L \neq 0 \text{ and } \tau_L \neq 0; \\ \bar{\delta} & \text{if } \sigma_L = 0; \\ \beta & \text{if } \tau_L = 0; \\ \emptyset & \text{if } \sigma_L = 0 \text{ and } \tau_L = 0, \end{cases}$$

- (2) If $\sigma = \rho \sigma_R \alpha$ and $\tau = \rho \tau_R \gamma$ then the mapping cone $M_{f^\bullet}^\bullet$ is isomorphic in $K^{b,-}(\text{proj}(\Lambda))$ to P_c^\bullet , where

$$c = \begin{cases} \bar{\gamma} \bar{\tau}_R \sigma_R \alpha & \text{if } \sigma_R \neq 0 \text{ and } \tau_R \neq 0; \\ \bar{\gamma} & \text{if } \sigma_R = 0; \\ \alpha & \text{if } \tau_R = 0; \\ \emptyset & \text{if } \sigma_R = 0 \text{ and } \tau_R = 0. \end{cases}$$

Remark 4.4. If in Theorem 4.3(1) the graph map right endpoint conditions (RG1) or (RG2) occur then in either case $\bar{\gamma}\bar{\tau}_R\sigma_R\alpha = \bar{\gamma}\bar{f}_R\alpha$.

Proof. We will prove statement (1) in detail. The proof of statement (2) is similar and we leave its proof to the reader.

We start by showing that $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$ is a direct summand of $M_{f^\bullet}^\bullet$. We do this by defining a split monomorphism $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and a retract $p_1^\bullet: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet$ by looking at the homotopy substrings on the right and using the right endpoint conditions. The maps $i_2^\bullet: P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and $p_2^\bullet: M_{f^\bullet}^\bullet \rightarrow P_{c_2}^\bullet$ are defined via a similar analysis of the homotopy substrings on the left and the corresponding left endpoint conditions. Putting these together will define a split monomorphism $i^\bullet = \begin{bmatrix} i_1^\bullet \\ i_2^\bullet \end{bmatrix}: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ with retract $p^\bullet = [p_1^\bullet \ p_2^\bullet]: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet \oplus P_{c_2}^\bullet$. In each case we will have $i_j^\bullet \circ p_j^\bullet = \text{id}_{P_{c_j}^\bullet}$ for $j = 1, 2$ (recalling our convention of composing maps from left to right).

If the endpoint condition (RG ∞) holds then $M_{f^\bullet}^\bullet$ has only one indecomposable summand, which arises from the dual analysis involving the left endpoint conditions. Indeed, the part of the complex corresponding to the maximal common homotopy substring occurring in (RG ∞) will turn out to be null-homotopic using the arguments in the second step below. Thus, we need only consider the right endpoint conditions (RG1), (RG2) and (RG3). Note further that if both (RG ∞) and (LG ∞) hold then the corresponding graph map is an isomorphism and therefore has trivial mapping cone.

In a case by case analysis of the (right) endpoint conditions we will now defined the maps $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and $p_1^\bullet: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet$.

Case (RG1): In this case, we have $\overline{\tau}_R\sigma_R = \overline{f}_R$ so that $c_1 = \bar{\gamma}\bar{f}_R\alpha$. The map i_1^\bullet is encoded in the following unfolded diagram, where we have only sketched the relevant part of the diagram.

$$\begin{array}{ccccccc}
 & P(x_R) & \xrightarrow{\alpha_R} & P(u_{k-1}) & \xrightarrow{\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) & \xrightarrow{\alpha_1} & P(u_0) \\
 & \downarrow +1 & \searrow f_R & & & & & & & \\
 & P(y_R) & \xrightarrow{\gamma_r} & P(v_{r-1}) & \xrightarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) & \xrightarrow{\gamma_1} & P(v_0) \\
 P(x_0) & \xrightarrow{-\sigma_R} & P(x_R) & \xrightarrow{-\alpha_k} & P(u_{k-1}) & \xrightarrow{-\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) & \xrightarrow{-\alpha_1} & P(u_0) \\
 & \searrow +1 & \searrow f_R & & & & & & & & \\
 & P(x_0) & \xrightarrow{\tau_R} & P(y_R) & \xrightarrow{\gamma_r} & P(v_{r-1}) & \xrightarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) & \xrightarrow{\gamma_1} & P(v_0)
 \end{array}$$

(Curved arrows with labels ± 1 connect the rows: $P(x_R) \xrightarrow{+1} P(y_R)$, $P(y_R) \xrightarrow{-1} P(u_{k-1})$, $P(u_{k-1}) \xrightarrow{+1} P(v_{r-1})$, $P(v_{r-1}) \xrightarrow{\pm 1} P(v_{r-2})$, $P(v_{r-2}) \xrightarrow{\pm 1} P(v_1)$, $P(v_1) \xrightarrow{+1} P(u_0)$, $P(u_0) \xrightarrow{+1} P(v_0)$, $P(x_0) \xrightarrow{+1} P(x_R)$, $P(x_R) \xrightarrow{+1} P(u_{k-1})$, $P(u_{k-1}) \xrightarrow{+1} P(v_{r-1})$, $P(v_{r-1}) \xrightarrow{+1} P(v_{r-2})$, $P(v_{r-2}) \xrightarrow{+1} P(v_1)$, $P(v_1) \xrightarrow{+1} P(u_0)$, $P(u_0) \xrightarrow{+1} P(v_0)$.)

Its retract $p^\bullet: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet$ is encoded in the following unfolded diagram:

$$\begin{array}{ccccccc}
 P(x_0) & \xrightarrow{-\sigma_R} & P(x_R) & \xrightarrow{-\alpha_R} & P(u_{k-1}) & \xrightarrow{-\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) & \xrightarrow{-\alpha_1} & P(u_0) \\
 & \searrow +1 & \searrow f_R & & & & & & & & \\
 & P(x_0) & \xrightarrow{\tau_R} & P(y_R) & \xrightarrow{\gamma_r} & P(v_{r-1}) & \xrightarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) & \xrightarrow{\gamma_1} & P(v_0) \\
 & \searrow \sigma_R & \searrow f_R & & & & & & & & \\
 & P(x_R) & \xrightarrow{\alpha_k} & P(u_{k-1}) & \xrightarrow{\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) & \xrightarrow{\alpha_1} & P(u_0) \\
 & & \searrow f_R & & & & & & & & \\
 & & P(y_R) & \xrightarrow{\gamma_r} & P(v_{r-1}) & \xrightarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) & \xrightarrow{\gamma_1} & P(v_0)
 \end{array}$$

(Curved arrows with labels ± 1 connect the rows: $P(x_R) \xrightarrow{+1} P(y_R)$, $P(y_R) \xrightarrow{-1} P(u_{k-1})$, $P(u_{k-1}) \xrightarrow{+1} P(v_{r-1})$, $P(v_{r-1}) \xrightarrow{\pm 1} P(v_{r-2})$, $P(v_{r-2}) \xrightarrow{\pm 1} P(v_1)$, $P(v_1) \xrightarrow{+1} P(u_0)$, $P(u_0) \xrightarrow{+1} P(v_0)$, $P(x_0) \xrightarrow{+1} P(x_R)$, $P(x_R) \xrightarrow{+1} P(u_{k-1})$, $P(u_{k-1}) \xrightarrow{+1} P(v_{r-1})$, $P(v_{r-1}) \xrightarrow{+1} P(v_{r-2})$, $P(v_{r-2}) \xrightarrow{+1} P(v_1)$, $P(v_1) \xrightarrow{+1} P(u_0)$, $P(u_0) \xrightarrow{+1} P(v_0)$.)

The unfolded diagram for i_1^\bullet is clearly commutative and therefore gives rise to a well-defined morphism $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$. We need to check the commutativity of the diagram defining p_1^\bullet .

We only need to check the commutativity of the diagram involving $P(x_0)$, $P(x_R)$ and $P(y_R)$; it is clear that the rest of the diagram commutes. At the level of complexes, considering only

the isomorphisms defined (in blue) in the unfolded diagram, we have

$$\begin{array}{ccccc}
 P(x_0) & \xrightarrow{[-\sigma_R + 1]} & P(x_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} f_R \\ \tau_R \end{bmatrix}} & P(y_R) \\
 & & \downarrow \begin{bmatrix} +1 \\ - \end{bmatrix} & & \downarrow [0 + 1] \\
 & & P(x_R) & \xrightarrow{[\alpha_k \ f_R]} & P(u_{k-1}) \oplus P(y_R)
 \end{array}$$

One can clearly see that adding the correction term σ_R for $-$ in the diagram above makes the left-hand square commute. For the right-hand square recall that $\sigma_R f_R = \tau_R$ by (RG1) and note that if α_k is inverse or zero then it does not occur in the right-hand diagram at all and therefore there is nothing to show. If α_k is direct, then $\sigma_R \alpha_k = 0$ by the definition of graph map, giving the required commutativity. This is more clearly seen at the level of unfolded diagrams shown below, where the correction terms are indicated in red, and morphisms factoring through zero are in green.

Case (RG2): Here we have $\overline{\tau_R} \sigma_R = \overline{w} \overline{w} f_R = \overline{f_R}$ so that $c_1 = \overline{\gamma} \overline{f_R} \alpha$. The split monomorphism $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and its retract $p_1: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet$ are encoded in the following unfolded diagrams, respectively.

As above, the unfolded diagram for p_1^\bullet is clearly commutative and therefore gives rise to a well-defined map $p_1^\bullet: M_{f^\bullet}^\bullet \rightarrow P_{c_1}^\bullet$. We check commutativity for i_1^\bullet . As above, the only places where the commutativity of i_1^\bullet is not immediately clear are those involving $P(x_0)$, $P(x_R)$ and

$P(y_R)$. At the level of complexes we have,

$$\begin{array}{ccc} P(x_R) & \xrightarrow{[f_R]} & P(y_R) \\ \downarrow [1] & & \downarrow [1 -] \\ P(x_R) & \xrightarrow{[f_R - \sigma_R]} & P(y_R) \oplus P(x_0) \xrightarrow{[\tau_R]} P(x_0) \end{array}$$

Analysing as in the case (RG1) above, one can see that substituting $-\omega$ for $-$ the diagram commutes.

Case (RG3): First assume that $\sigma_R \neq 0$ and $\tau_R \neq 0$. As above, the split monomorphism $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}$ and its retract $p_1^\bullet: M_{f^\bullet} \rightarrow P_{c_1}^\bullet$ are encoded in the following unfolded diagrams, respectively.

$$\begin{array}{ccccccc} P(x_R) & \xleftarrow{\alpha_k} & P(u_{k-1}) & \xleftarrow{\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) \xleftarrow{\alpha_1} P(u_0) \\ \downarrow +1 & \nwarrow \tau_R \sigma_R & \downarrow -1 & \nwarrow +1 & \downarrow +1 & \cdots & \downarrow \pm 1 \\ P(y_R) & \xleftarrow{\gamma_r} & P(v_{r-1}) & \xleftarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) \xleftarrow{\gamma_1} P(v_0) \\ \uparrow -\tau_R & \nearrow +1 & \downarrow +1 & \nearrow +1 & \downarrow +1 & \cdots & \downarrow +1 \\ P(x_0) & \xleftarrow{-\sigma_R} & P(x_R) & \xleftarrow{-\alpha_k} & P(u_{k-1}) & \xleftarrow{-\alpha_{k-1}} & P(u_{k-2}) \cdots P(u_1) \xleftarrow{-\alpha_1} P(u_0) \\ \downarrow +1 & & \downarrow \tau_R & & \downarrow \gamma_r & & \downarrow \gamma_{r-1} \\ P(x_0) & \xleftarrow{\tau_R} & P(y_R) & \xleftarrow{\gamma_r} & P(v_{r-1}) & \xleftarrow{\gamma_{r-1}} & P(v_{r-2}) \cdots P(v_1) \xleftarrow{\gamma_1} P(v_0) \end{array}$$

$$\begin{array}{ccccccc} P(x_0) & \xleftarrow{-\sigma_R} & P(x_R) & \xleftarrow{-\alpha_k} & P(u_{k-1}) & \xleftarrow{-\alpha_{k-1}} & P(u_{k-2}) \cdots P(u_1) \xleftarrow{-\alpha_1} P(u_0) \\ \downarrow +1 & \nwarrow \tau_R & \downarrow +1 & \nwarrow +1 & \downarrow +1 & \cdots & \downarrow \pm 1 \\ P(x_0) & \xleftarrow{\tau_R} & P(y_R) & \xleftarrow{\gamma_r} & P(v_{r-1}) & \xleftarrow{\gamma_{r-1}} & P(v_{r-2}) \cdots P(v_1) \xleftarrow{\gamma_1} P(v_0) \\ \uparrow +1 & \nearrow \sigma_R & \downarrow +1 & \nearrow +1 & \downarrow +1 & \cdots & \downarrow +1 \\ P(x_R) & \xleftarrow{\alpha_k} & P(u_{k-1}) & \xleftarrow{\alpha_{k-1}} & P(u_{k-2}) & \cdots & P(u_1) \xleftarrow{\alpha_1} P(u_0) \\ \downarrow \tau_R \sigma_R & & \downarrow \gamma_r & & \downarrow \gamma_{r-1} & & \downarrow \gamma_1 \\ P(y_R) & \xleftarrow{\gamma_r} & P(v_{r-1}) & \xleftarrow{\gamma_{r-1}} & P(v_{r-2}) & \cdots & P(v_1) \xleftarrow{\gamma_1} P(v_0) \end{array}$$

We check the commutativity of the diagram for i_1^\bullet . Like in the cases above, it is clearly commutative other than at $P(x_0)$, $P(x_R)$ and $P(y_R)$. At the level of complexes, the relevant part of the diagram is

$$\begin{array}{ccc} P(y_R) & \xrightarrow{[\tau_R \sigma_R]} & P(x_R) \\ \downarrow [+1 -] & & \downarrow [+1 0] \\ P(y_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} 0 & \tau_R \\ -\sigma_R & 1 \end{bmatrix}} & P(x_R) \oplus P(x_0) \end{array}$$

where substituting $-\tau_R$ for $-$ gives the required commutativity.

We now check commutativity for p_1^\bullet . As usual, we only need to check at $P(x_0)$, $P(x_R)$ and $P(y_R)$; the rest is clear. Firstly, consider the relevant part at the level of complexes.

$$\begin{array}{ccc} P(y_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} 0 & \tau_R \\ -\sigma_R & 1 \end{bmatrix}} & P(x_R) \oplus P(x_0) \\ \downarrow \begin{bmatrix} +1 \\ 0 \end{bmatrix} & & \downarrow \begin{bmatrix} +1 \\ - \end{bmatrix} \\ P(y_R) & \xrightarrow{[\tau_R \sigma_R]} & P(x_R) \end{array}$$

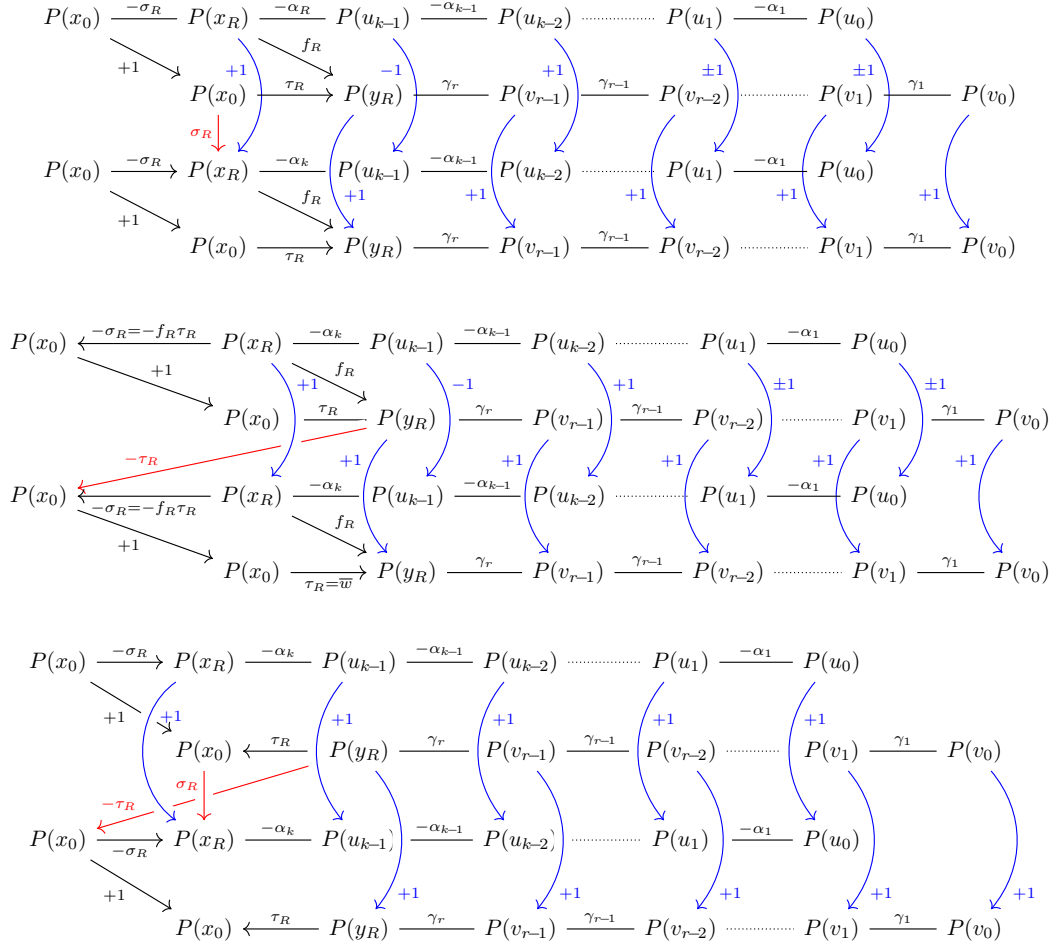


FIGURE 1. The composition $p_1^\bullet \circ i_1^\bullet$ on the level of the unfolded diagrams, case (RG1) top, (RG2) middle, and (RG3) bottom.

Substituting σ_R for $-$ gives the desired commutativity.

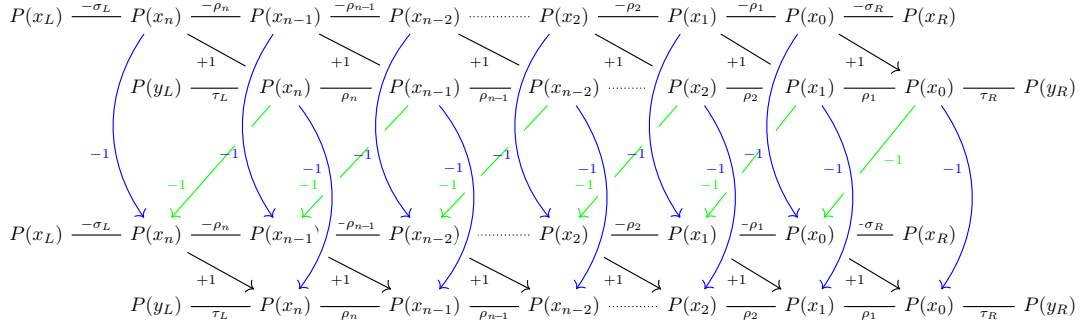
Note that if one of $\sigma_R = 0$ or $\tau_R = 0$, but not both, then the definition of the maps and the argument above goes through with the obvious modifications. If both $\sigma_R = 0$ and $\tau_R = 0$, then there are no maps i_1^\bullet or p_1^\bullet to define since c_1 is necessarily the empty string; indeed, we fall into case (2) of the statement of Theorem 4.3 in which ρ is finite.

We have now shown that $P_{c_1}^\bullet$ is a direct summand of M_{f^\bullet} by observing that $i_1^\bullet \circ p_1^\bullet = \text{id}_{P_{c_1}^\bullet}$. Similar arguments show that $P_{c_2}^\bullet$ is a direct summand of M_{f^\bullet} , whence $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$ is a direct summand of M_{f^\bullet} .

For the converse direction, i.e. showing that $M_{f^\bullet} \cong P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, we first note that $p^\bullet \circ i^\bullet \neq \text{id}_{M_{f^\bullet}}$; see Figure 1. However, we shall show that $p^\bullet \circ i^\bullet$ is homotopic to $\text{id}_{M_{f^\bullet}}$. In particular, we shall show $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}} \simeq 0$. In Figure 1 we draw the unfolded diagrams of the composites $p_1^\bullet \circ i_1^\bullet$ for each of the three right-sided graph map endpoint cases; the unfolded diagrams of the composites $p_2^\bullet \circ i_2^\bullet$ and hence $p^\bullet \circ i^\bullet$ can be drawn similarly.

Since $p^\bullet \circ i^\bullet$ is the identity map on the components of M_{f^\bullet} determined by the homotopy substrings α, β, γ and δ , without loss of generality we may assume that each of these substrings does not exist. In the next diagram we sketch the unfolded diagram of the map $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}}$ omitting some correction terms, which must be treated in a case by case analysis. Note

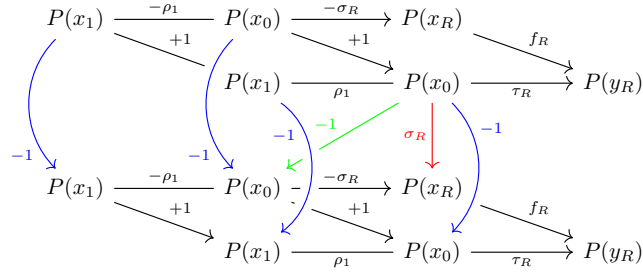
that, at the level of complexes, the components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ described in the (partial) unfolded diagram are entries lying on the diagonals of the matrices giving $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ in each cohomological degree.



Definition of the homotopy: Suppose that $P(x_i)$ sits in cohomological degree d_i in both P_σ^\bullet and P_τ^\bullet . Thus for $0 \leq i \leq n$ we have one copy of $P(x_i)$ in degree d_i and one copy of $P(x_i)$ in degree $d_i - 1$. Moreover, there is a component of the differential given by the identity map of $P(x_i)$ as a degree 1 map from degree $d_i - 1$ to degree d_i . To construct a (null)homotopy $h = \{h_i\}_{i=0}^n$, we define a degree -1 map as follows. For each $0 \leq i \leq n$, define a map by setting $h_i: P(x_i) \xrightarrow{-1} P(x_i)$ from degree d_i to $d_i - 1$. All other components of h are defined to be zero.

We now inductively show that h does indeed define a homotopy. We start at the beginning, i.e. right end, of the homotopy string. The base step of the induction requires us to treat each case of the right graph map endpoint conditions separately.

Base case for (RG1): Locally, the unfolded diagram of the map $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ is illustrated in the diagram below, where the components of h are indicated in green and the components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ are indicated in blue and red.



We show how the zeroth component of h , i.e. $h_0: P(x_0) \rightarrow P(x_0)$ constructs the components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ with source $P(x_0)$ in cohomological degrees d_0 and $d_0 - 1$. At the level of complexes we have, in the case that ρ_1 is direct,

$$\begin{array}{ccccccc}
 P(x_1) & \xrightarrow{[-\rho_1 \ 1]} & P(x_0) \oplus P(x_1) & \xrightarrow{\begin{bmatrix} -\sigma_R + 1 \\ 0 & \rho_1 \end{bmatrix}} & P(x_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} f_R \\ \tau_R \end{bmatrix}} & P(y_R) \\
 & & \downarrow B & \swarrow \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \downarrow A & \swarrow 0 & \downarrow [0] \\
 P(x_1) & \xrightarrow{[-\rho_1 \ 1]} & P(x_0) \oplus P(x_1) & \xrightarrow{\begin{bmatrix} -\sigma_R + 1 \\ 0 & \rho_1 \end{bmatrix}} & P(x_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} f_R \\ \tau_R \end{bmatrix}} & P(y_R)
 \end{array}$$

One sees immediately that $A = \begin{bmatrix} 0 & 0 \\ \sigma_R & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ -\rho_1 & 0 \end{bmatrix}$, giving rise to precisely the components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ with source in $P(x_0)$ in degrees d_0 and $d_0 - 1$ plus an unwanted off-diagonal term $-\rho_1: P(x_1) \rightarrow P(x_0)$ in degree $d_0 - 1 = d_1$, which will be removed inductively.

In the case that ρ_1 is inverse,

$$\begin{array}{ccccc}
P(x_0) & \xrightarrow{[-\rho_1 \ -\sigma_R \ +1]} & P(x_1) \oplus P(x_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} +1 & 0 \\ 0 & f_R \\ \rho_1 & \tau_R \end{bmatrix}} & P(x_1) \oplus P(y_R) \\
\downarrow B & & \downarrow A & \swarrow 0 & \\
P(x_0) & \xrightarrow{[-\rho_1 \ -\sigma_R \ +1]} & P(x_1) \oplus P(x_R) \oplus P(x_0) & \xrightarrow{\begin{bmatrix} +1 & 0 \\ 0 & f_R \\ \rho_1 & \tau_R \end{bmatrix}} & P(x_1) \oplus P(y_R)
\end{array}$$

We have $A = \begin{bmatrix} 0 & 0 & 0 \\ \rho_1 & \sigma_R & -1 \end{bmatrix}$ and $B = [-1]$, again giving precisely the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with source in $P(x_0)$ in degrees d_0 and $d_0 - 1$ plus an unwanted off-diagonal term $-\rho_1 : P(x_0) \rightarrow P(x_1)$ in degree $d_0 = d_1 - 1$, which again will be removed inductively.

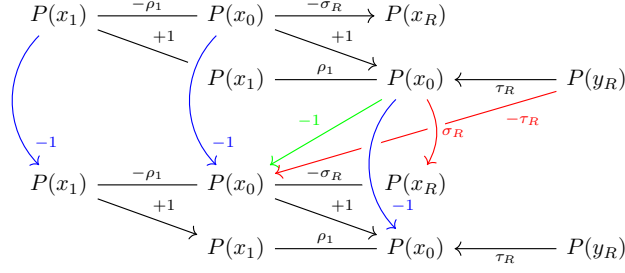
Base case for (RG2): Locally, the unfolded diagram of the map $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ is illustrated in the next diagram, where the components of h are indicated in green and the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ are indicated in blue and red.

As above, we show how h_0 constructs the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with source in $P(x_0)$ and $P(y_R)$ via the following diagram at the level of complexes. We only consider the case that ρ_1 is direct; the case that ρ_1 is inverse, is analogous and is left to the reader.

$$\begin{array}{ccccc}
P(x_1) \oplus P(x_R) & \xrightarrow{\begin{bmatrix} -\rho_1 & +1 & 0 \\ -\sigma_R & 0 & f_R \end{bmatrix}} & P(x_0) \oplus P(x_1) \oplus P(y_R) & \xrightarrow{\begin{bmatrix} +1 \\ \rho_1 \\ \tau_R \end{bmatrix}} & P(x_0) \\
\downarrow B & & \downarrow A & \swarrow [-1 \ 0 \ 0] & \\
P(x_1) \oplus P(x_R) & \xrightarrow{\begin{bmatrix} -\rho_1 & +1 & 0 \\ -\sigma_R & 0 & f_R \end{bmatrix}} & P(x_0) \oplus P(x_1) \oplus P(y_R) & \xrightarrow{\begin{bmatrix} +1 \\ \rho_1 \\ \tau_R \end{bmatrix}} & P(x_0)
\end{array}$$

We obtain $A = [-1]$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ -\rho_1 & 0 & 0 \\ -\tau_R & 0 & 0 \end{bmatrix}$, which again gives the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with source in $P(x_0)$ and $P(y_R)$ in degrees d_0 and $d_0 - 1$ plus an unwanted off-diagonal term $-\rho_1 : P(x_1) \rightarrow P(x_1)$ in degree $d_0 - 1 = d_1$, and we remove this in the induction step.

Base case for (RG3): As above, we assume that both $\sigma_R \neq 0$ and $\tau_R \neq 0$. In the case that one, but not both, is zero, the same argument will hold with the obvious modifications. Locally, the unfolded diagram of the map $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ is illustrated in the next diagram, where again the components of h are indicated in green and the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ are indicated in blue and red.



Again, re-encoding the information in the above figure shows how h_0 constructs the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with sources in $P(x_0)$ and $P(y_R)$ in cohomological degrees d_0 and $d_0 - 1$. As above, we only consider the case that ρ_1 is direct; ρ_1 inverse is similar.

$$\begin{array}{ccc}
 P(x_1) & \xrightarrow{[-\rho_1 \ 1 \ 0]} & P(x_0) \oplus P(x_1) \oplus P(y_R) \xrightarrow{\begin{bmatrix} -\sigma_R & +1 \\ 0 & \rho_1 \\ 0 & \tau_R \end{bmatrix}} P(x_R) \oplus P(x_0) \\
 & & \downarrow B \quad \swarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \downarrow A \\
 P(x_1) & \xrightarrow{[-\rho_1 \ 1 \ 0]} & P(x_0) \oplus P(x_1) \oplus P(y_R) \xrightarrow{\begin{bmatrix} -\sigma_R & +1 \\ 0 & \rho_1 \\ 0 & \tau_R \end{bmatrix}} P(x_R) \oplus P(x_0)
 \end{array}$$

We get $A = \begin{bmatrix} 0 & 0 \\ \sigma_R & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ -\rho_1 & 0 & 0 \\ -\tau_R & 0 & 0 \end{bmatrix}$, which again gives the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with source in $P(x_0)$ and $P(y_R)$ in degrees d_0 and $d_0 - 1$ plus an unwanted off-diagonal term $-\rho_1: P(x_1) \rightarrow P(x_1)$ in degree $d_0 - 1 = d_1$, removed in the induction.

Induction step: Suppose $0 \leq i < n$. By the base cases for (RG1), (RG2) and (RG3) the h_0 component of the candidate homotopy h constructs the components with source $P(x_0)$ and $P(y_R)$ in degrees d_0 and $d_0 - 1$. In addition, if ρ_1 is direct we acquire an unwanted ‘off-diagonal’ component $-\rho_1: P(x_1) \rightarrow P(x_0)$ in degree $d_1 = d_0 - 1$, and if ρ_1 is inverse, we have $-\rho_1: P(x_0) \rightarrow P(x_1)$ in degree $d_0 = d_1 - 1$.

Assume, by induction, we have constructed components $P(x_i) \xrightarrow{-1} P(x_i)$ in degrees d_i and $d_i - 1$ together with an unwanted off-diagonal component $-\rho_{i+1}: P(x_{i+1}) \rightarrow P(x_i)$ in degree $d_{i+1} = d_i - 1$ in the case ρ_{i+1} is direct, and $-\rho_{i+1}: P(x_i) \rightarrow P(x_{i+1})$ in degree $d_i = d_{i+1} - 1$ if ρ_{i+1} is inverse. We show now how the component h_{i+1} of the candidate homotopy h constructs the components of $p^\bullet \circ i^\bullet - \text{id}_{M_f^\bullet}$ with source in $P(x_{i+1})$ in degrees d_{i+1} and $d_{i+1} - 1$ while at the same time removing the unwanted component corresponding to $-\rho_{i+1}$ but adding a further unwanted component corresponding to $-\rho_{i+2}$. We assume that ρ_{i+1} is direct; the case that ρ_{i+1} is inverse is similar.

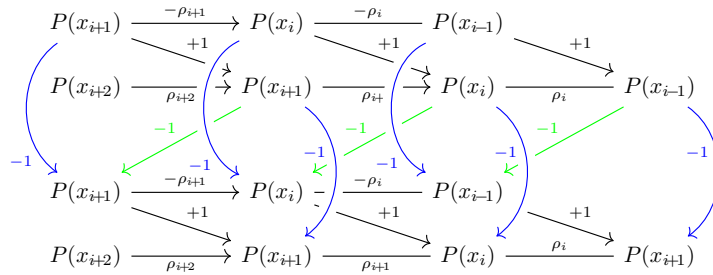


FIGURE 2. Unfolded diagram indicating the construction of the components $-1: P(x_{i+1}) \rightarrow P(x_{i+1})$ in degrees d_{i+1} and $d_{i+1} - 1$, the cancellation of the unwanted off-diagonal entry $-\rho_{i+1}$ and the creation of a new unwanted off-diagonal entry $-\rho_{i+2}$ in the induction step.

The setup at the level of unfolded diagrams is illustrated in Figure 2 above. At the level of complexes we have

$$\begin{array}{ccccc}
P(x_{i+1}) \oplus P(x_{i+2}) & \xrightarrow{\begin{bmatrix} -\rho_{i+1} & +1 \\ 0 & \rho_{i+2} \end{bmatrix}} & P(x_i) \oplus P(x_{i+1}) & \xrightarrow{\begin{bmatrix} +1 \\ \rho_{i+1} \end{bmatrix}} & P(x_i), \\
\downarrow A & \swarrow \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \downarrow B \begin{bmatrix} -1 & 0 \\ -\rho_{i+1} & 0 \end{bmatrix} & \swarrow \begin{bmatrix} -1 & 0 \end{bmatrix} = h_i & \downarrow [-1] \\
P(x_{i+1}) \oplus P(x_{i+2}) & \xrightarrow{\begin{bmatrix} -\rho_{i+1} & +1 \\ 0 & \rho_{i+2} \end{bmatrix}} & P(x_i) \oplus P(x_{i+1}) & \xrightarrow{\begin{bmatrix} +1 \\ \rho_{i+1} \end{bmatrix}} & P(x_i)
\end{array}$$

where the matrices labelling the vertical arrows are those constructed in the previous steps of the induction, in particular, including the unwanted ‘off-diagonal’ entry $-\rho_{i+1}$. We now see that

$$B = \begin{bmatrix} -1 & 0 \\ -\rho_{i+1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \rho_{i+1} & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -1 & 0 \\ -\rho_{i+2} & 0 \end{bmatrix},$$

i.e. in B we have removed the unwanted ‘off-diagonal’ entry $-\rho_{i+1}$ and are just left with components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ and in A we have added a component $P(x_{i+1}) \xrightarrow{-1} P(x_{i+1})$ in degree $d_{i+1} - 1$ but also the unwanted component $-\rho_{i+2}: P(x_{i+2}) \rightarrow P(x_{i+1})$ in degree $d_{i+1} - 1 = d_{i+2}$.

Termination step: Finally, by induction, we will have obtained all components of $p^\bullet \circ i^\bullet - \text{id}_{M_{f^\bullet}^\bullet}$ with sources in $P(x_i)$ for $0 \leq i < n$ and $P(y_R)$ together with an unwanted off-diagonal entry $-\rho_n: P(x_n) \rightarrow P(x_{n-1})$ in degree $d_n = d_{n-1} - 1$ when ρ_n is direct and $-\rho: P(x_{n-1}) \rightarrow P(x_n)$ in degree $d_{n-1} = d_n - 1$ when ρ_n is inverse. To complete the induction, one performs a case analysis for each of the left graph map endpoint conditions (LG1), (LG2) and (LG3) analogous to that for (RG1), (RG2) and (RG3) which began the induction, showing in particular that the unwanted off-diagonal entry corresponding to $-\rho_n$ is cancelled out at this stage and that the components $P(x_n) \xrightarrow{-1} P(x_n)$ in degrees d_n and $d_n - 1$ are constructed together with the components corresponding to the correction terms. This completes the argument in these cases.

Note that, in the case that the maximal common homotopy substring ρ is of length zero, then one proceeds directly from the base step to the termination step without passing through the induction step. In the case (LG ∞), one simply continues using the induction step as in a conventional induction. This completes the proof. \square

4.3. Mapping cones of graph maps involving a band complex. Suppose σ and τ are homotopy strings or bands, with at least one being a homotopy band. In this section we consider the mapping cones of graph maps $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$. The difference with Theorem 4.3 is that now $M_{f^\bullet}^\bullet$ has only one indecomposable summand. Moreover, this summand is a band complex precisely when both σ and τ are homotopy bands, and is a string complex otherwise.

We start with the situation that both σ and τ are homotopy bands. We impose the convention that the scalars λ and μ are placed on direct arrows of σ and τ , respectively.

Proposition 4.5. *Let $\sigma = \beta\sigma_L\rho\sigma_R\alpha$ and $\tau = \delta\tau_L\rho\tau_R\gamma$ be homotopy bands. Suppose $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ is a compatibly oriented graph map. Then*

$$M_{f^\bullet}^\bullet \cong \begin{cases} B_{c,\lambda\mu^{-1}}^\bullet & \text{if } f^\bullet \text{ has an even number of components;} \\ B_{c,-\lambda\mu^{-1}}^\bullet & \text{if } f^\bullet \text{ has an odd number of components,} \end{cases}$$

where $c = \beta\sigma_L\bar{\tau}_L\bar{\delta}\bar{\gamma}\bar{\tau}_R\sigma_R\alpha$ and where the scalar $\lambda\mu^{-1}$ is placed on a direct homotopy letter of α , β , $\bar{\gamma}$ or $\bar{\delta}$.

Proof. We first check that $c = \beta\sigma_L\bar{\tau}_L\bar{\delta}\bar{\gamma}\bar{\tau}_R\sigma_R\alpha$ is a homotopy band. To do this, we introduce some notation. Let

$$\partial(\sigma) = \# \text{ direct homotopy letters of } \sigma, \text{ and, } \iota(\sigma) = \# \text{ inverse homotopy letters of } \sigma.$$

We must show that $\partial(c) = \iota(c)$. Suppose, as usual, that $\sigma = \sigma_m \cdots \sigma_1$ and $\tau = \tau_n \cdots \tau_1$. Then we have $m = 2m'$ and $n = 2n'$, where $m' = \partial(\sigma) = \iota(\sigma)$ and $n' = \partial(\tau) = \iota(\tau)$.

We start by dealing with the case that $f^\bullet: B_{\sigma, \lambda}^\bullet \rightarrow B_{\tau, \mu}^\bullet$ does not satisfy the graph map endpoint conditions (RG3) or (LG3); not that f^\bullet cannot satisfy (RG ∞) or (LG ∞) because σ and τ are homotopy bands. In this case $c = \beta f_L \bar{\delta} \bar{\gamma} f_R \alpha$.

Suppose $\partial(\sigma_L \rho \sigma_R) = l = \partial(\tau_L \rho \tau_R)$ and $\iota(\sigma_L \rho \sigma_R) = l' = \iota(\tau_L \rho \tau_R)$. Then $\partial(\alpha) + \partial(\beta) = m' - l$ and $\iota(\alpha) + \iota(\beta) = m' - l'$. Similarly, $\partial(\gamma) + \partial(\delta) = n' - l$ and $\iota(\gamma) + \iota(\delta) = n' - l'$. It follows that

$$\begin{aligned} \partial(c) &= \partial(\alpha) + \partial(\beta) + \iota(\gamma) + \iota(\delta) + 1 = m' - l + n' - l' + 1, \text{ and,} \\ \iota(c) &= \iota(\alpha) + \iota(\beta) + \partial(\gamma) + \partial(\delta) + 1 = m' - l' + n' - l + 1. \end{aligned}$$

The remaining cases are:

- f^\bullet satisfies (RG3) but not (LG3) so that $c = \beta f_L \bar{\delta} \bar{\gamma} (\bar{\tau}_R \sigma_R) \alpha$, where since τ_R is an inverse homotopy letter then $(\bar{\tau}_R \sigma_R)$ is a direct homotopy letter;
- f^\bullet satisfies (LG3) but not (RG3) so that $c = \beta (\sigma_L \bar{\tau}_L) \bar{\delta} \bar{\gamma} \bar{f}_R \alpha$, where since σ_L is an inverse homotopy letter then $(\sigma_L \bar{\tau}_L)$ is also an inverse homotopy letter;
- f^\bullet satisfies (RG3) and (LG3) so that $c = \beta (\sigma_L \bar{\tau}_L) \bar{\delta} \bar{\gamma} (\bar{\tau}_R \sigma_R) \alpha$, with the same observations as above.

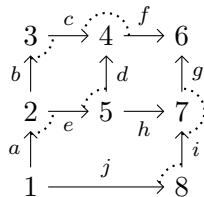
In each case, we obtain $\partial(c) = m' + n' - l - l' + 1 = \iota(c)$. Hence c is a homotopy band, as claimed.

The remainder of the proof is essentially the same as the proof of Theorem 4.3. Therefore, we only comment on the minor changes one needs to make in the proof of Theorem 4.3 in this case.

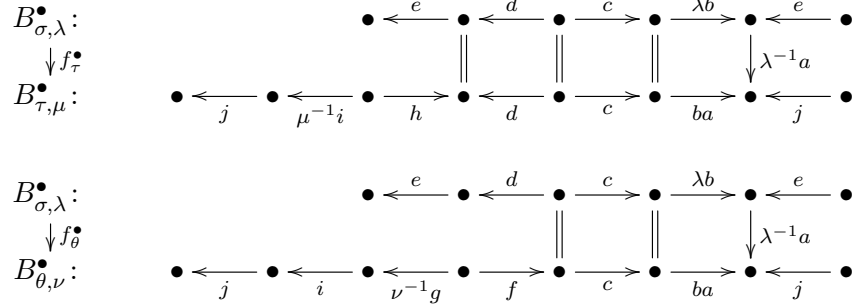
In the proof of Theorem 4.3 the maps $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f_\bullet}^\bullet$ and $i_2^\bullet: P_{c_2}^\bullet \rightarrow M_{f_\bullet}^\bullet$ could be defined independently of each other at either end of the unfolded diagram. Now we have only one map $i^\bullet: P_c^\bullet \rightarrow M_{f_\bullet}^\bullet$ which is essentially ‘glued together’ from i_1^\bullet and i_2^\bullet . In the definitions of i_1^\bullet and i_2^\bullet we had alternating signs on the identity maps between the indecomposable projective modules occurring in the homotopy substrings α and β in order to account for the minus sign required on the differential in the definition of the mapping cone. These alternating sequences of $\pm \text{id}$ were defined independently of one another in i_1^\bullet and i_2^\bullet in Theorem 4.3. Now, after gluing they are not, and the signs may not be definable in a compatible way. In the case that f^\bullet has an even number of components they are definable in a compatible way. However, in the case that f^\bullet has an odd number of components they are not. The remedy here is to introduce a minus sign on one (and only one) homotopy letter in c . In addition, the components of i^\bullet should be multiplied by the appropriate scalars $\lambda^{\pm 1}$ and $\mu^{\pm 1}$, depending on the position of λ and μ in the homotopy bands σ and τ , respectively. The definition of $p^\bullet: M_{f_\bullet}^\bullet \rightarrow P_c^\bullet$ is analogous. Finally, it is straightforward to collect all the scalars $\lambda^{\pm 1}$, $\mu^{\pm 1}$ and $(-1)^{|f^\bullet|}$ to be the coefficient of one homotopy letter of c . \square

It is useful to illustrate the proof of Proposition 4.5 in an example. This example is a modification of the one used in [4, §5.3] to illustrate maps involving band complexes.

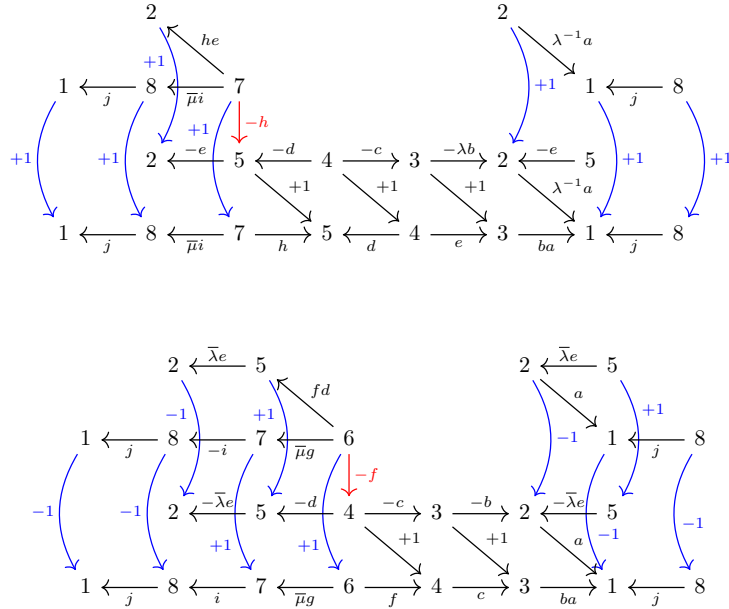
Example 4.6. Let Λ be the algebra given by the following quiver with relations.



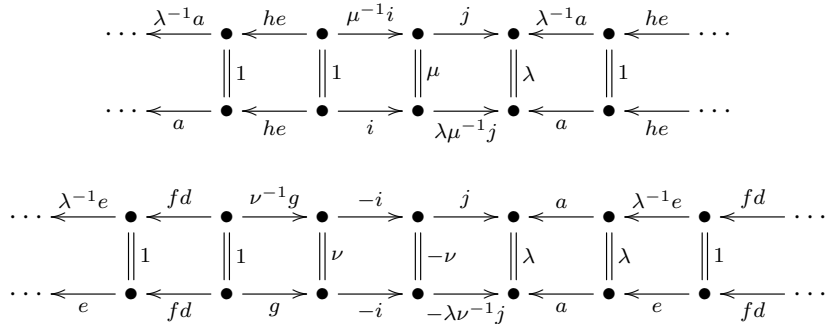
Consider the following homotopy bands: $\sigma = \bar{e}\bar{d}cb$, $\tau = \bar{j}\bar{i}h\bar{d}c\bar{b}a$ and $\theta = \bar{j}\bar{i}\bar{g}fcb\bar{a}$. We consider graph maps $f_\tau^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ and $f_\theta^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\theta,\nu}^\bullet$ defined by the following unfolded diagrams.



Let $c_\tau = \bar{h}\bar{e}i\bar{j}\bar{a}$ and $c_\theta = \bar{e}\bar{f}\bar{d}g\bar{i}\bar{j}\bar{a}$. In the figures below we illustrate the definition of maps $i_\tau^\bullet: B_{c_\tau,\lambda\mu^{-1}}^\bullet \rightarrow M_{f_\tau^\bullet}^\bullet$ and $i_\theta^\bullet: B_{c_\theta,-\lambda\nu^{-1}}^\bullet \rightarrow M_{f_\theta^\bullet}^\bullet$.



Finally, in the unfolded diagrams below, we indicate how to construct an isomorphism from the representatives of $B_{c_\tau,\lambda\mu^{-1}}^\bullet$ and $B_{c_\theta,-\lambda\nu^{-1}}^\bullet$ depicted in the figures above and more canonical representatives with all scalars collected into the coefficient of one homotopy letter.



We now examine the mapping cones of graph maps $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ where only one of σ or τ is a homotopy band. Again, $M_{f^\bullet}^\bullet$ is indecomposable, but in these cases it is a string complex. The proofs of the following propositions proceed exactly as in Theorem 4.3 with none of the subtleties of Proposition 4.5 and are thus omitted.

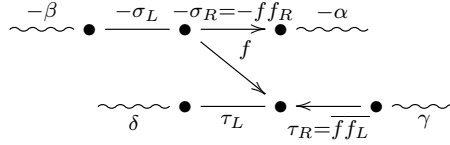
Proposition 4.7. Let $\sigma = \beta\sigma_L\rho\sigma_R\alpha$ be a homotopy band and $\tau = \delta\tau_L\rho\tau_R\gamma$ be a homotopy string. Suppose $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow P_\tau^\bullet$ is a compatibly oriented graph map. Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where $c = \delta\tau_L\bar{\sigma}_L\bar{\beta}\bar{\alpha}\bar{\sigma}_R\tau_R\gamma$.

Proposition 4.8. Let $\sigma = \beta\sigma_L\rho\sigma_R\alpha$ be a homotopy string and $\tau = \delta\tau_L\rho\tau_R\gamma$ be a homotopy band. Suppose $f^\bullet: P_\sigma^\bullet \rightarrow B_\tau^\bullet$ is a compatibly oriented graph map. Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where $c = \beta\sigma_L\bar{\tau}_L\bar{\delta}\bar{\gamma}\bar{\tau}_R\sigma_R\alpha$.

5. MAPPING CONES OF SINGLE MAPS

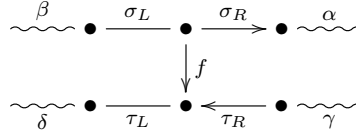
In this section, we describe the mapping cone calculus for single maps. As in Section 4, we first describe the unfolded diagram of the mapping cone and then describe the mapping cone calculus in the case of a single map between string complexes. We then deal with the cases in which (at least) one of the indecomposable complexes is a band complex.

5.1. The unfolded diagram of the mapping cone of a single map. Let $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ be a single map with single component f . We illustrate the unfolded diagram of $M_{f^\bullet}^\bullet$ in the generic situation (type (iv) of Definition 3.6.)



5.2. Mapping cones of single maps between string complexes. We start with a technical definition analogous to Definition 4.2.

Definition 5.1. Let σ and τ be homotopy strings and suppose $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ is a (not necessarily singleton) single map sitting in the following orientation.



We shall say that the homotopy strings σ and τ are *compatibly oriented* (for f^\bullet) if

- (i) the homotopy letters σ_L and τ_L do not contain f as a subletter;
- (ii) if σ_L is direct then $\sigma_L f = 0$ and if τ_L is inverse $f\tau_L = 0$;
- (iii) $\sigma_R = ff_R$ and $\tau_R = f\bar{f}_L$ for some (possibly trivial) paths f_R and f_L in (Q, I) .

We observe that if σ and τ are compatibly oriented for a single map $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ then such a single map is in one of the four situations in Definition 3.6, where we now allow f_R or f_L to be trivial. As such, the computation of the mapping cone of a single map has four cases, which are stated in the theorem below.

Theorem 5.2. Let $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ be a single map with single component f . Suppose that $\sigma = \beta\sigma_L\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_R\gamma$ are compatibly oriented for f . For cases (i) - (iv) of Definition 3.6, the mapping cone is given as follows.

- (1) If f^\bullet is of type (i) then $M_{f^\bullet}^\bullet$ is isomorphic to the indecomposable string complex P_c^\bullet , where $c = \sigma f \bar{\tau}$.
- (2) If f^\bullet is of type (ii) then $M_{f^\bullet}^\bullet$ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \beta\sigma_L f \bar{\tau}$ and $c_2 = \alpha$, where if $\sigma_R = \sigma_1$ then $c_2 = 1_{s(\sigma_1)}$ corresponding to the stalk complex of the indecomposable projective module $P(s(\sigma_1))$.
- (3) If f^\bullet is of type (iii) then $M_{f^\bullet}^\bullet$ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \sigma f \bar{\tau}_L \bar{\delta}$ and $c_2 = \gamma$, where if $\tau_R = \tau_1$ then $c_2 = 1_{s(\tau_1)}$ corresponding to the stalk complex of the indecomposable projective module $P(s(\tau_1))$.

(4) If f^\bullet is of type (iv) then $M_{f^\bullet}^\bullet$ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \beta\sigma_L f\bar{\tau}_L\bar{\delta}$ and $c_2 = \bar{\gamma}f_L f f_R \alpha$.

Remark 5.3. It is worth noting some small discrepancy between naive word combinatorics and the actual computation of the mapping cone that occurs in the computation of the summand $P_{c_2}^\bullet$ above. If f^\bullet is of type (iv), i.e. the generic case, then $c_2 = \bar{\gamma}\bar{\tau}_R\bar{f}\sigma_R\alpha = \bar{\gamma}\bar{f}\bar{f}_L\bar{f}f f_R\alpha = \bar{\gamma}f_L f f_R\alpha$. However, in type (ii) notice that $c_2 \neq \bar{f}f f_R\alpha = f_R\alpha$ as if type (ii) were a degeneration of type (iv). Similarly, for type (iii) $c_2 \neq f f_L f \gamma = \bar{f}_L\gamma$. In the cases that $\sigma_R = f$ and $\tau_R = \bar{f}$, i.e. a single map that is not singleton, then the mapping cones in types (ii) and (iii) are simply a degeneration of type (iv). Therefore, in the proof of Theorem 5.2 below, we treat singleton single maps of types (i) - (iv) and observe that the computation for type (iv) holds for an arbitrary single map.

Proof of Theorem 5.2. The strategy of the proof is the same as that in Theorem 4.3: we will show how to define a split monomorphism $i^\bullet: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and we will show that here i^\bullet is an isomorphism in $C^{-b}(\text{proj}(\Lambda))$, and hence in $K^{b,-}(\text{proj}(\Lambda))$.

Throughout the proof we work at the level of unfolded diagrams to construct the required isomorphism. While for word combinatorial reasons it is useful to state Theorem 5.2 in the setup of Definition 5.1, the unfolded diagrams of the mapping cones look ‘more like complexes’ and are thus easier to work with if the opposite orientation is taken for τ . Therefore, throughout the proof we work instead with $\bar{\tau}$. We shall draw a sketch (except for type (i), which is straightforward) to indicate how the map looks in each type with the opposite orientation for τ .

(1) In this case, the unfolded diagram of the mapping cone $M_{f^\bullet}^\bullet$ is,

$$\begin{array}{c} \bullet \xrightarrow{-\sigma_m} \cdots \bullet \xrightarrow{-\sigma_1} \bullet \xrightarrow{f} \bullet \xrightarrow{\bar{\tau}_1} \cdots \bullet \xrightarrow{\bar{\tau}_n} \bullet \end{array}$$

where, for convenience, we have uses the following relabelling, $\sigma_L = \sigma_1$, $\beta = \sigma_m \cdots \sigma_2$, $\tau_L = \tau_1$ and $\delta = \tau_n \cdots \tau_2$. Thus $M_{f^\bullet}^\bullet$ is isomorphic to P_c^\bullet , the isomorphism being given at the level of the unfolded diagrams by the ‘graph’ map i^\bullet with components $(+1, -1, \dots, (-1)^n, (-1)^n, \dots, (-1)^n)$, where the first n signs, corresponding to the homotopy letters of σ , alternate and are then fixed upon reaching the homotopy letters corresponding to f and τ .

(2) Now suppose f^\bullet is of type (ii), that is, it has the following unfolded diagram,

$$\begin{array}{c} \cdots \xrightarrow{\beta} \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{\sigma_R = f f_R} \bullet \xrightarrow{\alpha} \cdots \\ \downarrow f \\ \bullet \xrightarrow{\bar{\tau}_L} \bullet \xrightarrow{\bar{\delta}} \cdots \end{array}$$

For convenience, assume $\alpha = \sigma_{i-1} \cdots \sigma_1$, $\sigma_R = \sigma_i$, $\sigma_L = \sigma_{i+1}$, $\beta = \sigma_{i+2} \cdots \sigma_m$ and note that $\tau_L = \tau_1$ and $\delta = \tau_n \cdots \tau_2$. Then the unfolded diagram of the mapping cone $M_{f^\bullet}^\bullet$ is

$$\begin{array}{c} \bullet \xrightarrow{-\sigma_m} \cdots \bullet \xrightarrow{-\sigma_{i+1}} \bullet \xrightarrow{-\sigma_i = -f f_R} \bullet \xrightarrow{\cdots} \bullet \xrightarrow{-\sigma_1} \bullet \\ \downarrow f \\ \bullet \xrightarrow{\bar{\tau}_1} \cdots \bullet \xrightarrow{\bar{\tau}_n} \bullet \end{array}$$

We now describe an inductive procedure defining a map $i^\bullet = \begin{bmatrix} i_1^\bullet \\ i_2^\bullet \end{bmatrix}: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and we show that it is an isomorphism. For clarity we label the indecomposable projective modules in the unfolded diagrams as follows:

$$P_{\sigma}^{\bullet}: P(x_m) \xrightarrow{\sigma_m} P(x_{m-1}) \cdots P(x_1) \xrightarrow{\sigma_1} P(x_0)$$

$$P_{\tau}^{\bullet}: P(y_0) \xrightarrow{\overline{\tau_1}} P(y_1) \cdots P(y_{n-1}) \xrightarrow{\overline{\tau_n}} P(y_n)$$

The first components of $i_1^{\bullet}: P_{c_1}^{\bullet} \rightarrow M_{f,\bullet}^{\bullet}$ will be $P(x_i) \xrightarrow{+1} P(x_i)$, $P(y_0) \xrightarrow{+1} P(y_0)$ and a correction term $P(y_0) \xrightarrow{-f_R} P(x_{i-1})$. The first component of $i_2^{\bullet}: P_{c_2}^{\bullet} \rightarrow M_{f,\bullet}^{\bullet}$ will be $P(x_{i-1}) \xrightarrow{+1} P(x_{i-1})$. These are indicated in the unfolded diagram sketched in Figure 3, from which one can immediately observe commutativity for these components.

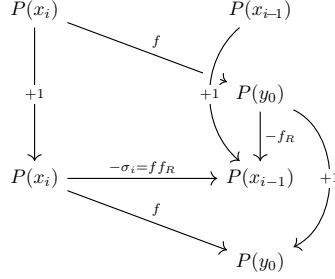


FIGURE 3. First components of $i_1^{\bullet}: P_{c_1}^{\bullet} \rightarrow M_{f,\bullet}^{\bullet}$ and $i_2^{\bullet}: P_{c_2}^{\bullet} \rightarrow M_{f,\bullet}^{\bullet}$ for f of type (ii).

Ensuring the commutativity of each square, the identity maps defining the first components of i_1^{\bullet} extend along the prefix $\sigma_{i-1} \cdots \sigma_1$ of both the string complex $P_{c_1}^{\bullet}$ and the mapping cone $M_{f,\bullet}^{\bullet}$ as in Figure 4. The only possibility for non-commutativity occurs with the component $-f_R: P(y_0) \rightarrow P(x_{i+1})$. If σ_{i-1} is inverse or zero, then there is nothing to show. If σ_{i-1} is direct, then since $\sigma_i = f f_R$ we have by the relations of the (gentle) algebra and the definition of a homotopy string that $-f_R \sigma_{i-1} = 0$, also giving the required commutativity.

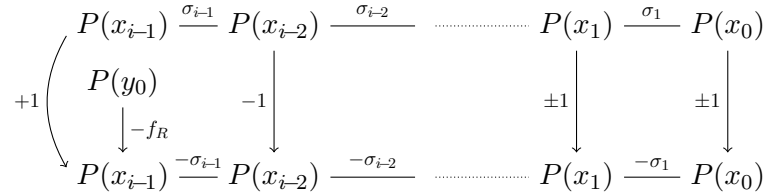


FIGURE 4. Identity maps defining the components of i_1^{\bullet} along to the start of σ .

The first components of i_1^{\bullet} extend along the suffix $\sigma_m \cdots \sigma_{i+1}$ of the string complex $P_{c_1}^{\bullet}$ as well as of the mapping cone $M_{f,\bullet}^{\bullet}$ as shown in Figure 5; in this case the commutativity is clear. This then completes the definition of i_1^{\bullet} .

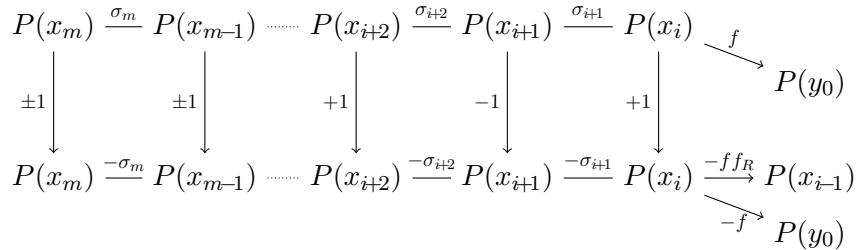


FIGURE 5. Extension of the identity maps defining the first components of i_1^{\bullet} along to the end of σ .

The first component of i_2^\bullet , namely $+1: P(y_0) \rightarrow P(y_0)$ trivially extends along the length of $\bar{\tau}$ defining the string complex $P_{c_2}^\bullet$ and $M_{f^\bullet}^\bullet$ as indicated in Figure 6; the commutativity is again clear.

$$\begin{array}{ccccccccc}
P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \cdots & P(y_{n-2}) & \xrightarrow{\bar{\tau}_{n-1}} & P(y_{n-1}) & \xrightarrow{\bar{\tau}_n} & P(y_n) \\
\downarrow +1 & & \downarrow +1 & & \downarrow +1 & & \downarrow +1 & & \downarrow +1 \\
P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \cdots & P(y_{n-2}) & \xrightarrow{\bar{\tau}_{n-1}} & P(y_{n-1}) & \xrightarrow{\bar{\tau}_n} & P(y_n)
\end{array}$$

FIGURE 6. Extension of the identity maps defining the first components of i_2^\bullet along to the end of τ .

By construction, this defines full rank matrices in each cohomological degree which commute with the differentials in both $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$ and $M_{f^\bullet}^\bullet$. Therefore $i^\bullet: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ is an isomorphism of complexes.

(3) Now suppose f^\bullet is of type (iii), that is it has the following unfolded diagram,

$$\begin{array}{c}
\beta \quad \bullet \xrightarrow{\sigma_L} \bullet \\
\downarrow f \\
\gamma \quad \bullet \xrightarrow{\bar{\tau}_R = f_L f} \bullet \xrightarrow{\bar{\tau}_L} \bullet \quad \delta
\end{array}$$

For convenience, assume $\gamma = \tau_j \dots \tau_{j-1}$, $\tau_R = \tau_j$, $\tau_L = \tau_{j+1}$, $\delta = \tau_n \dots \tau_{j+2}$ and note that $\beta = \sigma_m \dots \sigma_2$, $\sigma_L = \sigma_1$. Then the unfolded diagram of the mapping cone $M_{f^\bullet}^\bullet$ is

$$\begin{array}{ccccccc}
\bullet & \xrightarrow{-\sigma_m} & \bullet & \cdots & \bullet & \xrightarrow{-\sigma_1} & \bullet \\
& & & & \searrow f & & \\
\bullet & \xrightarrow{\bar{\tau}_1} & \bullet & \cdots & \bullet & \xrightarrow{\bar{\tau}_j = f_L f} & \bullet \xrightarrow{\bar{\tau}_{j+1}} \bullet \cdots \bullet \xrightarrow{\bar{\tau}_n} \bullet
\end{array}$$

We define the maps $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and $i_2^\bullet: P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ in the figures below. It is straightforward to check, using an analysis as in (2) above, that these induce well-defined morphisms of complexes. Moreover, $i^\bullet = \begin{bmatrix} i_1^\bullet \\ i_2^\bullet \end{bmatrix}: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ consists of full rank matrices in each degree, and is therefore an isomorphism of complexes.

$$\begin{array}{ccccccccccc}
P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \cdots & P(y_{j-2}) & \xrightarrow{\bar{\tau}_{j-1}} & P(y_{j-1}) & & & & \\
\downarrow +1 & & \downarrow +1 & & \downarrow +1 & & \downarrow +1 & & \downarrow -f_L & & \\
P(x_m) & \xrightarrow{-\sigma_m} & P(x_{m-1}) & \cdots & P(x_2) & \xrightarrow{-\sigma_2} & P(x_1) & \xrightarrow{-\sigma_1} & P(x_0) & \xrightarrow{f} & P(y_j) \xrightarrow{\bar{\tau}_{j+1}} P(y_{j+1}) \cdots P(y_{n-1}) \xrightarrow{\bar{\tau}_n} P(y_n)
\end{array}$$

$$\begin{array}{ccccccccccc}
P(x_m) & \xrightarrow{\sigma_m} & P(x_{m-1}) & \cdots & P(x_1) & \xrightarrow{\sigma_1} & P(x_0) & & & & \\
\downarrow \pm 1 & & \downarrow \mp 1 & & \downarrow -1 & & \downarrow +1 & & \downarrow +1 & & \\
P(x_m) & \xrightarrow{-\sigma_m} & P(x_{m-1}) & \cdots & P(x_1) & \xrightarrow{-\sigma_1} & P(x_0) & \xrightarrow{f} & P(y_j) & \xrightarrow{\bar{\tau}_{j+1}} & P(y_{j+1}) \cdots P(y_{n-1}) \xrightarrow{\bar{\tau}_n} P(y_n)
\end{array}$$

(4) Now suppose f^\bullet is of type (iv), that is it has the following unfolded diagram,

$$\begin{array}{ccccc}
 \beta & \bullet & \xrightarrow{\sigma_L} & \bullet & \xrightarrow{\sigma_R=ff_R} & \bullet & \xrightarrow{\alpha} \\
 & & & \downarrow f & & & \\
 \bar{\gamma} & \bullet & \xrightarrow{\bar{\tau}_R=f_L f} & \bullet & \xrightarrow{\bar{\tau}_L} & \bullet & \xrightarrow{\bar{\delta}}
 \end{array}$$

For convenience, assume $\beta = \sigma_m \dots \sigma_{i+2}$, $\sigma_L = \sigma_{i+1}$, $\sigma_R = \sigma_i$, $\alpha = \sigma_{i-1} \dots \sigma_1$ and $\gamma = \tau_1 \dots \tau_{j-1}$, $\tau_R = \tau_j$, $\tau_L = \tau_{j+1}$, $\delta = \tau_n \dots \tau_{j+2}$. Then the unfolded diagram of the mapping cone M_{f^\bullet} is

$$\begin{array}{ccccccc}
 \bullet & \xrightarrow{-\sigma_m} & \bullet & \dots & \bullet & \xrightarrow{-\sigma_{i+1}} & \bullet & \xrightarrow{-\sigma_i=ff_R} & \bullet & \xrightarrow{-\sigma_{i-1}} & \bullet & \dots & \bullet & \xrightarrow{-\sigma_1} & \bullet \\
 & & & & & & \searrow f & & & & & & & & \\
 \bullet & \xrightarrow{\bar{\tau}_1} & \bullet & \dots & \bullet & \xrightarrow{\bar{\tau}_{j-1}} & \bullet & \xrightarrow{\bar{\tau}_j=f_L f} & \bullet & \xrightarrow{\bar{\tau}_{j+1}} & \bullet & \dots & \bullet & \xrightarrow{\bar{\tau}_n} & \bullet
 \end{array}$$

As above, we simply define the maps $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}$ and $i_2^\bullet: P_{c_2}^\bullet \rightarrow M_{f^\bullet}$ in the figures below. It is straightforward to check that these induce well-defined morphisms of complexes. Moreover, $i^\bullet = \begin{bmatrix} i_1^\bullet \\ i_2^\bullet \end{bmatrix}: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}$ consists of full rank matrices in each degree, and is therefore an isomorphism of complexes.

$$\begin{array}{c}
 \begin{array}{ccccccc}
 P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \dots & P(y_{j-2}) & \xrightarrow{\bar{\tau}_{j-1}} & P(y_{j-1}) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(x_m) & \xrightarrow{-\sigma_m} & P(x_{m-1}) & \dots & P(x_{i+1}) & \xrightarrow{-\sigma_{i+1}} & P(x_i) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \dots & P(y_{j-2}) & \xrightarrow{\bar{\tau}_{j-1}} & P(y_{j-1})
 \end{array} \\
 \begin{array}{c}
 \nearrow f_L f f_R \\
 \downarrow \scriptstyle -f_L \\
 \searrow f
 \end{array} \\
 \begin{array}{ccccccc}
 P(x_{i-1}) & \xrightarrow{\sigma_{i-1}} & P(x_{i-2}) & \dots & P(x_1) & \xrightarrow{\sigma_1} & P(x_0) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle -1 & & \downarrow \scriptstyle \mp 1 & & \downarrow \scriptstyle \pm 1 \\
 P(x_{i-1}) & \xrightarrow{-\sigma_{i-1}} & P(x_{i-2}) & \dots & P(x_1) & \xrightarrow{-\sigma_1} & P(x_0) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(y_j) & \xrightarrow{\bar{\tau}_{j+1}} & P(y_{j+1}) & \dots & P(y_{n-1}) & \xrightarrow{\bar{\tau}_n} & P(y_n)
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccccc}
 P(x_m) & \xrightarrow{\sigma_m} & P(x_{m-1}) & \dots & P(x_{i+1}) & \xrightarrow{\sigma_{i+1}} & P(x_i) \\
 \downarrow \scriptstyle \pm 1 & & \downarrow \scriptstyle \mp 1 & & \downarrow \scriptstyle -1 & & \downarrow \scriptstyle +1 \\
 P(x_m) & \xrightarrow{-\sigma_m} & P(x_{m-1}) & \dots & P(x_{i+1}) & \xrightarrow{-\sigma_{i+1}} & P(x_i) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(y_0) & \xrightarrow{\bar{\tau}_1} & P(y_1) & \dots & P(y_{j-2}) & \xrightarrow{\bar{\tau}_{j-1}} & P(y_{j-1})
 \end{array} \\
 \begin{array}{c}
 \nearrow f \\
 \downarrow \scriptstyle -f_R \\
 \searrow f
 \end{array} \\
 \begin{array}{ccccccc}
 P(y_j) & \xrightarrow{\bar{\tau}_{j+1}} & P(y_{j+1}) & \dots & P(y_{n-1}) & \xrightarrow{\bar{\tau}_n} & P(y_n) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(x_{i-1}) & \xrightarrow{-\sigma_{i-1}} & P(x_{i-2}) & \dots & P(x_1) & \xrightarrow{-\sigma_1} & P(x_0) \\
 \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 & & \downarrow \scriptstyle +1 \\
 P(y_j) & \xrightarrow{\bar{\tau}_{j+1}} & P(y_{j+1}) & \dots & P(y_{n-1}) & \xrightarrow{\bar{\tau}_n} & P(y_n)
 \end{array}
 \end{array}$$

□

5.3. Mapping cones of single maps involving a band complex. Suppose σ and τ are homotopy strings or bands, with at least one being a homotopy band. We now consider the mapping cones of single maps $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$. As was the case in Section 4.3, M_{f^\bullet} is now indecomposable. Moreover, M_{f^\bullet} is a band complex if and only if σ and τ are homotopy bands. This is the situation with which we start.

If both σ and τ are homotopy bands, then any single map $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ is necessarily of type (iv). We again impose the convention that the scalars λ and μ are placed on direct arrows of σ and τ , respectively.

Proposition 5.4. *Suppose σ and τ are homotopy bands and $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ with single component f . Suppose that $\sigma = \beta \sigma_L \sigma_R \alpha$ and $\tau = \delta \tau_L \tau_R \gamma$ are compatibly oriented for f^\bullet . Then M_{f^\bullet} is isomorphic to a band complex $B_{c,\lambda\mu^{-1}}^\bullet$, where $c = \beta \sigma_L f \bar{\tau}_L \bar{\delta} \bar{\gamma} \bar{\tau}_R \bar{f} \sigma_R \alpha$ and where the scalar $\lambda\mu^{-1}$ is placed on a direct homotopy letter of α , β , $\bar{\gamma}$ or $\bar{\delta}$.*

Proof. The verification that c is indeed a homotopy band is similar to that in the proof of Proposition 4.5. The construction of an isomorphism $i^\bullet: B_{c, \lambda\mu^{-1}}^\bullet \rightarrow M_{f^\bullet}^\bullet$ proceeds exactly as in the proof of Theorem 5.2, with no extra signs required on the differential of $B_{c, \lambda\mu^{-1}}^\bullet$. \square

The following two propositions deal with the cases where precisely one of σ or τ is a homotopy band and the other is a homotopy string. In these cases $M_{f^\bullet}^\bullet$ is a string complex. The proofs proceed exactly as in Theorem 5.2 with no additional subtlety and are therefore omitted. Note that when σ is a homotopy band, any single map $f^\bullet: B_{\sigma, \lambda}^\bullet \rightarrow P_\tau^\bullet$ is necessarily of type (ii) or (iv) in Definition 3.6.

Proposition 5.5. *Suppose σ is a homotopy band and τ is a homotopy string. Suppose $f^\bullet: B_{\sigma, \lambda}^\bullet \rightarrow P_\tau^\bullet$ is a single map with single component f . Suppose that $\sigma = \beta\sigma_L\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_R\gamma$ is compatibly oriented for f^\bullet . Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where*

- (1) $c = \alpha\beta\sigma_L f \overline{\tau_L} \overline{\delta}$ when f^\bullet is of type (ii).
- (2) $c = \overline{\gamma} \overline{\tau_R} \overline{f} \sigma_R \alpha \beta \sigma_L f \overline{\tau_L} \overline{\delta}$ when f^\bullet is of type (iv).

When τ is a homotopy band, any single map $f^\bullet: P_\sigma^\bullet \rightarrow B_{\tau, \mu}^\bullet$ is necessarily of type (iii) or (iv) in Definition 3.6.

Proposition 5.6. *Suppose σ is a homotopy string and τ is a homotopy band. Suppose $f^\bullet: P_\sigma^\bullet \rightarrow B_{\tau, \mu}^\bullet$ is a single map with single component f . Suppose that $\sigma = \beta\sigma_L\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_R\gamma$ is compatibly oriented for f^\bullet . Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where*

- (1) $c = \beta\sigma_L f \overline{\tau_L} \overline{\delta} \overline{\gamma}$ when f^\bullet is of type (iii).
- (2) $c = \beta\sigma_L f \overline{\tau_L} \overline{\delta} \overline{\gamma} \overline{\tau_R} \overline{f} \sigma_R \alpha$ when f^\bullet is of type (iv).

6. MAPPING CONES OF DOUBLE MAPS

We now turn to the statement for double maps. Recall Definition 3.7 and the setup in (4) on page 11. Note that double maps are automatically ‘compatibly oriented’ and therefore we do not require such a definition in this case.

Theorem 6.1. *Let $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ be a double map with components (f_L, f_R) . Suppose that $\sigma = \beta\sigma_L\sigma_C\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_C\tau_R\gamma$. Then the mapping cone $M_{f^\bullet}^\bullet$ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \overline{\gamma} \overline{\tau_R} \overline{f_R} \sigma_R \alpha$ and $c_2 = \beta\sigma_L f_L \overline{\tau_L} \overline{\delta}$.*

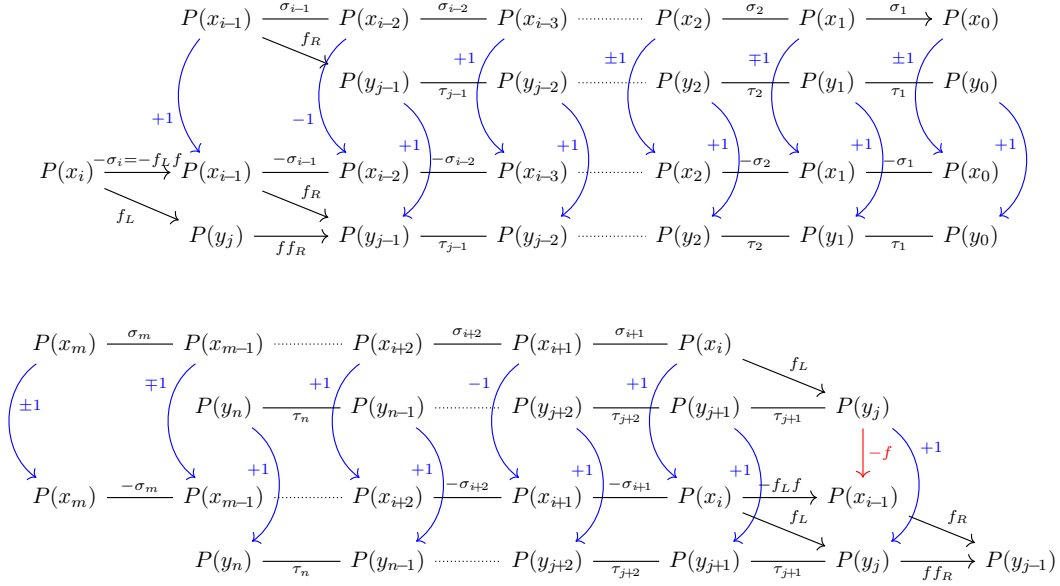
Proof. We treat the case that $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ is a singleton double map with components (f_L, f_R) . The argument when f^\bullet is not a singleton double map is the same with the component f replaced by a trivial path. In terms of unfolded diagrams, the map f^\bullet is of the following form

$$\begin{array}{ccccccc}
 \beta & \bullet & \xrightarrow{\sigma_L} & \bullet & \xrightarrow{\sigma_C = f_L f} & \bullet & \xrightarrow{\sigma_R} & \bullet & \xrightarrow{\alpha} & \\
 & & & \downarrow f_L & & \downarrow f_R & & & & \\
 \delta & \bullet & \xrightarrow{\tau_L} & \bullet & \xrightarrow{\tau_C = f f_R} & \bullet & \xrightarrow{\tau_R} & \bullet & \xrightarrow{\gamma} &
 \end{array}$$

For convenience, assume $\sigma_C = \sigma_i$ and $\tau_C = \tau_j$ etc. Then the unfolded diagram of the mapping cone $M_{f^\bullet}^\bullet$ is

$$\begin{array}{ccccccccccc}
 \bullet & \xrightarrow{-\sigma_m} & \bullet & \cdots & \bullet & \xrightarrow{-\sigma_{i+1}} & \bullet & \xrightarrow{-\sigma_i = -f_L f} & \bullet & \xrightarrow{\sigma_{i-1}} & \bullet & \cdots & \bullet & \xrightarrow{\sigma_1} & \bullet \\
 & & & & & & & \searrow f_L & & \searrow f_R & & & & & \\
 \bullet & \xrightarrow{\tau_n} & \bullet & \cdots & \bullet & \xrightarrow{\tau_{j+1}} & \bullet & \xrightarrow{\tau_j = f f_R} & \bullet & \xrightarrow{\tau_{j-1}} & \bullet & \cdots & \bullet & \xrightarrow{\tau_1} & \bullet
 \end{array}$$

The proof is the same as that of Theorem 5.2, therefore, we just write down the maps $i_1^\bullet: P_{c_1}^\bullet \rightarrow M_{f^\bullet}^\bullet$ and $i_2^\bullet: P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ at the level of unfolded diagrams in the figures below. It is then straightforward to check these define full rank matrices in each cohomological degree, so that $i^\bullet = \begin{bmatrix} i_1^\bullet \\ i_2^\bullet \end{bmatrix}: P_{c_1}^\bullet \oplus P_{c_2}^\bullet \rightarrow M_{f^\bullet}^\bullet$ is an isomorphism.



□

We now give the analogous statements for double maps $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ where at least one of σ or τ is a homotopy band. As was the case for graph maps and single maps before, $M_{f^\bullet}^\bullet$ is indecomposable and is a band complex if and only if both σ and τ are homotopy bands.

The proof of Theorem 6.1 carries through in each case below and therefore proofs are omitted. As in Proposition 5.4 there are no extra subtleties coming from additional signs required on homotopy letters. We start with the case that both σ and τ are homotopy bands.

Proposition 6.2. *Let $\sigma = \beta\sigma_L\sigma_C\sigma_R\alpha$ and $\tau = \delta\tau_L\tau_C\tau_R\gamma$ be homotopy bands. Suppose $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ is a double map with components (f_L, f_R) . Then $M_{f^\bullet}^\bullet$ is isomorphic to a band complex $B_{c,\lambda\mu^{-1}}^\bullet$, where $c = \beta\sigma_L f_L \bar{\tau}_L \bar{\delta} \bar{\gamma} \bar{\tau}_R \bar{f}_R \sigma_R \alpha$ and where the scalar $\lambda\mu^{-1}$ is placed on a direct homotopy letter of $\alpha, \beta, \bar{\gamma}$ or $\bar{\delta}$.*

Proposition 6.3. *Let $\sigma = \beta\sigma_L\sigma_C\sigma_R\alpha$ be a homotopy band and $\tau = \delta\tau_L\tau_C\tau_R\gamma$ be a homotopy string. Suppose $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow P_\tau^\bullet$ is a double map with components (f_L, f_R) . Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where $c = \delta\tau_L f_L \bar{\sigma}_L \bar{\beta} \bar{\alpha} \bar{\sigma}_R \bar{f}_R \tau_R \gamma$.*

Proposition 6.4. *Let $\sigma = \beta\sigma_L\sigma_C\sigma_R\alpha$ be a homotopy string and $\tau = \delta\tau_L\tau_C\tau_R\gamma$ be a homotopy band. Suppose $f^\bullet: P_\sigma^\bullet \rightarrow B_{\tau,\mu}^\bullet$ is a double map with components (f_L, f_R) . Then $M_{f^\bullet}^\bullet$ is isomorphic to the string complex P_c^\bullet , where $c = \beta\sigma_L f_L \bar{\tau}_L \bar{\delta} \bar{\gamma} \bar{\tau}_R \bar{f}_R \sigma_R \alpha$.*

7. MAPPING CONES AND QUASI-GRAPH MAPS

Single maps and double maps that are not singleton occur in a homotopy class that is determined by a quasi-graph map $P_\sigma^\bullet \rightsquigarrow \Sigma^{-1}P_\tau^\bullet$. In this case, it is possible to read off the mapping cone of any representative of the homotopy class from the quasi-graph map. Before we state how this is done, we need the following bookkeeping definition to cover the case when a quasi-graph map is supported in precisely one cohomological degree.

Definition 7.1. Let σ and τ be homotopy strings or bands and suppose $\varphi: Q_\sigma^\bullet \rightsquigarrow \Sigma^{-1}Q_\tau^\bullet$ is a quasi-graph map supported in exactly one degree, i.e. corresponds to the following diagram

$$\begin{array}{ccccccc}
 \beta & & \sigma_L & & \sigma_R & & \alpha \\
 \sim & \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet & \sim \\
 & & \parallel & & & & \\
 \delta & & \tau_L & & \tau_R & & \gamma \\
 \sim & \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet & \sim
 \end{array}$$

We say that the homotopy strings or bands σ and τ are *compatibly oriented on the left* if

- (1) If σ_L is direct then τ_L is either zero, inverse with $\sigma_L\tau_L \neq 0$ or direct with $\tau_L = \sigma'_L\sigma_L$ for some non-trivial σ'_L .
- (2) If σ_L is inverse then τ_L is inverse and $\sigma_L = \tau_L\tau'_L$ for some non-trivial τ'_L .
- (3) If σ_L is zero then τ_L is inverse.

Similarly, we say that the homotopy strings or bands σ and τ are *compatibly oriented on the right* if the following dual conditions hold:

- (1) If σ_R is inverse then τ_R is either zero, direct with $\sigma_R\tau_R \neq 0$ or inverse with $\tau_R = \sigma'_R\sigma_R$ for some non-trivial σ'_R .
- (2) If σ_R is direct then τ_R is direct and $\sigma_R = \tau_R\tau'_R$ for some non-trivial τ'_R .
- (3) If σ_R is zero then τ_R is direct.

We say that the homotopy strings or bands σ and τ are *compatibly oriented for φ* if they are compatibly oriented on the left and on the right.

Note that when the maximal common homotopy substring ρ determining a quasi-graph map $\varphi: Q_\sigma^\bullet \rightsquigarrow \Sigma^{-1}Q_\tau^\bullet$ is of length at least one, the homotopy strings or bands σ and τ are automatically compatibly oriented for φ in an unfolded diagram of φ .

Proposition 7.2. *Let σ and τ be homotopy strings or bands. Suppose $\varphi: P_\sigma^\bullet \rightsquigarrow \Sigma^{-1}P_\tau^\bullet$ is a quasi-graph map determined by a maximal common homotopy substring ρ , i.e. $\sigma = \beta\sigma_L\rho\sigma_R\alpha$ and $\tau = \delta\tau_L\rho\tau_R\gamma$. Assume further that σ and τ are compatibly oriented for φ . Suppose $f^\bullet: Q_\sigma^\bullet \rightarrow Q_\tau^\bullet$ is a representative of the homotopy set determined by φ .*

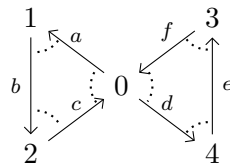
- (1) *If σ and τ are homotopy strings then M_{f^\bullet} is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \beta\sigma_L\rho\tau_R\gamma$ and $c_2 = \delta\tau_L\rho\sigma_R\alpha$.*
- (2) *If (σ, λ) and (τ, μ) are homotopy bands then M_{f^\bullet} is isomorphic to $B_{c, \lambda\mu^{-1}}^\bullet$, where $c = \beta\sigma_L\rho\tau_R\gamma\delta\tau_L\rho\sigma_R\alpha$ and where the scalar $\lambda\mu^{-1}$ is placed on a direct homotopy letter of α or β or an inverse homotopy letter of γ or δ .*
- (3) *If (σ, λ) is a homotopy band and τ is a homotopy string then M_{f^\bullet} is isomorphic to P_c^\bullet , where $c = \delta\tau_L\rho\sigma_R\alpha\beta\sigma_L\rho\tau_R\gamma$.*
- (4) *If σ is a homotopy string and (τ, μ) is a homotopy band then M_{f^\bullet} is isomorphic to P_c^\bullet , where $c = \beta\sigma_L\rho\tau_R\gamma\delta\tau_L\rho\sigma_R\alpha$.*

Proof. One simply chooses a representative of the homotopy class determined by the quasi-graph map $\varphi: Q_\sigma^\bullet \rightsquigarrow \Sigma^{-1}P_\tau^\bullet$ and carries out the computation in Theorem 5.2 or Theorem 6.1. Note the difference in the orientation of the homotopy strings σ and τ in this case with respect to the orientations in Theorems 5.2 and 6.1. \square

8. EXAMPLES

In this section we will illustrate the graphical mapping cone calculus in $D^b(\Lambda)$ developed in Sections 4-7 on some concrete examples. In the first example, we consider maps involving only string complexes and in the second example we consider band complexes.

Example 8.1. Let Λ be the gentle algebra given by the following quiver and relations:



(1) Consider the homotopy strings $\sigma = edcbad$ and $\tau = \bar{e}fcbaf e$, and the graph map $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ given by

$$\begin{array}{ccccccccccc} P_\sigma & & P(3) & \xrightarrow{e} & P(4) & \xrightarrow{dc} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xleftarrow{d} & P(4) \\ \downarrow & & & & \downarrow d & & \parallel & & \parallel & & \downarrow f & & \\ P_\tau & & P(4) & \xleftarrow{e} & P(3) & \xleftarrow{f} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{af} & P(3) & \xrightarrow{e} & P(4) \end{array}$$

By Theorem 4.3, the mapping cone $M_{f^\bullet}^\bullet$ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$ where $c_1 = dfe$ and $c_2 = edfe$ (cf. green and red boxes in the figure below).

$$\begin{array}{ccccccccccc} \boxed{P(3) \xrightarrow{e} P(4)} & \xrightarrow{dc} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & \boxed{P(0) \xleftarrow{d} P(4)} \\ & & \parallel & & \parallel & & \downarrow f \\ \boxed{P(4) \xleftarrow{e} P(3) \xleftarrow{f} P(0)} & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{af} & \boxed{P(3) \xrightarrow{e} P(4)} \end{array}$$

(2) Non-singleton double maps and single maps arise in the context of quasi-graph maps. As described in Section 3, a given quasi-graph map gives rise to a class of (single and double) maps, which are all homotopy equivalent to each other. In particular, they all have the same mapping cone, which is the ‘mapping cone of the quasi-graph map’. We will now illustrate this with an example. Consider a quasi-graph map $\varphi: P_\sigma^\bullet \rightsquigarrow \Sigma^{-1}P_\tau^\bullet$, for homotopy strings $\sigma = bacb$ and $\tau = \bar{f}cba$, given by

$$\begin{array}{ccccccc} P_\sigma & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) \\ \downarrow \wr & & & & & \parallel & & \parallel & & \parallel \\ \Sigma^{-1}P_\tau & P(3) & \xleftarrow{f} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) \end{array}$$

By Proposition 7.2(1), the mapping cone of any single or double map $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ in the homotopy set determined by φ is isomorphic to $P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = bacba$ and $c_2 = \bar{f}cb$ (cf. green and red boxes in the figure below).

$$\begin{array}{ccccccc} \boxed{P(2) \xrightarrow{b} P(1) \xrightarrow{a} P(0) \xrightarrow{c} P(2) \xrightarrow{b} P(1)} & & & & & & \\ & \parallel & & \parallel & & \parallel & \\ \boxed{P(3) \xleftarrow{f} P(0) \xrightarrow{c} P(2) \xrightarrow{b} P(1)} & \xrightarrow{a} & P(0) \end{array}$$

We now consider a single map $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ and a double map $g^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ in the homotopy set determined by the quasi-graph map φ .

(i) Let $f^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ be a single map in the homotopy set determined by φ given by

$$\begin{array}{ccccccc} P_\sigma & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) \\ \downarrow & & & & & \downarrow c & & & & \\ P_\tau & P(0) & \xleftarrow{a} & P(1) & \xleftarrow{b} & P(2) & \xleftarrow{c} & P(0) & \xrightarrow{f} & P(3) \end{array}$$

where we have drawn the unfolded diagram so that it is compatibly oriented (see Definition 5.1). By Theorem 5.2, the mapping cone is $M_{f^\bullet}^\bullet \cong P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = bacba$ and $c_2 = \bar{f}cb$ (cf. green and red boxes in the figure below).

$$\begin{array}{ccccccc} \boxed{P(2) \xrightarrow{b} P(1) \xrightarrow{a} P(0)} & \xrightarrow{c} & \boxed{P(2) \xrightarrow{b} P(1)} \\ & \parallel & \uparrow c \\ \boxed{P(0) \xleftarrow{a} P(1) \xleftarrow{b} P(2)} & \xleftarrow{c} & \boxed{P(0) \xrightarrow{f} P(3)} \end{array}$$

(ii) Let $g^\bullet: P_\sigma^\bullet \rightarrow P_\tau^\bullet$ be a double map in the homotopy set determined by φ given by

$$\begin{array}{ccccccc} P_\sigma & & & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) \\ \downarrow & & & & & \downarrow a & & \downarrow f & & & & \\ P_\tau & & P(0) & \xleftarrow{a} & P(1) & \xleftarrow{b} & P(2) & \xleftarrow{c} & P(0) & \xrightarrow{f} & P(3) \end{array}$$

By Theorem 6.1, its mapping cone is $M_{g^\bullet}^\bullet \cong P_{c_1}^\bullet \oplus P_{c_2}^\bullet$, where $c_1 = \bar{f}cb$ and $c_2 = bacba$ (cf. red and green boxes in the figure below).

$$\begin{array}{ccccccc} & & & P(2) & \xrightarrow{b} & P(1) & \xrightarrow{a} & P(0) & \xrightarrow{c} & P(2) & \xrightarrow{b} & P(1) \\ & & & & & \downarrow a & & \downarrow f & & & & \\ & & & & & & & & & & & \\ P(0) & \xleftarrow{a} & P(1) & \xleftarrow{b} & P(2) & \xleftarrow{c} & P(0) & \xrightarrow{f} & P(3) \end{array}$$

We finally give an example involving band complexes.

Example 8.2. For this example, we take Λ to be the algebra given in Example 4.6 because it has smaller homotopy bands than the example above. Let $\sigma = hg\bar{d}\bar{f}$ and $\tau = b\bar{e}\bar{d}c$ be homotopy bands with corresponding scalars λ and μ respectively. By Proposition 4.5, the mapping cone for the graph map $f^\bullet: B_{\sigma,\lambda}^\bullet \rightarrow B_{\tau,\mu}^\bullet$ given below is $M_{f^\bullet}^\bullet \cong B_{c,\lambda\mu}^\bullet$, where $c = hge\bar{b}\bar{c}\bar{f}$.

$$\begin{array}{ccccccc} & & & P(6) & \xrightarrow{\lambda hg} & P(5) & \xleftarrow{df} & P(6) & \xrightarrow{\quad} & \\ & & & & & \parallel & & \downarrow f & & \\ & & & & & & & & & \\ P(3) & \xrightarrow{b} & P(2) & \xleftarrow{e} & P(5) & \xleftarrow{d} & P(4) & \xrightarrow{\mu c} & P(3) & \end{array}$$

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