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THE EXT ALGEBRA OF A BRAUER GRAPH ALGEBRA

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Abstract. In this paper we study finite generation of the Ext algebra of a Brauer graph algebra by determining the degrees of the generators. As a consequence we characterize the Brauer graph algebras that are Koszul and those that are $\mathcal{K}_2$.

Introduction

This paper studies finite generation, and the corresponding degrees of the generators, of the Ext algebra of a Brauer graph algebra. We show that if the Brauer graph has no truncated edges then the Ext algebra of the associated Brauer graph algebra is finitely generated in degrees 0, 1 and 2. As a result we characterize those Brauer graph algebras that are $\mathcal{K}_2$ in the sense of Cassidy and Shelton [5]. Moreover, we determine the Koszul and the d-Koszul Brauer graph algebras.

Let K be a field, $\mathcal{Q}$ a finite quiver and I an ideal of $K\mathcal{Q}$. Let J be the ideal in $K\mathcal{Q}$ which is generated by the arrows of $\mathcal{Q}$ and assume that I is contained in J^2. Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional indecomposable algebra. Let $\mathbf{r}$ denote the Jacobson radical J/I of Λ. The Ext algebra (or cohomology ring) of Λ is given by $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) = \oplus_{n\geqslant 0} \mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$ with the Yoneda product. If the ideal I is generated by length homogeneous elements, then the length grading of $K\mathcal{Q}$ induces a grading $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$, where Λ_0 is the K-space spanned by the vertices of $\mathcal{Q}$. The graded Jacobson radical of Λ is $\mathbf{r} = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$, and $\Lambda_0 \cong \Lambda/\mathbf{r}$. In Section 1, we recall the definition of a Brauer graph and its associated Brauer graph algebra, as well as the concept of a truncated edge in a Brauer graph. The latter is of fundamental importance in determining the behaviour of the Ext algebra of a Brauer graph algebra.

This work was motivated by the study of the Koszul Brauer graph algebras. Koszul algebras play an important role in representation theory, and it is well-known that if $\Lambda = K\mathcal{Q}/I$ is a Koszul algebra then the Ext algebra $E(\Lambda)$ is finitely generated in degrees 0 and 1. Moreover the ideal I is quadratic, that is, I is generated by homogeneous

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elements of length 2. We determine the Koszul Brauer graph algebras in Theorem 2.4, before investigating several generalizations of this concept among Brauer graph algebras.

One such generalization is the class of $\mathcal{K}_2$ algebras, which was introduced by Cassidy and Shelton in [5]. They define a graded algebra A to be $\mathcal{K}_2$ if $E(A)$ is generated as an algebra in degrees 0, 1 and 2. In Theorem 8.1, we shows that if a Brauer graph has no truncated edges, then the Ext algebra of the associated Brauer graph algebra is finitely generated in degrees $0, 1$ and 2. We then characterize the $\mathcal{K}_2$ Brauer graph algebras in Theorem 9.4.

Another generalization of a Koszul algebra is that of a d-Koszul algebra, where $d \geqslant 2$; these algebras were introduced by Berger in [3]. A graded algebra $\Lambda = K\mathcal{Q}/I$ is a d-Koszul algebra if the n-th projective module in a minimal graded projective Λ-resolution of Λ_0 can be generated in degree $\delta(n)$, where the map $\delta : \mathbb{N} \to \mathbb{N}$ is defined by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd.} \end{cases}$$

In particular, the ideal I of a d-Koszul algebra $K\mathcal{Q}/I$ is generated by homogeneous elements of length d. Note that if $d = 2$, we recover the usual (quadratic) Koszul algebras. It was shown in [9] that the Ext algebra of a d-Koszul algebra is finitely generated in degrees 0, 1 and 2; thus d-Koszul algebras are $\mathcal{K}_2$ algebras. The d-Koszul Brauer graph algebras are fully determined in Theorem 2.4.

It is then natural to consider algebras $K\mathcal{Q}/I$ where the ideal I is generated by homogeneous elements of more than one length. We say that $\Lambda = K\mathcal{Q}/I$ is a 2-d-homogeneous algebra if I can be generated by homogeneous elements of degrees 2 and d. Green and Marcos introduced 2-d-determined and 2-d-Koszul algebras in [8]. We discuss these algebras in Section 9, but note here simply that a 2-d-Koszul algebra is 2-d-determined, which in turn is 2-d-homogeneous; moreover a 2-d-determined algebra is 2-d-Koszul if its Ext algebra is finitely generated. Having determined the 2-d-homogeneous Brauer graph algebras in Section 2, we return to these algebras in Section 9, where we give a positive answer for Brauer graph algebras to all three questions posed in [8]. As a consequence we are also able to give new classes of 2-d-Koszul algebras. In particular, the second question in [8] asks whether it is the case that the Ext algebra of a 2-d-Koszul algebra of infinite global dimension is necessarily generated in degrees 0, 1 and 2. This is indeed the case for Brauer graph algebras. Moreover, for a 2-d-homogeneous Brauer graph algebra $\mathcal{A}_\Gamma$ with Brauer graph Γ, we show in Theorem 9.6, that the following four conditions are equivalent: (1) Γ has no truncated edges, (2) $\mathcal{A}_\Gamma$ is 2-d-determined, (3) $\mathcal{A}_\Gamma$ is 2-d-Koszul, and (4) the Ext algebra of $\mathcal{A}_\Gamma$ is generated in degrees 0, 1 and 2 (that is, $\mathcal{A}_\Gamma$ is $\mathcal{K}_2$). It should be noted that these properties are not, in general, equivalent, as is demonstrated by Cassidy and Phan in [4].

This paper extends the work of Antipov and Generalov, who showed in [1] that the Ext algebra of a symmetric Brauer graph algebra is finitely generated. We remark that [1] used different methods, and gave no details of the degrees of the generators.

The body of the paper proves the necessary structural results to describe the uniserial modules, string modules, syzygies and projective resolutions of the simple modules. The paper uses the covering theory for Brauer graphs and Brauer graph algebras developed in [10]. We see at the start of Section 3, that [10] enables us to reduce to the case where the Brauer graph has no loops or multiple edges and where the multiplicity function is identically one; this vastly reduces the computations required to determine the Ext algebra. We then discuss the quantizing function $\mathfrak{q}$, and show, if K is algebraically closed, that we may further reduce to the case where $\mathfrak{q} \equiv 1$. Note that if the field is algebraically closed and if either the Brauer graph is a tree or the Brauer graph algebra is symmetric, then necessarily $\mathfrak{q} \equiv 1$.

We now introduce Brauer graph algebras, giving the definitions and notation which we use throughout the paper.

1. Background and Notation

Let Γ be a finite connected graph with at least one edge. We denote by Γ_0 the set of vertices of Γ and by Γ_1 the set of edges of Γ. We equip Γ with a *multiplicity function* $\mathfrak{m}\colon \Gamma_0 \to \mathbb{N} \setminus \{0\}$ and, for each vertex in Γ, we fix a cyclic ordering $\mathfrak{o}$ of the edges incident with this vertex. We call the triple $(\Gamma, \mathfrak{o}, \mathfrak{m})$ a *Brauer graph*. We may denote a Brauer graph by Γ, where the choice of cyclic ordering and multiplicity function are suppressed. In all examples a planar embedding of Γ is given and we choose the cyclic ordering to be the clockwise ordering of the edges around each vertex.

We say that an edge t in Γ is the *successor* of the edge s at the vertex α if both s and t are incident with α and edge t directly follows edge s in the cyclic ordering around α. For each $\alpha \in \Gamma_0$, let $\mathrm{val}(\alpha)$ denote the *valency* of α, that is, the number of edges incident with α where we count each loop as two edges. If $\mathrm{val}(\alpha) = 1$ with edge s incident with the vertex α then we say that s is its own successor. If α is a vertex with $\mathrm{val}(\alpha) = 1$ and $\mathfrak{m}(\alpha) = 1$ so that s is the only edge incident with α then we call s a *truncated edge* at the vertex α.

Following [2] and [13], we let K be a field and introduce the Brauer graph algebra of a Brauer graph Γ. We associate to Γ a quiver $\mathcal{Q}_\Gamma$ and a set of relations ρ_Γ in the path algebra $K\mathcal{Q}_\Gamma$, which we call the *Brauer graph relations*, defined below. Let I_Γ be the ideal of $K\mathcal{Q}_\Gamma$ which is generated by the set ρ_Γ. We define the *Brauer graph algebra* $\mathcal{A}_\Gamma$ of Γ to be the quotient $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma / I_\Gamma$. We keep the notation of [10] throughout this paper, and now define $\mathcal{Q}_\Gamma$ and ρ_Γ.

If the Brauer graph Γ is $\alpha \text{———} \beta$ with $\mathfrak{m}(\alpha) = \mathfrak{m}(\beta) = 1$ then $\mathcal{Q}_\Gamma$ is $\cdot \circlearrowleft x$ and $\rho_\Gamma = \{x^2\}$ so the Brauer graph algebra is $K[x]/(x^2)$.

We now define $\mathcal{Q}_\Gamma$ for a general Brauer graph (excluding the above case, so if edge s is truncated at vertex α and the endpoints of s are α and β then $\mathfrak{m}(\beta)\operatorname{val}(\beta) \geqslant 2$). The vertices of $\mathcal{Q}_\Gamma$ correspond to the edges of Γ, that is, for every edge $s \in \Gamma_1$ there is a corresponding vertex v_s in $\mathcal{Q}_\Gamma$. If edge t is the successor of edge s at the vertex α and edge s is not a truncated edge at α then there is an arrow from v_s to v_t in $\mathcal{Q}_\Gamma$. For each vertex α and edge s incident with α, let $s = s_0, s_1, \ldots, s_{\operatorname{val}(\alpha)-1}$ be the edges incident with α listed in the cyclic ordering, where the loops are listed twice and the other edges precisely once. We call this the *successor sequence of s at α*. We set $s_{\operatorname{val}(\alpha)} = s$, noting that s is the successor of $s_{\operatorname{val}(\alpha)-1}$. Observe that if $s = s_0, s_1, s_2, \ldots, s_{n-1}$ is the successor sequence for s at vertex α, then $s_1, s_2, \ldots, s_{n-1}, s_0$ is the successor sequence for s_1 at α.

In case Γ has at least one loop, care must be taken. In such circumstances, for each vertex α, we choose a distinguished edge, s_α, incident with α. If ℓ is a loop at α, ℓ occurs twice in the successor sequence of s_α. We distinguish the first and second occurrences of ℓ in this sequence and view the two occurrences as two edges in Γ_1. Thus, Γ_1 is the set of all edges with the proviso that loops are listed twice and have different successors.

In order to define the Brauer graph relations ρ_Γ we need a *quantizing function* $\mathfrak{q}$. Let $\mathcal{X}_\Gamma$ be the set of pairs (s, α) such that $s \in \Gamma_1$ is incident with $\alpha \in \Gamma_0$ and s is not truncated at either of its endpoints, and let $\mathfrak{q}\colon \mathcal{X}_\Gamma \to K \setminus \{0\}$ be a set function. We denote $\mathfrak{q}((s, \alpha))$ by $\mathfrak{q}_{s,\alpha}$. With this additional data we call $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ a *quantized Brauer graph.* We remark that if the Brauer graph Γ is $\alpha \text{———} \beta$ then $\mathcal{X}_\Gamma = \emptyset$. If the field is algebraically closed and if either the Brauer graph is a tree or the Brauer graph algebra is symmetric, then $\mathfrak{q} \equiv 1$ (see [2]).

There are three types of relations for $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$. Note that we write our paths from left to right.

Relations of type one. For each vertex α and edge s incident with α, which is not truncated at the vertex α, let $s = s_0, s_1, \ldots, s_{\operatorname{val}(\alpha)-1}$ be the successor sequence of s at α. From this we obtain a cycle $C_{s,\alpha} = a_0 a_1 \cdots a_{\operatorname{val}(\alpha)-1}$ in $\mathcal{Q}_\Gamma$ where the arrow a_r corresponds to the edge s_{r+1} being the successor of the edge s_r at the vertex α. With this notation, for each edge $s \in \Gamma$ with endpoints α and β so that s is not truncated at either α or β, ρ_Γ contains either $\mathfrak{q}_{s,\alpha} C_{s,\alpha}^{\mathfrak{m}(\alpha)} - \mathfrak{q}_{s,\beta} C_{s,\beta}^{\mathfrak{m}(\beta)}$ or $\mathfrak{q}_{s,\beta} C_{s,\beta}^{\mathfrak{m}(\beta)} - \mathfrak{q}_{s,\alpha} C_{s,\alpha}^{\mathfrak{m}(\alpha)}$. We call this a type one relation. Note that since one of these relations is the negative of the other, the ideal I_Γ does not depend on this choice.

Relations of type two. The second type of relation occurs if s is a truncated edge at the vertex α and the endpoints of s are α and β. Let $C_{s,\beta} = b_0 b_1 \cdots b_{\operatorname{val}(\beta)-1}$ be the cycle associated to edge s incident with vertex β. In this case we have a relation $C_{s,\beta}^{\mathfrak{m}(\beta)} b_0$.

Relations of type three. These relations are quadratic monomial relations of the form ab in $K\mathcal{Q}_\Gamma$ where ab is not a subpath of any $C_{s,\alpha}$.

We note that it is well-known that a Brauer graph algebra is special biserial and weakly symmetric.

Throughout this paper, all modules are right modules. We denote the Jacobson radical $\mathbf{r}_{\mathcal{A}_\Gamma}$ of the Brauer graph algebra $\mathcal{A}_\Gamma$ by $\mathbf{r}$ when no confusion can arise. We will use lower case letters such as s and t to denote edges in Γ, capital letters S and T to denote the corresponding simple $\mathcal{A}_\Gamma$-modules, and v_s and v_t to denote the corresponding vertices in $\mathcal{Q}_\Gamma$. If S is a simple $\mathcal{A}_\Gamma$-module, then we denote the projective $\mathcal{A}_\Gamma$-cover of S by $\pi_S\colon P_S \to S$.

Let P be an indecomposable projective $\mathcal{A}_\Gamma$-module corresponding to the vertex v_s in $\mathcal{Q}_\Gamma$ and edge s in Γ. If the edge s is not truncated, then P has both top and socle isomorphic to S, and $\operatorname{rad} P/\operatorname{Soc} P$ is a direct sum of two uniserial modules. Let the vertices of s be α and β and let $s, s_1, \dots, s_{\mathrm{val}(\alpha)-1}$ be the successor sequence for s at α, and $s, t_1, \dots, t_{\mathrm{val}(\beta)-1}$ be the successor sequence for s at β. Then $\operatorname{rad} P/\operatorname{Soc} P \cong U \oplus V$, where U and V have composition series

$$S_1, \dots, S_{\mathrm{val}(\alpha)-1}, S, S_1, \dots, S_{\mathrm{val}(\alpha)-1}, \dots, S, S_1, \dots, S_{\mathrm{val}(\alpha)-1}$$

and

$$T_1, \dots, T_{\mathrm{val}(\beta)-1}, S, T_1, \dots, T_{\mathrm{val}(\beta)-1}, \dots, S, T_1, \dots, T_{\mathrm{val}(\beta)-1}$$

respectively, such that, for $i = 1, \dots, \mathrm{val}(\alpha) - 1$, the simple module S_i occurs precisely $\mathfrak{m}(\alpha)$ times and is associated to the edge s_i in Γ, and, for $j = 1, \dots, \mathrm{val}(\beta) - 1$, the simple module T_j occurs precisely $\mathfrak{m}(\beta)$ times and is associated to the edge t_j in Γ.

In the case where s is truncated, then P is itself uniserial. Suppose that s is not truncated at vertex α and let $s, s_1, \dots, s_{\mathrm{val}(\alpha)-1}$ be the successor sequence for s at α. Then P has composition series

$$S, S_1, \dots, S_{\mathrm{val}(\alpha)-1}, S, S_1, \dots, S_{\mathrm{val}(\alpha)-1}, \dots, S, S_1, \dots, S_{\mathrm{val}(\alpha)-1}, S$$

where, for $i = 1, \dots, \mathrm{val}(\alpha) - 1$, the simple module S_i occurs precisely $\mathfrak{m}(\alpha)$ times and is associated to the edge s_i in Γ.

2. d-homogeneous and 2-d-homogeneous Brauer graph algebras

Suppose that $\Lambda = K\mathcal{Q}/I$ where I is a homogeneous ideal with respect to the length grading. Let ρ be a minimal set of generators for I; the elements in ρ are necessarily homogeneous. Let $d \geqslant 2$ and $d' \geqslant 2$ be distinct integers. We say that Λ is *d-homogeneous* (or *quadratic* when $d = 2$) if ρ contains homogeneous elements of length d only. We say that Λ is *d-d'-homogeneous* if Λ is not d-homogeneous or d'-homogeneous and ρ consists of homogeneous elements of length d or d'. Note that this does not depend on the choice of minimal generating set ρ for I.

In this section we investigate the d-homogeneous and 2-d-homogeneous Brauer graph algebras. Therefore we need to know a minimal generating set for I_Γ. Recall that, for an integer $n \geqslant 1$, $\mathbb{A}_n$ is the graph $\cdot \text{——} \cdot \text{——} \cdots\cdots \text{——} \cdot$ with n vertices and $\widetilde{\mathbb{A}}_n$ is the circular graph with $n+1$ vertices.

Lemma 2.1. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph such that Γ is not $\mathbb{A}_2$, and let $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$ be the associated Brauer graph algebra. Let $\rho \subset \rho_\Gamma$ be a minimal generating set for I_Γ. Then ρ contains all the relations of types one and three, and it contains the relation of type two associated to the edge s truncated at α if and only if the successor of s at its other endpoint β is also truncated.*

Proof. It is clear that relations of type one and three must be in ρ, so we must prove the condition on relations of type two. Since $\Gamma \neq \mathbb{A}_2$ and s is truncated at α, the edge s has a successor s_1 distinct from s at β. Let $s = s_0, s_1, \ldots, s_{\mathrm{val}(\beta)-1}$ be the successor sequence of s at β, and let R_2 be the relation of type two associated to s, so that $R_2 = C_{s,\beta}^{\mathfrak{m}(\beta)} b_0$ where $C_{s,\beta} = b_0 b_1 \cdots b_{\mathrm{val}(\beta)-1}$.

First assume that s_1 is not truncated at its other endpoint γ. We want to prove that the relation R_2 is not in ρ. Since s_1 is not truncated at either of its endpoints, there is a relation of type one associated to s_1, of the form $R_1 := \mathfrak{q}_{s_1,\beta} C_{s_1,\beta}^{\mathfrak{m}(\beta)} - \mathfrak{q}_{s_1,\gamma} C_{s_1,\gamma}^{\mathfrak{m}(\gamma)} \in \rho$. We have $C_{s_1,\beta} = b_1 \cdots b_{\mathrm{val}(\beta)-1} b_0$, and we let $C_{s_1,\gamma} = a_0 a_1 \cdots a_{\mathrm{val}(\gamma)-1}$. Therefore $R_2 = C_{s,\beta}^{\mathfrak{m}(\beta)} b_0 = b_0 C_{s_1,\beta}^{\mathfrak{m}(\beta)} = \mathfrak{q}_{s_1,\beta}^{-1} b_0 R_1 + \mathfrak{q}_{s_1,\beta}^{-1} \mathfrak{q}_{s_1,\gamma} R_3 a_1 \cdots a_{\mathrm{val}(\gamma)-1} C_{s_1,\gamma}^{\mathfrak{m}(\gamma)-1}$, where $R_3 := b_0 a_0 \in \rho$ is a relation of type three. Therefore $R_2 \notin \rho$.

Conversely, assume that s_1 is truncated at γ. Suppose that $R_2 \notin \rho$, so that we can write $R_2 = \sum_{i=1}^p \lambda_i R_1^{(i)} \mu_i + \sum_{j=1}^q \lambda'_j R_2^{(j)} \mu'_j + \sum_{k=1}^r \lambda''_k R_3^{(k)} \mu''_k$ for some $\lambda_i, \lambda'_j, \lambda''_k, \mu_i, \mu'_j, \mu''_k$ in $K\mathcal{Q}_\Gamma$ and relations $R_1^{(i)}, R_2^{(j)}, R_3^{(k)}$ of type one, two and three in ρ. We work in $K\mathcal{Q}_\Gamma$, which is graded by length.

The relation R_2 is monomial, hence must occur in one of the summands. By definition, the $R_3^{(k)}$ are not subpaths of R_2 (since the proper subpaths of R_2 are all subpaths of some $C_{s_i,\beta}$ and R_2 has length at least 3). Moreover, if $\lambda'_j R_2^{(j)} \mu'_j = R_2$, then $R_2^{(j)}$ is a product of (some of) the arrows b_ℓ for $0 \leqslant \ell \leqslant \mathrm{val}(\beta)-1$, so that $R_2^{(j)}$ must be a cyclic permutation of R_2 and hence equal to R_2, a contradiction. Finally, if R_2 is in the term $\lambda_i R_1^{(i)} \mu_i$, then $R_1^{(i)}$ is a K-linear combination of $C_{s_t,\beta}^{\mathfrak{m}(\beta)}$ for some t, and another cycle that does not contain a b_ℓ, and, for length reasons, we must have $t = 0$ or $t = 1$. But $s_0 = s$ and s_1 are truncated at α and γ respectively, so there are no relations of type one associated to these edges. We have again a contradiction, and therefore conclude that R_2 is in ρ. $\square$

We start by describing all d-homogeneous Brauer graph algebras for $d \geqslant 2$.

Proposition 2.2. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph and let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. Then $\mathcal{A}_\Gamma$ is quadratic if and only if $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is one of the following Brauer graphs.*

(1) $\Gamma = \mathbb{A}_2$ *with* $\mathfrak{m} \equiv 1$ *and* $\mathfrak{q} \equiv 1$.
(2) $\Gamma = \mathbb{A}_n$ *with* $n \geqslant 2$*, all multiplicities equal to* 1 *except at the first and last vertices which are equal to* 2.
(3) $\Gamma = \mathbb{A}_n$ *with* $n \geqslant 3$*, all multiplicities equal to* 1 *except at one end vertex which is equal to* 2.
(4) $\Gamma = \mathbb{A}_n$ *with* $n \geqslant 4$, $\mathfrak{m} \equiv 1$ *and* $\mathfrak{q} \equiv 1$.
(5) $\Gamma = \widetilde{\mathbb{A}}_n$ *with* $n \geqslant 1$ *and* $\mathfrak{m} \equiv 1$.
(6) $\Gamma =$  *with* $\mathfrak{m} \equiv 1$.

Proof. Suppose that $\mathcal{A}_\Gamma$ is quadratic. Let $\rho \subset \rho_\Gamma$ be a minimal generating set for I_Γ. If α is a vertex in Γ such that $\operatorname{val}(\alpha) > 2$, then there is a relation of type one or type two of length at least 3 in ρ. This contradicts the fact that $\mathcal{A}_\Gamma$ is quadratic. Therefore $\operatorname{val}(\alpha) \leqslant 2$ for all vertices α in Γ.

There are two cases to consider.

(i) We assume that there is a vertex α in Γ with $\operatorname{val}(\alpha) = 1$.

Then there is a unique edge s in Γ with endpoint α. Let β denote the other endpoint of s. There are two subcases to consider here.

First suppose that edge s is truncated at α, that is, $\mathfrak{m}(\alpha) = 1$. If $\operatorname{val}(\beta) = 1$, then, since Γ is connected, $\mathcal{A}_\Gamma = K[x]/(x^{\mathfrak{m}(\beta)+1})$ which is quadratic if and only if $\mathfrak{m}(\beta) = 1$. Thus $\Gamma = \mathbb{A}_2$ and $\mathfrak{m} \equiv 1$; this is **(1)**. Note that we can assume that $\mathfrak{q} \equiv 1$ since there are no relations of type one. On the other hand, if $\operatorname{val}(\beta) = 2$, then we have a relation of type two of length $\operatorname{val}(\beta)\mathfrak{m}(\beta) + 1 \geqslant 3$ in I_Γ associated to the edge s incident with vertex β. This relation cannot be in ρ, so the successor s_1 of s at β is not truncated at its other endpoint γ, by Lemma 2.1. Hence we have a relation of type one associated to s_1, of length $\mathfrak{m}(\beta)\operatorname{val}(\beta) = \mathfrak{m}(\gamma)\operatorname{val}(\gamma)$, in ρ. Therefore $\mathfrak{m}(\beta)\operatorname{val}(\beta) = \mathfrak{m}(\gamma)\operatorname{val}(\gamma) = 2$. If $\operatorname{val}(\gamma) = 1$ then $\mathfrak{m}(\gamma) = 2$ so that $\Gamma = \mathbb{A}_3$ and the multiplicities are $(1, 1, 2)$; this is part of **(3)**. If $\operatorname{val}(\gamma) = 2$, then $\mathfrak{m}(\gamma) = 1$ and we continue, to get $\mathbb{A}_n$ with $n \geqslant 4$ and multiplicities either $(1, 1, \ldots, 1, 1)$ or $(1, 1, \ldots, 1, 2)$ (the last edge can be truncated if $n \geqslant 4$). We have thus obtained **(3)** and **(4)**.

We may now assume that there are no truncated edges. Since the edge s is not truncated at either of its endpoints α and β, we have a relation of type one associated to s in ρ so that $\operatorname{val}(\beta)\mathfrak{m}(\beta) = 2 = \operatorname{val}(\alpha)\mathfrak{m}(\alpha)$. Since $\operatorname{val}(\alpha) = 1$, we have that $\mathfrak{m}(\alpha) = 2$. Moreover, either $\operatorname{val}(\beta) = 1$ in which case Γ is the graph $\mathbb{A}_2$

with $\mathfrak{m} \equiv 2$, or $\mathrm{val}(\beta) = 2$, $\mathfrak{m}(\beta) = 1$ and we continue to get the graph $\mathbb{A}_n$ with multiplicities $(2, 1, 1, \ldots, 1, 2)$. This gives **(2)**.

(ii) We assume that all vertices have valency 2.

Let α be a vertex in Γ. Since $\mathrm{val}(\alpha) = 2$, there is either a loop, a double edge or two single edges at α.

If there is a loop s at α, then, since Γ is connected, Γ is equal to $\begin{tikzcd} \alpha \arrow[loop right, dash, "s"] \end{tikzcd}$. Then there is a relation of type one associated to s so that $\mathrm{val}(\alpha)\mathfrak{m}(\alpha) = 2$ and therefore $\mathfrak{m}(\alpha) = 1$. This is the graph of **(6)**. If there is a double edge at α, then a similar argument shows that $\mathfrak{m}(\alpha) = 1$. If β is the other vertex of this double edge, then we have $\mathrm{val}(\beta) \geqslant 2$. However, all vertices in Γ have valency at most 2, so that $\mathrm{val}(\beta) = 2$. So Γ is $\alpha = \beta$ with $\mathfrak{m} \equiv 1$, and we have the graph $\widetilde{\mathbb{A}}_1$ of **(5)**. Finally, suppose there are two edges s and t which are incident with α. By assumption, neither s nor t is truncated, so that there is a relation of type one associated to both s and t, and therefore $\mathrm{val}(\alpha)\mathfrak{m}(\alpha) = 2$ and $\mathfrak{m}(\alpha) = 1$. Continuing this argument, shows that $\Gamma = \widetilde{\mathbb{A}}_n$ with $n \geqslant 2$ and $\mathfrak{m} \equiv 1$, which is **(5)**.

This gives all possible Brauer graphs $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$. We now give the associated Brauer graph algebras, which are all quadratic.

(1) $\mathcal{A}_\Gamma = K[x]/(x^2)$.

(2) $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$ where $\mathcal{Q}_\Gamma$ is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "b_0"] \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "\bar{a}_1"] \arrow[r, bend left, "a_2"] & 3 \arrow[l, bend left, "\bar{a}_2"] & \cdots & n-2 \arrow[r, bend left, "a_{n-2}"] & n-1 \arrow[l, bend left, "\bar{a}_{n-2}"] \arrow[loop right, "b_{n-1}"] \end{tikzcd}$$

and the ideal I_Γ is generated by $a_i\bar{a}_i - \bar{a}_{i-1}a_{i-1}$, $a_{i-1}a_i$ and $\bar{a}_i\bar{a}_{i-1}$ for $2 \leqslant i \leqslant n-2$, $a_1\bar{a}_1 - b_0^2$, $\bar{a}_{n-2}a_{n-2} - qb_{n-1}^2$, b_0a_1, $\bar{a}_1b_0$, $a_{n-2}b_{n-1}$ and $b_{n-1}\bar{a}_{n-2}$ for some nonzero $q \in K$. Note that we have scaled the arrows in the quiver so that the quantizing function $\mathfrak{q}$ is 1 except for one value which we have denoted by q; moreover, if q has a square root in K, then we can replace q by 1 (see [7]).

(3) $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$ where $\mathcal{Q}_\Gamma$ is the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "\bar{a}_1"] \arrow[r, bend left, "a_2"] & 3 \arrow[l, bend left, "\bar{a}_2"] & \cdots & n-2 \arrow[r, bend left, "a_{n-2}"] & n-1 \arrow[l, bend left, "\bar{a}_{n-2}"] \arrow[loop right, "b_{n-1}"] \end{tikzcd}$$

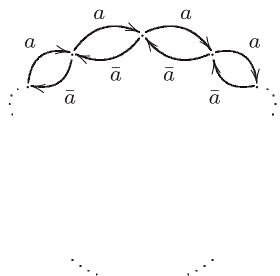
and the ideal I_Γ is generated by $a_i\bar{a}_i - \bar{a}_{i-1}a_{i-1}$, $a_{i-1}a_i$ and $\bar{a}_i\bar{a}_{i-1}$ for $2 \leqslant i \leqslant n-2$, $\bar{a}_{n-2}a_{n-2} - qb_{n-1}^2$, $a_{n-2}b_{n-1}$ and $b_{n-1}\bar{a}_{n-2}$ for some nonzero $q \in K$, which can be replaced by 1 if q has a square root in K.

(4) $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$ where $\mathcal{Q}_\Gamma$ is the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "\bar{a}_1"] \arrow[r, bend left, "a_2"] & 3 \arrow[l, bend left, "\bar{a}_2"] & \cdots & n-2 \arrow[r, bend left, "a_{n-2}"] & n-1 \arrow[l, bend left, "\bar{a}_{n-2}"] \end{tikzcd}$$

and the ideal I_Γ is generated by $a_i\bar{a}_i - \bar{a}_{i-1}a_{i-1}$, $a_{i-1}a_i$ and $\bar{a}_i\bar{a}_{i-1}$ for $2 \leqslant i \leqslant n-2$. Note that we have scaled the arrows in the quiver so that $\mathfrak{q} \equiv 1$.

(5) $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$ where $\mathcal{Q}_\Gamma$ is the quiver with n vertices and $2n$ arrows



and the ideal I_Γ is generated by a_ia_{i+1}, $\bar{a}_{i-1}\bar{a}_{i-2}$ and $a_i\bar{a}_i - q_i\bar{a}_{i-1}a_{i-1}$, for $i = 0, \ldots, m-1$, with $q_i \in K$, $q_i \neq 0$, where the subscripts are taken modulo n and where a_i denotes the arrow that goes from vertex i to vertex $i+1$ and $\bar{a}_i$ denotes the arrow that goes from vertex $i+1$ to vertex i. Again we can rescale the arrows so that $\mathfrak{q}$ is 1 at all but one place (see [6]).

(6) $\mathcal{A}_\Gamma = K\left[\,\beta \circlearrowleft \cdot \circlearrowright \alpha\,\right]/(\alpha\beta - q\beta\alpha, \alpha^2, \beta^2)$ for some nonzero $q \in K$. □

We now turn to d-homogeneous algebras with $d \geqslant 3$.

Proposition 2.3. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph and let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. Then $\mathcal{A}_\Gamma$ is d-homogeneous with $d \geqslant 3$ if and only if Γ is a star with n edges, for some $n \geqslant 1$, such that n divides $d-1$, the multiplicity of the central vertex is $\frac{d-1}{n}$ and the other multiplicities are equal to 1. The algebra $\mathcal{A}_\Gamma$ is uniquely determined by $(\Gamma, \mathfrak{o}, \mathfrak{m})$; it is the symmetric Nakayama algebra whose quiver is a cycle of length n and its ideal I_Γ is generated by all paths of length d.*

Proof. If $\mathcal{A}_\Gamma$ is d-homogeneous, then there are no relations of type three, so the quiver $\mathcal{Q}_\Gamma$ cannot contain distinct cycles at the same vertex. In terms of the graph Γ, this means that all edges in Γ are truncated at exactly one vertex. Therefore Γ is a star in which all the outer vertices have multiplicity 1. Let n be the number of edges in Γ and m the multiplicity of the central vertex. The only relations in the algebra $\mathcal{A}_\Gamma$ are of type two and are of length $nm+1$. Hence $nm = d-1$. Finally, since $\mathcal{A}_\Gamma$ is monomial, we may assume that $\mathfrak{q} \equiv 1$. □

It is well-known that all the Brauer graph algebras in Proposition 2.2**(1)**, **(2)**, **(5)**, **(6)** and in Proposition 2.3 are d-Koszul (see for instance [6, 7, 17] and the references therein).

However, it is easy to verify that the algebras of Proposition 2.2**(3)** and **(4)** are not Koszul, since the resolution of the simple module at the vertex 1 is not linear in either of these cases. This gives the following result.

Theorem 2.4. *Let* $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ *be a quantized Brauer graph and let* $\mathcal{A}_\Gamma$ *be the associated Brauer graph algebra. Then* $\mathcal{A}_\Gamma$ *is Koszul if and only if it is quadratic and either* $\Gamma = \mathbb{A}_2$ *or* Γ *has no truncated edges. For* $d \geqslant 3$*, the Brauer graph algebra* $\mathcal{A}_\Gamma$ *is* d*-Koszul if and only if it is* d*-homogeneous.*

Now, fix an integer $d > 2$. We describe the 2-d-homogeneous Brauer graph algebras.

Proposition 2.5. *Let* $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ *be a quantized Brauer graph and let* $\mathcal{A}_\Gamma$ *be the associated Brauer graph algebra. Then* $\mathcal{A}_\Gamma$ *is 2-*d*-homogeneous if and only if* $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ *satisfies one of the following conditions.*

(1) *For all vertices* α *in* Γ*, we have* $\mathfrak{m}(\alpha)\,\mathrm{val}(\alpha) = d$*.*
(2) Γ *has a truncated edge,* $\Gamma \neq \mathbb{A}_2$*, no two successors are truncated, and for every vertex* α *in* Γ *we have* $\mathfrak{m}(\alpha)\,\mathrm{val}(\alpha) \in \{1, d\}$*.*

Proof. Note that there must be at least one edge in Γ that is not truncated at either of its endpoints, otherwise we are in the situation of Proposition 2.2**(1)** or of Proposition 2.3, and the algebra $\mathcal{A}_\Gamma$ is quadratic or d-homogeneous. Let $\rho \subset \rho_\Gamma$ be a minimal set of generators for I_Γ. The proof has two cases.

(i) First assume that there is an edge s in Γ that is truncated at the vertex α in Γ. Let β be the other endpoint of s. If $\mathrm{val}(\beta) = 1$, then we only have relations of type two in ρ, all of the same length, so that $\mathcal{A}_\Gamma$ is homogeneous, which gives a contradiction.

We may therefore assume that there is an edge t in Γ, incident with β and such that $t \neq s$. There is a relation of type two associated to s at β of length $\mathrm{val}(\beta)\mathfrak{m}(\beta)+1 \geqslant 3$. If t is the successor of s at β and if t is truncated at its other endpoint, then, by Lemma 2.1, this relation of type two is in ρ, so that $\mathrm{val}(\beta)\mathfrak{m}(\beta) + 1 = d$. However, $\mathcal{A}_\Gamma$ is not d-homogeneous so there must be a nontruncated edge u incident with β so that $\mathrm{val}(\beta) \geqslant 3$. Thus there is a relation of type one associated to u of length $\mathrm{val}(\beta)\mathfrak{m}(\beta) = d - 1$ so that we have $d - 1 = 2$, since $\mathcal{A}_\Gamma$ is 2-d-homogeneous. But $\mathrm{val}(\beta) \geqslant 3$ so that we have a contradiction. This shows that no two successors are truncated and none of the relations of type two are in ρ.

Therefore the successor t of s at β is not truncated, and there is a relation of type one associated to t of length $\mathrm{val}(\beta)\mathfrak{m}(\beta) = \mathrm{val}(\gamma)\mathfrak{m}(\gamma)$, where γ is the other endpoint of t. Since $\mathcal{A}_\Gamma$ is not quadratic, we must have $\mathrm{val}(\beta)\mathfrak{m}(\beta) = \mathrm{val}(\gamma)\mathfrak{m}(\gamma) = d$. Continuing in this way, we see that every relation of type one must have length d and we get **(2)**. Moreover, if **(2)** is satisfied, then there are (quadratic) relations of type three since there are at least two adjacent cycles $C_{t,\beta}$ and $C_{t,\gamma}$ in $\mathcal{Q}_\Gamma$. Since there are no relations of type two in ρ and all relations of type one are of length d, it follows that $\mathcal{A}_\Gamma$ is indeed 2-d-homogeneous.

(ii) Now assume that there are no truncated edges in Γ. Therefore there are no relations of type two.

We suppose first that all vertices have valency at least 2. Then the relations of type one have length 2 or d. More precisely, for any edge s with endpoints α and β, we must have $\mathrm{val}(\alpha)\mathfrak{m}(\alpha) = \mathrm{val}(\beta)\mathfrak{m}(\beta) \in \{2, d\}$. Since Γ is connected and $\mathcal{A}_\Gamma$ is not quadratic, we must have $\mathrm{val}(\alpha)\mathfrak{m}(\alpha) = d$ for all vertices α, and we are in case **(1)**. Moreover, if all vertices α have valency at least 2 and $\mathrm{val}(\alpha)\mathfrak{m}(\alpha) = d$, then there are relations of type three so that $\mathcal{A}_\Gamma$ is 2-d-homogeneous.

Finally, we consider the case where there is a vertex α with $\mathrm{val}(\alpha) = 1$. Let s be the edge incident with α and let β be the other endpoint of s. Since s is not truncated at either endpoint, we have $\mathfrak{m}(\alpha) > 1$ and $\mathrm{val}(\beta)\mathfrak{m}(\beta) > 1$. If $\mathrm{val}(\beta) = 1$, then there are quadratic relations of type three, and a relation of type one associated to s of length $\mathfrak{m}(\alpha) = \mathfrak{m}(\beta)$, and so $\mathfrak{m}(\alpha) = \mathfrak{m}(\beta)$ must equal d. Thus $\Gamma = \mathbb{A}_2$ with multiplicity d at each vertex. It is easy to see that the corresponding algebra is 2-d-homogeneous. On the other hand, if $\mathrm{val}(\beta) > 1$, let t be another edge incident with β. By assumption, the edge t is not truncated at its other endpoint γ. Then there are quadratic relations of type three, a relation of type one associated to s of length $\mathfrak{m}(\alpha) = \mathrm{val}(\beta)\mathfrak{m}(\beta)$ and a relation of type one associated to t of length $\mathrm{val}(\beta)\mathfrak{m}(\beta) = \mathrm{val}(\gamma)\mathfrak{m}(\gamma)$. Therefore $\mathfrak{m}(\alpha) = \mathrm{val}(\beta)\mathfrak{m}(\beta) = \mathrm{val}(\gamma)\mathfrak{m}(\gamma) = d$. Continuing in this way, we have $\mathrm{val}(\varepsilon)\mathfrak{m}(\varepsilon) = d$ at every vertex ε in Γ, which completes **(1)**. The corresponding algebra is 2-d-homogeneous. □

We end this section with two corollaries which describe in more detail the 2-d-homogeneous Brauer graph algebras $\mathcal{A}_\Gamma$ in the cases where Γ is a star and where Γ is $\mathbb{A}_n$.

Corollary 2.6. *Suppose Γ is a star whose central vertex is α_0 and the other vertices are ordered cyclically $\alpha_1, \ldots, \alpha_n$; set $\alpha_{n+1} = \alpha_1$. Then the associated Brauer graph algebra is 2-d-homogeneous if and only if n divides d, the vertex α_0 has multiplicity $\frac{d}{n}$, and for every i with $1 \leqslant i \leqslant n$ we have $\mathfrak{m}(\alpha_i) \in \{1, d\}$ and $\mathfrak{m}(\alpha_i)\mathfrak{m}(\alpha_{i+1}) \in \{d, d^2\}$.*

Corollary 2.7. *Suppose $\Gamma = \mathbb{A}_n$. Then the associated generalized Brauer tree algebra is 2-d-homogeneous if and only if $n \geqslant 3$, d is even, the multiplicities of the first and last vertex are in $\{1, d\}$ with at least one of them equal to d if $n = 3$, and the multiplicities of the other vertices are all equal to $\frac{d}{2}$.*

We now look more generally at the Ext algebra of a Brauer graph algebra. We will return in Section 9 to 2-d-homogeneous Brauer graph algebras and the degrees in which the Ext algebra is generated.

3. The Ext algebra and coverings

In this section, we use the covering theory for Brauer graphs which was developed in [10] to simplify the calculation of the Ext algebra. We show that we may assume, without any loss of generality, that our quantized Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ has $\mathfrak{m} \equiv 1$ and contains no loops or multiple edges. We then discuss the quantizing function $\mathfrak{q}$, proving in Proposition 3.2, that if the field K is algebraically closed field and if the associated Brauer graph algebra $\mathcal{A}_\Gamma$ is length graded, then the number of generators of the Ext algebra $E(\mathcal{A}_\Gamma)$ and their degrees do not depend on $\mathfrak{q}$.

Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. We recall the following definitions from [10]. For each $\alpha \in \Gamma_0$, we define $\mathcal{Z}_\alpha$ to be the set $\mathcal{Z}_\alpha = \{(s,t) \mid s,t \in \Gamma_1, t \text{ is the successor of } s \text{ at vertex } \alpha\}$. Let $\mathcal{Z}_\Gamma$ be the disjoint union

$$\mathcal{Z}_\Gamma = \dot{\bigcup_{\alpha \in \Gamma_0}} \mathcal{Z}_\alpha.$$

Let G be a finite abelian group. A set function $W\colon \mathcal{Z}_\Gamma \to G$ is called a *successor weighting* of the Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$. For $\alpha \in \Gamma_0$ we define the *order of* α, denoted $\operatorname{ord}(\alpha)$, to be the order in G of the element $\omega_\alpha = \prod_{(s,t)\in\mathcal{Z}_\alpha} W(s,t)$. A successor weighting $W\colon \mathcal{Z}_\Gamma \to G$ of the Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ is called a *Brauer weighting* if $\operatorname{ord}(\alpha) \mid \mathfrak{m}(\alpha)$ for all $\alpha \in \Gamma_0$.

Let $W\colon \mathcal{Z}_\Gamma \to G$ be a Brauer weighting, and let $\mathcal{A}_{\Gamma_W}$ be the Brauer graph algebra associated to the quantized Brauer covering graph $(\Gamma_W, \mathfrak{o}_W, \mathfrak{m}_W, \mathfrak{q}_W)$. Let $\mathbf{r}_{\mathcal{A}_\Gamma}$ (respectively, $\mathbf{r}_{\mathcal{A}_{\Gamma_W}}$) be the Jacobson radical of $\mathcal{A}_\Gamma$ (respectively, $\mathcal{A}_{\Gamma_W}$). By [10, Theorem 4.3], $\mathcal{A}_{\Gamma_W}$ is the covering algebra associated to a weight function $W^*\colon (\mathcal{Q}_\Gamma)_1 \to G$. Hence the Ext algebra $\operatorname{Ext}^*_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma}, \mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma})$ is generated in degrees $d_1, \ldots, d_s$ if and only if the Ext algebra $\operatorname{Ext}^*_{\mathcal{A}_{\Gamma_W}}(\mathcal{A}_{\Gamma_W}/\mathbf{r}_{\mathcal{A}_{\Gamma_W}}, \mathcal{A}_{\Gamma_W}/\mathbf{r}_{\mathcal{A}_{\Gamma_W}})$ is generated in degrees $d_1, \ldots, d_s$. In fact, we have the following well-known result.

Proposition 3.1. *Keeping the notation above, the ring structure of* $\operatorname{Ext}^*_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma}, \mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma})$ *can be completely determined from the ring structure of* $\operatorname{Ext}^*_{\mathcal{A}_{\Gamma_W}}(\mathcal{A}_{\Gamma_W}/\mathbf{r}_{\mathcal{A}_{\Gamma_W}}, \mathcal{A}_{\Gamma_W}/\mathbf{r}_{\mathcal{A}_{\Gamma_W}})$.

Proof. The G-grading on $\mathcal{A}_\Gamma$ induced by the weight function $W^*\colon (\mathcal{Q}_\Gamma)_1 \to G$ induces a G-grading on $\operatorname{Ext}^*_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma}, \mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma})$ such that, if $g \in G$, and S and T are simple $\mathcal{A}_\Gamma$-modules then

$$\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(S,T)_g = \operatorname{Ext}^n_{\mathbf{Gr}}(S, \sigma(g)T),$$

where the right hand side is the graded Ext-group, S and T are viewed as graded modules with support in degree 1_G, and σ is the shift functor. The graded Yoneda product $\mathrm{Ext}^n_{\mathbf{Gr}}(T, \sigma(g)U) \times \mathrm{Ext}^n_{\mathbf{Gr}}(S, \sigma(h)T)$ is defined in the usual way after noting that $\mathrm{Ext}^n_{\mathbf{Gr}}(T, \sigma(g)U) \cong \mathrm{Ext}^n_{\mathbf{Gr}}(\sigma(h)T, \sigma(h)\sigma(g)U)$. Finally, using that the category of G-graded $\mathcal{A}_\Gamma$-modules is equivalent to the category of $\mathcal{A}_{\Gamma_W}$-modules we obtain the desired result. □

Now, in [10, Theorem 6.8], it was shown, for any quantized Brauer graph $(\Gamma_0, \mathfrak{o}_0, \mathfrak{m}_0, \mathfrak{q}_0)$, that there is a tower of quantized Brauer covering graphs $(\Gamma_0, \mathfrak{o}_0, \mathfrak{m}_0, \mathfrak{q}_0), (\Gamma_1, \mathfrak{o}_1, \mathfrak{m}_1, \mathfrak{q}_1), (\Gamma_2, \mathfrak{o}_2, \mathfrak{m}_2, \mathfrak{q}_2), (\Gamma_3, \mathfrak{o}_3, \mathfrak{m}_3, \mathfrak{q}_3)$ such that the topmost quantized Brauer covering graph $(\Gamma_3, \mathfrak{o}_3, \mathfrak{m}_3, \mathfrak{q}_3)$ has multiplicity function $\mathfrak{m}_3$ identically one, and the graph Γ_3 has no loops or multiple edges.

Hence, with Proposition 3.1, we may assume that $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ is a quantized Brauer graph with $\mathfrak{m} \equiv 1$ and with no loops or multiple edges.

We now consider the quantizing function $\mathfrak{q}$ in the case where $\mathcal{A}_\Gamma$ is length graded. It is known that if the field is algebraically closed and if either the Brauer graph is a tree or the Brauer graph algebra is symmetric, then we can always rescale the arrows so that $\mathfrak{q} \equiv 1$. The next result shows that we may also assume that $\mathfrak{q} \equiv 1$ in the case where $\mathcal{A}_\Gamma$ is length graded, since the number of generators of the Ext algebra $E(\mathcal{A}_\Gamma)$ and their degrees do not depend on $\mathfrak{q}$.

We begin by introducing some additional notation. If edge t is the successor of edge s in Γ at vertex α, we denote the corresponding arrow in $\mathcal{Q}_\Gamma$ from vertex v_s to vertex v_t by $a(s, t, \alpha)$. In fact, this arrow is uniquely determined by s and t. For, suppose there are vertices α and β in Γ such that $a(s, t, \alpha)$ and $a(s, t, \beta)$ are arrows. Since there are no loops in Γ, we have $s \neq t$. If $\alpha \neq \beta$, then s and t are distinct edges with endpoints α and β, contradicting the assumption that there are no multiple edges in Γ. Hence $\alpha = \beta$. So if edge t is the successor of edge s at vertex α in Γ, then the vertex α is unique. Thus we denote the arrow in $\mathcal{Q}_\Gamma$ from vertex v_s to vertex v_t simply by $a(s, t)$.

Proposition 3.2. *Let K be an algebraically closed field and let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph with $\mathfrak{m} \equiv 1$ and with no loops or multiple edges. Let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. Suppose that $\mathcal{A}_\Gamma$ is length graded. Then the number of generators of $\mathrm{Ext}_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma}, \mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma})$ and their degrees do not depend on $\mathfrak{q}$.*

Proof. Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph with $\mathfrak{m} \equiv 1$ and with no loops or multiple edges. We may assume that Γ is not a star ($\mathbb{A}_2$ included) since the associated Brauer graph algebras are all monomial and hence do not depend on a quantizing function $\mathfrak{q}$. Therefore there exists an edge s with endpoints α and β such that $v := \mathrm{val}(\alpha) > 1$ and $\mathrm{val}(\beta) > 1$.

Let A be the Brauer graph algebra associated to $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with quantizing function identically 1. We shall twist A by a graded algebra automorphism σ of A so that A^σ is the Brauer graph algebra associated to the quantized Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ with $\mathfrak{q}$ equal to 1 except at (s, α) and (s, β), then use [5] to see that the Ext algebras of A and A^σ have the same number of generators in the same degrees. This means that we will have changed precisely one relation in the generating set for I_Γ, namely $C_{s,\alpha} - C_{s,\beta}$ will become $\mathfrak{q}_{s,\alpha} C_{s,\alpha} - \mathfrak{q}_{s,\beta} C_{s,\beta}$ or, to simplify notation, $rC_{s,\alpha} - C_{s,\beta}$ where $r = \mathfrak{q}_{s,\alpha} \mathfrak{q}_{s,\beta}^{-1}$. None of the other relations will change.

Recall that the product in A^σ is given by $x \cdot y = x\sigma^{\ell(x)}(y)$ where x and y are length homogeneous elements in A and $\ell(x)$ is the length of x.

Let $s = s_0, s_1, s_2, \ldots, s_{v-1}$ be the successor sequence of s at α, and let $a_i = a(s_i, s_{i+1})$ be the corresponding arrows in the quiver $\mathcal{Q}_\Gamma$, for $i = 0, 1, \ldots, v-1$ (where $s_v = s$). In this notation, $C_{s,\alpha} = a_0 a_1 \cdots a_{v-1}$. We shall define a graded algebra automorphism σ of A by setting, for $i = 0, 1, \ldots, v-1$, $\sigma(a_i) = r_i a_i$ for some $r_i \in K \setminus \{0\}$ to be determined, and fixing all other arrows and vertices in $\mathcal{Q}_\Gamma$. Suppose we have such an automorphism σ. Then, in A^σ, the cycle $C_{s,\alpha}$ becomes $r_1 r_2^2 \cdots r_{v-1}^{v-1} a_0 a_1 \cdots a_{v-1}$. The arrows $a_0, \ldots, a_{v-1}$ occur in at most v relations of type one, involving the cyclic permutations of $C_{s,\alpha}$. Therefore we want

$$\begin{aligned}
&r_1 r_2^2 \cdots r_{v-2}^{v-2} r_{v-1}^{v-1} a_0 a_1 \cdots a_{v-2} a_{v-1} = r a_0 a_1 \cdots a_{v-2} a_{v-1} \\
&r_2 r_3^2 \cdots r_{v-1}^{v-2} r_0^{v-1} a_1 a_2 \cdots a_{v-1} a_0 = a_1 a_2 \cdots a_{v-1} a_0 \\
&\vdots \\
&r_0 r_1^2 \cdots r_{v-3}^{v-2} r_{v-2}^{v-1} a_{v-1} a_0 a_1 \cdots a_{v-2} = a_{v-1} a_0 a_1 \cdots a_{v-2},
\end{aligned}$$

so we must solve the system

$$\begin{aligned}
&r_1 r_2^2 \cdots r_{v-2}^{v-2} r_{v-1}^{v-1} = r \\
&r_2 r_3^2 \cdots r_{v-1}^{v-2} r_0^{v-1} = 1 \\
&\vdots \\
&r_0 r_1^2 \cdots r_{v-3}^{v-2} r_{v-2}^{v-1} = 1.
\end{aligned}$$

If $v = 2$, the system is immediately solved: $r_1 = r$ and $r_0 = 1$. If $v = 3$, it is easy to see that $r_0 = r_1^{-2}$, $r_2 = r_1^4$ and $r_1^9 = r$ so that choosing a 9-th root of r for r_1 defines σ. Now suppose that $v > 3$. Starting with the last equation, we can express r_0 in terms of $r_1, \ldots, r_{v-2}$ and then r_{v-1} also in terms of $r_1, \ldots, r_{v-2}$. At the next stage, r_{v-2} may be written in terms of $r_1, \ldots, r_{v-3}$, so that r_0 and r_{v-1} may also be written in terms of $r_1, \ldots, r_{v-3}$. Continuing in this way, we see that $r_1, \ldots, r_{v-2}$ must be equal up to v-th roots of unity so that

$$r_2 = \zeta_2 r_1,\ r_3 = \zeta_3 r_1,\ \dots, r_{v-2} = \zeta_{v-2} r_1$$

for some v-th roots of unity $\zeta_2, \dots, \zeta_{v-2}$. We then have $r_0 = \zeta_2^{-3} \cdots \zeta_{v-2}^{-(v-1)} r_1^{-\frac{(v+1)(v-2)}{2}}$ and $r_{v-1} = \zeta_2^2 \cdots \zeta_{v-2}^{(v-2)} r_1^{(v+1)(v-2)-\frac{(v+2)(v-3)}{2}}$. Therefore

$$r = r_1 r_2^2 \cdots r_{v-1}^{v-1} = \zeta_2^2 \cdots \zeta_{v-2}^{(v-2)} (\zeta_2^2 \cdots \zeta_{v-2}^{(v-2)})^{v-1} r_1^{\varphi(v)} = r_1^{\varphi(v)}$$

where $\varphi(v) = \frac{v^2(v-1)}{2}$. Choosing a $\varphi(v)$-th root of r for r_1 and $\zeta_2 = \cdots = \zeta_{v-2} = 1$ defines an automorphism σ as required. Note that $\varphi(3) = 9$, so that we have defined the same automorphism σ in the case $v = 3$.

We now use [5], where the authors show that the Ext algebra of A^σ is obtained from the Ext algebra of A by twisting (they consider a connected graded algebra, but the proof and result are easily adapted to a quotient of a path algebra by a length homogeneous ideal). Twisting does not change the number of generators of the Ext algebra or their degrees.

Proceeding in this way for each relation of type one, we see that the number of generators of the Ext algebra and their degrees do not depend on $\mathfrak{q}$. □

As a corollary of the proof, we may relax the condition that K is algebraically closed.

Corollary 3.3. *Let K be a field and let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph with $\mathfrak{m} \equiv 1$ and with no loops or multiple edges. Let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. We assume that one of the following conditions holds:*

(i) the valency of every vertex in Γ is at most two, or

(ii) there is an integer k such that the valency of every vertex in Γ is either 1 or k and the field K contains a root of the polynomial $X^{k^2(k-1)/2} - r$ for any $r \in K$.

Then the number of generators of $\mathrm{Ext}_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma}, \mathcal{A}_\Gamma/\mathbf{r}_{\mathcal{A}_\Gamma})$ and their degrees do not depend on $\mathfrak{q}$.

Proof. In both cases, the ideal I_Γ is length homogeneous, and hence $\mathcal{A}_\Gamma$ is length graded. It then follows from the proof of Proposition 3.2 that the result holds. □

From now on, we assume that $\mathfrak{q} \equiv 1$, and write $(\Gamma, \mathfrak{o}, \mathfrak{m})$ for a Brauer graph with $\mathfrak{q} \equiv 1$. We assume that $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph with no loops or multiple edges and $\mathfrak{m} \equiv 1$. The next sections describe the structure of certain classes of modules of a Brauer graph algebra.

4. Structure of indecomposable modules

Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph with no loops or multiple edges and $\mathfrak{m} \equiv 1$. Let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. We assume $\mathfrak{q} \equiv 1$. The following result is used in Sections 5 and 6 where we determine the structure of uniserial and string $\mathcal{A}_\Gamma$-modules.

Proposition 4.1. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Let S and T be simple $\mathcal{A}_\Gamma$-modules associated to the edges s and t in Γ.*

(1) *If $S \ncong T$, then* $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) \leqslant 1$.
(2) *If $S \cong T$, then* $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) = 2$.
(3) *We have that* $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) = 1$ *if, and only if, $S \ncong T$ and there is a uniserial module with top S and socle T. In this case, the uniserial module is unique up to isomorphism.*

Proof. **(1)** Suppose that $S \ncong T$ and assume for contradiction that $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) \geqslant 2$. Denote the endpoints of edge t in Γ by α and β. By our no loops assumption, $\alpha \neq \beta$. Since $\mathcal{A}_\Gamma$ is a special biserial selfinjective algebra, $\operatorname{rad}(P_T)/\operatorname{Soc}(P_T)$ is either a uniserial module U or a direct sum of two uniserial modules $L_1 \oplus L_2$. Since $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) \geqslant 2$ and $\operatorname{Soc}(P_T) = T$, the simple S occurs at least twice as a composition factor of $\operatorname{rad}(P_T)/\operatorname{Soc}(P_T)$. If S occurs as a composition factor of either U or one of the L_i's at least two times, then either $\mathfrak{m}(\alpha) \geqslant 2$, $\mathfrak{m}(\beta) \geqslant 2$ or s is a loop, which all contradict our hypothesis. On the other hand, suppose that S occurs as a composition factor of both L_1 and L_2. Then s occurs in the successor sequences of t at both vertices α and β. Hence, s also has endpoints α and β. But then s and t are distinct edges between α and β, which contradicts the hypothesis that there are no multiple edges. Thus $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) \leqslant 1$.

(2) This is proved by a similar argument to that in **(1)**.

(3) First assume that $S \ncong T$ and there is a uniserial module V having top S and socle T. Then V embeds in the injective module P_T. Hence S must be a composition factor of P_T. Then $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) = 1$ by **(1)**. Next suppose that $\dim_K(\operatorname{Hom}_{\mathcal{A}_\Gamma}(P_S, P_T)) = 1$. By **(2)**, $S \ncong T$. If P_T is uniserial, then it follows that there is a uniserial submodule of P_T with top S and socle T since S is a composition factor of P_T. Otherwise, we may suppose that $\operatorname{rad}(P_T)/\operatorname{Soc}(P_T) = L_1 \oplus L_2$, and, by assumption, S is a composition factor of exactly one of L_1 or L_2. So suppose that S is a composition factor of L_1 and g is the composition of the canonical surjections $\operatorname{rad}(P_T) \to \operatorname{rad}(P_T)/T$ and $\operatorname{rad}(P_T)/T \to L_1$. Let $V = g^{-1}(L_1)$. Then V is a uniserial submodule of P_T having S as a composition factor. Hence, there is a uniserial module with top S and socle T.

It remains to show that if V_1 and V_2 are uniserial modules with top S and socle T, then $V_1 \cong V_2$. Note that V_1 and V_2 both embed in P_T. If $V_1 \ncong V_2$ then S would occur at least twice as a composition factor of P_T, contradicting **(1)**. This completes the proof.

□

Corollary 4.2. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Let S and T be simple $\mathcal{A}_\Gamma$-modules associated to the edges s and t in Γ. Then we have the following.*

(1) *If $f, g\colon P_S \to \operatorname{rad}(P_T)$ are nonzero morphisms, then $\operatorname{Im}(f) = \operatorname{Im}(g)$.*
(2) *There are only a finite number of submodules of an indecomposable projective $\mathcal{A}_\Gamma$-module.*
(3) *A submodule M of an indecomposable projective $\mathcal{A}_\Gamma$-module is determined by the simple $\mathcal{A}_\Gamma$-modules occurring in the top of M.*
(4) *There are only a finite number of quotient modules of an indecomposable projective $\mathcal{A}_\Gamma$-module.*

Proof. We see that **(1)** follows from Proposition 4.1**(1)** and **(2)**.

To show **(2)** and **(3)**, let P_T be an indecomposable projective $\mathcal{A}_\Gamma$-module and let M be a nonprojective, nonsimple submodule of P_T. Then we have an inclusion $f\colon M \to P_T$. If P_S is an indecomposable projective $\mathcal{A}_\Gamma$-module and $g\colon P_S \to M$ is a module homomorphism such that the induced map $\bar{g}\colon P_S \to M/\operatorname{rad} M$ is nonzero, then there is a nonzero map $h = f \circ g\colon P_S \to \operatorname{rad}(P_T)$. By **(1)**, $\operatorname{Im}(h)$ is unique. Now P_T is either uniserial or biserial. If P_T is uniserial, then $M = \operatorname{Im}(h)$ and, by Proposition 4.1, both **(2)** and **(3)** follow.

Now suppose that P_T is biserial with $\operatorname{rad}(P_T)/\operatorname{Soc}(P_T) = L_1 \oplus L_2$. By Proposition 4.1**(1)**, we see $\operatorname{Im}(h)/\operatorname{Soc}(P_T) \subseteq L_1$ or $\operatorname{Im}(h)/\operatorname{Soc}(P_T) \subseteq L_2$. Assume, without loss of generality, that $\operatorname{Im}(h)/\operatorname{Soc}(P_T) \subseteq L_1$. If $M/\operatorname{Soc}(P_T) \subseteq L_1$ then $M = \operatorname{Im}(h)$ and there are only a finite number of such submodules M. So suppose that $M/\operatorname{Soc}(P_T) \not\subseteq L_1$ so that $M \neq \operatorname{Im}(h)$. Note that $M/\operatorname{Soc}(P_T) \not\subseteq L_2$. Then, since $\operatorname{rad}(P_T)/\operatorname{Soc}(P_T) = L_1 \oplus L_2$, we have $M/\operatorname{rad} M \cong S \oplus S'$, for some simple $\mathcal{A}_\Gamma$-module S'. By Proposition 4.1**(1)** and **(2)**, $S \ncong S'$. Defining $h'\colon P_{S'} \to P_T$ in a similar fashion to the definition of h, we see that $\operatorname{Im}(h')/\operatorname{Soc}(P_T) \subseteq L_2$ and $M/\operatorname{Soc}(P_T) = \operatorname{Im}(h)/\operatorname{Soc}(P_T) \oplus \operatorname{Im}(h')/\operatorname{Soc}(P_T)$. Parts **(2)** and **(3)** now follow.

The proof of **(4)** follows from **(2)**. □

Let S be the simple $\mathcal{A}_\Gamma$-module associated to the edge s in Γ. We remarked at the end of Section 1 that P_S is uniserial if and only if s is a truncated edge. The next result is more specific on the structure of the indecomposable projective $\mathcal{A}_\Gamma$-modules, in the case where $\mathfrak{m} \equiv 1$ and Γ has no loops or multiple edges; its proof is straightforward and we leave it to the reader.

Lemma 4.3. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Assume S is a simple $\mathcal{A}_\Gamma$-module such that P_S is biserial and U is a simple $\mathcal{A}_\Gamma$-module such that P_U is uniserial. If s is the edge in Γ associated to S and s has endpoints α and β, then let $s = s_0, s_1, s_2, \dots, s_{m-1}$*

and $s = t_0, t_1, t_2, \ldots, t_{n-1}$ *be the successor sequences for* s *at vertices* α *and* β *respectively. Let* S_i *(resp.* T_i*) be the simple* $\mathcal{A}_\Gamma$*-module associated to the edge* s_i *(resp.* t_i*). If* u *is the edge in* Γ *associated to* U *and* u *has endpoints* α' *and* β' *with* u *truncated at* β'*, then let* $u = u_0, u_1, u_2, \ldots, u_{p-1}$ *be the successor sequence for* u *at* α'*. Let* U_i *be the simple* $\mathcal{A}_\Gamma$*-module associated to the edge* u_i*.*

(1) *The composition factors of* P_S *are* $\{S, S_1, \ldots, S_{m-1}, T_1, \ldots, T_{n-1}, S\}$*.*
(2) *The composition factors of* P_U *are* $\{U, U_1, \ldots, U_{p-1}, U\}$*.*
(3) *For* $i = 1, \ldots, m-1$ *and* $j = 1, \ldots, n-1$*,* $S_i \ncong T_j$*.*
(4) *For* $0 \leqslant i < j \leqslant m-1$*,* $S_i \ncong S_j$*.*
(5) *For* $0 \leqslant i < j \leqslant n-1$*,* $T_i \ncong T_j$*.*
(6) *For* $0 \leqslant i < j \leqslant p-1$*,* $U_i \ncong U_j$*.*

5. Structure of uniserial modules

In this section we describe the structure of the uniserial modules of a Brauer graph algebra.

Proposition 5.1. *Let* $(\Gamma, \mathfrak{o}, \mathfrak{m})$ *with* $\mathfrak{m} \equiv 1$ *be a Brauer graph with no loops or multiple edges and let* $\mathcal{A}_\Gamma$ *denote the associated Brauer graph algebra. Assume* S *is a simple* $\mathcal{A}_\Gamma$*-module such that* P_S *is biserial and* U *is a simple* $\mathcal{A}_\Gamma$*-module such that* P_U *is uniserial. If* s *is the edge in* Γ *associated to* S *and* s *has endpoints* α *and* β*, then let* $s = s_0, s_1, s_2, \ldots, s_{m-1}$ *and* $s = t_0, t_1, t_2, \ldots, t_{n-1}$ *be the successor sequences for* s *at vertices* α *and* β *respectively. Let* S_i *(resp.* T_i*) be the simple* $\mathcal{A}_\Gamma$*-module associated to the edge* s_i *(resp.* t_i*) and* $S_m = T_n = S$*. If* u *is the edge in* Γ *associated to* U *and* u *has endpoints* α' *and* β' *with* u *truncated at* β'*, then let* $u = u_0, u_1, u_2, \ldots, u_{p-1}$ *be the successor sequence for* u *at* α'*. Let* U_i *be the simple* $\mathcal{A}_\Gamma$*-module associated to the edge* u_i *and* $U_p = U_0 = U$*. Let* L *be a nonzero uniserial* $\mathcal{A}_\Gamma$*-module with composition series for* L*,* $(0) = L_{k+1} \subset L_k \subset \cdots \subset L_1 \subset L_0 = L$*.*

(1) *If the socle of* L *is isomorphic to* S*, then either* $0 \leqslant k \leqslant m-1$ *and, for* $j = 0, \ldots, k$*,* $L_j/L_{j+1} \cong S_{m-k+j}$ *or* $0 \leqslant k \leqslant n-1$ *and, for* $j = 0, \ldots, k$*,* $L_j/L_{j+1} \cong T_{n-k+j}$*.*
(2) *If the socle of* L *is isomorphic to* U*, then* $0 \leqslant k \leqslant p$ *and* $j = 0, \ldots, k$*,* $L_j/L_{j+1} \cong U_{p-k+j}$*.*

Proof. We prove **(1)** and leave **(2)** to the reader. If L is a simple module, then $L \cong S = S_m$, and taking $k = 0$ we see that **(1)** follows. Now assume that L is a nonsimple uniserial module with socle S. It follows that L is isomorphic to a submodule of P_S since P_S is the injective envelope of S. Since $\mathfrak{m} \equiv 1$ and Γ has no loops or multiple edges, $\mathrm{rad}(P_S)$ is the sum of two uniserial modules X and Y such that

(i) $X \cap Y = \mathrm{Soc}(P_S)$,

(ii) if $0 = X_m \subset X_{m-1} \subset \cdots \subset X_0 = X$ is the composition series for X, then, for $j = 0, \ldots, m-1$, $X_j/X_{j+1} \cong S_{j+1}$,

(iii) if $0 = Y_n \subset Y_{n-1} \subset \cdots \subset Y_0 = Y$ is the composition series for Y, then, for $j = 0, \ldots, n-1$, $Y_j/Y_{j+1} \cong T_{j+1}$.

By Corollary 4.2**(3)** and Lemma 4.3**(5)**, the uniserial module L must be isomorphic to a submodule of either X or Y and the result follows. $\square$

An immediate consequence of the above result is the following.

Corollary 5.2. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. If L and L' are two nonsimple, nonprojective uniserial $\mathcal{A}_\Gamma$-modules such that* $\mathrm{Soc}(L) \cong \mathrm{Soc}(L')$ *and* $\mathrm{Top}(L) \cong \mathrm{Top}(L')$, *then $L \cong L'$.*

6. Structure of string modules

Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. We now classify the string modules for $\mathcal{A}_\Gamma$ and begin with uniserial modules. Let L be a (nonzero) uniserial $\mathcal{A}_\Gamma$-module. There are 3 cases to consider. The first case is that L is a projective-injective module. There is no special notation for this case. The second case is that L is isomorphic to a simple $\mathcal{A}_\Gamma$-module, S. Let s be the edge in Γ associated to S. In this case, we denote L (up to isomorphism) by $\mathbf{str}(s^+)$, where $\mathbf{str}(s^+) \cong S$. The final case is that L is a nonsimple nonprojective uniserial module with top T and socle S. Let s and t be the edges in Γ associated to the simple modules S and T respectively. By Corollary 5.2, T and S completely determine L up to isomorphism. We denote L by either $\mathbf{str}(t^+, s^-)$ or $\mathbf{str}(s^-, t^+)$.

If s and t are edges in Γ, and the successor sequence for s at vertex α is $s = s_0, s_1, \ldots, s_{m-1}$, then we say t *occurs in the successor sequence for s at vertex α* if α is a nontruncated endpoint of s and $t = s_i$, for some $1 \leqslant i \leqslant m-1$. Clearly t occurs in the successor sequence for s at vertex α if and only if s occurs in the successor sequence for t at vertex α. Note also that t cannot occur in the successor sequence for s at both endpoints α and β, for otherwise $\alpha = \beta$ which contradicts the assumption that there are no loops in Γ. Let S and T be the simple $\mathcal{A}_\Gamma$-modules associated to s and t respectively. By Proposition 5.1 and its proof, we see that there is a uniserial module L, unique up to isomorphism, with top T and socle S, if and only if $s = t$ or t occurs in the successor sequence for s at some vertex α.

Thus, summarizing the description of uniserial string modules, we have the projective-injective uniserial modules, the simple modules $\mathbf{str}(s^+)$, and the modules of the form $\mathbf{str}(t^+, s^-)$ where t occurs in the successor sequence for s at some vertex α.

We now describe the nonuniserial string modules for $\mathcal{A}_\Gamma$ in terms of sequences of weighted edges in the Brauer graph Γ.

Definition 6.1. For $n \geqslant 2$, let $\hat{s}_1, \ldots, \hat{s}_n$ be edges in Γ. We assign either $+$ or $-$ to each edge $\hat{s}_i$ and denote this assignment by either $\hat{s}_i^+$ or $\hat{s}_i^-$. Consider the sequence $\sigma = \hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \ldots, \hat{s}_n^{e_n}$, where $e_i \in \{+,-\}$ for $i = 1, \ldots, n$. We say σ is an *acceptable sequence of weighted edges* if the following hold.

(ST1) For $i = 1, \ldots, n-1$, $e_i \neq e_{i+1}$.
(ST2) For $i = 1, \ldots, n-1$, there are vertices α_i in Γ, such that $\hat{s}_{i+1}$ occurs in the successor sequence for $\hat{s}_i$ at vertex α_i.
(ST3) For $i = 1, \ldots, n-2$, $\alpha_i \neq \alpha_{i+1}$.

We note that, for $i = 1, \ldots, n-1$, the vertex α_i is uniquely determined by $\hat{s}_i$ and $\hat{s}_{i+1}$ by our assumption that Γ has no loops or multiple edges. We use the notation $\hat{s}$ in order to distinguish acceptable sequences and successor sequences, but it may happen that $\hat{s}_{i+1}$ is in fact successor of $\hat{s}_i$ at one of its vertices.

The following result is straightforward and the proof is left to the reader.

Lemma 6.2. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and suppose that $\sigma = \hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \ldots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ with $n \geqslant 2$. Then*

(1) $\hat{s}_n^{e_n}, \hat{s}_{n-1}^{e_{n-1}}, \ldots, \hat{s}_1^{e_1}$ *is an acceptable sequence of weighted edges in Γ,*
(2) $\hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \ldots, \hat{s}_i^{e_i}$ *is an acceptable sequence of weighted edges in Γ, for $i = 2, \ldots, n$, and,*
(3) *for $i = 1, \ldots, n$, if $e_i^* = +$ when $e_i = -$, and $e_i^* = -$ when $e_i = +$, then $\hat{s}_1^{e_1^*}, \hat{s}_2^{e_2^*}, \ldots, \hat{s}_n^{e_n^*}$ is an acceptable sequence of weighted edges in Γ.*

Suppose that $\sigma = \hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \cdots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ. For $i = 1, \ldots, n$, let $\hat{S}_i$ be the simple $\mathcal{A}_\Gamma$-module associated to the edge $\hat{s}_i$. We define $\mathbf{str}(\sigma)$ inductively such that the top of $\mathbf{str}(\sigma)$ is $\oplus_{\{i \,|\, e_i = +\}} \hat{S}_i$ and the socle of $\mathbf{str}(\sigma)$ is $\oplus_{\{i \,|\, e_i = -\}} \hat{S}_i$. We say $\mathbf{str}(\sigma)$ satisfies the *top and socle condition.*

Definition 6.3. Let $\sigma = \hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \ldots, \hat{s}_n^{e_n}$ be an acceptable sequence of weighted edges in Γ, and let $n \geqslant 2$. For $n = 2$, $\mathbf{str}(\hat{s}_1^{e_1}, \hat{s}_2^{e_2})$ was defined above, and clearly satisfies the top and socle condition.

Assume $n \geqslant 3$ and suppose we have defined $\mathbf{str}(\hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \ldots, \hat{s}_{n-1}^{e_{n-1}})$ satisfying the top and socle condition. There are two cases: $e_{n-1} = -$ and $e_{n-1} = +$.

(i) Suppose $e_{n-1} = -$, so that $e_n = +$. We set

$$\mathbf{str}(\sigma) = \operatorname{coker}(\mu \colon \hat{S}_{n-1} \to \mathbf{str}(\hat{s}_1^{e_1}, \ldots, \hat{s}_{n-2}^{e_{n-2}}, \hat{s}_{n-1}^{-}) \oplus \mathbf{str}(\hat{s}_{n-1}^{-}, \hat{s}_n^{+})),$$

where μ is induced from the map on socles given by

$$\begin{cases} \hat{S}_{n-1} \to (\hat{S}_2 \oplus \hat{S}_4 \oplus \cdots \oplus \hat{S}_{n-1}) \oplus \hat{S}_{n-1} \text{ with } x \mapsto ((0,\dots,0,x),x) & \text{if } n \text{ is odd} \\ \hat{S}_{n-1} \to (\hat{S}_1 \oplus \hat{S}_3 \oplus \cdots \oplus \hat{S}_{n-1}) \oplus \hat{S}_{n-1} \text{ with } x \mapsto ((0,\dots,0,x),x) & \text{if } n \text{ is even.} \end{cases}$$

(ii) Now suppose that $e_{n-1} = +$. Then $e_n = -$. We define $\mathbf{str}(\sigma)$ to be the kernel of the composition

$$\mathbf{str}(\hat{s}_1^{e_1},\dots,\hat{s}_{n-2}^{e_{n-2}},\hat{s}_{n-1}^{+}) \oplus \mathbf{str}(\hat{s}_{n-1}^{+},\hat{s}_n^{-}) \to (\oplus_{\{i\,|\,e_i=+,1\leqslant i\leqslant n-1\}}\hat{S}_i) \oplus \hat{S}_{n-1} \xrightarrow{\nu} \hat{S}_{n-1},$$

where the first map is given by canonical surjection onto the tops of $\mathbf{str}(\hat{s}_1^{e_1},\dots,\hat{s}_{n-1}^{e_{n-1}})$ and $\mathbf{str}(\hat{s}_{n-1}^{e_{n-1}},\hat{s}_n^{e_n})$ and ν is given by

$$\begin{cases} ((y_2,y_4,\dots,y_{n-1}),y'_{n-1}) \mapsto y_{n-1} - y'_{n-1} & \text{if } n \text{ is odd} \\ ((y_1,y_3,\dots,y_{n-1}),y'_{n-1}) \mapsto y_{n-1} - y'_{n-1} & \text{if } n \text{ is even,} \end{cases}$$

where $y_i \in \hat{S}_i$ and $y'_{n-1} \in \hat{S}_{n-1}$.

The reader may check that the top and socle condition is satisfied by $\mathbf{str}(\sigma)$ in all cases. The next proposition gives an alternative definition for $\mathbf{str}(\sigma)$.

Proposition 6.4. *Let $(\Gamma,\mathfrak{o},\mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose that $\sigma = \hat{s}_1^{e_1},\hat{s}_2^{e_2},\dots,\hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ with $n \geqslant 3$ and let $\hat{S}_i$ be the simple $\mathcal{A}_\Gamma$-module associated to $\hat{s}_i$. Let $2 \leqslant k \leqslant n-1$.*

(1) *If $e_k = -$, then we set*

$$X = \operatorname{coker}(\mu\colon \hat{S}_k \to \mathbf{str}(\hat{s}_1^{e_1},\dots,\hat{s}_k^{e_k}) \oplus \mathbf{str}(\hat{s}_k^{e_k},\dots,\hat{s}_n^{e_n})),$$

where μ is induced from the map on socles given by

$$\begin{cases} \hat{S}_k \to (\hat{S}_2 \oplus \hat{S}_4 \oplus \cdots \oplus \hat{S}_k) \oplus (\hat{S}_k \oplus \hat{S}_{k+2} \oplus \cdots) & \\ \qquad \textit{with } x \mapsto ((0,\dots,0,x),(x,0,\dots,0)) & \textit{if } k \textit{ is even} \\ \hat{S}_k \to (\hat{S}_1 \oplus \hat{S}_3 \oplus \cdots \oplus \hat{S}_k) \oplus (\hat{S}_k \oplus \hat{S}_{k+2} \oplus \cdots) & \\ \qquad \textit{with } x \mapsto ((0,\dots,0,x),(x,0,\dots,0)) & \textit{if } k \textit{ is odd.} \end{cases}$$

(2) *If $e_k = +$, then we set X to be the kernel of the composition of canonical surjections onto tops*

$$\mathbf{str}(\hat{s}_1^{e_1},\dots,\hat{s}_k^{e_k}) \oplus \mathbf{str}(\hat{s}_k^{e_k},\dots,\hat{s}_n^{e_n}) \to (\oplus_{\{e_i=+,1\leqslant i\leqslant k\}}\hat{S}_i) \oplus (\oplus_{\{e_i=+,k\leqslant i\leqslant n\}}\hat{S}_i)$$

with

$$\nu\colon (\oplus_{\{e_i=+,1\leqslant i\leqslant k\}}\hat{S}_i) \oplus (\oplus_{\{e_i=+,k\leqslant i\leqslant n\}}\hat{S}_i) \to \hat{S}_k,$$

where ν is given by

$$\begin{cases} ((y_2,y_4,\dots,y_k),(y'_k,y_{k+2},\dots)) \mapsto y_k - y'_k & \textit{if } k \textit{ is even} \\ ((y_1,y_3,\dots,y_k),(y'_k,y_{k+2},\dots)) \mapsto y_k - y'_k & \textit{if } k \textit{ is odd,} \end{cases}$$

with $y_i \in \hat{S}_i$ and $y'_k \in \hat{S}_k$.

Then $\mathbf{str}(\sigma) \cong X$.

Proof. We proceed by induction on n. For $n = 3$, k must be $2 = n-1$ and the result follows from the definition of $\mathbf{str}(\sigma)$. Now assume the result is true for all m, $3 \leqslant m \leqslant n-1$. Fix $2 \leqslant k \leqslant n-1$. If $k = n-1$, then the result again follows from the definition of $\mathbf{str}(\sigma)$. Assume $k < n-1$. There are many cases to consider: e_k equal to $+$ or $-$, k even or odd, and n even or odd. The cases all have similar proofs. We prove one case and leave the others to the reader.

The case we prove is for n even, k even, and $e_k = +$. Note that we then have, for i odd, $e_i = -$ and, for i even, $e_i = +$. For ease of notation, we set $Z_1 = \mathbf{str}(\hat{s}_1^-, \ldots, \hat{s}_{n-1}^-)$, $Z_2 = \mathbf{str}(\hat{s}_{n-1}^-, \hat{s}_n^+)$, $U_1 = \mathbf{str}(\hat{s}_1^-, \ldots, \hat{s}_k^+)$, $U_2 = \mathbf{str}(\hat{s}_k^+, \ldots, \hat{s}_n^+)$, and $V_1 = \mathbf{str}(\hat{s}_k^+, \ldots, \hat{s}_{n-1}^-)$.

From the definition of $\mathbf{str}(\sigma)$, we have a short exact sequence of $\mathcal{A}_\Gamma$-modules

$$0 \to \hat{S}_{n-1} \xrightarrow{\mu} Z_1 \oplus Z_2 \to \mathbf{str}(\sigma) \to 0.$$

By induction, we have a short exact sequence

$$0 \to Z_1 \xrightarrow{g} U_1 \oplus V_1 \xrightarrow{h} \hat{S}_k \to 0.$$

From this short exact sequence we obtain

$$0 \to Z_1 \oplus Z_2 \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & \mathrm{Id}_{Z_2} \end{pmatrix}} (U_1 \oplus V_1) \oplus Z_2 \to \hat{S}_k \oplus 0 \to 0.$$

Also by induction, we have a short exact sequence

$$0 \to \hat{S}_{n-1} \to V_1 \oplus Z_2 \xrightarrow{\phi} U_2 \to 0.$$

From this short exact sequence we obtain

$$0 \to 0 \oplus \hat{S}_{n-1} \to U_1 \oplus (V_1 \oplus Z_2) \xrightarrow{\begin{pmatrix} \mathrm{Id}_{U_1} & 0 \\ 0 & \phi \end{pmatrix}} U_1 \oplus U_2 \to 0.$$

Using the above sequences and that $U_1 \oplus V_1 \oplus Z_2 = (U_1 \oplus V_1) \oplus Z_2 = U_1 \oplus (V_1 \oplus Z_2)$, we obtain an exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \hat{S}_{n-1} \arrow[r, "\mu"] \arrow[d, "\cong"] & Z_1 \oplus Z_2 \arrow[r] \arrow[d, "\begin{pmatrix} g & 0 \\ 0 & \mathrm{Id}_{Z_2} \end{pmatrix}"'] & \mathbf{str}(\sigma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \oplus \hat{S}_{n-1} \arrow[r] & U_1 \oplus V_1 \oplus Z_2 \arrow[r, "\begin{pmatrix} \mathrm{Id}_{U_1} & 0 \\ 0 & \phi \end{pmatrix}"] \arrow[d, "\begin{pmatrix} h \\ 0 \end{pmatrix}"'] & U_1 \oplus U_2 \arrow[r] \arrow[d] & 0 \\
 & & \hat{S}_k \oplus 0 \arrow[r, "\cong"] \arrow[d] & \hat{S}_k \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The reader may check that the exact sequence that appears as the last column in the diagram above proves that $X \cong \mathbf{str}(\sigma)$ in this case. ☐

Corollary 6.5. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose that $\sigma = \hat{s}_1^{e_1}, \hat{s}_2^{e_2}, \dots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ with $n \geqslant 2$. Then $\tau = \hat{s}_n^{e_n}, \hat{s}_{n-1}^{e_{n-1}}, \dots, \hat{s}_1^{e_1}$ is also an acceptable sequence of weighted edges in Γ and*

$$\mathbf{str}(\sigma) \cong \mathbf{str}(\tau).$$

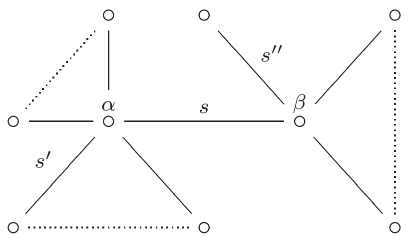
Proof. By Lemma 6.2**(1)**, τ is an acceptable sequence of weighted edges in Γ. For $n = 2$, the result follows from the definitions. If $n \geqslant 3$, taking $k = 2$ in Proposition 6.4, and induction yields the result. ☐

By Corollary 6.5, we see that there are 3 types of string modules over $\mathcal{A}_\Gamma$. Suppose $\sigma = \hat{s}_1^{e_1}, \dots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ. We say that $\mathbf{str}(\sigma)$ is a *positive string module* if $e_1 = + = e_n$, a *negative string module* if $e_1 = - = e_n$, and a *mixed string module* if $e_1 \neq e_n$. We will see that the positive string modules play an important role in the cohomology theory of $\mathcal{A}_\Gamma$.

7. Syzygies and resolutions in a Brauer graph algebra with no truncated edges

In this section we assume that $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ is a Brauer graph with no loops or multiple edges, and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. The main result here is Theorem 7.4, where we give a minimal projective resolution of a simple $\mathcal{A}_\Gamma$-module, in the case where Γ has no truncated edges.

We begin by fixing a nontruncated edge s in Γ with endpoints α and β. Since $\mathfrak{m} \equiv 1$, both α and β have valency at least 2. Let edge s' be in the successor sequence of s at vertex α. We say edge s'' *follows* (s', s) if s'' is the successor of s at vertex β. The following diagram illustrates this definition:



As in Section 6, if $e = +$, we let $e^* = -$, and if $e = -$, we let $e^* = +$. We are now in a position to describe the first syzygy of a positive string module. For this, we introduce the following notational conventions. The string module $\mathbf{str}(s^+, t^-, u^+)$ will be schematically represented by

$$\begin{tikzcd} S \arrow[dr, dotted, no head] & & U \arrow[dl, dotted, no head] \\ & T & \end{tikzcd}$$

For a nontruncated edge s, the indecomposable projective $\mathcal{A}_\Gamma$-module with top S will be schematically represented by

$$\begin{tikzcd} & S \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & \\ T \arrow[dr, dotted, no head] & & U \arrow[dl, dotted, no head] \\ & S & \end{tikzcd}$$

where the edge t in Γ is in the successor sequence for s at one endpoint of s, and the edge u is in the successor sequence for s at the other endpoint of s. If a solid line

$$\begin{tikzcd} S \arrow[d, no head] \\ T \end{tikzcd}$$

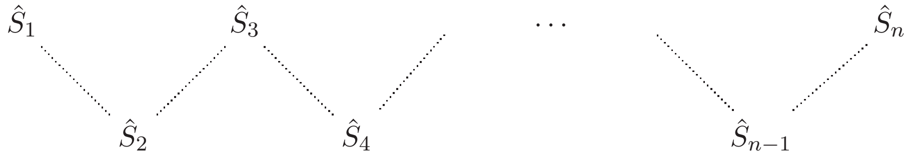
appears, then that signifies that not only is t in the successor sequence of s at some vertex of Γ, but t is the successor of s at that vertex.

Proposition 7.1. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges, and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose that $M = \mathbf{str}(\sigma)$ is a positive string module, where $\sigma = \hat{s}_1^{e_1}, \ldots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted* *edges in Γ, and $n \geqslant 3$. Suppose also, for each i with $e_i = +$, that $\hat{s}_i$ is a nontruncated edge. Then the first syzygy of M is isomorphic to the positive string module $\mathbf{str}(\tau)$, where*

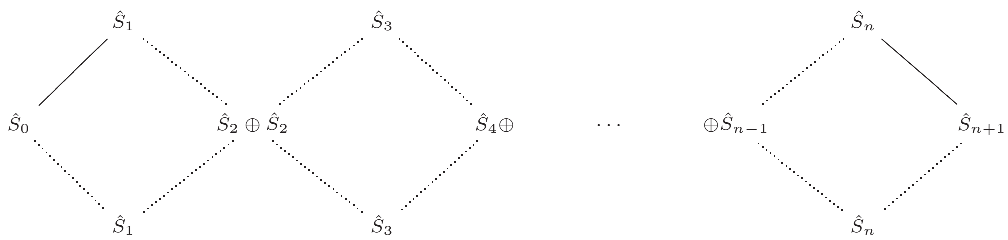
$$\tau = \hat{s}_0^{e_1}, \hat{s}_1^{e_1^*}, \ldots, \hat{s}_n^{e_n^*}, \hat{s}_{n+1}^{e_n},$$

and where $\hat{s}_0$ follows $(\hat{s}_2, \hat{s}_1)$ and $\hat{s}_{n+1}$ follows $(\hat{s}_{n-1}, \hat{s}_n)$.

Proof. Since $\mathbf{str}(\sigma)$ is a positive string module, n is an odd integer, say $n = 2m+1$. By assumption, $m \geqslant 1$. The projective cover of M is $P_{\hat{S}_1} \oplus P_{\hat{S}_3} \oplus \cdots \oplus P_{\hat{S}_{2m+1}}$ and the socle of M is given by $\hat{S}_2 \oplus \hat{S}_4 \oplus \cdots \oplus \hat{S}_{2m}$. Thus M structurally looks like:



Since $\hat{s}_1, \hat{s}_3, \ldots, \hat{s}_{2m+1}$ are all nontruncated edges, the corresponding indecomposable projectives $P_{\hat{S}_1}, P_{\hat{S}_3}, \ldots, P_{\hat{S}_{2m+1}}$ are biserial. Thus, from the definition of 'follows', $P_{\hat{S}_1} \oplus P_{\hat{S}_3} \oplus \cdots \oplus P_{\hat{S}_{2m+1}}$ looks like:



From these diagrams, the reader can easily provide the remaining details of the proof. $\square$

We assume for the rest of this section that Γ contains no truncated edges.

To describe projective resolutions of simple $\mathcal{A}_\Gamma$-modules, we will need further notation. For s an edge in Γ, we represent the simple $\mathcal{A}_\Gamma$-module S by $v_s(\mathcal{A}_\Gamma/\mathbf{r})$, where v_s is the vertex in $\mathcal{Q}_\Gamma$ associated to the edge s. Here we are viewing v_s as the idempotent in $\mathcal{A}_\Gamma$ corresponding to the edge s in Γ. We also set the projective $\mathcal{A}_\Gamma$-module P_S to be $v_s(\mathcal{A}_\Gamma)$.

Let $\hat{s}_0$ be an edge in Γ. We now present a minimal projective $\mathcal{A}_\Gamma$-resolution of the simple module $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$,

$$(Q^\bullet, f^\bullet): \quad \cdots \to Q^2 \xrightarrow{f^2} Q^1 \xrightarrow{f^1} Q^0 \xrightarrow{f^0} v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r}) \to 0.$$

We see that $Q^0 = v_{\hat{s}_0}(\mathcal{A}_\Gamma)$ with f^0 being the canonical surjection and the first syzygy is $\mathbf{str}(\hat{s}_{-1}^+, \hat{s}_0^-, \hat{s}_1^+)$, where $\hat{s}_{-1}$ and $\hat{s}_1$ are the successors of $\hat{s}_0$ at its endpoints. Applying Proposition 7.1 repeatedly, we see that, if n is odd, then the n-th syzygy of $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$ is

$$\Omega^n_{\mathcal{A}_\Gamma}(v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})) = \mathbf{str}(\hat{s}_{-n}^+, \hat{s}_{-n+1}^-, \hat{s}_{-n+2}^+, \ldots, \hat{s}_{-1}^+, \hat{s}_0^-, \hat{s}_1^+, \ldots, \hat{s}_{n-1}^-, \hat{s}_n^+),$$

and, if n is even, then the n-th syzygy of $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$ is

$$\Omega^n_{\mathcal{A}_\Gamma}(v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})) = \mathbf{str}(\hat{s}^+_{-n}, \hat{s}^-_{-n+1}, \hat{s}^+_{-n+2}, \dots, \hat{s}^-_{-1}, \hat{s}^+_0, \hat{s}^-_1, \dots, \hat{s}^-_{n-1}, \hat{s}^+_n),$$

where, for $i = 2, \dots, n$, $\hat{s}_{-i}$ follows $(\hat{s}_{-i+2}, \hat{s}_{-i+1})$, and $\hat{s}_i$ follows $(\hat{s}_{i-2}, \hat{s}_{i-1})$. From this we obtain the next result.

Proposition 7.2. *Keeping the above notation, let Q^n be the n-th projective in a minimal projective $\mathcal{A}_\Gamma$-resolution of the simple module $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$, and let $n > 0$. If n is odd,*

$$Q^n = P_{\hat{S}_{-n}} \oplus P_{\hat{S}_{-n+2}} \oplus \cdots \oplus P_{\hat{S}_{-1}} \oplus P_{\hat{S}_1} \oplus \cdots \oplus P_{\hat{S}_{n-2}} \oplus P_{\hat{S}_n},$$

and, if n is even,

$$Q^n = P_{\hat{S}_{-n}} \oplus P_{\hat{S}_{-n+2}} \oplus \cdots \oplus P_{\hat{S}_{-2}} \oplus P_{\hat{S}_0} \oplus P_{\hat{S}_2} \oplus \cdots \oplus P_{\hat{S}_{n-2}} \oplus P_{\hat{S}_n},$$

where, $\hat{s}_{-1}$ and $\hat{s}_1$ are the successors of $\hat{s}_0$ at its endpoints, and, for $i = 2, \dots, n$, $\hat{s}_{-i}$ follows $(\hat{s}_{-i+2}, \hat{s}_{-i+1})$, and $\hat{s}_i$ follows $(\hat{s}_{i-2}, \hat{s}_{i-1})$.

It remains to describe the maps f^n in the projective resolution. We recall from Section 3 that, if edge t is the successor of edge s in Γ at vertex α, then we denote the corresponding arrow in $\mathcal{Q}_\Gamma$ from vertex v_s to vertex v_t by $a(s,t)$, since there are no loops or multiple edges in Γ. Suppose that $s = s_0, s_1, s_2, \dots, s_{n-1}$ is the successor sequence for s at the vertex α in Γ, and set $s_n = s_0$. If $1 \leqslant k \leqslant n-1$, we denote the path $a(s_0, s_1)a(s_1, s_2) \cdots a(s_{k-1}, s_k)$, from v_{s_0} to v_{s_k} in $\mathcal{Q}_\Gamma$, by $p(s_0, s_k)$. Note that our assumptions on Γ show that $p(s_0, s_k)$ is well-defined.

Lemma 7.3. *Suppose that s is an edge in Γ with endpoints α and β and that $s = s_0, s_1, \dots, s_{n-1}$ is the successor sequence for s at α. Let $s_n = s_0$ since s_0 is the successor of s_{n-1}. Assume that t is in the successor sequence of s at β.*

(1) *If $0 \leqslant i < j < k \leqslant n$, then $p(s_i, s_j)p(s_j, s_k) = p(s_i, s_k) \neq 0$.*
(2) *If $1 \leqslant i \leqslant n$, then $p(t, s_0)p(s_0, s_i) = 0$.*
(3) *If $0 \leqslant i \leqslant n-1$, then $p(s_i, s_n)p(s_0, t) = 0$.*

Proof. To prove **(1)**, we note that $p(s_i, s_j) = a(s_i, s_{i+1}) \cdots a(s_{j-1}, s_j)$ and $p(s_j, s_k) = a(s_j, s_{j+1}) \cdots a(s_{k-1}, s_k)$. Hence $p(s_i, s_j)p(s_j, s_k) = p(s_i, s_k)$. That $p(s_i, s_k) \neq 0$ follows from the relations defining I_Γ and the fact that $p(s_i, s_k)$ is a factor of $C_{s,\alpha}$.

The other parts follow from the relations defining I_Γ, and the fact that $p(s_0, s_i)$ and $p(s_i, s_n)$ are associated to the successor sequence for s at vertex α, whereas the paths $p(t, s_0)$ and $p(s_0, t)$ are associated to the successor sequence for s at vertex β. $\square$

We are now in a position to define the maps $f^n \colon Q^n \to Q^{n-1}$, for $n \geqslant 0$. The map $f^0 \colon v_{\hat{s}_0}(\mathcal{A}_\Gamma) \to v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$ is the canonical surjection. Recall that, for each edge $s \in \Gamma$, we are setting $P_S = v_s(\mathcal{A}_\Gamma)$. Using our description of the projective module Q^n in

Proposition 7.2, for $n \geqslant 1$, we write Q^n as an $(n+1) \times 1$ column vector. Then f^n will be given as an $n \times (n+1)$ matrix with the (i,j)-th entry in $v_{\hat{s}_{-n+2i-1}}(\mathcal{A}_\Gamma)v_{\hat{s}_{-n+2j-2}}$, representing a map from $P_{\hat{S}_{-n+2j-2}} = v_{\hat{s}_{-n+2j-2}}(\mathcal{A}_\Gamma)$ to $P_{\hat{S}_{-n+2i-1}} = v_{\hat{s}_{-n+2i-1}}(\mathcal{A}_\Gamma)$.

The map $f^1\colon Q^1 \to Q^0$ is given by the 1×2 matrix

$$\begin{pmatrix} p(\hat{s}_0, \hat{s}_{-1}) & p(\hat{s}_0, \hat{s}_1) \end{pmatrix},$$

where $\hat{s}_{-1}$ and $\hat{s}_1$ are the successors of $\hat{s}_0$ at its endpoints, so that $p(\hat{s}_0, \hat{s}_{-1}) = a(\hat{s}_0, \hat{s}_{-1})$ and $p(\hat{s}_0, \hat{s}_1) = a(\hat{s}_0, \hat{s}_1)$.

For $n \geqslant 2$, f^n is given by the matrix

$$\begin{pmatrix} (-1)^{n-1}p(\hat{s}_{-n+1}, \hat{s}_{-n}) & p(\hat{s}_{-n+1}, \hat{s}_{-n+2}) & 0 & \cdots & 0 & 0 \\ 0 & (-1)^{n-1}p(\hat{s}_{-n+3}, \hat{s}_{-n+2}) & p(\hat{s}_{-n+3}, \hat{s}_{-n+4}) & & 0 & 0 \\ 0 & 0 & (-1)^{n-1}p(\hat{s}_{-n+5}, \hat{s}_{-n+4}) & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & p(\hat{s}_{n-3}, \hat{s}_{n-2}) & 0 \\ 0 & 0 & 0 & \cdots & (-1)^{n-1}p(\hat{s}_{n-1}, \hat{s}_{n-2}) & p(\hat{s}_{n-1}, \hat{s}_n) \end{pmatrix}.$$

We now come to the main result of this section, which shows that we have indeed described a minimal projective $\mathcal{A}_\Gamma$-resolution of the simple module $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$.

Theorem 7.4. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges and no truncated edges, and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Let $\hat{s}_0$ be an edge in Γ and*

$$(Q^\bullet, f^\bullet): \quad \cdots \to Q^2 \xrightarrow{f^2} Q^1 \xrightarrow{f^1} Q^0 \xrightarrow{f^0} v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r}) \to 0$$

be as given above. Then $(Q^\bullet, f^\bullet)$ is a minimal projective $\mathcal{A}_\Gamma$-resolution of $v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$.

Proof. We begin by showing that $f^{n-1} \circ f^n = 0$, for $n \geqslant 1$. For $n = 1$ this is clear since, from the definitions of f^0 and f^1, we see that $\mathrm{Im}(f^1) = v_{\hat{s}_0}\mathbf{r} = \mathbf{str}(\hat{s}_{-1}^+, \hat{s}_0^-, \hat{s}_1^+) = \mathrm{Ker}(f^0)$. That $f^1 \circ f^2 = 0$ can be proved directly from the matrices. So assume $n \geqslant 3$.

Let A be the matrix representing f^n

$$\begin{pmatrix} (-1)^{n-1}p(\hat{s}_{-n+1}, \hat{s}_{-n}) & p(\hat{s}_{-n+1}, \hat{s}_{-n+2}) & 0 & \cdots & 0 & 0 \\ 0 & (-1)^{n-1}p(\hat{s}_{-n+3}, \hat{s}_{-n+2}) & p(\hat{s}_{-n+3}, \hat{s}_{-n+4}) & & 0 & 0 \\ 0 & 0 & (-1)^{n-1}p(\hat{s}_{-n+5}, \hat{s}_{-n+4}) & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & p(\hat{s}_{n-3}, \hat{s}_{n-2}) & 0 \\ 0 & 0 & 0 & \cdots & (-1)^{n-1}p(\hat{s}_{n-1}, \hat{s}_{n-2}) & p(\hat{s}_{n-1}, \hat{s}_n) \end{pmatrix}$$

and B be the matrix representing f^{n-1}

$$\begin{pmatrix} (-1)^{n-2}p(\hat{s}_{-n+2}, \hat{s}_{-n+1}) & p(\hat{s}_{-n+2}, \hat{s}_{-n+3}) & 0 & \cdots & 0 & 0 \\ 0 & (-1)^{n-2}p(\hat{s}_{-n+4}, \hat{s}_{-n+3}) & p(\hat{s}_{-n+4}, \hat{s}_{-n+5}) & & 0 & 0 \\ 0 & 0 & (-1)^{n-2}p(\hat{s}_{-n+6}, \hat{s}_{-n+5}) & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & p(\hat{s}_{n-4}, \hat{s}_{n-3}) & 0 \\ 0 & 0 & 0 & \cdots & (-1)^{n-2}p(\hat{s}_{n-2}, \hat{s}_{n-3}) & p(\hat{s}_{n-2}, \hat{s}_{n-1}) \end{pmatrix}.$$

We show BA is the zero matrix. The $(1,1)$-entry of BA is $-p(\hat{s}_{-n+2},\hat{s}_{-n+1})p(\hat{s}_{-n+1},\hat{s}_{-n)}$. But $\hat{s}_{-n+2}$ and $\hat{s}_{-n}$ are in the successor sequences for $\hat{s}_{-n+1}$ at different vertices. Hence $p(\hat{s}_{-n+2},\hat{s}_{-n+1})p(\hat{s}_{-n+1},\hat{s}_{-n)} = 0$ by Lemma 7.3**(3)**. The remaining entries of the first column in BA are all 0.

The $(1,2)$-entry of BA is

$$(-1)^{n-2}p(\hat{s}_{-n+2},\hat{s}_{-n+1})p(\hat{s}_{-n+1},\hat{s}_{-n+2}) + (-1)^{n-1}p(\hat{s}_{-n+2},\hat{s}_{-n+3})p(\hat{s}_{-n+3},\hat{s}_{-n+2}).$$

Suppose the endpoints of $\hat{s}_{-n+2}$ are α and β in Γ. If $\hat{s}_{-n+1}$ is in the successor sequence of $\hat{s}_{-n+2}$ at the vertex α, then $p(\hat{s}_{-n+2},\hat{s}_{-n+1})p(\hat{s}_{-n+1},\hat{s}_{-n+2}) = C_{\hat{s}_{-n+2},\alpha}$. We must also have that $\hat{s}_{-n+3}$ is in the successor sequence of $\hat{s}_{-n+2}$ at the vertex β, and $p(\hat{s}_{-n+2},\hat{s}_{-n+3})p(\hat{s}_{-n+3},\hat{s}_{-n+2}) = C_{\hat{s}_{-n+2},\beta}$. Hence we see that the $(1,2)$-entry of BA is $(-1)^{n-2}(C_{\hat{s}_{-n+2},\alpha} - C_{\hat{s}_{-n+2},\beta})$ which is a scalar multiple of a relation of type one in I_Γ and hence 0.

The $(2,2)$-entry of BA is $(-1)^{n-2}p(\hat{s}_{-n+4},\hat{s}_{-n+3})(-1)^{n-1}p(\hat{s}_{-n+3},\hat{s}_{-n+2})$. But $\hat{s}_{-n+4}$ and $\hat{s}_{-n+2}$ are in the successor sequences for $\hat{s}_{-n+3}$ at different vertices. Hence $p(\hat{s}_{-n+4},\hat{s}_{-n+3})p(\hat{s}_{-n+3},\hat{s}_{-n+2}) = 0$ by Lemma 7.3. The remaining entries of the second column in BA are all 0.

This alternating pattern continues for the remaining columns and we have shown BA is the zero matrix. Thus $\mathrm{Im}(f^n) \subseteq \mathrm{Ker}(f^{n-1})$.

To show equality, we note that the top of Q^n maps into $\mathrm{Ker}(f^{n-1})$. Inductively, we may assume that $\mathrm{Ker}(f^{n-1})$ is isomorphic to $\Omega^n_{\mathcal{A}_\Gamma}(v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r}))$. The uniqueness of the simple composition factors of indecomposable projective $\mathcal{A}_\Gamma$-modules given in Lemma 4.3**(3)**, together with the structure of the syzygies given in Proposition 7.1, show that the top of Q^n maps isomorphically to the top of $\mathrm{Ker}(f^{n-1})$. Hence $\mathrm{Im}(f^n) = \mathrm{Ker}(f^{n-1})$.

Since, for $n \geqslant 1$, the image of f^n is contained in $Q^{n-1}\mathbf{r}$, the resolution is minimal and the proof is complete. □

8. The Ext algebra of a Brauer graph algebra with no truncated edges

In this section, we assume that $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph with no truncated edges. We prove one of the main results of this paper, showing that the Ext algebra of the associated Brauer graph algebra $\mathcal{A}_\Gamma$ is finitely generated in degrees 0, 1 and 2.

Let G be a finite abelian group and let $W\colon \mathcal{Z}_\Gamma \to G$ be a Brauer weighting such that the Brauer covering graph $(\Gamma_W, \mathfrak{o}_W, \mathfrak{m}_W)$ has $\mathfrak{m}_W \equiv 1$ and no loops or multiple edges (see Section 3). Suppose that s is an edge in Γ incident with vertex α in Γ, and s_g is an edge in Γ_W incident with vertex α_g in Γ_W, such that s_g lies over s and α_g lies over α for some $g \in G$. Then, by [10, Proposition 3.4 and Definition 3.5], $\mathfrak{m}(\alpha)\,\mathrm{val}_\Gamma(\alpha) = \mathfrak{m}_W(\alpha_g)\,\mathrm{val}_{\Gamma_W}(\alpha_g)$, where $\mathrm{val}_\Gamma(\alpha)$ and $\mathrm{val}_{\Gamma_W}(\alpha_g)$ denote the valencies of α and α_g respectively. It follows that edge s in Γ is truncated at vertex α in Γ if and only if edge s_g in Γ_W is truncated at vertex α_g in Γ_W. Thus, to study the Ext algebra of a Brauer graph algebra associated to a Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with no truncated edges, it follows from the above discussion and Proposition 3.1 that we may assume $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph with $\mathfrak{m} \equiv 1$, with no loops or multiple edges and no truncated edges.

Let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra and let $\mathbf{r}$ denote the Jacobson radical of $\mathcal{A}_\Gamma$. The Ext algebra of $\mathcal{A}_\Gamma$ is $E(\mathcal{A}_\Gamma) = \oplus_{n\geqslant 0} \operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}, \mathcal{A}_\Gamma/\mathbf{r})$ with the Yoneda product. Let $\hat{s}_0$ be an edge in Γ and $\hat{S}_0 = v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r})$ the associated simple $\mathcal{A}_\Gamma$-module, where $v_{\hat{s}_0}$ is the idempotent in $\mathcal{A}_\Gamma$ associated to $\hat{s}_0$. Let

$$(Q^\bullet, f^\bullet): \quad \cdots \to Q^2 \xrightarrow{f^2} Q^1 \xrightarrow{f^1} Q^0 \xrightarrow{f^0} v_{\hat{s}_0}(\mathcal{A}_\Gamma/\mathbf{r}) \to 0$$

be the minimal projective $\mathcal{A}_\Gamma$-resolution of $\hat{S}_0$ given in Theorem 7.4, with

$$Q^n = \begin{cases} P_{\hat{S}_{-n}} \oplus P_{\hat{S}_{-n+2}} \oplus \cdots \oplus P_{\hat{S}_{-2}} \oplus P_{\hat{S}_0} \oplus P_{\hat{S}_2} \oplus \cdots \oplus P_{\hat{S}_n}, & \text{for } n \text{ even} \\ P_{\hat{S}_{-n}} \oplus P_{\hat{S}_{-n+2}} \oplus \cdots \oplus P_{\hat{S}_{-1}} \oplus P_{\hat{S}_1} \oplus \cdots \oplus P_{\hat{S}_n}, & \text{for } n \text{ odd.} \end{cases}$$

Since each $P_{\hat{S}}$ is an indecomposable projective $\mathcal{A}_\Gamma$-module, we choose a K-basis $G^n_i(\hat{S}_0)$, where $i \in \{-n, -n+2, \ldots, n-2, n\}$, for $\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, \mathcal{A}_\Gamma/\mathbf{r})$, where $G^n_i(\hat{S}_0)$ represents the element in $\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, \mathcal{A}_\Gamma/\mathbf{r})$ given by the composition

$$Q^n \to P_{\hat{S}_i} \to \hat{S}_i \to \mathcal{A}_\Gamma/\mathbf{r},$$

where the first map is the projection map, the second map is the canonical surjection, and the third map is inclusion. We call $\{G^n_i(\hat{S}_0)\}_{i\in\{-n,-n+2,\ldots,n-2,n\}}$ the *canonical basis of* $\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, \mathcal{A}_\Gamma/\mathbf{r})$. Since $\mathcal{A}_\Gamma/\mathbf{r} = \oplus_{\hat{s}_0\in\Gamma_1} \hat{S}_0$, so that

$$\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}, \mathcal{A}_\Gamma/\mathbf{r}) = \bigoplus_{\hat{s}_0\in\Gamma_1} \operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, \mathcal{A}_\Gamma/\mathbf{r}),$$

we abuse notation and view

$$\mathcal{G}^n = \bigcup_{\hat{s}_0\in\Gamma_1} \{G^n_i(\hat{S}_0)\}_{i\in\{-n,-n+2,\ldots,n-2,n\}}$$

as a K-basis of $\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\mathcal{A}_\Gamma/\mathbf{r}, \mathcal{A}_\Gamma/\mathbf{r})$.

We now present the main result of this section.

Theorem 8.1. *Let* $(\Gamma, \mathfrak{o}, \mathfrak{m})$ *be a Brauer graph with no truncated edges, and let* $\mathcal{A}_\Gamma$ *denote the associated Brauer graph algebra. Then the Ext algebra,* $E(\mathcal{A}_\Gamma)$*, is finitely generated in degrees* $0, 1$ *and* 2*.*

Proof. From the above discussion, we may assume $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph with $\mathfrak{m} \equiv 1$, with no loops or multiple edges and no truncated edges. Fix an edge $\hat{s}_0$ in Γ with associated simple $\mathcal{A}_\Gamma$-module $\hat{S}_0$. We keep the previous notation. Since $\mathcal{A}_\Gamma/\mathbf{r} = \oplus_{t\in\Gamma_1} T$, we have $\operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, \mathcal{A}_\Gamma/\mathbf{r}) = \oplus_{t\in\Gamma_1} \operatorname{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, T)$, and hence, for $i \in \{-n, -n+2, \ldots, n-2, n\}$, we may view $G^n_i(\hat{S}_0)$ as a map $G^n_i(\hat{S}_0)\colon Q^n \to \hat{S}_i$.

First we suppose that $n \geqslant 2$ is an even integer. Let $i \in \{2, 4, \dots, n-2, n\}$, and let

$$(R^\bullet, g^\bullet): \quad \cdots \to R^2 \stackrel{g^2}{\to} R^1 \stackrel{g^1}{\to} R^0 \stackrel{g^0}{\to} v_{\hat{s}_{i-1}}(\mathcal{A}_\Gamma/\mathbf{r}) \to 0$$

be the minimal projective $\mathcal{A}_\Gamma$-resolution of $\hat{S}_{i-1}$ given in Theorem 7.4. We show that

$$G_i^n(\hat{S}_0) = G_{i-1}^{n-1}(\hat{S}_0) * G_i^1(\hat{S}_{i-1}),$$

where the right hand side is viewed in the Yoneda product $\mathrm{Ext}^1_{\mathcal{A}_\Gamma}(\hat{S}_{i-1}, \hat{S}_i) \times \mathrm{Ext}^{n-1}_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_{i-1})$. For ease of notation and consistency, we set $\hat{t}_0 = \hat{s}_{i-1}$ (noting that $i \neq 0$). Let $\hat{t}_1$ and $\hat{t}_{-1}$ be the edges in Γ that are the successors of $\hat{t}_0$ at the endpoints of $\hat{t}_0$. Define the sequence $\hat{t}_{-n}, \hat{t}_{-n+1}, \dots, \hat{t}_{n-1}, \hat{t}_n$ recursively:

$$\text{for } i > 1,\ \hat{t}_i \text{ follows } (\hat{t}_{i-2}, \hat{t}_{i-1}) \text{ and } \hat{t}_{-i} \text{ follows } (\hat{t}_{-i+2}, \hat{t}_{-i+1}).$$

With this notation,

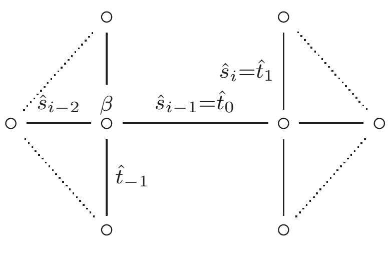
$$R^m = \begin{cases} P_{\hat{T}_{-m}} \oplus P_{\hat{T}_{-m+2}} \oplus \cdots \oplus P_{\hat{T}_{-2}} \oplus P_{\hat{T}_0} \oplus P_{\hat{T}_2} \oplus \cdots \oplus P_{\hat{T}_m}, & \text{for } m \text{ even} \\ P_{\hat{T}_{-m}} \oplus P_{\hat{T}_{-m+2}} \oplus \cdots \oplus P_{\hat{T}_{-1}} \oplus P_{\hat{T}_1} \oplus \cdots \oplus P_{\hat{T}_m}, & \text{for } m \text{ odd,} \end{cases}$$

and the maps $g^m : R^m \to R^{m-1}$ are given in a similar fashion to the maps f^n in the resolution of $\hat{S}_0$.

We begin by finding maps ψ_0 and ψ_1 such that the following diagram commutes.

$$(*) \qquad \begin{tikzcd} Q^n \arrow[r, "f^n"] \arrow[d, "\psi_1"] & Q^{n-1} \arrow[d, "\psi_0"] \arrow[dr, "G_{i-1}^{n-1}(\hat{S}_0)"] & \\ R^1 \arrow[r, "g^1"] & R^0 \arrow[r, "g^0"] & \hat{S}_{i-1} \end{tikzcd}$$

Since $\hat{t}_0 = \hat{s}_{i-1}$ and since $\hat{s}_i$ follows $(\hat{s}_{i-2}, \hat{s}_{i-1})$, we see that $\hat{s}_i$ is the successor of $\hat{s}_{i-1}$ at one of its endpoints. Furthermore, $\hat{s}_{i-2}$ is in the successor sequence of $\hat{s}_{i-1}$ at the other endpoint β of $\hat{s}_{i-1}$. Thus, after reordering, we may assume that $\hat{t}_1 = \hat{s}_i$ and that both $\hat{s}_{i-2}$ and $\hat{t}_{-1}$ are in the successor sequence of $\hat{s}_{i-1}$ at the vertex β, with $\hat{t}_{-1}$ being the successor of $\hat{s}_{i-1}$ at β. The following diagram illustrates this.



From this we see that $R^0 = P_{\hat{T}_0} = P_{\hat{S}_{i-1}}$ and $R^1 = P_{\hat{T}_{-1}} \oplus P_{\hat{T}_1} = P_{\hat{T}_{-1}} \oplus P_{\hat{S}_i}$. Define $\psi_0 : Q^{n-1} \to R^0$ by $\psi_0(x_{-n+1}, x_{-n+3}, \dots, x_{n-3}, x_{n-1}) = x_{i-1}$ and define $\psi_1 : Q^n \to R^1$ by the $2 \times n$ matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -p(\hat{t}_{-1}, \hat{s}_{i-2}) & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & v_{\hat{s}_i} & 0 & \cdots & 0 \end{pmatrix}$$

where the first nonzero column represents the map from $P_{\hat{S}_{i-2}}$ to $P_{\hat{T}_{-1}} \oplus P_{\hat{S}_i}$ and the next nonzero column represents the map from $P_{\hat{S}_i}$ to $P_{\hat{T}_{-1}} \oplus P_{\hat{S}_i}$. The reader may now check the commutativity of $(*)$.

Next we see from the diagram

$$\begin{tikzcd}
Q^n \arrow[r, "f^n"] \arrow[d, "\psi_1"] & Q^{n-1} \arrow[d, "\psi_0"] \arrow[dr, "G^{n-1}_{i-1}(\hat{S}_0)"] & \\
R^1 \arrow[r, "g^1"] \arrow[dr, "G^1_i(\hat{S}_{i-1})"'] & R^0 \arrow[r, "g^0"] & \hat{S}_{i-1} \\
& \hat{S}_i &
\end{tikzcd}$$

that $G^1_i(\hat{S}_{i-1}) \circ \psi_1 = G^{n-1}_{i-1}(\hat{S}_0) * G^1_i(\hat{S}_{i-1}) = G^n_i(\hat{S}_0)$.

Next suppose $i \in \{-n, -n+2, \ldots, -2\}$. The proof that $G^{n-1}_{i+1}(\hat{S}_0) * G^1_i(\hat{S}_{i+1}) = G^n_i(\hat{S}_0)$ is similar and is left to the reader. In fact, interchanging $\hat{s}_i$ with $\hat{s}_{-i}$ for $i \in \{2, \ldots, n-2, n\}$ in the above proof, gives the result.

For n even, it remains to consider the case when $i = 0$. Here we show that

$$G^{n-2}_0(\hat{S}_0) * G^2_0(\hat{S}_0) = G^n_0(\hat{S}_0).$$

This is clear if $n = 2$ so assume $n \geqslant 4$. We find explicit maps θ_0, θ_1, and θ_2 such that the following diagram commutes.

$$(**) \qquad \begin{tikzcd}
Q^n \arrow[r, "f^n"] \arrow[d, "\theta_2"] & Q^{n-1} \arrow[r, "f^{n-1}"] \arrow[d, "\theta_1"] & Q^{n-2} \arrow[d, "\theta_0"] \arrow[dr, "G^{n-2}_0(\hat{S}_0)"] & \\
Q^2 \arrow[r, "f^2"] & Q^1 \arrow[r, "f^1"] & Q^0 \arrow[r, "f^0"] & \hat{S}_0
\end{tikzcd}$$

We have that $Q^i = P_{\hat{S}_{-i}} \oplus P_{\hat{S}_{-i+2}} \oplus \cdots \oplus P_{\hat{S}_{i-2}} \oplus P_{\hat{S}_i}$. Thus, for $i \geqslant 3$, and $j = i - 2k$, with $k \geqslant 0$,

$$Q^i = P_{\hat{S}_{-i}} \oplus P_{\hat{S}_{-i+2}} \oplus \cdots \oplus P_{\hat{S}_{-j-2}} \oplus Q^j \oplus P_{\hat{S}_{j+2}} \oplus \cdots \oplus P_{\hat{S}_i}.$$

We then take θ_0, θ_1, and θ_2 to be the projections $Q^i \to Q^j$, for the appropriate i's and j's. The reader may check that $(**)$ commutes. Noting that the composition $G^2_0(\hat{S}_0) \circ \theta_2$ is just $G^n_0(\hat{S}_0)$, we have that $G^{n-2}_0(\hat{S}_0) * G^2_0(\hat{S}_0) = G^n_0(\hat{S}_0)$. This completes the study of the case when n is even.

Now suppose that n is odd, $n \geqslant 3$. We claim that, for $i \in \{1, 3, \ldots, n-2, n\}$,

$$G^{n-1}_{i-1}(\hat{S}_0) * G^1_i(\hat{S}_{i-1}) = G^n_i(\hat{S}_0),$$

and for $i \in \{-1, -3, \ldots, -n+2, -n\}$,

$$G^{n-1}_{i+1}(\hat{S}_0) * G^1_i(\hat{S}_{i+1}) = G^n_i(\hat{S}_0).$$

For $i \neq -1, 1$, the proof is analogous to the n even case. For $i = -1$ or $i = +1$, keeping θ_0 and θ_1 as above, we get a commutative diagram

$$\begin{tikzcd} Q^n \arrow[r, "f^n"] \arrow[d, "\theta_1"] & Q^{n-1} \arrow[d, "\theta_0"] \arrow[dr, "G_0^{n-1}(\hat{S}_0)"] & \\ Q^1 \arrow[r, "f^1"] & Q^0 \arrow[r, "f^0"] & \hat{S}_0 \end{tikzcd}$$

and it is now immediate that

$$G_0^{n-1}(\hat{S}_0) * G_i^1(\hat{S}_0) = G_i^n(\hat{S}_0).$$

Thus we have shown that every basis element in $\mathrm{Ext}^n_{\mathcal{A}_\Gamma}(\hat{S}_0, T)$ is the product of an element in some $\mathrm{Ext}^{n-1}_{\mathcal{A}_\Gamma}(\hat{S}_i, T)$ with an element in $\mathrm{Ext}^1_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_i)$, or is the product of an element in some $\mathrm{Ext}^{n-2}_{\mathcal{A}_\Gamma}(\hat{S}_i, T)$ with an element in $\mathrm{Ext}^2_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_i)$. This completes the proof. □

We end this section with an immediate application to $\mathcal{K}_2$ algebras. The concept of a $\mathcal{K}_2$ algebra was introduced by Cassidy and Shelton in [5, Definition 1.1], where they defined a graded algebra A to be $\mathcal{K}_2$ if the Ext algebra, $E(A)$, is generated as an algebra in degrees 0, 1 and 2. This class of algebras is a natural generalization of the class of Koszul algebras.

Corollary 8.2. *Let K be an algebraically closed field and let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph with no truncated edges. Let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose $\mathcal{A}_\Gamma$ is length graded. Then $\mathcal{A}_\Gamma$ is a $\mathcal{K}_2$ algebra.*

Proof. From the discussion at the beginning of this section, we may assume that $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ is a quantized Brauer graph with $\mathfrak{m} \equiv 1$, with no loops or multiple edges and no truncated edges. From Proposition 3.2, we may assume further that $\mathfrak{q} \equiv 1$ since the number of generators of the Ext algebra, $E(\mathcal{A}_\Gamma)$, and their degrees do not depend on $\mathfrak{q}$. The result now follows from Theorem 8.1 and the fact that $\mathcal{A}_\Gamma$ is a graded algebra. □

We consider $\mathcal{K}_2$ algebras and other generalizations of Koszul algebras further in the next section.

9. Length graded Brauer graph algebras

In this section, $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph and $\mathcal{A}_\Gamma$ denotes the associated Brauer graph algebra.

Our first results lead to Theorem 9.4, where we characterise the Brauer graph algebras $\mathcal{A}_\Gamma$ where the Ext algebra, $E(\mathcal{A}_\Gamma)$, is finitely generated in degrees 0, 1 and 2. This provides a converse to Theorem 8.1. In the remainder of the section we consider length graded Brauer graph algebras $\mathcal{A}_\Gamma$. We recall the definition from [8] of a 2-d-Koszul algebra. We then complete our study of generalizations of Koszul algebras with Theorem 9.6, where we classify the Brauer graph algebras that are 2-d-Koszul. Indeed, Theorem 9.6 shows that a Brauer graph algebra is a 2-d-Koszul algebra if and only if it is 2-d-homogeneous and a $\mathcal{K}_2$ algebra.

We begin with a result on syzygies of string modules where the Brauer graph Γ may have both truncated and nontruncated edges.

Proposition 9.1. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges, and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Let $\sigma = \hat{s}_1^{e_1}, \ldots, \hat{s}_n^{e_n}$ be an acceptable sequence of weighted edges in Γ, and $n \geqslant 2$. Let $M = \mathbf{str}(\sigma)$.*

(1) *Suppose that $e_1 = +$.*
- *If $\hat{s}_1$ is a nontruncated edge, then the first syzygy of M is isomorphic to the string module $\mathbf{str}(\tau)$, where τ begins as in Proposition 7.1.*
- *If $\hat{s}_1$ is a truncated edge, then the first syzygy of M is isomorphic to the string module $\mathbf{str}(\tau)$, where τ begins $\tau = \hat{s}_1^{e_1^*}, \hat{s}_2^{e_2^*}, \ldots$*

(2) *Suppose that $e_1 = -$.*
- *If $\hat{s}_1$ and $\hat{s}_2$ are incident at the vertex α and if $\hat{s}_2$ is the successor of $\hat{s}_1$ at α, then the first syzygy of M is isomorphic to the string module $\mathbf{str}(\tau)$, where τ begins $\tau = \hat{s}_2^{e_2^*}, \hat{s}_3^{e_3^*}, \ldots$*
- *If $\hat{s}_1$ and $\hat{s}_2$ are incident at the vertex α and if $\hat{s}_2$ is not the successor of $\hat{s}_1$ at α, then the first syzygy of M is isomorphic to the string module $\mathbf{str}(\tau)$, where τ begins $\tau = \hat{t}^+, \hat{s}_2^{e_2^*}, \hat{s}_3^{e_3^*}, \ldots$, and where $\hat{t}$ is the successor of $\hat{s}_1$ at α.*

(3) *The first syzygy of M may be fully determined using* **(1)**, **(2)** *and Corollary 6.5.*

The proof is similar to that of Proposition 7.1; note that these syzygies also appear in [1] where they consider the Ext algebra of a symmetric Brauer graph algebra.

For a string module $\mathbf{str}(\sigma)$, where $\sigma = \hat{s}_1^{e_1}, \ldots, \hat{s}_n^{e_n}$ is an acceptable sequence of weighted edges in Γ, we see that the only $\hat{s}_i$ that can be truncated edges are $\hat{s}_1$ and $\hat{s}_n$. Thus we may use Proposition 9.1 to find all syzygies of the simple $\mathcal{A}_\Gamma$-modules. In the case where S is the simple $\mathcal{A}_\Gamma$-module corresponding to a truncated edge s in Γ, we can simplify Proposition 9.1, and have the following corollary.

Corollary 9.2. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges, and assume that $\Gamma \neq \mathbb{A}_2$. Let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose that edge s in Γ is truncated at vertex α. Define edges $\hat{s}_i$ recursively, so that $\hat{s}_0 = s$, $\hat{s}_1$ is the successor of s at the endpoint $\beta \neq \alpha$, and, for $m \geqslant 1$, if $\hat{s}_m$ is not truncated at the vertex which is not incident with $\hat{s}_{m-1}$, then $\hat{s}_{m+1}$ follows $(\hat{s}_{m-1}, \hat{s}_m)$. Then,*

(1) *there is some $n \geqslant 1$ so that $\hat{s}_n$ is a truncated edge in Γ;*
(2) *for $1 \leqslant m \leqslant n$, the m-th syzygy of S is $\mathbf{str}(\hat{s}_m^+, \hat{s}_{m-1}^-)$;*
(3) *the $(n+1)$-st syzygy of S is $\mathbf{str}(\hat{s}_n^+)$.*

Moreover, S is a periodic $\mathcal{A}_\Gamma$-module.

The sequence $(\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_n)$ of Corollary 9.2 is precisely the Brauer walk for $s = \hat{s}_0$; see [14] and [11]. We note that it is also known from [15, Corollary 2.7] that S is a periodic $\mathcal{A}_\Gamma$-module when s is a truncated edge in Γ.

For the next result, we recall that $\underline{\mathrm{Hom}}_{\mathcal{A}_\Gamma}(-,-)$ denotes the $\mathcal{A}_\Gamma$-module homomorphisms modulo those homomorphisms which factor through a projective $\mathcal{A}_\Gamma$-module.

Theorem 9.3. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ with $\mathfrak{m} \equiv 1$ be a Brauer graph with no loops or multiple edges. Let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. If Γ has both truncated and nontruncated edges, then the Ext algebra, $E(\mathcal{A}_\Gamma)$, is not generated in degrees 0, 1 and 2 alone.*

In particular, if we have a sequence of edges $\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_n$ with $n \geqslant 2$, where

(i) $\hat{s}_0$ and $\hat{s}_n$ are truncated edges,
(ii) $\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{n-1}$ are nontruncated edges,
(iii) $\hat{s}_1$ is the successor of $\hat{s}_0$,
(iv) $\hat{s}_{i+1}$ follows $(\hat{s}_{i-1}, \hat{s}_i)$ for $1 \leqslant i \leqslant n-1$,

then there is an element of $\mathrm{Ext}^{n+1}_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_n)$ which is not in the subalgebra of $E(\mathcal{A}_\Gamma)$ generated by the elements of degree at most n.

Proof. Note that we continue to assume that the Brauer graph Γ is connected. We shall use Corollary 9.2 repeatedly without comment. Recall that $\mathrm{Ext}^k_{\mathcal{A}_\Gamma}(\hat{S}_0, T) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^k(\hat{S}_0), T) \cong \underline{\mathrm{Hom}}_{\mathcal{A}_\Gamma}(\Omega^{k+\ell}(\hat{S}_0), \Omega^\ell(T))$ for any simple module T, and that for simple modules S, T and U, and integers k and ℓ, the Yoneda product $\mathrm{Ext}^\ell_{\mathcal{A}_\Gamma}(T, U) \times \mathrm{Ext}^k_{\mathcal{A}_\Gamma}(S, T) \to \mathrm{Ext}^{k+\ell}_{\mathcal{A}_\Gamma}(S, U)$ can be identified with the composition of maps $\underline{\mathrm{Hom}}_{\mathcal{A}_\Gamma}(\Omega^{k+\ell}(S), \Omega^\ell(T)) \times \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^\ell(T), U) \to \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^{k+\ell}(S), U)$.

Since $\hat{s}_0$ is a truncated edge, we know that $\mathrm{Ext}^{n+1}_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_n) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^{n+1}(\hat{S}_0), \hat{S}_n) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\hat{S}_n, \hat{S}_n) \cong K$. We must prove that homomorphisms from $\hat{S}_n$ to itself cannot be written as sums of Yoneda products of elements of $\mathrm{Ext}^i_{\mathcal{A}_\Gamma}(\hat{S}_0, T)$ and $\mathrm{Ext}^j_{\mathcal{A}_\Gamma}(T, \hat{S}_n)$ with T a simple $\mathcal{A}_\Gamma$-module, $i + j = n + 1$ and i and j nonzero.

First note that $\mathrm{Ext}^i_{\mathcal{A}_\Gamma}(\hat{S}_0, T) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^i(\hat{S}_0), T) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\mathrm{Top}(\Omega^i(\hat{S}_0)), T)$, and we have $\mathrm{Top}(\Omega^i(\hat{S}_0)) = \hat{S}_i$ for some i with $1 \leqslant i \leqslant n$. Therefore $T = \hat{S}_i$.

Moreover, $\mathrm{Ext}^i_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_i) \cong \underline{\mathrm{Hom}}_{\mathcal{A}_\Gamma}(\Omega^{i+j}(\hat{S}_0), \Omega^j(\hat{S}_i)) = \underline{\mathrm{Hom}}_{\mathcal{A}_\Gamma}(\Omega^{n+1}(\hat{S}_0), \Omega^j(\hat{S}_i)) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\hat{S}_n, \Omega^j(\hat{S}_i))$ and $\mathrm{Ext}^j_{\mathcal{A}_\Gamma}(\hat{S}_i, \hat{S}_n) \cong \mathrm{Hom}_{\mathcal{A}_\Gamma}(\Omega^j(\hat{S}_i), \hat{S}_n)$. These two spaces must be nonzero; therefore $\hat{S}_n$ must be in the socle and in the top of $\Omega^j(\hat{S}_i)$. Since $\hat{s}_n$ is truncated, this means that $\Omega^j(\hat{S}_i)$ is a mixed string module, of the form $\mathbf{str}(\hat{s}_n^-, \ldots, \hat{s}_n^+)$, by Proposition 9.1. But then the composition $\hat{S}_n \to \Omega^j(\hat{S}_i) \to \hat{S}_n$ must be zero, unless $\Omega^j(\hat{S}_i) = \hat{S}_n$, since the first map must go into the socle and the second map comes from the top. Hence assume that $\Omega^j(\hat{S}_i) = \hat{S}_n$. Since $\hat{S}_n = \Omega^{n+1}(\hat{S}_0)$ and $\mathcal{A}_\Gamma$ is selfinjective, this implies that $\hat{S}_i = \Omega^{n+1-j}(\hat{S}_0) = \Omega^i(\hat{S}_0)$ so that $\Omega^i(\hat{S}_0)$ is simple. However, $\Omega^i(\hat{S}_0)$ is the nonsimple uniserial module with top $\hat{S}_i$ and socle $\hat{S}_{i-1}$, so that we have a contradiction. Therefore any Yoneda product $\mathrm{Ext}^{n+1-i}_{\mathcal{A}_\Gamma}(T, \hat{S}_n) \times \mathrm{Ext}^i_{\mathcal{A}_\Gamma}(\hat{S}_0, T) \to \mathrm{Ext}^{n+1}_{\mathcal{A}_\Gamma}(\hat{S}_0, \hat{S}_n)$ with $1 \leqslant i \leqslant n$ and T simple is zero, and we have the required result. □

The next theorem characterizes the Brauer graph algebras $\mathcal{A}_\Gamma$ where the Ext algebra, $E(\mathcal{A}_\Gamma)$, is finitely generated in degrees 0, 1 and 2, providing a converse to Theorem 8.1.

Theorem 9.4. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Then the Ext algebra, $E(\mathcal{A}_\Gamma)$, is finitely generated in degrees 0, 1 and 2 if and only if Γ does not have both truncated and nontruncated edges.*

Proof. If Γ has no truncated edges, then it follows from Theorem 8.1 that $E(\mathcal{A}_\Gamma)$ is finitely generated in degrees 0,1 and 2. So suppose that Γ has at least one truncated edge. If all the edges are truncated then Γ is a star (including the case $\Gamma = \mathbb{A}_2$) and the associated Brauer graph algebra $\mathcal{A}_\Gamma$ is a Nakayama algebra. It is well-known that such an algebra is d-Koszul (for $d \geqslant 2$) and hence, by [9], its Ext algebra is generated in degrees (at most) 0, 1 and 2. (Indeed, a d-Koszul algebra is also length graded and so is $\mathcal{K}_2$.)

Thus we may assume that Γ has both truncated and nontruncated edges. From the discussion at the start of Section 8 and Proposition 3.1, we may assume that $(\Gamma, \mathfrak{o}, \mathfrak{m})$ is a Brauer graph with $\mathfrak{m} \equiv 1$, with no loops or multiple edges and which also has both truncated and nontruncated edges. It is now immediate from Theorem 9.3, that $E(\mathcal{A}_\Gamma)$ cannot be generated only in degrees 0, 1 and 2. □

Corollary 9.5. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose $\mathcal{A}_\Gamma$ is length graded. Then $\mathcal{A}_\Gamma$ is $\mathcal{K}_2$ if and only if Γ does not have both truncated and nontruncated edges.*

We now introduce 2-d-Koszul algebras, a class of graded algebras which includes the Koszul algebras. Recall from Section 2 that an algebra $\Lambda = KQ/I$ is a 2-d-homogeneous algebra if I can be generated by homogeneous elements of lengths 2 and d.

Let $\Lambda = KQ/I$ where I is generated by homogeneous elements, so that Λ is length graded with $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$. Following Green and Marcos in [8], and for a function $F\colon \mathbb{N} \to \mathbb{N}$, the algebra Λ is said to be *F-determined* (respectively, *weakly F-determined*) if the n-th projective module in a minimal graded projective resolution of Λ_0 (viewed as a graded Λ-module in degree 0) can be generated in degree $F(n)$ (respectively, $\leqslant F(n)$), for all $n \in \mathbb{N}$. Let $\delta\colon \mathbb{N} \to \mathbb{N}$ be the map given by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd.} \end{cases}$$

An algebra $\Lambda = KQ/I$ is then said to be a *2-d-determined* algebra if I can be generated by homogeneous elements of degrees 2 and d, and if Λ is weakly δ-determined. Thus a 2-d-determined algebra is a 2-d-homogeneous algebra. Furthermore, a 2-d-determined algebra is said to be *2-d-Koszul* if its Ext algebra is finitely generated.

Assume that $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ is a quantized Brauer graph and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Suppose that $\mathcal{A}_\Gamma$ is graded with the length grading. Then the conclusions of Theorems 8.1 and 9.4 are still true for $\mathcal{A}_\Gamma$, provided the field K satisfies one of the two conditions in Corollary 3.3. This is the case if K is algebraically closed, for which see Proposition 3.2 and also Corollary 8.2. This allows us to classify the 2-d-homogeneous Brauer graph algebras which are 2-d-determined.

Theorem 9.6. *Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantized Brauer graph, let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra, and assume that either $\mathfrak{q} \equiv 1$ or the field K satisfies the conditions in Corollary 3.3. Let $d \geqslant 3$ and suppose that $\mathcal{A}_\Gamma$ is 2-d-homogeneous. Then the following are equivalent:*

(1) *Γ has no truncated edges,*
(2) *$\mathcal{A}_\Gamma$ is 2-d-determined,*
(3) *$\mathcal{A}_\Gamma$ is 2-d-Koszul,*
(4) *$\mathcal{A}_\Gamma$ is $\mathcal{K}_2$.*

Proof. The equivalence between **(1)** and **(4)** follows from Corollary 9.5, since a 2-d-homogeneous Brauer graph algebra cannot have only truncated edges. By definition, **(3)** implies **(2)**. We prove that **(1)** and **(2)** are equivalent, and then that **(2)** implies **(3)**. Suppose throughout that $\mathcal{A}_\Gamma$ is 2-d-homogeneous.

(2) $\Rightarrow$ **(1)**. Suppose that Γ has at least one truncated edge s. By Proposition 2.5, we know that no two successors are truncated and that for any vertex α in Γ we have $\mathfrak{m}(\alpha)\operatorname{val}(\alpha) \in \{1, d\}$. There are edges $\hat{s}_0 = s, \hat{s}_1, \ldots, \hat{s}_{n-1}, \hat{s}_n$ such that $\hat{s}_n$ is truncated, $n \geqslant 2$, $\hat{s}_1, \ldots, \hat{s}_{n-1}$ are not truncated, $\hat{s}_1$ is the successor of $\hat{s}_0$ at the vertex α_1 and for each integer m with $1 \leqslant m < n$ the edge $\hat{s}_{m+1}$ follows $(\hat{s}_{m-1}, \hat{s}_m)$ at the vertex α_{m+1}, as in Corollary 9.2. Note that when $\hat{s}_m$ is a loop, it occurs twice in the successor sequence of $\hat{s}_{m-1}$; when we say that $\hat{s}_{m+1}$ follows $(\hat{s}_{m-1}, \hat{s}_m)$ in this case, we have considered one instance of the loop being in the successor sequence of $\hat{s}_{m-1}$, and $\hat{s}_{m+1}$ is the successor of $\hat{s}_{m-1}$ at the other instance of this loop in the successor sequence of $\hat{s}_{m-1}$. We shall now prove that the third projective in a minimal projective resolution of the simple $\mathcal{A}_\Gamma$-module $\hat{S}_{n-1}$ has a generator in degree $d+2 > \delta(3)$ so that $\mathcal{A}_\Gamma$ cannot be 2-d-determined.

Let t be the successor of $\hat{s}_{n-1}$ at α_{n-1} and, if t is not truncated, let u follow $(\hat{s}_{n-1}, t)$. Then the indecomposable projective module $P_{\hat{S}_n}$ is the uniserial module of length d whose top and socle are $\hat{S}_n$, which we represent by $P_{\hat{S}_n} = \begin{smallmatrix}\hat{S}_n\\ Y\\ \hat{S}_{n-1}\\ \hat{S}_n\end{smallmatrix}$. The indecomposable projective module $P_{\hat{S}_{n-1}}$ is biserial, we represent it by $P_{\hat{S}_{n-1}} = \begin{smallmatrix} & \hat{S}_{n-1} & \\ T & & \hat{S}_n\\ X & & Y\\ & \hat{S}_{n-1} & \end{smallmatrix}$, where $\begin{smallmatrix}T\\ X\\ \hat{S}_{n-1}\end{smallmatrix}$ is the uniserial module of length $d-1$ with top T and socle $\hat{S}_{n-1}$ (defined from the successor sequence of t at α_{n-1}). We shall also need P_T and P_U which we represent as follows:

$$P_T = \begin{cases} \begin{smallmatrix}T\\ X\\ \hat{S}_{n-1}\\ T\end{smallmatrix} & \text{if } t \text{ is truncated}\\[2ex] \begin{smallmatrix} & T & \\ U & & X\\ Z & & \hat{S}_{n-1}\\ & T & \end{smallmatrix} & \text{if } t \text{ is not truncated}\end{cases} \qquad \text{and } P_U = \begin{cases} \begin{smallmatrix}U\\ Z\\ T\\ U\end{smallmatrix} & \text{if } u \text{ is truncated}\\[2ex] \begin{smallmatrix} & U & \\ Z & & L\\ T & & \\ & U & \end{smallmatrix} & \text{if } u \text{ is not truncated,}\end{cases}$$

where U, Z and L are uniserial modules defined as before from the appropriate successor sequences.

Let $(Q^\bullet, f^\bullet)$ be a minimal projective resolution of $\hat{S}_{n-1}$. Then Q^1 is generated in degree 1 (by arrows in the quiver). If the edge t is truncated, then $\Omega^2(\hat{S}_{n-1}) = \begin{smallmatrix} & \hat{S}_{n-1} & \\ T & & \hat{S}_n\end{smallmatrix}$, the module Q^2 is generated in degree d, the next syzygy is $\Omega^3(\hat{S}_{n-1}) = \begin{smallmatrix}X & & Y\\ & \hat{S}_{n-1} & \end{smallmatrix}$ and Q^3 is generated in degree $d+2 > \delta(3)$. If the edge t not is truncated, then $\Omega^2(\hat{S}_{n-1}) = \begin{smallmatrix}U & & & \\ Z & & \hat{S}_{n-1} & \\ & T & & \hat{S}_n\end{smallmatrix}$, and the module Q^2 is generated in degrees 2 and d. If the edge u is truncated, the next syzygy is $\Omega^3(\hat{S}_{n-1}) = \begin{smallmatrix} & T & & & \\ U & & X & & Y\\ & & & \hat{S}_{n-1} & \end{smallmatrix}$ and Q^3 is generated in degrees $d+1$ and $d+2 > \delta(3)$. Finally, if the edge u is not truncated, the next syzygy is $\Omega^3(\hat{S}_{n-1}) = \begin{smallmatrix}L & & T & & & \\ & U & & X & & Y\\ & & & & \hat{S}_{n-1} & \end{smallmatrix}$ and Q^3 is generated in degrees 3, $d+1$ and $d+2 > \delta(3)$.

Therefore, if Γ has a truncated edge, then $\mathcal{A}_\Gamma$ is not 2-d-determined. Hence **(2)** implies **(1)**.

(1) $\Rightarrow$ **(2)**. Now assume that Γ does not have any truncated edges. Then by Proposition 2.5, we know that for any vertex α in Γ we have $\mathfrak{m}(\alpha)\operatorname{val}(\alpha) = d$. To prove that $\mathcal{A}_\Gamma$ is 2-d-determined, we shall follow the lines of Proposition 7.2 and give a projective resolution of the simple module $\hat{S}_0$, in which the projectives will be generated in appropriate degrees. We need to be more precise in our notation, since we are no longer assuming that there are no multiple edges; in particular, in the notation $a(s,t,\alpha)$ defined in Section 3, the vertex α must be specified.

Let $\hat{s}_0$ be an edge in Γ, and let α_0 and β_0 be its endpoints. Let $\hat{s}_1$ be the successor of $\hat{s}_0$ at α_0 and let α_1 be the other endpoint of $\hat{s}_1$. Let $\hat{s}_{-1}$ be the successor of $\hat{s}_0$ at β_0 and let β_1 be the other endpoint of $\hat{s}_{-1}$. For an integer $n \geqslant 2$ and for any integer i with $2 \leqslant i \leqslant n$, let $\hat{s}_i$ be the the edge that follows $(\hat{s}_{i-2}, \hat{s}_{i-1})$ at α_{i-1}, let α_i be the other endpoint of $\hat{s}_i$, let $\hat{s}_{-i}$ be the the edge that follows $(\hat{s}_{-i+2}, \hat{s}_{-i+1})$ at β_{i-1}, let β_i be the other endpoint of $\hat{s}_{-i}$.

Then the projectives Q^n in a minimal projective resolution of $\hat{S}_0$ are as in Proposition 7.2. The map $f^1 : Q^1 \to Q^0$ is given by the matrix

$$(a(\hat{s}_0, \hat{s}_{-1}, \beta_0) \quad a(\hat{s}_0, \hat{s}_1, \alpha_0)).$$

In order to define the maps $f^n : Q^n \to Q^{n-1}$ for $n \geqslant 2$, we need to define, for $0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$, the following paths of length $d-1$:

$$\tilde{p}(\hat{s}_{n-2j-1}, \hat{s}_{n-2j-2}, \alpha_{n-2j-2}) := p(\hat{s}_{n-2j-1}, \hat{s}_{n-2j-2}, \alpha_{n-2j-2}) C^{m(\alpha_{n-2j-2})-1}_{\hat{s}_{n-2j-2}, \alpha_{n-2j-2}} \quad \text{and}$$

$$\tilde{p}(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j+2}, \beta_{n-2j-2}) := p(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j+2}, \beta_{-n+2j+2}) C^{m(\beta_{n-2j-2})-1}_{\hat{s}_{-n+2j+2}, \beta_{n-2j-2}}.$$

If $n \geqslant 2$ is even and $0 \leqslant j \leqslant \frac{n}{2} - 1$, the matrix of $f^n : Q^n \to Q^{n-1}$ is described as follows,

$$\begin{cases} \text{its } (j,j)\text{-entry is } -a(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j}, \beta_{n-2j-1}) \\ \text{its } (j,j+1)\text{-entry is } \tilde{p}(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j+2}, \beta_{n-2j-2}) \\ \text{its } (n-1-j, n-1-j)\text{-entry is } -\tilde{p}(\hat{s}_{n-2j-1}, \hat{s}_{n-2j-2}, \alpha_{n-2j-2}) \\ \text{its } (n-1-j, n-j)\text{-entry is } a(\hat{s}_{n-2j-1}, \hat{s}_{n-2j}, \alpha_{n-2j-1}). \end{cases}$$

If $n \geqslant 3$ is odd and $0 \leqslant j \leqslant \frac{n-3}{2}$, the matrix of $f^n : Q^n \to Q^{n-1}$ is described as follows,

$$\begin{cases} \text{its } (j,j)\text{-entry is } a(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j}, \beta_{n-2j-1}) \\ \text{its } (j,j+1)\text{-entry is } \tilde{p}(\hat{s}_{-n+2j+1}, \hat{s}_{-n+2j+2}, \beta_{n-2j-2}) \\ \text{its } (n-1-j, n-1-j)\text{-entry is } \tilde{p}(\hat{s}_{n-2j-1}, \hat{s}_{n-2j-2}, \alpha_{n-2j-2}) \\ \text{its } (n-1-j, n-j)\text{-entry is } a(\hat{s}_{n-2j-1}, \hat{s}_{n-2j}, \alpha_{n-2j-1}) \\ \text{its } (\frac{n-1}{2}, \frac{n-1}{2})\text{-entry is } a(\hat{s}_0, \hat{s}_{-1}, \beta_0) \\ \text{its } (\frac{n-1}{2}, \frac{n+1}{2})\text{-entry is } a(\hat{s}_0, \hat{s}_1, \alpha_0). \end{cases}$$

The projective Q^1 is generated in degree 1 (by arrows in the quiver $\mathcal{Q}_\Gamma$), the projective Q^2 is generated in degrees 2 and d (by some elements in the set of minimal generators of the ideal I_Γ, but this can also be seen directly). It can then be seen inductively, using the resolution given above, that Q^n is generated in degrees at most $\delta(n)$; more precisely, the modules $P_{\hat{S}_{n-2j}}$ and $P_{\hat{S}_{-n+2j}}$ are generated in degrees $n + j(d-2)$ with $0 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$. This is true of the resolution of any simple $\mathcal{A}_\Gamma$-module, and therefore $\mathcal{A}_\Gamma$ is 2-d-determined. Thus **(1)** implies **(2)**, and we have that **(1)**, **(2)** and **(4)** are equivalent.

(2) $\Rightarrow$ **(3)**. Suppose $\mathcal{A}_\Gamma$ is 2-d-determined. Then, by the equivalence of **(2)** and **(4)**, we know that $\mathcal{A}_\Gamma$ is $\mathcal{K}_2$. Hence the Ext algebra, $E(\mathcal{A}_\Gamma)$, is generated in degrees 0, 1 and 2 and so is finitely generated. Thus $\mathcal{A}_\Gamma$ is 2-d-Koszul and hence **(3)** holds. This completes the proof. □

This theorem gives a positive answer for Brauer graph algebras to all three questions asked by Green and Marcos in [8, Section 5]. Specifically, we have shown, for a Brauer graph algebra $\mathcal{A}_\Gamma$ which is 2-d-homogeneous, that

(1) if $\mathcal{A}_\Gamma$ is a 2-d-determined algebra, then $E(\mathcal{A}_\Gamma)$ is finitely generated;

(2) if $\mathcal{A}_\Gamma$ is a 2-d-determined algebra and if $E(\mathcal{A}_\Gamma)$ is finitely generated, then $E(\mathcal{A}_\Gamma)$ is generated in degrees 0, 1 and 2;

(3) if $E(\mathcal{A}_\Gamma)$ is generated in degrees 0, 1 and 2, then $\mathcal{A}_\Gamma$ is a 2-d-determined algebra.

In addition, note that algebras Λ_N of [16], where $N \geqslant 1$, are all Brauer graph algebras, where Γ is the oriented cycle with every vertex having multiplicity N. Moreover, these algebras are 2-2N-homogeneous and, using Theorem 9.6, we see that they are also 2-2N-Koszul. This example gives a new class of 2-d-Koszul algebras.

In contrast to Theorem 9.6, a negative answer was given by Cassidy and Phan to the first two questions posed by Green and Marcos in [8]. In [4], Cassidy and Phan give specific infinite-dimensional algebras A and B such that A is 2-4-determined but $E(A)$ is not finitely generated, and B is 2-4-determined of infinite global dimension, $E(B)$ is finitely generated, but $E(B)$ is not generated in degrees 0, 1 and 2. Another generalization of Koszul is given by Herscovich and Rey in [12], where they study multi-Koszul algebras. In particular, they remark that a left $\{2, d\}$-multi-Koszul algebra is a 2-d-Koszul algebra in the sense of [8], though the converse does not hold. However, they showed that the Ext algebra of a multi-Koszul algebra A is generated in degrees 0, 1 and 2, so that the algebra A itself is $\mathcal{K}_2$.

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