TRIVIAL EXTENSIONS OF GENTLE ALGEBRAS AND BRAUER GRAPH ALGEBRAS

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Abstract. We show that two well-studied classes of tame algebras coincide: namely, the class of symmetric special biserial algebras coincides with the class of Brauer graph algebras. We then explore the connection between gentle algebras and symmetric special biserial algebras by explicitly determining the trivial extension of a gentle algebra by its minimal injective co-generator. This is a symmetric special biserial algebra and hence a Brauer graph algebra of which we explicitly give the Brauer graph. We further show that a Brauer graph algebra gives rise, via admissible cuts, to many gentle algebras and that the trivial extension of a gentle algebra obtained via an admissible cut is the original Brauer graph algebra.

As a consequence we prove that the trivial extension of a Jacobian algebra of an ideal triangulation of a Riemann surface with marked points in the boundary is isomorphic to the Brauer graph algebra with Brauer graph given by the arcs of the triangulation.

1. Introduction

One of the cornerstones of the representation theory of finite dimensional algebras is Drozd's classification of algebras in terms of their representation type, which is either finite, tame or wild. Algebras of tame representation type are of great interest since they have infinitely many isomorphism classes of indecomposable modules, yet they do usually still exhibit discernible patterns in their representation theory.

One of the most important classes of tame algebras are special biserial algebras. This class comprises many well-studied classes of algebras such as gentle algebras, string algebras and Brauer graph algebras.

In this paper we start by proving that the classes of Brauer graph algebras and that of symmetric special biserial algebras coincide. We then determine that the trivial extension of a gentle algebra by its minimal injective co-generator is a Brauer graph algebra, for which we explicitly construct the Brauer graph.

While trivial extensions are a way to obtain new algebras from existing ones by increasing the dimension of the underlying vector space, admissible cuts, as introduced in [10], are a way of obtaining new algebras from existing ones by decreasing the dimension of the underlying vector space. We show that in the setting of symmetric special biserial

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and gentle algebras, taking admissible cuts and taking trivial extensions are inverse operations in the following sense. We show that an admissible cut of a symmetric special biserial algebra (with simple cycles) or equivalently of a Brauer graph algebra (without multiplicities) is a gentle algebra and that the original algebra is recovered from this gentle algebra via a trivial extension.

This is of particular interest since it can be directly applied to Jacobian algebras of triangulations of Riemann surfaces with marked points in the boundary to show that the trivial extension of a Jacobian algebra is the Brauer graph algebra with Brauer graph given by the arcs of the triangulation.

Historically, the study of Brauer graph algebras was initiated by Richard Brauer in the context of modular representation theory of finite groups. They play a pivotal role as they appear in the form of Brauer tree algebras as blocks of finite groups with cyclic defect group. However, since their inception, Brauer graph algebras have also been extensively studied outside the context of the representation theory of finite groups, see for example [14, 15, 22, 24], for a small selection of this literature. In parallel and largely independently, special biserial algebras and in particular, symmetric special biserial algebras have been extensively studied. For example, their Auslander-Reiten structure has been determined in [9], and, for an example of more recent papers on the cohomological structure of these algebras see [7, 8, 27].

Roggenkamp has shown in [24] that if the quiver of a symmetric special biserial algebra Λ has no two arrows in the same direction then Λ is a Brauer graph algebra. We show that this result holds in general. That is any symmetric special biserial algebra is a Brauer graph algebra. We do this by associating a graph with local structure to a symmetric special biserial algebra and we show that if this graph is considered as a Brauer graph then the corresponding Brauer graph algebra is isomorphic to the symmetric special biserial algebra. More precisely, we show the following.

Theorem 1.1. *Let k be an algebraically closed field and let $\Lambda = kQ/I$ be a finite dimensional symmetric special biserial algebra and let G_Λ be its graph with local structure. Let B be the Brauer graph algebra associated to G_Λ. Then B is isomorphic to Λ.*

Given a gentle algebra A, the trivial extension $T(A) = A \ltimes D(A)$ of A by its minimal co-generator $D(A)$ is a symmetric special biserial algebra [21, 23, 26]. Thus it is a Brauer graph algebra. In order to identify its Brauer graph we construct a graph Γ_A, equipped with a cyclic ordering of the edges around each vertex induced by the maximal paths in A and we show that Γ_A is the Brauer graph of $T(A)$.

Theorem 1.2. *Let k be an algebraically closed field and let $A = kQ/I$ be a finite dimensional gentle algebra and let Γ_A be its graph. Let B be the Brauer graph algebra defined on Γ_A with cyclic ordering induced by the maximal paths in A. Then the trivial extension $T(A)$ of A is isomorphic to B and $G_{T(A)} = \Gamma_A$ where $G_{T(A)}$ is the graph of $T(A)$ as symmetric special biserial algebra.*

We show that every Brauer graph algebra with multiplicity identically one (or alternatively every symmetric special biserial algebra in which all cycles in the relations are no power of a proper sub-cycle) is the trivial extension of a gentle algebra. For this we define the notion of an admissible cut of a Brauer graph algebra. Admissible cuts of this form were first considered in [10, 12].

Theorem 1.3. *Let k be an algebraically closed field and let $\Lambda = kQ_\Lambda/I_\Lambda$ be a Brauer graph algebra with multiplicity one at all vertices in the associated Brauer graph. Let A be an admissible cut of Λ. Then A is gentle and $T(A)$ and Λ are isomorphic.*

In fact, there are many different gentle algebras that can be obtained from a given Brauer graph algebra resulting from different admissible cuts. In general, while having the same number of isomorphism classes of simple modules these algebras are non-isomorphic and usually they are not even derived equivalent. However, as shown in Theorem 1.3, their trivial extensions are all isomorphic.

Furthermore, we have the following consequence of Theorem 1.3.

Corollary 1.4. *Every Brauer graph algebra with multiplicity function identically one is the trivial extension of a gentle algebra.*

Connections between Brauer graph algebras and Jacobian algebras of triangulations of marked Riemann surfaces have recently been established by several authors. In particular, the connection of mutation, flip of diagonals in triangulations and derived equivalences have been studied in [17, 18], [2] and [20]. In particular, a flip of a diagonal always gives rise to a derived equivalence of the corresponding Brauer graph algebras [20], however, the corresponding Jacobian algebras are not necessarily derived equivalent [17].

Here we give another connection of Brauer graph algebras and Jacobian algebras of marked surfaces. Namely, let (S, M) be a bordered Riemann surface and M a set of marked points in the boundary of S. Given an ideal triangulation T of (S, M), let (Q, I) be the associated bound quiver as defined in [4, 5] and let $A = kQ/I$ be the associated finite dimensional gentle algebra [1]. The orientation of S induces a cyclic ordering of the arcs of T around each marked point and thus T can be viewed as a Brauer graph.

Corollary 1.5. *Let A be a gentle algebra over an algebraically closed field arising from an ideal triangulation T of a Riemann surface with a set of marked points in the boundary. Then the trivial extension $T(A)$ of A is isomorphic to the Brauer graph algebra with Brauer graph T.*

Given a triangulation T of a Riemann surface with marked points in the boundary (S, M), in [6] surface algebras were defined by cutting T at internal triangles. This gives rise to a partial triangulation P of (S, M). A surface algebra A is a finite dimensional gentle algebra and it can be constructed by associating a bound quiver (Q, I) to P such that $A = kQ/I$, see [6]. We show that P together with the cyclic ordering of arcs around each marked point induced by the orientation of the surface is the Brauer graph associated to the trivial extension of A.

Corollary 1.6. *Let k be an algebraically closed field and let the k-algebra A be a surface algebra of a partial triangulation P of a Riemann surface with a set of marked points in the boundary. Then the trivial extension $T(A)$ of A is isomorphic to the Brauer graph algebra with Brauer graph P.*

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2. Gentle algebras, symmetric special biserial algebras and Brauer graph algebras

Let k be an algebraically closed field and let Q be a finite connected quiver. Let I be an admissible ideal in the path algebra kQ such that kQ/I is a finite dimensional algebra. A path in Q is in the bound quiver (Q, I) if it avoids the relations in I.

For a finite dimensional k-algebra A, let $D = Hom_k(-, k)$ denote the standard duality of the module category $A - mod$ of finitely generated A-modules. The algebra A is symmetric if it is isomorphic to $D(A)$ as an A-$A-$bimodule. Let $A^e \simeq A \otimes_k A^{op}$ be the enveloping algebra of A. The socle $\mathrm{soc}(M)$ of a right A-module M is the largest semisimple submodule of M and the radical of M is defined by $\mathrm{rad}(M) = \mathrm{rad}(A)M$ where $\mathrm{rad}(A)$ is the Jacobson radical of A. Unless otherwise stated all algebras considered are indecomposable and all modules are right modules.

2.1. **Special biserial and gentle algebras.** We say that a finite dimensional algebra A is *special biserial* if it is Morita equivalent to an algebra of the form kQ/I where

(S1) Each vertex of Q is the starting point of at most two arrows and is the end point of at most two arrows.

(S2) For each arrow α in Q there is at most one arrow β in Q such that $\alpha\beta$ is not in I and there is at most one arrow γ such that $\gamma\alpha$ is not in I.

We say that an algebra A is *gentle* if it is Morita equivalent to an algebra kQ/I satisfying (S1), (S2) and

(S3) I is generated by paths of length 2.

(S4) For each arrow α in Q there is at most one arrow δ in Q such that $\alpha\delta$ is in I and there is at most one arrow ε in Q such that $\varepsilon\alpha$ is in I.

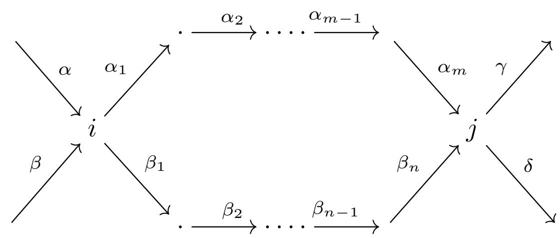
An arrow α in Q starts at the vertex $s(\alpha)$ and ends at the vertex $t(\alpha)$. If $p = \alpha_1\alpha_2 \dots \alpha_n$, for arrows α_i, $1 \leq i \leq n$, is a path in Q then $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$.

Let $\mathcal{P}$ be the set of paths in (Q, I). Let $\mathcal{M}$ be the set of maximal elements in $\mathcal{P}$, that is all paths p in $\mathcal{P}$ such that for all arrows α in Q, $\alpha p \notin \mathcal{P}$ and $p\alpha \notin \mathcal{P}$. In general, for any finite dimensional algebra every path is a subpath of some maximal path and for a gentle algebra that maximal path is unique. Furthermore, a non-trivial path p in Q is in $\mathcal{P}$ if and only if it is a subpath of a (unique if A gentle) maximal path $m \in \mathcal{M}$ where $m = qpq'$ with q, q' paths in Q.

Suppose kQ/I is a gentle algebra. Then every arrow is contained in a unique maximal path, and there are at most two maximal paths starting at any given vertex, and at most two maximal paths ending at any given vertex. It follows that two distinct maximal paths cannot have a common arrow. Hence maximal paths only intersect at a vertex of Q.

Lemma 2.1. *Let $A = kQ/I$ be a gentle algebra. Then the maximal paths in (Q, I) form a basis of* $\mathrm{soc}_{\mathrm{A^e}}\mathrm{A}$.

Proof: We have $\mathrm{soc}_{A^e} A = \bigoplus_{i,j \text{ vertices in } Q} e_i(\mathrm{soc}_{A^e} A)e_j$ as k-vector spaces where e_i and e_j are the trivial paths at vertices i and j respectively. We will show that the maximal paths from i to j form a basis of $e_i(\mathrm{soc}_{A^e} A)e_j$. Recall that a path p in (Q, I) is maximal if for all arrows α in Q, $p\alpha, \alpha p \in I$ or equivalently if $p\alpha = 0 = \alpha p$ in A. Set $R = \mathrm{soc}_{A^e} A$. Since A is gentle there are at most two non-zero paths from i to j, denote them by $p = \alpha_1\alpha_2 \dots \alpha_m$ and $q = \beta_1\beta_2 \dots \beta_n$. Let $\alpha, \beta, \gamma, \delta$ be arrows in Q such that $t(\alpha) = t(\beta) = i$ and $s(\gamma) = s(\delta) = j$ as in the diagram below.



A case by case analysis will give a basis of e_iRe_j in terms of p and q. Suppose first that none of the arrows $\alpha, \beta, \gamma, \delta$ exist in Q, that is the only arrows starting at i are α_1 and β_1 and there are no arrows ending at i and the only arrows ending at j are α_m and β_n and there are no arrows starting at j. Then p and q are maximal and $\{p, q\}$ is a basis of e_iRe_j.

Now suppose that β, γ, δ do not exist in Q but that α is an arrow in Q. Then either $\alpha p \notin I$ or $\alpha q \notin I$. Suppose that $\alpha p \notin I$. Then p is not maximal and q is maximal and $\{q\}$ is a basis of e_iRe_j. Similarly, if α, β, γ do not exist in Q but δ is an arrow in Q.

Suppose now that β and δ do not exist in Q but that α and γ are arrows in Q. Then there are two cases: (i) if $\alpha\alpha_1 \in I$ and $\alpha_m\gamma \in I$ then $\{p\}$ is a basis of e_iRe_j, (ii) if $\alpha\alpha_1 \in I$ and $\beta_n\gamma \in I$ then p and q are not maximal and no linear combination of p and q is such that it is zero by left and right multiplication by all arrows in Q. Hence $e_iRe_j = \{0\}$.

If α and β are arrows in Q and γ but δ do not exist in Q then either (i) $\alpha\alpha_1 \in I$ and $\beta\beta_1 \in I$ or (ii) $\alpha\beta_1 \in I$ and $\beta\alpha_1 \in I$. Assume without loss of generality that (i) holds. Then p and q are not maximal and as above $e_iRe_j = \{0\}$.

Suppose that α, β, and γ are arrows in Q and that δ does not exist. Then as above we can assume without loss of generality that $\alpha_m\gamma \in I$ and hence that p is maximal but q is not and $\{p\}$ is a basis of e_iRe_j.

If all arrows α, β, γ and δ exist in Q then both p and q cannot be maximal and $e_iRe_j = \{0\}$.

All remaining cases are covered by similar arguments. $\square$

2.2. **Brauer graph algebras.** In this section we define symmetric Brauer graph algebras.

We call a finite connected graph Γ (possibly containing loops and multiple edges) a *Brauer graph* if Γ is equipped with a cyclic ordering of the edges around each vertex and if for every vertex ν in Γ there is an associated strictly positive integer $e(\nu)$ called the *multiplicity of* ν.

For an edge E in Γ with vertices ν and μ, we say that E is a *leaf* if the valency of either ν or μ is equal to one, and we call a vertex of valency one a *leaf vertex*.

To a Brauer graph Γ we associate a quiver Q_Γ where the vertices of Q_Γ correspond to the edges in Γ. Let i and j be two distinct vertices in Q_Γ corresponding to edges E_i and E_j in Γ. Then there is an arrow $i \xrightarrow{\alpha} j$ in Q_Γ if the edge E_j is a direct successor of the edge E_i in the cyclic ordering around a common vertex in Γ. If the edge E_i is a leaf with leaf vertex ν with $e(\nu) \geq 2$ then E_i is its own successor in the cyclic ordering and there is an arrow $i \xrightarrow{\alpha} i$. If $e(\nu) = 1$ then no such arrow exists.

It follows from the construction of Q_Γ that every vertex ν of Γ gives rise to an oriented cycle in Q_Γ, unless ν is a leaf with leaf vertex ν where $e(\nu) = 1$. Furthermore, no two cycles in Q_Γ corresponding to distinct vertices in Γ have a common arrow.

Define on Q_Γ a set of relations ρ_Γ as follows. Let E_i be an edge in Γ with vertices ν and ν' where if ν (respectively ν') is a leaf vertex then $e(\nu) \neq 1$ (respectively $e(\nu') \neq 1$). Denote by $C_{\nu,i}$ and $C_{\nu',i}$ the corresponding cycles in Q_Γ starting at vertex i in Q_Γ. Let $C_{\nu,i}^{e(\nu)}$ be the $e(\nu)$-th power of $C_{\nu,i}$. Then $C_{\nu,i}^{e(\nu)} - C_{\nu',i}^{e(\nu')} \in \rho_\Gamma$. Now suppose the edge E_i is such that ν is a leaf with $e(\nu) = 1$. Then $C_{\nu',i}\alpha \in \rho_\Gamma$ where α is the arrow in $C_{\nu',i}$ starting at i. Finally if α, β are two arrows in Q_Γ such that $t(\alpha) = s(\beta) = i$ where α is in $C_{\nu,j}$ for $s(\alpha) = j$ and β is in $C_{\nu',i}$ with $\nu \neq \nu'$ then $\alpha\beta \in \rho_\Gamma$.

The algebra $B_\Gamma = kQ_\Gamma/I_\Gamma$ where I_Γ is the ideal generated by ρ_Γ is called the *Brauer graph algebra associated to the Brauer graph* Γ. Note that B_Γ is a finite dimensional symmetric special biserial algebra, that is B_Γ satisfies (S1) and (S2). Furthermore, it immediately follows from the definition of Brauer graph algebras that they satisfy condition (S4).

We remark that the above definition of Brauer graph algebras has to be slightly adjusted to include the algebras $k[x]/x^2$ and k. The Brauer graph of $k[x]/x^2$ would then correspond to a single edge with both vertices of multiplicity one and that of the algebra k to a single vertex. However, in order to keep the above notation and for clarity of exposition we exclude these two algebras in what follows.

2.3. **Symmetric special biserial algebras.** Let $\Lambda = kQ/I$ be a finite dimensional symmetric special biserial algebra. Since k is algebraically closed, without loss of generality we can assume that a set of relations ρ generating I contains only zero relations and commutativity relations of the form $p - q$ for p, q paths in Q such that $p, q \notin \rho$.

Since Λ is symmetric special biserial the projective indecomposable modules are uniserial or biserial. We adopt the following notation. Let i be a vertex in Q. Then if the projective indecomposable P_i at i is uniserial there exists a unique non-trivial maximal path p in (Q, I) with $s(p) = t(p) = i$. Let e_i be the trivial path at i. Then we write $P_i = P_i(p, e_i) = P_i(p)$ or $P(p)$ for short. If P_i is biserial then there exist two distinct non-trivial paths p, q in Q with $s(p) = s(q) = t(p) = t(q) = i$ such that $p - q \in I$. We write $P_i = P_i(p,q)$ or $P(p,q)$ for short. Since Λ is symmetric, the projective indecomposable at i is also the injective indecomposable at i and hence the paths p and q are maximal in (Q,I). It is a direct consequence of (S2) that p and q do not start or end with a common arrow. Furthermore, for any two non-trivial paths p,q in Q, if there is a relation $p-q \in I$ where $p,q \notin I$ then $P_i = P(p,q)$ where $i = s(p)$. This follows directly from (S2) and the fact that $\operatorname{rad} P_i/\operatorname{soc} P_i$ is a direct sum of two uniserial modules.

Remark. Suppose that B is a Brauer graph algebra with Brauer graph Γ. With the notation above let $C_{\nu,i}^{e(\nu)}, C_{\nu',i}^{e(\nu')}$ be the two maximal paths defined by two vertices ν,ν' of Γ connected by an edge E_i corresponding to a vertex $i \in Q_\Gamma$. Then the projective indecomposable B-module at i is given by $P_i(C_{\nu,i}^{e(\nu)}, C_{\nu',i}^{e(\nu')})$ where $C_{\nu,i}^{e(\nu)}$ or $C_{\nu',i}^{e(\nu')}$ might be the trivial path at i if ν (respectively ν') is a leaf vertex with $e(\nu)=1$ (respectively $e(\nu')=1$). In the latter case the projective indecomposable is uniserial.

Lemma 2.2. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ and $\Delta = kQ_\Delta/I_\Delta$ be symmetric special biserial algebras. Suppose $Q = Q_\Lambda = Q_\Delta$ and suppose that for every vertex in Q, the projective indecomposable modules of Λ and Δ have k-bases given by the same paths in Q. Then the algebras Λ and Δ are isomorphic.*

Proof: We will show that $I_\Lambda = I_\Delta$. Denote the projective indecomposable module of Λ (respectively Δ) at vertex i by P_i^Λ (respectively P_i^Δ). It is enough to show that every generating relation for I_Λ must also be a generating relation for I_Δ. Suppose $\alpha_1 \dots \alpha_n$ is a path in Q with $\alpha_1 \dots \alpha_n \in I_\Lambda$. Now assume that $\alpha_1 \dots \alpha_n \notin I_\Delta$. Then there is a path p_1 in Q such that $p = \alpha_1 \dots \alpha_n p_1$ and $P_i^\Delta = P(p,q)$ for some possibly trivial path q and where $i = s(\alpha_1)$. But $p \in I_\Lambda$ since $\alpha_1 \dots \alpha_n \in I_\Lambda$. Therefore $P_i^\Lambda \not\simeq P(p,q)$, a contradiction and thus $\alpha_1 \dots \alpha_n \in I_\Delta$. By symmetry of the argument this implies that $\alpha_1 \dots \alpha_n \in I_\Lambda$ if and only if $\alpha_1 \dots \alpha_n \in I_\Delta$. Now suppose that p,q are paths in Q with $p,q \notin I_\Lambda$ and $0 \neq p-q \in I_\Lambda$. By (S2) p and q do not start with the same arrow (since otherwise $p=q$). But then the projective P_i^Λ with $i = s(p) = s(q)$ is biserial. Hence $P_i^\Lambda \simeq P(p,q)$. But then $P(p,q) \simeq P_i^\Delta$ and thus $p-q \in I_\Delta$. $\square$

2.3.1. *Graph of a symmetric special biserial algebra.* In [24] Roggenkamp showed that if the quiver of a symmetric biserial algebra Λ has no double arrows then Λ is a Brauer graph algebra. We will adapt Roggenkamp's construction of the Brauer graph associated to a symmetric special biserial algebra to show that this result holds also for quivers with double arrow, thus holding in general.

Let $\Lambda = kQ/I$ be a symmetric special biserial algebra. We now use the projective indecomposable Λ-modules to define a graph G_Λ with a cyclic ordering of the edges around each vertex. As mentioned above we follow Roggenkamp's construction for this. However, instead of using the Loewy structure of projective indecomposables as in [24], we consider the arrows and paths defining the projective indecomposables. This eliminates any ambiguity arising from the existence of double arrows. Recall that the projective indecomposable Λ-modules are either uniserial or biserial and that they are denoted by $P_i(p)$ and $P_i(p,q)$ respectively where p,q are maximal cyclic paths from a vertex i in Q to itself. For the purpose of this construction we consider the trivial path e_i at vertex i to be a maximal cyclic path at i if the projective indecomposable at i is uniserial. This implies that there are two maximal cyclic paths at every vertex (one of them possibly being the trivial path). For non-trivial maximal cyclic paths p and q

at vertex i there are strictly positive integers m and n such that $p = p_0^m$ and $q = q_0^n$ where p_0 and q_0 are cycles starting and ending at i which are no proper power of cycles of shorter length. We call p_0 and q_0 *simple cycles*. Set $e(p) = m$ and $e(q) = n$ and call it the *multiplicity* of p and respectively of q. Trivial maximal paths always have multiplicity one.

Suppose $p = p_0^m$ is a maximal path as above where $p_0 = i \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} i_2 \cdots \xrightarrow{\alpha_{k-1}} i_k \xrightarrow{\alpha_k} i$. Define the *$p$-cycle of p* to be the sequence $\mu(p) = (i, i_1, i_2, \ldots, i_k)$ of vertices in Q. If the trivial path e_i at a vertex i is maximal, we set $\mu(e_i) = i$.

Remark. (1) If $\tilde{p}$ is a cyclic rotation of p then $e(\tilde{p}) = e(p)$.

(2) The vertices of Q occurring in $\mu(p)$ need not all be different. However, since Λ is special biserial, each one can occur no more than twice.

(3) If $\tilde{p}$ is a cyclic rotation of p then $\mu(\tilde{p})$ is a cyclic rotation of $\mu(p)$.

Set $p \sim \tilde{p}$ if $\tilde{p}$ is a cyclic rotation of p. This defines an equivalence relation on the set of cyclic rotations of p. Denote the equivalence class of p by $\nu(p)$ and call it a *vertex*. For later use, we also introduce the following closely related terminology, if $p = p_0^{e(p)}$ with p_0 a simple cycle then we call the rotation class of p_0 a *vertex cycle*.

The set of vertices V of G_Λ consists of the equivalence classes $V = \{\nu(p) \mid p \text{ maximal path in } (Q, I)\}$. To each vertex $\nu(p)$ we associate its multiplicity $e(p)$. Note that this is well-defined since $e(p) = e(\tilde{p})$ for $p \sim \tilde{p}$. To the vertex $\nu(p)$ we now attach *germs of edges* labelled by the vertices of Q contained in $\mu(p)$. Note that the arrows in p_0 induce a linear order on the vertices in $\mu(p)$ by setting $i_n < i_{n+1}$ for $0 \leq n \leq k-1$ and we can complete this to a cyclic order by setting $i_k < i$. This defines a cyclic ordering of the germs around $\nu(p)$. Call this the *local structure of* G_Λ.

Remark. (1) The local structure of G_Λ is not changed if we replace $\mu(p)$ by $\mu(\tilde{p})$ where $p \sim \tilde{p}$. Hence denote the p-cycle $\mu(p)$ by μ_ν where $\nu = \nu(p)$.

(2) In the set $\{\mu_\nu | \nu \in V\}$ each vertex of Q appears exactly twice. Thus for every vertex in Q there are exactly two germs labelled by that vertex. Note that a vertex of Q can appear twice in one μ_ν, labelling two distinct germs.

Definition 2.3. *The graph G_Λ of a symmetric special biserial algebra Λ is given by the vertices $\nu \in V$ with cyclic ordering given by $\mu(p)$ for some maximal path p such that $\mu(p) = \mu_\nu$. There is an edge from ν to ν' if the cycles μ_ν and $\mu_{\nu'}$ have a common vertex of Q. In this case we join the two corresponding germs to a genuine edge.*

Note that if a vertex of Q occurs twice in μ_ν then there is a loop in G_Λ.

The graph G_Λ is a graph with a cyclic ordering of the edges at each vertex. Thus together with the multiplicity of the maximal paths (defined above) associated to the corresponding vertices of G_Λ, the graph G_Λ is a Brauer graph. Let B be the corresponding Brauer graph algebra. It follows immediately from the definition of the cyclic ordering in G_Λ which is induced by the arrows in Q that the projective indecomposable B-modules have a k-basis given by the same paths in Q as the corresponding projective indecomposable Λ-modules. Combining this with Lemma 2.2 proves Theorem 1.1 which for the convenience of the reader we restate here.

Theorem 1.1 *Let k be an algebraically closed field and let $\Lambda = kQ/I$ be a symmetric special biserial algebra and let G_Λ be its graph with local structure as defined above. Let B be the symmetric Brauer graph algebra associated to G_Λ, then B is isomorphic to Λ.*

3. Trivial extensions of gentle algebras

Let A be a finite dimensional k-algebra. Recall that throughout we assume that the field k is algebraically closed. The *trivial extension* $T(A)$ of A by its minimal injective co-generator $D(A)$ is the algebra $T(A) = A \ltimes D(A)$. As a k-vector space $T(A)$ is given by $A \oplus D(A)$ with the multiplication defined by $(a, f)(b, g) = (ab, ag + fb)$, for $a, b \in A$ and $f, g \in D(A)$. It is well-known that the trivial extension algebra $T(A)$ is a symmetric algebra (see, for example, [25, Proposition 6.5]).

As recalled in the introduction it is shown in [21, 23] and [26] that A is gentle if and only if $T(A)$ is special biserial. Therefore the trivial extension of a gentle algebra is a symmetric special biserial algebra. Hence it follows from Theorem 1.1 that $T(A)$ is a Brauer graph algebra.

Let $A = kQ_A/I_A$. It is proven in [11, 2.2] that the vertices of the quiver $Q_{T(A)}$ of $T(A)$ correspond to the vertices of the quiver Q_A of A and that the number of arrows from a vertex i to a vertex j in $T(A)$ is equal to the number of arrows from i to j in Q_A plus the dimension of the k-vector space $e_j(\mathrm{soc}_{A^e} A)e_i$.

The map $D(A) \longrightarrow T(A)$ given by $x \mapsto (0, x)$ is an injective $T(A)$-bimodule homomorphism resulting in a short exact sequence of $T(A)$-bimodules

$$0 \longrightarrow D(A) \longrightarrow T(A) \longrightarrow A \longrightarrow 0.$$

For every vertex i in $Q_{T(A)}$, let $e_i \in T(A)$ be the corresponding primitive idempotent. Then by left multiplication with e_i we obtain a short exact sequence of right $T(A)$-modules

$$0 \longrightarrow e_i D(A) \longrightarrow e_i T(A) \longrightarrow e_i A \longrightarrow 0$$

where $e_i T(A)$ is the projective-injective indecomposable $T(A)$-module at vertex i, $e_i A$ is the projective A-module at i and $e_i D(A)$ is the injective A-module at i (see, for example, [25, Section 6.2]).

3.1. **Construction of the graph of a gentle algebra.**

Given an indecomposable gentle algebra $A = kQ/I$, we will define a graph Γ_A associated to A.

As in the construction of the graph associated to a symmetric special biserial algebra, we will now extend the set $\mathcal{M}$ of maximal paths in (Q, I). Let $\overline{\mathcal{M}}$ be the set containing $\mathcal{M}$ and all those trivial paths associated to vertices i of Q that are a sink with a single arrow ending at i, a source with a single arrow starting from i or such that there is a single arrow α ending at i and a single arrow β starting at i and $\alpha\beta \notin I$. We will still call the elements of $\overline{\mathcal{M}}$ maximal paths. As a consequence of the above definition every vertex of Q lies in exactly two distinct maximal paths in $\overline{\mathcal{M}}$.

Let Γ_A be the graph with vertex set V_0 given by the (extended) set of maximal paths $\overline{\mathcal{M}}$, and with edge set V_1 given by the set of vertices of Q. Denote by $\nu(m)$ the vertex of Γ_A corresponding to the maximal path m in $\overline{\mathcal{M}}$ and denote by E_i the edge in Γ_A corresponding to the vertex i in Q. Recall that by the definition of $\overline{\mathcal{M}}$ every vertex of Q lies in exactly two maximal paths. Then for a vertex i in Q lying in the maximal paths m_i and n_i of $\overline{\mathcal{M}}$, let E_i be the edge in Γ_A connecting $\nu(m_i)$ and $\nu(n_i)$.

Lemma 3.1. *Each maximal path m in $\mathcal{M}$ gives rise to a linear order of the edges connected to the corresponding vertex $\nu(m)$ in Γ_A.*

Proof: Let $m = i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} i_{n+1}$ be a maximal path in $\mathcal{M}$ and let $\nu(m)$ be the corresponding vertex in Γ_A. Then each i_j corresponds to an edge E_{i_j} in Γ_A connecting $\nu(m)$ and $\nu(n_j)$, where n_j is the second maximal path in $\overline{\mathcal{M}}$ going through the vertex i_j of Q. Thus as edges in Γ_A, all E_{i_j}, for $1 \leq j \leq n+1$, are connected to $\nu(m)$. Then setting $E_{i_j} < E_{i_l}$ if there exists a subpath p of m with $s(p) = i_j$ and $t(p) = i_l$ defines a linear order on the set $\{E_{i_1}, E_{i_2}, \ldots, E_{i_{n+1}}\}$. □

Note that maximal paths in A correspond to maximal fans in Γ_A. A fan is a subgraph of Γ_A such that all edges in the subgraph are connected to a common vertex.

An example of a graph associated to a gentle algebra is given by the arcs of a triangulation of a marked Riemann surface: Let T be an ideal triangulation of a bordered Riemann surface with marked points in the boundary and let A be the associated gentle algebra [1]. Then $\Gamma_A = T$ (see Section 4 for more details and an example).

We will now construct a quiver Q_E extending the quiver Q_A. It follows from the proof of Lemma 3.1 that if $i_1 < i_2 < \cdots < i_{n+1}$ is the linear order defined by a maximal path $m \in \mathcal{M}$ then this linear order can be completed to a cyclic order by adding a single arrow $i_{n+1} \xrightarrow{\beta_m} i_1$.

Define Q_E to be the quiver with

- set of vertices given by the vertices in Q and
- set of arrows given by the arrows of Q together with a set of new arrows $\{\beta_m |$ for every $m \in \mathcal{M}$ where $s(\beta_m) = t(m)$ and $t(\beta_m) = s(m)\}$.

Lemma 3.2. *With the notations above the quiver Q_E is the quiver of the trivial extension algebra $T(A)$ of A.*

Proof: By [11] and Lemma 2.1 the arrows of $Q_{T(A)}$ are given by the arrows of Q_A plus for every maximal path $m \in \mathcal{M}$ a new arrow β_m such that $s(\beta_m) = t(m)$ and $t(\beta_m) = s(m)$. □

The following Lemma follows directly from the definitions of Q_E and the definition of the quiver of a Brauer graph algebra.

Lemma 3.3. *With the notations above the quiver Q_E is the quiver of the Brauer graph algebra with associated Brauer graph Γ_A and with cyclic ordering of the edges of Γ_A around each vertex induced by the arrows in Q_E.*

We can now prove Theorem 1.2 which we restate here for convenience.

Theorem 1.2 *Let $A = kQ/I$ be a gentle algebra over an algebraically closed field k and let Γ_A be its graph. Let B be the Brauer graph algebra defined on Γ_A with cyclic* *ordering induced by the maximal paths in A and with multiplicity equal to one at every vertex. Then the trivial extension $T(A)$ of A is isomorphic to B and $G_{T(A)} = \Gamma_A$ where $G_{T(A)}$ is the graph of $T(A)$ as symmetric special biserial algebra.*

Proof: The Brauer graph algebra associated to Γ_A and the algebra $T(A)$ are symmetric special biserial algebras. The projective indecomposable modules of the Brauer graph algebra can be read of the graph Γ_A as described above. The projective indecomposable $T(A)$-modules are given by short exact sequences as described in Section 3 above. That is they are induced by the short exact sequences

$$0 \longrightarrow e_i D(A) \longrightarrow e_i T(A) \longrightarrow e_i A \longrightarrow 0$$

for every vertex i in Q. Suppose that i is a vertex such that there are two non-trivial maximal paths m_i and n_i of A going through i where

$$m_i = i_0 \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{k-1}} i_k = i \xrightarrow{\alpha_k} \cdots \xrightarrow{\alpha_{m-1}} i_m$$

and

$$n_i = j_0 \xrightarrow{\gamma_0} j_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{l-1}} j_l = i \xrightarrow{\gamma_l} \cdots \xrightarrow{\gamma_{n-1}} j_n.$$

The projective right A-module e_iA has a k-basis given by all paths starting at i. By (S1) there are at most two arrows starting at i, which are α_k and γ_l (if they exist). By (S2) there is at most one arrow β such that $\alpha_k\beta$ is non-zero in A, so $\beta = \alpha_{k+1}$ and there is at most one arrow δ such that $\alpha_{k+1}\delta$ is non-zero, so $\delta = \alpha_{k+2}$ and so forth, until we reach α_{m-1}. Then by maximality of m_i there exists no arrow ε in Q such that $\alpha_{m-1}\varepsilon$ is non-zero in A. Similarly, $i \xrightarrow{\gamma_l} \cdots \xrightarrow{\gamma_{n-1}} j_n$ is the unique non-zero path starting with γ_l and there exists no arrow β in Q such that $\gamma_l \dots \gamma_{n-1}\beta$ is non-zero in A. Therefore, e_iA corresponds to the string module $M(w)$ with string $w = i_m \xleftarrow{\alpha_{m-1}} \cdots \xleftarrow{\alpha_{k+1}} i_{k+1} \xleftarrow{\alpha_k} i \xrightarrow{\gamma_l} j_{l+1} \xrightarrow{\gamma_{l+1}} \cdots \xrightarrow{\gamma_{n-1}} j_n$. Note that if neither the arrow α_k nor the arrow γ_l exists then e_iA is the simple module at i. Similar arguments on the arrows ending at i show that $e_iD(A)$ is isomorphic to the string module $M(v)$ with string $v = i_0 \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{k-2}} i_{k-1} \xrightarrow{\alpha_{k-1}} i \xleftarrow{\gamma_{l-1}} j_{l-1} \xleftarrow{\gamma_{l-2}} \cdots \xleftarrow{\gamma_1} j_1 \xleftarrow{\gamma_0} j_0$. In $Q_{T(A)}$ there are arrows $i_m \xrightarrow{\beta_{m_i}} i_0$ and $j_n \xrightarrow{\beta_{n_i}} j_0$ such that the projective-injective indecomposable $T(A)$-module $e_iT(A)$ is given by the biserial module that has the simple at i as top and socle and whose heart $\operatorname{rad} e_iT(A)/\operatorname{soc} e_iT(A)$ is given by a direct sum of uniserial modules given by the direct strings $i_{k+1} \xrightarrow{\alpha_{k+1}} \cdots \xrightarrow{\alpha_{m-1}} i_m \xrightarrow{\beta_{m_i}} i_0 \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{k-2}} i_{k-1}$ and $j_{l+1} \xrightarrow{\gamma_{l+1}} \cdots \xrightarrow{\gamma_{n-1}} j_n \xrightarrow{\beta_{n_i}} j_0 \xrightarrow{\gamma_0} j_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{l-2}} j_{l-1}$. Denote by E_i the edge of Γ_A corresponding to the vertex i of Q and let $\nu(m_i)$ and $\nu(n_i)$ be the vertices at either end of E_i corresponding to the maximal paths m_i and n_i. We have seen that the arrows in m_i and n_i give rise to a cyclic ordering of the edges i_s around $\nu(m_i)$ and j_t around $\nu(n_i)$. Then the projective-injective indecomposable B-module P_i^B at vertex i has a k-basis given by paths in Q_E and $e_iT(A)$ has a k-basis given by the same paths in Q_E.

We will now consider the situation where either n_i or m_i correspond to the trivial path at i. Assume without loss of generality that $n_i = e_i$. Then $e_iT(A)$ is uniserial corresponding to the string $i_k = i \xrightarrow{\alpha_k} i_{k+1} \xrightarrow{\alpha_{k+1}} \cdots \xrightarrow{\alpha_{m-1}} i_m \xrightarrow{\beta_{m_i}} i_0 \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{k-1}} i_k = i$ and the edge E_i in Γ_A corresponding to i is a leaf. By an argument similar to the one above we see that P_i^B, the projective indecomposable B-module at vertex i, and $e_iT(A)$ both have a basis given by the same paths in Q_E.

By Lemma 2.2, the algebras $T(A)$ and B are isomorphic. From the structure of the projective indecomposable modules for B and $T(A)$ it immediately follows that the graphs Γ_A and $G_{T(A)}$ are isomorphic. That the cyclic orderings of the edges around each vertex in Γ_A and $G_{T(A)}$ coincide, follows from the fact that both cyclic orderings are induced by the arrows in Q_E. □

4. Admissible Cuts in Symmetric Special Biserial Algebras

One way of deleting arrows in quivers and constructing new algebras in this way is through the notion of admissible cuts of finite dimensional algebras which has been studied for example in [3, 10, 12], and in the context of cluster algebras, see for example, [6], and also [19].

Given a Brauer graph algebra $\Lambda = kQ_\Lambda/I_\Lambda$ with multiplicity one at all vertices of the corresponding Brauer graph, we define a *cutting set D of Q_Λ* to be a subset of arrows in Q_Λ formed of exactly one arrow in every vertex cycle in Q_Λ (see section 2.3.1 for the definition of a vertex cycle). Note that this corresponds to the cutting set defined in [12]. An *admissible cut of Λ with respect to the cutting set D* is the algebra kQ_Λ/J_Λ where J_Λ is the ideal generated by $I_\Lambda \cup D$.

We now prove Theorem 1.3 which is in a similar vein as the results in [12] (see also [10]) and which states:

Theorem 1.3 *Let k be an algebraically closed field and let $\Lambda = kQ_\Lambda/I_\Lambda$ be a Brauer graph algebra with multiplicity one at all vertices in the associated Brauer graph. Let A be an admissible cut of Λ. Then A is gentle and $T(A)$ and Λ are isomorphic.*

Proof: Let Q be the quiver such that $Q_0 = (Q_\Lambda)_0$ and $Q_1 = (Q_\Lambda)_1 \setminus D$ where D is a cutting set of Q_Λ. Define I to be the ideal $I_\Lambda \cap kQ$ and let J_Λ be the ideal of Λ generated by $I_\Lambda \cup D$. Then it follows from the second isomorphism theorem that $A = kQ_\Lambda/J_\Lambda \simeq kQ/I$.

We first show that kQ/I is gentle. Since $Q \subset Q_\Lambda$, it clearly satisfies (S1). All zero relations in I_Λ are of the form $p - q$ and $p\alpha$ for p, q simple cycles and α an arrow in Q_Λ or are monomial relations $\alpha\beta$ of lengths two where α and β belong to two distinct vertex cycles. By definition of D, any maximal cyclic path p appearing in a relation generating I_Λ, contains exactly one arrow in D. Thus it is not in $I = I_\Lambda \cap kQ$ and therefore I is generated by paths of length two only. In order to show that (S4) holds, suppose that α, β, β' are arrows in $Q \subset Q_\Lambda$ such that $\alpha\beta \in I$ and $\alpha\beta' \in I$. Then $\alpha\beta \in I_\Lambda \supset I$ and also $\alpha\beta' \in I_\Lambda \supset I$ and $\beta = \beta'$, since (S4) holds for Λ. A similar argument holds for arrows preceding α and thus (S4) holds for kQ/I. To show (S2), suppose α, β, β' are arrows in Q such that $\alpha\beta$ and $\alpha\beta'$ are non-zero in kQ. Suppose further that $\alpha\beta \notin I$ and $\alpha\beta' \notin I$. Since $I = I_\Lambda \cap kQ$ this implies $\alpha\beta \notin I_\Lambda$ and $\alpha\beta' \notin I_\Lambda$. Because (S2) holds for Λ this implies $\beta = \beta'$. By a symmetric argument for arrows preceding α, it follows that (S2) holds for kQ/I and therefore $A \simeq kQ/I$ is gentle.

There is a bijection between vertex cycles in Λ and maximal paths in kQ/I. Namely let $i \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} i_n \xrightarrow{\alpha_n} s$ be a vertex cycle ν in Λ. Suppose $\alpha_j \in D$ for some $0 \leq j \leq n$. Then no other arrow in ν is in D and $p = i_{j+1} \xrightarrow{\alpha_{j+1}} i_{j+2} \dots i_n \xrightarrow{\alpha_n} i \xrightarrow{\alpha_0} i_1 \dots i_{j-1} \xrightarrow{\alpha_{j-1}} i_j$ is a path in (Q, I). As subpaths of p, we have $\alpha_{j-1}\alpha_j \notin I$ and $\alpha_j\alpha_{j+1} \notin I$. If there exists an arrow β in Q with $s(\beta) = t(\alpha_{j-1})$ then by (S2) $\alpha_{j-1}\beta \in I$. And hence $p\beta \in I$. Similarly, if there exists an arrow $\gamma \neq \alpha_j$ in Q such that $t(\gamma) = s(\alpha_{j+1})$ then $\gamma p \in I$. Hence p is a maximal path in (Q, I). Conversely, every non-zero path in Λ is a subpath of a unique vertex cycle. Thus if p is a maximal path in kQ/I then p is a subpath of a vertex cycle ν. But since no two distinct maximal paths in kQ/I can have a common arrow and since we have cut exactly one arrow in each vertex cycle of Λ, p is the path that starts at the end of the cut arrow in ν and ends at the start of the cut arrow in ν. Thus the vertices of the graph G_Λ of the symmetric special biserial algebra Λ and the vertices of the graph $\Gamma_{kQ/I}$ of the gentle algebra kQ/I coincide.

Furthermore, the vertices of Q_Λ in a p-cycle associated to a vertex cycle ν correspond up to rotation to the vertices of the corresponding maximal path p in (Q, I). Recall further that the edges in G_Λ are given by connecting the two p-cycles containing the same vertex of Q_Λ and that the edges in $\Gamma_{kQ/I}$ are given by connecting the two maximal paths containing the same vertex of Q. The cyclic order of the edges around a vertex in G_Λ is induced by the arrows in the corresponding vertex cycle and the maximal paths in (Q, I) induce the linear order of the edges in $\Gamma_{kQ/I}$. As described above the latter can be extended to a cyclic ordering. With this induced cyclic ordering on $\Gamma_{kQ/I}$, the graphs G_Λ and $\Gamma_{kQ/I}$ are isomorphic and have the same cyclic ordering of the edges around each vertex. Thus by Theorem 1.1 and Theorem 1.2, the algebras $T(A) \simeq T(kQ/I)$ and Λ are isomorphic. □

Note that a Brauer graph algebra with multiplicity function identically equal to one corresponds to a symmetric special biserial algebra kQ/I where all relations in I that are not monomial of length 2, are given by simple cycles.

The following Corollary immediately follows from Theorem 1.3.

Corollary 1.4 *Every Brauer graph algebra over an algebraically closed field with multiplicity function identically one is the trivial extension of a gentle algebra.*

Finally we end the paper with two applications to gentle algebras associated to marked Riemann surfaces.

Let S be a connected 2-dimensional Riemann surface with boundary ∂S and let M be a non-empty finite set of points in the boundary of S. Let T be an ideal triangulation of (S, M) as defined in [13]. That is T consists of a maximal collection of arcs given by isotopy classes of pairwise non-intersecting curves connecting two marked points and such that the curves are not isotopic to a curve lying in the boundary connecting two adjacent marked points on the same boundary component of S. In [1] and in [16], a finite dimensional algebra, the so-called Jacobian algebra, is associated to the triple (S, M, T). In [1] this algebra is shown to be gentle.

Corollary 1.5 *Let k be an algebraically closed field and let the k-algebra A be the Jacobian algebra associated to an ideal triangulation T of a marked Riemann surface*

(S, M) with $M \subset \partial S$. Then the trivial extension $T(A)$ of A is isomorphic to the Brauer graph algebra with Brauer graph T where the cyclic ordering of the edges around each vertex in T is induced by the orientation of S.

Proof: Let $\Lambda = kQ_\Lambda/I_\Lambda$ be the Brauer graph algebra associated to T and let $A = kQ_A/I_A$ be the (gentle) Jacobian algebra associated to the triangulation T of (S, M). We start by constructing an admissible cut $J = kQ_J/I_J$ of Λ. For this embed the quiver Q_Λ into (S, M, T) via T. That is every arc of T corresponds to a vertex of Q_Λ and the arrows of Q_Λ correspond to the ordering of the edges of T in the cyclic ordering induced by the orientation of S. Then each vertex of T that is not a leaf vertex, corresponds to a cycle in Q_Λ and there is exactly one arrow in every such cycle crossing either one or two boundary segments of T (crossing one boundary segment precisely when the vertex under consideration is the only marked point in its boundary component of (S, M)). The collection of these arrows is a cutting set D of Q_Λ and $Q_A = Q_\Lambda \setminus D = Q_J$. For an example, see figure 1 below.

Let ρ_Λ be the set of relations generating I_Λ as described in Section 2.2. Then $\rho_\Lambda \cap kQ_J$ is a generating set of relations for I_J. Let ρ_A be a set of relations generating I_A. Then ρ_A consists of paths $\alpha\beta$ of lengths two where α and β are two consecutive arrows in an internal triangle of T. As arrows of Q_Λ, α and β are in two distinct vertex cycles of Λ and $\alpha\beta \in \rho_\Lambda$. Hence, since $Q_A = Q_J$, we have $\alpha\beta \in \rho_\Lambda \cap kQ_J$. Conversely, any non-zero path $\alpha\beta$ in ρ_J is given by two arrows α and β belonging to two distinct vertex cycles of Λ. Thus $\alpha\beta$ corresponds to two consecutive arrows in an internal triangle of T. From this we conclude that the generating sets of I_J and I_A coincide and hence A is isomorphic to J. The fact that $T(A)$ is isomorphic to Λ then follows from Theorem 1.3. □

Example. We consider the gentle Jacobian algebra given by the triangulation of the marked annulus in figure 1.

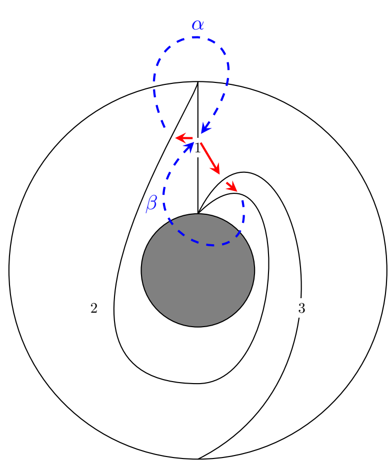


Figure 1: Triangulation of an annulus and associated arrows as described in Corollary 1.5 and its proof.

The quiver given by the solid (red) arrows is the quiver of the gentle Jacobian algebra associated to the triangulation T of the annulus in figure 1. Note that the vertices of this quiver correspond to the three internal arcs of T, marked 1,2, and 3. The graph given by these arcs together with the marked points constitutes precisely the graph of T. The quiver given by the solid (red) arrows together with the dashed (blue) arrows is the quiver of the trivial extension of A or equivalently the quiver of the Brauer graph algebra associated to T as a Brauer graph (with clockwise ordering of the edges around each vertex induced by the clockwise orientation of the annulus). The dashed (blue) arrows form the cutting set D described in the proof of Corollary 1.5 above. Note that α corresponds to an arrow crossing two distinct boundary segments of (S, M) and β is an arrow crossing a single boundary segment of (S, M).

Given a triangulation T of (S, M), in [6] surface algebras were defined by cutting T at internal triangles. This gives rise to a partial triangulation of (S, M). A surface algebra A is again a finite dimensional gentle algebra and it can be constructed by associating a bound quiver (Q, I) to the partial triangulation of (S, M) such that $A = kQ/I$ (see [6]).

Corollary 1.6 *Let k be an algebraically closed field and let the k-algebra A be a surface algebra of a partial triangulation $\mathcal{P}$ of a Riemann surface (S, M) with set of marked points $M \subset \partial S$. Then the trivial extension $T(A)$ of A is isomorphic to the Brauer graph algebra with Brauer graph $\mathcal{P}$ with cyclic orientation induced by the orientation of S.*

Proof: Again, as in the proof of Corollary 1.5, we construct an admissible cut J of the Brauer graph algebra associated to $\mathcal{P}$ and we show that J is isomorphic to A. However, in A not all relations result from paths of lengths two lying in an internal triangle. There are also relations generated by paths of lengths two lying in a quadrilateral resulting from the 'cut' of an internal triangle (see [6]). However, these relations also appear in the corresponding Brauer graph algebra and its admissible cut. Taking this into account, the proof then is similar to the proof of Corollary 1.5. □

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