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Supporting Information for

No unique scaling law for igneous dikes

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S1: Dike growth and relaxation

S1A. Kinetics-dominated versus toughness-dominated growth for linear pressure model

Spence and Turcotte (1985) developed a first-order approximate model for the growth of a 2D dike of length L and thickness T in a linear elastic host with plane strain modulus $E' = \frac{E}{1-\nu^2}$, fracture toughness K_{Ic}, and magma viscosity η. This analysis is re-evaluated here for the purposes of dike scaling interpretation, and to investigate the criteria for transition between toughness and kinetic-controlled dike formation.

Stress intensity at the dike tip

The mode I stress intensity at the tip of a crack subject to an internal pressure distribution $p(x)$ is given by

$$K_I = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} p(x) \sqrt{\frac{a+x}{a-x}} dx$$

(A.1)

where $L = 2a$. A simple linear approximation to the pressure distribution is proposed such that $p(x) = P + \Delta P \left|\frac{x}{a}\right|$. In this case equation (A.1) gives

$$K_I = \sqrt{\frac{a}{\pi}} (\pi P + 2\Delta P)$$

(A.2)

The condition for fracture propagation is $K_I = K_{Ic}$. Given the central magma pressure P, it is then required that

$$\Delta P = \frac{\pi}{2}\left[\frac{K_{IC}}{\sqrt{\pi a}} - P\right]$$

(A.3)

for dike propagation to occur. In the kinetic-controlled limit we can assume that the material resistance of the host rock is negligible ($K_{Ic} = 0$) which yields the result that $\Delta P = -\frac{\pi}{2} P$.

Spence and Turcotte (1985) analysis

The volume (area) of the 2D dike is assumed to evolve as a prescribed function of time, $V = Qt$, where Q (m^2/s) is a constant, and the magma pressure is assumed to be linear, such that $p(x) = P\left[1 - \frac{\pi}{2}\left|\frac{x}{a}\right|\right]$ as above. [Note the problem with the exact pressure distribution was solved numerically by Spence and Sharp (1985) for self-similar dike evolution with $V(t) = Qt^{\alpha}$]. From their analysis we have the following parameters

$$\gamma = \frac{2K}{(6Q\eta E')^{\frac{1}{4}}} = \left(\frac{4}{3\pi}\right)^{\frac{1}{4}} \frac{(6A_0^3 - 1)}{(1 + 12A_0^3)^{\frac{3}{4}}}$$

$$k = \frac{(1 + 12A_0^3)^{\frac{1}{2}}}{\sqrt{12\pi}A_0^2}$$

$$K = \frac{K_{IC}}{\sqrt{\pi}}$$

(A.4)

such that the parameter introduced in equation (5) is defined in terms of these as

$$\lambda = \frac{K_{Ic}}{(Q\eta E')^{\frac{1}{4}}} = \frac{6^{\frac{1}{4}}\sqrt{\pi}}{2}\gamma = \left(\frac{\pi}{2}\right)^{\frac{1}{4}} \frac{(6A_0^3 - 1)}{(1 + 12A_0^3)^{\frac{3}{4}}}$$

(A.5)

The dike length and thickness are given in terms of the parameters in (A.4) by equations (27) and (28) in Spence and Turcotte (1985)

$$L = \frac{2}{6^{\frac{1}{6}}}.kQ^{\frac{1}{2}}\left(\frac{E'}{\eta}\right)^{\frac{1}{6}} t^{\frac{2}{3}}$$

$$T = 2.6^{\frac{1}{6}}.kA_0.Q^{\frac{1}{2}}\left(\frac{\eta}{E'}\right)^{\frac{1}{6}} t^{\frac{1}{3}}$$

(A.6)

where t is time. These can be combined to give

$$T = 24^{\frac{1}{4}}k^{\frac{1}{2}}A_0\left(\frac{Q\eta}{E'}\right)^{\frac{1}{4}} L^{\frac{1}{2}}$$

(A.7)

Following equation (4) we write this as

$$T = f.\sqrt{\frac{8}{\pi}\frac{K_{Ic}}{E'}}L^{\frac{1}{2}}$$

(A.8)

such that

$$f = \sqrt{\frac{\pi}{8}}.24^{\frac{1}{4}}\frac{k^{\frac{1}{2}}A_0}{\lambda} = \frac{(1 + 12A_0^3)}{2(6A_0^3 - 1)} \tag{A.9}$$

This is the blue line shown in Figure 2.

Now, in the *kinetic-controlled limit* ($\lambda \to 0$) we have $\gamma \to 0$ so (A.1) gives $6A_0^3 = 1$ such that

$$f = \sqrt{\frac{3\pi}{32}}\lambda^{-1} \tag{A.10}$$

This is the red line shown in Figure 2 and shows that the viscous terms work well for $\lambda < 0.2$. Given 2D volume (area) $V_{2D} = Qt$ we can also write this as

$$L = \sqrt{\frac{6}{\pi}}\left(\frac{E'}{Q\eta}\right)^{\frac{1}{6}} V_{2D}^{\frac{2}{3}} = \sqrt{\frac{6}{\pi}}\left(\frac{V_{2D}^2}{L_\eta}\right)^{\frac{1}{3}} \qquad T = \sqrt{\frac{6}{\pi}}\left(\frac{Q\eta}{E'}\right)^{\frac{1}{6}} V_{2D}^{\frac{1}{3}} = \sqrt{\frac{6}{\pi}}\left(L_\eta V_{2D}\right)^{\frac{1}{3}} \tag{A.11}$$

and

$$T = \left(\frac{6}{\pi}\right)^{\frac{1}{4}} \left(L_\eta L\right)^{\frac{1}{2}} \tag{A.12}$$

which only depends on the kinetic length scale L_η as expected.

In the *toughness-controlled limit* ($\lambda \to \infty$) we note that equations (34) and (35) in Spence and Turcotte (1985) are wrong, as they show a η dependence which should not be there in this regime. Carrying out the algebraic substitutions correctly, the actual result is as follows. In the limit of $\lambda \to \infty$ we get

$$\lambda = \left(\frac{3\pi}{8}\right)^{\frac{1}{4}} A_0^{\frac{3}{4}} \tag{A.13}$$

then

$$k = \left(\frac{3}{8\pi^2}\right)^{\frac{1}{6}} . \lambda^{-\frac{2}{3}} \tag{A.14}$$

giving, from (A.6),

$$L = \left(\frac{2}{\pi}\right)^{\frac{1}{3}} \lambda^{-\frac{2}{3}} Q^{\frac{1}{2}} \left(\frac{E'}{\eta}\right)^{\frac{1}{6}} t^{\frac{2}{3}} = \left(\frac{2}{\pi}\right)^{\frac{1}{3}} \left(\frac{QE'}{K_{IC}}\right)^{\frac{2}{3}} t^{\frac{2}{3}} \tag{A.15}$$

and

$$T = 2\left(\frac{2}{\pi}\right)^{\frac{2}{3}} \lambda^{\frac{2}{3}} Q^{\frac{1}{2}} \left(\frac{\eta}{E'}\right)^{\frac{1}{6}} t^{\frac{1}{3}} = 2\left(\frac{2}{\pi}\right)^{\frac{2}{3}} \left(\frac{Q^{\frac{1}{2}} K_{Ic}}{E'}\right)^{\frac{2}{3}} t^{\frac{1}{3}} \tag{A.16}$$

Writing this in terms of $V_{2D} = Qt$ gives

$$L = \left(\frac{2}{\pi}\right)^{\frac{1}{3}} \left(\frac{E'}{K_{IC}}\right)^{\frac{2}{3}} V_{2D}^{\frac{2}{3}} = \left(\frac{2}{\pi}\right)^{\frac{1}{3}} \left(\frac{V_{2D}^2}{L_K}\right)^{\frac{1}{3}} \qquad T = 2\left(\frac{2}{\pi}\right)^{\frac{2}{3}} \left(\frac{K_{Ic}}{E'}\right)^{\frac{2}{3}} V_{2D}^{\frac{1}{3}} = 2\left(\frac{2}{\pi}\right)^{\frac{2}{3}} (L_K V_{2D})^{\frac{1}{3}} \tag{A.17}$$

Combining these gives equation (3)

$$T = \sqrt{\frac{8}{\pi}} (L_K L)^{\frac{1}{2}} = 1.60 (L_K L)^{\frac{1}{2}}$$

(A.18)

which only depends on the toughness length scale as expected, and yields $f = 1$ in this case as required.

S1B. Non-linear inflation and relaxation model

The aim here is to extend the analysis of Spence and Turcotte (1985), which has been re-evaluated in appendix A, to be applicable to the general case where $V(t)$ is a general function of time. Here an approximate analytical solution is derived using a variational method for kinetic processes defined by Cocks et al. (1998). This postulates that the best estimate of a kinetic field minimises a variational function

$$\Pi = \Psi + \dot{G}$$

(B.1)

where Ψ is a dissipation potential and $\dot{G}$ is the rate of change of Gibbs free energy. In this case, the dissipation is due to magma flow. The Gibbs free energy is the driving force for this flow. It has two contributions

$$G = U_e + 2\Gamma L$$

(B.2)

where U_e is the change in elastic strain energy in the host rock due to changes in the dike geometry and/or magma pressure (equivalent to the energy release rate for crack growth), and $2\Gamma L$ is the fracture energy, where $\Gamma \approx \frac{K_{Ic}^2}{2E'}$ is the (constant) energy per unit area of fracture and $2L$ is the area of the crack face created as two crack faces are produced by splitting. Here the analysis is limited to the kinetic-controlled regime such that the second term is omitted, i.e. $\Gamma = 0$.

Gibbs free energy, G

Here we utilise the fact that in linear elasticity the change in elastic strain energy, $U_e = \frac{1}{2}\Omega$, is half the work done by the applied load. Here, this is the work done by the internal pressure, $p(x)$, in generating an opening thickness, $h(x)$

$$\Omega = \int_{-a}^{a} p(x)h(x)dx \tag{B.3}$$

The deformed shape for the assumed kinetic-controlled pressure profile, $p(\xi) = P\left[1 - \frac{\pi}{2}\ |\xi|\right]$ is given by equation (20) in Spence and Turcotte (1985) as

$$h(\xi) = \frac{2PL}{E'}\left[\sqrt{1-\xi^2} + \frac{1}{2}\xi^2 \ln\left(\frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}}\right)\right] \tag{B.4}$$

where $\xi = x/a$. Note that the definition of maximum thickness, $T = h(0)$, recovers equation (2). The volume of magma-filled crack is

$$V(t) = \frac{L}{2}\int_{-1}^{1} h(\xi)d\xi = 1.051\frac{PL^2}{E'} = 0.525\ LT \tag{B.5}$$

Evaluation of the integral gives

$$\Omega = 0.527\frac{P^2L^2}{E'} = \beta PV \tag{B.6}$$

where $\beta = 0.502$. This scaling is universal, with only the exact value of the pre-factor β depending on the choice of pressure distribution within the dike. Note that for a uniform pressure of $p(x) = P$ the pressure term can be moved out of the integral in (B.3) such that $\Omega = PV$ in this case. Hence it is expected that the actual distribution will produce a pre-factor somewhere between these two cases, i.e. β is between 0.5 and 1.0.

Dissipation potential, Ψ

The average magma flux through the dike at a distance x from the centre assumes laminar flow such that magma flows down the pressure gradient

$$j(x) = -kf \tag{B.7}$$

where $k(x) = \frac{h(x)^3}{12\eta}$ is the permeability of the magma channel and $f = \frac{\partial p}{\partial x}$ is the driving force for flow per unit volume. Following Cocks et al. (1998) we write this in terms of a dissipation rate per unit volume ψ such that

$$f = -\frac{\partial \psi}{\partial j} \tag{B.8}$$

The total dissipation can then be determined from (B.7) and (B.8) to be

$$\Psi = \int_{-a}^{a} \psi dx = \frac{1}{2}\int_{-a}^{a} \frac{j^2}{k} dx \tag{B.9}$$

The flux is related to the dike shape. For $0 \leq x \leq a$ we have

$$j(x) = -\int_{0}^{x} \frac{\partial h}{\partial t} dx + j_0 \tag{B.10}$$

where the flux at the centre of the dike is $j(0) = j_0$. It is tempting to determine the flux using the dike profile defined by (B.4) for the linear pressure gradient, but this is not possible as the chosen pressure distribution is not an exact solution. In practice the pressure gradient f at the tip must be infinite to generate a finite flux where the dike thickness h is zero (Rubin, 1995). If (B.4) is used then the dissipation potential is infinite at the tip. To simply generate an estimate of the dissipation, we therefore assume a simple rectangular dike shape, whereby the dike is of length L and average thickness $\tilde{T}$ where volume conservation requires that $V = L\tilde{T}$. This will provide the correct scaling, and the contribution from the actual shape can be calibrated later from the solution of Spence and Turcotte (1985). Equation (B.5) yields the relation $\tilde{T} = cT$, where $c = 0.525$. Now (B.10) becomes

$$j(x) = -\int_0^x \frac{\partial \tilde{T}}{\partial t} dx + j_0 = -c\dot{T}x + j_0$$

(B.11)

where $j_0 = \frac{1}{2}\dot{V} > 0$ is the rate of change of half the magma volume in the growing dike. Given $\dot{V} = c(\dot{T}L + T\dot{L})$ we can write this as

$$j(x) = cT\dot{a}\left(\frac{x}{a}\right) + j_0\left(1 - \frac{x}{a}\right)$$

(B.12)

To calculate (B.9) we assume an average permeability $\tilde{k} = d.\frac{\tilde{T}^3}{12\eta}$ where the pre-factor d is to be determined based on the dike shape. The dissipation potential is therefore

$$\Psi = \frac{L}{12\tilde{k}}\left[\left(cT\dot{L}\right)^2 + cT\dot{L}\dot{V} + \dot{V}^2\right]$$

(B.13)

Variational functional, Π

We write $P = \frac{E'T}{2L} = \frac{E'V}{2cL^2}$ from equation (2) such that (B.6) becomes $\Omega = \frac{\beta E' V^2}{2cL^2}$ and hence

$$\dot{G} = \frac{1}{2}\dot{\Omega} = \frac{\beta E' V^2}{2cL^2}\left[\frac{\dot{V}}{V} - \frac{\dot{L}}{L}\right]$$

(B.14)

The variational functional (B.1) can therefore be written as

$$\Pi = \frac{L}{12\tilde{k}}\left[\left(cT\dot{L}\right)^2 + cT\dot{L}\dot{V} + \dot{V}^2\right] + \left[\frac{\dot{V}}{V} - \frac{\dot{L}}{L}\right]$$

(B.15)

As $\dot{V}$ is prescribed in this analysis, the only kinetic degree-of-freedom is $\dot{L}$ whose optimal solution minimises (B.15) such that $\frac{\partial \Pi}{\partial \dot{L}} = 0$. This gives

$$\frac{L}{12\tilde{k}}\left[2c^2T^2\dot{L} + cT\dot{V}\right] = \frac{\beta E' V^2}{2cL^3}$$

(B.16)

Now, as we have already seen, $\dot{V} = c\left[T\dot{L} + L\dot{T}\right]$. To make simple analytical progress, we follow Spence and Turcotte (1985) by looking for power law solutions where $T = hL^m$, where h and m are constants. This yields $L\dot{T} = mT\dot{L}$ and thus $\dot{V} = c(m+1)T\dot{L}$ such that (B.16) becomes

$$\frac{(3+m)c^2T^2L}{12\tilde{k}}\dot{L} = \frac{\beta E' V^2}{2cL^3}$$

(B.17)

Substituting for $\tilde{k} = d.\frac{(cT)^3}{12\mu}$ and $T = \frac{V}{cL}$ we get

$$\dot{L} = \frac{d\beta E' V^3}{2c(3+m)\mu L^5}$$

(B.18)

Rearranging and integrating over time gives

$$L(t) = A\left(\frac{E'S}{\eta}\right)^{\frac{1}{6}}$$

(B.19)

where the pre-factor $A = \left(\frac{3d\beta}{4c(3+m)}\right)^{\frac{1}{6}}$ and we have introduced the variable

$$S(t) = 4\int_0^t V(t)^3 dt$$

(B.20)

The pre-factor is calibrated using the linear growth case of $V = Qt$ examined by Spence and Turcotte (1985) for which $m = \frac{1}{2}$ and $A = 1.38$.

General solution

We therefore determine how the length, maximum thickness and maximum pressure in the dike evolve over time for a general volumetric time evolution $V(t)$ as

$$L(t) = 1.38\left(\frac{E'S}{\eta}\right)^{\frac{1}{6}} \qquad T(t) = 1.38V\left(\frac{\eta}{E'S}\right)^{\frac{1}{6}} \qquad P(t) = 0.94V\left(\frac{E'^2\eta}{S}\right)^{\frac{1}{3}}$$

(B.21)

This assumes a self-similar shape for the dike during growth, although this will not necessarily be completely true during the relaxation phase, which is complicated by freezing.

Inflation stage solution for power law magma injection

If we assume power law growth during the inflation phase, such that $V(t) = Qt^{\alpha}$, then (B.21) can be expressed as

$$L(t) = 1.38Q^{\frac{1}{2}}\left(\frac{\hat{\beta}E'}{\eta}\right)^{\frac{1}{6}} t^{\frac{3\alpha+1}{6}} \qquad T(t) = 1.38Q^{\frac{1}{2}}\left(\frac{\eta}{\hat{\beta}E'}\right)^{\frac{1}{6}} t^{\frac{3\alpha-1}{6}} \qquad P(t) = 0.94\left(\frac{E'^2\eta}{\hat{\beta}}\right)^{\frac{1}{3}} t^{-\frac{1}{3}}$$

(B.22)

where $\hat{\beta} = \frac{4}{3\alpha+1}$. Note that the scaling is identical to the exact solution of Spence and Sharp (1985). Now, the thickness–length relationship can be written in a more general form as

$$T = 1.38^{1-m}\left(\frac{\eta}{\hat{\beta}E'}\right)^{\frac{1+m}{6}} L^m$$

(B.23)

where $m = \frac{3\alpha-1}{3\alpha+1}$. Note that Equation 6 is recovered if $\alpha = 1$ when the exponent reduces to $m = \frac{1}{2}$.

Relaxation stage solution

If we assume growth ends at time $t = t_0$ then the final volume is $V(t_0) = V_0 = Qt_0^{\alpha}$. The dike can still evolve over time even without magma emplacement, although at a much-reduced rate. If this evolution occurs for an additional relaxation time t_r such that $t = t_0 + t_r$, then in (B.21) we have

$$S = 4V_0^3\left(t_r + \frac{t_0}{3\alpha + 1}\right)$$

(B.24)

We can see that during volumetric growth the length increases as $t^{\frac{1}{6}+\frac{\alpha}{2}}$ and during constant volume relaxation it increases more slowly, tending towards $t^{\frac{1}{6}}$ when $t_r \gg t_0$. Similarly, the thickness increases during volumetric growth as $t^{-\frac{1}{6}+\frac{\alpha}{2}}$ and decreases during constant volume relaxation, tending towards $t^{-\frac{1}{6}}$ when $t_r \gg t_0$.

S1C. Solidification time

An upper estimate for the time for relaxation (before freezing) is obtained from equation (9)

$$t_r = \frac{T(t_r)^2}{16\kappa\beta^2}$$

(C.1)

where the final thickness $T(t_r)$ is given by (B.21) and (B.24). These equations can be re-arranged into a quartic in t_r

$$t_r^3\left(t_r+\frac{t_0}{3\alpha+1}\right)=\frac{(1.38V_0)^6}{4V_0^3(16\kappa\beta^2)^3}\left(\frac{\eta}{E'}\right)$$

(C.2)

which can be solved numerically.

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S2. Dike and vein scaling data presented in Figures 1, 3, and 6. Full references provided in the main article.

S3. Calculation of toughness versus viscous growth from Equations A9 and A10 in supplement S1, based on analysis by Spence and Turcotte (1985).

S4. Calculation of dike inflation and relaxation for different parameters: **(A)** $Q = 1m^2/s$, $s = 0.65$ (i.e., $Q = 1m^2s^{-0.65}$), $E' = 1\ GPa$, $\mu = 10^8\ Pa.s$ and $\kappa = 10^{-6}\ m^2/s$. Plots in B–F show effects of changing individual parameters relative to (A), with: **(B)** reduced $\mu = 10^6\ Pa.s$; **(C)** higher growth rate $Q = 10\ m^2s^{-0.65}$; **(D)** higher growth exponent (and rate) with $Q = 1$ and $s = 1$ (i.e., $Q = 1m^2/s$); **(E)** lower growth exponent $s = 0.5$ with increased $Q = 10m^2s^{-0.5}$ (note that without increasing Q, growth takes hundreds of years); **(F)** lower thermal diffusivity (which affects cooling rate) with $\kappa = 10^{-7}m^2/s$.