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On the 2-compact group $DI(4)$

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Abstract. Besides the simple connected compact Lie groups there exists one further simple connected 2-compact group, constructed by Dwyer and Wilkerson, the group $DI(4)$. The mod-2 cohomology of the associated classifying space $BDI(4)$ realizes the rank 4 mod-2 Dickson invariants. We show that mod-2 cohomology determines the homotopy type of the space $BDI(4)$ and that the maximal torus normalizer determines the isomorphism type of $DI(4)$ as 2-compact group. We also calculate the set of homotopy classes of self maps of $BDI(4)$.

1. Introduction

As introduced by Dwyer and Wilkerson in their influential paper [10], a p-compact group is a pair (X, BX) of spaces such that X is $\mathbb{F}_p$-finite, i.e. $H^*(X;\mathbb{F}_p)$ is a finite vector space, BX is pointed, both spaces are p-complete (in the sense of [3]) and the loop space ΩBX and X are homotopy equivalent. This is a homotopy theoretic generalization of the concept of compact Lie groups. For a connected compact Lie group G, the pair $(G_p^\wedge, BG_p^\wedge)$ together with the natural equivalence $\Omega BG_p^\wedge \simeq G_p^\wedge$ is a p-compact group, which, by abuse of notation, we also denote by G. For odd primes, there exist many connected p-compact groups which do not come from compact Lie groups, e.g. Sullivan showed that, if n divides $p-1$, the sphere S^{2n-1} gives rise to a connected p-compact group $((S^{2n-1})_p^\wedge, B(S^{2n-1})_p^\wedge)$ [29]. At the prime 2, Dwyer and Wilkerson constructed the only known example $DI(4)$ of a connected 2-compact group, which is not a Lie group [9]. The classifying space $BDI(4)$ realizes the rank 4 mod-2 Dickson invariants, namely the ring of invariants of the natural action of $Gl(4,\mathbb{F}_2)$ on the rank 4 polynomial algebra $H^*((B\mathbb{Z}/2)^4;\mathbb{F}_2)$. In fact, they conjectured that every connected 2-compact group is a product of a connected compact Lie group and copies of $DI(4)$. In this work we will approach this conjecture by showing that the 2-compact group $DI(4)$ satisfies some uniqueness results. We also show that this conjecture is at least rationally true (see Theorem 1.7).

p-compact groups behave astonishingly similar as compact Lie groups. Every p-compact group X has a maximal torus $T_X \to X$, a Weyl group W_X, a maximal torus normalizer $N_X \to X$ and a center $Z(X) \to X$, all with basically the same properties as in the case of compact Lie groups. And two p-compact groups X and Y are isomorphic

if the classifying spaces BX and BY are homotopy equivalent. For further details about p-compact groups see the survey article [17] or the references mentioned there.

For p-compact groups, there are two classification conjectures around. A map between algebras over the Steenrod algebra is called a $\mathscr{K}$-map.

Conjecture 1.1. (i) *Let X be a simply connected or centerfree p-compact group. A p-complete space A is equivalent to BX if and only if there exists a $\mathscr{K}$-isomorphism $H^*(A;\mathbb{F}_p) \cong H^*(BX;\mathbb{F}_p)$.*

(ii) *Two connected p-compact groups X and Y are isomorphic, i.e. the classifying spaces BX and BY are equivalent, if and only if the maximal torus normalizer N_X and N_Y are isomorphic.*

At the prime two, both conjectures are proved for special unitary groups [6], [7], [19], [22], for the exceptional Lie group G_2 [32] and for the special orthogonal groups $SO(2n+1)$ [20], [26], [28]. For the spinor groups $Spin(2n+1)$ [26], for $Sp(n)$ [30] and the exceptional Lie group F_4 [31] only the second part is known.

We will show that both conjectures are also valid for the Dwyer-Wilkerson example $DI(4)$.

Theorem 1.2. *A 2-complete space A is homotopy equivalent to $BDI(4)$ if and only if there exists a $\mathscr{K}$-isomorphism $H^*(A;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$.*

Theorem 1.3. *A 2-compact group X is isomorphic to $DI(4)$ if and only if the maximal torus normalizer N_X and $N_{DI(4)}$ are isomorphic.*

If $H^*(A;\mathbb{F}_2) \cong H^*\big((B\mathbb{Z}/2)^4;\mathbb{F}_2\big)^{Gl(4,\mathbb{F}_2)}$, standard methods show that ΩA is $\mathbb{F}_2$-finite (see [9]). The pair $(\Omega A, A)$ gives rise to a 2-compact group. The next proposition shows that Theorem 1.2 and Theorem 1.3 are equivalent statements.

Proposition 1.4. *For a 2-compact group X, the maximal torus normalizer N_X and $N_{DI(4)}$ are isomorphic if and only if there exists a $\mathscr{K}$-isomorphism $H^*(BX;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$.*

It turns out that we can get away with less assumptions to prove a version of Theorem 1.3. We only need the rational Weyl group data of a 2-compact group to compare with $DI(4)$.

Every p-compact group X has an associated p-adic lattice $L_X := \pi_1(T_X)$ which carries an action of the Weyl group W_X. If X is connected, the representation $W_X \to Gl(L_X \otimes \mathbb{Q})$ represents W_X as a pseudo reflection group, i.e. W_X is finite and generated by elements fixing a hyperplane of codimension 1. The above representation gives the rational Weyl group data of X. We say that two p-compact groups X and Y have the same rational Weyl group data if the two representations $W_X \to Gl(L_X \otimes \mathbb{Q})$ and $W_Y \to Gl(L_Y \otimes \mathbb{Q})$ are weakly isomorphic. That is there exists an abstract isomorphism $W_X \xrightarrow{\cong} W_Y$ such that $L_X \otimes \mathbb{Q}$ and $L_Y \otimes \mathbb{Q}$ are isomorphic as $\mathbb{Z}_p^\wedge[W_X]$-modules.

Proposition 1.5. *Let X be a 2-compact group which has the same rational Weyl group data as $DI(4)$. Then, the normalizers N_X and $N_{DI(4)}$ are isomorphic.*

We get the following corollary which is a slightly stronger result than Theorem 1.3.

Corollary 1.6. *A 2-compact group X is isomorphic to $DI(4)$ if and only if X and $DI(4)$ have the same rational Weyl group data.*

Proof. This follows from Proposition 1.5 and Theorem 1.3. □

The next theorem shows that the above mentioned conjecture of Dwyer and Wilkerson is rationally true.

Theorem 1.7. *Every connected 2-compact group X splits into a product $X \cong X_1 \times X_2$ such that X_1 is isomorphic to some copies of $DI(4)$ and X_2 has the same rational Weyl group data as a suitable connected compact Lie group G.*

Finally we characterize the homotopy classes of self maps of $BDI(4)$.

Theorem 1.8. *Let $i: T_{DI(4)} \subset DI(4)$ be a maximal torus of $DI(4)$. Then, for two self maps $f, g: BDI(4) \to BDI(4)$, the following conditions are equivalent.*

(i) *f and g are homotopic.*

(ii) *The compositions $f \circ Bi$ and $g \circ Bi$ are homotopic.*

(iii) $H^*(f; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q} = H^*(g; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q}$.

In fact, we will show that every self $f: BDI(4) \to BDI(4)$ is an Adams operation (see Section 6). That is there exists a $k_f \in \mathbb{Z}_2^\wedge$ such that f induces on $H^{2i}(BDI(4); \mathbb{Z}_2^\wedge) \otimes \mathbb{Q}$ multiplication by $(k_f)^i$. The 2-adic integer k_f is called the degree of f. It will turn out that k_f is either 0 or a 2-adic unit, that all these values can be realized and that k_f determines f up to homotopy (Theorem 3.5).

The strategy for the proofs of Theorem 1.2 and Theorem 1.3 is clear and known to experts. It uses the construction of the classifying space $BDI(4)$ of Dwyer and Wilkerson. They constructed $BDI(4)$ as a homotopy colimit of a diagram based on the Rector category of $H^*((B\mathbb{Z}/2)^4; \mathbb{F}_2)^{Gl(4, \mathbb{F}_2)}$, which is equivalent to the Quillen category of $DI(4)$, namely the category of elementary abelian 2-subgroups of $DI(4)$. One of the spaces in that diagram is the classifying space $BSpin(7)$ of the 2-compact group $Spin(7)$, which plays a major role in the construction. The missing gap for completing the proof is a uniqueness result for $Spin(7)$ which is provided in [26]. Having this at hand we construct a map from the homotopy colimit approximating $BDI(4)$ to the classifying space BX of a p-compact group satisfying Proposition 1.4. And this map will turn out to be an equivalence after completion.

Remark. As Wilkerson pointed out to us, one can get away with weaker assumptions as in Theorem 1.3 or Corollary 1.6. If the Weyl groups of a 2-compact group X and $DI(4)$ are isomorphic as abstract groups, then both have isomorphic normalizers as

well as isomorphic mod-2 cohomology. In fact, this applies to several 2-compact groups. A detailed proof will appear in future work by Dwyer and Wilkerson.

There is also some overlap with work by Dwyer and Wilkerson. Proposition 1.4, Corollary 1.6 and Theorem 1.8 are also known to them and were presented by Wilkerson at conference talks in Seattle and Orlando [34].

The paper is organized as follows. In Section 2 we recall the construction of $BDI(4)$ of Dwyer and Wilkerson. In Section 3 we discuss self maps of $DI(4)$ and prove Theorem 1.8. Section 4 contains the proof of Proposition 1.4, Section 5 the proofs of the theorems 1.2 and 1.3, Section 6 the proof of Proposition 1.5 and the final section the proof of Theorem 1.7.

We will use the language of p-compact groups all over the places. For references we refer the reader to the survey article [17] or the references mentioned there.

Mostly, cohomology is taken with $\mathbb{F}_2$ as coefficients. And $H^*_{\mathbb{Q}^\wedge_2}(-)$ denotes the cohomology groups $H^*(-;\mathbb{Z}^\wedge_2)\otimes\mathbb{Q}$.

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2. The construction of $DI(4)$

In this section we recall the construction of $DI(4)$ and other material from [9]. The general linear group $Gl(4,\mathbb{F}_2)$ acts on the 4-dimensional $\mathbb{F}_2$-vector space $E_4=(\mathbb{F}_2)^4$ and therefore on the algebra of formal polynomial functions $\mathbb{F}_2[E_4]$ on E_4 with coefficients in $\mathbb{F}_2$. That is the symmetric part of the tensor algebra of the dual of E. We can identify this algebra with the algebra $H^*(BE_4;\mathbb{F}_2)$. The ring of invariants $H^*(BE_4)^{Gl(4,\mathbb{F}_2)}\cong\mathbb{F}_2[c_8,c_{12},c_{14},c_{15}]$ carries an action of the Steenrod algebra $\mathscr{A}_2$. The indices denote the degree of the classes. The action is determined by $Sq^4(c_8)=c_{12}$, $Sq^2(c_{12})=c_{14}$ and $Sq^1(c_{14})=c_{15}$.

The Bockstein spectral sequence collapses at the E_2-page and shows that $H^*_{\mathbb{Q}}(BDI(4);\mathbb{Z}^\wedge_p)\cong\mathbb{Q}^\wedge_p[x_8,x_{12},x_{28}]$. This is also true for any space A such that $H^*(A)\cong H^*(BE)^{Gl(4,\mathbb{F}_2)}$.

$DI(4)$ is a 2-compact group of rank 3 with maximal torus $T_{DI(4)}\simeq(S^1)^3\to DI(4)$. The Weyl group $W_{DI(4)}\cong\mathbb{Z}/2\times Gl(3,\mathbb{F}_2)$ arises as the pseudo reflection group No. 24 in the Clark-Ewing list of irreducible rational pseudo reflection groups [4]. The representation

$$W_{DI(4)}\cong\mathbb{Z}/2\times Gl(3,\mathbb{F}_2)\to Gl(L_{DI(4)})\cong Gl(3,\mathbb{Z}^\wedge_2)$$

maps the generator of $\mathbb{Z}/2$ on $-I$ where I is the identity. The composition $W_{DI(4)}\to Gl(3,\mathbb{Z}^\wedge_2)\to Gl(3,\mathbb{F}_2)$ is the projection on the second factor [9], Theorem 5.1. In particular $L_{DI(4)}/2:=L_{DI(4)}\otimes\mathbb{F}_2$ is an irreducible $W_{DI(4)}$-module.

Lemma 2.1. *Every $W_{DI(4)}$-lattice of $L_{DI(4)} \otimes \mathbb{Q}$ is isomorphic to $L_{DI(4)}$ as $\mathbb{Z}_p^\wedge[W_{DI(4)}]$-module.*

Proof. Let $L \subset L_{DI(4)} \otimes \mathbb{Q}$ be a $W_{DI(4)}$-lattice. There exists a $W_{DI(4)}$-equivariant monomorphism $L \to L_{DI(4)}$ such that the cokernel K is a torsion group and such that rank of $K/2 := K \otimes \mathbb{F}_2$ is smaller than 3. Tensoring with $\mathbb{Z}/2$ yields an exact sequence

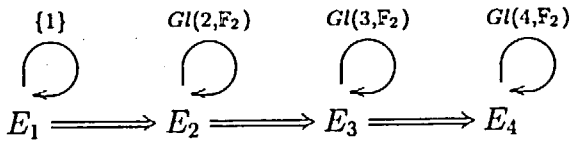
$$0 \to Tor(K, \mathbb{Z}/2) \to L/2 \to L_{DI(4)}/2 \to K/2 \to 0$$

of $W_{DI(4)}$-modules. Since $L_{DI(4)}/2$ is irreducible, $K/2 = 0$, hence $K = 0$ and $L \cong L_{DI(4)}$. □

Next we describe the Quillen category of $DI(4)$. Since

$$H^*(BDI(4); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$$

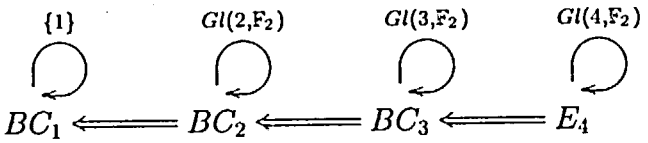
every elementary abelian 2-subgroup $E \subset DI(4)$ is subconjugated to E_4 and every pair of subgroups $E, E' \subset E_4$ of the same dimension are conjugate in $DI(4)$. Essentially, for $i = 1, \ldots, 4$, there is exactly one elementary abelian 2-subgroup $E_i \subset DI(4)$ of dimension i. The Quillen category is equivalent to the full subcategory of the following shape



where the groups above the objects denote the self maps and where $\Rightarrow$ denotes the set of morphisms from E_i to E_{i+1}.

We have $C_{DI(4)}(E_4) = E_4$. And $C_3 := C_{DI(4)}(E_3) \cong T_{DI(4)} \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on $T_{DI(4)}$ via an Adams operation ψ^{-1} of degree -1, i.e. ψ^{-1} induces on $\pi_2(BT_{DI(4)})$ multiplication by -1. The group E_3 is contained in $T_{DI(4)}$ and therefore toral in $DI(4)$ as well as E_2 and E_1. The induced action of $Gl(3, \mathbb{F}_2)$ on $T_{DI(4)}$ comes from the representation $Gl(3, \mathbb{F}_2) \to Gl(3, \mathbb{Z}_2^\wedge)$ given by restricting the 2-adic Weyl group representation $W_{DI(4)} \to Gl(3, \mathbb{Z}_2^\wedge)$ to the second component. Moreover, $C_2 := C_{DI(4)}(E_2) \cong SU(2)^3/\Delta$, where Δ is the image of the diagonal embedding of $\mathbb{Z}/2 \to SU(2)^3$. And $C_1 := C_{DI(4)}(E_1) \cong Spin(7)$.

The decomposition diagram of $BDI(4)$ given by the centralizers of the elementary abelian subgroups of $DI(4)$ [8], [11] has the form



Let **Top** denote the category of topological spaces and let $\Theta\colon \mathbf{A}_p(DI(4)) \to \mathbf{Top}$ be the functor given by the above diagram. Then, evaluation at basepoints establishes a map

$$\operatorname{hocolim}_{\mathbf{A}_p(DI(4))} \Theta \to BDI(4).$$

Theorem 2.2 ([8], [9]). *The map* $\operatorname{hocolim}_{\mathcal{A}_p(DI(4))} \Theta \to BDI(4)$ *is a* mod-2 *equivalence.*

The 2-compact groups C_1 and $Spin(7)$ are isomorphic. We will switch between them. For example, for the above decomposition diagram it is more appropriate to work with C_1 since we have to think of it as a centralizer, and for describing self maps of $BDI(4)$ we prefer to work with $Spin(7)$.

Remark 2.3. We do need some more detailed information about the subgroup $Spin(7) \subset DI(4)$. Let $E_1 \to E_4$ be the inclusion of the first coordinate. The isotropy subgroup $\Gamma := Iso(E_1) \subset Gl(4, \mathbb{F}_2)$ is the subgroup of all matrices of the form $\begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix}$ such that $A \in Gl(3, \mathbb{F}_2)$. The classifying space $BC_{DI(4)}(E_1) \simeq BSpin(7)^\wedge_2$ has mod-2 cohomology

$$H^*(BSpin(7); \mathbb{F}_2) \cong H^*(BC_{DI(4)}(E_1); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^\Gamma.$$

This ring of invariants is a polynomial algebra and $H^*(BSpin(7); \mathbb{F}_2) \cong \mathbb{F}_2[d_4, d_6, d_7, d_8]$ where subindices denote degrees. In particular, $H^*(BDI(4); \mathbb{F}_2) \to H^*(BSpin(7); \mathbb{F}_2)$ is a monomorphism. Moreover, since E_1 is subconjugated to $T_{DI(4)}$, $Spin(7) \subset DI(4)$ is a subgroup of maximal rank and $W_{Spin(7)} \subset W_{DI(4)}$. The Euler characteristic $\chi(DI(4)/Spin(7))$ of the homotopy fiber of $BSpin(7) \to BDI(4)$ is the index of $[W_{DI(4)} : W_{Spin(7)}]$ of the Weyl groups. The order of $W_{DI(4)}$ is 336, the order of $W_{Spin(7)}$ is 48 and $\chi(DI(4)/Spin(7)) = 7$. We explained this calculation in detail since, unfortunately, there is a misprint in [9], Theorem 1.8. This calculation always works for subgroups of p-compact groups of maximal rank [11], [18].

The maximal torus normalizer $N_{DI(4)}$ contains the subgroup

$$T_{DI(4)} \rtimes \mathbb{Z}/2 \cong C_{DI(4)}(E_3).$$

And therefore a unique subgroup $E_4 \subset T_{DI(4)} \rtimes \mathbb{Z}/2 \subset N_{DI(4)}$. Since $Spin(7) \cong C_{DI(4)}(E_1)$ and since $E_1 \subset T_{DI(4)}$, the Weyl group $W_{Spin(7)} \subset W_{DI(4)}$ is the isotropy subgroup of E_1 for the $W_{DI(4)}$-action on $T_{DI(4)}$ [11].

We finish this section with some statements necessary for later purpose.

Proposition 2.4. *For any elementary abelian 2-group E, the map*

$$[BE, BE_4] \to [BE, BDI(4)]$$

factors through the bijection

$$[BE, BE_4]/Gl(4, \mathbb{F}_2) \to [BE, BDI(4)].$$

Proof. Since $H^*(BDI(4); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{Gl(4, \mathbb{F}_2)}$, this statement is well known. It follows from facts about Lannes' T-functor, namely that $H^*(\operatorname{map}(BE, BX); \mathbb{F}_2) \cong T^E(H^*(BDI(4); \mathbb{F}_2))$ and that

$$T^E(H^*(BE_4; \mathbb{F}_2))^{Gl(4, \mathbb{F}_2)} \cong (T^E(H^*(BE_4; \mathbb{F}_2)))^{Gl(4, \mathbb{F}_2)} \cong H^*(\operatorname{map}(BE, BE_4); \mathbb{F}_2)^{Gl(4, \mathbb{F}_2)}$$

[15], [10] (for the latter sequence of isomorphisms see also [9]). □

Lemma 2.5. *There exist exactly one monomorphism*

$$R^* := H^*(BDI(4); \mathbb{F}_2) \to S^* := H^*(BSpin(7); \mathbb{F}_2)$$

of algebras over the Steenrod algebra, such that S becomes a finitely generated R-module.

Proof. Let $A^* := H^*(BE_4; \mathbb{F}_2)$ and $E_4 \subset Spin(7)$ the standard inclusion. This inclusion induces a monomorphism $S^* \to A^*$. Therefore it suffices to show the statement if we replace S^* by A^*. Let α: $R^* \to A^*$ be the map induced by the inclusion $BE_4 \to BDI(4)$. Let β: $R^* \to A^*$ be any map which makes A^* to a finitely generated module over R^*. In particular, β is a monomorphism. Then there exists an automorphism γ: $A^* \to A^*$ such that $\beta\gamma = \alpha$. Since $Gl(4, \mathbb{F}_2)$ is the Galois group of the integral extension $R^* \to A^*$, the maps α and β have the same image. □

The classifying space $BN_{DI(4)}$ of the normalizer $N_{DI(4)}$ of $DI(4)$ fits into a fibration $BT_{DI(4)} \to BN_{DI(4)} \to BW_{DI(4)}$ where the $W_{DI(4)}$-action on $BT_{DI(4)}$ is described by the Weyl group action of $W_{DI(4)}$ on $L_{DI(4)}$. We say that the normalizer splits, if the projection $BN_{DI(4)} \to BW_{DI(4)}$ has a right inverse. Passing to classifying spaces, the semi direct product $T_{DI(4)} \rtimes W_{DI(4)}$ always produces a fibration, which splits.

Such fibrations, considered as extensions of $BT_{DI(4)}$ by $BW_{DI(4)}$, are classified by elements of $H^3(W_{DI(4)}; \pi_3(BSHE(BT_{DI(4)}))) \cong H^3(W_{DI(4)}; L_{DI(4)})$ where $SHE(BT_{DI(4)})$ denotes the set of self equivalences of $BT_{DI(4)}$ homotopic to the identity, where $W_{DI(4)}$ acts on $L_{DI(4)}$ via the Weyl group action, and where the latter cohomology groups denote group cohomology with twisted coefficients (see [23]).

Lemma 2.6. *The normalizer $N_{DI(4)}$ does not split.*

Proof. Let $E_2 \subset T_{DI(4)}$ be the elementary abelian subgroup of $T_{DI(4)}$ described above. Let $H \subset \mathbb{Z}/2 \times Gl(2, \mathbb{F}_2)$ be the subgroup generated by $\mathbb{Z}/2$ and the matrices of the form $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$. This is an elementary abelian subgroup isomorphic to $\mathbb{Z}/2^3$ and acts trivially on E_2. If the normalizer $N_{DI(4)}$ does split it contains the subgroup generated by E_2 and H which is isomorphic to $\mathbb{Z}/2^5$. And so does $DI(4)$. This is a contradiction since every elementary abelian subgroup of $DI(4)$ is subconjugated to $E_4 \subset DI(4)$ (see Proposition 2.4). □

Lemma 2.7. $H^3(W_{DI(4)}; L_{DI(4)}) \cong \mathbb{F}_2$.

For the proof we do need a description of the mod-2 cohomology of the dihedral group D_8 of order 8. We use the presentation of D_8 given as $D_8 := \{x, y : x^2 = y^2 = 1 = (xy)^4\}$. There exists a rank 2 elementary abelian 2-subgroup $E_{xz} \subset D_*$ generated by x and $z := xy$. The cohomology of D_8 is given by $H^*(D_8; \mathbb{F}_2) \cong \mathbb{F}_2[v_x, v_y, w_z]/(v_x, v_y)$ where $\deg v_x = \deg v_y = 1$ and $\deg w_z = 2$. The mod-2 group cohomology of E_{xz} is given by $H^*(E_{xz}; \mathbb{F}_2) \cong \mathbb{F}_2[u_x, u_z]$ where both classes have degree 1. Restriction to E_{xz} maps v_x to u_x, w_z to u_z, and u_y to 0. All this can be found in [1].

Proof of Lemma 2.7. For abbreviation we set $G := Gl(3, \mathbb{F}_2)$ and denote by $\Gamma \subset G$ the subgroup of G generated by all matrices of the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$.

Since $W_{DI(4)} = \mathbb{Z}/2 \times G$ we can use the Lyndon-Hochschild-Serre spectral sequence to calculate $H^3(W_{DI(4)}; L_{DI(4)}) \cong \mathbb{Z}/2$. The following terms in the E_2-page have total degree 3: $H^3(G; H^0(\mathbb{Z}/2; L_{DI(4)}))$, $H^2(G; H^1(\mathbb{Z}/2; L_{DI(4)}))$, $H^1(G; H^2(\mathbb{Z}/2; L_{DI(4)}))$ and $H^0(G; H^3(\mathbb{Z}/2; L_{DI(4)}))$. Since $\mathbb{Z}/2$ acts via multiplication by -1 on $L_{DI(4)}$, the cohomology groups $H^*(\mathbb{Z}/2; L_{DI(4)})$ vanish in even degrees, and are isomorphic to $\mathbb{F}_2^3$ in odd degrees. Moreover, we have $(\mathbb{F}_2^3)^G = 0$. This shows that the first and the last two cohomology groups are trivial, and that $H^2(G; H^1(\mathbb{Z}/2; L_{DI(4)})) \cong H^2(G; \mathbb{F}_2^3)$. Since there exists a non trivial extension of $T_{DI(4)}$ by $W_{DI(4)}$ (Lemma 2.6), the cohomology group $H^3(W_{DI(4)}; L_{DI(4)})$ as well as $H^2(G; \mathbb{F}_2^3)$ contain at least two elements.

The G-representation $\mathbb{F}_2^3$ is induced from the 1-dimensional trivial representation for the subgroup $\Gamma \subset G$. Hence, by the Shapiro Lemma $H^2(G; \mathbb{F}_2^3) \cong H^2(\Gamma; \mathbb{F}_2)$. The subgroup of upper triangular matrices of G is a 2-Sylow subgroup for G as well as for Γ and isomorphic to D_8. We will use our above description and identify x with $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the element y with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and the element z with $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since $H^*(\Gamma; \mathbb{Z}/2) \cong H^*(D_8; \mathbb{Z}/2)^{\text{stable}}$ is isomorphic to the stable elements of the cohomology of a 2-Sylow subgroup, we have to calculate the latter group. The subgroup $K := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in Gl(2, \mathbb{F}_2) \right\} \cong Gl(2; \mathbb{F}_2) \subset G$ normalizes $E_{xz} \subset D_8$ and acts on E_{xz} via matrix multiplication from the right applied to a single row. This representation is isomorphic to the dual representation of the standard representation on E_{xz}. Hence, on $H^*(E_{xz}; \mathbb{Z}/2)$, the subgroup K acts via the standard representation on $H^1(E_{xz}; \mathbb{Z}/2) \cong Hom(E_{xz}, \mathbb{F}_2)$. A short calculation shows that $H^2(E_{xz}; \mathbb{F}_2)^K \cong \mathbb{F}_2$ is generated by the element $u_x^2 + u_x u_z + u_z^2$. In particular this shows that $v_x^2, w_z \in H^2(D_8; \mathbb{Z}/2)$ are not stable elements and that $H^2(H; \mathbb{Z}/2) \cong H^2(D_8; \mathbb{F}_2)^{\text{stable}}$ contains at most two elements. And therefore, $H^3(W_{DI(4)}; L_{DI(4)}) \cong H^2(G; \mathbb{F}_2^3) \cong H^*(\Gamma; \mathbb{F}_2) \cong \mathbb{F}_2$. □

3. Self maps of *BDI*(4)

In this section we want to prove Theorem 1.8. We use the same notation as in Section 2.

Remark 3.1. Every elementary abelian 2-subgroup $E \subset DI(4)$ of rank k, $1 \leqq k \leqq 4$, is conjugate to the inclusion $E_k \subset E_4 \subset DI(4)$. We denote by i_k: $BC_k := BC_{DI(4)}(E_k) \to BDI(4)$ the inclusion of the centralizer of E_k. Let ϕ: $C_k \to DI(4)$ be a monomorphism. Since $E_4 \subset C_k$, there exists a homomorphism ρ: $E_4 \to E_4$ such that the diagram

$$\begin{array}{ccc} E_4 & \xrightarrow{\subset} & C_k \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \phi} \\ E_4 & \xrightarrow{\subset} & DI(4) \end{array}$$

commutes up to conjugation (Proposition 2.4). Actually, ρ is a monomorphism, because ϕ is one [10], and therefore an isomorphism. Since the Weyl group of $E_4 \subset DI(4)$ is $Gl(4;\mathbb{F}_2)$ we can assume that $\rho = \mathrm{id}$ and that the restriction $\phi|_{E_k}$ is the inclusion $E_k \subset DI(4)$. Passing to centralizers establishes a monomorphism $\phi'\colon C_k \cong C_{C_k}(E_k) \to C_{DI(4)}(E_k) = C_k$. In fact, this is an isomorphism [18].

Because $E_k \subset C_k$ is central there exists a homomorphism $\mu\colon C_k \times E_k \to C_k$. By construction, $B\phi'\colon BC_k \to \mathrm{map}(BE_k, BDI(4))_{B\phi|_{BE_k}} \simeq BC_k$ is the adjoint of the composition $BC_k \times BE_k \xrightarrow{B\mu} BC_k \xrightarrow{B\phi} BDI(4)$ and an equivalence. If $i = 1$ or 2, the centralizer C_k is connected and $E_k \subset C_k$ is subconjugated to the maximal torus of C_k.

All this together establishes the assumptions made in Proposition 4.6 of [25]. The conclusion we can draw is the following. If $B\psi\colon BC_k \to BDI(4)$ is another map such that $H^*_{\mathbb{Q}}(B\phi) = H^*_{\mathbb{Q}}(B\psi)$, then $B\phi$ and $B\psi$ are homotopic.

Lemma 3.2. *A map $g\colon BSpin(7)^\wedge_2 \to BDI(4)$ is either homotopic to the constant map or there exists a self equivalence $h\colon BSpin(7)^\wedge_2 \to BSpin(7)^\wedge_2$ such that $Bi_1 \circ h \simeq g$.*

Proof. Let $\phi\colon Spin(7) \to DI(4)$ be the underlying homomorphism such that $g \simeq B\phi$. The homomorphism ϕ is either conjugate to the constant map or the kernel of ϕ is a central subgroup of $Spin(7)$ [21], hence a subgroup of $\mathbb{Z}/2$. If the kernel is the trivial group, then ϕ is a monomorphism. If the kernel equals $\mathbb{Z}/2$, then ϕ factors through a monomorphism $\bar{\phi}\colon SO(7) = Spin(7)/(\mathbb{Z}/2) \to DI(4)$ [21]. This gives a contradiction, since $SO(7)$ contains an elementary abelian 2-group of rank 6 and $DI(4)$ does not. Therefore, ϕ is either the constant map or a monomorphism. In the latter case, the existence of h follows from Remark 3.1. □

Proposition 3.3. *Let $f\colon BDI(4) \to BDI(4)$ be a self map. Then the following holds:*

(i) *The map f is either homotopic to the constant map or an equivalence.*

(ii) *There exists a self map $h\colon BSpin(7)^\wedge_2 \to BSpin(7)^\wedge_2$ such that the diagram*

$$\begin{array}{ccc} BSpin(7)^\wedge_2 & \xrightarrow{h} & BSpin(7)^\wedge_2 \\ {\scriptstyle Bi_1}\downarrow & & \downarrow{\scriptstyle Bi_1} \\ BDI(4) & \xrightarrow{f} & BDI(4) \end{array}$$

commutes up to homotopy.

Proof. If $f \circ Bi_1$ is null homotopic we can choose the constant map for h. Hence, the second part is a consequence of Lemma 3.2.

The map h is either null homotopic or an equivalence [14]. If h is an equivalence, then f induces a monomorphism in mod-2 cohomology, therefore an isomorphism, and is an equivalence.

Since the Weyl group index $[W_{DI(4)} : W_{Spin(7)}]$ is odd, $Spin(7)$ and $DI(4)$ have the same 2-Sylow subgroup $P \subset Spin(7) \subset DI(4)$. Thus, if h is null homotopic, the restriction $f|_{BP}$ is null homotopic and so is f itself [24]. □

For a p-compact group X, we call a self map $f\colon BX \to BX$ an Adams operation of degree $k \in \mathbb{Z}_p^\wedge$ if, for any class $x \in H^{2i}_{\mathbb{Q}_p^\wedge}(BX)$, $f^*(x) = k^i x$. We denote the units of $\mathbb{Z}_2^\wedge$ by $\mathbb{Z}_2^{\wedge *}$.

Corollary 3.4. *Every self equivalence of $BDI(4)$ is an Adams operation whose degree is a 2-adic unit.*

Proof. Every self equivalence of $BSpin(7)_2^\wedge$ has this property [14]. Since $H^*_{\mathbb{Q}_2^\wedge}(BDI(4)) \to H^*_{\mathbb{Q}_2^\wedge}(BSpin(7))$ is a monomorphism, the statement is a consequence of Proposition 3.3. □

Considering the constant map as an Adams operation of degree 0, the above corollary establishes a map

$$D\colon [BDI(4), BDI(4)] \to \{0\} \coprod \mathbb{Z}_2^{\wedge *}.$$

Theorem 3.5. *The map $D\colon [BDI(4), BDI(4)] \to \{0\} \coprod \mathbb{Z}_2^{\wedge *}$ is a bijection.*

We postpone the proof of Theorem 3.5 to the end of this section.

Proof of Theorem 1.8. If, for two maps $f, g\colon BDI(4) \to BDI(4)$, the restrictions to $BT_{DI(4)}$ are homotopic, then both induce the same map in rational cohomology, are Adams operations of the same degree and therefore homotopic by Theorem 3.5. □

As in Section 2 we denote by $\mathbf{A} := \mathbf{A}_p(DI(4))$ the Quillen category of $DI(4)$ and by $\Theta := \Theta_{DI(4)}\colon \mathbf{A} \to \mathbf{Top}$ the functor describing the Dwyer-Wilkerson decomposition. Since $\operatorname{hocolim}_{\mathbf{A}} \Theta \to BDI(4)$ induces an equivalence after 2-adic completion (Theorem 2.2) we get an equivalence

$$\operatorname{map}(BDI(4), BDI(4)) \simeq \operatorname{holim}_{\mathbf{A}} \operatorname{map}(\Theta, BDI(4)).$$

The equivalence establishes a map

$$[BDI(4), BDI(4)] \to \lim_{\leftarrow \mathbf{A}} [\Theta, BDI(4)].$$

For $\phi \in \lim_{\leftarrow \mathbf{A}} [\Theta, BDI(4)]$ and an object $E \subset X$ of $\mathbf{A}$ we denote by $\operatorname{map}(BC(E), BDI(4))_\phi$ the component determined by ϕ and by

$$\operatorname{map}(BDI(4), BDI(4))_\phi := \operatorname{holim}_{\mathbf{A}} \operatorname{map}(\Theta, BDI(4))_\phi$$

the union of all components whose image is ϕ.

The homotopy groups of the last space can be calculated with the Bousfield-Kan spectral sequence [3]. The E^2-term has the form

$$E^2_{i,j} \cong \lim^i_{\leftarrow \mathbf{A}} \pi_j(\operatorname{map}(\widetilde{BC}, BDI(4)))_\phi =: \lim^i_{\leftarrow \mathbf{A}} \Pi_j(\phi).$$

If ϕ is the image of the constant map, then, for all E_k,

$$\operatorname{map}(BC_{DI(4)}(E_k), BDI(4))_\phi \simeq BDI(4)$$

and the functor $\Pi_j(\phi)$ is the constant functor. If ϕ is the image of a self equivalence $f: BDI(4) \to BDI(4)$, then composition with f induces a natural equivalence

$$\Pi_j := \Pi_j(\mathrm{id}) \xrightarrow{\cong} \Pi_j(\phi).$$

Since for any abelian p-compact group A and any p-compact group X

$$BZ\big(C_X(A)\big) \simeq \mathrm{map}\big(BC_X(A), BC_X(A)\big)_{\mathrm{id}} \simeq \mathrm{map}\big(BC_X(A), BX\big)_{Bi}$$

where i denotes the inclusion of the centralizer [11], [25], the functor Π_j is naturally equivalent to the inclusion functor $J: \mathbf{A} \to \mathcal{A}b$ into the category of abelian groups.

In both cases the higher derived limits all vanish [9], the Bousfield spectral sequence collapses and is concentrated in a singular line. This allows to prove the following proposition:

Proposition 3.6. (i) *The map* $[BDI(4), BDI(4)] \to \lim_{\leftarrow \mathbf{A}} [\Theta, BDI(4)]$ *is a bijection.*

(ii) *For all non trivial self maps* $f: BDI(4) \to BDI(4)$, *the mapping space* $\mathrm{map}\big(BDI(4), BDI(4)\big)_f$ *is contractible, and* $\mathrm{map}\big(BDI(4), BDI(4)\big)_{\mathrm{const}} \simeq BDI(4)$.

Proof. The first part follows from the above considerations. They also show that, for all non trivial self maps $f: BDI(4) \to BDI(4)$, we have $\mathrm{map}\big(BDI(4), BDI(4)\big)_{\mathrm{id}} \simeq \mathrm{map}\big(BDI(4), BDI(4)\big)_f$. The first mapping space is equivalent to the classifying space of the center of $DI(4)$ [11], which is trivial [9]. The component of the constant map is calculated in [11]. □

Proof of Theorem 3.5. Let $\mathbf{A}' \subset \mathbf{A}$ denote the full subcategory whose objects are elementary abelian 2-subgroups of $DI(4)$ of rank less or equal to 2. Then, the results of [27] say that the second arrow of

$$[BDI(4), BDI(4)] \xrightarrow{\cong} \lim_{\leftarrow \mathbf{A}} [\Theta, BDI(4)] \to \lim_{\leftarrow \mathbf{A}'} [\Theta, BDI(4)]$$

is an isomorphism. The first isomorphism follows from Proposition 3.7. We denote by $i_E: C_{DI(4)}(E) \to DI(4)$ the inclusion of the centralizer of E.

We first show that D is an injection. The degree of a self map of $BDI(4)$ is determined by the induced map in rational cohomology. Let $f, g: BDI(4) \to BDI(4)$ be two self maps. Since a self map is null homotopic if and only if it induces the trivial map in rational cohomology [16], the map f is null homotopic if and only if g is. Moreover, if f is not null homotopic it is an equivalence as well as g. The compositions $f \circ Bi_E$ and $g \circ Bi_E$ induce the same map in rational cohomology. If the rank of E is less or equal to 2, then Remark 3.1 shows that both compositions are actually homotopic. By Proposition 3.6, this implies that $f \simeq g$. Hence, D is an injection.

The degree 0 is realized by the constant map. For $k \in \mathbb{Z}_2^{\wedge *}$ and $E \subset DI(4)$ of rank $\leqq 2$, the centralizer $C_{DI(4)}(E)$ is the completion of a connected compact Lie group. There exists an Adams operation $f'_E: BC_{DI(4)}(E) \to BC_{DI(4)}(E)$ of degree k [14]. Let $f_E := Bi_E \circ f'_E: BC_{DI(4)}(E) \to BDI(4)$ denote the composition. We want to show that the

tuple $(f_E)_{E\in\mathcal{A}'}$ gives an element of $\lim_{\leftarrow \mathcal{A}'}[\Theta, BDI(4)]$. That is that for every morphism $E_0 \to E_1$ the composition $BC(E_1) \to BC(E_0) \xrightarrow{f_{E_0}} BDI(4)$ is homotopic to f_{E_1}. By construction, the composition and f_{E_1} induce the same map in rational cohomology. Remark 3.1 implies that both maps are homotopic. Therefore, $(f_E)_{E\in\mathcal{A}'} \in \lim_{\leftarrow \mathcal{A}'}[\Theta, BDI(4)]$ and D is surjective (Proposition 3.7), which finishes the proof. □

4. The proof of Proposition 1.4

First we assume that X is a 2-compact group, such that

$$H^*(BX;\mathbb{F}_2) \cong H^*\big(BDI(4);\mathbb{F}_2\big) \cong H^*(BE_4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}.$$

Lemma 4.1. (i) *Every elementary abelian 2-subgroup $E \subset X$ is subconjugated to $E_4 \subset X$.*

(ii) *The 2-compact group X is connected, centerfree and simple. In particular, the rational Weyl group representation $W_X \to Gl(L_X \otimes \mathbb{Q})$ is irreducible.*

(iii) *X and $DI(4)$ have the same rational Weyl group data.*

Proof. The $\mathcal{K}$-map $H^*(BX;\mathbb{F}_2) \to H^*(BE_4;\mathbb{F}_2)$ can be realized by a topological map $BE_4 \to BX$ [15] which makes $E_4 \subset X$ to a subgroup of X. Moreover, since $H^*(BE;\mathbb{F}_2)$ is an injective object in the category $\mathcal{K}$ of algebras over the Steenrod algebra, every map $BE \to BX$ factors through BE_4. This proves (i).

For p-compact groups, the T-functor calculates the cohomology of classifying spaces of centralizers of elementary abelian p-subgroups [10]. Since the T-functor commutes with taking invariants, a central subgroup $E \subset E_4 \subset X$ has to be fixed under the action of $Gl(4,\mathbb{F}_2)$. There are no nontrivial fixed points, and therefore X is centerfree.

As a connected centerfree 2-compact group, X splits into a product of simple centerfree 2-compact groups [12]. A non trivial splitting of X would establish a non trivial splitting of $H^*(BX;\mathbb{F}_2)$ into a tensor product. But such a splitting does not exist for $H^*(BE_4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$. Hence, X is simple. Since, for p-compact groups, simple just means that the associated rational Weyl group is irreducible [12], [24], the last fact of (ii) is obvious.

As already mentioned in Section 2, the $\mathcal{A}_2$-action on $H^*(BX;\mathbb{F}_2)$ implies that $H^*_{\mathbb{Q}_2^\wedge}(BX) \cong \mathbb{Q}_2^\wedge[y_8, y_{12}, y_{28}] \cong H^*_{\mathbb{Q}_2^\wedge}\big(BDI(4)\big)$. Since $H^*_{\mathbb{Q}_2^\wedge}(BX) \cong H^*_{\mathbb{Q}_2^\wedge}(BT_X)^{W_X}$ [10], the 2-compact groups X and $DI(4)$ have the same rank and the Weyl groups W_X and $W_{DI(4)}$ the same degrees. Looking at the classification of all irreducible rational pseudo reflection groups [4] shows that both Weyl groups are weakly isomorphic. That is that X and $DI(4)$ have the same rational Weyl group data. □

Proof of Proposition 1.4 (first half). Let X be a 2-compact group, such that $H^*(X;\mathbb{F}_2) \cong H^*\big(BDI(4);\mathbb{F}_2\big)$. We want to show that both have isomorphic maximal torus normalizer. By Lemma 4.1, we can identify W_X and $W_{DI(4)}$ via the rational Weyl group data. Then L_X and $L_{DI(4)}$ are $W_{DI(4)}$-lattices of $L_{DI(4)} \otimes \mathbb{Q}$. By Lemma 2.1, both lattices

are isomorphic. That is we can identify BT_X and $BT_{DI(4)}$ as $W_{DI(4)}$-spaces. Hence, both normalizers are extensions of $T_{DI(4)}$ by $W_{DI(4)}$. Using Lemma 4.1, the argument in the proof of Lemma 2.6 shows that N_X also does not split. By Lemma 2.8, there exists only one non splitting extension. This shows that N_X and $N_{DI(4)}$ are isomorphic. □

Now we prove the other half of Proposition 1.4, which is nothing but part (vi) of the following proposition.

Proposition 4.2. *For a 2-compact group X, such that N_X and $N_{DI(4)}$ are isomorphic, the following holds:*

(i) *The 2-compact group X is connected.*

(ii) *There exists a monomorphism $Spin(7) \to X$ such that $Spin(7)$ is a subgroup of maximal rank.*

(iii) *The Euler characteristique $\chi(X/Spin(7))$ is odd.*

(iv) *The map $j^*: H^*(BX; \mathbb{F}_2) \to H^*(BSpin(7); \mathbb{F}_2)$ is a monomorphism.*

(v) *The subgroup $E_4 \subset N_{DI(4)} \subset X$ is self centralizing and*

$$H^*(BX; \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{\overline{W}_X},$$

where $\overline{W}_X$ is the Weyl group of $E_4 \subset X$.

(vi) *$H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$ as algebras over the Steenrod algebra.*

Here, the Weyl group $\overline{W}_X$ is the group of all elements $w \in Gl(4, \mathbb{F}_2)$ such that the inclusion $i_X: E_4 \to X$ and the composition $i_X w$ are conjugate in X.

Proof. By [10], for any p-compact group Y, the map $W_Y = \pi_0(N_Y) \to \pi_0(Y)$ is an epimorphism and the kernel is the Weyl group of the component Y_0 of the unit. Hence, there exists an epimorphism $W_X = \mathbb{Z}/2 \times Gl(3, \mathbb{F}_2) \to \pi_0(X)$. The target is a 2-group and the kernel W_0 is the Weyl group of X_0. In particular, since X and X_0 have both rank 3, the rational representation, $W_0 \to Gl(3, \mathbb{Q}_2^\wedge)$ represents W_0 as a pseudo reflection group. The order of W_0 divides $336 = 2^4 \cdot 3 \cdot 7$ which is the order of $W_{DI(4)}$. We have to find all pseudo reflection groups of rank 3 for which the product of the degrees is divisible by 3 and 7. Only the group No. 24 satisfies this property, which is the $W_{DI(4)}$. Hence, $\pi_0(X)$ is trivial and X connected. This proves part (i).

The maximal torus normalizer $N_X \cong N_{DI(4)}$ contains a unique subgroup $E_4 \subset N_{DI(4)}$ (Remark 2.3). Let $E_1 \subset E_4$ be the inclusion of the first coordinate. Since the Weyl group of $C_X(E_1)$ and of the component of the unit $C_X(E_1)_0$ can be read off from the maximal torus normalizer N_X [11], Theorem 7.6, the centralizers $C_X(E_1)$ and $C_{DI(4)}(E_1)$ have isomorphic maximal torus normalizer as well as $C_X(E_1)_0$ and $C_{DI(4)}(E_1)_0$. Since $Spin(7) \cong C_{DI(4)}(E_1)$ is connected, $C_X(E_1)$ and $C_X(E)_0$ have isomorphic Weyl groups. Hence, $C_X(E_1)$ is connected and has the same maximal torus normalizer as $Spin(7)$. By [26], this implies that both are isomorphic, which proves part (ii).

The subgroup $Spin(7) \subset X$ is of maximal rank. As explained in Remark 2.3, the Euler characteristique of $X/Spin(7)$ equals the Weyl group index $[W_X : W_{Spin(7)}] = [W_{DI(4)} : W_{Spin(7)}]$ which is odd (Remark 2.3). This proves part (iii).

There exists a transfer $tr: H^*(BSpin(7); \mathbb{F}_2) \to H^*(BX; \mathbb{F}_2)$ such that the composition with the restriction map $j^*: H^*(BX; \mathbb{F}_2) \to H^*(BSpin(7); \mathbb{F}_2)$ is multiplication by $\chi(X/Spin(7))$ [5]. Since this is an odd number, the composition is the identity and the map j^* of (iv) a monomorphism as desired. We also notice that the transfer is a linear map of $H^*(BX)$-modules.

Since E_4 centralizes E_1, we have $E_4 \subset Spin(7)$. And since $E_1 \subset E_4$, we have $C_X(E_4) \cong C_{Spin(7)}(E_4) \cong E_4$. The last identity follows because $E_4 \subset Spin(7)$ is self centralizing.

Let $R^* := H^*(BX; \mathbb{F}_2)$, $S^* := H^*(BSpin(7); \mathbb{F}_2)$ and $A^* := H^*(BE_4; \mathbb{F}_2)$. Then the composition $i^*: R^* \to S^* \to A^*$ is a monomorphism and the Adams-Wilkerson embedding. Let $D^* \subset A^*$ be the smallest Hopf algebra containing R^* which is also an algebra over the Steenrod algebra. By [7], Theorem 3.6, we have $D^* = T_{i^*}^{E_4}(R^*) = H^*(BC_X(E_4); \mathbb{F}_2) = A^*$. The last identity follows since $E_4 \subset X$ is self centralizing.

Since A^* is a finitely generated R^*-algebra, the fields of fractions $F(R^*) \subset F(A^*)$ establish an algebraic extension. We can apply [33], Theorem II. In our situation it tells us that there exists a 'Galois' group $W \subset Gl(E_4)$ such that $R^* \subset Un(F(R^*)) = (A^*)^W \subset F(R^*)$, where $Un(F(R^*))$ denotes the unstable part of $F(R^*)$. Since the homotopy classes of maps $BE_4 \to BX$ can be identified with the maps $H^*(BX; \mathbb{F}_2) \to H^*(BE_4; \mathbb{F}_2)$ of algebras over the Steenrod algebra, $W = \overline{W}_X$ is the Weyl group of $E_4 \subset X$. This establishes a chain of monomorphisms

$$R^* \to (A^*)^{\overline{W}_X} \to (A^*)^\Gamma = S^*,$$

where Γ, as explained in Section 2, denotes the Weyl group of $E_4 \subset Spin(7)$. Let $x \in (A^*)^{\overline{W}_X}$. Then, $x = a/b$ with $a, b \in R^*$ and $b \cdot tr(x) = tr(bx) = tr(a) = a$. Since we are doing calculations in a subring of a polynomial ring this shows that $tr(x) \in R^*$ is divisible by b, that $R^* \to (A^*)^{\overline{W}_X}$ is an epimorphism and that $R^* \cong (A^*)^{\overline{W}_X}$. This proves part (v).

Finally we have to prove part (vi). By construction, $\Gamma \subset \overline{W}_X \subset Gl(4, \mathbb{F}_2)$. The first inclusion is proper. Otherwise, the map $BSpin(7)_2^\wedge \to BX$ induces an isomorphism in mod-2 cohomology and is an equivalence. This is a contradiction, since $Spin(7)$ and X have non isomorphic maximal torus normalizer, and are therefore non isomorphic [18].

Lemma 4.3 below shows that $\Gamma \subset Gl(4, \mathbb{F}_2)$ is a maximal subgroup. This implies that $\overline{W}_X = Gl(4, \mathbb{F}_2)$ and that $H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$. This proves the last part and finishes the proof. □

The stabilizer subgroup of a subspace $U' \subset U := (\mathbb{F}_p)^n$ is the subgroup of $Gl(n, \mathbb{F}_p)$ of all elements which map U' onto itself. For $p = 2$ and for a 1-dimensional subspace, the isotropy subgroup and the stabilizer subgroup are equal.

Lemma 4.3. *For any prime p, the stabilizer subgroup $\Gamma \subset Gl(n, \mathbb{F}_p)$ of any 1-dimensional subspace of U is a maximal subgroup.*

Proof. We can assume that Γ is the stabilizer subgroup of the subspace generated by the first standard basis vector e_1. We write every element of $B \in Gl(n, \mathbb{F}_p)$ in the form $B = \begin{pmatrix} \mu & c \\ b & M \end{pmatrix}$, where $\mu \in \mathbb{F}_p$, b and c have $n-1$ components and where M is an $(n-1) \times (n-1)$ matrix. For elements of Γ, we have $\mu \neq 0$, $b = 0$ and $M \in Gl(n-1, \mathbb{F}_p)$.

Let $\Gamma \subsetneq \Gamma' \subseteq Gl(n, \mathbb{F}_p)$. We have to show that $\Gamma' = Gl(n, \mathbb{F}_p)$. Since $\Gamma \backslash Gl(n, \mathbb{F}_p) \cong U' := U \backslash \{0\}$, the quotient $\Gamma' \backslash Gl(n, \mathbb{F}_p)$ is a quotient of U'. Therefore, it is sufficient to show that Γ' acts transitively on U'.

Let $B = \begin{pmatrix} \mu & c \\ b & M \end{pmatrix} \notin \Gamma$; i.e. $b \neq 0$. Multiplying B by elements of Γ from the left shows that, for every vector $u \in U$, there exists a matrix $B' \in \Gamma'$ such that the first column (μ', b') equals u. Thus, the Γ'-orbit of e_1 contains all vectors of U' and Γ' acts transitively on U'. □

5. Proof of the Theorems 1.2 and 1.3

Because of Proposition 1.4, both theorems are a consequence of the following statement.

Theorem 5.1. *Let X be a 2-compact group, such that X and $DI(4)$ have isomorphic maximal torus normalizer and such that $H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$ (as algebras over the Steenrod algebra). Then, X and $DI(4)$ are isomorphic as 2-compact groups.*

The rest of this section is devoted to the proof of Theorem 5.1. And X will always denote a 2-compact group satisfying the assumptions of the theorem. Before we start with the proof we fix some notation, which partly was already introduced in Section 2 and Section 3. Any compact Lie group G appearing in the decomposition diagram of $BDI(4)$ is the centralizer $C_{DI(4)}(E_k) =: C_k$ of a k-dimensional elementary abelian 2-subgroup of $DI(4)$. Actually, for all k, C_k is already contained in $C_1 \cong Spin(7)$. We denote by $j_k: BC_k \to BC_1$ and $j_k: BC_k \to BDI(4)$ the maps associated to these inclusions. In particular, $i_1: BC_1 \to BDI(4)$ comes from the standard inclusion $C_1 \subset DI(4)$. We denote by T the maximal torus of C_1 and denote by $j_T: BT \to BC_1$ the associated map of the inclusion. The composition $T \subset C_1 \subset DI(4)$ is also a maximal torus and establishes the map $i_T: BT \to BDI(4)$. By $W := W_{DI(4)}$ we denote the Weyl group of $DI(4)$ and by $L_T := \pi_1(T)$ the 'dual weight lattice'.

Lemma 5.2. (i) *There exist a monomorphism $f_1: BC_1 \to BX$ and an isomorphism $H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$ such that the diagram*

$$\begin{array}{ccc} & H^*(BC_1; \mathbb{Z}/2) & \\ {\scriptstyle f_1^*}\nearrow & & \nwarrow{\scriptstyle i_1^*} \\ H^*(BX; \mathbb{F}_2) & \xrightarrow{\cong} & H^*(BDI(4); \mathbb{F}_2) \end{array}$$

commutes.

(ii) *The composition* $T \xrightarrow{j_T} BC_1 \xrightarrow{f_1} X$ *is a maximal torus and the associated Weyl group representation* $W_X \to Gl(L_T)$ *is weakly isomorphic to the Weyl group representation* $W \to Gl(L_T)$.

Proof. The composition of inclusions $E_1 \subset E_4 \subset T \rtimes \mathbb{Z}/2 \cong C_2 \subset N_{DI(4)}4(T) \subset X$ factors through the maximal torus T. By Proposition 4.3, $C_X(E_1) \cong Spin(7) \cong C_1$ which establishes the monomorphism $f_1: BC_1 \to BX$. This map induces a monomorphism $f_1^*: H^*(BX; \mathbb{F}_2) \to H^*(BC_1; \mathbb{F}_2)$ in mod-2 cohomology (Proposition 4.3) and makes the target to a finitely generated module over the source. By Lemma 2.5, both maps, f_1^* and i_1^* have the same image. Therefore the desired isomorphism $H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$ exists. This proves the first part.

The composition $T \subset C_1 \subset X$ is obviously a maximal torus. The weak isomorphism between the two Weyl group representations follows from Lemma 2.1. □

Using Part (ii) of this lemma we can identify the maximal tori and Weyl groups of $DI(4)$ and X. The inclusion $T \subset X$ establishes a map $f_T: BT \to BX$ and $f_T = f_1 j_T$.

Let $\mathbf{A}_p(DI(4))$ denote the Quillen category of the $DI(4)$ and let

$$\Theta: \mathbf{A}_p(DI(4)) \to \mathbf{Top}$$

be the functor described in Section 2, which gives the Dwyer-Wilkerson decomposition for $DI(4)$. This functor maps E_k on the space $BC_k := BC_{DI(4)}(E_k)$. For each space BC_k we use the inclusion $j_k BC_k \to BC_1$ (see Section 2), to define $f_k: BC_k \to BX$ by the composition $f_k := f_1 j_k$.

Proposition 5.3. *Let* $\alpha: E_l \to E_k$ *be any morphism of* $\mathbf{A}_p(DI(4))$. *Then,* $f_l\Theta(\alpha)$ *and* f_k *are homotopic maps.*

Proof. Since $i_k = i_1 j_k$, we can assume that $l = 1$. By assumption, $i_1\Theta(\alpha)$ and i_k are homotopic. Now we check the statement for each value of i.

For $k = 1$ there is nothing to show since the automorphism group of E_1 is trivial.

Let $E_4 \subset C_1$ be the standard inclusion. Then, $BC_4 \simeq BE_4$ and, by Lemma 5.2, the triangle

$$\begin{array}{ccc} & H^*(BE_4; \mathbb{Z}/2) & \\ {\scriptstyle j_{E_4}^*}\nearrow & & \nwarrow{\scriptstyle i_{E_4}^*} \\ H^*(BX; \mathbb{F}_2) & \xrightarrow{\cong} & H^*(BDI(4); \mathbb{F}_2) \end{array}$$

commutes. Since homotopy classes of maps $BE_4 \to BX$ are determined by the induced map in mod-2 cohomology, the statement also follows for $k = 4$.

For $k = 2, 3$ we first notice that the maximal torus $T \subset C_1$ also establishes a maximal torus for C_k.

Claim. The restrictions $f_1\Theta(\alpha)|_{BT}$ and $f_k|_{BT}$ are homotopic.

Proof. The restriction $\Theta(\alpha)|_{BT}$ factors through a self map $w: BT \to BT$. Since $i_1\Theta(\alpha) \simeq i_1$, we have $i_T w \simeq i_T$ which shows that $w \in W$. Hence,

$$f_1\Theta(\alpha)|_{BT} \simeq f_1 j_T w \simeq f_T w \simeq f_T \simeq f_1 j_T \simeq f_k|_{BT},$$

which proves the claim. □

Now we consider the cases $k = 2$ and $k = 3$ separately. The centralizer C_2 comes from a compact connected Lie group with center $E_2 \subset T \subset C_2$ and the inclusion $C_2 \subset X$ induces a homotopy equivalence $BC_2 \simeq BC_X(E_2)$. Therefore all assumptions of Proposition 4.6 of [25] are satisfied (for $X = DI(4)$ this is explained in more detail in Remark 3.1). Part (c) of that proposition and the above claim imply that $f_1\Theta(\alpha) \simeq f_2$. This proves the statement for $k = 2$.

The final case $k = 3$ is a consequence of the above Claim and the next Proposition. □

Proposition 5.4. *Let $g, h: BC_3 \to BX$ be two maps such that the restrictions $g|_{BT}$ and $h|_{BT}$ are homotopic and monomorphisms. Then g and h are homotopic.*

Proof. The 2-compact group C_2 establishes a fibration $BT \to BC_2 \to B\mathbb{Z}/2$. The group action of $\mathbb{Z}/2$ on $L := \pi_2(BT)$ is given by multiplication by -1. Let $g' := g|_{BT}$ and let $M_{g'}$ denote the union of all components of $\mathrm{map}(BC_2, BX)$ whose restriction to BT is homotopic to g'. We make $BT \to BC_2$ into a fibration respectively into a principal $\mathbb{Z}/2$-bundle. Then $M_{g'}$ can be described as the homotopy fixed-point set $\left(\mathrm{map}(BT, BX)_{g'}\right)^{h\mathbb{Z}/2}$ [13]. We only have to show that this homotopy fixed-point set is connected. Since g' is a monomorphism, it can serve as a maximal torus and $\mathrm{map}(BT, BX)'_g \simeq BT$. Hence, the Borel construction establishes a fibration $BT \to E \to B\mathbb{Z}/2$ and the space of sections is homotopy equivalent to the homotopy fixed-point set. In this case, up to homotopy, the sections are classified by the obstruction group $H^2(B\mathbb{Z}/2; L)$ where $\mathbb{Z}/2$ acts on L as described above. Since $H^2(\mathbb{Z}/2; \mathbb{Z}_2^\wedge) = 0$ for the nontrivial action of $\mathbb{Z}/2$ on $\mathbb{Z}_2^\wedge$, this obstruction group vanishes and the space of sections of the above fibration is connected as well as the homotopy fixed-point set. This proves the statement. □

Let $K := \mathrm{hocolim}_{\mathbf{A}_p(DI(4))} \Theta$. This is the realization of a simplicial space. By $K^{(n)}$ we denote the realization of the n-skeleton of this simplicial space. By the definition of the homotopy colimit, Proposition 5.3 establishes a map

$$\phi_1: K^{(1)} \to BX.$$

Proof of Theorem 5.1. We want to extend the map $\phi_1: K^{(1)} \to BX$ to K by induction over the skeleton. There exists an obstruction theory for such extension problems [35], whose obstruction groups we describe next.

Let $\Pi_i\colon \mathbf{A}_p(DI(4)) \to \mathcal{A}b$ be the functor taking values in the category of abelian groups, which is defined by $\Pi_i(E_k) := \pi_k(\mathrm{map}(BC_k, BX)_{f_k})$. The above mentioned obstruction groups are given by higher derived limits of this functor. Since $BE_k \simeq \mathrm{map}(BC_k, BX)_{f_k}$, the functor Π_i is trivial for $i \neq 1$ and nothing but the inclusion functor $J\colon \mathbf{A}_p(DI(4)) \to \mathcal{A}b$ for $i = 1$. Here, J is defined by $J(E_k) := E_K$. The higher derived limits of J are calculated in [9] and shown to vanish. Therefore, we can solve the extension problems and there exists a map $\phi\colon K \to BX$. By construction $K \to BDI(4)$ is a mod-2 equivalence. And this establishes a homotopy equivalence $BDI(4) \simeq BX$ and finishes the proof of Theorem 5.1. □

6. The proof of Proposition 1.5

In this section we will prove Proposition 1.5. Throughout we denote by X a 2-compact group which has rationally the same Weyl group data as $DI(4)$. We want to show that the maximal torus normalizer N_X is isomorphic to $N_{DI(4)}$. From Lemma 2.1 we know that X and $DI(4)$ have the same integral Weyl group data; i.e. there exists an abstract isomorphism $W_X \cong W_{DI(4)}$ and a $W_{DI(4)}$-equivariant isomorphism $L_X \cong L_{DI(4)}$ where $W_{DI(4)}$ acts on L_X via the abstract isomorphism. That is we can identify the maximal tori, the lattices as well as the Weyl groups and denote them by T, L and W. Let $\mathbb{Z}/2 \cong E_1 \subset T$ be the inclusion into the first coordinate as described in Section 2. Let $Y := C_X(E_1)$ denote the centralizer of E_1. Then $T \subset Y$ is a maximal torus and, since the Weyl group of the centralizer of a toral subgroup can be calculated from the Weyl group data [11] (Theorem 7.6), the Weyl group action of W_Y on L is the same as for $Spin(7)$ which is $C_{DI(4)}(E_1)$.

Since $E_1 \subset T \subset Y$ is a central subgroup, we can pass to the quotient $\overline{Y} := Y/\mathbb{Z}/2$. Since $E_1 \subset T \subset Y_0$ both groups have isomorphic group of components and isomorphic Weyl groups, which we identify and which we denote by W'. In fact, $W' \cong \mathbb{Z}/2 \wr \Sigma_3$ is isomorphic to the Weyl group of $Spin(7)$ respectively $SO(7)$. The maximal torus $\overline{T} := T_{\overline{Y}}$ fits into an exact sequence $1 \to \mathbb{Z}/2 \cong E_1 \to T \to \overline{T} \to 1$ and the lattice $\bar{L}$ into a short exact sequence $0 \to L \to \bar{L} \to \mathbb{Z}/2 \to 0$ of W'-modules which is the same as for the lattices associated to the extension $1 \to \mathbb{Z}/2 \to Spin(7) \to SO(7) \to 1$. Hence $\overline{Y}$ has 2-adically the same Weyl group data as $SO(7)$.

We can identify several maximal tori, associated lattices and Weyl groups. In this section we will denote by T and L the maximal tori and associated lattices of $DI(4)$, X, $Spin(7)$ and Y, by $\overline{T}$ and $\bar{L}$ the maximal tori and associated lattices of $SO(7)$, $O(6)$ and $\overline{Y}$, by W the Weyl groups of $DI(4)$ and X, and by W' the Weyl groups of $Spin(7)$, Y, $SO(7)$, $O(6)$ and $\overline{Y}$.

The above considerations are collected in the following lemma.

Lemma 6.1. *Let X be a 2-compact group with the same rational Weyl group data as $DI(4)$. Then there exists 2-compact groups Y and $\overline{Y}$ such that the following holds:*

(i) *X has 2-adically the same Weyl group data as $DI(4)$.*

(ii) *Y has the 2-adically the same Weyl group data as $Spin(7)$ and $\overline{Y}$ as $SO(7)$.*

(iii) *There exists a short exact sequence*

$$1 \to \mathbb{Z}/2 \to Y \to \overline{Y} \to$$

of 2-compact groups. Moreover, the classifying map $B\overline{Y} \to B^2\mathbb{Z}/2$ *of the principal fibration between the classifying spaces is non trivial.*

(iv) *The 2-compact group* $Y \subset X$ *is a subgroup of maximal rank.*

(v) *If the maximal torus normalizer* N_X *splits, then* N_Y *splits as well as* $N_{\overline{Y}}$ *and* $N_{\overline{Y}}$ *and* $N_{SO(7)}$ *are isomorphic.*

Proof. The claims (i), (ii), (iv) and the first half of (iii) we already discussed. Since Y and $\overline{Y}$ have the same components, we have $BY \neq B\overline{Y} \times B\mathbb{Z}/2$. That is that the classifying map $B\overline{Y} \to B^2\mathbb{Z}/2$ is non trivial. The maximal torus normalizer N_Y and N_X fit into a diagram of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & N_Y & \longrightarrow & W' & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T & \longrightarrow & N_X & \longrightarrow & W & \longrightarrow & 1. \end{array}$$

Hence, the normalizer N_Y is isomorphic to the pull back, induced by the inclusion $W' \subset W$, of the extension N_X of T by W. This shows that, if N_X splits, then N_Y splits. Moreover, if N_Y splits then $N_{\overline{Y}}$ does it, too. It can be constructed by a push out from N_Y. Since $N_{SO(7)}$ also splits, this proves the last claim. □

The Weyl group $W' \cong W_{SO(7)}$ of $SO(7)$ is isomorphic to the wreath product $\mathbb{Z}/2 \wr \Sigma_3$ and acts on $\mathbb{Z}_2^{\wedge 3} \cong L$ in the obvious way. Let $V_2 := \{(a_1, a_2, a_3) \in \mathbb{Z}/2^3 : a_1 + a_2 + a_3 = 0\}$. This is an Σ_3-equivariant subgroup of $\mathbb{Z}/2^3 =: V_3$ and $W'' := V_2 \rtimes \Sigma_3 \subset W \cong V_3 \rtimes \Sigma_3$ is a normal subgroup. Actually, W'' is the Weyl group of $SO(6)$ and the inclusion $W'' \subset W'$ is induced by the inclusion $SO(6) \subset SO(7)$.

We are interested in the homotopy type of $B\overline{Y}$. We say that two 2-compact groups X and Y have the same N-type, if there exist isomorphisms $N_X \cong N_Y$ and $\pi_0(X) \cong \pi_0(Y)$ such that the diagram

$$\begin{array}{ccc} N_X & \xrightarrow{\cong} & N_Y \\ \downarrow & & \downarrow \\ \pi_0(X) & \xrightarrow{\cong} & \pi_0(Y) \end{array}$$

commutes. For a more detailed description of this notion, see [26].

Lemma 6.2. *If the maximal torus normalizer* N_X *splits, then the following holds*:

(i) *The 2-compact group* $\overline{Y}$ *has the same N-type as* $SO(7)$ *or as* $O(6)$.

(ii) *The 2-compact group* $\overline{Y}$ *is either isomorphic to* $SO(7)$ *or to* $O(6)$.

For the proof we will need the following lemma.

Lemma 6.3. *Let π be a finite 2-group and ρ: $W' \cong \mathbb{Z}/2 \wr \Sigma_3 \to \pi$ be an epimorphism, such that the kernel of ρ is generated by elements of order 2. Then either π is the trivial group or $\pi \cong \mathbb{Z}/2$ and W'' is the kernel of ρ.*

Proof. Let $\sigma \in \Sigma_3$ denote an element of order 3. And let K denote the kernel of ρ. Then $\sigma \in K$ and since $K \subset W'$ is a normal subgroup, every element of the form $(v,1)(0,\sigma)(v,1)(0,\sigma^2) = (v+\sigma(v),1)$ is contained in K, where $v \in V_3$. The elements $v+\sigma(v)$ generate the subgroup V_2. Moreover, since K is generated by elements of order two and since any element of the form (v,σ) has order divisible by three, there exists an element $(w,\tau) \in K$ of order two, where $\tau \in \Sigma_3$ is a transposition. The order condition implies that $w \in V_2$, and hence that $(0,\tau) \in K$, that $\Sigma_3 \subset K$ and that $W' \subset K$. Since the index of $W' \in W$ is equal to 2 this proves the claim. □

Proof of Lemma 6.2. For any p-compact group Z, the map $N_Z \to \pi_0(Z)$ is onto and factors through the Weyl group W_Z. The kernel of the map $W_Z \to \pi_0(Z)$ is the Weyl group and the kernel of $N_Z \to \pi_0(Z)$ the maximal torus normalizer of the connected component of Z. For the prime 2, the Weyl group of a connected 2-compact group is generated by reflection, in particular by elements of order two.

By Lemma 6.1 (v) we know that $N_{\overline{Y}}$ and $N_{SO(7)}$ are isomorphic. If $\overline{Y}$ is connected, then $\overline{Y}$ and $SO(7)$ have the same N-type. If $\overline{Y}$ is not connected, the group of components $\pi_0(\overline{Y})$ is a finite 2-group. By Lemma 6.3, this shows that $\pi_0(\overline{Y}) \cong \mathbb{Z}/2$ and that the kernel of $W' \to \pi_0(\overline{Y})$ equals W''. Therefore, $\overline{Y}$ and $O(6)$ have the same N-type. This proves part (i). For $SO(2n+1)$ and $O(n)$, the N-type determines the isomorphism type of these 2-compact groups [26]. This proves the second part. □

For the proof of Proposition 1.5 we need one further lemma.

Lemma 6.4. *For any section s: $W_{O(n)} \to N_{O(n)}$ of the projection $N_{O(n)} \to W_{O(n)}$ of the maximal torus normalizer of the orthogonal $O(n)$ to the Weyl group, the composition Bs': $BW \to BN \to BO(6)$ induces in* mod-2 *cohomology a map which is injective in degree* 1 *and* 2.

Proof. Let $\overline{T} := T_{O(n)}$ denote the maximal torus, $W := W_{O(n)}$ the Weyl group and $N := N_{O(n)}$ the maximal torus normalizer of $O(n)$. We have a diagram of fibrations

$$\begin{array}{ccccc}
T & \longrightarrow & O(n)/W & \longrightarrow & O(n)/N \\
\downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\
T & \longrightarrow & BW & \xrightarrow{Bs} & BN \\
 & & \downarrow{\scriptstyle Bs'} & & \downarrow \\
 & & BO(n) & \xrightarrow{=} & BO(n).
\end{array}$$

The top row shows that $H^1(O(n)/N;\mathbb{F}_2) \cong H^1(O(n)/W;\mathbb{F}_2)$. Since

$$H^*(BO(n);\mathbb{F}_2) \to H^*(BN;\mathbb{F}_2)$$

is a monomorphism, there are no differentials in the Serre spectral sequence of the right column which end at the horizontal edge. Therefore, in the Serre spectral sequence of the middle column there is no differential ending at $H^*(BO(n);\mathbb{F}_2)$ for $* = 1,2$, which proves the statement and $H^k(BO(n);\mathbb{F}_2) \to H^k(BW;\mathbb{F}_2)$ is a monomorphism for $k = 1,2$. □

Now we are in the situation to prove Proposition 1.5.

Proof of Proposition 1.5. Since we know that X and $DI(4)$ have the same 2-adic Weyl group data (Lemma 2.1) and since there exist only two extensions of $T_{DI(4)}$ by $W_{DI(4)}$ (Lemma 2.7), we only have to prove that N_X does not split.

There exists a non trivial principal fibration $B\mathbb{Z}/2 \to BY \to B\overline{Y} \to B^2\mathbb{Z}/2$ and a monomorphism $BY \to BX$ of maximal rank with the properties mentioned in Lemma 6.1. Hence the classifying map $B\overline{Y} \to B^2\mathbb{Z}/2$ describes a non trivial two dimensional mod-2 cohomology class.

Let us assume that N_X does split. If $\overline{Y}$ is connected, then $B\overline{Y} \simeq BSO(7)^\wedge_2$ (Lemma 6.2) and $BY \simeq BSpin(7)^\wedge_2$, since $H^2(BSO(7);\mathbb{F}_2) \cong \mathbb{F}_2$. Since the maximal torus normalizer of $Spin(7)$ does not split, this gives a contradiction.

If $\overline{Y}$ is not connected then $B\overline{Y} \simeq BO(6)^\wedge_2$ (Lemma 6.2). By Lemma 6.4, for any section $Bs\colon BW \to BN_{O(6)}$ the composition $BW \to BN_{O(6)} \to BO(6) \to B^2\mathbb{Z}/2$ is not null homotopic. In particular there exist no lift of this composition to BY. Hence, N_Y is a non splitting maximal torus normalizer as well as N_X (Lemma 6.1), which is again a contradiction. Therefore N_X is a non splitting extension and isomorphic to $N_{DI(4)}$. □

7. Proof of Theorem 1.7

We start directly with the proof. For any connected 2-compact group X there exists a short exact sequence $Z \to X \to \overline{X}$ of 2-compact groups respectively a fibration $BZ \to BX \to B\overline{X}$ for the associated classifying spaces, such that $Z \subset X$ is a central subgroup, in particular a product of a finite abelian 2-group and a torus, and such that $\overline{X}$ is centerfree [18]. In fact, the fibration is principal and has a classifying map $B\overline{X} \to B^2Z$. By [12], $\overline{X} \cong \prod_i Y_i$ is congruent to a finite product of simple centerfree 2-compact groups Y_i. In particular, for all i, the rational Weyl group representation $W_{Y_i} \to Gl(L_{Y_i} \otimes \mathbb{Q})$ is irreducible. (Actually, all this is true for any prime.) If Y_i has the same rational Weyl group data as $DI(4)$, then Y_i and $DI(4)$ have isomorphic maximal torus normalizer (Remark 4.3) and are isomorphic by Theorem 1.3.

Let X_1 be the product of all factors Y_i with the same rational Weyl group data as $DI(4)$ and $\overline{X}_2$ the product of all the others. At the prime 2, the rational Weyl group representation of $DI(4)$ is the only irreducible pseudo reflection group over $\mathbb{Q}^\wedge_2$ which is not weakly isomorphic to the 2-adic rational Weyl group representation of a simple connected

compact Lie group. This follows by checking the Clark-Ewing list [4]. Therefore $\overline{X}_2$ has the same rational Weyl group data as a suitable connected compact Lie group, and X_1 is congruent to a product of copies of $DI(4)$ (Corollary 1.6).

Since $BDI(4)$ is 7-connected, the composition $BX_1 \to BX_1 \times B\overline{X}_2 \to B^2Z$ is null homotopic and $BX \simeq BX_1 \times BX_2$ where BX_2 is the fiber of the composition $B\overline{X}_2 \to BX_1 \times B\overline{X}_2 \to B^2Z$. Since BX_2 fits into a fibration $BZ \to BX_2 \to B\overline{X}_2$, we can think of it as the classifying space of a 2-compact group X_2 . And, up to some trivial summands, BX_2 has the same rational Weyl group data as $B\overline{X}$. In particular, BX_2 has the same rational Weyl group data as a suitable connected compact Lie group. This proves Theorem 1.7. □

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