Canonicity and Bi-Approximation in Non-Classical Logics

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Abstract

Non-classical logics, or variants of non-classical logics, have rapidly been developed together with the progress of computer science since the 20th century. Typically, we have found that many variants of non-classical logics are represented as ordered algebraic structures, more precisely as lattice expansions. From this point of view, we can think about the study of ordered algebraic structures, like lattice expansions or more generally poset expansions, as a universal approach to non-classical logics. Towards a general study of non-classical logics, in this dissertation, we discuss canonicity and bi-approximation in non-classical logics, especially in lattice expansions and poset expansions. Canonicity provides us with a connection between logical calculi and space-based semantics, e.g. relational semantics, possible world semantics or topological semantics. Note that these results are traditionally considered over bounded distributive lattice-based logics, because they are based on Stone representation. Today, thanks to the recent generalisation of canonical extensions, we can talk about the canonicity over poset expansions. During our investigation of canonicity over poset expansions, we will find the notion of bi-approximation, and apply it to non-classical logics, especially with resource sensitive logics.

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Contents

Chapter 1

About this dissertation

1.1 Background and motivation

Since the early 20th century, the study area of mathematical logic, a.k.a. symbolic logic, has widely and rapidly been spreading out based on the foundation of mathematics or closely related to the formal analysis of computer systems. As a part of branches of the large stream, we can find non-classical logics: modal logics, conditional logics, intuitionistic logic, many-valued logics, relevance logics, fuzzy logics, linear logic, substructural logic and those with additional modalities or quantifications, e.g. [5, 66, 11, 12, 70, 8, 34, 74, 63, 68, 23]. For example, substructural logic originally appeared as an analysis of meanings of structural rules in Gentzen's sequent calculi LK and LJ, and today it is also known as a uniform study of resource sensitive logics like relevance logic, fuzzy logics, intuitionistic logic, many-valued logics and linear logic.

As semantic models of (especially) propositional non-classical logics, we often consider classes of ordered algebras, or lattice expansions, e.g. modal algebras, Heyting algebras, residuated algebras, etc. Note that, in the middle of 19th century when mathematical logic emerged, the original idea itself already appeared in the works of Boole, De Morgan, etc. Thanks to Lindenbaum-Tarski algebras, these (propositional) non-classical logics naturally behave with those algebraic models, hence it is a recent trend to introduce those logics as classes of lattice-based algebras. We may call those logics *lattice-based logics* or algebraic logics. We mention that for predicate (quantified) logics like first-order logic, first-order modal logics, higher-order logics, etc., there are also various discussions of algebraic models, or more generally categorical models: cylindric algebras, polyadic algebras, hyperdoctrins, fibrations or topoi, e.g. [61, 39, 40, 56, 47, 55, 9, 59, 7, 37].

On the other hand, the invention of relational semantics, the so-called Kripke semantics [54], had a great impact on modal logic. Since the invention, modal logic has been applied to various areas particularly in computer science. The completeness property for modal logics with respect to Kripke semantics is called *Kripke completeness*. It tells us that Kripke complete modal logics characterise a class of relational structures. We often prove the completeness results by Henkin's canonical model construction. Nowadays, thanks to Sahlqvist theorem [71], we can obtain many Kripke complete modal logics and classes of the corresponding relational structures which are first-order definable. Note that Sahlqvist theorem was proved based on the following works [22, 57]: see e.g. [36]. Sahlqvist theorem has been rephrased as the algebraic analysis of *perfect extensions (canonical extensions)* of Boolean algebras with operators [50, 51], based on Stone representation for Boolean algebras [77]. Later, the theorem is reformulated and extended in the light of canonical extensions, for example, in [72, 73, 14, 38]. Therefore, an algebraic study of Kripke completeness, canonicity which is the representation property via canonical extensions, or more generally duality between lattice expansions and ordered topological spaces, e.g. Stone duality [78], attracts our interests. Recently, the study of canonicity, representation and duality is also featured from the point of applications to computer science. For example, in [1, p.5], we can find the following phrase.

> The importance of Stone duality for Computer Science is that *it provides the right framework for understanding the relationship between denotational semantics and program logic.*

In process algebra (e.g. [24]), program logics (variants of modal logics), for example, Hennessy-Milner logic [45], computation tree logic [20] or linear temporal logic [65] have used to analyse several transition systems of computer processes, the so-called *model checking.*

Nowadays, we can also find many variants of non-classical logics or resource sensitive logics applied to computer science, for example, linear logic [19], Hoare logic [46], dynamic logic [42], separation logic [69], Kleene algebra with tests [52, 53], etc. "Can we also obtain *the right framework for understanding the relationship between relational-based semantics and non-classical logics*?" More precisely, "which types of non-classical logics can have a Stone-type representation (closed under canonical extensions)?" In the literature, there are many contributions for specific non-classical logics: e.g. sequent systems [15, 16], BCK logics [64], relevant logic [70, 87], poset-based substructural logics [18], intuitionistic modal logic [33], distributive lattice expansions [49], relevant modal logics [76]. In this dissertation, we present a general theory of canonical representations for propositional non-classical logics which universally subsumes the above results.

To extend the representation theorem between relational semantics and, not necessarily distributive, lattice-based logics (or poset-based logics), it is necessary to generalise Stone representation for bounded distributive lattices to a representation for posets or lattices. To do so, we introduce the construction of canonical extensions of posets via Dedekind-MacNeille completions, whose construction has already appeared in [4]. Nowadays, canonical extensions are also universally characterised over posets by topological terms as *compact dense completions* [29, 27, 18]. In this dissertation, however, we define exactly the same structures, i.e. Dedekind-MacNeille completions and canonical extensions, in a different manner from their original construction in [33], which allows us to think about those completions as a collection of *bi-directionally approximated points*. We call the property that every point is approximated both from above and from below *bi-approximation*, which is also known as *denseness* in the topological characterisation [18]. We will show how reasonably the bi-approximation works for canonicity and how naturally it fits to non-classical logics, especially substructural logic (reasoning logical consequences) in this dissertation.

1.2 Overview

This dissertation mainly consists of the current author's works during his PhD research: canonicity of lattice expansions (Chapters 2, 3 and 4) is in [84], canonicity of poset expansions (Chapters 2, 5 and 6) is in [79] and bi-approximation semantics, a relational-type semantics, for substructural logic (Chapter 8) is in [83].

Thinking about logical consequences, we often accept the following two reasonings:

1. "if A then A" always holds,

2. "if A then B" and "if B then C" imply "if A then C."

When we look at each logical consequence "if A then B" as a binary relation $\leq$ on the set of all propositions, i.e. $A \leq B$. The above conditions tell us that the set of all propositions and the binary relation $\leq$ form a *preorder set*. From this point of view, we can see order theory as a mathematical tool to probe logical reasonings. In Chapter 2, we give preliminary definitions in order theory, e.g. posets, lattices, filters, ideals, etc., which are mainly discussed in this dissertation. Moreover, we also provide the main constructions of completions of posets and distributive lattices, i.e. Stone representations, Dedekind-MacNeille completions and canonical extensions. Furthermore, we define *bi-approximation* and *bases* with help of the canonical extension of posets.

Given a propositional logic, the Lindenbaum-Tarski algebra (free algebra) gives us an algebraic characterisation of the logic. When we consider well known logics like classical logic, modal logic, intuitionistic logic or substructural logic, the algebraic counterparts are described with *lattice expansions*, which consist of an underlying lattice and operations on it. In other words, lattice expansions are a universal machinery to study lattice-based logics. In Chapter 3, we introduce lattice expansions and give examples of ordered algebraic structures which are seen as algebraic counterparts of non-classical logics, e.g. modal logic, distributive modal logic and substructural logic. And we discuss canonical extensions of lattice expansions based on the canonical extension of posets. In addition, we explain Ghilardi and Meloni's parallel computation [33] and extend it from Heyting algebras with a unary modality to lattice expansions in general. In Chapter 4, as applications to non-classical logics, we show our canonicity results to substructural logic, relevant modal logics and distributive modal logic, and compare with existing results.

The argument of canonical extensions are nowadays generalised up to posets in general. Surprisingly, we can universally characterise canonical extensions over posets, i.e. including lattices, bounded distributive lattices and Boolean algebras, as compact dense completions, and they are unique up to isomorphism. One question that arise with this generalisation is that the canonicity property over poset expansions in general. More precisely, how does the lack of the lattice operations $\vee$ and $\wedge$ affect our canonicity argument? In Chapter 5, we discuss a way to extend Ghilardi and Meloni's canonicity methodology to poset expansions. Then we notice that even the simple version of our canonicity method for lattice expansions does not hold on poset expansions in general. Nevertheless, we show that we can still use the framework by carefully removing the problematic cases and obtain reasonable canonicity results for poset expansions as well. During the discussion, we also notice that the presence of the empty bases directly affects our methodology for poset expansions and consider how to deal with the presence of the empty bases. Furthermore, in Chapter 6, we illustrate that our canonicity results can still cover many canonical inequalities.

In Chapter 7, we give other perspectives of canonical extensions. Here we explain canonical extensions as compact dense completions based on the terminology [29] and the descendants [27, 18]. We summarise Ghilardi and Meloni's canonicity methodology for lattice expansions in the light of the topological characterisation of canonical extensions, as compact dense completions, and illustrate it with giving a concrete example from substructural logic. Furthermore, we also introduce another aspect of canonical extensions "an estimation of the perfect information from the observable data." And, we show a *Unschärferelation (uncertainty principle)* in the canonical extension.

When we think about bounded distributive lattice based logics, e.g. intuitionistic logic, modal logic, relevance logic or distributive modal logic, the Kripke-type semantics, or Routley-Meyer semantics, is obtained as the Stone-dual space of lattice expansions of those logics. However, without the distributivity, the same technique does not work anymore. This is because, if we interpret conjunctions and disjunctions as follows:

1. $w \Vdash \phi \wedge \psi \iff w \Vdash \phi$ and $w \Vdash \psi$,

2. $w \Vdash \phi \vee \psi \iff w \Vdash \phi$ or $w \Vdash \psi$,

$w \Vdash \phi \wedge (\psi \vee \chi)$ always implies $w \Vdash (\phi \wedge \psi) \vee (\phi \wedge \psi)$. That is, the above interpretation always validates the distributive law. Then, how can we consider a space-based semantics, or a relational-type semantics, for lattice-based logics in general? To give a possible answer to this question, in Chapter 8, we introduce *bi-approximation semantics*, a two sorted relational-type semantic, for substructural logic via the canonical extensions of lattice expansions to characterise Ghilardi and Meloni's parallel computation. As a result, we can also prove the first-order definability, which completes Sahlqvist argument for substructural logic [80].

Finally, we summarise the results in this dissertation and give some future works in Chapter 9.

1.3 Main results

The main results in this dissertation are the following.

In Chapter 3, we extend Ghilardi and Meloni's canonicity methodology to lattice expansions and inequalities.

Main Theorem (for lattice expansions)**.** *Let s, t be terms over lattice expansions. An inequality $s \leq t$ is canonical, if it has consistent variable occurrence.*

In Chapter 5, we prove a canonicity result for poset expansions.

Main Theorem (for poset expansions)**.** *Let s, t be terms over poset expansions. An inequality $s \leq t$ is canonical, whenever it satisfies the following two conditions:*

1. *$s \leq t$ has consistent variable occurrence,*

2. *each variable in $s \leq t$ is uniquely signed either in the $-$-signed construction tree of s or in the $+$-signed construction tree of t. Note that these construction trees are not pruned.*

In Chapter 8, we show that substructural logic extended by the canonical inequalities obtained by Theorem 3.3.22 is complete with respect to bi-approximation semantics, which is actually first-order definable [80].

Main Theorem (Sahlqvist-type completeness for substructural logic)**.** *Let Ω be a set of sequents which have consistent variable occurrence (see Main Theorem 3.3.22 and Section 4.1). A substructural logic extended by Ω is complete with respect to a class of p-frames.*

Chapter 2

Ordered structures and canonical extensions

How can we present logical reasonings and compute them? When we consider logical consequences, we quite often accept the following two reasonings:

1. "if A then A" always holds,
2. "if A then B" and "if B then C" imply "if A then C."

It tells us that, if we interpret a logical consequence "if A then B" as a binary relation $A \leq B$, the tuple of the set of propositions and this binary relation forms a *preordered set*. In this chapter, we briefly recall the basic terminology of order theory, algebras relating to some non-classical logics, and Stone representation, Dedekind-MacNeille completions and canonical extensions of ordered sets, which are also strongly connected to non-classical logics.

2.1 Preordered sets, posets and lattices

As preliminaries, we recall the basic terminology of *order theory* for this dissertation, see e.g. [4, 10, 13].

Let P be a set. A binary relation $\leq$ on P is a *preorder*, if it satisfies the following conditions:

1. for each $a \in P$. $a \leq a$, (reflexivity)

2. for all $a, b, c \in P$. if $a \leq b$ and $b \leq c$ then $a \leq c$. (transitivity)

We call a pair $\langle P, \leq \rangle$ of an underlying set P and a preorder $\leq$ on P a *preordered set*. Furthermore, if a preorder $\leq$ on P also satisfies the following condition:

3. for all $a, b \in P$. if $a \leq b$ and $b \leq a$ then $a = b$, (anti-symmetry)

we call $\leq$ a *partial order*, and the pair $\langle P, \leq \rangle$ a *partially ordered set* or a *poset* for short.

Given a poset $\langle P, \leq \rangle$ and a subset $S \subseteq P$, an element $\sup S$ satisfying the following conditions (Items 4 and 5), if it exists in P, is called *the least upper bound of S* or *the supremum of S*:

4. for all $s \in S$. $s \leq \sup S$,

5. for all $a \in P$. if $s \leq a$ for all $s \in S$ then $\sup S \leq a$.

Order dually, an element $\inf S$ satisfying the following conditions (Items 6 and 7), if it exists in P, is called *the greatest lower bound* or *the infimum of S*:

6. for all $s \in S$. $\inf S \leq s$,

7. for all $a \in P$. if $a \leq s$ for all $s \in S$ then $a \leq \inf S$.

The supremum of the empty set $\sup \emptyset$, if it exists in P, we denote it as a constant $\bot$, i.e. $\bot = \sup \emptyset$, called *bottom*. And, the infimum of the empty set $\inf \emptyset$, if it exists in P, we denote it as a constant $\top$, i.e. $\top = \inf \emptyset$, called *top*. If a poset $\langle P, \leq \rangle$ has both bottom and top, we call it *bounded* and sometimes denote the constants clearly as $\langle P, \leq, \top, \bot \rangle$.

Given a poset $\langle P, \leq \rangle$ and arbitrary elements $a, b \in P$, the element $\sup\{a, b\}$, if it exists in P, is *the (binary) join of* a *and* b denoted by $a \vee b$, and the element $\inf\{a, b\}$, if it exists in P, is *the (binary) meet of* a *and* b, denoted by $a \wedge b$. A poset $\langle P, \leq \rangle$ is a *lattice*, if $a \vee b$ and $a \wedge b$ exist in P for arbitrary $a, b \in P$. Furthermore, if a poset $\langle P, \leq \rangle$ has the supremum and the infimum for arbitrary subsets of P, we call it a *complete lattice*.

We sum up our ordered structures as follows.

Preordered set : a set with a reflexive and transitive binary relation

Poset : a set with a reflexive, transitive and anti-symmetric binary relation

Bounded poset : a poset with top and bottom

Lattice : a poset with all binary joins and all binary meets

Bounded lattice : a poset with all finite supremums, which are supremums for finite subsets, and all finite infimums, which are infimums for finite subsets

Complete lattice : a poset with all supremums and all infimums for arbitrary (possibly infinite) subsets

Lattices can be also introduced as algebraic structures as follows.

Definition 2.1.1 (Lattice)**.** A triple $\langle L, \vee, \wedge \rangle$ is a *lattice*, where L is a set, and $\vee$ and $\wedge$ are binary operations, the so-called *lattice operations*, on L satisfying

1. $a \vee b = b \vee a$, $\qquad a \wedge b = b \wedge a$, (commutativity)
2. $a \vee (b \vee c) = (a \vee b) \vee c$, $\qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c$, (associativity)
3. $a \vee (a \wedge b) = a$, $\qquad a \wedge (a \vee b) = a$, (absorption)

for all $a, b, c \in L$.

Remark 2.1.2. To define lattices, we often assume the idempotency for joins and meets, i.e. $a \vee a = a$ and $a \wedge a = a$. However, they are deducible from the absorption laws as follows.

$$a \vee a = a \vee (a \wedge (a \vee b)) = a \qquad a \wedge a = a \wedge (a \vee (a \wedge b)) = a$$

It is known that a lattice $\mathbb{L} = \langle L, \vee, \wedge \rangle$ can be seen as a poset with the following partial order $\leq$ induced by lattice operations: for all $a, b \in L$, we let

$$a \leq b \iff a \vee b = b \iff a \wedge b = a.$$

Next we introduce classes of lattices and lattice expansions which appear in later sections (see Section 3.1). These structures are e.g. in [4, 10, 11, 12, 13, 25].

Definition 2.1.3 (Distributive lattice)**.** A lattice $\mathbb{L} = \langle L, \vee, \wedge \rangle$ is *distributive*, if it satisfies the following conditions, the so-called *distributive laws*:

1. for all $a, b, c \in L$. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
2. for all $a, b, c \in L$. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 2.1.4 (Boolean algebra)**.** A *Boolean algebra* is a bounded distributive lattice $\langle L, \vee, \wedge, \top, \bot \rangle$ with a unary operation $\neg$, the so-called *complement*, which satisfies

1. for all $a \in L$. $a \vee (\neg a) = \top$ and $a \wedge (\neg a) = \bot$,
2. for all $a \in L$. $\neg(\neg a) = a$,
3. for all $a, b \in L$. $\neg(a \wedge b) = (\neg a) \vee (\neg b)$ and $\neg(a \vee b) = (\neg a) \wedge (\neg b)$.

Definition 2.1.5 (Modal algebra)**.** A *modal algebra* $\langle L, \vee, \wedge, \neg, \top, \bot, \Diamond \rangle$ is a Boolean algebra $\langle L, \vee, \wedge, \neg, \top, \bot \rangle$ with a unary operation $\Diamond$ which satisfies

1. for all $a, b \in L$. $\Diamond(a \vee b) = (\Diamond a) \vee (\Diamond b)$,
2. $\Diamond \bot = \bot$.

Definition 2.1.6 (Heyting algebra)**.** A *Heyting algebra* is a bounded distributive lattice $\langle L, \vee, \wedge, \top, \bot \rangle$ with a binary operation $\rightarrow$ which satisfies

$$a \wedge b \leq c \iff b \leq a \rightarrow c, \text{ for all } a, b, c \in L.$$

Definition 2.1.7 (Full Lambek algebra)**.** A *full Lambek algebra*, *FL-algebra* for short, is a 8-tuple $\langle L, \vee, \wedge, \circ, \backslash, /, 1, 0 \rangle$, where $\langle L, \vee, \wedge \rangle$ is a lattice, $\langle L, \circ, 1 \rangle$ is a monoid, 0 is a constant in L, and the binary operations $\circ$, $\backslash$ and $/$ satisfy *the residuation law*: for all $a, b, c \in L$,

$$a \circ b \leq c \iff b \leq a \backslash c \iff a \leq c / b.$$

Definition 2.1.8 (Many-valued algebra)**.** A *many-valued algebra*, *MV-algebra* for short, is a commutative monoid $\langle L, \oplus, e \rangle$ with a unary operation $\neg$ which satisfies

1. for all $a \in L$. $\neg(\neg a) = a$,

2. for all $a \in L$. $a \oplus (\neg e) = \neg e$,

3. for all $a, b \in L$. $(\neg((\neg a) \oplus b)) \oplus b = (\neg((\neg b) \oplus a)) \oplus a$.

Note that there are two ways to prove that MV-algebras are term-equivalent to specific FL-algebras (see [12]).

Hereinafter, to avoid nesting brackets, we sometimes omit brackets with respecting the tightness of operations as usual. That is, unary operations are the most tight, lattice operations are the second most, and the others follow. For example, $\neg a \vee b \backslash c \wedge \neg d$ is a shorthand for $((\neg a) \vee b) \backslash (c \wedge (\neg d))$.

Finally, we recall some fundamental terms in order theory. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. A subset S of P is an *upset*, if it is upward closed, i.e. if $a \leq b$ and $a \in S$ then $b \in S$ for all $a, b \in P$. Order-dually, a subset S of P is a *downset*, if it is downward closed. Note that the empty set $\emptyset$ is an upset and a downset, and the underlying set P is also an upset and a downset. A subset S of P is *down-directed*, if there exists at least one lower bound in S for each pair of elements in S, i.e. for all $a, b \in S$, there exists $c \in S$ such that $c \leq a$ and $c \leq b$. Order-dually, a subset S of P is *up-directed*, if there exists at least one upper bound in S for each pair of elements in S. We call non-empty, down-directed upsets *filters* and non-empty, up-directed downsets *ideals*. Over lattices, our definitions of filters and ideals correspond to the lattice-theoretic filters and ideals. That is, given a lattice $\langle L, \vee, \wedge \rangle$, a subset S of L is a *filter* (an *ideal*), if S is non-empty and upward closed (downward closed), and for all $a, b \in S$, we have $a \wedge b \in S$ ($a \vee b \in S$). Furthermore, over lattices, a filter F (an ideal I) is *prime*, if for all $a, b \in L$. if $a \vee b \in F$ ($a \wedge b \in I$) then $a \in F$ or $b \in F$ ($a \in I$ or $b \in I$). Over Boolean algebras, prime filters (prime ideals) coincide with *maximal filters* (*maximal ideals*). We sometimes call maximal filters *ultrafilters*.

2.2 Stone representation, Dedekind-MacNeille completion and the canonical extension

In this section, we present the main constructions of *Dedekind-MacNeille completions* and *canonical extensions* of ordered structures that we study in this dissertation. In order theory, a *completion* $\overline{\mathbb{P}}$ of an ordered structure $\mathbb{P} = \langle P, \leq \rangle$, e.g. posets or lattices, is a complete lattice on which $\mathbb{P}$ is order-embeddable, i.e. there exists an injective order-preserving map from $\mathbb{P}$ to $\overline{\mathbb{P}}$. Whenever we consider ordered algebraic structures as poset expansions (or lattice expansions), we assume that a completion is not only an order-embeddable but also homomorphic as an algebraic structure. That is, there exists an injective *order-preserving homomorphism* from the original algebraic structure to the completion. For example, a complete lattice $\overline{\mathbb{L}}$ is a completion of a lattice $\mathbb{L}$, if there exists an injective order-preserving homomorphism h which satisfies $h(a \vee_{\mathbb{L}} b) = h(a) \vee_{\overline{\mathbb{L}}} h(b)$ and $h(a \wedge_{\mathbb{L}} b) = h(a) \wedge_{\overline{\mathbb{L}}} h(b)$ for all $a, b \in L$.

Stone representation Let $\mathbb{L} = \langle L, \vee, \wedge, \top, \bot \rangle$ be a bounded distributive lattice, and $\mathcal{P}(\mathbb{L})$, $\mathcal{P}$ for short, the set of all prime filters of $\mathbb{L}$ ordered by inclusion. Then, the set $\mathcal{U}(\mathcal{P})$ of all upsets of $\mathcal{P}$ is a completion of $\mathbb{L}$. We sometimes refer to $\mathcal{P}$ as *the dual space of* $\mathbb{L}$ and denote $\mathcal{P}$ as $\mathbb{L}_+$, and sometimes refer to $\mathcal{U}(\mathcal{P})$ as *the dual algebra of* $\mathcal{P}$ and denote $\mathcal{U}(\mathcal{P})$ as $\mathcal{P}^+$.

Theorem 2.2.1 (Stone representation for bounded distributive lattices [78])**.** *Every bounded distributive lattice* $\mathbb{L}$ *has a completion* $(\mathbb{L}_+)^+$*. The embedding* $\hat{\_} : \mathbb{L} \to (\mathbb{L}_+)^+$

is given by the following: for each $a \in L$, we let

$$\hat{a} := \{P \in \mathcal{P} \mid a \in P\}.$$

Remark 2.2.2. In duality theory between lattices and (ordered) topological spaces, one can find dualities based on Stone representation: Stone duality for Boolean algebras [77], see also [48], Priestley duality for distributive lattices [67], see also [13], and Esakia duality for Heyting algebras [21], see also [3].

Dedekind-MacNeille completion Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. The original *Dedekind-MacNeille completion* $\overline{\mathbb{P}}$ of $\mathbb{P}$ is given by the collection of all subsets S of P satisfying $(S^u)^l = S$, where $\_^u$ and $\_^l$ are defined as follows [60, 4, 13]:

1. $S^u := \{a \in P \mid \forall s \in S.\ s \leq a\}$, (the set of upper bounds of S)

2. $S^l := \{a \in P \mid \forall s \in S.\ a \leq s\}$. (the set of lower bounds of S)

The order on $\overline{\mathbb{P}}$ is the set-inclusion. Intuitively speaking, the Dedekind-MacNeille completion $\overline{\mathbb{P}}$ is the sublattice of $(\_^u)^l$-stable subsets of the powerset complete lattice $\wp(P)$.

However, to focus on a different aspect of the Dedekind-MacNeille completion, we *redefine* it as an abstract (point-free) structure as follows. Let $\mathcal{D}(\mathbb{P})$ be the set of all downsets of $\mathbb{P}$ ordered by inclusion $\subseteq$, and $\mathcal{U}(\mathbb{P})^\partial$ the set of all upsets of $\mathbb{P}$ ordered by the reverse-inclusion $\supseteq$. Note that the superscript $\_^\partial$ points out that $\mathcal{U}(\mathbb{P})^\partial$ is the order dual structure of $\mathcal{U}(\mathbb{P})$ ordered by inclusion $\subseteq$. Between $\mathcal{D}(\mathbb{P})$ and $\mathcal{U}(\mathbb{P})$, we introduce two order-preserving maps $\lambda : \mathcal{D}(\mathbb{P}) \to \mathcal{U}(\mathbb{P})^\partial$ and $\upsilon : \mathcal{U}(\mathbb{P})^\partial \to \mathcal{D}(\mathbb{P})$ as follows: for all $D \in \mathcal{D}(\mathbb{P})$ and all $U \in \mathcal{U}(\mathbb{P})^\partial$, we let

1. $\lambda(D) := \{a \in P \mid \forall d \in D.\ d \leq a\}$, (the set of upper bounds of D)

2. $\upsilon(U) := \{a \in P \mid \forall u \in U.\ a \leq u\}$. (the set of lower bounds of U)

Remark 2.2.3. Note that these maps λ (*lambda*) and υ (*upsilon*) are exactly the same as $\_^u$ and $\_^l$, respectively. Then one may feel that λ should correspond to $\_^l$ not to $\_^u$. However, this is just a matter of taste: in the original notation, S^u means "taking *u*pper bounds of S", whereas $\lambda(D)$ states "constructing the *l*east upper bound of D", which is also explained as "approximation from below (the lower-side)" later. Analogously, $\upsilon(U)$ says "constructing the greatest lower bound of U by approximating from the *u*pper-side." See also Section 7.1.

Proposition 2.2.4 (Galois connection $\lambda \dashv \upsilon$). *λ and υ form a Galois connection, $\lambda \dashv \upsilon$. That is, for all $D \in \mathcal{D}(\mathbb{P})$ and all $U \in \mathcal{U}(\mathbb{P})^\partial$, we have*

$$\lambda(D) \supseteq U \iff D \subseteq \upsilon(U).$$

Recall that the order on $\mathcal{U}(\mathbb{P})^\partial$ is the reverse-inclusion $\supseteq$.

$$\begin{tikzcd} \mathcal{D}(\mathbb{P}) \arrow[r, bend left, "\lambda"] \arrow[r, phantom, "\perp"] & \mathcal{U}(\mathbb{P})^\partial \arrow[l, bend left, "\upsilon"] \end{tikzcd}$$

By a fact in category theory, we obtain the following: see e.g. [55, 6, 58].

Proposition 2.2.5. *The images $\upsilon[\mathcal{U}(\mathbb{P})^\partial]$ and $\lambda[\mathcal{D}(\mathbb{P})]$ are isomorphic via the Galois connection $\lambda \dashv \upsilon$, i.e. $\upsilon[\mathcal{U}(\mathbb{P})^\partial] \cong \lambda[\mathcal{D}(\mathbb{P})]$.*

The following lemma is also obtained by a fact of Galois connections.

Lemma 2.2.6. *The original Dedekind-MacNeille completion $(\mathbb{P}^u)^l$ is isomorphic to $\upsilon[\mathcal{U}(\mathbb{P})^\partial]$ and to $\lambda[\mathcal{D}(\mathbb{P})]$.*

Thanks to Lemma 2.2.6, we can *re-define* Dedekind-MacNeille completions as follows.

Definition 2.2.7 (Dedekind-MacNeille completion)**.** An abstract ordered structure $\overline{\mathbb{P}}$ is a *Dedekind-MacNeille completion of* $\mathbb{P}$, when there exist two isomorphisms $\_^{\downarrow} : \overline{\mathbb{P}} \to \upsilon[\mathcal{U}(\mathbb{P})^{\partial}]$ and $\_{\uparrow} : \overline{\mathbb{P}} \to \lambda[\mathcal{D}(\mathbb{P})]$ which make the following diagram commute.

$$
\begin{tikzcd}
 & \overline{\mathbb{P}} \arrow[dl, "\_^{\downarrow}"'] \arrow[dr, "\_{\uparrow}"] & \\
\upsilon[\mathcal{U}(\mathbb{P})^{\partial}] \arrow[rr, bend left, "\lambda"] \arrow[rr, phantom, "\cong"] & & \lambda[\mathcal{D}(\mathbb{P})] \arrow[ll, bend left, "\upsilon"]
\end{tikzcd}
$$

As we saw in Lemma 2.2.6, the two definitions of Dedekind-MacNeille completions coincide. But, our definition of Dedekind-MacNeille completions claims "every element of $\overline{\mathbb{P}}$ is *approximated* both from above and from below (*bi-approximation*)."

Remark 2.2.8. The Dedekind-MacNeille completions in Definition 2.2.7 are, by definition, unique up to isomorphism. Therefore, hereinafter, we call them *the Dedekind-MacNeille completion.* Moreover, for every poset $\mathbb{P}$, the existence of the Dedekind-MacNeille completion $\overline{\mathbb{P}}$ is guaranteed by $\upsilon[\mathcal{U}(\mathbb{P})^{\partial}]$ and $\lambda[\mathcal{D}(\mathbb{P})]$.

Canonical extensions The canonical extension of ordered algebraic structures is a completion which is originally given by Stone representation and is also closely related to Henkin's canonical model in modal logic. More precisely, given a modal logic $\mathbf{L}$, the Stone space (dual space) $\mathcal{L}_+$ of the Lindenbaum-Tarski algebra $\mathcal{L}$ for $\mathbf{L}$ corresponds to Henkin's canonical model for $\mathbf{L}$. The study of canonical extensions of Boolean algebras with operators has already appeared in [50, 51] based on Stone representation for Boolean algebras. In this paragraph, to apply this type of completions to poset expansions in general, we introduce canonical extensions of posets. Recall that Stone representation only works over bounded distributive lattices with Axiom of Choice. One can find that the construction itself has already appeared in [4]. But, we mention that the same (or the closely related) construction is also in [85, 35, 86, 2, 33, 44, 41, 27, 18]. Here, to introduce *Ghilardi and Meloni's parallel computation*, which is the main technique of our canonicity methodology, we focus especially on the construction of the canonical extension in [33].

Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. We denote by $\mathcal{F}(\mathbb{P})$, $\mathcal{F}$ for short, and $\mathcal{I}(\mathbb{P})$, $\mathcal{I}$ for short, the set of all filters of $\mathbb{P}$ and the set of all ideals of $\mathbb{P}$, respectively. On the union $\mathcal{F} \cup \mathcal{I}$, of $\mathcal{F}$ and $\mathcal{I}$, we define a binary relation $\sqsubseteq$ as follows: for each $F, G \in \mathcal{F}$ and $I, J \in \mathcal{I}$, we let

1. $F \sqsubseteq G \iff F \supseteq G$,
2. $I \sqsubseteq J \iff I \subseteq J$,
3. $F \sqsubseteq I \iff F \cap I \neq \emptyset$,
4. $I \sqsubseteq F \iff \forall i \in I, \forall f \in F.\ i \leq f$.

It is straightforwardly proved that $\sqsubseteq$ is a partial order on both $\mathcal{F}$ and $\mathcal{I}$ but *not* on $\mathcal{F} \cup \mathcal{I}$, because $\sqsubseteq$ is not anti-symmetric on $\mathcal{F} \cup \mathcal{I}$. More precisely, for each $a \in \mathbb{P}$, the principal filter $\uparrow a := \{b \mid a \leq b\}$ and the principal ideal $\downarrow a := \{b \mid b \leq a\}$ satisfy $\uparrow a \sqsubseteq \downarrow a$ and $\downarrow a \sqsubseteq \uparrow a$, but $\uparrow a \neq \downarrow a$. To make the binary relation $\sqsubseteq$ on $\mathcal{F} \cup \mathcal{I}$ anti-symmetric, we define the equivalence relation $\sim$ as follows: for all $X, Y \in \mathcal{F} \cup \mathcal{I}$, we let $X \sim Y \iff X \sqsubseteq Y$ and $Y \sqsubseteq X$. Note that the equivalence relation identifies each principal filter with the principal ideal generated by the same element.

Definition 2.2.9 (Intermediate level)**.** Let $\mathbb{P}$ be a poset. The quotient poset of $\mathcal{F} \cup \mathcal{I}$ with respect to $\sim$ is *the intermediate level*, denoted by $\mathcal{F} +_{\mathbb{P}} \mathcal{I}$.

Next, to construct the canonical extension of $\mathbb{P}$, we take the Dedekind-MacNeille completion of $\mathcal{F} +_{\mathbb{P}} \mathcal{I}$. That is, we define a Galois connection $\lambda \dashv \upsilon$ between the set $\mathcal{D}(\mathcal{F})$ of all downsets of $\langle \mathcal{F}, \sqsubseteq \rangle$ and the set $\mathcal{U}(\mathcal{I})^\partial$ of all upsets of $\langle \mathcal{I}, \sqsubseteq \rangle$ as follows. For each $\mathfrak{F} \in \mathcal{D}(\mathcal{F})$ and each $\mathfrak{I} \in \mathcal{U}(\mathcal{I})^\partial$, we let

1. $\lambda(\mathfrak{F}) := \{I \in \mathcal{I} \mid \forall F \in \mathfrak{F}.\ F \sqsubseteq I\}$,

2. $\upsilon(\mathfrak{I}) := \{F \in \mathcal{F} \mid \forall I \in \mathfrak{I}.\ F \sqsubseteq I\}$.

$$\begin{tikzcd}
\mathcal{D}(\mathcal{F}) \arrow[r, bend left, "\lambda"] \arrow[r, phantom, "\perp"] & \mathcal{U}(\mathcal{I})^\partial \arrow[l, bend left, "\upsilon"]
\end{tikzcd}$$

Henceforward, we denote the image of λ and the image of υ as $\mathbb{U}_\lambda$ and $\mathbb{D}_\upsilon$, i.e. $\mathbb{D}_\upsilon = \upsilon[\mathcal{U}(\mathcal{I})^\partial]$ and $\mathbb{U}_\lambda = \lambda[\mathcal{D}(\mathcal{F})]$. Then, we define the canonical extension of posets as follows.

Definition 2.2.10 (Canonical extension)**.** Let $\mathbb{P}$ be a poset. A triple $\langle \overline{\mathbb{P}}, \_^{\downarrow}, \_{}_{\uparrow} \rangle$, $\overline{\mathbb{P}}$ for short, is *the canonical extension of* $\mathbb{P}$, if $\overline{\mathbb{P}}$ is the Dedekind-MacNeille completion of the intermediate level $\mathcal{F} +_{\mathbb{P}} \mathcal{I}$. In other words, there exist two isomorphisms $\_^{\downarrow} : \overline{\mathbb{P}} \to \mathbb{D}_\upsilon$ and $\_{}_{\uparrow} : \overline{\mathbb{P}} \to \mathbb{U}_\lambda$, and $\langle \overline{\mathbb{P}}, \_^{\downarrow}, \_{}_{\uparrow} \rangle$ makes the following diagram commute.

$$\begin{tikzcd}
& \overline{\mathbb{P}} \arrow[dl, "_^{\downarrow}"'] \arrow[dr, "_{}_{\uparrow}"] & \\
\mathbb{D}_\upsilon \arrow[rr, bend left, "\lambda"] \arrow[rr, phantom, "\cong"] & & \mathbb{U}_\lambda \arrow[ll, bend left, "\upsilon"]
\end{tikzcd}$$

Note that the existence of canonical extensions is trivial, because we can take $\langle \mathbb{D}_\upsilon, id, \lambda \rangle$ or $\langle \mathbb{U}_\lambda, \upsilon, id \rangle$ as a canonical extension.

Remark 2.2.11.

1. The reason we define a triple $\langle \overline{\mathbb{P}}, \_^{\downarrow}, \_{\uparrow} \rangle$ as the canonical extension is that it gives us the possibility to calculate simultaneously in $\mathbb{D}_{\upsilon}$ and in $\mathbb{U}_{\lambda}$: see also Section 7.2.

2. When we consider the class of bounded distributive lattices, we can define two adjoint pairs among $\mathcal{D}(\mathcal{F})$, $\mathcal{U}(\mathcal{P})$ and $\mathcal{U}(\mathcal{I})^{\partial}$, where $\mathcal{P}$ is the set of all prime filters and $\mathcal{U}(\mathcal{P})$ is the set of all upset of $\langle \mathcal{P}, \subseteq \rangle$, as follows: for each $\mathfrak{F} \in \mathcal{D}(\mathcal{F})$, each $\mathfrak{P} \in \mathcal{U}(\mathcal{P})$ and each $\mathfrak{I} \in \mathcal{U}(\mathcal{I})^{\partial}$, we let

   (a) $\lambda_1(\mathfrak{F}) := \{P \in \mathcal{P} \mid \exists F \in \mathfrak{F}.\ F \subseteq P\}$,

   (b) $\lambda_2(\mathfrak{P}) := \{I \in \mathcal{I} \mid \forall P \in \mathcal{P}.\ P \in \mathfrak{P} \Longrightarrow P \cap I \neq \emptyset\}$,

   (c) $\upsilon_1(\mathfrak{P}) := \{F \in \mathcal{F} \mid \forall P \in \mathcal{P}.\ F \subseteq P \Longrightarrow P \in \mathfrak{P}\}$,

   (d) $\upsilon_2(\mathfrak{I}) := \{P \in \mathcal{P} \mid \forall I \in \mathfrak{I}.\ P \cap I \neq \emptyset\}$.

$$\begin{tikzcd} \mathcal{D}(\mathcal{F}) \arrow[r, bend left, "\lambda_1"] \arrow[r, phantom, "\perp"] & \mathcal{U}(\mathcal{P}) \arrow[l, bend left, "\upsilon_1"] \arrow[r, bend left, "\lambda_2"] \arrow[r, phantom, "\perp"] & \mathcal{U}(\mathcal{I})^{\partial} \arrow[l, bend left, "\upsilon_2"] \end{tikzcd}$$

   Then, with the Prime filter theorem, e.g. [13], or equivalently the Axiom of Choice, we can prove that the images $\mathbb{D}_{\upsilon}$ and $\mathbb{U}_{\lambda}$ are isomorphic to the standard canonical extension given by Stone representation, e.g. [29].

Remark 2.2.12. With the terminology in [29, 41] taken over from [50, 51], $\alpha^{\downarrow} \in \mathbb{D}_{\upsilon}$ and $\alpha_{\uparrow} \in \mathbb{U}_{\lambda}$ are explained as *a join of closed elements* and *a meet of open elements*, respectively: see also Section 7.1. However, to build the parallel computation, it is necessary for us to make the clear distinction of these two directions of the approximation, e.g. $\alpha^{\downarrow}$ (approximated from above) and $\alpha_{\uparrow}$ (approximated from below).

We sum up the construction and relations between structures as follows.

$$
\begin{tikzcd}
& \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{D}(\mathcal{F}) \arrow[dd, bend left, "\lambda"] \arrow[dd, phantom, "\vdash"] & \mathbb{D}_\upsilon \arrow[l, hook'] \arrow[dd, bend left, "\lambda"] \arrow[dd, phantom, "\cong"] & \\
\mathbb{P} \arrow[ur, hook, "\uparrow"] \arrow[rr, hook] \arrow[dr, hook, "\downarrow"'] & & \mathcal{F} +_{\mathbb{P}} \mathcal{I} & & & \overline{\mathbb{P}} \arrow[ul, bend right, ".\downarrow"'] \arrow[dl, bend left, "\dot{}\uparrow"] \arrow[l, phantom, "\cong"] \\
& \mathcal{I} \arrow[rr] \arrow[ur] & & \mathcal{U}(\mathcal{I})^\partial \arrow[uu, bend left, "\upsilon"] & \mathbb{U}_\lambda \arrow[l, hook'] \arrow[uu, bend left, "\upsilon"] &
\end{tikzcd}
$$

We can define the following two canonical embeddings $\grave{\_} : \mathbb{P} \to \mathbb{D}_\upsilon$ and $\acute{\_} : \mathbb{P} \to \mathbb{U}_\lambda$ as follows. For each $a \in \mathbb{P}$, we let

1. $\grave{a} := \{F \in \mathcal{F} \mid a \in F\}$,

2. $\acute{a} := \{I \in \mathcal{I} \mid a \in I\}$.

2.3 Bi-approximation and bases

It is a fact that, for each poset, we can characterise the canonical extensions of posets with a topological terminology and prove they are unique up to isomorphism (see Section 7.1). Nevertheless, in Definition 2.2.10, we introduced the canonical extensions of posets as an abstract (point-free) complete lattice which is isomorphic to both $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$. This is because, to prove canonicity, we would like to come always back to $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$ to compute term functions on canonical extensions. We call this property that canonical extensions are isomorphic to both $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$ *bi-directional approximation*, or *bi-approximation* for short. Note that this property is topologically explained as *denseness* (see Section 7.1).

Bi-approximation in canonical extensions Given a poset $\mathbb{P}$, the canonical extension $\overline{\mathbb{P}}$ is an abstract structure which is isomorphic to both $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$.

$$\begin{tikzcd}
 & \overline{\mathbb{P}} \arrow[dl, "\text{_}^{\downarrow}"'] \arrow[dr, "\text{_}_{\uparrow}"] & \\
\mathbb{D}_\upsilon \arrow[rr, bend left=10, "\lambda"] \arrow[rr, phantom, "\cong"] & & \mathbb{U}_\lambda \arrow[ll, bend left=10, "\upsilon"]
\end{tikzcd}$$

Our setting allows us to reason about $\overline{\mathbb{P}}$ both in $\mathbb{D}_\upsilon$ and in $\mathbb{U}_\lambda$ in parallel. That is, every element $\alpha \in \overline{\mathbb{P}}$ can be seen as an element $\alpha^{\downarrow} \in \mathbb{D}_\upsilon$, that is a downset of filters, and as an element $\alpha_{\uparrow} \in \mathbb{U}_\lambda$, that is an upset of ideals. Since each element in $\mathbb{D}_\upsilon$ is an image of an upset $\mathfrak{I}$ of ideals, namely $\alpha^{\downarrow} = \upsilon(\mathfrak{I})$, we call $\mathfrak{I}$ a *(ideal) basis of* α, and we also say that α is *approximated by* $\mathfrak{I}$. Analogously, if $\alpha_{\uparrow} = \lambda(\mathfrak{F})$ for some downset $\mathfrak{F}$ of filters, we call $\mathfrak{F}$ a *(filter) basis of* α, and we also say that α is *approximated by* $\mathfrak{F}$. Note that the superscript $\text{\_}^{\downarrow}$ and the subscript $\text{\_}_{\uparrow}$ mean that $\alpha^{\downarrow}$ is approximated from the upper-side and $\alpha_{\uparrow}$ is approximated from the lower-side: see also Remark 2.2.3 and Section 7.1.

For every poset, we can prove the following.

Proposition 2.3.1.

1. *For each $\alpha \in \overline{\mathbb{P}}$, we have $\alpha^{\downarrow} = \upsilon(\alpha_{\uparrow})$ and $\alpha_{\uparrow} = \lambda(\alpha^{\downarrow})$.*

2. *For all $\alpha, \beta \in \overline{\mathbb{P}}$, we have that $\alpha \leq \beta \iff \alpha^{\downarrow} \subseteq \beta^{\downarrow} \iff \alpha_{\uparrow} \supseteq \beta_{\uparrow}$.*

3. *For any $\mathfrak{F} \in \mathcal{D}(\mathcal{F})$ and any $\mathfrak{I} \in \mathcal{U}(\mathcal{I})^{\partial}$, if $\alpha_{\uparrow} = \lambda(\mathfrak{F})$ ($\mathfrak{F}$ is a basis of α) and $\beta^{\downarrow} = \upsilon(\mathfrak{I})$ ($\mathfrak{I}$ is a basis of β), we have*

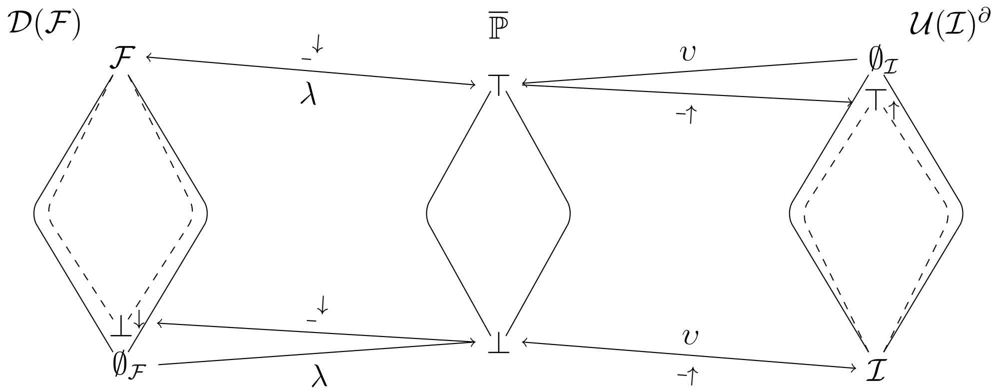
$$\alpha \leq \beta \iff \forall F \in \mathfrak{F}, \forall I \in \mathfrak{I}.\ F \sqsubseteq I.$$

4. *For each $\alpha \in \overline{\mathbb{P}}$, we have that $\alpha^{\downarrow}$ is a $\sqsubseteq$-downset and $\alpha_{\uparrow}$ is an $\sqsubseteq$-upset.*

Remark 2.3.2. In addition to Item 4 in Proposition 2.3.1, if $\mathbb{P}$ is a lattice, we can also state that, for every $\alpha \in \overline{\mathbb{P}}$, $\alpha^{\downarrow}$ is an ideal of $\langle \mathcal{F}, \sqsubseteq \rangle$, and $\alpha_{\uparrow}$ is a filter of $\langle \mathcal{I}, \sqsubseteq \rangle$, where joins on $\mathcal{F}$ and meets on $\mathcal{I}$ are the set-theoretical intersection, see also Definition 3.2.1.

The non-empty basis and boundedness When we consider the canonical extension $\overline{\mathbb{P}}$ of a poset $\mathbb{P}$, we notice the presence of *empty bases*, i.e. the empty set $\emptyset_{\mathcal{F}}$ of filters (the empty filter basis) and the empty set $\emptyset_{\mathcal{I}}$ of ideals (the empty ideal basis). In general, the empty filter basis is a basis of the bottom $\perp$ in $\overline{\mathbb{P}}$ and the empty ideal basis is a basis of the top $\top$ in $\overline{\mathbb{P}}$, but they may not be unique. That is, there may be some *non-empty* other downsets of filters, especially $\perp^{\downarrow}$ and some other *non-empty* upsets of ideals, especially $\top_{\uparrow}$ (Fig. 2.1).

Figure 2.1: Top and bottom in the canonical extension



In our method, the presence of empty bases makes our proofs complex. Not only that, it sometimes affects our technique critically (Chapter 5). Here we give two typical classes of posets where we can assume the non-emptiness of bases.

Bounded posets : for an arbitrary bounded poset $\langle P, \leq, \top, \perp \rangle$, the principal filter

$\uparrow\perp$ intersects with all ideals in $\mathcal{I}$, hence $\lambda(\{\uparrow\perp\}) = \mathcal{I} = \perp_{\uparrow}$. And, the principal ideal $\downarrow\top$ intersects with all filters in $\mathcal{F}$, hence $\upsilon(\{\downarrow\top\}) = \mathcal{F} = \top^{\downarrow}$.

Lattices : for an arbitrary lattice $\langle L, \vee, \wedge\rangle$, the whole underlying set L is non-empty, upward closed, downward closed, closed under finite joins and closed under finite meets. Therefore, L is a filter and an ideal. Because filters and ideals are non-empty, every ideal intersects with L and every filter intersects with L, i.e. $\lambda(\{L\}) = \mathcal{I} = \perp_{\uparrow}$ and $\upsilon(\{L\}) = \mathcal{F} = \top^{\downarrow}$.

Therefore, hereinafter, when we think about bounded posets and lattices, we assume that every basis is non-empty. But, for posets in general, we cannot assume the non-emptiness of bases.

Chapter 3

Canonicity of lattice expansions

In this chapter, we first introduce *lattice expansions*, which uniformly subsume well-known algebraic counterparts of substructural and lattice-based logics. Moreover, we consider canonical extensions of lattice expansions. In this end, we generalise *Ghilardi and Meloni's canonicity methodology* to lattice expansions in general.

3.1 Lattice expansions

Let $\mathbb{P}$ be a poset. The order dual structure is denoted by adding the superscript $\_^{\partial}$ as $\mathbb{P}^{\partial}$. To distinguish the original poset $\mathbb{P}$ from the order dual structure $\mathbb{P}^{\partial}$ clearly, we sometimes denote the original poset by adding the superscript $\_^{1}$ like $\mathbb{P}^{1}$. A n-ary function f on $\mathbb{P}$ is a ϵ*-operation on* $\mathbb{P}$, if there exists a list, called *order-type*, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{1, \partial\}$ for each $i \in \{1, \ldots, n\}$, such that f is a monotone map from the product domain.

$$f : \mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n} \to \mathbb{P}$$

We call a pair of an underlying poset $\mathbb{P}$ and a set of ϵ-operations on $\mathbb{P}$ a *poset expansion*. For example, the lattice operations $\vee : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}$ and $\wedge : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}$ are $(1,1)$-operations on $\mathbb{P}$, hence every lattice is a poset expansion. Furthermore, the lattice operations are uniquely defined by the order $\leq$: for arbitrary $a, b \in L$, $a \leq b \iff a \vee b = b \iff a \wedge b = a$, e.g. [10]. In this chapter, we focus only on *lattice expansions*, which consists of an underlying lattice $\mathbb{L}$ and a set of ϵ-operations on $\mathbb{L}$.

A n-ary ϵ-operation $f : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ is a *ϵ-join preserving operation (ϵ-meet preserving operation)*, if f is a ϵ-operation which is join-preserving (meet-preserving) from the product domain $\mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n}$. Note that this is different from preserving joins (meets) in each coordinate.

Example 3.1.1 (Modal algebra, e.g. [5])**.** A modal algebra $\mathfrak{A} = \langle \mathbb{L}, \neg, \Diamond, \bot \rangle$ is a lattice expansion, where $\langle \mathbb{L}, \neg, \bot \rangle$ is a Boolean algebra, and $\Diamond : \mathbb{L}^1 \to \mathbb{L}$ is a 1-join preserving operation satisfying $\Diamond \bot = \bot$.

Example 3.1.2 (Distributive modal algebra, [30])**.** A distributive modal algebra $\mathfrak{A} = \langle \mathbb{L}, \bot, \top, \Diamond, \Box, \rhd, \lhd \rangle$ is a lattice expansion, where $\langle \mathbb{L}, \bot, \top \rangle$ is a bounded distributive lattice, and

1. $\Diamond : \mathbb{L}^1 \to \mathbb{L}$ is a 1-join preserving operation satisfying $\Diamond \bot = \bot$,

2. $\Box : \mathbb{L}^1 \to \mathbb{L}$ is a 1-meet preserving operation satisfying $\Box \top = \top$,

3. $\rhd : \mathbb{L}^\partial \to \mathbb{L}$ is a ∂-meet preserving operation satisfying $\rhd \bot = \top$,

4. $\lhd : \mathbb{L}^\partial \to \mathbb{L}$ is a ∂-join preserving operation satisfying $\lhd \top = \bot$.

A ϵ-operation is a *ϵ-additive operation (ϵ-multiplicative operation)*, if it is join-preserving (meet-preserving) in each coordinate. Note that joins in order dual structures are meets and meets in order dual structures are joins. For example, $f : \mathbb{L}^{\partial} \times \mathbb{L}^{1} \to \mathbb{L}$ is $(\partial, 1)$-additive, if we have the following two equations: for all $a, x, y \in L$,

1. $f(x \wedge y, a) = f(x, a) \vee f(y, a)$,

2. $f(a, x \vee y) = f(a, x) \vee f(a, y)$.

Among ϵ-additive operations and ϵ-multiplicative operations, we are interested in the following pairs of ϵ-additive operations and ϵ-multiplicative operations.

Definition 3.1.3 (Adjoint pair)**.** Let l be a $(\mu_1, \ldots, \mu_n)$-additive (n-ary) operation and r a $(\nu_1, \ldots, \nu_n)$-multiplicative (n-ary) operation, where, for a fixed coordinate i, $\mu_i = \nu_i = 1$ and, for the other coordinates $k (\neq i)$, μ_k and ν_k are the reverse order, i.e. $\mu_k = \partial$ and $\nu_k = 1$, or $\mu_k = 1$ and $\nu_k = \partial$. l and r form an *adjoint pair with respect to the i-th coordinate*, or simply *adjoint pair*, denoted by $l \dashv^{i} r$, if l and r satisfy the following: for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, x, y \in L$,

$$l(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \leq y \iff x \leq r(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n). \quad (3.1)$$

As in category theory, if a pair of maps l and r satisfies the condition (3.1), we say that l is a *left-adjoint to r* and r is a *right-adjoint to l.*

Proposition 3.1.4 (e.g. [55])**.** *Let l and r be n-ary maps. If l and r form an adjoint pair $l \dashv^{i} r$, l is join-preserving with respect to the i-th coordinate, and r is meet-preserving with respect to the i-th coordinate.*

Remark 3.1.5. Proposition 3.1.4 does not state that l and r are either join-preserving or meet-preserving *from the product domain.*

Note that our adjoint pairs are a parametrised version of the standard *adjointness* in category theory, because we also assume that l is *μ-additive* and r is *ν-multiplicative*, not only for the i-th coordinate. It is necessary later: see Lemma 3.3.8.

When we consider lattice-based logics, the parametrised version of adjoint pairs with μ-additivity and ν-multiplicativity is fundamental (see the following examples).

Example 3.1.6 (Heyting algebra, e.g. [11])**.** A Heyting algebra $\mathfrak{A} = \langle \mathbb{L}, \to, \top, \bot \rangle$ is a lattice expansion, where $\langle \mathbb{L}, \top, \bot \rangle$ is a bounded distributive lattice, and $\wedge$ and $\to$ form an adjoint pair; $\to : \mathbb{L}^{\partial} \times \mathbb{L}^1 \to \mathbb{L}$ is a right-adjoint to $\wedge : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$.

Example 3.1.7 (FL-algebra, e.g. [63])**.** A FL-algebra $\mathfrak{A} = \langle \mathbb{L}, \circ, \backslash, /, 1, 0 \rangle$ is a lattice expansion, where $\mathbb{L}$ is the underlying lattice, $\langle L, \circ, 1 \rangle$ a monoid, 0 an arbitrary constant, and $(1,1)$-additive operation $\circ$, $(\partial, 1)$-multiplicative operation $\backslash$ and $(1, \partial)$-multiplicative operation $/$ form adjoint pairs $\circ \dashv^2 \backslash$ and $\circ \dashv^1 /$.

Example 3.1.8 (B.C$_{\Box\Diamond}$-algebra, [75, 76])**.** We can consider each B.C$_{\Box\Diamond}$-algebra $\mathfrak{A} = \langle \mathbb{L}, \circ, \to, \neg, \Diamond, \Box, \boxdot, \Diamond\!\!\!\cdot\,, 1 \rangle$ as a lattice expansion as follows. $\mathbb{L}$ is the underlying lattice, 1 the left identity element of $\circ$, and

1. $\circ : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$ and $\to : \mathbb{L}^{\partial} \times \mathbb{L}^1 \to \mathbb{L}$ form an adjoint pair $\circ \dashv^1 \to$,
2. $\neg : \mathbb{L}^{\partial} \to \mathbb{L}$ is a ∂-join preserving and ∂-meet preserving operation satisfying $\neg\neg a = a$ for each $a \in L$,
3. $\Diamond : \mathbb{L}^1 \to \mathbb{L}$ is a 1-join preserving operation,
4. $\Box : \mathbb{L}^1 \to \mathbb{L}$ is a 1-meet preserving operation,
5. $\boxdot : \mathbb{L}^1 \to \mathbb{L}$ is a 1-meet preserving operation defined by $\boxdot a = \neg \Diamond \neg a$ for each $a \in L$,

6. $\Diamond : \mathbb{L}^1 \to \mathbb{L}$ is a 1-join preserving operation defined by $\Diamond a = \neg\Box\neg a$ for each $a \in L$.

Note that we should write $\to: \mathbb{L}^1 \times \mathbb{L}^\partial \to \mathbb{L}$ because of $\circ \dashv^1 \to$. But, we adopt the conventional notation $\to: \mathbb{L}^\partial \times \mathbb{L}^1 \to \mathbb{L}$ here.

Many algebraic structures of lattice-based logics are included in our general framework: especially, lattice expansions consisting of ϵ-join preserving operations, ϵ-meet preserving operations, ϵ-additive operations, ϵ-multiplicative operations, adjoint pairs and constants.

3.2 Canonical extensions of lattice expansions

In this section, based on the approach of [33], we extend ϵ-operations on a lattice $\mathbb{L}$ to the canonical extension $\overline{\mathbb{L}}$ *in two steps*. That is, we firstly extend ϵ-operations onto the intermediate level. Then we extend those operations onto $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$ to make them isomorphic. Otherwise, we cannot define the canonical extension of lattice expansions: see Definition 2.2.10.

The canonicity approach of [18] or [27] also uses the extended basic operations on the intermediate level to define the canonical extension of operations. However, as distinct from the approach there, our interest is to lift up *term functions* from a lattice onto the intermediate level, not only basic operations. Then, we face with the fact that the intermediate level is *two-sorted*: $\mathcal{F}$ and $\mathcal{I}$. We achieve to lift up term functions on the two phases, by introducing *the parallel computation* with the following notation: $x \| y$. See also Section 7.2.

To save space, within this section, except Proposition 3.2.10, Proposition 3.2.11 and Abbreviation 3.2.4, we discuss only two types of ϵ-operations $f : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$

and $g : \mathbb{L}^\partial \times \mathbb{L}^1 \to \mathbb{L}$. That is, we focus on a lattice expansion $\langle \mathbb{L}, f, g\rangle$. However, the argument is straightforwardly extended to arbitrary ϵ-operations.

Based on given ϵ-operations, we inductively define terms as usual.

$$term ::= p_i \mid f(term, term) \mid g(term, term),$$

where p_i is a propositional variable. Next we interpret each term t as a function, called a *term function*, $t\colon \mathbb{L} \times \cdots \times \mathbb{L} \to \mathbb{L}$ as follows. For all $x_1, \ldots, x_n \in L$, we let

1. $p_i(x_1, \ldots, x_n) := x_i$,
2. $f(t_1, t_2)(x_1, \ldots, x_n) := f(t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n))$,
3. $g(t_1, t_2)(x_1, \ldots, x_n) := g(t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n))$.

Note that we interpret each propositional variable p_i as the i-th projection map.

Term functions on the intermediate level To construct the parallel computation on the intermediate level, we firstly extend the basic operations f and g to *two types of partial functions on* $\mathcal{F} +_{\mathbb{L}} \mathcal{I}$:

1. $f : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and $f : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$,
2. $g : \mathcal{I} \times \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{F} \times \mathcal{I} \to \mathcal{I}$,

as follows.

Definition 3.2.1 (ϵ-operations on the intermediate level)**.** For all $F, G \in \mathcal{F}$ and $I, J \in \mathcal{I}$, we define

1. $f(F, G) := \{x \in L \mid \exists a \in F, \exists b \in G.\ f(a, b) \leq x\}$,

2. $f(I, J) := \{y \in L \mid \exists a \in I, \exists b \in G.\ y \leq f(a, b)\}$,

3. $g(I, F) := \{x \in L \mid \exists a \in I, \exists b \in F.\ g(a, b) \leq x\}$,

4. $g(F, I) := \{y \in L \mid \exists a \in F, \exists b \in I.\ y \leq g(a, b)\}$.

We can check that these partial functions are well-defined, which is a special case of Proposition 5.2.2. The lattice operations $\vee : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$ and $\wedge : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$ are also extended to the intermediate level, by Definition 3.2.1, as follows: for all $F, G \in \mathcal{F}$ and $I, J \in \mathcal{I}$, we let

1. $F \vee G := \{x \in L \mid \exists a \in F, \exists b \in G.\ a \vee b \leq x\} \ \ (= F \cap G)$,

2. $I \vee J := \{y \in L \mid \exists a \in I, \exists b \in J.\ y \leq a \vee b\} \ \ (= {\downarrow}(I \cup J))$,

3. $F \wedge G := \{x \in L \mid \exists a \in F, \exists b \in G.\ a \wedge b \leq x\} \ \ (= {\uparrow}(F \cup G))$,

4. $I \wedge J := \{y \in L \mid \exists a \in I, \exists b \in J.\ y \leq a \wedge b\} \ \ (= I \cap J)$,

where ${\downarrow}(I \cup J)$ is the ideal generated by $I \cup J$ and ${\uparrow}(F \cup G)$ is the filter generated by $F \cup G$.

Remark 3.2.2. These lattice operations on the intermediate level are partial functions. Therefore, the intermediate level may not be a lattice. For example, we do not define $F \vee I$ nor $I \wedge F$ for any non-principal filter F and any non-principal ideal I. On the other hand, whenever we restrict these operations on $\mathcal{F}$ or $\mathcal{I}$, $\langle \mathcal{F}, \vee, \wedge \rangle$ and $\langle \mathcal{I}, \vee, \wedge \rangle$ form lattices.

If our interest were only to define the canonical extension of basic operations, some functions in Definition 3.2.1 would be redundant. For example, if we want to take f^σ (σ-extension) and g^π (π-extension), they are defined only by Items 1 and

4, see Remark 3.2.7, the definitions of $f_{\uparrow}$ (3.7) and (3.8), and the definition of $g^{\downarrow}$ (3.9) and (3.10). However, in our case, because the main interest is to calculate *term functions on the intermediate level*, all functions defined in Definition 3.2.1 are essential, e.g. see Proposition 3.3.2.

Additionally, to introduce term (partial) functions on the intermediate level, we also introduce the following notation, $P \| N$ where $P, N \in \mathcal{F} +_{\mathbb{L}} \mathcal{I}$, and P and N are in the different sorts. Namely, if $P \in \mathcal{F}$ then $N \in \mathcal{I}$, and conversely, if $P \in \mathcal{I}$ then $N \in \mathcal{F}$. The notation $P \| N$ means P is assigned to *the positive occurrences* and N is assigned to *the negative occurrences*. Then, term (partial) functions on the intermediate level are defined in parallel: for all $F_1, \ldots, F_n \in \mathcal{F}$ and $I_1, \ldots, I_n \in \mathcal{I}$,

$$
\begin{array}{l|l}
p_i(\underline{F\|I}) := F_i & p_i(\underline{I\|F}) := I_i \\
f(t_1,t_2)(\underline{F\|I}) := f(t_1(\underline{F\|I}), t_2(\underline{F\|I})) & f(t_1,t_2)(\underline{I\|F}) := f(t_1(\underline{I\|F}), t_2(\underline{I\|F})) \\
g(t_1,t_2)(\underline{F\|I}) := g(t_1(\underline{I\|F}), t_2(\underline{F\|I})) & g(t_1,t_2)(\underline{I\|F}) := g(t_1(\underline{F\|I}), t_2(\underline{I\|F}))
\end{array}
$$

where $(\underline{F\|I})$ and $(\underline{I\|F})$ are $(F_1\|I_1, \ldots, F_n\|I_n)$ and $(I_1\|F_1, \ldots, I_n\|F_n)$. For example, $g(g(p_1,p_2), f(p_1,p_2))$ is calculated on the two phases in parallel as follows: for all $F, G \in \mathcal{F}$ and $I, J \in \mathcal{I}$,

1. $g(g(p_1,p_2), f(p_1,p_2))(F\|I, G\|J) = g(g(F,J), f(F,G))$,

2. $g(g(p_1,p_2), f(p_1,p_2))(I\|F, J\|G) = g(g(I,G), f(I,J))$.

On the intermediate level, we can straightforwardly prove the monotonicity lemma.

Lemma 3.2.3 (Monotonicity on the intermediate level)**.** *Let t be a term. For all $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathcal{F}$ and $I_1, \ldots, I_n, J_1, \ldots, J_n \in \mathcal{I}$, if $F_i \sqsubseteq G_i$ and $I_i \sqsubseteq J_i$ for* *each* $i \in \{1, \ldots, n\}$*, we have*

$$t(F_1 \| J_1, \ldots, F_n \| J_n) \sqsubseteq t(G_1 \| I_1, \ldots, G_n \| I_n),$$

$$t(I_1 \| G_1, \ldots, I_n \| G_n) \sqsubseteq t(J_1 \| F_1, \ldots, J_n \| F_n).$$

Abbreviation 3.2.4 (Parallel notations)**.** Before moving further, we introduce some abridged notations relating to the parallel notation $\|$. Hereafter, we often encounter arguments with parallel notations, especially in Section 3.3. Moreover, the generality of our theory requires these abbreviations to simplify our discussion.

Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be arbitrary elements. We let

$$\underline{X \| Y} = X_1 \| Y_1, \ldots, X_n \| Y_n. \tag{3.2}$$

Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_n, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_n$ be sets, and f is a ϵ-operation. In $f(Z_1, \ldots, Z_n)$, we assume the following. For each coordinate $k \in \{1, \ldots, n\}$,

$$Z_k \in (\mathfrak{X}_k \| \mathfrak{Y}_k) \iff \begin{cases} Z_k \in \mathfrak{X}_k & \text{if } \epsilon_k = 1 \\ Z_k \in \mathfrak{Y}_k & \text{if } \epsilon_k = \partial. \end{cases} \tag{3.3}$$

In later sections, we introduce term types, e.g. $\cup$-term and $\cap$-term in Definition 3.3.3. For all terms (term functions) $t_1, \ldots, t_n$, if we have a ϵ-operation f, and term types S and T, we let, in $f(t_1, \ldots, t_n)$,

$$t_k \text{ is a } (S \| T)\text{-term} \iff \begin{cases} t_k \text{ is a } S\text{-term} & \text{if } \epsilon_k = 1 \\ t_k \text{ is a } T\text{-term} & \text{if } \epsilon_k = \partial. \end{cases} \tag{3.4}$$

Similarly, $f(t_{(S\|T)}, \ldots, t_{(S\|T)})$ means that the k-th term is a S-term if $\epsilon_k = 1$ and the k-th term is a T-term if $\epsilon_k = \partial$ for every k. Moreover, $f(c, \ldots, t_{(S\|T)}, \ldots, c)$ means that there exists only one coordinate k substituted by t_S if $\epsilon_k = 1$ and by t_T if $\epsilon_k = \partial$, and other coordinates are fixed by constants (constant terms).

Parallel computation on lattices To prove canonicity, it is necessary to build a firm connection between term functions on the original lattice and term functions on the intermediate level. However, since term functions on the intermediate level are two-sorted with the parallel notation $\|$, it is not easy to directly connect them to term functions on the original lattice in general. Here, by introducing the parallel computation for term functions on the original lattice, we obtain an indirect connection between term functions on the original lattice and term functions on the intermediate level: see Lemma 3.2.6.

For all $x_1, \ldots, x_n, y_1, \ldots, y_n \in L$, we let

1. $p_i(\underline{x\|y}) := x_i$,
2. $f(t_1, t_2)(\underline{x\|y}) := f(t_1(\underline{x\|y}), t_2(\underline{x\|y}))$,
3. $g(t_1, t_2)(\underline{x\|y}) := g(t_1(\underline{y\|x}), t_2(\underline{x\|y}))$.

That is, only when we take the order dual elements, e.g. the first argument in g, we swap the left-hand side and the right-hand side. Hence, in $t(x_1\|y_1, \ldots, x_n\|y_n)$, the positive occurrences of p_i are replaced by x_i and the negative occurrences of p_i are replaced by y_i for each propositional variable p_i. In general, we have that $t(x_1, \ldots, x_n) = t(x_1\|y_1, \ldots, x_n\|y_n)$, if all variables appear positively in t. But, whenever we use the same element for both sides, e.g. $x\|x$, we can state that $t(x_1, \ldots, x_n) = t(x_1\|x_1, \ldots, x_n\|x_n)$. We can prove the following monotonicity lemma straightforwardly.

Lemma 3.2.5 (Monotonicity on $\mathbb{L}$)**.** *Let t be a term. For all $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$, $w_1, \ldots, w_n \in L$, if $x_i \le y_i$ and $z_i \le w_i$ for each $i \in \{1, \ldots, n\}$, we have*

$$t(x_1\|w_1, \ldots, x_n\|w_n) \le t(y_1\|z_1, \ldots, y_n\|z_n).$$

The connection between term functions on $\mathbb{L}$ and on the intermediate level is given by the following lemma.

Lemma 3.2.6. *Let t be a term. For all $F_1, \ldots, F_n \in \mathcal{F}$, $I_1, \ldots, I_n \in \mathcal{I}$ and $x, y \in L$, we have*

1. $x \in t(\underline{F\|I}) \iff \forall l \in \{1, \ldots, n\}, \exists a_l \in F_l, \exists b_l \in I_l.\ t(a_1\|b_1, \ldots, a_n\|b_n) \le x,$
2. $y \in t(\underline{I\|F}) \iff \forall l \in \{1, \ldots, n\}, \exists c_l \in I_l, \exists d_l \in F_l.\ y \le t(c_1\|d_1, \ldots, c_n\|d_n).$

Proof. Simultaneous induction on t. Basic cases are trivial. Here, we only check the induction step of Item 1 for $g(t_1, t_2)$. Assume that t_1 and t_2 satisfy Items 1 and 2.

($\Rightarrow$). If $x \in g(t_1, t_2)(\underline{F\|I}) = g(t_1(\underline{I\|F}), t_2(\underline{F\|I}))$, there exist $a \in t_1(\underline{I\|F})$ and $b \in t_2(\underline{I\|F})$ such that $g(a, b) \le x$. By induction hypothesis, for each $l \in \{1, \ldots, n\}$, there exist $f_l, g_l \in F_l$ and $i_l, j_l \in I_l$ such that $a \le t_1(\underline{i\|f})$ and $t_2(\underline{g\|j}) \le b$. Since each F_l is a filter and each I_l is an ideal, there exist $h_l \in F_l$ and $k_l \in I_l$ such that $h_l \le f_l$, $h_l \le g_l$, $i_l \le k_l$ and $j_l \le k_l$. By Lemma 3.2.5, we have that $a \le t_1(\underline{k\|h})$ and $t_2(\underline{h\|k}) \le b$. Therefore, $g(t_1, t_2)(\underline{h\|k}) \le g(a, b) \le x$.

($\Leftarrow$). If there exist $f_l \in F_l$ and $i_l \in I_l$ such that $g(t_1, t_2)(\underline{f\|i}) \le x$, by definition, we have $x \in g(t_1, t_2)(\underline{F\|I})$, because $t_1(\underline{i\|f}) \in t_1(\underline{I\|F})$ and $t_2(\underline{f\|i}) \in t_2(\underline{F\|I})$ by induction hypothesis. $\square$

Term functions on the canonical extension Next we extend ϵ-operations to the canonical extension $\overline{\mathbb{L}}$. Since the canonical extension $\overline{\mathbb{L}}$ is isomorphic to both $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$, it is necessary to extend each ϵ-operation onto $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$ to *make them isomorphic*. Otherwise, we cannot define the canonical extension, see Definition 2.2.10.

In general, we have two types of the extensions, $\_^{\downarrow}$ *approximated from above* and $\_{\uparrow}$ *approximated from below* for each ϵ-operation (*approximation*: see Section 2.3). In other words, since the canonical extension $\overline{\mathbb{L}}$ is isomorphic to two structures $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$, for every ϵ-operation f we have two natural extensions $f^{\downarrow}$ defined on $\mathbb{D}_\upsilon$ (and copied to $\mathbb{U}_\lambda$), and $f_{\uparrow}$ defined on $\mathbb{U}_\lambda$ (and copied to $\mathbb{D}_\upsilon$).

For the $(1,1)$-operation $f : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$, the extension $f^{\downarrow}$ is defined on $\mathbb{D}_\upsilon$:

$$(f^{\downarrow}(\alpha,\beta))^{\downarrow} := \upsilon(\{f(I,J) \mid I \in \alpha_{\uparrow}, J \in \beta_{\uparrow}\}), \tag{3.5}$$

and copied onto $\mathbb{U}_\lambda$:

$$(f^{\downarrow}(\alpha,\beta))_{\uparrow} := \lambda((f^{\downarrow}(\alpha,\beta))^{\downarrow}). \tag{3.6}$$

We mention that $(f^{\downarrow}(\alpha,\beta))^{\downarrow}$ and $(f^{\downarrow}(\alpha,\beta))_{\uparrow}$ are the same operations, but the values are evaluated in the different sorts, $(f^{\downarrow}(\alpha,\beta))^{\downarrow} \in \mathbb{D}_\upsilon$ and $(f^{\downarrow}(\alpha,\beta))_{\uparrow} \in \mathbb{U}_\lambda$, see also Section 2.3. On the other hand, the extension $f_{\uparrow}$ is defined on $\mathbb{U}_\lambda$:

$$(f_{\uparrow}(\alpha,\beta))_{\uparrow} := \lambda(\{f(F,G) \mid F \in \alpha^{\downarrow}, G \in \beta^{\downarrow}\}), \tag{3.7}$$

and copied onto $\mathbb{D}_\upsilon$:

$$(f_{\uparrow}(\alpha,\beta))^{\downarrow} := \upsilon((f_{\uparrow}(\alpha,\beta))_{\uparrow}). \tag{3.8}$$

For the $(\partial, 1)$-operation $g : \mathbb{L}^\partial \times \mathbb{L}^1 \to \mathbb{L}$, the extension $g^\downarrow$ is defined on $\mathbb{D}_\upsilon$:

$$(g^\downarrow(\alpha, \beta))^\downarrow := \upsilon(\{g(F, I) \mid F \in \alpha^\downarrow, I \in \beta_\uparrow\}), \tag{3.9}$$

and copied onto $\mathbb{U}_\lambda$:

$$(g^\downarrow(\alpha, \beta))_\uparrow := \lambda((g^\downarrow(\alpha, \beta))^\downarrow). \tag{3.10}$$

On the other hand, the extension $g_\uparrow$ is defined on $\mathbb{U}_\lambda$:

$$(g_\uparrow(\alpha, \beta))_\uparrow := \lambda(\{g(I, F) \mid I \in \alpha_\uparrow, F \in \beta^\downarrow\}), \tag{3.11}$$

and copied onto $\mathbb{D}_\upsilon$:

$$(g_\uparrow(\alpha, \beta))^\downarrow := \upsilon((g_\uparrow(\alpha, \beta))_\uparrow). \tag{3.12}$$

Note that, by definition, $f^\downarrow$, $f_\uparrow$, $g^\downarrow$ or $g_\uparrow$ are two pairs of two functions (one is on $\mathbb{D}_\upsilon$, e.g. (3.5) and the other is on $\mathbb{U}_\lambda$, e.g. (3.6)) which are always the same functions for *any* ϵ-operation regardless of their properties, like (ϵ-)join preserving, etc.

Conversely, in general, we cannot show that the two types of the extensions $f^\downarrow$ and $f_\uparrow$ agree, nor that the two types of the extensions $g^\downarrow$ and $g_\uparrow$ do. That is, for example, the following two Equations (3.13) and (3.14) may not hold:

$$\upsilon(\{f(I, J) \mid I \in \alpha_\uparrow, J \in \beta_\uparrow\}) = \upsilon(\lambda(\{f(F, G) \mid F \in \alpha^\downarrow, G \in \beta^\downarrow\})), \tag{3.13}$$

$$\lambda(\{f(F, G) \mid F \in \alpha^\downarrow, G \in \beta^\downarrow\}) = \lambda(\upsilon(\{f(I, J) \mid I \in \alpha_\uparrow, J \in \beta_\uparrow\})). \tag{3.14}$$

Therefore, to define the canonical extension of lattice expansions, it is necessary to choose the appropriate extension (the approximating direction $\_^\downarrow$ or $\_{}_\uparrow$) for each operation first.

Remark 3.2.7. The extension $f^{\downarrow}$ coincides with f^{π} (π-extension) and the extension $f_{\uparrow}$ corresponds to f^{σ} (σ-extension) in e.g. [18].

Remark 3.2.8. We cannot define the canonical extensions of *arbitrary* lattice expansions in a uniform way. Namely, for an arbitrary ϵ-operation f, we do not know, in general, which extension $f^{\downarrow}$ or $f_{\uparrow}$ of f is appropriate. But, in substructural logic, the canonical extensions of fusion $\circ$ and residuals $\backslash$ and $/$ have to be $\circ_{\uparrow}$, $\backslash^{\downarrow}$ and $/^{\downarrow}$. Otherwise, the adjointness does not hold on the canonical extension: see [31] or [32].

Once we have obtained canonical extensions of lattice expansions, based on these operations, we inductively define term functions on $\overline{\mathbb{L}}$. Let $\tilde{f}$ be either $f^{\downarrow}$ or $f_{\uparrow}$, and $\tilde{g}$ either $g^{\downarrow}$ or $g_{\uparrow}$. Recall, once more that, before we define term functions on the canonical extension, we *must* decide which extension is chosen for each ϵ-operation. Let $\langle \overline{\mathbb{L}}, \tilde{f}, \tilde{g}\rangle$ be the canonical extension of $\langle \mathbb{L}, f, g\rangle$. Notice that the lattice operations are special cases of f, see also Proposition 3.2.10 and Proposition 3.2.11. For all $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{L}}$, we let

$\mathbb{D}_{\upsilon}$-1 : $(p_i(\alpha_1, \ldots, \alpha_n))^{\downarrow} := {\alpha_i}^{\downarrow}$,

$\mathbb{D}_{\upsilon}$-2 : $(\tilde{f}(t_1, t_2)(\alpha_1, \ldots, \alpha_n))^{\downarrow} := (\tilde{f}(t_1(\alpha_1, \ldots, \alpha_n), t_2(\alpha_1, \ldots, \alpha_n)))^{\downarrow}$,

$\mathbb{D}_{\upsilon}$-3 : $(\tilde{g}(t_1, t_2)(\alpha_1, \ldots, \alpha_n))^{\downarrow} := (\tilde{g}(t_1(\alpha_1, \ldots, \alpha_n), t_2(\alpha_1, \ldots, \alpha_n)))^{\downarrow}$

$\mathbb{U}_{\lambda}$-1 : $(p_j(\alpha_1, \ldots, \alpha_n))_{\uparrow} := \alpha_{i\uparrow}$,

$\mathbb{U}_{\lambda}$-2 : $(\tilde{f}(t_1, t_2)(\alpha_1, \ldots, \alpha_n))_{\uparrow} := (\tilde{f}(t_1(\alpha_1, \ldots, \alpha_n), t_2(\alpha_1, \ldots, \alpha_n)))_{\uparrow}$,

$\mathbb{U}_{\lambda}$-3 : $(\tilde{g}(t_1, t_2)(\alpha_1, \ldots, \alpha_n))_{\uparrow} := (\tilde{g}(t_1(\alpha_1, \ldots, \alpha_n), t_2(\alpha_1, \ldots, \alpha_n)))_{\uparrow}$.

Hereafter, we focus on the canonical extension of constants, ϵ-join preserving operations, ϵ-meet preserving operations, ϵ-additive operations, ϵ-multiplicative operations, and adjoint pairs. For constants, we can straightforwardly prove the following proposition.

Proposition 3.2.9. *Let $\mathbb{L}$ be a lattice, and c a constant in $\mathbb{L}$. Then, the two types of canonical extensions are, on $\mathbb{D}_\upsilon$, $c^{\downarrow} := \{F \in \mathcal{F} \mid c \in F\}$ and, on $\mathbb{U}_\lambda$, $c_{\uparrow} := \{I \in \mathcal{I} \mid c \in I\}$. Moreover, we have that $c^{\downarrow} = \upsilon(c_{\uparrow})$ and $c_{\uparrow} = \lambda(c^{\downarrow})$. If a constant $1 \in \mathbb{L}$ is the identity of a binary $(1,1)$-operation $\circ : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$, then, regardless of the canonical extension of $\circ$ (either of $\circ^{\downarrow}$ and $\circ_{\uparrow}$), $1^{\downarrow}$ is the identity of $\circ$ on $\mathbb{D}_\upsilon$ and $1_{\uparrow}$ is the identity of $\circ$ on $\mathbb{U}_\lambda$.*

The following proposition states that all ϵ-join preserving operations and ϵ-meet preserving operations are *smooth*, or *continuous*, i.e. two types of extensions, σ-extension (approximated from below) and π-extension (approximated from above) coincide: see also [27].

Proposition 3.2.10. *Let $\mathbb{L}$ be a lattice, $f : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ a ϵ-join preserving operation, and $g : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ a ϵ-meet preserving operation. Then, for all $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{L}}$, we have (recall the abbreviation Equation (3.3))*

$$\upsilon(\{f(Y_1, \ldots, Y_n) \mid Y_k \in (\alpha_{k\uparrow} \| \alpha_k{}^{\downarrow})\}) = \upsilon(\lambda(\{f(X_1, \ldots, X_n) \mid X_k \in (\alpha_k{}^{\downarrow} \| \alpha_{k\uparrow})\})), \tag{3.15}$$

$$\lambda(\{g(X_1, \ldots, X_n) \mid X_k \in (\alpha_k{}^{\downarrow} \| \alpha_{k\uparrow})\}) = \lambda(\upsilon(\{g(Y_1, \ldots, Y_n) \mid Y_k \in (\alpha_{k\uparrow} \| \alpha_k{}^{\downarrow})\})). \tag{3.16}$$

Proof. To save space, we assume that $f : \mathbb{L}^\partial \times \mathbb{L}^1 \to \mathbb{L}$ is a ϵ-join preserving operation. That is, we claim Equation (3.15):

$$\upsilon(\{f(F,J) \mid F \in \alpha^{\downarrow}, J \in \beta_{\uparrow}\}) = \upsilon(\lambda(\{f(I,G) \mid I \in \alpha_{\uparrow}, G \in \beta^{\downarrow}\})).$$

But, we can easily generalise for arbitrary cases.

Firstly, we notice that

$$f(I,G) \sqsubseteq f(F,J), \tag{3.17}$$

for all $F \in \alpha^{\downarrow}$, $I \in \alpha_{\uparrow}$, $G \in \beta^{\downarrow}$ and $J \in \beta_{\uparrow}$, since we have that $F \sqsubseteq I$ and $G \sqsubseteq J$ (Proposition 2.3.1). By Equation (3.17), we have

$$f(F,J) \in \lambda(\{f(I,G) \mid I \in \alpha_{\uparrow}, G \in \beta^{\downarrow}\}).$$

Therefore, $\upsilon(\lambda(\{f(I,G) \mid I \in \alpha_{\uparrow}, G \in \beta^{\downarrow}\})) \subseteq \upsilon(\{f(F,J) \mid F \in \alpha^{\downarrow}, J \in \beta_{\uparrow}\})$.

To prove the converse direction, it suffices to show that, there exist $F \in \alpha^{\downarrow}$ and $J \in \beta_{\uparrow}$ such that $f(F,J) \sqsubseteq Y$, for each $Y \in \lambda(\{f(I,G) \mid I \in \alpha_{\uparrow}, G \in \beta^{\downarrow}\})$. Let Y be in $\lambda(\{f(I,G) \mid I \in \alpha_{\uparrow}, G \in \beta^{\downarrow}\})$. Then, for arbitrary $K \in \alpha_{\uparrow}$ and $H \in \beta^{\downarrow}$, we have $f(K,H) \sqsubseteq Y$. Now we define the following two sets.

$$f_1^{-1}(Y,H) := \{x \mid \exists y \in Y, \exists h \in H.\ f(x,g) \leq y\}$$

$$f_2^{-1}(K,Y) := \{x \mid \exists k \in K, \exists y \in Y.\ f(k,x) \leq y\}$$

Next we prove that $f_1^{-1}(Y,H)$ is a filter and $f_2^{-1}(K,Y)$ is an ideal. Firstly $f_1^{-1}(Y,H)$ and $f_2^{-1}(K,Y)$ are non-empty, because of $f(K,H) \sqsubseteq Y$. Since the domain of f is $\mathbb{L}^{\partial} \times \mathbb{L}^{1}$, it is trivial that $f_1^{-1}(Y,H)$ is an upset and $f_2^{-1}(K,Y)$ is a downset.

For all $x_1, x_2 \in f_1^{-1}(Y, H)$ and $x_3, x_4 \in f_2^{-1}(K, Y)$, there exist $y_1, y_2, y_3, y_4 \in Y$, $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $f(x_1, h_1) \leq y_1$, $f(x_2, h_2) \leq y_2$, $f(k_1, x_3) \leq y_3$ and $f(k_2, x_4) \leq y_4$. By the ϵ-join preservability of f, (recall that the domain of f is $\mathbb{L}^\partial \times \mathbb{L}^1$), we have

$$f(x_1 \wedge x_2, h_1 \vee h_2) = f(x_1, h_1) \vee f(x_2, h_2) \leq y_1 \vee y_2,$$

$$f(k_1 \wedge k_2, x_3 \vee x_4) = f(k_1, x_3) \vee f(k_2, x_4) \leq y_3 \vee y_4.$$

Therefore, $x_1 \wedge x_2 \in f_1^{-1}(Y, H)$ and $x_3 \vee x_4 \in f_2^{-1}(K, Y)$. Besides, we can also prove that $f_1^{-1}(Y, H) \sqsubseteq K'$ and $H' \sqsubseteq f_2^{-1}(K, Y)$ for all $K' \in \alpha_\uparrow$ and $H' \in \beta^\downarrow$, because, by definition, we have $f(K, H') \sqsubseteq Y$ and $f(K', H) \sqsubseteq Y$. So, $f_1^{-1}(Y, H) \in \alpha^\downarrow$ and $f_2^{-1}(K, Y) \in \beta_\uparrow$. Finally, we show that

$$f(f_1^{-1}(Y, H), f_2^{-1}(K, Y)) \sqsubseteq Y.$$

Let $a \in f(f_1^{-1}(Y, H), f^{-1}(K, Y))$. By Lemma 3.2.6, there exist $x_1 \in f_1^{-1}(Y, H)$ and $x_2 \in f_2^{-1}(K, Y)$ such that $a \leq f(x_1, x_2)$. Since $x_1 \in f^{-1}(Y, H)$ and $x_2 \in f_2^{-1}(K, Y)$, there exist $y_1, y_2 \in Y$, $h \in H$ and $k \in K$ such that $f(x_1, h) \leq y_1$ and $f(k, x_2) \leq y_2$. By the ϵ-join preservability of f and the monotonicity of f (recall that the domain of f is $\mathbb{L}^\partial \times \mathbb{L}^1$), we have

$$a \leq f(x_1, x_2) \leq f(x_1 \wedge k, h \vee x_2) = f(x_1, h) \vee f(k, x_2) \leq y_1 \vee y_2.$$

Therefore, $a \in Y$. ϵ-meet preserving operations are analogous. $\square$

Moreover, for ϵ-join preserving operations and ϵ-meet preserving operations, we obtain the following proposition.

Proposition 3.2.11. *Let $\mathbb{L}$ be a lattice, $f : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ a ϵ-join preserving operation, and $g : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ a ϵ-meet preserving operation. Then, the extension $f_\uparrow$ of f, approximated from below, is a ϵ-join preserving operation on $\overline{\mathbb{L}}$, and the extension $f^\downarrow$ of f, approximated from above, is a ϵ-meet preserving operation on $\overline{\mathbb{L}}$.*

Proof. To save space, we treat only a $(\partial, 1)$-multiplicative operation $g : \mathbb{L}^\partial \times \mathbb{L}^1 \to \mathbb{L}$. For arbitrary $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \overline{\mathbb{L}}$, we can trivially prove

$$g^\downarrow(\alpha_1 \vee \alpha_2, \beta_1 \wedge \beta_2) \leq g^\downarrow(\alpha_1, \beta_1) \wedge g^\downarrow(\alpha_2, \beta_2).$$

For the converse inequality, by Proposition 2.3.1, it suffices to show that, for each $X \in (g^\downarrow(\alpha_1, \beta_1))^\downarrow \cap (g^\downarrow(\alpha_2, \beta_2))^\downarrow$, we have $X \in (g^\downarrow(\alpha_1 \vee \alpha_2, \beta_1 \wedge \beta_2))^\downarrow$. For arbitrary $X' \in (\alpha_1 \vee \alpha_2)^\downarrow$ and $Y' \in (\beta_1 \wedge \beta_2)_\uparrow$, there exist $F \in {\alpha_1}^\downarrow$, $G \in {\alpha_2}^\downarrow$, $I \in {\beta_1}_\uparrow$ and $J \in {\beta_2}_\uparrow$ such that $X' \sqsubseteq F \vee G$ and $I \wedge J \sqsubseteq Y'$. Furthermore, by $X \sqsubseteq g(F, I)$ and $X \sqsubseteq g(G, J)$, we obtain

$$X \sqsubseteq g(F, I) \wedge g(G, J) = g(F \vee G, I \wedge J) \sqsubseteq g(X', Y').$$

□

For ϵ-additive operations and ϵ-multiplicative operations, we can prove the following.

Proposition 3.2.12. *Let $\mathbb{L}$ be a lattice, $l : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ an ϵ-additive operation, and $r : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$ a ϵ-multiplicative operation. Then, the* *extension $l_\uparrow$ of l, approximated from below, is ϵ-additive on $\overline{\mathbb{L}}$, the extension $r^\downarrow$ of r, approximated from above, is ϵ-multiplicative on $\overline{\mathbb{L}}$.*

Proof. To save space, we focus only on a $(1, \partial)$-additive operation $l : \mathbb{L}^1 \times \mathbb{L}^\partial \to \mathbb{L}$. For arbitrary $\alpha, \beta, \gamma \in \overline{\mathbb{L}}$, we trivially have

$$l_\uparrow(\alpha \vee \beta, \gamma) \geq l_\uparrow(\alpha, \gamma) \vee l_\uparrow(\beta, \gamma).$$

Conversely, by Proposition 2.3.1, we need to show that

$$(l_\uparrow(\alpha \vee \beta, \gamma))_\uparrow \supseteq (l_\uparrow(\alpha, \gamma))_\uparrow \cap (l_\uparrow(\beta, \gamma))_\uparrow.$$

Let Y be an arbitrary element of $(l_\uparrow(\alpha, \gamma))_\uparrow \cap (l_\uparrow(\beta, \gamma))_\uparrow$. For an arbitrary $K \in \gamma_\uparrow$, we define a set

$$l^{-1}(Y, K) := \{x \mid \exists y \in Y, \exists k \in K.\ l(x, k) \leq y\}.$$

We can prove that $l^{-1}(Y, K)$ is an ideal. Moreover, by assumption, we also have that $l^{-1}(Y, K) \in \alpha_\uparrow \cap \beta_\uparrow$, hence, for each $X \in (\alpha \vee \beta)^\downarrow$, we have $X \sqsubseteq l^{-1}(Y, K)$. By definition, it follows that $l(X, K) \sqsubseteq Y$.

$l_\uparrow(\alpha, \beta \wedge \gamma) = l_\uparrow(\alpha, \beta) \wedge l_\uparrow(\alpha, \gamma)$ is analogous. □

Furthermore, for adjoint pairs, we can show the following.

Proposition 3.2.13. *Let $\mathbb{L}$ be an underlying lattice, and a μ-additive operation $l : \mathbb{L}^{\mu_1} \times \cdots \times \mathbb{L}^{\mu_n} \to \mathbb{L}$ and a ν-multiplicative operation $r : \mathbb{L}^{\nu_1} \times \cdots \times \mathbb{L}^{\nu_n} \to \mathbb{L}$ form an adjoint pair with respect to the i-th coordinate $l \dashv^i r$. Then, the extension $l_\uparrow$ of l, approximated from below, and the extension $r^\downarrow$ of r, approximated from above, form* *an adjoint pair with respect to the i-th coordinate on $\overline{\mathbb{L}}$, $l_{\uparrow} \dashv^i r^{\downarrow}$. In other words, left-adjoints defined on $\mathbb{U}_{\lambda}$, approximated from below, and right-adjoints defined on $\mathbb{D}_{\upsilon}$, approximated from above, preserve adjointness on the canonical extension.*

Proof. To save space, we consider $l : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$ and $r : \mathbb{L}^{\partial} \times \mathbb{L}^1 \to \mathbb{L}$ satisfying $l \dashv^2 r$. But, we can easily extend the argument to arbitrary adjoint pairs.

By Proposition 2.3.1, it suffices to prove $(l_{\uparrow}(\gamma, \alpha))_{\uparrow} \supseteq \beta_{\uparrow} \iff \alpha^{\downarrow} \subseteq (r^{\downarrow}(\gamma, \beta))^{\downarrow}$, for all $\alpha, \beta, \gamma \in \overline{\mathbb{L}}$.

We claim that, for all $F, G \in \mathcal{F}$ and $I \in \mathcal{I}$,

$$l(G, F) \sqsubseteq I \iff F \sqsubseteq r(G, I). \tag{3.18}$$

But, it is almost direct from the adjointness on $\mathbb{L}$.

($\Rightarrow$). Let F be an arbitrary element of $\alpha^{\downarrow}$. For all $G \in \gamma^{\downarrow}$ and $I \in \beta_{\uparrow}$, by the assumption, we have that $f(G, F) \sqsubseteq I$. By the condition (3.18), we obtain $F \sqsubseteq r(G, I)$, hence $F \in (r^{\downarrow}(\gamma, \beta))^{\downarrow}$. The reverse direction is analogous. $\square$

3.3 Ghilardi & Meloni's canonicity methodology

In this section, we firstly generalise the approach in [33] from Heyting algebras with unary modalities to arbitrary lattice expansions and from equalities to inequalities, and show the simple but strong key technique of their method. Secondly, we syntactically describe a class of canonical inequalities of lattice expansions which consists of constants, ϵ-join preserving operations, ϵ-meet preserving operations, ϵ-additive operations, ϵ-multiplicative operations and adjoint pairs. We remind that one can apply our technique only after the canonical extensions of all operations in a target lattice expansion are given. That is, we assume that for each ϵ-operation f, its canonical extension is already defined either by $f^{\downarrow}$ or by $f_{\uparrow}$.

Now we define the *canonicity* of inequalities. Hereinafter, we call both terms and term functions simply terms.

Definition 3.3.1 (Canonicity)**.** Let s, t be terms. An inequality $s \leq t$ is *canonical* with respect to a class of lattice expansions, if, for every lattice expansion $\mathbb{L}$ in the class and all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$, $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$ whenever $s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N)$ for all $x_1, \ldots, x_N \in L$.

Thanks to the parallel computation, we can straightforwardly obtain the following proposition, for each term t.

Proposition 3.3.2 (Rough basis)**.** *Let t be an arbitrary term and $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$. We have*

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} \supseteq \{t(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\},$$

$$t(\alpha_1, \ldots, \alpha_N)_{\uparrow} \supseteq \{t(I_1 \| F_1, \ldots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^{\downarrow}\}.$$

Proposition 3.3.2 claims that, for each term t, the set of filters of the form $t(\underline{F \| I})$ is always in $t(\underline{\alpha})^{\downarrow}$ and the set of ideals of the form $t(\underline{I \| F})$ is always in $t(\underline{\alpha})_{\uparrow}$, where $F_k \in {\alpha_k}^{\downarrow}$ and $I_k \in \alpha_{k\uparrow}$: see also Section 7.2.

The following is the central definition, introduced in [33], to obtain the canonicity results.

Definition 3.3.3 ($\cup$-term and $\cap$-term)**.** Let t be a term. t is a $\cup$-*term*, if, for all

$\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$,

$$t(\alpha_1, \ldots, \alpha_N)_\uparrow = \lambda(\{t(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}).$$

t is a $\cap$-*term*, if, for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$,

$$t(\alpha_1, \ldots, \alpha_N)^\downarrow = \upsilon(\{t(I_1 \| F_1, \ldots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^\downarrow\}).$$

$\cup$-terms and $\cap$-terms can be explained with the approximation as follows: for each $\cup$-term t, the value $t(\alpha_1, \ldots, \alpha_N)$ can be approximated from below by the set of filters $t(\underline{F \| I})$, where $F_k \in {\alpha_k}^\downarrow$ and $I_k \in \alpha_{k\uparrow}$. Conversely, if a term t is not a $\cup$-term, the set of filters $t(\underline{F \| I})$, where $F_k \in {\alpha_k}^\downarrow$ and $I_k \in \alpha_{k\uparrow}$, may not be enough to reconstruct the limiting point $t(\alpha_1, \ldots, \alpha_N)$, cf. Proposition 3.3.2. Analogously for $\cap$-terms. See also Section 7.2.

Remark 3.3.4. The important reason to introduce $\cup$-terms and $\cap$-terms is that we can simply prove the canonicity of inequalities $s \leq t$ whenever s is a $\cup$-term and t is a $\cap$-term.

With $\cup$-terms and $\cap$-terms, we obtain the following canonicity results for arbitrary lattice expansions. Note that this is a simple version of Theorem 3.3.22.

Theorem 3.3.5. *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s is a $\cup$-term and t is a $\cap$-term.*

Proof. Let $\mathbb{L}$ be an arbitrary lattice expansion. We need to show the following: whenever $s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N)$ for all $x_1, \ldots, x_N \in \mathbb{L}$, we have that $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$ for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$.

Suppose that $s(x_1, \dots, x_N) \leq t(x_1, \dots, x_N)$ for all $x_1, \dots, x_N \in \mathbb{L}$. For arbitrary $\alpha_1, \dots, \alpha_N \in \overline{\mathbb{L}}$, since s is a $\cup$-term and t is a $\cap$-term, we have

$$s(\alpha_1, \dots, \alpha_N)_{\uparrow} = \lambda(\{s(F_1 \| I_1, \dots, F_N \| I_N) \mid F_k \in \alpha_k{}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}),$$

$$t(\alpha_1, \dots, \alpha_N)^{\downarrow} = \upsilon(\{t(I_1 \| F_1, \dots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in \alpha_k{}^{\downarrow}\}).$$

By Proposition 2.3.1, it suffices to show, for arbitrary $F_k, G_k \in \alpha_k{}^{\downarrow}$ and $I_k, J_k \in \alpha_{k\uparrow}$,

$$s(F_1 \| I_1, \dots, F_N \| I_N) \sqsubseteq t(J_1 \| G_1, \dots, J_N \| G_N).$$

Since each $\alpha_k{}^{\downarrow}$ is an ideal and each $\alpha_{k\uparrow}$ is a filter by Proposition 2.3.1, for all $F_k, G_k \in \alpha_k{}^{\downarrow}$ and $I_k, J_k \in \alpha_{k\uparrow}$, there exist $F_k \vee G_k \in \alpha_k{}^{\downarrow}$ and $I_k \wedge J_k \in \alpha_{k\uparrow}$. By Lemma 3.2.3, we obtain that

$$s(F_1 \| I_1, \dots, F_N \| I_N) \sqsubseteq s((F_1 \vee G_1) \| (I_1 \wedge J_1), \dots, (F_N \vee G_N) \| (I_N \wedge J_N)),$$

$$t((I_1 \wedge J_1) \| (F_1 \vee G_1), \dots, (I_N \wedge J_N) \| (F_N \vee G_N)) \sqsubseteq t(J_1 \| G_1, \dots, J_N \| G_N).$$

Here, for each $k \in \{1, \dots, N\}$, we have $F_k \vee G_k \sqsubseteq I_k \wedge J_k$, i.e. $(F_k \vee G_k) \cap (I_k \wedge J_k) \neq \emptyset$, so let x_k be an element in $(F_k \vee G_k) \cap (I_k \wedge J_k)$. Then, by assumption, we obtain

$$s(x_1 \| x_1, \dots, x_N \| x_N) = s(x_1, \dots, x_N) \leq t(x_1, \dots, x_N) = t(x_1 \| x_1, \dots, x_N \| x_N).$$

Therefore, we conclude $s(F_1 \| I_1, \dots, F_N \| I_N) \sqsubseteq t(J_1 \| G_1, \dots, J_N \| G_N)$. $\square$

Remark 3.3.6. Theorem 3.3.5 is a general fact for arbitrary lattice expansions. However, we have not discussed how to recognise $\cup$-terms and $\cap$-terms, yet. Now, we start to focus on lattice expansions which consist of constants, ϵ-join preserving operations, ϵ-meet preserving operations, ϵ-additive operations, ϵ-multiplicative operations and adjoint pairs. Then, we can syntactically describe large classes of $\cup$-terms and $\cap$-terms. Furthermore, we can also extend Theorem 3.3.5 to cover a larger class of canonical inequalities, to Theorem 3.3.22.

Syntactic description of canonical inequalities Hereafter, we consider a lattice expansion $\langle \mathbb{L}, f, g, \neg, l, r, c \rangle$, where

1. $f : \mathbb{L}^{\delta_1} \times \cdots \times \mathbb{L}^{\delta_m} \to \mathbb{L}$ is an arbitrary δ-join preserving operation on $\mathbb{L}$,
2. $g : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_{m'}} \to \mathbb{L}$ is an arbitrary ϵ-meet preserving operation on $\mathbb{L}$,
3. $\neg : \mathbb{L}^{\partial} \to \mathbb{L}$ is an arbitrary ∂-join-preserving and ∂-meet-preserving operation satisfying $\neg\neg x = x$ for each $x \in L$, so-called an *involution*,
4. $l : \mathbb{L}^{\mu_1} \times \cdots \times \mathbb{L}^{\mu_n} \to \mathbb{L}$ is an arbitrary μ-additive operation,
5. $r : \mathbb{L}^{\nu_1} \times \cdots \times \mathbb{L}^{\nu_n} \to \mathbb{L}$ is an arbitrary ν-multiplicative operation,
6. c is an arbitrary constant in $\mathbb{L}$,
7. the arities m,m' and n are less than a natural number N, and we assume that N is always large enough for any arity.

Note that the lattice operations $\vee$ and $\wedge$ are special cases of f and g, respectively, and an involution is a special case of both f and g. Recall that adjoint pairs are special cases of μ-additive operations and ν-multiplicative operations. When we need to assume that l and r form an adjoint pair for a fixed coordinate i, we emphasise it by $l \dashv^i r$. This is the only reason we introduce l and r with the same arity

n. By Propositions 3.2.9, 3.2.10, 3.2.11, 3.2.12 and 3.2.13, we define the canonical extension of the lattice expansion as $\langle \overline{\mathbb{L}}, \vee_{\uparrow}, \wedge^{\downarrow}, f_{\uparrow}, g^{\downarrow}, \neg, l_{\uparrow}, r^{\downarrow}, c \rangle$. Here, we denote neither $\neg_{\uparrow}$, $c_{\uparrow}$ nor $\neg^{\downarrow}$, $c^{\downarrow}$, because $\neg$ and c are not only smooth but also *unbiased*: cf. $\vee$, $\wedge$, f or g are smooth but *biased* (see Proposition 3.2.10). In fact, without the adjointness of $l \dashv^{i} r$, we have not had any counterexample to show that $l^{\downarrow}$ and $r_{\uparrow}$ should not be canonical extensions, yet. However, Proposition 3.2.12 supports to define $\langle \overline{\mathbb{L}}, \vee_{\uparrow}, \wedge^{\downarrow}, f_{\uparrow}, g^{\downarrow}, \neg, l_{\uparrow}, r^{\downarrow}, c \rangle$ as *the canonical extension* without the adjointness of l and r. In other words, even if l and r do not form any adjoint pair, we call it the canonical extension.

By definition, we obtain the following lemma.

Lemma 3.3.7. *All constants and propositional variables are both $\cup$-terms and $\cap$-terms. Moreover, every term built up only from constants (without variables) is also both a $\cup$-term and a $\cap$-term.*

We call terms without variables *constant terms*. It is trivial that every constant term is a constant. Therefore, we sometimes do not distinguish constant terms from constants.

Lemma 3.3.8. *Let $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$, $\mathfrak{F}_1, \ldots, \mathfrak{F}_N \in \wp(\mathcal{F})$, and $\mathfrak{I}_1, \ldots, \mathfrak{I}_N \in \wp(\mathcal{I})^{\partial}$. If, for each $k \in \{1, \ldots, N\}$, $\mathfrak{F}_k$ and $\mathfrak{I}_k$ are bases of α_k, then we have (recall Abbreviation 3.2.4)*

1. $(\vee_{\uparrow}(\alpha_1, \alpha_2))_{\uparrow} = \lambda(\{F_1 \vee F_2 \mid F_1 \in \mathfrak{F}_1, F_2 \in \mathfrak{F}_2\})$,
2. $(\wedge^{\downarrow}(\alpha_1, \alpha_2))^{\downarrow} = \upsilon(\{I_1 \wedge I_2 \mid I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\})$,
3. $(f_{\uparrow}(\alpha_1, \ldots, \alpha_m))_{\uparrow} = \lambda(\{f(X_1, \ldots, X_m) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$,
4. $(g^{\downarrow}(\alpha_1, \ldots, \alpha_{m'}))^{\downarrow} = \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\mathfrak{I}_k \| \mathfrak{F}_k)\})$,

5. $\neg(\alpha_1)_\uparrow = \lambda(\{\neg I_1 \mid I_1 \in \mathfrak{I}_1\})$,

6. $\neg(\alpha_1)^\downarrow = \upsilon(\{\neg F_1 \mid F_1 \in \mathfrak{F}_1\})$,

7. $(l_\uparrow(\alpha_1, \ldots, \alpha_n))_\uparrow = \lambda(\{l(X_1, \ldots, X_n) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$,

8. $(r^\downarrow(\alpha_1, \ldots, \alpha_n))^\downarrow = \upsilon(\{r(Y_1, \ldots, Y_n) \mid Y_k \in (\mathfrak{I}_k \| \mathfrak{F}_k)\})$.

Proof. By definition, every basis $\mathfrak{F}$ ($\mathfrak{I}$) is a subset of $\alpha^\downarrow$ ($\alpha_\uparrow$). Therefore, the $\subseteq$-direction is trivial for each case. Items 1, 2, 5, and 6 are special cases of Items 3 and 4. But, with the lattice structures, we can show Items 1 and 2 much easier than Items 3 and 4. So, we firstly prove Item 1.

Let I be an arbitrary element in $\lambda(\{F_1 \vee F_2 \mid F_1 \in \mathfrak{F}_1, F_2 \in \mathfrak{F}_2\})$. For arbitrary $F_1 \in \mathfrak{F}_1$ and $F_2 \in \mathfrak{F}_2$, we have $F_1 \vee F_2 \sqsubseteq I$. We can easily show that

$$F_1 \vee F_2 \sqsubseteq I \iff F_1 \sqsubseteq I \text{ and } F_2 \sqsubseteq I. \tag{3.19}$$

Therefore, $I \in \alpha_{1\uparrow}$ and $I \in \alpha_{2\uparrow}$. Then, for arbitrary $G_1 \in \alpha_1{}^\downarrow$ and $G_2 \in \alpha_2{}^\downarrow$, $G_1 \sqsubseteq I$ and $G_2 \sqsubseteq I$. Again, by (3.19) we have $G_1 \vee G_2 \sqsubseteq I$, hence $I \in (\alpha_1 \vee_\uparrow \alpha_2)_\uparrow$.

(Item 3). Let Y be an element of $\lambda(\{f(X_1, \ldots, X_m) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$. By definition, for each $X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)$, we have

$$f(X_1, \ldots, X_m) \sqsubseteq Y.$$

Then, we define the following sets: for each $k \in \{1, \ldots, m\}$, we let

$$Y_k := \{x_k \mid \exists x_1 \in X_1, \ldots, \exists x_m \in X_m, \exists y \in Y.\ f(x_1, \ldots, x_m) \leq y\}.$$

Then, we can show that Y_k is an ideal if $\delta_k = 1$, otherwise, Y_k is a filter if $\delta_k = \partial$: see also the proof of Proposition 3.2.10. By the definition of Y_k, we have $Y_k \in (\alpha_{k\uparrow} \| \alpha_k{}^{\downarrow})$. By the δ-join preservability of f, we also obtain

$$f(Y_1, \ldots, Y_m) \sqsubseteq Y.$$

It follows that $Y \in (f_\uparrow(\alpha_1, \ldots, \alpha_m))_\uparrow$. Item 4 is analogous.

Finally, we check Item 7. To simplify our proof, we here assume that l and r form an adjoint pair, i.e. $l \dashv^i r$. But, we can prove the same results without the adjointness.

Let $Y \in \lambda(\{l(X_1, \ldots, X_n) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$. For each $X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)$, we have

$$l(X_1, \ldots, X_n) \sqsubseteq Y.$$

By the adjointness of l and r, we can straightforwardly obtain

$$l(X_1, \ldots, X_n) \sqsubseteq Y \iff X_i \sqsubseteq r(X_1, \ldots, Y, \ldots, X_n). \tag{3.20}$$

Since X_i is arbitrary, $r(X_1, \ldots, Y, \ldots, X_n) \in \alpha_{i\uparrow}$. Hence, for an arbitrary $X'_i \in \alpha_i{}^{\downarrow}$, $X'_i \sqsubseteq r(X_1, \ldots, Y, \ldots, X_n)$. Again by (3.20), we have

$$l(X_1, \ldots, X'_i, \ldots, X_n) \sqsubseteq Y.$$

If l has a right-adjoint r' for a coordinate k, i.e. $l \dashv^k r$, we could repeat to replace each $X \in (\mathfrak{F} \| \mathfrak{I})$ with $X' \in (\alpha^{\downarrow} \| \alpha_\uparrow)$ with the same method, and then we could conclude $Y \in (l_\uparrow(\alpha_1, \ldots, \alpha_n))_\uparrow$. However, we do not assume that, for each coordinate, $l^{(i)}$ has a right-adjoint. Therefore, it may not work. Nevertheless, we can somehow imitate the technique. Namely, even if l does not have right-adjoints for some coordinates, we can define the lacking right-adjoints on the intermediate level. This is exactly the reason we assume that l is additive.

Let k be a coordinate for which l does not have a right-adjoint. Whenever $l(X_1, \ldots, X_n) \sqsubseteq Y$ holds, we can define the following set:

$$Y_k := \{x_k \mid \exists x_1 \in X_1, \ldots, \exists x_n \in X_n, \exists y \in Y.\ l(x_1, \ldots, x_n) \leq y\}.$$

Since l is additive, we can prove that, if $\mu_k = 1$, then Y_k is an ideal satisfying

$$l(X_1, \ldots, X_n) \sqsubseteq Y \iff X_k \sqsubseteq Y_k, \tag{3.21}$$

and, if $\mu_k = \partial$, then Y_k is a filter satisfying

$$l(X_1, \ldots, X_n) \sqsubseteq Y \iff Y_k \sqsubseteq X_k. \tag{3.22}$$

Note that these conditions (3.21) and (3.22) coincide with the covariant adjointness and contravariant adjointness, respectively. This is the end of the proof. Item 8 is analogous. □

Remark 3.3.9. Lemma 3.3.8 looks similar to the definition of the canonical extensions of each operation. However, Lemma 3.3.8 proposes an *effective evaluation*. For example, let $\circ : \mathbb{L}^1 \times \mathbb{L}^1 \to \mathbb{L}$ be a non-associative fusion-like $(1, 1)$-additive operation. By definition, the value $(\alpha \circ_{\uparrow} (\beta \circ_{\uparrow} \gamma))$ is calculated as follows: firstly, we calculate

$$(\beta \circ_\uparrow \gamma)^\downarrow = \upsilon(\lambda(\{G \circ H \mid G \in \beta^\downarrow, H \in \gamma^\downarrow\})),$$

then we obtain

$$(\alpha \circ_\uparrow (\beta \circ_\uparrow \gamma))_\uparrow = \lambda(\{F \circ F' \mid F \in \alpha^\downarrow, F' \in \upsilon(\lambda(\{G \circ H \mid G \in \beta^\downarrow, H \in \gamma^\downarrow\}))\})$$

On the other hand, thanks to Lemma 3.3.8 (Item 7), we can evaluate the same value as follows:

$$(\alpha \circ_\uparrow (\beta \circ_\uparrow \gamma))_\uparrow = \lambda(\{F \circ (G \circ H) \mid F \in \alpha^\downarrow, G \in \beta^\downarrow, H \in \gamma^\downarrow\}). \tag{3.23}$$

Note that, in the latter case, $\upsilon(\lambda(\_))$ does not appear.

From Lemma 3.3.8, we can straightforwardly prove the following lemma: see e.g. Equation (3.23).

Lemma 3.3.10. *Let $t_1, \ldots, t_N$ be terms. Then, we have (recall Abbreviation 3.2.4)*

1. *$t_1 \vee t_2$ is a $\cup$-term whenever t_1 and t_2 are $\cup$-terms,*
2. *$t_1 \wedge t_2$ is a $\cap$-term whenever t_1 and t_2 are $\cap$-terms,*
3. *$f(t_1, \ldots, t_m)$ is a $\cup$-term whenever t_k is a $(\cup\|\cap)$-term for each $k \in \{1, \ldots, m\}$,*
4. *$g(t_1, \ldots, t_{m'})$ is a $\cap$-term whenever t_k is a $(\cap\|\cup)$-term for each $k \in \{1, \ldots, m'\}$,*
5. *$\neg t_1$ is a $\cup$-term whenever t_1 is a $\cap$-term,*
6. *$\neg t_1$ is a $\cap$-term whenever t_1 is a $\cup$-term,*
7. *$l(t_1, \ldots, t_n)$ is a $\cup$-term whenever t_k is a $(\cup\|\cap)$-term for each $k \in \{1, \ldots, n\}$,*

8. $r(t_1, \ldots, t_n)$ *is a* $\cap$*-term whenever* t_k *is a* $(\cap \| \cup)$*-term for each* $k \in \{1, \ldots, n\}$.

Furthermore, we obtain the following $\cup$-terms and $\cap$-terms.

Lemma 3.3.11. *Every term of type* $t_\vee$ *is a* $\cap$*-term and every term of type* $t_\wedge$ *is a* $\cup$*-term (recall Abbreviation 3.2.4), where* $t_\vee$ *and* $t_\wedge$ *are defined as follows.*

$$t_\vee ::= p \mid c \mid t_\vee \vee t_\vee \mid f(t_{(\vee\|\wedge)}, \ldots, t_{(\vee\|\wedge)}) \mid \neg t_\wedge \mid l(c, \ldots, t_{(\vee\|\wedge)}, \ldots, c),$$

$$t_\wedge ::= p \mid c \mid t_\wedge \wedge t_\wedge \mid g(t_{(\wedge\|\vee)}, \ldots, t_{(\wedge\|\vee)}) \mid \neg t_\vee \mid r(c, \ldots, t_{(\wedge\|\vee)}, \ldots, c).$$

With the distributive law, on distributive lattice expansions, we can also add $t_\vee \wedge c$ *and* $c \wedge t_\vee$ *to type* $t_\vee$*, and* $t_\wedge \vee c$ *and* $c \vee t_\wedge$ *to type* $t_\wedge$*, where* c *is a constant or a constant term.*

Proof. This proof will be divided into the following two parts: firstly, by the definition of type $t_\vee$ and type $t_\wedge$, we can obtain the following by induction. For all $x_1, \ldots, x_N$, $y_1, \ldots, y_N$, $z_1, \ldots, z_N$, $w_1, \ldots, w_N \in L$, we have

$$\begin{aligned} &t_\vee((x_1 \vee y_1)\|(z_1 \wedge w_1), \ldots, (x_N \vee y_N)\|(z_N \wedge w_N)) \\ &\qquad = t_\vee(x_1\|z_1, \ldots, x_N\|z_N) \vee t_\vee(y_1\|w_1, \ldots, y_N\|w_N), \end{aligned}$$

$$\begin{aligned} &t_\wedge((x_1 \wedge y_1)\|(z_1 \vee w_1), \ldots, (x_N \wedge y_N)\|(z_N \vee w_N)) \\ &\qquad = t_\wedge(x_1\|z_1, \ldots, x_N\|z_N) \wedge t_\wedge(y_1\|w_1, \ldots, y_N\|w_N). \end{aligned}$$

Secondly, we show $t_\vee$ is a $\cap$-term and $t_\wedge$ is a $\cup$-term. Here we check only that $t_\vee$ is a $\cap$-term. However, it is order dually shown that $t_\wedge$ is a $\cup$-term. By the definitions of type $t_\vee$ and type $t_\wedge$ and Lemmata 3.3.7 and 3.3.10, it is straightforward that every term of type $t_\vee$ is a $\cup$-term and every term of type $t_\wedge$ is a $\cap$-term.

Let t be a term of type $t_\vee$. Then, we have, for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$,

$$t(\alpha_1, \ldots, \alpha_N)_\uparrow = \lambda(\{t(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in \alpha_k{}^\downarrow, I_k \in \alpha_{k\uparrow}\}).$$

To prove that t is a $\cap$-term, we need to check that

$$\upsilon(t(\alpha_1, \ldots, \alpha_N)_\uparrow) = \upsilon(\{t(I_1 \| F_1, \ldots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in \alpha_k{}^\downarrow\})$$

$\subseteq$ is trivial. To prove the converse $\supseteq$, it suffices to show that, for an arbitrary $Y \in t(\alpha_1, \ldots, \alpha_N)_\uparrow$, there exist $I_k \in \alpha_{k\uparrow}$ and $F_k \in \alpha_k{}^\downarrow$ such that

$$t(I_1 \| F_1, \ldots, I_N \| F_N) \sqsubseteq Y.$$

Let Y be an arbitrary element of $t(\alpha_1, \ldots, \alpha_N)_\uparrow$. For arbitrary $F_k \in \alpha_k{}^\downarrow$ and $I_k \in \alpha_{k\uparrow}$, by definition, we have $t(F_1 \| I_1, \ldots, F_N \| I_N) \sqsubseteq Y$. Now we replace $\|$ with , (comma). We obtain

$$t(F_1, I_1, \ldots, F_N, I_N) \sqsubseteq Y. \tag{3.24}$$

Then, we notice that

$$t : \mathbb{L}^1 \times \mathbb{L}^\partial \times \cdots \times \mathbb{L}^1 \times \mathbb{L}^\partial \to \mathbb{L}$$

can be seen as a ϵ-join preserving operation on $\mathbb{L}$, as we checked firstly. Then, we can use exactly the same technique to find the appropriate ideals and filters with the proof of Proposition 3.2.10. $\square$

Remark 3.3.12. The important technique in the proof of Lemma 3.3.11 is to show the existence of ideals and filters satisfying the condition (3.24). Although additivity (multiplicativity) is enough to define these ideals and filters, *the join-preservability (meet-preservability) from the product domain* is necessary to prove the condition (3.24). Since l and r may be neither join-preserving nor meet-preserving *from the product domain*, it is difficult to use the same methodology without fixing the other coordinates with constants or constant terms, like $l(c, \ldots, t_{(\vee\|\wedge)}, \ldots, c)$.

From the above Lemmata 3.3.7, 3.3.10 and 3.3.11, we directly obtain the following theorem.

Theorem 3.3.13. *Each term of type $t_\cup$ is a $\cup$-term and each term of type $t_\cap$ is a $\cap$-term, where $t_\cup$ and $t_\cap$ are defined as follows.*

$$t_\cup ::= p \mid c \mid t_\cup \vee t_\cup \mid f(t_{(\cup\|\cap)}, \ldots, t_{(\cup\|\cap)}) \mid \neg t_\cap \mid l(t_{(\cup\|\cap)}, \ldots, t_{(\cup\|\cap)}) \mid t_\wedge,$$

$$t_\cap ::= p \mid c \mid t_\cap \wedge t_\cap \mid g(t_{(\cap\|\cup)}, \ldots, t_{(\cap\|\cup)}) \mid \neg t_\cup \mid r(t_{(\cap\|\cup)}, \ldots, t_{(\cap\|\cup)}) \mid t_\vee.$$

Over distributive lattice expansions, with the distributive law, we can also add $t_\cup \wedge t_\cup$ to type $t_\cup$, and $t_\cap \vee t_\cap$ to type $t_\cap$.

Now, with Theorems 3.3.5 and 3.3.13, we can syntactically obtain a class of canonical inequalities. However, canonical inequalities $t_\cup \leq t_\cap$ do not cover the standard Sahlqvist formulae, e.g. [11] or [5], even if we consider $t_\cup \leq t_\cap$ on modal algebras. For example, although any positive formula is Sahlqvist in modal logic, our canonical inequalities cannot cover some positive formulae. Therefore, to extend our results, we analyse other terms not of type $t_\cup$ and $t_\cap$. We start with the construction trees of terms. Let t be an arbitrary term. We draw the standard construction tree of t in which the root is t and every leaf is either a variable or a constant. Now we add labels ($\cup$, $\cap$ or ?) to the construction tree of t in the following manner ruled by Theorem 3.3.13. Note that the labelling starts at the root not leaves.

$\cap$ ($\cup$) **labelling algorithm**

1. The root is labelled with $\cap$ ($\cup$).

2. If the node is either a constant or a variable, then we have finished labelling the branch. Otherwise, we label the children along the following rule. If the node is

   (a) $t_1 \vee t_2$ and labelled with $\cup$, then label t_1 and t_2 with $\cup$,

   (b) $t_1 \wedge t_2$ and labelled with $\cap$, then label t_1 and t_2 with $\cap$,

   (c) either $f(t_1, \ldots, t_m)$ or $l(t_1, \ldots, t_n)$ and labelled with $\cup$, then, for each coordinate k, label t_k with $\cup$ if $\delta_k = 1$ or $\mu_k = 1$, and label t_k with $\cap$ if $\delta_k = \partial$ or $\mu_k = \partial$,

   (d) either $g(t_1, \ldots, t_{m'})$ or $r(t_1, \ldots, t_n)$ and labelled with $\cap$, then, for each coordinate k, label t_k with $\cap$ if $\epsilon_k = 1$ or $\nu_k = 1$, and label t_k with $\cup$ if $\epsilon_k = \partial$ or $\nu_k = \partial$,

   (e) $\neg t$ labelled with $\cup$, then label t with $\cap$,

   (f) $\neg t$ labelled with $\cap$, then label t with $\cup$,

   (g) either $t_\wedge$ labelled with $\cup$ or $t_\vee$ labelled with $\cap$, then label every node (not only the children) below the current node with $\cup$ or $\cap$, respectively and stop labelling the branch,

   (h) not satisfying any of (a) - (g), label the all nodes below the current node with ?.

3. Move to every child and repeat Item 2 until every node in the tree is labelled.

Over distributive lattice expansions, we can replace Items (a) and (b) in Item 2 with the following two Items (a') and (b'):

(a') either $t_1 \vee t_2$ or $t_1 \wedge t_2$ labelled with $\cup$, then label t_1 and t_2 with $\cup$,

(b') either $t_1 \wedge t_2$ or $t_1 \vee t_2$ labelled with $\cap$, then label t_1 and t_2 with $\cap$.

We call this labelled tree *the $\cap$-labelled construction tree of t* because of the labelling the root with $\cap$. We can define *the $\cup$-labelled construction tree of t* by the same algorithm with the only exception that we start labelling the root with $\cup$, instead of $\cap$. In Section 4.4, we can find an example of $\cap$-labelling in Fig. 4.2 and an example of $\cup$-labelling in Fig. 4.1. Note that these labellings are based on distributive lattice expansions. By Theorem 3.3.13, we obtain the following proposition.

Proposition 3.3.14. *Let t be a term. t is a $\cup$-term of type $t_\cup$, if and only if there is no node labelled with ? in the $\cup$-labelled construction tree of t. t is a $\cap$-term of type $t_\cap$, if and only if there is no node labelled with ? in the $\cap$-labelled construction tree of t.*

We also introduce *the $\wedge$-labelled construction tree of t* and *the $\vee$-labelled construction tree of t*: see Proposition 4.4.3. These labelling rules are similar to $\cap$-labelled construction trees or $\cup$-labelled construction trees, but they are ruled by Lemma 3.3.11.

$\wedge$ ($\vee$) labelling algorithm

1. The root is labelled with $\wedge$ ($\vee$).

2. If the node is either a constant or a variable, then we have finished labelling the branch. Otherwise, we label the children along the following rule. If the node is

   (a) $t_1 \vee t_2$ labelled with $\vee$, then label t_1 and t_2 with $\vee$,

   (b) $t_1 \wedge t_2$ labelled with $\wedge$, then label t_1 and t_2 with $\wedge$,

   (c) $f(t_1, \ldots, t_m)$ labelled with $\vee$, then, for each coordinate k, label t_k with $\vee$ if $\delta_k = 1$, and label t_k with $\wedge$ if $\delta_k = \partial$,

   (d) $g(t_1, \ldots, t_{m'})$ labelled with $\wedge$, then, for each coordinate k, label t_k with $\wedge$ if $\epsilon_k = 1$, and label t_k with $\vee$ if $\epsilon_k = \partial$,

   (e) $\neg t$ labelled with $\vee$, then label t with $\wedge$,

   (f) $\neg t$ labelled with $\wedge$, then label t with $\vee$,

   (g) $l(c_1, \ldots, t_k, \ldots, c_n)$ labelled with $\vee$ where $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ are constants or constant terms, label every node below $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ (not only for the children) with $\vee$, and label t_k with $\vee$ if $\mu_k = 1$ and label t_k with $\wedge$ if $\mu_k = \partial$,

   (h) $r(c_1, \ldots, t_k, \ldots, c_n)$ labelled with $\wedge$ where $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ are constants or constant terms, label each node below $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ (not only for the children) with $\wedge$, and label t_k with $\wedge$ if $\nu_k = 1$ and label t_k with $\vee$ if $\nu_k = \partial$,

   (i) not satisfying (a) - (h), label the all nodes below the current node with ?.

3. Move to every child and repeat Item 2 until every node in the tree is labelled.

Over distributive lattice expansions, we can add the following Items (a') and (b') to Item 2:

(a') either $t \wedge c$ or $c \wedge t$, where c is a constant (term), labelled with $\vee$, then label all nodes below c with $\vee$ and label t with $\vee$,

(b') either $t \vee c$ or $c \vee t$, where c is a constant (term), labelled with $\wedge$, then label all nodes below c with $\wedge$ and label t with $\wedge$.

By Lemma 3.3.11, we have the following proposition.

Proposition 3.3.15. *Let t be a term. t is a $\cup$-term of type $t_\wedge$, if and only if there is no node labelled with ? in the $\wedge$-labelled construction tree of t. t is a $\cap$-term of type $t_\vee$, if and only if there is no node labelled with ? in the $\vee$-labelled construction tree of t.*

Based on $\cap$-labelled construction trees and $\cup$-labelled construction trees, we define the following.

Definition 3.3.16 (Critical subterm)**.** Let t be a term. A subterm of t is $\cap$-*critical* ($\cup$-*critical)*, if it is both a node labelled by either $\cup$ or $\cap$ and a parent of nodes labelled with ? in the $\cap$-labelled ($\cup$-labelled) construction tree of t.

We can find examples of critical subterms in Fig. 4.1 and Fig. 4.2, in Section 4.4

There are two technical reasons to introduce critical subterms. The first one is that, if a term contains $\cup$-critical ($\cap$-critical) subterms, it is not of type $t_\cup$ ($t_\cap$). Another is that every term can be seen as a $\cup$-term ($\cap$-term), if each $\cup$-critical ($\cap$-critical) subterm is replaced by a new variable: see Definition 3.3.20 and Proposition 3.3.21 Next, to state the main theorem for lattice expansions, Theorem 3.3.22, clearly, we define the following.

Definition 3.3.17 (Well-pruned tree)**.** Let s, t be terms. A (possibly empty) subtree of the construction tree of t is *pruned*, if it is obtained by pruning some branches away. A pruned tree of the construction tree of t is $\cup$*-well-pruned* ($\cap$*-well-pruned)*, if every leaf of the tree is a propositional variable and each path from a leaf to the root contain a $\cup$-critical ($\cap$-critical) subterm. *The* $\cup$*-well-pruned (the* $\cap$*-well-pruned)* tree of t is the largest $\cup$-well-pruned ($\cap$-well-pruned) tree of t. Especially, if t is a $\cup$-term ($\cap$-term), the $\cup$-well-pruned (the $\cap$-pruned) tree of t is an empty tree. For an inequality $s \leq t$, *the well-pruned pair of tree for* $s \leq t$ is a pair of the $\cup$-well-pruned tree of s and the $\cap$-well-pruned tree of t.

We can find an example of the $\cup$-well-pruned tree in Fig. 4.1 and an example of the $\cap$-well-pruned tree in Fig. 4.2 in Section 4.4 In those figures, the dashed lines are pruned to obtain the well-pruned trees.

On the well-pruned pair of trees for $s \leq t$, we label every node with a sign ($+$ or $-$) in the following manner. Note that we can do the same labelling before pruning.

Signing algorithm

1. Label the root of the $\cup$-well-pruned tree of s with $-$ and the root of the $\cap$-well-pruned tree of t with $+$.

2. If the node does not have any child, we stop labelling. Otherwise, we label $+$ or $-$ for each child based on the following step.

   (a) If the node is either $t_1 \vee t_2$ or $t_1 \wedge t_2$, then label t_1 and t_2 with the same sign of the current node.

   (b) If the node is $\neg t$, then label t with the converse sign of the current node.

(c) Otherwise, the node is one of $f(t_1, \ldots, t_m)$, $g(t_1, \ldots, t_{m'})$, $l(t_1, \ldots, t_n)$ and $r(t_1, \ldots, t_n)$. Then, for each coordinate k, label t_k with the same sign of the current node if the k-th order type (δ_k, ϵ_k, μ_k or ν_k) is 1, and label t_k with the converse node if the k-th order type (δ_k, ϵ_k, μ_k or ν_k) is ∂.

3. Move to every child and repeat Item 2 until every node is labelled.

The well-pruned pair of trees for $s \leq t$ is *signed*, if it is labelled along with the signing algorithm. For example, if s is a negative term, in which every variable occurs negatively, and t is a positive term, in which every variable occurs positively, every variable in the signed well-pruned pair of trees for $s \leq t$ is labelled with $+$. We can find an example of the $-$-signed tree in Fig. 4.1 and an example of the $+$-signed tree in Fig. 4.2, in Section 4.4

Definition 3.3.18 (Consistent variable occurrence)**.** Let s, t be terms. An inequality $s \leq t$ has *consistent variable occurrence*, if there exists no propositional variable signed with both $-$ and $+$ in the signed well-pruned pair of trees for $s \leq t$.

Remark 3.3.19. We want to apply an analogous argument of the proof of Theorem 3.3.5 not only for $\cup$-terms and $\cap$-terms but also for all terms and inequalities (recall that Theorem 3.3.5 states canonicity of inequalities $s \leq t$, where s is a $\cup$-term and t is a $\cap$-term). It is crucial to introduce pseudoterms to analyse all terms and inequalities: see Theorem 3.3.22 We mention that [33] did not define pseudoterms, and hence it is unclear whether their main theorem [33, Theorem 7.2] can be applied for non-$\cap$-terms.

Definition 3.3.20 (Pseudo-$\cup$-term and pseudo-$\cap$-term)**.** Let t be a term. A term t' is *the pseudo-$\cap$-term of t (the pseudo-$\cup$-term of t)*, if every $\cap$-critical ($\cup$-critical) subterms of t is replaced with a *new* variable. Note that, even if a $\cap$-critical ($\cup$-critical) subterm appears in t more than once, we replace each occurrence with distinct variables.

Clearly, for each term t of type $t_\cap$ (type $t_\cup$), the pseudo-$\cap$-term of t (the pseudo-$\cup$-term of t) is t itself.

In general, if t' is the pseudoterm of t, then t' and t are different. However, the next proposition provides us with a meaningful connection between terms and the pseudoterms. The proof is straightforward from a fact that every pseudo-$\cap$-term is a $\cap$-term and every pseudo-$\cup$-term is a $\cup$-term.

Proposition 3.3.21. *Let s, t be terms. We denote by $t(p_1, \ldots, p_N)$ (or $t(\underline{p})$, for short) that each variable in t is a variable p_k, analogously $s(p_1, \ldots, p_N)$ or $s(\underline{p})$. Let $t_1, \ldots, t_a$ be all $\cap$-critical subterms of t, $s_1, \ldots, s_b$ all $\cup$-critical subterms of s, and $t'(\underline{p}, q_1, \ldots, q_a)$ the pseudo-$\cap$-term of t, where each t_k in t is replaced by q_k, and $s'(\underline{p}, q'_1, \ldots, q'_b)$ the pseudo-$\cup$-term of s, where each s_k in s is replaced with q'_k. Then, we have*

$$t(\underline{p}) = t'(\underline{p}, q_1, \ldots, q_a)[t_1(\underline{p})/q_1, \ldots, t_a(\underline{p})/q_a],$$

$$s(\underline{p}) = s'(\underline{p}, q'_1, \ldots, q'_b)[s_1(\underline{p})/q'_1, \ldots, s_b(\underline{p})/q'_b].$$

Moreover, we also have, for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$,

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} = \upsilon(\{t'(I_1 \| F_1, \ldots, I_N \| F_N, Y_1, \ldots, Y_a)\}),$$

$$s(\alpha_1, \ldots, \alpha_N)_{\uparrow} = \lambda(\{s'(F_1 \| I_1, \ldots, F_N \| I_N, X_1, \ldots, X_b)\}),$$

where all $I_k \in \alpha_{k\uparrow}$, $F_k \in \alpha_k{}^{\downarrow}$, *and*[1]

$$Y_k \in (t_k(\alpha_1, \ldots, \alpha_N)_{\uparrow} \| t_k(\alpha_1, \ldots, \alpha_N)^{\downarrow}),$$

$$X_k \in (s_k(\alpha_1, \ldots, \alpha_N)^{\downarrow} \| s_k(\alpha_1, \ldots, \alpha_N)_{\uparrow}).$$

Now we state the main theorem for lattice expansions.

Main Theorem 3.3.22 (for lattice expansions)**.** *Let s, t be terms over lattice expansions. An inequality $s \leq t$ is canonical, if it has consistent variable occurrence.*

Proof. Recall Abbreviation 3.2.4. Let $p_1, \ldots, p_M, p_{M+1}, \ldots, p_N$ be all variables in $s \leq t$, $s_1^-, \ldots, s_a^-$ $\cup$-critical subterms of s signed with $-$, $s_1^+, \ldots, s_b^+$ $\cup$-critical subterms of s signed with $+$, $t_1^+, \ldots, t_c^+$ $\cap$-critical subterms signed with $+$, and $t_1^-, \ldots, t_d^-$ $\cap$-critical subterms signed with $-$ in the signed well-pruned pair of trees for $s \leq t$. Now, since $s \leq t$ has consistent variable occurrence, without loss of generality, we can assume that all $p_1, \ldots, p_M$ are signed with $+$ and all $p_{M+1}, \ldots, p_L$ are signed with $-$ in the signed well-pruned pair of trees for $s \leq t$. Let s' be the pseudo-$\cup$-term of s and t' the pseudo-$\cap$-term of t. By Proposition 3.3.21, we have

$$s(\alpha_1, \ldots, \alpha_N)_{\uparrow} = \lambda(\{s'(F_1' \| I_1', \ldots, F_N' \| I_N', G_1, \ldots, G_a, J_1, \ldots, J_b)\}),$$

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} = \upsilon(\{t'(I_1'' \| F_1'', \ldots, I_N'' \| F_N'', K_1, \ldots, K_c, H_1, \ldots, H_d)\}),$$

where $F_u' \in \alpha_u{}^{\downarrow}$, $F_u'' \in \alpha_u{}^{\downarrow}$ and $I_u' \in \alpha_{u\uparrow}$, $I_u'' \in \alpha_{u\uparrow}$ for each $u \in \{1, \ldots, N\}$, $G_g \in s_g^-(\alpha_1, \ldots, \alpha_N)^{\downarrow}$ for each $g \in \{1, \ldots, a\}$, $J_j \in s_j^+(\alpha_1, \ldots, \alpha_N)_{\uparrow}$ for each $j \in \{1, \ldots, b\}$, $K_k \in t_k^+(\alpha_1, \ldots, \alpha_N)_{\uparrow}$ for each $k \in \{1, \ldots, c\}$, and $H_h \in t_h^-(\alpha_1, \ldots, \alpha_N)^{\downarrow}$

[1] We do not use the parallel notation $\|$ for Y_k and X_k, because each critical subterm is replaced with a *fresh* variable, hence it occurs only once, positively or negatively.

for each $h \in \{1, \ldots, d\}$.[2] By Proposition 2.3.1, it suffices to show that, for each $F'_u, F''_u, I'_u, I''_u, G_g, J_j, K_k, H_h$,

$$s'(\underline{F'\|I'}, G_1, \ldots, G_a, , J_1, \ldots, J_b) \sqsubseteq t'(\underline{I''\|F''}, K_1, \ldots, K_c, H_1, \ldots, H_d),$$

Since every $\alpha^{\downarrow}$ is an ideal of filters and every $\alpha_{\uparrow}$ is a filter of ideals, there exist $F_u \in {\alpha_u}^{\downarrow}$ and $I_u \in \alpha_{u\uparrow}$ for each $u \in \{1, \ldots, N\}$ such that, by Lemma 3.2.3, we have

$$s'(\underline{F'\|I'}, G_1, \ldots, G_a, J_1, \ldots, J_b) \sqsubseteq s'(\underline{F\|I}, G_1, \ldots, G_a, J_1, \ldots, J_b),$$

$$t'(\underline{I\|F}, K_1, \ldots, K_c, H_1, \ldots, H_d) \sqsubseteq t'(\underline{I''\|F''}, K_1, \ldots, K_c, H_1, \ldots, H_d),$$

Therefore, hereafter, we will show

$$s'(\underline{F\|I}, G_1, \ldots, G_a, J_1, \ldots, J_b) \sqsubseteq t'(\underline{I\|F}, K_1, \ldots, K_c, H_1, \ldots, H_d),$$

By Proposition 3.3.2, for each $g \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b\}$, $k \in \{1, \ldots, c\}$ $h \in \{1, \ldots, d\}$, we have

$$G_g \sqsubseteq s_g^-(I_1\|F_1, \ldots, I_N\|F_N), \tag{3.25}$$

$$s_j^+(F_1\|I_1, \ldots, F_N\|I_N) \sqsubseteq J_j, \tag{3.26}$$

$$t_k^+(F_1\|I_1, \ldots, F_N\|I_N) \sqsubseteq K_k, \tag{3.27}$$

$$H_h \sqsubseteq t_h^-(I_1\|F_1, \ldots, I_N\|F_N). \tag{3.28}$$

[2] The reason we do not use $\|$ notations for G_g, J_j, K_k, and H_h is that we never use the other side elements by occurrences of s^-, s^+, t^+ and t^-. That is, even if the same critical subterms occur more than twice in a term, we replace it with a *fresh* variable each time.

Because, $p_1, \ldots, p_M$ are signed with $+$ and $p_{M+1}, \ldots, p_N$ are signed with $-$ in the signed well-pruned pair of trees for $s \leq t$, we can rewrite (3.25) - (3.28) as follows:

$$G_g \sqsubseteq s_g^-(F_1, \ldots, F_M, I_{M+1}, \ldots, I_N),$$

$$s_j^+(F_1, \ldots, F_M, I_{M+1}, \ldots, I_N) \sqsubseteq J_j,$$

$$t_k^+(F_1, \ldots, F_M, I_{M+1}, \ldots, I_N) \sqsubseteq K_k,$$

$$H_h \sqsubseteq t_h^-(F_1, \ldots, F_M, I_{M+1}, \ldots, I_N).$$

By Lemma 3.2.6, there exist $f_1^p, \ldots, f_{a+b+c+d}^p \in F_p$ and $i_1^q, \ldots, i_{a+b+c+d}^q \in I_q$ for each $p \in \{1, \ldots, M\}$ and each $q \in \{M+1, \ldots, N\}$ such that, for each g, j, k, h,

$$s_g^-(f_g^1, \ldots, f_g^M, i_g^{M+1}, \ldots, i_g^N) \in G_g,$$

$$s_j^+(f_{j+a}^1, \ldots, f_{j+a}^M, i_{j+a}^{M+1}, \ldots, i_{j+a}^N) \in J_j,$$

$$t_k^+(f_{k+a+b}^1, \ldots, f_{k+a+b}^M, i_{k+a+b}^{M+1}, \ldots, i_{k+a+b}^N) \in K_k,$$

$$t_h^-(f_{h+a+b+c}^1, \ldots, f_{h+a+b+c}^M, i_{h+a+b+c}^{M+1}, \ldots, i_{h+a+b+c}^N) \in H_h.$$

For each $u \in \{1, \ldots, N\}$, since $F_u \sqsubseteq I_u$ (recall that $F_u \in {\alpha_u}^{\downarrow}$ and $I_u \in \alpha_{u\uparrow}$), there exists $x'_u \in F_u \cap I_u$. Then, for each $g \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b\}$, $k \in \{1, \ldots, c\}$,

$h \in \{1, \ldots, d\}$, we have

$$\begin{array}{ccccc}
F_1 \cap I_1 & \ni & x_1 & = & x'_1 \wedge f^1_1 \wedge \cdots \wedge f^1_{a+b+c+d}, \\
\vdots & & \vdots & & \vdots \\
F_M \cap I_M & \ni & x_M & = & x'_M \wedge f^M_1 \wedge \cdots \wedge f^M_{a+b+c+d}, \\
F_{M+1} \cap I_{M+1} & \ni & x_{M+1} & = & x'_{M+1} \vee i^{M+1}_1 \vee \cdots \vee i^{M+1}_{a+b+c+d}, \\
\vdots & & \vdots & & \vdots \\
F_N \cap I_N & \ni & x_N & = & x'_N \vee i^N_1 \vee \cdots \vee i^N_{a+b+c+d}.
\end{array}$$

Moreover, by Lemma 3.2.5, we have $s^-_g(x_1, \ldots, x_N) \in G_g$, $s^+_j(x_1, \ldots, x_N) \in J_j$, $t^+_k(x_1, \ldots, x_N) \in K_k$ and $t^-_h(x_1, \ldots, x_N) \in H_h$ for each $g \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b\}$, $k \in \{1, \ldots, c\}$, $h \in \{1, \ldots, d\}$. (Recall our assumption; $p_1, \ldots, p_M$ are signed with $+$, hence they are positive in s^+_j and t^+_k, and negative in s^-_g and t^-_h. Conversely, $p_{M+1}, \ldots, p_N$ are signed with $-$, hence they are positive in s^-_g and t^-_h, and negative in s^+_j and t^+_k.) Therefore, we have

$$s(x_1, \ldots, x_N) = s(x_1 \| x_1, \ldots, x_N \| x_N) \in s'(\underline{F \| I}, G_1, \ldots, G_a, J_1, \ldots, J_b), \tag{3.29}$$

$$t(x_1, \ldots, x_N) = t(x_1 \| x_1, \ldots, x_N \| x_N) \in t'(\underline{I \| F}, K_1, \ldots, K_c, H_1, \ldots, H_d), \tag{3.30}$$

Since $s(x_1, \ldots, x_N) \leq t(x_1, \ldots x_N)$ for each $x_1, \ldots, x_N \in L$, and $s(\underline{F \| I})$ is a filter and $t(\underline{I \| F})$ is an ideal, by (3.29) and (3.30), we have

$$s'(\underline{F \| I}, G_1, \ldots, G_a, J_1, \ldots, J_b) \sqsubseteq t'(\underline{I \| F}, K_1, \ldots, K_c, H_1, \ldots, H_d)$$

Hence,

$$s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N).$$

$\square$

Remark 3.3.23. This theorem extends [33, Theorem 7.2] from Heyting algebras to lattice expansions which may be neither bounded nor distributive, and from equalities to inequalities. Therefore, we can apply our method to several algebraic logics: for example, substructural logic or lattice-based logics.

Chapter 4

Applications to lattice-based logics

In the previous chapter, we have generalised Ghilardi and Meloni's canonicity methodology to lattice expansions in general and shown the canonicity results in Theorem 3.3.22. In this chapter, we will show how to interpret our canonicity results to specific lattice-based logics, especially to variants of substructural logics and distributive modal logic, and show that our canonicity results not only uniformly subsume known canonicity results of those logics, but also account for many new canonicity results.

4.1 Application 1: Substructural logic

In this section, we apply Theorem 3.3.22 to substructural logic. Firstly, we recall substructural logic: see [25]. Our language consists of propositional variables, two constants $\mathbf{t}$ and $\mathbf{f}$, and five logical connectives $\vee, \wedge, \circ, \rightarrow, \leftarrow$. Formulae are inductively defined as follows:

$$Fr ::= p \mid \mathbf{t} \mid \mathbf{f} \mid Fr \vee Fr \mid Fr \wedge Fr \mid Fr \circ Fr \mid Fr \rightarrow Fr \mid Fr \leftarrow Fr.$$

Definition 4.1.1 (Substructural logic)**.** A set $\mathbf{L}$ of formulae is a *substructural logic*, if it satisfies

1. $\mathbf{L}$ contains all formulae which are provable in the sequent system FL (see Fig. 8.1),
2. if $\phi \in \mathbf{L}$ and $\phi \to \psi \in \mathbf{L}$, then $\psi \in \mathbf{L}$,
3. if $\phi \in \mathbf{L}$ and $\phi \wedge \mathbf{t} \in \mathbf{L}$,
4. if $\phi \in \mathbf{L}$, then, for an arbitrary formula ψ, we have that $\psi \to (\phi \circ \psi) \in \mathbf{L}$ and $(\psi \circ \phi) \leftarrow \psi \in \mathbf{L}$,
5. $\mathbf{L}$ is closed under substitution.

Replacing in Item 1 the sequent system FL with the sequent system DFL [62, 63], see also Section 8.2, defines *distributive substructural logics.* Moreover, adding the following two initial sequents, $\Gamma \Rightarrow \mathbf{T}$ and $\Gamma, \mathbf{F}, \Sigma \Rightarrow \varphi$, defines *bounded distributive substructural logics*, where $\mathbf{T}$ and $\mathbf{F}$ are constants additionally introduced to consider bounded distributive substructural logics: see Section 4.2. Note that our bounded distributive substructural logics are different from extensions of FL_w, which is bounded with 1 and 0.

Hereafter, $\mathbf{t}, \mathbf{f}, \vee, \wedge, \circ, \to$ and $\leftarrow$ are denoted by $1, 0, \vee, \wedge, \circ, \backslash, /$, respectively. As algebraic semantics of substructural logic, we introduce FL-algebras.

Definition 4.1.2 (FL-algebra)**.** A tuple $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0\rangle$ is a *FL-algebra*, if $\langle A, \vee, \wedge\rangle$ is a lattice, $\langle A, \circ, 1\rangle$ a monoid, 0 an arbitrary constant in A, and $\mathfrak{A}$ satisfies the residuation law: $a \circ b \leq c \iff b \leq a\backslash c \iff a \leq c/b$ for each $a, b, c \in A$. Moreover, if $\langle A, \vee, \wedge\rangle$ is a distributive lattice, $\mathfrak{A}$ is called a *DFL-algebra.*

As we mentioned in Example 3.1.7, every FL-algebra is a lattice expansion having constants 1 and 0 and adjoint pairs $\circ \dashv^2 \backslash$ and $\circ \dashv^1 /$. The canonical extension of FL-algebras $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0\rangle$ is $\langle \overline{\mathfrak{A}}, \vee_\uparrow, \wedge^\downarrow, \circ_\uparrow, \backslash^\downarrow, /^\downarrow, 1, 0\rangle$, by Proposition 3.2.13.

Theorem 4.1.3 (Canonical extension of FL-algebras)**.** *The canonical extension of FL-algebras is also a FL-algebra.*

Now, we define canonicity in substructural logic.

Definition 4.1.4 (Canonicity of term)**.** On the class of FL-algebras, a term t is *canonical*, if the inequality $1 \leq t$ is canonical.

As a corollary of Theorem 3.3.5, we obtain the following.

Corollary 4.1.5. *In substructural logic, every* $\cap$*-term is canonical.*

On our language for substructural logic, the terms of type $t_\cup$, type $t_\cap$, type $t_\vee$ and type $t_\wedge$ in Theorem 3.3.13 are interpreted as follows:

$$t_\cup ::= p \mid 1 \mid 0 \mid t_\cup \vee t_\cup \mid t_\cup \circ t_\cup \mid t_\wedge,$$

$$t_\cap ::= p \mid 1 \mid 0 \mid t_\cap \wedge t_\cap \mid t_\cup \backslash t_\cap \mid t_\cap / t_\cup \mid t_\vee,$$

$$t_\vee ::= p \mid 1 \mid 0 \mid t_\vee \vee t_\vee \mid t_\vee \circ C \mid C \circ t_\vee,$$

$$t_\wedge ::= p \mid 1 \mid 0 \mid t_\wedge \wedge t_\wedge \mid t_\vee \backslash C \mid C \backslash t_\wedge \mid t_\wedge / C \mid C / t_\vee,$$

where C is a constant or a constant term.

Then, applying Theorem 3.3.22, we find that our technique covers most of canonical formulae of substructural logic: for example, the following inequalities are canonical.

1. (Commutativity): $p_1 \circ p_2 \leq p_2 \circ p_1$.

2. (Square-increase): $p \leq p \circ p$.

3. (Right-lower-bound): $p_1 \circ p_2 \leq p_2$.

4. (A non-commutative version of Peirce's law): $((p_1 \backslash p_2) \backslash p_1) \leq p_1$.

5. (A non-commutative version of double negation): $((p \backslash 0) \backslash 0) \leq p$.

[18] covers 1, 2 and 3, while it does not prove the others. [25] proves that the canonicity of inequalities $s \leq t$, where s and t are finite compositions of $\vee$ and $\circ$. This is also a consequence of our approach: if s and t are finite compositions of $\vee$ and $\circ$, we have that s is a $\cup$-term and t does not have any negative occurrences, which means $s \leq t$ has consistent variable occurrence. So, $s \leq t$ is canonical. Furthermore, against the deficiency "As soon as we begin mixing multiplication (fusion) with divisions (residuals), things go wrong" in [25, p.302], Theorem 3.3.22 supplies many canonical inequalities containing both the fusion $\circ$ and the residuals $\backslash$ and $/$. For example, the following inequalities are consequences of Theorem 3.3.22:

6. $1 \leq (p_2/(p_2 \backslash p_1)) \backslash (p_1 \circ 0)$,

7. $p_1 \circ (p_2 \backslash p_1) \leq p_2 \circ (p_2/p_1)$,

8. $p_1 \circ (p_1/p_2) \leq (p_2 \circ p_2)/(p_1 \circ p_2)$,

9. $(p_2/p_1) \circ (p_1 \backslash p_3) \leq p_1 \backslash (p_2 \wedge p_3)$,

10. $p_3 \backslash (p_1 \circ p_2) \leq (p_1/p_3) \backslash p_2$.

4.2 Application 2: Completeness

One of the advantages of proving canonicity is that canonicity immediately provides Kripke completeness for bounded distributive lattice-based logics, e.g. intuitionistic logic or modal logic. Note that the argument of relational-type semantics for lattice-based logics in general is not completely done, yet. This is because the relational-type semantics over lattice-based logics, without Stone representation, is not thoroughly explained. We will come back to this discussion and one possible answer in Chapter 8. In this section, however, we focus only on bounded distributive substructural logics. In bounded distributive substructural logics, a relational semantics is spelled out via Stone representation in [81, 82].

In bounded distributive substructural logics, our language consists of propositional variables, *four* constants **t**, **f**, **T** and **F**, and five logical connectives. The algebraic counterparts of bounded distributive substructural logics are *bounded DFL-algebras* $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot \rangle$, where $\langle A, \vee, \wedge, \top, \bot \rangle$ is a bounded distributive lattice. Here, $\top$ and $\bot$ are constants corresponding to **T** and **F**, respectively. Firstly, we introduce the standard canonical extension of bounded DFL-algebras, via Stone representation and prove that it corresponds to the canonical extension $\langle \overline{\mathfrak{A}}, \vee_{\uparrow}, \wedge^{\downarrow}, \circ_{\uparrow}, \backslash^{\downarrow}, /^{\downarrow}, 1, 0, \top, \bot \rangle$.

Definition 4.2.1 (Stone representation of bounded DFL-algebras)**.** For a bounded DFL-algebra $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot \rangle$, the canonical extension of $\mathfrak{A}$ given by Stone representation is $(\mathfrak{A}_+)^+ = \langle \mathcal{U}(\mathcal{P}(\mathfrak{A})), \cup, \cap, *, \searrow, \swarrow\ \mathcal{P}_1(\mathfrak{A}), \mathcal{P}_0(\mathfrak{A}), \mathcal{P}(\mathfrak{A}), \emptyset \rangle$, where $\mathcal{P}(\mathfrak{A})$ is the set of all prime filters of $\mathfrak{A}$, $\mathcal{P}_1(\mathfrak{A})$ the set of all prime filters containing 1, $\mathcal{P}_0(\mathfrak{A})$ the set of all prime filters containing 0, $\mathcal{U}(X)$ the set of all upsets of X with respect to the set-theoretical inclusion, and binary operations

$*, \searrow, \swarrow$ on $\mathcal{U}(\mathcal{P}(\mathfrak{A}))$ are defined as follows; for each $X, Y \in \mathcal{U}(\mathcal{P}(\mathfrak{A}))$,

1. $X * Y := \{P \in \mathcal{P}(\mathfrak{A}) \mid \exists F \in X, \exists G \in Y.\ F \circ G \subseteq P\}$,
2. $X \searrow Y := \{P \in \mathcal{P}(\mathfrak{A}) \mid \forall F, G \in \mathcal{P}(\mathfrak{A}).\ F \circ P \subseteq G,\ F \in X \Longrightarrow G \in Y\}$,
3. $Y \swarrow X := \{P \in \mathcal{P}(\mathfrak{A}) \mid \forall F, G \in \mathcal{P}(\mathfrak{A}).\ P \circ F \subseteq G,\ F \in X \Longrightarrow G \in Y\}$,

where $F \circ G := \{x \in A \mid \exists f \in F, \exists g \in G.\ f \circ g \leq x\}$. Notice that this is the same as $\circ$ on the intermediate level in Definition 3.2.1. Here, it is restricted only for prime filters.

Note that prime filter frames $\mathfrak{A}_+$ of bounded DFL-algebras correspond to relational (Kripke) semantics, DFL-frames.

Definition 4.2.2 (DFL-frame)**.** A tuple $\mathbb{F} = \langle W, W_t, W_f, R_\circ \rangle$ is a *DFL-frame*, if W is a non-empty set, W_t a non-empty subset of W, W_f a subset of W, $R_\circ$ a ternary relation, and $\mathbb{F}$ satisfies the following. For each $w, v, u, s, w', v', u' \in W$, we have

1. $\exists t \in W_t.\ R_\circ(w, t, w)$ and $\exists t' \in W_t . R_\circ(w, w, t')$,
2. $R_\circ(w, v, u), w \preceq w', v' \preceq v, u' \preceq u \Rightarrow R_\circ(w', v', u')$,
3. $\exists x \in W.\ (R_\circ(w, x, s)$ and $R_\circ(x, v, u))$
$$\Longleftrightarrow \exists y \in W.\ (R_\circ(w, v, y) \text{ and } R_\circ(y, u, s)),$$
4. $w \preceq v$ and $w \in W_t$ imply $v \in W_t$,
5. $w \preceq v$ and $w \in W_f$ imply $v \in W_f$,

where $w \preceq v \iff \exists t \in W_t.\ R_\circ(v, t, w)$ or $R_\circ(v, w, t)$.

A valuation is a function from the set of propositional variables to the set of all $\preceq$-upsets of W. Then, we prove the following theorem. Note that, in the proof, we use the Prime filter theorem, see e.g. [13], and the Squeeze lemma in [17], which depend on the Axiom of Choice.

Theorem 4.2.3. *Let $\mathfrak{A}$ be a bounded DFL-algebra. With the Axiom of Choice, the canonical extension of $\mathfrak{A}$ is isomorphic to the canonical extension $(\mathfrak{A}_+)^+$ of $\mathfrak{A}$ given by Stone representation.*

Proof. Recall Item (b) in Remark 2.2.11. It states that the underlying set of $(\mathfrak{A}_+)^+$ and $\overline{\mathfrak{A}}$ are isomorphic. What we need to prove is that every operation and constant in bounded DFL-algebras coincides. The constant part is rather trivial. We here only check $\vee_\uparrow$ and $\backslash^\downarrow$.

For each $\alpha, \beta \in \overline{\mathfrak{A}}$, we claim that

$$(\alpha \vee_\uparrow \beta)_\uparrow = \lambda_2(\upsilon_2(\alpha_\uparrow) \cup \upsilon_2(\beta_\uparrow)), \tag{4.1}$$

$$(\alpha \backslash^\downarrow \beta)^\downarrow = \upsilon_1(\lambda_1(\alpha^\downarrow) \searrow \lambda_1(\beta^\downarrow)). \tag{4.2}$$

(4.1) Let P be an arbitrary prime filter in $\upsilon_2(\alpha_\uparrow) \cup \upsilon_2(\beta_\uparrow)$. Then, either, for each $I \in \alpha_\uparrow$, $P \cap I \neq \emptyset$, or, for each $J \in \beta_\uparrow$, $P \cap J \neq \emptyset$. By the way, since $\alpha \leq \alpha \vee_\uparrow \beta$ and $\beta \leq \alpha \vee_\uparrow \beta$, by Proposition 2.3.1, we have $(\alpha \vee_\uparrow \beta)_\uparrow \subseteq \alpha_\uparrow$ and $(\alpha \vee_\uparrow \beta)_\uparrow \subseteq \beta_\uparrow$. It follows the $\subseteq$-direction. Conversely, suppose that $I \notin (\alpha \vee_\uparrow \beta)_\uparrow$. Then, there exist $F \in \alpha^\downarrow$ and $G \in \beta^\downarrow$ such that $F \vee G \nsubseteq I \iff (F \cap G) \cap I = \emptyset$. By the Prime filter theorem, we can obtain a prime filter P satisfying that $F \vee G \subseteq P$ and $P \cap I = \emptyset$. Therefore, $I \notin \lambda_2(\upsilon_2(\alpha_\uparrow) \cup \upsilon_2(\beta_\uparrow))$.

(4.2) Let X be an arbitrary element in $(\alpha \backslash^\downarrow \beta)^\downarrow$, P an arbitrary prime filter satisfying

$X \subseteq P$. Since $X \in (\alpha\backslash^{\downarrow}\beta)^{\downarrow}$, for each $F \in \alpha^{\downarrow}$ and each $J \in \beta_{\uparrow}$, we have $X \sqsubseteq F\backslash J$. By the adjointness on the intermediate level, we also obtain that $F \circ X \in \beta^{\downarrow}$. For arbitrary prime filters P_1, P_2, if $P_1 \circ P \subseteq P_2$. Then, there exists $F \in \alpha^{\downarrow}$ such that $F \circ X \subseteq F \circ P \subseteq P_2 \in \beta^{\downarrow}$, hence the $\subseteq$-direction holds. On the other hand, suppose that $X \notin (\alpha\backslash^{\downarrow}\beta)^{\downarrow}$. There exist $F \in \alpha^{\downarrow}$, $J \in \beta_{\uparrow}$ such that $X \cap (F\backslash J) = \emptyset$. It follows that $(F \circ X) \cap J = \emptyset$. By the Prime filter theorem, there exists a prime filter P_2 such that $F \circ X \subseteq P_2$. By the Squeeze lemma, there exist prime filters P, P_1 such that $X \subseteq P$, $F \subseteq P_1$, and $P \circ P_1 \subseteq P_2$. Therefore, the $\supseteq$-direction is completed.

$\square$

In bounded distributive substructural logics, by Lemma 3.3.11 and Theorem 3.3.13, we can obtain the following terms of type $t_{\cup}$, $t_{\cap}$, $t_{\vee}$ and $t_{\wedge}$:

$$t_{\cup} ::= p \mid 1 \mid 0 \mid \top \mid \bot \mid t_{\cup} \vee t_{\cup} \mid t_{\cup} \wedge t_{\cup} \mid t_{\cup} \circ t_{\cup} \mid t_{\wedge},$$

$$t_{\cap} ::= p \mid 1 \mid 0 \mid \top \mid \bot \mid t_{\cap} \vee t_{\cap} \mid t_{\cap} \wedge t_{\cap} \mid t_{\cup}\backslash t_{\cap} \mid t_{\cap}/t_{\cup} \mid t_{\vee},$$

$$t_{\vee} ::= p \mid 1 \mid 0 \mid \top \mid \bot \mid t_{\vee} \vee t_{\vee} \mid t_{\vee} \wedge C \mid C \wedge t_{\vee} \mid t_{\vee} \circ C \mid C \circ t_{\vee},$$

$$t_{\wedge} ::= p \mid 1 \mid 0 \mid \top \mid \bot \mid t_{\wedge} \vee C \mid C \vee t_{\wedge} \mid t_{\wedge} \wedge t_{\wedge} \mid t_{\vee}\backslash C \mid C\backslash t_{\wedge} \mid t_{\wedge}/C \mid C/t_{\vee},$$

where C is a constant or a constant term. Based on these terms of type $t_{\cup}$, $t_{\cap}$, $t_{\vee}$ and $t_{\wedge}$. We can obtain canonical inequalities with Theorem 3.3.22. For example,

1. $1 \leq (p_1\backslash(0/p_2))\backslash(p_1\backslash 0)$,

2. $p_1\backslash(p_1\backslash 0) \leq 0$,

3. $(p_1 \wedge p_2)\backslash p_3 \leq (p_2\backslash 0)/(p_1 \wedge ((p_3\backslash 0)\backslash 0))$,

4. $1 \leq p_1 \vee (p_1\backslash 0)$,

5. $p_1 \wedge (p_1\backslash p_2) \leq p_2$,

Items 1 and 2 are derived without the distributivity, whereas the distributivity is essential for Items 3, 4 and 5.

Proposition 4.2.4. *Given a bounded distributive substructural logic* $\boldsymbol{L}$*, if the class of bounded DFL-algebras validating* $\boldsymbol{L}$ *is closed under the canonical extension,* $\boldsymbol{L}$ *is Kripke complete.*

As a corollary, together with Theorem 3.3.22, we obtain the following.

Theorem 4.2.5. *Let* Γ *be a set of canonical formulae as in Theorem 3.3.22. A bounded distributive substructural logic extended by* Γ *is Kripke complete.*

4.3 Application 3: Relevant modal logics

In this section, we compare our method with a Sahlqvist theorem for relevant modal logic in [76]. Our language of relevant modal logics consists of propositional variables, one constant 1, nine logical connectives $\vee, \wedge, \circ, \rightarrow, \neg, \Diamond, \Box, \boxdot$ and $\diamond\!\!\!\cdot$. Formulae are inductively defined as follows:

$$Fr ::= p \mid 1 \mid Fr \vee Fr \mid Fr \wedge Fr \mid Fr \circ Fr \mid Fr \rightarrow Fr \mid \Diamond Fr \mid \Box Fr \mid \boxdot Fr \mid \diamond\!\!\!\cdot Fr.$$

In [75, 76], the following algebra is given as the algebraic semantics of relevance modal logics.

Definition 4.3.1 (B.C$_{\Box\Diamond}$-algebra)**.** A tuple $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \rightarrow, \neg, \Diamond, \Box, \boxdot, \Diamonddot, 1\rangle$ is a *B.C$_{\Box\Diamond}$-algebra*, if $\langle A, \vee, \wedge\rangle$ is a distributive lattice, and $\mathfrak{A}$ satisfies, for all $a, b, c \in A$,

1. $a \leq b$ implies $a \circ c \leq b \circ c$,

2. $a \leq b$ implies $c \circ a \leq c \circ b$,

3. $a \circ b \leq c \iff a \leq b \rightarrow c$,

4. $a \vee b = \neg(\neg a \wedge \neg b)$,

5. $\Box(a \wedge b) = \Box a \wedge \Box b$,

6. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,

7. $1 \circ a = a$,

8. $\boxdot a = \neg\Diamond\neg a$,

9. $\Diamonddot a = \neg\Box\neg a$.

Remark 4.3.2. In B.C$_{\Box\Diamond}$-algebras, $\circ$ may not be associative nor commutative. In this sense, B.C$_{\Box\Diamond}$-algebras are weaker than FL-algebras or commutative FL-algebras.

Every B.C$_{\Box\Diamond}$-algebra can be seen as a lattice expansion. So, the canonical extension of B.C$_{\Box\Diamond}$-algebras is defined as $\langle \overline{\mathfrak{A}}, \vee_{\uparrow}, \wedge^{\downarrow}, \circ_{\uparrow}, \rightarrow^{\downarrow}, \neg, \Diamond_{\uparrow}, \Box^{\downarrow}, \boxdot^{\downarrow}, \Diamonddot_{\uparrow}, 1\rangle$: see Example 3.1.8 It is necessary to check the following theorem.

Theorem 4.3.3 (Canonical extension of B.C$_{\Box\Diamond}$-algebras)**.** *The canonical extension of a B.C$_{\Box\Diamond}$-algebra is also a B.C$_{\Box\Diamond}$-algebra.*

Proof. It is analogous to the proof of Theorem 4.1.3, except Item 4 in Definition 4.3.1.

Let α, β be arbitrary elements in $\overline{\mathfrak{A}}$. Since both $\alpha \vee_{\uparrow} \beta$ and $\neg(\neg\alpha \wedge^{\downarrow} \neg\beta)$ are $\cup$-term, see terms of type $t_{\cup}$, $t_{\cap}$, $t_{\vee}$ and $t_{\wedge}$ (4.3) - (4.6), we have

$$(\alpha \vee_{\uparrow} \beta)_{\uparrow} = \lambda(\{F \vee G \mid F \in \alpha^{\downarrow}, G \in \beta^{\downarrow}\}),$$

$$\neg(\neg\alpha \wedge^{\downarrow} \neg\beta)_{\uparrow} = \lambda(\{\neg(\neg F \wedge \neg G) \mid F \in \alpha^{\downarrow}, G \in \beta^{\downarrow}\}).$$

Then, we notice that it suffices to show that, for each $F \in \alpha^{\downarrow}$ and each $G \in \beta^{\downarrow}$, $F \vee G = \neg(\neg F \wedge \neg G)$, which is straightforward. $\square$

Definition 4.3.4 (Canonicity of term)**.** On the class of B.C$_{\Box\Diamond}$-algebras, a formula (term) t is *canonical*, if the inequality $1 \leq t$ is canonical.

As a corollary of Theorem 3.3.5, we obtain the following.

Corollary 4.3.5. *In relevant modal logics, every $\cap$-term is canonical.*

Now, we syntactically describe a class of canonical formulae. By Lemma 3.3.11 and Theorem 3.3.13, we obtain the following terms of type $t_{\cup}$, $t_{\cap}$, $t_{\vee}$ and $t_{\wedge}$.

$$t_{\cup} ::= p \mid 1 \mid t_{\cup} \vee t_{\cup} \mid t_{\cup} \wedge t_{\cup} \mid t_{\cup} \circ t_{\cup} \mid \neg t_{\cap} \mid \Diamond t_{\cup} \mid \Diamond\!\!\!\cdot\ t_{\cup} \mid t_{\wedge}, \tag{4.3}$$

$$t_{\cap} ::= p \mid 1 \mid t_{\cap} \vee t_{\cap} \mid t_{\cap} \wedge t_{\cap} \mid t_{\cup} \to t_{\cap} \mid \neg t_{\cup} \mid \Box t_{\cap} \mid \boxdot t_{\cap} \mid t_{\vee}, \tag{4.4}$$

$$t_{\vee} ::= p \mid 1 \mid t_{\vee} \vee t_{\vee} \mid t_{\vee} \wedge C \mid C \wedge t_{\vee} \mid t_{\vee} \circ C \mid C \circ t_{\vee} \mid \neg t_{\wedge} \mid \Diamond t_{\vee} \mid \Diamond\!\!\!\cdot\ t_{\vee}, \tag{4.5}$$

$$t_{\wedge} ::= p \mid 1 \mid t_{\wedge} \wedge t_{\wedge} \mid t_{\wedge} \vee C \mid C \vee t_{\wedge} \mid t_{\vee} \to C \mid C \to t_{\wedge} \mid \neg t_{\vee} \mid \Box t_{\wedge} \mid \boxdot t_{\wedge}, \tag{4.6}$$

where C is a constant or a constant term.

As a corollary of Theorem 3.3.22, together with terms of type $t_{\cup}$, $t_{\cap}$, $t_{\vee}$ and $t_{\wedge}$ (4.3) - (4.6), we obtain canonical logics of relevance modal logics. We briefly check that [76] is a consequence of our approach. [76] proves that a finite conjunction of $\blacksquare^n(B \to C)$ is canonical, where $\blacksquare^n$ is a n-composition of $\Box$ or $\boxdot$, C is a positive formula, and B is untied: $B := p \mid N \mid B \wedge B \mid \Diamond B \mid \diamonddot B$ where N is a negative formula. As our method is not closed under conjunctions, we think about a set of $\blacksquare^n(B \to C)$ instead of conjunctions. Let us consider the inequality $1 \leq \blacksquare^n(B \to C)$. There is only one possibility that $\blacksquare^n(B \to C)$ has $\cap$-critical subterms; as positive formulae, in C, and as negative formulae, in B. It follows that the inequality has consistent variable occurrence. Therefore, all Sahlqvist formulae in [76] are consequences of our theorem. On the other hand, for example, formulae in the following list

1. $(p_1 \circ (p_2 \to p_1)) \to ((p_1 \circ p_2) \to (p_2 \circ p_2))$.

2. $((\Diamond p_1 \to p_2) \circ (p_1 \to \neg p_3)) \to \Box(p_1 \to (p_2 \wedge \neg p_3))$.

3. $\Box(p_1 \circ (p_2 \to p_1)) \to \Box(\Diamond p_2 \circ (\Box p_1 \to p_2))$.

are not of type $\blacksquare^n(B \to C)$ but they are consequences of Theorem 3.3.22, together with the terms of type $t_\cup$, $t_\cap$, $t_\vee$ and $t_\wedge$ (4.3) - (4.6).

4.4 Application 4: Distributive modal logic

In this section, we give a comparison between Ghilardi and Meloni's method and the technique of [30]. Firstly, we fix the language for the setting in [30]. A distributive modal algebra is a tuple $\mathbb{A} = \langle A, \vee, \wedge, \perp, \top, \Diamond, \Box, \rhd, \lhd \rangle$. As we saw in Example 3.1.2, together with Proposition 3.2.10, distributive modal algebras are smooth lattice expansions. Based on the language, we obtain that the syntactically described terms of types $t_\cup$, $t_\cap$, $t_\vee$ and $t_\wedge$ are,

$$t_\cup ::= p \mid \bot \mid \top \mid t_\cup \vee t_\cup \mid t_\cup \wedge t_\cup \mid \Diamond t_\cup \mid \lhd\, t_\cap \mid t_\wedge,$$

$$t_\cap ::= p \mid \bot \mid \top \mid t_\cap \vee t_\cap \mid t_\cap \wedge t_\cap \mid \Box t_\cap \mid \rhd\, t_\cup \mid t_\vee,$$

$$t_\vee ::= p \mid \bot \mid \top \mid t_\vee \vee t_\vee \mid t_\vee \wedge C \mid C \wedge t_\vee \mid \Diamond t_\vee \mid \lhd\, t_\wedge,$$

$$t_\wedge ::= p \mid \bot \mid \top \mid t_\wedge \vee C \mid C \vee t_\wedge \mid t_\wedge \wedge t_\wedge \mid \Box t_\wedge \mid \rhd\, t_\vee,$$

where C is a constant term: see Lemma 3.3.11 and Theorem 3.3.13.

Now, we briefly recall the canonicity algorithm shown in [30]. In [30, p.79], there is a slogan "the main feature that may make non-Sahlqvist formulae ill-behaved is that the 'outside' connectives are 'universal' (boxes), while the 'inside' connectives are 'choice' connectives (that is, diamonds or disjunction)."

Let $s \leq t$ be an inequality. On the construction (generation) trees of s and t, we label every node with a *sign* ($+$ or $-$) in the following manner.[1] This labelling also starts with the root.

Signing algorithm in [30]

1. Label the root of the construction tree of s with $-$ and the root of the construction tree of t with $+$.

2. If the current note does not have any child, we stop labelling. Otherwise, we label $+$ or $-$ for each child based on the following step.

   (a) If the node is either $\rhd\, t$ or $\lhd\, t$, then label t with the converse sign of the

[1] This labelling is the converse of the original work [30]. Reversing $+$ and $-$, these signs directly correspond to our signs on the well-pruned pair of trees for $s \leq t$.

current node.

(b) Otherwise, label every child with the same sign of the current node.

3. Move every child and repeat Item 2 until every node is labelled.

Based on these signs, we define *universal nodes* and *choice nodes* as follows.

1. A node is *choice*, if either the node is signed with $+$ and the outermost connective is $\wedge, \Box$ or $\rhd$, or the node is signed with $-$ and the outermost connective is $\vee, \Diamond$ or $\lhd$.

2. A node is *universal*, if either the node is signed with $+$ and the outermost connective is $\Diamond$ or $\lhd$, or the node is signed with $-$ and the outermost connective is $\Box$ or $\rhd$.

Moreover, a universal node is called *the first universal node*, if there is not any universal node on the path from it to the root, except itself. This is not defined in [30], but it allows us to compare the two method simply.

Systematic analysis of two types of terminology We compare these labelling rules, and our terminology ($\cup$-terms, $\cap$-terms, critical terms) with choice nodes and universal nodes in [30]. We can straightforwardly understand that the signing algorithm ($+$ and $-$) in [30] is exactly the same as our signing algorithm *before* pruning, see Definition 3.3.17[2] Let us start with an example, an inequality $s \leq t$, where

$$s = (\rhd\ (p \wedge (\lhd\ q))) \vee (\lhd\ \Box(p \wedge q)),$$

$$t = (\Diamond\ \rhd\ ((\rhd\ p) \wedge q)) \wedge (\Box((\lhd\ p) \wedge (q \vee p))).$$

[2] Our signing is introduced after pruning. But, we can naturally consider the same algorithm without pruning.

Figure 4.1: The ∪-labelled and −-signed construction tree of s

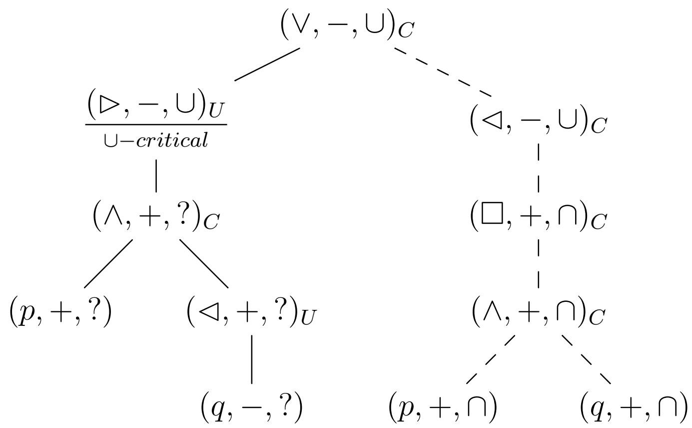
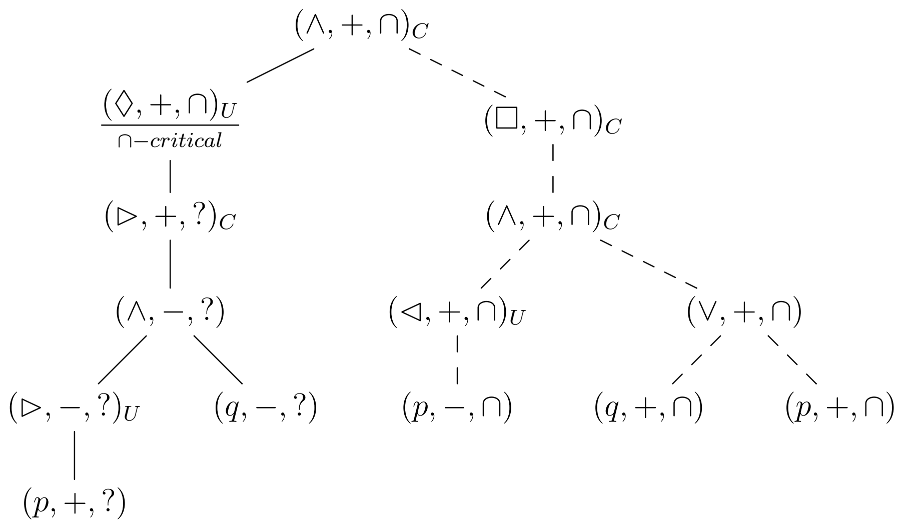


Figure 4.2: The ∩-labelled and +-signed construction tree of t



To compare the two methods clearly, we draw the following labelled and signed construction trees of s and t (Fig. 4.1 and Fig. 4.2), where each node is denoted by a tuple (*the outermost connective*, *the sign of the node*, *the label of the node*), universal nodes and choice nodes are denoted by subscripts U and C, and dashed lines are pruned when we consider the well-pruned trees.

At first, we may feel that the signs $-$ and $+$ are similar to the labels $\cup$ and $\cap$ respectively. However, it is apparent from the above example that they are not precisely the same. Next, we check the details of the labels ($\cup$, $\cap$ and ?) and the signs (+ and $-$). Our discussion is separated into two parts: above the first universal nodes and below the first universal nodes, if they exist.

We prove the following proposition for every node above the first universal node, from which there is no universal node on the path to the root.

Proposition 4.4.1. *Above the first universal nodes (if they exist), every node is either labelled with* $\cup$ *and signed with* $-$*, called type* $(-, \cup)$*, or labelled* $\cap$ *and signed with* $+$*, called type* $(+, \cap)$*. Hence, above the first universal nodes (if they exist),* $\cup$ *and* $\cap$ *correspond to* $-$ *and* $+$*, respectively.*

Proof. Induction on the construction trees of s and t. The basic steps are the roots. It is straightforward by definition, because the root of s is labelled with $\cup$ and signed with $-$, hence it is of type $(-, \cup)$, and the root of t is labelled with $\cap$ and signed with $+$, hence it is of type $(+, \cap)$.

To consider the inductive steps, we compare the labelling algorithms and the signing algorithm. Then, we can sum up as follows:

1. if a outermost connective of a node of type $(-, \cup)$ is one of $\vee, \wedge, \Diamond$, we label the children of the node with the same label $\cup$ and the same sign $-$; that is, every child is of type $(-, \cup)$,

2. if a outermost connective of a node of type $(-, \cup)$ is $\lhd$, we label the child of the node with the label $\cap$ and the converse sign $+$; that is, the child is of type $(+, \cap)$,

3. if a node of type $(-, \cup)$ is neither Item 1 nor Item 2, the outermost connective of the node must be either of $\Box$ and $\rhd$,

4. if a outermost connective of a node of type $(+, \cap)$ is one of $\vee, \wedge, \Box$, we label the children of the node with the same label $\cap$ and the same sign $+$; that is, every child is of type $(+, \cap)$,
5. if a outermost connective of a node of type $(+, \cap)$ is $\rhd$, we label the child of the node with the label $\cup$ and the converse sign $-$; that is, the child is of type $(-, \cup)$,
6. If a node of type $(+, \cap)$ is neither Item 4 nor Item 5, the outermost connective of the node must be either of $\Diamond$ and $\lhd$.

Then, whenever we consider the cases of Items 1, 2, 4 and 5, they prove the inductive steps, hence $\cup$ and $\cap$ correspond to $-$ and $+$, respectively. Now, we focus on the cases of Items 3 and 6. If a node satisfies Item 3 or 6 and above the first universal node, by the definition of universal nodes, the node is the first universal node. □

Remark 4.4.2. Let $s \leq t$ be an inequality. If the construction trees of s and t have no universal node, s is a term of type $t_\cup$ containing no subterms of type $t_\vee$ and $t_\wedge$, and t is a term of type $t_\cap$ containing no subterms of type $t_\vee$ and $t_\wedge$.

We look back to the conditions of Items 3 and 6 in the proof of Proposition 4.4.1. In our $\cup$-labelling and $\cap$-labelling algorithms, if a node satisfies one of Items 3 and 6, we must have a break of the labelling from the root to the leaves, see Section 3.3, because, if a node is of type $(-, \cup)$ and satisfies Item 3, we have two choices to label:

(i) if the node is a term of type $t_\wedge$, all nodes below the node, not only the children, are labelled with $\cup$,

(ii) otherwise, all nodes below the node are labelled with ?.

Analogously, if a node is of type $(+, \cap)$ and satisfies Item 6, we have two choices to label:

(i) if the node is a term of type $t_\vee$, all nodes below the node, not only the children, are labelled with $\cap$,

(ii) otherwise, all nodes below the node are labelled with ?.

Note that any term is a *critical* subterm, if Item (ii) holds.

Next, we compare the labelling algorithms under the first universal nodes. Let A be the first universal node of type $(-, \cup)$, and B the first universal node of type $(+, \cap)$. To compare the labelling under the first universal nodes A and B, we recall the $\wedge$-labelled construction tree of A and the $\vee$-labelled construction tree of B in Section 3.3. Note that, if the $\wedge$-labelled ($\vee$-labelled) construction tree of A (B) has no nodes labelled with ?, A is a term of type $t_\wedge$ ($t_\vee$), hence A (B) is not critical. In the $\wedge$-labelled and $-$-signed construction tree of A and the $\vee$-labelled and $+$-signed construction tree of B (note that signs are the same as the above ones), we can prove the following proposition. In the proof of Proposition 4.4.3, we compare the signing algorithm in [30] and the $\wedge$-labelling $\vee$-labelling algorithm based on *non-distributive* cases. We will shortly come back to the comparison with $\wedge$-labelling and $\vee$-labelling on *distributive* cases in Remark 4.4.5.

Proposition 4.4.3. *In the $\wedge$-labelled and $-$-signed construction tree of A, every node above any choice node is either of type $(-, \wedge)$ or of type $(+, \vee)$. Analogously, in the $\vee$-labelled and $+$-signed construction tree of B, every node above any choice node is either of type $(+, \vee)$ or of type $(-, \wedge)$.*

Proof. Induction on the construction trees of A and B. The basic cases are rather trivial, by definition. To discuss the induction steps, we compare our labelling algorithms and the signing algorithm. Remind that we here compare the signing in [30] with the $\wedge$-labelling and $\vee$-labelling based on *non-distributive cases*, i.e. without (a') and (b'): see $\wedge$ ($\vee$) labelling algorithm.

1. If a connective of a node of type $(-, \wedge)$ is either of $\wedge$ and $\Box$, we label the children with the same label $\wedge$ and the same sign $-$; that is, every child of the node is of type $(-, \wedge)$.

2. If a connective of a node of type $(-, \wedge)$ is $\rhd$, we label the child with the label $\vee$ and the converse sign $+$; that is, the child is of type $(+, \vee)$.

3. If a node of type $(-, \wedge)$ is neither item 1 nor item 2, a connective of the node must be one of $\vee, \Diamond, \lhd$.

4. If a connective of a node of type $(+, \vee)$ is either of $\vee$ and $\Diamond$, we label the children with the same label $\vee$ and the same sign $+$; that is, every child is of type $(+, \vee)$.

5. If a connective of a node of type $(+, \vee)$ is $\lhd$, we label the child with the label $\wedge$ and the converse sign $-$; that is, the child is of type $(-, \wedge)$.

6. If a node of type $(+, \vee)$ is neither item 4 nor item 5, a connective of the node must be one of $\wedge, \Box, \rhd$.

Then, when we see item 1, 2, 4 and 5, they prove the induction steps. Moreover, if a node satisfies item 3 or item 4, it is a choice node by the definition of choice nodes. $\Box$

As a corollary, we straightforwardly obtain the following.

Corollary 4.4.4. *Critical subterms are the first universal nodes which contain choice nodes in their scopes.*

Remark 4.4.5. The converse of Corollary 4.4.4 is not true in general. That is, there are some terms which have the first universal nodes containing choice nodes in their scopes but are not critical. In the inequality $\Box((\Box\bot) \vee (\rhd\, p)) \leq \Diamond((\rhd\, p) \vee (\lhd\, p))$, for example, the (sub)term $\Box((\Box\bot) \vee (\rhd\, p))$ is the first universal node containing a choice node $(\Box\bot) \vee (\rhd\, p)$, but it is not $\cup$-critical. The difference comes from the $\wedge$-labelling algorithm on the distributive-cases: see (a') and (b') in the $\wedge$-labelling algorithm in Section 3.3.

Comparison of two results By Corollary 4.4.4, we can conclude that canonicity results in [30] are also consequences of Theorem 3.3.22. On the other hand, as we saw in Remark 4.4.5, the class of canonical inequalities syntactically obtained by our approach is slightly larger than the one in [30]. The difference comes from the definition of choice nodes and the $\wedge$-labelling and $\vee$-labelling algorithms based on the distributive-cases. Concretely, the definitions that $\vee$ signed with $-$ and $\wedge$ signed with $+$ are choice are slightly stronger conditions than what we assume. On the other hand, both results coincide, if we modify the definition of choice nodes of [30] as follows: A node is *choice*, if *every branch of the node contains at least one propositional variable*, and either the node is signed with $+$ and the outermost connective is $\wedge$, $\Box$ or $\rhd$, or the node is signed with $-$ and the outermost connective is $\vee$, $\Diamond$ or $\lhd$. Then, we can summarise the correspondence as in Table 4.1.[3]

[3] In the table, the signs $+$ and $-$ are the converse to the original notation in [30]

Table 4.1: Summary of the Correspondence

Our terminology	Terminology in [30]
$+$	$+$
$-$	$-$
$\cup$	signed with $-$ and not below the first universal node
$\cap$	signed with $+$ and not below the first universal node
critical	the first universal node containing choice nodes in the scope

Chapter 5

Canonicity of poset expansions

The argument of canonical extensions are nowadays generalised up to posets in general. We can universally characterise canonical extensions over posets, i.e. including lattices, bounded distributive lattices and Boolean algebras, as compact dense completions and they are unique up to isomorphism (see Section 7.1). With this recent generalisation of canonical extensions, one of the questions which naturally arise is the canonicity property over poset expansions in general. More precisely, how does the lack of the lattice operations $\vee$ and $\wedge$ affect our canonicity argument?

In this chapter, we will show that the main technique of Ghilardi and Meloni's canonicity methodology does not work over poset expansions in general. However, we will also show that, by removing ill-behaved parts carefully, we can still account for reasonably many canonical results of poset expansions.

5.1 Poset expansions

In this section, we recall poset expansions: see Section 3.1. Let $\mathbb{P}$ be a poset. A n-ary function on $\mathbb{P}$ is a ϵ-operation, if there exists a list, *order-type*, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, in which each ϵ_i is either 1 or ∂, such that f is monotone from the product poset $\mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n}$ to the codomain $\mathbb{P}$. Among ϵ-operations, we define the following properties for poset expansions. Note that these properties are the generalised versions of the case of lattice expansions.

Definition 5.1.1 (Join-preservability and meet-preservability)**.** Let $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$ be posets. A monotone map $f : \mathbb{P} \to \mathbb{Q}$ is *join-preserving*, if, for all $p_1, p_2 \in P$ and each $q \in Q$ satisfying $f(p_1) \leq_Q q$ and $f(p_2) \leq_Q q$, there exists $p \in P$ such that $p_1 \leq_P p$, $p_2 \leq_P p$ and $f(p) \leq_Q q$. Order-dually, a monotone map $g : \mathbb{P} \to \mathbb{Q}$ is *meet-preserving*, if, for all $p_1, p_2 \in P$ and each $q \in Q$ satisfying $q \leq_Q g(p_1)$ and $q \leq_Q g(p_2)$, there exists $p \in P$ such that $p \leq_P p_1$, $p \leq_P p_2$ and $q \leq_Q g(p)$. Especially, if a ϵ-operation f is join-preserving (meet-preserving) from the product domain $\mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n}$, it is called *$\epsilon$-join-preserving* (*$\epsilon$-meet-preserving*).

Definition 5.1.2 (Strictness)**.** Let $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$ be posets. A monotone map $f : \mathbb{P} \to \mathbb{Q}$ is *$\bot$-strict*, if, for each $q \in Q$, there exists $p \in P$ such that $f(p) \leq_Q q$. Analogously, a monotone map $g : \mathbb{P} \to \mathbb{Q}$ is *$\top$-strict*, if, for each $q \in Q$, there exists $p \in P$ such that $q \leq_Q g(p)$.

Definition 5.1.3 (Additivity and multiplicativity)**.** Let $\mathbb{P}$ be a poset. A ϵ-operation $l : \mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n} \to \mathbb{P}$ is *ϵ-additive*, if l is join-preserving for each coordinate. Additionally, if l satisfies $\bot$-strictness for each coordinate, l is *$\epsilon_\bot$-additive*. Analogously, a ϵ-operation $r : \mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n} \to \mathbb{P}$ is *ϵ-multiplicative*, if r is meet-preserving for each coordinate. Moreover, if r satisfies $\top$-strictness for each coordinate as well, r is *$\epsilon^\top$-multiplicative*. If the order-type is clear, we sometimes refer them simply to *additive*, *$\bot$-additive*, *multiplicative* and *$\top$-multiplicative*.

Example 5.1.4. Let $\mathbb{P}$ be a poset and $r : \mathbb{P}^1 \times \mathbb{P}^\partial \to \mathbb{P}$ a $(1, \partial)$-operation. r

is $(1, \partial) - \textit{multiplicative}$, if r satisfies the following Items 1 and 2: for arbitrary $p_1, p_2, a, b, q \in P$,

1. if $q \leq r(p_1, b)$ and $q \leq r(p_2, b)$, there exists $p \in P$ such that $p \leq p_1$, $p \leq p_2$ and $q \leq r(p, b)$,

2. if $q \leq r(a, p_1)$ and $q \leq r(a, p_2)$, there exists $p \in P$ such that $p_1 \leq p$, $p_2 \leq p$ and $q \leq r(a, p)$. (Note that the order of the second coordinate is reversed.)

Moreover, if r also satisfies the following Items 3 and 4, r is $(1, \partial)^\top$-multiplicative. For all $a, b, q \in P$,

3. there exists $p_1 \in P$ such that $q \leq r(p_1, b)$,

4. there exists $p_2 \in P$ such that $q \leq r(a, p_2)$.

Definition 5.1.5 (Adjoint pair)**.** Let $\mathbb{P}$ be a poset, $l : \mathbb{P}^{\mu_1} \times \cdots \times \mathbb{P}^{\mu_n} \to \mathbb{P}$ a n-ary μ-additive operation, $r : \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_n} \to \mathbb{P}$ a n-ary ν-multiplicative operation, and i be a fixed coordinate in $\{1, \ldots, n\}$. A pair l and r forms an *adjoint pair with respect to the coordinate* i, denoted by $l \dashv^i r$, if $\mu_i = \nu_i = 1$ and, for all the other coordinates the order-types are opposite, i.e. either $\mu_k = 1$ and $\nu_k = \partial$ or $\mu_k = \partial$ and $\nu_k = 1$ for each $k \in \{1, \ldots, i-1, i+1, \ldots, n\}$, and they satisfy the following adjointness condition: for all $x, y, c_1, \ldots, c_n \in P$, we have

$$l(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n) \leq y \iff x \leq r(c_1, \ldots, c_{i-1}, y, c_{i+1}, \ldots, c_n).$$

Remark 5.1.6. As in the case of lattice expansions, our definition of adjoint pairs require the join-preservability and the meet-preservability for each coordinate, not only for the i-th coordinate. It is necessary to prove Theorems 5.5.16 and 5.5.17.

We call a pair of a poset $\mathbb{P}$ and a set of ϵ-operations on $\mathbb{P}$ a *poset expansion*, denoted by $\langle \mathbb{P}, f_1, \dots \rangle$.[1] In the rest of this section and the following section (Section 5.2), we consider just a poset expansion $\langle \mathbb{P}, f, g \rangle$, where f is a $(1,1)$-operation and g is a $(\partial, 1)$-operation. But, it is straightforwardly extended to arbitrary poset expansions. Based on the poset expansion $\langle \mathbb{P}, f, g \rangle$, we inductively define terms as usual.

$$term ::= p_i \mid f(term, term) \mid g(term, term),$$

where p_i is a propositional variable. We think about terms as term functions as follows: for all $x_1, \dots, x_N \in P$, we let

1. $p_i(x_1, \dots, x_N) := x_i$,
2. $f(t_1, t_2)(x_1, \dots, x_N) := f(t_1(x_1, \dots, x_N), t_2(x_1, \dots, x_N))$,
3. $g(t_1, t_2)(x_1, \dots, x_N) := g(t_1(x_1, \dots, x_N), t_2(x_1, \dots, x_N))$,

where, and hereinafter, we always assume that N is a finite number which is large enough to cover any arity as in the case of lattice expansions.

5.2 Canonical extension of poset expansions

In this section, we introduce canonical extensions of poset expansions on two steps, via the intermediate level. We note that the main technique invented in [33] is not only to extend each operation on a poset onto the canonical extension but also to lift up the evaluation of all term functions onto the intermediate level, which is used to *approximate* the values of term functions on the canonical extension, see also [84].

[1] Each operation can have its own order-type, e.g. $\langle \mathbb{P}, f, g \rangle$ where f is $(1,1)$-operation and g is $(\partial, 1)$-operation. In other words, the order type ϵ is not uniform for all operations.

We extend ϵ-operations on the intermediate level as follows. Note that the following definition looks like the same as Definition 3.2.1. But, in the following definition, filters and ideals are based on posets.

Definition 5.2.1 (ϵ-operation on the intermediate level)**.** For all $F, G \in \mathcal{F}$ and all $I, J \in \mathcal{I}$, we let

1. $f(F, G) := \{x \in P \mid \exists a \in F, \exists b \in G.\ f(a, b) \leq x\}$,
2. $f(I, J) := \{y \in P \mid \exists a \in I, \exists b \in J.\ y \leq f(a, b)\}$,
3. $g(I, F) := \{x \in P \mid \exists a \in I, \exists b \in F.\ g(a, b) \leq x\}$,
4. $g(F, I) := \{y \in P \mid \exists a \in F, \exists b \in I.\ y \leq g(a, b)\}$.

Proposition 5.2.2. *Each ϵ-operation on the intermediate level is well-defined. That is,f is extended to both $f : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ (Item 1) and $f : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ (Item 2). g is extended to both $g : \mathcal{I} \times \mathcal{F} \to \mathcal{F}$ (Item 3) and $g : \mathcal{F} \times \mathcal{I} \to \mathcal{I}$ (Item 4).*

Proof. We check only Item 3, here. But, the others are analogously proved. For each filter $F \in \mathcal{F}$ and each ideal $I \in \mathcal{I}$, since every filter and every ideal are non-empty, $g(I, F)$ is also non-empty. By definition, $g(I, F)$ is upward closed. For all $x_1, x_2 \in g(I, F)$, there exist $a_1, a_2 \in I$ and $b_1, b_2 \in F$ such that $g(a_1, b_1) \leq x_1$ and $g(a_2, b_2) \leq x_2$. As I is an ideal and F is a filter, there exist $a \in I$ and $b \in F$ such that $a_1 \leq a$, $a_2 \leq a$, $b \leq b_1$ and $b \leq b_2$. Since g is a $(\partial, 1)$-operation, we obtain

$$g(a, b) \leq g(a_1, b_1) \leq x_1 \text{ and } g(a, b) \leq g(a_2, b_2) \leq x_2.$$

Since $g(a, b) \in g(I, F)$ and $g(a, b)$ is a lower bound of $\{x_1, x_2\}$, $g(I, F)$ is down-directed. Hence, it is a filter. $\square$

Remark 5.2.3. On the intermediate level, we naturally have two types of extensions of ϵ-operations: extending on filters and extending on ideals. Since they are order-dually related, i.e. every filter on $\mathbb{P}^1$ is an ideal on $\mathbb{P}^\partial$ vice versa, every order-dual coordinate should be the opposite-type. For example, each first coordinate of a $(\partial, 1)$-operation g is the opposite sort. $g : \mathcal{I} \times \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{F} \times \mathcal{I} \to \mathcal{I}$.

Remark 5.2.4. Since our target is to evaluate every term function on the intermediate level and to use them to approximate the value on the canonical extension, the both types of extensions are mandatory. Otherwise, we would have some term functions which cannot be computed on the intermediate level (see Section 7.2).

Parallel computation on the intermediate level To computer all term functions on the intermediate level, we first introduce the $\|$-notation, e.g. $Pos\|Neg$, which means that positive occurrences are replaced by Pos and negative occurrences are replaced by Neg (see [33]). Based on the $\|$-notation, we inductively evaluate term functions on the intermediate level as follows: for all $F_1, \ldots, F_N \in \mathcal{F}$ and all $I_1, \ldots, I_N \in \mathcal{I}$, we let

$$
\begin{array}{l|l}
p_i(\underline{F\|I}) := F_i & p_i(\underline{I\|F}) := I_i \\
f(t_1,t_2)(\underline{F\|I}) := f(t_1(\underline{F\|I}), t_2(\underline{F\|I})) & f(t_1,t_2)(\underline{I\|F}) := f(t_1(\underline{I\|F}), t_2(\underline{I\|F})) \\
g(t_1,t_2)(\underline{F\|I}) := g(t_1(\underline{I\|F}), t_2(\underline{F\|I})) & g(t_1,t_2)(\underline{I\|F}) := g(t_1(\underline{F\|I}), t_2(\underline{I\|F}))
\end{array}
$$

where $(\underline{F\|I}) = (F_1\|I_1, \ldots, F_N\|I_N)$ and $(\underline{I\|F}) = (I_1\|F_1, \ldots, I_N\|F_N)$.

Example 5.2.5. For each $F \in \mathcal{F}$ and each $I \in \mathcal{I}$, the term $g(p_1, p_1)$ is calculated in parallel as follows:

1. $g(p_1, p_1)(F\|I) = g(p_1(I\|F), p_1(F\|I)) = g(I, F)$,

2. $g(p_1, p_1)(I\|F) = g(p_1(F\|I), p_1(I\|F)) = g(F, I)$.

As in the case of lattice expansions, we straightforwardly obtain the following monotonicity lemma, by the parallel induction.

Lemma 5.2.6 (Monotonicity on the intermediate level)**.** *Let t be a term. For all $F_1, \ldots, F_N, G_1, \ldots, G_N \in \mathcal{F}$ and all $I_1, \ldots, I_N, J_1, \ldots, J_N \in \mathcal{I}$, if $F_k \sqsubseteq G_k$ and $I_k \sqsubseteq J_k$ for each $k \in \{1, \ldots, N\}$, we have*

$$t(F_1\|J_1, \ldots, F_N\|J_N) \sqsubseteq t(G_1\|I_1, \ldots, G_N\|I_N),$$

$$t(I_1\|G_1, \ldots, I_N\|G_N) \sqsubseteq t(J_1\|F_1, \ldots, J_N\|F_N).$$

Next we introduce the $\|$-notation on the poset expansion $\langle \mathbb{P}, f, g \rangle$, which allows us to build up a stable relationship between term functions on the poset expansion and those on the intermediate level (see Lemma 5.2.8). On the poset expansion $\langle \mathbb{P}, f, g \rangle$, we define the (partial) term functions as follows: for all $x_1, \ldots, x_N, y_1, \ldots, y_N \in P$, we let

1. $p_i(x_1\|y_1, \ldots, x_N\|y_N) := x_i$,

2. $f(t_1, t_2)(x_1\|y_1, \ldots, x_N\|y_N) := f(t_1(x_1\|y_1, \ldots, x_N\|y_N), t_2(x_1\|y_1, \ldots, x_N\|y_N))$,

3. $g(t_1, t_2)(x_1\|y_1, \ldots, x_N\|y_N) := g(t_1(y_1\|x_1, \ldots, y_N\|x_N), t_2(x_1\|y_1, \ldots, x_N\|y_N))$.

We note that, whenever we put the same variable for each occurrence, e.g. $x_k\|x_k$, for each coordinate, we obtain $t(x_1, \ldots, x_N) = t(x_1\|x_1, \ldots, x_N\|x_N)$. On $\mathbb{P}$, we obtain the monotonicity lemma.

Lemma 5.2.7. *Let t be a term. For arbitrary $x_1, \ldots, x_N$, $y_1, \ldots, y_N$, $z_1, \ldots, z_N$, $w_1, \ldots, w_N \in P$, if $x_k \leq y_k$ and $z_k \leq w_k$ for each $k \in \{1, \ldots, N\}$, we have*

$$t(x_1 \| w_1, \ldots, x_N \| w_N) \leq t(y_1 \| z_1, \ldots, y_N \| z_N).$$

As in the case of lattice expansions, we also obtain the following lemma. Note that the statement in Lemma 5.2.8 is exactly the same as Lemma 3.2.6, but filters and ideals are based on posets.

Lemma 5.2.8. *Let t be a term. For all $F_1, \ldots, F_N \in \mathcal{F}$, all $I_1, \ldots, I_N \in \mathcal{I}$ and all $x, y \in P$, we have*

1. $x \in t(\underline{F \| I}) \iff \forall k \in \{1, \ldots, N\}, \exists a_k \in F_k, \exists b_k \in I_k.\ t(\underline{a \| b}) \leq x,$
2. $y \in t(\underline{I \| F}) \iff \forall k \in \{1, \ldots, N\}, \exists c_k \in I_k, \exists d_k \in F_k.\ y \leq t(\underline{c \| d}).$

Proof. Parallel induction. Base cases are straightforward (recall that every filter and every ideal are non-empty). Here we check only the inductive step of g for Item 2. But the others are analogous.

($\Rightarrow$). For each $y \in g(t_1, t_2)(\underline{I \| F}) = g(t_1(\underline{F \| I}), t_2(\underline{I \| F}))$, by definition, there exist $z_1 \in t_1(\underline{F \| I})$ and $z_2 \in t_2(\underline{I \| F})$ such that $y \leq g(z_1, z_2)$. By induction hypothesis, for each $k \in \{1, \ldots, N\}$, there exist $a_k, d_k \in F_K$ and $b_k, c_k \in I_k$ such that $t_1(\underline{a \| b}) \leq z_1$ and $z_2 \leq t_2(\underline{c \| d})$. Since each F_k is a filter and I_k is an ideal, there exist $f_k \in F_k$ and $i_k \in I_k$ such that $f_k \leq a_k$, $f_k \leq d_k$, $b_k \leq i_k$ and $c_k \leq i_k$. By Lemma 5.2.7, we obtain

$$y \leq g(z_1, z_2) \leq g(t_1(\underline{a \| b}), t_2(\underline{c \| d})) \leq g(t_1(\underline{f \| i}), t_2(\underline{i \| f})) = g(t_1, t_2)(\underline{i \| f}).$$

($\Leftarrow$). Suppose that, for each $k \in \{1, \ldots, N\}$ there exist $c_k \in I_k$ and $d_k \in F_k$

such that $y \leq g(t_1, t_2)(\underline{c\|d}) = g(t_1(\underline{d\|c}), t_2(\underline{c\|d}))$. By induction hypothesis, we have $t_1(\underline{d\|c}) \in t_1(\underline{F\|I})$ and $t_2(\underline{c\|d}) \in t_2(\underline{I\|F})$. Therefore, by the definition of g on the intermediate level, we conclude $y \in g(t_1, t_2)(\underline{I\|F})$. $\square$

Canonical extensions of ϵ-operations Next we extend ϵ-operations to the canonical extension. As in the lattice expansion case, we have two natural choices of extensions for each operation. Namely, one is *approximated from below*, denoted by adding the subscript $\_{\uparrow}$, e.g. $f_{\uparrow}$, and the other is *approximated from above*, denoted by adding the superscript $\_^{\downarrow}$, e.g. $f^{\downarrow}$. Furthermore, since the canonical extension $\overline{\mathbb{P}}$ is a *point-free* structure which is isomorphic to both $\mathbb{D}_{\upsilon}$ and $\mathbb{U}_{\lambda}$, we have two ways of the evaluation for each value: on $\mathbb{D}_{\upsilon}$, denoted by $\_^{\downarrow}$, and on $\mathbb{U}_{\lambda}$, denoted by $\_{\uparrow}$.

Remark 5.2.9. We list both definitions, approximated from below and approximated from above, for each ϵ-operation, below. However, in general, these two types of extensions do not coincide (see [31] or [32]). Therefore, before discussing canonicity of poset expansions, we must decide how to choose the extension. We will come back to the discussion in Section 5.5.

The extension $f_{\uparrow}$ (approximated from below) of the $(1,1)$-operation f is defined as follows: for all $\alpha, \beta \in \overline{\mathbb{P}}$, we let

1. $(f_{\uparrow}(\alpha, \beta))_{\uparrow} := \lambda(\{f(F, G) \mid F \in \alpha^{\downarrow}, G \in \beta^{\downarrow}\})$,

2. $(f_{\uparrow}(\alpha, \beta))^{\downarrow} := \upsilon((f_{\uparrow}(\alpha, \beta))_{\uparrow})$.

The extension $g_{\uparrow}$ (approximated from below) of the $(\partial, 1)$-operation g is defined as follows: for all $\alpha, \beta \in \overline{\mathbb{P}}$, we let

1. $(g_{\uparrow}(\alpha, \beta))_{\uparrow} := \lambda(\{g(I, F) \mid I \in \alpha_{\uparrow}, F \in \beta^{\downarrow}\})$,

2. $(g_\uparrow(\alpha,\beta))^\downarrow := \upsilon((g_\uparrow(\alpha,\beta))_\uparrow)$.

Note that each Item 1 is evaluated on $\mathbb{U}_\lambda$ and each Item 2 is evaluated on $\mathbb{D}_\upsilon$ (pay attention to the subscript $\__\uparrow$ and the superscript $\_^\downarrow$).

The extension $f^\downarrow$ (approximated from above) of the $(1,1)$-operation f is defined as follows: for all $\alpha,\beta \in \overline{\mathbb{P}}$, we let

1. $(f^\downarrow(\alpha,\beta))^\downarrow := \upsilon(\{f(I,J) \mid I \in \alpha_\uparrow, J \in \beta_\uparrow\})$,

2. $(f^\downarrow(\alpha,\beta))_\uparrow := \lambda((f^\downarrow(\alpha,\beta))^\downarrow)$.

The extension $g^\downarrow$ (approximated from above) of the $(\partial,1)$-operation g is defined as follows: for all $\alpha,\beta \in \overline{\mathbb{P}}$, we let

1. $(g^\downarrow(\alpha,\beta))^\downarrow := \upsilon(\{g(F,I) \mid F \in \alpha^\downarrow, I \in \beta_\uparrow\})$,

2. $(g^\downarrow(\alpha,\beta))_\uparrow := \lambda((g^\downarrow(\alpha,\beta))^\downarrow)$.

Note that each Item 1 is evaluated on $\mathbb{D}_\upsilon$ and each Item 2 is evaluated on $\mathbb{U}_\lambda$ (pay attention to the superscript $\_^\downarrow$ and the subscript $\__\uparrow$). Based on these extensions, we inductively define term functions on $\overline{\mathbb{P}}$. Recall once more that we have to choose the extension type for each operation before defining term functions. Here, we use that $\tilde{f}$, instead of either $f_\uparrow$ or $f^\downarrow$, and $\tilde{g}$, instead of either $g_\uparrow$ or $g^\downarrow$. For arbitrary $\alpha_1,\ldots,\alpha_N \in \overline{\mathbb{P}}$, we let

$$
\begin{array}{l|l}
(p_i(\underline{\alpha}))^\downarrow := \alpha_i{}^\downarrow & (p_i(\underline{\alpha}))_\uparrow := \alpha_{i\uparrow} \\
(\tilde{f}(t_1,t_2)(\underline{\alpha}))^\downarrow := (\tilde{f}(t_1(\underline{\alpha}),t_2(\underline{\alpha})))^\downarrow & (\tilde{f}(t_1,t_2)(\underline{\alpha}))_\uparrow := (\tilde{f}(t_1(\underline{\alpha}),t_2(\underline{\alpha})))_\uparrow \\
(\tilde{g}(t_1,t_2)(\underline{\alpha}))^\downarrow := (\tilde{g}(t_1(\underline{\alpha}),t_2(\underline{\alpha})))^\downarrow & (\tilde{g}(t_1,t_2)(\underline{\alpha}))_\uparrow := (\tilde{g}(t_1(\underline{\alpha}),t_2(\underline{\alpha})))_\uparrow
\end{array}
$$

where $(\underline{\alpha}) = (\alpha_1,\ldots,\alpha_N)$.

5.3 Problem of extending Ghilardi and Meloni's methodology for poset expansions

In this section and the next section (Sections 5.3 and 5.4), we will discuss canonicity of *arbitrary* poset expansions. Within these two sections, we assume that we have already defined the canonical extension of each operation f, either $f_{\uparrow}$ or $f^{\downarrow}$. We here summarise Ghilardi and Meloni's canonicity methodology for lattice expansions (Section 3.3), and bring up a problem to generalise it to poset expansions. Firstly, We define canonicity of inequalities for poset expansions as follows.

Definition 5.3.1 (Canonicity)**.** Let s, t be terms. An inequality $s \leq t$ is *canonical*, if, for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$ whenever $s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N)$ for all $x_1, \ldots x_N \in P$.

We also define $\cup$-terms and $\cap$-terms which are the same as Definition 3.3.3.

Definition 5.3.2 ($\cup$-term and $\cap$-term)**.** Let t be a term. A term t is a $\cup$-*term*, if, for each $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have

$$t(\alpha_1, \ldots, \alpha_N)_{\uparrow} = \lambda(\{t(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}).$$

And, a term t is a $\cap$-*term*, if, for each $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} = \upsilon(\{t(I_1 \| F_1, \ldots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^{\downarrow}\}).$$

Remark 5.3.3. $\cup$-terms and $\cap$-terms are explained with the bi-directional approximation and bases as follows. As we saw in Section 2.3, the canonical extension is a *point-free* structure on which every value is always evaluated by the bi-directional approximation, $\_^{\downarrow}$ and $\_{\uparrow}$. Conversely, every element on the canonical extension has at least one, not necessarily unique, filter basis and at least one, not necessarily unique, ideal basis. If a term t is a $\cup$-term, it claims that we have a *reasonable* filter basis of $t(\alpha_1, \ldots, \alpha_N)$: namely, the set of filters which form $t(\underline{F \| I})$, for each $F_k \in {\alpha_k}^{\downarrow}$ and each $I_k \in \alpha_{k\uparrow}$, for each coordinate $k \in \{1, \ldots, N\}$. Analogously, if a term t is a $\cap$-term, it says that we have a *reasonable* ideal basis of $t(\alpha_1, \ldots, \alpha_N)$: namely, the set of ideals which form $t(\underline{I \| F})$, for each $I_k \in \alpha_{k\uparrow}$ and each $F_k \in {\alpha_k}^{\downarrow}$, for each coordinate $k \in \{1, \ldots, N\}$.

Remark 5.3.4. Unlike what happens in the setting of lattice expansions, we need to mention clearly how the empty elements are treated. Let t be a term (function). The set $\{t(\underline{F \| I}) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}$ is empty, if at least one of the filters or the ideals which are necessary to compute t on the intermediate level is missing. For example, the set $\{p_2(F_1 \| I_1, F_2 \| I_2) \mid F_1 \in {\alpha_1}^{\downarrow}, F_2 \in {\alpha_2}^{\downarrow}, I_1 \in \alpha_{1\uparrow}, I_2 \in \alpha_{2\uparrow}\}$ is empty, if and only if ${\alpha_2}^{\downarrow} = \emptyset$.

The most important reason to introduce these terms is that, whenever we consider lattice expansions, we can prove Theorem 3.3.5. Now we look a sketch of the proof.

Theorem 5.3.5. *Let s, t be terms. Over lattice expansions, an inequality $s \leq t$ is canonical, whenever s is a $\cup$-term and t is a $\cap$-term.*

(Sketch). Let s be a $\cup$-term, and t a $\cap$-term. We assume that, for all $x_1, \ldots, x_N \in L$, we have $s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N)$. By Definition 5.3.2, for all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{L}}$, we have

$$s(\alpha_1, \ldots, \alpha_N)_{\uparrow} = \lambda(\{s(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}),$$

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} = \upsilon(\{t(I_1 \| F_1, \ldots, I_N \| F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in \alpha_k{}^{\downarrow}\}).$$

Now what we want to show is $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$. By Proposition 2.3.1, it is equivalent to proving that, for all $F_k, G_k \in \alpha_k{}^{\downarrow}$ and all $I_k, J_k \in \alpha_{k\uparrow}$, we have

$$s(F_1 \| I_1, \ldots, F_N \| I_N) \sqsubseteq t(J_1 \| G_1, \ldots, J_N \| G_N). \tag{5.1}$$

Over lattice expansions, to verify the condition (5.1), we use the following fact: for each $k \in \{1, \ldots, N\}$, if $F_k, G_k \in \alpha_k{}^{\downarrow}$ and $I_k, J_k \in \alpha_{k\uparrow}$, then $F_k \cap G_k \cap I_k \cap J_k \neq \emptyset$. This is because every filter in $\alpha_k{}^{\downarrow}$ and every ideal in $\alpha_{k\uparrow}$ has non-empty intersection, i.e. $F_k \cap I_k \neq \emptyset$, $F_k \cap J_k \neq \emptyset$, $G_k \cap I_k \neq \emptyset$ and $G_k \cap J_k \neq \emptyset$. So, we take $a \in F_k \cap I_k$, $b \in F_k \cap J_k$, $c \in G_k \cap I_k$, $d \in G_k \cap J_k$. Since F_k and G_k are filters, and I_k and J_k are downward closed, we have $a \wedge b \in F_k \cap I_k \cap J_k$ and $c \wedge d \in G_k \cap I_k \cap J_k$. Moreover, as F_k and G_k are upward closed, and I_k and J_k are ideals, we obtain

$$(a \wedge b) \vee (c \wedge d) \in F_k \cap G_k \cap I_k \cap J_k.$$

Note that the uniqueness of the join of $\{a \wedge b, c \wedge d\}$ is essentially working. Therefore, $F_k \cap G_k \cap I_k \cap J_k \neq \emptyset$, hence there is $x_k \in F_k \cap G_k \cap I_k \cap J_k$. Then, together with the assumption $s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N)$, we have

$$s(x_1 \| x_1, \ldots, x_N \| x_N) = s(x_1, \ldots, x_N) \leq t(x_1, \ldots, x_N) = t(x_1 \| x_1, \ldots, x_N \| x_N). \tag{5.2}$$

By Lemma 5.2.8, we have

$$s(x_1 \| x_1, \ldots, x_N \| x_N) \in s(F_1 \| I_1, \ldots, F_N \| I_N), \tag{5.3}$$

Figure 5.1: $\mathbb{P}_c$

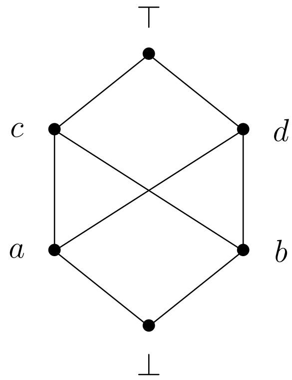
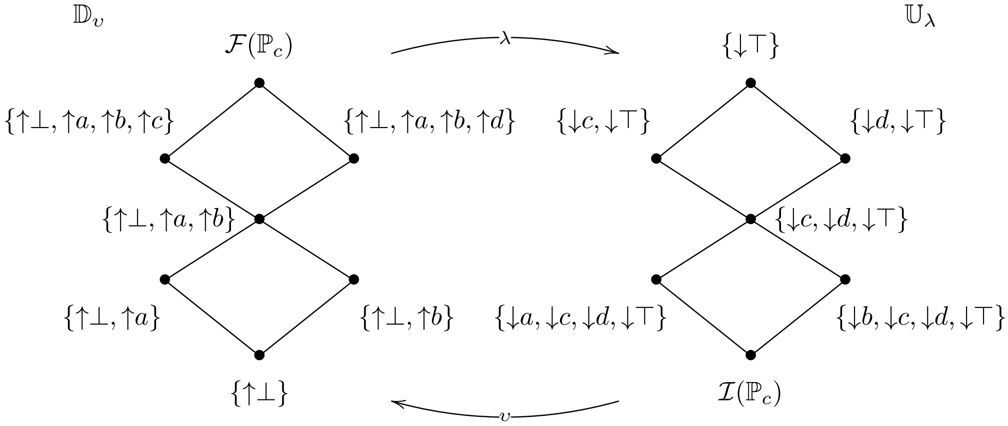


Figure 5.2: The canonical extension $\overline{\mathbb{P}_c}$ of $\mathbb{P}$



$$t(x_1\|x_1, \ldots, x_N\|x_N) \in t(J_1\|G_1, \ldots, J_N\|G_N). \tag{5.4}$$

It derives $s(F_1\|I_1, \ldots, F_N\|I_N) \sqsubseteq t(J_1\|G_1, \ldots, J_N\|G_N)$. □

However, over poset expansions, the same methodology does not work in general. Let us consider a poset $\mathbb{P}_c$ given by the Hasse diagram Fig. 5.1. On the poset $\mathbb{P}_c$, every filter and every ideal are principal: for each $x \in P$, $\uparrow x := \{y \in P \mid x \leq y\}$ and $\downarrow x := \{y \in P \mid y \leq x\}$. That is, we have $\mathcal{F}(\mathbb{P}_c) = \{\uparrow\bot, \uparrow a, \uparrow b, \uparrow c, \uparrow d, \uparrow\top\}$ and $\mathcal{I}(\mathbb{P}_c) = \{\downarrow\bot, \downarrow a, \downarrow b, \downarrow c, \downarrow d, \downarrow\top\}$. Therefore, $\mathcal{F} +_{\mathbb{P}_c} \mathcal{I} \cong \{\bot, a, b, c, d, \top\}$. Then, we obtain $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$ expressed by the Hasse diagram Fig. 5.2. Since the canonical extension of $\mathbb{P}_c$ is isomorphic to $\mathbb{D}_\upsilon$ and $\mathbb{U}_\lambda$, there must exist $\alpha \in \overline{\mathbb{P}_c}$ such that

$\alpha^{\downarrow} = \{\uparrow\bot, \uparrow a, \uparrow b\}$ and $\alpha_{\uparrow} = \{\downarrow c, \downarrow d, \downarrow\top\}$. We can affirm two facts about this α. Firstly, unlike over lattice expansions, $\alpha^{\downarrow}$ is not an ideal of filters: see Proposition 2.3.1 (Item 4). In this case, there is no lower bound of $\{\uparrow a, \uparrow b\}$ in $\alpha^{\downarrow}$. Analogously, $\alpha_{\uparrow}$ is not a filter of ideals. Secondly, if we take $\uparrow a, \uparrow b \in \alpha^{\downarrow}$ and $\downarrow c, \downarrow d \in \alpha_{\uparrow}$, unlike in the case of lattice expansions, we obtain

$$\uparrow a \cap \uparrow b \cap \downarrow c \cap \downarrow d = \emptyset.$$

Recall that, over lattice expansions, to find a tuple of elements satisfying all condition (5.2), (5.3) and (5.4), we can take a tuple $(x_1 \| x_1, \ldots, x_N \| x_N)$ where, for each coordinate k, $x_k \in F_k \cap G_k \cap I_k \cap J_k$. However, over poset expansions, $F_k \cap G_k \cap I_k \cap J_k$ may be *empty*, as we saw in the above example. Therefore, we cannot directly apply Ghilardi and Meloni's method to arbitrary poset expansions.

5.4 A solution: canonicity of poset expansions

In the previous section, we found a problem to generalise Ghilardi and Meloni's technique to poset expansions. Nevertheless, we can still leverage our insight to find canonical inequalities over arbitrary poset expansions. In this section, we carefully remove problematic conditions to extend Ghilardi and Meloni's method to poset expansions, and systematically obtain canonicity results for *arbitrary* poset expansions.

We recall the signing algorithm for construction trees of terms in Section 3.3. However, we now label the signs without prunings. Let t be a term. On the construction tree of t, we label each node with a sign $+$ or $-$ in the following manner.

Note that our labelling start at the root.

Signing algorithm

1. Label the root with $+$.

2. If the node does not have any child, we have finished labelling the branch. Otherwise, the node is labelled with a ϵ-operation $f : \mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_n} \to \mathbb{P}$ by $f(t_1, \ldots, t_n)$. Then, for each coordinate k, we label the child t_k with the same sign of the current node if $\epsilon_k = 1$, and label the child t_k with the converse sign of the current node if $\epsilon_k = \partial$.

3. Move to each child and repeat Item 2 until every node is labelled.

We call the construction tree of t $+$-*signed*. We also define the $-$-signed construction tree of t with the same labelling rule except Item 1: we start labelling the root with $-$: see e.g. Fig. 5.5 and Fig. 5.6 in Section 5.5.

Now, we show the following canonicity theorem for arbitrary poset expansions.

Theorem 5.4.1. *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s is a $\cup$-term, t a $\cap$-term, and there is no propositional variable in $s \leq t$ satisfying the following two conditions:*

1. *it is signed with $+$ and $-$ in the $-$-signed construction tree of s,*

2. *it is signed with $+$ and $-$ in the $+$-signed construction tree of t.*

Proof. For arbitrary $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, since s is a $\cup$-term and t is a $\cap$-term, we have

$$s(\alpha_1, \ldots, \alpha_N)_\uparrow = \lambda(\{s(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}),$$

$$t(\alpha_1, \ldots, \alpha_N)^{\downarrow} = \upsilon(\{t(I_1\|F_1, \ldots, I_N\|F_N) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^{\downarrow}\}).$$

By Proposition 2.3.1, $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$ is equivalent to the following: for all $F_1, G_1 \in {\alpha_1}^{\downarrow}, \ldots, F_N, G_N \in {\alpha_N}^{\downarrow}$ and all $I_1, J_1 \in \alpha_{1\uparrow}, \ldots, I_N, J_N \in \alpha_{N\uparrow}$,

$$s(F_1\|I_1, \ldots, F_N\|I_N) \sqsubseteq t(J_1\|G_1, \ldots, J_N\|G_N). \tag{5.5}$$

As distinct from the lattice case, there may be ${\alpha_k}^{\downarrow} = \emptyset$ or $\alpha_{k\uparrow} = \emptyset$ for some coordinate k. In this case, if the emptiness makes the set $\{s(\underline{F\|I}) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}$ empty, $s(\alpha_1, \ldots, \alpha_N) = \bot$. Analogously, we obtain $t(\alpha_1, \ldots, \alpha_N) = \top$, whenever the emptiness makes the set $\{t(\underline{I\|F}) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^{\downarrow}\}$ empty. Hence, the statement trivially holds. Hereafter, we consider the case that both the set $\{s(\underline{F\|I}) \mid F_k \in {\alpha_k}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}$ and the set $\{t(\underline{I\|F}) \mid I_k \in \alpha_{k\uparrow}, F_k \in {\alpha_k}^{\downarrow}\}$ are non-empty.

To save space, we assume that $s \leq t$ contains only four propositional variables p_1, p_2, p_3, p_4. By assumption, without loss of the generality, we can assume that there is no p_1 (p_2) signed with $+$ ($-$) in the $-$-signed construction tree of s and there is no p_3 (p_4) signed with $-$ ($+$) in the $+$-signed construction tree of t. In other words, in s, p_1 appear only positively and p_2 appear only negatively (recall that s is $-$-signed), and, in t, p_3 appear only positively and p_4 does only negatively.

Here, by our assumption, the condition (5.5) can be simplified as follows: for all $F_1, G_1 \in {\alpha_1}^{\downarrow}, \ldots, F_4, G_4 \in {\alpha_4}^{\downarrow}$ and all $I_1, J_1 \in \alpha_{1\uparrow}, \ldots, I_4, J_4 \in \alpha_{4\uparrow}$,

$$s(F_1, I_2, F_3\|I_3, F_4\|I_4) \sqsubseteq t(J_1\|G_1, J_2\|G_2, J_3, G_4). \tag{5.6}$$

So, we will verify the condition (5.6). As J_1 is an element of $\alpha_{1\uparrow}$, we obtain $F_1 \sqsubseteq J_1$

and $G_1 \sqsubseteq J_1$, hence there exist $x_1' \in F_1 \cap J_1$ and $x_1'' \in G_1 \cap J_1$. Since J_1 is an ideal, there exists $x_1 \in J_1$ such that $x_1' \leq x_1$ and $x_1'' \leq x_1$. Then, we obtain $x_1 \in F_1 \cap G_1 \cap J_1$ because F_1 and G_1 are upsets. Analogously, we can obtain elements x_2, x_3, x_4 satisfying $x_2 \in I_2 \cap J_2 \cap G_2$, $x_3 \in F_3 \cap I_3 \cap J_3$, and $x_4 \in F_4 \cap G_4 \cap I_4$. By Lemma 5.2.8, we have

$$s(x_1, x_2, x_3, x_4) = s(x_1, x_2, x_3 \| x_3, x_4 \| x_4) \in s(F_1, I_2, F_3 \| I_3, F_4 \| I_4),$$

$$t(x_1, x_2, x_3, x_4) = t(x_1 \| x_1, x_2 \| x_2, x_3, x_4) \in t(J_1 \| G_1, J_2 \| G_2, J_3, G_4).$$

By assumption, $s(x_1, \dots, x_4) \leq t(x_1, \dots, x_4)$. Hence, $s(\alpha_1, \dots, \alpha_4) \leq t(\alpha_1, \dots, \alpha_4)$.

□

Remark 5.4.2. In Theorem 5.4.1, we mainly used the following fact: for arbitrary $F, G \in \alpha^{\downarrow}$ and $I, J \in \alpha_{\uparrow}$, $F \cap G \cap I \cap J$ may be empty over posets. However, on posets, we still have that all intersections of three out of four (F, G, I, J) are always non-empty, e.g $F \cap G \cap I \neq \emptyset$.

In addition to Theorem 5.4.1, we also show the following two theorems for arbitrary poset expansions.

Theorem 5.4.3. *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s and t are $\cup$-terms, all propositional variables in t also appear in s, and every propositional variable is uniformly signed either only by $+$ or only by $-$ in both the $+$-signed construction tree of s and the $+$-signed construction tree of t.*[2]

Proof. We assume that all propositional variables in $s \leq t$ are $p_1, \dots, p_a$ signed with

[2]This condition looks like *uniform formulae*. However, it is not the same. This is because the signs in s and in t are dually related. For example, in FL-algebras, $s \leq t \iff 1 \leq s \backslash t$, hence every positive proposition in s occurs negatively in $s \backslash t$.

$+$ and $p_{a+1}, \ldots, p_{a+b}$ signed with $-$ in the $+$-signed construction tree of s and $+$-signed construction tree of t. In other words, all $p_1, \ldots, p_a$ occur positively both in s and in t, and all $p_{a+1}, \ldots, p_{a+b}$ occur negatively both in s and in t.

By definition, for arbitrary $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have

$$s(\alpha_1, \ldots, \alpha_N)_\uparrow = \lambda(\{s(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}),$$

$$t(\alpha_1, \ldots, \alpha_N)_\uparrow = \lambda(\{t(F_1 \| I_1, \ldots, F_N \| I_N) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}).$$

By Proposition 2.3.1, the condition $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$ is equivalent to the following condition $s(\alpha_1, \ldots, \alpha_N)_\uparrow \supseteq t(\alpha_1, \ldots, \alpha_N)_\uparrow$. Hence, it suffices to show that, for all $F_1 \in {\alpha_1}^\downarrow, \ldots, F_N \in {\alpha_N}^\downarrow$ and all $I_1 \in \alpha_{1\uparrow}, \ldots, I_N \in \alpha_{N\uparrow}$,

$$s(F_1 \| I_1, \ldots, F_N \| I_N) \sqsubseteq t(F_1 \| I_1, \ldots, F_N \| I_N). \tag{5.7}$$

If there is an element ${\alpha_k}^\downarrow = \emptyset$ or $\alpha_{k\uparrow} = \emptyset$ for some coordinate k and it forces that the set $\{t(\underline{F \| I}) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}$ is empty. By our assumption, the fact derives that the set $\{s(\underline{F \| I}) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}$ is also empty. It follows that $s(\alpha_1, \ldots, \alpha_N) = t(\alpha_1, \ldots, \alpha_N) = \bot$, hence the statement holds. This is exactly why we assume that every propositional variable in t occurs in s as well (see also Theorem 5.4.5). Hereafter, we assume that the sets $\{s(\underline{F \| I}) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}$ and $\{t(\underline{F \| I}) \mid F_k \in {\alpha_k}^\downarrow, I_k \in \alpha_{k\uparrow}\}$ are non-empty.

By our assumption, the condition (5.7) is equivalent to

$$s(F_1, \ldots, F_a, I_{a+1}, \ldots, I_{a+b}) \sqsubseteq t(F_1, \ldots, F_a, I_{a+1}, \ldots, I_{a+b}). \tag{5.8}$$

For an arbitrary $x \in t(F_1, \ldots, F_a, I_{a+1}, \ldots, I_{a+b})$, there exist $f_1 \in F_1, \ldots, f_a \in F_a$ and $i_{a+1} \in I_{a+1}, \ldots, i_{a+b} \in I_{a+b}$ such that $t(f_1, \ldots, f_a, i_{a+1}, \ldots, i_{a+b}) \leq x$. By assumption, we have

$$s(f_1, \ldots, f_a, i_{a+1}, \ldots, i_{a+b}) \leq t(f_1, \ldots, f_a, i_{a+1}, \ldots, i_{a+b}) \leq x.$$

It follows the condition (5.8). $\square$

We also obtain the following canonicity result. The proof is analogous to the proof of Theorem 5.4.3.

Theorem 5.4.4. *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s and t are $\cap$-terms, all propositional variables in s appear in t, and every propositional variable is uniformly signed either only by $+$ or only by $-$ in both the $+$-signed construction tree of s and the $+$-signed construction tree of t.*

Theorem 5.4.1 is an analog of Theorem 3.3.5: over lattice expansions, any inequality $s \leq t$ is canonical, whenever s is a $\cup$-term and t is a $\cap$-term, regardless of *signs of variables.* But, over poset expansions, the keystone of Ghilardi and Meloni's technique does not work (Section 5.3). Therefore, while we can extend our method from lattice expansions to poset expansions, we can obtain restricted results. Moreover, we can also prove Theorems 5.4.3 and 5.4.4 for arbitrary poset expansions.[3] In fact, for bounded poset expansions, we can weaken the variable conditions in Theorem 5.4.3 and Theorem 5.4.4 as follows.

Theorem 5.4.5 (Theorem 5.4.3 for bounded poset expansions)**.** *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s and t are $\cup$-terms, and every proposi-*

[3] These theorems are also provable over lattice expansions.

tional variable is uniformly signed either only by + or only by − in both the +-signed construction tree of s and the +-signed construction tree of t.

Theorem 5.4.6 (Theorem 5.4.4 for bounded poset expansions)**.** *Let s, t be terms. An inequality $s \leq t$ is canonical, whenever s and t are $\cap$-terms, and every propositional variable is uniformly signed either only by + or only by − in both the +-signed construction tree of s and the +-signed construction tree of t.*

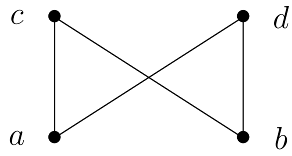
This is because, over bounded poset expansions, we do not need to take into account the emptiness of $\alpha^{\downarrow}$ or $\alpha_{\uparrow}$ (see Section 5.5). In the following section, we will give a syntactic characterisation of canonical inequalities of certain poset expansions. There again, unlike what happens in the setting of lattice expansions, we will encounter a sensitive argument of emptiness, boundedness and ϵ-operations.

Remark 5.4.7. Whereas the above five theorems are general results for arbitrary (bounded) poset expansions, we have to mention two things. Firstly, we have not discussed *the canonical extension* of arbitrary poset expansions. Namely, in practice, we have to define how to extend ϵ-operations and justify it for each poset expansion. Secondly, those theorem are validated only when we obtained classes of $\cup$-terms and $\cap$-terms.

5.5 A syntactic description of canonical inequalities

In the previous section, we carefully removed problematic conditions to extend Ghilardi and Meloni's technique from lattice expansions to arbitrary poset expansions.

Figure 5.3: The poset $\mathbb{P}_b$

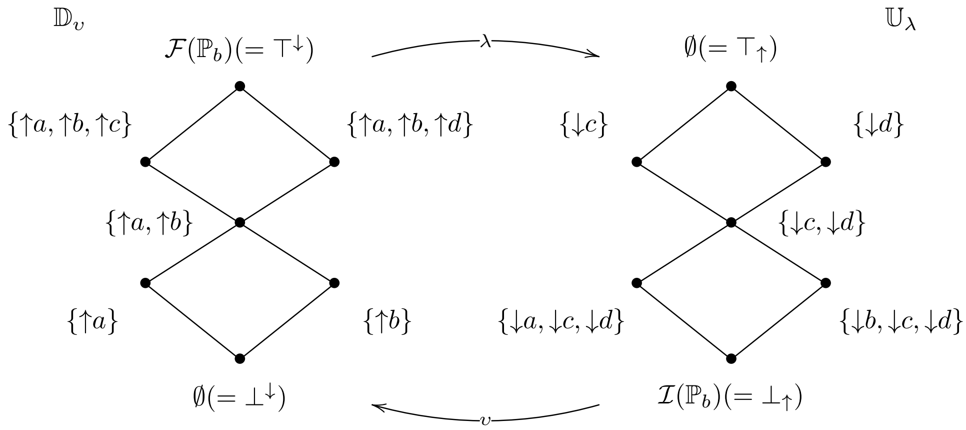


In this section, we will syntactically describe classes of $\cup$-terms and $\cap$-terms to obtain a syntactic characterisation of canonical inequalities of poset expansions.

Our technique looks similar to the case of lattice expansions. But, as distinct from the lattice case, we must seriously take into consideration the possibility of the emptiness of α (as a set of filters or a set of ideals) in the canonical extensions, which generates a considerable complexity of our argument. For example, as we saw in the previous section, there are some theorems which statements are simplified by assuming the boundedness of posets, or emptiness of bases (see Theorem 5.4.3 and Theorem 5.4.5).

Before going further, we recall where the emptiness comes and how the boundedness relates to the non-emptiness. As the canonical extension of poset is a complete lattice, it is closed arbitrary joins and arbitrary meets. Hence, as the join of the empty set and the meet of the empty set, it has $\perp$ and $\top$. Moreover, $\perp_{\uparrow} = \mathcal{I}$ on $\mathbb{U}_{\lambda}$ and $\top^{\downarrow} = \mathcal{F}$ on $\mathbb{D}_{\upsilon}$ for any poset. On the other hand, we do not have the uniform characterisations of $\perp^{\downarrow}$ on $\mathbb{D}_{\upsilon}$ and $\top_{\uparrow}$ on $\mathbb{U}_{\lambda}$, in general. For example, let us consider the poset $\mathbb{P}_b$ given by the Hasse diagram Fig. 5.3. Then, we obtain the canonical extension $\overline{\mathbb{P}_b}$ of $\mathbb{P}_b$ in the Hasse diagram Fig. 5.4. In the canonical extension $\overline{\mathbb{P}_b}$ of $\mathbb{P}_b$, we obtain $\perp^{\downarrow} = \emptyset$ and $\top_{\uparrow} = \emptyset$ (cf. the canonical extension $\overline{\mathbb{P}_c}$ in Fig. 5.2). By comparing the posets $\mathbb{P}_c$ and $\mathbb{P}_b$, we notice that the emptiness of elements in $\overline{\mathbb{P}_b}$ comes from the values $\lambda(\mathcal{F})$ and $\upsilon(\mathcal{I})$. More precisely, in general, we may not

Figure 5.4: The canonical extension $\overline{\mathbb{P}_b}$ of $\mathbb{P}_b$



have any filter which intersects with arbitrary ideals, and any ideal which does with arbitrary filters, on posets. However, over bounded posets, e.g. $\mathbb{P}_c$, thanks to the bounded constants $\bot$ and $\top$, we can always assume the non-emptiness of $\bot^\downarrow$ and $\top_\uparrow$, because the principal filter $\uparrow\bot$ intersects with arbitrary ideals, and the principal ideal $\downarrow\top$ intersects with arbitrary filters.

For future use, we also show that how the binary joins and the binary meets are computed on the canonical extension of posets.

Proposition 5.5.1 (Joins and Meets on the canonical extension)**.** *Let $\mathbb{P}$ a poset. For all $\alpha, \beta \in \overline{\mathbb{P}}$, we have*

1. $(\alpha \vee \beta)_\uparrow = \lambda(\alpha^\downarrow \cup \beta^\downarrow) = \alpha_\uparrow \cap \beta_\uparrow$ *and* $(\alpha \vee \beta)^\downarrow = \upsilon(\alpha_\uparrow \cap \beta_\uparrow)$,

2. $(\alpha \wedge \beta)^\downarrow = \upsilon(\alpha_\uparrow \cup \beta_\uparrow) = \alpha^\downarrow \cap \beta^\downarrow$ *and* $(\alpha \wedge \beta)_\uparrow = \lambda(\alpha^\downarrow \cap \beta^\downarrow)$.

Proof. For an arbitrary element $\gamma \in \overline{\mathbb{P}}$, we need to prove the following: (recall Proposition 2.3.1)

1. $\gamma_\uparrow \subseteq \alpha_\uparrow$ and $\gamma_\uparrow \subseteq \beta_\uparrow$ if and only if $\gamma_\uparrow \subseteq \alpha_\uparrow \cap \beta_\uparrow$,

2. $\gamma^\downarrow \subseteq \alpha^\downarrow$ and $\gamma^\downarrow \subseteq \beta^\downarrow$ if and only if $\gamma^\downarrow \subseteq \alpha^\downarrow \cap \beta^\downarrow$.

However, this is straightforward. $\square$

A syntactic description of canonical extension of ϵ-operations Here, we will show basic properties of canonical extensions of certain ϵ-operations and define the canonical extension of certain (bounded) poset expansions. Since we mainly run two types of discussion (poset expansions and bounded poset expansions) in parallel, to avoid possible confusion, we use the distinct notations for those operations. That is, on posets, we focus on the following four types of ϵ-operations (recall Definition 5.1.3):

1. $\lfloor l \rfloor : \mathbb{P}^{\mu_1} \times \cdots \times \mathbb{P}^{\mu_n} \to \mathbb{P}$ is a $\mu_\perp$-additive operation,

2. $\lceil r \rceil : \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_n} \to \mathbb{P}$ is a $\nu^\top$-multiplicative operation,

3. $\Diamond : \mathbb{P}^1 \to \mathbb{P}$ is a 1-additive operation,

4. $\Box : \mathbb{P}^1 \to \mathbb{P}$ is a 1-multiplicative operation,

where *diamond* $\Diamond$ and *box* $\Box$ form an adjoint pair, i.e. $\Diamond \dashv^1 \Box$. We mention that $\Diamond$ and $\Box$ are $\perp$-strict and $\top$-strict, which follow the adjointness of unary operations, since for every $p \in P$, we have $\Box p \leq \Box p \iff \Diamond\Box p \leq p$. And, on bounded posets, we consider the following four types of ϵ-operations (recall Definition 5.1.1 and Definition 5.1.3):

1. $f : \mathbb{P}^{\delta_1} \times \cdots \times \mathbb{P}^{\delta_m} \to \mathbb{P}$ is a δ-join-preserving operation,

2. $g : \mathbb{P}^{\epsilon_1} \times \cdots \times \mathbb{P}^{\epsilon_{m'}} \to \mathbb{P}$ is a ϵ-meet-preserving operation,

3. $l : \mathbb{P}^{\mu_1} \times \cdots \times \mathbb{P}^{\mu_n} \to \mathbb{P}$ is a μ-additive operation,

4. $r : \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_n} \to \mathbb{P}$ is a ν-multiplicative operation.

Whenever we assume that $\lfloor l \rfloor$ and $\lceil r \rceil$ or l and r form adjoint pairs with respect to a fixed coordinate i, we explicitly denote it by $\lfloor l \rfloor \dashv^i \lceil r \rceil$ or $l \dashv^i r$. This is the only reason we assume that $\lfloor l \rfloor$ and $\lceil r \rceil$ (l and r) have the same arity. Note that we assume that, not just $\perp^{\downarrow}$ and $\top_{\uparrow}$, but also every basis is non-empty, whenever we discuss bounded poset expansions.

The following poset expansion is introduced in [18].

Example 5.5.2 (Residuated algebra)**.** A *residuated algebra* is a poset expansion $\langle \mathbb{P}, \circ, \to, \leftarrow \rangle$ where $\mathbb{P}$ is a poset, $\circ$ is a $(1,1)_{\perp}$-additive operation, $\to$ is a $(\partial, 1)^{\top}$-multiplicative operation, $\leftarrow$ is a $(1, \partial)^{\top}$-multiplicative operation, and $\circ$, $\to$ and $\leftarrow$ form adjoint pairs $\circ \dashv^2 \to$ and $\circ \dashv^1 \leftarrow$.

Remark 5.5.3. In residuated algebras, the $\perp$-strictness of $\circ$ and the $\top$-strictness of $\to$ and $\leftarrow$ follows from the residuation law (adjointness $\circ \dashv^2 \to$ and $\circ \dashv^1 \leftarrow$). For example, for all $a, b \in \mathbb{P}$, we have $a \circ (a \to b) \leq b$ and $(b \leftarrow a) \circ a \leq b$.

The following proposition shows the smoothness properties of ϵ-join-preserving operations and ϵ-meet-preserving operations on bounded poset expansions.

Proposition 5.5.4. *On a bounded poset $\mathbb{P}$, the δ-join-preserving operation f is smooth, and the ϵ-meet-preserving operation g is smooth. Namely, for arbitrary $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{m'} \in \overline{\mathbb{P}}$, we have*

1. $(f_{\uparrow}(\alpha_1, \ldots, \alpha_m))^{\downarrow} = (f^{\downarrow}(\alpha_1, \ldots, \alpha_m))^{\downarrow}$,

2. $(g^{\downarrow}(\beta_1, \ldots, \beta_{m'}))_{\uparrow} = (g_{\uparrow}(\beta_1, \ldots, \beta_{m'}))_{\uparrow}$.

Proof. Here, we check only Item 2. For arbitrary $\beta_1, \ldots, \beta_{m'} \in \overline{\mathbb{P}}$, we will show

$$\lambda(\upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})\})) = \lambda(\{g(X_1, \ldots, X_{m'}) \mid X_k \in (\beta_k{}^{\downarrow} \| \beta_{k\uparrow})\}). \tag{5.9}$$

($\subseteq$). For each coordinate $k \in \{1, \ldots, m'\}$, if $X_k \in (\beta_k{}^{\downarrow} \| \beta_{k\uparrow})$ and $Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})$, we have that $X_k \sqsubseteq Y_k$ when $\epsilon_k = 1$ and $Y_k \sqsubseteq X_k$ when $\epsilon_k = \partial$. Therefor, $g(X_1, \ldots, X_{m'}) \sqsubseteq g(Y_1, \ldots, Y_{m'})$. Since $X_1, \ldots, X_{m'}, Y_1, \ldots, Y_{m'}$ are arbitrary, we obtain

$$\{g(X_1, \ldots, X_{m'}) \mid X_k \in (\beta_k{}^{\downarrow} \| \beta_{k\uparrow})\} \subseteq \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})\}), \tag{5.10}$$

which concludes the $\subseteq$-direction.

($\supseteq$). It suffices to show that, for each $X \in \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})\})$, there exist, for each coordinate k, $X_k \in (\beta_k{}^{\downarrow} \| \beta_{k\uparrow})$ such that $X \sqsubseteq g(X_1, \ldots, X_{m'})$. For each $X \in \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})\})$ and arbitrary $Y_k \in (\beta_{k\uparrow} \| \beta_k{}^{\downarrow})$, we define X_k for each $k \in \{1, \ldots, m'\}$ as follows:

$$X_k := \{y_k \in P \mid \forall j \in \{1, \ldots, m'\} \setminus \{k\}, \exists y_j \in Y_j, \exists x \in X.\ x \leq g(y_1, \ldots, y_{m'})\}. \tag{5.11}$$

On bounded posets, as every element in the canonical extension is non-empty, we can prove that each X_k is non-empty. By ϵ-meet-preservability of g, we have that X_k is a filter if $\epsilon_k = 1$, and X_k is an ideal if $\epsilon_k = \partial$ for each coordinate k. Furthermore, by definition, we have $X_k \in (\beta_k{}^{\downarrow} \| \beta_{k\uparrow})$. Finally, we show $X \sqsubseteq g(X_1, \ldots, X_{m'})$ ($g(X_1, \ldots, X_{m'}) \subseteq X$). For an arbitrary $a \in g(X_1, \ldots, X_{m'})$, for each coordinate $k \in \{1, \ldots, m'\}$, there exist $z_k \in X_k$ such that $a \leq g(z_1, \ldots, z_{m'})$. By definition of X_k, there exist $x_1, \ldots, x_{m'} \in X$ and $y^i_j \in Y_j$ (for $j \in \{1, \ldots, k-1, k+1, \ldots, m'\}$ and $i \in \{1, \ldots, j-1, j+1, \ldots, m'\}$) such that $x_k \leq g(y^k_1, \ldots, y^k_{k-1}, z_k, y^k_{k+1}, \ldots, y^k_{m'})$. Because X is a filter, there exists $x \in X$ such that $x \leq g(y^k_1, \ldots, y^k_{k-1}, z_k, y^k_{k+1}, \ldots, y^k_{m'})$. By ϵ-meet-preservability of g, we can find $w_1, \ldots, w_{m'} \in P$ satisfying $x \leq g(w_1, \ldots, w_{m'})$, and $w_k \leq z_k$ (if $\epsilon_k = 1$) and $z_k \leq w_k$ (if $\epsilon_k = \partial$) for each coordinate k. Hence, we obtain $x \leq g(w_1, \ldots, w_{m'}) \leq a$, by monotonicity of g. Therefore, $a \in X$. $\square$

Next we prove the properties of canonical extensions of additive operations, $\perp$-additive operations, multiplicative operations and $\top$-multiplicative operations.

Proposition 5.5.5. *The extensions $\lfloor l \rfloor_\uparrow$, $\lceil r \rceil^\downarrow$, $\Diamond_\uparrow$, $\Box^\downarrow$, $l_\uparrow$ and $r^\downarrow$ are a $\perp$-additive operation, a $\top$-multiplicative operation, an additive operation, a multiplicative operation, an additive operation and a multiplicative operation on the canonical extension. Furthermore, the adjointness is also preserved. Namely, $\Diamond_\uparrow \dashv^1 \Box^\downarrow$, and if $\lfloor l \rfloor \dashv^i \lceil r \rceil$, then $\lfloor l \rfloor_\uparrow \dashv^i \lceil r \rceil^\downarrow$, and if $l \dashv^i r$, then $l_\uparrow \dashv^i r^\downarrow$.*

Proof. Here we check only $\mu_\perp$-additivity of $\lfloor l \rfloor_\uparrow$ and the adjointness $\lfloor l \rfloor_\uparrow \dashv^i \lceil r \rceil$. But, the other cases are analogous.

($\perp$-strictness of $\lfloor l \rfloor_\uparrow$). Since the canonical extension is a complete lattice, it suffices to show the following condition for each coordinate $k \in \{1, \ldots, n\}$: for arbitrary $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{P}}$,

1. if the order-type $\mu_k = 1$, then $\lfloor l \rfloor_\uparrow(\alpha_1, \ldots \alpha_{k-1}, \perp, \alpha_{k+1}, \ldots, \alpha_n) \leq \perp$,

2. if the order-type $\mu_k = \partial$, then $\lfloor l \rfloor_\uparrow(\alpha_1, \ldots \alpha_{k-1}, \top, \alpha_{k+1}, \ldots, \alpha_n) \leq \perp$.

Firstly, $\perp_\uparrow = \emptyset$ does not happen. Secondly, whenever there exists a coordinate $j \in \{1, \ldots, k-1, k+1, \ldots, n\}$ such that the order-type $\mu_j = 1$ and ${\alpha_j}^\downarrow = \emptyset$ or the order-type $\mu_j = \partial$ and $\alpha_{j\uparrow} = \emptyset$, Items 1 and 2 trivially hold. Otherwise, for an arbitrary $Y \in \bot_\uparrow$ and all $X_j \in (\alpha_j{}^\downarrow \| \alpha_{j\uparrow})$ (recall Abbreviation 3.2.4), we define

$$Y_k := \{x_k \in P \mid \forall j \in \{1, \ldots, n\} \setminus \{k\}, \exists x_j \in X_j, \exists y \in Y.\ \lfloor l \rfloor(x_1, \ldots, x_n) \leq y\}.$$

Then, we can straightforwardly obtain that Y_k is an ideal if $\mu_k = 1$, and if Y_k is a filter if $\mu_k = \partial$. Note that the non-emptiness comes from $\bot$-strictness of $\lfloor l \rfloor$ and the directed-ness is from μ-additivity. Because $\bot^\downarrow = \upsilon(\mathcal{I})$ and $\top_\uparrow = \lambda(\mathcal{F})$, we obtain that $X_k \sqsubseteq Y_k$ if $\mu_k = 1$, and $Y_k \sqsubseteq X_k$ if $\mu_k = \partial$, for each $X_k \in (\bot^\downarrow \| \top_\uparrow)$. Therefore, Items 1 and 2 hold.

(μ-additivity of $\lfloor l \rfloor_\uparrow$). By Proposition 5.5.1, it suffices to show the following condition for each coordinate $k \in \{1, \ldots, n\}$: for arbitrary $\alpha, \beta, \gamma, \gamma_1, \ldots, \gamma_n \in \overline{\mathbb{P}}$, if $\mu_k = 1$,

$$\lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \alpha, \ldots, \gamma_n) \leq \gamma \text{ and } \lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \beta, \ldots, \gamma_n) \leq \gamma$$
$$\Longrightarrow \lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \alpha \vee \beta, \ldots, \gamma_n) \leq \gamma,$$

and if $\mu_k = \partial$,

$$\lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \alpha, \ldots, \gamma_n) \leq \gamma \text{ and } \lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \beta, \ldots, \gamma_n) \leq \gamma$$
$$\Longrightarrow \lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \alpha \wedge \beta, \ldots, \gamma_n) \leq \gamma.$$

If $\gamma_\uparrow = \emptyset$, which means $\gamma = \top$, it is trivial. And, if there exists a coordinate $j \in \{1, \ldots, k-1, k+1, \ldots, n\}$ such that the order-type $\mu_j = 1$ and $\gamma_j{}^\downarrow = \emptyset$ or the order-type $\mu_j = \partial$ and $\gamma_{j\uparrow} = \emptyset$, we obtain $\lfloor l \rfloor_\uparrow(\gamma_1, \ldots, \gamma_n) = \bot$ (regardless of α, β, $\alpha \vee \beta$ or $\alpha \wedge \beta$). Hence, it trivially holds. Otherwise, for arbitrary $Y \in \gamma_\uparrow$ and

$X_j \in ({\alpha_j}^{\downarrow} \| \alpha_{j\uparrow})$, we define

$$Y_k := \{x_k \in P \mid \forall j \in \{1, \ldots, n\} \setminus \{k\}, \exists x_j \in X_j, \exists y \in Y.\ \lfloor l \rfloor(x_1, \ldots, x_n) \leq y\}.$$

As above, we can prove that Y_k is an ideal if $\mu_k = 1$ and Y_k is a filter if $\mu_k = \partial$. Furthermore, this ideal (filter) Y_k simulates an adjointness on the intermediate level. Namely, for an arbitrary filter (ideal) X_k, we have

$$\begin{aligned} X_k \sqsubseteq Y_k &\iff \lfloor l \rfloor(X_1, \ldots, X_n) \sqsubseteq Y \\ (Y_k \sqsubseteq X_k &\iff \lfloor l \rfloor(X_1, \ldots, X_n) \sqsubseteq Y). \end{aligned} \tag{5.12}$$

Therefore, by $\lfloor l \rfloor_{\uparrow}(\gamma_1, \ldots, \alpha, \ldots, \gamma_n) \leq \gamma$ and $\lfloor l \rfloor_{\uparrow}(\gamma_1, \ldots, \beta, \ldots, \gamma_n) \leq \gamma$, we have $Y_k \in ((\alpha \vee \beta)_{\uparrow} \| (\alpha \wedge \beta)^{\downarrow})$, which concludes with Equation (5.12) that, for each $X_k \in ((\alpha \vee \beta)^{\downarrow} \| (\alpha \wedge \beta)_{\uparrow})$, we obtain $\lfloor l \rfloor(X_1, \ldots, X_n) \sqsubseteq Y$.

($\lfloor l \rfloor \dashv^i \lceil r \rceil$). We prove that, if $\lfloor l \rfloor$ and $\lceil r \rceil$ form an adjoint pair, the extensions $\lfloor l \rfloor_{\uparrow}$ and $\lceil r \rceil^{\downarrow}$ also form an adjoint pair on the canonical extension $\overline{\mathbb{P}}$. Namely, for all $\alpha, \beta, \gamma_1, \ldots, \gamma_n \in \overline{\mathbb{P}}$,

$$\lfloor l \rfloor_{\uparrow}(\gamma_1, \ldots, \alpha, \ldots, \gamma_n) \leq \beta \iff \alpha \leq \lceil r \rceil^{\downarrow}(\gamma_1, \ldots, \beta, \ldots, \gamma_n).$$

If there exist a γ_j which is either ${\gamma_j}^{\downarrow} = \emptyset$ or $\gamma_{j\uparrow} = \emptyset$, it trivially holds (see Items 1 and 2 above). Moreover, if either $\alpha^{\downarrow} = \emptyset$ or $\beta_{\uparrow} = \emptyset$, this is also trivial. Hereafter, we treat the non-empty case. For arbitrary $F \in \alpha^{\downarrow}$, $I \in \beta_{\uparrow}$, and all $X_k \in ({\gamma_k}^{\downarrow} \| \gamma_{k\uparrow})$ and all $Y_k \in (\gamma_{k\uparrow} \| {\gamma_k}^{\downarrow})$, we claim

$$\lfloor l \rfloor(X_1, \ldots, F, \ldots, X_n) \sqsubseteq I \iff F \sqsubseteq \lceil r \rceil(Y_1, \ldots, I, \ldots, Y_n).$$

($\Rightarrow$). If there exist $a \in F$, $b \in I$, $x_k \in X_k$ for each coordinate k such that $\lfloor l \rfloor(x_1, \ldots, a, \ldots, x_n) \leq b$. Furthermore, for each coordinate k, as $X_k \sqsubseteq Y_k$, there exists an element $y_k \in X_k \cap Y_k$. Since X_k is a filter and Y_k is downward closed, there exists an element z_k such that $z_k \in X_k \cap Y_k$, $z_k \leq x_k$ and $z_k \leq y_k$. Moreover, by monotonicity of $\lfloor l \rfloor$, we have $\lfloor l \rfloor(z_1, \ldots, a, \ldots, z_n) \leq b$. By the adjointness $\lfloor l \rfloor \dashv^i \lceil r \rceil$, we obtain $a \leq \lceil r \rceil(z_1, \ldots, b, \ldots, z_n)$, which means $F \sqsubseteq \lceil r \rceil(Y_1, \ldots, I, \ldots, Y_n)$.

($\Leftarrow$). This is analogous. Therefore, $\lfloor l \rfloor_\uparrow \dashv^i \lceil r \rceil^\downarrow$. $\square$

Proposition 5.5.4 and Proposition 5.5.5 authorise the following definition.

Definition 5.5.6 (Canonical extension of poset expansions)**.** *The canonical extension* of a poset expansion $\langle \mathbb{P}, \lfloor l \rfloor, \lceil r \rceil, \Diamond, \Box, c \rangle$, where c is a constant, is the 6-tuple $\langle \overline{\mathbb{P}}, \lfloor l \rfloor_\uparrow, \lceil r \rceil^\downarrow, \Diamond_\uparrow, \Box^\downarrow, c \rangle$. The 6-tuple $\langle \overline{\mathbb{P}}, f_\uparrow, g^\downarrow, l_\uparrow, r^\downarrow, c \rangle$ is *the canonical extension* of a bounded poset expansion $\langle \mathbb{P}, f, g, l, r, c \rangle$, where $\mathbb{P}$ is a bounded poset and c is a constant.

Remark 5.5.7. (a) Proposition 5.5.5 tells us that canonical extensions of poset expansions with constants, $\bot$-additive operations, $\top$-multiplicative operations, diamond, box and strict adjoint pairs satisfy the necessary conditions of canonical extensions. Moreover, this is a unique way to preserve adjointness, in general (see [31]). However, we do not know, whether it is sufficient or not. For example, a full algebra for linear logic in [2, p.516] is satisfying $\circ +$ associativity axioms. But, we do not know whether these axioms are satisfied on the canonical extension.[4]

(b) By Proposition 5.5.5, the canonical extension of a bounded poset expan-

[4]Note that the canonical extensions of $\circ$ and $+$ are uniquely defined, since $\circ$ is a left adjoint to both implications ($\rightarrow$ and $\leftarrow$), and $+$ is a right adjoint to both coimplications ($\rightharpoondown$ and $\leftharpoondown$).

sion with constants, additive operations, multiplicative operations and adjoint pairs satisfy the essential requirements of canonical extensions. But, as distinct from the lattice case, we do not know whether the canonical extensions of ϵ-join-preserving operations (ϵ-meet-preserving operations) satisfy ϵ-join-preservability (ϵ-meet-preservability) on the canonical extension or not. Nevertheless, by Proposition 5.5.4, we justify our canonical extension of a bounded poset expansion with constants, ϵ-join-preserving operations, ϵ-meet-preserving operations, additive operations, multiplicative operations and adjoint pairs.

(c) Without the adjointness of $\lfloor l \rfloor$ and $\lceil r \rceil$ (l and r), we also assume that the canonical extensions of $\lfloor l \rfloor$ and $\lceil r \rceil$ (l and r) are $\lfloor l \rfloor_{\uparrow}$ and $\lceil r \rceil^{\downarrow}$ ($l_{\uparrow}$ and $r^{\downarrow}$), because the additivity of $\lfloor l \rfloor_{\uparrow}$ ($l_{\uparrow}$) and the multiplicativity of $\rho^{\downarrow}$ ($r^{\downarrow}$) are naturally proved.

(d) For bounded lattice expansions, we do not need to assume $\perp$-strictness or $\top$-strictness. It is used only in the proof of Proposition 5.5.5 to prove the non-emptiness of Y_k. Note that, for $\Diamond$ and $\Box$, we can take $\Box Y$ and $\Diamond Y$ as Y_k.

A syntactic characterisation of $\cup$-terms and $\cap$-terms As in the case of lattice expansions, we will describe a class of $\cup$-terms and a class of $\cap$-terms. Hereafter, we consider a poset expansion $\langle \mathbb{P}, \lfloor l \rfloor, \lceil r \rceil, \Diamond, \Box, c \rangle$, a bounded poset expansion $\langle \mathbb{P}, f, g, l, r, c \rangle$, and their canonical extensions $\langle \overline{\mathbb{P}}, \lfloor l \rfloor_{\uparrow}, \lceil r \rceil^{\downarrow}, \Diamond_{\uparrow}, \Box^{\downarrow}, c \rangle$ and $\langle \overline{\mathbb{P}}, f_{\uparrow}, g^{\downarrow}, l_{\uparrow}, r^{\downarrow}, c \rangle$.[5]

Firstly, we straightforwardly notice the following for propositional variables, con-

[5]Since a constant c is trivially smooth, i.e. $c_{\uparrow} = c^{\downarrow}$, and *unbiased*, we do not add $\_{\uparrow}$ or $\_^{\downarrow}$. On the other hand,while f and g are also smooth, they are *biased* (see Proposition 5.5.4).

stants and constant terms, which are terms containing no propositional variable.

Lemma 5.5.8. *All propositional variables are both $\cup$-terms and $\cap$-terms. Each constant is a $\cup$-term and a $\cap$-term. Every constant term is a $\cup$-term and a $\cap$-term.*

Proof. Here we show just that a propositional variable p_i is a $\cup$-term. That is, for arbitrary $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have

$$(p_{i\uparrow}(\alpha_1, \ldots, \alpha_N))_{\uparrow} = \lambda(\{p_i(\underline{F\|I}) \mid \forall k \in \{1, \ldots, N\}, F_k \in \alpha_k{}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}).$$

However, this is straightforward, because it means $\alpha_{i\uparrow} = \lambda(\{F_i \mid F_i \in \alpha_i{}^{\downarrow}\})$ (recall the empty condition of the right hand side in Remark 5.3.4). This is exactly the definition of filter bases. $\square$

On the poset expansion $\langle \mathbb{P}, \lfloor l \rfloor, \lceil r \rceil, \Diamond, \Box, c\rangle$, we obtain the following.

Lemma 5.5.9. *Let $\alpha, \alpha_1, \ldots, \alpha_n \in \overline{\mathbb{P}}$. For arbitrary $\mathfrak{F}, \mathfrak{F}_1, \ldots, \mathfrak{F}_n \in \wp(\mathcal{F})$ and $\mathfrak{I}, \mathfrak{I}_1, \ldots, \mathfrak{I}_n \in \wp(\mathcal{I})^{\partial}$, if $\mathfrak{F}_k$ and $\mathfrak{I}_k$ are bases of α_k for each index $k \in \{\_, 1, \ldots, n\}$, we have*

1. $(\lfloor l \rfloor_{\uparrow}(\alpha_1, \ldots, \alpha_n))_{\uparrow} = \lambda(\{\lfloor l \rfloor(X_1, \ldots, X_n) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$,

2. $(\lceil r \rceil^{\downarrow}(\alpha_1, \ldots, \alpha_n))^{\downarrow} = \upsilon(\{\lceil r \rceil(Y_1, \ldots, Y_n) \mid Y_k \in (\mathfrak{I}_k \| \mathfrak{I}_k)\})$,

3. $(\Diamond_{\uparrow}(\alpha))_{\uparrow} = \lambda(\{\Diamond F \mid F \in \mathfrak{F}\})$,

4. $(\Box^{\downarrow}(\alpha))^{\downarrow} = \upsilon(\{\Box I \mid I \in \mathfrak{I}\})$.

Proof. We check only Item 1. ($\subseteq$). This is straightforward.

($\supseteq$). If there exists a coordinate k such that the order-type $\mu_k = 1$ and $\mathfrak{F}_k = \emptyset$, or the order-type $\mu_k = \partial$ and $\mathfrak{I}_k = \emptyset$. Then, $\alpha_k = \bot$ ($\mu_k = 1$) or $\alpha_k = \top$ ($\mu_k = \partial$). Therefore, Item 1 holds (recall Items 1 and 2 in the proof of Proposition 5.5.5).

Otherwise, for each coordinate k, for all $X_j \in (\mathfrak{F}_j \| \mathfrak{I}_j)$ and an arbitrary ideal $I \in \lambda(\{\lfloor l \rfloor(X_1, \ldots, X_n) \mid X_j \in (\mathfrak{F}_j \| \mathfrak{I}_j)\})$, we define

$$Y_k := \{x_k \in P \mid \forall j \in \{1, \ldots, n\} \setminus \{k\}, \exists x_j \in X_j, \exists i \in I.\ \lfloor l \rfloor(x_1, \ldots, x_n) \leq i\}.$$

If $\mu_k = 1$, Y_k is an ideal, and if $\mu_k = \partial$, it is a filter. By definition, we also have that $Y_k \in (\alpha_{k\uparrow} \| \alpha_k{}^{\downarrow})$. Hence, we have $\lfloor l \rfloor(X_1, \ldots, X_{k-1}, X'_k, X_{k+1}, \ldots, X_n) \sqsubseteq I$, for every $X'_k \in (\alpha_k{}^{\downarrow} \| \alpha_{k\uparrow})$ (recall Equation (5.12) in the proof of Proposition 5.5.5). Notice that the k-th coordinate is changed from an arbitrary element of *basis* to an arbitrary element of α_k. We repeat this replacement for each coordinate, which concludes Item 1. Note that, for Item 3, we can take $\Box I$ as Y_k, and Item 4 is analogous to Item 3. $\Box$

As a corollary of Lemma 5.5.9, we obtain the following.

Corollary 5.5.10. *Let $t, t_1, \ldots, t_n$ be terms. Then, we have (recall Abbreviation 3.2.4)*

1. *$\lfloor l \rfloor(t_1, \ldots, t_n)$ is a $\cup$-term, if t_k is a $(\cup \| \cap)$-term for each $k \in \{1, \ldots, n\}$,*

2. *$\lceil r \rceil(t_1, \ldots, t_n)$ is a $\cap$-term, if t_k is a $(\cap \| \cup)$-term for each $k \in \{1, \ldots, n\}$,*

3. *$\Diamond t$ is a $\cup$-term if t is a $\cup$-term,*

4. *$\Box t$ is a $\cap$-term if t is a $\cap$-term.*

On the bounded poset expansion $\langle \mathbb{P}, f, g, l, r, c \rangle$, we obtain the following. Recall that we assume non-emptiness of bases.

Lemma 5.5.11. *Let $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, $\mathfrak{F}_1, \ldots, \mathfrak{F}_N \in \wp(\mathcal{F})$ and $\mathfrak{I}_1, \ldots, \mathfrak{I}_N \in \wp(\mathcal{I})^\partial$. If $\mathfrak{F}_k$ and $\mathfrak{I}_k$ are non-empty bases of α_k, then we have*

1. $(f_{\uparrow}(\alpha_1, \ldots, \alpha_m))_{\uparrow} = \lambda(\{f(X_1, \ldots, X_m) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$,

2. $(g^{\downarrow}(\alpha_1, \ldots, \alpha_{m'}))^{\downarrow} = \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_k \in (\mathfrak{I}_k \| \mathfrak{F}_k)\})$,

3. $(l_{\uparrow}(\alpha_1, \ldots, \alpha_n))_{\uparrow} = \lambda(\{l(X_1, \ldots, X_n) \mid X_k \in (\mathfrak{F}_k \| \mathfrak{I}_k)\})$,

4. $(r^{\downarrow}(\alpha_1, \ldots, \alpha_n))^{\downarrow} = \upsilon(\{r(Y_1, \ldots, Y_n) \mid Y_k \in (\mathfrak{I}_k \| \mathfrak{F}_k)\})$.

Proof. Items 3 and 4 are analogous to Items 1 and 2 in Lemma 5.5.9. We show only Item 2, here. (Actually, this is also analogous to Item 2 in Lemma 5.5.9). ($\subseteq$). This is straightforward.

($\supseteq$). Let k be each coordinate. For any $X \in \upsilon(\{g(Y_1, \ldots, Y_{m'}) \mid Y_j \in (\mathfrak{I}_j \| \mathfrak{F}_j)\})$ and all $Y_j \in (\mathfrak{I}_j \| \mathfrak{F}_j)$, we define

$$X_k := \{y_k \in P \mid \forall j \in \{1, \ldots, m'\} \setminus \{k\}, \exists y_j \in Y_j, \exists x \in X.\ x \leq g(y_1, \ldots, y_{m'})\}.$$

If $\epsilon_k = 1$ ($\epsilon_k = \partial$), then X_k is a filter (an ideal). Note that the non-emptiness comes from non-empty bases. Namely, as every basis is non-empty, there always exist $Y_j \in \mathcal{I}_j$ such that $X \sqsubseteq g(Y_1, \ldots, Y_{m'})$, hence there is $y_k \in X \cap g(Y_1, \ldots, Y_{m'})$, which is a witness of the non-emptiness of X_k. Moreover, by definition, $X_k \in ({\alpha_k}^{\downarrow} \| \alpha_{k\uparrow})$. Therefore, for an arbitrary $Y'_k \in \alpha_{k\uparrow}$, we have $X \sqsubseteq g(Y_1, \ldots, Y'_k, \ldots, Y_{m'})$. We repeat this replacement for all coordinates. $\square$

As a corollary of Lemma 5.5.11, we obtain the following.

Corollary 5.5.12. *Let $t_1, \ldots, t_N$ be terms. Then, we have*

1. *$f(t_1, \ldots, t_m)$ is a $\cup$-term, if t_k is a $(\cup \| \cap)$-term for each $k \in \{1, \ldots, m\}$,*

2. *$g(t_1, \ldots, t_{m'})$ is a $\cap$-term, if t_k is a $(\cap \| \cup)$-term for each $k \in \{1, \ldots, m'\}$,*

3. *$l(t_1, \ldots, t_n)$ is a $\cup$-term, if t_k is a $(\cup\|\cap)$-term for each $k \in \{1, \ldots, n\}$,*

4. *$r(t_1, \ldots, t_n)$ is a $\cap$-term, if t_k is a $(\cap\|\cup)$-term for each $k \in \{1, \ldots, n\}$.*

Furthermore, we also have the following lemmata.

Lemma 5.5.13. *On posets, every term of type $t_{\underline{\vee}}$ is a $\cap$-term and every term of type $t_{\bar{\wedge}}$ is a $\cup$-term, where $t_{\underline{\vee}}$ and $t_{\bar{\wedge}}$ are defined as follows (recall Abbreviation 3.2.4):*

$$t_{\underline{\vee}} ::= p \mid \lfloor l \rfloor(c, \ldots, t_{(\underline{\vee}\|\bar{\wedge})}, \ldots, c) \mid \Diamond t_{\underline{\vee}},$$

$$t_{\bar{\wedge}} ::= p \mid \lceil r \rceil(c, \ldots, t_{(\bar{\wedge}\|\underline{\vee})}, \ldots, c) \mid \Box t_{\bar{\wedge}}.$$

Proof. By Corollary 5.5.10, every term t of type $t_{\underline{\vee}}$ is a $\cup$-term, hence for arbitrary $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we have

$$t(\alpha_1, \ldots, \alpha_N)_{\uparrow} = \lambda(\{t(\underline{F\|I}) \mid F_k \in \alpha_k{}^{\downarrow}, I_k \in \alpha_{k\uparrow}\}). \tag{5.13}$$

Besides, we claim that each $t_{\underline{\vee}}$ is a unary $\perp$-additive operation and each $t_{\bar{\wedge}}$ is a unary $\top$-multiplicative operation. This is by parallel induction.

Base cases are rather trivial. Suppose that $t_{\underline{\vee}}$ is a unary $\perp$-additive operation and $t_{\bar{\wedge}}$ is a unary $\top$-multiplicative operation. Let $\lfloor l \rfloor(c, \ldots, t_{(\underline{\vee}\|\bar{\wedge}), \ldots, c})$ be a term function where the k-th coordinate is substituted by $t_{(\underline{\vee}\|\bar{\wedge})}$.

(Case $\mu_k = 1$). For arbitrary $x, y, z \in P$, if $\lfloor l \rfloor(c, \ldots, t_{\underline{\vee}}, \ldots, c)(x) \leq z$ and $\lfloor l \rfloor(c, \ldots, t_{\underline{\vee}}, \ldots, c)(y) \leq z$, by $\perp$-additivity of $\lfloor l \rfloor$, there exists $z' \in P$ such that $t_{\underline{\vee}}(x) \leq z'$, $t_{\underline{\vee}}(y) \leq z'$ and $\lfloor l \rfloor(c, \ldots, t_{\underline{\vee}}, \ldots, c)(z') \leq z$. By inductive hypothesis, there exists $z'' \in P$ such that $x \leq z''$, $y \leq z''$ and $t_{\underline{\vee}}(z'') \leq z'$, hence $\lfloor l \rfloor(c, \ldots, t_{\underline{\vee}}, \ldots, c)(z'') \leq z$. (If the domain of $t_{\underline{\vee}}$ is $\mathbb{P}^{\partial}$, then there is $z'' \in P$ such that $z'' \leq x$ and $z'' \leq y$).

For an arbitrary $y \in P$, because $\lfloor l \rfloor$ is $\bot$-strict, there exists $x \in P$ such that $\lfloor l \rfloor(c, \ldots, x, \ldots, c) \leq y$. By induction hypothesis, there is also $x' \in P$ such that $t_{\underline{\vee}}(x') \leq x$, hence $\lfloor l \rfloor(c, \ldots, t_{\underline{\vee}}, \ldots, c)(x') \leq y$.

(Case $\mu_k = \partial$). This is analogous.

Therefore, as $\Diamond$ is a special case of $\lfloor l \rfloor$, $t_{\underline{\vee}}$ is a unary $\bot$-additive operation. It is analogously proved that $t_{\overline{\wedge}}$ is a unary $\top$-multiplicative operation.

Next we show here that every term t of type $t_{\underline{\vee}}$ is a $\cap$-term. It suffices to show that, for any $X \in \upsilon(\{t(I \| F) \mid I \in \alpha_{\uparrow}, F \in \alpha^{\downarrow}\})$, there is a filter $F \in \alpha^{\downarrow}$ such that $X \sqsubseteq t(F)$. (If the domain of t is $\mathbb{P}^{\partial}$, then we find an ideal $I \in \alpha_{\uparrow}$ such that $X \sqsubseteq t(I)$). For each $X \in \upsilon(\{t(I \| F) \mid I \in \alpha_{\uparrow}, F \in \alpha^{\downarrow}\})$, we define

$$Y := \{y \in P \mid \exists x \in X.\ x \leq t(y)\}. \tag{5.14}$$

Then, Y is a filter (Y is an ideal) which we need. Note that, to prove the non-emptiness of Y, we use strictness of t. □

Lemma 5.5.14. *On bounded posets, each term of type $t_{\vee}$ is a $\cap$-term and each term of type $t_{\wedge}$ is a $\cup$-term, where $t_{\vee}$ and $t_{\wedge}$ are defined as follows:*

$$t_{\vee} ::= p \mid f(c, \ldots, t_{(\vee \| \wedge)}, \ldots, c) \mid l(c, \ldots, t_{(\vee \| \wedge)}, \ldots, c),$$

$$t_{\wedge} ::= p \mid g(c, \ldots, t_{(\wedge \| \vee)}, \ldots, c) \mid r(c, \ldots, t_{(\wedge \| \vee)}, \ldots, c).$$

Proof. The proof is analogous to the proof of Lemma 5.5.13. The difference is only how to guarantee the non-emptiness of Y. □

Remark 5.5.15. We mention that the condition of Lemma 5.5.13 is restricted than the case of lattice expansions. In [84], the term types $t_\vee$ and $t_\wedge$ are defined as follows:

$$t_\vee ::= p \mid f(t_{(\vee\|\wedge)}, \ldots, t_{(\vee\|\wedge)}) \mid l(c, \ldots, t_{(\vee\|\wedge)}, \ldots, c),$$

$$t_\wedge ::= p \mid g(t_{(\wedge\|\vee)}, \ldots, t_{(\wedge\|\vee)}) \mid r(c, \ldots, t_{(\wedge\|\vee)}, \ldots, c).$$

This is because, over lattice expansions, we can prove that every term of type $t_\vee$ is join-preserving from the product domain, and every term of type $t_\wedge$ is meet-preserving from the product domain. However, over poset expansions, we cannot prove the same result without fixing other coordinates of f and g with constants,[6] because the existence of the *unique* least upper bound and the *unique* greatest lower bound is not guaranteed on posets.

Finally, we obtain the following theorems, which describe classes of $\cup$-terms and $\cap$-terms syntactically.

Theorem 5.5.16 ($\cup$-term and $\cap$-term)**.** *On posets, every term of type $t_\sqcup$ is a $\cup$-term and every term of type $t_\sqcap$ is a $\cap$-term, where $t_\sqcup$ and $t_\sqcap$ are defined as follows:*

$$t_\sqcup ::= p \mid c \mid \lfloor l \rfloor(t_{(\sqcup\|\sqcap)}, \ldots, t_{(\sqcup\|\sqcap)}) \mid \Diamond t_\sqcup \mid t_{\bar{\wedge}},$$

$$t_\sqcap ::= p \mid c \mid \lceil r \rceil(t_{(\sqcap\|\sqcup)}, \ldots, t_{(\sqcap\|\sqcup)}) \mid \Box t_\sqcap \mid t_{\underline{\vee}}.$$

Theorem 5.5.17 ($\cup$-term and $\cap$-term)**.** *On bounded posets, each term of type $t_\cup$ is a $\cup$-term and every term of type $t_\cap$ is a $\cap$-term, where $t_\cup$ and $t_\cap$ are defined as*

[6] In fact, we can weaken this restriction: $f(p, \ldots, t_{(\vee\|\wedge)}, \ldots p)$ where p is propositional variables not appearing in $t_{(\vee\|\wedge)}$, etc. For example, if f is a $(1, 1)$-join-preserving operation, a term function $f(p_1, f(p_2, p_3))$ is $(1, 1, 1)$-join-preserving operation.

follows:

$$t_{\cup} ::= p \mid c \mid f(t_{(\cup\|\cap)}, \ldots, t_{(\cup\|\cap)}) \mid l(t_{(\cup\|\cap)}, \ldots, t_{(\cup\|\cap)}) \mid t_{\wedge},$$

$$t_{\cap} ::= p \mid c \mid g(t_{(\cap\|\cup)}, \ldots, t_{(\cap\|\cup)}) \mid r(t_{(\cap\|\cup)}, \ldots, t_{(\cap\|\cup)}) \mid t_{\vee}.$$

Example 5.5.18. Over residuated algebras $\langle \mathbb{P}, \circ, \rightarrow, \leftarrow \rangle$, the types $t_{\sqcup}$ and $t_{\sqcap}$ are obtained as follows:

$$t_{\sqcup} ::= p \mid t_{\sqcup} \circ t_{\sqcup},$$

$$t_{\sqcap} ::= p \mid t_{\sqcup} \rightarrow t_{\sqcap} \mid t_{\sqcap} \leftarrow t_{\sqcup}.$$

Note that, as residuated algebras do not have any constant, we do not need $t_{\underline{\vee}}$ nor $t_{\bar{\wedge}}$.

By Theorems 5.5.16 and 5.5.17, together with Theorems 5.4.1, 5.4.3, 5.4.4, 5.4.5 and 5.4.6, we can syntactically characterise a class of canonical inequalities. However, our results obtained so far are still restricted, because our technique can check only the inequalities of type $s \leq t$ where s is a $\cup$-term and t is a $\cap$-term. In other words, our methodology so far does not tell how to apply our technique for arbitrary terms containing non-$\cup$-terms or non-$\cap$-terms. Therefore, in the following paragraph, we will expand our scope with introducing new notions, like *pruned trees*, *critical subterms*, *pseudo-$\cup$-terms* or *pseudo-$\cap$-terms*. Finally, we will prove Theorem 5.5.25, which uniformly subsume all theorems in Section 5.4.

The main theorem for poset expansions In this paragraph, we basically discuss the bounded poset expansion $\langle \mathbb{P}, f, g, l, r, c \rangle$. However, whenever we apply the following argument for the poset expansion $\langle \mathbb{P}, \lfloor l \rfloor, \lceil r \rceil, \Diamond, \Box, c \rangle$, we just forget about f and g and replace l with $\lfloor l \rfloor$ and $\Diamond$, r with $\lceil r \rceil$ and $\Box$, $t_{\vee}$ with $t_{\underline{\vee}}$, $t_{\wedge}$ with $t_{\bar{\wedge}}$, $t_{\cup}$

with $t_\sqcup$ and $t_\cap$ with $t_\sqcap$.

Based on the syntactic description of $\cup$-terms and $\cap$-terms (Theorems 5.5.16 and 5.5.17), we label construction trees of term with $\cup$, $\cap$ and ? as follows:

1. Label the root with $\cap$.

2. If the node does not have any child, then we have already finished labelling the branch. Otherwise, we label each child with the following rule.

   (a) If the node is $f(t_1, \ldots, t_m)$ and labelled with $\cup$, then we label t_k with $\cup$ if $\delta_k = 1$, and t_k with $\cap$ if $\delta_k = \partial$, for each coordinate $k \in \{1, \ldots, m\}$.

   (b) If the node is $g(t_1, \ldots, t_{m'})$ and labelled with $\cap$, then we label t_k with $\cap$ if $\epsilon_k = 1$, and t_k with $\cup$ if $\epsilon_k = \partial$, for each coordinate $k \in \{1, \ldots, m'\}$.

   (c) If the node is $l(t_1, \ldots, t_n)$ and labelled with $\cup$, then we label t_k with $\cup$ if $\mu_k = 1$, and t_k with $\cap$ if $\mu_k = \partial$, for each coordinate $k \in \{1, \ldots, n\}$.

   (d) If the node is $r(t_1, \ldots, t_n)$ and labelled with $\cap$, then we label t_k with $\cap$ if $\nu_k = 1$, and t_k with $\cup$ if $\nu_k = \partial$, for each coordinate $k \in \{1, \ldots, n\}$.

   (e) If the node is $t_\vee$ and labelled with $\cap$, then we label all nodes below the current node with $\cap$.

   (f) If the node is $t_\wedge$ and labelled with $\cup$, then we label all nodes below the current node with $\cup$.

   (g) Otherwise, we label all nodes below the current node with ?.

3. Move to every child and repeat 2 until every node is labelled.

We call the construction tree $\cap$-*labelled. The* $\cup$-*labelled construction tree* is defined by the same algorithm but labelling the root with $\cup$. Then, we can straightforwardly claim the following proposition.

Proposition 5.5.19. *A term t is of type $t_\cup$, hence a $\cup$-term, if there is no node labelled with ? in the $\cup$-labelled construction tree of t, and analogously, a term t is of type $t_\cap$, hence a $\cap$-term, if there is no node labelled with ? in the $\cap$-labelled construction tree of t.*

Now, as in the case of lattice expansions, we need the following definitions.

Definition 5.5.20 (Critical subterms)**.** Let t be a term. A subterm of t is $\cap$-*critical* ($\cup$-*critical*), if it is both a node labelled with either $\cup$ or $\cap$, and a parent of nodes labelled with ? in the $\cap$-labelled construction tree of t (in the $\cup$-labelled construction tree of t).

In the following definition, subtrees are embedding into the original trees. That is, the roots are preserved.

Definition 5.5.21 (Well-pruned tree)**.** Let s, t be terms. A subtree t' of the construction tree of t is $\cup$-*well-pruned* ($\cap$-*well-pruned*) if, in t', each path from a propositional variable to the root contains a $\cup$-critical ($\cap$-critical) subterm. *The* $\cup$-*well-pruned* (*the* $\cap$-*well-pruned*) tree of t is the largest $\cup$-well-pruned ($\cap$-well-pruned) subtree of t. Especially, if t is a propositional variable, the $\cup$-well-pruned ($\cap$-well-pruned) subtree of t is the empty tree. For an inequality $s \leq t$, *the well-pruned pair of trees for* $s \leq t$ is a pair of the $\cup$-well-pruned tree of s and the $\cap$-well-pruned tree of t.

Definition 5.5.22 (Consistent variable occurrence)**.** Let s, t be terms. We say that *an inequality $s \leq t$ has consistent variable occurrence*, when there is no variable in $s \leq t$ signed with both signs ($+$ and $-$) in the well-pruned pair of trees for $s \leq t$, where s is $-$-signed and t is $+$-signed.

Figure 5.5: The $\cup$-labelled, $-$-signed construction tree of s

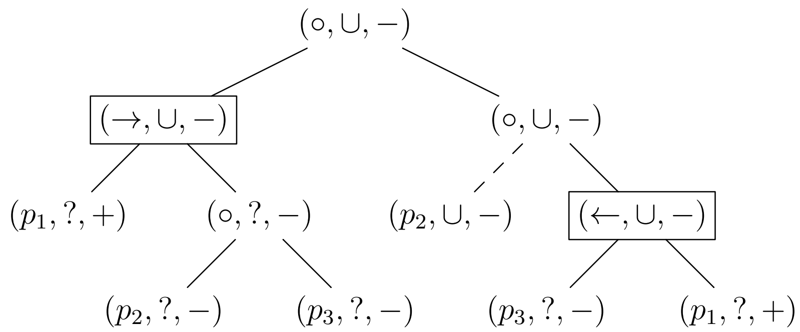
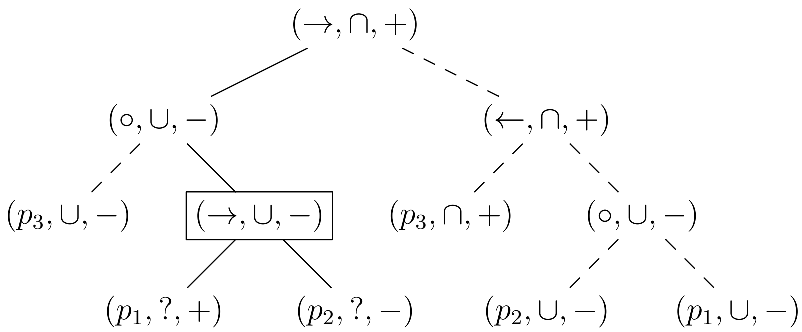


Figure 5.6: The $\cap$-labelled, $+$-signed construction tree of t



Let us take an example from residuated algebras. For the following terms s and t, we will illustrate that the inequality $s \leq t$ has consistent variable occurrence.

$$s = (p_1 \rightarrow (p_2 \circ p_3)) \circ (p_2 \circ (p_3 \leftarrow p_1))$$

$$t = (p_3 \circ (p_1 \rightarrow p_2)) \rightarrow (p_3 \leftarrow (p_2 \circ p_1))$$

Firstly, we draw the labelled and singed construction trees of s and t in Fig. 5.5 and Fig. 5.6. On these trees, each node shows (*the outermost connective*, *the label*, *the sign*). All dashed lines are pruned, when we consider the well-pruned trees. And, framed nodes are critical: recall Example 5.5.18. Then, we notice that the inequality $s \leq t$ has consistent variable occurrence, because each variable, in the scope of critical subterms (i.e, we only see the solid lines and ignore all dashed lines), is uniquely signed: p_1 is $+$, and p_2 and p_3 are $-$.

Definition 5.5.23 (Pseudo-$\cup$-term and pseudo-$\cap$-term)**.** Let t be a term. A term t' is *the pseudo-$\cap$-term of t* (*the pseudo-$\cup$-term of t*), if every $\cap$-critical ($\cup$-critical) subterms of t is replaced with a *fresh* variable. Note that, even if a $\cap$-critical ($\cup$-critical) subterm appears in t more than once, we replace each occurrence with distinct variables.

For the above example, we say that the following term $s' = q_1 \circ (p_2 \circ q_2)$ is the pseudo-$\cup$-term of $s = (p_1 \to (p_2 \circ p_3)) \circ (p_2 \circ (p_3 \leftarrow p_1))$: the $\cup$-critical subterms $p_1 \to (p_2 \circ p_3)$ and $p_3 \leftarrow p_1$ are replaced by fresh variables q_1 and q_2, respectively. We also say that the following term $t' = (p_3 \circ q_3) \to (p_3 \leftarrow (p_2 \circ p_1))$ is the pseudo-$\cap$-term of $t = (p_3 \circ (p_1 \to p_2)) \to (p_3 \leftarrow (p_2 \circ p_1))$: the $\cap$-critical subterm $p_1 \to p_2$ is replaced by a fresh variable q_3.

It is straightforward that, for each term t of type $t_\cap$ (type $t_\cup$), the pseudo-$\cap$-term of t (the pseudo-$\cup$-term of t) is t itself. But, in general, pseudoterms are not the same as the original terms. However, the next proposition provides us with a meaningful connection between terms and the pseudoterms, which allows us to generalise Theorem 5.4.1 from $\cup$-terms and $\cap$-terms to arbitrary terms, because all terms can be seen as both a pseudo-$\cup$-term and a pseudo-$\cap$-term. The proof is straightforward from the fact that every pseudo-$\cap$-term is a $\cap$-term and every pseudo-$\cup$-term is a $\cup$-term.

Proposition 5.5.24. *Let s, t be terms. We denote by $t(p_1, \dots, p_N)$ (or $t(\underline{p})$, for short) that each propositional variable in t is one of p_k in the list $(p_1, \dots, p_N)$, analogously $s(p_1, \dots, p_N)$ or $s(\underline{p})$. Let $t_1, \dots, t_a$ be all $\cap$-critical subterms of t, $s_1, \dots, s_b$ all $\cup$-critical subterms of s, and $t'(\underline{p}, q_1, \dots, q_a)$ the pseudo-$\cap$-term of t, where each*

t_k in t is replaced by q_k, and $s'(\underline{p}, r_1, \ldots, r_b)$ the pseudo-$\cup$-term of s, where each s_k in s is replaced with r_k. Then, we have

$$t(\underline{p}) = t'(\underline{p}, q_1, \ldots, q_a)[t_1(\underline{p})/q_1, \ldots, t_a(\underline{p})/q_a],$$

$$s(\underline{p}) = s'(\underline{p}, r_1, \ldots, r_b)[s_1(\underline{p})/r_1, \ldots, s_b(\underline{p})/r_b].$$

Moreover, we also have, for each $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$,

$$t(\underline{\alpha})^{\downarrow} = \upsilon(\{t'(\underline{I \| F}, Y_1, \ldots, Y_a) \mid I_k \in \alpha_{k\uparrow}, F_k \in \alpha_k{}^{\downarrow}, Y_j \in (t_j(\underline{\alpha})_{\uparrow} \| t_j(\underline{\alpha})^{\downarrow})\}),$$

$$s(\underline{\alpha})_{\uparrow} = \lambda(\{s'(\underline{F \| I}, X_1, \ldots, X_b) \mid F_k \in \alpha_k{}^{\downarrow}, I_k \in \alpha_{k\uparrow}, X_j \in (s_j(\underline{\alpha})^{\downarrow} \| s_j(\underline{\alpha})_{\uparrow})\}),$$

where $(\underline{\alpha}) = (\alpha_1, \ldots, \alpha_N)$.[7]

Finally, we obtain the main theorem for poset expansions.

Main Theorem 5.5.25 (for poset expansions)**.** *Let s, t be terms over poset expansions. An inequality $s \leq t$ is canonical, whenever it satisfies the following two conditions:*

1. *$s \leq t$ has consistent variable occurrence,*

2. *each variable in $s \leq t$ is uniquely signed either in the $-$-signed construction tree of s or in the $+$-signed construction tree of t. Note that these construction trees are not pruned.*

Proof. To save space, but to discuss precisely enough, we assume that all variables in $s \leq t$ are p_1, p_2, p_3 and p_4 satisfying the following condition.[8]

[7] We do not use the pair of elements notation for Y_k and X_k, because each critical subterm is replaced with a *fresh* variable, hence it occurs only once, either positively or negatively.

[8] In fact, this table does not cover all the cases, because the signs of critical subterms do not

	p_1	p_2	p_3	p_4
s	$-$	$+$	$+/-$	$+/-$
t	$+/-$	$+/-$	$+$	$-$
critical	$-$	$+$	$+$	$-$

In the above table, these signs are the same as the signed construction trees of s and t, where s is $-$-signed and t is $+$-signed, and, by $+/-$, we denote that a variable may be signed with both. For example, p_3 may occur both positively and negatively in s,[9] but it is signed only by $+$ in t and in the scope of $\cup$-critical subterms of s. In the third row, named critical, the signs are only in the well-pruned pair of trees for $s \leq t$. Moreover, we assume that, in s, there are two $\cup$-critical subterms s^- signed with $-$ and s^+ signed with $+$ in the $-$-signed construction tree of s, and, in t, there are two $\cap$-critical subterms t^+ signed with $+$ and t^- signed with $-$ in the $+$-signed construction tree of t.

Let s' be the pseudo-$\cup$-term of s, and t' the pseudo-$\cap$-term of t: that is,

$$s(p_1, \ldots, p_4) = s'(p_1, \ldots, p_4, s^-(p_1, \ldots, p_4), s^+(p_1, \ldots, p_4)),$$

$$t(p_1, \ldots, p_4) = t'(p_1, \ldots, p_4, t^+(p_1, \ldots, p_4), t^-(p_1, \ldots, p_4)).$$

For arbitrary $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, by Proposition 5.5.24, we have

$$s(\underline{p})(\underline{\alpha})_\uparrow = \lambda(\{s'(\underline{F'\|I'}, G, J) \mid F'_k \in \alpha_k{}^\downarrow, I'_k \in \alpha_{k\uparrow}, G \in s^-(\underline{p})(\underline{\alpha})^\downarrow, J \in s^+(\underline{p})(\underline{\alpha})_\uparrow\}),$$

need to correspond to the signs in s or t. For example, we can also consider a propositional variable p_5 signed with $-$ in s (but not under critical subterms) and signed with $+$ under critical subterms. But, the proof is analogous.

[9] But, if it is signed with $-$, it is not in the scope of $\cup$-critical subterms of s. Otherwise, $s \leq t$ does not have consistent variable occurrence.

$$t(\underline{p})(\underline{\alpha})^{\downarrow} = \upsilon(\{t'(\underline{I''\|F''}, K, H) \mid I''_k \in \alpha_{k\uparrow}, F''_k \in \alpha_k{}^{\downarrow}, K \in t^+(\underline{p})(\underline{\alpha})_{\uparrow}, H \in t^-(\underline{p})(\underline{\alpha})^{\downarrow}\}),$$

where $(\underline{p}) = (p_1, \ldots, p_4)$ and $(\underline{\alpha}) = (\alpha_1, \ldots, \alpha_N)$. Note that $s(\underline{p})(\underline{\alpha}) = s(\alpha_1, \ldots, \alpha_4)$ and $t(\underline{p})(\underline{\alpha}) = t(\alpha_1, \ldots, \alpha_4)$.

If there is some α_k ($k \in \{1, \ldots, 4\}$) which makes the filter basis of $s(\alpha_1, \ldots, \alpha_4)$ empty, then $s(\alpha_1, \ldots, \alpha_4) = \bot$, hence the statement trivially holds. And, if there exists some α_k ($k \in \{1, \ldots, 4\}$) which makes the ideal basis of $t(\alpha_1, \ldots, \alpha_4)$ empty, then $t(\alpha_1, \ldots, \alpha_4) = \top$, hence the statement trivially holds.

Otherwise, by Proposition 2.3.1, it suffices to show that, for any $F'_u, F''_u \in \alpha_u{}^{\downarrow}$ and $I'_u, I''_u \in \alpha_{u\uparrow}$ for each $u \in \{1, \ldots, 4\}$, $G \in s^-(\alpha_1, \ldots, \alpha_4)^{\downarrow}$, $J \in s^+(\alpha_1, \ldots, \alpha_4)_{\uparrow}$, $K \in t^+(\alpha_1, \ldots, \alpha_4)_{\uparrow}$ and $H \in t^-(\alpha_1, \ldots, \alpha_4)^{\downarrow}$,

$$s'(F'_1\|I'_1, \ldots, F'_4\|I'_4, G, J) \sqsubseteq t'(I''_1\|F''_1, \ldots, I''_4\|F''_4, K, H).$$

By our assumption, it is equivalent to prove

$$s'(F'_1, I'_2, F'_3\|I'_3, F'_4\|I'_4, G, J) \sqsubseteq t'(I''_1\|F''_1, I''_2\|F''_2, I''_3, F''_4, K, H).$$

As $G \in s^-(\alpha_1, \ldots, \alpha_4)^{\downarrow}$, $J \in s^+(\alpha_1, \ldots, \alpha_4)_{\uparrow}$, $K \in t^+(\alpha_1, \ldots, \alpha_4)_{\uparrow}$, $H \in t^-(\alpha_1, \ldots, \alpha_4)^{\downarrow}$, and $s \leq t$ has consistent variable occurrence, we have

$$G \sqsubseteq s^-(I''_1, F''_2, F'_3, I'_4),$$

$$s^+(I''_1, F''_2, F'_3, I'_4) \sqsubseteq J,$$

$$t^+(I''_1, F''_2, F'_3, I'_4) \sqsubseteq K,$$

$$H \sqsubseteq t^-(I_1'', F_2'', F_3', I_4').$$

By Lemma 5.2.8, there exist $i_1^g, i_1^j, i_1^k, i_4^h \in I_1''$, $f_2^g, f_2^j, f_2^k, f_2^h \in F_2''$, $f_3^g, f_3^j, f_3^k, f_3^h \in F_3'$ and $i_4^g, i_4^j, i_4^k, i_4^h \in I_4'$ such that

$$s^-(i_1^g, f_2^g, f_3^g, i_4^g) \in G,$$

$$s^+(i_1^j, f_2^j, f_3^j, i_4^j) \in J,$$

$$t^+(i_1^k, f_2^k, f_3^k, i_4^k) \in K,$$

$$t^-(i_1^h, f_2^h, f_3^h, i_4^h) \in H.$$

Since $F_1', F_1'' \in \alpha_1{}^{\downarrow}$ and $I_1'' \in \alpha_{1\uparrow}$, we have $F_1' \cap F_1'' \cap I_1'' \neq \emptyset$, hence these exists $x_1' \in F_1' \cap F_1'' \cap I_1''$. Analogously, $x_2' \in I_2' \cap F_2'' \cap I_2''$, $x_3' \in F_3' \cap I_3' \cap I_3''$ and $x_4' \in F_4' \cap I_4' \cap F_4''$. Then, we obtain that

$$\exists x_1 \in F_1' \cap F_1'' \cap I_1''.\ x_1' \leq x_1, i_1^g \leq x_1, i_1^j \leq x_1, i_1^k \leq x_1, \text{ and } i_1^h \leq x_1,$$

$$\exists x_2 \in I_2' \cap F_2'' \cap I_2''.\ x_2 \leq x_2', x_2 \leq f_2^g, x_2 \leq f_2^j, x_2 \leq f_2^k, \text{ and } x_2 \leq f_2^h,$$

$$\exists x_3 \in F_3' \cap I_3' \cap I_3''.\ x_3 \leq x_3', x_3 \leq f_3^g, x_3 \leq f_3^j, x_3 \leq f_3^k, \text{ and } x_3 \leq f_3^h,$$

$$\exists x_4 \in F_4' \cap I_4' \cap F_4''.\ x_4' \leq x_4, i_4^g \leq x_4, i_4^j \leq x_4, i_4^k \leq x_4, \text{ and } i_4^h \leq x_4.$$

It follows that $s^-(x_1, \ldots, x_4) \in G$, $s^+(x_1, \ldots, x_4) \in J$, $t^+(x_1, \ldots, x_4) \in K$ and $t^-(x_1, \ldots, x_4) \in H$. Then, we obtain

$$s(x_1, \ldots, x_4) \in s'(F_1' \| I_1', \ldots, F_4' \| I_4', G, J),$$

$$t(x_1, \ldots, x_4) \in t'(I_1''\|F_1'', \ldots, I_4''\|F_4'', K, H).$$

By assumption, $s(x_1, \ldots, x_4) \leq t(x_1, \ldots, x_4)$. Therefore, the statement holds. $\square$

Remark 5.5.26. The proof of Theorem 5.5.25 looks similar to the proof of lattice expansions (Theorem 3.3.22). However, in the lattice case, we need to care only for the signs in the scopes of all critical subterms. That is, it is necessary to check just whether $s \leq t$ has consistent variable occurrence. On the other hand, as we saw in Section 5.3, we cannot prove that $t_\cup \leq t_\cap$ is canonical for poset expansions. Then, we also need to keep our eyes on variables in $\cup$-terms and $\cap$-terms. Theorem 5.5.25 tells us what types of combinations of signs between variables in $\cup$-terms and $\cap$-terms and variables in the well-pruned pair of trees are acceptable under our method.

Chapter 6

Application to poset-based residuated algebras

In this chapter, we will apply Theorem 5.5.25 to (poset-based) residuated algebras introduced in [18], and show how to interpret our canonicity results on this setting. In this application, we can also notice that our canonicity results account for reasonably many canonical inequalities.

The following list shows canonical inequalities proved in [18].

1. (Associativity): $(p_1 \circ p_2) \circ p_3 \leq p_1 \circ (p_2 \circ p_3)$ and $p_1 \circ (p_2 \circ p_3) \leq (p_1 \circ p_2) \circ p_3$.

2. (Commutativity): $p_1 \circ p_2 \leq p_2 \circ p_1$.

3. (Square-increasingness): $p \leq p \circ p$.

4. (Right-lower-boundedness): $p_1 \circ p_2 \leq p_2$.

All these results are consequences of our main theorem, Theorem 5.5.25. Furthermore, we obtain many new canonical inequalities: in the following list, we can find some remarkable examples.

1. (Uniform inequality): $s \leq t$ where every propositional variable in s and t uniformly signed with either $+$ or $-$ both in the $-$-signed construction tree of s and in the $+$-signed construction tree of t. For example, the following inequality in residuated algebras is of this type:

$$((p_2 \circ p_2) \to p_1) \circ ((p_1 \leftarrow p_2) \circ p_1) \leq ((p_2 \leftarrow p_1) \circ p_2) \leftarrow ((p_2 \to p_1) \circ p_1).$$

2. (Simple Sahlqvist inequality [5, p.161]): $s \leq t$, where s is a term of type $t_\cup$, and t is positive (every variable is signed with $+$ in the $+$-signed construction tree of t). For example, the following inequality in residuated algebras is of this type:

$$p_1 \circ (p_2 \circ p_3) \leq (p_1 \circ p_2) \circ (p_1 \circ p_3).$$

3. (Simple Sahlqvist-like inequality 1): $s \leq t$, where s is a negative term (every variable is signed with $-$ in the $-$-signed construction tree of s), and t is a term of type $t_\cap$.[1]

4. (Simple Sahlqvist-like inequality 2): $s \leq t$, where s is a term of type $t_\cup$, and t is uniform (there is no variable in t singed with both $+$ and $-$ in the $+$-signed construction tree of t). For example, the following inequality in residuated algebras is of this type:

$$p_1 \circ (p_2 \circ p_3) \leq (p_2 \leftarrow (p_1 \circ p_3)) \circ (p_2 \circ (p_1 \to p_2)).$$

5. (Simple Sahlqvist-like inequality 3): $s \leq t$, where t is a term of type $t_\cap$, and s

[1] We do not have any example in residuated algebras, because we cannot have any negative term based on the language.

is uniform. For example, the following inequality in residuated algebras is of this type:

$$(p_2 \circ p_1) \leftarrow (p_1 \to (p_3 \circ p_3)) \leq (p_1 \circ p_2) \to (p_1 \leftarrow (p_2 \circ p_3)).$$

6. The following inequality in residuated algebras is not any of the above types, namely non-uniform nor non-simple Sahlqvist(-like):

$$(p_1 \to (p_2 \circ p_3)) \circ (p_2 \circ (p_3 \leftarrow p_1)) \leq (p_3 \circ (p_1 \to p_2)) \to (p_3 \leftarrow (p_2 \circ p_1)).$$

Nevertheless, it is canonical (see Fig. 5.5 and Fig. 5.6).

Chapter 7

Canonical extensions from other perspectives

So far, we have studied canonical extensions of lattice expansions, poset expansions and the canonicity results, based on the construction given in [33], mainly to discuss Ghilardi and Meloni's canonicity methodology. In this chapter, we will explain Ghilardi and Meloni's canonicity methodology, in particular the parallel computation, in the light of the topological characterisation of canonical extensions. In the end, we will propose a new perspective of canonical extensions as a machinery to describe continuous properties from observable data, and a *Unschärferelation* of order theory.

7.1 Canonical extensions as compact dense completions

The study of canonical extensions based on Stone representation [78] in [50, 51, 49] is reformulated by introducing topological terminology, *closed elements*, *open elements*, *denseness* or *compactness*, in [29]. The topological characterisation of canonical extensions is taken, for example, from [27, 28, 18, 88]. Thanks to the universality of the topological characterisation of canonical extensions, nowadays we can introduce *the* canonical extension, unique up to isomorphism, over posets [18]. In this section, we characterise our canonical extension of posets by the topological terminology, which is used in the later sections. Note that, for poset expansions including lattice expansions, we do not know whether there is a unique characterisation of the canonical extensions. This is because we are still discussing how the canonical extension of ϵ-operations should be in general: see Remarks 3.2.8 and 5.2.9

Let $\mathbb{P}$ be a poset. A completion $\overline{\mathbb{P}}$ of $\mathbb{P}$ is a complete lattice in which $\mathbb{P}$ is embeddable. An element $k \in \overline{\mathbb{P}}$ is *closed*, if there exists a filter F of $\mathbb{P}$ such that k is the greatest lower bound of F, i.e. $k = \bigwedge F$, which always exists in $\overline{\mathbb{P}}$ as it is a complete lattice. An element $o \in \overline{\mathbb{P}}$ is *open*, if there exists an ideal I of $\mathbb{P}$ such that o is the least upper bound of I, i.e. $o = \bigvee I$, which always exists in $\overline{\mathbb{P}}$ as it is a complete lattice. $\mathcal{K}(\mathbb{P})$, or simply $\mathcal{K}$, is the set of all closed elements of $\overline{\mathbb{P}}$. $\mathcal{O}(\mathbb{P})$, or simple $\mathcal{O}$, is the set of all open elements of $\overline{\mathbb{P}}$.

Remark 7.1.1. $\mathcal{K}(\mathbb{P}) \cong \mathcal{F}(\mathbb{P})$ and $\mathcal{O}(\mathbb{P}) \cong \mathcal{I}(\mathbb{P})$. Hence, we can consider every filter as a closed element in $\overline{\mathbb{P}}$ and every ideal as an open element in $\overline{\mathbb{P}}$.

We state that the completion $\overline{\mathbb{P}}$ is *dense*, if every element α of $\overline{\mathbb{P}}$ is both the least upper bound of the set of all closed elements below α, namely,

$$\alpha = \bigvee\{k \in \mathcal{K}(\mathbb{P}) \mid k \leq \alpha\},$$

and the greatest lower bound of the set of all open elements above α, namely,

$$\alpha = \bigwedge\{o \in \mathcal{O}(\mathbb{P}) \mid \alpha \leq o\}.$$

Remark 7.1.2. $\alpha_{\uparrow}$ and $\alpha^{\downarrow}$ coincide with $\bigvee\{k \in \mathcal{K} \mid k \leq \alpha\}$ and $\bigwedge\{o \in \mathcal{O} \mid \alpha \leq o\}$, respectively. That is, λ constructs the least upper bounds for subsets of closed elements (filters), and υ constructs the greatest lower bounds for subsets of open elements (ideals). In other words, λ approximates elements in $\overline{\mathbb{P}}$ from the lower-sides, and υ approximates elements in $\overline{\mathbb{P}}$ from the upper-sides. The definition of canonical extensions in Definition 2.2.10 allows us to keep the approximation directions by adding the subscript $\_{\uparrow}$ and the superscript $\_^{\downarrow}$.

We say that the completion $\overline{\mathbb{P}}$ is *compact*, if for an arbitrary pair of a closed element $k \in \mathcal{K}$ and an open element $o \in \mathcal{O}$, there exists an element a in $\mathbb{P}$ between k and o whenever $k \leq o$. That is, for every $k \in \mathcal{K}$ and each $o \in \mathcal{O}$, if $k \leq o$ then there exists $a \in \mathbb{P}$ such that $k \leq a \leq o$.

Remark 7.1.3. Since every closed element is a filter F and every open element is an ideal I, the *compactness* is explained as $F \sqsubseteq I$ $(F \cap I \neq \emptyset)$.

Theorem 7.1.4 (Uniqueness, [18])**.** *For every poset $\mathbb{P}$, if there exist compact dense completions of $\mathbb{P}$, they are isomorphic. In other words, for every poset $\mathbb{P}$, a compact dense completion of $\mathbb{P}$ is unique up to isomorphism.*

Theorem 7.1.5 (Existence, [18])**.** *Every poset $\mathbb{P}$ has a compact dense completion.*

Remark 7.1.6. The construction of a compact dense completion in the proof [18, Theorem2.6] is exactly the same as construction of the canonical extension: see Definition 2.2.10.

Corollary 7.1.7. *The canonical extension of a poset is a compact dense completion of the poset.*

7.2 Parallel computation and canonicity

In this section, Ghilardi and Meloni's canonicity methodology for lattice expansions is outlined in the light of the topological characterisation.

The main technical points of Ghilardi and Meloni's approach can be summarised as follows.

1. The order relation on the canonical extension is exchanged for the relationships between closed elements (filters) and open elements (ideals): see Item 3 in Proposition 2.3.1

2. For each term function, the parallel computation on the intermediate level, between closed elements (filters) and open elements (ideals), is introduced: see Section 3.2 As a result, the parallel computation allows us to discuss the relationship between term functions on the canonical extension and on the intermediate level, see Proposition 3.3.2 and Definition 3.3.3 Furthermore, a tight connection between the parallel computation on the intermediate level and term functions on the algebra is provided by introducing the parallel computation on the algebra: see Lemma 3.2.6.

The order on the canonical extension and the intermediate level Let $\mathbb{L}$ be a lattice. Since the canonical extension $\overline{\mathbb{L}}$ is a compact dense completion, every element α can be seen as a join of closed elements $\alpha_{\uparrow}$ and as a meet of open elements $\alpha^{\downarrow}$, Remark 7.1.2.

Let α and β be arbitrary elements in the canonical extension $\overline{\mathbb{L}}$, K_a a closed basis of α, i.e. $\alpha_{\uparrow} = \lambda(K_a)$, and O_b an open basis of β, i.e. $\alpha^{\downarrow} = \upsilon(O_b)$. By Proposition 2.3.1 (Item 3), we have

$$\alpha \leq \beta \iff \forall k \in K_a, \forall o \in O_b.\ k \leq o.$$

This fact shows that, whenever we have *any* closed basis of α and *any* open basis of β, the order $\alpha \leq \beta$ on the canonical extension can be verified by those bases *without taking limits.*

To prove canonicity for term functions s and t, and arbitrary tuples of elements $(\alpha_1, \ldots, \alpha_N)$ in the canonical extension, we want to justify the order relation $s(\alpha_1, \ldots, \alpha_N) \leq t(\alpha_1, \ldots, \alpha_N)$. To syntactically characterise canonical inequalities, we need to give explicit characterisations of closed bases of $s(\alpha_1, \ldots, \alpha_N)$ and open bases of $t(\alpha_1, \ldots, \alpha_N)$. To this end, as we will discuss below, Ghilardi and Meloni's parallel computation provides us with an *inductive* characterisations of (rough) bases.

Ghilardi and Meloni's parallel computation on the intermediate level To simplify our discussion, we fix our language to substructural logic, namely $\circ$, $\rightarrow$ and $\leftarrow$ (see Section 4.1), in the rest of this section. Since fusion $\circ$ is additive, we take $\circ^{\sigma}$ (σ-extension, which is the same as $\circ_{\uparrow}$) as the fusion on the canonical extension. And, since residuals $\rightarrow$ and $\leftarrow$ are multiplicative, we take $\rightarrow^{\pi}$ and $\leftarrow^{\pi}$ (π-extensions, which are the same as $\rightarrow^{\downarrow}$ and $\leftarrow^{\downarrow}$) as the residuals on the canonical extension: see Remark 3.2.7. In other words, for all elements α and β in the canonical extension, we define

1. $\alpha \circ^\sigma \beta := \bigvee\{k_a \circ k_b \mid k_a \leq \alpha, k_b \leq \beta, k_a, k_b \in \mathcal{K}\}$,

2. $\alpha \rightarrow^\pi \beta := \bigwedge\{k \rightarrow o \mid k \leq \alpha, \beta \leq o, k \in \mathcal{K}, o \in \mathcal{O}\}$,

3. $\beta \leftarrow^\pi \alpha := \bigwedge\{o \leftarrow k \mid k \leq \alpha, \beta \leq o, k \in \mathcal{K}, o \in \mathcal{O}\}$.

Therefore, to describe the canonical extension of $\circ$, $\rightarrow$ and $\leftarrow$, we need to define only the following operations on the intermediate level, $\circ : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$, $\rightarrow: \mathcal{K} \times \mathcal{O} \rightarrow \mathcal{O}$ and $\leftarrow: \mathcal{O} \times \mathcal{K} \rightarrow \mathcal{O}$. Additionally, by definition, we notice that the sets

1. $\{k_a \circ k_b \mid k_a \leq \alpha, k_b \leq \beta, k_a, k_b \in \mathcal{K}\}$,

2. $\{k \rightarrow o \mid k \leq \alpha, \beta \leq o, k \in \mathcal{K}, o \in \mathcal{O}\}$,

3. $\{o \leftarrow k \mid k \leq \alpha, \beta \leq o, k \in \mathcal{K}, o \in \mathcal{O}\}$,

are a closed (filter) basis of $\alpha \circ^\sigma \beta$, an open (ideal) basis of $\alpha \rightarrow^\pi \beta$ and an open (ideal) basis of $\beta \leftarrow^\pi \alpha$, respectively. We mention that the canonical extensions of fusion and residuals are the same as in [18] or [27].

However, as distinct from [18] or [27], our target is to calculate each term function on the intermediate level and to characterise bases, to construct a relation with all term functions on the canonical extension and their (rough) bases: see Proposition 3.3.2 To achieve our goal, we require introducing two additional features: *the opposite-type* operations and *the* $\|$*-notation* on the intermediate level. Namely, we also define the operations $\circ : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$, $\rightarrow: \mathcal{O} \times \mathcal{K} \rightarrow \mathcal{K}$ and $\leftarrow: \mathcal{K} \times \mathcal{O} \rightarrow \mathcal{K}$. Note that we never claim that $\{o_a \circ o_b \mid \alpha \leq o_a \in \mathcal{O}, \beta \leq o_b \in \mathcal{O}\}$ is an open (ideal) basis of $\alpha \circ^\sigma \beta$, which contradicts to existing results, see MV-algebras in [31].[1] But, if we do not introduce the opposite-type operations, some term functions on the

[1]This is actually an open (ideal) basis of $\alpha \circ^\pi \beta$ in general. But, there is no chance to use this fact in our arguments, because we do not define $\circ^\pi$ for substructural logic.

intermediate level could not be computed, e.g. $p_1 \to (p_2 \circ p_3)$, because the second argument of $\to$ is of type $\mathcal{O}$ which would be a mismatch with the codomain of $\circ$. With the $\|$-notation, we can assign each variable with distinct elements in different sorts, depending on either it appears positively or negatively. For example, $x\|y$ means that x is for positive occurrences and y is for negative occurrences. Without the $\|$-notation, we could not manage the terms in which a propositional variable appears both positively and negatively, e.g. $p_1 \to (p_1 \circ p_2)$.

Hereafter, instead of closed elements and open elements, we use filters and ideals. This is because the parallel computation is a calculus in *two-sorts*: filters and ideals. For example, the constant 1 is clopen. But, we can clearly know that $\uparrow 1$ is a filter (counted as a closed element) and $\downarrow 1$ is an ideal (counted as an open element).

On the intermediate level, the parallel computation does compute all term functions with positive occurrences as filters and negative occurrences as ideals on one hand as $t(F_1\|I_1, \ldots, F_n\|I_n)$. On the other hand, all term functions are calculated with positive occurrences as ideals and negative occurrences as filters as $t(I_1\|F_1, \ldots, I_n\|F_n)$: see Section 3.2.

Example 7.2.1. A term function $p_1 \to (p_1 \circ p_2)$ is calculated in parallel as follows: for all filters F, G and all ideals I, J,

$$(p_1 \to (p_1 \circ p_2))(F\|I, G\|J) = I \to (F \circ G),$$

$$(p_1 \to (p_1 \circ p_2))(I\|F, J\|G) = F \to (I \circ J).$$

Based on the parallel computation on the intermediate level, for every term function t and all elements $\alpha_1 \ldots, \alpha_N$ in the canonical extension, we obtain the following rough basis property of $t(\alpha_1, \ldots, \alpha_N)$:

$$t(F_1 \| I_1, \ldots, F_n \| I_n) \leq t(\alpha_1, \ldots, \alpha_n), \tag{7.1}$$

$$t(\alpha_1, \ldots, \alpha_n) \leq t(I_1 \| F_1, \ldots, I_n \| F_n), \tag{7.2}$$

for all $F_k \leq \alpha_k$ and all $I_k \geq \alpha_k$ for each $k \in \{1, \ldots, n\}$, see Proposition 3.3.2. Actually, the parallel computation also allow us to characterise not only *rough bases* but also *bases* for all term functions: see Proposition 3.3.21.[2] However, the rough description of bases should be already enough here to show our main idea and the difference from the approach in [18], [27] or [30]. The next paragraph presents a worked out example illustrating the use of rough bases.

An example from substructural logic In Section 4.1, we list an inequality $1 \leq (p_2/(p_2 \backslash p_1)) \backslash (p_1 \circ 0)$ as a canonical inequality (Item 6 in the list). Here, we concretely discuss the canonicity along with the above explanation.

Let $\mathbb{L}$ be an FL-algebra and $\overline{\mathbb{L}}$ the canonical extension of $\mathbb{L}$. Assume that, for all elements $a, b \in \mathbb{L}$, we have

$$1 \leq (b/(b \backslash a)) \backslash (a \circ 0). \tag{7.3}$$

For arbitrary $\alpha, \beta \in \overline{\mathbb{L}}$, we want to show that $1 \leq (\beta/(\beta \backslash \alpha)) \backslash (\alpha \circ 0)$. Since 1 is in $\mathbb{L}$, (hence 1 is clopen), 1 is approximated by $\{1\}$. Here, to think about this singleton set $\{1\}$ as a set of closed elements, we denote it as $\{\uparrow 1\}$, where $\uparrow 1$ is the principal filter generated by 1. On the other hand, $(\beta/(\beta \backslash \alpha)) \backslash (\alpha \circ 0)$ is approximated by the

[2] Note that this characterisation of bases is not as simple as equation (7.1) or equation (7.2), in general.

following set of open elements by definition of $\backslash^{\pi}$ ($\backslash^{\downarrow}$):

$$\{k\backslash o \mid k \leq (\beta/(\beta\backslash\alpha)),\ (\alpha \circ 0) \leq o\}. \tag{7.4}$$

Hereafter, we consider k as a filter F and o as an ideal I. Hence, $F \leq (\beta/(\beta\backslash\alpha))$ and $(\alpha \circ 0) \leq I$. Thanks to *the rough basis property*, we also have the following conditions: for each $G \leq \alpha$ and each $J \geq \beta$ (and arbitrary $J' \geq \alpha$ and $G' \leq \beta$),

$$\beta/(\beta\backslash\alpha) = (p_2/(p_2\backslash p_1))(\alpha, \beta) \leq (p_2/(p_2\backslash p_1))(J'\|G, J\|G') = J/(J\backslash G),$$

by equation (7.2) and, by equation (7.1),

$$G \circ {\uparrow}0 = (p_1 \circ 0)(G\|J', G'\|J) \leq (p_1 \circ 0)(\alpha, \beta) = \alpha \circ 0,$$

where ${\uparrow}0$ is the principal filter generated by 0. Therefore, we have ($\sqsubseteq$ is the partial order which is restricted on the intermediate level)

$$F \sqsubseteq J/(J\backslash G) \iff F \cap (J/(J\backslash G)) \neq \emptyset, \tag{7.5}$$

$$G \circ {\uparrow}0 \sqsubseteq I \iff (G \circ {\uparrow}0) \cap I \neq \emptyset, \tag{7.6}$$

By equation (7.5) and Lemma 3.2.6, there exist $j \in J$ and $g_1 \in G$ such that $j/(j\backslash g_1) \in F$. By equation (7.6) and Lemma 3.2.6, there exists g_2 such that $g_2 \circ 0 \in I$. Since G is a filter, the meet g of g_1 and g_2 is also in G. Furthermore, by monotonicity (Lemma 3.2.5), we also have $j/(j\backslash g) \in F$ and $g \circ 0 \in I$, hence $(j/(j\backslash g))\backslash(g \circ 0) \in F\backslash I$, by Lemma 4.23. Finally, by our assumption (7.3), we have $\uparrow 1 \sqsubseteq F \backslash I$, which concludes $1 \leq (\beta/(\beta\backslash\alpha))\backslash(\alpha \circ 0)$, hence the inequality is canonical.

Therefore, based on Ghilardi and Meloni's parallel computation, we can not just account for the canonicity results of smooth lattice expansions in [27] or [30], but also obtain canonical inequalities of lattice expansions with *non-smooth operations*.

7.3 Unschärferelation in the canonical extension

In this section we propose an *Unschärferelation (uncertainty principle)* on the canonical extension of a poset expansion.[3]

We look back to the canonicity of poset expansions first. It states that for an inequality $s \leq t$, we have

$$\mathbb{P} \models s \leq t \iff \overline{\mathbb{P}} \models s \leq t. \tag{7.7}$$

That is, the inequality $s \leq t$ is valid on $\mathbb{P}$ if and only if it is valid on $\overline{\mathbb{P}}$. In Equation (7.7), ($\Leftarrow$)-direction is rather trivial, because $\mathbb{P}$ is a subalgebra of $\overline{\mathbb{P}}$: we can consider $\mathbb{P}$ as a part of $\overline{\mathbb{P}}$. Therefore, the real problem of the canonicity is to prove ($\Rightarrow$)-direction in Equation (7.7). We may explain ($\Rightarrow$)-direction as follows: if we are interested in whether a property expressed by an inequality $s \leq t$ holds on $\overline{\mathbb{P}}$, it is enough to investigate only on $\mathbb{P}$. That is, "a part of information, or a piece of information, for an inequality $s \leq t$ on $\mathbb{P}$ can describe all the information, or the perfect information, for the inequality $s \leq t$ on $\overline{\mathbb{P}}$."

However, in the real world, we can rarely observe all the perfect information or

[3]The reason we choose the German word is to avoid the term "uncertainty" in a mathematical setting.

facts. For example, many observable data in natural science are discrete, whereas we often characterise those properties as continuous maps on continuous spaces. In other words, we often *estimate* the perfect information from a piece of *observable* information or data. From this point of view, we can think about the framework of the canonical extension as follows.

Hypothesis 1. *A poset $\mathbb{P}$ is a collection of observable ordered data, and the canonical extension $\overline{\mathbb{P}}$ is the complete lattice of the perfect information.*

Usually, we accumulate the observable data by repeating experiments or observations again and again. For each set S of observable ordered data, we can take two types of representatives of those data: *the lower representatives of S* and *the upper representatives of S*. That is, the lower representatives of S are the data which provide lower boundaries of S and the upper representatives of S are the data which provide upper boundaries of S.

Hypothesis 2. *For each set S of observable ordered data, the lower representatives of S are filters F (closed elements) including S, i.e. $S \subseteq F$, and the upper representatives of S are ideals I (open elements) including S, i.e. $S \subseteq I$.*

Therefore, the set $\mathcal{F}(\mathbb{P})$, or $\mathcal{K}(\mathbb{P})$, is the collection of all possible lower representative data, and the set $\mathcal{I}(\mathbb{P})$, or $\mathcal{O}(\mathbb{P})$, is the collection of all possible upper representative data.

Remark 7.3.1. If we have few observable data, the representatives are quite rough and they could be far from keen estimations. However, by repeating observations, we must collect many representatives which describe observable data sensibly.

Hypothesis 3. *All the perfect information is obtained by a colimit of a set of lower representatives and by a limit of a set of upper representatives. In other words, every information in the complete lattice of perfect information is approximated by a set of lower representatives of from below and approximated by a set of upper representatives from above.*

Therefore, in our setting, all the observable (ordered) data, if we collect them, form a poset $\mathbb{P}$, and the complete structure of all the perfect information lies as the canonical extension $\overline{\mathbb{P}}$ of $\mathbb{P}$. Now let us consider to denote a property t like a movement of particles or a information flow based on our setting.

Hypothesis 4. *Every property t can be described by a combination of ϵ-operations on $\mathbb{P}$. That is, every property t which we consider here is expressed by a term function t of a lattice expansion based on the underlying poset $\mathbb{P}$.*

From this point of view, we can think about the canonicity problem as a question of type "can we obtain the perfect description of a property t only by the observable data?" The answer is, of course, "not always." However, by using Ghilardi and Meloni's parallel computation, we can give a more detailed answer.

Theorem 7.3.2 (Unschärferelation)**.** *A property t may not be perfectly described by observable data, even if we observe infinitely many times. However, if we repeat the observation infinitely many times, the observable data of the property t would always distribute in a certain range, in which the perfect information of the property t is also lying.*

Proof. Let t be a property. For all $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{P}}$, we want to know the perfect description of $t(\alpha_1, \ldots, \alpha_N)$. Thanks to the parallel computation, for each coordinate

$j \in \{1, \ldots, N\}$, each closed element $k_j \in \mathcal{K}$ which is below α_j, i.e. $k_j \leq \alpha_j$, and each open elements $o_j \in \mathcal{O}$ which is above α_j, i.e. $\alpha_j \leq o_j$, we have

$$t(k_1 \| o_1, \ldots, k_N \| o_N) \leq t(\alpha_1, \ldots, \alpha_N) \leq t(o_1 \| k_1, \ldots, o_N \| k_N). \tag{7.8}$$

By the way, since the canonical extension $\overline{\mathbb{P}}$ is compact, there exists an observable data $x_j \in \mathbb{P}$ satisfying

$$k_j \leq x_j \leq o_j,$$

for each coordinate $j \in \{1, \ldots, N\}$. By Lemma 3.2.6, we obtain

$$t(k_1 \| o_1, \ldots, k_N \| o_N) \leq t(x_1, \ldots, x_N) \leq t(o_1 \| k_1, \ldots, o_N \| k_N). \tag{7.9}$$

We state that the gap between $t(x_1, \ldots, x_N)$ and $t(\alpha_1, \ldots, \alpha_N)$ is closer than the one between $t(k_1 \| o_1, \ldots, k_N \| o_N)$ and $t(o_1 \| k_1, \ldots, o_N \| k_N)$, by Equations (7.8) and (7.9). Hence, if we take more and more precise data by repeating the observation infinitely many times, the observable data of the property t distribute in the following range.

$$\left[\bigvee \{ t(k_1 \| o_1, \ldots, k_N \| o_N) \mid k_j \leq \alpha_j \leq o_j \}, \bigwedge \{ t(o_1 \| k_1, \ldots, o_N \| k_N) \mid k_j \leq \alpha_j \leq o_j \} \right].$$

By Equation (7.8), the perfect description of the property $t(\alpha_1, \ldots, \alpha_N)$ is also in the range. $\square$

Chapter 8

Bi-approximation semantics for substructural logic

In this chapter, we will discuss a relational-type semantics, or a space-based semantics, for substructural logic. Unlike what happens in the setting of modal logic, substructural logic is not necessarily distributive, namely $\phi \wedge (\psi \vee \chi)$ may not imply $(\phi \wedge \psi) \vee (\phi \wedge \chi)$. If we interpret conjunctions and disjunctions as follows:

1. $w \Vdash \phi \wedge \psi \iff w \Vdash \phi$ and $w \Vdash \psi$,

2. $w \Vdash \phi \vee \psi \iff w \Vdash \phi$ or $w \Vdash \psi$,

$w \Vdash \phi \wedge (\psi \vee \chi)$ always implies $w \Vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)$. To avoid this problem, in this dissertation, we will introduce *bi-approximation semantics*, a two sorted relational-type semantics, via the canonical extension of lattice expansions, to reasons about not formulae but logical consequences, or sequents. That is, we reason about premises and conclusions separately on each sort, and evaluate logical consequences as a relation between these two sorts. Moreover, by introducing the bi-directional approximation and bases, we track down a connection to Kripke-type semantics for distributive substructural logics in Section 4.2 through a relationship between basis and the existential quantifier. Based on the framework, we prove a soundness theorem and a completeness theorem via a representation theorem plus invariance of validity along a back-and-force correspondences.

8.1 Discussions on relational semantics for substructural logic

What is a natural relational semantics for substructural logic or resource sensitive logics? Unlike Kripke semantics for modal logic, we can find several types of relational semantics for substructural logic based on their philosophy or on their mathematical frameworks. For example, the study of relational semantics for distributive substructural logics has led to an operational semantics for relevant implication [85]. In [70], a ternary relational semantics, a.k.a. Routley-Meyer semantics, has been introduced by a different interpretation of relevant implication. For distributive substructural logics, we can also find other relational semantics, see e.g. [68]. Reasoning about relational-type semantics for non-distributive substructural logics, one encounters the interpretation problem of disjunction, namely how to avoid the distributivity of *conjunction* and *disjunction*. For orthologic, one can solve the problem on Goldblatt frames [35], by introducing a non-standard interpretation of disjunction. With Dedekind-MacNeille frames and the closure operator interpretation in [43] and [44], one can also solve the problem by using a closure operator. Generalized Kripke frames [26], which are introduced by characterising the intermediate level of canonical extensions of lattice expansions (see e.g. [18] or [27]), provide another semantics in which one can avoid the disjunction problem by a Galois connection.

The aim of the current chapter is to propose another possible relational-type semantics for substructural logic. To achieve our goal, we introduce a two-sorted relational-type semantics, called *bi-approximation semantics*, and describe Ghilardi and Meloni's parallel computation on the intermediate level [33], see also [84]. Our framework is closely related to the works [43], [44] and [26]. On the other hand, bi-approximation semantics has novel aspects: *bi-directional approximation*, *bases*, and *doppelgänger valuations* which allow us to evaluate *sequents* (Section 8.3). Based on our setting, we will come across one possible interpretation of the two sorts, *premises* and *conclusions*, and discover a relationship to Kripke-type semantics for distributive substructural logics through *bases* and *the existential quantifier* (Section 8.4). Furthermore, the connection between *bases* and *the existential quantifier* provides an *effective evaluation of sequents* in bi-approximation semantics, which is useful to prove the soundness theorem (Theorem 8.6.1). In Section 8.5, we prove the representation theorem of FL-algebras via p-frames, which is used to show the completeness theorem in Section 8.6.

8.2 Substructural logic

In this section, we denote propositional variables by $p, q, r, p_1, \ldots$, the set of all propositional variables by Φ, and $\mathbf{t}$ and $\mathbf{f}$ are logical constants representing *true* and *false*, respectively. As logical connectives, we use *disjunction* $\vee$, *conjunction* $\wedge$, *fusion (multiplication)* $\circ$, *implications (residuals)* $\rightarrow$ and $\leftarrow$. Formulae of substructural logic are denoted by ϕ, ψ, $\phi_1, \ldots$ and $\psi_1, \ldots$, and the set of all formulae is denoted by Λ. The following BNF generates formulae of substructural logic.

$$\phi ::= p \mid \mathbf{t} \mid \mathbf{f} \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \circ \phi \mid \phi \to \phi \mid \phi \leftarrow \phi$$

$\Gamma, \Delta, \Sigma, \Pi$ are (possibly empty) finite lists of formulae, and φ is a list of at most one formula. Then, we call $\Gamma \Mapsto \varphi$ a *sequent*.

Gentzen's sequent system for substructural logic Let ϕ, ψ be arbitrary formulae, $\Gamma, \Delta, \Sigma, \Pi$ arbitrary (possibly empty) finite lists of formulae, φ a list of at most one formula: see e.g. [63]. The sequent system FL is in Fig. 8.1. In the sequent system FL, a formula ϕ is *provable in FL* if the sequent $\Mapsto \phi$ is derivable in FL. The substructural logic **FL** is the set of all provable formulae in FL.

Proposition 8.2.1 ([63])**.** *For all formulae ϕ and ψ, we have*

1. *ϕ is provable if and only if $\mathbf{t} \Mapsto \phi$ is derivable,*

2. *$\phi \Mapsto \psi$ is derivable if and only if $\phi \to \psi$ is provable in FL if and only if $\psi \leftarrow \phi$ is provable in FL,*

3. *$\phi_1, \ldots, \phi_n \Mapsto \varphi$ is derivable in FL if and only if $\phi_1 \circ \cdots \circ \phi_n \Mapsto \varphi$ is derivable in FL.*

The algebraic counterparts of substructural logic **FL** are known as FL-algebras [25].

Definition 8.2.2 (FL-algebra)**.** An 8-tuple $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ is a *FL-algebra*, if $\langle A, \vee, \wedge \rangle$ is a lattice, $\langle A, *, 1 \rangle$ is a monoid, 0 is a constant in A, and for all $a, b, c \in A$,

$$a * b \leq c \iff b \leq a \backslash c \iff a \leq c/b.$$

Figure 8.1: The sequent system FL

Initial sequents :

$$\phi \Rrightarrow \phi \qquad\qquad \Rrightarrow \mathbf{t} \qquad\qquad \mathbf{f} \Rrightarrow$$

Cut rule :

$$\frac{\Gamma \Rrightarrow \phi \qquad \Sigma, \phi, \Pi \Rrightarrow \varphi}{\Sigma, \Gamma, \Pi \Rrightarrow \varphi}\,(\mathrm{cut})$$

Rules for constants :

$$\frac{\Gamma, \Delta \Rrightarrow \varphi}{\Gamma, \mathbf{t}, \Delta \Rrightarrow \varphi}\,(\mathbf{t}\mathrm{w}) \qquad\qquad \frac{\Gamma \Rrightarrow}{\Gamma \Rrightarrow \mathbf{f}}\,(\mathbf{f}\mathrm{w})$$

Rules for logical connectives :

$$\frac{\Gamma, \phi, \Delta \Rrightarrow \varphi \qquad \Gamma, \psi, \Delta \Rrightarrow \varphi}{\Gamma, \phi \vee \psi, \Delta \Rrightarrow \varphi}\,(\vee \Rrightarrow)$$

$$\frac{\Gamma \Rrightarrow \phi}{\Gamma \Rrightarrow \phi \vee \psi}\,(\Rrightarrow \vee_1) \qquad\qquad \frac{\Gamma \Rrightarrow \psi}{\Gamma \Rrightarrow \phi \vee \psi}\,(\Rrightarrow \vee_2)$$

$$\frac{\Gamma, \phi, \Delta \Rrightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rrightarrow \varphi}\,(\wedge_1 \Rrightarrow) \qquad\qquad \frac{\Gamma, \psi, \Delta \Rrightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rrightarrow \varphi}\,(\wedge_2 \Rrightarrow)$$

$$\frac{\Gamma \Rrightarrow \phi \qquad \Gamma \Rrightarrow \psi}{\Gamma \Rrightarrow \phi \wedge \psi}\,(\Rrightarrow \wedge)$$

$$\frac{\Gamma, \phi, \psi, \Delta \Rrightarrow \varphi}{\Gamma, \phi \circ \psi, \Delta \Rrightarrow \varphi}\,(\circ \Rrightarrow) \qquad\qquad \frac{\Gamma \Rrightarrow \phi \qquad \Sigma \Rrightarrow \psi}{\Gamma, \Sigma \Rrightarrow \phi \circ \psi}\,(\Rrightarrow \circ)$$

$$\frac{\Gamma \Rrightarrow \phi \qquad \Sigma, \psi, \Pi \Rrightarrow \varphi}{\Sigma, \Gamma, \phi \to \psi, \Pi \Rrightarrow \varphi}\,(\to\Rrightarrow) \qquad\qquad \frac{\phi, \Gamma \Rrightarrow \psi}{\Gamma \Rrightarrow \phi \to \psi}\,(\Rrightarrow\to)$$

$$\frac{\Gamma \Rrightarrow \phi \qquad \Sigma, \psi, \Pi \Rrightarrow \varphi}{\Sigma, \psi \leftarrow \phi, \Gamma, \Pi \Rrightarrow \varphi}\,(\leftarrow\Rrightarrow) \qquad\qquad \frac{\Gamma, \phi \Rrightarrow \psi}{\Gamma \Rrightarrow \psi \leftarrow \phi}\,(\Rrightarrow\leftarrow)$$

By Proposition 8.2.1, we sometimes state that **FL** is the set of all sequents derivable in FL. On FL-algebras, each sequent $\phi_1, \ldots, \phi_n \mapsto \varphi$ is interpreted as an inequality $\phi_1 * \cdots * \phi_n \leq \varphi$.

8.3 Bi-approximation semantics

In this section, we firstly introduce a *polarity*, see [4] or [89], which is the foundation of *bi-approximation semantics*.

Polarity and bi-directional approximation

Definition 8.3.1 (Polarity)**.** A triple $\langle X, Y, B\rangle$ is a *polarity*, if X and Y are non-empty sets, and B a binary relation on $X \times Y$, i.e. $B \subseteq X \times Y$.

Given a polarity $\langle X, Y, B\rangle$, we induce a preorder $\leq_B$ on $X \cup Y$ as follows, see [26]: for all $x_1, x_2 \in X$ and all $y_1, y_2 \in Y$, we let

1. $x_1 \leq_B x_2 \iff$ for each $y \in Y$, $x_2 B y$ implies $x_1 B y$,
2. $y_1 \leq_B y_2 \iff$ for each $x \in X$, $x B y_1$ implies $x B y_2$,
3. $x_1 \leq_B y_1 \iff x_1 B y_1$,
4. $y_1 \leq_B x_1 \iff$ for each $x' \in X$ and each $y' \in Y$, $x' B y'$ if $x' B y_1$ and $x_1 B y'$.

Hereinafter, we sometimes omit the subscript $_B$ from the induced preorder $\leq_B$, and refer to the triple $\langle X, Y, \leq\rangle$ as the polarity. That is, a polarity $\langle X, Y, \leq\rangle$ is a preordered set $\langle X \cup Y, \leq\rangle$.

Next, we introduce two approximation functions for polarities. Let $\langle X, Y, \leq\rangle$ be a polarity, $\wp(X)$ the poset of all subsets of X ordered by inclusion $\subseteq$, and $\wp(Y)^\partial$ the poset of all subsets of Y ordered by reverse-inclusion $\supseteq$. We define two functions $\lambda : \wp(X) \to \wp(Y)^\partial$ and $\upsilon : \wp(Y)^\partial \to \wp(X)$ as follows: for each $\mathfrak{X} \in \wp(X)$ and each $\mathfrak{Y} \in \wp(Y)^\partial$,

1. $\lambda(\mathfrak{X}) := \{y \in Y \mid \forall x \in \mathfrak{X}.\ x \leq y\}$,

2. $\upsilon(\mathfrak{Y}) := \{x \in X \mid \forall y \in \mathfrak{Y}.\ x \leq y\}$.

The functions λ and υ form a *Galois connection*, i.e. $\lambda \dashv \upsilon$. Hence, the images $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$ are isomorphic. Hereafter, we denote the image $\lambda[\wp(X)]$ by $\mathbb{U}$ and the image $\upsilon[\wp(Y)^\partial]$ by $\mathbb{D}$. We mention that the images are *the Dedekind-MacNeille completion* of the quotient poset of $\langle X, Y, \leq \rangle$ with respect to the equivalence relation associated with $\leq$, see [4] or [13]. We call each element in $\mathbb{D}$ a *Galois stable X-set* and refer to each Galois stable X-set by adding the superscript $\_^{\downarrow}$, e.g. $\alpha^{\downarrow}$. We call each element in $\mathbb{U}$ a *Galois stable Y-set* and refer to each Galois stable Y-set by adding the subscript $\_{\uparrow}$, e.g. $\alpha_{\uparrow}$. Since every Galois stable X-set is an image of some (not necessarily unique) subset of Y, and every Galois stable Y-set is an image of some (not necessarily unique) subset of X, we introduce the following terminology.

Definition 8.3.2 (Approximation and basis)**.** Let $\mathfrak{X} \in \wp(X)$, $\mathfrak{Y} \in \wp(Y)^\partial$, $\alpha^{\downarrow} \in \mathbb{D}$ and $\beta_{\uparrow} \in \mathbb{U}$. An element $\alpha^{\downarrow}$ is *approximated from above by* $\mathfrak{Y}$ and $\mathfrak{Y}$ is a *(Y-)basis of* α, if $\alpha^{\downarrow} = \upsilon(\mathfrak{Y})$. An element $\beta_{\uparrow}$ is *approximated from below by* $\mathfrak{X}$ and $\mathfrak{X}$ is a *(X-)basis of* β, if $\beta_{\uparrow} = \lambda(\mathfrak{X})$.

Later, we will construct two isomorphic FL-algebras on $\mathbb{D}$ and $\mathbb{U}$: see Section 8.5. Namely, we will take the abstract algebra whose underlying poset is isomorphic to both $\mathbb{D}$ and $\mathbb{U}$. Then, we can see every point α as $\alpha^{\downarrow}$ and as $\alpha_{\uparrow}$. In other words, every point in an abstract algebra is approximated from both *above* and *below*. The main concept of *bi-approximation semantics* is to keep the two directions of approximation: see e.g. Proposition 8.4.7.

Bi-approximation model Based on a polarity, we introduce *bi-approximation semantics* for substructural logic.

Definition 8.3.3 (P-frame for substructural logic)**.** A *p-frame for substructural logic*, *p-frame* for short, is a 8-tuple $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$, where the triple $\langle X, Y, \leq \rangle$ is a polarity, $R \subseteq X \times X \times Y$ a ternary relation, O_X a non-empty Galois stable X-set, N_X a Galois stable X-set, O_Y and N_Y are Galois stable Y-sets, and $\mathbb{F}$ satisfies the following.

R-order: For all $x, x' \in X$, $x' \leq x$ if and only if

$$\exists o \in O_X.[\forall y \in Y.[R(x, o, y) \Rightarrow x' \leq y] \text{ or } \forall y \in Y.[R(o, x, y) \Rightarrow x' \leq y]],$$

R-identity: For each $x \in X$, $[\exists o_2 \in O_X, \forall y \in Y.[R(x, o_2, y) \Rightarrow x \leq y]$

$$\text{and } \exists o_1 \in O_X, \forall y \in Y.[R(o_1, x, y) \Rightarrow x \leq y]],$$

R-transitivity: For all $x_1, x_1', x_2, x_2' \in X$ and $y, y' \in Y$,

$$x_1' \leq x_1, x_2' \leq x_2, y \leq y' \text{ and } R(x_1, x_2, y) \Rightarrow R(x_1', x_2', y'),$$

R-associativity: For all $x_1, x_2, x_3, x \in X$,

$$\exists x' \in X.[\forall y \in Y.(R(x_1, x', y) \Rightarrow x \leq y) \text{ and } \forall y' \in Y.(R(x_2, x_3, y') \Rightarrow x' \leq y')]$$

if and only if

$$\exists x' \in X.[\forall y \in Y.(R(x', x_3, y) \Rightarrow x \leq y) \text{ and } \forall y' \in Y.(R(x_1, x_2, y') \Rightarrow x' \leq y')],$$

O-isom: $O_X = \upsilon(O_Y)$ and $O_Y = \lambda(O_X)$,

N-isom: $N_X = \upsilon(N_Y)$ and $N_Y = \lambda(N_X)$,

$\circ$-tightness: For all $x_1, x_2 \in X$, there exists $x \in X$ such that

$$\forall y \in Y.[R(x_1, x_2, y) \text{ if and only if } x \leq y],$$

$\to$-tightness: For each $x_1 \in X$ and each $y \in Y$, there exists $y_2 \in Y$ such that

$$\forall x_2 \in X.[R(x_1, x_2, y) \text{ if and only if } x_2 \leq y_2],$$

$\leftarrow$-tightness: For each $x_2 \in X$ and each $y \in Y$, there exists $y_1 \in Y$ such that

$$\forall x_1 \in X.[R(x_1, x_2, y) \text{ if and only if } x_1 \leq y_1].$$

A p-frame $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$ is intuitively explained as follows: the Galois stable sets O_X, O_Y, N_X and N_Y define the worlds where we assume $\mathbf{t}$, conclude $\mathbf{t}$, assume $\mathbf{f}$ and conclude $\mathbf{f}$. The conditions O-isom and N-isom guarantee that every $x \in X$ where we assume the formula $\mathbf{t}$ ($\mathbf{f}$), if and only if every $y \in Y$ where we conclude the formula $\mathbf{t}$ ($\mathbf{f}$) have the consequence relation $x \leq y$. The ternary relation R is another consequence relation which allows us to reason about logical consequences between two premises and one conclusion. The R-order condition says that the induced relation on X, $x' \leq x$ is also obtained by the ternary consequence relation R. The tightness conditions guarantee that the ternary consequence relation R respects $\leq$.

Remark 8.3.4. In Definition 8.3.3 one may feel that the conditions R-order, R-identity and R-associativity look too complicated. However, we reformulate them in Remark 8.4.3.

Our framework is similar to generalized Kripke frames in [26]. However, we do not assume neither Separation axioms nor Reduced axioms, hence p-frames may not be RS-frames. Our current purpose is to characterise Ghilardi and Meloni's parallel computation on the intermediate level [33], see also [84]. The main points of difference are how to evaluate formulae on bi-approximation semantics, i.e. the valuation on two-sorted frames by introducing *doppelgänger valuation*, and how to interpret the satisfaction relation $\Vdash$ on each sort, X and Y.

Definition 8.3.5 (Doppelgänger valuation)**.** On a p-frame $\mathbb{F}$, a pair $V = \langle V^{\downarrow}, V_{\uparrow} \rangle$ of two functions $V^{\downarrow} : \Phi \to \mathbb{D}$ and $V_{\uparrow} : \Phi \to \mathbb{U}$ is a *doppelgänger valuation*, if $V^{\downarrow}(p)$ and $V_{\uparrow}(p)$ coincide for every propositional variable $p \in \Phi$. That is, $V^{\downarrow}(p) = \upsilon(V_{\uparrow}(p))$ and $V_{\uparrow}(p) = \lambda(V^{\downarrow}(p))$ for each propositional variable $p \in \Phi$.

Definition 8.3.6 (Bi-approximation model)**.** Given a p-frame $\mathbb{F}$ and a doppelgänger valuation V, we call the pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ a *bi-approximation model.*

On a bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V \rangle$, we inductively define a *satisfaction relation* $\Vdash$ as follows: for each $x \in X$, we let

X-1: $\mathbb{M}, x \Vdash p \iff x \in V^{\downarrow}(p)$ for each $p \in \Phi$,

X-2: $\mathbb{M}, x \Vdash \mathbf{t} \iff x \in O_X$,

X-3: $\mathbb{M}, x \Vdash \mathbf{f} \iff x \in N_X$,

X-4: $\mathbb{M}, x \Vdash \phi \vee \psi \iff \forall y \in Y.\ [\mathbb{M}, y \Vdash \phi \vee \psi \Rightarrow x \leq y]$,

X-5: $\mathbb{M}, x \Vdash \phi \wedge \psi \iff \mathbb{M}, x \Vdash \phi$ and $\mathbb{M}, x \Vdash \psi$,

X-6: $\mathbb{M}, x \Vdash \phi \circ \psi \iff \forall y \in Y.\ [\mathbb{M}, y \Vdash \phi \circ \psi \Rightarrow x \leq y]$,

X-7: $\mathbb{M}, x \Vdash \phi \to \psi \iff \forall x' \in X, y \in Y.\ [\mathbb{M}, x' \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi \Rightarrow R(x', x, y)]$,

X-8: $\mathbb{M}, x \Vdash \psi \leftarrow \phi \iff \forall x' \in X, y \in Y.\ [\mathbb{M}, x' \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi \Rightarrow R(x, x', y)]$.

For each $y \in Y$, we let

Y-1: $\mathbb{M}, y \Vdash p \iff y \in V_{\uparrow}(p)$ for each $p \in \Phi$,

Y-2: $\mathbb{M}, y \Vdash \mathbf{t} \iff y \in O_Y$,

Y-3: $\mathbb{M}, y \Vdash \mathbf{f} \iff y \in N_Y$,

Y-4: $\mathbb{M}, y \Vdash \phi \vee \psi \iff \mathbb{M}, y \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi$,

Y-5: $\mathbb{M}, y \Vdash \phi \wedge \psi \iff \forall x \in X.\ [\mathbb{M}, x \Vdash \phi \wedge \psi \Rightarrow x \leq y]$,

Y-6: $\mathbb{M}, y \Vdash \phi \circ \psi \iff \forall x_1, x_2 \in X.\ [\mathbb{M}, x_1 \Vdash \phi$ and $\mathbb{M}, x_2 \Vdash \psi \Rightarrow R(x_1, x_2, y)]$,

Y-7: $\mathbb{M}, y \Vdash \phi \rightarrow \psi \iff \forall x \in X.\ [\mathbb{M}, x \Vdash \phi \rightarrow \psi \Rightarrow x \leq y]$,

Y-8: $\mathbb{M}, y \Vdash \psi \leftarrow \phi \iff, \forall x \in X.\ [\mathbb{M}, x \Vdash \psi \leftarrow \phi \Rightarrow x \leq y]$.

In bi-approximation models, the satisfaction relation $\Vdash$ has two distinct interpretations depending on the domains X and Y. On X, we comprehend $\mathbb{M}, x \Vdash \phi$ as the formula ϕ is *assumed at* x, and on Y, $\mathbb{M}, y \Vdash \phi$ as the formula ϕ is *concluded at* y. Moreover, we also define $\mathbb{F}, x \Vdash \phi$ and $\mathbb{F}, y \Vdash \phi$ as usual: for every doppelgänger valuation V, we have $\mathbb{F}, V, x \Vdash \phi$ and $\mathbb{F}, V, y \Vdash \phi$, respectively.

An interpretation of the two-sorted semantics To reason about resource sensitive logics, we make a clear distinction between premises and conclusions, and evaluate logical consequences as relations between premises and conclusions. On p-frames, we think about X as a set of *premise worlds* where we evaluate only premises, and about Y as a set of *conclusion worlds* where we evaluate just conclusions. One may feel that the satisfaction relation $\mathbb{M}, y \Vdash \phi$, which says "the formula ϕ is *concluded* at the conclusion world y", is the same as "the formula ϕ is *true* at

y." However, these two concepts are not the same. This is because, even if we conclude a formula ϕ at y, we cannot logically judge whether the formula is true or not. For example, if we conclude a formula ϕ meaning "tomorrow is Sunday" at a conclusion world y, we do not have any clue to justify that the formula is a fact. In other words, we may explain $\mathbb{M}, y \Vdash \phi$ as someone is just claiming "ϕ should be concluded" without any reason. Of course, we cannot consider it as logical reasoning. Only when we also have a reasonable premise like "today is Saturday" or "tomorrow is Sunday," we can justify that the logical consequence is *true*. More precisely, only when we have a pair of a premise and a conclusion, we can justify the logical consequence.

Formally the concept of *truth* of logical consequences on bi-approximation models is defined as follows. To reason about truth on bi-approximation models, it is necessary to extend the satisfaction relation $\Vdash \subseteq (X \times \Lambda) \cup (Y \times \Lambda)$ to a relation between $X \times Y$ and pairs of two formulae $\Lambda \times \Lambda$, or *sequents*. For our purpose, we fix the interpretation between sequents and pairs of two formulae. Given a sequent $\phi_1, \ldots, \phi_n \mapsto \varphi$, we translate it to $(\phi_1 \circ \cdots \circ \phi_n, \varphi)$. If $n = 0$, the left-hand side is empty and we write $(\mathbf{t}, \varphi)$. If the right-hand side is empty, we write $(\phi_1 \circ \ldots \circ \phi_n, \mathbf{f})$. But, whenever it is not confusing, we do not make any distinction between sequents and pairs of two formulae. So, both are called just *sequents* and are denoted by $\Gamma \mapsto \varphi$.

Definition 8.3.7 (Truth)**.** Let $\mathbb{M} = \langle \mathbb{F}, V \rangle$ be a bi-approximation model and $\Gamma \mapsto \varphi$ a sequent. We let

1. $\mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi \iff x \leq y$ whenever $\mathbb{M}, x \Vdash \Gamma$ and $\mathbb{M}, y \Vdash \varphi$,

2. $\mathbb{F}, (x, y) \Vdash \Gamma \mapsto \varphi \iff \langle \mathbb{F}, V \rangle, (x, y) \Vdash \Gamma \mapsto \varphi$ for each doppelgänger

valuation V,

3. $\mathbb{M} \Vdash \Gamma \mapsto \varphi \iff \mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi$ for all $x \in X$ and $y \in Y$,

4. $\mathbb{F} \Vdash \Gamma \mapsto \varphi \iff \langle \mathbb{F}, V \rangle, (x, y) \Vdash \Gamma \mapsto \varphi$ for all $x \in X$ and $y \in Y$, and every doppelgänger valuation V.

We interpret $\mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi$ as the sequent $\Gamma \mapsto \varphi$ is *true at the pair* (x, y), and $\mathbb{F} \Vdash \Gamma \mapsto \varphi$ as the sequent $\Gamma \mapsto \varphi$ is *valid on* $\mathbb{F}$.

Remark 8.3.8. Unlike what happens in the setting of the normal Kripke semantics, in bi-approximation models we reason about *sequents* but not *formulae*, in general. But, thanks to Proposition 8.2.1, this distinction is not critical when we consider substructural logic.

Hereinafter, we sometimes write $(x, y) \Vdash \phi \mapsto \psi$ instead of $\mathbb{M}, (x, y) \Vdash \phi \mapsto \psi$.

External reasoning and internal reasoning on p-frames Before we show preliminary results for bi-approximation semantics, we explain how to evaluate premises, conclusions and logical consequences on p-frames.

Recall the satisfaction relation $\Vdash$ in (X-1) - (X-8) and (Y-1) - (Y-8). We notice that there are two types of reasoning: *internal* and *external.* Namely, there is the reasoning on X, e.g. (X-4), or on Y, e.g. (Y-5), and there is the reasoning given by the relation $\leq$ or R between X and Y, e.g. (X-4) or (Y-6). Intuitively speaking, the internal reasoning derives a premise from premises, or a conclusion from conclusions, e.g. we assume $\phi \wedge \psi$ at x if and only if we assume ϕ and ψ at x (X-5). On the other hand, the external reasoning evaluates logical consequences. That is, we describe a premise world by conclusion worlds, and vise versa. For example, a conclusion world y where we conclude $\phi \wedge \psi$ is described by all premise worlds where we assume ϕ and ψ (Y-5). We also say that the conclusion world y is *approximated* by the corresponding premise worlds. Analogously, e.g. (X-4), a premise world is *approximated* by the corresponding conclusion worlds. See also Proposition 8.3.10. This is what we call *bi-approximation* in our framework.

Whereas the external reasoning is fundamental in bi-approximation models, we also have the internal reasoning as well. One may feel that the internal reasoning (Y-4) is far from our intuition. However, we can also explain it as follows. Recall the sequent calculus LK. In LK, we consider a sequent as a pair of a finite list of premises and a finite list of conclusions, $\phi_1, \ldots, \phi_m \Rightarrow \psi_1, \ldots, \psi_n$. The intuitive interpretation of this sequent is "if we assume *all* premises $\phi_1, \ldots, \phi_m$ then we conclude *one of these* conclusions $\psi_1, \ldots, \psi_n$." In other words, premises are *compulsory* and conclusions are *elective*. Therefore, it is natural to consider (Y-4) as "ϕ and ψ are possible conclusions at y if and only if $\phi \vee \psi$ is a possible conclusion at y."

Preliminary results for bi-approximation semantics In this paragraph, we show basic properties on bi-approximation semantics. The following proposition corresponds to Hereditary property in Kripke semantics for intuitionistic logic, e.g. [11]. But, it is two-sorted in our case.

Proposition 8.3.9 (Hereditary)**.** *Let $\mathbb{M}$ be a bi-approximation model and ϕ a formula. For all elements $x, x' \in X$ and $y, y' \in Y$, we have*

1. *if $x' \leq x$ and ϕ is assumed at x, $x \Vdash \phi$, then it is also assumed at x', $x' \Vdash \phi$,*
2. *if $y \leq y'$ and ϕ is concluded at y, $y \Vdash \phi$, then it is also concluded at y', $y' \Vdash \phi$.*

Proof. Parallel induction. Base cases are straightforward, since every Galois stable X-set is a downset and every Galois stable Y-set is an upset.

Inductive steps: $\vee$**:** Assume $y \Vdash \phi \vee \psi$. By definition, $y \Vdash \phi$ and $y \Vdash \psi$. By induction hypothesis, we obtain $y' \Vdash \phi$ and $y' \Vdash \psi$, hence $y' \Vdash \phi \vee \psi$. Suppose $x \Vdash \phi \vee \psi$. For each $y \Vdash \phi \vee \psi$, we have $x \leq y$. Because of $x' \leq x$, we obtain $x' \leq y$, hence $x' \Vdash \phi \vee \psi$.

$\wedge$**:** Assume $x \Vdash \phi \wedge \psi$. By definition, $x \Vdash \phi$ and $x \Vdash \psi$. By induction hypothesis, we obtain $x' \Vdash \phi$ and $x' \Vdash \psi$, hence $x' \Vdash \phi \wedge \psi$. Suppose $y \Vdash \phi \wedge \psi$. For each $x \Vdash \phi \wedge \psi$, we have $x \leq y$. Because of $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \phi \wedge \psi$.

$\circ$**:** Assume $y \Vdash \phi \circ \psi$. If $x_1 \Vdash \phi$ and $x_2 \Vdash \psi$, then we have $R(x_1, x_2, y)$. Since $y \leq y'$, by R-transitivity, we obtain $R(x_1, x_2, y')$, hence $y' \Vdash \phi \circ \psi$. Suppose $x \Vdash \phi \circ \psi$. For each $y \Vdash \phi \circ \psi$, we have $x \leq y$. Because of $x' \leq x$, we obtain $x' \leq y$, hence $x' \Vdash \phi \circ \psi$.

$\rightarrow$**:** Assume $x \Vdash \phi \rightarrow \psi$. For each $x_1 \Vdash \phi$ and each $y \Vdash \psi$, we have $R(x_1, x, y)$. By R-transitivity, we have $R(x_1, x', y)$, hence $x' \Vdash \phi \rightarrow \psi$. Suppose $y \Vdash \phi \rightarrow \psi$. For each $x \Vdash \phi \rightarrow \psi$, we have $x \leq y$. Since $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \phi \rightarrow \psi$.

$\leftarrow$**:** Assume $x \Vdash \psi \leftarrow \phi$. For each $x_2 \Vdash \phi$ and each $y \Vdash \psi$, we have $R(x, x_2, y)$. By R-transitivity, we have $R(x', x_2, y)$, hence $x' \Vdash \psi \leftarrow \phi$. Suppose $y \Vdash \psi \leftarrow \phi$. For each $x \Vdash \psi \leftarrow \phi$, we have $x \leq y$. Because of $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \psi \leftarrow \phi$.

□

Proposition 8.3.10. *For each bi-approximation model $\mathbb{M}$, each $x \in X$, each $y \in Y$, and every formula ϕ, if $\mathbb{M}, x \Vdash \phi$ and $\mathbb{M}, y \Vdash \phi$, then $x \leq y$. Furthermore, we have*

1. $\mathbb{M}, x \Vdash \phi \iff$ *for every $y \in Y$. if $\mathbb{M}, y \Vdash \phi$ then $x \leq y$,*

2. $\mathbb{M}, y \Vdash \phi \iff$ *for every $x \in X$. if $\mathbb{M}, x \Vdash \phi$ then $x \leq y$.*

Remark 8.3.11. Proposition 8.3.10 tells us initial sequents $\phi \Mapsto \phi$ is valid on every p-frame $\mathbb{F}$. Intuitively, if ϕ is assumed at x, then it should be concluded everywhere in Y above x. Conversely, if ϕ is concluded at y, then it should be assumed everywhere in X below y.

As a corollary of Proposition 8.3.10, we obtain the following.

Corollary 8.3.12. *For every p-frame $\mathbb{F}$, each doppelgänger valuation V is naturally extended from the set of all propositional variables Φ to the set of all formulae Λ, i.e. for each formula ϕ, we let*

1. $\tilde{V}^{\downarrow}(\phi) := \{x \in X \mid \mathbb{F}, V, x \Vdash \phi\}$,

2. $\tilde{V}_{\uparrow}(\phi) := \{y \in Y \mid \mathbb{F}, V, y \Vdash \phi\}$.

$$
\begin{tikzcd}
 & \Lambda \arrow[d, dash, dashed] \arrow[ddl, bend right, "\tilde{V}^{\downarrow}"'] \arrow[ddr, bend left, "\tilde{V}_{\uparrow}"] & \\
 & \Phi \arrow[dl, "V^{\downarrow}"'] \arrow[dr, "V_{\uparrow}"] & \\
\mathbb{D} \arrow[rr, bend left, "\lambda"] \arrow[rr, phantom, "\cong"] & & \mathbb{U} \arrow[ll, bend left, "\upsilon"]
\end{tikzcd}
$$

8.4 Bi-approximation, bases and the existential quantifier

In Kripke semantics, we have a simple interpretation of modal operators $\Diamond$ and $\Box$ as follows: for each Kripke model $\mathbb{M}$ and each possible world w, we let

(i) $\mathbb{M}, w \Vdash \Diamond\phi \iff \exists v \in W$ such that $R(w, v)$ and $\mathbb{M}, v \Vdash \phi$,

(ii) $\mathbb{M}, w \Vdash \Box\phi \iff \forall v \in W.$ if $R(w, v)$ then $\mathbb{M}, v \Vdash \phi$,

whereas, in bi-approximation models, all logical connectives are interpreted uniformly with *conjunction*, *implication* and *universal quantifier* $\forall$. For example, if we introduce $\Diamond$ on bi-approximation semantics, it is interpreted as follows:

(iii) $\mathbb{M}, x \Vdash \Diamond\phi \iff \forall y \in Y.$ if $\mathbb{M}, y \Vdash \Diamond\phi$ then $x \leq y$,

(iv) $\mathbb{M}, y \Vdash \Diamond\phi \iff \forall x \in X.$ if $\mathbb{M}, x \Vdash \phi$ then $R(x, y)$,

where R is a binary relation on $X \times Y$. This is because it is essential to set up our interpretation to return Galois stable sets. Note that item (iv) gives the definition of $\Diamond$ on $\mathbb{U}$, and item (iii) copies the same value to $\mathbb{D}$: see also Section 8.5. As we saw in Corollary 8.3.12, this setting allows us to assign the corresponding Galois stable X-set and Y-set for every formula between $\mathbb{D}$ and $\mathbb{U}$. On the other hand, to evaluate any formula on bi-approximation models, we encounter the universal quantifier $\forall$ and an implication in each step, which generates considerable complexity.

However, in this section, we will show that we can reduce the complexity in specific cases by introducing *auxiliary relations* for R. In other words, some logical connectives are translated into other simpler conditions with *the existential quantifier*, which may not be equivalent to the original conditions anymore. Through these simpler conditions, we will find the relationship between relational semantics and bi-approximation semantics. Furthermore, we will also unearth a connection among bi-approximation, bases and the existential quantifier.

Definition 8.4.1 (Auxiliary relations)**.** For every bi-approximation model $\mathbb{M}$ and the ternary relation $R \subseteq X \times X \times Y$, we let the following three ternary relations

$R^\circ \subseteq X \times X \times X$, $R^\rightarrow \subseteq X \times Y \times Y$ and $R^\leftarrow \subseteq Y \times X \times Y$:

1. $R^\circ(x_1, x_2, x) \iff$ for every $y \in Y$. if $R(x_1, x_2, y)$ then $x \leq y$,

2. $R^\rightarrow(x_1, y_2, y) \iff$ for every $x_2 \in X$. if $R(x_1, x_2, y)$ then $x_2 \leq y_2$,

3. $R^\leftarrow(y_1, x_2, y) \iff$ for every $x_1 \in X$. if $R(x_1, x_2, y)$ then $x_1 \leq y_1$.

Note that R° is related to $R^\downarrow$ in [26], but we also introduce $R^\rightarrow$ and $R^\leftarrow$ to show Theorem 8.6.1. Thanks to the tightness conditions in p-frames, see Definition 8.3.3, we obtain the following.

Lemma 8.4.2. *For an arbitrary bi-approximation model* $\mathbb{M}$ *and the ternary relation* $R \subseteq X \times X \times Y$*,*

1. $R(x_1, x_2, y) \iff$ *for every* $x \in X$*. if* $R^\circ(x_1, x_2, x)$ *then* $x \leq y$*,*

2. $R(x_1, x_2, y) \iff$ *for every* $y_2 \in Y$*. if* $R^\rightarrow(x_1, y_2, y)$ *then* $x_2 \leq y_2$*,*

3. $R(x_1, x_2, y) \iff$ *for every* $y_1 \in Y$*. if* $R^\leftarrow(y_1, x_2, y)$ *then* $x_1 \leq y_1$*.*

Proof. Item 1. ($\Rightarrow$). Suppose $R^\circ(x_1, x_2, x)$, i.e. if $R(x_1, x_2, y')$ then $x \leq y'$ for every $y' \in Y$. By assumption, we obtain $R(x_1, x_2, y)$, which derives $x \leq y$.

($\Leftarrow$). Contraposition. Namely, we claim that there exists $x \in X$ such that $R^\circ(x_1, x_2, x)$ and $x \not\leq y$, under the assumption that $R(x_1, x_2, y)$ does not hold. Suppose that $R(x_1, x_2, y)$ does not hold. By $\circ$-tightness, there exists $x \in X$ such that, $R^\circ(x_1, x_2, x)$, and, for each $y' \in Y$, if $x \leq y'$, then $R(x_1, x_2, y')$. Since $R(x_1, x_2, y)$ does not hold, we have $x \not\leq y$. Item 2 and item 3 are analogous to item 1. $\square$

Remark 8.4.3. By Definition 8.4.1, we can reformulate R-order, R-identity and R-associativity in Definition 8.3.3 as follows:

R-order: for all $x, x' \in X$, $x' \leq x \iff \exists o \in O_X.\ [R^\circ(x, o, x') \text{ or } R^\circ(o, x, x')]$,

R-identity: for every $x \in X.\ [\exists o_2 \in O_X.\ R^\circ(x, o_2, x) \text{ and } \exists o_1 \in O_X.\ R^\circ(o_1, x, x)]$,

R-associativity: for all $x_1, x_2, x_3, x \in X$.

$$\exists x' \in X.\ [R^\circ(x_1, x', x) \text{ and } R^\circ(x_2, x_3, x')]$$
$$\iff \exists x' \in X.\ [R^\circ(x', x_3, x) \text{ and } R^\circ(x_1, x_2, x')].$$

We note that similar conditions for R-order, R-identity and R-associativity can be found in a relational semantics for distributive substructural logics, e.g. [82, Definition 6].[1] Thanks to the auxiliary relations R°, $R^\rightarrow$ and $R^\leftarrow$, we obtain other interpretations of formulae on bi-approximation semantics.

Theorem 8.4.4. *For every bi-approximation model* $\mathbb{M}$ *and all formulae* ϕ, ψ, *we have*

1. $y \Vdash \phi \circ \psi \iff \forall x_1 \in X, y_2 \in Y.$ *if* $x_1 \Vdash \phi$ *and* $R^\rightarrow(x_1, y_2, y)$ *then* $y_2 \Vdash \psi$,
2. $y \Vdash \phi \circ \psi \iff \forall x_2 \in X, y_1 \in Y.$ *if* $x_2 \Vdash \psi$ *and* $R^\leftarrow(y_1, x_2, y)$ *then* $y_1 \Vdash \phi$,
3. $x_2 \Vdash \phi \rightarrow \psi \iff \forall x_1, x \in X.$ *if* $x_1 \Vdash \phi$ *and* $R^\circ(x_1, x_2, x)$ *then* $x \Vdash \psi$,
4. $x_2 \Vdash \phi \rightarrow \psi \iff \forall y_1, y \in Y.$ *if* $y \Vdash \psi$ *and* $R^\leftarrow(y_1, x_2, y)$ *then* $y_1 \Vdash \phi$,
5. $x_1 \Vdash \psi \leftarrow \phi \iff \forall x_2, x \in X.$ *if* $x_2 \Vdash \phi$ *and* $R^\circ(x_1, x_2, x)$ *then* $x \Vdash \psi$,
6. $x_1 \Vdash \psi \leftarrow \phi \iff \forall y_2, y \in Y.$ *if* $y \Vdash \psi$ *and* $R^\rightarrow(x_1, y_2, y)$ *then* $y_2 \Vdash \phi$,
7. $x \Vdash \phi \circ \psi \Longleftarrow \exists x_1, x_2 \in X$ *such that* $x_1 \Vdash \phi$, $x_2 \Vdash \psi$ *and* $R^\circ(x_1, x_2, x)$,
8. $y_2 \Vdash \phi \rightarrow \psi \Longleftarrow \exists x_1 \in X$, $\exists y \in Y$ *such that* $x_1 \Vdash \phi$, $y \Vdash \psi$ *and* $R^\rightarrow(x_1, y_2, y)$,

[1] The order of the ternary relation is different. That is, $R^\circ(x_1, x_2, x)$ in this chapter is the same as $R_\circ(x, x_1, x_2)$ in [82].

9. $y_1 \Vdash \psi \leftarrow \phi \Longleftarrow \exists x_2 \in X,\ \exists y \in Y$ *such that* $x_2 \Vdash \phi$, $y \Vdash \psi$ *and* $R^{\leftarrow}(y_1, x_2, y)$.

Proof. Items 1 - 5 are analogous to item 6. And, item 8 and item 9 are analogous to item 7.

6. By Proposition 8.3.10, Definition 8.4.1 and Lemma 8.4.2, we can prove as follows.

$$\begin{aligned}
x_1 \Vdash \psi \leftarrow \phi &\iff \forall x_2 \in X, \forall y \in Y.[x_2 \Vdash \phi, y \Vdash \psi \Rightarrow R(x_1, x_2, y)] \\
&\iff \forall x_2 \in X, \forall y_2, y \in Y.[x_2 \Vdash \phi, y \Vdash \psi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow x_2 \leq y_2] \\
&\iff \forall y_2, y \in Y.[y \Vdash \psi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow y_2 \Vdash \phi]
\end{aligned}$$

7. Suppose that there exist $x_1, x_2 \in X$ such that $x_1 \Vdash \phi$, $x_2 \Vdash \psi$ and $R^{\circ}(x_1, x_2, x)$. We claim that every element $y \in Y$ at which $\phi \circ \psi$ is concluded is above x. If $y \Vdash \phi \circ \psi$ holds, then, by definition, $R(x_1, x_2, y)$ holds. By Definition 8.4.1, we also obtain that $x \leq y'$, whenever $R(x_1, x_2, y')$ holds for every $y' \in Y$. Hence, $x \leq y$ holds, which derives $x \Vdash \phi \circ \psi$.

□

In Theorem 8.4.4, item 3 and item 5 correspond to the normal interpretations in Kripke semantics. The same results for item 3 and item 5 are obtained by generalized Kripke frames [26]. Moreover, item 7 looks similar to the interpretation on ternary-relational semantics of distributive substructural logics. Item 7 must be closely related to the discussion in [26, p.264]. However, unlike what happens in the setting of generalized Kripke frames, the conditions of item 7, item 8 and item 9 are *more beneficial* to evaluate formulae in our framework. More precisely, the auxiliary relations R°, $R^{\rightarrow}$ and $R^{\leftarrow}$ provide bases of $V(\phi \circ \psi)$, $V(\phi \rightarrow \psi)$ and $V(\psi \leftarrow \phi)$: see Theorem 8.4.6 and Proposition 8.4.7.

Related to Theorem 8.4.4, we also obtain the following results for $\vee$ and $\wedge$.

Theorem 8.4.5. *Let $\mathbb{M}$ be an arbitrary bi-approximation model, ϕ, ψ be all formulae. For each $x \in X$ and each $y \in Y$,*

1. $\mathbb{M}, x \Vdash \phi \vee \psi \Longleftarrow \mathbb{M}, x \Vdash \phi$ *or* $\mathbb{M}, x \Vdash \psi$,

2. $\mathbb{M}, y \Vdash \phi \wedge \psi \Longleftarrow \mathbb{M}, y \Vdash \phi$ *or* $\mathbb{M}, y \Vdash \psi$.

Proof. Item 1. Suppose that $x \Vdash \phi$ or $x \Vdash \psi$. For an arbitrary $y \in Y$, if $y \Vdash \phi \vee \psi$, by definition, $y \Vdash \phi$ and $y \Vdash \psi$. By Proposition 8.3.10, $x \Vdash \phi$ or $x \Vdash \psi$, either way, $x \leq y$ holds. Therefore, $x \Vdash \phi \vee \psi$. Item 2 is analogous to item 1. $\square$

Items 7 - 9 in Theorem 8.4.4 and Theorem 8.4.5 indicate that, when we reason about formulae with the existential quantifier and disjunction, we may not accumulate all worlds in X (in Y) where the formulae are assumed (concluded). However, as we will see below, we can still collect *essential worlds* in X (in Y) to gather *all worlds* in Y (in X) where the formulae are concluded (assumed): see Theorem 8.4.6. Hereinafter, to discuss the connection between the existential quantifier and the bi-approximation clearly, we introduce an auxiliary relation $\Vdash_{\mathfrak{bs}}$ of $\Vdash$ as follows (the subscript $_{-\mathfrak{bs}}$ refers to *bases*, see Theorem 8.4.6):

1. $x \Vdash_{\mathfrak{bs}} \phi \vee \psi \iff x \Vdash_{\mathfrak{bs}} \phi$ or $x \Vdash_{\mathfrak{bs}} \psi$,

2. $y \Vdash_{\mathfrak{bs}} \phi \wedge \psi \iff y \Vdash_{\mathfrak{bs}} \phi$ or $y \Vdash_{\mathfrak{bs}} \psi$,

3. $x \Vdash_{\mathfrak{bs}} \phi \circ \psi \iff \exists x_1, x_2 \in X$ s.t. $x_1 \Vdash_{\mathfrak{bs}} \phi$, $x_2 \Vdash_{\mathfrak{bs}} \psi$ and $R^{\circ}(x_1, x_2, x)$,

4. $y_2 \Vdash_{\mathfrak{bs}} \phi \to \psi \iff \exists x_1 \in X, \exists y \in Y$ s.t. $x_1 \Vdash_{\mathfrak{bs}} \phi$, $y \Vdash_{\mathfrak{bs}} \psi$ and $R^{\to}(x_1, y_2, y)$,

5. $y_1 \Vdash_{\mathfrak{bs}} \psi \leftarrow \phi \iff \exists x_2 \in X, \exists y \in Y$ s.t. $x_2 \Vdash_{\mathfrak{bs}} \phi$, $y \Vdash_{\mathfrak{bs}} \psi$ and $R^{\leftarrow}(y_1, x_2, y)$.

6. $x \Vdash_{\mathfrak{bs}} \phi \iff x \Vdash \phi$, whenever ϕ is a propositional variable or a constant, or the outermost connective of ϕ is either $\wedge$, $\rightarrow$ or $\leftarrow$,

7. $y \Vdash_{\mathfrak{bs}} \psi \iff y \Vdash \psi$, whenever ψ is a propositional variable or a constant, or the outermost connective of ψ is either $\vee$ or $\circ$.

By parallel induction, we obtain the following straightforwardly. For every formula ϕ, each $x \in X$ and each $y \in Y$, we have

1. if $x \Vdash_{\mathfrak{bs}} \phi$, then $x \Vdash \phi$,

2. if $y \Vdash_{\mathfrak{bs}} \phi$, then $y \Vdash \phi$.

Furthermore, we also obtain the following.

Theorem 8.4.6. *Let $\mathbb{M}$ be an arbitrary bi-approximation model and ϕ each formula. Then, we have the following (recall $\tilde{V}$ in Corollary 8.3.12 and basis in Definition 8.3.2):*

1. *the set $\{x \in X \mid \mathbb{M}, x \Vdash_{\mathfrak{bs}} \phi\}$ is a basis of $\tilde{V}_{\uparrow}(\phi)$,*

2. *the set $\{y \in Y \mid \mathbb{M}, y \Vdash_{\mathfrak{bs}} \phi\}$ is a basis of $\tilde{V}^{\downarrow}(\phi)$.*

Proof. Parallel induction. Base cases are trivial.

1. $\upsilon(\{y \in Y \mid y \Vdash_{\mathfrak{bs}} \phi \wedge \psi\}) = \tilde{V}^{\downarrow}(\phi \wedge \psi)$. $(\subseteq)$. For each x, suppose that $x \leq y$, if $y \Vdash_{\mathfrak{bs}} \phi$ or $y \Vdash_{\mathfrak{bs}} \psi$ for every y. It is equivalent to both $x \leq y$ if $y \Vdash_{\mathfrak{bs}} \phi$ and $x \leq y$ if $y \Vdash_{\mathfrak{bs}} \psi$. By induction hypothesis, we have $x \Vdash \phi$ and $x \Vdash \psi$, hence $x \Vdash \phi \wedge \psi$. $(\supseteq)$. trivial.

2. $\lambda(\{x \in X \mid x \Vdash_{\mathfrak{bs}} \phi \circ \psi\}) = \tilde{V}_{\uparrow}(\phi \circ \psi)$. For each $y \in Y$, by Theorem 8.4.4,
$$y \Vdash \phi \circ \psi \iff \forall x_1, x_2.[x_1 \Vdash_{\mathfrak{bs}} \phi, x_2 \Vdash_{\mathfrak{bs}} \psi \Rightarrow R(x_1, x_2, y)]$$

$$\iff \forall x_1, x_2, y_2.[x_1 \Vdash_{\mathfrak{bs}} \phi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow (x_2 \Vdash_{\mathfrak{bs}} \psi \Rightarrow x_2 \leq y_2)]$$

$$\iff \forall x_1, x_2' \in X.[x_1 \Vdash_{\mathfrak{bs}} \phi, x_2' \Vdash \psi \Rightarrow R(x_1, x_2', y)].$$

Note that $x_2 \Vdash_{\mathfrak{bs}} \psi$ changes to $x_2' \Vdash \psi$. Repeat the same replacement for x_1. The other cases are analogous. □

Theorem 8.4.6 tells us that, in bi-approximation semantics, bases are (partly) inductively characterised by the existential quantifier and disjunction: see also ∪-terms, ∩-terms, pseudo-∪-terms and pseudo-∩-terms in Section 3.3. Moreover, this property works beneficially together with the following proposition: see Remark 8.6.2.

Proposition 8.4.7. *Let* $\mathbb{M}$ *be an arbitrary bi-approximation model and* ϕ, ψ *all formulae. Then, we have*

$$\mathbb{M} \Vdash \phi \mapsto \psi \iff \forall x \in X, \forall y \in Y. \textit{ if } \mathbb{M}, x \Vdash_{\mathfrak{bs}} \phi \textit{ and } \mathbb{M}, y \Vdash_{\mathfrak{bs}} \psi, \textit{ then } x \leq y.$$

Proof. ($\Rightarrow$). Since $x \Vdash \phi$ ($y \Vdash \psi$) whenever $x \Vdash_{\mathfrak{bs}} \phi$ ($y \Vdash_{\mathfrak{bs}} \psi$), this is trivial. ($\Leftarrow$). Let x be an arbitrary element where ϕ is premised, y an arbitrary element where ψ is concluded. By our assumption, for an arbitrary $x_B \Vdash_{\mathfrak{bs}} \phi$, we have $x_B \leq y_B$ for every $y_B \Vdash_{\mathfrak{bs}} \psi$. By Theorem 8.4.6, we obtain $x_B \Vdash \psi$, hence $x_B \leq y$ (Proposition 8.3.10). As x_B is arbitrary, by Theorem 8.4.6, $y \Vdash \phi$ also holds. Therefore, $x \leq y$ (Proposition 8.3.10). □

8.5 The Representation theorem

In this section, to prove Theorem 8.6.3, we show that FL-algebras can be represented by p-frames. By analogy to the situation in modal logic (e.g. [5]), we will show that the dual frames of FL-algebras are p-frames and the dual algebras of p-frames are FL-algebras. Moreover, the validity relations between p-frames and FL-algebras are also proved as in the case of modal logic: see Theorem 8.5.3 and Theorem 8.5.5.

Dual algebra of p-frame For each p-frame $\mathbb{F}$, we construct two isomorphic FL-algebras *in parallel* based on the isomorphic posets $\mathbb{D}$ and $\mathbb{U}$. Namely, we define the operations $\vee$, $\wedge$, $*$, $\backslash$ and $/$, and the constants 1 and 0 on both $\mathbb{D}$ and $\mathbb{U}$, as they are isomorphic FL-algebras, i.e. $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, 1, 0\rangle \cong \langle \mathbb{U}, \vee, \wedge, *, \backslash, /, 1, 0\rangle$.

Since $\mathbb{D}$ and $\mathbb{U}$ are isomorphic through the Galois connection $\lambda \dashv \upsilon$, we have two natural ways to define each operation, in general. That is, an operation on $\mathbb{U}$ is *approximated from below*, and take the copy to the other side via $\upsilon : \mathbb{U} \to \mathbb{D}$. Or, an operation on $\mathbb{D}$ is *approximated from above*, and take the copy to the other side via $\lambda : \mathbb{D} \to \mathbb{U}$. In our case, the additive operations $\vee$ and $*$ are defined on $\mathbb{U}$, approximated from below, and the multiplicative operations $\wedge$, $\backslash$ and $/$ are defined on $\mathbb{D}$, approximated from above. Otherwise, we cannot prove the residuality (see [31] and [32]).

For each p-frame $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y\rangle$, we define $\vee$, $\wedge$, $*$, $\backslash$ and $/$ are defined as follows: on $\mathbb{D}$, for all $\alpha^{\downarrow}, \beta^{\downarrow} \in \mathbb{D}$,

$\mathbb{D}$-1: $\alpha^{\downarrow} \vee \beta^{\downarrow} := \upsilon(\alpha_{\uparrow} \vee \beta_{\uparrow})$,

$\mathbb{D}$-2: $\alpha^{\downarrow} \wedge \beta^{\downarrow} := \alpha^{\downarrow} \cap \beta^{\downarrow}$,

$\mathbb{D}$-3: $\alpha^{\downarrow} * \beta^{\downarrow} := \upsilon(\alpha_{\uparrow} * \beta_{\uparrow})$,

$\mathbb{D}$-4: $\alpha^{\downarrow}\backslash\beta^{\downarrow} := \{x_2 \in X \mid \forall x_1 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}.\ R(x_1, x_2, y)\}$,

$\mathbb{D}$-5: $\beta^{\downarrow}/\alpha^{\downarrow} := \{x_1 \in X \mid \forall x_2 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}.\ R(x_1, x_2, y)\}$.

On $\mathbb{U}$, for all $\alpha_\uparrow, \beta_\uparrow \in \mathbb{U}$,

$\mathbb{U}$-1: $\alpha_\uparrow \vee \beta_\uparrow := \alpha_\uparrow \cap \beta_\uparrow$,

$\mathbb{U}$-2: $\alpha_\uparrow \wedge \beta_\uparrow := \lambda(\alpha^\downarrow \wedge \beta^\downarrow)$,

$\mathbb{U}$-3: $\alpha_\uparrow * \beta_\uparrow := \{y \in Y \mid \forall x_1 \in \alpha^\downarrow, \forall x_2 \in \beta^\downarrow.\ R(x_1, x_2, y)\}$,

$\mathbb{U}$-4: $\alpha_\uparrow \backslash \beta_\uparrow := \lambda(\alpha^\downarrow \backslash \beta^\downarrow)$,

$\mathbb{U}$-5: $\beta_\uparrow / \alpha_\uparrow := \lambda(\beta^\downarrow / \alpha^\downarrow)$.

Based on these operations, we can show the following.

Theorem 8.5.1. *Both $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X\rangle$ and $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y\rangle$ are FL-algebras, and they are isomorphic.*

Proof. Firstly, we need to check well-definedness of each operation. Namely, it is necessary to show that every value returns a Galois stable set. The copying parts are trivial, hence we need to check the following definition parts.

$\vee$: We claim that $\alpha_\uparrow \cap \beta_\uparrow = \lambda(\alpha^\downarrow \cup \beta^\downarrow)$. ($\subseteq$). For each $y \in \alpha_\uparrow \cap \beta_\uparrow$, since $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$, $x \leq y$ for each $x \in \alpha^\downarrow \cup \beta^\downarrow$. ($\supseteq$). If $y \in \lambda(\alpha^\downarrow \cup \beta^\downarrow)$, for arbitrary $x_a \in \alpha^\downarrow$ and $x_b \in \beta^\downarrow$, we have $x_a \leq y$ and $x_b \leq y$, hence $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$.

$\wedge$: We claim that $\alpha^\downarrow \cap \beta^\downarrow = \upsilon(\alpha_\uparrow \cup \beta_\uparrow)$. ($\subseteq$). For each $x \in \alpha^\downarrow \cap \beta^\downarrow$, since $x \in \alpha^\downarrow$ and $x \in \beta^\downarrow$, $x \leq y$ for each $y \in \alpha_\uparrow \cup \beta_\uparrow$. ($\supseteq$). If $x \in \upsilon(\alpha_\uparrow \cup \beta_\uparrow)$, for arbitrary $y_a \in \alpha_\uparrow$ and $y_b \in \beta_\uparrow$, we have $x \leq y_a$ and $x \leq y_b$, hence $x \in \alpha^\downarrow$ and $x \in \beta^\downarrow$.

$*$: We claim that $\alpha_\uparrow * \beta_\uparrow = \lambda(\{x \in X \mid x_1 \in \alpha^\downarrow, x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x)\})$.

$$\begin{aligned}
\alpha_\uparrow * \beta_\uparrow &= \{y \in Y \mid \forall x_1 \in \alpha^\downarrow, \forall x_2 \in \beta^\downarrow, R(x_1, x_2, y)\} \\
&= \{y \in Y \mid \forall x \in X, \forall x_1 \in \alpha^\downarrow, \forall x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x) \Rightarrow x \leq y\} \\
&= \lambda(\{x \in X \mid x_1 \in \alpha^\downarrow, x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x)\})
\end{aligned}$$

$\backslash$: We claim that $\alpha^{\downarrow}\backslash\beta^{\downarrow} = \upsilon(\{y_2 \in Y \mid x_1 \in \alpha^{\downarrow}, y \in \beta_{\uparrow}, R^{\rightarrow}(x_1, y_2, y)\})$.

$$\begin{aligned}
\alpha^{\downarrow}\backslash\beta^{\downarrow} &= \{x_2 \in X \mid \forall x_1 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}, R(x_1, x_2, y)\} \\
&= \{x_2 \in X \mid \forall y_2 \in Y, \forall x_1 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}, R^{\rightarrow}(x_1, y_2, y) \Rightarrow x_2 \leq y_2\} \\
&= \upsilon(\{y_2 \in Y \mid x_1 \in \alpha^{\downarrow}, y \in \beta_{\uparrow}, R^{\rightarrow}(x_1, y_2, y)\})
\end{aligned}$$

$/$ is analogous to $\backslash$.

Therefore, all operations are well-defined. Furthermore, these two algebras are isomorphic by definition. Next, we prove they are FL-algebras.

$\langle \mathbb{D}, \vee, \wedge \rangle$ and $\langle \mathbb{U}, \vee, \wedge \rangle$ are lattices. For all α, β, γ, we claim that[2]

$$\alpha \leq \gamma \text{ and } \beta \leq \gamma \iff \alpha \vee \beta \leq \gamma, \tag{8.1}$$

$$\gamma \leq \alpha \text{ and } \gamma \leq \beta \iff \gamma \leq \alpha \wedge \beta. \tag{8.2}$$

($\Rightarrow$) of the condition (8.1). For each $y \in \gamma_{\uparrow}$, since $\alpha_{\uparrow} \supseteq \gamma_{\uparrow}$ and $\beta_{\uparrow} \supseteq \gamma_{\uparrow}$, we have $y \in \alpha_{\uparrow}$ and $y \in \beta_{\uparrow}$, hence $y \in \alpha_{\uparrow} \cap \beta_{\uparrow}$. ($\Leftarrow$) of the condition (8.1). For each $y \in \gamma_{\uparrow}$, since $\alpha_{\uparrow} \cap \beta_{\uparrow} \supseteq \gamma_{\uparrow}$, we obtain $y \in \alpha_{\uparrow}$ and $y \in \beta_{\uparrow}$. The condition (8.2) is analogous.

$\langle \mathbb{D}, *, O_X \rangle$ and $\langle \mathbb{U}, *, O_Y \rangle$ are monoids. For all α, β, γ, we claim that

$$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma, \tag{8.3}$$

$$\alpha * O = \alpha = O * \alpha, \tag{8.4}$$

where O is either O_X or O_Y depending on the domain. The condition (8.3). Let y be an arbitrary element in $\alpha_{\uparrow} * (\beta_{\uparrow} * \gamma_{\uparrow})$. By Theorem 8.4.6, for all $x_1, x_2, x_3, x', x \in X$, if $x_1 \in \alpha^{\downarrow}$, $x_2 \in \beta^{\downarrow}$, $x_3 \in \gamma^{\downarrow}$, $R^{\circ}(x_1, x', x)$ and $R^{\circ}(x_2, x_3, x')$, then $x \leq y$ holds. By R-

[2]Recall that the order $\leq$ is $\subseteq$ on $\mathbb{D}$ and $\supseteq$ on $\mathbb{U}$.

associativity (see Remark 8.4.3), the condition is equivalent to that, for each element x, for all $x'' \in X$, if $x_1 \in \alpha^{\downarrow}$, $x_2 \in \beta^{\downarrow}$, $x_3 \in \gamma^{\downarrow}$, $R^{\circ}(x'', x_3, x)$ and $R^{\circ}(x_1, x_2, x'')$, then $x \leq y$, which concludes $y \in (\alpha_{\uparrow} * \beta_{\uparrow}) * \gamma_{\uparrow}$.

The left equality of the condition (8.4). ($\subseteq$). For each $x_1 \in \alpha^{\downarrow}$, by R-identity, there exists $o_2 \in O_X$ such that $R^{\circ}(x_1, o_2, x_1)$. By definition, for every $y' \in Y$, if $R(x_1, o_2, y')$, then $x_1 \leq y'$ holds. Now, for every $y \in \alpha_{\uparrow} * O_Y$, by definition, $R(x_1, o_2, y)$ holds, hence $x_1 \leq y$. Since x_1 is arbitrary in $\alpha^{\downarrow}$, which derives $y \in \alpha_{\uparrow}$. ($\supseteq$). For arbitrary $x \in \alpha^{\downarrow}$ and $o \in O_X$, by $\circ$-tightness, there exists $x' \in X$ such that $R^{\circ}(x, o, x')$ and $x' \leq y' \Rightarrow R(x, o, y')$ for each $y' \in Y$. For every $y \in \alpha_{\uparrow}$, we have $x \leq y$, because of $x \in \alpha^{\downarrow}$. Furthermore, by R-order, $x' \leq x$ holds. Since $\leq$ is transitive, we obtain $x' \leq y$, hence $R(x, o, y)$. The right equality of the condition (8.4) is analogous.

Finally, we will show the residuality: for all α, β, γ,

$$\alpha * \beta \leq \gamma \iff \beta \leq \alpha \backslash \gamma \iff \alpha \leq \gamma / \beta. \tag{8.5}$$

($\Rightarrow$) of the first equivalence in the condition (8.5). Let x_2 be an arbitrary element in $\beta^{\downarrow}$. For arbitrary $x_1 \in \alpha^{\downarrow}$ and $y \in \gamma_{\uparrow}$, since $\alpha_{\uparrow} * \beta_{\uparrow} \supseteq \gamma_{\uparrow}$, we have $R(x_1, x_2, y)$. Hence, $x_2 \in \alpha^{\downarrow} \backslash \gamma^{\downarrow}$. ($\Leftarrow$) of the first equivalence in the condition (8.5). Let y be an arbitrary element in $\gamma_{\uparrow}$. For arbitrary $x_1 \in \alpha^{\downarrow}$ and $x_2 \in \beta^{\downarrow}$, since $\beta^{\downarrow} \subseteq \alpha^{\downarrow} \backslash, \gamma^{\downarrow}$, we obtain $R(x_1, x_2, y)$. Hence, $y \in \alpha_{\uparrow} * \beta_{\uparrow}$. The other equivalence is analogous. $\square$

By Theorem 8.5.1, we naturally define the dual FL-algebras of p-frames.

Definition 8.5.2 (Dual algebra)**.** Let $\mathbb{F}$ be a p-frame. *The dual algebra of* $\mathbb{F}$ is an algebra $\mathbb{F}^{+} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ which is isomorphic to $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X \rangle$

and $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y \rangle$.

Along with the definition of dual algebras, we obtain the equivalence of validity, as usual.

Theorem 8.5.3. *For every p-frame $\mathbb{F}$ and each sequent $\Gamma \mapsto \varphi$, the sequent $\Gamma \mapsto \varphi$ is valid on $\mathbb{F}$ if and only if it is valid on the dual algebra $\mathbb{F}^+$.*

$$\mathbb{F} \Vdash \Gamma \mapsto \varphi \iff \mathbb{F}^+ \models \Gamma \leq \varphi$$

Dual frame of FL-algebras Here we construct the dual frames of FL-algebras. We mention that the dual frame corresponds to *the intermediate level* introduced in [33] but see also [18] and [84].

Let $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ be a FL-algebra. The set of all filters and the set of all ideals are denoted by $\mathcal{F}$ and $\mathcal{I}$. On $\mathcal{F} \cup \mathcal{I}$, we define a binary relation $\sqsubseteq$ as follows: for all filters $F, F_1, F_2 \in \mathcal{F}$ and all ideals $I, I_1, I_2 \in \mathcal{I}$,

1. $F_1 \sqsubseteq F_2 \iff F_1 \supseteq F_2$,
2. $F \sqsubseteq I \iff F \cap I \neq \emptyset$,
3. $I \sqsubseteq F \iff \forall a \in I, \forall b \in F.\ a \leq b$,
4. $I_1 \sqsubseteq I_2 \iff I_1 \subseteq I_2$.

Next, on the triple $\langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$, we build a ternary relation R, and subsets $O_{\mathcal{F}}$, $O_{\mathcal{I}}$, $N_{\mathcal{F}}$ and $N_{\mathcal{I}}$ as follows: for all $F_1, F_2 \in \mathcal{F}$ and each $I \in \mathcal{I}$,

1. $R(F_1, F_2, I) \iff F_1 * F_2 \sqsubseteq I$,

   where $F_1 * F_2 := \{a \in A \mid \exists f_1 \in F_1, \exists f_2 \in F_2.\ f_1 * f_2 \leq a\}$,

2. $O_{\mathcal{F}}$ is the set of all filters containing 1,

3. $O_{\mathcal{I}}$ is the set of all ideals containing 1,

4. $N_{\mathcal{F}}$ is the set of all filters containing 0,

5. $N_{\mathcal{I}}$ is the set of all ideals containing 0.

Then, the 8-tuple $\mathbb{A}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq, R, O_{\mathcal{F}}, O_{\mathcal{I}}, N_{\mathcal{F}}, N_{\mathcal{I}} \rangle$ is *the dual frame of* $\mathbb{A}$. To prove the following theorems, we here mention that, for all $F, F_1, F_2 \in \mathcal{F}$ and each $I, I_1, I_2 \in \mathcal{I}$,

1. $F_1 * F_2$ is a filter,

2. $F \backslash I := \{a \in A \mid \exists f \in F, \exists i \in I.\ a \leq f \backslash i\}$ is an ideal,

3. $I/F := \{a \in A \mid \exists i \in I, \exists f \in F.\ a \leq i/f\}$ is an ideal.

4. $R^{\circ}(F_1, F_2, F) \iff F \sqsubseteq F_1 * F_2$,

5. $R^{\rightarrow}(F_1, I_2, I) \iff F_1 \backslash I \sqsubseteq I_2$,

6. $R^{\leftarrow}(I_1, F_2, I) \iff I/F_2 \sqsubseteq I_1$.

Then, we can prove the following.

Theorem 8.5.4. *For any FL-algebra $\mathbb{A}$, the dual frame $\mathbb{A}_+$ is a p-frame.*

Proof. By definition, $\langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$ is a polarity.

R-order: Let F, F' be arbitrary filters. Suppose $F' \sqsubseteq F$. Since $F = F * {\uparrow}1$, we obtain $F' \sqsubseteq F * {\uparrow}1$. Conversely, if $F' \sqsubseteq F * O$ or $F' \sqsubseteq O * F$ for some $O \in O_{\mathcal{F}}$, because $1 \in O$, we obtain $F * O \sqsubseteq F$ or $O * F \sqsubseteq F$, hence $F' \sqsubseteq F$.

R-identity: Let $\uparrow 1$ be the principal filter generated by 1. For each filter F, we have $F * \uparrow 1 = \uparrow 1 * F = F$, hence $R^{\circ}(F, \uparrow 1, F)$ and $R^{\circ}(\uparrow 1, F, F)$.

R-transitivity: For all $F_1, F_1', F_2, F_2' \in \mathcal{F}$ and all $I, I' \in \mathcal{I}$, if $F_1' \sqsubseteq F_1$, $F_2' \sqsubseteq F_2$, $I \sqsubseteq I'$ and $F_1 * F_2 \sqsubseteq I$, then there exist $f_1 \in F_1$, $f_2 \in F_2$ and $i \in I$ such that $f_1 * f_2 \leq i$. Since $f_1 \in F_1'$, $f_2 \in F_2'$ and $i \in I'$, we also have $F_1' * F_2' \sqsubseteq I'$.

R-associativity: For all $F_1, F_2, F_3 \in \mathcal{F}$, we have $F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3$, by the associativity of $*$ on $\mathbb{A}$. If $F \sqsubseteq F_1 * F'$ and $F' \sqsubseteq F_2 * F_3$, we obtain $F \sqsubseteq F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3$. Let $F'' = F_1 * F_2$. Then, $F \sqsubseteq F'' * F_3$ and $F'' \sqsubseteq F_1 * F_2$ hold.

O-isom (N-isom): For each $F \in O_{\mathcal{F}}$ $(N_{\mathcal{F}})$ and each $I \in O_{\mathcal{I}}$ $(N_{\mathcal{I}})$, they have 1 (0) in common.

$\circ$-tightness: For all $F_1, F_2 \in \mathcal{F}$, it is trivially true that $R(F_1, F_2, I)$ if and only if $F_1 * F_2 \sqsubseteq I$ for every $I \in \mathcal{I}$. The other is analogous.

$\rightarrow$-tightness: For each $F_1 \in \mathcal{F}$ and each $I \in \mathcal{I}$, by definition, for each $F_2 \in \mathcal{F}$, $R(F_1, F_2, I)$ if and only if $F_2 \sqsubseteq F_1 \backslash I$.

$\leftarrow$-tightness: For each $F_2 \in \mathcal{F}$ and each $I \in \mathcal{I}$, by definition, for each $F_1 \in \mathcal{F}$, $R(F_1, F_2, I)$ if and only if $F_1 \sqsubseteq I / F_2$.

$\square$

We prove the validity relationship between FL-algebras and the dual p-frames.

Theorem 8.5.5. *Let $\mathbb{A}$ be every FL-algebra and $\Gamma \mapsto \varphi$ each sequent. If the sequent* *is valid on the dual frame $\mathbb{A}_+$, it is also valid on the original FL-algebra $\mathbb{A}$.*

$$\mathbb{A} \models \Gamma \leq \varphi \Longleftarrow \mathbb{A}_+ \Vdash \Gamma \mapsto \varphi$$

Proof. Let $f : \Phi \to \mathbb{A}$ be an arbitrary assignment. We also denote the normally extended assignment $f : \Lambda \to \mathbb{A}$ by f. Then, we define a doppelgänger valuation V based on f as follows: for each proposition $p \in \Phi$,

1. $V^{\downarrow}(p) := \{F \in \mathcal{F} \mid f(p) \in F\} = \upsilon(\{\downarrow f(p)\})$,

2. $V_{\uparrow}(p) := \{I \in \mathcal{I} \mid f(p) \in I\} = \lambda(\{\uparrow f(p)\})$.

We claim that $f(\phi) \in F \iff \mathbb{A}_+, V, F \Vdash \phi$ and $f(\phi) \in I \iff \mathbb{A}_+, V, I \Vdash \phi$, for each filter F, each ideal I and each formula ϕ. Base cases are trivial. Inductive steps. For each filter $F \in \mathcal{F}$ and each ideal $I \in \mathcal{I}$,

$\vee$: Suppose that $f(\phi) \vee f(\psi) = f(\phi \vee \psi) \in I$. It is equivalent to $f(\phi) \in I$ and $f(\psi) \in I$. By induction hypothesis, it is also equivalent to $I \Vdash \phi$ and $I \Vdash \psi$, which, by definition, $I \Vdash \phi \vee \psi$.

If $f(\phi \vee \psi) \in F$, then F has non-empty intersection with all ideals containing $f(\phi \vee \psi)$. We obtain $F \Vdash \phi \vee \psi$, because every ideal I satisfying $I \Vdash \phi \vee \psi$ contains $f(\phi \vee \psi)$. Conversely, if $F \Vdash \phi \vee \psi$, then it must have non-empty intersection with $\downarrow f(\phi \vee \psi)$ as well. Therefore, $f(\phi \vee \psi) \in F$.

$\wedge$: Suppose that $f(\phi) \wedge f(\psi) = f(\phi \wedge \psi) \in F$. It is equivalent to $f(\phi) \in F$ and $f(\psi) \in F$. By induction hypothesis, it is also equivalent to $F \Vdash \phi$ and $F \Vdash \psi$, which $F \Vdash \phi \wedge \psi$ by definition.

If $f(\phi \wedge \psi) \in I$, then I has non-empty intersection with all filters containing

$f(\phi \wedge \psi)$. We obtain $I \Vdash \phi \wedge \psi$, because every filter F satisfying $F \Vdash \phi \wedge \psi$ contains $f(\phi \wedge \psi)$. Conversely, if $I \Vdash \phi \wedge \psi$, then it must have non-empty intersection with $\uparrow f(\phi \wedge \psi)$ as well. Therefore, $f(\phi \wedge \psi) \in I$.

$\circ$: Suppose that $f(\phi) * f(\psi) = f(\phi \circ \psi) \in I$. For any $F_1, F_2 \in \mathcal{F}$, if $F_1 \Vdash \phi$ and $F_2 \Vdash \psi$, by induction hypothesis, $f(\phi) \in F_1$ and $f(\psi) \in F_2$. Therefore, we have $f(\phi) * f(\psi) \in F_1 * F_2$, which derives $F_1 * F_2 \sqsubseteq I$, i.e. $R(F_1, F_2, I)$. Conversely, assume that $I \Vdash \phi \circ \psi$. By definition, for arbitrary $F_1 \Vdash \phi$ and $F_2 \Vdash \psi$, $F_1 * F_2 \sqsubseteq I$ holds. Then, $\uparrow f(\phi) * \uparrow f(\psi) \sqsubseteq I$ must hold, hence $f(\phi \circ \psi) \in I$.

If $f(\phi \circ \psi) \in F$, then F has non-empty intersection with all ideals containing $f(\phi \circ \psi)$. Since every ideal I satisfying $I \Vdash \phi \circ \psi$ contains $f(\phi \circ \psi)$, we have $F \Vdash \phi \circ \psi$. Conversely, if $F \Vdash \phi \circ \psi$, then it must have non-empty intersection with $\downarrow f(\phi \circ \psi)$ as well. Therefore, $f(\phi \circ \psi) \in F$.

$\rightarrow$: Suppose that $f(\phi) \backslash f(\psi) = f(\phi \rightarrow \psi) \in F$. For arbitrary $F' \in \mathcal{F}$ and $I \in \mathcal{I}$, if $F' \Vdash \phi$ and $I \Vdash \psi$, by induction hypothesis, $f(\phi) \in F'$ and $f(\psi) \in I$, hence $f(\phi) \backslash f(\psi) \in F' \backslash I$. By the residuality on $\mathbb{A}$, we obtain $F' * F \sqsubseteq I$, i.e. $R(F', F, I)$ holds. Conversely, assume that $F \Vdash \phi \rightarrow \psi$. By definition, for arbitrary $F' \Vdash \phi$ and $I \Vdash \psi$, we have $F' * F \sqsubseteq I$. Then, $\uparrow f(\phi) * F \sqsubseteq \downarrow f(\psi)$ must hold as well. So, there exists $x \in F$ such that $x \leq f(\phi) \backslash f(\psi) = f(\phi \rightarrow \psi)$, hence $f(\phi \rightarrow \psi) \in F$.

If $f(\phi \rightarrow \psi) \in I$, then I has non-empty intersection with all filters containing $f(\phi \rightarrow \psi)$. Since every filter F satisfying $F \Vdash \phi \rightarrow \psi$ contains $f(\phi \rightarrow \psi)$, we have $I \Vdash \phi \rightarrow \psi$. Conversely, if $I \Vdash \phi \rightarrow \psi$, then it must have non-empty intersection with $\uparrow f(\phi \rightarrow \psi)$ as well. Therefore, $f(\phi \rightarrow \psi) \in I$.

$\leftarrow$: Suppose that $f(\psi)/f(\phi) = f(\psi \leftarrow \phi) \in F$. For arbitrary $F' \in \mathcal{F}$ and $I \in \mathcal{I}$, if $F' \Vdash \phi$ and $I \Vdash \psi$, by induction hypothesis, $f(\phi) \in F'$ and $f(\psi) \in I$, hence $f(\psi)/f(\phi) \in I/F'$. By the residuality on $\mathbb{A}$, we obtain $F * F' \sqsubseteq I$, i.e. $R(F, F', I)$ holds. Conversely, assume that $F \Vdash \psi \leftarrow \phi$. By definition, for arbitrary $F' \Vdash \phi$ and $I \Vdash \psi$, we have $F * F' \sqsubseteq I$. Then, $F * {\uparrow} f(\phi) \sqsubseteq {\downarrow} f(\psi)$ must hold as well. So, there exists $x \in F$ such that $x \leq f(\psi)/f(\phi) = f(\psi \leftarrow \phi)$, hence $f(\psi \leftarrow \phi) \in F$.

If $f(\psi \leftarrow \phi) \in I$, then I has non-empty intersection with all filters containing $f(\psi \leftarrow \phi)$. Since every filter F satisfying $F \Vdash \psi \leftarrow \phi$ contains $f(\psi \leftarrow \phi)$, we have $I \Vdash \psi \leftarrow \phi$. Conversely, if $I \Vdash \psi \leftarrow \phi$, then it must have non-empty intersection with ${\uparrow} f(\psi \leftarrow \phi)$ as well. Therefore, $f(\psi \leftarrow \phi) \in F$.

Finally, we finish up the proof. Assume $\Gamma \mapsto \varphi$ is not valid on $\mathbb{A}$. Then, there exists an assignment $f : \Phi \to \mathbb{A}$ such that $f(\Gamma) \not\leq f(\varphi)$. We have that ${\uparrow} f(\Gamma) \in \mathcal{F}$ and ${\downarrow} f(\varphi) \in \mathcal{I}$. Moreover, we also have $\mathbb{A}_+, V, {\uparrow} f(\Gamma) \Vdash \Gamma$ and $\mathbb{A}_+, V, {\downarrow} f(\varphi) \Vdash \varphi$. However, since $f(\Gamma) \not\leq f(\varphi)$, ${\uparrow} f(\Gamma) \not\sqsubseteq {\downarrow} f(\varphi)$. Therefore, $\mathbb{A}_+ \nVdash \Gamma \mapsto \varphi$. $\square$

8.6 Soundness and Completeness

In this section, we will show that p-frames are a sound and complete semantics for the substructural logic **FL**. Unlike what happens in the setting of relational semantics for distributive substructural logics, soundness is not straightforward. This is because bi-approximation models evaluate formulae through *the Galois connection* $\lambda \dashv \upsilon$. To avoid this complex argument, we can use the relationship between the *bi-approximation* and the *bases*: recall Proposition 8.4.7.

Theorem 8.6.1 (Soundness)**.** *Let* $\Gamma \Rightarrow \varphi$ *be an arbitrary sequent. If the sequent* $\Gamma \Rightarrow \varphi$ *is derivable in FL, it is valid on every p-frame* $\mathbb{F}$.

Proof. Let $\mathbb{F}$ be an arbitrary p-frame and V an arbitrary doppelgänger valuation on $\mathbb{F}$. On the bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V \rangle$, all initial sequents are true, by Proposition 8.3.10. Note that we use Proposition 8.4.7 to prove the inductive steps. We mention that (**f**w) and ($\circ \Rightarrow$) are trivial, and ($\Rightarrow \vee_2$), ($\wedge_2 \Rightarrow$), ($\Rightarrow\rightarrow$) and ($\leftarrow\Rightarrow$) are analogous to ($\Rightarrow \vee_1$), ($\wedge_1 \Rightarrow$), ($\Rightarrow\leftarrow$) and ($\rightarrow\Rightarrow$), respectively.

(cut): For arbitrary $x \in X$ and $y \in Y$, let $\mathbb{M}, x \Vdash_{\mathfrak{bs}} \Sigma \circ \Gamma \circ \Pi$ and $\mathbb{M}, y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Sigma$, $x_2 \Vdash_{\mathfrak{bs}} \Gamma$, $x_3 \Vdash_{\mathfrak{bs}} \Pi$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$. By induction hypothesis, $\Gamma \Rightarrow \phi$ is true on $\mathbb{M}$. By Proposition 8.3.10, we obtain that $x_2 \Vdash \phi$, hence $x \Vdash \Sigma \circ \phi \circ \Pi$. Again, by induction hypothesis, $\mathbb{M} \Vdash \Sigma \circ \phi \circ \Pi \Rightarrow \varphi$, which concludes $x \leq y$.

(tw): For arbitrary $x \in X$ and $y \in Y$, let $\mathbb{M}, x \Vdash_{\mathfrak{bs}} \Gamma \circ \mathbf{t} \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash_{\mathfrak{bs}} \mathbf{t}$, $x_3 \Vdash_{\mathfrak{bs}} \Delta$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$. Because $x_2 \in O_X$ and $R^\circ(x_2, x_3, x')$, we obtain $x' \leq x_3$ by R-order. By Hereditary (Proposition 8.3.9), we also have $x' \Vdash \Delta$, hence $x \Vdash \Gamma \circ \Delta$ holds. Finally, by induction hypothesis, $\mathbb{M} \Vdash \Gamma \circ \Delta \Rightarrow \varphi$. Therefore, $x \leq y$.

($\vee \Rightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ (\phi \vee \psi) \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \circ \phi \circ \Delta \Rightarrow \varphi$ and $\mathbb{M} \Vdash \Gamma \circ \psi \circ \Delta \Rightarrow \varphi$. So, we obtain $y \Vdash \Gamma \circ \phi \circ \Delta$ and $y \Vdash \Gamma \circ \psi \circ \Delta$. With repeating Definition 8.4.1 and Lemma 8.4.2, we obtain the following:

$y \Vdash \Gamma \circ \phi \circ \Delta \iff \forall y', y_2 \in Y, \forall x_1, x_3 \in X.$

$$x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \phi,$$

$$y \Vdash \Gamma \circ \psi \circ \Delta \iff \forall y', y_2 \in Y, \forall x_1, x_3 \in X.$$

$$x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \psi,$$

$$y \Vdash \Gamma \circ (\phi \vee \psi) \circ \Delta \iff \forall y', y_2 \in Y, \forall x_1, x_3 \in X.$$

$$x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \phi \vee \psi.$$

Therefore, we obtain $y \Vdash \Gamma \circ (\phi \vee \psi) \circ \Delta$, hence $x \leq y$.

($\Rrightarrow \vee_1$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash \phi \vee \psi$. By definition, $y \Vdash \phi$. By induction hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rrightarrow \phi$, hence $x \leq y$.

($\wedge_1 \Rrightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ (\phi \wedge \psi) \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash \phi \wedge \psi$, $x_3 \Vdash_{\mathfrak{bs}} \Delta$, $R^{\circ}(x_1, x', x)$ and $R^{\circ}(x_2, x_3, x')$. By definition, we also have $x_2 \Vdash \phi$, hence $x \Vdash \Gamma \circ \phi \circ \Delta$. By induction hypothesis, $\mathbb{M} \Vdash \Gamma \circ \phi \circ \Delta \Rrightarrow \varphi$, hence $x \leq y$.

($\Rrightarrow \wedge$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash_{\mathfrak{bs}} \phi \wedge \psi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rrightarrow \phi$ and $\mathbb{M} \Vdash \Gamma \Rrightarrow \psi$. Therefore, we obtain that $x \Vdash \phi$ and $x \Vdash \psi$, which derives $x \Vdash \phi \wedge \psi$. Then, $x \leq y$.

($\Rrightarrow \circ$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ \Sigma$ and $y \Vdash \phi \circ \psi$. Then, there exist $x_1, x_2 \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash_{\mathfrak{bs}} \Sigma$ and $R^{\circ}(x_1, x_2, x)$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rrightarrow \phi$ and $\mathbb{M} \Vdash \Sigma \Rrightarrow \psi$. We obtain $x_1 \Vdash \phi$ and $x_2 \Vdash \psi$. By definition, since $y \Vdash \phi \circ \psi$, $R(x_1, x_2, y)$ holds. Because of Definition 8.4.1, we conclude $x \leq y$.

($\rightarrow \Rrightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Sigma \circ \Gamma \circ (\phi \to \psi) \circ \Pi$ and $y \Vdash_{\mathfrak{bs}} \varphi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Sigma \circ \psi \circ \Pi \Rrightarrow \varphi$, hence $y \Vdash \Sigma \circ \psi \circ \Pi$. Moreover, there exist $x_1, x_2, x_3, x_4, x', x'' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Sigma$, $x_2 \Vdash_{\mathfrak{bs}} \Gamma$,

$x_3 \Vdash_{\mathfrak{bs}} \phi \to \psi$, $x_4 \Vdash_{\mathfrak{bs}} \Pi$, $R^\circ(x_2, x_3, x')$, $R^\circ(x', x_4, x'')$ and $R^\circ(x_1, x'', x)$. By inductive hypothesis, $\mathbb{M} \Vdash \Gamma \Rightarrow \phi$, hence $x_2 \Vdash \phi$. Furthermore, because $x_2 \Vdash \phi$ and $x_3 \Vdash \phi \to \psi$, we have that, for each $x''' \in X$, if $R^\circ(x_2, x_3, x''')$ holds, then $x''' \Vdash \psi$ (Theorem 8.4.4). Because of $R^\circ(x_2, x_3, x')$, we obtain $x' \Vdash \psi$. Hence, we derive $x \Vdash \Sigma \circ \psi \circ \Pi$. Therefore, $x \leq y$.

($\Rightarrow\leftarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash_{\mathfrak{bs}} \psi \leftarrow \phi$. Then, there exist $x_2 \in X$ and $y' \in Y$ such that $x_2 \Vdash_{\mathfrak{bs}} \phi$, $y' \Vdash_{\mathfrak{bs}} \psi$ and $R^{\leftarrow}(y, x_2, y')$. By induction hypothesis, we have $\mathbb{M} \Vdash \Gamma \circ \phi \Rightarrow \psi$, hence $y' \Vdash \Gamma \circ \phi$. By Theorem 8.4.4, for every $y'' \in Y$, if $R^{\leftarrow}(y'', x_2, y')$, then $y'' \Vdash \Gamma$. Finally, since $x \Vdash \Gamma$, we conclude $x \leq y$.

□

Remark 8.6.2. We mention that, in the proof of Theorem 8.6.1, we effectively use the bi-approximation, bases and the existential quantifier, i.e. Theorem 8.4.4, Theorem 8.4.6 and Proposition 8.4.7, to *stay away from taking the Galois connection.*

Theorem 8.6.3 (Completeness)**.** *Let $\Gamma \Rightarrow \varphi$ be an arbitrary sequent. If the sequent $\Gamma \Rightarrow \varphi$ is valid on every p-frame $\mathbb{F}$, then it is derivable in FL.*

Proof. Let $\mathbb{L}$ be Lindenbaum-Tarski algebra of substructural logic **FL**. If $\Gamma \Rightarrow \varphi$ is not derivable in FL, then $\Gamma \Rightarrow \varphi$ is not valid on $\mathbb{L}$. By Theorem 8.5.4, the dual frame $\mathbb{L}_+$ of $\mathbb{L}$ is a p-frame. Furthermore, by theorem 8.5.5, the sequent $\Gamma \Rightarrow \varphi$ is not valid on $\mathbb{L}_+$. □

Therefore, combined with the canonicity results for substructural logic in Section 4.1, we obtain the following.

Main Theorem 8.6.4 (Sahlqvist-type completeness for substructural logic)**.** *Let Ω be a set of sequents which have consistent variable occurrence (see Main Theorem 3.3.22 and Section 4.1). A substructural logic extended by Ω is complete with respect to a class of p-frames.*

8.7 Conclusive remarks on bi-approximation semantics

We introduced bi-approximation semantics to describe Ghilardi and Meloni's parallel computation on the intermediate level of the canonical extension of lattice expansions. Unlike what happens in the setting of standard relational semantics, like Kripke semantics or Routley-Meyer semantics, bi-approximation semantics is two-sorted. However, we claim that this is a natural framework for the study of logic, because logic is *a priori* two-sorted: premises and conclusions. In other words, logic is the study of consequence relations.

From this point of view, bi-approximation semantics is a reasonable relational-type semantics for lattice-based logics. This framework could be valuable when we think about resource sensitive logics, since there we explicitly distinguish premises from conclusions. Even over distributive lattice-based logics like intuitionistic logic, our two-sorted semantics may be worthwhile. For example, the first-order definability for intuitionistic modal logic on Kripke semantics is still open (see the footnote in [33, p.2]), whereas the first-order definability on bi-approximation semantics is effectively solved [80].

Chapter 9

Summary

How can we present logical reasonings, and compute them? In this dissertation, we have considered ordered algebraic structures, i.e. lattice expansions and poset expansions, as mathematical presentations of logical reasonings, and studied the canonical representation theorems of them, which provide the right framework between logical calculi and space-based semantics, e.g. operational semantics and denotational semantics in computer science. As a summary, we list the results in this dissertation and possible future work in this direction.

9.1 The results in this dissertation

In this dissertation, we have

Main Theorem 3.3.22: generalised

- generalised Ghilardi and Meloni's canonicity methodology from Heyting algebra with unary modalities to lattice expansions: see **Main Theorem 3.3.22**,

- applied our canonicity method to non-classical logics: substructural logic, relevant modal logics and distributive modal logic: see **Chapter 4**,

- compared our canonicity results with a Sahlqvist theorem for distributive modal logic in [30]: see **Section 4.4**,
- brought up a problem to extend our canonicity technique for poset expansions: see **Section 5.3**,
- discussed a way to carefully remove the problematic cases and proved canonicity results for poset expansions: see **Section 5.4**,
- described the canonical extension of poset expansions consisting of $\epsilon_\perp$-additive operations, $\epsilon^\top$-multiplicative operations, diamonds $\Diamond$, boxes $\Box$, adjoint pairs and constant, and the canonical extension of bounded poset expansions consisting of ϵ-join preserving operations, ϵ-meet preserving operations, ϵ-additive operations, ϵ-multiplicative operations, adjoint pairs and constants: see **Section 5.5**, and syntactically characterised canonical inequalities: see **Main Theorem 5.5.25**,
- applied the canonicity results for poset expansions to residuated algebras in [18] and illustrated that our results still account for many canonical inequalities: see **Chapter 6**,
- spelled out the canonical extension of posets with the topological terms provided in [29] and the descendants [27] or [18]: see **Section 7.1**,
- explained Ghilardi and Meloni's parallel computation in the light of the characterisation with topological terms, and illustrated how to use the parallel computation on the intermediate level to obtain canonicity results by giving

an example from substructural logic: see **Section 7.2**,

- given another perspective of the canonicity problem and the canonical extension as "the estimation of the perfect information from the observable data," and shown a *Unschärferelation* which says "even if we accumulate the observable data infinitely many times, it may not enough to perfectly describe a property, but the observable data of the property distribute in a certain range,": see **Section 7.3**

- introduced *bi-approximation semantics* for substructural logic based on the parallel computation on the intermediate level: see **Section 8.3**,

- tracked down the connection between bi-approximation semantics and Kripke-style semantics through bi-approximation, bases and the existential quantifier: see **Section 8.4**,

- proved a representation theorem between FL-algebras and bi-approximation semantics: see **Section 8.5**, and shown that a soundness theorem and a completeness theorem for substructural logic **FL** via the representation theorem plus invariance of validity along a back-and-force correspondences: see **Section 8.6**,

- stated a completeness theorem for extensions of **FL** by combining with our canonicity results in Main Theorem 3.3.22: see **Main Theorem 8.6.3**.

9.2 Future work

In this dissertation, we have studied quite a few canonicity results of lattice expansions and poset expansions by generalising Ghilardi and Meloni's canonicity methodology. Moreover, we have also thought about a space-based semantics of lattice-based logics, and shown that our canonical representation theorems consistently provide the right framework between logical calculi and the space-based semantics, e.g. operational semantics and denotational semantics in computer science. However, we must accept that the study of canonical representations, or duality theories, is still under development. This is because, in this dissertation, we are dealing only with propositional lattice-based (poset-based) logics. But, this is not enough to capture logics applied to computer science, natural science, economics or law. Towards a universal study of these logics via representations, or duality theories, we list some possible future works as follows.

1. Can we syntactically describe a wider class of canonical inequalities?

2. How can we explain the notion of the bi-approximation in logical reasonings?

3. Can we have a universal representation theorem subsuming existing lattice-based predicate logics?

4. Can we apply our representation technique to other topics in computer science, like program logics, automata theory or language theory, or more widely to natural science, e.g. quantum mechanics or bioinformatics?

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