

# **Fiscal Policy in Models of Economic Growth**

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by

**Massimo Antonini**

Department of Economics

University of Leicester

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# Abstract

This thesis analyses fiscal policy in four models of economic growth. The first model is a variant of Jones [61]; overlapping generations are introduced and it is shown that the allocation is dynamically inefficient. As in Diamond [42], a debt financed transfer to current generations can lead to a Pareto improvement; interestingly, the improvement is achieved not by discouraging capital accumulation but through a reallocation of labour between sectors. The second is a two-sector model of growth with public capital. It is shown that perpetual fiscal deficit cannot be sustained. The first best allocation is examined and for the log-utility case an explicit solution can be found. Implementation of the optimal allocation is discussed. The third model features disembodied technological progress as in Solow [100], but it is assumed dependent on public investment. Conditions under which perpetual deficits are sustainable are discussed. The fourth and last model introduces excludable and congestible public services. The optimal fiscal policy, including optimal user charges, is studied. It is shown that in the long-run the optimal income tax is zero and that revenues from user charges is more than sufficient to finance public investment in infrastructures.

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# Introduction

This thesis is a contribution to the study of fiscal policy in dynamic general equilibrium models of growing economies. The seminal work of Solow [100] and Ramsey [87] has inspired a large literature on the determinants of economic growth and the analysis of optimal intertemporal policies. Cass [28] and Koopmans [71] are two crucial contributions on the latter. Understanding the determinants of growth is of obvious importance: if sustained for decades, even small differences in growth rates lead to large differences in output levels. Unsurprisingly, many economists have written on the determinants of economic growth and on how policies impact on growth rates. This literature has received an additional impetus with the emergence of the endogenous growth models of Romer [92], [93], Lucas [74], Grossman and Helpman [59] and Aghion and Howitt [2] just to name a few.

While all of these models are rather stylised descriptions of the evolution of any real economy, their relative simplicity allow us to focus on some important principles that are important to help the design of good policies. We believe that this type of analysis can be of practical interest to policy makers. The planning of fiscal policy or the designing of fiscal rules such as the Growth and Stability pact in the Eurozone and the Fiscal Framework in the UK require an understanding of the concepts of fiscal sustainability and intergenerational fairness. The study of aggregate but rigorously microfounded models of economic growth can help significantly this discussion by taking to the fore the interactions between decisions of governments and private sector agents over time.

This analysis requires a detailed description of savings and investment decisions. Two main approaches are used in the literature: infinitely-lived agents and overlapping generations. An example of a thorough analysis of the former is Brock and Turnovsky [23] while Diamond [42] is the seminal reference for the latter. Blanchard [21] suggested an overlapping generations model of "perpetual youth" that has the advantage of tractability and admits the infinitely lived agents framework as a special case.

Some of the main themes highlighted by this literature are dynamic efficiency, sustainability of fiscal deficits, and optimal investment and taxation policies. These three themes are very closely related. Both dynamic efficiency and the sustainability of perpetual fiscal deficit depend on the relative magnitude of the rate of growth to the interest rate (See Cass [29], [30] on dynamic efficiency; most advanced macroeconomics texts -e.g. Turnovsky [106]- on sustainability). It has been shown that when an economy is dynamically inefficient, fiscal deficits are feasible and welfare improving (Tirole [104]), although the converse is not necessarily true (Grossman and Yanagawa [60]). On the other hand, an economy that invests optimally cannot suffer from dynamic inefficiency.

The issue of dynamic efficiency has been investigated extensively in models of exogenous growth and in fully endogenous growth models. To the best of our knowledge, however, not in the semi-endogenous growth model of Jones [61]. This is an important omission because, as pointed out by Solow [101] amongst others, endogenous growth models rely on a "knife-edge" assumption that may be unwarranted. In the first chapter we fill this gap in the literature by analysing a model of growth through R&D that is very close to that of Jones except for the assumption of overlapping genera-

tions *à la* Blanchard [21] instead of the infinitely-lived agent framework. We investigate conditions under which fiscal policy can lead to a Pareto superior allocation.

However, the uncoordinated actions of economic agents is ensured to achieve an efficient allocation of an existing stock of resources only under restrictive assumptions; for example any form of externality and public goods may imply inefficiency. The presence of these market failures justifies government intervention beyond the strictly intergenerational redistributive policies justified by dynamic inefficiency. It becomes then important to develop models that may inform the discussion on the main principles that these policies should follow, without loosing sight of the dynamic aspects of the problem. In this sense the literature on intertemporal optimal taxation pioneered by Chamley [34] and Judd [66] is the offspring of the literature on static optimal taxation pioneered by Ramsey [86] and Diamond and Mirrlees [43], [44].

In the second chapter we introduce government spending on public goods. We analyse a two-sector growth model with overlapping generations *à la* Samuelson-Diamond (Samuelson [97], Diamond [42]). On the production side the model is an extension of the classic two-sector model of Uzawa [108]. In our version, firms benefit from public services produced using infrastructures generated by the capital sector. We investigate the sustainability of fiscal policy and we characterise the optimal intertemporal fiscal policy.

The main theme of the third chapter is again fiscal policy sustainability in a framework that analyses public investment explicitly. We present a one-sector model where the rate of growth depends on accumulation of public infrastructures. We discuss conditions under which the rate of economic growth may exceed the rate of interest, which allows the government to run Ponzi-finance of public investment.

The fourth chapter presents an analysis of public investment and optimal fiscal policy in an intertemporal model with infinitely-lived agents. The main novelty is the assumption that public services are excludable, which allows for an explicit analysis of user fees. Some important properties of the optimal fiscal mix are discussed.

# 1 Semi-endogenous growth with finite lifetimes

There is a vast literature that analyses the effects of different fiscal policies in growing economies. One important question concerns dynamic efficiency. An economy is dynamically inefficient if, given the way available resources are allocated in a given period, consumption can be increased in all period by modifying the *intertemporal* allocation of resources.<sup>1</sup> Cass [29],[30] showed that in a competitive economy, if an equilibrium path is inefficient, asymptotically the rate of interest is below the growth rate. It is well known that the Ramsey-Cass-Koopmans neoclassical growth model is dynamically efficient (Shell [99], Blanchard and Fischer [22]), but when the model is modified to allow an overlapping generations structure, dynamic inefficiency becomes a possibility (Diamond [42], Blanchard [21], Cremers [37]).

In the one-sector neoclassical model with overlapping generations, it is shown that dynamic inefficiency arises because the saving that households make for life-cycle reasons may lead to an overaccumulation of capital. That is, the capital stock is so large that the marginal product of capital is depressed to the point that it is below the investment required for maintaining the marginal unit of capital; in this situation, were the economy to save less, consumption in all periods could actually increase. This is fundamentally a coordination problem. If all agents could coordinate their actions, they would agree to save less; but in a competitive economy each individual responds to the

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<sup>1</sup> For many authors, static efficiency is a pre-requisite for dynamic efficiency. It seems to us that the two sources of inefficiency are in general distinct and that even after having recognised the existence of a static distortion the discussion of whether overaccumulation occurs is still of interest. In our model we need to keep the two distinct as the monopolistic structure of the goods market implies that the static allocation of resources is inefficient.

incentives given by prices, and although from the social point of view there may be overaccumulation, each households is optimising given the prices it faces.<sup>2</sup>

Once it is recognised that the economy may be dynamically inefficient, the next natural question is whether fiscal policy may be used to improve the allocation. It is known that with overlapping generations -whether with finite or infinite horizons- Ricardian equivalence may fail, and government bonds are considered net wealth (Barro [11], Weil [109], Buiter [25]). Under these circumstances, fiscal policy can indeed improve the intertemporal allocation of resources. One possible way of doing so is to finance lump-sum transfers to generations currently alive, with issuing of bonds that are constantly rolled over. This "Ponzi finance" is made feasible by the overlapping demographic structure (O'Connell and Zeldes [81]). These transfers will make each generation feel wealthier and induce them to consume more, resolving the coordination problem. The scheme will induce a smaller accumulation of capital, increase the marginal product of capital and hence the interest rate. As long as the interest rate is not pushed above the growth rate asymptotically, the scheme will be feasible (Tirole [104]).

With the explosion of the literature on *endogenous growth* in the 80s/90s, many of the old questions analysed in the older economic growth literature have been re-examined, including the dynamic efficiency. Saint-Paul [96] and Grossman and Yanagawa [60] show that the one-sector model with a *learning-by-doing* externality à la Romer [92] and overlapping generations is dynamically efficient. In fact because of the externality, the interest rate does not reflect the social marginal product of capital, and if anything the problem is one of capital *under-* not *over-*accumulation. While Ponzi

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<sup>2</sup> A solution to this coordination problem through Coasian bargaining is clearly not feasible, as it would not only involve bargain between literally an infinity of individuals, but even between individuals who can never meet because their lives do not overlap.

finance is still feasible, it cannot be Pareto improving. In fact public debt would reduce savings and permanently reduce the growth rate, thus harming future generations. King and Ferguson [69] clarified that the result that the economy is always dynamically efficient, depends on the assumption that there is only one type of capital. With several capital goods and externalities, the economy may be dynamically inefficient, but the problem is not the *scale* of the capital stock, but its *composition*. It does remain true that Ponzi finance, as long as it affects the scale but not the composition, cannot be Pareto improving. Khan, Lim and Rhee [68] analyse an endogenous growth model with human capital à la Lucas [74] and overlapping generations, and also find that dynamic inefficiency is possible; but again the problem is one of the capital mix (too little human capital is accumulated), and Ponzi finance only makes matters worse by permanently decreasing the growth rate.

Another strand of models endogenises the growth rate by explicitly incorporating R&D activities. These models typically assume imperfect competition, and hence the equilibrium is not even statically efficient. However, it is still interesting to ask whether, *given* the way existing resources are allocated by the market *within* any given period, an *intertemporal* reallocation may be Pareto improving. In early versions of the model (Romer [93], Grossman and Helpman [59], Aghion and Howit [2]), the decentralised economy tends to underinvest in R&D, and therefore one would expect the problem to be again one of *underaccumulation*, and Ponzi finance to be counterproductive. Olivier [82] introduces overlapping generations in a R&D model à la Romer [93] and confirms this intuition. Interestingly, although the result is similar to that found by Saint-Paul [96], Grossman and Yanagawa [60] and King and Ferguson [69], the mechanism is different. In the former class of models, public debt crowds out physical capital, without

affecting the interest rate; in the latter instead, as agents feel wealthier, they spend more, which increases the interest rate. As the interest rate increases future profits are discounted more heavily, which discourages the allocation of labour to the R&D sector thus permanently reducing the growth rate.<sup>3</sup>

Crucial for Olivier's result is the fact that the long-run growth rate depends on the share of labour allocated to R&D activity. As noted by Jones [61], this is a result that depends crucially on *knife-edge* assumptions on the R&D technology, and at odds with empirical observations. Once diminishing social returns to labour in R&D are allowed, the long-run growth rate is independent from the share of labour allocated to that sector. In this model, therefore, Ponzi finance cannot permanently depress the growth rate, and one wonders whether it may be Pareto improving. In this chapter we confirm that this may indeed be the case. We modify the model of Jones by introducing overlapping generations. We show that long-run equilibria exist that are characterised by the interest rate being asymptotically below the growth rate. We show that Ponzi finance can be Pareto improving. Our overlapping generations framework is identical to Olivier's, therefore any difference in results must be driven by the different assumptions on technology. We argue that the intuition is a mixture of the observations made above. In this model there are two factors that are accumulated, physical capital and knowledge. It is the mix of these two factors, rather than just the scale, that can be inefficient. The mechanism through which Ponzi finance may improve the allocation is similar to that in Olivier's model. Here as there, the effect is through the increase in the interest rate, which encourages a reallocation of labour from the R&D sector to the consumption sector. Only while in the Olivier-Romer model allocation to R&D is always insufficiently

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<sup>3</sup> Interestingly, Olivier finds that a bubble on equity will have the opposite effect by encouraging R&D activity.

low, in the Jones model it can be too high. When this is the case the issuing of debt ameliorates the physical capital-knowledge mix, without affecting long-run growth, and can result in a Pareto improvement.

The next section describes the structure of the model. Section 1.2 shows that the equilibrium is locally determined. Section 1.3 introduces fiscal policy and shows that debt financed transfers may increase aggregate consumption in all periods and thus be Pareto improving. As the dynamics of the system are rather complicated, we resort to numerical simulation to establish this result. Section 1.5 summarises and concludes.

## 1.1 Model

We consider an economy that produces a single consumption good, which is taken as the numeraire. Agents in this economy are overlapping generations of households (dynasties), firms and a government.

### 1.1.1 Households

The demographic structure is as in Blanchard [21] and Buiter [25]. Each generation is composed of a continuum of households. At any time  $t$ ,  $\beta > 0$  new households are born (formed). Each household faces a constant probability of death (breaking)  $p > 0$ . We assume  $\beta > p$ , that is positive population growth. Each household only cares about its own consumption, and supply labour inelastically. Preferences of a household born at time  $s$  are represented by

$$E_t \left[ \int_t^{+\infty} u(c(s, v)) e^{\theta(t-v)} dv \right], \quad \theta > 0,$$

where  $\theta$  is the discount factor;  $c(s, v)$  indicates consumption at time  $v$  of a household born at time  $s$ . The term in square brackets captures the idea that *as long as it survives* the household enjoys a flow of utility from consumption  $u(c(s, v))$ ; the parameter  $\theta$  measures how future utility is discounted, that is how impatient the household is. The expectation sign is needed because the household does not know for how long it will survive; in other words there is a chance that consumption planned in the future will not be enjoyed because the household disappears in the meantime. The utility function is assumed to be logarithmic

$$u(c(s, v)) = \ln c(s, v).$$

Over time, households accumulate financial assets. Uncertain lifetimes implies that households may leave unintended bequests. However, although lifetimes are uncertain from the point of view of the single households, on aggregate there is no uncertainty on the number of deaths and births. It is then possible and convenient to assume that there is a competitive insurance market. It is easy to show that households will find it convenient to stipulate the following contract: the insurance company pays the household  $p$  units of the consumption good at time  $t$ , to receive one unit of a financial asset at the time of the household's death.<sup>4</sup> To avoid unintended bequests, households will contract all of their wealth.

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<sup>4</sup> By stipulating this contract, the household receives  $p$  in addition to  $r(t)$  in each period until death.

Given the assumptions listed above, at time  $t$ , maximisation of the expected utility is equivalent to maximisation of<sup>5</sup>

$$\int_t^{+\infty} \ln c(s, t) e^{(\theta+p)(t-v)} dv. \quad (1)$$

The household's budget constraint is

$$\dot{a}(s, t) = (r(t) + p) a(s, t) + l(s, t) w(t) - c(s, t). \quad (2)$$

We add a condition, known in the literature as *no Ponzi games condition*,<sup>6,7</sup> that constrains the growth of household's debt to be below the interest rate asymptotically:

$$\lim_{v \rightarrow +\infty} e^{-\int_t^v (r(\mu) + p) d\mu} a(s, t) \geq 0.$$

where  $c(s, t)$  is consumption,  $l(s, t)$  is the endowment of labour,  $a(s, t)$  the stock of financial assets owned,  $r(t)$  the real rate of return,  $w(t)$  the real wage per unit of labour supplied. We assume that the endowment of labour decreases with age, according to  $l(s, t) = l_0 e^{-\varepsilon(t-s)}$ .

Call  $N(t)$  the mass of households alive at time  $t$ . At any time  $t$ ,  $\beta N(t)$  new households are born. Of the households born at time  $v$ ,  $\beta N(v) e^{-p(t-v)}$  are still alive at time  $t$ . Population grows at the rate  $\beta - p$ . Normalising  $N(0) = 1$ ,  $N(t) = e^{(\beta-p)t}$ .

---

<sup>5</sup> The probability of being alive at time  $v$ , conditional on being alive at time  $t$ , is  $e^{-p(v-t)}$ . Then  $E_t \left[ \int_t^{+\infty} u(c(s, v)) e^{\theta(t-v)} dv \right]$   
 $= \int_t^{+\infty} u(c(s, v)) e^{\theta(t-v)} e^{-p(v-t)} dv$   
 $= \int_t^{+\infty} u(c(s, v)) e^{(\theta+p)(t-v)} dv.$

<sup>6</sup> Without this condition, the household would borrow as much as possible each period and use these funds to finance consumption and interest payments on previous borrowing, without ever repaying neither the principal nor interest. The condition takes its name from Charles Ponzi, who made a quick fortune in the 1920s by using loans from new lenders to repay old lenders (Blanchard and Fischer [22]).

<sup>7</sup> This is not innocuous; when a government can run a Ponzi scheme, a private agent could in principle initiate one too. We follow much of the literature and disregard equilibria with households running such schemes.

The total labour supply at time  $t$  is

$$\begin{aligned} L(t) &= \int_{-\infty}^t l(s, t) \beta N(s) e^{-p(t-s)} ds \\ &= \int_{-\infty}^t [l_0 e^{\varepsilon(s-t)}] [\beta e^{(\beta-p)s}] e^{-p(t-s)} ds \\ &= \frac{l_0 \beta}{\beta + \varepsilon} e^{(\beta-p)t}. \end{aligned}$$

Thus  $\varepsilon$  affects the level but not the growth rate of the aggregate labour endowment.

Maximisation of (1) s.t. (2) implies<sup>8</sup>

$$c(s, t) = (\theta + p) (a(s, t) + h(s, t)), \quad (3)$$

where

$$h(s, t) = \int_t^{+\infty} l(s, v) w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv. \quad (4)$$

is the human wealth of a household born at  $s$ . We can rewrite the expression for human wealth as

$$\begin{aligned} h(s, t) &= \int_t^{+\infty} l_0 e^{\varepsilon(s-v)} w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \\ &= l_0 e^{\varepsilon s} \int_t^{+\infty} e^{-\varepsilon v} w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \end{aligned}$$

Note that the value of the last integral is identical for all households, independently of their date of birth.

For any individual variable  $x(s, t)$ , the corresponding aggregate is

$$X(t) = \beta e^{-pt} \int_{-\infty}^t x(s, t) e^{\beta s} ds.$$

---

<sup>8</sup> Furthermore the following transversality condition must hold:

$$\lim_{t \rightarrow +\infty} e^{-(\theta+p)t} \frac{a(s, t)}{c(s, t)} = 0.$$

Then we can write<sup>9</sup> the aggregate human wealth as

$$\begin{aligned}
H(t) &= \beta e^{-pt} \left[ \int_t^{+\infty} e^{-\varepsilon v} w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \right] l_0 \int_{-\infty}^t e^{\varepsilon s} e^{\beta s} ds \\
&= \beta e^{-pt} \left[ \int_t^{+\infty} e^{-\varepsilon v} w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \right] l_0 \int_{-\infty}^t e^{(\beta+\varepsilon)s} ds \\
&= \left[ \int_t^{+\infty} e^{-\varepsilon v} w(v) e^{-\int_t^v (r(\mu)+p)d\mu} dv \right] l_0 \frac{\beta}{\beta+\varepsilon} e^{(\beta-p+\varepsilon)t} \\
&= \left[ \int_t^{+\infty} w(v) e^{-\int_t^v (r(\mu)+p+\varepsilon)d\mu} dv \right] L(t).
\end{aligned} \tag{5}$$

Differentiating (5) with respect to  $t$  we have

$$\dot{H} = (\beta + \varepsilon + r(t)) H(t) - w(t) L(t). \tag{6}$$

Aggregating, (3) and (2) can be written

$$C(t) = (\theta + p)(A(t) + H(t)), \tag{7}$$

and

$$\dot{A}(t) = r(t) A(t) + w(t) L(t) - C(t), \tag{8}$$

respectively. The equations (6)-(8) describe the aggregate behaviour of the households.

Differentiating (7) and eliminating  $H(t)$ , we can write more compactly:

$$\dot{C}(t) = (r(t) - \theta + \beta - p + \varepsilon) C(t) - (\theta + p)(\beta + \varepsilon) A(t). \tag{9}$$

## 1.1.2 Corporate sector

The production side of the economy is very similar to Jones [61]. The final good, which is taken as the numeraire, is produced with labour and a combination of intermediate goods according to

$$Y(t) = L_y(t)^{1-\alpha} \int_0^{m(t)} x_i(t)^\alpha di, \tag{10}$$

---

<sup>9</sup> It will be verified later that the last intergral converges. When  $H(t)$  converges,  $h(s, t) < \infty \forall s < t$ .

where  $L_y(t)$  is the fraction of the aggregate labour endowment allocated to the production of final goods,  $x_i(t)$  is the quantity of intermediate good  $i$  used and  $m(t)$  is the mass of available intermediate goods. Intermediate goods depreciate completely with production.<sup>10</sup>

The production function (10) tries to capture the distinction between accumulation of ideas and accumulation of objects (Romer [94]). In traditional growth theory, all capital goods are aggregated together in a single capital stock,  $K$ . This seems reasonable as long as the new object acquired are similar to those already in stock; but the crucial feature of innovation is the introduction of new processes and activities and new tools to carry them out. Typically technological progress involves a increasing degree of specialisation, made possible by the invention of new techniques and machines. Equation (10) is an attempt to formalise this.<sup>11,12</sup>

A look at (10) confirms that production of final goods is characterised by constant returns to scale; we assume perfect competition in this sector. If  $\nu_i(t)$  is the price of the  $i$ -th intermediate good at time  $t$ , profits will be  $Y(t) - w(t)L_y(t) - \int_0^{m(t)} \nu_i(t)x_i(t)di$  and profit maximisation implies

$$L_y(t) = \frac{(1 - \alpha)Y(t)}{w(t)}, \quad (11)$$

---

<sup>10</sup> Alternatively one could work with the assumption that intermediates are durable goods, at the cost of complicating the algebra slightly. In particular the equilibrium condition  $K(t) = m(t)x(t)$  would become  $K(t) = \dot{m}(t)x(t) + m(t)\dot{x}(t)$ , which is more awkward to work with.

<sup>11</sup> It applies the "love for variety" models, introduced in consumer choice theory by Spence [102] and Dixit and Stiglitz [46], to production. This was done first in a model of international trade by Ethier [51], and in growth models by Romer[93] and Grossman and Helpman [59].

<sup>12</sup> To see how the distinction between accumulation of objects and ideas is incorporated, assume that  $x_i$  is constant for all  $i$  (as it will be in equilibrium); then the production function will become  $Y = mL_y^{1-\alpha}x^\alpha = (mL_y)^{1-\alpha}(mx)^\alpha = (mL_y)^{1-\alpha}K^\alpha$ . Then  $mx$  measures the quantity of objects that the economy has accumulated, while  $m$  is a measure of ideas. Then it becomes clear that growth comes from increasing physical inputs ( $L_y$  and  $K$ ) and ideas ( $m$ ). It is also clear from the equation above that for a given  $K$ ,  $m$  measures labour productivity.

and

$$x_i(t) = \left( \frac{\alpha}{\nu_i(t)} \right)^{\frac{1}{1-\alpha}} L_y(t). \quad (12)$$

Intermediate goods are produced using capital rented from households by firms that have purchased a patent from the R&D sector at a price  $P_m(t)$ . A patent gives the exclusive right to produce a given variety, therefore each producer acts as a monopolist. We assume for simplicity that one unit of capital is needed to produce one unit of an intermediate good and that capital does not depreciate. Calling  $r(t)$  the rental rate at time  $t$ , intermediate goods producers solve

$$\max_{\nu_i} \nu_i(t) x_i(t) - r(t) x_i(t)$$

subject to (12). The inverse demand function for intermediate good  $i$  is  $\nu_i(t) = \alpha L_y(t)^{1-\alpha} x_i(t)^{\alpha-1}$ . Using this, the problem can be written

$$\max_{x_i} \alpha L_y(t)^{1-\alpha} x_i(t)^{\alpha} - r(t) x_i(t),$$

the first order condition gives

$$x_i(t) = x(t) = (\alpha^2/r(t))^{1/(1-\alpha)} L_y(t), \quad (13)$$

and therefore

$$\nu_i(t) = \nu(t) = \frac{r(t)}{\alpha}. \quad (14)$$

Since capital is used only to produce intermediate goods, and in a one-to-one fashion, we must have that the aggregate capital stock satisfies

$$K(t) = \int_0^{m(t)} x_i(t) di = m(t) x(t). \quad (15)$$

Now, since  $x_i(t) = x(t) \forall t$ , using (15)  $Y(t) = L_y(t)^{1-\alpha} m(t) x(t)^{\alpha} = (m(t) L_y(t))^{1-\alpha} K(t)^{\alpha}$ .

Then  $Y(t)/K(t) = (L_y(t)/x(t))^{1-\alpha}$ , and using (13) one obtains

$$r(t) = \alpha^2 \frac{Y(t)}{K(t)}. \quad (16)$$

Calling  $\pi(t)$  the profit for one existing intermediate goods firm (profits are identical for all existing firms),

$$\begin{aligned}
\pi(t) &= \nu(t)x(t) - r(t)x(t) = \left(\frac{1-\alpha}{\alpha}\right)r(t)\left(\frac{\alpha^2}{r(t)}\right)^{\frac{1}{1-\alpha}}L_y(t) \\
&= \left(\frac{1-\alpha}{\alpha}\right)r(t)\left(\frac{\alpha^2}{r(t)}\right)^{\frac{1}{1-\alpha}}x(t)\left(\frac{\alpha^2}{r(t)}\right)^{-\frac{1}{1-\alpha}} \\
&= \left(\frac{1-\alpha}{\alpha}\right)\alpha^2\frac{Y(t)}{K(t)}x(t) = \alpha(1-\alpha)\left(\frac{Y(t)}{K(t)}\right)^{-\frac{\alpha}{1-\alpha}}L_y(t) \\
&= \frac{\alpha(1-\alpha)Y(t)}{m(t)},
\end{aligned} \tag{17}$$

where we used (13) and (14) to derive the third equality.

The last activity to be considered is the invention of new varieties, the R&D sector. As in Jones [61], we assume that the individual firm perceives that the labour required to discover new innovations is given by

$$\dot{m}(t) = \bar{\delta}(t)L_m(t),$$

where  $L_m(t)$  is the amount of labour allocated to R&D activity. However, there is a spillover effect, so that  $\bar{\delta}(t) = \delta L_m(t)^{\psi-1} m(t)^\phi$ , so effectively

$$\dot{m}(t) = \delta L_m(t)^\psi m(t)^\phi. \tag{18}$$

The term  $m(t)^\phi$  is meant to capture the spillover between research activities; in other words, the body of research done previously ( $m(t)$ ), makes the discovery of new ideas easier. The parameter  $\phi$ , measures the strength of this effect; Romer [93] assumes  $\phi = 1$ , which makes growth fully endogenous (but also explosive in a model with population growth). Here we follow Jones [61] and assume  $0 < \phi < 1$ . The term  $L_m(t)^{\psi-1}$  captures the idea that there may be duplications in research activities, so that doubling the economy wide number of researchers may not double the research output.

The parameter  $0 < \psi \leq 1$  measures the strength of this effect. When  $\psi = 1$ , duplication is absent, when  $\psi < 1$  it is present.

In equilibrium, an individual must be indifferent between working in the consumption good sector (getting a wage  $w(t)$  per unit of labour supplied), and devoting their time to research (earning the value of the innovation produced). We must then have<sup>13</sup>

$$w(t) = P_m(t) \bar{\delta}(t) = P_m(t) \delta L_m^{\psi-1} m(t)^\phi. \quad (19)$$

The maximum an intermediate producer will be willing to pay for a patent is the present value of future profits,  $\int_t^{+\infty} \pi(v) e^{-\int_t^v r(\mu) d\mu} dv$ . Competition between potential producers will bid the price of a patent to that level, i.e.

$$P_m(t) = \int_t^{+\infty} \pi(v) e^{-\int_t^v r(\mu) d\mu} dv. \quad (20)$$

The ownership of the patent gives the exclusive right to produce the intermediate good and thus allows the owner to obtain the stream of profit  $\pi(t)$  as long as one retains it. Furthermore the owner of the patent has always the option to sell it. Hence  $P_m(t)$  must obey the standard no arbitrage condition<sup>14</sup>:

$$r(t) = \frac{\dot{P}_m(t)}{P_m(t)} + \frac{\pi(t)}{P_m(t)}, \quad (21)$$

where the left-hand side is the instantaneous return of an alternative investment, the right-hand side is the instantaneous return of a patent, given by the "dividend"  $\pi/P_m$  and the "capital gain"  $\dot{P}_m/P_m$ .

Assuming that capital is produced under perfect competition with a linear technology that transforms one unit of final good into one unit of capital, and that capital

<sup>13</sup> Recall that  $P_m(t)$  is the price of a patent for a newly discovered variety. This equation can equivalently be read as a free entry/zero profit condition for the R&D sector.

<sup>14</sup> Obtained differentiating (20) w.r.t. time.

does not depreciate, market clearing in the final goods sector requires

$$\dot{K}(t) = Y(t) - C(t). \quad (22)$$

The financial market clears if  $A(t) = K(t)$ , so (9) becomes

$$\dot{C}(t) = (r(t) - \theta + \beta - p + \varepsilon) C(t) - (\theta + p)(\beta + \varepsilon)(K(t) + P_m(t)m(t)) \quad (23)$$

The final equilibrium condition is that the labour market clears. This requires the sum of employment in the R&D and final good sector to be equal to total labour supply:

$$L_m(t) + L_y(t) = L(t). \quad (24)$$

## 1.2 Dynamics

Arnold [3] studies the dynamics of the original model by Jones [61] with infinite lifetimes and  $\psi = 1$ . We study the dynamics allowing for finite lifetimes and  $\psi \leq 1$ . We first show that the dynamics of the system can be represented by the evolution of four suitably defined variables that are stationary in a balanced path. The transformed system is shown to always admit a steady-state. A full characterisation of stability is unfortunately elusive, but we show with a combination of numerical exercises and a sufficient condition that saddlepath stability obtains for a non-negligible subset of possible parameter combinations. We then discuss under what circumstances the dynamics of the transformed system determine a path for the original variables that can be sustained as an equilibrium. Finally, we note that equilibria exist in which the growth rate of output exceeds the interest rate.

First we prove that the dynamics of the economy can be described by a system of four differential equations in suitably defined stationary variables. In particular we

observe that the real interest rate  $r(t)$  will be constant in the long run. This suggest choosing  $\rho(t) = (m(t) L_y(t) / K(t))^{1-\alpha} = Y(t) / K(t) = \alpha^2 r(t)$  as one of the stationary variables. Similarly the observation that technological progress will be stationary suggest defining  $\zeta(t) = L(t)^\psi / m(t)^{1-\phi}$  as another stationary variable; note that  $\zeta(t) = \lambda(t)^\psi \dot{m}(t) / m(t)$ . In a balanced growth path aggregate consumption and aggregate physical capital stock will grow at the same rate, which suggests choosing  $u(t) = C(t) / K(t)$ . Looking at (19) one can guess that the ratio between output and the value of production in the R&D sector will converge. Hence we define  $q(t) = (Y(t) / P_m(t) m(t))$ . Finally as the share of labour employed in the consumption sector must be constant in a balanced path, we define  $\lambda(t) = L_y(t) / L(t)$ .

**Proposition 1.1** *Let  $\rho(t) = (m(t) L_y(t) / K(t))^{1-\alpha}$ ,  $\zeta(t) = L(t)^\psi / m(t)^{1-\phi}$ ,  $u(t) = C(t) / K(t)$ ,  $q(t) = (Y(t) / P_m(t) m(t))$ ,  $\lambda(t) = L_y(t) / L(t)$ . Then the dynamics of the economy are described by the following system of differential equations:*

$$\dot{u}(t) = \left[ u(t) - \frac{(\theta + p)(\beta + \varepsilon)}{u(t)} \left( 1 + \frac{\rho(t)}{u(t)} \right) - (1 - \alpha^2) \rho(t) - \theta + \beta - p + \varepsilon \right] u(t), \quad (25)$$

$$\dot{\zeta}(t) = \left[ \psi(\beta - p) - (1 - \phi) \delta (1 - \Lambda(q(t) / \zeta(t)))^\psi \zeta(t) \right] \zeta(t), \quad (26)$$

$$\dot{q}(t) = \left\{ (1 - \alpha)(1 - \psi)(\beta - p) - \alpha u(t) + \alpha(1 - \alpha)(q(t) + \rho(t)) + (1 - \alpha)(1 - \phi) \frac{(1 - \lambda(t))^{\psi+1}}{1 - \psi\lambda(t)} \delta \zeta(t) \right\} \frac{q(t)}{D(t)}, \quad (27)$$

$$\dot{\rho}(t) = \left\{ \frac{(1 - \alpha)(1 - \psi)}{1 - \psi\lambda(t)} (\beta - p) + (\alpha + (1 - \alpha)(1 - \alpha^2) E(t)) \rho(t) - (\alpha + (1 - \alpha) E(t)) u(t) + \alpha(1 - \alpha)^2 E(t) q(t) - \phi(1 - \alpha) \frac{(1 - \lambda(t))^{\psi+1}}{1 - \psi\lambda(t)} \delta \zeta(t) \right\} \frac{\rho(t)}{D(t)}, \quad (28)$$

where

$$E(t) \equiv \frac{1 - \lambda(t)}{1 - \psi \lambda(t)},$$

$$D(t) \equiv 1 - (1 - \alpha) E(t),$$

and  $\Lambda(q(t)/\zeta(t))$  is defined implicitly by

$$\Lambda(q(t)/\zeta(t)) = \frac{(1 - \alpha) q(t)}{\delta \zeta(t)} (1 - \Lambda(q(t)/\zeta(t)))^{1-\psi}.$$

**Proof** See section 1.6. ■

The following corollary shows that the system simplifies somewhat if one assumes  $\psi = 1$ .

**Corollary 1.1** *If  $\psi = 1$ , the system (25)-(28) simplifies to (25) and*

$$\frac{\dot{\zeta}(t)}{\zeta(t)} = (\beta - p) - (1 - \phi) (\delta \zeta(t) - (1 - \alpha) q(t)),$$

$$\begin{aligned} \frac{\dot{q}(t)}{q(t)} &= -u(t) - \left(1 - \frac{(1 - \alpha)(1 - \phi)}{\alpha}\right) \delta \zeta(t) \\ &\quad + ((1 - \alpha)(1 - \phi) + 2\alpha) \frac{(1 - \alpha)}{\alpha} q(t) + (1 - \alpha) \rho(t), \end{aligned}$$

$$\frac{\dot{\rho}(t)}{\rho(t)} = \frac{(1 - \alpha)}{\alpha} [(1 - \phi) \delta \zeta(t) - (1 - \alpha) \phi q(t) + (1 - \alpha) \rho(t)].$$

**Proof** If  $\psi = 1$ , then (38) in section 1.6 simplifies to  $\lambda(t) = [(1 - \alpha) q(t)] / [\delta \zeta(t)]$ .

The expressions in the corollary are obtained substituting the latter and  $\psi = 1$  into (25)-(28). ■

One important issue is the existence of a balanced path for the economy.

**Proposition 1.2** *The system of differential equations (25)-(28) has a unique steady-state with  $u, \zeta, q, \rho > 0, \lambda \in (0, 1)$ .*

**Proof** See section 1.6. ■

Next we show that the converse of proposition 1.2 is also true, all convergent solutions to (25)-(28) are equilibria for the economy.

**Proposition 1.3** *If the solution to the system (25)-(28) converges to a steady state, then the path  $\{Y(t), K(t), C(t), L_y(t), L_m(t), x(t), m(t), w(t), p(t), P_m(t)\}_{t=0}^{+\infty}$  with*

$$\begin{aligned} L_y(t) &= \lambda(t) L(t), \quad L_m(t) = (1 - \lambda(t)) L(t), \quad r(t) = \alpha^2 \rho(t), \\ p(t) &= \alpha \rho(t), \quad x(t) = L_y(t) / \rho(t)^{1/(1-\alpha)}, \quad m(t) = L(t)^{\psi/(1-\phi)} / \zeta(t)^{1/(1-\phi)}, \\ Y(t) &= m(t) L_y(t)^{1-\alpha} x(t)^\alpha, \quad w(t) = (1 - \alpha) m(t) (x(t) / L_y(t))^\alpha, \\ K(t) &= m(t) x(t), \quad C(t) = u(t) K(t), \quad P_m = w(t) L_m(t)^{1-\psi} / (\delta m(t)^\phi). \end{aligned}$$

*constitutes an equilibrium for the economy. In the balanced growth path,*

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \left(1 + \frac{\psi}{1-\phi}\right) (\beta - p), \\ \frac{\dot{m}(t)}{m(t)} &= \frac{\psi}{1-\phi} (\beta - p), \\ \frac{\dot{P}_m(t)}{P_m(t)} &= \frac{\dot{x}(t)}{x(t)} = \beta - p. \end{aligned}$$

**Proof** *The expressions for the various variable are obtained by working back from the definitions of  $u(t), \zeta(t), q(t), \rho(t)$ , and  $\lambda(t)$ . It is left to show that  $H(t) < +\infty$ , this is done in section 1.6. ■*

In the balanced path, aggregate final good production and consumption and the aggregate capital stock all grow at the rate  $(1 + \psi / (1 - \phi)) (\beta - p)$ , in this sense growth is semi-endogenous: the engine of growth is research activity, accumulation of ideas, which is modelled explicitly; but in the long-run growth is limited by population growth, an exogenous variable. Per capita variables grow at the rate  $(\psi / (1 - \phi)) (\beta - p)$ , but one should be careful that when  $\varepsilon > 0$ , per capita and per worker variables are different in levels (although they grow at the same rate). And, of course, in this model asset holding and therefore consumption varies across households, with wealthier older households consuming more than younger ones.

So far we have shown that the dynamics of the economy can be studied by analysing the behaviour of a system of four autonomous differential equations that admits a unique steady state. One would like to establish general conditions under which the system is stable. Unfortunately the expressions for the Jacobian (even in the slightly simpler case  $\psi = 1$ ) prove too complicated to find truly general conditions. We had therefore to resort to numerical computations (as Eicher and Turnovsky [48]) and always found the Jacobian to have two stable and two explosive roots. We generally find the stable roots to be complex, in the rest of the analysis we assume that the parameters are such that this result holds. Of the four variables  $u(t)$ ,  $\zeta(t)$ ,  $q(t)$ , and  $\rho(t)$ , one is a state variable ( $\zeta(t)$ ), and three are jump variables ( $q(t)$ ,  $u(t)$  and  $\rho(t)$ ). However, the market clearing equilibrium  $m(t)x(t) = K(t)$  gives a restriction between jump variables:

$$\frac{\Lambda(q(t)/\zeta(t))}{\rho(t)^{1/(1-\alpha)}} = \frac{K(t)}{m(t)L(t)}. \quad (29)$$

We now show that the equilibrium is locally determined; that is, if the system starts sufficiently close to the balanced path, there is a unique equilibrium path converging there.

Denote the stable eigenvalues of the Jacobian of the system (calculated at the steady state) with  $\mu_1$  and  $\mu_2$ . We focus on the case that we found most common, where  $\mu_j$  are two complex conjugate numbers, say  $\mu \pm \sigma i$ , where  $\mu$  indicates the real part and  $\sigma$  the imaginary part of  $\mu_j$ ,  $j = 1, 2$ . In the proximity of the steady-state, the solution to (25)-(28) converging to the long-run equilibrium is approximated by<sup>15</sup>

$$u(t) - u = B_1 v_{11} e^{\mu_1 t} + B_2 v_{12} e^{\mu_2 t},$$

$$\zeta(t) - \zeta = B_1 v_{21} e^{\mu_1 t} + B_2 v_{22} e^{\mu_2 t},$$

$$q(t) - q = B_1 v_{31} e^{\mu_1 t} + B_2 v_{32} e^{\mu_2 t},$$

$$\rho(t) - \rho = B_1 v_{41} e^{\mu_1 t} + B_2 v_{42} e^{\mu_2 t},$$

where  $[v_{1j}, v_{2j}, v_{3j}, v_{4j}]$  is the (complex) eigenvectors associated with  $\mu_j$ ,  $j = 1, 2$ ;  $B_1$  and  $B_2$  are (complex) constants to be determined. It is computationally more convenient to transform the complex numbers in trigonometric form and rewrite the system as<sup>16</sup>

$$u(t) - u = e^{\mu t} [(D_1 \cos(\sigma t) - D_2 \sin(\sigma t)) v_{11}^r - (D_1 \sin(\sigma t) + D_2 \cos(\sigma t)) v_{11}^{im}],$$

$$\zeta(t) - \zeta = e^{\mu t} [(D_1 \cos(\sigma t) - D_2 \sin(\sigma t)) v_{21}^r - (D_1 \sin(\sigma t) + D_2 \cos(\sigma t)) v_{21}^{im}],$$

$$q(t) - q = e^{\mu t} [(D_1 \cos(\sigma t) - D_2 \sin(\sigma t)) v_{31}^r - (D_1 \sin(\sigma t) + D_2 \cos(\sigma t)) v_{31}^{im}],$$

$$\rho(t) - \rho = e^{\mu t} [(D_1 \cos(\sigma t) - D_2 \sin(\sigma t)) v_{41}^r - (D_1 \sin(\sigma t) + D_2 \cos(\sigma t)) v_{41}^{im}],$$

where for any element of the a eigenvector,  $v_{kj}$ , we indicated the real part with  $v_{kj}^r$  and the imaginary part with  $v_{kj}^{im}$ , and  $D_j = 2B_j$ ,  $j = 1, 2$ . Setting  $t = 0$ , we find two

<sup>15</sup> Using standard linearization methods. See, for example, de la Fuente [39] and Turnovsky [106].

<sup>16</sup> Again, the proof can be found in de la Fuente [39].

conditions that determine the arbitrary constant. The first is given by the fact that the initial value of  $\zeta$  is given:

$$\zeta(0) - \zeta = D_1 v_{21}^r - D_2 v_{21}^{im},$$

The second condition is given by the restriction between jump variables (29) evaluated at time 0:

$$\Lambda(q(0)/\zeta(0))m(0)L(0) = K(0)\rho(0)^{1/(1-\alpha)},$$

where  $q(0)$  and  $\rho(0)$  are functions of  $D_1$  and  $D_2$ . The above is a non-linear system of two equations in two unknowns, so it may have one unique, multiple or no solutions. In the case of no solution one would have to conclude that the steady-state is not stable. In the case of multiple solutions one would have multiple equilibrium paths converging to the steady state. Numerical simulations suggest that for plausible parameter values the system admits a solution.

### Example

Assume  $\alpha = 0.33$ ,  $\phi = 0.6$ ,  $\psi = 1$ ,  $\delta = 1$ ,  $\theta = 0.02$ ,  $\beta = 0.233$ ,  $p = 0.0133$ ,  $\varepsilon = 0.05$ . Then computations show that the eigenvalues of the Jacobian at the steady-state are approximately  $[0.14803942, -0.031984309 + 0.053778338i, -0.031984309 - 0.053778338i, 0.019592400]$ , thus the system is saddlepath stable. Furthermore, on the balanced path  $\dot{C}/C = 0.025$ , and  $r = 0.015$ . Note that these parameters imply a rate of growth of 2.5%, an (riskless) interest rate of 1.5%, a rate of population growth of 1% and life expectancy of about 75 years. Thus, although this choice of parameters is purely for illustration, they do come reasonably close to stylised facts for the US economy.<sup>17</sup> We emphasise, though, that these parameters are chosen

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<sup>17</sup> We emphasise, though, that these parameters are chosen simply to illustrate the possible dynamics in this model (and in the next section the possibility of dynamic inefficiency), we do not claim that there is any strong evidence that these are empirically sound." At least some of the parameters, though, are

simply to illustrate the possible dynamics in this model (and in the next section the possibility of dynamic inefficiency), we do not claim that there is any strong evidence that these are empirically sound.

To illustrate the dynamics along convergence towards the steady state, we assume that at time 0 the stock of ideas is at a level compatible with balanced growth, but the capital stock is 10% lower. We compute the path of the stationary variables  $u(t)$ ,  $\zeta(t)$ ,  $q(t)$  and  $\rho(t)$ , and the implied path for the aggregate variables. Figure 1.1 shows the growth rates for per capita consumption and capital stock and for the stock of ideas,  $m(t)$ . Initially, a larger proportion of the labour force is allocated to the final good sector than in the LR which allows the economy to accumulate capital faster, at the expenses of a lower accumulation of ideas. Over time though, the growth rates of all variables converge to the long-run values, although with damped oscillations.

The careful reader will have noticed that in the example  $\dot{C}/C = \dot{Y}/Y > r$ . This proves the following proposition.

**Proposition 1.4**      *There exist equilibria that converge to a balanced growth path characterised by  $\dot{C}/C = \dot{Y}/Y > r$ .*

While the search for restriction on parameters that ensures the equilibrium is dynamically inefficient has proven fruitless, some intuitive remarks can be offered. In the long run,  $\dot{C}/C = (1 + \psi / (1 - \phi)) (\beta - p)$ . An increase in the rate at which individ-

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reasonable, to explain our choice further we added the following footnote in the same page: "In particular we have chosen a value of  $\alpha = 0.33$  in order to replicate the well known fact that the ratio of wage earning to GDP is approximately  $2/3$ . The values of  $\beta$  and  $p$  are chosen to replicate population growth and life expectancy in the US. The choice of  $\theta = 0.02$  is fairly standard in this type of exercise in the literature. For the remaining parameters we could not find reliable estimates and therefore we chose the values arbitrarily.

ual labour supply declines,  $\varepsilon$ , increase the life cycle motive for saving, which tends to decrease the interest rate, and thus makes it more likely that  $\dot{C}/C > r$ .

As discussed in the introduction, in perfectly competitive models, when asymptotically  $\dot{C}/C > r$ , the economy is *dynamically inefficient* (Cass [29], [30]). In this case, the government can use public debt to achieve a Pareto improvement (Diamond [42], Tirole [104]). It is well known that in AK models, such policies are feasible, but not Pareto improving (King and Ferguson [69], Saint-Paul [96]). In the next section we analyse the effect of introducing debt in the model presented in this chapter.

### 1.3 Fiscal policy

In this section we analyse the effects of a simple type of fiscal policy. Let us assume that the government finances lump-sum transfers to households alive a time  $t$ ,  $T(s, t)$ , by issuing bonds,  $B(t)$ , offering an instantaneous rate of return  $r(t)$ . New bonds issued must equal interest payment on old bonds plus the aggregate transfer. The government budget constraint is

$$\dot{B}(t) = r(t) B + T(t), \quad (30)$$

where the aggregate transfer  $T(t)$  is  $T(t) = \beta e^{-\rho t} \int_{-\infty}^t T(s, t) e^{\beta s} ds$ .

We will assume that the government aims at maintaining a fixed debt to capital ratio<sup>18</sup>:  $B(t)/K(t) = b \geq 0$ . The presence of transfers changes the household's budget constraint to

$$\dot{a}(s, t) = (r(t) + p) a(s, t) + l(s, t) w(t) - c(s, t) + T(s, t),$$

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<sup>18</sup> We choose this policy because it is consistent with a balanced growth path and it is relatively simple to analyse.

and the solution still implies (3) provided that households' human wealth  $h(s, t)$  is redefined as

$$h(s, t) = \int_t^{+\infty} [l(s, v) w(v) + T(s, v)] e^{-\int_t^v [r(\mu) + p] d\mu} dv$$

to include the present value of government transfers. Repeating the same steps as in section 1.1.1, one establishes that (9) holds, whereas (8) becomes

$$\dot{A}(t) = r(t) A(t) + w(t) L(t) + T(t) - C(t). \quad (31)$$

The transfer  $T(s, t)$  has the effect of increasing an individual's human wealth. However, since all individuals have the same marginal propensity to consume out of wealth,  $(\theta + p)$ , the way the aggregate transfer is distributed has no consequence for the evolution of aggregate variables, although it matters for welfare.

There are now two types of assets in our economy, capital stock,  $K$  and public bonds  $B$ . The financial market clears if  $A(t) = K(t) + B(t)$ , hence we can write (9) as

$$\frac{\dot{C}(t)}{C(t)} = (r(t) - \theta + \beta - p + \varepsilon) - (\theta + p)(\beta + \varepsilon) \left( \frac{K(t)}{C(t)} + \frac{B(t)}{C(t)} \right).$$

Just repeating the same line of proof as for proposition 1.1 and lemma 1.2, one obtains the following results.

**Proposition 1.5** *The dynamics of the economy are described by the of differential equations given by*

$$\dot{u}(t) = \left[ \alpha^2 \rho(t) - \theta + \beta - p + \varepsilon - \frac{(\theta + p)(\beta + \varepsilon)}{u(t)} \left( 1 + b + \frac{\rho(t)}{q(t)} \right) - \rho(t) + u(t) \right] u(t), \quad (32)$$

and (26)-(28), where the variables are defined as in proposition 1.1.

**Proof** See section 1.6. ■

Again, the existence of a unique balanced path can be proven.

**Proposition 1.6** *The system (32), (26)-(28) always admits a unique steady state with  $\gamma, \lambda, z, \nu > 0$ .*

**Proof** See section 1.6. ■

**Remark.** When  $\beta = \varepsilon = 0$ , that is when no new households are ever formed, equation (32) is identical to (25) whatever the value of  $b$ , the debt to capital ratio. As one would expect, fiscal policy has no effect on the equilibrium allocation, Ricardian equivalence holds (cf Buiters [25]). Note also that any value of  $b$  is sustainable; but high values of  $b$  will be associated with negative transfers. Manipulation of (30) gives  $T(t)/B(t) = \left( \dot{K}(t)/K(t) \right) - r(t)$  in a balanced growth path. It will be shown below that as  $b$  increases, the long-run value of  $r$  increases, whereas the growth rate of capital is unaffected, so eventually the long-run value of the transfer will be negative.

### 1.3.1 The effect of an increase in $b$ .

We now wish to analyse the impact of an increase in  $b$ . Let us assume that the economy is initially in a balanced growth path, and that the government suddenly and unexpectedly increases  $b$  marginally. By continuity the old steady-state will be in the neighbourhood of the new one. To simplify expressions, assume the policy change occurs at time  $t = 0$ .

**Proposition 1.7** *The effect of an increase in  $b$  is to increase  $u, \zeta, q, \rho$  and  $\lambda$ . In particular*

$$\frac{\partial u}{\partial b} = \frac{(\theta + p)(\beta + \varepsilon)}{\alpha^2 u^2 + (\theta + p)(p + \varepsilon)(1 + b)} > 0,$$

$$\begin{aligned}\frac{\partial \rho}{\partial b} &= \frac{\partial u}{\partial b} > 0, \\ \frac{\partial q}{\partial b} &= \frac{\alpha}{1-\alpha} \frac{\partial \rho}{\partial b} > 0, \\ \frac{\partial \lambda}{\partial b} &= \frac{(1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)q + \psi(\beta-p)} \frac{\partial q}{\partial b} > 0, \\ \frac{\partial \zeta}{\partial b} &= \frac{\psi(\beta-p)}{(1-\phi)\delta(1-\lambda)^{\psi+1}} \frac{\partial \lambda}{\partial b} = \frac{\zeta}{1-\lambda} \frac{\partial \lambda}{\partial b} > 0.\end{aligned}$$

**Proof.** Following the steps in the proof of proposition 1.6 it can easily be established that an increase in  $b$  increases  $u$ , which in turn causes all other variables to increase. The first equality is established by implicit differentiation of (45); the following ones by differentiation of (46), (38) and (48). ■

Since  $\rho = Y/K$ , and  $u = C/K$ ,  $u/\rho = C/Y$ . Then one can easily see<sup>19</sup> that  $d(u/\rho)/db$  is proportional to  $(\rho - u) > 0$ : an increase in  $b$  causes the saving ratio to fall.

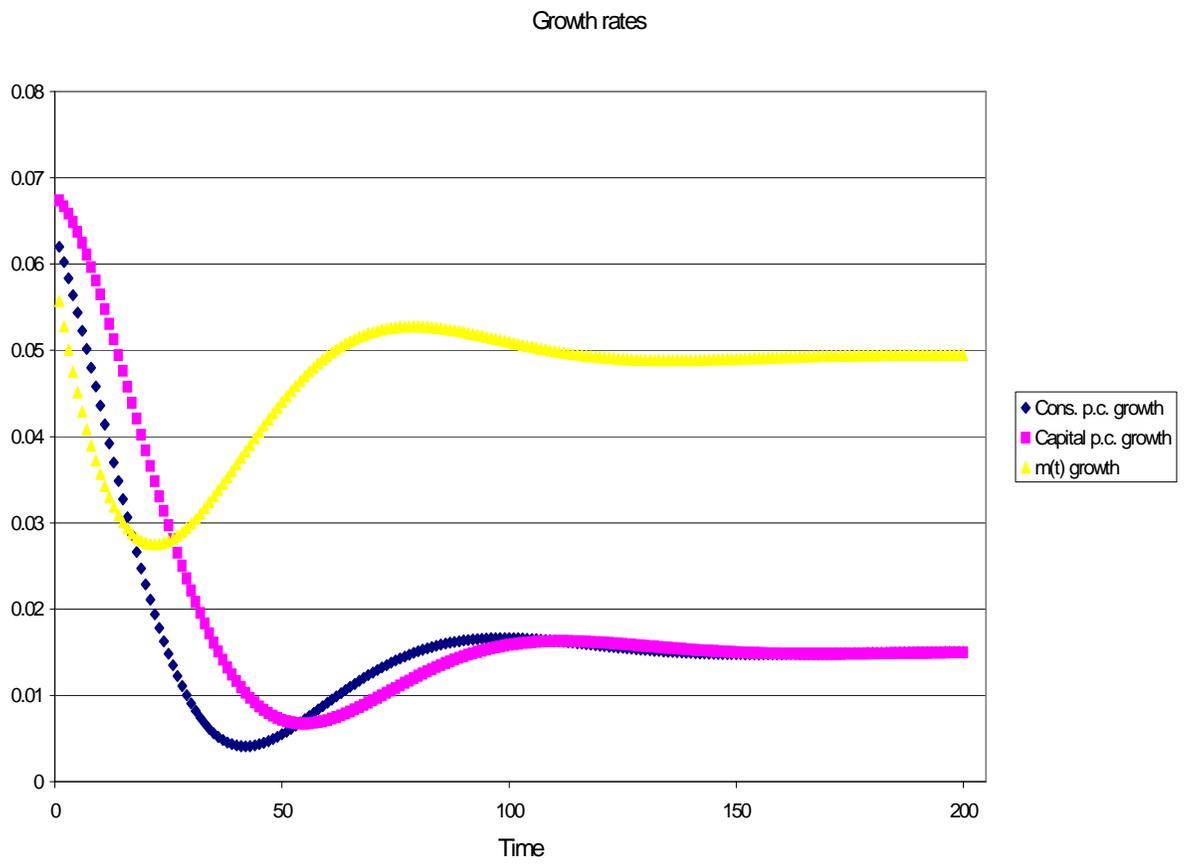
According to proposition 1.7, the long run effect of an increase in public debt is to increase  $\rho$  and hence the interest rate (remember that  $r = \alpha^2\rho$ , see section 1.6) and  $\lambda$ . The intuition for this is that as the interest rate increases, future profits are discounted more heavily, which reduce the value of an innovation (see equation (20)). Therefore less of the workforce is allocated to R&D; although this does not affect the long run *growth* of labour productivity ( $\dot{m}/m \rightarrow (\beta - p) / (1 - \phi)$ ), it will affect the *level*, which will be lower. The overall effect on consumption goods production is ambiguous, as, on one hand, the level of labour productivity is lower, but on the other hand, a larger share of the workforce is allocated to that sector. We show with a numerical example that aggregate output and consumption can increase.

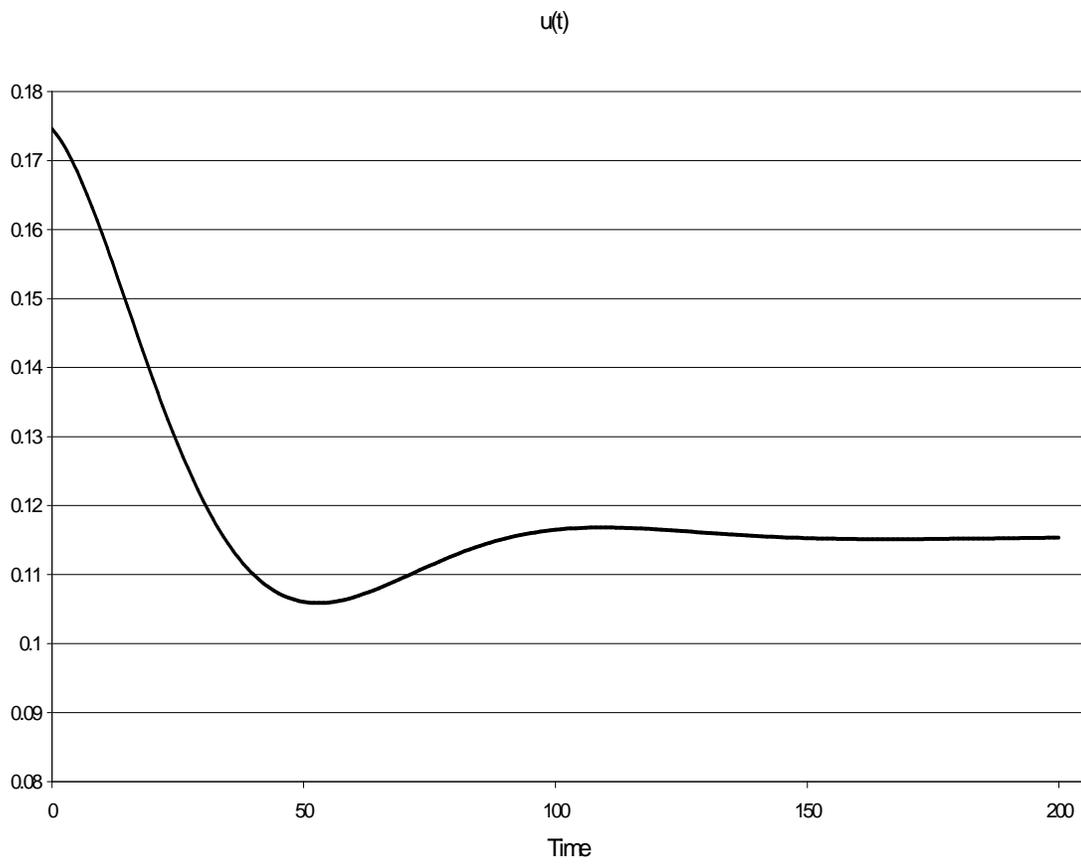
<sup>19</sup>  $d(u/\rho)/du = [\rho du - u d\rho] / \rho^2 = (\rho - u) du / \rho^2$ , since  $d\rho = du$ .

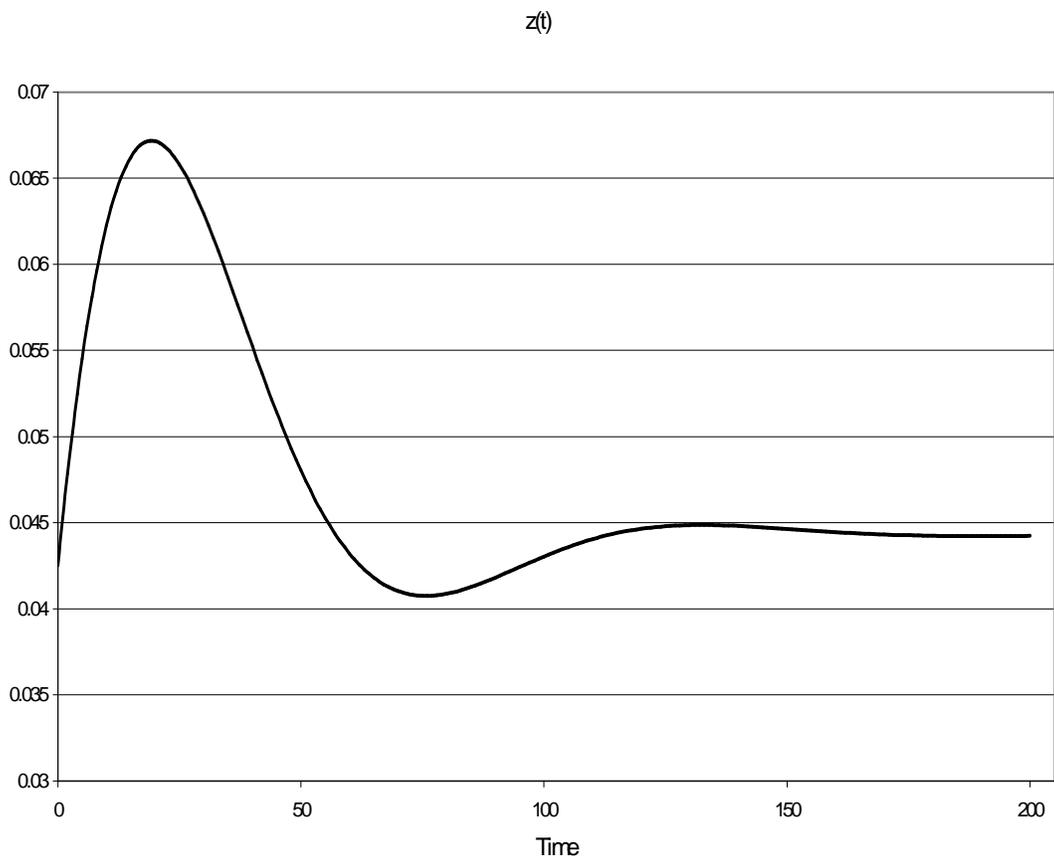
We assume the same parameter values as in the example of section 1.2, and we consider the effect of going from  $b = 0$  to  $b = 0.1$ . We computed the long-run values for  $u$ ,  $\zeta$ ,  $q$  and  $\rho$  for both values of  $b$ . We started by assuming that the economy would be on a balanced path with  $b = 0$  and computed the evolution<sup>20</sup> of  $C(t)$  and  $K(t)$  if the government kept  $b$  at 0. We then computed the adjustment that would ensue an announcement that  $b$  is to be raised once and for all to  $b = 0.1$ . Figures 1.2 to 1.5 show the evolution of the transformed stationary variables. What can be noted is that convergence is relatively slow, and characterised by overshooting and damped oscillations (as one would expect given that the eigenvalues are complex). But of course the important question is what these dynamics imply for aggregate variables. We computed the evolution of  $C(t)$  and  $K(t)$  under the two scenarios, indicating with a prime the values of a variable after the policy change. Figure 1.6 shows the evolution of the ratio  $C'(t)/C(t)$ . As it can be observed, consumption immediately jumps up as the new policy is announced and implemented, keeps on growing much faster for a long period, so that the ratio overshoots considerably its long run variable, and then converges with damped oscillations. Remarkably, the ratio always stays above 1, indicating that aggregate consumption under the policy change scenario is always larger than along the initial balanced path. Figure 1.7 is also remarkable. It shows the ratio between the capital stock after the policy change and what it would have been along the initial balanced path. Somewhat surprisingly, the capital stock after the policy change is always higher than what it would otherwise have been. This observation emphasises how the Pareto improvement arises not from the crowding out of the capital stock, as in Diamond [42], but through the reallocation of labour, similarly to Olivier [82].

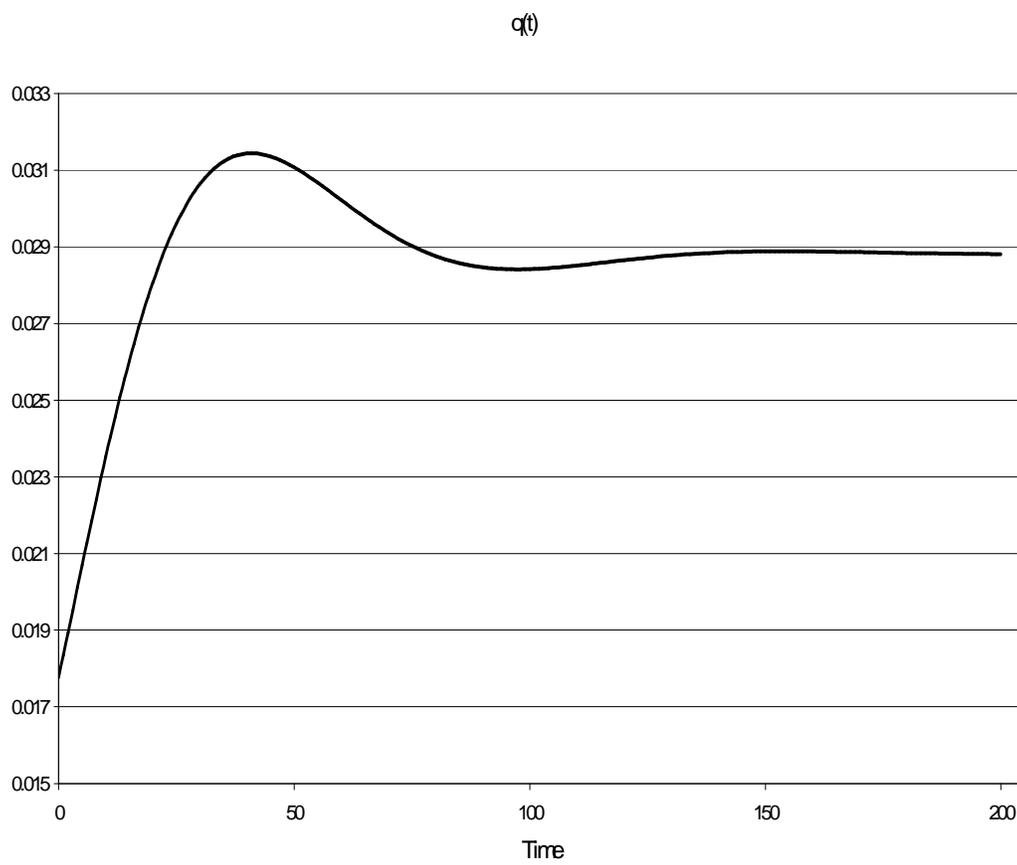
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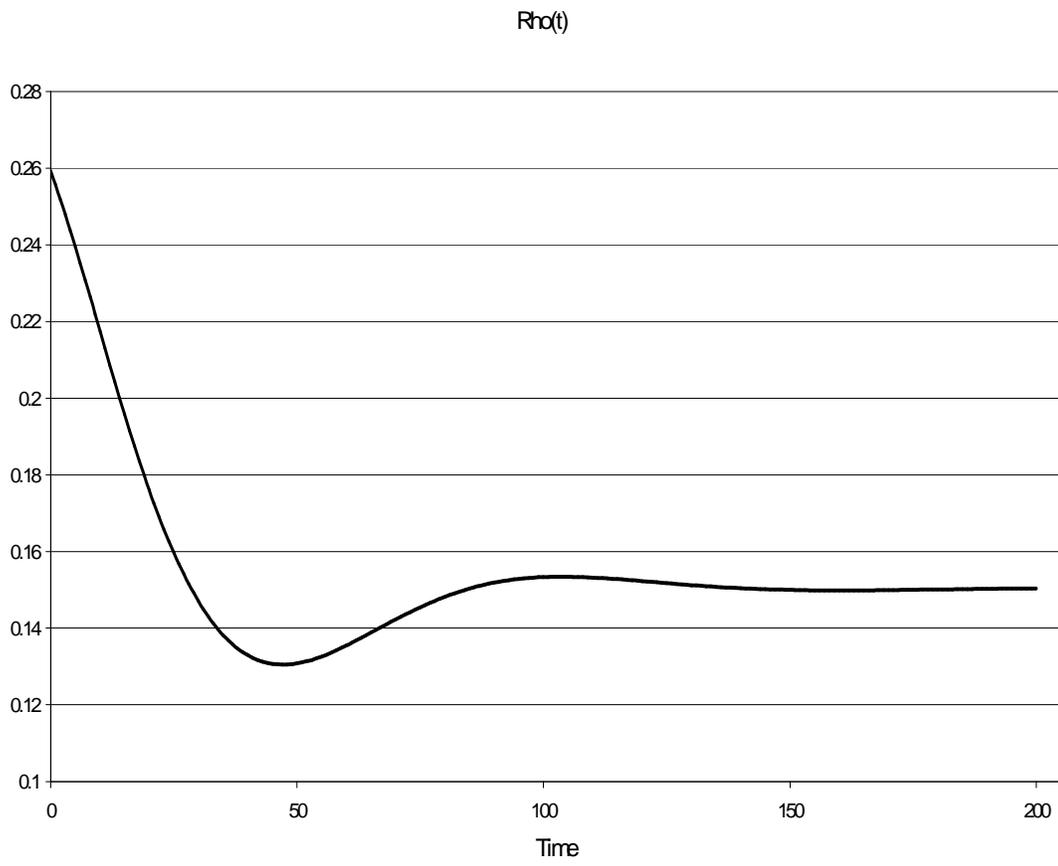
<sup>20</sup> Given an arbitrary value for  $L(0)$  (1 in our example), the balanced path restrictions determine the

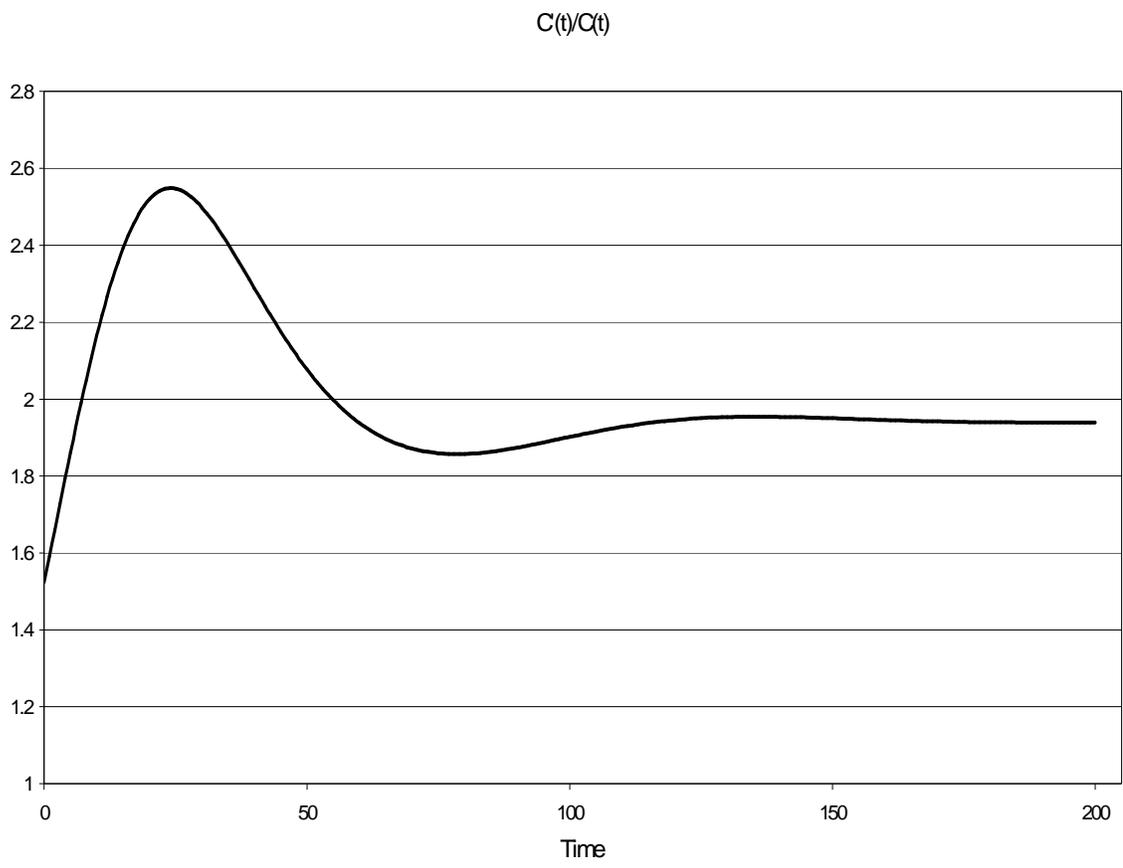


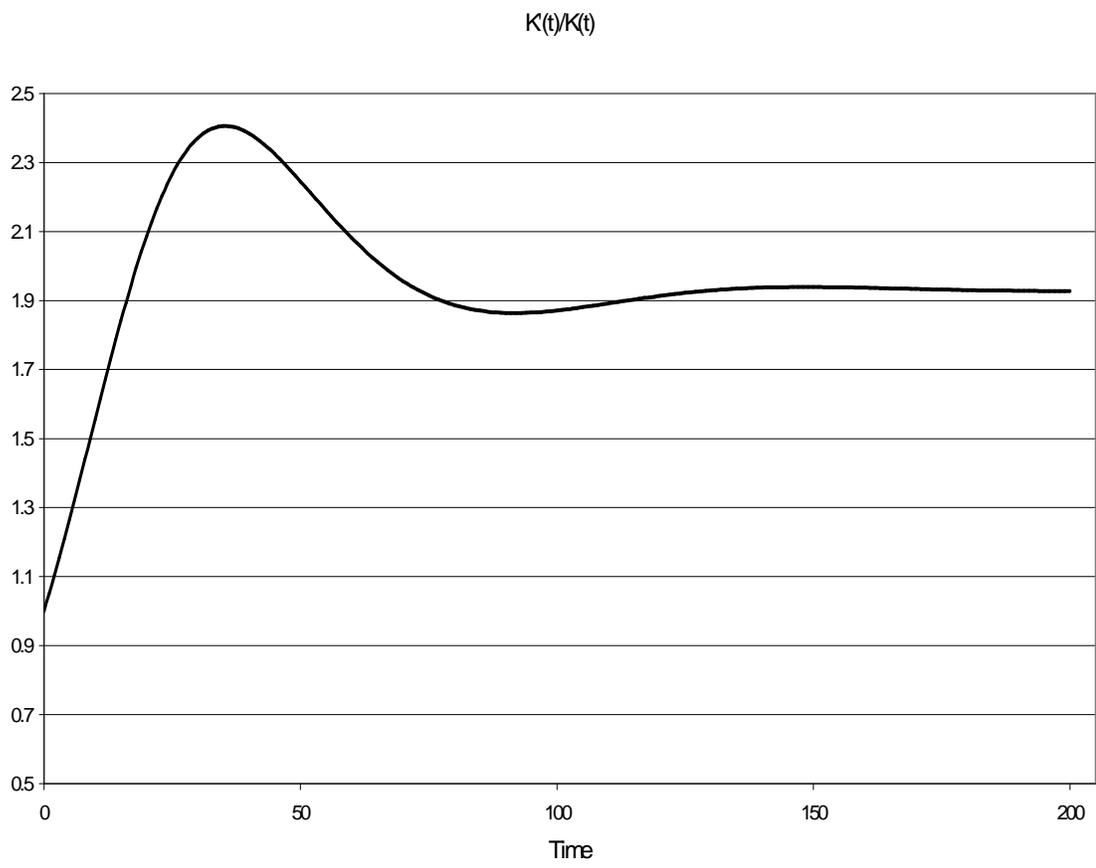












### 1.3.2 Intuition

What is the intuition behind the numerical experiment? Imagine we are on an initial balanced growth path without debt and  $r < \dot{C}/C$ . If the government announces a series of transfers for today and the future to be financed by debt, households currently alive will feel wealthier and desire to consume more. In the short-run this is only possible if more labour is allocated to final good consumption, i.e.  $L_y$  increases. Since the initial stocks of capital and ideas are given, the reallocation requires wages to fall and the interest rate to increase. The first increases labour demand in the consumption sector, the second depresses the value of innovation because future profits are discounted more heavily, thus lowering labour demand (or equivalently entry) in the R&D sector. Both the decrease in the wage rate and the increase in the interest rate tend to depress the value of human wealth,  $h(t)$ ; however, at least on average, this must be more than compensated by the increase due to the transfer, since consumption has increased. In the short-run the lower share of labour employed in R&D reduces the rate of technological progress, so  $m(t)$  is lower than it would have been. However this is more than compensated by the increase in the amount of labour, so that output is always larger. This allows consumption to be permanently higher. In the long run the growth of labour productivity returns to  $(\beta - p) / (1 - \phi)$ , although the level is permanently lower.

We have thus shown that transfers perpetually financed with issuing bonds can increase aggregate consumption, as in Diamond [42]; however, there are important differences. Whereas in Diamond's model the fundamental problem is one of overaccumulation of capital, and the policy works by lowering the capital stock per capita, here, despite higher consumption, the per capita capital stock is actually *increased* by the pol-

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initial values for  $K$  and  $m$ .

icy change. There are here two types of capital, physical capital,  $K(t)$  and the stock of ideas,  $m(t)$ . The decrease in the saving rate affect not only the accumulation of physical capital, but through its effect on the allocation of labour, it affects the accumulation of ideas. In other words, there is a composition effects as well as a size effect. As observed by King and Ferguson [69], economies with several capital goods may be inefficient not only or even not at all because their capital stock is too large, but because of its composition. But in the types of model analysed by King and Ferguson, policies that affect the saving rate would affect the size but not the composition of the capital stock, and hence cannot increase aggregate consumption. In the model examined here, however, composition *is* affected, and aggregate consumption can be larger. Clearly this is true for most R&D based models exhibiting scale effects, but in those model the allocation of labour affects the long-run growth rate; there the improvement in aggregate consumption cannot be sustained, as it will affect negatively the long-run growth rate, which will eventually offset the gains coming from allocating more labour to consumption goods production, as in Olivier [82]. In non-scale models as ours, the long-run growth rate is not affected, and a sustained increased in aggregate consumption can be achieved through Ponzi finance.

## 1.4 Tax on profits

In the previous section we argued that the Pareto improvement occurs to the reallocation of labour. One may therefore expect that other policies that induce a reallocation of labour will have similar effects. One example that confirm this intuition is a tax on profits. In our model the only sector that makes positive profits is the intermediate

goods sector, which is -as it will be recalled- assumed monopolistically competitive. Let then assume that the government imposes a tax  $\tau$  on profits and rebates the proceeds lump-sum to households. Then the maximum a producer will now be willing to pay for a patent (and therefore the patent's price) is

$$P_m(t) = \int_t^{+\infty} (1 - \tau) \pi(v) e^{-\int_t^v r(\mu) d\mu} dv,$$

which should be compared with (20). Then the no-arbitrage condition (21) becomes

$$r(t) = \frac{\dot{P}_m(t)}{P_m(t)} + \frac{(1 - \tau) \pi(t)}{P_m(t)}.$$

Assuming a balanced budget, the aggregate transfer must be

$$T(t) = \tau \pi(t) m(t)$$

Following the same steps as in section 1.2, we find that the dynamics of the economy could be studied by studying the system of differential equations (25), (26),

$$\begin{aligned} \dot{q}(t) = & \left\{ (1 - \alpha)(1 - \psi)(\beta - p) - \alpha u(t) + \alpha(1 - \alpha)((1 - \tau)q(t) + \rho(t)) \right. \\ & \left. + (1 - \alpha)(1 - \phi) \frac{(1 - \lambda(t))^{\psi+1}}{1 - \psi\lambda(t)} \delta\zeta(t) \right\} \frac{q(t)}{D(t)}, \end{aligned}$$

and

$$\begin{aligned} \dot{\rho}(t) = & \left\{ \frac{(1 - \alpha)(1 - \psi)}{1 - \psi\lambda(t)} (\beta - p) + (\alpha + (1 - \alpha)(1 - \alpha^2) E(t)) \rho(t) \right. \\ & - (\alpha + (1 - \alpha) E(t)) u(t) + \alpha(1 - \alpha)^2 E(t) (1 - \tau) q(t) \\ & \left. - \phi(1 - \alpha) \frac{(1 - \lambda(t))^{\psi+1}}{1 - \psi\lambda(t)} \delta\zeta(t) \right\} \frac{\rho(t)}{D(t)}; \end{aligned}$$

note the appearance of  $(1 - \tau)$  next to  $q(t)$ . Using the same initial numerical example as before, we studied numerically the effect of the introduction of a tax. The resulting dynamics were very similar, with both aggregate consumption and capital stock

increasing at all time when compared with the original balanced path without policy change.

We think that similar forces are at play here and in the example discussed in the previous section. Here the tax reduces future profits and hence the value of patents. As a consequence labour demand in the R&D sector falls. The current and future transfers make household feel richer, and hence increases their demand for consumption and capital goods. The increase demand for final goods is met by increasing the labour force in that sector.

As an alternative policy, one may think of taxing the output of the R&D sector. Suppose the government imposed a tax proportional to the value of the R&D output. Then R&D firms would earn  $(1 - \tau) P_m(t) m(t)$ . Then equation (19) that gave us the labour demand in the R&D sector would become

$$w(t) = P_m(t) (1 - \tau) \delta L_m^{\psi-1} m(t)^\phi.$$

Therefore the effect of introducing the tax is similar to the effect of a reduction in the technological parameter  $\delta$ . However the *long run* value of  $\lambda$  is independent of  $\delta$ , as it can be verified from the derivation of equation (47) in the proof of propositions 1.2 and 1.6.<sup>21</sup> In fact the long run share of labour allocated to R&D is independent from  $\delta$  in the original model by Jones [61]. Therefore in terms of the transformed variables, the policy has only transitional effects; in term of the original variables, the economy will converge to a balanced path with a lower labour productivity level (i.e. a lower  $m$ , due to the lower labour force in R&D during the transition). Again we found that introducing a tax can increase consumption at all times, although quantitatively the magnitudes are smaller, due to the fact that the reallocation of labour is only temporary.

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<sup>21</sup> The parameter  $\delta$  does not appear in any of the equations used to derive (19).

## 1.5 Conclusions

In this chapter we modified the model of Jones [61] by introducing overlapping generation of households as in Blanchard [21] and Buiter [25]. We proved that a balanced path exists, and showed that the dynamics can be represented by a system of 4 stationary transformed variables. It was not possible to find general conditions that guarantee that the equilibrium is locally determined, but numerical simulation confirmed that for plausible parameters it is so. Those simulations also showed that convergence to steady-state are generally characterised by damped oscillations. We also showed with numerical examples that the balanced path may be characterised by the interest rate being lower than the growth rate. We then analysed the introduction of debt-financed lump sum transfers and showed with a numerical example that it may induce higher consumption in every current and future period. In this case it is possible for the government to devise a transfer scheme that increases the consumption of all household every period. This shows that the policy may be Pareto improving.

Crucial to our result is the fact that in this model Ricardian equivalence fails because of the overlapping generation of unconnected households, so that the debt policy affects the saving behaviour of agents. As in the neoclassical exogenous growth literature and in contrast to endogenous growth models with learning-by-doing externalities, the policy affects the long-run interest rate but not the long-run growth rate. However, in sharp contrast with the exogenous growth literature, the Pareto improvement occurs not through crowding out of physical capital but through the crowding out of ideas obtained through a reallocation of labour caused by the change in factor prices. If the reallocation of labour were to be prevented, for example by hypothesising that at each moment

in time two type of workers are born, some with the ability to work in R&D and some with the skills to work in the consumption sector, the model would become very similar to a standard exogenous growth model.<sup>22</sup> If the life-cycle motif for saving is high, over-accumulation will occur, and a debt finance transfer can be Pareto improving as in Diamond [42] and Blanchard [21]; in this case, though, the improvement would be obtained by reducing physical capital accumulation, while all the example above were characterised by an increase in physical capital accumulation.

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<sup>22</sup> Growth would not be any more exogenous than in the present version, but preference parameters and policy changes would not effect nor the level nor the growth of labour productivity, nor in the short nor in the long run.

## 1.6 Proofs of propositions in chapter 1

### 1.6.1 Proof of propositions 1.1 and 1.5

We prove proposition 1.5, the proof of proposition 1.1 is identical once one imposes  $b = 0$ .

First note that in a symmetric equilibrium,

$$Y(t) = m(t) L_y(t)^{1-\alpha} x(t)^{1-\alpha} = K^\alpha (m(t) L_y(t))^{1-\alpha}, \quad (33)$$

which implies  $Y(t)/K(t) = (m(t) L_y(t)/K(t))^{1-\alpha} = \rho(t)$ . Then using (16)

$$r(t) = \alpha^2 \rho(t). \quad (34)$$

Then (23) becomes

$$\frac{\dot{C}(t)}{C(t)} = (\alpha^2 \rho(t) - \theta + \beta - p + \varepsilon) - \frac{(\theta + p)(\beta + \varepsilon)(1 + b)}{u(t)}. \quad (35)$$

The market clearing condition (22) can be written

$$\frac{\dot{K}(t)}{K(t)} = \frac{Y(t)}{K(t)} - \frac{C(t)}{K(t)} = \rho(t) - u(t), \quad (36)$$

and so

$$\frac{\dot{u}(t)}{u(t)} = (\alpha^2 \rho(t) - \theta + \beta - p + \varepsilon) - \frac{(\theta + p)(\beta + \varepsilon)(1 + b)}{u(t)} - \rho(t) + u(t),$$

which proves (25).

Next, the labour demand equations (11) and (19) yield

$$L_y(t) = \frac{(1 - \alpha) Y(t)}{w(t)} = \frac{(1 - \alpha) Y(t) L_m(t)^{1-\psi}}{\delta P_m(t) m(t) m(t)^{\phi-1}} = \frac{(1 - \alpha) q(t) L_m(t)^{1-\psi}}{\delta} m(t)^{1-\phi},$$

or

$$\frac{L_y(t)}{L(t)} = \frac{(1 - \alpha) q(t) L_m(t)^{1-\psi}}{\delta} \frac{L_m(t)^{1-\psi}}{L} m(t)^{1-\phi}. \quad (37)$$

The latter can be written using (24) and the definition of  $\lambda(t)$  as

$$\lambda(t) = \frac{(1 - \alpha) q(t)}{\delta \zeta(t)} (1 - \lambda(t))^{1-\psi}. \quad (38)$$

It can be easily shown that this last equation always has an unique solution  $\lambda(t) \in [0, 1]$ , and thus it implicitly defines  $\lambda(t)$  as a function<sup>23</sup> of  $q(t)/\zeta(t)$ . Let us indicate this function with  $\Lambda(q(t)/\zeta(t))$ . Then log-differentiating with respect to  $t$  the identity  $\lambda(t) = \Lambda(q(t)/\zeta(t))$ ,

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \left[ \frac{\Lambda'(q(t)/\zeta(t)) q(t)}{\Lambda(q(t)/\zeta(t)) \zeta(t)} \right] \left( \frac{\dot{q}(t)}{q(t)} - \frac{\dot{\zeta}(t)}{\zeta(t)} \right).$$

Implicitly differentiating (38), we can show that the elasticity of  $\Lambda$  with respect to  $q/\zeta$  (the term in square brackets above) is

$$\frac{\Lambda'(q(t)/\zeta(t)) q(t)}{\Lambda(q(t)/\zeta(t)) \zeta(t)} = \frac{1 - \lambda(t)}{1 - \psi \lambda(t)}.$$

Therefore

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \left( \frac{1 - \lambda(t)}{1 - \psi \lambda(t)} \right) \left( \frac{\dot{q}(t)}{q(t)} - \frac{\dot{\zeta}(t)}{\zeta(t)} \right). \quad (39)$$

Next, with (18) and (24)

$$\frac{\dot{m}(t)}{m(t)} = \frac{\delta L_m(t)^\psi}{m(t)^{1-\phi}} = \frac{\delta (1 - \lambda(t))^\psi L(t)^\psi}{m(t)^{1-\phi}} = \delta (1 - \lambda(t))^\psi \zeta(t), \quad (40)$$

and since by the definition of  $\zeta(t)$ ,  $\dot{\zeta}(t)/\zeta(t) = \psi \dot{L}(t)/L(t) - (1 - \phi) \dot{m}(t)/m(t)$ ,

$$\frac{\dot{\zeta}(t)}{\zeta(t)} = \psi(\beta - p) - (1 - \phi) \delta (1 - \lambda(t))^\psi \zeta(t),$$

which proves (26).

From (17)

$$\frac{\pi(t)}{P_m(t)} = \alpha(1 - \alpha) \frac{Y(t)}{P_m(t) m(t)} = \alpha(1 - \alpha) q(t).$$

<sup>23</sup> It can be shown that this function is strictly increasing and strictly convex.

Then the no arbitrage condition (21) can be written (using (34))

$$\frac{\dot{P}_m(t)}{P_m(t)} = \alpha^2 \rho(t) - \alpha(1 - \alpha) q(t). \quad (41)$$

To prove (27) and (28) we note that from (10) we have

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{m}(t)}{m(t)} + \alpha \frac{\dot{K}(t)}{K(t)} + (1 - \alpha) \left[ \frac{\dot{\lambda}(t)}{\lambda(t)} + \frac{\dot{L}(t)}{L(t)} \right]. \quad (42)$$

From  $\rho(t) = Y(t)/K(t)$  we obtain

$$\frac{\dot{\rho}(t)}{\rho(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{K}(t)}{K(t)}. \quad (43)$$

Finally, from the definition of  $q(t)$

$$\frac{\dot{q}(t)}{q(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{P}_m(t)}{P_m(t)} - \frac{\dot{m}(t)}{m(t)}. \quad (44)$$

Now, equations (39), (42), (43) and (44) constitute a system of four equations that can be solved for  $\dot{\lambda}(t)/\lambda(t)$ ,  $\dot{Y}(t)/Y(t)$ ,  $\dot{\rho}(t)/\rho(t)$  and  $\dot{q}(t)/q(t)$  as functions of  $\dot{K}(t)/K(t)$ ,  $\dot{m}(t)/m(t)$ ,  $\dot{\zeta}(t)/\zeta(t)$  and  $\dot{P}_m(t)/P_m(t)$  for which we found expressions above. In matrix form the system can be written

$$\begin{pmatrix} 1 & 0 & 0 & -E(t) \\ -(1 - \alpha) & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\lambda}(t)/\lambda(t) \\ \dot{Y}(t)/Y(t) \\ \dot{\rho}(t)/\rho(t) \\ \dot{q}(t)/q(t) \end{pmatrix} = \begin{pmatrix} -E(t) \\ \frac{\dot{m}(t)}{m(t)} + \alpha \frac{\dot{K}(t)}{K(t)} + (1 - \alpha)(\beta - p) \\ -\dot{K}(t)/K(t) \\ -\frac{\dot{P}_m(t)}{P_m(t)} - \frac{\dot{m}(t)}{m(t)} \end{pmatrix},$$

where  $E(t) \equiv (1 - \lambda(t))/(1 - \psi\lambda(t))$ . Solving the system and rearranging we find

(27) and (28). ■

## 1.6.2 Proof of propositions 1.2 and 1.6

We prove proposition 1.6, the proof of proposition 1.2 is identical once one sets  $b = 0$ .

First note that in a balanced path,  $\dot{m}/m = \psi(\beta - p)/(1 - \phi)$ ; to see this, call  $g_m(t) \equiv \dot{m}(t)/m(t)$ , and use (18) to write

$$\frac{\dot{g}_m(t)}{g_m(t)} = \psi \frac{\dot{L}_m}{L_m} - (1 - \phi) \frac{\dot{m}}{m},$$

then the result follows from the facts that in a balanced path the growth of the rate of growth of  $m$  is constant (i.e.  $\dot{g}_m = 0$ ) and employment in research grows at the same rate as population.

Furthermore, in a balanced path,  $\dot{Y}/Y = \dot{K}/K = \dot{C}/C$ . From (33),  $\dot{Y}/Y = (1 + \psi/(1 - \phi))(\beta - p)$ , which is the common long-run growth rate of *all* aggregate variables. Then (35) becomes

$$(\alpha^2 \rho - \theta + \beta - p + \varepsilon) - \frac{(\theta + p)(\beta + \varepsilon)(1 + b)}{u} = \left(1 + \frac{\psi}{1 - \phi}\right)(\beta - p).$$

Similarly (36) gives

$$\rho = \left(1 + \frac{\psi}{1 - \phi}\right)(\beta - p) + u.$$

The last two can be used to compute  $\rho$  and  $u$ . Substitute the latter in the former:

$$\alpha^2 u - \frac{(\theta + p)(\beta + \varepsilon)(1 + b)}{u} = (1 - \alpha^2) \left(1 + \frac{\psi}{1 - \phi}\right)(\beta - p) + \theta - (\beta - p) - \varepsilon. \quad (45)$$

Clearly the left-hand side is a continuous, strictly increasing function of  $u$ , taking values from  $-\infty$  to  $+\infty$  as  $u$  goes from 0 to  $+\infty$ , whereas the right hand side is a constant; a unique  $u > 0$  must exist that satisfies the equation. Once  $u$  is found, one can compute  $\rho$ , which will also be strictly positive.

Next, from the definition of  $q$ ,  $\dot{q}/q = 0$  implies

$$0 = \frac{\dot{Y}}{Y} - \frac{\dot{P}_m}{P_m} - \frac{\dot{m}}{m},$$

which establishes

$$\frac{\dot{P}_m}{P_m} = \beta - p.$$

But then (41) reduces to

$$q = \frac{\alpha}{1 - \alpha} \rho > 0. \quad (46)$$

Next,  $\dot{m}/m = \psi(\beta - p)/(1 - \phi)$  and (38) yield

$$\lambda = \frac{(1 - \phi)(1 - \alpha)q}{(1 - \phi)(1 - \alpha)q + \psi(\beta - p)}, \quad (47)$$

which gives the long-run  $\lambda \in (0, 1)$ . Finally to find  $\zeta$ , set  $\dot{\rho}/\rho = 0$  to find

$$\delta(1 - \lambda)^\psi \zeta - \rho + u + \beta - p = 0,$$

or

$$\zeta = \frac{\rho - u - (\beta - p)}{\delta(1 - \lambda)^\psi} = \frac{\psi(\beta - p)}{(1 - \phi)\delta(1 - \lambda)^\psi} > 0, \quad (48)$$

where we used  $\rho - u = \dot{K}/K = (1 + \psi/(1 - \phi))(\beta - p)$  in the second equality. ■

### 1.6.3 Proof of proposition 1.3

By (11)

$$w(t) = (1 - \alpha)Y(t)/L_y(t) = (1 - \alpha)Y(t)/(\lambda(t)L(t))$$

The economy is assumed to converge to a path where  $\lambda(t)$  is constant asymptotically.

Then the growth rate of wages converges to

$$\frac{\dot{w}(t)}{w(t)} \rightarrow \frac{\dot{Y}(t)}{Y(t)} - (\beta - p) = \frac{\psi}{1 - \phi}(\beta - p),$$

where we have used proposition 1.2. From the definition of  $H(t)$ , (5), it is clear that convergence requires

$$\frac{\psi}{1 - \phi}(\beta - p) - r - p - \varepsilon < 0.$$

From proposition 1.2,  $\dot{C}/C = (1 + \psi/(1 - \phi))(\beta - p)$ , which using (35) can be written, after rearranging, as

$$\frac{\psi}{1 - \phi}(\beta - p) - r - p - \varepsilon = -\theta - p - \frac{(\theta + p)(p + \varepsilon)(1 + b)}{u} < 0.$$

Hence convergence of  $H(t)$  is always ensured. ■

## 2 Two-sector Model

In this chapter we develop and analyse an overlapping-generations competitive general equilibrium two-sector model for a closed economy. The two sectors produce consumption goods and capital goods. We first analyse conditions for existence and uniqueness of a perfect foresight equilibrium. We then discuss arbitrary fiscal policy with proportional tax rates and investigate whether sustained primary deficits are sustainable. Finally we investigate optimal policies for the case of constant elasticity of intertemporal substitution. We show that the fiscal instrument considered are sufficient to decentralise the first-best optimum.

### 2.1 Introduction

It has been long recognized by economists that fiscal policy can have effects on growth rates in the long-run. This is one of the main contribution of the endogenous growth literature. An important part of this literature devotes its attention to the role of the government in correcting, through a system of taxes and subsidies, inefficiencies due to the presence of externalities (Romer, [92]).

Another widely investigated issue is the optimal structure of tax rates. Both in the case of unproductive public expenditures and in the case of productive public expenditures, attention has been devoted to establishing the optimal level of taxation and its composition. When public spending is assumed to finance public consumption, most papers argue that taxes are detrimental to growth and income taxes more so than consumption taxes. Jones and Manuelli [62] show that taxes not only reduce growth but

can even prevent the possibility of sustained growth; Rebelo [88] also concludes that income taxes reduce the growth rate whereas consumption taxes have level but not growth effects. Devereux and Love [41] also support the view that income taxes are detrimental to growth, and so do King and Rebelo [70].

A different view is taken in model such as those in Barro [13] and Turnovsky [105], where productive public expenditures are allowed. Barro establishes the existence of a sort of “Laffer” curve, with the effect of an increase in taxation on growth being positive below a certain critical value and negative thereafter. In models with congestion, the optimal capital income tax may be positive. This is because when congestion is linked with the size of the private capital stock relative to the size of public services, the tax on capital acts like an user fee and (at least partially) induces firms to internalise the externality that their capital accumulation exercises on other firms through congestion (Barro and Sala-i-Martin [14] and Turnovsky [105]).

Most models consider balanced budget policies only, ruling out deficit finance by definition. Others, like Judd [67], allow for deficits; but in his paper, which assumes a demographic structure with infinitely lived agents, the government cannot sustain a policy with a positive net present value of the fiscal deficit. However in the case of overlapping generations, it might be feasible for the government to sustain permanent deficits. This means that in an overlapping generations model government fiscal policy is not necessarily subject to the constraint that the present value of expenditures must be equal to the present value of taxes. This leaves space not only for the possibility of a continuous roll-over of the debt, but even to perpetual primary deficits.

The overlapping generation model of Diamond [42] has been an important tool for the analysis of fiscal policy- see for example Auerbach and Kotlikoff [7] - but has

received far less attention by the endogenous growth literature, which focuses more often on the infinitely lived representative agent. A noticeable exception is Saint-Paul [96], who considers an AK model with overlapping generation in continuous time à la Blanchard [21].

We believe that the more common infinitely lived agent framework limits the set of feasible policies in an important way. In this setting the present value of expenditures must equal the present value of taxes, “Ponzi schemes” are not feasible. With overlapping generations they are at least in principle feasible. The sustainability in endogenous growth models of a simple Ponzi scheme (a “Bubble” in the sense of Tirole, [104], and O’Connell and Zeldes, [81]), has been investigated in Grossman and Yanagawa[60], King and Ferguson [69] and Azariadis and Reichlin [10]. All these papers conclude that although feasible, policies with a positive present value of debt are always growth and welfare reducing. In all these papers, however, there is no public investment.

There has been growing interest in the issue of the desirability of imposing restrictions on government budget deficits. This is not surprising since, in reality, sustained deficit policies are more the norm than the exception. Ghiglini and Shell [56] show that even though such constraints do not matter if the government can use lump sum taxation, they do reduce the set of feasible allocations if only proportional taxation is allowed. Their paper however, considers an exchange economy without production, and has nothing to say on long-run effects on growth. In the exogenous growth context, Uhlig [107], Buitert and Kletzer [26] and Chalk [33] analyses the sustainability of perpetual primary deficits. None of these papers, though, consider endogenous growth and productive public expenditures.

We present a model in which the government supplies a public good, which affects the productivity of private inputs. All markets are assumed competitive and there are two sectors: the consumption sector produces the good that enters the utility function; the capital sector produces capital that can be used in production; we also assume, for simplicity, that the government can costlessly transform private capital into public capital.<sup>24</sup> The public good acts as an externality in the latter sector, and it is assumed to be subject to congestion. Both from the aggregate point of view and the point of view of the individual firm, the technology has constant returns to scale. The consumption sector adopts a technology with the usual neoclassical properties. The assumptions of the model guarantee that sustained growth is possible with a convex technology. In this sense the model is closely related to Jones and Manuelli [62] and Rebelo [88], although this way of introducing public capital in a two-sector model is -to the best of our knowledge- novel.

We first analyse the dynamic equilibrium under arbitrary choice of the fiscal instruments. As long as the saving rate is not too responsive to the interest rate, the dynamic equilibrium is unique. There are no transitional dynamics, the economy settles immediately on a balanced growth path. In this respect the dynamics of the model resemble those of the model in Glomm and Ravikumar [57]. It is also shown that in this economy the interest rate exceeds the growth rate. This eliminates the possibility of the government engaging in "Ponzi finance" (O'Connell and Zeldes [81]); in this respect our model differs sharply from most basic endogenous growth models, for which

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<sup>24</sup> Or in other words, we are implicitly assuming that public and private capital are not physically different, it is where they are allocated that matters. This is clearly an oversimplification; one could introduce in the model a technology to transform private into public capital; if this technology has also constant returns to scale, most of qualitative results would not change much.

it such schemes are typically possible (Saint-Paul [96], Grossman and Yanagawa [60] and King and Ferguson [69]).

Next we examine the first best allocation and the optimal fiscal policy. For the constant intertemporal elasticity of substitution utility function, we prove that an optimal allocation exists as long as the maximal growth rate is not too high, and it is characterised by a constant growth rate (i.e. again there are no transitional dynamics). We then show how the first best can be decentralised. This implies maintaining an optimal ratio of public to private inputs in the capital sector. There is more than one set of taxes and borrowing policies that decentralise the first best. We concentrate on the case where the government use different proportional taxes on capital and wage income. We show that -due to the overlapping generations framework- the optimal capital income tax is not zero (analogously to Erosa and Gervais [50] and in contrast with Chamley [34]).

In the next sections we describe the environment, the objectives and constraints of the players and the equilibrium concept adopted. The players are: the firms in the two sectors, the infinitely lived government and the individual consumers. We show that if the rate of saving is not too sensitive with respect to the real interest rate, then for a given (stationary) fiscal policy, the equilibrium, exists, and it is unique. For more general preferences, multiplicity of equilibria cannot be ruled out. Section 2.3 discusses the sustainability of primary deficits in the long-run. Section 2.4 analyses the intertemporal allocation that would be chosen by an all powerful central planner. Section 2.5 shows that the first best optimum can be decentralised with appropriate tax/subsidies and public debt. Finally section 2.6 concludes.

## 2.2 The model

We consider a two-sector growth model with overlapping generations. Time is discrete and indexed with  $t = \{0, 1, \dots\}$ . There are two kinds of private goods: a consumption good, which enters individuals' utility function, and a capital good, that is not an argument of the utility function, but is necessary to produce the consumption good. Except for an initial generation that lives only one period and is born with an endowment of capital and government bonds, agents are all identical and live for two periods. They are endowed with one unit of labour -that they supply inelastically- in the first period of their life and none in the second. Population grows at a constant rate  $n \geq 0$ . There is an infinitely lived government that supplies a public good. This public good is produced with capital goods and produces services that are essential in the production of new capital<sup>25</sup>; however these services are subject to congestion. The model does not feature increasing returns to scale in any of the sectors nor at the private, or social levels. The technological set is convex.<sup>26</sup> Public expenditures are financed with proportional taxes and public debt.

There is no uncertainty in this model, and we assume that all actors have perfect foresight.

### 2.2.1 Agents

We shall call the set of agents born at time  $t \geq 1$  as *generation t*. For time 0, we need to distinguish between the set of agents 'born young' (i.e. who will be alive at time 1) who we will call *generation 0*, and those 'born old', *generation -1*.

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<sup>25</sup> To the best of our knowledge, public capital has never been introduced in a growth model in this particular way.

<sup>26</sup> In fact one can easily show that the production set is a convex cone.

Except for generation -1, which is born old and is endowed with an initial amount of capital  $K_0 > 0$  and government bonds  $B_0 \geq 0$ , each individual is endowed with one unit of labour when young, none when old. Each agent, then, supplies its endowment of labour to firms, earns the competitively determined wage,  $w_t$ , part of which is consumed at the end of the period. The remaining is invested in bonds and capital. Consumption in old age is financed from interest earnings from this investment. Generation -1 simply consumes interest earning on its endowment.

We assume that the preferences of agents born at time  $t \geq 0$  are described by a utility function  $U_t : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}$

$$U_t \equiv u(c_t^y, c_{t+1}^o), \quad \forall t \geq 0; \quad (1)$$

except for generation  $-1$ , who has an utility function  $U_{-1} : \mathfrak{R}_+ \rightarrow \mathfrak{R}$

$$U_{-1} \equiv u_{-1}(c_0^o);$$

where  $c_i^j$  is consumption at time  $i$  of an agent of age  $j$  ( $j = y, o$ , where  $y$  stands for young,  $o$  stands for old). We assume that  $u(\cdot, \cdot)$  is an increasing, concave utility function, with  $\partial u / \partial c^j \rightarrow -\infty$  for  $c^j \rightarrow 0$ , and  $\partial u / \partial c^j \rightarrow 0$  for  $c^j \rightarrow +\infty$ ,  $j = y, o$ ; and that  $c_t^y$  and  $c_{t+1}^o$  are normal goods. Similarly  $\partial u_{-1} / \partial c_0^o \rightarrow -\infty$  for  $c_0^o \rightarrow 0$ , and  $\partial u_{-1} / \partial c_0^o \rightarrow 0$  for  $c_0^o \rightarrow +\infty$ . The only further assumption we require is that  $u(\cdot)$  is homothetic. Although it could be relaxed, this assumption greatly simplifies the analysis of the dynamics of the model.

Let us indicate with  $R_t$  the *gross* real interest rate, with  $w_t$  the real wage, with  $\tau_t$  the labour income tax and with  $\theta_t$  the interest income tax at time  $t$ . Then the budget

constraint for individuals of generation  $t \geq 0$  is

$$c_t^y + \frac{c_{t+1}^o}{(1 - \theta_{t+1}) R_{t+1}} \leq (1 - \tau_t) w_t, \quad \forall t \geq 0; \quad (2)$$

whereas individuals from generation  $-1$  face the constraint

$$c_0^o = (1 - \theta_0) [R_0 B_0 + R_0 (K_0 - G_0)] / L_{-1}, \quad (3)$$

where  $(K_0 - G_0) / L_{-1}$  measures the capital endowment owned by a generation  $-1$  individual:  $K_0$  is the aggregate capital stock at time 0, of which  $G_0$  is owned by the government;  $B_0$  is the initial stock of public debt, of which each individual holds a fraction  $1/L_{-1}$ . Young agents will save as to maximize  $U_t$  subject to the above constraints. By the homotheticity assumption, the young generation's saving rate depends only on the *after tax* real interest rate.<sup>27</sup> Calling  $s((1 - \theta_{t+1}) R_{t+1})$  this saving rate, we have

$$c_t^y = [1 - s((1 - \theta_{t+1}) R_{t+1})] (1 - \tau_t) w_t,$$

$$c_{t+1}^o = (1 - \theta_{t+1}) R_{t+1} s((1 - \theta_{t+1}) R_{t+1}) (1 - \tau_t) w_t.$$

For any  $R_{t+1} > 0$  the assumption that marginal utility goes to infinity as consumption goes to zero ensures that the solutions to the agents' problem is such that  $s((1 - \theta_{t+1}) R_{t+1}) \in (0, 1) \forall R_{t+1} > 0$ . Young agents demand assets to ensure positive consumption in their old age.

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<sup>27</sup> A sketch of the proof is as following. The first order condition for a max implies that the MRS between  $c_t^y$  and  $c_{t+1}^o$  ought to be equal to  $(1 - \theta_{t+1}) R_{t+1}$ . Homotheticity implies that the MRS depends on the  $c_t^y / c_{t+1}^o$  ratio only. Then we obtain  $c_{t+1}^o / c_t^y = f((1 - \theta_{t+1}) R_{t+1})$  for some function  $f$ . Substituting in the budget constraint we obtain  $c_t^y$  as a function of  $(1 - \theta_{t+1}) R_{t+1}$  only, as desired.

### 2.2.2 Firms

There is an indeterminate<sup>28</sup> number of firms in both sectors. Firms in the consumption good sector produce the final output combining capital and labour. Firms in the capital good sector need to employ directly only capital; however the level of public capital supplied determines the level of total productivity. All markets are assumed perfectly competitive. Young individuals supply labour inelastically, which means that the units of labour supplied will always be equal to the number of individuals in the young generation. To simplify expressions, we also assume complete capital depreciation in both sectors. We use the consumption good as numeraire. We now describe each sector in turn.

#### Consumption good sector

The consumption good,  $Y_t$ , is produced combining labour and capital. We assume a Cobb-Douglas function.<sup>29</sup>

$$Y_t = AH_t^\alpha L_t^{1-\alpha}, \quad (4)$$

where  $H_t$  is the amount of capital and  $L_t$  the labour employed at time  $t$ ;  $A \in (0, +\infty)$  and  $\alpha \in (0, 1)$  are technological parameters. Firms maximize profits taking factor prices as given. The price of a unit of capital is  $r_t$ , the price of a unit of labour is  $w_t$ . Then profits are  $Y_t - r_t H_t - w_t L_t$ . Profit maximisation implies that firms will employ capital and labour so that their price equals their marginal products

$$r_t = \alpha AH_t^{\alpha-1} L_t^{1-\alpha}, \quad (5)$$

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<sup>28</sup> Because of the assumption of constant returns to scale. See below.

<sup>29</sup> In fact an argument similar to the one in corollary 3.1 of Fisher [53] applies here, so the Cobb-Douglas case is the only one compatible with long run growth from the class of CES function.

$$w_t = (1 - \alpha) AH_t^\alpha L_t^{-\alpha}. \quad (6)$$

### Capital good sector

The only private input in capital good production is the capital good, indicated by  $X_t$ .

The production function is assumed linear for simplicity<sup>30</sup>

$$K_{t+1} = MX_t g_t, \quad (7)$$

where  $K_{t+1}$  is the capital produced, capital that can be used in the next period to produce the consumption good or more capital.  $g_t$  is the flow of service from the public good, and is considered exogenous by producers.  $M \in (1, +\infty)$  is a technological parameter. Indicating with  $p_t$  the price of capital in terms of the consumption good, profits are given by  $p_t MX_t g_t - r_t X_t$ . The profit maximization condition yields

$$r_t = p_t M g_t. \quad (8)$$

Equations (5) and (8), imply

$$p_t = \frac{\alpha A}{M g_t} H_t^{\alpha-1} L_t^{1-\alpha}. \quad (9)$$

The price of the capital good is proportional to the *private* marginal rate of transformation between the capital good and the consumption good. It should be noted that the price of capital is inversely proportional to the capital/labour ratio in the consumption sector; therefore as the economy grows the price of the capital good decreases.

This property of the model allows for sustained growth. In fact a feature of overlapping

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<sup>30</sup> It could be a more general function  $F(X_t, L_t; g_t)$ ; as Rebelo [88] and Fisher [53] showed, what is necessary in order to obtain sustained growth is that the non-accumulated factor (labour here) is not necessary in production, and that the marginal productivity never falls below the depreciation factor (that we assume to be 1). That is what we need is  $F(X_t, 0; g_t) > 0$  and  $\lim_{K_t \rightarrow \infty} \partial F / \partial K_t > 1$ . To have growth in per capita variables, one needs  $\lim_{K_t \rightarrow \infty} \partial F / \partial K_t > 1 + n$ .

generations models is that every period the whole capital stock must be purchased by the young generation, whose only source of income is wage earning. If the production function in the consumption good sector is concave, as the economy grows, the ratio between wages and capital stock goes to zero. Unless the price of capital declines sufficiently rapidly, there will be a point in which wage earnings are not sufficient to buy the capital stock, thus putting a limit to growth. See Fisher [53] for a thorough discussion of these issues. Note as well, that in this model, the price of capital is inversely proportional to the stream of public services,  $g$ ; this is because a higher level of public provision increases the productivity of the capital good sector, thus decreasing the price of its output.

As explained in the previous paragraph, sustained growth requires the price of capital,  $p_t$ , to decrease over time, i.e.  $p_t/p_{t+1} > 1$ . Now, using (9)

$$\frac{p_t}{p_{t+1}} = \frac{g_{t+1}}{g_t} \left[ \frac{(H_{t+1}/H_t)}{1+n} \right]^{1-\alpha}, \quad (10)$$

therefore the rate at which the price of the capital good falls depends proportionally on the growth rates of public services and capital allocated to the consumption sector and inversely on the rate of population growth.

We now describe the government actions and constraints.

### 2.2.3 Government

The government provides a certain amount of public good,  $G_t$ . As already mentioned in the previous section, this public good produces a stream of services,  $g_t$ , which enter the production function in the capital good sector. In this sense, our analysis differs from the large part of the literature that consider public expenditures as a pure waste, i.e. goods that do not enter either the production or the utility functions. It is also different

from some more recent approaches that are also cast in multi-sector models. In those models, for example Lin [72] or Judd [67], the capital good is interpreted as human capital, and it is inseparable from labour. This implies that the *service* from human capital is tradable, but not human capital itself. That analysis is interesting because it does capture one activity that governments typically embark on, that is provision of public education, but it is ill-suited to analyze different issues, as for example the provision of infrastructures. In this case it seems more appropriate to assume that the public expenditures benefit the production of market goods.

The public good is made out of capital good, and for simplicity we assume that one unit of public good is produced with one unit of capital good; it would be possible to consider a more general specification in which public capital is produced with a given technology employing private capital and possibly labour; as long as this technology has constant returns to scale, the results would not qualitatively change much.

We also assume that there is a certain degree of congestion, which we regard, in general, as quite realistic an assumption. In most cases, the quality of the service provided by public goods decreases with the degree of utilization (for example, one generally travels slower the more congested a road is). We therefore assume that  $g_t = (G_t/X_t)^\beta$ , with  $\beta \in [0, 1]$ . We shall indicate the ratio of public to private capital good input,  $G_t/X_t$ , with  $\mu_t$ . We can then write

$$g_t = (G_t/X_t)^\beta = \mu_t^\beta \quad (11)$$

The government needs to purchase capital goods to provide the public good. It finances its expenditures with proportional taxes and debt.<sup>31</sup> The government can issue

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<sup>31</sup> Of course this is an important assumption: most of the analysis depends on the set of fiscal instruments available to the government. Things would be quite different, for example, if lump-sum taxation would also be available. Buiter and Kletzer [26], for example, discuss how the feasibility of Ponzi finance

bonds,  $B_t$ , every period; we assume that one bond costs one unit of consumption good and promises to pay  $R_{t+1}$  units of consumption good at the end of the next period, after production has taken place. The budget constraint of the government is

$$B_{t+1} + T_t = R_t B_t + p_t G_{t+1}, \quad (12)$$

where  $T_t$  are tax revenues. Note that the timing for  $G$  refers to when it becomes effective, not when the transaction occurs; for each unit of public capital to be in use at time  $t + 1$ , the government has to purchase one unit of private capital at time  $t$ ; that is why  $G_{t+1}$  belongs to time  $t$  budget constraint.

Our justification for considering only proportional taxes is a common one: although given the assumptions of the model there is nothing to preclude the government from using it, lump sum taxation is generally thought to be politically infeasible while proportional taxes constitute the principal instruments in actual tax systems. Having said that, since labour is supplied inelastically, the proportional tax on wages is in effect a lump-sum on the young.<sup>32</sup> We believe that allowing for non-balanced budget policies is interesting, because much of the existing literature on the link between public expenditures and growth assumes away deficit finance, with the remarkable exception of Judd [67] and Turnovsky [105]. Cavalcanti Ferreira [31] also departs from the balanced budget hypothesis, but goes to the other extreme, with zero taxes.

Total tax revenues are given by

$$T_t = \tau_t w_t L_t + \theta_t r_t (X_t + H_t) + \theta_t R_t B_t. \quad (13)$$

We can completely describe the fiscal policy with the triplet  $(\tau_t, \theta_t, \mu_t)$ ; in fact, for given  $\tau$ ,  $\theta$  and  $\mu$ , the budget constraint uniquely determines the amount of new debt issues

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depends on the constraints on taxation.

<sup>32</sup> We will make some further comments on the choice of fiscal instruments at the end of section 2.5.

necessary, given a sequence of interest rates. We make the reasonable assumption that there are upper bounds on the government's ability to tax.

**Assumption 2.1**  $\tau_t, \theta_t < 1 \forall t$ .

**Definition 2.1 (Fiscal Policy)** *A fiscal policy is a sequence  $\{\mu_t, \theta_t, \tau_t\}_{t=0}^{+\infty}$  of tax rates and public/private capital ratio. A stationary fiscal policy is a fiscal policy such that  $\forall t \geq 0$ ,  $\mu_t = \mu$ ,  $\theta_t = \theta$  and  $\tau_t = \tau$  for some  $(\tau, \theta, \mu)$ .*

Next we describe the equilibrium.

## 2.2.4 Equilibrium

An equilibrium for a policy  $\{\mu_t, \theta_t, \tau_t\}_{t=0}^{+\infty}$  and initial conditions  $K_0, B_0$ , is a sequence of quantities  $\{K_t, H_t, X_t, G_t, B_t, c_t^y, c_t^o\}_{t=0}^{+\infty}$  and a sequence of prices  $\{R_t, r_t, w_t, p_t\}_{t=0}^{+\infty}$ , such that in every period all agents are solving their maximization problems, all markets clear and all factors and goods have the same prices across markets. Formally, we have the following definition.

**Definition 2.2 (Equilibrium)** *Given a fiscal policy  $\{\mu_t, \theta_t, \tau_t\}_{t=0}^{+\infty}$ , and an initial amount of capital  $K_0$  and debt  $B_0$ , an equilibrium is a sequence  $\{K_t, H_t, X_t, G_t, B_t, c_t^y, c_t^o\}_{t=0}^{+\infty}$  of quantities, and a sequence  $\{R_t, r_t, w_t, p_t\}_{t=0}^{+\infty}$  of prices, such that  $\forall t \geq 0$ ,*

1.  $G_t/X_t = \mu_t$ ;
2.  $R_{t+1} = r_{t+1}/p_t$  if  $B_{t+1} \neq 0$ ;
3.  $X_t + H_t + G_t = K_t$ ;
4.  $B_{t+1} = (1 - \theta_t) R_t B_t + p_t G_{t+1} - \tau_t w_t L_t - \theta_t r_t (H_t + X_t)$ ;
5.  $L_t s ((1 - \theta_{t+1}) R_{t+1}) (1 - \tau_t) w_t = p_t (K_{t+1} - G_{t+1}) + B_{t+1}$ .

6.  $\{c_t^y, c_{t+1}^o\}$  solve the maximization problems of generations  $t \geq 0$  individuals, i.e.  $(c_t^y, c_{t+1}^o) = \arg \max U_t$  s.t. (2);  $c_0^o$  solve generation  $-1$ 's maximization problem, i.e.  $c_0^o = (1 - \theta_0) [R_0 B_0 + R_0 (K_0 - G_0)] / L_{-1}$ .
7.  $C_t \equiv L_t c_t^y + L_{t-1} c_t^o = Y_t$ .
8.  $(H_t, L_t) = \arg \max [A H_t^\alpha L_t^{1-\alpha} - r_t H_t - w_t L_t]$  and  $X_t = \arg \max [p_t M X_t g_t - r_t X_t]$ .

The first condition in definition 2.2 simply states that the government provision of the public good follows the fiscal policy over time. The second condition in definition 2.2 is an arbitrage condition.<sup>33</sup> The intuition is simple: a young agent who has just received his wage and wants to have positive consumption in the following period, must decide how to invest her savings; if she buys bonds, for each unit of consumption forgone, she will receive  $(1 - \theta_{t+1}) R_{t+1}$  units of consumption next period; if she buys capital, for any units of consumption foregone she will receive  $(1 - \theta_{t+1}) r_{t+1}/p_t$  units from the firms. In an equilibrium with positive levels of debt and capital, the rates of return of the two assets must be the same; this condition must be satisfied for agents to be willing to hold both capital and bonds in their portfolios. A negative value of  $B_t$  implies that the government owned part of the *private* capital stock  $X_t$ ; on this it would earn the same return than a private agent would, hence the equality must still hold. Only when there is no public debt, its rate of return is undefined. This condition implies, using (8), (10) and (11)

$$\begin{aligned} R_{t+1} &= \frac{p_{t+1}}{p_t} M \mu_{t+1}^\beta \\ &= \left[ \frac{1+n}{(H_{t+1}/H_t)} \right]^{1-\alpha} M \mu_t^\beta. \end{aligned} \quad (14)$$

<sup>33</sup> The condition has this particularly simple form because of the assumption of complete depreciation. In a more general case, with a rate of depreciation  $\delta \in [0, 1]$ , it would be the more familiar  $R_{t+1} = \frac{p_{t+1}(1-\delta)+r_{t+1}}{p_t}$ .

The third condition in definition 2.2 imposes market clearing in the capital good market: the total stock of capital in the economy,  $K_t$ , must be equal to the sum of its uses: capital employed in the consumption sector, capital employed in the capital sector and public capital. Note that the condition can be rewritten like  $(1 + \mu_t) X_t + H_t = K_t$ . Furthermore, it will be convenient to define a new variable  $u_t \equiv H_t/K_t$ , that is the fraction of the capital stock devoted to consumption good production. Then we can write

$$H_t = u_t K_t,$$

and

$$X_t = \frac{1 - u_t}{1 + \mu_t} K_t. \quad (15)$$

The fourth condition in definition 2.2 states that the government budget constraint is satisfied. Using the first order conditions (5), (6), (9) and the production functions (4), (7), and the above definitions, the expression for tax revenues (13) becomes

$$T_t = [\alpha \theta_t + (1 - \alpha) \tau_t] A u_t^\alpha K_t^\alpha L_t^{1-\alpha} + \theta_t p_t K_{t+1}.$$

Substituting this last expression in the government budget constraint (12) we obtain

$$B_{t+1} - p_t G_{t+1} = (1 - \theta_t) R_t B_t - [\alpha \theta_t + (1 - \alpha) \tau_t] A u_t^\alpha K_t^\alpha L_t^{1-\alpha} - \theta_t p_t K_{t+1}. \quad (16)$$

The fifth condition is the market clearing condition for the assets market. The demand for assets is given by the aggregate savings of the currently young,  $s((1 - \theta_{t+1}) R_{t+1}) (1 - \tau_t) w_t L_t$ ; the supply is given by the value of capital goods not purchased by the government plus the issue of government bonds,  $p_t (K_{t+1} - G_{t+1}) + B_{t+1}$ . The condition can be rearranged as

$$B_{t+1} - p_t G_{t+1} = s((1 - \theta_{t+1}) R_{t+1}) (1 - \tau_t) (1 - \alpha) A u_t^\alpha K_t^\alpha L_t^{1-\alpha} - p_t K_{t+1}. \quad (17)$$

Finally the sixth, seventh and eighth conditions in definition 2.2 requires all agents to respond optimally to prevailing prices and the consumption goods market to clear.

In the next section we analyze the properties of equilibria; in particular we will show that under certain conditions, associated to any stationary policy there is a unique balanced growth equilibrium.

### 2.2.5 Properties of equilibria

First note that in an equilibrium, using (7), (11) and (15),

$$\frac{K_{t+1}}{K_t} = \frac{1 - u_t}{1 + \mu_t} M \mu_t^\beta, \quad (18)$$

and using (9), (11) and (18),

$$p_t K_{t+1} = \left( \frac{1 - u_t}{1 + \mu_t} \right) \alpha A u_t^{\alpha-1} K_t^\alpha L_t^{1-\alpha} = \frac{\alpha}{1 + \mu_t} \frac{1 - u_t}{u_t} Y_t. \quad (19)$$

It is easier to describe the equilibrium in terms of ratios that do not grow unboundedly. A convenient way to analyse the model is to divide all quantities by  $p_t K_{t+1}$ . Call  $b_{t+1} \equiv B_{t+1}/p_t K_{t+1}$ . Then dividing both sides of the equations by  $p_t K_{t+1}$ , we can rewrite the equilibrium condition<sup>34</sup> (16) as

$$b_{t+1} - \frac{\mu_{t+1}}{1 + \mu_{t+1}} (1 - u_{t+1}) = \frac{(1 + \mu_t)(1 - \theta_t)}{1 - u_t} b_t + [\alpha \theta_t + (1 - \alpha) \tau_t] \frac{1 + \mu_t}{\alpha} \frac{u_t}{1 - u_t} - \theta_t, \quad (20)$$

where we used the fact that<sup>35</sup>

$$\begin{aligned} \frac{R_t B_t}{p_t K_{t+1}} &= R_t \frac{p_{t-1}}{p_t} \frac{K_t}{K_{t+1}} b_t \\ &= \frac{(1 + \mu_t)}{(1 - u_t)} b_t, \end{aligned}$$

<sup>34</sup> We use the fact that  $p_t G_{t+1}/p_t K_{t+1} = (1 - u_{t+1}) \mu_{t+1}/(1 + \mu_{t+1})$ .

<sup>35</sup> The first equality is obtained multiplying and dividing by  $p_{t-1} K_t$ ; the second is obtained exploiting (14) and (18).

and that from (19)

$$\frac{Y_t}{p_t K_{t+1}} = \frac{A u_t^\alpha K_t^\alpha L_t^{1-\alpha}}{p_t K_{t+1}} = \frac{1 + \mu_t}{\alpha} \frac{u_t}{1 - u_t}.$$

Similarly we can write (17) as

$$b_{t+1} - \frac{\mu_{t+1}}{1 + \mu_{t+1}} (1 - u_{t+1}) = s ((1 - \theta_{t+1}) R_{t+1}) \frac{(1 + \mu_t) (1 - \alpha) (1 - \tau_t)}{\alpha} \frac{u_t}{1 - u_t} - 1. \quad (21)$$

Combining (20) and (21) we obtain

$$b_t = \left[ \frac{s ((1 - \theta_{t+1}) R_{t+1}) (1 - \alpha) (1 - \tau_t) + \alpha \theta_t + (1 - \alpha) \tau_t}{\alpha (1 - \theta_t)} \right] u_t - \frac{1 - u_t}{1 + \mu_t} \quad (22)$$

Substituting back into (21) we obtain

$$\begin{aligned} & \frac{s ((1 - \theta_{t+2}) R_{t+2}) (1 - \tau_{t+1}) (1 - \alpha) + \tau_{t+1} (1 - \alpha) + \theta_{t+1} \alpha}{(1 - \theta_{t+1})} \\ &= (1 - \alpha) (1 - \tau_t) s ((1 - \theta_{t+1}) R_{t+1}) (1 + \mu_t) \frac{u_t}{(1 - u_t) u_{t+1}}. \end{aligned} \quad (23)$$

But observing that (18) implies

$$\frac{H_{t+1}}{H_t} = \frac{u_{t+1} K_{t+1}}{u_t K_t} = \frac{M \mu_t^\beta (1 - u_t) u_{t+1}}{1 + \mu_t u_t},$$

from (14) we have

$$\frac{u_t}{u_{t+1} (1 - u_t)} = \frac{R_{t+1}^{\frac{1}{1-\alpha}}}{(1 + n) (1 + \mu_t) \left( M \mu_t^\beta \right)^{\frac{\alpha}{1-\alpha}}}; \quad (24)$$

therefore (23) becomes

$$\begin{aligned} & \frac{s ((1 - \theta_{t+2}) R_{t+2}) (1 - \tau_{t+1}) (1 - \alpha) + \tau_{t+1} (1 - \alpha) + \theta_{t+1} \alpha}{(1 - \theta_{t+1})} \\ &= \frac{(1 - \alpha) (1 - \tau_t) s ((1 - \theta_{t+1}) R_{t+1}) R_{t+1}^{\frac{1}{1-\alpha}}}{(1 + n) \left( M \mu_t^\beta \right)^{\frac{\alpha}{1-\alpha}}} \end{aligned} \quad (25)$$

### Stationary fiscal policies

Let us first look at equilibria with *stationary policies*. When the tax rates  $\tau$  and  $\theta$  and the public/private capital ratio  $\mu$  are kept constant over time, (25) becomes

$$\begin{aligned} & \frac{s((1-\theta)R_{t+2})(1-\tau)(1-\alpha) + \tau(1-\alpha) + \theta\alpha}{(1-\theta)} \\ &= \frac{(1-\alpha)(1-\tau)s((1-\theta)R_{t+1})R_{t+1}^{\frac{1}{1-\alpha}}}{(1+n)(M\mu^\beta)^{\frac{\alpha}{1-\alpha}}}. \end{aligned} \quad (26)$$

or

$$s((1-\theta)R_{t+2}) = \frac{s((1-\theta)R_{t+1})R_{t+1}^{\frac{1}{1-\alpha}}}{(1+n)(1-\theta)(M\mu^\beta)^{\frac{\alpha}{1-\alpha}}} - \frac{\tau}{1-\tau} - \frac{\theta\alpha}{(1-\alpha)(1-\tau)}. \quad (27)$$

This equation gives a "law of motion" for  $R_{t+1}$ . In studying this dynamic equation, one should keep in mind that  $R_{t+1}$  is a price, so we cannot take its initial value as given. The question then becomes what, if any, initial values are consistent with a perfect-foresight rational equilibrium. The easiest case to analyse is the logarithmic utility case.

### Constant saving rate

If the utility function is assumed to be

$$U_t = \ln c_t^y + \sigma \ln c_{t+1}^o,$$

the saving rate is  $\sigma/(1+\sigma) \equiv s$  irrespective of  $R_{t+1}$ . Hence (27) simplifies further to

$$R_{t+1} = \left[ \left( s + \frac{\tau}{(1-\tau)} + \frac{\theta\alpha}{(1-\tau)(1-\alpha)} \right) \frac{(1+n)(1-\theta)}{s} \right]^{1-\alpha} (M\mu^\beta)^\alpha, \quad (28)$$

which is clearly constant over time. Hence in the case of stationary policy and log utility, there is only one initial value  $R_0$  consistent with a perfect foresight equilibrium; this unique value is maintained for all  $t$ .

**Proposition 2.1**      Assume  $s((1 - \theta)R_{t+1}) = \sigma / (1 + \sigma) \equiv s \forall R_{t+1} > 0$ , and a stationary fiscal policy such that

$$\frac{s(1 + \mu)}{\left[ s + \frac{\tau}{(1 - \tau)} + \frac{\theta\alpha}{(1 - \tau)(1 - \alpha)} \right] (1 - \theta)} < 1. \quad (29)$$

Then there exists an unique non-trivial equilibrium. This equilibrium is characterised by a constant  $R_t$ :

$$R = \left[ \left( s + \frac{\tau}{(1 - \tau)} + \frac{\theta\alpha}{(1 - \tau)(1 - \alpha)} \right) \frac{(1 + n)(1 - \theta)}{s} \right]^{1 - \alpha} (M\mu^\beta)^\alpha,$$

and constant  $u_t$ :

$$u = 1 - \frac{s(1 + \mu)}{\left[ s + \frac{\tau}{(1 - \tau)} + \frac{\theta\alpha}{(1 - \tau)(1 - \alpha)} \right] (1 - \theta)}.$$

**Proof.**      The expression for  $R$  follows directly from (28), that for  $u$  by substituting the result into (24). We must also make sure that  $u > 0$ , hence the condition (29) in the proposition. ■

Note that the non-trivial equilibrium is characterised by a constant value of  $u$ , and therefore constant values of  $b$ , growth factor for capital, and aggregate consumption. Therefore there is no transitional dynamics: the economy starts in any of the two equilibria, and stays there forever. We should emphasise that existence of a non-trivial equilibrium does not mean that positive growth is assured. In fact if the resulting value for  $u$  is too small<sup>36</sup>, the economy shrinks at a constant rate. It is interesting to note that the equilibrium values of  $R$  and  $u$  are unaffected by the initial level of debt,  $B_0$ . The

<sup>36</sup> Positive growth of per capita consumption requires

$$\frac{M\mu^\beta(1 - u)}{(1 + \mu)(1 + n)} > 1.$$

explanation is that at all times we must have (see (22))

$$b = \left[ \frac{s(1-\alpha)(1-\tau) + \alpha\theta + (1-\alpha)\tau}{\alpha(1-\theta)} \right] u - \frac{1-u}{1+\mu}.$$

At time zero  $b_0 = B_0/(p_0K_0)$ ;  $B_0$  and  $K_0$  are predetermined, but  $p_0$  jumps to its long-run value in consequence of the jump of  $u_0$  to its long run value.

Finally, looking at (29), we note that an equilibrium will fail to exist when  $\mu$  is too high or when  $\tau$  and  $\theta$  are too small; in both cases the government is trying to run excessively high deficits. We will return to this point in section 2.3.

### **Variable saving rate**

When the saving rate varies with the interest rate, neither the existence nor the instability of a non-zero steady-state for (27) can be established in general. This is reminiscent of a general problem with overlapping generations models: standard assumptions on preferences allow a very wide variety of behaviours of the saving function, which in turn allows a large variety of qualitative dynamics to the economic system. (See Galor and Ryder [55]). Numerical analysis of the specific case of constant intertemporal elasticity of substitution, always confirmed the existence of a unique unstable steady-state for the dynamic equation (27), and hence the existence of a unique equilibrium for the economy. We also experimented with other arbitrary forms for the saving function, and generally find a unique unstable steady-state for (27) when the saving rate function is assumed monotonic (whether increasing or decreasing) in  $r$ . In fact as long as a steady-state exists, a sufficient condition for uniqueness and instability is that the saving rate is non decreasing in the after tax interest rate. To see this, implicit differentiation of (27)

gives (at a steady state)

$$\frac{dR_{t+2}}{dR_{t+1}} = \left[ \frac{R^{\frac{1}{1-\alpha}}}{(1+n)(1-\theta)(M\mu^\beta)^{\frac{\alpha}{1-\alpha}}} \right] \left( 1 + \frac{1}{(1-\alpha)\varepsilon_s} \right),$$

where  $\varepsilon_s \equiv [ds(\cdot)/d(1-\theta)R] [(1-\theta)R/s(\cdot)]$  is the elasticity of the saving rate with respect to the after tax gross interest rate. Using (24), the last equation becomes

$$\frac{dR_{t+2}}{dR_{t+1}} = \frac{1+\mu}{(1-\theta)(1-u)} \left( 1 + \frac{1}{(1-\alpha)\varepsilon_s} \right).$$

Then clearly  $\varepsilon_s \geq 0$  is a sufficient condition for an unique unstable steady-state.<sup>37</sup>

However, counter examples are also easy to construct. We found cases where the steady-state is still unique, but stable; then since  $r_0$  is not given, there is a continuum of possible initial values and correspondingly a continuum of intertemporal equilibria, all converging to the balanced path. We found cases where more than one steady-state exist, some stable some not. In conclusion, for a large set of preferences, technological and policy parameters, an equilibrium exists and it is unique. However, robust cases of multiplicity of equilibria can also be found.

## 2.3 The sustainability of fiscal deficits

In this section we investigate the sustainability of fiscal deficits, maintaining the assumptions of stationary policy and unique equilibrium. We distinguish between primary deficits, which are shown to be unsustainable in the long run, and conventional deficits that may be sustainable. Primary deficits are defined as

$$D_t^p \equiv p_t G_{t+1} - T_t,$$

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<sup>37</sup> Obviously the case of a logarithmic utility function is a special case with  $\varepsilon_s = 0$ .

i.e. the difference between non-interest public spending and tax revenues. Conventional deficits are defined as

$$D_t = R_t B_t + p_t G_{t+1} - T_t,$$

that is the primary deficit plus interest spending.

**Lemma 2.1** *In an equilibrium the government cannot sustain positive primary deficit unless it is a net creditor to the private sector.*

**Proof** Call  $d_t^p \equiv D_t^p / (p_t K_{t+1})$ . From (16) we have

$$b_{t+1} = \frac{R_t B_t}{p_t K_{t+1}} + d_t^p;$$

we can write (see (20))

$$b_{t+1} = \frac{1 + \mu}{1 - u} b_t + d_t^p.$$

In an equilibrium  $b_{t+1} = b_t = b$ , and  $d_t^p = d^p$ . Then we must have

$$b = \frac{d^p}{1 - \frac{1+\mu}{1-u}}. \quad (30)$$

But  $\frac{1+\mu}{1-u} > 1$ , so  $b$  and  $d^p$  must be of opposite sign. ■

In fact ordinary fiscal deficits cannot be sustained either. A necessary condition for deficit finance is that the interest rate is smaller than the rate of growth of income (O'Connell and Zeldes[81]); however this cannot happen in this model. To see this take (24) and assume that a steady-state has been achieved. Re-arranging the equation we get

$$R = \left( \frac{1-u}{1+\mu} M \mu^\beta \right)^\alpha (1+n)^\alpha \frac{1+\mu}{1-u}.$$

But one can easily see that

$$\frac{Y_{t+1}}{Y_t} = \left( \frac{1-u}{1+\mu} M \mu^\beta \right)^\alpha (1+n)^\alpha,$$

and since  $(1 + \mu) / (1 - u) > 1$ , we must have

$$R - 1 > \frac{Y_{t+1} - Y_t}{Y_t}.$$

This results contrasts with results typically obtained in endogenous growth models with externalities in factor accumulation. In those models the interest rate reflects the private marginal product of factors, while the growth rate is determined by the social marginal product. The presence of the externality implies that the former is always below the latter. This allow for the possibility that the growth rate exceeds the interest rate, and thus opens the door to debt finance (Grossman and Yanagawa [60], Saint-Paul [96], King and Ferguson [69]). In our model, though, this cannot happen; as shown above, here the interest rate always exceeds the growth rate. Therefore sustained deficits are only possible if the government is a net creditor.

## 2.4 Optimal allocation

In this section we try to answer the following question: what is the optimal fiscal policy? We start by looking at the solution that an all powerful central planner would choose, and then show that this solution can be replicated by a government with the same policy tools as the one analysed in the previous sections.

We shall concentrate exclusively on the case of constant intertemporal elasticity of substitution utility function. It will be shown that in this case the optimal policy is stationary. In the case of log utility, one can derive closed form solutions.

Let us assume that the utility function for any agent born at time  $t$  is

$$u(c_t^y, c_{t+1}^o) = \frac{(c_t^y)^{1-\gamma}}{1-\gamma} + \sigma \frac{(c_{t+1}^o)^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \sigma > 0,$$

except for the generation that was "born old", whose utility is given simply by  $\sigma (c_0^o)^{1-\gamma} / (1 - \gamma)$ .

We first need to specify the objective function of the planner; this is problematic because in overlapping generations models, there is an infinity of agents, and inescapable trade offs, therefore we must somehow judge how to value different distributions. We assume that the planner would wish to maximise the discounted sum of individual utilities:

$$W \equiv \frac{\sigma (c_0^o)^{1-\gamma}}{\delta (1 - \gamma)} + \sum_{t=0}^{+\infty} \delta^t \left( \frac{(c_t^y)^{1-\gamma} + \sigma (c_{t+1}^o)^{1-\gamma}}{1 - \gamma} \right).$$

Note that  $\sigma$  is the rate at which an individual discount the future, whereas  $\delta$  is the planner's discount factor; the two may or may not be the same. The welfare function  $W$  can be rewritten more conveniently putting contemporaneous terms together<sup>38</sup>

$$W = \sum_{t=0}^{+\infty} \delta^t \left( \frac{(c_t^y)^{1-\gamma} + \frac{\sigma}{\delta} (c_t^o)^{1-\gamma}}{1 - \gamma} \right). \quad (31)$$

The constraints the planner faces are

$$L_t c_t^y + L_{t-1} c_t^o \leq Y_t = A H_t^\alpha L^{1-\alpha}, \quad (32)$$

$$K_{t+1} \leq M X_t^{1-\beta} G_t^\beta, \quad (33)$$

$$X_t + H_t + G_t \leq K_t, \quad (34)$$

$$K_0 \text{ given.} \quad (35)$$

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<sup>38</sup> This is only possible thank to the assumption that the individual utility functions are time separable. It is still possible to analyse the social optimum if we relax this assumption, and the solution may present more complicated dynamics, as demonstrated for a one-sector model by Michel and Venditti [78].

**Lemma 2.2** *The function  $W$  is maximised subject to (32)-(35) if and only if the function  $\bar{W} \equiv \sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma)$  is maximised subject to the same constraints, and*

$$c_t^y = \frac{(Y_t/L_t)}{1 + (1+n)^{\frac{1-\gamma}{\gamma}} \left(\frac{\sigma}{\delta}\right)^{\frac{1}{\gamma}}}, \quad (36)$$

$$c_t^o = \frac{(1+n)^{\frac{1-\gamma}{\gamma}} \left(\frac{\sigma}{\delta}\right)^{\frac{1}{\gamma}}}{1 + (1+n)^{\frac{1-\gamma}{\gamma}} \left(\frac{\sigma}{\delta}\right)^{\frac{1}{\gamma}}} (Y_t/L_{t-1}). \quad (37)$$

**Proof** It is immediately apparent that the planning problem can be split in two subproblems. The first one is how to allocate inputs across the two sectors, the second is how to allocate the consumption sector output between the generations alive. The second problem can be easily solved: the first-order conditions can be rearranged to give

$$\frac{\sigma}{\delta} \left( \frac{c_t^o}{c_t^y} \right)^{-\gamma} = 1 + n,$$

which has a straightforward economic interpretation. The left hand side is the planner's marginal rate of substitution between consumption by the currently old and currently young. One unit of consumption by the young can be converted into  $1 + n$  units of consumption for the old, thus the right-hand side gives the planner's marginal rate of transformation. Optimality requires the two to be equal. Using this condition and (32), we obtain the sharing rules (36) and (37). Substituting these rules back into (31) and rearranging one obtains

$$W = [\phi^{1-\gamma} + (1-\phi)^{1-\gamma} (1+n)^{1-\gamma}] \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma},$$

where

$$\phi \equiv \left[ 1 + (1+n)^{\frac{1-\gamma}{\gamma}} \left(\frac{\sigma}{\delta}\right)^{\frac{1}{\gamma}} \right]^{-1}.$$

Therefore, (31) is maximised if and only if

$$\bar{W} \equiv \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma} \quad (38)$$

is maximised. ■

A standard method of proof (see Lucas and Stokey [103]) establishes the following lemma.

**Lemma 2.3** Call  $\mu^* \equiv \beta / (1 - \beta) = \underset{\mu}{\operatorname{argmax}} \left( M \frac{\mu^\beta}{1+\mu} X_t^\beta \right)$ . Assume

$$\delta \left( M \mu^{*\beta} / (1 + \mu^*) \right)^{\alpha(1-\gamma)} (1+n)^{(1-\alpha)(1-\gamma)} < 1.$$

A feasible path  $\{Y_t, K_t, H_t, G_t\}_{t=0}^{\infty}$ , i.e. a path that satisfies (32)-(35), maximises  $\bar{W}$  if and only if it satisfies the Bellman equation

$$\bar{W}^*(K_t) = \max \left\{ \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma} + \delta \bar{W}^*(K_{t+1}) \right\}, \quad \text{s.t. (32) - (35)}, \quad (39)$$

where  $\bar{W}^*(K_\tau) = \max \sum_{t=\tau}^{+\infty} \delta^t \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma}$  s.t. (32)-(33) and  $K_\tau = K_t$ .

**Proof** See section 2.7. ■

Equation (39) is the Bellman equation. Applying lemma 2.3, we find the optimal solution, as summarised in the following proposition.

**Proposition 2.2** For any  $\gamma > 0$ , the optimal plan is characterised by  $H_t = u^* K_t$ ,  $X_t = (1 - u^*) K_t / (1 + \mu^*)$ ,  $G_t = \mu^* X_t$ , (36), (37), where  $u^*$  is a constant and  $\mu^* = \beta / (1 - \beta)$ . If  $\gamma = 1$  (log utility), then  $u^* = 1 - \delta$ .

**Proof** The proposition is proved by guessing that the value function  $\bar{W}^*(K)$  belongs to the family of functions  $F (K_t/L_t)^{1-\gamma} / (1 - \gamma)$ , where  $F$  is a constant to be determined. Using this guess to find the optimal policy function for any  $F$ , and then use

the method of undetermined coefficients to compute  $F$ . Then the policy function gives the equations for  $H_t$ ,  $X_t$ , and  $G_t$ . In the case of log utility, a closed form solution for  $u$  can be found, for  $\gamma \neq 1$ , we can only show that a  $u^*$  exists, but we cannot give an explicit formula for it. The details of the proof are in section 2.7. ■

One can note that  $\mu$ , i.e. the allocation of capital stock devoted to the capital sector between  $X$  and  $G$ , is always chosen in the way that maximises the output of the capital sector. This is intuitive as any other allocation will result in less capital stock in the future, thus reducing the production possibilities for the next generations, without any advantage for anyone currently alive. A second observation concerns the optimal  $u^*$ ;  $u$  determines the amount of capital devoted to the production of consumer goods. A higher  $u$  means higher consumption today, at the expenses of lower consumption tomorrow. The optimal  $u$  depends on the marginal rates of transformation and substitution between consumption at different points in time. The former depends on the production functions in the two sectors and the growth of the labour supply; the latter depends on the elasticity of intertemporal substitution and the planner's discount rate as well as the growth rate of consumption. So in general  $u$  depends on all parameters in the utility and production functions as well as on population growth. Looking at (41), it is clear, for example, that the effect on  $u$  of an increase in  $M$  depends on the sign of  $1 - \gamma$ . The intuition is that an increase in productivity has an income and a substitution effect. If  $1 - \gamma > 0$  the income effect dominates, and *vice versa*. In the logarithmic case the two effects cancel out exactly; then we obtain a very simple closed form solution that relates the optimal  $u$  to the degree of impatience of the planner only. A similar result was obtained by Radner [85]. One can also draw a comparison with the one-sector growth model: with log-utility, Cobb-Douglas production function and full depreciation, it is

well known<sup>39</sup> that the optimal saving rate is equal to the product of the discount factor and the output elasticity with respect to capital. In our model the elasticity with respect to capital of the capital sector production function is unity. Thus since both the saving rate in the one-sector model and  $1 - u$  in our model measure the fraction of consumption forgone to allow future capital accumulation, there is a clear analogy between the two results.

A final remark is that the allocation of capital between consumption and capital sectors,  $u$ , is a function of the planner's discount factor  $\delta$  but not of the discount factor of the households,  $\sigma$ . The latter does influence how the planner divide any given amount of output between old and young agents (see the sharing rules (36) and (37)), but not the intertemporal allocation of resources. For an analogous result, see Calvo and Obstfeld [27] and De La Croix and Michel [38].

## 2.5 Implementation of the optimal policy

In this section we show that the fiscal instruments considered in section 2.2 are sufficient to decentralise the optimal policy. More precisely, we show that given quantities  $\{H_t, X_t, G_t, c_t^y, c_t^o\}_{t=0}^{+\infty}$ , there exist prices  $\{w_t, R_t, p_t\}_{t=0}^{+\infty}$ , tax rates<sup>40</sup>  $\{\tau_t, \theta_t\}$  and a sequence of debt stocks  $\{B_t\}_{t=0}^{+\infty}$ , that constitute an equilibrium. That is any feasible path is decentralisable as a competitive equilibrium, including obviously the optimal path.

Firstly we observe that the factor prices must obey the firms' first order conditions

(5) and (6), so  $\forall t \geq 0$

$$r_t = \alpha A H_t^{\alpha-1} L_t^{1-\alpha},$$

<sup>39</sup> See for example Ljungqvist and Sargent [73] who attribute the result to Brock and Mirman [24].

<sup>40</sup> It should be noted that we do not impose any limit on the size of the tax rates. The implied taxes may turn out to be negative, in which case we interpret them as subsidies.

$$w_t = (1 - \alpha) AH_t^\alpha L_t^{-\alpha}.$$

The price of capital must obey (9), so  $\forall t \geq 0$

$$p_t = \frac{\alpha AH_t^{\alpha-1} L_t^{1-\alpha}}{MG_t^\beta X_t^{-\beta}}.$$

Then the arbitrage condition in definition 2.2 gives  $\forall t \geq 0$

$$R_{t+1} = r_{t+1}/p_t;$$

note that this leaves  $R_0$  still undetermined. The tax on capital earning can be derived from the first order condition of a young household<sup>41</sup>, so  $\forall t \geq 0$

$$\theta_{t+1} = 1 - \frac{u_y(c_t^y, c_{t+1}^o)}{u_o(c_t^y, c_{t+1}^o) R_{t+1}},$$

where  $u_y(c_t^y, c_{t+1}^o) \equiv \partial u(c_t^y, c_{t+1}^o) / \partial c_t^y$ ,  $u_o(c_t^y, c_{t+1}^o) \equiv \partial u(c_t^y, c_{t+1}^o) / \partial c_{t+1}^o$ . Note that again  $\theta_0$  is left undetermined. Then rearranging the household budget constraint, we have  $\forall t \geq 0$

$$\tau_t = 1 - \left[ c_t^y + \frac{c_{t+1}^o}{(1 - \theta_{t+1}) R_{t+1}} \right] \frac{1}{w_t}.$$

The sequence of debt stocks is then given by the government budget constraint

$$B_{t+1} = (1 - \theta_t) R_t B_t + p_t G_{t+1} - \tau_t w_t L_t - \theta_t r_t K_t.$$

As we noted,  $\theta_0$  and  $R_0$  are undetermined, but we must have

$$c_0^o = (1 - \theta_0) [r_0 K_0 + R_0 B_0].$$

In conclusion we have shown that the fiscal instruments considered in the first part of this chapter are sufficient to implement the first best optimum. We showed in the previous section that the optimal allocation is characterised by constant  $\mu$  and  $u$ ,

<sup>41</sup> That is

$$\frac{u_o(c_t^y, c_{t+1}^o)}{u_y(c_t^y, c_{t+1}^o)} = \frac{1}{(1 - \theta_{t+1}) R_{t+1}}.$$

and therefore constant growth for all variables. It is easy to see from the preceding derivations, that this implies constant tax rates  $\tau$  and  $\theta$ .

For the specific case of log utility, we can derive closed form solutions, which are appealing because they allow us to highlight the economic intuition behind the optimal policy more clearly. In this case from the sharing rules (36) and (37)

$$\frac{c_{t+1}^o}{c_t^y} = \left(\frac{\sigma}{\delta}\right) (Y_{t+1}/Y_t) = \left(\frac{\sigma}{\delta}\right) \left(\frac{\delta M \mu^{*\beta}}{1 + \mu^*}\right)^\alpha (1 + n)^{1-\alpha}$$

or

$$R = (M \mu^{*\beta})^\alpha (1 + \mu^*)^{1-\alpha} \left(\frac{1 + n}{\delta}\right)^{1-\alpha}.$$

Following the steps highlighted above, we find

$$1 - \tau = \left(\frac{\delta}{\delta + \sigma}\right) \left(\frac{1 + \sigma}{1 - \alpha}\right),$$

$$\theta = \beta;$$

Therefore the optimal labour income tax depends on the planner's and households' discount factors as well as the labour share in the consumption sector; One can note that the optimal tax may be negative, i.e. implementation of the first best may require a subsidy to labour income. The capital income tax, instead, is a function of the elasticity with respect to public capital of the capital sector aggregate production function,  $\beta$ .<sup>42</sup>

In contrast with Jones and Manuelli [62], Rebelo [88], Devereux and Love [41] and King and Rebelo [70], in this model the optimal level of capital income taxation is strictly positive. The reason for this result has to do with the overlapping generations structure. In fact one can show that  $R_{t+1}$  equals the marginal rate at which society can

<sup>42</sup> Remember that our assumptions imply that at the aggregate level the capital sector production function can be written:  $K_{t+1} = M X_t^{1-\beta} G_t^\beta$ .

transform consumption at time  $t$  into consumption at time  $t + 1$ .<sup>43</sup> So in this model -in contrast with Barro [13] and Turnovsky [105]- the capital income tax is not needed to equalise the social and marginal rates of transformations between consumption in different time periods. But at the optimum the planner's marginal rate of substitution between consumption in different periods differs from that of a given household; this is a crucial difference between the overlapping generation setup and the representative agent framework of Chamley [34]. Hence in an overlapping generation model, a positive capital income tax is generally optimal (Erosa and Gervais [50], Mathieu-Bolh [76]).

We conclude this section with two remarks. The ability to decentralise the first best optimum depends on having a sufficient number of instruments. In turn, the number of instruments needed depends on the number of margins to be controlled. For example, if the labour supply were elastic, the wage income tax would have to be chosen to elicit the optimal amount of labour and a further fiscal instrument would be needed to induce the first best level of consumption (e.g. a tax on consumption).

The second remark is that in general the set of instruments that are sufficient to decentralise the first best is not unique. For example, in an earlier version of this paper we considered the case of different taxes in the two sectors rather than different taxes

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$$\begin{aligned} \frac{dY_{t+1}}{dY_t} &= \frac{\partial Y_{t+1}}{\partial K_{t+1}} \frac{dK_{t+1}}{dY_t} \\ &= -MPK_{Y_{t+1}} (MPK_{K_t} / MPK_{Y_t}), \end{aligned}$$

where  $MPK$  stands for marginal product of capital and the subscript indicates the sector and timeperiod. Using (5), (9), (7), (11) and the no arbitrage condition 2 of definition 2.2 one indeed finds

$$R_{t+1} = -\frac{dY_{t+1}}{dY_t}.$$

on different sources of income; again we found that set of fiscal instruments sufficient for the decentralisation of the first best.

## 2.6 Conclusions

The effect of different fiscal policies on long-run growth is an issue that has interested both academics and policy makers for a very long time. Two aspects of this issue that we feel are very important are: (i) whether a certain deficit policy is sustainable in the long-run; (ii) whether allowing for deficit finance enlarges the set of feasible allocations.

We have presented an overlapping generations model in which the government supplies a public good which acts as an externality in the capital sector, and it is subject to congestion. The technology has constant returns to scale. We showed that the economy is capable of sustained endogenous growth. Given some simplifying assumptions, in particular that both kind of capital depreciate completely in one period, there is no transitional dynamics, the economy settles immediately on the balanced path whenever one exists.

Concerning the first question, therefore, we reached a negative answer: in this model debt finance is possible only if the government is a net creditor. Clearly the result depends crucially on the technological assumptions: we showed that our assumptions on technologies imply that the rate of return dominates the growth rate, making it impossible for the government to sustain positive primary deficit in the long-run, unless it is a net creditor in the economy. In chapter 3 we present a model with the same demographic structure but different technological assumptions and show that in that framework positive primary deficit might be sustainable in the long-run.

Concerning the second question, Ghigliano and Shell [56] showed in a pure exchange economy that imposing limits on the size of the deficit the government can maintain does not matter if the government can use lump-sum taxation, but would reduce the set of feasible allocation if only proportional taxation is allowed. But being based on a pure exchange economy, their model has nothing to say on the effects on growth. We have shown that any feasible allocation can be decentralised given the fiscal instruments we considered. It would not have been possible to establish this result if we eliminated public debt.<sup>44</sup> In fact the decentralisation of the first best requires a determinate path of public debt. It should be emphasised, however, that there are alternative fiscal tools that can also be used to decentralise the same allocation. For example in an earlier version of this chapter we considered sector specific taxes and also showed that all feasible allocations could be decentralised. In future research we intend to investigate these issues further.

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<sup>44</sup> Unless of course we give the government other instruments. We noted in the previous section that there is more than one set of instruments that allow the decentralisation of the optimum. Debt can undoubtedly be replaced with some other fiscal instrument, but unless we expand the set of available taxes, it is necessary to decentralise the first best.

## 2.7 Proofs of propositions in chapter 2

### 2.7.1 Proof of lemma 2.3

The method of proof is standard (see Lucas and Stokey [103] and De La Croix and Michel [38]) and composed of 4 main steps. First we show that for all feasible allocation  $\sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma) \in \Re \cup \{-\infty\}$ . Then we show that the function  $V(K, L_t) = \sup \left\{ \sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma) \right\}$  is defined and satisfies  $V(K_t, L_t) = \sup \left\{ (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1}) \right\}$ . The third step is to show that a given path is optimal if and only if  $V(K_t^*, L_t) = \left\{ (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}^*, L_{t+1}) \right\}$ . The fourth and final step is to show that the supremum is reached, so we can substitute *max* for *sup*.

**Lemma 2.4**      Call  $\mu^* \equiv \beta / (1 - \beta) = \underset{\mu}{\operatorname{argmax}} \left( M \frac{\mu^\beta}{1+\mu} X_t^\beta \right)$ . If

$$\delta \left( M \mu^{*\beta} / (1 + \mu^*) \right)^{\alpha(1-\gamma)} (1+n)^{(1-\alpha)(1-\gamma)} < 1,$$

then for any feasible sequence  $\{Y_t, K_t, H_t, X_t, G_t\}_{t=0}^{+\infty}$ ,

$$\sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma) < +\infty.$$

**Proof**      Let us indicate with  $\{K_t^{acc}\}_{t=0}^{+\infty}$  the path that satisfies  $K_0^{acc} = K_0$ ,  $K_{t+1}^{acc} = M \frac{\mu^{*\beta}}{1+\mu^*} K_t^{acc}$ ; we shall call this the *pure accumulation path*. This path gives an upper bound for  $K_t$  for any given  $t$ . Consider the sequence  $\{Y_t^{acc}/L_t\}_{t=0}^{+\infty}$  obtained from  $Y_t^{acc}/L_t \equiv A (K_t^{acc}/L_t)^\alpha$ . As  $K_t^{acc} = K_0 (M \mu^{*\beta} / (1 + \mu^*))^t$ ,

$$Y_t^{acc} = Y_0^{acc} \left( M \mu^{*\beta} / (1 + \mu^*) \right)^{\alpha t} (1+n)^{(1-\alpha)t}.$$

For any feasible path,  $Y_t \leq Y_t^{acc}$  with strict inequality for at least all but one  $t$ ; hence

$$\begin{aligned}
& \sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma) < \sum_{t=0}^{+\infty} \delta^t (Y_t^{acc}/L_t)^{1-\gamma} / (1-\gamma) \\
& = \sum_{t=0}^{+\infty} \delta^t \left( Y_0^{acc} (M\mu^{*\beta} / (1+\mu^*))^{\alpha t} (1+n)^{(1-\alpha)t} / L_0 \right)^{1-\gamma} / (1-\gamma) \\
& = \frac{Y_0^{acc}/L_0}{1-\gamma} \sum_{t=0}^{+\infty} \left( \delta (M\mu^{*\beta} / (1+\mu^*))^{\alpha(1-\gamma)} (1+n)^{(1-\alpha)(1-\gamma)} \right)^t \\
& < +\infty. \blacksquare
\end{aligned}$$

Define  $V(K)$  the function defined by  $V(K) = \sup \left\{ \sum_{t=0}^{+\infty} \delta^t (Y_t/L_t)^{1-\gamma} / (1-\gamma) \right\}$ , where the supremum is taken over all sequences feasible from  $K$ .

**Lemma 2.5**      *The function  $V(K)$  is defined for all  $K$  and satisfies*

$$V(K_t, L_t) = \sup \left\{ (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1}) \right\}.$$

**Proof**      First note that for all  $K$  steps similar to those taken in the previous lemma show that  $V(K_t, L_t) < +\infty$ . Furthermore  $V(K_t, L_t) > -\infty$ , as one feasible path can always be found by taking arbitrary and constant  $u$  and  $\mu$ . These paths will be characterised by positive consumption in all periods and give a value to the social objective function that is a real number. We next need to show that for any  $K_t$

$$V(K_t, L_t) \geq (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1})$$

for any  $Y_t$  and  $K_{t+1}$  feasible from  $K_t$ . And that for any  $\varepsilon > 0$

$$V(K_t, L_t) \leq (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1}) + \varepsilon$$

for some  $Y_t, K_{t+1}$  feasible from  $K_t$ . To show the first, note that by the definition of  $V$ , for any  $\varepsilon > 0$  there exist  $\{Y'_\tau, K'_{\tau+1}\}_{\tau=t+1}^{+\infty}$  feasible from  $K_{t+1}$ , such that

$$\sum_{\tau=t+1}^{+\infty} \delta^\tau (Y'_\tau/L_\tau)^{1-\gamma} / (1-\gamma) \geq V(K_{t+1}, L_{t+1}) - \varepsilon.$$

Hence

$$\begin{aligned} V(K_t, L_t) &\geq (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta \sum_{\tau=t+1}^{+\infty} \delta^\tau (Y'_\tau/L_\tau)^{1-\gamma} / (1-\gamma) \\ &\geq (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1}) - \delta\varepsilon, \end{aligned}$$

and since  $\varepsilon$  was arbitrary, we established that  $V(K_t, L_t) \geq (Y_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}, L_{t+1})$  as required. Next for any  $K_t$  and  $\varepsilon > 0$ , one can choose a path  $\{Y''_\tau, K''_{\tau+1}\}_{\tau=t}^{+\infty}$  feasible from  $K_t$  such that

$$\begin{aligned} V(K_t, L_t) &\leq \sum_{\tau=t}^{+\infty} \delta^\tau (Y''_\tau/L_\tau)^{1-\gamma} / (1-\gamma) + \varepsilon \\ &= (Y''_t/L_t)^{1-\gamma} / (1-\gamma) + \delta \sum_{\tau=t+1}^{+\infty} \delta^\tau (Y''_\tau/L_\tau)^{1-\gamma} / (1-\gamma) + \varepsilon. \end{aligned}$$

But from the definition of  $V$ , it follows that

$$V(K_t, L_t) \geq (Y''_t/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K''_{t+1}, L_{t+1}) + \varepsilon,$$

which is what we wanted to show. ■

We have completed the first two steps of our proof. The third is accomplished by the next lemma.

**Lemma 2.6** *A feasible path  $\{Y_t^*, H_t^*, X_t^*, G_t^*, K_t^*\}_{t=0}^{+\infty}$  is optimal if and only if for any  $t$*

$$V(K_t^*, L_t) = (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}^*, L_{t+1}).$$

**Proof** We first prove necessity. If  $\{Y_t^*, H_t^*, X_t^*, G_t^*, K_t^*\}_{t=0}^{+\infty}$  is optimal, then

$$\begin{aligned} V(K_t^*, L_t) &= \sum_{\tau=t}^{+\infty} \delta^\tau (Y_\tau^*/L_\tau)^{1-\gamma} / (1-\gamma) = (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) \\ &\quad + \sum_{\tau=t+1}^{+\infty} \delta^\tau (Y_\tau^*/L_\tau)^{1-\gamma} / (1-\gamma) \\ &= (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}^*, L_{t+1}). \end{aligned}$$

To prove sufficiency, assume that  $\{Y_t^*, H_t^*, X_t^*, G_t^*, K_t^*\}_{t=0}^{+\infty}$  is such that  $V(K_t^*, L_t) = (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) + \delta V(K_{t+1}^*, L_{t+1})$  for all  $t$ . By induction

$$V(K_t^*, L_t) = \sum_{t=0}^T \delta^t (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) + \delta^T V(K_{T+1}^*, L_{T+1}).$$

Taking the limit at  $T \rightarrow +\infty$

$$V(K_t^*, L_t) = \sum_{t=0}^{\infty} \delta^t (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) + \lim_{T \rightarrow \infty} \delta^T V(K_{T+1}^*, L_{T+1}).$$

We therefore need to show that  $\lim_{T \rightarrow \infty} \delta^T V(K_{T+1}^*, L_{T+1}) = 0$ . To see this note that in lemma 2.4 we showed that

$$\begin{aligned} V(K_t, L_t) &\leq \frac{(AK_t^\alpha L_t^{-\alpha})^{1-\gamma}}{1-\gamma} \sum_{t=0}^{+\infty} \delta^t \left( (M\mu^\beta / (1+\mu))^{\alpha(1-\gamma)} (1+n)^{(1-\alpha)(1-\gamma)} \right)^t \\ &\equiv J(K_t, L_t) < +\infty. \end{aligned}$$

But then

$$\delta^t V(K_t, L_t) \leq \delta^t J(K_t, L_t),$$

and clearly  $\lim_{t \rightarrow +\infty} \delta^t V(K_t, L_t) = 0$ . ■

Finally, we prove that the supremum is achieved.

**Lemma 2.7** *There exists a feasible path  $\{Y_t^*, H_t^*, X_t^*, G_t^*, K_t^*\}_{t=0}^{+\infty}$  such that*

$$\sum_{t=0}^{+\infty} \delta^t (Y_t^*/L_t)^{1-\gamma} / (1-\gamma) = V(K_0, L_0),$$

*i.e. the supremum is achieved.*

**Proof** We first show that the feasible set is convex, and that  $\bar{W}$  is concave and therefore continuous. Then use Weierstrass theorem, to show that the supremum is attained. Take any two feasible paths  $\{Y_t^i, H_t^i, X_t^i, G_t^i, K_t^i\}_{t=0}^{+\infty}$ ,  $i = 1, 2$ . Consider  $K_0^\lambda = \lambda K_0^1 + (1 - \lambda) K_0^2$ ,  $\lambda \in [0, 1]$ . The allocation  $X_0^\lambda = \lambda X_0^1 + (1 - \lambda) X_0^2$ ,  $H_0^\lambda = \lambda H_0^1 + (1 - \lambda) H_0^2$ ,  $G_0^\lambda = \lambda G_0^1 + (1 - \lambda) G_0^2$  is feasible and produces  $Y_0^\lambda = \lambda Y_1^1 + (1 - \lambda) Y_1^2$ , and  $K_1^\lambda = \lambda K_1^1 + (1 - \lambda) K_1^2$ . By induction, any  $\{Y_t^\lambda, H_t^\lambda, X_t^\lambda, G_t^\lambda, K_t^\lambda\}_{t=0}^{+\infty}$  is feasible, hence the set of all feasible path is a convex set.

Now from the definition of  $\bar{W}$ , for any two feasible paths  $\{Y_t^i, H_t^i, X_t^i, G_t^i, K_t^i\}_{t=0}^{+\infty}$ ,  $i = 1, 2$ , we have

$$\begin{aligned} \bar{W}(K_0^\lambda) &= \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t^\lambda/L_t)^{1-\gamma}}{1-\gamma} \geq \lambda \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t^1/L_t)^{1-\gamma}}{1-\gamma} + (1-\lambda) \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t^2/L_t)^{1-\gamma}}{1-\gamma} \\ &= \lambda \bar{W}(K_0^1) + (1-\lambda) \bar{W}(K_0^2). \end{aligned}$$

that is  $\bar{W}$  is a concave function and therefore it is continuous. The set of all feasible path is the product of closed, bounded sets, and therefore is compact in the product topology by Tychonov theorem. By Weierstrass theorem a continuous function defined on a compact set attains its maximum in that set. ■

The lemmas in this section imply that a function  $\bar{W}^*(K_t, L_t)$  exists such that  $\bar{W}^*(K_t, L_t) = \max \sum_{t=0}^{+\infty} \delta^t \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma}$  and it satisfies the Bellman equation (39). Furthermore if a feasible path is such that

$$\bar{W}^*(K_t) = \frac{(Y_t/L_t)^{1-\gamma}}{1-\gamma} + \delta \bar{W}^*(K_{t+1}),$$

then the path is optimal. QED.

### 2.7.2 Proof of proposition 2.2

We guess that the function that satisfies the Bellman equation (39) is of the form<sup>45</sup>

$$\bar{W}^*(K_t, L_t) = \frac{F}{1-\gamma} \left( \frac{K_t}{L_t} \right)^{\alpha(1-\gamma)}.$$

Then the Bellman equation (39) is

$$\begin{aligned} \frac{F}{1-\gamma} \left( \frac{K_t}{L_t} \right)^{\alpha(1-\gamma)} &= \max \left\{ \left( \frac{Y_t}{L_t} \right)^{1-\gamma} + \delta \frac{F}{1-\gamma} \left( \frac{K_{t+1}}{L_{t+1}} \right)^{\alpha(1-\gamma)} \right\} \\ &= \max \left\{ \frac{(A u_t^\alpha K_t^\alpha L_t^{-\alpha})^{1-\gamma}}{1-\gamma} + \delta \frac{F}{1-\gamma} \left( M \mu^\beta \frac{(1-u_t)}{1+\mu} \frac{K_t}{(1+n)L_t} \right)^{\alpha(1-\gamma)} \right\}. \end{aligned}$$

Maximisation with respect to  $\mu_t$  yields

$$\mu_t = \frac{\beta}{1-\beta} = \mu^*.$$

Maximisation with respect to  $u_t$  yields

$$u_t = \left( 1 + \chi F^{\frac{1}{1-\alpha(1-\gamma)}} \right)^{-1} \equiv u(F), \quad (40)$$

where

$$\chi \equiv \left[ \frac{\delta}{A^{1-\gamma}} \left( \frac{M \mu^{*\beta}}{(1+\mu^*)(1+n)} \right)^{\alpha(1-\gamma)} \right]^{\frac{1}{1-\alpha(1-\gamma)}}.$$

Note that  $u(F)$  is a decreasing function, with  $u(0) = 1$ ,  $u(+\infty) = 0$ . Then substituting back into the maximand, we find after simplifications

$$F = A^{1-\gamma} u(F) + \left( \frac{M \mu^{*\beta}}{(1+\mu^*)(1+n)} \right)^{\alpha(1-\gamma)} \delta F (1-u(F))^{\alpha(1-\gamma)}. \quad (41)$$

As functions of  $F$ , the left-hand side is a straight line through the origin with unitary slope. The right-hand side is a function that tends to  $A^{1-\gamma}$  as  $F$  goes to 0 and tends to  $\left( \frac{M \mu^{*\beta}}{(1+\mu^*)(1+n)} \right)^{\alpha(1-\gamma)} \delta F$  as  $F$  goes to infinity. Since we assume

$$\delta \left( M \mu^\beta / (1+\mu) \right)^{\alpha(1-\gamma)} (1+n)^{(1-\alpha)(1-\gamma)} < 1$$

<sup>45</sup> The guess is obtained by conjecturing that the optimal policy is to keep  $u_t$  and  $\mu_t$  constant over time and then computing the implied value for  $\bar{W}$ .

and  $n \geq 0$ ,  $\delta \left( \frac{M\mu^{*\beta}}{(1+\mu^*)(1+n)} \right)^{\alpha(1-\gamma)} < 1$ . But then the right-hand side must eventually be below the left-hand side, thus an intersection must exist. Therefore the equation can in principle be solved for  $F$ , although an explicit solution cannot be found in general. Substituting the value for  $F$  into (40) we find the candidate for a solution. Note that by construction the candidate solution and  $\bar{W}^*$  satisfy (39) and thus by lemma 2.3 we found the optimal policy.

In the case of log utility, we guess that the value function will be of the family

$$\bar{W}^*(K_t, L_t) = E + F \ln \left( \frac{K_t}{L_t} \right)^\alpha,$$

for some  $E$  and  $F$ . Then the Bellman equation is

$$E + F \ln \left( \frac{K_t}{L_t} \right)^\alpha = \max \left\{ \ln A u_t^\alpha K_t^\alpha L_t^{-\alpha} + \delta E + \delta F \ln \left( \frac{M\mu^\beta (1-u_t) K_t}{(1+\mu)(1+n) L_t} \right)^\alpha \right\}.$$

Maximisation with respect to  $\mu$  still yields  $\mu^*$ . Maximisation with respect to  $u_t$  yields

$$u^* = \frac{1}{1 + \delta F}.$$

Substituting back into the Bellman equation we find

$$F = \frac{1}{1 - \delta}$$

and so

$$u^* = 1 - \delta. \blacksquare$$

### 3 One-sector Model

It should not be too controversial that: (i) economic development needs adequate infrastructures; (ii) in many developing countries, the government is severely limited in its capability of borrowing from abroad. In this chapter we try to investigate the interaction between these two observations. We consider an economy in which the stock of public capital (infrastructures) determines technological progress. The government finances public expenditures through taxation and domestic borrowing. We, therefore, abstract from seigniorage and external borrowing.

The seminal papers of Romer [92] and Lucas [74] on endogenous growth have stimulated an impressive amount of new research on the causes of economic growth and development. But whereas some undoubtedly interesting issues (such as, just to give an example, the convergence-divergence debate) have been extensively investigated both theoretically (Azariadis and Drazen, [9]) and empirically (Barro and Sala-i-Martin, [16]), others have received far less attention that they deserve. For example, one of the most important contribution of the endogenous growth theory is that it has allowed the development of a theory in which government intervention can have effects on growth rates in the long run as well as in the short run. However, whereas there is a rich literature that analyses fiscal and monetary policies in the context of exogenous growth, endogenous growth models often assume balanced budget policy rules (there are of course exceptions; recently there has been an increasing interest in debt finance, as documented below).

In an important line of research, endogenous growth is possible because public capital enters the aggregate production function. However, with few exceptions (on which more later), not enough attention has been devoted to the study of the mode in which public investment is financed.

The contribution of public capital to development is an old theme, going back at least to Rosenstein-Rodan [95]. More recent interest has followed Murphy et al. [79] and Barro's [13] model of endogenous growth with government expenditures as an input in production. Glomm and Ravikumar [58] is another recent important paper on the topic.

On the other hand, the effects of public debt on growth is also an extensively studied topic. The obvious quotation is, of course, the Diamond [42] model of capital accumulation and growth with overlapping generation. Tirole [104] helped clarifying some interesting issues connected to the feasibility of Ponzi scheme; a theme re-examined in the endogenous growth context by King and Ferguson [69] and Grossman and Yanagawa [60]. More recently, Chalk [33] showed that sustainability of a permanent deficit requires more than simply the rate of interest to be less than the growth rate. Far less attention has been devoted to the possibility that public investment could be financed (at least in part) through domestic borrowing.<sup>46</sup> Rioja [90] notes that in Latin America fiscal restraint in the mid-1980s (called for by stabilization plans) was in part responsible for the decline in public investment that has not yet recovered 1970s levels. Although developing countries' debt is mainly external debt, there are cases with significant level of domestic indebtedness (e.g. Mexico. See Agenor and Montiel, [1]).

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<sup>46</sup> In their seminal work, Arrow and Kurz [5] do consider this possibility.

In none of the papers quoted in the previous paragraph, however, do public expenditures enter the production function. We believe that this omission is crucial; although it is undeniable that part (maybe even a large part) of public expenditure are unproductive (being mere redistribution of income<sup>47</sup> if not even purely wasted resources), at least part of it is devoted to the creation, maintenance and updating of infrastructures. Cavalcanti Ferreira [31], presents a very similar model compared with ours. But there is seigniorage, not debt, the financial resource for the government, and it is the flow, not the stock of capital that matters; furthermore there is no taxation and the consequences of an imposition of limits on the size of the fiscal deficit are not analysed.

Our analysis shows that when the rate of public investment is sufficiently high, there exist a steady-state in which the government runs perpetual primary deficits. For any initial level of the capital stock, there is a unique level of public debt such that the economy converges to that steady-state. Any bigger initial level of debt means that the combination of income tax and public investment is not sustainable; for lower levels the economy converges to a different steady state with primary surpluses. An analogous point was made by Chalk [33] in an exogenous growth context. However in our model some policy changes may have unexpected results. Specifically, if initially we are at the steady-state with primary deficits, a tax increase will make the policy combination unsustainable. There is no analogous result in Chalk's paper. We consider this result unexpected as one would expect a reduction in the deficit to imply that the policy should be more sustainable, not less. What explain this result is that the steady state we start from is characterised by a large level of debt. The tax increase reduces

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<sup>47</sup> One should note, however, that in certain frameworks a redistribution of income can have consequences for long-run growth. Jones and Manuelli [63] argue that in a one-sector overlapping generations model with convex technologies, sustained growth is possible only with an appropriate redistribution of income.

private capital accumulation. This tends to push up the marginal product of capital and therefore the interest rate. This increases the government's interest payment by more than the increase tax reduces the primary deficit. Hence the overall fiscal position of the government worsen rather than improve.

A crucial feature of the model we analyse is the way in which public investment affects technological progress. An obvious alternative to our approach is to consider public capital entering directly the production function. Chapter 2 analysed such a model in a two-sector framework. There we showed that primary deficits could not be sustained, hence none of the points made in the paragraph above are valid. In a one-sector model à la Barro [13] things are likely to be different. If the government maintains a constant public to private capital ratio, the marginal product of capital and hence the interest rate will be constant and the interest payment effects we discussed in the previous section will be absent. Therefore we conjecture that a tax increase should have the expected effect of improving sustainability. A related analysis has been provided in a recent contribution by Yakita [111]. Given the contributions of Cazzavillan [32] and Azariadis and Reichlin [10] one may also expect complex dynamics to arise.

### 3.1 The model

We consider an overlapping generation economy in which individuals live two periods, supply labour inelastically in the first, and retire in the second. Assume, furthermore, that the utility function of an individual born at time  $t$  is

$$U_t \equiv u(c_t^y, c_{t+1}^o), \quad (1)$$

where the subscript indexes the time at which consumption occurs, the superscript the age of the consumer. The intertemporal budget constraint is

$$c_t^y + \frac{c_{t+1}^o}{R_{t+1}} \leq \tilde{W}_t, \quad (2)$$

where  $\tilde{W}_t$  is the after tax wage earned at time  $t$ ,  $R_{t+1}$  the after tax gross interest rate between time  $t$  and  $t + 1$ . For simplicity, we will assume that

$$U_t = \ln c_t^y + \beta \ln c_{t+1}^o,$$

where  $\beta > 0$  is the discount factor. This implies, of course, a considerable loss in generality, but on the other hand, the fact that then the saving rate is independent from the interest rate<sup>48</sup> makes the derivation of the results very simple. Although the assumption of logarithmic function for utility is often defended on the grounds that empirical studies found little evidence of sensitivity of saving rates to interest rate, we regard this as a simplification useful to obtain a first approximation, but in later research a more general functional form has to be allowed. With this assumption, however, the saving rate is going to be  $s = \beta / (1 + \beta)$ .

Next we characterize the production side of the economy. We assume that the production function can be written as

$$Y_t = F(K_t, A_t L_t), \quad (3)$$

where  $Y$  stands for output,  $K_t$  for the stock of private capital,  $A_t$  measures labour productivity and  $L_t$  the labour employed. Since labour is supplied inelastically, the economy will be always in full employment. We will abstract from population growth, therefore  $L_t = L, \forall t$ . We will assume that  $F(\cdot, \cdot)$  is homogeneous of degree 1, and in

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<sup>48</sup> This result depends on the assumption of log-utility and the assumption that households do not have any source of income in old age except for the return on their young age savings.

general has all the properties of a neoclassical production function as summarized in the following assumption.

**Assumption 1.** *The production function  $F : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}_+$  is concave and homogeneous of degree 1 in its arguments, and has the following properties:*

$$\begin{aligned} F(0, AL) &= F(K, 0) = 0; \\ F_K &\equiv \partial F(.,.) / \partial K \geq 0 \forall K \geq 0; \\ F_L &\equiv \partial F(.,.) / \partial L \geq 0 \forall L \geq 0; \\ F_K(0, AL) &= F_L(K, 0) = +\infty; \\ F_K(+\infty, AL) &= F_L(K, +\infty) = 0; \end{aligned}$$

Technological progress is not exogenous, but depends on the stock of publicly provided infrastructures. In particular we assume that  $A_{t+1} = \Gamma(G_t/A_t L_t) A_t$ ; where  $G_t$  is the *stock* of public capital. Assumptions regarding  $\Gamma$  are that it is a concave function, increasing in the public capital stock in efficiency units.

**Assumption 2.** *The function  $\Gamma : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ , has the following properties:*

$$\begin{aligned} \Gamma(0) &= \Gamma_{\min} \geq 0; \\ \Gamma(+\infty) &= \Gamma_{\max} > 1; \\ \Gamma'(G/AL) &\equiv d\Gamma(G/AL)/d(G/AL) > 0; \\ \Gamma''(G/AL) &\equiv d^2\Gamma(G/AL)/d(G/AL)^2 < 0. \end{aligned}$$

A brief discussion of the assumptions on  $\Gamma$  is necessary. That technological progress should depend exclusively on public capital stock will undoubtedly sound an

odd assumption. A more realistic scenario would display some sort of complementarity between private investment and public capital stock. Here, however, we are not trying to give a theory of the sources of technological progress, but we focus on public policies when public capital expenditures have direct consequences for the aggregate production function. Perhaps, we could interpret the above assumption as an environment in which technological progress is exogenous, but new technologies can be adopted only if the country is supplied with adequate infrastructures.  $\Gamma_{\min} = 0$ , would mean that with no infrastructure it would be impossible to have any production at all; allowing for a  $\Gamma_{\min}$  positive but smaller than 1 implies that with no infrastructures some production is possible, but the economy shrinks over time. Finally, the function  $\Gamma$  could be bounded or not, but  $\Gamma_{\max}$  has to be bigger than 1 for sustained growth to be possible. There are no strong technical reasons to assume that  $\Gamma$  is bounded, but boundedness could be a natural assumption, as it seems intuitive that there should be a limit to the rate of growth that can be induced by public investment.

The role played by  $G$  in this model is different than that in Barro [13]; not only because it is the stock, not the flow, of public capital that matters<sup>49</sup>, but because here public capital is not directly an input in production, but it is something that makes possible or accelerates technological progress. The same assumption is taken by Cavalcanti Ferreira [31], but there is again the flow, not the stock that matters. We assume that the stock is what enters the function  $\Gamma$  to add realism. As it should be clear from the analysis of the dynamics below, none of the main results depend crucially on this assumption.

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<sup>49</sup> The consequences of substituting stock to flow in the Barro model have already been analyzed in Futagami et al.[54].

We are saying that infrastructures allow adoption of more modern technologies. As an example, building roads and railways allows the use of modern means of transport (trucks and train). Similarly we think that there would be very little innovation in a society without schools. And we are also saying -looking at the variables that enter as argument  $\Gamma$ - that as the technological level increases, a higher level of infrastructures is needed to enhance further progress. That is the level of infrastructures per capita that allowed an increased in productivity of, say, 2% yesterday, will allow a smaller further increase today because we start from an already higher level. That is, to sustain a certain growth in productivity, infrastructures need not only to be maintained, but to be continuously updated.

There are different possible reasons why the stock of infrastructures should influence the rate of increase of labour productivity. If we interpret  $A_t$  as human capital, and  $G_t$  as the stock of public investment in education -in a broad definition (schools, labs, etc.)- then the interpretation could be that people trained in better schools develop a better ability to accumulate experiences and knowledge that allow them to increase their productivity on the job.

Alternatively one could think of a country with better infrastructures as one in which communication, travel and consequently the exchange of information and experiences is facilitated. Again, through this channel infrastructures contribute to faster productivity growth.

The provision of public capital  $G$ , must be financed somehow. We will assume that public expenditures are financed through taxes on wage and capital earnings and domestic borrowing; we abstract from seigniorage and external borrowing (we consider a real closed economy). We also assume that the tax rates are fixed for all periods. Al-

though this is clearly an oversimplification, it is also true that in general modifying the tax rate might be a more complicated matter than issuing bonds. Constraints on the tax rate can be political, legal or ideological. Furthermore, often increases in the tax rate can be pointless, as they are offset by increases in tax evasion. All these reasons and others that we might have neglected, make tax rates quite stable; we make the extreme assumption that the government tries to fix them once and for all. We then study the combinations of tax and investment policies that are sustainable, i.e. mutually consistent. If an arbitrarily chosen combination of policy is not sustainable it will have to be changed sooner or later. In the final section we derive some interesting implications from this observation.

Indicating by  $B_t$  the stock of public debt,  $D_t$  the investment in public capital and  $X_t$  investment in private capital, the evolution of the stocks of private and public capital are given respectively by

$$K_{t+1} = (1 - \delta_k) K_t + X_t, \quad (4)$$

$$G_{t+1} = (1 - \delta_g) G_t + D_t, \quad (5)$$

where  $\delta_k$ ,  $\delta_g$  are the depreciation rates of private and public capital respectively. Fiscal policy must satisfy the following budget constraint

$$D_t = B_{t+1} - (1 + r_t)B_t + \tau (W_t L_t + r_t (K_t + B_t)). \quad (6)$$

Here  $\tau$  is the tax rate on wages,  $W_t$  is the before tax wage rate at time  $t$ , and  $r_t$  is the net interest rate before tax. Note that the tax on capital earnings is proportional to  $r$  not  $R$ , i.e. net rather than gross earnings are taxed; more importantly, this means that there is a deduction for depreciation (as it will be clear below).<sup>50</sup>

<sup>50</sup> This is not essential for the analysis, one could work with the alternative hypothesis that there is no

Equilibrium requires all markets to clear. In this economy there are a labour market, and a market for assets: bonds and private capital. Factor market equilibrium requires

$$r_t + \delta_k = \partial F(K_t, A_t L) / \partial K_t, \quad (7)$$

$$W_t = A_t \partial F(K_t, A_t L) / \partial L. \quad (8)$$

Asset market clearing imposes that the sum of private capital stock and the stock of debt must equal saving:

$$K_{t+1} + B_{t+1} = s(1 - \tau) W_t L. \quad (9)$$

The behaviour of the system can be more easily illustrated expressing all variable in efficiency units. We will employ the common convention of indicating with lower case the value of a variable in efficiency units, that is for any variable  $Z_t$  the corresponding lowercase indicates  $z_t \equiv Z_t / A_t L$ . We shall use also the abbreviation  $\Gamma_t \equiv \Gamma(G_t / A_t L) = \Gamma(g_t)$ .

Using the homogeneity property of the production function we can write

$$y_t = f(k_t), \quad (10)$$

where  $f(k_t) \equiv F(K_t, A_t L) / A_t L = F(k_t, 1)$ ; which in turn implies that (7) and (8) can be written<sup>51</sup>

$$r_t + \delta_k = f'(k_t), \quad (11)$$

$$w_t = f(k_t) - k_t f'(k_t), \quad (12)$$

---

depreciation allowance.

<sup>51</sup> Apostrophes indicate derivatives. So  $f'(k) \equiv \partial f(k) / \partial k$ ;  $f''(k) \equiv \partial^2 f(k) / \partial k^2$ .

where  $w_t$  is the wage rate for effective unit of labour ( $W_t = A_t w_t$ ). Furthermore, the equations (4)-(6) become

$$\Gamma_t k_{t+1} = (1 - \delta_k) k_t + x_t, \quad (13)$$

$$\Gamma_t g_{t+1} = (1 - \delta_g) g_t + d_t, \quad (14)$$

$$d_t = \Gamma_t b_{t+1} - (1 + (1 - \tau) r_t) b_t + \tau w_t + \tau r_t k_t. \quad (15)$$

Equation (9) can be written

$$\Gamma_t k_{t+1} = s (1 - \tau) w_t - \Gamma_t b_{t+1}, \quad (16)$$

while (14), (15) and  $w_t + r_t k_t = f(k_t)$  give

$$\Gamma_t g_{t+1} = (1 - \delta_g) g_t + \Gamma_t b_{t+1} - (1 + (1 - \tau) r_t) b_t + \tau f(k_t). \quad (17)$$

A look at (17) can already illustrate the effect of changing the composition of finance, that is reducing (increasing) taxes and increasing (reducing) issues of new debt maintaining the same level of public investment  $d_t$  (and therefore the same  $g_t \Gamma_{t+1}$ ). A switch from taxes to borrowing that leave unchanged the level of public investment must satisfy (from (17))

$$\Gamma_t db_{t+1} + (f(k_t) + r_t k_t) d\tau = 0 \quad (18)$$

that is

$$(f(k_t) + r_t k_t) d\tau = -\Gamma_t db_{t+1}. \quad (19)$$

From (16) this implies

$$\Gamma_t dk_{t+1} = -\Gamma_t db_{t+1} - s w_t d\tau = (f(k_t) - s w_t + \tau r_t b_t) d\tau, \quad (20)$$

and therefore an increase in debt (a tax cut) reduces capital accumulation and vice versa (crowding out effect).<sup>52</sup> It is easy to verify that this conclusion does not depend on the assumption of log-utility. It is also clear that the same conclusion will hold in steady state (assuming that one exists). The above discussion seems to imply that the government can choose the mix of tax and deficit finance freely. We will show below that this is not always true, that is there are cases in which a reduction of the long-run level of deficit requires lower public investment levels.

## 3.2 Fixed public investment policy

In this section we analyse the equilibrium outcome under the hypothesis that the government maintains fixed the rate of public investment in efficiency units,  $d$ . Then equations (16), (14) and (15) become<sup>53</sup>

$$k_{t+1} = [s(1 - \tau)w_t - \Gamma_t b_{t+1}] / \Gamma_t, \quad (21)$$

$$g_{t+1} = [(1 - \delta_g)g_t + d] / \Gamma_t, \quad (22)$$

$$b_{t+1} = [d + (1 + (1 - \tau)f'(k_t))b_t - \tau f(k_t)] / \Gamma_t. \quad (23)$$

First we want to establish whether this system admits a steady-state. A steady-state is characterized by  $k_t = k$ ,  $g_t = g$  and  $b_t = b \forall t$ . Then the above system can be rewritten

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<sup>52</sup> In an exogenous growth model, crowding out can be welfare improving if the economy is dynamically inefficient, which occurs when the growth rate is bigger than the interest rate. It will be shown below that in this model there are equilibria characterized by a growth rate that is bigger than the interest rate. This opens the question of whether crowding out is desirable in this setting.

<sup>53</sup> Where  $w_t = f(k_t) - k_t f'(k_t)$ .

as

$$k = [s(1 - \tau)w - \Gamma b] / \Gamma, \quad (24)$$

$$g = [(1 - \delta_g)g + d] / \Gamma, \quad (25)$$

$$b = [d + (1 + (1 - \tau)f'(k))b - \tau f(k)] / \Gamma. \quad (26)$$

We can solve the last for  $b$  to find

$$b = \frac{d - \tau f(k)}{\Gamma - R}; \quad (27)$$

where  $R$ , it will be recalled, is  $1 + (1 - \tau)f'(k)$ .

Equation (27) has an important and intuitive interpretation: steady-state debt will be positive only if the sign of the primary deficit (the numerator) is the same as the sign of the difference between the growth rate and the interest rate. In other words, if the government is a net debtor, a sustained primary deficit can be positive only if the interest rate is less than the growth rate. Conversely, if the interest rate exceeds the growth rate, the only way to sustain a positive debt level is to have primary surplus. Chalk [33] showed that this is a necessary but not sufficient condition; we will show below that the same is true in our model.

### 3.2.1 Existence

To prove existence of a balanced path equilibrium we exploit the recursive nature of the dynamic system. Note that (22) depends on  $g$  and  $d$  but not on  $k$ . It certainly has a unique fixed point, as proven in the following lemma.

**Lemma 3.1** *Under assumption 2, the difference equation (22) has a unique steady-state  $g$ , which is increasing in  $d$ .*

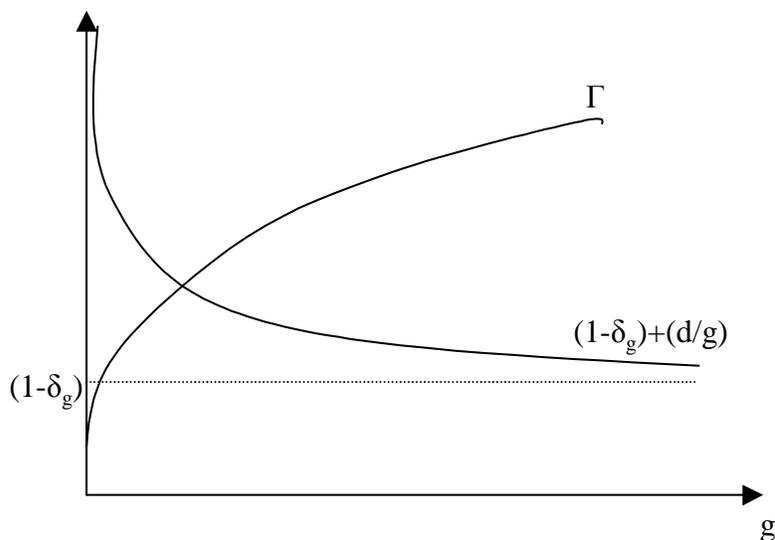
**Proof** Assuming  $g \neq 0$ , we can rewrite (25) as

$$\Gamma = (1 - \delta_g) + \frac{d}{g}.$$

The left hand side is concave, increasing, starts from  $\Gamma_{\min}$ , and approaching  $\Gamma_{\max}$  as  $g \rightarrow +\infty$ ; the right hand side is a decreasing, convex function, going to  $+\infty$  as  $g \rightarrow 0$ , and to  $(1 - \delta_g)$  as  $g \rightarrow +\infty$ ; since  $\Gamma_{\max} > (1 - \delta_g)$  by assumption 2, there must be at least one intersection; since one function is strictly increasing and the other is strictly decreasing there will be only one intersection. Figure 3.1 illustrates. How does the fixed point of (25) changes with  $d$ ? By implicit differentiation we obtain

$$\frac{dg}{dd} = \frac{1}{\Gamma'g + \Gamma - (1 - \delta_g)} > 0;$$

the above derivative is certainly positive because for any  $g$  satisfying (25),  $\Gamma > (1 - \delta_g)$  (see figure 3.1). If  $\Gamma_{\max} = +\infty$ , then as  $d \rightarrow +\infty$  then  $g \rightarrow +\infty$  and  $\Gamma \rightarrow +\infty$ . Therefore an increase in  $d$ , results in an increase in  $g$  and therefore  $\Gamma$ . In general, as  $d \rightarrow +\infty$ ,  $g \rightarrow +\infty$  and  $\Gamma \rightarrow \Gamma_{\max}$ . ■



Once the steady-state value of  $g$  has been found, we can look at the system formed by (24) and (26). We can solve them both for  $b$ . From (24) we have

$$b = \frac{s(1-\tau)w - \Gamma k}{\Gamma} \equiv \psi_\pi(k), \quad (28)$$

from (26) we have

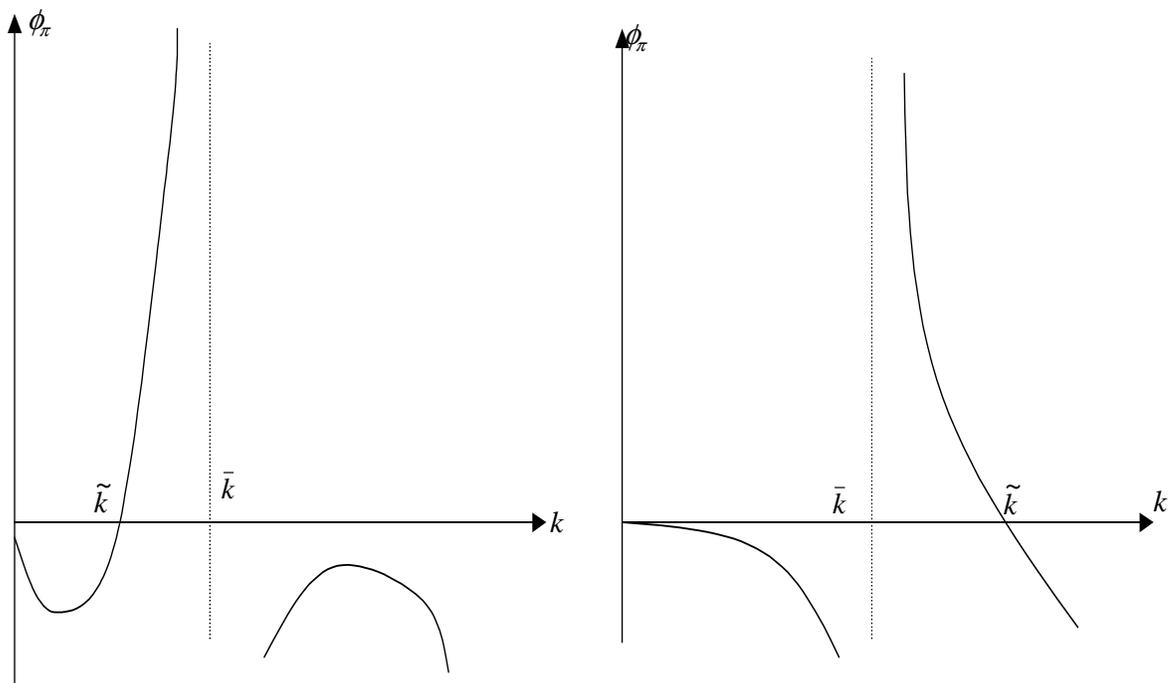
$$b = \frac{d - \tau f(k)}{\Gamma - R} \equiv \phi_\pi(k). \quad (29)$$

Let us call  $\bar{k}_\pi$  the value of  $k$  such that  $R = \Gamma$ , and  $\tilde{k}_\pi$  the value of  $k$ , such that  $d - \tau f(k) = 0$ . It is clear that  $\phi_\pi$  has a discontinuity at  $\bar{k}_\pi$ . Note also that  $\phi_\pi(+\infty) = -\infty$ . If  $\tilde{k}_\pi < \bar{k}_\pi$  ( $\tilde{k}_\pi > \bar{k}_\pi$ ), then  $\lim_{k \rightarrow \bar{k}_\pi^-} \phi_\pi(k) = +\infty$  ( $-\infty$ ). Furthermore as  $k \rightarrow 0$  the numerator goes to  $d$  while the denominator goes to  $-\infty$ ; it follows that  $\phi(0) = 0$  and that  $\phi(k) < 0$  in a right-neighbourhood of 0. Differentiating with respect to  $k$ , we find

$$\phi'(k) \equiv \frac{d\phi(k)}{dk} = \frac{-\tau f'(k) + \phi(k)(1-\tau)f''(k)}{\Gamma - R}.$$

Note that when  $\phi(k) \geq 0$ ,  $\text{sign}(\phi'(k)) = -\text{sign}(\Gamma - R)$ .

Figure 3.2 illustrates two possible shapes that  $\phi_\pi(k)$  may have. The left-hand side illustrate a case for which  $\tilde{k}_\pi < \bar{k}_\pi$ ; we established that for  $k$  close enough to 0,  $\phi(k) < 0$ . On the other hand  $\phi(\bar{k}_\pi^-) = +\infty$ . The intersection with the horizontal axes at  $\tilde{k}_\pi$  must be unique as there  $\phi'(\tilde{k}_\pi) = -\tau f'(\tilde{k}_\pi) / (\Gamma - R) > 0$ , since  $\phi(\tilde{k}_\pi) = 0$  by definition and  $\Gamma - R < 0$  for  $k < \bar{k}_\pi$ . Since  $\phi(\bar{k}_\pi^+) = \phi(+\infty) = -\infty$ ,  $\phi(k)$  must have a shape similar to that in the left and side of figure 3.2. The right-hand side of figure 3.2 illustrate the case  $\tilde{k}_\pi > \bar{k}_\pi$ . Clearly in this case  $\phi(0) = 0$ ,  $\phi(k) < 0 \forall k \in (0, \bar{k})$ , and  $\phi(\bar{k}_\pi^-) = -\infty$  as in the figure. While  $\phi(\bar{k}_\pi^+) = +\infty$ ,  $\phi'(k) < 0 \forall k \left( \bar{k}_\pi, \tilde{k}_\pi \right]$  and  $\phi(+\infty) = -\infty$ . Since  $\phi'(\tilde{k}_\pi) < 0$ ,  $\tilde{k}_\pi$  is again unique. Therefore the graph of  $\phi$  will look as in the right-hand side of figure 3.2. When considering the figures one should note that the second derivative of  $\phi$  depends in a rather complicated way on the



3.2

first three derivative of  $f$ ,<sup>54</sup> therefore the concavity/convexity properties of  $\phi$  may differ from the figure.

The above discussion has proved the following lemma.

**Lemma 3.2** For a given  $\pi$ , call  $\Phi_\pi \subset \mathfrak{R}_+$  the set of  $k > 0$  such that  $\phi_\pi(k) \geq 0$ ;

- then
- (i) if  $\tilde{k}_\pi < \bar{k}_\pi$  then  $\Phi_\pi = [\tilde{k}_\pi, \bar{k}_\pi)$ ;
  - (ii) if  $\tilde{k}_\pi > \bar{k}_\pi$  then  $\Phi_\pi = (\bar{k}_\pi, \tilde{k}_\pi]$ .
  - (iii) if  $\tilde{k}_\pi = \bar{k}_\pi$  then  $\Phi_\pi = \emptyset$ .

<sup>54</sup> One finds

$$\phi'' = -\frac{f''}{\Gamma - R} + 2\frac{\phi(1-\tau)^2(f'')^2}{(\Gamma - R)^2} + \frac{\phi(1-\tau)f'''}{\Gamma - R}.$$

Let us turn to  $\psi_\pi(k)$ . Note that  $\psi_\pi(0) = 0$ . Let us call  $\psi_{\pi\infty} \equiv \psi_\pi(+\infty)$ ; now since  $w = f(k) - kf'(k)$ , we have

$$\begin{aligned}\psi_\pi(k) &= \frac{s(1-\tau)[f(k) - kf'(k)] - \Gamma k}{\Gamma} \\ &= \frac{s(1-\tau)f(k) - k[s(1-\tau)f' + \Gamma]}{\Gamma}.\end{aligned}$$

The the assumption that the marginal product of capital tends to zero as the capital stock goes to infinity (see assumption 1) gurantees that  $s(1-\tau)f(k) - \Gamma k \rightarrow -\infty$  as  $k \rightarrow +\infty$ ; that is  $\psi_{d\infty} = -\infty$ . Note also that

$$\begin{aligned}\psi'_d(k) &\equiv d\psi_d(k)/dk = [(1-\tau)s(dw/dk) - \Gamma]/\Gamma \\ &= \frac{(1-\tau)s(-kf''(k)) - \Gamma}{\Gamma}\end{aligned}$$

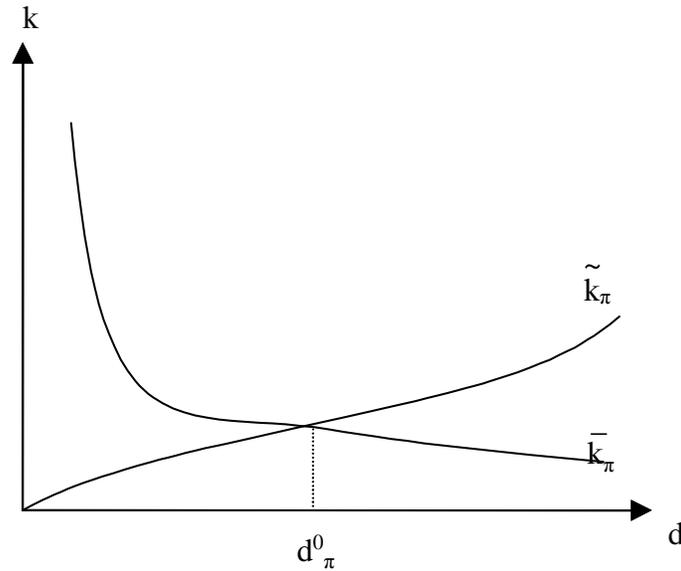
can be positive or negative depending on the value of  $f''$  relative to  $\Gamma$ . If  $\psi'_\pi(0) > 0$  there is a right neighbourhood of 0 such that for all  $k$  belonging to it,  $\psi_\pi(k) > 0$ . When  $\Gamma$  is so high that  $\psi'_\pi(0) < 0$  there cannot be any steady-state with a positive level of public debt.

We need to establish when we will have  $\tilde{k}_\pi > \bar{k}_\pi$  or  $\tilde{k}_\pi < \bar{k}_\pi$ . The following lemma prove that, for given tax rates, there is critical value for the investment rate,  $d_\pi^0$  such that  $\tilde{k}_\pi < (>) \bar{k}_\pi$  for  $d < (>) d_\pi^0$ .

**Lemma 3.3** *For any tax rate  $\tau \in (0, 1)$  there is a  $d_\pi^0$  such that  $\tilde{k}_\pi < \bar{k}_\pi$  if  $d < d_\pi^0$ , and  $\tilde{k}_\pi > \bar{k}_\pi$  if  $d > d_\pi^0$ .*

**Proof** Since  $\tilde{k}_\pi$  is defined by  $d = \tau f(\tilde{k}_\pi)$ , it is clear that  $\lim_{d \rightarrow 0} \tilde{k}_\pi = 0$  and  $\lim_{d \rightarrow +\infty} \tilde{k}_\pi = +\infty$ . Furthermore by implicit differentiation  $\partial \tilde{k}_\pi / \partial d = 1/\tau f'(\tilde{k}_\pi) > 0$ . That is  $\tilde{k}_\pi$  increases monotonically with  $d$  from 0 to  $+\infty$ .  $\bar{k}_\pi$  is defined by  $\Gamma - (1 + (1-\tau)f'(\bar{k}_\pi)) = 0$ . But  $\Gamma$  is an increasing function of  $g$  which in turn was

shown to be an increasing function of  $d$  in lemma 3.1. There we have also shown that  $\lim_{d \rightarrow 0} \Gamma = \Gamma_{min}$ , and  $\lim_{d \rightarrow +\infty} \Gamma = \Gamma_{max}$ . Then the limit of  $\bar{k}_\pi$  for  $d \rightarrow 0$  is defined implicitly by  $\Gamma_{min} = (1 + (1 - \tau) f'(\bar{k}_\pi))$ , while the limit of  $\bar{k}_\pi$  for  $d \rightarrow +\infty$  is defined implicitly by  $\Gamma_{max} = (1 + (1 - \tau) f'(\bar{k}_\pi))$ . Furthermore by implicit differentiation we have  $\partial \bar{k}_\pi / \partial d = (\partial \Gamma / \partial d) / (1 - \tau) f''(\bar{k}_\pi) < 0$ . Thus  $\bar{k}_\pi$  is monotonically decreasing. It follows that there can be only one  $d$  such that  $\tilde{k}_\pi = \bar{k}_\pi$ . Figure 3.3 illustrates. ■



3.3

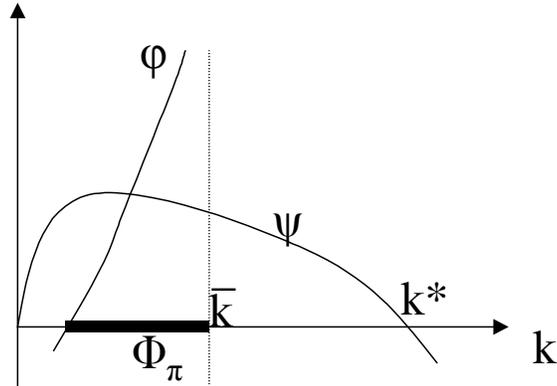
### 3.3 The $\Gamma \leq R$ case

We first look briefly at fiscal policies for which  $d < d_\pi^0$ . Our treatment of this case is kept brief because we are mostly interested in the analysis of adjustment from a situation

of sustained primary deficits, but this is only possible when the growth rate exceeds the interest rate.

**Lemma 3.4**      Suppose  $d < d_\pi^0$  and call  $k_\pi^*$  the value of  $k$  such that  $s(1 - \tau)w = \Gamma k$ . A steady-state with  $b > 0$  exists if and only if  $\psi'(0) > 0$  and  $k_\pi^* > \tilde{k}_\pi$ .

**Proof**      If  $d < d_\pi^0$ ,  $\phi(k)$  will look as in the left-hand side of figure 3.2. If an intersection with  $\psi(k)$  in the positive quadrant exists, it must be for a  $k \in \Phi_\pi$ . Figure 3.4 illustrates. ■



3.4

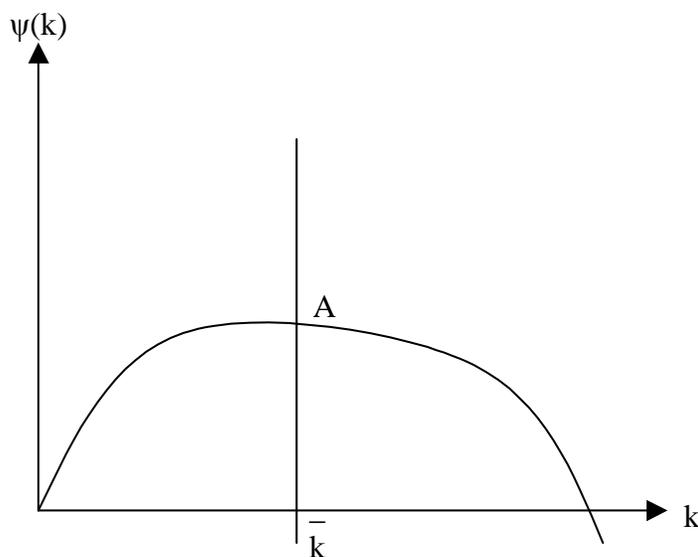
Next we examine the case where  $d = d^0$  and therefore  $\tilde{k}_\pi = \bar{k}_\pi$ . In this case a unique steady-state exists, because from (26) we have

$$(\Gamma - R)b = 0 = d - \tau f(k),$$

which implies  $k = \tilde{k}_\pi$ . Then from (24) we have

$$b = \frac{s(1 - \tau) \left[ f(\tilde{k}_\pi) - \tilde{k}_\pi f'(\tilde{k}_\pi) \right] - \Gamma \tilde{k}_\pi}{\Gamma}.$$

Figure 3.5 illustrates. Point A in the figure is the unique steady-state. Note that the figure illustrate the case for which the resulting  $b$  is positive, but if  $\tilde{k}_\pi > k^*$  then the intersection would be for a negative  $b$ .



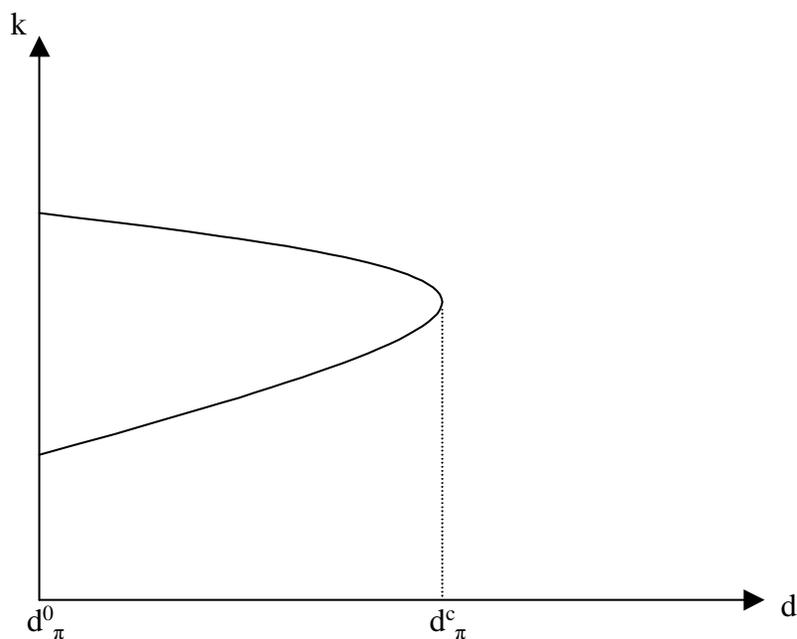
3.5

### 3.4 The $\Gamma > R$ case

When  $\Gamma > R$  there is the possibility that the government can sustain positive deficits in the long-run. Chalk [33] has shown, in an exogenous growth model, that even in this favourable case, the fiscal policy of the government is not unlimited. Analogous restrictions will hold in our set up as well, with additional complications given by the more general fiscal policy that we consider (Chalk abstracts from taxation), and, more importantly, the endogeneity of the long-run growth rate. From the above discussion we have that for  $d > d_\pi^0$ , if a steady state with positive public debt exists, it will be

characterised by  $\Gamma > R$ . In this section we consider this case and study the existence and dynamic properties of such a steady-state.

We argue that under fairly general conditions, for values of  $d$  arbitrarily close to  $d_\pi^0$  there are at least two steady-states, one of which characterized by a positive value and the other with a negative value of public debt. As  $d$  increases the two steady-states get arbitrarily close in the following sense: the corresponding  $(k, b)$  converge (coordinate-wise) to the same couple of values. There is a critical value  $d_\pi^c$  (which depends on the tax rates), such that there is a unique steady-state (at this point we do not know whether this is characterized by net indebtedness of the government). For any  $d > d_\pi^c$  there is no steady-state. Formally, we will prove that the system undergoes a *saddle-node bifurcation* at  $d_\pi^c$ . We also prove that of the two steady-states, one is a saddle the other a sink. Graphically, we can represent the bifurcation diagram as in figure 3.6 (note that in this figure we only consider  $d \in (d_\pi^0, +\infty)$ ).



3.6

The above discussion is summarized in the following proposition (that we consider the main result of this section).

**Proposition 3.1** *For each tax rate  $\tau \in (0, 1)$  there exists a  $d_\pi^c$  such that  $\forall d \in (d_\pi^0, d_\pi^c)$  there are two distinct steady-state with  $k > \bar{k}$ , one of which is a saddle, the other is a sink; for  $d = d_\pi^c$  there is a unique steady-state;  $\forall d > d_\pi^c$  there is no steady-state with  $k > \bar{k}$ . Although the intuition behind proposition 1 can easily be*

grasped from the diagrammatic exposition once one has understood the way the curves move with  $d$ , the formal proof is rather tedious. We shall devote the rest of this section to proving proposition one; the reader more interested in the economic implication of the proposition than the technicalities involved can safely skip the rest of this section in a first reading.

We structure our proof as follows. We first prove the existence of a steady-state for values of  $d$  arbitrarily close to  $d_\pi^0$ . We then consider a reparameterisation of the fundamental equation such that the steady-state is fixed at  $(0, 0)$ .<sup>55</sup> We then prove that the newly defined map undergoes a saddle-node bifurcation at  $d_\pi^c$ . Because the two maps are topologically conjugate, they will always have the same number of fixed points. Finally, we will use the Jacobian calculated at the two steady-states to establish which is stable and which is a saddle.

$$\mathbf{d} \simeq \mathbf{d}_\pi^0$$

For values of  $d$  bigger than but arbitrarily close to  $d_\pi^0$  there exists at least one steady-state. That is because in this case  $\tilde{k}$  is bigger but arbitrarily close to  $\bar{k}$ ; it follows

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<sup>55</sup> Technically this involves constructing a topological conjugacy.

that  $\phi(k)$  is arbitrarily close to the vertical line in figure 3.4. A steady-state in proximity of point A must therefore exist.

### Reparameterisation

Established the existence of a steady-state in the neighbourhood of  $d_\pi^0$ , we now want to establish how many there are and how they change with  $d$ . Call  $k_{ss}$  the steady-state value for  $k$  if it exists. Clearly  $k_{ss}$  must satisfy the following equation

$$\psi_\pi(k_{ss}) - \phi_\pi(k_{ss}) = 0; \quad (30)$$

let us define the function  $h : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  as

$$h_\pi(x) - x \equiv \psi_\pi(k) - \phi_\pi(k),$$

where  $x = k - k_{ss}$ . We have obtained the desired reparameterisation. To every steady-state for the original parameterization corresponds a steady-state for the new one. In fact when  $k = k_{ss}$ ,  $x = 0$  so the equation  $x = h_\pi(x)$  is identically satisfied. Similarly if there exists a value  $k'_{ss} \neq k_{ss}$  such that  $\psi_\pi(k'_{ss}) - \phi_\pi(k'_{ss}) = 0$ , then there exists a  $x' \neq 0$  such that  $x' = h_\pi(x')$ .<sup>56</sup> In fact the map  $h$  is topologically conjugate with the map that describe the original system. What is left to to is to show that the map  $h_\pi$  undergoes a saddle-node bifurcation for a critical value  $d_\pi^c$ . To apply the saddle-node bifurcation theorem, we need to show that there exists a critical value for  $d$  such that the curves  $\phi$  and  $\psi$  are tangent. This is done in the following lemma.

**Lemma 3.5** *There exists a  $d_\pi^c$  such that*

$$h'(0) \equiv \left. \frac{dh}{dx} \right|_{x=0} = 1.$$

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<sup>56</sup> This is always true by definition.

**Proof** We need to show that there exists a  $d_\pi^c$  such that a steady-state  $k_{ss}$  exists and  $h'(0) = 1$ . Now,  $h'(0) = 1$  if and only if

$$\psi'_\pi(k_{ss}) - \phi'_\pi(k_{ss}) = 0,$$

where  $\psi'(\cdot) \equiv \partial\psi/\partial k$ ,  $\phi'(\cdot) \equiv \partial\phi/\partial k$ . Our proof is by contradiction. Assume that  $d_\pi^c$  does not exist, i.e. that  $\psi'_\pi(k_{ss}) - \phi'_\pi(k_{ss}) \neq 0 \forall d \in (d_\pi^0, +\infty) \equiv \Delta$ . Then for all  $d \in \Delta$ , the condition of the implicit function theorem are satisfied, and a  $k_{ss}$  that satisfies (30) exists and changes continuously with  $d$ . We now show that for  $d$  sufficiently high there cannot be such a solution, which provide us the contradiction we were looking for. We will then have to conclude that there must be a point in  $\Delta$  where the implicit function is not applicable. To show that a maximum sustainable  $d$  exists, assume

$$d > \max_k \left\{ \frac{\Gamma_{max} - R}{\Gamma_{max}} s(1 - \tau)w + \tau f(k) - (\Gamma_{max} - R)k \right\}.$$

Then

$$\begin{aligned} d &> \max_k \left\{ \frac{\Gamma_{max} - R}{\Gamma_{max}} s(1 - \tau)w + \tau f(k) - (\Gamma_{max} - R)k \right\} \\ &> \frac{\Gamma_{max} - R}{\Gamma_{max}} s(1 - \tau)w + \tau f(k) - (\Gamma_{max} - R)k \\ &> \frac{\Gamma - R}{\Gamma} s(1 - \tau)w + \tau f(k) - (\Gamma - R)k, \end{aligned}$$

or

$$d - \tau f(k) > \frac{\Gamma - R}{\Gamma} s(1 - \tau)w - (\Gamma - R)k$$

If  $0 < k < \bar{k}$  then  $\Gamma < R$ . Then rearranging the last inequality

$$\frac{d - \tau f(k)}{\Gamma - R} < \frac{s(1 - \tau)w - \Gamma k}{\Gamma},$$

i.e.

$$\phi_\pi(k) < \psi_\pi(k).$$

If  $k > \bar{k}$  then  $\Gamma > R$ , and we have

$$\frac{d - \tau f}{\Gamma - R} > \frac{s(1 - \tau)w - \Gamma k}{\Gamma},$$

i.e.

$$\phi_\pi(k) > \psi_\pi(k).$$

Either way it is clear that  $\phi_\pi(k) \neq \psi_\pi(k) \forall k$ . ■

We are now ready to apply the saddle-node bifurcation theorem.

**Lemma 3.6** *Call  $\pi_c$  a fiscal policy such that  $h'_{\pi_c}(0) = 1$ , and  $d_{\pi_c}^c$  the corresponding value for  $d$ . Then, keeping fixed  $\tau$ , there is a neighbourhood of  $d_{\pi_c}^c$  such that (i) for  $d > d_{\pi_c}^c$  there are no steady-state equilibria; (ii) for  $d = d_{\pi_c}^c$  there is one unique equilibrium; (iii) for  $d < d_c$  there are two equilibria, one is a saddle, the other is a sink.*

**Proof** We have to show that the map  $h_\pi$  satisfies the conditions for the saddle-node bifurcation theorem (Cfr. Devaney [40], theorem 12.6, p.88): (i)  $h_{\pi_c}(0) = 0$ ; (ii)  $h'_{\pi_c}(0) = 1$ ; (iii)  $h''_{\pi_c}(0) \neq 0$ ; (iv)  $\partial h_{\pi_c} / \partial d_c \neq 0$ . (i) trivially holds by definition ( $x = 0 \iff k = k_{ss}$ ).

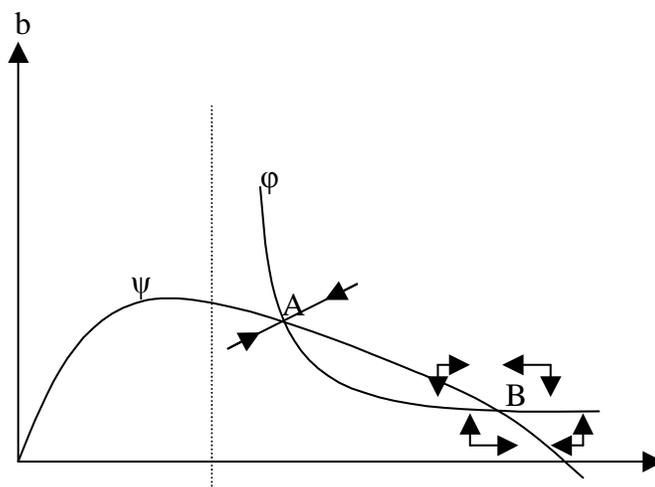
(ii) by hypothesis.

$$(iii) h''_{\pi_c} = \psi''_{\pi_c} - \phi''_{\pi_c} \neq 0.$$

$$(iv) \partial h_{\pi_c} / \partial d = -\frac{s(1-\tau)w}{\Gamma^2} \frac{d\Gamma}{dd} - \frac{1}{(\Gamma-R)^2} + \frac{\phi_{\pi_c}}{\Gamma-R} \frac{d\Gamma}{dd} \neq 0.$$

Thus all the conditions of the saddle-node bifurcation theorem are satisfied. The stability-unstability result becomes from the observation that the eigenvalues of the Jacobian are real, and from the fact that<sup>57</sup>  $\phi' > \psi'$  is a necessary and sufficient condition for a saddle, and  $\phi' < \psi'$  for a sink. ■

Figure 3.7 illustrates the result of proposition 1.



3.7

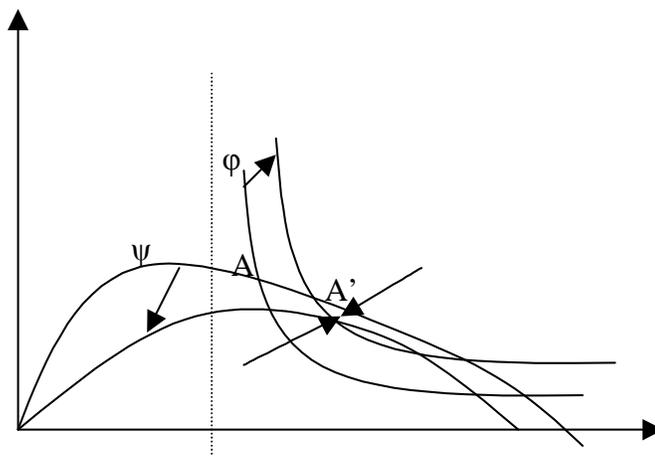
The fundamental message of the proposition is that if there is a crossing between  $\phi$  and  $\psi$ , in general there must be two of them. The steady-state with higher  $k$  is stable whereas the other is unstable. As the public investment rate increases (i.e.  $d$  increases),  $\phi$  moves in the north-east direction, whereas  $\psi$  shrinks; the two steady-states get closer until the point of tangency, after which there is no steady-state.

We next illustrates some examples that we find interesting.

<sup>57</sup> Where all derivatives are calculate at the steady-state values.

### Examples

Suppose that the economy is at point A in figure 3.8, with zero primary deficit. There is a positive amount of debt, but tax revenues are enough to cover public investment and interest payments. If the government tries to increase investment without touching the tax rate, i.e. tries to run positive primary deficits,  $\phi$  moves in the north-east direction, whereas  $\psi$  shrinks as illustrated in figure 3.8.

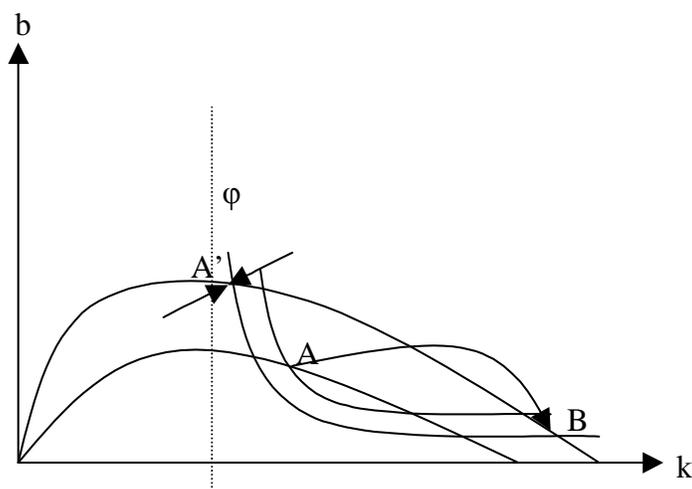


3.8

Even under the assumption that a new steady-state  $A'$  exists,  $A$  will certainly not lay into its basin of attraction. Actually  $A$  is, after the change in policy, in the unsustainable area. This shows that even a small change can transform a sustainable policy into an unsustainable one. One might object that of the two equilibria the saddle is the most unlikely, since it requires special initial condition for the economy to converge there. It can be argued, however, that if it was optimal for the government to converge there (given its preferences before the change), then the economy would have converged there. The change then must be the result of an unforeseen shock for the

planner. Alternatively, one could think of a situation in which the economy is not at A, but in proximity of it, converging to the stable equilibrium but still far away from it. If the change in policy is big enough, a similar result would again hold: the economy was on a sustainable path and now it is on a unsustainable one.

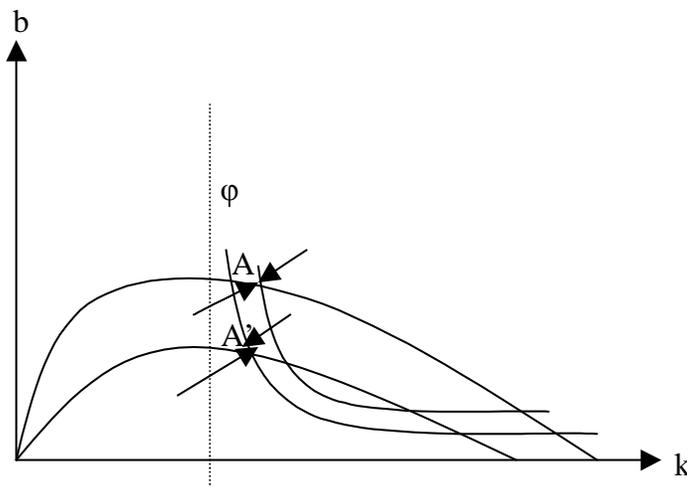
Clearly, if in the previous example the shift in policy is towards a decrease in investment, the result would be that now the economy will start moving towards point B as illustrated in figure 3.9. The lesson from this example is that even marginal changes in the policy design can cause dramatic changes in the asymptotic behaviour of the economy.



3.9

As a final example consider figure 3.10, but suppose that now in A, the government is running a primary deficit. Again this policy is sustainable in the long-run by definition, since we are in a steady-state. Suppose that now the government is forced to eliminate or even slightly reduce the primary deficit by increasing taxation (in order to join the common currency, for example, or because a constitutional bill has been

passed as sometimes called for in the USA). The consequence is that  $\phi$  moves in the south-west direction, whereas  $\psi$  shrinks. As it should be cleared by the figure, A is now in the non-sustainable area; this means that the government simply cannot maintain the same level of investment, but has to reduce it. Even though the initial policy was sustainable, the supposedly "virtuous" shift towards a more balanced budget necessarily involves cutting down public investment, which in turn means slower growth in the long-run.



3.10

The conclusions reached here depend on the assumption that all public expenditures are treated as productive investment. In reality is clear that government spending includes a share of consumption expenditures; if we would model public consumption by imposing a fixed ratio between consumption and investment, the main conclusions would be basically the same. If however we allowed the share to vary then we would have two possibility: on one hand the reduction in deficit spending might induce the government to reduce consumption and wasteful expenditures, which could counter bal-

ance and even subvert our story. On the other hand, if the government has less control on consumption than on investment (for example because of strong lobbying activity), the point would emerge even more strongly.

## 4 A dynamic analysis of user charges and public investment

As it has been mentioned in preceding chapters, there is a vast literature, both theoretical and empirical, on the effects of public investment on the growth of an economy. This literature has produced a number of interesting results on the effects of different fiscal policies on the dynamics of an economy, and on the principles that should guide the design of the optimal fiscal mix. This last chapter is a further contribution to this literature. The aim is to introduce a characteristic that seems to be shared by most if not all public inputs: rejectability. To illustrate, take the example of a road. It seems plausible that a well developed road network is likely to have a positive impact on the productive capability of an economy. But it seems also clear to us that the impact that a given set of roads has will depend crucially on the extent to which firms and households decide to exploit them. This for two reasons: on one hand, the construction of a new road will bring little benefit to my firm if I decide not to use it. Similarly, the construction of a lane for fast vehicles will have little *direct* impact if I do not own a fast vehicle and decide not to buy any. Secondly, the decisions of all potential users to use the existing roads, and how intensively, will have effects on the degree of congestion present and therefore on the benefit that each user can get from the existing stock of infrastructures. So, returning to the creation of the fast lane, even if I do not plan to use it, I may *indirectly* benefit if enough other users switch to use it thus reducing congestion on the slowest lanes too. This aspect of public inputs has received surprisingly little attention in the literature on optimal intertemporal fiscal policy.<sup>58</sup> While Arrow

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<sup>58</sup> There is a large literature concentrating on static analysis. See Berglas and Pines [20] and Cornes

and Kurz [5], McMillan [77], Pestieu [84] Weitzman [110] and Barro [13] are all examples of intertemporal models with public inputs, a common assumption is that public goods are not rejectable. The only paper we are aware of that explicitly analyses rejectability in a dynamic setting is Ott and Turnovsky [83]. Our model, though, differs from theirs in a number of important respects. First, Ott and Turnovsky assume that the degree of congestion perceived by a firm is related to the ratio of its capital stock relative to the aggregate capital stock. This implies that the level of congestion will depend on the number of firms. In contrast, we assume that congestion is given by the ratio of aggregate usage relative to the existing stock of infrastructures as in much of the literature on congestible facilities. Second, they assume the number of firms is exogenously given; we allow for free entry, so the number of firms is endogenously determined. As we shall see, in our model, in contrast with most of the literature on public investment and growth, the social and private returns to capital coincide, which has important implications for the optimal fiscal mix, and in particular for the optimal income tax.

Much of the literature on optimal taxation argues that the optimal tax on capital earning at least tends asymptotically to zero when it is not actually equal to zero after an initial transitional period (Chamley [34], Judd [66],[67]; see Atkeson et Al. [6] for a review). The reason for this is that a positive tax on the return from current savings makes consumption in the future more expensive. In a steady-state or balanced growth path, however, the elasticity of demand for consumption is constant; therefore having a positive capital income tax violates the principle that tax rates should be inversely proportional to demand elasticities (Ramsey [86], Baumol and Bradford [18]). There are, however, important exceptions. For a start, positive optimal capital income taxes

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and Sandler [35] for an overview.

are generally found in models with overlapping generations unless fairly restrictive assumptions on preferences are made (Erosa and Gervais[50], Mathieu-Bolh [76]). The reason is that in life-cycle models, individual consumption and leisure typically vary over time even when the aggregate counterparts achieve a steady-state. But then the elasticities are not in general constant over the lifetime of an individual and therefore taxing his consumption differently over time may be optimal (Erosa and Gervais [49]). Another exception is found in models with public capital with congestion (Turnovsky [105]). In this case, the reason for the departure is that private investors neglect the externality that their accumulation of capital exercises on others through increased congestion. A positive capital income tax is required to internalise this externality. Similar results follow if there are non-discretionary public expenditures linked to the current level of output and/or a spillover externality from private capital accumulation (Marrero and Novales [75]). The former would call for a positive tax on capital income in order to reduce the crowding out of private consumption due to the increase in government consumption caused by economic growth. The latter may actually call for a negative tax (a subsidy) to correct the externality when the externality is positive (as, for example, with learning-by-doing externalities as in Arrow [4] and Romer [92]).

Relative to the literature, the model developed in this chapter is closest to those in Turnovsky [105] and Ott and Turnovsky [83].<sup>59</sup> Like them, we look at a growth model with congestion prone public infrastructures and we look at the optimal pricing of public services. While those two studies find that the optimal capital income tax in the long-run is positive, we find it to be zero in the main specification of our model. Crucial to this difference is how we model utilisation of public services. In Turnovsky [105] and

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<sup>59</sup> Although some important differences with the latter have already been noted at the end of the first paragraph.

Ott and Turnovsky [83], as in much of the literature<sup>60</sup>, congestion is proportional to the capital stock of the firm. This is to a large extent realistic: if the capital stock is given by the number of trucks owned by the firm, one would expect that the more trucks firms buy, the more congestion we will see on the roads. However, for any given numbers of trucks, the firms can still make choices that will determine how intensively it will use the road network. For example the firm may decide to organise its delivery system so to reduce the amount of miles its trucks have to run to pick up materials and deliver goods. This would reduce the amount of congestion. We are not aware of any model that tries to include these considerations in an otherwise standard model of economic growth. We do this in this chapter. In our model, firms decide both how much capital to employ in production and how intensively to use the stock of public infrastructures. This assumption has an important consequence. Given that aggregate utilisation of the public capital stock is not automatically proportional to the capital stock, the wedge between the social and marginal product of private capital that is key in the result of Turnovsky [105] and Ott and Turnovsky [83] depends on the user charge. When the latter is chosen to obtain static efficiency, the optimal capital income tax, at least in the long-run, is zero.

We see the main contributions of this chapter to the literature to be the following. (i) A part from Ott and Turnovsky [83], we are not aware of any analysis of rejectable public goods in an intertemporal contest; we consider this to be an important gap in the existing literature. Our models differ in important ways from that of Ott and Turnovsky, and a comparison between the two helps clarifying further the intuition behind Chamley's famous result. (ii) We assume that investment is irreversible and show

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<sup>60</sup> Starting from the seminal papers of Barro [13] and Barro and Sala-i-Martin [15] to Glomm and Ravikumar [57] and Eicher and Turnovsky [47], just to cite a few.

that this may have important consequences for the optimal fiscal policy. To the best of our knowledge, this issues has not received attention in the existing optimal taxation literature.

## 4.1 The model

We first discuss the production side of the model. The next subsection offers the micro-foundations for assuming that the aggregate production function is of the form

$$Y = f(K, V, V/G),$$

where  $K$  is the private capital stock,  $V$  the aggregate utilisation of public services and  $G$  the public capital stock. The term  $V/G$  is meant to capture a congestion externality and it is often denote by  $\Gamma$ . The reader mostly interested in the result concerning the policy implications and less with the technicalities of the models may skim through the following subsection and move quickly to the next section.

### 4.1.1 Microfoundations

On the production side of the model we have a continuum of identical firms indexed by  $i \in [0, +\infty)$ . Firms use private capital and public services to produce an homogenous good. Their production set is indicated by  $\Psi(\Gamma)$  and it is described by

$$\Psi(\Gamma) = \{(y_i, k_i, v_i) : \psi(k_i, v_i, \Gamma) - y_i \geq 0, k_i \geq \bar{k}, v_i \geq 0\} \cup \{(0, 0, 0)\},$$

where  $y_i$  is the level of output,  $k_i$  the capital stock,  $\bar{k} > 0$  is a minimum capital requirement in production,  $v_i$  the level of usage of public services chosen by the firm and  $\Gamma$  is the level of congestion. We assume that  $\psi(\cdot, \cdot, \cdot)$  is strictly increasing and strictly

concave in its first two arguments, strictly decreasing in its third argument, and that  $\lambda\psi(k, v, \Gamma) \geq \psi(\lambda k, \lambda v, \Gamma) \forall \lambda \geq 0$  (decreasing returns to scale.)

The interpretation of  $\Psi(\Gamma)$  is the following. In order to produce a given level of output, firms require capital and public infrastructure services. The latter are rejectable, each firm can decide how intensively to use them; the level of utilisation is indicated by  $v_i$ . The contribution to production of utilisation of the public services depends inversely on congestion, hence the assumption  $\psi_\Gamma(k, v, \Gamma) \equiv \partial\psi(k, v, \Gamma)/\partial\Gamma < 0$ . We find it also reasonable to assume that there is a limit to the degree of substitutability between private capital and public services. We capture this by assuming that for  $v$  large enough the marginal product of  $v$  becomes null. Formally, for any  $k$  and  $\Gamma$  there is a  $\bar{v}$  such that  $\psi_v(k, \bar{v}, \Gamma) \equiv \partial\psi(k, \bar{v}, \Gamma)/\partial v = 0$ .

We indicate with  $K \equiv \int k_i di$  the existing aggregate stock of private capital, with  $G$  the existing stock of public capital and with  $V \equiv \int v_i di$  the aggregate utilisation of public services. At the level of the individual firm the production set is not convex due to the presence of the minimum capital requirement  $\bar{k}$ . However one can show that the aggregate production set, given by the integral of the correspondence that associates to each  $i$  the production set, is convex (Aumann [8]). Furthermore, since we have assumed that all  $i$ 's have the same production set, the aggregate production set is a convex cone (Novshek and Sonnenschein [80]); the aggregate production set is the convex cone generated<sup>61</sup> by  $\Psi(\Gamma)$ . This convex cone can be regarded as the hypograph of a concave, function  $f(K, V; \Gamma)$  homogenous in  $(K, V)$ . The following lemma describes the relationship between  $f$  and  $\psi$ .

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<sup>61</sup> The convex cone generated by  $\Psi(\Gamma)$  is the set of all positive linear combinations of elements of  $\Psi(\Gamma)$  (Rockafellar [91], p.14). It is the smallest convex cone that contains  $\Psi(\Gamma)$ .

**Lemma 4.1** Call  $\hat{k}(v, \Gamma)$  the function that satisfies<sup>62</sup>

$$\psi(\hat{k}(v, \Gamma), v, \Gamma) = \psi_k(\hat{k}(v, \Gamma), v, \Gamma) \hat{k}(v, \Gamma) + \psi_v(\hat{k}(v, \Gamma), v, \Gamma) v.$$

For any given  $(k, v)$ , call  $\hat{v}(k/v, \Gamma)$  the solution for  $\hat{v}$  to  $k/v = \hat{k}(\hat{v}, \Gamma) / \hat{v}$ . Then we have

$$f(k, v, \Gamma) = \frac{v}{\hat{v}(k/v, \Gamma)} \psi(\hat{k}(\hat{v}(k/v, \Gamma), \Gamma), \hat{v}(k/v, \Gamma), \Gamma).$$

**Proof** See section 4.6. ■

So one could start from reasonable assumptions on the firms' production set and then derive the relevant properties of the aggregate production set. For example, if one takes the following Cobb-Douglas specification

$$\psi(k, v, \Gamma) = (k - \bar{k})^\alpha v^\beta - \Gamma v,$$

then applying lemma 4.1 one finds

$$f(k, v, \Gamma) = \left( \frac{\alpha}{1 - \alpha - \beta} \right)^\alpha \frac{\bar{k}^\alpha}{\hat{k}^{1-\beta}} k^{1-\beta} v^\beta - \Gamma v,$$

where<sup>63</sup>  $\hat{k} = (1 - \beta) \bar{k} / (1 - \alpha - \beta)$ . A slight generalisation is given by assuming

$$\psi(k, v, \Gamma) = \hat{\psi}(k, v) - \Gamma v,$$

where  $\hat{\psi}$  is concave. Then it can be shown that the aggregate production function will be of the form

$$f(k, v, \Gamma) = \hat{f}(k, v) - \Gamma v,$$

<sup>62</sup> As explained in more detail in appendix 4.6, the function  $\hat{k}(v, \Gamma)$  gives all the pairs  $(k, v)$  such that at  $(\hat{k}(v, \Gamma), v)$ , the straight line going through  $(0, 0, 0)$  and  $(\hat{k}(v, \Gamma), v, \Gamma)$  is tangent to the production set. Essentially it is the projection on the  $(k, v)$  plane of the points of contact between the boundaries of the production set and the cone the latter generates.

<sup>63</sup> In this particular case  $\hat{k}(\cdot)$  turns out to be constant.

where  $\hat{f}$  is homogeneous and concave. It can then be easily verified<sup>64</sup> that the function

$$Y = f(K, V, V/G)$$

is concave in  $(K, V, G)$ . This fact is important because as we will discuss shortly, the properties of this function are important to the study of the equilibria in the economy and to characterise the optimal policy.

For more general specifications, though, it seems rather difficult to infer the properties of  $f$  from  $\psi$  beyond concavity and homogeneity w.r.t.  $(k, v)$ . We take a shortcut instead, by making some assumptions directly on  $f$ . For any aggregate production set, and therefore any  $f(K, V, \Gamma)$ , there will be a family of production sets  $\Psi(\Gamma)$  (not unique) that generates it. We therefore make the following assumptions:  $\forall (K, V, \Gamma)$

$$f_{kv}(K, V, \Gamma) \equiv \frac{\partial^2 f(K, V, \Gamma)}{\partial K \partial V} > 0, \quad (1)$$

$$f_{k\Gamma}(K, V, \Gamma) \equiv \frac{\partial^2 f(K, V, \Gamma)}{\partial K \partial \Gamma} < 0, \quad (2)$$

$$f_{v\Gamma}(K, V, \Gamma) \equiv \frac{\partial^2 f(K, V, \Gamma)}{\partial V \partial \Gamma} < 0, \quad (3)$$

$$f_{\Gamma\Gamma}(K, V, \Gamma) \equiv \frac{\partial^2 f(K, V, \Gamma)}{\partial \Gamma^2} \leq 0.$$

We next define a *static equilibrium* and show that it can be related to the aggregate production function. To understand the definition, one should think of the supplies of private and public capital as given (by past investment), and the user charge as exogenously given (chosen by the government); then we look for a price for capital,  $r$ , such that the aggregate demand for capital equals supply; and a level of congestion  $\Gamma$  such

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<sup>64</sup> By checking the signs of the leading principal minors of the Hessian matrix.

that if expected, firms would choose a level of utilisation that would result in exactly that level of congestion.

**Definition 4.1 (Static equilibrium)** *A for a given pair of stocks  $(K, G) \in \mathfrak{R}_{++}^2$ , and a user charge  $p \in \mathfrak{R}_+$ , a static equilibrium is a price for capital  $r \in \mathfrak{R}_+$ , a pair<sup>65</sup> of firm's decisions  $(k, v) \in \mathfrak{R}_+^2$ , and a number of active firms  $n \in \mathfrak{R}_+$  such that*

1. *(Profit maximisation) Given  $(r, p)$ ,  $(k, v) = \operatorname{argmax} \psi(k, v, \Gamma) - rk - pv$ .*
2. *(Market clearing)  $nk = K$ .*
3. *(Free entry)  $\psi(k, v, \Gamma) - rk - pv = 0$*
4. *(Expected congestion = actual congestion)  $\Gamma = nv/G$ .*

The static equilibrium can be characterised in terms of the aggregate production function  $f$ .

**Lemma 4.2** *If for given  $(K', G', p')$ , the pair  $(r', V')$  satisfies*

$$f_k(K', V', V'/G') \equiv \partial f(K', V', V'/G') / \partial K = r',$$

$$f_v(K', V', V'/G') \equiv \partial f(K', V', V'/G') / \partial V = p,$$

*then there exist  $n'$ ,  $k'$  and  $v'$  such that  $r'$  and  $(n', k', v')$  constitute a temporary equilibrium given  $(K', G', p')$ .*

**Proof** Consider the triple  $(Y', K', V')$ , where  $Y' = F(K', V', V'/G')$ . This triple belongs to the boundary of the aggregate production set for  $\Gamma' = V'/G'$ . Since

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<sup>65</sup> We thus impose a high degree of symmetry: all active firms are assumed to choose the same plan. This is clearly over restrictive and could be relaxed, but it simplifies the analysis.

the aggregate production set is the cone generated by  $\Psi (V'/G')$ , there exists a triple,  $(y', k', v')$ , such that  $y' = \psi (k', v', V'/G')$  and  $(n'y', n'k', n'v') = (Y', K', V')$ , for some  $n' > 0$ . Furthermore, since the aggregate production set is a cone, if  $(Y', K', V')$  belongs to its boundary, so does  $(y', k', v')$ . So we must have  $y' = f (k', v', V'/G')$ . Now  $f$  and  $\phi$  are both concave,  $f (\cdot, \cdot, \cdot) \geq \psi (\cdot, \cdot, \cdot)$  and  $f (k', v', V'/G') = \psi (k', v', V'/G')$ ; by lemma 1 in Benveniste and Scheinkman [19] their gradients at  $(k', v', V'/G')$  must coincide. Therefore we have

$$\psi_k (k', v', V'/G') = f_k (k', v', V'/G') = f_k (K', V', V'/G') = r',$$

$$\psi_v (k', v', V'/G') = f_v (k', v', V'/G') = f_v (K', V', V'/G') = p',$$

where the equalities  $f_j (k', v', V'/G') = f_j (K', V', V'/G')$   $j = k, v$  follow from the fact that if  $f$  is homogeneous of degree 1 in  $(K, V)$ , its derivatives are homogenous of degree 0. Then, given  $(r', p')$ , the pair  $(k', v')$  satisfies the first order condition for profit maximisation. Given the concavity of  $\psi$ , these conditions are sufficient for a local maximum. The only other candidate for a global maximum is  $(0, 0, 0)$ , which yields zero profit. But so does  $(y', k', v')$ , since

$$\begin{aligned} n' (y' - r'k' - p'v') &= Y' - r'K' - p'V' \\ &= f (K', V', V'/G') - f_k (K', V', V'/G') K' - f_v (k', v', V'/G') V' \\ &= 0, \end{aligned}$$

where the last equality follows from Euler's theorem for homogenous functions. ■

Next we define a fiscal policy. This is a vector of functions of time,  $(I_g (t), B (t), \tau (t), p (t))_{t=0}^{+\infty}$ , where  $I_g (t) \geq 0$  is public investment which determines the evolution

of the public capital stock according to

$$\dot{G}(t) = I_g(t);$$

$B(t)$  is the stock of public bonds,  $0 \leq \tau(t) \leq 1$  is the income tax and  $p(t)$  the user charge at time  $t$ .

The economy is inhabited by an infinitely-lived representative agent that choose a path for consumption,  $c(t)$ , and assets,  $A(t)$ , so as to solve the problem

$$\max \int_0^{+\infty} \frac{c(t)^{1-\gamma} - 1}{1-\gamma} e^{-\rho t} dt \quad (4)$$

subject to

$$\dot{A}(t) = (1 - \tau(t)) r(t) A(t) - c(t), \quad (5)$$

$$A(0) \text{ given.} \quad (6)$$

We are now ready to define an intertemporal equilibrium.

**Definition 4.2 (Intertemporal equilibrium)** *An intertemporal equilibrium is a vector of functions of time,  $(K(t), G(t), B(t), V(t), r(t), p(t), \tau(t), k(t), v(t), n(t))_{t=0}^{+\infty}$ , such that*

1. *(Static equilibrium)  $\forall t$   $(K(t), G(t)), (r(t), p(t)), (k(t), v(t))$  and  $n(t)$  constitute a static equilibrium and  $V(t) = n(t) v(t)$ .*
2. *(Utility maximisation)  $c(t)$  and  $A(t)$  solve (4) s.t. (5)-(6).*
3. *(Government budget constraint)  $\forall t$   $B(t)$  satisfies*

$$\dot{B}(t) = (1 - \tau(t)) r(t) B(t) + I_g(t) - \tau(t) r(t) K(t) - p(t) V(t). \quad (7)$$

4. (*Public capital accumulation*)  $\forall t$   $G(t)$  satisfies

$$\dot{G}(t) = I_g(t) \geq 0. \quad (8)$$

5. (*Resource constraint*)  $\forall t$   $K(t)$ ,  $G(t)$  and  $V(t)$  satisfy

$$\dot{K}(t) = f(K(t), V(t), V(t)/G(t)) - I_g(t) - c(t) \geq 0. \quad (9)$$

6. (*Asset market clearing*)  $\forall t$  we have

$$K(t) + B(t) = A(t). \quad (10)$$

In other words an intertemporal equilibrium is a sequence of static equilibria such that all markets clear, the representative agent maximise his welfare and the government budget constraint is satisfied. The latter requires that new debt,  $\dot{B}(t)$ , is issued to cover the difference between expenses (that is interest payments and public investment) and tax revenues (of course when revenues exceed expenses the stock of debt is falling).

Note that (4) and (5) in the definition introduce the assumption of irreversible investment. Without this assumption capital goods could be costlessly be transformed into consumption goods, public capital could be converted into private capital and *vice versa*. All this does not seem very realistic, we compel us to assume irreversibility. Having said that we emphasise that most result do not depend crucially on this assumption. The exceptions are: (i) in proposition 4.1 the initial public to private capital ratio will not matter anymore: since the planner can convert public into private capital and *vice versa*, the optimal ratio is always chosen and then either there is growth in the long run (if the marginal product of capital is high enough), or there is not, just as in Barro [13]. (ii) Propositions 4.3 and 4.4 that deal with the cases where the irreversible investment constraints bind would no longer make sense.

Note also that we have assumed no depreciation. This is purely for simplicity, nothing of substance depends on this. In fact one could always interpret the production functions as giving output net of depreciation.

We have now fully described all the elements of the model. In the next sections we will first look at the first best intertemporal allocation and then at the optimal fiscal policy.

## 4.2 Command optimum

In this section we look at the allocation that a social planner with full control of the economy's resources would choose. The planner's problem is

$$\max \int_0^{+\infty} \frac{c(t)^{1-\gamma} - 1}{1-\gamma} e^{-\rho t} dt \quad (11)$$

subject to

$$\dot{K}(t) = f(K(t), V(t), V(t)/G(t)) - c(t) - I_g(t) \geq 0, \quad (12)$$

$$\dot{G}(t) = I_g(t) \geq 0, \quad (13)$$

$$K(0) > 0, G(0) > 0 \text{ given.} \quad (14)$$

Before characterising the solution to this problem, it is useful to analyse the problem of choosing  $V(t)$  to maximise aggregate output; one would expect the planner to do so at all times, and the following proposition will confirm that this is indeed optimal. But it will also allow us to define the ratio  $G/K$  to which the optimal path will converge when there is sustained growth. Let us then call  $f^*(K, G) =$

$\max_{V(t)} f(K(t), V(t), V(t)/G(t))$ . Given that  $f$  is homogenous of degree 1 in  $(K, V, G)$ , it can be shown<sup>66</sup> that  $f^*$  is homogenous of degree 1 in  $(K, G)$ . Then we can write  $f^*(K, G) = f^*(1, G/K)K$ . We shall indicate with  $x^*$  the solution to  $f_k^*(1, x^*) = f_g^*(1, x^*)$ . We shall refer to  $x^*$  as the optimal  $G/K$  ratio, as a planner that could transform  $K$  into  $G$  and vice versa costlessly, would always choose  $G/K = x^*$ . In general  $x^*$  needs not be unique. However, if  $f(K(t), V(t), V(t)/G(t))$  is concave in  $(K(t), V(t), G(t))$ , then  $f^*(K(t), G(t))$  is also concave<sup>67</sup> (Fiacco and Kyriaris [52]) In what follows we assume that  $f^*(K(t), G(t))$  is strictly concave<sup>68</sup> which implies a unique  $x^*$ .

**Proposition 4.1** *If  $(K^p(t), V^p(t), G^p(t), c^p(t))_{t=0}^{+\infty}$  solve (11) s.t. (12)-(14) and  $f^*(K(t), G(t))$  is concave in  $(K(t), G(t))$  then we have the following cases:*

1. *If  $\max\{f_k^*(K(0), G(0)), f_g^*(K(0), G(0))\} \leq \rho$ , then  $\forall t \dot{c}^p(t) = \dot{K}^p(t) = \dot{G}^p(t) = 0$  and  $c^p(t) = f^*(K(0), G(0))$ .*
2. *If  $f_k^*(K(0), G(0)) > \rho > f_g^*(K(0), G(0))$  and  $f_k^*(1, x^*) < \rho$  then  $\forall t G^p(t) = G^p(0)$ ,  $\dot{c}^p(t)/c^p(t) = [f_k^*(K^p, G(0)) - \rho]/\gamma$ ,  $\dot{K}^p(t) =$*

<sup>66</sup>  $f^*(\varepsilon K, \varepsilon G) = \max_{V_1} f(\varepsilon K, V_1, V_1/(\varepsilon G)) = \max_{V_2} f(\varepsilon K, \varepsilon V_2, \varepsilon V_2/(\varepsilon G)) = \max_V \varepsilon f(K, V, V/G) = \varepsilon \max_V f(K, V, V/G) = \varepsilon f^*(K, G)$ . The first equality follows by definition; the second follows from the fact that the two maximands are identical except that in the second we have  $\varepsilon V_2$  rather than  $V_1$ ; then clearly if  $V_1^*$  and  $V_2^*$  solve the first and second problems respectively, we will have  $V_1^* = \varepsilon V_2^*$ , but the value of the maximand will be the same; the third equality follows from homogeneity of  $f(\cdot, \cdot, \cdot)$ ; the fourth is trivial and the last equality follows again from the definition of  $f^*(\cdot, \cdot)$ .

<sup>67</sup>  $f^*(\lambda K_1 + (1-\lambda)K_2, \lambda G_1 + (1-\lambda)G_2) = \max_V f(\lambda K_1 + (1-\lambda)K_2, V, V/(\lambda G_1 + (1-\lambda)G_2)) = \max_{V_1, V_2} f(\lambda K_1 + (1-\lambda)K_2, \lambda V_1 + (1-\lambda)V_2, \lambda V_1 + (1-\lambda)V_2/(\lambda G + (1-\lambda)G)) \geq \max_{V_1, V_2} \{\lambda f(K_1, V_1, V_1/G_1) + (1-\lambda)f(K_2, V_2, V_2/G_2)\} = \lambda f^*(K_1, G_1) + (1-\lambda)f^*(K_2, G_2)$ .

The second equality follows from the fact that the maximand are identical except that in the second we write  $\lambda V_1 + (1-\lambda)V_2$ ; so if  $V^*$  solves the first problem, any  $V_1^*$  and  $V_2^*$  such that  $\lambda V_1^* + (1-\lambda)V_2^* = V^*$  will be optimal for the second and achieve the same value for the maximand. Other equalities follow by definition or from properties of concave functions,

<sup>68</sup> The examples discussed in p. 127 generate a strictly concave  $f^*$ .

$f^*(K^p(t), G^p(0)) - c^p(t)$ , and calling  $K_\infty^p \equiv \lim_{t \rightarrow +\infty} K^p(t)$ , and  $c_\infty^p \equiv \lim_{t \rightarrow +\infty} c^p(t)$ ,

we have

$$f_k^*(K_\infty^p, G(0)) = \rho,$$

$$c_\infty^p = f^*(K_\infty^p, G(0)).$$

3. If  $f_g^*(K(0), G(0)) > \rho > f_k^*(K(0), G(0))$  then  $\forall t \ K^p(t) = K(0)$ ,

$$\dot{c}^p(t)/c^p(t) = [f_g^*(K^p(0), G^p(t)) - \rho]/\gamma, \text{ and } \dot{G}^p(t) = f^*(K^p(0), G^p(t)) -$$

$c^p(t)$  and calling  $G_\infty^p \equiv \lim_{t \rightarrow +\infty} G^p(t)$ , and  $c_\infty^p \equiv \lim_{t \rightarrow +\infty} c^p(t)$ , we have

$$f_g^*(K(0), G_\infty^p) = \rho,$$

$$c_\infty^p = f^*(K(0), G_\infty^p).$$

4. If  $f_k^*(K(0), G(0)) > f_g^*(K(0), G(0))$  and  $f_k^*(1, x^*) > \rho$  then  $\exists T > 0$  such

that  $\forall t \in [0, T) \ \dot{G}^p(t) = 0$ ,  $\dot{c}^p(t)/c^p(t) = [f_k^*(K^p(t), G^*(0)) - \rho]/\gamma$ , and

$$\dot{K}^p(t) = f^*(K^p(t), G^p(0)) - c^p(t). \forall t \geq T$$

$$\dot{K}^p(t)/K^p(t) = \dot{G}^p(t)/G^p(t) = \dot{c}^p(t)/c^p(t) = [f_k^*(1, x^*) - \rho]/\gamma$$

and

$$c^p(T) = f^*(K^p(T), G^p(T)) - \frac{f_k^*(1, x^*) - \rho}{\gamma} (K^p(T) + G^p(T)).$$

5. If  $f_k^*(K(0), G(0)) = f_g^*(K(0), G(0)) > \rho$ , then<sup>69</sup>  $\forall t$

$$\dot{K}^p(t)/K^p(t) = \dot{G}^p(t)/G^p(t) = \dot{c}^p(t)/c^p(t) = [f_k^*(1, x^*) - \rho]/\gamma$$

and

$$c^p(0) = f^*(K(0), G(0)) - \frac{f_k^*(1, x^*) - \rho}{\gamma} (K(0) + G(0)).$$

<sup>69</sup> It may be useful to note that  $f_k^* = f_g^*$  implies that  $G/K = x^*$ . In this case  $f_k^*(K(0), G(0)) = f_k^*(1, G(0)/K(0)) = f_k^*(1, x^*)$ , where the first equality comes from the fact that if a function is homogeneous of degree 1, its derivative is homogeneous of degree 0.

6. If  $f_g^*(K(0), G(0)) > f_k^*(K(0), G(0))$  and  $f_k^*(1, x^*) > \rho$  then  $\exists T > 0$  such that  $\forall t \in [0, T)$   $\dot{K}^p(t) = 0$ ,  $\dot{c}^p(t)/c^p(t) = [f_g^*(K(0), G^p(t)) - \rho]/\gamma$ , and  $\dot{G}^p(t) = f^*(K^p(t), G^p(t)) - c^p(t)$ .  $\forall t \geq T$

$$\dot{K}^p(t)/K^p(t) = \dot{G}^p(t)/G^p(t) = \dot{c}^p(t)/c^p(t) = [f_k^*(1, x^*) - \rho]/\gamma$$

and

$$c^p(T) = f^*(K^p(T), G^p(T)) - \frac{f_k^*(1, x^*) - \rho}{\gamma} (K^p(T) + G^p(T)).$$

**Proof** See section 4.6. ■

The intuition for this proposition is as follows. First of all, to maximise welfare any investment should be channel to the type of capital the yields the higher return. So investment is specialised except when the marginal products of the two types of capital are equal. When they differ, the return to investment is the greater between the two marginal product. Secondly, if a unit of output is not allocated to investment, it is allocated to consumption. The return to consumption is<sup>70</sup>  $\rho + \gamma\dot{c}/c$ . Optimisation requires the equality between the rates of return on investment and consumption, i.e.

$$\rho + \gamma\frac{\dot{c}}{c} = \max(f_k^*, f_g^*),$$

or rearranging

$$\frac{\dot{c}}{c} = \frac{\max(f_k^*, f_g^*) - \rho}{\gamma},$$

the Keynes-Ramsey rule. Intuitively the plan chosen by the planner depends on the relative marginal productivities of  $K$  and  $G$  and their relationship to the intertemporal

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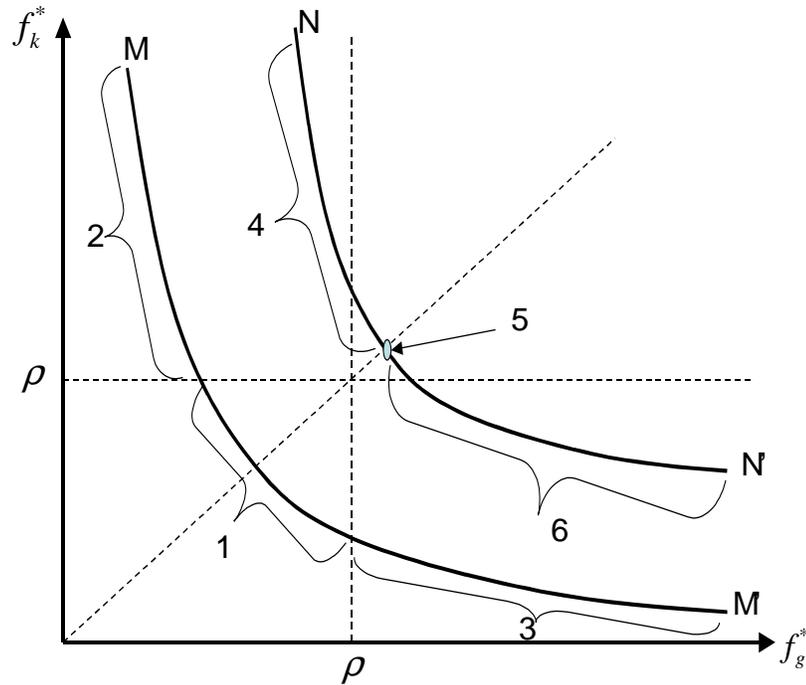
<sup>70</sup> If we indicate with  $u_c(t)$  the instantaneous marginal utility of consumption at time  $t$ , the utility rate of return on consumption is

$$\frac{u_c(t) e^{-\beta t} - u_c(t+dt) e^{-\beta(t+dt)}}{u_c(t) e^{-\beta t} dt};$$

taking the limit for  $dt \rightarrow 0$ , we obtain  $\rho - \dot{u}_c/u_c$ . But the last term is (with a CEIS utility function)  $-\gamma\dot{c}/c$ .

discount factor. The six cases can then be understood with the help of figure 4.1. Since  $f^*$  is constant returns to scale, the marginal products are determined by the  $G/K$  ratio. As this ratio increases (falls)  $f_k^*$  increases (falls) and  $f_g^*$  falls (increases). The curves  $MM'$  and  $NN'$  represent two different loci for two different technologies. Case 1 occurs when the return on investment is below the return on consumption; if possible the planner would like to convert some of the capital stock into consumption goods, but the irreversibility constraints prevent this. The best option is therefore to have no investment at all and consume all output every period and the economy stagnates. In case 2 (3) the return on private (public) capital is initially high enough to induce the planner to accumulate more of it. This however reduces (increases) its marginal product and the economy moves down (up) the  $MM'$  locus until the return to investment equals the discount factor  $\rho$  at which point no further accumulation takes place. Case 4 (6) is analogous except that in this case the economy converges to the optimal  $G/K$  ratio and then keeps on growing on a balanced path with investment in both types of capital. Finally, case 5 is the special case in which the initial  $G/K$  ratio happens to be the optimal one and the economy grows on the balanced path from the beginning.

If one were to remove the irreversibility assumption, the planner would always convert public capital into public capital or *vice versa* to obtain  $G/K = x^*$ , therefore we would always be on the  $45^\circ$  locus in the figure. Then depending on technology we would have either stagnation (on  $MM'$ ) or balanced growth from the start (on  $NN'$ ).



### 4.3 Optimal fiscal policy

In this section we infer some important properties of an optimal fiscal policy. By optimal fiscal policy we mean a vector of functions of time  $\{\tau(t), I_g(t), p(t), B(t)\}_{t=0}^{+\infty}$  such that they can be part of an intertemporal equilibrium that maximises the representative agent's utility. We find convenient to define  $\bar{r}(t) \equiv (1 - \tau(t))r(t)$ . As we shall see presently, one can equivalently think of the government fixing the income tax  $\tau$  or the after tax return to capital  $\bar{r}$ . The following lemma helps in formulating the problem.

**Lemma 4.3** *An intertemporal allocation  $\{c(t), K(t), G(t), V(t)\}_{t=0}^{+\infty}$  is decentralisable as an intertemporal equilibrium if and only if there are  $\{\bar{r}(t), I_g(t),$*

$p(t), B(t), \lambda(t)\}_{t=0}^{+\infty}$  such that  $\forall t$

$$\dot{K}(t) = f(K(t), V(t), V(t)/G(t)) - c(t) - I_g(t) \geq 0, \quad (15)$$

$$\dot{G}(t) = I_g(t) \geq 0, \quad (16)$$

$$c(t)^{-\gamma} = \lambda_c(t), \quad (17)$$

$$\dot{\lambda}_c(t) = (\rho - \bar{r}(t)) \lambda_c(t), \quad (18)$$

$$\dot{B}(t) = \bar{r}(t)(B(t) + K(t)) + I_g(t) - f(K(t), V(t), V(t)/G(t)), \quad (19)$$

$$\bar{r}(t) \geq 0, \quad (20)$$

$$\lim_{t \rightarrow +\infty} \lambda_c(t) e^{-\rho t} K(t) = \lim_{t \rightarrow +\infty} \lambda_c(t) e^{-\rho t} B(t) = 0. \quad (21)$$

**Proof** The details of the proof are in section 4.6, but essentially to be decentralisable an allocation must satisfy the technological constraint (15) and (16); it must be chosen optimally by the representative agents, hence it must satisfy (17) and (18) (which together give the Keynes-Ramsey equation), as well as the transversality conditions (21); using the relationships between prices and marginal products and Euler's theorem for homogeneous functions, we can rewrite the government budget constraint as (19). ■

The the problem of identifying the optimal fiscal policy can be written as

$$\max \int_0^{+\infty} \frac{c(t)^{1-\gamma} - 1}{1-\gamma} e^{-\rho t} dt \quad (22)$$

subject to (15)-(21) and  $K(0), G(0)$  given. Henceforth, to de-clutter the notation, we stop indicating explicitly the dependence of variable on time when no confusion should arise. To derive the first order conditions, define the Hamiltonian:

$$\begin{aligned} H(K, G, \lambda_c, B, c, I_g, V, \lambda_k, \lambda_g, \phi_{\lambda_c}, \mu) &\equiv \frac{c^{1-\gamma} - 1}{1-\gamma} \\ &+ \lambda_k [f(K, V, V/G) - c - I_g] + \lambda_g I_g + \phi_{\lambda_c} (\rho - \bar{r}) \lambda_c \\ &+ \mu [\bar{r} (B + K) + I_g - f(K, V, V/G)], \end{aligned}$$

and the Lagrangian

$$\begin{aligned} L(K, G, \lambda_c, B, c, I_g, V, \lambda_k, \lambda_g, \phi_{\lambda_c}, \mu, \eta_{\bar{r}}, \eta_{\lambda_c}, \eta_k, \eta_g) \\ \equiv H + \eta_{\lambda_c} [c^{-\gamma} - \lambda_c] + \eta_{\bar{r}} \bar{r} + \eta_k [f(K, V, V/G) - c - I_g] + \eta_g I_g. \end{aligned}$$

**Lemma 4.4** *If  $\{c^{op}, K^{op}, G^{op}, V^{op}, I_g^{op}\}_{t=0}^{+\infty}$  are the solution to (22) subject to (15)-(21) and  $K(0), G(0)$  given, then there are  $\{\lambda_k^{op}, \lambda_g^{op}, \phi_{\lambda_c}^{op}, \mu^{op}, \eta_{\lambda_c}^{op}, \eta_{\bar{r}}^{op}, \eta_k^{op}, \eta_g^{op}\}_{t=0}^{+\infty}$  such that  $c^{op}, V^{op}, I_g^{op}, \bar{r}$  maximise  $H$  s.t.  $I_g \geq 0$  and  $f(K, V, V/G) - c - I_g \geq 0, \bar{r} \geq 0$ .*

$$(c^{op})^{-\gamma} = \lambda_k^{op} + \gamma \eta_{\lambda_c}^{op} (c^{op})^{-\gamma-1} + \eta_k^{op}, \quad (23)$$

$$[\lambda_k^{op} - \mu^{op} + \eta_k^{op}] [f_v(K^{op}, V^{op}, V^{op}/G^{op}) + f_{\Gamma}(K^{op}, V^{op}, V^{op}/G^{op})/G^{op}] = 0, \quad (24)$$

$$-\phi_{\lambda_c}^{op} \lambda_c^{op} + \mu^{op} (B^{op} + K^{op}) + \eta_{\bar{r}}^{op} = 0, \quad (25)$$

$$\lambda_g^{op} - \lambda_k^{op} + \mu^{op} - \eta_k^{op} + \eta_g^{op} = 0, \quad (26)$$

$$\dot{\lambda}_k^{op} = \rho \lambda_k^{op} - [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] f_k(K^{op}, V^{op}, V^{op}/G^{op}) - \mu^{op} \bar{r}^{op}, \quad (27)$$

$$\dot{\lambda}_g^{op} = \rho \lambda_g^{op} + [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{(G^{op})^2} \quad (28)$$

$$\dot{\phi}_{\lambda_c}^{op} = \rho \phi_{\lambda_c}^{op} - \phi_{\lambda_c}^{op} (\rho - \bar{r}^{op}) + \eta_{\lambda_c}^{op}, \quad (29)$$

$$\dot{\mu}^{op} = \mu^{op} [\rho - \bar{r}^{op}]. \quad (30)$$

$$\eta_k^{op} [f(K^{op}, V^{op}, V^{op}/G^{op}) - c^{op} - I_g^{op}] = 0, \quad \eta_k^{op} \geq 0, \quad (31)$$

$$\eta_g^{op} I_g^{op} = 0, \quad \eta_g^{op} \geq 0. \quad (32)$$

**Proof** These first order conditions are standard results in optimal control theory, see Arrow and Kurz [5] and Seierstad and Sydsaeter [98]. ■

First note that  $\lambda_k$ , the shadow value of private capital  $K$ , will be strictly positive given non-satiation and the fact that the marginal product of capital is always positive. If lump-sum taxes were allowed, the level of the stock of public debt could be altered without the deadweight losses associated with distortionary taxation. The costate variable  $\mu^{op}$  associated to the government budget constraint therefore measures the cost of distortionary taxation, the so-called *marginal excess burden of taxation*, and is clearly non-positive. Then using (24) we have

$$f_v(K^{op}, V^{op}, V^{op}/G^{op}) + f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op})/G^{op} = 0.$$

Since in a static equilibrium we must have  $f_v(K, V, V/G) = p$ , then the optimal fiscal policy implies that the government chooses a user charge  $p$  that satisfies

$$p = -f_\Gamma(K, V, V/G)/G. \quad (33)$$

The interpretation of this result is straightforward: in choosing  $v$ , the individual level of utilisation, each firm only considers the contribution that the marginal unit of consumption of public services brings thus aligning the private benefit to the private cost:  $\psi_v = f_v = p$ ; this way the social cost, created by the congestion externality is neglected. The optimal pricing policy is to correct for this externality by imposing an user fee that reflects the external diseconomy created by the marginal usage of public services; the marginal unit of utilisation decreases aggregate output by  $f_\Gamma/G$ , so this is its social cost. The optimal user fee makes firms internalise this external effect. It follows that the pricing of public services should reflect only static efficiency considerations, not the need to finance investment in public infrastructures.<sup>71</sup> In the notation of the previous section, we have just obtained the result that if the government adopts the optimal fiscal policy, then  $\forall t$

$$f(K^{op}, V^{op}, V^{op}/G^{op}) = f^*(K^{op}, G^{op}). \quad (34)$$

Let us express the marginal burden of taxation  $\mu^{op}$  in terms of consumption goods by defining  $m = -\mu^{op}/\lambda_c^{op}$ . Barro [12] argued that the optimal debt policy smooths out the excess burden over time. The following lemma shows that this result remains valid in our framework. An analogous result was established by Chamley [34] and Judd [67].

<sup>71</sup> Batina [17] obtained a similar result in an overlapping generations model where current government expenditures (not the stock) affects households' utilities (not firms' production sets).

**Lemma 4.5**  $m \equiv -\mu^{op}/\lambda_c^{op}$  is constant over time.

**Proof** Equations (18) and (30) imply

$$\frac{\dot{\mu}^{op}}{\mu^{op}} = \rho - \bar{r}^{op} = \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}},$$

i.e.  $\mu^{op}/\lambda_c^{op}$  is constant. ■

The following proposition shows that the optimal income tax is zero along a balanced growth path.

**Proposition 4.2** Call  $\Theta^{op} \equiv \{t : \dot{K}^{op} > 0, \dot{G}^{op} > 0\}$ . Then for all  $t \in \Theta^{op}$  we have EITHER  $\bar{r}^{op} = f_k(K^{op}, V^{op}, V^{op}/G^{op})$  (or equivalently  $\tau = 0$ ) OR  $\bar{r}^{op} = 0$  (or equivalently  $\tau^{op} = 1$ ). Furthermore  $t \in \Theta^{op}$  only if  $G/K = x^*$ .

**Proof** See section 4.6. ■

So, as in the first best, positive investment in both types of capital occurs only if their social marginal products coincide. If the economy converges to a balanced path, therefore, it converges to one that has the same  $G/K$  ratio as the first best, and -in virtue of  $\dot{c}^{op}/c^{op} = (1/\gamma)[\bar{r}^{op} - \rho] = (1/\gamma)[f_k^* - \rho] = \dot{c}^p/c^p$ - the same growth rate. This does not mean, however, that the first best can be replicated; the transitional dynamics will in general<sup>72</sup> be different, and therefore the *levels* of consumption and capital stocks will in general be different as well. On a balanced path the income tax is either 100% or 0. But since the growth rate is  $(1/\gamma)(\bar{r}^{op} - \rho)$ , the optimal tax cannot be 100% unless we have negative growth.

Once realised that the optimal user charge internalises the congestion externality and therefore that  $f(K, V, V/G) = f^*(K, G)$ , this "zero tax" result is completely

<sup>72</sup> One can show that the first best can be replicated if the initial level of the stock of debt  $B(0)$  has a particular value, depending on  $K(0)$  and  $G(0)$ .

analogous to the result obtained by Chamley [34] and Judd [67]. Manipulating the expression in the proof of the proposition, one can easily show that<sup>73</sup>

$$f_k(K^{op}, V^{op}, V^{op}/G^{op}) - \bar{r}^{op} = -\frac{m}{\Lambda} \frac{d}{dt} \varepsilon_{MU},$$

where  $m$  -it will be recalled- is the marginal burden of taxation,  $\Lambda$  is the marginal social value of government wealth holding private wealth constant<sup>74</sup> and  $\varepsilon_{MU}$  is the elasticity of the marginal utility of consumption.<sup>75</sup> This formula, valid when  $\bar{r} > 0$ , shows that the optimal wedge between  $f_k$  (the social return to investment) and  $\bar{r}$  (the private after tax return) is proportional to the inverse of the elasticity of consumption demand,<sup>76</sup> which corresponds to the inverse elasticity result found in the static optimal taxation literature (see for example Baumol and Bradford [18]). We have already shown that  $m$  is constant,  $\varepsilon_{MU}$  is always constant with the CIES utility function we have chosen<sup>77</sup>.

Before the reader discounts the result of the proposition as trivial, however, we would like to emphasise that it might not have been entirely expected however. In fact even Judd [67] to whom we owe this observation takes the case of congestion as one example where departure from the zero tax rule is to be expected. And we have already noted that Turnovsky [105] and Ott and Turnovsky [83] also find non-zero tax results based on the presence of congestion. The latter is particularly remarkable given that their model also have an user charge. We will expand on this in section 4.4.

<sup>73</sup> An analogous expression was first derived by Judd [67].

<sup>74</sup> See the proof of the proposition for more detail.

<sup>75</sup> I.e.  $\varepsilon_{MU} \equiv cu_{cc}/u_c$ , with obvious meaning of the symbols.

<sup>76</sup> From the household first order conditions  $u_c = \lambda_c$ . Then

$$\frac{dc}{d\lambda_c} \frac{\lambda_c}{c} = \frac{u_c}{u_{cc}c}.$$

<sup>77</sup> And must be constant in a steady-state, which is why Chamley could derive the result for more general utility functions.

Another feature of the optimal plan is worthy of being highlighted. Given that  $p^{op} = -f_{\Gamma}(K^{op}, V^{op}, V^{op}/G^{op})/G^{op}$ , we have

$$p^{op}V^{op} = -f_{\Gamma}(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{G^{op}}. \quad (35)$$

On the other hand by (34) and the envelope theorem,

$$f_g^*(K^{op}, G^{op}) = -f_{\Gamma}(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{(G^{op})^2};$$

comparing this last equation with (35) we obtain

$$f_g^*(K^{op}, G^{op}) = \frac{p^{op}V^{op}}{G^{op}},$$

or

$$f_g^*(K^{op}, G^{op}) G^{op} = p^{op}V^{op}. \quad (36)$$

In other words, through user charge revenues, the government appropriates a share of total output just equal to the contribution of the stock of infrastructures provided. The optimal policy essentially mimics the situation where we have both types of capital provided privately and competitively. This result is of interest because it indicates that the same allocation could be decentralised in an equilibrium where public infrastructures are privately owned but the user charge is administratively set equal to  $p^{op}$  by the government. Furthermore, this result can be used to demonstrate that if the optimal policy converges to the balanced growth path, the government runs primary surpluses in the long run. This is interesting, because for a broad class of models, one typically finds that the optimal policy entails the government running large fiscal surplus in initial periods and then use the income stream from the accumulated wealth to finance fiscal deficits in the long-run when taxes are kept low and the primary budget is in deficit (Jones, Manuelli and Rossi [65]). The reason why things are different here is that in our

model one element of the optimal policy mix, the optimal user charge, creates a flow of revenues that is more than sufficient to cover for the financing of public investment.

**Corollary 4.2** *If  $\exists T$  such that  $t > T \Rightarrow t \in \Theta^{op}$  (where  $\Theta^{op}$  is as defined in proposition 4.2) and  $\dot{c}^{op}/c^{op} = \dot{K}^{op}/K = \dot{G}^{op}/G^{op}$ , then  $p^{op}V^{op} > \dot{G}^{op} = I_g$ .*

**Proof** Convergence to a balanced growth path and the transversality conditions (21) imply

$$f_k^*(K^{op}, G^{op}) = f_g^*(K^{op}, G^{op}) > \frac{f_k^*(K^{op}, G^{op}) - \rho}{\gamma} = \frac{f_g^*(K^{op}, G^{op}) - \rho}{\gamma},$$

i.e. that the economy is dynamically efficient. Then

$$f_g^*(K^{op}, G^{op}) G > \frac{f_g^*(K^{op}, G^{op}) - \rho}{\gamma} G^{op}.$$

Balanced growth means

$$\dot{G}^{op} = \frac{f_g^*(K^{op}, G^{op}) - \rho}{\gamma} G^{op}.$$

Then using (36) and the last two equations, we have

$$p^{op}V^{op} > \dot{G}^{op} = I_g.$$

QED. ■

So far we have concentrated on balanced growth. But the solution the first best allocation problem showed that typically there will be periods where investment is specialised in one type of capital only. For these cases, we have the following interesting results.

**Proposition 4.3**  *$\forall t$  such that  $\dot{K}^{op} > 0$ ,  $\dot{G}^{op} = 0$ , we must have either  $\bar{r}^{op} = 0$  or  $\bar{r}^{op} = f_k^*(K^{op}, G^{op})$ .*

**Proof** The proof is virtually identical to that of the first part of proposition 4.2. ■

**Proposition 4.4**  $\forall t$  such that  $\dot{K}^{op} = 0$ ,  $\dot{G}^{op} > 0$  we must have either  $\bar{r}^{op} = 0$  or  $\bar{r}^{op} = f_g^*(K^{op}, G^{op})$ .

**Proof** See section 4.6 ■

These last two propositions deal with the transitional dynamics when the irreversibility constraints hold. The two are formally very similar, but while there is probably not much to notice in the first one, the second show that when the initial  $G/K$  ratio is below the optimal  $x^*$  the optimal tax is not zero when it is not 100% as in the previous two proposition. To see this note that since  $\bar{r} = (1 - \tau) f_k(K^{op}, V^{op}, V^{op}/G^{op})$  and  $f_k(K^{op}, V^{op}, V^{op}/G^{op}) = f_k^*(K^{op}, G^{op})$  when the optimal user charge is set, we have

$$(1 - \tau) f_k^*(K^{op}, G^{op}) = f_g^*(K^{op}, G^{op}),$$

or

$$\tau = \frac{f_k^*(K^{op}, G^{op}) - f_g^*(K^{op}, G^{op})}{f_k^*(K^{op}, G^{op})}.$$

But since it is only optimal to specialise in public capital accumulation when its marginal product exceeds that of private capital, it follows that the optimal tax is negative, i.e. a subsidy.

One way to understand this result is that in this second best scenario the government does not control saving decision directly. If the government set  $\tau = 0$ , the private sector would take  $f_k^*$  as the return on investment, but from the social point of view the rate of return is  $f_g^*$ . It is clear that this divergence between the marginal product of private and public capital can only occur when the irreversibility constraint on public investment is present and binding. When the binding constraint is the irreversibility of

private capital, there is no need to correct the rate of return perceived by the private sector, because in this case  $f_k^*$  is also the correct rate of return from the social point of view.

#### **4.4 Non-optimal user charges and alternative congestion function**

In this section we discuss two modifications to the model. The first is to impose non-optimal user charges, the second is to consider an alternative modelling of congestion. The purpose of this section is to clarify what drives the main results of the previous section and to link our analysis to the existing literature. The zero capital tax result of proposition 4.2 is driven by the fact that the optimal user charge acts as a Pigouvian tax correcting the congestion externality. To show this we analyse how the optimal tax would look like if the optimal user charge cannot be selected. We shall see that then the optimal tax is generally different from zero, although it may be negative (should the arbitrary user charge be too high). Next we consider an alternative congestion function; in particular we shall follow Turnovsky [105] and Ott and Turnovsky [83] and assume that congestion is a function of the private/public capital stock ratio. In this case the optimal tax is positive. This last conclusion is obviously a mere re-statement of a result of the afore mentioned papers, but it is reported here so that putting it side by side with the analysis in the previous section will clarify where the differences come from.

### 4.4.1 Non-optimal user charges

Let us then arbitrarily fix<sup>78</sup> the user charge at some level  $p \geq 0$  for all  $t$ . For any pair of capital stocks  $(K, G)$ , we will have a static equilibrium as described in lemma 4.2; in particular we will have

$$f_v(K, V, V/G) = p,$$

which determines  $V$  as a function of  $(K, G; p)$ . Substituting this value of  $V$  in the aggregate production function we obtain the output produced. Let us then indicate<sup>79</sup>

$$\hat{f}(K, G) \equiv f(K, V(K, G; p), V(K, G; p)/G).$$

One may usefully compare  $\hat{f}$  with  $f^*$  defined at p. 134. The latter indicates the maximum output achievable with given capital stocks  $(K, G)$ ; to achieve that level of output the decentralised economy requires  $p$  to be set optimally.  $\hat{f}$  is the output obtained when the user charge is arbitrarily set, and obviously  $\hat{f} \leq f^*$ . It is easily verified that  $\hat{f}$  is homogenous of degree 1 in  $(K, G)$ , and that

$$\hat{f}_k(K, G) = f_k(K, V(K, G; p), V(K, G; p)/G) + \quad (37)$$

$$[f_v(K, V(K, G; p), V(K, G; p)/G)] V_k(K, G; p) \quad (38)$$

$$+ [f_\Gamma(K, V(K, G; p), V(K, G; p)/G)/G] V_k(K, G; p), \quad (39)$$

where

$$V_k(K, G; p) \equiv \partial V(K, G; p) / \partial K.$$

Furthermore

$$rK + pV = \hat{f}(K, G)$$

<sup>78</sup> One can of course also consider an arbitrary but time varying user charge. Nothing of substance would change except that the aggregate production function would become time-varying. We would therefore have to invest in some more notation without much to gain in terms of intuition.

<sup>79</sup> Since in this section  $p$  will be kept constant, we drop it as a term in  $\hat{f}$ .

remains valid. Then following the same steps as in lemma 4.3, we find that the optimal fiscal policy can be derived from the solution to the following problem:

$$\begin{aligned}
 & \max \int_0^{+\infty} \frac{c(t)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt \quad s.t. \\
 & \dot{K}(t) = \hat{f}(K(t), G(t)) - c(t) - I_g(t) \geq 0, \\
 & \dot{G}(t) = I_g(t) \geq 0, \\
 & c(t)^{-\gamma} = \lambda_c(t), \\
 & \dot{\lambda}_c(t) = (\rho - \bar{r}(t)) \lambda_c(t), \\
 & \dot{B}(t) = \bar{r}(t) (B(t) + K(t)) + I_g(t) - \hat{f}(K(t), G(t)), \\
 & \bar{r}(t) \geq 0, \\
 & \lim_{t \rightarrow +\infty} \lambda_c(t) e^{-\rho t} K(t) = \lim_{t \rightarrow +\infty} \lambda_c(t) e^{-\rho t} B(t) = 0.
 \end{aligned} \tag{40}$$

**Proposition 4.5** *Assume that the government chooses an optimal fiscal policy given the constraint  $p(t) = p \forall t$ . On a balanced growth path with  $\dot{K}(t) > 0$  and  $\dot{G}(t) > 0$  the optimal tax is*

$$\tau = \frac{f_k(K, V(K, G; p), V(K, G; p)/G) - \hat{f}_k(K, G)}{f_k}$$

**Proof** See section 4.6. ■

Using (37) and dropping the arguments to make the result more easily readable, we have found that the optimal tax is

$$\tau = -\frac{(f_v + f_\Gamma/G) V_k}{f_k}.$$

This formula is rather intuitive: when the user charge is insufficient to internalise the congestion externality (i.e.  $f_v + f_\Gamma/G < 0$ ) the income tax should be positive, and *vice versa*. It follows that the optimal tax is zero either when the user charge is chosen optimally (and therefore  $f_v + f_\Gamma/G = 0$ ) or when an increase in the private capital stock does not affect the choice of  $V$  (i.e. when  $V_k = 0$ ). This would happen if the production function is such that  $f_v$  is independent of  $K$ .

One can draw a parallel between this result and that in Correia [36]. The latter finds that the Chamley result not to hold when the tax system is incomplete in the sense that there is one factor of production that cannot be taxed; then unless there is a strong separability between taxable and non taxable factors in the production function, the tax is positive (negative) when factors are complements (substitute). In this section we have established a similar result: it is only optimal to alter the intertemporal margin if there are not enough instruments to correct all distortions at the intratemporal margins.

#### 4.4.1 An alternative modelling of congestion

In this subsection we depart from the assumptions on technology that we have worked with so far. The reason for doing so is to shed some further insight on the results obtained and to link these results with the existing literature. Let us assume that the production side of the economy is characterised by a single representative consumer acting competitively, with production function

$$f(K, V, \Gamma),$$

where now

$$\Gamma = K/G,$$

i.e. congestion depends on the private/public capital ratio rather than the aggregate utilisation/public capital ratio.

**Proposition 4.6** *When congestion is a function of  $K/G$ , the optimal user charge is zero and the optimal tax on a balanced growth path is generally positive.*

**Proof** See section 4.6 ■

This result is hardly surprising at this point. We shown in section 4.3 that the optimal user charge is chosen to force firms to internalise the congestion externality implied by their choice of utilisation,  $v$ . However, in the model of this subsection the choice of  $v$  has no effect on congestion, hence there is no need to correct the private choice of  $v$ . But now the private choice of  $K$  does create a negative congestion externality; it therefore becomes optimal to use the income tax to internalise it.

We conclude this sub-section with a comparison between our main model and that in Ott and Turnovsky [83]. They assume that the economy is populated by  $n$  firms with identical production functions:

$$y = f(k, E_s),$$

where  $k$  is the firm's capital stock while  $E_s$  is the flow of public services.<sup>80</sup> The latter is given by

$$E_s = E \left( \frac{k}{K} \right)^\varepsilon,$$

where  $E$  measures government expenditures and  $K = nk$  is the aggregate capital stock. It is therefore assumed that each firms benefits from a given level of public expenditures proportionally to it capital stock relative to the aggregate stock. The parameter  $\varepsilon \in$

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<sup>80</sup> In fact Ott and Turnovsky formulation has two types of public goods, one excludable and one non-excludable. We simplify by dropping the latter, since we have not included it in our analysis.

$[0, 1]$  measures the degree of congestion:  $\varepsilon = 0$  means that any unit of  $E$  affects all firms independently from their number, i.e. there is no congestion;  $\varepsilon = 1$  means that  $E_s = E/n$ , i.e. public services are essentially a private good.. Ott and Turnovsky make the assumption that the user charge is chosen as to equate the demand and supply of government spending, i.e.  $p$  solves

$$p = f_E (K/n, En^{-\varepsilon}) n^{-\varepsilon}.$$

They then show that when the optimal fiscal policy implies a positive tax. The reason for this result is that the user charge in this case does not internalise the congestion externality, and therefore in this model the private and social marginal product of capital differ. The income tax is needed to correct this divergence.

## 4.5 Conclusions

We have analysed a model of endogenous economic growth driven by investment in public infrastructures. To an otherwise relatively standard framework, we have added two realistic features: that public services are excludable and rejectable and that investment is irreversible. We have analysed the first best allocation, that is the allocation chosen by an all powerful and benevolent social planner. The qualitative characteristic of this allocation resemble those found in simpler models: investment is specialised in the form of capital (private or public) with the highest marginal product. Static efficiency is attained at all times.

We then analysed the optimal fiscal policy, i.e. the best allocation that can be decentralised as a competitive equilibrium with distortionary taxes and public services fees. We showed that in contrast with other existing models with congestion external-

ities, the zero capital tax result typically found in models with infinitely lived agents remains valid in ours.

There are many interesting directions for future research. First of all, we have seen that when the social marginal product of public capital exceed the social marginal product of private capital, the qualitative differences between the first and second best allocation appear sharper. It seems that the irreversibility of investment can have important effects on the design of the optimal fiscal policy. We think it will be interesting to investigate this issue further probably with the aid of numerical computation.

We have not considered the possibility that households as well as firms may be users of public services. One can easily verify that if usage of public services enter the utility function in an additive way, most of the results found in the previous section still hold although the optimal user charge will in general be different. In particular, the income tax will still have the "all or nothing" characteristic found above. However, the case in which the utilisation of public services affects the marginal utility of consumption of the household is more complicated, but also potentially more interesting.

Recently Rioja [89] and Dioikitopoulos and Kalyvitis [45] have studied growth models where the depreciation rate of public capital depends on maintenance expenditures. It would be interesting to expand their analysis by allowing depreciation to depend on utilisation of public services.

Finally, one major issue with our approach so far is that the representative agent approach does not allow us to address the issue of equity. Particularly if households are consumer of public services together with firms, the user charge that maximise aggregate output may cut off poorer household from public services altogether. We conjecture that the optimal fiscal mix may look very differently in this set up.

All these interesting avenues are left to be explored in future research.

## 4.6 Proofs of propositions in chapter 4

### 4.6.1 Proof of lemma 4.1

The proof is constructive. We first look for all the points on the boundary of  $\Psi(\Gamma)$  such that  $(0, 0, 0)$  belongs to the tangent hyperplane at that point.<sup>81</sup> Then the union of all rays going from  $(0, 0, 0)$  to this points gives us the boundary of the aggregate production set. This fact is used to give the relationship between  $f$  and  $\psi$  given in the lemma.

If  $y_0 = \psi(k_0, v_0, \Gamma)$ , then the tangent hyperplane has equation

$$y - y_0 = \psi_k(k_0, v_0, \Gamma)(k - k_0) + \psi_v(k_0, v_0, \Gamma)(v - v_0).$$

If  $(0, 0, 0)$  belongs to this hyperplane, we must then have

$$y_0 = \psi_k(k_0, v_0, \Gamma)k_0 + \psi_v(k_0, v_0, \Gamma)v_0,$$

or

$$\psi(k_0, v_0, \Gamma) = \psi_k(k_0, v_0, \Gamma)k_0 + \psi_v(k_0, v_0, \Gamma)v_0.$$

Pairs  $(k, v)$  that satisfies this equation are therefore those for which the tangent hyperplane goes through  $(0, 0, 0)$  as wished, hence the definition of  $\hat{k}(v)$  given in the lemma. Now for an arbitrary  $(k, v)$ , let us call  $\xi \equiv k/v$ . The solution for  $\hat{v}$  to  $\xi\hat{v} = \hat{k}(\hat{v})$  gives the function  $\hat{v}(k/v)$  as in the lemma. Now by construction the pair  $(\hat{k}(\hat{v}(k/v)), \hat{v}(k/v))$  is such that it belongs to the locus of tangency between  $f$  and  $\psi$  and  $\hat{k}(\hat{v}(k/v))/\hat{v}(k/v) = \xi$ . At this point, then

$$\hat{y} \equiv \psi(\hat{k}(\hat{v}(k/v)), \hat{v}(k/v)) = f(\hat{k}(\hat{v}(k/v)), \hat{v}(k/v)).$$

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<sup>81</sup> This is analogous to finding the point where the average and marginal product coincide for a function of one variable.

Given homogeneity of  $f$ , if  $y = f(k, v, \Gamma)$ , then  $y/\hat{y} = v/\hat{v}$ . Then we can write

$$f(k, v, \Gamma) = \frac{v}{\hat{v}(k/v, \Gamma)} \psi \left( \hat{k}(\hat{v}(k/v, \Gamma), \Gamma), \hat{v}(k/v, \Gamma), \Gamma \right),$$

QED. ■

### 4.6.2 Proof of proposition 4.1

We apply theorem 1 p. 276 and theorem 5 p.287 of Seierstad and Sydsaeter [98].

These guarantee that  $(K^p(t), V^p(t), G^p(t), c^p(t))_{t=0}^{+\infty}$  are a solution to the problem if and only if there exist continuous and piecewise continuously differentiable functions

$(\lambda_k^p(t), \lambda_g^p(t))_{t=0}^{+\infty}$  and non-negative and piecewise continuous functions  $(\eta_k^p(t), \eta_g^p(t))_{t=0}^{+\infty}$

such that calling

$$\begin{aligned} H(c(t), V(t), I_g(t), K(t), G(t), \lambda_k^p(t), \lambda_g^p(t)) &\equiv \frac{c(t)^{1-\gamma} - 1}{1-\gamma} \\ &+ \lambda_k^p(t) [f(K(t), V(t), V(t)/G(t)) - c(t) - I_g(t)] + \lambda_g^p(t) I_g(t), \end{aligned}$$

we have that for any  $(c(t), V(t), I_g(t)) \geq 0$ ,

$$\begin{aligned} &H(c^p(t), V^p(t), I_g^p(t), K^p(t), G^p(t), \lambda_k^p(t), \lambda_g^p(t)) \quad (41) \\ &\geq H(c(t), V(t), I_g(t), K^p(t), G^p(t), \lambda_k^p(t), \lambda_g^p(t)), \end{aligned}$$

$$c^p(t)^{-\gamma} = \lambda_k^p(t) + \eta_k^p(t), \quad (42)$$

$$f_v(K^p(t), V^p(t), V^p(t)/G^p(t)) = -\frac{f_\Gamma(K^p(t), V^p(t), V^p(t)/G^p(t))}{G^p(t)}, \quad (43)$$

$$\lambda_g^p(t) + \eta_g^p(t) = \lambda_k^p(t) + \eta_k^p(t), \quad (44)$$

$$\dot{\lambda}_k^p(t) = \rho \lambda_k^p(t) - (\lambda_k^p(t) + \eta_k^p(t)) f_k(K^p(t), V^p(t), V^p(t)/G^p(t)), \quad (45)$$

$$\dot{\lambda}_g^p(t) = \rho \lambda_g^p(t) + (\lambda_k^p(t) + \eta_k^p(t)) f_\Gamma(K^p(t), V^p(t), V^p(t)/G^p(t)) \frac{V^p(t)}{G^p(t)^2}, \quad (46)$$

$$\eta_k^p(t) [f(K^p(t), V^p(t), V^p(t)/G^p(t)) - c^p(t) - I_g^p(t)] = 0, \quad (47)$$

$$\eta_g^p(t) I_g^p(t) = 0, \quad (48)$$

where  $\lambda_k^p(t)$  is the costate variable associated with  $K^p(t)$ ,  $\lambda_g^p(t)$  the costate variable associated with  $G^p(t)$ ,  $\eta_k^p(t)$  the multiplier associated with the constraint  $\dot{K}^p(t) \geq 0$  and  $\eta_g^p(t)$  the multiplier associated with  $\dot{G}^p(t) \geq 0$ .

Observe that (41) implies that the planner will choose

$$V(t) = \operatorname{argmax} f(K(t), V(t), V(t)/G(t)).$$

Therefore  $\forall t$  we have

$$f(K^p(t), V^p(t), V^p(t)/G^p(t)) = f^*(K^p(t), G^p(t)).$$

By the envelope theorem

$$f_k(K^p(t), V^p(t), V^p(t)/G^p(t)) = f_k^*(K^p(t), G^p(t)). \quad (49)$$

and

$$-f_\Gamma(K^p(t), V^p(t), V^p(t)/G^p(t)) \frac{V^p(t)}{G^p(t)^2} = f_g^*(K^p(t), G^p(t)). \quad (50)$$

To prove part 1 of the proposition, consider the system

$$\begin{aligned}\lambda_k^p + \eta_k^p - f^*(K(0), G(0))^{-\gamma} &= 0, \\ \lambda_g^p + \eta_g^p - \lambda_k^p - \eta_k^p &= 0, \\ (\rho - f_k^*(K(0), G(0))) \lambda_k^p - \eta_k^p f_k^*(K(0), G(0)) &= 0, \\ \rho \lambda_g^p - \lambda_k^p f_g^*(K(0), G(0)) - \eta_k^p f_g^*(K(0), G(0)) &= 0.\end{aligned}$$

This is a linear system of 4 equations in 4 unknowns:  $\lambda_k^p$ ,  $\eta_k^p$ ,  $\lambda_g^p$  and  $\eta_g^p$ . It can be solved to yield

$$\begin{aligned}\lambda_k^p &= f^*(K(0), G(0))^{-\gamma} \frac{f_k^*(K(0), G(0))}{\rho} > 0, \\ \eta_k^p &= f^*(K(0), G(0))^{-\gamma} \frac{\rho - f_k^*(K(0), G(0))}{\rho} > 0, \\ \lambda_g^p &= f^*(K(0), G(0))^{-\gamma} \frac{f_g^*(K(0), G(0))}{\rho} > 0, \\ \eta_g^p &= f^*(K(0), G(0))^{-\gamma} \frac{\rho - f_g^*(K(0), G(0))}{\rho} > 0.\end{aligned}$$

Now if we choose  $\lambda_k^p(t) = \lambda_k^p$ ,  $\eta_k^p(t) = \eta_k^p$ ,  $\lambda_g^p(t) = \lambda_g^p$  and  $\eta_g^p(t) = \eta_g^p \forall t$ ,  $V^p(t) = \operatorname{argmax} f(K(0), V^p(t), V^p(t)/G(0))$ ,  $c^p(t) = f^*(K(0), G(0))$ ,  $I_g^p(t) = 0 \forall t$ , one can easily verify that the proposed control, state and costate variables satisfy all of the necessary and sufficient conditions for an optimum and therefore constitute an optimal plan.

For part 2, choose  $\eta_k^p(t) = \eta_g^p(t) = 0 \forall t$ ,

$$\lambda_k^p(0) = \lambda_g^p(0) = \left[ f^*(K(0), G(0)) - \frac{f_k^*(1, x^*) - \rho}{\gamma} (K(0) + G(0)) \right]^{-\gamma},$$

$\lambda_j^p(t) = \lambda_j^p(0) e^{[\rho - f_k^*(1, x^*)]t} \forall t \ j = k, g$ . Again it is straightforward to see that the control, state and costate variables suggested satisfies all necessary and sufficient conditions and therefore are optimal.

For part 3, set  $\eta_k^p(t) = 0 \forall t$ . Call  $C(K)$  the solution to the following differential equation<sup>82</sup>

$$\frac{dc}{dK} = \frac{[f_k^*(K, G(0)) - \rho] / \gamma}{f_k^*(K, G(0)) - c(K)},$$

with boundary solution  $(c_\infty^p, K_\infty^p)$ . For  $K^p(t)$  choose the solution of the differential equation  $\dot{K}^p(t) = f^*(K(t), G(0)) - c(K^p(t))$  with initial condition  $K(0)$ . Set  $\lambda_k^p(t) = c(K^p(t))^{-\gamma}$ ; for  $\lambda_g^p(t)$  choose the solution to the differential equation  $\dot{\lambda}_g^p(t) = \rho \lambda_g^p(t) - \lambda_k^p(t) f_g^*(K^p(t), G(0))$  is

$$\lambda_g^p(t) = e^{\rho t} \left[ \lambda_g^p(0) - \int_0^{+\infty} f_g^*(K(\xi), G(0)) \lambda_k^p(\xi) e^{-\rho(t-\xi)} d\xi \right].$$

If we choose  $\lambda_g^p(0) = \int_0^{+\infty} \lambda_k^p f_g^*(K^p(t), G(0)) e^{-\rho t} dt$  then the transversality condition

$$\lim_{t \rightarrow +\infty} \lambda_g^p(t) G^p(t) e^{-\rho t} = 0$$

will be met; finally choose  $\eta_g^p(t) = \lambda_k^p(t) - \lambda_g^p(t)$ . Then the proposed path will satisfy all necessary and sufficient conditions and it is therefore optimal.

For part 4, set  $\eta_k^p(t) = 0 \forall t$ . Call  $C(K)$  the solution to the following differential equation

$$\frac{dc}{dK} = \frac{[f_k^*(K, G(0)) - \rho] / \gamma}{f_k^*(K, G(0)) - c(K)}$$

with boundary condition  $(c(T), K(T))$ , where  $K(T) = G(0) / x^*$ ,  $c(T) = f^*(K(T), G(0))$

$-\frac{f_k^*(1, x^*) - \rho}{\gamma} (K(T) + G(0))$ . For  $K^p(t)$  choose the solution of the differential equation  $\dot{K}^p(t) = f^*(K(t), G(0)) - C(K^p(t))$  with initial condition  $K(0)$ . Set  $\lambda_k^p(t) = C(K^p(t))^{-\gamma}$ ; for  $\lambda_g^p(t)$  choose the solution to the differential equation  $\dot{\lambda}_g^p(t) = \rho \lambda_g^p(t) - \lambda_k^p(t) f_g^*(K^p(t), G(0))$  with boundary condition  $\lambda_g^p(T) = \lambda_k^p(T)$ ; finally choose  $\eta_g^p(t) = \lambda_g^p(t) - \lambda_k^p(t)$ . Then the proposed path will satisfy all necessary and sufficient conditions and it is therefore optimal.

<sup>82</sup> The graph of the function  $C(K)$  is the path converging to the steady-state in a  $(c, K)$  phase diagram.

For part 5, set  $\eta_g^p(t) = 0 \forall t$ . Call  $C(G)$  the solution to the following differential equation

$$\frac{dc}{dG} = \frac{[f_g^*(K(0), G) - \rho] / \gamma}{f_g^*(K(0), G) - c(G)}$$

with boundary condition  $(c_\infty^p, G_\infty^p)$  Set  $G^p(t)$  equal to the solution to  $\dot{G}^p(t) = f^*(K(0), G(t))$

$-C(G^p(t))$  with initial condition  $G(0)$ . Set  $I_g^p(t) = \dot{G}^p(t)$ . Then choose  $\lambda_k^p(t) = c^p(t)^{-\gamma}$ , and for  $\lambda_g^p(t)$  the solution to  $\dot{\lambda}_g^p(t) = \rho\lambda_g^p(t) - \lambda_k^p(t) f_g^*(K^p(t), G(0))$  with initial condition  $\lambda_g^p(0) = \int_0^{+\infty} \lambda_k^p f_g^*(K^p(t), G(0)) e^{-\rho t} dt$ ; finally choose  $\eta_g^p(t) = \lambda_g^p(t) - \lambda_k^p(t)$ . Then the proposed path will satisfy all necessary and sufficient conditions and it is therefore optimal.

For part 6, set  $\eta_g^p(t) = 0 \forall t$ . Call  $C(G)$  the solution to the following differential equation

$$\frac{dc}{dG} = \frac{[f_g^*(K(0), G) - \rho] / \gamma}{f_g^*(K(0), G) - c(G)}$$

with boundary condition  $(c(T), G(T))$  where  $G(T) = K(0) x^*$ ,  $c(T) = f^*(K(0), G(T))$

$-\frac{f_k^*(1, x^*) - \rho}{\gamma} (K(0) + G(T))$ . Set  $I_g^p(t) = f^*(K(0), G(t)) - C(G^p(t))$ . Set  $G^p(t)$  equal to the solution to  $\dot{G}^p(t) = f^*(K(0), G(t)) - C(G^p(t))$  with initial condition  $G(0)$ . Set  $I_g^p(t) = \dot{G}^p(t)$ . Then choose  $\lambda_k^p(t) = c^p(t)^{-\gamma}$ , and for  $\lambda_g^p(t)$  the solution to  $\dot{\lambda}_g^p(t) = \rho\lambda_g^p(t) - \lambda_k^p(t) f_g^*(K^p(t), G(0))$  with initial condition  $\lambda_g^p(0) = \int_0^{+\infty} \lambda_k^p f_g^*(K^p(t), G(0)) e^{-\rho t} dt$ ; finally choose  $\eta_g^p(t) = \lambda_g^p(t) - \lambda_k^p(t)$ . Then the proposed path will satisfy all necessary and sufficient conditions and it is therefore optimal.

### 4.6.3 Proof of lemma 4.3

We first show that if all the conditions in the lemma are satisfied, the allocation can be decentralised as an intertemporal equilibrium. Equations (8) and (16) are identical,

as are equations (9) and (15), so condition 4 and 5 of the definition of intertemporal equilibrium are automatically satisfied. If we set  $r = f_k(K, V, V/G)$ , and set  $V$  such that it solves  $f_v(K, V, V/G) = p$ , then by lemma 4.2, there are  $n, k$  and  $v$  such that  $K, G, r, p, n, k$ , and  $v$  constitute a static equilibrium. Summing (15) to (19) and using  $\bar{r} = (1 - \tau)r$  we obtain

$$\dot{K} + \dot{B} = (1 - \tau)r(K + B) - c,$$

i.e. (5) (in view of (10)). Then interpreting  $\lambda_c$  as the costate variable in the household's utility maximisation problem, we see that (17), (18) and (21) are necessary and sufficient conditions for a solution. Finally, choose  $\tau(t)$  such that  $\bar{r} = (1 - \tau)f_k(K, V, V/G)$ ; then from (19) we have

$$\begin{aligned} \dot{B} &= (1 - \tau)rB + (1 - \tau)f_k(K, V, V/G)K + I_g - f(K, V, V/G) \\ &= (1 - \tau)rB + I_g - \tau rK - [f(K, V, V/G) - f_k(K, V, V/G)K] \\ &= (1 - \tau)rB + I_g - \tau rK - [f_v(K, V, V/G)V] \\ &= (1 - \tau)rB + I_g - \tau rK - pV, \end{aligned}$$

i.e. (7); the third equality follows from Euler's theorem for homogenous functions.

The converse is also easily established. If  $\{c, K, G, V\}_{t=0}^{+\infty}$  are part of an equilibrium, we must have  $r = f_k(K, V, V/G)$ ,  $p = f_v(K, V, V/G)$ . (17), (18) and (21) are deduced from the solution to the representative agent's maximisation problem. Finally

(19) is deduced from (7) as follows

$$\begin{aligned}
\dot{B} &= (1 - \tau) rB + I_g - \tau rK - pV \\
&= \bar{r}B + I_g - \tau f_k(K, V, V/G) K - f_v(K, V, V/G) V \\
&\quad + f_k(K, V, V/G) K - f_k(K, V, V/G) K \\
&= \bar{r}B + I_g + (1 - \tau) f_k(K, V, V/G) K \\
&\quad - f(K, V, V/G) \\
&= \bar{r}(B + K) + I_g - f(K, V, V/G),
\end{aligned}$$

where again we used Euler's theorem for homogeneous functions. ■

#### 4.6.4 Proof of proposition 4.2

First note that  $\dot{K} > 0$ ,  $\dot{G} > 0$ , then by (31)-(32),  $\eta_k^{op} = \eta_g^{op} = 0$ .

We first show that if  $\bar{r}^{op} > 0$  then  $\bar{r}^{op} = f_k(K^{op}, V^{op}, V^{op}/G^{op})$ .

$\bar{r}^{op} > 0$  implies  $\eta_{\bar{r}}^{op} = 0$ . Use (25) to write

$$\phi_{\lambda_c}^{op} = -m(B^{op} + K^{op}). \quad (51)$$

Differentiate this last equation w.r.t. time and use (15) and (19) to obtain

$$\begin{aligned}
\dot{\phi}_{\lambda_c}^{op}(t) &= -m(\dot{B}^{op} + \dot{K}^{op}) \\
&= -m[\bar{r}^{op}(B^{op} + K^{op}) - c^{op}].
\end{aligned}$$

Then using (29),

$$-m\bar{r}^{op}(B^{op} + K^{op}) + mc^{op} = \eta_{\lambda_c}^{op} + \phi_{\lambda_c}^{op}\bar{r}^{op};$$

which in light of (51) can be simplified to

$$mc^{op} = \eta_{\lambda_c}^{op}. \quad (52)$$

From (17) and (23)

$$c^{op} = -\frac{\gamma\eta_{\lambda_c}^{op}\lambda_c^{op}}{\lambda_k^{op} - \lambda_c^{op}},$$

so

$$m c^{op} = \frac{\gamma\mu^{op}\eta_{\lambda_c}^{op}}{\lambda_k^{op} - \lambda_c^{op}}.$$

Setting the right hand side of this last equation equal to the right hand side of (52) and rearranging we obtain

$$\frac{\lambda_c^{op} - \lambda_k^{op}}{\mu^{op}} = \gamma,$$

or<sup>83</sup>

$$\Lambda = \frac{\lambda_k^{op} - \mu^{op}}{\lambda_c^{op}} = 1 + m(\gamma + 1).$$

Since  $\Lambda$  is constant,

$$\dot{\Lambda} = \frac{\dot{\lambda}_k^{op} - \dot{\mu}^{op}}{\lambda_c^{op}} - \Lambda \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}} = 0.$$

Using (27), (30) and (18) we get (after some simplifications)

$$\Lambda(\rho - f_k(K^{op}, V^{op}, V^{op}/G^{op})) = \Lambda(\rho - \bar{r}^{op}),$$

which given that  $\Lambda > 0$  implies

$$f_k(K^{op}, V^{op}, V^{op}/G^{op}) = \bar{r}^{op}.$$

Next we show that -analogously to the first best optimum- the optimal policy generates positive investment in both types of capital only when their social marginal products coincide, i.e. when  $f_k^*(K^{op}, G^{op}) = f_g^*(K^{op}, G^{op})$ .

We can use (26) to write

$$\dot{\lambda}_g^{op} = \dot{\lambda}_k^{op} - \dot{\mu}^{op},$$

---

<sup>83</sup> Judd [67] calls  $\Lambda$  the *marginal social value of government wealth holding private wealth constant*. Private wealth is  $K + B$ ; if one increases the government wealth (or equivalently reduces the stock of debt) by one unit, one must increase the stock of private capital by one unit to keep private wealth constant. At the margin the overall effect is given by  $\Lambda$ . The constancy of  $\Lambda$  is obviously a consequence of the constancy of  $m$ . Intuitively, if the excess burden of taxation has been smoothed out completely, an increase in government wealth has the same effect irrespective of when it occurs.

which using (27), (28) and (30) gives

$$\begin{aligned} & \rho \lambda_g^{op} + [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{(G^{op})^2} \\ = & \{ \rho \lambda_k^{op} - [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] f_k(K^{op}, V^{op}, V^{op}/G^{op}) - \mu^{op} \bar{r}^{op} \} - \mu^{op} [\rho - \bar{r}^{op}]. \end{aligned}$$

After simplifications we have

$$\begin{aligned} & \rho (\lambda_g^{op} - \lambda_k^{op} + \mu^{op}) \\ = & - (\lambda_k^{op} - \mu^{op}) \left[ f_k(K^{op}, V^{op}, V^{op}/G^{op}) + f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{G^2} \right], \end{aligned}$$

which given (26), the positivity of  $\lambda_k^{op}$  and non-positivity of  $\mu^{op}$  yields

$$f_k(K^{op}, V^{op}, V^{op}/G^{op}) = -f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) \frac{V^{op}}{G^2}.$$

There may be, however, intervals in which the government finds it optimal to tax capital at the maximal rate.<sup>84</sup>

But given that  $V^{op}$  is chosen to maximise  $f(K^{op}, V^{op}, V^{op}/G^{op})$  (see (34)), the last equation is satisfied if and only if  $G^{op}/K^{op} = x^*$ . ■

#### 4.6.5 Proof of proposition 4.4

The proof follows exactly the same steps as in the proof for proposition 4.2.

$\bar{r}^{op} > 0$  implies  $\eta_{\bar{r}}^{op} = 0$ . Proceeding as before we can derive (52)

From (17) and (23)

$$c^{op} = - \frac{\gamma \eta_{\lambda_c}^{op} \lambda_c^{op}}{\lambda_k^{op} - \lambda_c^{op} + \eta_k^{op}},$$

so

$$m_C^{op} = - \frac{\gamma \mu^{op} \eta_{\lambda_c}^{op}}{\lambda_k^{op} - \lambda_c^{op} + \eta_k^{op}}.$$

<sup>84</sup> Chamley [34] and Jones et Al. [64] show that with assumptions typically encountered in growth theory the optimal policy entails keeping the capital tax at its maximum for an initial period and switching to zero capital taxation afterwards.

Setting the right hand side of this last equation equal to the right hand side of (52) and rearranging we obtain

$$\frac{\lambda_k^{op} - \lambda_c^{op} + \eta_k^{op}}{\mu^{op}} = \gamma,$$

or

$$\hat{\Lambda} \equiv \frac{\lambda_k^{op} - \mu^{op} + \eta_k^{op}}{\lambda_c^{op}} = 1 + m(1 - \gamma).$$

Differentiating with respect to time we obtain

$$\dot{\hat{\Lambda}} = \frac{\dot{\lambda}_k^{op} - \dot{\mu}^{op} + \dot{\eta}_k^{op}}{\lambda_c^{op}} - \hat{\Lambda} \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}}.$$

Differentiating (26) with respect to time, we have  $\dot{\lambda}_k^{op} - \dot{\mu}^{op} + \dot{\eta}_k^{op} = \dot{\lambda}_g$ , and hence

$$\dot{\hat{\Lambda}} = \frac{\dot{\lambda}_g}{\lambda_c^{op}} - \hat{\Lambda} \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}}.$$

Using (27), (30) and (18) we get (after some simplifications)

$$\hat{\Lambda} (\rho + f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) V^{op} / (G^{op})^2) = \hat{\Lambda} (\rho - \bar{r}^{op}),$$

which given that  $\Lambda > 0$  implies

$$-f_\Gamma(K^{op}, V^{op}, V^{op}/G^{op}) V^{op} / (G^{op})^2 = \bar{r}^{op},$$

which looking at (50) implies

$$f_g^*(K^{op}, G^{op}) = \bar{r}^{op}.$$

■

#### 4.6.6 Proof of proposition 4.5

The first order conditions are virtually identical to those in lemma 4.4, specifically

$$(c^{op})^{-\gamma} = \lambda_k^{op} + \gamma \eta_{\lambda_c}^{op} (c^{op})^{-\gamma-1} + \eta_k^{op}, \quad (53)$$

$$-\phi_{\lambda_c}^{op} \lambda_c^{op} + \mu^{op} (B^{op} + K^{op}) + \eta_{\bar{r}}^{op} = 0, \quad (54)$$

$$\lambda_g^{op} - \lambda_k^{op} + \mu^{op} - \eta_k^{op} + \eta_g^{op} = 0, \quad (55)$$

$$\dot{\lambda}_k^{op} = \rho \lambda_k^{op} - [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] \hat{f}_k(K^{op}, G^{op}) - \mu^{op} \bar{r}^{op}, \quad (56)$$

$$\dot{\lambda}_g^{op} = \rho \lambda_g^{op} - [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] \hat{f}_g(K^{op}, G^{op}) \quad (57)$$

$$\dot{\phi}_{\lambda_c}^{op} = \rho \phi_{\lambda_c}^{op} - \phi_{\lambda_c}^{op} (\rho - \bar{r}^{op}) + \eta_{\lambda_c}^{op}, \quad (58)$$

$$\dot{\mu}^{op} = \mu^{op} [\rho - \bar{r}^{op}]. \quad (59)$$

$$\eta_k^{op} \left[ \hat{f}(K^{op}, G^{op}) - c^{op} - I_g^{op} \right] = 0, \quad \eta_k^{op} \geq 0, \quad (60)$$

$$\eta_g^{op} I_g^{op} = 0, \quad \eta_g^{op} \geq 0. \quad (61)$$

In the proof of proposition 4.2 in section 4.6.4 we used equations (17), (23), (25), (29) together with  $\eta_k^{op} = \eta_g^{op} = 0$  to establish

$$\Lambda \equiv \frac{\lambda_k^{op} - \mu^{op}}{\lambda_c^{op}} = 1 + m(\gamma + 1).$$

and

$$\dot{\Lambda} = \frac{\dot{\lambda}_k^{op} - \dot{\mu}^{op}}{\lambda_c^{op}} - \Lambda \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}} = 0.$$

Noting that the corresponding equations -(40) (53), (54), (58)- are formally identical, it must follow that the two equations above are valid for the problem we are now looking

at. Using (56) and (59) we have

$$\frac{\dot{\lambda}_k^{op} - \dot{\mu}^{op}}{\lambda_c^{op}} = \left( \rho - \hat{f}_k(K, G) \right) \frac{\lambda_k^{op} - \mu^{op}}{\lambda_c} = \left( \rho - \hat{f}_k(K, G) \right) \Lambda.$$

Equation (40) implies

$$\frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}} = (\rho - \bar{r}).$$

Therefore combining the last three equations we get

$$\left( \rho - \hat{f}_k(K, G) \right) \Lambda = (\rho - \bar{r}) \Lambda,$$

and since  $\Lambda > 0$ ,

$$\hat{f}_k(K, G) = \bar{r}.$$

But  $\bar{r} = (1 - \tau) f_k(K, V(K, G; p), V(K, G; p)/G)$  while

$$\begin{aligned} \hat{f}_k(K, G) &= f_k(K, V(K, G; p), V(K, G; p)/G) + \\ & f_v(K, V(K, G; p), V(K, G; p)/G) \frac{\partial V(K, G; p)}{\partial K} + \\ & f_\Gamma(K, V(K, G; p), V(K, G; p)/G) \cdot \frac{\partial V(K, G; p)}{\partial K} \end{aligned}$$

In other words (dropping the arguments of the functions)

$$\tau = - \frac{(f_v + f_\Gamma/G) V_k}{f_k}.$$

■

#### 4.6.7 Proof of proposition 4.6

We first show that the optimal tax is generally positive. Assume  $\dot{K} > 0$ ,  $\dot{G} > 0$ . The first order conditions imply yet again that

$$\frac{\dot{\lambda}_k^{op} - \dot{\mu}^{op}}{\lambda_c^{op}} - \Lambda \frac{\dot{\lambda}_c^{op}}{\lambda_c^{op}} = 0.$$

However in this case<sup>85</sup>

$$\dot{\lambda}_k^{op} = \rho \lambda_k^{op} - [\lambda_k^{op} - \mu^{op} + \eta_k^{op}] [f_k(K^{op}, V^{op}, K^{op}/G^{op}) + f_\Gamma(K^{op}, V^{op}, K^{op}/G^{op})/G^{op}] - \mu^{op} \bar{r}^{op}.$$

Proceeding as usual we then find

$$f_k(K^{op}, V^{op}, K^{op}/G^{op}) + f_\Gamma(K^{op}, V^{op}, K^{op}/G^{op})/G^{op} = (1 - \tau) f_k(K^{op}, V^{op}, K^{op}/G^{op}),$$

i.e.

$$\tau = - \frac{f_\Gamma(K^{op}, V^{op}, K^{op}/G^{op})/G^{op}}{f_k(K^{op}, V^{op}, K^{op}/G^{op})} \geq 0.$$

So the tax is positive unless on the balance growth path the marginal effect of congestion is null.

Finally we show that the optimal user charge is zero. The analogous to (24) is now

$$[\lambda_k^{op} - \mu^{op} + \eta_k^{op}] f_v(K^{op}, V^{op}, K^{op}/G^{op}) = 0,$$

which implies

$$f_v(K^{op}, V^{op}, K^{op}/G^{op}) = 0.$$

The result follows from the fact that in a competitive equilibrium

$$f_v(K^{op}, V^{op}, K^{op}/G^{op}) = p.$$

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<sup>85</sup> The reader will find it useful to compare this equation with (27).

# Conclusions

In this thesis we attempted to contribute to the literature on fiscal policy in theoretical models of economic growth. There is no single model that can be said to incontrovertibly dominate all the others; there is therefore no alternative but analysing the same issues in all plausible models, trying to gauge how robust a given result is. We analysed four very different models and investigated the issues of dynamic efficiency, debt sustainability and optimal design of fiscal policy.

The first model, presented in chapter 1, is a semi-endogenous growth model. Here the engine of growth is investment in the accumulation of new ideas. Technological progress is determined endogenously but, in stark contrast with earlier endogenous growth models, it cannot be influenced by fiscal policy. Nevertheless, we have shown that the allocation can be dynamically inefficient and that a debt policy can unambiguously improve the allocation, as in the strictly exogenous growth model. We have also emphasised an important difference, however. While in the neoclassical model the main problem is one of overaccumulation of physical capital, here it is the allocation to inventive activity that is the main issue. By allocating more workers to the consumption sector, the economy achieves a better mix between the stock of physical capital and the stock of knowledge (or human capital).

In chapter 2 we presented a two-sector model of economic growth. In this case, growth is driven by investment in private and public capital. The crucial assumption for the feasibility of sustained economic growth is that the aggregate capital goods production function is linearly homogeneous in the two types of capital. This assumption guarantees that the marginal productivities of the two inputs do not fall as the avail-

able amounts of the inputs increase. At the same time, though, this implies that the rate of interest does not fall either. It is shown that the rate of growth is always below the rate of interest, i.e. the economy is dynamically efficient and perpetual fiscal deficits are precluded. We then looked at the optimal allocation and showed that it can be decentralised.

The dynamic efficiency result obtained in chapter 2 depends crucially on the assumptions on technology. Chapter 3 introduces disembodied, labour-augmenting technological progress as in the basic exogenous growth model; but crucially, it is assumed that productivity growth is dependent on public investment rather than being purely exogenous. It then follows that fiscal policy can affect both the rate of growth through public investment and the rate of interest through the financing policy. It is shown that in this model perpetual deficits may be sustained when the rate of public investment is high enough. It is also shown that what could be presumed to be a "virtuous" policy, such as an increase in taxation to reduce the fiscal deficit may have the paradoxical effect of making unsustainable an investment policy that was initially sustainable.

The fourth and final chapter is a contribution to the literature on optimal intertemporal taxation. We presented a growth models in which firms benefit from congestible public services, the provision of which requires building a stock of public infrastructures. The main novelty is the assumption that public services are excludable, which allows us to analyse explicitly user fees. We show that the optimal user fee policy fully internalise the congestion externality. Furthermore, the optimal long-run tax on capital is zero, i.e. the famous Chamley-Judd result is valid in this model; this is in contrast with most other models with congestible public capital.

# Appendix A

## Tables of Symbols

### A.1 Chapter 1

Symbol	Description	Page
$\beta$	Birth rate	9
$p$	Probability of death	9
$\theta$	Discount factor	10
$c(s, v)$	Consumption at time $v$ of household born at time $s$	10
$l(s, t) = l_0 e^{-\varepsilon(t-s)}$	Labour endowment at time $t$ of household born at $s$	11
$\varepsilon$	Rate of decrease of household's labour endowment	11
$a(s, t)$	Financial assets at time $t$ of household born at $s$	11
$r(t)$	Real rate of return	11
$w(t)$	Real wage per unit of labour supplied	11
$N(t)$	Mass of households alive at time $t$	11
$L(t)$	Total labour supply at time $t$	12
$h(, t)$	Human capital at time $t$ of household born at $s$	13
$H, C, A, \text{ etc.}$	Aggregate $h, c, a, \text{ etc.}$	12
$Y(t)$	Aggregate production of consumption goods	13
$1 - \alpha$	Elasticity w.r.t. $L_y$ of the final goods prod fct.	13
$L_y(t)$	Labour allocated to production of $Y$	14
$x_i(t)$	Quantity of variety $i$ of intermediate good	14
$m(t)$	Number (mass) of existing intermediate goods	14
$\nu_i(t)$	Price of $i$ -th intermediate good	14
$P_m(t)$	Price of a patent at time $t$	15
$K(t)$	$\int_0^{m(t)} x_i(t) di = m(t) x(t)$ , cap. in prod. of int.	15

*Continues overleaf.*

Symbol	Description	Page
$\pi_i(t)$	Profit for the firm producing $i$ -th variety	16
$\delta$	Total factor productivity in research	16
$L_m(t)$	Labour allocated to research	16
$\psi$	Elasticity of research prod fct w.r.t. $L_m$	16
$\phi$	Elasticity of research prod fct w.r.t. $m$	16
$\rho(t)$	$= (m(t) L_y(t) / K(t))^{1-\alpha} = Y(t) / K(t)$	19
$\zeta(t)$	$= L(t)^\psi / m(t)^{1-\phi} = \dot{m}(t) / \delta m(t)$	19
$u(t)$	$= C(t) / K(t)$	19
$q(t)$	$= (Y(t) / P_m(t) m(t))$	19
$\lambda(t)$	$= L_y(t) / L(t)$	19
$T(s, v)$	Transfer at time $v$ to households born at $s$	26
$B(t)$	Stock of public bonds	26
$T(t)$	Aggregate transfer	26
$b$	$= B(t) / K(t)$	26

## A.2 Chapter 2

Symbol	Description	Page
$n$	Population growth rate	55
$K_t$	Aggregate private capital at time $t$	56
$B_t$	Stock of public bonds at time $t$	56
$w_t$	Real wage after tax	56
$u(c_t^y, c_{t+1}^o)$	Utility function	56
$c_t^y$	Consumption of a young agent at time $t$	56
$c_t^o$	Consumption of an old agent at time $t$	56
$R_t$	$= 1 + r_t$	56
$\tau_t$	Labour income tax at time $t$	56
$\theta_t$	Interest income tax at time $t$	56
$s(\cdot)$	Saving rate	57
$H_t$	Capital employed in cons. sector	58
$L_t$	Labour supply	58
$A$	Total factor productivity cons. sect.	58
$\alpha$	Capital elasticity cons. sect.	58
$X_t$	Capital employed in capital sect.	59
$g_t$	$= (G_t/X_t)^\beta$ , flow of public services	59, 61
$M$	Total factor productivity cap. sect.	59
$p_t$	Price of capital	59
$\beta$	Elasticity of public services w.r.t. $G/X$	61
$\mu_t$	$= G_t/X_t$	61
$T_t$	Tax revenues at time $t$	62
$u_t$	$= H_t/K_t$	65
$b_{t+1}$	$= B_{t+1}/p_t K_{t+1}$	66
$s$	Saving rate, logarithmic utility	68
$\gamma$	Intertemporal elasticity of substitution	73
$\sigma$	Households' discount factor	73
$\delta$	Planner's Discount factor	74

### A.3 Chapter 3

Symbol	Description	Page
$c_t^y$	Consumption of a young agent at time $t$	94
$c_t^o$	Consumption of an old agent at time $t$	94
$\beta$	Discount factor	95
$s$	$= \beta / (1 + \beta)$ , saving rate	95
$Y_t$	Aggregate output	95
$K_t$	Capital stock	95
$A_t$	Labour productivity	95
$L_t$	Labour supply	95
$F(\cdot, \cdot)$	Aggregate production function	95
$G_t$	Stock of public capital	96
$\Gamma(\cdot)$	Rate of technological progress	96
$B_t$	Stock of public debt	99
$D_t$	Investment in public capital	99
$X_t$	Investment in private capital	99
$\delta_k, \delta_g$	Depreciation rates of private and public capital	99
$\tau$	Labour income tax	99
$W_t$	Before tax real wage	99
$r_t, R_t$	Net and gross interest rates ( $R_t = 1 + r_t$ )	99
$y_t, k_t, g_t, x_t, d_t, w_t, b_t$	Upper case variable in efficiency units	100
$k_\pi$	$k$ for which $\Gamma = R$	105
$k_\pi$	$k$ for which $d = \tau f(k)$	105
$k^*$	$k$ for which $s(1 - \tau)w = \Gamma k$	105

## A.4 Chapter 4

Symbol	Description	Page
$i$	Index for firms	125
$\Gamma$	Congestion	125
$\Psi(\Gamma)$	Firm $i$ 's production set	125
$\psi(k, v, \Gamma)$	Firm $i$ 's production function	125
$y_i$	Firm $i$ 's output	125
$k_i$	Firm $i$ 's capital	125
$k$	Minimum capital for active firms	125
$v_i$	Firm $i$ 's utilisation of public services	125
$f(K, V, \Gamma)$	Aggregate production function	126
$K$	Aggregate private capital stock	126
$G$	Public capital stock	126
$V$	Aggregate utilisation of public services	126
$r$	Real rate of interest	129
$p$	User fee	129
$n$	Number of active firms	129
$I_g$	Public investment	130
$B$	Stock of public bonds	131
$\tau$	Income tax	131
$c$	Consumption	131
$A$	Representative agent's total assets	131
$f^*(K, G)$	$f^*(K, G) = \max_{V(t)} f(K(t), V(t), V(t)/G(t))$	134
$x^*$	Optimal $G/K$	134
$K^p, G^p$ , etc.	Variables from solution to planner's problem	134

*Continues overleaf.*

Symbol	Description	Page
$\bar{r}(t)$	After tax return to capital	138
$K^{op}, G^{op}, \text{etc.}$	Variables from solution to optimal fiscal policy problem	140
$\lambda_k^{op}$	Shadow price of $K$ under optimal fiscal policy	140
$\lambda_g^{op}$	Shadow price of $G$ under optimal fiscal policy	140
$\lambda_c^{op}$	Marginal utility of consumption under opt. fisc. pol.	140
$\phi_{\lambda_c}^{op}$	Costate variable associated with repr. agent's Euler eq.	140
$\mu^{op}$	Costate variable associated with gov. budget constr.	140
$\eta_{\lambda_c}^{op}$	Lagrange multiplier for $c^{-\gamma} = \lambda_c$	140
$\eta_{\bar{r}}^{op}$	Lagrange multiplier for $\bar{r} \geq 0$	140
$\eta_K^{op}$	Lagrange multiplier for $\dot{K} \geq 0$	140
$\eta_g^{op}$	Lagrange multiplier for $\dot{G} \geq 0$	140
$m$	$-\mu^{op}/\lambda_c^{op}$ , excess burden of taxation	142
$\Theta^{op}$	$\{\forall t : \dot{K}(t) > 0, \dot{G}(t) > 0\}$	143
$\Lambda$	$(\lambda_k^{op} - \mu) / \lambda_c^{op}$ , marginal social value of government wealth	164

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