Hochschild cohomology and periodicity of tame weakly symmetric algebras

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Abstract

In this thesis we study the second Hochschild cohomology group of all tame weakly symmetric algebras having simply connected Galois coverings and only periodic modules. These algebras have been determined up to Morita equivalence by Białkowski, Holm and Skowroński in [4] where they give finite dimensional algebras $A_1(\lambda), A_2(\lambda), A_3$, $\ldots, A_{16}$ which are a full set of representatives of the equivalence classes. Hochschild cohomology is invariant under Morita equivalence, and this thesis describes $\mathrm{HH}^2(\Lambda)$ for each algebra $\Lambda = A_1(\lambda), A_2(\lambda), A_3, \ldots, A_{16}$ in this list. We also find the periodicity of the simple modules for each of these algebras. Moreover, for the algebra $A_1(\lambda)$ we find the minimal projective bimodule resolution of $A_1(\lambda)$ and discuss the periodicity of this resolution.

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Contents

Introduction

Hochschild cohomology is an important invariant in the representation theory of algebras, and it is well known from work of [24, Proposition 2.5] (see also [19, Theorem 4.2]) that it is invariant under Morita equivalence and derived equivalence. In this thesis we study the second Hochschild cohomology group of a finite dimensional algebra over an algebraically closed field K. This is linked to the theory of deformations which we describe in Chapter 2.

The theory of the cohomology of algebras was initiated by Hochschild in [21] and evolved in tandem with group cohomology. The nth Hochschild cohomology group of a finite dimensional algebra Λ is defined by $\mathrm{HH}^n(\Lambda) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$ where Λ^e is the enveloping algebra of Λ. The Hochschild cohomology ring of Λ is $\mathrm{HH}^*(\Lambda) := \oplus_{n\geqslant 0} \mathrm{HH}^n(\Lambda)$ with the Yoneda product. Chapters 1 and 2 introduce the second Hochschild cohomology group and explore the link between the bar resolution and deformation theory. Chapter 3 investigates a minimal projective bimodule resolution of Λ. We start by discussing the beginning of a minimal projective bimodule resolution of Λ of Green and Snashall from [16] and the minimal projective resolution of $\Lambda/\mathfrak{r}$ of Green, Solberg and Zacharia in [18]. From this we can determine the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$.

We are interested in tame weakly symmetric algebras having simply connected Galois coverings and only $\tau-$periodic modules where τ is Auslander-Reiten translate. The class of finite dimensional algebras over an algebraically closed field K may be divided into two disjoint classes wild and tame as shown in [8]. One class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional $K-$ algebras. The second class consists of tame algebras for which the indecomposable modules occur in each dimension d, in a finite number of discrete and a finite number of one-parameter families. So a classification of the finite dimensional modules is only feasible for tame algebras (see [25]). The tame weakly symmetric algebras having simply connected Galois coverings and only $\tau-$periodic modules have been determined up to Morita equivalence by Białkowski and Skowroński in [5] and up to derived equivalence by Białkowski, Holm and Skowroński in [4]. A complete list of representatives of the Morita equivalence classes is given by algebras $A_1(\lambda), A_2(\lambda), A_3, \ldots, A_{16}$ which we describe explicitly by quiver and relations in the main body of the thesis. These algebras are precisely the weakly symmetric algebras of tubular type and nonsingular Cartan matrix. We recall here the following two results from [5].

Theorem 4.6. [5, Theorem 2] Let Λ be a basic connected finite dimensional algebra over an algebraically closed field K. Then Λ is weakly symmetric of tubular type and nonsingular Cartan matrix if and only if Λ is isomorphic to one of the algebras $A_1(\lambda), A_2(\lambda), \lambda \in K \setminus \{0,1\}, A_3$ (if $\operatorname{char} K = 2$), or $A_i, 4 \leqslant i \leqslant 16$.

Note that the algebra A_3 is also weakly symmetric of tubular type and nonsingular Cartan matrix if $\operatorname{char} K \neq 2$.

Theorem 4.7. [5, Corollary 3] Let Λ be a weakly symmetric algebra of tubular type and with nonsingular Cartan matrix. Then Λ has at most four simple modules and its stable Auslander-Reiten quiver consists of tubes of rank $\leqslant 4$.

After a brief introduction to derived equivalence and Morita equivalence in Chapter 4, we discuss the derived equivalence classes of [4].

Theorem 4.10. [4]

(1) The algebras A_5 and A_6 are derived equivalent.

(2) The algebras $A_{12}, A_{13}, A_{14}, A_{15}$ and A_{16} are derived equivalent.

(3) The algebras $A_4, A_7, A_8, A_9, A_{10}$ and A_{11} are derived equivalent.

Chapters 5-20 of this thesis study the second Hochschild cohomology group for the algebras $A_1(\lambda), A_2(\lambda), A_3, \ldots, A_{16}$ by finding the basis and the dimension of this group. The second Hochschild cohomology group of the algebra A_3 was determined by [1] and [13], details are given in Chapter 7. The results of these chapters are summarized in the following Theorem.

Theorem 20.2. Let $\lambda \in K \setminus \{0, 1\}$.

For the algebra $\Lambda = A_1(\lambda)$, we have $\dim \mathrm{HH}^2(\Lambda) = 3$,

the algebra $\Lambda = A_2(\lambda)$ has $\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 6 \text{ if } \operatorname{char} K = 2 \\ 4 \text{ if } \operatorname{char} K \neq 2, \end{cases}$

the algebra $\Lambda = A_i$ where $i = 4, 7, 8, 9, 10, 11$ has $\dim \mathrm{HH}^2(\Lambda) = 2$,

the algebra $\Lambda = A_j$ where $j = 5, 6$ has $\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 3 \text{ if } \operatorname{char} K = 2 \\ 4 \text{ if } \operatorname{char} K = 3 \\ 3 \text{ if } \operatorname{char} K \neq 2, 3, \end{cases}$

and for the algebra $\Lambda = A_k$ where $k = 12, 13, 14, 15, 16$, we have

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 3 \text{ if } \operatorname{char} K = 2 \\ 2 \text{ if } \operatorname{char} K \neq 2. \end{cases}$$

In Chapters 21-26 we show that the simple modules for the algebras $A_1(\lambda), A_2(\lambda), A_5$, A_6, A_7 and A_{12} are $\Omega-$periodic and find the periodicity. Recall that $A_1(\lambda), A_2(\lambda), A_5, A_7$, A_{12} are a complete set of representatives of the derived equivalence classes. However, despite having A_5 and A_6 in the same derived equivalence class, we study both of them and show that the periodicity of their simple modules is different.

We also discuss the periodicity of the algebra $A_1(\lambda)$. Chapter 21 finds a minimal projective bimodule resolution of the algebra $A_1(\lambda)$. This done by defining projectives P^m and maps $d^m : P^m \to P^{m-1}$, and proving that (P^m, d^m) is a complex, then by using an argument in [16] prove that (P^m, d^m) is a minimal projective bimodule resolution of $A_1(\lambda)$. The main results in this chapter are:

Theorem 21.6. Let $\Lambda = A_1(\lambda)$ where $\lambda \in K \setminus \{0, 1\}$. If there exists some $n \geqslant 1$ such that $(-\lambda)^n = 1$ then $\Omega^{4n}_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules. Moreover Λ has a periodic projective $\Lambda - \Lambda-$bimodule resolution.

Theorem 21.7. For $\Lambda = A_1(\lambda)$ we have that $\mathrm{HH}^*(\Lambda)/\mathcal{N} = K$ or $K[x]$. If there is $n \geqslant 1$ with $(-\lambda)^n = 1$ then $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x]$ where x is in degree m, and m is minimal such that $\Omega^{m}_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules. In this case m divides $4n$.

Remark. (1) The algebras $A_1(\lambda)$ and $A_2(\lambda)$ are of "quaternion" type as defined in [10], where it is proved in [11] that almost all algebras of quaternion type are periodic as bimodules. However the small parameters are not covered by the results in [11]. Theorem 21.1 is dealing with one of the small parameter cases.

(2) The algebra A_4 is the smallest "mesh algebra" $\Lambda(\mathbf{C}_n)$ of Dynkin type $\mathbf{C}_n$, as introduced in the survey article in [12]. This algebra is also amongst the ones studied in a recent paper by Dugas [9].

1. Background Definitions

1.1. **Projective Modules.**

In this section R is a ring and all modules are right modules. The definitions and results are standard and can be found in [2].

Definition 1.1. *Let A, B, C be R-modules and $f : A \to B$, $g : B \to C$ be R-module homomorphisms. The sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if $\operatorname{Im} f = \operatorname{Ker} g$. A short exact sequence is a sequence of R-modules and R-module homomorphisms of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which is exact at A, B and C.

Equivalently, this sequence is exact at A if and only if f is $1-1$, it is exact at B if and only if $\operatorname{Im} f = \operatorname{Ker} g$ and it is exact at C if and only if g is onto.

Definition 1.2. *A long exact sequence is a sequence of R-modules and R-module homomorphisms of the form*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \qquad \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} 0$$

such that it is exact at $A_1, A_2, \ldots, A_n$.

Definition 1.3. *Let P, M and N be R-modules. Then P is a projective R-module if for each epimorphism $g : M \to N$ and each R-homomorphism $\nu : P \to N$ there is an R-homomorphism $\overline{\nu} : P \to M$ such that the diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\overline{\nu}"'] \arrow[d, "\nu"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

Definition 1.4. *Let M be a right R-module. Then a projective resolution of M is an exact sequence of projective R-modules P^i and R-module homomorphisms d^i such that*

$$\cdots \longrightarrow P^{n+1} \xrightarrow{d^{n+1}} P^n \xrightarrow{d^n} \cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} R \longrightarrow 0$$

is exact at P^i and R for all $i \geqslant 0$.

Proposition 1.5. *Let R be a ring. Then R is a projective right R-module.*

Proposition 1.6. *Let R be a ring and e an idempotent ($e^2 = e$) in R. Then eR is a projective right R-module.*

1.2. **Quivers and Relations.**

The definitions and results in this section are taken from [10]. Throughout this section we assume K is a field.

Definition 1.7. *A quiver $\mathcal{Q}$ is a directed graph $\mathcal{Q} = \{\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{o}, \mathfrak{t}\}$ where $\mathcal{Q}_0$ is the set of vertices, $\mathcal{Q}_1$ is the set of arrows, and $\mathfrak{o}, \mathfrak{t}$ are maps $\mathcal{Q}_1 \to \mathcal{Q}_0$. Given an arrow $\alpha \in \mathcal{Q}_1$, we say it starts at vertex $\mathfrak{o}(\alpha)$ and terminates at vertex $\mathfrak{t}(\alpha)$. The quiver is said to be finite provided both $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are finite sets.*

Definition 1.8. *(a): Given $v, w \in \mathcal{Q}_0$, then a path of length $l \geqslant 1$ from v to w is a path $\alpha_1\alpha_2\cdots\alpha_l$ where α_i is an arrow satisfying $\mathfrak{o}(\alpha_1) = v, \mathfrak{t}(\alpha_i) = \mathfrak{o}(\alpha_{i+1})$ and $\mathfrak{t}(\alpha_l) = w$. In addition, we also define for any vertex v of $\mathcal{Q}$, a path of length zero (from v to itself) denoted by e_v. We write $\mathfrak{o}(e_v) = v = \mathfrak{t}(e_v)$.*
(b): The path algebra $K\mathcal{Q}$ of $\mathcal{Q}$ is defined to be the K-vector space with basis the set of all paths in $\mathcal{Q}$. The product of two paths is taken to be the composition if it exists and zero otherwise.

Definition 1.9. *Let $\mathcal{Q}$ be a finite quiver and $R_{\mathcal{Q}}$ be the arrow ideal of the path algebra $K\mathcal{Q}$. A two-sided ideal I of $K\mathcal{Q}$ is said to be admissible if there exists $m \geqslant 2$ such that $R_{\mathcal{Q}}^m \subseteq I \subseteq R_{\mathcal{Q}}^2$* [3].

We write paths from left to right and deal with right modules.

1.3. **Radical.**

Throughout this section K is a field and Λ is a finite dimensional algebra. All the definitions in this section can be found in [2].

Definition 1.10. *Let Λ be a finite dimensional algebra and M a Λ-module, the radical of M, denoted by* $\operatorname{rad} M$, *is the smallest submodule of M such that $M/\operatorname{rad} M$ is semisimple.*

Remark.

(1) $\operatorname{rad} M$ is also the intersection of all the maximal submodules of ([2, 9.13]).

(2) $\operatorname{rad} M$ is the submodule of M such that $M/\operatorname{rad} M$ is the largest semisimple quotient of M.

(3) For a finite dimensional algebra $\Lambda = K\mathcal{Q}/I$, then $\operatorname{rad}(\Lambda)$ is the ideal of Λ generated by all the arrows of $\mathcal{Q}$, and we write $\operatorname{rad} \Lambda = \mathfrak{r}$.

Now we define the top and the socle of the right R-module M.

Definition 1.11. [10] *Let M be a right R-module. Then the top of M is the largest semisimple factor R-module of M, that is,* $\operatorname{top} M = M/\operatorname{rad}(M)$. *If $\mathcal{Q}$ has n vertices then* $\operatorname{top} \Lambda = \Lambda/\operatorname{rad} \Lambda = Ke_1 \oplus \cdots \oplus Ke_n$.

Definition 1.12. [10] *Let M be a right module. Then the socle of M is the largest semisimple submodule of M. The socle of M is denoted by* $\operatorname{soc} M$.

1.4. **Homological Algebra.**
Suppose that Λ is a finite dimensional algebra, K is a field, M is a right Λ-module and P^n are Λ-modules for $n \geqslant 0$. Then if we have a map $d^n : P^n \to P^{n-1}$ then we have an induced map d^{n*}, that is, $d^{n*} : \mathrm{Hom}(P^{n-1}, M) \to \mathrm{Hom}(P^n, M)$ is given by $f \mapsto f \cdot d^n$:

$$P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{f} M \ .$$

Definition 1.13. *Let Λ be a finite dimensional algebra over K. Then the opposite algebra Λ^{op} is defined as the vector space Λ with new multiplication $a * b := ba$ where $a, b \in \Lambda$. The enveloping algebra Λ^e of Λ is defined as the vector space $\Lambda^{op} \otimes_K \Lambda$ with the multiplication given as follows:*

$$\begin{aligned}(a \otimes b)(c \otimes d) &= (a * c) \otimes bd \\ &= ca \otimes bd.\end{aligned}$$

Proposition 1.14. *If M is a $\Lambda - \Lambda$-bimodule then M is a right Λ^e-module via*

$$m(a \otimes b) = (am)b = a(mb)$$

for $a, b \in \Lambda, m \in M$.

Throughout this thesis we use the notation of a $\Lambda - \Lambda$-bimodule interchangeably with the notation of a right Λ^e-module.

Suppose P^n, P^{n-1} are $\Lambda - \Lambda-$bimodules and $d^n : P^n \to P^{n-1}$ is a $\Lambda - \Lambda-$bimodule homomorphism. Then $d^{n*} : \mathrm{Hom}_{\Lambda^e}(P^{n-1}, \Lambda) \to \mathrm{Hom}_{\Lambda^e}(P^n, \Lambda)$ is also a $\Lambda - \Lambda$-bimodule homomorphism.

Consider a projective bimodule resolution of Λ

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{d^{n-1}} \cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0 \qquad (1)$$

so the P^i are projective $\Lambda - \Lambda-$bimodules. Apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \longrightarrow \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \xrightarrow{d^{1*}} \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \xrightarrow{d^{2*}} \cdots$$

$$\longrightarrow \mathrm{Hom}_{\Lambda^e}(P^{n-2}, \Lambda) \xrightarrow{d^{n-1*}} \mathrm{Hom}_{\Lambda^e}(P^{n-1}, \Lambda) \xrightarrow{d^{n*}} \mathrm{Hom}_{\Lambda^e}(P^n, \Lambda) \longrightarrow \ldots \qquad (2)$$

Definition 1.15. *A sequence of modules and homomorphisms*

$$X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} X_3 \xrightarrow{d_3} \cdots$$

with $d^2 = 0$, that is, $d_{i+1}d_i = 0$ for all i, is called a complex.

Proposition 1.16. *The sequence* (2) *is a complex.*

Thus we can construct the factor (quotient) module $\mathrm{Ker}\, d^{n+1*} / \mathrm{Im}\, d^{n*}$ for each n.

Definition 1.17. *The projective resolution* (1) *of Λ as a $\Lambda - \Lambda-$bimodule is minimal if $\mathrm{Im}\, d^n \subseteq \mathrm{rad}(P^{n-1})$ for all $n \geqslant 1$.*

If M is a right Λ-module, consider a projective resolution of M

$$\cdots \longrightarrow Q^2 \xrightarrow{\partial^2} Q^1 \xrightarrow{\partial^1} Q^0 \longrightarrow M \longrightarrow 0$$

so the Q^i are projective right Λ-modules. We will apply $\mathrm{Hom}_\Lambda(-,\Lambda)$ to get the complex

$$0 \longrightarrow \mathrm{Hom}(Q^0,\Lambda) \xrightarrow{\partial^{1*}} \mathrm{Hom}(Q^1,\Lambda) \xrightarrow{\partial^{2*}} \cdots$$

and so $\mathrm{Im}\,\partial^{n*} \subseteq \mathrm{Ker}\,\partial^{n+1*}$.

Definition 1.18. *With the above notation* $\mathrm{Ext}^n_\Lambda(M,\Lambda) = \mathrm{Ker}\,\partial^{n+1*}/\,\mathrm{Im}\,\partial^{n*}$ *and* $\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda) = \mathrm{Ker}\,d^{n+1*}/\,\mathrm{Im}\,d^{n*}$.

Note that in the case (P^*,d^*) is a projective resolution of Λ as a bimodule, we write $\delta^n = d^{n+1*}$. So $\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda) = \mathrm{Ker}\,\delta^n/\,\mathrm{Im}\,\delta^{n-1}$.

Definition 1.19. *Suppose that* Λ *is a finite dimensional algebra over a field* K*. Then the nth Hochschild cohomology group is* $\mathrm{HH}^n(\Lambda) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda)$*. The Hochschild cohomology ring of* Λ *is* $\mathrm{HH}^*(\Lambda) := \oplus_{n\geqslant 0}\mathrm{HH}^n(\Lambda)$ *with Yoneda product.*

Note that we can consider the elements of $\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda)$ as exact sequences

$$0 \longrightarrow \Lambda \longrightarrow E^n \longrightarrow E^{n-1} \longrightarrow \cdots \qquad \longrightarrow E^1 \longrightarrow \Lambda \longrightarrow 0$$

where Λ, E^i are all $\Lambda-\Lambda$-bimodules.

Theorem 1.20. [22] $\mathrm{Ext}^n_\Lambda(M,\Lambda)$ *is independent of the choice of projective resolution of* M *as a* Λ*-module.* $\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda)$ *is independent of the choice of projective resolution of* Λ *as a* Λ^e*-module.*

2. The bar resolution and deformation theory

In this chapter, we describe the link between the deformation theory of Λ and the cohomology group $\mathrm{HH}^2(\Lambda)$ for a finite dimensional algebra Λ over a field K.

2.1. **The bar resolution.**

Throughout this section we will let A be a $\Lambda - \Lambda$-bimodule and use left modules. We follow [6] and [19].

Definition 2.1. *Let $S_n(\Lambda) = \Lambda^{\otimes(n+2)}$ be the left $\Lambda \otimes_K \Lambda^{op}$-module which has $n+2$ copies of Λ, $n \geqslant -1$, and Λ^e acts via*

$$(\mu \otimes \nu^*)(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = (\mu\lambda_0) \otimes \lambda_1 \otimes \cdots \otimes (\lambda_{n+1}\nu).$$

Definition 2.2. *Let $b'_n : S_n(\Lambda) \to S_{n-1}(\Lambda)$ be defined via*

$$b'_n : (\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) \mapsto \sum_{i=0}^{n} (-1)^i \lambda_0 \otimes \cdots \otimes \lambda_i\lambda_{i+1} \otimes \lambda_{n+1}.$$

Proposition 2.3. *Let*

$$S_*(\Lambda) = \cdots \longrightarrow S_{n+1}(\Lambda) \xrightarrow{b'_{n+1}} S_n(\Lambda) \xrightarrow{b'_n} S_{n-1}(\Lambda) \xrightarrow{b'_{n-1}}$$

$$\cdots \longrightarrow S_1(\Lambda) \xrightarrow{b'_1} \Lambda \longrightarrow 0.$$

Then $S_*(\Lambda)$ is a chain complex.

Definition 2.4. [7] *A chain contraction $f : C \longrightarrow C$ for a chain complex $C = (C_n, \partial_n)$ is a sequence of maps $f_n : C_n \to C_{n+1}$*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathrm{id}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathrm{id}"'] \arrow[dl, "f_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\mathrm{id}"'] \arrow[dl, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"'] & C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"'] & \cdots
\end{tikzcd}$$

satisfying

$$\partial_{n+1} \circ f_n + f_{n-1} \circ \partial_n = \mathrm{id}_{C_n}.$$

Proposition 2.5. *Let $s_n : S_{n-1}(\Lambda) \to S_n(\Lambda)$ be given by*

$$s_n(\lambda_0 \otimes \cdots \otimes \lambda_n) \mapsto \lambda_0 \otimes \cdots \otimes \lambda_n \otimes 1.$$

Then $b'_{n+1} \circ s_{n+1} - s_n \circ b'_n = (-1)^{n+1} \mathrm{id}_{S_n}$.

From Definition 2.4 and Proposition 2.5, if we substitute ∂_{n+1} by b'_{n+1} and f_n by $(-1)^{n+1}s_{n+1}$, so that ∂_n is replaced by b'_n and f_{n+1} by $(-1)^n s_n$, then we have that $b'_{n+1} \circ (-1)^{n+1} s_{n+1} + (-1)^n s_n \circ b'_n = (-1)^{n+1}(b'_{n+1} \circ s_{n+1} - s_n \circ b'_n) = (-1)^{n+1}(-1)^{n+1} \mathrm{id} = (-1)^{2n+2} \mathrm{id} = \mathrm{id}_{S_n}$. Hence $(-1)^n s_n$ is a chain contraction of $S_*(\Lambda)$.

Proposition 2.6. *The chain complex $S_*(\Lambda) = (S_n(\Lambda), b'_n)$ is exact.*

Proposition 2.7. *Let $\tilde{S_n}(\Lambda) = \Lambda^{\otimes n}$ with $n \geqslant 0$. Then we have an isomorphism $S_n(\Lambda) \cong (\Lambda \otimes \Lambda^{op}) \otimes \tilde{S_n}(\Lambda)$ given by*

$$\lambda_0 \otimes \cdots \otimes \lambda_{n+1} \leftrightarrow (\lambda_0 \otimes \lambda_{n+1}^*) \otimes (\lambda_1 \otimes \cdots \otimes \lambda_n)$$

as Λ^e-modules.

So, since $\tilde{S_n}(\Lambda)$ is a projective K-module, $S_n(\Lambda)$ is a projective Λ^e-module and $S_*(\Lambda)$ is exact, it follows that $S_*(\Lambda)$ is a projective resolution of Λ as a Λ^e-module.

Definition 2.8. *The projective resolution $S_*(\Lambda)$ of Λ as a Λ^e-module is called the* acyclic Hochschild complex*.*

Definition 2.9. *Let $S^n(\Lambda, A) = \mathrm{Hom}_{\Lambda^e}(S_n(\Lambda), A)$.*

Proposition 2.10.

$$\begin{aligned} S^n(\Lambda, A) &\cong \mathrm{Hom}_{\Lambda^e}(\Lambda^e \otimes \tilde{S_n}(\Lambda), A) \\ &\cong \mathrm{Hom}_K(\tilde{S_n}(\Lambda), A). \end{aligned}$$

Definition 2.11. [19] *Let b^n be the map*

$$b^n : \mathrm{Hom}_K(\tilde{S_n}(\Lambda), A) \to \mathrm{Hom}_K(\tilde{S}_{n+1}(\Lambda), A)$$

given by

$$\begin{aligned} (b^n f)(\lambda_1 \otimes \cdots \otimes \lambda_{n+1}) &= \lambda_1 f(\lambda_2 \otimes \cdots \otimes \lambda_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(\lambda_1 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1}) \\ &+ (-1)^{n+1} f(\lambda_1 \otimes \cdots \otimes \lambda_n)\lambda_{n+1}, \end{aligned}$$

for all $n \geqslant 0, \lambda_1 \otimes \cdots \otimes \lambda_{n+1} \in \tilde{S}_{n+1}(\Lambda)$ and $f \in \mathrm{Hom}_K(\tilde{S_n}(\Lambda), A)$.

Proposition 2.12. *The complex $S^*(\Lambda, A) = (S^n(\Lambda, A), b^n)$ is a cochain complex.*

Definition 2.13. *Let A be a $\Lambda - \Lambda$-bimodule. Then $H^n(\Lambda, A)$ is the nth cohomology group of the cochain complex $S^*(\Lambda, A)$. It is called the nth Hochschild cohomology group of Λ with coefficients in bimodule A.*

A Hochschild n-cochain is a K-linear map $\Lambda^{\otimes n} \to A$ and the group of all n-cochains is $\mathrm{Hom}_K(\tilde{S_n}(\Lambda), A)$. The kernel of b^n in $\mathrm{Hom}_K(\tilde{S_n}(\Lambda), A)$ is called the group of n-cocycles. The image of b^{n-1} in $\mathrm{Hom}_K(\tilde{S_n}(\Lambda), A)$ is called the group of n-coboundaries. The nth Hochschild cohomology group $H^n(\Lambda, A)$ is defined to be $\mathrm{Ker}\, b^n / \mathrm{Im}\, b^{n-1}$.

Using the facts that $S_*(\Lambda)$ is a projective resolution of Λ as a Λ^e-module and that $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is independent of choice of projective resolution of Λ [22], it follows that

$$H^n(\Lambda, A) \cong \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$$

is indeed the nth Hochschild cohomology group as described in chapter 1. Recall that we write $\mathrm{HH}^n(\Lambda)$ for $H^n(\Lambda, \Lambda)$.

2.2. **The centre of Λ and Derivations of Λ.**

Definition 2.14. [19] *Let K be a field and let A be a $\Lambda - \Lambda-$bimodule. Let*

$$\mathrm{Der}(\Lambda, A) = \{\delta \in \mathrm{Hom}_K(\Lambda, A) \mid \delta(ab) = a\delta(b) + \delta(a)b, \text{ for all } a, b \in \Lambda\}$$

be the K-vector space of derivations of Λ on A. We write $\mathrm{Der}^0(\Lambda, A)$ to denote the subspace of inner derivations, that is,

$$\mathrm{Der}^0(\Lambda, A) = \{\delta_x : \Lambda \to A \mid \delta_x(a) = ax - xa, x \in A \text{ and } a \in \Lambda\}.$$

Proposition 2.15. *Let Λ be a finite dimensional K-algebra where K is a field and let A be a $\Lambda - \Lambda-$bimodule. Then the following hold:*
(i) $\mathrm{Hom}_\Lambda(\Lambda, A) \cong A$.
(ii) If $\Lambda = A$ *then* $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$, *the centre of* Λ.
(iii) $H^1(\Lambda, A) = \mathrm{Der}(\Lambda, A)/\,\mathrm{Der}^0(\Lambda, A)$.

Definition 2.16. [19] *Suppose that $\delta \in \mathrm{Der}(\Lambda, A)$. Then δ is called an outer derivation if the residue class of δ in $H^1(\Lambda, A)$ is different from zero.*

2.3. **Deformation Theory.**
Gerstenhaber in [15] introduced algebraic deformation theory for associative algebras. All definitions and theorems in this section can be found in [14].

Let K be a field. A one-parameter algebraic deformation of a finite dimensional K-algebra Λ, may be considered informally as a family of algebras $\{\Lambda_t\}$ parameterized by K such that $\Lambda_0 \cong \Lambda$ and the multiplicative structure of Λ_t varies algebraically with t.

Definition 2.17. *Let $\Lambda[[t]]$ be the $K[[t]]$-module of power series with coefficients in Λ, so that $\Lambda[[t]] = \Lambda \otimes_K K[[t]]$ as a module.*

Now we will give the formal definition of a deformation.

Definition 2.18. *A one-parameter formal deformation of a K-algebra Λ is a formal power series $F = \Sigma_{n=0}^{\infty} f_n t^n$ with coefficients in $\mathrm{Hom}_k(\Lambda \otimes \Lambda, \Lambda)$ and for all $a, b \in \Lambda, f_0(a, b) = ab$. The deformation $\Lambda[[t]]$ with the multiplication defined by F can be written as $\Lambda[[t]]_F$ or Λ_F. If F is finite, or at least finite for each pair $(a, b) \in \Lambda \otimes \Lambda$, the multiplication may be defined on $\Lambda[t]$ over $K[t]$.*

Definition 2.19. *Let Λ be an associative K-algebra. Then the deformation Λ_F is called associative if*

$$F(F(a, b), c) = F(a, F(b, c)) \text{ for all } a, b, c \in \Lambda.$$

If we expand $F(F(a, b), c)$ we will get $F(F(a, b), c) = F(\Sigma_{r=0}^{\infty} f_r(a, b)t^r, c)$
$= \Sigma_{r=0}^{\infty} F(f_r(a, b)t^r, c) = \Sigma_{r=0}^{\infty}\Sigma_{s=0}^{\infty} f_s(f_r(a, b)t^r, c)t^s = \Sigma_{r=0}^{\infty}\Sigma_{s=0}^{\infty} f_s(f_r(a, b), c)t^{r+s}$. Also

we have $F(a, F(b, c)) = F(a, \Sigma_{r=0}^{\infty} f_r(b, c)t^r) = \Sigma_{r=0}^{\infty} F(a, f_r(b, c))t^r$
$= \Sigma_{r=0}^{\infty}\Sigma_{s=0}^{\infty} f_s(a, f_r(b, c))t^{r+s}$. Now if we collect the coefficients of t^n we have

$$\Sigma_{i=0}^{n} f_i(f_{n-i}(a, b), c) = \Sigma_{i=0}^{n} f_i(a, f_{n-i}(b, c)) \qquad (1)$$

Definition 2.20. *Let f_n be the first non zero coefficient after f_0 in the expansion $F = \Sigma f_m t^m$. Then f_n is called the infinitesimal of F.*

Theorem 2.21. [14] *If F is an associative deformation of Λ then the infinitesimal f_n of F is a Hochschild 2-cocycle.*

Proof. Let F be an associative deformation of Λ and f_n be the infinitesimal of F. We can write (1) as

$$f_0(f_n(a, b), c) + f_n(f_0(a, b), c) = f_0(a, f_n(b, c)) + f_n(a, f_0(b, c)).$$

Since f_0 is multiplication in Λ we will get

$$f_n(ab, c) + f_n(a, b)c = af_n(b, c) + f_n(a, bc)$$

and then

$$af_n(b, c) - f_n(ab, c) + f_n(a, bc) - f_n(a, b)c = 0. \qquad (2)$$

From the definition of b^2 the left hand side of (2) is $b^2(f_n)(a, b, c)$ and therefore $b^2 f_n = 0$. So $f_n \in \operatorname{Ker} b^2$. Thus f_n is a Hochschild 2-cocycle. $\square$

For arbitrary m, (1) may be written as
$f_0(f_m(a, b), c) + \Sigma_{i=1}^{m-1} f_i(f_{m-i}(a, b), c) + f_m(f_0(a, b), c) = f_0(a, f_m(b, c))$
$+ \Sigma_{i=1}^{m-1} f_i(a, f_{m-i}(b, c)) + f_m(a, f_0(b, c))$ and so

$$b^2 f_m(a, b, c) = \Sigma_{i=1}^{m-1} f_i(f_{m-i}(a, b), c) - \Sigma_{i=1}^{m-1} f_i(a, f_{m-i}(b, c)). \qquad (3)$$

The right hand side of (3) is the obstruction to finding f_m that extends the deformation.

Theorem 2.22. [14] *The obstruction is a Hochschild 3-cocycle.*

Corollary 2.23. [14] *If $\mathrm{HH}^3(\Lambda) = 0$ then every 2-cocycle of Λ may be extended to an associative deformation of Λ.*

2.4. **Equivalent and trivial deformations.**
Given associative deformations Λ_F and Λ_G of Λ, we want to know when there is an isomorphism $\Lambda_F \to \Lambda_G$ which keeps Λ fixed.

Definition 2.24. *A formal isomorphism $\Psi : \Lambda_F \to \Lambda_G$ is a $K[[t]]$-linear map $\Lambda[[t]]_F \to \Lambda[[t]]_G$ which is written in the form*

$$\Psi(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \psi_3(a)t^3 + \cdots$$

where $\psi_0(a) = a$ *and* $a \in \Lambda$. *Note that it is enough to consider* $a \in \Lambda$, *since* Ψ *is defined over* $K[[t]]$. *We consider that each* $\psi_n \in \mathrm{Hom}_K(\Lambda, \Lambda)$. *If* Ψ *is multiplication preserving, we say it is an algebraic isomorphism, if*

$$G(\Psi(a), \Psi(b)) = \Psi(F(a, b))$$

for all $a, b \in \Lambda$.

Definition 2.25. *The two deformations* Λ_F *and* Λ_G *are said to be equivalent if there exists a formal isomorphism* $\Psi : \Lambda_F \to \Lambda_G$, *that is,* $\Lambda_F \cong \Lambda_G$.

Proposition 2.26. *If* f_n *and* g_n *are the infinitesimals of* F *and* G *respectively, then they are in the same cohomology class, that is, they represent the same element of* $\mathrm{HH}^2(\Lambda)$.

Theorem 2.27. [14] *If* $\mathrm{HH}^2(\Lambda) = 0$, *then all deformations of* Λ *are isomorphic.*

Definition 2.28. *A deformation* Λ_F *is called a trivial deformation if* $\Lambda_F \cong \Lambda$, *that is,* $F = f_0$.

Definition 2.29. *An algebra* Λ *is called rigid if it has only trivial deformations.*

So if $\mathrm{HH}^2(\Lambda) = 0$ then Λ is rigid.

3. Projective bimodule resolution of Λ

In this chapter we will look at the projective $\Lambda-\Lambda$-bimodule resolution of Green and Snashall [16] and the minimal projective resolution of Green, Solberg and Zacharia [18]. We use a minimal right projective resolution of $\Lambda/\mathfrak{r}$ over Λ from [18] to construct the first four projective $\Lambda-\Lambda-$bimodules P^i and maps T_i for $i = 0, 1, 2, 3$ of the minimal projective $\Lambda-\Lambda-$bimodule resolution of Λ.

3.1. **The minimal projective resolution of $\Lambda/\mathfrak{r}$ of Green, Solberg and Zacharia.**

In this section we assume that Λ is a finite dimensional algebra over a field K. Let $\Lambda = K\mathcal{Q}/I$ and $\mathfrak{r}$ be the Jacobson radical of Λ. Let g_i^n denote an element in the set g^n where the sets g^n are defined as follows in [18]: then we let $Q^n = \oplus_{g_i \in g^n} t(g_i^n)\Lambda$ and the map $d^n : Q^n \to Q^{n-1}$ (where the maps d^n are define g^n). Then [18] proves that (Q^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module. In [16] they use the same sets g^n and define $P^n = \oplus_{g_i^n \in g^n} \Lambda\mathfrak{o}(g_i^n) \otimes_K \mathfrak{t}(g_i^n)\Lambda$ and maps $T_i : P^i \to P^{i-1}$. So that the sequence $\cdots \to P^3 \to P^2 \to P^1 \to P^0 \to \Lambda \to 0$ is the start of a minimal projective bimodule resolution of Λ.

Note that we can get the right $\Lambda-$module resolution of $\Lambda/\mathfrak{r}$ from the bimodule resolution of Λ. The good thing is that the right $\Lambda-$module resolution of $\Lambda/\mathfrak{r}$ still contains all the information needed to construct and go back to the bimodule resolution of Λ.

Let $F = \oplus_{i=1}^n e_i K\mathcal{Q}$. Then we have an epimorphism

$$F \to \Lambda/\mathfrak{r} \to 0$$

and F is a projective $K\mathcal{Q}-$module.

Now we want to look at [18] to see how they constructed the filtration of F by $K\mathcal{Q}$-submodules. This filtration is

$$\cdots \subset F^m \subset F^{m-1} \subset \cdots \subset F^1 \subset F^0$$

where F^i's are right projective $K\mathcal{Q}$-modules. We introduce the notation of [18] which is required to define the submodules F^i. Let $R = K\mathcal{Q}$ and we can choose a family $\{f_i^0\}_{i \in A}$ of elements of $K\mathcal{Q}$ such that the projective $K\mathcal{Q}$-module $\oplus_{i \in A} f_i^0 R / \oplus_{i \in A} f_i^0 I$ maps onto $\Lambda/\mathfrak{r} = S_1 \oplus \cdots \oplus S_n$. Suppose that $f^0 = \{f_i^0\}_{i=1}^n = \{e_1, \ldots, e_n\}$ then we have $0 \to \Omega_R^1(\Lambda/\mathfrak{r}) \to \oplus_{i=1}^n f_i^0 R \to \Lambda/\mathfrak{r} \to 0$ and $F^0 = \oplus_{i=1}^n f_i^0 R = \oplus_{i=1}^n f_i^0 K\mathcal{Q} = \oplus_{i=1}^n e_i K\mathcal{Q}$.

We have $0 \to \Omega_R^1(\Lambda/\mathfrak{r}) \to \oplus_{i=1}^n e_i R \to \Lambda/\mathfrak{r} \to 0$ where $\Omega_R^1(\Lambda/\mathfrak{r})$ is a submodule of $\oplus_{i=1}^n e_i R$ and therefore $\Omega_R^1(\Lambda/\mathfrak{r})$ is projective. Choose a set $\{f_i^{1*}\}$ of elements of $\oplus_{i=1}^n f_i^0 R$ such that $\Omega_R^1(\Lambda/\mathfrak{r}) = \oplus_i f_i^{1*} R$. Now we can discard those elements f_i^* that are in $\oplus_i f_i^0 I = \oplus_{i=1}^n e_i I$ and denote by $\{f_i^1\}$ those which are not elements of $\oplus_i f_i^0 I$. This gives us a set $f^1 = \{f_i^1\}$. So $F^1 = \oplus_i f_i^1 R$ and $F^1 \subseteq F^0$. In [18] the sets f^m are defined inductively and $F^{m+1} = \oplus f_i^{m+1} R$, so we have a filtration of submodules

$$\cdots \subseteq F^{m+1} \subseteq F^m \subseteq \cdots \subseteq F^1 \subseteq F^0.$$

Suppose we have already $f^m = f_i^m$. Then consider the intersection $(\oplus_i f_i^m R) \cap (\oplus_j f_j^{m-1} I)$. We may write this intersection as $\oplus_e f_e^{m+1*} R$. This gives the set $\{f_e^{m+1*}\}$. We discard those elements of the set $\{f_e^{m+1*}\}$ in $\oplus_i f_i^m I$ and denote the rest of the elements by $f^{m+1} = \{f_e^{m+1}\}$. We stop if the intersection is zero and we set it equal to some $\oplus_i f_i^{m+1*} R$ otherwise.

Remark. [18] Note that for each $m > 0$, we have a representation of f_k^m in $\oplus_i f_i^{m-1} R$ as follows

$$f_k^m = \Sigma_i f_i^{m-1} h_{i,k}^{m-1,m}$$

for scalars $h_{i,k}^{m-1,m} \in R$.

Definition 3.1. [18, 1.1] *For $m \geqslant 0$ let $P^m = \oplus_i f_i^m R / \oplus_i f_i^m I$, and let $\delta^m : P^m \to P^{m-1}$ be the homomorphism induced by the inclusion $\oplus_i f_i^m R \subset \oplus_j f_j^{m-1} R$.*

Lemma 3.2. *We have an isomorphism $f_1^m R / f_1^m I \cong \mathfrak{t}(f_1^m)\Lambda$ as Λ-modules.*

Theorem 3.3. [18, Theorem 1.2] *The resolution*

$$(\mathcal{P}, \delta) : \cdots \longrightarrow P^m \xrightarrow{\delta^m} P^{m-1} \xrightarrow{\delta^{m-1}} \cdots \longrightarrow P^1 \xrightarrow{\delta^1} P^0 \longrightarrow \Lambda/\mathfrak{r} \longrightarrow 0$$

is a projective resolution of $\Lambda/\mathfrak{r}$ over Λ.

Theorem 3.4. [18, Theorem 2.4] *Let $\Lambda/\mathfrak{r}$ be a Λ-module and let $(\mathcal{P}, \delta)$ be the projective resolution of $\Lambda/\mathfrak{r}$ as in Theorem 3.3, where the representatives $\{f^m\}$ are chosen in such a way, that for each m, no proper K-linear combination of a subset of $\{f^m\}$ lies in $\oplus f^{m-1} I + \oplus f^{m*} J$. Then, the resolution $(\mathcal{P}, \delta)$ is minimal.*

3.2. **The first four terms in a projective $\Lambda - \Lambda$-bimodule resolution of Λ of Green and Snashall.**

In general, Theorem [16, 2.9] tells us the beginning of a minimal projective bimodule resolution of Λ.

Definition 3.5. [16, 1.1] *An element x in $K\mathcal{Q}$ is uniform if $x = e_i x e_j$ for some i, j. We say $\mathfrak{o}(x) = e_i$ and $\mathfrak{t}(x) = e_j$.*

Definition 3.6. [16, 1.1] *Let $\Lambda = K\mathcal{Q}/I$ be finite dimensional algebra over a field K where $\mathcal{Q}$ is a quiver and I is an admissible ideal. We let R denote the path algebra $R = K\mathcal{Q}$. Define the following subsets of R:*

g^0 is the set of vertices v of $\mathcal{Q}$,

g^1 is the set of arrows a of $\mathcal{Q}$,

g^2 is a minimal set of uniform relations x in the generating set of I.

Definition 3.7. [16, 2.5] *Suppose that the elements of* g^2 *are* $\{g^2_1, \dots, g^2_{m_2}\}$. *From* [18], *each element of* g^3 *is in* $\oplus_i g^2_i R \cap \oplus_a aI$ *where* $a \in g^1$. *Let* y *denote an arbitrary element of* g^3; *then* $y = \Sigma_{i=1}^{m_2} g^2_i p_i = \Sigma_{i=1}^{m_2} q_i g^2_i r_i$ *where* p_i, q_i, r_i *are elements in* R *with* q_i *in the arrow ideal of* R.

Notation. [16, 1.1] For $x \in g^2$, we have that x is a uniform relation in the minimal generating set for the ideal I. Moreover x is a linear combination of a finite number of paths in R, say of r paths. So we write $x = \Sigma_{j=1}^{r} c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ for $j = 1, \dots, r$, $c_j \in K$ and a_{ij} is an arrow for all i, j.

Definition 3.8. [16, Section 2] *Let* P^0, P^1, P^2 *and* P^3 *be the projective* $\Lambda{-}\Lambda{-}$*bimodules given by:*

$$P^0 = \oplus_{v \in g^0} \Lambda v \otimes v\Lambda,$$
$$P^1 = \oplus_{a \in g^1} \Lambda \mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda,$$
$$P^2 = \oplus_{x \in g^2} \Lambda \mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda,$$
$$P^3 = \oplus_{y \in g^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda.$$

Definition 3.9. [16, Section 2] *Let* $f : P^0 \to \Lambda$ *be the bimodule map given by* $v \otimes v \mapsto v$. *Let* T_1 *be the map* $T_1 : P^1 \to P^0$ *given by*

$$\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$$

for each arrow a. *Define the map* $T_2 : P^2 \to P^1$ *by*

$$\mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \Sigma_{j=1}^{r} c_j (\Sigma_{k=1}^{s_j} a_{1j} \dots a_{(k-1)j} \otimes a_{(k+1)j} \dots a_{s_j j})$$

for each $x \in g^2$ *and* $a_{1j} \dots a_{(k-1)j} \otimes a_{(k+1)j} \dots a_{s_j j} \in \Lambda \mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$. *Define the map* $T_3 : P^3 \to P^2$ *by*

$$\mathfrak{o}(y) \otimes \mathfrak{t}(y) \mapsto \Sigma_{i=1}^{m_2} (\mathfrak{o}(g^2_i) \otimes p_i - q_i \otimes r_i)$$

where $y = \Sigma_{i=1}^{m_2} g^2_i p_i = \Sigma_{i-1}^{m_2} q_i g^2_i r_i$ *and* $p_i, q_i r_i$ *are elements in* R *with* q_i *in the arrow ideal of* R.

Notation. Throughout this thesis we will write $\Lambda e_i \otimes_\alpha e_j \Lambda$ to express the summand of P^1 corresponding to the arrow α between the idempotents e_i and e_j and $\Lambda e_i \otimes_{g^2_k} e_j \Lambda$ to express the summand of P^2 corresponding to the relation g^2_k between the idempotents e_i and e_j.

Theorem 3.10. [16, Theorem 2.9] *With the above definitions, the following sequence forms part of a minimal projective resolution of* Λ *over* Λ^e:

$$P^3 \xrightarrow{T_3} P^2 \xrightarrow{T_2} P^1 \xrightarrow{T_1} P^0 \xrightarrow{f} \Lambda \longrightarrow 0$$

with maps $T_i : P^i \to P^{i-1}$ *for* $i = 1, 2, 3$ *and* $f : P^0 \to \Lambda$.

4. Tame weakly symmetric algebras having only periodic modules

In this chapter we will look at the tame weakly symmetric algebras having simply connected Galois coverings and only $\tau-$periodic modules where τ is the Auslander-Reiten translate. These algebras were classified in [4] and [5] up to Morita equivalence and derived equivalence. We will let K be an algebraically closed field and Λ a finite dimensional $K-$algebra with identity.

Definition 4.1. [10, I.3] *Let* $\operatorname{mod}\Lambda$ *be the category of finite dimensional left* $\Lambda-$*modules and let* $D : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ *be the standard duality* $\operatorname{Hom}_K(-,K)$. *So* $D(\Lambda)$ *is a right* $\Lambda-$*module.*

Definition 4.2. [10, I.3] *An algebra* Λ *is called selfinjective if* $\Lambda \cong D(\Lambda)$ *as right* $\Lambda-$*modules, that is, the projective right* $\Lambda-$*modules are injective.*

We also need $D(\Lambda)$ to have a left $\Lambda-$module structure.

Definition 4.3. [10, I.3] *An algebra* Λ *is called symmetric if* Λ *and* $D(\Lambda)$ *are isomorphic as* $\Lambda-\Lambda-$*bimodules.*

Definition 4.4. [10, I.3] *An algebra* Λ *is called weakly symmetric if for any indecomposable projective* $\Lambda-$*module* P *the socle* $\operatorname{soc} P \cong \operatorname{top} P / \operatorname{rad} P$.

Note that every symmetric algebra is weakly symmetric and every weakly symmetric algebra is selfinjective.

Definition 4.5. [10, I.8] *For a selfinjective algebra* Λ, *we denote by* Γ^s_Λ *the stable Auslander-Reiten quiver of* Λ, *obtained from the Auslander-Reiten quiver* Γ_Λ *of* Λ *by removing all projective modules and arrows attached to them.*

Theorem 4.6. [5, Theorem 2] *Let* Λ *be a basic connected finite dimensional algebra over an algebraically closed field* K. *Then* Λ *is weakly symmetric of tubular type and nonsingular Cartan matrix if and only if* Λ *is isomorphic to one of the algebras* $A_1(\lambda), A_2(\lambda), \lambda \in K \setminus \{0,1\}, A_3$ *(if* $\operatorname{char} K = 2$), *or* $A_i, 4 \leqslant i \leqslant 16$.

These algebras A_i where $i = 1, \dots, 16$ are explicitly described in Chapters 5 to 20 of this thesis.

Theorem 4.7. [5, Corollary 3] *Let* Λ *be a weakly symmetric algebra of tubular type and with nonsingular Cartan matrix* C_Λ. *Then* Λ *has at most four simple modules and* Γ^s_Λ *consists of tubes of rank* $\leqslant 4$.

So each simple module S satisfies $\tau^k(S) \ncong S$ where τ is the Auslander-Reiten translate and $k \leqslant 4$. If Λ is symmetric then $\tau \cong \Omega^2_\Lambda$ so $\Omega^m_\Lambda(S) \cong S$ for $m \leqslant 8$.

Now we define Morita equivalence and derived equivalence (see [25] and [24]).

Definition 4.8. *Two algebras R and S are said to be Morita equivalent if their module categories* $\operatorname{mod} R$ *and* $\operatorname{mod} S$ *are equivalent.*

Definition 4.9. *Let Λ and Γ be finite dimensional algebras and let $\mathcal{A} = \operatorname{mod}\Lambda$ and $\mathcal{B} = \operatorname{mod}\Gamma$. If there is a triangle equivalence $D(\mathcal{A}) \to D(\mathcal{B})$ between the derived categories $D(\mathcal{A})$ and $D(\mathcal{B})$ then we say Λ and Γ are derived equivalent.*

Hochschild cohomology is Morita invariant, so if R and S are Morita equivalent algebras then $\mathrm{HH}^*(R) \cong \mathrm{HH}^*(S)$. Rickard in [24] showed that Hochschild cohomology is also invariant under derived equivalence. It is well-known that if two algebras are Morita equivalent then they are derived equivalent (see [25]).

Białkowski, Holm and Skowroński in their paper [4] describe the tame weakly symmetric algebras having simply connected Galois coverings and only $\tau-$periodic modules up to derived equivalence.

Theorem 4.10.

(i) [4, Lemma 2.1] *The algebras A_5 and A_6 are derived equivalent.*

(ii) [4, Lemma 3.1, 3.2, 3.3, 3.4] *The algebras $A_{12}, A_{13}, A_{14}, A_{15}$ and A_{16} are derived equivalent.*

(iii) [4, Lemma 4.1, 4.2, 4.3, 4.4, 4.5] *The algebras $A_4, A_7, A_8, A_9, A_{10}$ and A_{11} are derived equivalent.*

We describe and determine the second Hochschild cohomology group of these algebras A_i where $i = 1, \ldots, 16$ in Chapters 5 to 20. Also in these chapters, when we find the minimal projective resolutions of the simple Λ-modules S_n of the A_i algebras we mean the beginning of the minimal projective resolutions of the simple Λ-modules S_n of the A_i algebras. And when we refer to periodicity we mean $\Omega-$periodic.

5. The algebra $A_1(\lambda)$

Throughout the following chapters we will let K be an algebraically closed field and Λ a finite dimensional $K-$algebra.

Definition 5.1. *Let $A_1(\lambda)$ be the algebra KQ/I where Q is the quiver*

$$\begin{tikzcd} 0 \arrow[r, shift left, "\alpha"] & 1 \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\sigma"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and

$$I = \langle \alpha\gamma\alpha - \alpha\sigma\beta, \beta\gamma\alpha - \lambda\beta\sigma\beta, \gamma\alpha\gamma - \sigma\beta\gamma, \gamma\alpha\sigma - \lambda\sigma\beta\sigma \rangle$$

where $\lambda \in K \setminus \{0, 1\}$.

5.1. **The structure of the indecomposable projectives.**

We start by considering the elements of length 4. For $\alpha\gamma\alpha\sigma$, we have $\alpha\gamma\alpha\sigma = \alpha(\gamma\alpha\sigma) = \lambda\alpha\sigma\beta\sigma = \lambda\alpha\gamma\alpha\sigma$ so that $(1-\lambda)\alpha\gamma\alpha\sigma = 0$; since $\lambda \neq 1$ and $\lambda \neq 0$ we have $\alpha\gamma\alpha\sigma = 0$. Similarly we can show that $\beta\gamma\alpha\gamma$ and $\alpha\sigma\beta\sigma$ are zero elements. Also we can show that all the elements of length 5 are zero elements such as $\alpha\gamma\alpha\gamma\alpha$. The non zero elements of length 4 are $\alpha\gamma\alpha\gamma, \gamma\alpha\gamma\alpha$, and $\beta\sigma\beta\sigma$.

So the indecomposable projective Λ-modules are $e_0\Lambda, e_1\Lambda, e_2\Lambda$ where
$e_0\Lambda = sp\{e_0, \alpha, \alpha\sigma, \alpha\gamma, \alpha\gamma\alpha, \alpha\gamma\alpha\gamma\}$,
$e_1\Lambda = sp\{e_1, \sigma, \gamma, \sigma\beta, \gamma\alpha, \gamma\alpha\gamma, \sigma\beta\sigma, \gamma\alpha\gamma\alpha\}$,
$e_2\Lambda = sp\{e_2, \beta, \beta\gamma, \beta\sigma, \beta\sigma\beta, \beta\sigma\beta\sigma\}$.

So we have for $e_0\Lambda$

$$\begin{tikzcd}[column sep=small] & 0 \arrow[d, no head] & \\ & 1 \arrow[dl, no head] \arrow[dr, no head] & \\ 2 \arrow[dr, no head] & & 0 \arrow[dl, no head] \\ & 1 \arrow[d, no head] & \\ & 0 & \end{tikzcd}$$

for $e_1\Lambda$

$$\begin{tikzcd}
& 1 \arrow[dl, dash] \arrow[dr, dash] & \\
2 \arrow[d, dash] & & 0 \arrow[d, dash] \\
1 \arrow[d, dash] & & 1 \arrow[d, dash] \\
2 \arrow[dr, dash] & & 0 \arrow[dl, dash] \\
& 1 &
\end{tikzcd}$$

and for $e_2\Lambda$

$$\begin{tikzcd}
& 2 \arrow[d, dash] & \\
& 1 \arrow[dl, dash] \arrow[dr, dash] & \\
0 \arrow[dr, dash] & & 2 \arrow[dl, dash] \\
& 1 \arrow[d, dash] & \\
& 2 &
\end{tikzcd}$$

Also we can express $e_0\Lambda, e_1\Lambda$ and $e_2\Lambda$ as follows:
for $e_0\Lambda$

$$\begin{tikzcd}
& e_0 \arrow[d, dash] & \\
& \alpha \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha\sigma \arrow[dr, dash] & & \alpha\gamma \arrow[dl, dash] \\
& \alpha\gamma\alpha \arrow[d, dash] & \\
& \alpha\gamma\alpha\gamma &
\end{tikzcd}$$

for $e_1\Lambda$

$$\begin{tikzcd}[arrows=dash]
 & e_1 \arrow[dl] \arrow[dr] & \\
\sigma \arrow[d] & & \gamma \arrow[d] \\
\sigma\beta \arrow[d] & & \gamma\alpha \arrow[d] \\
\sigma\beta\sigma \arrow[dr] & & \gamma\alpha\gamma \arrow[dl] \\
 & \gamma\alpha\gamma\alpha &
\end{tikzcd}$$

and for $e_2\Lambda$

$$\begin{tikzcd}[arrows=dash]
 & e_2 \arrow[d] & \\
 & \beta \arrow[dl] \arrow[dr] & \\
\beta\gamma \arrow[dr] & & \beta\sigma \arrow[dl] \\
 & \beta\sigma\beta \arrow[d] & \\
 & \beta\sigma\beta\sigma &
\end{tikzcd}$$

We may indicate the structures of the indecomposable projectives as follows:

$$\begin{array}{ccc}
e_0\Lambda & e_1\Lambda & e_2\Lambda \\
\\
\begin{array}{ccc} & 0 & \\ & 1 & \\ 0 & & 2 \\ & 1 & \\ & 0 & \end{array} &
\begin{array}{ccc} & 1 & \\ 0 & & 2 \\ 1 & & 1 \\ 0 & & 2 \\ & 1 & \end{array} &
\begin{array}{ccc} & 2 & \\ & 1 & \\ 0 & & 2 \\ & 1 & \\ & 2 & \end{array}
\end{array}$$

5.2. **The minimal projective resolutions of the simple Λ-modules S_0, S_2.** We have the minimal projective resolutions of the simple Λ-modules S_0 and S_2:

For S_0 we have;

$$\cdots \longrightarrow e_0\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_0\Lambda \longrightarrow S_0 \longrightarrow 0$$

where

$$\partial^1 : e_1\zeta \mapsto \alpha e_1\zeta,$$
$$\partial^2 : e_1\zeta \mapsto (\gamma\alpha - \sigma\beta)e_1\zeta,$$

$$\partial^3 : e_0\nu \mapsto \gamma e_0\nu,$$
for $\zeta, \nu \in \Lambda$.

For S_2:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$$\partial^1 : e_1\zeta \mapsto \beta e_1\zeta,$$
$$\partial^2 : e_1\zeta \mapsto (\gamma\alpha - \lambda\sigma\beta)e_1\zeta,$$
$$\partial^3 : e_2\eta \mapsto \sigma e_2\eta,$$
for $\zeta, \eta \in \Lambda$.

5.3. **The minimal projective resolution of the simple Λ-module S_1.**
Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3. For the simple Λ-module S_1, the minimal projective resolution of S_1 begins:

$$\cdots \longrightarrow e_0\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : (e_0\nu, e_2\eta) \mapsto \gamma e_0\nu + \sigma e_2\eta$, for $\nu, \eta \in \Lambda$.

5.3.1. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_0\nu = c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma$ and $e_2\eta = c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma$ with $c_i \in K$. Assume that $(e_0\nu, e_2\eta) \in \operatorname{Ker}\partial^1$ then $\gamma e_0\nu + \sigma e_2\eta = 0$ so $\gamma(c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma) + \sigma(c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma) = 0$, that is, $c_0\gamma + c_1\gamma\alpha + c_2\gamma\alpha\sigma + c_3\gamma\alpha\gamma + c_4\gamma\alpha\gamma\alpha + c_6\sigma + c_7\sigma\beta + c_8\sigma\beta\gamma + c_9\sigma\beta\sigma + c_{10}\sigma\beta\sigma\beta = 0$. So $c_0\gamma + c_1\gamma\alpha + c_6\sigma + c_7\sigma\beta + (c_2 + c_9\lambda^{-1})\sigma\beta\sigma + (c_3 + c_8)\gamma\alpha\gamma + (c_4 + c_{10}\lambda^{-1})\gamma\alpha\gamma\alpha = 0$ which implies that $c_0 = c_1 = 0 = c_6 = c_7$ and $c_9 = -c_2\lambda, c_8 = -c_3, c_{10} = -c_4\lambda$. Thus $e_0\nu = c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma$ and $e_2\eta = -c_2\lambda\beta\sigma - c_3\beta\gamma - c_4\lambda\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma, -c_2\lambda\beta\sigma - c_3\beta\gamma - c_4\lambda\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma, -c_2\lambda\beta\sigma - c_3\beta\gamma - c_4\lambda\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma)$, that is, $x = (\alpha\gamma, -\beta\gamma)(c_3e_0 + c_4\alpha + c_5\alpha\gamma) + (\alpha\sigma, -\lambda\beta\sigma)(c_2e_2 - c_{11}\lambda^{-1}\beta\sigma)$ since $\alpha\sigma\beta\sigma = 0$ and $\beta\gamma\alpha\gamma = 0$. Thus $x \in (\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda$.

On the other hand, let $y = (\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta \in (\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \gamma(\alpha\gamma e_0\nu + \alpha\sigma e_2\eta) - \sigma(\beta\gamma e_0\nu + \lambda\beta\sigma e_2\eta) = (\gamma\alpha\gamma - \sigma\beta\gamma)e_0\nu + (\gamma\alpha\sigma - \lambda\sigma\beta\sigma)e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha\gamma, -\beta\gamma)e_0\Lambda + (\alpha\sigma, -\lambda\beta\sigma)e_2\Lambda$. ∎

So $\partial^2 : e_0\Lambda \oplus e_2\Lambda \to e_0\Lambda \oplus e_2\Lambda$ is given by $(e_0\nu, e_2\eta) \mapsto (\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta$, for $\nu, \eta \in \Lambda$.

5.3.2. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Assume that $(e_0\nu, e_2\eta) \in \operatorname{Ker}\partial^2$. Then $(\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta = (0,0)$. Write $e_0\nu = c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma$ and $e_2\eta = c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma$ with $c_i \in K$. So $(\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta = (\alpha\gamma, -\beta\gamma)(c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma) + (\alpha\sigma, -\lambda\beta\sigma)(c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma) = (c_0\alpha\gamma + c_1\alpha\gamma\alpha + c_3\alpha\gamma\alpha\gamma, -c_0\beta\gamma - c_1\beta\gamma\alpha - c_2\beta\gamma\alpha\sigma) + (c_6\alpha\sigma + c_7\alpha\sigma\beta + c_8\alpha\sigma\beta\gamma, -c_6\lambda\beta\sigma - c_7\lambda\beta\sigma\beta - c_9\lambda\beta\sigma\beta\sigma) = (0,0)$. Thus $c_0\alpha\gamma + c_1\alpha\gamma\alpha + c_3\alpha\gamma\alpha\gamma + c_6\alpha\sigma + c_7\alpha\sigma\beta + c_8\alpha\sigma\beta\gamma = 0$ and $-c_0\beta\gamma - c_1\beta\gamma\alpha - c_2\beta\gamma\alpha\sigma - c_6\lambda\beta\sigma - c_7\lambda\beta\sigma\beta - c_9\lambda\beta\sigma\beta\sigma = 0$, that is, $c_0\alpha\gamma + c_6\alpha\sigma + (c_1 + c_7)\alpha\gamma\alpha + (c_3 + c_8)\alpha\gamma\alpha\gamma = 0$ and $-c_0\beta\gamma - c_6\lambda\beta\sigma - (c_1 + c_7)\lambda\beta\sigma\beta - (c_2 + c_9)\lambda\beta\sigma\beta\sigma = 0$. So from the two sequences we have $c_0 = 0 = c_6, c_7 = -c_1$, $c_8 = -c_3$ and $c_9 = -c_2$. Thus $e_0\nu = c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma$ and $e_2\eta = -c_1\beta - c_2\beta\sigma - c_3\beta\gamma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma$. Hence $\operatorname{Ker}\partial^2 = \{(c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma, -c_1\beta - c_2\beta\sigma - c_3\beta\gamma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha, -\beta)e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma, -c_1\beta - c_2\beta\sigma - c_3\beta\gamma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma)$ so $u = (\alpha, -\beta)(c_1e_1 + c_2\sigma + c_3\gamma + c_5\gamma\alpha\gamma - c_{11}\sigma\beta\sigma) + (\alpha\gamma, \beta\sigma)(c_4\alpha + c_{10}\beta)$. Now we can show that $(\alpha\gamma, \beta\sigma)(c_4\alpha + c_{10}\beta) \subseteq (\alpha, -\beta)\Lambda$, since $(\alpha\gamma, \beta\sigma)(c_4\alpha + c_{10}\beta) = (\alpha, -\beta)(1-\lambda)^{-1}((c_4 + c_{10})\gamma\alpha - (\lambda c_4 + c_{10})\sigma\beta)$. So $(\alpha\gamma, \beta\sigma)(c_4\alpha + c_{10}\beta) \subseteq (\alpha, -\beta)e_1\Lambda$. Hence $u \in (\alpha, -\beta)e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha, -\beta)e_1\Lambda$.

On the other hand, let $v = (\alpha, -\beta)e_1\zeta \in (\alpha, -\beta)e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha, -\beta)e_1\zeta) = ((\alpha\gamma, -\beta\gamma)\alpha + (\alpha\sigma, -\lambda\beta\sigma)(-\beta)) = (\alpha\gamma\alpha - \alpha\sigma\beta) - (\beta\gamma\alpha - \lambda\beta\sigma\beta) = 0$. Therefore $(\alpha, -\beta)e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha, -\beta)e_1\Lambda$. □

So the map $\partial^3 : e_1\Lambda \to e_0\Lambda \oplus e_2\Lambda$ is given by $e_1\zeta \mapsto (\alpha, -\beta)e_1\zeta$, for $\zeta \in \Lambda$.

Thus the minimal projective resolution of the simple Λ-module S_1 starts

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_0\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_0\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$$\begin{aligned}
&\partial^1 : (e_0\nu, e_2\eta) \mapsto \gamma e_0\nu + \sigma e_2\eta,\\
&\partial^2 : (e_0\nu, e_2\eta) \mapsto (\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta,\\
&\partial^3 : e_1\zeta \mapsto (\alpha, -\beta)e_1\zeta,
\end{aligned}$$

for $\nu, \eta, \zeta \in \Lambda$.

5.4. g^3 **for** S_0, S_1 **and** S_2.

Now we want to find the elements of g^3; these are paths in KQ.

For S_0 we have

$$e_0 \xrightarrow{\partial^3} \gamma \xrightarrow{\partial^2} (\gamma\alpha - \sigma\beta)\gamma \xrightarrow{\partial^1} \alpha(\gamma\alpha - \sigma\beta)\gamma = \alpha\gamma\alpha\gamma - \alpha\sigma\beta\gamma \ ,$$

so $\alpha\gamma\alpha\gamma - \alpha\sigma\beta\gamma \in g^3$.

For S_1

$e_1 \overset{\partial^3}{\to} (\alpha, -\beta) \overset{\partial^2}{\to} (\alpha\gamma, -\beta\gamma)\alpha - (\alpha\sigma, -\lambda\beta\sigma)\beta = (\alpha\gamma\alpha - \alpha\sigma\beta, -\beta\gamma\alpha + \lambda\beta\sigma\beta) \overset{\partial^1}{\to}$ $\gamma(\alpha\gamma\alpha - \alpha\sigma\beta) - \sigma(\beta\gamma\alpha - \lambda\beta\sigma\beta)$, so $(\gamma\alpha\gamma\alpha - \gamma\alpha\sigma\beta) - (\sigma\beta\gamma\alpha - \lambda\sigma\beta\sigma\beta) \in g^3$.

For S_2

$$e_2 \xrightarrow{\partial^3} \sigma \xrightarrow{\partial^2} (\gamma\alpha - \lambda\sigma\beta)\sigma \xrightarrow{\partial^1} \beta(\gamma\alpha - \lambda\sigma\beta)\sigma = \beta\gamma\alpha\sigma - \lambda\beta\sigma\beta\sigma \ ,$$

so $\beta\gamma\alpha\sigma - \lambda\beta\sigma\beta\sigma \in g^3$.

Let $g_1^3 = \alpha\gamma\alpha\gamma - \alpha\sigma\beta\gamma, g_2^3 = (\gamma\alpha\gamma\alpha - \gamma\alpha\sigma\beta) - (\sigma\beta\gamma\alpha - \lambda\sigma\beta\sigma\beta)$, and $g_3^3 = \beta\gamma\alpha\sigma - \lambda\beta\sigma\beta\sigma$. So $g^3 = \{g_1^3, g_2^3, g_3^3\}$.

We know that $g^2 = \{\alpha\gamma\alpha - \alpha\sigma\beta, \beta\gamma\alpha - \lambda\beta\sigma\beta, \gamma\alpha\gamma - \sigma\beta\gamma, \gamma\alpha\sigma - \lambda\sigma\beta\sigma\}$. Denote

$g_1^2 = \alpha\gamma\alpha - \alpha\sigma\beta$,
$g_2^2 = \gamma\alpha\gamma - \sigma\beta\gamma$,
$g_3^2 = \gamma\alpha\sigma - \lambda\sigma\beta\sigma$ and
$g_4^2 = \beta\gamma\alpha - \lambda\beta\sigma\beta$.

So we have

$g_1^3 = g_1^2\gamma = \alpha g_2^2$,
$g_2^3 = g_2^2\alpha - g_3^2\beta = \gamma g_1^2 - \sigma g_4^2$ and
$g_3^3 = g_4^2\sigma = \beta g_3^2$.

5.5. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0 \ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \overset{\delta^0}{\to} \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \overset{\delta^1}{\to} \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda) \overset{\delta^2}{\to} \mathrm{Hom}_{\Lambda^e}(P^3, \Lambda) \to \dots$$

5.5.1. $\mathrm{Ker}\, \delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\, \delta^2$ and $\mathrm{Im}\, \delta^1$. Let $\theta \in \mathrm{Ker}\, \delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned} &\theta : P^2 \to \Lambda \\ &e_0 \otimes_{g_1^2} e_1 \mapsto j_1\alpha + j_2\alpha\gamma\alpha \\ &e_1 \otimes_{g_2^2} e_0 \mapsto j_3\gamma + j_4\gamma\alpha\gamma \\ &e_1 \otimes_{g_3^2} e_2 \mapsto j_5\sigma + j_6\sigma\beta\sigma \\ &e_2 \otimes_{g_4^2} e_1 \mapsto j_7\beta + j_8\beta\sigma\beta, \end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_0 \otimes_{g_1^3} e_0 \mapsto e_0 \otimes_{g_1^2} \gamma - \alpha \otimes_{g_2^2} e_0$

$e_1 \otimes_{g_2^3} e_1 \mapsto e_1 \otimes_{g_2^2} \alpha - e_1 \otimes_{g_3^2} \beta - \gamma \otimes_{g_1^2} e_1 + \sigma \otimes_{g_4^2} e_1$

$e_2 \otimes_{g_3^3} e_2 \mapsto e_2 \otimes_{g_4^2} \sigma - \beta \otimes_{g_3^2} e_2$.

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_0 \otimes_{g_1^3} e_0) = \theta((e_0 \otimes_{g_1^2} e_1)\gamma - \alpha(e_1 \otimes_{g_2^2} e_0)) = (j_1\alpha + j_2\alpha\gamma\alpha)\gamma - \alpha(j_3\gamma + j_4\gamma\alpha\gamma) = (j_1 - j_3)\alpha\gamma + (j_2 - j_4)\alpha\gamma\alpha\gamma = 0$ so $j_1 = j_3$ and $j_2 = j_4$.

Also $\theta d^3(e_1 \otimes_{g_2^3} e_1) = \theta(e_1 \otimes_{g_2^2} \alpha - e_1 \otimes_{g_3^2} \beta - \gamma \otimes_{g_1^2} e_1 + \sigma \otimes_{g_4^2} e_1) = \theta((e_1 \otimes_{g_2^2} e_0)\alpha - (e_1 \otimes_{g_3^2} e_2)\beta - \gamma(e_0 \otimes_{g_1^2} e_1) + \sigma(e_2 \otimes_{g_4^2} e_1)) = j_3\gamma\alpha + j_4\gamma\alpha\gamma\alpha - j_5\sigma\beta - j_6\sigma\beta\sigma\beta - j_1\gamma\alpha - j_2\gamma\alpha\gamma\alpha + j_7\sigma\beta + j_8\sigma\beta\sigma\beta = (j_3 - j_1)\gamma\alpha + (j_4 - j_2)\gamma\alpha\gamma\alpha + (j_7 - j_5)\sigma\beta + (j_8 - j_6)\sigma\beta\sigma\beta = (j_3 - j_1)\gamma\alpha + (j_7 - j_5)\sigma\beta + (\lambda^{-1}(j_2 - j_4) + j_8 - j_6)\sigma\beta\sigma\beta = 0$, so $j_3 = j_1, j_7 = j_5$ and $j_8 = \lambda^{-1}(j_4 - j_2) + j_6$.

And $\theta d^3(e_2 \otimes_{g_3^3} e_2) = \theta(e_2 \otimes_{g_4^2} \sigma - \beta \otimes_{g_3^2} e_2) = \theta(e_2 \otimes_{g_4^2} e_1)\sigma - \beta(e_1 \otimes_{g_3^2} e_2) = (j_7\beta + j_8\beta\sigma\beta)\sigma - \beta(j_5\sigma + j_6\sigma\beta\sigma) = (j_7 - j_5)\beta\sigma + (j_8 - j_6)\beta\sigma\beta\sigma = 0$, so $j_7 = j_5$ and $j_8 = j_6$.

So $\theta \in \operatorname{Ker} \delta^2$ is given by

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_0 \otimes_{g_1^2} e_1 \mapsto j_1\alpha + j_2\alpha\gamma\alpha \\
&e_1 \otimes_{g_2^2} e_0 \mapsto j_1\gamma + j_2\gamma\alpha\gamma \\
&e_1 \otimes_{g_3^2} e_2 \mapsto j_5\sigma + j_6\sigma\beta\sigma \\
&e_2 \otimes_{g_4^2} e_1 \mapsto j_5\beta + j_6\beta\sigma\beta,
\end{aligned}$$

where $j_i \in K$. Hence $\dim \operatorname{Ker} \delta^2 = 4$.

5.5.2. $\operatorname{Im} \delta^1$.

Now to find $\operatorname{Im} \delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_0 \otimes_\alpha e_1 \to z_0\alpha + z_1\alpha\gamma\alpha$

$e_1 \otimes_\sigma e_2 \to z_2\sigma + z_3\sigma\beta\sigma$

$e_2 \otimes_\beta e_1 \to z_4\beta + z_5\beta\sigma\beta$

$e_1 \otimes_\gamma e_0 \to z_6\gamma + z_7\gamma\alpha\gamma$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_0 \otimes_{g_1^2} e_1 \mapsto e_0 \otimes_\alpha \gamma\alpha + \alpha \otimes_\gamma \alpha + \alpha\gamma \otimes_\alpha e_1 - e_0 \otimes_\alpha \sigma\beta - \alpha \otimes_\sigma \beta - \alpha\sigma \otimes_\beta e_1$

$e_1 \otimes_{g_2^2} e_0 \mapsto e_1 \otimes_\gamma \alpha\gamma + \gamma \otimes_\alpha \gamma + \gamma\alpha \otimes_\gamma e_0 - e_1 \otimes_\sigma \beta\gamma - \sigma \otimes_\beta \gamma - \sigma\beta \otimes_\gamma e_0$

$e_1 \otimes_{g_3^2} e_2 \mapsto e_1 \otimes_\gamma \alpha\sigma + \gamma \otimes_\alpha \sigma + \gamma\alpha \otimes_\sigma e_2 - e_1 \otimes_\sigma \lambda\beta\sigma - \lambda\sigma \otimes_\beta \sigma - \lambda\sigma\beta \otimes_\sigma e_2$

$e_2 \otimes_{g_4^2} e_1 \mapsto e_2 \otimes_\beta \gamma\alpha + \beta \otimes_\gamma \alpha + \beta\gamma \otimes_\alpha e_1 - e_2 \otimes_\beta \lambda\sigma\beta - \lambda\beta \otimes_\sigma \beta - \lambda\beta\sigma \otimes_\beta e_1$.

Then the map φd^2 is given by

$\varphi d^2(e_0 \otimes_{g_1^2} e_1) = (z_0\alpha + z_1\alpha\gamma\alpha)\gamma\alpha + \alpha(z_6\gamma + z_7\gamma\alpha\gamma)\alpha + \alpha\gamma(z_0\alpha + z_1\alpha\gamma\alpha) - (z_0\alpha + z_1\alpha\gamma\alpha)\sigma\beta - \alpha(z_2\sigma + z_3\sigma\beta\sigma)\beta - \alpha\sigma(z_4\beta + z_5\beta\sigma\beta) = z_0\alpha\gamma\alpha + z_6\alpha\gamma\alpha + z_0\alpha\gamma\alpha - z_0\alpha\sigma\beta - z_2\alpha\sigma\beta - z_4\alpha\sigma\beta = (z_0 - z_2 - z_4 + z_6)\alpha\gamma\alpha$,

$\varphi d^2(e_1 \otimes_{g_2^2} e_0) = (z_6\gamma + z_7\gamma\alpha\gamma)\alpha\gamma + \gamma(z_0\alpha + z_1\alpha\gamma\alpha)\gamma + \gamma\alpha(z_6\gamma + z_7\gamma\alpha\gamma) - (z_2\sigma + z_3\sigma\beta\sigma)\beta\gamma - \sigma(z_4\beta + z_5\beta\sigma\beta)\gamma - \sigma\beta(z_6\gamma + z_7\gamma\alpha\gamma) = z_6\gamma\alpha\gamma + z_0\gamma\alpha\gamma + z_6\gamma\alpha\gamma - z_2\sigma\beta\gamma - z_4\sigma\beta\gamma - z_6\sigma\beta\gamma = (z_0 - z_2 - z_4 + z_6)\gamma\alpha\gamma$,

$\varphi d^2(e_1 \otimes_{g_3^2} e_2) = (z_6\gamma + z_7\gamma\alpha\gamma)\alpha\sigma + \gamma(z_0\alpha + z_1\alpha\gamma\alpha)\sigma + \gamma\alpha(z_2\sigma + z_3\sigma\beta\sigma) - (z_2\sigma + z_3\sigma\beta\sigma)\lambda\beta\sigma - \lambda\sigma(z_4\beta + z_5\beta\sigma\beta)\sigma - \lambda\sigma\beta(z_2\sigma + z_3\sigma\beta\sigma) = z_6\gamma\alpha\sigma + z_0\gamma\alpha\sigma + z_2\gamma\alpha\sigma - z_2\lambda\sigma\beta\sigma - z_4\lambda\sigma\beta\sigma - z_2\lambda\sigma\beta\sigma = (z_0 - z_2 - z_4 + z_6)\lambda\sigma\beta\sigma$, and

$\varphi d^2(e_2 \otimes_{g_4^2} e_1) = (z_4\beta + z_5\beta\sigma\beta)\gamma\alpha + \beta(z_6\gamma + z_7\gamma\alpha\gamma)\alpha + \beta\gamma(z_0\alpha + z_1\alpha\gamma\alpha) - (z_4\beta + z_5\beta\sigma\beta)\lambda\sigma\beta - \lambda\beta(z_2\sigma + z_3\sigma\beta\sigma)\beta - \lambda\beta\sigma(z_4\beta + z_5\beta\sigma\beta) = z_4\beta\gamma\alpha + z_6\beta\gamma\alpha + z_0\beta\gamma\alpha - z_4\lambda\beta\sigma\beta - z_2\lambda\beta\sigma\beta - z_4\lambda\beta\sigma\beta = (z_0 - z_2 - z_4 + z_6)\lambda\beta\sigma\beta$. We will write $z = (z_0 - z_2 - z_4 + z_6)$ for $z_i \in K$.

Thus φd^2 is given by

$$\begin{aligned} &P^2 \to \Lambda \\ &e_0 \otimes_{g_1^2} e_1 \mapsto z\alpha\gamma\alpha \\ &e_1 \otimes_{g_2^2} e_0 \mapsto z\gamma\alpha\gamma \\ &e_1 \otimes_{g_3^2} e_2 \mapsto z\lambda\sigma\beta\sigma \\ &e_2 \otimes_{g_4^2} e_1 \mapsto z\lambda\beta\sigma\beta, \end{aligned}$$

for some $z \in K$. Therefore $\dim \operatorname{Im} \delta^1 = 1$.

5.5.3. $\mathrm{HH}^2(\Lambda)$.

From 5.5.1 and 5.5.2 we have that $\dim \mathrm{HH}^2(\Lambda) = 3$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{aligned} &P^2 \to \Lambda \\ &e_0 \otimes_{g_1^2} e_1 \mapsto d_0\alpha + d_1\alpha\gamma\alpha \\ &e_1 \otimes_{g_2^2} e_0 \mapsto d_0\gamma + d_1\gamma\alpha\gamma \\ &e_1 \otimes_{g_3^2} e_2 \mapsto d_2\sigma \\ &e_2 \otimes_{g_4^2} e_1 \mapsto d_2\beta \end{aligned} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, u\}$ where

$$\begin{aligned} &x : P^2 \to \Lambda \\ &e_0 \otimes_{g_1^2} e_1 \mapsto \alpha \\ &e_1 \otimes_{g_2^2} e_0 \mapsto \gamma \\ &\text{else} \mapsto 0, \end{aligned}$$

$$\begin{aligned} &y : P^2 \to \Lambda \\ &e_1 \otimes_{g_3^2} e_2 \mapsto \sigma \\ &e_2 \otimes_{g_4^2} e_1 \mapsto \beta \\ &\text{else} \mapsto 0, \end{aligned}$$

$$\begin{aligned} &u : P^2 \to \Lambda \\ &e_0 \otimes_{g_1^2} e_1 \mapsto \alpha\gamma\alpha \\ &e_1 \otimes_{g_2^2} e_0 \mapsto \gamma\alpha\gamma \\ &\text{else} \mapsto 0. \end{aligned}$$

Note that u represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_0 \otimes_{g_1^2} e_1 \mapsto -\sigma\beta\sigma\\
&e_1 \otimes_{g_2^2} e_0 \mapsto -\beta\sigma\beta\\
&\text{else} \mapsto 0.
\end{aligned}$$

So we can write $\mathrm{HH}^2(\Lambda)$ as

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l}
P^2 \to \Lambda\\
e_0 \otimes_{g_1^2} e_1 \mapsto d_0\alpha\\
e_1 \otimes_{g_2^2} e_0 \mapsto d_0\gamma\\
e_1 \otimes_{g_3^2} e_2 \mapsto d_2\sigma + d_3\sigma\beta\sigma\\
e_2 \otimes_{g_4^2} e_1 \mapsto d_2\beta + d_3\beta\sigma\beta
\end{array} \right\}$$

with $d_i \in K$.

6. The algebra $A_2(\lambda)$

Definition 6.1. [5] *Let $A_2(\lambda)$ be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\sigma"] & 2 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and

$$I = \langle \alpha^2 - \sigma\gamma, \lambda\beta^2 - \gamma\sigma, \gamma\alpha - \beta\gamma, \sigma\beta - \alpha\sigma \rangle$$

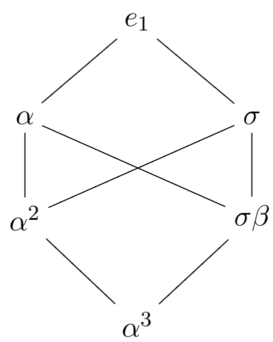
where $\lambda \in K \setminus \{0, 1\}$.

6.1. **The structure of the indecomposable projectives.**
We start by considering the elements of length three and four, for $\alpha\sigma\beta$, we have $\alpha\sigma\beta = \alpha^2\sigma = \sigma\gamma\sigma = \lambda\sigma\beta^2 = \lambda\alpha\sigma\beta$ so that $(1-\lambda)\alpha\sigma\beta = 0$ since $\lambda \neq 1$ and $\lambda \neq 0$ we have $\alpha\sigma\beta = 0$. Similarly we can show that $\alpha^2\sigma, \sigma\gamma\sigma, \sigma\beta^2, \beta\gamma\alpha, \beta^2\gamma, \gamma\sigma\gamma$ and $\gamma\alpha^2$ are the zero elements. Also we can show that all the elements of length four are zero elements such as α^4. The non zero elements of length three are $\alpha^3 = \sigma\beta\gamma = \alpha\sigma\gamma = \sigma\gamma\alpha$ and $\beta^3 = \lambda^{-1}\gamma\alpha\sigma = \lambda^{-1}\gamma\sigma\beta = \lambda^{-1}\beta\gamma\sigma$.

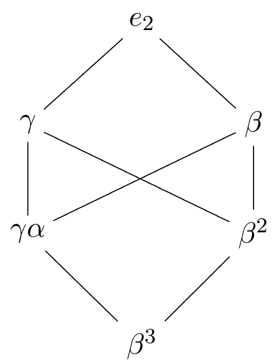
So the indecomposable projective Λ-modules are $e_1\Lambda$ and $e_2\Lambda$ where

$e_1\Lambda = sp\{e_1, \alpha, \sigma, \sigma\beta, \alpha^2, \alpha^3\}$,
$e_2\Lambda = sp\{e_2, \gamma, \beta, \gamma\alpha, \beta^2, \beta^3\}$.

for $e_1\Lambda$



and for $e_2\Lambda$



6.2. **The minimal projective resolutions of the simple Λ-modules S_1 and S_2.**

6.2.1. *The minimal projective resolution of the simple* Λ*-module* S_1*.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where
$\partial^1 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\zeta, e_2\eta) \to \alpha e_1\zeta + \sigma e_2\eta$, for $\zeta, \eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

6.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_1\zeta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$ with $c_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^1$ then $\alpha e_1\zeta + \sigma e_2\eta = 0$ so $\alpha(c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3) + \sigma(c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) = 0$, that is, $c_1\alpha + c_2\alpha^2 + c_3\alpha\sigma + c_5\alpha^3 + c_7\sigma + c_8\sigma\gamma + c_9\sigma\beta + c_{10}\sigma\gamma\alpha = 0$, that is, $c_1\alpha + c_2\alpha^2 + c_3\sigma\beta + c_5\alpha^3 + c_7\sigma + c_8\alpha^2 + c_9\sigma\beta + c_{10}\alpha^3 = 0$. So $c_1\alpha + (c_2 + c_8)\alpha^2 + (c_3 + c_9)\sigma\beta + (c_5 + c_{10})\alpha^3 + c_7\sigma = 0$ which implies that $c_1 = 0 = c_7$ and $c_8 = -c_2, c_9 = -c_3$ and $c_{10} = -c_5$. Thus $e_1\zeta = c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = -c_2\gamma - c_3\beta - c_5\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_2\gamma - c_3\beta - c_5\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_2\gamma - c_3\beta - c_5\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3)$, that is, $x = (\alpha, -\gamma)(c_2e_1 + c_5\alpha + c_6\alpha^2) + (-\sigma, \beta)(-c_3e_2 - c_4\beta - c_{12}\beta^2) + (0, \beta^2)(c_4e_2 + c_{11}e_2)$. However we can show that $(0, \beta^2)(c_4e_2 + c_{11}e_2) \subseteq (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$, since $(0, \beta^2)(c_4e_2 + c_{11}e_2) = (\alpha, -\gamma)(1-\lambda)^{-1}\sigma(c_4e_2 + c_{11}e_2) + (-\sigma, \beta)(1-\lambda)^{-1}\beta(c_4e_2 + c_{11}e_2)$. Thus $x \in (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$.

On the other hand, let $y = (\alpha, -\gamma)e_1\zeta + (-\sigma, \beta)e_2\eta \in (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \alpha(\alpha e_1\zeta - \sigma e_2\eta) + \sigma(-\gamma e_1\zeta + \beta e_2\eta) = (\alpha^2 - \sigma\gamma)e_1\zeta + (\sigma\beta - \alpha\sigma)e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\gamma)e_1\Lambda + (-\sigma, \beta)e_2\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (-\sigma, \beta)e_2\eta$, for $\zeta, \eta \in \Lambda$.

6.2.3. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^2$. Then $(\alpha, -\gamma)e_1\zeta + (-\sigma, \beta)e_2\eta = (0, 0)$. We know that $e_1\zeta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$ with $c_i \in K$. So $(\alpha, -\gamma)e_1\zeta +$

$(-\sigma,\beta)e_2\eta = (\alpha,-\gamma)(c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3) + (-\sigma,\beta)(c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\sigma + c_5\alpha^3, -c_1\gamma - c_2\gamma\alpha - c_3\gamma\sigma - c_4\gamma\sigma\beta) + (-c_7\sigma - c_8\sigma\gamma - c_9\sigma\beta - c_{10}\sigma\gamma\alpha, c_7\beta + c_8\beta\gamma + c_9\beta^2 + c_{11}\beta^3) = (0,0)$. Thus $c_1\alpha + c_2\alpha^2 + c_3\sigma\beta + c_5\alpha^3 - c_7\sigma - c_8\alpha^2 - c_9\sigma\beta - c_{10}\alpha^3 = 0$ which implies that $c_1\alpha + (c_2 - c_8)\alpha^2 + (c_3 - c_9)\sigma\beta + (c_5 - c_{10})\alpha^3 - c_7\sigma = 0$ and therefore $c_1 = 0, c_7 = 0, c_8 = c_2, c_9 = c_3$ and $c_5 = c_{10}$. Also $-c_1\gamma - c_2\gamma\alpha - c_3\lambda\beta^2 - c_4\lambda\beta^3 + c_7\beta + c_8\gamma\alpha + c_9\beta^2 + c_{11}\beta^3 = 0$, that is $-c_1\gamma + (-c_2 + c_8)\gamma\alpha + (-c_3\lambda + c_9)\beta^2 + (-c_4\lambda + c_{11})\beta^3 + c_7\beta = 0$ and therefore $c_1 = 0, c_7 = 0$, , $c_8 = c_2$, $c_9 = \lambda c_3$ and $c_{11} = \lambda c_4$. Note that from the two sequences we have that $c_3 = 0 = c_9$. Thus $e_1\zeta = c_2\alpha + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = c_2\gamma + c_5\gamma\alpha + c_4\lambda\beta^2 + c_{12}\beta^3$. Hence $\operatorname{Ker}\partial^2 = \{(c_2\alpha + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, c_2\gamma + c_5\gamma\alpha + c_4\lambda\beta^2 + c_{12}\beta^3) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha,\gamma)e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\alpha + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, c_2\gamma + c_5\gamma\alpha + c_4\lambda\beta^2 + c_{12}\beta^3)$ so $u = (\alpha,\gamma)(c_2e_1 + c_4\sigma + c_5\alpha + c_6\alpha^2 + c_{12}\lambda^{-1}\alpha\sigma)$. Hence $u \in (\alpha,\gamma)e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha,\gamma)e_1\Lambda$.

On the other hand, let $v = (\alpha,\gamma)e_1\zeta \in (\alpha,\gamma)e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha,\gamma)e_1\zeta) = ((\alpha,-\gamma)\alpha + (-\sigma,\beta)\gamma) = (\alpha^2 - \sigma\gamma) - (\gamma\alpha - \beta\gamma) = 0$. Therefore $(\alpha,\gamma)e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha,-\beta)e_1\Lambda$. □

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_1\zeta \mapsto (\alpha,\gamma)e_1\zeta$, for $\zeta \in \Lambda$.

6.2.4. *The minimal projective resolution of the simple Λ-module S_2.*
Now the minimal projective resolution of the simple Λ-module S_2 starts by:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where
$\partial^1 : e_1\Lambda \oplus e_2\Lambda \to e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto \gamma e_1\zeta + \beta e_2\eta$, for $\zeta,\eta \in \Lambda$.

Now we want to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ for S_2.

6.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$, let $e_1\zeta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$ with $c_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^1$ then $\gamma e_1\zeta + \beta e_2\eta = 0$ so $\gamma(c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3) + \beta(c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) = 0$, that is, $c_1\gamma + c_2\gamma\alpha + c_3\gamma\sigma + c_4\gamma\sigma\beta + c_7\beta + c_8\beta\gamma + c_9\beta^2 + c_{11}\beta^3 = 0$, that is, $c_1\alpha + (c_2 + c_8)\gamma\alpha + (c_3\lambda + c_9)\beta^2 + (c_4\lambda + c_{11})\beta^3 + c_7\beta = 0$. So $c_1 = 0 = c_7$, $c_8 = -c_2, c_9 = -c_3\lambda$ and $c_{11} = -c_4\lambda$. Thus $e_1\zeta = c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = -c_2\gamma - c_3\lambda\beta + c_{10}\gamma\alpha - c_4\lambda\beta^2 + c_{12}\beta^3$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_2\gamma - c_3\lambda\beta + c_{10}\gamma\alpha - c_4\lambda\beta^2 + c_{12}\beta^3) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha,-\gamma)e_1\Lambda + (\sigma,-\lambda\beta)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_2\gamma - c_3\lambda\beta + c_{10}\gamma\alpha - c_4\lambda\beta^2 + c_{12}\beta^3)$, that is, $x = (\alpha, -\gamma)(c_2e_1 + c_5\alpha + c_6\alpha^2) + (\sigma, -\lambda\beta)(c_3e_2 + c_4\beta - c_{12}\lambda^{-1}\beta^2) + (0, \gamma\alpha)(c_{10}e_1 + c_5e_1)$. However we can show that $(0, \gamma\alpha)(c_{10}e_1 + c_5e_1) \subseteq (\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda$, since $(c_{10}e_1 + c_5e_1) = (\alpha, -\gamma)(\lambda - 1)^{-1}\alpha(c_{10}e_1 + c_5e_1) + (\sigma, -\lambda\beta)(\lambda-1)^{-1}(-\gamma)(c_{10}e_1 + c_5e_1)$. Thus $x \in (\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda$.

On the other hand, let $y = (\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta \in (\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \gamma(\alpha e_1\zeta + \sigma e_2\eta) + \beta(-\gamma e_1\zeta - \lambda\beta e_2\eta) = (\gamma\alpha - \beta\gamma)e_1\zeta - (\lambda\beta^2 - \gamma\sigma)e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\gamma)e_1\Lambda + (\sigma, -\lambda\beta)e_2\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta$, for $\zeta, \eta \in \Lambda$.

6.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^2$. Then $(\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta = (0,0)$. We know that $e_1\zeta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$ with $c_i \in K$. So $(\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta = (\alpha, -\gamma)(c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3) + (\sigma, -\lambda\beta)(c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\sigma + c_5\alpha^3, -c_1\gamma - c_2\gamma\alpha - c_3\gamma\sigma - c_4\gamma\sigma\beta) + (c_7\sigma + c_8\sigma\gamma + c_9\sigma\beta + c_{10}\sigma\gamma\alpha, -c_7\lambda\beta - c_8\lambda\beta\gamma - c_9\lambda\beta^2 - c_{11}\lambda\beta^3) = (0,0)$. Thus $c_1\alpha + c_2\alpha^2 + c_3\sigma\beta + c_5\alpha^3 + c_7\sigma + c_8\alpha^2 + c_9\sigma\beta + c_{10}\alpha^3 = 0$ which implies that $c_1\alpha + (c_2 + c_8)\alpha^2 + (c_3 + c_9)\sigma\beta + (c_5 + c_{10})\alpha^3 + c_7\sigma = 0$ and therefore $c_1 = 0, c_7 = 0, c_8 = -c_2, c_9 = -c_3$ and $c_{10} = -c_5$. Also $-c_1\gamma - c_2\gamma\alpha - c_3\lambda\beta^2 - c_4\lambda\beta^3 - c_7\lambda\beta - c_8\lambda\gamma\alpha - c_9\lambda\beta^2 - c_{11}\lambda\beta^3 = 0$, that is, $-c_1\gamma + (-c_2 - c_8\lambda)\gamma\alpha + (-c_3 - c_9)\lambda\beta^2 + (-c_4 - c_{11})\lambda\beta^3 - c_7\lambda\beta = 0$ and therefore $c_1 = 0, c_7 = 0$, $c_8 = -c_2\lambda^{-1}$, $c_9 = -c_3$ and $c_{11} = -c_4$. Note that from the two sequences we have that $c_2 = 0 = c_8$. Thus $e_1\zeta = c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ and $e_2\eta = -c_3\beta - c_5\gamma\alpha - c_4\beta^2 + c_{12}\beta^3$. Hence $\operatorname{Ker}\partial^2 = \{(c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_3\beta - c_5\gamma\alpha - c_4\beta^2 + c_{12}\beta^3) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\sigma, \beta)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3, -c_3\beta - c_5\gamma\alpha - c_4\beta^2 + c_{12}\beta^3)$ so $u = (\sigma, -\beta)(c_3e_2 + c_4\beta + c_5\gamma + c_6\gamma\alpha - c_{12}\beta^2)$. Hence $u \in (\sigma, -\beta)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\sigma, -\beta)e_2\Lambda$.

On the other hand, let $v = (\sigma, -\beta)e_2\eta \in (\sigma, -\beta)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\sigma, -\beta))e_2\eta) = ((\alpha, -\gamma)\sigma + (\sigma, -\lambda\beta)(-\beta)) = -(-\alpha\sigma + \sigma\beta) + (\lambda\beta^2 - \gamma\sigma) = 0$. Therefore $(\sigma, -\beta)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\sigma, -\beta)e_2\Lambda$. □

So the map $\partial^3 : e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_2\eta \mapsto (\sigma, -\beta)e_2\eta$, for $\eta \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \alpha e_1\zeta + \sigma e_2\eta,$$

$$\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (-\sigma, \beta)e_2\eta,$$
$$\partial^3 : e_1\zeta \mapsto (\alpha, \gamma)e_1\zeta,$$

for $\zeta, \eta \in \Lambda$.

And the maps for S_2 are:

$$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \gamma e_1\zeta + \beta e_2\eta,$$
$$\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta,$$
$$\partial^3 : e_2\eta \mapsto (\sigma, -\beta)e_2\eta,$$

for $\zeta, \eta \in \Lambda$.

6.3. **g^3 for S_1 and S_2.**

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_1 \xrightarrow{\partial^3} (\alpha, \gamma) \xrightarrow{\partial^2} (\alpha, -\gamma)\alpha + (-\sigma, \beta)\gamma = (\alpha^2, -\gamma\alpha) + (-\sigma\gamma, \beta\gamma) = (\alpha^2 - \sigma\gamma, -\gamma\alpha + \beta\gamma) \xrightarrow{\partial^1} \alpha(\alpha^2 - \sigma\gamma) - \sigma(\gamma\alpha - \beta\gamma)$, so $(\alpha^3 - \alpha\sigma\gamma) - (\sigma\gamma\alpha - \sigma\beta\gamma) \in g^3$.

For S_2

$e_2 \xrightarrow{\partial^3} (\sigma, -\beta) \xrightarrow{\partial^2} (\alpha, -\gamma)\sigma + (\sigma, -\lambda\beta)(-\beta) = (\alpha\sigma, -\gamma\sigma) + (-\sigma\beta, \lambda\beta^2) = (-(-\alpha\sigma + \sigma\beta), \lambda\beta^2 - \gamma\sigma) \xrightarrow{\partial^1} -(\gamma\sigma\beta - \gamma\alpha\sigma) + (\lambda\beta^3 - \beta\gamma\sigma)$, so $-\gamma(\sigma\beta - \alpha\sigma) + \beta(\lambda\beta^2 - \gamma\sigma) \in g^3$.

Let $g_1^3 = \alpha^3 - \alpha\sigma\gamma - \sigma\gamma\alpha + \sigma\beta\gamma$ and $g_2^3 = -\gamma\sigma\beta + \gamma\alpha\sigma + \lambda\beta^3 - \beta\gamma\sigma$. So $g^3 = \{g_1^3, g_2^3\}$.

We know that $g^2 = \{\alpha^2 - \sigma\gamma, \lambda\beta^2 - \gamma\sigma, \gamma\alpha - \beta\gamma, \sigma\beta - \alpha\sigma\}$. Denote

$g_1^2 = \alpha^2 - \sigma\gamma$,
$g_2^2 = \lambda\beta^2 - \gamma\sigma$,
$g_3^2 = \gamma\alpha - \beta\gamma$ and
$g_4^2 = \sigma\beta - \alpha\sigma$.

So we have

$g_1^3 = g_1^2\alpha + g_4^2\gamma = \alpha g_1^2 - \sigma g_3^2$ and
$g_2^3 = g_2^2\beta + g_3^2\sigma = \beta g_2^2 - \gamma g_4^2$.

6.4. $\mathrm{HH}^2(\Lambda)$**.**

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3, \Lambda) \to \dots$$

6.4.1. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda)$.

So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 \\
&e_2 \otimes_{g_2^2} e_2 \mapsto j_5 e_2 + j_6\beta + j_7\beta^2 + j_8\beta^3 \\
&e_2 \otimes_{g_3^2} e_1 \mapsto j_9\gamma + j_{10}\gamma\alpha \\
&e_1 \otimes_{g_4^2} e_2 \mapsto j_{11}\sigma + j_{12}\sigma\beta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \alpha + e_1 \otimes_{g_4^2} \gamma - \alpha \otimes_{g_1^2} e_1 + \sigma \otimes_{g_3^2} e_1$

$e_2 \otimes_{g_2^3} e_2 \mapsto e_2 \otimes_{g_2^2} \beta + e_2 \otimes_{g_3^2} \sigma - \beta \otimes_{g_2^2} e_2 + \gamma \otimes_{g_4^2} e_2$.

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = \theta((e_1 \otimes_{g_1^2} e_1)\alpha + (e_1 \otimes_{g_4^2} e_2)\gamma - \alpha(e_1 \otimes_{g_1^2} e_1) + \sigma(e_2 \otimes_{g_3^2} e_1)) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\alpha + (j_{11}\sigma + j_{12}\sigma\beta)\gamma - \alpha(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3) + \sigma(j_9\gamma + j_{10}\gamma\alpha) = j_{11}\alpha^2 + j_{12}\alpha^3 + j_9\alpha^2 + j_{10}\alpha^3 = (j_9 + j_{11})\alpha^2 + (j_{10} + j_{12})\alpha^3 = 0$, so $j_{11} = -j_9$ and $j_{12} = -j_{10}$.

And $\theta d^3(e_2 \otimes_{g_2^3} e_2) = \theta((e_2 \otimes_{g_2^2} e_2)\beta + (e_2 \otimes_{g_3^2} e_1)\sigma - \beta(e_2 \otimes_{g_2^2} e_2) + \gamma(e_1 \otimes_{g_4^2} e_2)) = (j_5e_2 + j_6\beta + j_7\beta^2 + j_8\beta^3)\beta + (j_9\gamma + j_{10}\gamma\alpha)\sigma - \beta(j_5e_2 + j_6\beta + j_7\beta^2 + j_8\beta^3) + \gamma(j_{11}\sigma + j_{12}\sigma\beta) = j_9\gamma\sigma + j_{10}\gamma\alpha\sigma + j_{11}\gamma\sigma + j_{12}\gamma\sigma\beta = (j_9 + j_{11})\lambda\beta^2 + (j_{10} + j_{12})\lambda\beta^3 = 0$, so $j_9 = -j_{11}$ and $j_{12} = -j_{10}$, for $j_i \in K$.

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 \\
&e_2 \otimes_{g_2^2} e_2 \mapsto j_5 e_2 + j_6\beta + j_7\beta^2 + j_8\beta^3 \\
&e_2 \otimes_{g_3^2} e_1 \mapsto j_9\gamma + j_{10}\gamma\alpha \\
&e_1 \otimes_{g_4^2} e_2 \mapsto -j_9\sigma - j_{10}\sigma\beta,
\end{aligned}$$

where $j_i \in K$. Hence $\dim \operatorname{Ker}\delta^2 = 10$.

6.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_1 \to z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3$

$e_1 \otimes_\sigma e_2 \to z_4\sigma + z_5\sigma\beta$

$e_2 \otimes_\gamma e_1 \to z_6\gamma + z_7\gamma\alpha$

$e_2 \otimes_\beta e_2 \to z_8e_2 + z_9\beta + z_{10}\beta^2 + z_{11}\beta^3$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \alpha + \alpha \otimes_\alpha e_1 - e_1 \otimes_\sigma \gamma - \sigma \otimes_\gamma e_1$

$e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \otimes_\beta \lambda\beta + \lambda\beta \otimes_\beta e_2 - e_2 \otimes_\gamma \sigma - \gamma \otimes_\sigma e_2$

$e_2 \otimes_{g_3^2} e_1 \mapsto e_2 \otimes_\gamma \alpha + \gamma \otimes_\alpha e_1 - e_2 \otimes_\beta \gamma - \beta \otimes_\gamma e_1$

$e_1 \otimes_{g_4^2} e_2 \mapsto e_1 \otimes_\sigma \beta + \sigma \otimes_\beta e_2 - e_1 \otimes_\alpha \sigma - \alpha \otimes_\sigma e_2$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\alpha + \alpha(z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) - (z_4\sigma + z_5\sigma\beta)\gamma - \sigma(z_6\gamma + z_7\gamma\alpha) = 2z_0\alpha + 2z_1\alpha^2 + 2z_2\alpha^3 - z_4\alpha^2 - z_5\alpha^3 - z_6\alpha^2 - z_7\alpha^3 = 2z_0\alpha + (2z_1 - z_4 - z_6)\alpha^2 + (2z_2 - z_5 - z_7)\alpha^3$,

$\varphi d^2(e_2 \otimes_{g_2^2} e_2) = (z_8e_2 + z_9\beta + z_{10}\beta^2 + z_{11}\beta^3)\lambda\beta + \lambda\beta(z_8e_2 + z_9\beta + z_{10}\beta^2 + z_{11}\beta^3) - (z_6\gamma + z_7\gamma\alpha)\sigma - \gamma(z_4\sigma + z_5\sigma\beta) = 2z_8\lambda\beta + (2z_9 - z_6 - z_4)\lambda\beta^2 + (2z_{10} - z_5 - z_7)\lambda\beta^3$,

$\varphi d^2(e_2 \otimes_{g_3^2} e_1) = (z_6\gamma + z_7\gamma\alpha)\alpha + \gamma(z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) - (z_8e_2 + z_9\beta + z_{10}\beta^2 + z_{11}\beta^3)\gamma - \beta(z_6\gamma + z_7\gamma\alpha) = (z_0 - z_8)\gamma + (z_1 - z_9)\gamma\alpha$, and

$\varphi d^2(e_1 \otimes_{g_4^2} e_2) = (z_4\sigma + z_5\sigma\beta)\beta + \sigma(z_8e_2 + z_9\beta + z_{10}\beta^2 + z_{11}\beta^3) - (z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\sigma - \alpha(z_4\sigma + z_5\sigma\beta) = (z_8 - z_0)\sigma + (z_9 - z_1)\sigma\beta$.

Thus φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_4 - z_6)\alpha^2 + (2z_2 - z_5 - z_7)\alpha^3\\
&e_2 \otimes_{g_2^2} e_2 \mapsto 2z_8\lambda\beta + (2z_9 - z_4 - z_6)\lambda\beta^2 + (2z_{10} - z_5 - z_7)\lambda\beta^3\\
&e_2 \otimes_{g_3^2} e_1 \mapsto (z_0 - z_8)\gamma + (z_1 - z_9)\gamma\alpha\\
&e_1 \otimes_{g_4^2} e_2 \mapsto (z_8 - z_0)\sigma + (z_9 - z_1)\sigma\beta,
\end{aligned}$$

where $z_i \in K$.

Now we need to consider two cases if $\operatorname{char} K = 2$ and if $\operatorname{char} K \neq 2$.

If $\operatorname{char} K = 2$ then φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto (z_4 + z_6)\alpha^2 + (z_5 + z_7)\alpha^3\\
&e_2 \otimes_{g_2^2} e_2 \mapsto (z_4 + z_6)\lambda\beta^2 + (z_5 + z_7)\lambda\beta^3\\
&e_2 \otimes_{g_3^2} e_1 \mapsto (z_0 + z_8)\gamma + (z_1 + z_9)\gamma\alpha\\
&e_1 \otimes_{g_4^2} e_2 \mapsto (z_0 + z_8)\sigma + (z_1 + z_9)\sigma\beta,
\end{aligned}$$

where $z_i \in K$ and therefore $\dim \operatorname{Im} \delta^1 = 4$.

If $\operatorname{char} K \neq 2$ then φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_4 - z_6)\alpha^2 + (2z_2 - z_5 - z_7)\alpha^3\\
&e_2 \otimes_{g_2^2} e_2 \mapsto 2z_8\lambda\beta + (2z_9 - z_4 - z_6)\lambda\beta^2 + (2z_{10} - z_5 - z_7)\lambda\beta^3\\
&e_2 \otimes_{g_3^2} e_1 \mapsto (z_0 - z_8)\gamma + (z_1 - z_9)\gamma\alpha\\
&e_1 \otimes_{g_4^2} e_2 \mapsto -(z_0 - z_8)\sigma - (z_1 - z_9)\sigma\beta,
\end{aligned}$$

where $z_i \in K$. Note that $z_1 - z_9 = \frac{1}{2}[(2z_1 - z_4 - z_6) - (2z_9 - z_4 - z_6)]$. Then $\dim \operatorname{Im} \delta^1 = 6$.

6.4.3. $\mathrm{HH}^2(\Lambda)$.

From 6.4.1 and 6.4.2 we have that if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 6$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l}
P^2 \to \Lambda\\
e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha + d_3\alpha^2 + d_4\alpha^3\\
e_2 \otimes_{g_2^2} e_2 \mapsto d_5e_2 + d_6\beta\\
e_2 \otimes_{g_3^2} e_1 \mapsto 0\\
e_1 \otimes_{g_4^2} e_2 \mapsto 0
\end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x_0, x_1, x_2, x_3, x_4, x_5\}$ where

$$\begin{aligned}&x_0 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto e_1\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&x_1 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto \alpha\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&x_2 : P^2 \to \Lambda\\ &e_2 \otimes_{g_2^2} e_2 \mapsto e_2\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&x_3 : P^2 \to \Lambda\\ &e_2 \otimes_{g_2^2} e_2 \mapsto \beta\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&x_4 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto \alpha^2\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&x_5 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto \alpha^3\\ &\text{else} \mapsto 0.\end{aligned}$$

Note that x_4 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}&P^2 \to \Lambda\\ &e_2 \otimes_{g_2^2} e_2 \mapsto -\lambda\beta^2\\ &\text{else} \mapsto 0,\end{aligned}$$

and x_5 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}&P^2 \to \Lambda\\ &e_2 \otimes_{g_2^2} e_2 \mapsto -\lambda\beta^3\\ &\text{else} \mapsto 0.\end{aligned}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 4$ and we have

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda\\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha^2 + d_3\alpha^3\\ e_2 \otimes_{g_2^2} e_2 \mapsto d_4e_2\\ e_2 \otimes_{g_3^2} e_1 \mapsto 0\\ e_1 \otimes_{g_4^2} e_2 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{y_1, y_2, y_3, y_4\}$ where

$$\begin{aligned}&y_1 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto e_1\\ &\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&y_2 : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_1 \mapsto \alpha^2\\ &\text{else} \mapsto 0,\end{aligned}$$

$$
\begin{aligned}
&y_3 : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto \alpha^3 \\
&\text{else} \mapsto 0,
\end{aligned}
$$

$$
\begin{aligned}
&y_4 : P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\
&\text{else} \mapsto 0.
\end{aligned}
$$

Note that y_2 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$
\begin{aligned}
&P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -\lambda\beta^2 \\
&\text{else} \mapsto 0,
\end{aligned}
$$

and y_3 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$
\begin{aligned}
&P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -\lambda\beta^3 \\
&\text{else} \mapsto 0.
\end{aligned}
$$

7. The algebra A_3

Definition 7.1. [5] *Let A_3 be the algebra KQ/I where Q is the quiver*

$$\begin{tikzcd}
 & 2 \arrow[d, shift left, "\gamma"] & \\
 & 4 \arrow[u, shift left, "\delta"] \arrow[dl, shift left, "\beta"] \arrow[dr, shift left, "\varepsilon"] & \\
1 \arrow[ur, shift left, "\alpha"] & & 3 \arrow[ul, shift left, "\zeta"]
\end{tikzcd}$$

and

$$I = \langle \beta\alpha + \delta\gamma + \varepsilon\zeta, \alpha\beta, \zeta\varepsilon, \gamma\delta \rangle.$$

This is the preprojective algebra of type D_4.

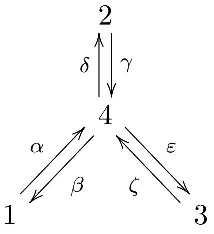
Theorem 7.2. [1, Theorem 4.1] *Let $\Lambda = A_3$. Assume* $\operatorname{char} K = 2$. *Then* $\dim \mathrm{HH}^2(\Lambda) = 3$.

Theorem 7.3. [13, Theorem 3.3 and Lemma 3.5] *Let $\Lambda = A_3$. Assume* $\operatorname{char} K \neq 2$. *Then* $\mathrm{HH}^2(\Lambda) = 0$.

Theorem 7.4. [13, Section 2.10] *For the algebra A_3, the preprojective algebra of type D_4, we have $\Omega^3(S_i) \cong S_i$ for all $i = 1, 2, 3, 4$. Moreover, A_3 is periodic as a bimodule of period 3 if* $\operatorname{char} K = 2$, *and of period 6 otherwise.*

8. The algebra A_4

Definition 8.1. [5] *Let A_4 be the algebra KQ/I where $\mathcal{Q}$ is the quiver*



and

$$I = \langle \beta\alpha + \delta\gamma + \varepsilon\zeta, \alpha\beta, \gamma\varepsilon, \zeta\delta \rangle.$$

8.1. **The structure of the indecomposable projectives.**

The zero elements of length two are $\alpha\beta, \gamma\varepsilon, \zeta\delta$ and the non zero elements of length two are $\beta\alpha, \delta\gamma, \varepsilon\zeta, \alpha\delta, \alpha\varepsilon, \gamma\delta, \gamma\beta, \zeta\varepsilon, \zeta\beta$. Note that:

$\delta\gamma = -\beta\alpha - \varepsilon\zeta$,
$\beta\alpha = -\delta\gamma - \varepsilon\zeta$,
$\varepsilon\zeta = -\delta\gamma - \beta\alpha$.

Also the zero elements of length three are: $\alpha\beta\alpha, \gamma\varepsilon\zeta, \beta\alpha\beta, \varepsilon\zeta\delta, \zeta\delta\gamma, \delta\gamma\varepsilon$ but the non zero elements of length three are:

$\alpha\delta\gamma = -\alpha\varepsilon\zeta$,
$\delta\gamma\beta = -\varepsilon\zeta\beta$,
$\delta\gamma\delta = -\beta\alpha\delta$,
$\gamma\delta\gamma = -\gamma\beta\alpha$,
$\varepsilon\zeta\varepsilon = -\beta\alpha\varepsilon$,
$\zeta\varepsilon\zeta = -\zeta\beta\alpha$.

For the elements of length four the zero elements are: $\alpha\delta\gamma\delta, \alpha\varepsilon\zeta\varepsilon, \gamma\beta\alpha\varepsilon, \gamma\varepsilon\zeta\beta$, $\zeta\delta\gamma\delta, \zeta\varepsilon\zeta\beta, \alpha\beta\alpha\varepsilon, \gamma\delta\gamma\beta, \zeta\delta\gamma\beta$. The non zero elements of length four are:

$\alpha\delta\gamma\beta = -\alpha\varepsilon\zeta\beta$,
$\gamma\delta\gamma\delta = -\gamma\beta\alpha\delta$,
$\zeta\varepsilon\zeta\varepsilon = -\zeta\beta\alpha\varepsilon$,
$\beta\alpha\delta\gamma = \delta\gamma\beta\alpha = \varepsilon\zeta\varepsilon\zeta = -\beta\alpha\varepsilon\zeta = -\delta\gamma\delta\gamma = -\varepsilon\zeta\beta\alpha$.

Note that all elements of length five are zero elements. So the indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$e_1\Lambda = sp\{e_1, \alpha, \alpha\delta, \alpha\varepsilon, \alpha\delta\gamma, \alpha\delta\gamma\beta\}$,
$e_2\Lambda = sp\{e_2, \gamma, \gamma\beta, \gamma\delta, \gamma\delta\gamma, \gamma\delta\gamma\delta\}$,
$e_3\Lambda = sp\{e_3, \zeta, \zeta\varepsilon, \zeta\beta, \zeta\varepsilon\zeta, \zeta\varepsilon\zeta\varepsilon\}$,
$e_4\Lambda = sp\{e_4, \beta, \delta, \varepsilon, \beta\alpha, \delta\gamma, \beta\alpha\varepsilon, \delta\gamma\delta, \varepsilon\zeta\beta, \beta\alpha\varepsilon\zeta\}$.

For $e_1\Lambda$

$$\begin{tikzcd}
& e_1 \arrow[d, dash] & \\
& \alpha \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha\delta \arrow[dr, dash] & & \alpha\varepsilon \arrow[dl, dash] \\
& \alpha\delta\gamma \arrow[d, dash] & \\
& \alpha\delta\gamma\beta &
\end{tikzcd}$$

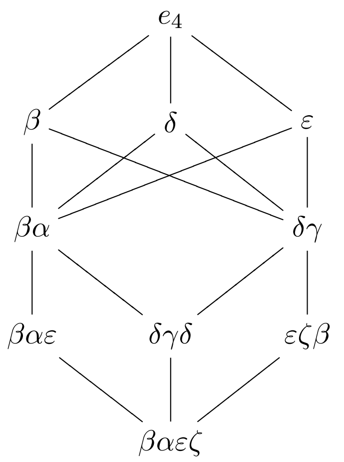
For $e_2\Lambda$

$$\begin{tikzcd}
& e_2 \arrow[d, dash] & \\
& \gamma \arrow[dl, dash] \arrow[dr, dash] & \\
\gamma\beta \arrow[dr, dash] & & \gamma\delta \arrow[dl, dash] \\
& \gamma\delta\gamma \arrow[d, dash] & \\
& \gamma\delta\gamma\delta &
\end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{tikzcd}
& e_3 \arrow[d, dash] & \\
& \zeta \arrow[dl, dash] \arrow[dr, dash] & \\
\zeta\beta \arrow[dr, dash] & & \zeta\varepsilon \arrow[dl, dash] \\
& \zeta\varepsilon\zeta \arrow[d, dash] & \\
& \zeta\varepsilon\zeta\varepsilon &
\end{tikzcd}$$

And for $e_4\Lambda$



8.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

8.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_4\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_4\Lambda \to e_1\Lambda$ is given by $e_4\mu \mapsto \alpha e_4\mu$, for $\mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

8.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_4\mu = t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^1$. Then $\alpha e_4\mu = 0$ so $\alpha(t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta) = 0$, that is, $t_1\alpha + t_3\alpha\delta + t_4\alpha\varepsilon + t_6\alpha\delta\gamma + t_9\alpha\varepsilon\zeta\beta = 0$ and then $t_1 = t_3 = t_4 = t_6 = t_9 = 0$. Thus $e_4\mu = t_2\beta + t_5\beta\alpha + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_{10}\beta\alpha\varepsilon\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{t_2\beta + t_5\beta\alpha + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_{10}\beta\alpha\varepsilon\zeta : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = t_2\beta + t_5\beta\alpha + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_{10}\beta\alpha\varepsilon\zeta$, that is, $x = \beta(t_2e_1 + t_5\alpha + t_7\alpha\varepsilon - t_8\alpha\delta + t_{10}\alpha\varepsilon\zeta)$. Thus $x \in \beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \beta e_1\Lambda$.

On the other hand, let $y = \beta e_1\nu \in \beta e_1\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \alpha(\beta e_1\nu) = \alpha\beta e_1\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \beta e_1\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \to e_4\Lambda$ is given by: $e_1\nu \mapsto \beta e_1\nu$, for $\nu \in \Lambda$.

8.2.3. Ker ∂^2.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\nu = c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta$. Assume that $e_1\nu \in \operatorname{Ker}\partial^2$. Then $\beta e_1\nu = 0$. So $\beta e_1\nu = \beta(c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta) = c_1\beta + c_2\beta\alpha + c_3\beta\alpha\delta + c_4\beta\alpha\varepsilon + c_5\beta\alpha\delta\gamma + c_6\beta\alpha\delta\gamma\beta = 0$. So $c_1\beta + c_2\beta\alpha + c_3\beta\alpha\delta + c_4\beta\alpha\varepsilon + c_5\beta\alpha\delta\gamma = 0$ which implies that $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Thus $e_1\nu = c_6\alpha\delta\gamma\beta$ and therefore $\operatorname{Ker}\partial^2 = \{c_6\alpha\delta\gamma\beta : c_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha\delta\gamma\beta e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = c_6\alpha\delta\gamma\beta$ so $u = \alpha\delta\gamma\beta(c_6e_1)$. Hence $u \in \alpha\delta\gamma\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \alpha\delta\gamma\beta e_1\Lambda$.

On the other hand, let $v = \alpha\delta\gamma\beta e_1\nu \in \alpha\delta\gamma\beta e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha\delta\gamma\beta e_1\nu) = \beta\alpha\delta\gamma\beta e_1\nu = 0$. Therefore $\alpha\delta\gamma\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \alpha\delta\gamma\beta e_1\Lambda$. □

Note that $\operatorname{Ker}\partial^2 \cong S_1$ and so $\Omega^3(S_1) \cong S_1$.

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\nu \mapsto \alpha\delta\gamma\beta e_1\nu$, for $\nu \in \Lambda$.

8.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.

The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_4\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_4\Lambda \to e_2\Lambda$ is given by $e_4\mu \to \gamma e_4\mu$, for $\mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

8.2.5. Ker ∂^1.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$, let $e_4\mu = t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^1$. Then $\gamma e_4\mu = 0$ so $\gamma(t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta) = 0$, that is, $t_1\gamma + t_2\gamma\beta + t_3\gamma\delta + t_5\gamma\beta\alpha + t_6\gamma\delta\gamma + t_8\gamma\delta\gamma\delta = t_1\gamma + t_2\gamma\beta + t_3\gamma\delta - t_5\gamma\delta\gamma + t_6\gamma\delta\gamma + t_8\gamma\delta\gamma\delta = 0$. So $t_1 = t_2 = t_3 = t_8 = 0$ and $t_6 - t_5 = 0$ which implies that $t_6 = t_5$. Thus $e_4\mu = t_4\varepsilon + t_5(\beta\alpha + \delta\gamma) + t_7\beta\alpha\varepsilon + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{t_4\varepsilon + t_5(\beta\alpha + \delta\gamma) + t_7\beta\alpha\varepsilon + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \varepsilon e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = t_4\varepsilon + t_5(\beta\alpha + \delta\gamma) + t_7\beta\alpha\varepsilon + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$. So $x = \varepsilon(t_4e_3 - t_5\zeta - t_7\zeta\varepsilon + t_9\zeta\beta - t_{10}\zeta\varepsilon\zeta)$. Thus $x \in \varepsilon e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \varepsilon e_3\Lambda$.

On the other hand, let $y = \varepsilon e_3\lambda \in \varepsilon e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \gamma(\varepsilon e_3\lambda) = \gamma\varepsilon e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\varepsilon e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \varepsilon e_3\Lambda$. □

So $\partial^2 : e_3\Lambda \to e_4\Lambda$ is given by: $e_3\lambda \mapsto \varepsilon e_3\lambda$, for $\lambda \in \Lambda$.

8.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_3\lambda = d_1e_3 + d_2\zeta + d_3\zeta\varepsilon + d_4\zeta\beta + d_5\zeta\varepsilon\zeta + d_6\zeta\varepsilon\zeta\varepsilon$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^2$. Then $\varepsilon e_3\lambda = 0$. So $\varepsilon e_3\lambda = \varepsilon(d_1e_3 + d_2\zeta + d_3\zeta\varepsilon + d_4\zeta\beta + d_5\zeta\varepsilon\zeta + d_6\zeta\varepsilon\zeta\varepsilon) = d_1\varepsilon + d_2\varepsilon\zeta + d_3\varepsilon\zeta\varepsilon + d_4\varepsilon\zeta\beta + d_5\varepsilon\zeta\varepsilon\zeta = 0$ which implies that $d_1 = d_2 = d_3 = d_4 = d_5 = 0$. Thus $e_3\lambda = d_6\zeta\varepsilon\zeta\varepsilon$ and therefore $\operatorname{Ker}\partial^2 = \{d_6\zeta\varepsilon\zeta\varepsilon : d_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \zeta\varepsilon\zeta\varepsilon e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = d_6\zeta\varepsilon\zeta\varepsilon$ so $u = \zeta\varepsilon\zeta\varepsilon(d_6e_3)$. Hence $u \in \zeta\varepsilon\zeta\varepsilon e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \zeta\varepsilon\zeta\varepsilon e_3\Lambda$.

On the other hand, let $v = \zeta\varepsilon\zeta\varepsilon e_3\lambda \in \zeta\varepsilon\zeta\varepsilon e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\zeta\varepsilon\zeta\varepsilon e_3\lambda) = \varepsilon\zeta\varepsilon\zeta\varepsilon e_3\lambda = 0$. Therefore $\zeta\varepsilon\zeta\varepsilon e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \zeta\varepsilon\zeta\varepsilon e_3\Lambda$. □

Note that $\operatorname{Ker}\partial^2 \cong S_3$, so $\Omega^3(S_2) \cong S_3$.

So the map $\partial^3 : e_3\Lambda \to e_3\Lambda$ is given by $e_3\lambda \mapsto \zeta\varepsilon\zeta\varepsilon e_3\lambda$, for $\lambda \in \Lambda$.

8.2.7. *The minimal projective resolution of the simple Λ-module S_3.*

The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_4\Lambda \to e_3\Lambda$ is given by $e_4\mu \to \zeta e_4\mu$, for $\mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

8.2.8. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$, let $e_4\mu = t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^1$. Then $\zeta e_4\mu = 0$ so $\zeta(t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta) = 0$, that is, $t_1\zeta + t_2\zeta\beta + t_4\zeta\varepsilon + t_5\zeta\beta\alpha + t_7\zeta\beta\alpha\varepsilon = 0$. So $t_1 = t_2 = t_4 = t_5 = t_7 = 0$. Thus $e_4\mu = t_3\delta + t_6\delta\gamma + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$, that is, $e_4\mu = t_3\delta + t_6\delta\gamma + t_8\delta\gamma\delta - t_9\delta\gamma\beta - t_{10}\delta\gamma\delta\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{t_3\delta + t_6\delta\gamma + t_8\delta\gamma\delta - t_9\delta\gamma\beta - t_{10}\delta\gamma\delta\gamma : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \delta e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = t_3\delta + t_6\delta\gamma + t_8\delta\gamma\delta - t_9\delta\gamma\beta - t_{10}\delta\gamma\delta\gamma$. So $x = \delta(t_3e_2 + t_6\gamma + t_8\gamma\delta - t_9\gamma\beta - t_{10}\gamma\delta\gamma)$. Thus $x \in \delta e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \delta e_2\Lambda$.

On the other hand, let $y = \delta e_2\eta \in \delta e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \zeta(\delta e_2\eta) = \zeta\delta e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\delta e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \delta e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_4\Lambda$ is given by: $e_2\eta \mapsto \delta e_2\eta$, for $\eta \in \Lambda$.

8.2.9. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_2\eta = f_1e_2 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta$. Assume that $e_2\eta \in \operatorname{Ker}\partial^2$. Then $\delta e_2\eta = 0$. So $\delta e_2\eta = \delta(f_1e_2 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta) = f_1\delta + f_2\delta\gamma + f_3\delta\gamma\beta + f_4\delta\gamma\delta + f_5\delta\gamma\delta\gamma = 0$ which implies that $f_1 = f_2 = f_3 = f_4 = f_5 = 0$. Thus $e_2\eta = f_6\gamma\delta\gamma\delta$ and therefore $\operatorname{Ker}\partial^2 = \{f_6\gamma\delta\gamma\delta : f_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \gamma\delta\gamma\delta e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_6\gamma\delta\gamma\delta$, that is, $u = \gamma\delta\gamma\delta(f_6e_2)$. Hence $u \in \gamma\delta\gamma\delta e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \gamma\delta\gamma\delta e_2\Lambda$.

On the other hand, let $v = \gamma\delta\gamma\delta e_2\eta \in \gamma\delta\gamma\delta e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\gamma\delta\gamma\delta e_2\eta) = \delta\gamma\delta\gamma\delta e_2\eta = 0$. Therefore $\gamma\delta\gamma\delta e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \gamma\delta\gamma\delta e_2\Lambda$. ☐

We remark that $\operatorname{Ker}\partial^2 \cong S_2$, so $\Omega^3(S_3) \cong S_2$.

So the map $\partial^3 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\eta \mapsto \gamma\delta\gamma\delta e_2\eta$, for $\eta \in \Lambda$.

8.2.10. *The minimal projective resolution of the simple* Λ*-module* S_4.
The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \to e_4\Lambda$ is given by $(e_1\nu, e_2\eta, e_3\lambda) \mapsto \beta e_1\nu + \delta e_2\eta + \varepsilon e_3\lambda$, for $\nu, \lambda, \eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

8.2.11. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$, let $e_1\nu = c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta$, $e_2\eta = f_1e_2 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta$ and $e_3\lambda = d_1e_3 + d_2\zeta + d_3\zeta\varepsilon + d_4\zeta\beta + d_5\zeta\varepsilon\zeta + d_6\zeta\varepsilon\zeta\varepsilon$, with $c_i, f_i, d_i \in K$. Assume that $(e_1\nu, e_2\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\beta e_1\nu + \delta e_2\eta + \varepsilon e_3\lambda = 0$ so $\beta(c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta) + \delta(f_1e_2 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta) + \varepsilon(d_1e_3 + d_2\zeta + d_3\zeta\varepsilon + d_4\zeta\beta + d_5\zeta\varepsilon\zeta + d_6\zeta\varepsilon\zeta\varepsilon) = 0$, that is, $c_1\beta + c_2\beta\alpha + c_3\beta\alpha\delta + c_4\beta\alpha\varepsilon + c_5\beta\alpha\delta\gamma + f_1\delta + f_2\delta\gamma + f_3\delta\gamma\beta + f_4\delta\gamma\delta + f_5\delta\gamma\delta\gamma + d_1\varepsilon + d_2\varepsilon\zeta + d_3\varepsilon\zeta\varepsilon + d_4\varepsilon\zeta\beta + d_5\varepsilon\zeta\varepsilon\zeta = c_1\beta + c_2\beta\alpha - c_3\delta\gamma\delta - c_4\varepsilon\zeta\varepsilon - c_5\delta\gamma\delta\gamma + f_1\delta + f_2\delta\gamma - f_3\varepsilon\zeta\beta + f_4\delta\gamma\delta + f_5\delta\gamma\delta\gamma + d_1\varepsilon + d_2\varepsilon\zeta + d_3\varepsilon\zeta\varepsilon + d_4\varepsilon\zeta\beta + d_5\varepsilon\zeta\varepsilon\zeta = 0$. Then $c_1\beta + c_2\beta\alpha + (-c_3 + f_4)\delta\gamma\delta + (-c_4 + d_3)\varepsilon\zeta\varepsilon + (-c_5 + f_5)\delta\gamma\delta\gamma + f_1\delta + f_2\delta\gamma + (-f_3 + d_4)\varepsilon\zeta\beta + d_1\varepsilon + d_2\varepsilon\zeta + d_5\varepsilon\zeta\varepsilon\zeta = 0$. Therefore $c_1 = c_2 = d_1 = d_2 = d_5 = f_1 = f_2 = 0, c_3 = f_4, c_4 = d_3, c_5 = f_5$ and $f_3 = d_4$. Thus $e_1\nu = c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta, e_2\eta = f_3\gamma\beta + c_3\gamma\delta + c_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta$ and $e_3\lambda = c_4\zeta\varepsilon + d_4\zeta\beta + d_6\zeta\varepsilon\zeta\varepsilon$.

Hence $\operatorname{Ker}\partial^1 = \{(c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta, f_3\gamma\beta + c_3\gamma\delta + c_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta, c_4\zeta\varepsilon + d_4\zeta\beta + d_6\zeta\varepsilon\zeta\varepsilon) : c_i, f_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha,\gamma,\zeta)e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_3\alpha\delta + c_4\alpha\varepsilon + c_5\alpha\delta\gamma + c_6\alpha\delta\gamma\beta, f_3\gamma\beta + c_3\gamma\delta + c_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta, c_4\zeta\varepsilon + d_4\zeta\beta + d_6\zeta\varepsilon\zeta\varepsilon)$, that is, $x = (\alpha,\gamma,\zeta)(c_3\delta + c_4\varepsilon + c_5\delta\gamma + c_6\delta\gamma\beta + f_3\beta + f_6\delta\gamma\delta + d_6\varepsilon\zeta\varepsilon)$. Thus $x \in (\alpha,\gamma,\zeta)e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha,\gamma,\zeta)e_4\Lambda$.

On the other hand, let $y = (\alpha,\gamma,\zeta)e_4\mu \in (\alpha,\gamma,\zeta)e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha,\gamma,\zeta)e_4\mu) = (\beta\alpha+\delta\gamma+\varepsilon\zeta)e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha,\gamma,\zeta)e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha,\gamma,\zeta)e_4\Lambda$. □

So $\partial^2 : e_4\Lambda \to e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ is given by: $e_4\mu \mapsto (\alpha,\gamma,\zeta)e_4\mu$, for $\mu \in \Lambda$.

8.2.12. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_4)$. Let $e_4\mu = t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta$. Assume that $e_4\mu \in \operatorname{Ker}\partial^2$. Then $(\alpha,\gamma,\zeta)e_4\mu = 0$. So $(\alpha,\gamma,\zeta)e_4\mu = (\alpha,\gamma,\zeta)(t_1e_4 + t_2\beta + t_3\delta + t_4\varepsilon + t_5\beta\alpha + t_6\delta\gamma + t_7\beta\alpha\varepsilon + t_8\delta\gamma\delta + t_9\varepsilon\zeta\beta + t_{10}\beta\alpha\varepsilon\zeta) = (t_1\alpha + t_3\alpha\delta + t_4\alpha\varepsilon + t_6\alpha\delta\gamma + t_9\alpha\varepsilon\zeta\beta, t_1\gamma + t_2\gamma\beta + t_3\gamma\delta + (-t_5 + t_6)\gamma\delta\gamma + t_8\gamma\delta\gamma\delta, t_1\zeta + t_2\zeta\beta + t_4\zeta\varepsilon + t_5\zeta\beta\alpha + t_7\zeta\beta\alpha\varepsilon) = (0,0,0)$. Then $t_1\alpha + t_3\alpha\delta + t_4\alpha\varepsilon + t_6\alpha\delta\gamma + t_9\alpha\varepsilon\zeta\beta = 0$ which implies that $t_1 = t_3 = t_4 = t_6 = t_9 = 0$. And $t_1\gamma + t_2\gamma\beta + t_3\gamma\delta + (-t_5 + t_6)\gamma\delta\gamma + t_8\gamma\delta\gamma\delta = 0$, so $t_1 = t_2 = t_3 = t_8 = 0$ and $t_6 = t_5$. Also $t_1\zeta + t_2\zeta\beta + t_4\zeta\varepsilon + t_5\zeta\beta\alpha + t_7\zeta\beta\alpha\varepsilon = 0$ so $t_1 = t_2 = t_4 = t_5 = t_7 = 0$. Hence $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = t_9 = 0$. Thus $e_4\mu = t_{10}\beta\alpha\varepsilon\zeta$ and therefore $\operatorname{Ker}\partial^2 = \{t_{10}\beta\alpha\varepsilon\zeta : t_{10} \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\alpha\varepsilon\zeta e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = t_{10}\beta\alpha\varepsilon\zeta$ so $u = \beta\alpha\varepsilon\zeta(t_{10}e_4)$. Hence $u \in \beta\alpha\varepsilon\zeta e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\alpha\varepsilon\zeta e_4\Lambda$.

On the other hand, let $v = \beta\alpha\varepsilon\zeta e_4\mu \in \beta\alpha\varepsilon\zeta e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\alpha\varepsilon\zeta e_4\mu) = (\alpha,\gamma,\zeta)\beta\alpha\varepsilon\zeta e_4\mu = (0,0,0)$. Therefore $\beta\alpha\varepsilon\zeta e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\alpha\varepsilon\zeta e_4\Lambda$. □

Note that $\operatorname{Ker}\partial^2 \cong S_4$ and so $\Omega^3(S_4) \cong S_4$.

So the map $\partial^3 : e_4\Lambda \to e_4\Lambda$ is given by $e_4\mu \mapsto \beta\alpha\varepsilon\zeta e_4\mu$, for $\mu \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : e_4\mu \mapsto \alpha e_4\mu,$$
$$\partial^2 : e_1\nu \mapsto \beta e_1\nu,$$
$$\partial^3 : e_1\nu \mapsto \alpha\delta\gamma\beta e_1\nu,$$

for $\nu, \mu \in \Lambda$.

The maps for S_2 are:

$$\partial^1 : e_4\mu \mapsto \gamma e_4\mu,$$
$$\partial^2 : e_3\lambda \mapsto \varepsilon e_3\lambda,$$

$\partial^3 : e_3\lambda \mapsto \zeta\varepsilon\zeta\varepsilon e_3\lambda,$

for $\mu, \lambda \in \Lambda$.

The maps for S_3 are:

$\partial^1 : e_4\mu \mapsto \zeta e_4\mu,$

$\partial^2 : e_2\eta \mapsto \delta e_2\eta,$

$\partial^3 : e_2\eta \mapsto \gamma\delta\gamma\delta e_2\eta,$

for $\eta, \mu \in \Lambda$.

The maps for S_4 are:

$\partial^1 : (e_1\nu, e_2\eta, e_3\lambda) \mapsto \beta e_1\nu + \delta e_2\eta + \varepsilon e_3\lambda,$

$\partial^2 : e_4\mu \mapsto (\alpha, \gamma, \zeta)e_4\mu,$

$\partial^3 : e_4\mu \mapsto \beta\alpha\varepsilon\zeta e_4\mu,$

for $\nu, \eta, \mu, \lambda \in \Lambda$.

8.3. g^3 **for** S_1, S_2, S_3 **and** S_4**.**

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$$e_1 \xrightarrow{\partial^3} \alpha\delta\gamma\beta \xrightarrow{\partial^2} \beta(\alpha\delta\gamma\beta) \xrightarrow{\partial^1} \alpha\beta(\alpha\delta\gamma\beta), \text{ so } \alpha\beta(\alpha\delta\gamma\beta) \in g^3.$$

For S_2

$$e_3 \xrightarrow{\partial^3} \zeta\varepsilon\zeta\varepsilon \xrightarrow{\partial^2} \varepsilon(\zeta\varepsilon\zeta\varepsilon) \xrightarrow{\partial^1} \gamma\varepsilon(\zeta\varepsilon\zeta\varepsilon), \text{ so } \gamma\varepsilon(\zeta\varepsilon\zeta\varepsilon) \in g^3.$$

For S_3

$$e_2 \xrightarrow{\partial^3} \gamma\delta\gamma\delta \xrightarrow{\partial^2} \delta(\gamma\delta\gamma\delta) \xrightarrow{\partial^1} \zeta\delta(\gamma\delta\gamma\delta), \text{ so } \zeta\delta(\gamma\delta\gamma\delta) \in g^3.$$

For S_4

$$e_4 \xrightarrow{\partial^3} \beta\alpha\varepsilon\zeta \xrightarrow{\partial^2} (\alpha, \gamma, \zeta)\beta\alpha\varepsilon\zeta = (\alpha\beta\alpha\varepsilon\zeta, \gamma\beta\alpha\varepsilon\zeta, \zeta\beta\alpha\varepsilon\zeta) \xrightarrow{\partial^1} \beta(\alpha\beta\alpha\varepsilon\zeta) + \delta(\gamma\beta\alpha\varepsilon\zeta)$$
$$+ \varepsilon(\zeta\beta\alpha\varepsilon\zeta), \text{ so } \beta\alpha\beta\alpha\varepsilon\zeta + \delta\gamma\beta\alpha\varepsilon\zeta + \varepsilon\zeta\beta\alpha\varepsilon\zeta = (\beta\alpha + \delta\gamma + \varepsilon\zeta)\beta\alpha\varepsilon\zeta \in g^3.$$

Let $g_1^3 = \alpha\beta\alpha\delta\gamma\beta$, $g_2^3 = \gamma\varepsilon\zeta\varepsilon\zeta\varepsilon$, $g_3^3 = \zeta\delta\gamma\delta\gamma\delta$, and $g_4^3 = \beta\alpha\beta\alpha\varepsilon\zeta + \delta\gamma\beta\alpha\varepsilon\zeta + \varepsilon\zeta\beta\alpha\varepsilon\zeta$.
So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3\}$.

We know that $g^2 = \{\beta\alpha + \delta\gamma + \varepsilon\zeta, \alpha\beta, \gamma\varepsilon, \zeta\delta\}$. Denote

$g_1^2 = \alpha\beta,$

$g_2^2 = \gamma\varepsilon,$

$g_3^2 = \zeta\delta$ and

$g_4^2 = \beta\alpha + \delta\gamma + \varepsilon\zeta.$

So we have

$g_1^3 = g_1^2\alpha\delta\gamma\beta = \alpha g_4^2\delta\gamma\beta - \alpha\varepsilon g_3^2\gamma\beta + \alpha\delta g_2^2\zeta\beta - \alpha\delta\gamma g_4^2\beta + \alpha\delta\gamma\beta g_1^2,$

$g_2^3 = g_2^2\zeta\varepsilon\zeta\varepsilon = \gamma g_4^2\varepsilon\zeta\varepsilon - \gamma\delta g_2^2\zeta\varepsilon + \gamma\beta g_1^2\alpha\varepsilon - \gamma\beta\alpha g_4^2\varepsilon + \gamma\beta\alpha\delta g_2^2,$

$$g_3^3 = g_3^2\gamma\delta\gamma\delta = \zeta g_4^2\delta\gamma\delta - \zeta\varepsilon g_3^2\gamma\delta + \zeta\beta g_1^2\alpha\delta - \zeta\beta\alpha g_4^2\delta + \zeta\beta\alpha\varepsilon g_3^2 \text{ and}$$
$$\begin{aligned} g_4^3 = g_4^2\beta\alpha\varepsilon\zeta &= \beta g_1^2\alpha\varepsilon\zeta - \delta g_2^2\zeta\varepsilon\zeta + \varepsilon g_3^2\gamma\delta\gamma + \delta\gamma g_4^2\varepsilon\zeta - \varepsilon\zeta g_4^2\delta\gamma - \delta\gamma\delta g_2^2\zeta \\ &- \varepsilon\zeta\beta g_1^2\alpha + \varepsilon\zeta\varepsilon g_3^2\gamma + \varepsilon\zeta\beta\alpha g_4^2. \end{aligned}$$

8.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\,.$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots$$

8.4.1. $\operatorname{Ker}\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\operatorname{Ker}\delta^2$ and $\operatorname{Im}\delta^1$. Let $\theta \in \operatorname{Ker}\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned} &\theta : P^2 \to \Lambda \\ &e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\alpha\delta\gamma\beta \\ &e_2 \otimes_{g_2^2} e_3 \mapsto 0 \\ &e_3 \otimes_{g_3^2} e_2 \mapsto 0 \\ &e_4 \otimes_{g_4^2} e_4 \mapsto j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta, \end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$\begin{aligned} &e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \alpha\delta\gamma\beta \\ &\quad - [\alpha \otimes_{g_4^2} \delta\gamma\beta - \alpha\varepsilon \otimes_{g_3^2} \gamma\beta + \alpha\delta \otimes_{g_2^2} \zeta\beta - \alpha\delta\gamma \otimes_{g_4^2} \beta + \alpha\delta\gamma\beta \otimes_{g_1^2} e_1], \\ &e_2 \otimes_{g_2^3} e_3 \mapsto e_2 \otimes_{g_2^2} \zeta\varepsilon\zeta\varepsilon \\ &\quad - [\gamma \otimes_{g_4^2} \varepsilon\zeta\varepsilon - \gamma\delta \otimes_{g_2^2} \zeta\varepsilon + \gamma\beta \otimes_{g_1^2} \alpha\varepsilon - \gamma\beta\alpha \otimes_{g_4^2} \varepsilon + \gamma\beta\alpha\delta \otimes_{g_2^2} e_3], \\ &e_3 \otimes_{g_3^3} e_2 \mapsto e_3 \otimes_{g_3^2} \gamma\delta\gamma\delta \\ &\quad - [\zeta \otimes_{g_4^2} \delta\gamma\delta - \zeta\varepsilon \otimes_{g_3^2} \gamma\delta + \zeta\beta \otimes_{g_1^2} \alpha\delta - \zeta\beta\alpha \otimes_{g_4^2} \delta + \zeta\beta\alpha\varepsilon \otimes_{g_3^2} e_2], \\ &e_4 \otimes_{g_4^3} e_4 \mapsto e_4 \otimes_{g_4^2} \beta\alpha\varepsilon\zeta \\ &\quad - [\beta \otimes_{g_1^2} \alpha\varepsilon\zeta - \delta \otimes_{g_2^2} \zeta\varepsilon\zeta + \varepsilon \otimes_{g_3^2} \gamma\delta\gamma + \delta\gamma \otimes_{g_4^2} \varepsilon\zeta - \varepsilon\zeta \otimes_{g_4^2} \delta\gamma \\ &\qquad - \delta\gamma\delta \otimes_{g_2^2} \zeta - \varepsilon\zeta\beta \otimes_{g_1^2} \alpha + \varepsilon\zeta\varepsilon \otimes_{g_3^2} \gamma + \varepsilon\zeta\beta\alpha \otimes_{g_4^2} e_4]. \end{aligned}$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\alpha\delta\gamma\beta)\alpha\delta\gamma\beta - [\alpha(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\delta\gamma\beta - \alpha\delta\gamma(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\beta + \alpha\delta\gamma\beta(j_1e_1 + j_2\alpha\delta\gamma\beta)] = 0$,

for $\theta d^3(e_2 \otimes_{g_2^3} e_3) = -[\gamma(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\varepsilon\zeta\varepsilon + \gamma\beta(j_1e_1 + j_2\alpha\delta\gamma\beta)\alpha\varepsilon - \gamma\beta\alpha(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\varepsilon] = 0$,

also for $\theta d^3(e_3 \otimes_{g_3^3} e_2) = -[\zeta(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\delta\gamma\delta + \zeta\beta(j_1e_1 + j_2\alpha\delta\gamma\beta)\alpha\delta - \zeta\beta\alpha(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\delta] = 0$,

and for $\theta d^3(e_4 \otimes_{g_4^3} e_4) = (j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\beta\alpha\varepsilon\zeta - [\beta(j_1e_1 + j_2\alpha\delta\gamma\beta)\alpha\varepsilon\zeta + \delta\gamma(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\varepsilon\zeta - \varepsilon\zeta(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)\delta\gamma - \varepsilon\zeta\beta(j_1e_1 + j_2\alpha\delta\gamma\beta)\alpha + \varepsilon\zeta\beta\alpha(j_3e_4 + j_4\beta\alpha + j_5\delta\gamma + j_6\beta\alpha\varepsilon\zeta)] = 0$.

So $\theta \in \operatorname{Ker} \delta^2$ is given by

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2 \alpha\delta\gamma\beta \\
&e_2 \otimes_{g_2^2} e_3 \mapsto 0 \\
&e_3 \otimes_{g_3^2} e_2 \mapsto 0 \\
&e_4 \otimes_{g_4^2} e_4 \mapsto j_3 e_4 + j_4 \beta\alpha + j_5 \delta\gamma + j_6 \beta\alpha\varepsilon\zeta,
\end{aligned}$$

where $j_i \in K$. Hence $\dim \operatorname{Ker} \delta^2 = 6$.

8.4.2. $\operatorname{Im} \delta^1$.

Now to find $\operatorname{Im} \delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_4 \to z_1 \alpha + z_2 \alpha\delta\gamma$

$e_4 \otimes_\beta e_1 \to z_3 \beta + z_4 \varepsilon\zeta\beta$

$e_2 \otimes_\gamma e_4 \to z_5 \gamma + z_6 \gamma\delta\gamma$

$e_4 \otimes_\delta e_2 \to z_7 \delta + z_8 \delta\gamma\delta$

$e_3 \otimes_\zeta e_4 \to z_9 \zeta + z_{10} \zeta\varepsilon\zeta$

$e_4 \otimes_\varepsilon e_3 \to z_{11} \varepsilon + z_{12} \beta\alpha\varepsilon$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \beta + \alpha \otimes_\beta e_1$

$e_2 \otimes_{g_2^2} e_3 \mapsto e_2 \otimes_\gamma \varepsilon + \gamma \otimes_\varepsilon e_3$

$e_3 \otimes_{g_3^2} e_2 \mapsto e_3 \otimes_\zeta \delta + \zeta \otimes_\delta e_2$

$e_4 \otimes_{g_4^2} e_4 \mapsto e_4 \otimes_\beta \alpha + \beta \otimes_\alpha e_4 + e_4 \otimes_\delta \gamma + \delta \otimes_\gamma e_4 + e_4 \otimes_\varepsilon \zeta + \varepsilon \otimes_\zeta e_4$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1\alpha + z_2\alpha\delta\gamma)\beta + \alpha(z_3\beta + z_4\varepsilon\zeta\beta) = (z_2 - z_4)\alpha\delta\gamma\beta$,

$\varphi d^2(e_2 \otimes_{g_2^2} e_3) = (z_5\gamma + z_6\gamma\delta\gamma)\varepsilon + \gamma(z_{11}\varepsilon + z_{12}\beta\alpha\varepsilon) = 0$,

$\varphi d^2(e_3 \otimes_{g_3^2} e_2) = (z_9\zeta + z_{10}\zeta\varepsilon\zeta)\delta + \zeta(z_7\delta + z_8\delta\gamma\delta) = 0$, and

$\varphi d^2(e_4 \otimes_{g_4^2} e_4) = (z_3\beta + z_4\varepsilon\zeta\beta)\alpha + \beta(z_1\alpha + z_2\alpha\delta\gamma) + (z_7\delta + z_8\delta\gamma\delta)\gamma + \delta(z_5\gamma + z_6\gamma\delta\gamma) + (z_{11}\varepsilon + z_{12}\beta\alpha\varepsilon)\zeta + \varepsilon(z_9\zeta + z_{10}\zeta\varepsilon\zeta) = (z_1 + z_3 - z_9 - z_{11})\beta\alpha + (z_5 + z_7 - z_9 - z_{11})\delta\gamma + (-z_2 + z_4 + z_6 + z_8 - z_{10} + z_{12})\beta\alpha\varepsilon\zeta$. We will write $c_1 = z_2 - z_4, c_2 = z_1 + z_3 - z_9 - z_{11}, c_3 = z_5 + z_7 - z_9 - z_{11}$ and $c_4 = -z_2 + z_4 + z_6 + z_8 - z_{10} + z_{12}$, for $c_i \in K$.

Thus φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto c_1 \alpha\delta\gamma\beta \\
&e_2 \otimes_{g_2^2} e_3 \mapsto 0 \\
&e_3 \otimes_{g_3^2} e_2 \mapsto 0 \\
&e_4 \otimes_{g_4^2} e_4 \mapsto c_2 \beta\alpha + c_3 \delta\gamma + c_4 \beta\alpha\varepsilon\zeta,
\end{aligned}$$

where $c_1, c_2, c_3, c_4 \in K$. Therefore $\dim \operatorname{Im} \delta^1 = 4$.

8.4.3. $\mathrm{HH}^2(\Lambda)$.

From 8.4.1 and 8.4.2 we have that $\dim \mathrm{HH}^2(\Lambda) = 2$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_2 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_3 \otimes_{g_3^2} e_2 \mapsto 0 \\ e_4 \otimes_{g_4^2} e_4 \mapsto d_2 e_4 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ \text{else} \mapsto 0 \end{array}$$

and

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_4 \otimes_{g_4^2} e_4 \mapsto e_4 \\ \text{else} \mapsto 0. \end{array}$$

8.5. **The period of the simple A_4−modules.**

Theorem 8.2. *For the algebra A_4, we have $\Omega^3(S_1) \cong S_1, \Omega^3(S_2) \cong S_3, \Omega^3(S_3) \cong S_2$ and $\Omega^3(S_4) \cong S_4$. Hence $\Omega^6(S_i) \cong S_i$ for all $i = 1, 2, 3, 4$.*

9. The algebra A_5

Definition 9.1. [5] *Let A_5 be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[loop left, "\gamma"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\alpha"] \end{tikzcd}$$

and

$$I = \langle \gamma^2 - \beta\alpha, \alpha\gamma\beta \rangle.$$

9.1. The structure of the indecomposable projectives.

The indecomposable projective Λ-modules are $e_1\Lambda$ and $e_2\Lambda$ where

$$e_1\Lambda = sp\{e_1, \gamma, \beta, \gamma\beta, \gamma^2, \gamma^2\beta, \gamma^3, \gamma^4\},$$
$$e_2\Lambda = sp\{e_2, \alpha, \alpha\gamma, \alpha\beta, \alpha\gamma^2, \alpha\beta\alpha\beta\}.$$

We have $e_1\Lambda$

$$\begin{tikzcd}
 & & e_1 \arrow[dl, no head] \arrow[dr, no head] & \\
 & \gamma \arrow[dl, no head] \arrow[dr, no head] & & \beta \arrow[dl, no head] \\
\gamma\beta \arrow[dr, no head] & & \gamma^2 \arrow[dl, no head] \arrow[dr, no head] & \\
 & \gamma^3 \arrow[dr, no head] & & \gamma^2\beta \arrow[dl, no head] \\
 & & \gamma^4 &
\end{tikzcd}$$

and for $e_2\Lambda$

$$\begin{tikzcd}
 & e_2 \arrow[d, no head] & \\
 & \alpha \arrow[dl, no head] \arrow[dr, no head] & \\
\alpha\gamma \arrow[dr, no head] & & \alpha\beta \arrow[dl, no head] \\
 & \alpha\gamma^2 \arrow[d, no head] & \\
 & \alpha\beta\alpha\beta &
\end{tikzcd}$$

9.2. The minimal projective resolutions of the simple Λ-modules S_1 and S_2.

9.2.1. *The minimal projective resolution of the simple* Λ*-module* S_1.

Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\zeta, e_2\eta) \to \gamma e_1\zeta + \beta e_2\eta$, for $\zeta, \eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

9.2.2. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$ and $e_2\eta = d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^1$. Then $\gamma e_1\zeta + \beta e_2\eta = 0$ so $\gamma(c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4) + \beta(d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta) = 0$, that is, $c_1\gamma + c_2\gamma^2 + c_3\gamma\beta + c_4\gamma^2\beta + c_5\gamma^3 + c_7\gamma^4 + d_1\beta + d_2\beta\alpha + d_3\beta\alpha\gamma + d_4\beta\alpha\beta + d_5\beta\alpha\gamma^2 = 0$, that is, $c_1\gamma + c_2\gamma^2 + c_3\gamma\beta + c_4\gamma^2\beta + c_5\gamma^3 + c_7\gamma^4 + d_1\beta + d_2\gamma^2 + d_3\gamma^3 + d_4\gamma^2\beta + d_5\gamma^4 = 0$. So $c_1\gamma + d_1\beta + (c_2+d_2)\gamma^2 + c_3\gamma\beta + (c_5+d_3)\gamma^3 + (c_4+d_4)\gamma^2\beta + (c_7+d_5)\gamma^4 = 0$ which implies that $c_1 = c_3 = 0 = d_1$ and $d_2 = -c_2, d_3 = -c_5, d_4 = -c_4$ and $d_5 = -c_7$. Thus $e_1\zeta = c_2\gamma + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$ and $e_2\eta = -c_2\alpha - c_4\alpha\beta - c_5\alpha\gamma - c_7\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\gamma + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4, -c_2\alpha - c_4\alpha\beta - c_5\alpha\gamma - c_7\alpha\gamma^2 + d_6\alpha\beta\alpha\beta) : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\gamma, -\alpha)e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\gamma + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4, -c_2\alpha - c_4\alpha\beta - c_5\alpha\gamma - c_7\alpha\gamma^2 + d_6\alpha\beta\alpha\beta)$, that is, $x = (\gamma, -\alpha)(c_2e_1 + c_4\beta + c_5\gamma + c_6\gamma\beta + c_7\gamma^2 + c_8\gamma^3 - d_6\beta\alpha\beta)$. Thus $x \in (\gamma, -\alpha)e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\gamma, -\alpha)e_1\Lambda$.

On the other hand, let $y = (\gamma, -\alpha)e_1\zeta \in (\gamma, -\alpha)e_1\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \gamma(\gamma e_1\zeta) + \beta(-\alpha e_1\zeta) = (\gamma^2 - \beta\alpha)e_1\zeta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\gamma, -\alpha)e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\gamma, -\alpha)e_1\Lambda$. □

So $\partial^2 : e_1\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by: $e_1\zeta \mapsto (\gamma, -\alpha)e_1\zeta$, for $\zeta \in \Lambda$.

9.2.3. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Assume that $e_1\zeta \in \operatorname{Ker}\partial^2$. Then $(\gamma, -\alpha)e_1\zeta = (0, 0)$. We know that $e_1\zeta = c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$ with $c_i \in K$. So $(\gamma, -\alpha)e_1\zeta = (\gamma, -\alpha)(c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4) = (c_1\gamma + c_2\gamma^2 + c_3\gamma\beta + c_4\gamma^2\beta + c_5\gamma^3 + c_7\gamma^4, -c_1\alpha - c_2\alpha\gamma - c_3\alpha\beta - c_5\alpha\gamma^2 - c_6\alpha\gamma^2\beta) = (0, 0)$. So $c_1\gamma + c_2\gamma^2 + c_3\gamma\beta + c_4\gamma^2\beta + c_5\gamma^3 + c_7\gamma^4 = 0$, which implies that, $c_1 = c_2 = c_3 = c_4 = c_5 = c_7 = 0$ and $-c_1\alpha - c_2\alpha\gamma - c_3\alpha\beta - c_5\alpha\gamma^2 - c_6\alpha\gamma^2\beta = 0$, that is, $c_1 = c_2 = c_3 = c_5 = c_6 = 0$. Thus $e_1\zeta = c_8\gamma^4$ and therefore $\operatorname{Ker}\partial^2 = \{c_8\gamma^4 : c_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \gamma^4 e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = c_8\gamma^4$ so $u = \gamma^4(c_8 e_1)$. Hence $u \in \gamma^4 e_1\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \gamma^4 e_1 \Lambda$.

On the other hand, let $v = \gamma^4 e_1 \zeta \in \gamma^4 e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\gamma^4 e_1\zeta) = (\gamma, -\alpha)(\gamma^4 e_1 \zeta) = (\gamma^5, -\alpha\gamma^4)e_1\zeta = (0,0)$. Therefore $\gamma^4 e_1\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \gamma^4 e_1\Lambda$. □

We remark that $\operatorname{Ker} \partial^2 \cong S_1$ so that $\Omega^3(S_1) = S_1$.

The map $\partial^3 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\zeta \mapsto \gamma^4 e_1 \zeta$, for $\zeta \in \Lambda$.

9.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2*.*
Now the minimal projective resolution of the simple Λ-module S_2 starts by:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\zeta \mapsto \alpha e_1\zeta$, for $\zeta \in \Lambda$.

Now we want to find $\operatorname{Ker} \partial^1$ and $\operatorname{Ker} \partial^2$ for S_2.

9.2.5. $\operatorname{Ker} \partial^1$.
To find $\operatorname{Ker} \partial^1 = \Omega^2(S_2)$, let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker} \partial^1$. Then $\alpha e_1 \zeta = 0$ so $\alpha(c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4) = 0$, that is, $c_1\alpha + c_2\alpha\gamma + c_3\alpha\beta + c_5\alpha\gamma^2 + c_6\alpha\gamma^2\beta = 0$. So $c_1 = c_2 = c_3 = c_5 = c_6 = 0$. Thus $e_1\zeta = c_4\gamma\beta + c_7\gamma^3 + c_8\gamma^4$.

Hence $\operatorname{Ker} \partial^1 = \{c_4\gamma\beta + c_7\gamma^3 + c_8\gamma^4 : c_i \in K\}$.

Claim. $\operatorname{Ker} \partial^1 = \gamma\beta e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^1$; then $x = c_4\gamma\beta + c_7\gamma^3 + c_8\gamma^4$, that is, $x = \gamma\beta(c_4e_2 + c_7\alpha + c_8\alpha\gamma)$. Thus $x \in \gamma\beta e_2\Lambda$ and therefore $\operatorname{Ker} \partial^1 \subseteq \gamma\beta e_2\Lambda$.

On the other hand, let $y = \gamma\beta e_2\eta \in \gamma\beta e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \alpha(\gamma\beta e_2\eta) = \alpha\gamma\beta e_2\eta = 0$. Therefore $y \in \operatorname{Ker} \partial^1$ and so $\gamma\beta e_2\Lambda \subseteq \operatorname{Ker} \partial^1$.

Hence $\operatorname{Ker} \partial^1 = \gamma\beta e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_1\Lambda$ is given by: $e_2\eta \mapsto \gamma\beta e_2\eta$, for $\eta \in \Lambda$.

9.2.6. $\operatorname{Ker} \partial^2$.
Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_2)$. Assume that $e_2\eta \in \operatorname{Ker} \partial^2$. Then $\gamma\beta e_2\eta = 0$. We know that $e_2\eta = d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$ with $d_i \in K$. So $\gamma\beta(d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta) = d_1\gamma\beta + d_2\gamma\beta\alpha + d_3\gamma\beta\alpha\gamma = 0$, that is, $d_1\gamma\beta + d_2\gamma^3 + d_3\gamma^4 = 0$ so $d_1 = d_2 = d_3 = 0$. Thus $e_2\eta = d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker} \partial^2 = \{d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta : d_4, d_5, d_6 \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = \alpha\beta e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$ so $u = \alpha\beta(d_4e_2 + d_5\alpha + d_6\alpha\beta)$. Hence $u \in \alpha\beta e_2\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \alpha\beta e_2\Lambda$.

On the other hand, let $v = \alpha\beta e_2\eta \in \alpha\beta e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha\beta e_2\eta) = \gamma\beta(\alpha\beta e_2\eta) = \gamma^3\beta e_2\eta = 0$. Therefore $\alpha\beta e_2\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \alpha\beta e_2\Lambda$. □

The map $\partial^3 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\eta \mapsto \alpha\beta e_2\eta$, for $\eta \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \gamma e_1\zeta + \beta e_2\eta,$$
$$\partial^2 : e_1\zeta \mapsto (\gamma, -\alpha)e_1\zeta,$$
$$\partial^3 : e_1\zeta \mapsto \gamma^4 e_1\zeta,$$

for $\zeta, \eta \in \Lambda$.

The maps for S_2 are:

$$\partial^1 : e_1\zeta \mapsto \alpha e_1\zeta,$$
$$\partial^2 : e_2\eta \mapsto \gamma\beta e_2\eta,$$
$$\partial^3 : e_2\eta \mapsto \alpha\beta e_2\eta,$$

for $\zeta, \eta \in \Lambda$.

9.3. g^3 for S_1 and S_2.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$$e_1 \xrightarrow{\partial^3} \gamma^4 \xrightarrow{\partial^2} (\gamma, -\alpha)\gamma^4 = (\gamma^5, -\alpha\gamma^4) \xrightarrow{\partial^1} \gamma^6 - \beta\alpha\gamma^4 \text{ , so } \gamma^6 - \beta\alpha\gamma^4 \in g^3.$$

For S_2

$$e_2 \xrightarrow{\partial^3} \alpha\beta \xrightarrow{\partial^2} \gamma\beta\alpha\beta \xrightarrow{\partial^1} \alpha\gamma\beta\alpha\beta \text{ , so } \alpha\gamma\beta\alpha\beta \in g^3.$$

Let $g_1^3 = \gamma^6 - \beta\alpha\gamma^4$ and $g_2^3 = \alpha\gamma\beta\alpha\beta$. So $g^3 = \{g_1^3, g_2^3\}$.

We know that $g^2 = \{\gamma^2 - \beta\alpha, \alpha\gamma\beta\}$. Denote

$$g_1^2 = \gamma^2 - \beta\alpha \text{ and}$$
$$g_2^2 = \alpha\gamma\beta.$$

So we have

$$\begin{aligned} g_1^3 &= \gamma^6 - \beta\alpha\gamma^4 = g_1^2\gamma^4 \\ &= \gamma g_1^2\gamma^3 + \gamma\beta g_2^2\alpha - \beta\alpha\gamma g_1^2\gamma - \beta g_2^2\alpha\gamma + \gamma\beta\alpha\gamma g_1^2 \text{ and} \\ g_2^3 &= \alpha\gamma\beta\alpha\beta = g_2^2\alpha\beta = \alpha g_1^2\gamma\beta - \alpha\gamma g_1^2\beta + \alpha\beta g_2^2. \end{aligned}$$

9.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\operatorname{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \to \operatorname{Hom}_{\Lambda^e}(P^0, \Lambda) \xrightarrow{\delta^0} \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda) \xrightarrow{\delta^1} \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda) \xrightarrow{\delta^2} \operatorname{Hom}_{\Lambda^e}(P^3, \Lambda) \to \ldots$$

9.4.1. $\operatorname{Ker}\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\operatorname{Ker}\delta^2$ and $\operatorname{Im}\delta^1$. Let $\theta \in \operatorname{Ker}\delta^2$; then $\theta \in \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned} e_1 \otimes_{g_1^2} e_1 &\mapsto j_1 e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4 \\ e_2 \otimes_{g_2^2} e_2 &\mapsto j_6 e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta, \end{aligned}$$

for some $j_i \in K$.

The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \gamma^4 - [\gamma \otimes_{g_1^2} \gamma^3 - \beta \otimes_{g_2^2} \alpha\gamma + \gamma\beta \otimes_{g_2^2} \alpha - \beta\alpha\gamma \otimes_{g_1^2} \gamma + \gamma\beta\alpha\gamma \otimes_{g_1^2} e_1]$

$e_2 \otimes_{g_2^3} e_2 \mapsto e_2 \otimes_{g_2^2} \alpha\beta - \alpha \otimes_{g_1^2} \gamma\beta + \alpha\gamma \otimes_{g_1^2} \beta - \alpha\beta \otimes_{g_2^2} e_2$.

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4)\gamma^4 - \gamma(j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4)\gamma^3 + \beta(j_6e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta)\alpha\gamma - \gamma\beta(j_6e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta)\alpha + \beta\alpha\gamma(j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4)\gamma - \gamma\beta\alpha\gamma(j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4) = 0$.

And $\theta d^3(e_2 \otimes_{g_2^3} e_2) = (j_6e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta)\alpha\beta - \alpha(j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4)\gamma\beta + \alpha\gamma(j_1e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4)\beta - \alpha\beta(j_6e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta) = 0$.

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$\begin{aligned} &P^2 \to \Lambda \\ &e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\gamma + j_3\gamma^2 + j_4\gamma^3 + j_5\gamma^4 \\ &e_2 \otimes_{g_2^2} e_2 \mapsto j_6 e_2 + j_7\alpha\beta + j_8\alpha\beta\alpha\beta, \end{aligned}$$

where $j_i \in K$. Hence $\dim\operatorname{Ker}\delta^2 = 8$ and therefore $\operatorname{Ker}\delta^2 = \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$.

9.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\gamma e_1 \to z_0e_1 + z_1\gamma + z_2\gamma^2 + z_3\gamma^3 + z_4\gamma^4$

$e_1 \otimes_\beta e_2 \to z_5\beta + z_6\gamma\beta + z_7\gamma^2\beta$

$e_2 \otimes_\alpha e_1 \to z_8\alpha + z_9\alpha\gamma + z_{10}\alpha\gamma^2$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\gamma \gamma + \gamma \otimes_\gamma e_1 - e_1 \otimes_\beta \alpha - \beta \otimes_\alpha e_1$

$e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \otimes_\alpha \gamma\beta + \alpha \otimes_\gamma \beta + \alpha\gamma \otimes_\beta e_2$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_0e_1 + z_1\gamma + z_2\gamma^2 + z_3\gamma^3 + z_4\gamma^4)\gamma + \gamma(z_0e_1 + z_1\gamma + z_2\gamma^2 + z_3\gamma^3 + z_4\gamma^4) - (z_5\beta + z_6\gamma\beta + z_7\gamma^2\beta)\alpha - \beta(z_8\alpha + z_9\alpha\gamma + z_{10}\alpha\gamma^2) = 2z_0\gamma + (2z_1 - z_5 - z_8)\gamma^2 + (2z_2 - z_6 - z_9)\gamma^3 + (2z_3 - z_7 - z_{10})\gamma^4$,

$\varphi d^2(e_2 \otimes_{g_2^2} e_2) = (z_8\alpha + z_9\alpha\gamma + z_{10}\alpha\gamma^2)\gamma\beta + \alpha(z_0e_1 + z_1\gamma + z_2\gamma^2 + z_3\gamma^3 + z_4\gamma^4)\beta + \alpha\gamma(z_5\beta + z_6\gamma\beta + z_7\gamma^2\beta) = z_0\alpha\beta + (z_2 + z_6 + z_9)\alpha\beta\alpha\beta$.

Therefore φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\gamma + (2z_1 - z_5 - z_8)\gamma^2 + (3z_2 - c)\gamma^3 + (2z_3 - z_7 - z_{10})\gamma^4 \\ e_2 \otimes_{g_2^2} e_2 \mapsto z_0\alpha\beta + c\alpha\beta\alpha\beta, \end{array}$$

where $c = z_2 + z_6 + z_9$ and $z_i \in K$. Note that the 2 in $\operatorname{Im}\delta^1$ occurs because of the power of γ in the relation $\gamma^2 - \beta\alpha$.

So if $\operatorname{char} K = 2$ then $\dim \operatorname{Im}\delta^1 = 5$ and φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_5 + z_8)\gamma^2 + (z_6 + z_9)\gamma^3 + (z_7 + z_{10})\gamma^4 \\ e_2 \otimes_{g_2^2} e_2 \mapsto z_0\alpha\beta + c\alpha\beta\alpha\beta, \end{array}$$

where $c = z_2 + z_6 + z_9$ and $z_i \in K$.

If $\operatorname{char} K = 3$ then $\dim \operatorname{Im}\delta^1 = 4$ and φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\gamma + (2z_1 - z_5 - z_8)\gamma^2 - c\gamma^3 + (2z_3 - z_7 - z_{10})\gamma^4 \\ e_2 \otimes_{g_2^2} e_2 \mapsto z_0\alpha\beta + c\alpha\beta\alpha\beta, \end{array}$$

where $c = z_2 + z_6 + z_9$ and $z_i \in K$.

Now if $\operatorname{char} K \neq 2, 3$ then $\dim \operatorname{Im}\delta^1 = 5$ and φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\gamma + (2z_1 - z_5 - z_8)\gamma^2 + (3z_2 - c)\gamma^3 + (2z_3 - z_7 - z_{10})\gamma^4 \\ e_2 \otimes_{g_2^2} e_2 \mapsto z_0\alpha\beta + c\alpha\beta\alpha\beta, \end{array}$$

where $c = z_2 + z_6 + z_9$ and $z_i \in K$.

9.4.3. $\mathrm{HH}^2(\Lambda)$.

From 9.4.1 and 9.4.2 we have that if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2\gamma \\ e_2 \otimes_{g_2^2} e_2 \mapsto d_3 e_2 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, u\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \gamma \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} u : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\ \text{else} \mapsto 0. \end{array}$$

If $\operatorname{char} K = 3$ then $\dim \mathrm{HH}^2(\Lambda) = 4$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2\gamma + d_3\gamma^3 \\ e_2 \otimes_{g_2^2} e_2 \mapsto d_4 e_2 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{y_1, y_2, y_3, y_4\}$ where

$$\begin{array}{l} y_1 : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y_2 : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \gamma \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y_3 : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \gamma^3 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y_4 : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\ \text{else} \mapsto 0. \end{array}$$

Note that y_2 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto -\alpha\beta \\ \text{else} \mapsto 0, \end{array}$$

and y_3 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto -\alpha\beta\alpha\beta \\ \text{else} \mapsto 0. \end{array}$$

If $\operatorname{char} K \neq 2, 3$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2 \gamma \\ e_2 \otimes_{g_2^2} e_2 \mapsto d_3 e_2 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, w\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \gamma \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} w : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\ \text{else} \mapsto 0. \end{array}$$

Note that y represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto -\alpha\beta \\ \text{else} \mapsto 0. \end{array}$$

10. The algebra A_6

Definition 10.1. [5] *Let A_6 be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\gamma"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and

$$I = \langle \alpha^3 - \gamma\beta, \beta\gamma, \beta\alpha^2, \alpha^2\gamma \rangle.$$

10.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda$ and $e_2\Lambda$ where

$$e_1\Lambda = sp\{e_1, \alpha, \gamma, \alpha\gamma, \alpha^2, \alpha^3, \alpha^4\},$$
$$e_2\Lambda = sp\{e_2, \beta, \beta\alpha, \beta\alpha\gamma\}.$$

We have $e_1\Lambda$

$$\begin{tikzcd}[arrows=dash]
 & & e_1 \arrow[dl] \arrow[drr] & & \\
 & \alpha \arrow[dl] \arrow[dr] & & & \gamma \arrow[ddl] \\
\alpha\gamma \arrow[ddrr] & & \alpha^2 \arrow[dr] & & \\
 & & & \alpha^3 \arrow[dl] & \\
 & & \alpha^4 & &
\end{tikzcd}$$

and for $e_2\Lambda$

$$\begin{tikzcd}[arrows=dash]
e_2 \arrow[d] \\
\beta \arrow[d] \\
\beta\alpha \arrow[d] \\
\beta\alpha\gamma
\end{tikzcd}$$

10.2. **The minimal projective resolutions of the simple Λ-modules S_1 and S_2.**

10.2.1. *The minimal projective resolution of the simple Λ-module S_1.*

Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\zeta, e_2\eta) \to \alpha e_1\zeta + \gamma e_2\eta$, for $\zeta, \eta \in \Lambda$.

Now we need to find $\operatorname{Ker} \partial^1$ and $\operatorname{Ker} \partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

10.2.2. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^1$. Then $\alpha e_1\zeta + \gamma e_2\eta = 0$. So $\alpha(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + \gamma(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = 0$, that is, $c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + c_4\alpha^3 + c_5\alpha^4 + d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha = 0$. Then $c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + (c_4 + d_2)\alpha^3 + (c_5 + d_3)\alpha^4 + d_1\gamma = 0$. So $c_1 = c_2 = c_3 = 0 = d_1, d_2 = -c_4$ and $d_3 = -c_5$. Thus $e_1\zeta = c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{(c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma)$, that is, $x = (\alpha^2, -\beta)(c_4e_1 + c_5\alpha + c_6\alpha^2 - d_4\alpha\gamma) + (\alpha\gamma, 0)(c_7e_2)$. Thus $x \in (\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda$.

On the other hand, let $y = (\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta \in (\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha^2e_1\zeta + \alpha\gamma e_2\eta, -\beta e_1\zeta) = \alpha(\alpha^2e_1\zeta + \alpha\gamma e_2\eta) + \gamma(-\beta e_1\zeta) = (\alpha^3 - \gamma\beta)e_1\zeta + \alpha^2\gamma e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha^2, -\beta)e_1\Lambda + (\alpha\gamma, 0)e_2\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by: $(e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta$, for $\zeta, \eta \in \Lambda$.

10.2.3. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. We assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^2$. Then $(\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta = (0, 0)$. So $(\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta = (\alpha^2, -\beta)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + (\alpha\gamma, 0)(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) + (d_1\alpha\gamma + d_2\alpha\gamma\beta, 0) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + d_1\alpha\gamma + d_2\alpha\gamma\beta, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0, 0)$. So $(c_1\alpha^2 + c_3\alpha^3 + (c_4 + d_2)\alpha^4 + d_1\alpha\gamma, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0, 0)$. Thus $c_1 = c_3 = c_7 = d_1 = 0$ and $d_2 = -c_4$. Hence $e_1\zeta = c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4$ and $e_2\eta = -c_4\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta + d_3\beta\alpha + d_4\beta\alpha\gamma)$ so $u = (\alpha^2, -\beta)(c_4e_1 + c_5\alpha + c_6\alpha^2 - d_3\alpha - d_4\alpha\gamma) + (\gamma, 0)(c_2e_2 + d_3\beta) + (0, \beta)(c_5\alpha)$. However we can show $(0, \beta)(c_5\alpha) \subseteq (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$, since $(0, \beta)(c_5\alpha) = (\alpha^2, -\beta)e_1\nu + (\gamma, 0)e_2\mu$ where $e_1\nu = -c_5\alpha$ and $e_2\mu = c_5\beta$. So $u \in (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

On the other hand, let $v = (\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta \in (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta) = \partial^2(\alpha^2e_1\zeta +$

$\gamma e_2\eta, -\beta e_1\zeta) = (\alpha^2, -\beta)(\alpha^2 e_1\zeta + \gamma e_2\eta) + (\alpha\gamma, 0)(-\beta e_1\zeta) = ((\alpha^4 + \alpha^2\gamma e_2\eta, -\beta\alpha^2 e_1\zeta - \beta\gamma e_2\eta) + (-\alpha\gamma\beta e_1\zeta, 0) = ((\alpha^4 - \alpha\gamma\beta)e_1\zeta + \alpha^2\gamma e_2\eta, -\beta\alpha^2 e_1\zeta - \beta\gamma e_2\eta) = (0, 0)$. Therefore $(\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha^2, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$. □

So the map $\partial^3 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta$, for $\zeta, \eta \in \Lambda$.

10.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.
Now the minimal projective resolution of the simple Λ-module S_2 starts by:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\zeta \mapsto \beta e_1\zeta$, for $\zeta \in \Lambda$.

Now we want to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ for S_2.

10.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker}\partial^2$. Then $\beta e_1\zeta = 0$. So $\beta e_1\zeta = \beta(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) = c_1\beta + c_3\beta\alpha + c_7\beta\alpha\gamma = 0$. Thus $c_1 = c_3 = c_7 = 0$. Hence $e_1\zeta = c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4$. Therefore $\operatorname{Ker}\partial^2 = \{c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha^2 e_1\Lambda + \gamma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4$. So $x = \alpha^2(c_4e_1 + c_5\alpha + c_6\alpha^2) + \gamma(c_2e_2)$. Thus $x \in \alpha^2 e_1\Lambda + \gamma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \alpha^2 e_1\Lambda + \gamma e_2\Lambda$.

On the other hand, let $y = \alpha^2 e_1\zeta + \gamma e_2\eta \in \alpha^2 e_1\Lambda + \gamma e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha^2 e_1\zeta + \gamma e_2\eta) = \beta(\alpha^2 e_1\zeta + \gamma e_2\eta) = \beta\alpha^2 e_1\zeta + \beta\gamma e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\alpha^2 e_1\Lambda + \gamma e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \alpha^2 e_1\Lambda + \gamma e_2\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by: $(e_1\zeta, e_2\eta) \mapsto \alpha^2 e_1\zeta + \gamma e_2\eta$, for $\zeta, \eta \in \Lambda$.

10.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^2$. Then $\alpha^2 e_1\zeta + \gamma e_2\eta = 0$. So $\alpha^2 e_1\zeta + \gamma e_2\eta = \alpha^2(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + \gamma(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha = 0$ so $c_1\alpha^2 + (c_3 + d_2)\alpha^3 + (c_4 + d_3)\alpha^4 + d_1\gamma = 0$. Thus $c_1 = 0 = d_1$, $d_2 = -c_3$ and $d_3 = -c_4$. Hence $e_1\zeta = c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = (c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma)$ so $u = (\alpha, -\beta)(c_3e_1 + c_4\alpha + c_5\alpha^2 + c_6\alpha^3 + c_7\gamma - d_4\alpha\gamma) + (\gamma, 0)(c_2e_2)$. Hence $u \in (\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq (\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

On the other hand, let $v = (\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta \in (\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta) = \partial^2(\alpha e_1\zeta + \gamma e_2\eta, -\beta e_1\zeta) = \alpha^2(\alpha e_1\zeta + \gamma e_2\eta) + \gamma(-\beta e_1\zeta) = (\alpha^3 - \gamma\beta)e_1\zeta + \alpha^2\gamma e_2\eta = 0$. Therefore $(\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = (\alpha, -\beta)e_1\Lambda + (\gamma, 0)e_2\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto (\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta$, for $\zeta, \eta \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \alpha e_1\zeta + \gamma e_2\eta,$$
$$\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta,$$
$$\partial^3 : (e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta,$$

for $\zeta, \eta \in \Lambda$.

The maps for S_2 are:

$$\partial^1 : e_1\zeta \mapsto \beta e_1\zeta,$$
$$\partial^2 : (e_1\zeta, e_2\eta) \mapsto \alpha^2 e_1\zeta + \gamma e_2\eta,$$
$$\partial^3 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta,$$

for $\zeta, \eta \in \Lambda$.

10.3. g^3 for S_1 and S_2.

Now we want to find the elements of g^3; these are paths in $K\mathcal{Q}$.

For S_1

$(e_1, 0) \xrightarrow{\partial^3} (\alpha^2, -\beta) \xrightarrow{\partial^2} (\alpha^2, -\beta)\alpha^2 + (\alpha\gamma, 0)(-\beta) = (\alpha^4, -\beta\alpha^2) + (-\alpha\gamma\beta, 0) = (\alpha^4 - \alpha\gamma\beta, -\beta\alpha^2) \xrightarrow{\partial^1} \alpha(\alpha^4 - \alpha\gamma\beta) + \gamma(-\beta\alpha^2) = \alpha^5 - \alpha^2\gamma\beta - \gamma\beta\alpha^2$, so $\alpha^5 - \alpha^2\gamma\beta - \gamma\beta\alpha^2 \in g^3$.

$(0, e_2) \xrightarrow{\partial^3} (\gamma, 0) \xrightarrow{\partial^2} (\alpha^2, -\beta)\gamma = (\alpha^2\gamma, -\beta\gamma) \xrightarrow{\partial^1} \alpha^3\gamma - \gamma\beta\gamma$, so $\alpha^3\gamma - \gamma\beta\gamma \in g^3$.

For S_2

$(e_1, 0) \xrightarrow{\partial^3} (\alpha, -\beta) \xrightarrow{\partial^2} \alpha^3 - \gamma\beta \xrightarrow{\partial^1} \beta\alpha^3 - \beta\gamma\beta$, so $\beta\alpha^3 - \beta\gamma\beta \in g^3$.

$(0, e_2) \xrightarrow{\partial^3} (\gamma, 0) \xrightarrow{\partial^2} \alpha^2\gamma \xrightarrow{\partial^1} \beta\alpha^2\gamma$, so $\beta\alpha^2\gamma \in g^3$.

Let $g_1^3 = \alpha^5 - \alpha^2\gamma\beta - \gamma\beta\alpha^2, g_2^3 = \alpha^3\gamma - \gamma\beta\gamma, g_3^3 = \beta\alpha^3 - \beta\gamma\beta$ and $g_4^3 = \beta\alpha^2\gamma$ So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3\}$.

We know that $g^2 = \{\alpha^3 - \gamma\beta, \beta\gamma, \beta\alpha^2, \alpha^2\gamma\}$. Denote

$g_1^2 = \alpha^3 - \gamma\beta$,
$g_2^2 = \beta\gamma$,
$g_3^2 = \beta\alpha^2$ and
$g_4^2 = \alpha^2\gamma$.

So we have

$$\begin{aligned}
g_1^3 &= \alpha^5 - \alpha^2\gamma\beta - \gamma\beta\alpha^2 = g_1^2\alpha^2 - g_4^2\beta = \alpha^2 g_1^2 - \gamma g_3^2,\\
g_2^3 &= \alpha^3\gamma - \gamma\beta\gamma = g_1^2\gamma = \alpha g_4^2 - \gamma g_2^2,\\
g_3^3 &= \beta\alpha^3 - \beta\gamma\beta = g_3^2\alpha - g_2^2\beta = \beta g_1^2,\\
g_4^3 &= \beta\alpha^2\gamma = g_3^2\gamma = \beta g_4^2.
\end{aligned}$$

10.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\,.$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots$$

10.4.1. $\operatorname{Ker}\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\operatorname{Ker}\delta^2$ and $\operatorname{Im}\delta^1$. Let $\theta \in \operatorname{Ker}\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
e_1 \otimes_{g_1^2} e_1 &\mapsto j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4\\
e_2 \otimes_{g_2^2} e_2 &\mapsto j_6e_2 + j_7\beta\alpha\gamma\\
e_2 \otimes_{g_3^2} e_1 &\mapsto j_8\beta + j_9\beta\alpha\\
e_1 \otimes_{g_4^2} e_2 &\mapsto j_{10}\gamma + j_{11}\alpha\gamma,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \alpha^2 - e_1 \otimes_{g_4^2} \beta - \alpha^2 \otimes_{g_1^2} e_1 + \gamma \otimes_{g_3^2} e_1$

$e_1 \otimes_{g_2^3} e_2 \mapsto e_1 \otimes_{g_1^2} \gamma - \alpha \otimes_{g_4^2} e_2 + \gamma \otimes_{g_2^2} e_2$

$e_2 \otimes_{g_3^3} e_1 \mapsto e_2 \otimes_{g_3^2} \alpha - e_2 \otimes_{g_2^2} \beta - \beta \otimes_{g_1^2} e_1$

$e_2 \otimes_{g_4^3} e_2 \mapsto e_2 \otimes_{g_3^2} \gamma - \beta \otimes_{g_4^2} e_2.$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4)\alpha^2 - (j_{10}\gamma + j_{11}\alpha\gamma)\beta - \alpha^2(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4) + \gamma(j_8\beta + j_9\beta\alpha) = (j_8 - j_{10})\alpha^3 + (j_9 - j_{11})\alpha^4 = 0$. So $j_{10} = j_8, j_{11} = j_9$.

For $\theta d^3(e_1 \otimes_{g_2^3} e_2) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4)\gamma - \alpha(j_{10}\gamma + j_{11}\alpha\gamma) + \gamma(j_6e_2 + j_7\beta\alpha\gamma) = (j_1 + j_6)\gamma + (j_2 - j_{10})\alpha\gamma = 0$. Hence $j_6 = -j_1$ and $j_{10} = j_2$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_1) = (j_8\beta + j_9\beta\alpha)\alpha - (j_6e_2 + j_7\beta\alpha\gamma)\beta - \beta(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4) = -(j_1 + j_6)\beta + (j_8 - j_2)\beta\alpha = 0$. Hence $j_6 = -j_1$ and $j_8 = j_2$.

And $\theta d^3(e_2 \otimes_{g_4^3} e_2) = (j_8\beta + j_9\beta\alpha)\gamma - \beta(j_{10}\gamma + j_{11}\alpha\gamma) = (j_9 - j_{11})\beta\alpha\gamma = 0$. So $j_{11} = j_9$.

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$\begin{aligned}
e_1 \otimes_{g_1^2} e_1 &\mapsto j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 + j_5\alpha^4\\
e_2 \otimes_{g_2^2} e_2 &\mapsto -j_1e_2 + j_7\beta\alpha\gamma\\
e_2 \otimes_{g_3^2} e_1 &\mapsto j_2\beta + j_9\beta\alpha\\
e_1 \otimes_{g_4^2} e_2 &\mapsto j_2\gamma + j_9\alpha\gamma,
\end{aligned}$$

where $j_i \in K$. Hence $\dim \operatorname{Ker}\delta^2 = 7$.

10.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_1 \to z_1 e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4$

$e_1 \otimes_\gamma e_2 \to z_6\gamma + z_7\alpha\gamma$

$e_2 \otimes_\beta e_1 \to z_8\beta + z_9\beta\alpha,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \alpha^2 + \alpha \otimes_\alpha \alpha + \alpha^2 \otimes_\alpha e_1 - e_1 \otimes_\gamma \beta - \gamma \otimes_\beta e_1$

$e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \otimes_\beta \gamma + \beta \otimes_\gamma e_2$

$e_2 \otimes_{g_3^2} e_1 \mapsto e_2 \otimes_\beta \alpha^2 + \beta \otimes_\alpha \alpha + \beta\alpha \otimes_\alpha e_1$

$e_1 \otimes_{g_4^2} e_2 \mapsto e_1 \otimes_\alpha \alpha\gamma + \alpha \otimes_\alpha \gamma + \alpha^2 \otimes_\gamma e_2.$

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4)\alpha^2 + \alpha(z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4)\alpha + \alpha^2(z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4) - (z_6\gamma + z_7\alpha\gamma)\beta - \gamma(z_8\beta + z_9\beta\alpha) = 3z_1\alpha^2 + (3z_2 - z_6 - z_8)\alpha^3 + (3z_3 - z_7 - z_9)\alpha^4,$

$\varphi d^2(e_2 \otimes_{g_2^2} e_2) = (z_8\beta + z_9\beta\alpha)\gamma + \beta(z_6\gamma + z_7\alpha\gamma) = (z_7 + z_9)\beta\alpha\gamma,$

$\varphi d^2(e_2 \otimes_{g_3^2} e_1) = (z_8\beta + z_9\beta\alpha)\alpha^2 + \beta(z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4)\alpha + \beta\alpha(z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4) = 2z_1\beta\alpha,$

$\varphi d^2(e_1 \otimes_{g_4^2} e_2) = (z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4)\alpha\gamma + \alpha(z_1e_1 + z_2\alpha + z_3\alpha^2 + z_4\alpha^3 + z_5\alpha^4)\gamma + \alpha^2(z_6\gamma + z_7\alpha\gamma) = 2z_1\alpha\gamma.$

So φd^2 is given by

$P^2 \to \Lambda$

$e_1 \otimes_{g_1^2} e_1 \mapsto 3z_1\alpha^2 + (3z_2 - z_6 - z_8)\alpha^3 + (3z_3 - z_7 - z_9)\alpha^4$

$e_2 \otimes_{g_2^2} e_2 \mapsto (z_7 + z_9)\beta\alpha\gamma$

$e_2 \otimes_{g_3^2} e_1 \mapsto 2z_1\beta\alpha$

$e_1 \otimes_{g_4^2} e_2 \mapsto 2z_1\alpha\gamma,$

where $z_i \in K$. Note that the 3 in $\operatorname{Im}\delta^1$ occurs because of the power of α in the relation $\alpha^3 - \gamma\beta$. Thus we need to consider three cases, namely $\operatorname{char} K = 2, \operatorname{char} K = 3, \operatorname{char} K \neq 2, 3$.

If $\operatorname{char} K = 2$ then $\dim \operatorname{Im}\delta^1 = 4$ and φd^2 is given by

$P^2 \to \Lambda$

$e_1 \otimes_{g_1^2} e_1 \mapsto z_1\alpha^2 + (z_2 + z_6 + z_8)\alpha^3 + (z_3 + z_7 + z_9)\alpha^4$

$e_2 \otimes_{g_2^2} e_2 \mapsto (z_7 + z_9)\beta\alpha\gamma$

$e_2 \otimes_{g_3^2} e_1 \mapsto 0$

$e_1 \otimes_{g_4^2} e_2 \mapsto 0,$

where $z_i \in K$.

If $\operatorname{char} K = 3$ then $\dim \operatorname{Im} \delta^1 = 3$ and φd^2 is given by

$$\begin{aligned}&P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto -(z_6+z_8)\alpha^3 - (z_7+z_9)\alpha^4\\&e_2 \otimes_{g_2^2} e_2 \mapsto (z_7+z_9)\beta\alpha\gamma\\&e_2 \otimes_{g_3^2} e_1 \mapsto 2z_1\beta\alpha\\&e_1 \otimes_{g_4^2} e_2 \mapsto 2z_1\alpha\gamma,\end{aligned}$$

where $z_i \in K$.

Finally, if $\operatorname{char} K \neq 2, 3$ then $\dim \operatorname{Im} \delta^1 = 4$ and φd^2 is given by

$$\begin{aligned}&P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto 3z_1\alpha^2 + (3z_2 - z_6 - z_8)\alpha^3 + (3z_3 - z_7 - z_9)\alpha^4\\&e_2 \otimes_{g_2^2} e_2 \mapsto (z_7+z_9)\beta\alpha\gamma\\&e_2 \otimes_{g_3^2} e_1 \mapsto 2z_1\beta\alpha\\&e_1 \otimes_{g_4^2} e_2 \mapsto 2z_1\alpha\gamma,\end{aligned}$$

where $z_i \in K$.

10.4.3. $\mathrm{HH}^2(\Lambda)$.

If $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and

$$\mathrm{HH}^2(\Lambda) = \left\{\begin{aligned}&P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha\\&e_2 \otimes_{g_2^2} e_2 \mapsto -d_1e_2\\&e_2 \otimes_{g_3^2} e_1 \mapsto d_2\beta + d_3\beta\alpha\\&e_1 \otimes_{g_4^2} e_2 \mapsto d_2\gamma + d_3\alpha\gamma\end{aligned}\right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{aligned}&x : P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto e_1\\&e_2 \otimes_{g_2^2} e_2 \mapsto -e_2\\&\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&y : P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto \alpha\\&e_2 \otimes_{g_3^2} e_1 \mapsto \beta\\&e_1 \otimes_{g_4^2} e_2 \mapsto \gamma\\&\text{else} \mapsto 0,\end{aligned}$$

$$\begin{aligned}&z : P^2 \to \Lambda\\&e_2 \otimes_{g_3^2} e_1 \mapsto \beta\alpha\\&e_1 \otimes_{g_4^2} e_2 \mapsto \alpha\gamma\\&\text{else} \mapsto 0.\end{aligned}$$

If $\operatorname{char} K = 3$ then $\dim \mathrm{HH}^2(\Lambda) = 4$ and

$$\mathrm{HH}^2(\Lambda) = \left\{\begin{aligned}&P^2 \to \Lambda\\&e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha + d_3\alpha^2 + d_4\alpha^4\\&e_2 \otimes_{g_2^2} e_2 \mapsto -d_1e_2\\&e_2 \otimes_{g_3^2} e_1 \mapsto d_2\beta\\&e_1 \otimes_{g_4^2} e_2 \mapsto d_2\gamma\end{aligned}\right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x_0, x_1, x_2, x_3\}$ where

$$\begin{aligned}
&x_0 : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -e_2 \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&x_1 : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\
&e_2 \otimes_{g_3^2} e_1 \mapsto \beta \\
&e_1 \otimes_{g_4^2} e_2 \mapsto \gamma \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&x_2 : P^2 \to \Lambda \\
&e_2 \otimes_{g_1^2} e_1 \mapsto \alpha^2 \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&x_3 : P^2 \to \Lambda \\
&e_2 \otimes_{g_1^2} e_1 \mapsto \alpha^4 \\
&\text{else} \mapsto 0.
\end{aligned}$$

Note that x_3 represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -\beta\alpha\gamma \\
&\text{else} \mapsto 0.
\end{aligned}$$

Finally, if $\operatorname{char} K \neq 2, 3$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2\alpha + d_3\alpha^2 \\ e_2 \otimes_{g_2^2} e_2 \mapsto -d_1 e_2 \\ e_2 \otimes_{g_3^2} e_1 \mapsto d_2\beta \\ e_1 \otimes_{g_4^2} e_2 \mapsto d_2\gamma \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{aligned}
&x : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -e_2 \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&y : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\
&e_2 \otimes_{g_3^2} e_1 \mapsto \beta \\
&e_1 \otimes_{g_4^2} e_2 \mapsto \gamma \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&z : P^2 \to \Lambda \\
&e_2 \otimes_{g_1^2} e_1 \mapsto \alpha^2 \\
&\text{else} \mapsto 0.
\end{aligned}$$

Note that z represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$
\begin{aligned}
&z : P^2 \to \Lambda \\
&e_2 \otimes_{g_3^2} e_1 \mapsto -\tfrac{3}{2}\beta\alpha \\
&e_1 \otimes_{g_4^2} e_2 \mapsto -\tfrac{3}{2}\alpha\gamma \\
&\text{else} \mapsto 0.
\end{aligned}
$$

11. The algebra A_7

Definition 11.1. [5] *Let A_7 be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\delta"] & 3 \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\varepsilon"] & 4 \arrow[l, shift left, "\zeta"] \end{tikzcd}$$

and

$$I = \langle \beta\alpha - \delta\gamma, \gamma\delta - \varepsilon\zeta, \alpha\delta\varepsilon, \zeta\gamma\beta \rangle.$$

11.1. The structure of the indecomposable projectives.

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$$\begin{aligned} e_1\Lambda &= sp\{e_1, \alpha, \alpha\beta, \alpha\delta, \alpha\beta\alpha, \alpha\beta\alpha\beta\}, \\ e_2\Lambda &= sp\{e_2, \beta, \delta, \beta\alpha, \delta\varepsilon, \beta\alpha\beta, \delta\varepsilon\zeta, \beta\alpha\beta\alpha\}, \\ e_3\Lambda &= sp\{e_3, \gamma, \varepsilon, \gamma\beta, \varepsilon\zeta, \gamma\beta\alpha, \varepsilon\zeta\varepsilon, \varepsilon\zeta\varepsilon\zeta\}, \\ e_4\Lambda &= sp\{e_4, \zeta, \zeta\gamma, \zeta\varepsilon, \zeta\varepsilon\zeta, \zeta\varepsilon\zeta\varepsilon\}. \end{aligned}$$

So we have for $e_1\Lambda$

$$\begin{tikzcd} & e_1 \arrow[d, dash] & \\ & \alpha \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha\beta \arrow[dr, dash] & & \alpha\delta \arrow[dl, dash] \\ & \alpha\beta\alpha \arrow[d, dash] & \\ & \alpha\beta\alpha\beta & \end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd} & e_2 \arrow[dl, dash] \arrow[dr, dash] & & \\ \beta \arrow[dr, dash] & & \delta \arrow[dl, dash] \arrow[dr, dash] & \\ & \beta\alpha \arrow[dl, dash] \arrow[dr, dash] & & \delta\varepsilon \arrow[dl, dash] \\ \beta\alpha\beta \arrow[dr, dash] & & \delta\varepsilon\zeta \arrow[dl, dash] & \\ & \beta\alpha\beta\alpha & & \end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{tikzcd}
 & & e_3 \arrow[dl, dash] \arrow[dr, dash] & \\
 & \gamma \arrow[dl, dash] \arrow[dr, dash] & & \varepsilon \arrow[dl, dash] \\
\gamma\beta \arrow[dr, dash] & & \varepsilon\zeta \arrow[dl, dash] \arrow[dr, dash] & \\
 & \gamma\beta\alpha \arrow[dr, dash] & & \varepsilon\zeta\varepsilon \arrow[dl, dash] \\
 & & \varepsilon\zeta\varepsilon\zeta &
\end{tikzcd}$$

And for $e_4\Lambda$

$$\begin{tikzcd}
 & e_4 \arrow[d, dash] & \\
 & \zeta \arrow[dl, dash] \arrow[dr, dash] & \\
\zeta\gamma \arrow[dr, dash] & & \zeta\varepsilon \arrow[dl, dash] \\
 & \zeta\varepsilon\zeta \arrow[d, dash] & \\
 & \zeta\varepsilon\zeta\varepsilon &
\end{tikzcd}$$

11.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

11.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \alpha e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

11.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\alpha e_2\nu = 0$, so $\alpha(d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha) = 0$, that is, $d_1\alpha + d_2\alpha\beta + d_3\alpha\delta + d_4\alpha\beta\alpha + d_6\alpha\beta\alpha\beta = 0$ and then $d_1 = d_2 = d_3 = d_4 = d_6 = 0$. Thus $e_2\nu = d_5\delta\varepsilon + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$.

Hence $\operatorname{Ker}\partial^1 = \{d_5\delta\varepsilon + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \delta\varepsilon e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = d_5\delta\varepsilon + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$, that is, $x = \delta\varepsilon(d_5 e_4 + d_7\zeta + d_8\zeta\gamma)$. Thus $x \in \delta\varepsilon e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \delta\varepsilon e_4\Lambda$.

On the other hand, let $y = \delta\varepsilon e_4\mu \in \delta\varepsilon e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \alpha(\delta\varepsilon e_4\mu) = \alpha\delta\varepsilon e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\delta\varepsilon e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \delta\varepsilon e_4\Lambda$. □

So $\partial^2 : e_4\Lambda \to e_2\Lambda$ is given by $e_4\mu \mapsto \delta\varepsilon e_4\mu$, for $\mu \in \Lambda$.

11.2.3. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_4\mu = t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$. Assume that $e_4\mu \in \operatorname{Ker}\partial^2$. Then $\delta\varepsilon e_4\mu = 0$. So $\delta\varepsilon e_4\mu = \delta\varepsilon(t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon) = t_1\delta\varepsilon + t_2\delta\varepsilon\zeta + t_3\delta\varepsilon\zeta\gamma = 0$ so $t_1 = t_2 = t_3 = 0$. Thus $e_4\mu = t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$ and therefore $\operatorname{Ker}\partial^2 = \{t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \zeta\varepsilon e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$ so $u = \zeta\varepsilon(t_4e_4 + t_5\zeta + t_6\zeta\varepsilon)$. Hence $u \in \zeta\varepsilon e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \zeta\varepsilon e_4\Lambda$.

On the other hand, let $v = \zeta\varepsilon e_4\mu \in \zeta\varepsilon e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\zeta\varepsilon e_4\mu) = \delta\varepsilon\zeta\varepsilon e_4\mu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\zeta\varepsilon e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \zeta\varepsilon e_4\Lambda$. □

So the map $\partial^3 : e_4\Lambda \to e_4\Lambda$ is given by $e_4\mu \mapsto \zeta\varepsilon e_4\mu$, for $\mu \in \Lambda$.

11.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \to \beta e_1\eta + \delta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

11.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ and $e_3\lambda = f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\beta e_1\eta + \delta e_3\lambda = 0$ so $\beta(c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta) + \delta(f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta) = 0$, that is, $c_1\beta + c_2\beta\alpha + c_3\beta\alpha\beta + c_4\beta\alpha\delta + c_5\beta\alpha\beta\alpha + f_1\delta + f_2\delta\gamma + f_3\delta\varepsilon + f_4\delta\gamma\beta + f_5\delta\varepsilon\zeta + f_6\delta\gamma\beta\alpha = c_1\beta + (c_2 + f_2)\beta\alpha + (c_3 + f_4)\beta\alpha\beta + (c_4 + f_5)\delta\varepsilon\zeta + (c_5 + f_6)\beta\alpha\beta\alpha + f_1\delta + f_3\delta\varepsilon = 0$. So $c_1 = f_1 = f_3 = 0, c_2 + f_2 = 0, c_3 + f_4 = 0, c_4 + f_5 = 0$ and $c_5 + f_6 = 0$. So $f_2 = -c_2, f_4 = -c_3, f_5 = -c_4$ and $f_6 = -c_5$. Thus $e_2\eta = c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ and $e_3\lambda = -c_2\gamma - c_3\gamma\beta - c_4\varepsilon\zeta - c_5\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha+c_3\alpha\beta+c_4\alpha\delta+c_5\alpha\beta\alpha+c_6\alpha\beta\alpha\beta, -c_2\gamma-c_3\gamma\beta-c_4\varepsilon\zeta-c_5\gamma\beta\alpha+f_7\varepsilon\zeta\varepsilon+f_8\varepsilon\zeta\varepsilon\zeta) : c_i, f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, -\gamma)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta, -c_2\gamma - c_3\gamma\beta - c_4\varepsilon\zeta - c_5\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta)$. So $x = (\alpha, -\gamma)(c_2e_2 + c_3\beta + c_4\delta + c_5\beta\alpha + c_6\beta\alpha\beta - f_7\delta\varepsilon - f_8\delta\varepsilon\zeta)$. Thus $x \in (\alpha, -\gamma)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\gamma)e_2\Lambda$.

On the other hand, let $y = (\alpha, -\gamma)e_2\nu \in (\alpha, -\gamma)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = (\beta\alpha - \delta\gamma)e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, -\gamma)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\gamma)e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_2\nu \mapsto (\alpha, -\gamma)e_2\nu$, for $\nu \in \Lambda$.

11.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta+d_7\delta\varepsilon\zeta+d_8\beta\alpha\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^2$. Then $(\alpha, -\gamma)e_2\nu = 0$. So $(\alpha, -\gamma)e_2\nu = (\alpha, -\gamma)(d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha) = (d_1\alpha+d_2\alpha\beta+d_3\alpha\delta+d_4\alpha\beta\alpha+d_6\alpha\beta\alpha\beta, -d_1\gamma-d_2\gamma\beta-d_3\gamma\delta-d_4\gamma\beta\alpha-d_5\gamma\delta\varepsilon-d_7\gamma\delta\varepsilon\zeta) = (0, 0)$. So $d_1\alpha+d_2\alpha\beta+d_3\alpha\delta+d_4\alpha\beta\alpha+d_6\alpha\beta\alpha\beta = 0$ and then $d_1 = d_2 = d_3 = d_4 = d_6 = 0$. Also $-d_1\gamma - d_2\gamma\beta - d_3\gamma\delta - d_4\gamma\beta\alpha - d_5\gamma\delta\varepsilon - d_7\gamma\delta\varepsilon\zeta = 0$, that is, $d_1 = d_2 = d_3 = d_4 = d_5 = d_7 = 0$. Thus $e_2\nu = d_8\beta\alpha\beta\alpha$ and therefore $\operatorname{Ker}\partial^2 = \{d_8\beta\alpha\beta\alpha : d_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\alpha\beta\alpha e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = d_8\beta\alpha\beta\alpha$ so $u = \beta\alpha\beta\alpha(d_8e_2)$. Hence $u \in \beta\alpha\beta\alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\alpha\beta\alpha e_2\Lambda$.

On the other hand, let $v = \beta\alpha\beta\alpha e_2\nu \in \beta\alpha\beta\alpha e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\alpha\beta\alpha e_2\nu) = (\alpha\beta\alpha\beta\alpha, -\gamma\beta\alpha\beta\alpha)e_2\nu = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta\alpha\beta\alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\alpha\beta\alpha e_2\Lambda$. □

Note that $\operatorname{Ker}\partial^2 \cong S_2$, so $\Omega^3(S_2) \cong S_2$.

So the map $\partial^3 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\nu \mapsto \beta\alpha\beta\alpha e_2\nu$, for $\nu \in \Lambda$.

11.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3.

The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \oplus e_4\Lambda \to e_3\Lambda$ is given by $(e_2\nu, e_4\mu) \mapsto \gamma e_2\nu + \varepsilon e_4\mu$, for $\nu, \mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

11.2.8. Ker ∂^1.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$ and $e_4\mu = t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$ with $d_i, t_i \in K$. Assume that $(e_2\nu, e_4\mu) \in \operatorname{Ker}\partial^1$. Then $\gamma e_2\nu + \varepsilon e_4\mu = 0$ so $\gamma(d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha) + \varepsilon(t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon) = 0$, that is, $d_1\gamma + d_2\gamma\beta + d_3\gamma\delta + d_4\gamma\beta\alpha + d_5\gamma\delta\varepsilon + d_7\gamma\delta\varepsilon\zeta + t_1\varepsilon + t_2\varepsilon\zeta + t_3\varepsilon\zeta\gamma + t_4\varepsilon\zeta\varepsilon + t_5\varepsilon\zeta\varepsilon\zeta = d_1\gamma + d_2\gamma\beta + (d_3 + t_2)\varepsilon\zeta + (d_4 + t_3)\varepsilon\zeta\gamma + (d_5 + t_4)\varepsilon\zeta\varepsilon + (d_7 + t_5)\varepsilon\zeta\varepsilon\zeta + t_1\varepsilon = 0$. So $d_1 = d_2 = t_1 = 0, t_2 = -d_3, t_3 = -d_4, t_4 = -d_5$ and $t_5 = -d_7$. Thus $e_2\nu = d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$ and $e_4\mu = -d_3\zeta - d_4\zeta\gamma - d_5\zeta\varepsilon - d_7\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$.

Hence $\operatorname{Ker}\partial^1 = \{(d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha, -d_3\zeta - d_4\zeta\gamma - d_5\zeta\varepsilon - d_7\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon) : d_i, t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\delta, -\zeta)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha, -d_3\zeta - d_4\zeta\gamma - d_5\zeta\varepsilon - d_7\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon)$. So $x = (\delta, -\zeta)(d_3e_3 + d_4\gamma + d_5\varepsilon + d_6\gamma\beta + d_7\varepsilon\zeta + d_8\gamma\beta\alpha - t_6\varepsilon\zeta\varepsilon)$. Thus $x \in (\delta, -\zeta)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\delta, -\zeta)e_3\Lambda$.

On the other hand, let $y = (\delta, -\zeta)e_3\lambda \in (\delta, -\zeta)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = (\gamma\delta - \varepsilon\zeta)e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\delta, -\zeta)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\delta, -\zeta)e_3\Lambda$. □

So $\partial^2 : e_3\Lambda \to e_2\Lambda \oplus e_4\Lambda$ is given by: $e_3\lambda \mapsto (\delta, -\zeta)e_3\lambda$, for $\lambda \in \Lambda$.

11.2.9. Ker ∂^2.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_3\lambda = f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^2$. Then $(\delta, -\zeta)e_3\lambda = 0$. So $(\delta, -\zeta)e_3\lambda = (\delta, -\zeta)(f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta) = (f_1\delta + f_2\delta\gamma + f_3\delta\varepsilon + f_4\delta\gamma\beta + f_5\delta\varepsilon\zeta + f_6\delta\gamma\beta\alpha, -f_1\zeta - f_2\zeta\gamma - f_3\zeta\varepsilon - f_5\zeta\varepsilon\zeta - f_7\zeta\varepsilon\zeta\varepsilon) = (0, 0)$. So $f_1\delta + f_2\delta\gamma + f_3\delta\varepsilon + f_4\delta\gamma\beta + f_5\delta\varepsilon\zeta + f_6\delta\gamma\beta\alpha = 0$, that is, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$. Also $-f_1\zeta - f_2\zeta\gamma - f_3\zeta\varepsilon - f_5\zeta\varepsilon\zeta - f_7\zeta\varepsilon\zeta\varepsilon = 0$ and then $f_1 = f_2 = f_3 = f_5 = f_7 = 0$. Thus $e_3\lambda = f_8\varepsilon\zeta\varepsilon\zeta$ and therefore $\operatorname{Ker}\partial^2 = \{f_8\varepsilon\zeta\varepsilon\zeta : f_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \varepsilon\zeta\varepsilon\zeta e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_8\varepsilon\zeta\varepsilon\zeta$, that is, $u = \varepsilon\zeta\varepsilon\zeta(f_8e_3)$. Hence $u \in \varepsilon\zeta\varepsilon\zeta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \varepsilon\zeta\varepsilon\zeta e_3\Lambda$.

On the other hand, let $v = \varepsilon\zeta\varepsilon\zeta e_3\lambda \in \varepsilon\zeta\varepsilon\zeta e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\varepsilon\zeta\varepsilon\zeta e_3\lambda) = (\delta, -\zeta)\varepsilon\zeta\varepsilon\zeta e_3\lambda = (\delta\varepsilon\zeta\varepsilon\zeta, -\zeta\varepsilon\zeta\varepsilon\zeta)$
$e_3\lambda = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\varepsilon\zeta\varepsilon\zeta e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \varepsilon\zeta\varepsilon\zeta e_3\Lambda$. □

We remark that $\operatorname{Ker}\partial^2 \cong S_3$, so $\Omega^3(S_3) \cong S_3$.

So the map $\partial^3 : e_3\Lambda \to e_3\Lambda$ is given by $e_3\lambda \mapsto \varepsilon\zeta\varepsilon\zeta e_3\lambda$, for $\lambda \in \Lambda$.

11.2.10. *The minimal projective resolution of the simple* Λ*-module* S_4.
The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_4\Lambda$ is given by $e_3\lambda \mapsto \zeta e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

11.2.11. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$. Let $e_3\lambda = f_1e_3+f_2\gamma+f_3\varepsilon+f_4\gamma\beta+f_5\varepsilon\zeta+f_6\gamma\beta\alpha+f_7\varepsilon\zeta\varepsilon+f_8\varepsilon\zeta\varepsilon\zeta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\zeta e_3\lambda = 0$ so $\zeta(f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta) = 0$, that is, $f_1\zeta + f_2\zeta\gamma + f_3\zeta\varepsilon + f_5\zeta\varepsilon\zeta + f_7\zeta\varepsilon\zeta\varepsilon = 0$. Therefore $f_1 = f_2 = f_3 = f_5 = f_7 = 0$. Thus $e_3\lambda = f_4\gamma\beta + f_6\gamma\beta\alpha + f_8\varepsilon\zeta\varepsilon\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{f_4\gamma\beta + f_6\gamma\beta\alpha + f_8\varepsilon\zeta\varepsilon\zeta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \gamma\beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = f_4\gamma\beta + f_6\gamma\beta\alpha + f_8\varepsilon\zeta\varepsilon\zeta$, that is, $x = \gamma\beta(f_4e_1 + f_6\alpha + f_8\alpha\delta)$. Thus $x \in \gamma\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \gamma\beta e_1\Lambda$.

On the other hand, let $y = \gamma\beta e_1\eta \in \gamma\beta e_1\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\gamma\beta e_1\eta) = \zeta\gamma\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\gamma\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \gamma\beta e_1\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \to e_3\Lambda$ is given by $e_1\eta \mapsto \gamma\beta e_1\eta$, for $\eta \in \Lambda$.

11.2.12. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_4)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^2$. Then $\gamma\beta e_1\eta = 0$. So $\gamma\beta e_1\eta = \gamma\beta(c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta) = 0$. Then $c_1\gamma\beta + c_2\gamma\beta\alpha + c_4\gamma\beta\alpha\delta = 0$, that is, $c_1 = c_2 = c_4 = 0$. Thus $e_1\eta = c_3\alpha\beta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ and therefore $\operatorname{Ker}\partial^2 = \{c_3\alpha\beta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha\beta e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = c_3\alpha\beta+c_5\alpha\beta\alpha+c_6\alpha\beta\alpha\beta$ so $u = \alpha\beta(c_3e_1+c_5\alpha+c_6\alpha\beta)$. Hence $u \in \alpha\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \alpha\beta e_1\Lambda$.

On the other hand, let $v = \alpha\beta e_1\eta \in \alpha\beta e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha\beta e_1\eta) = \gamma\beta\alpha\beta e_1\eta = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\alpha\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \alpha\beta e_1\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \alpha\beta e_1\eta$, for $\eta \in \Lambda$.

Thus the maps for S_1 are:

$\partial^1 : e_2\nu \mapsto \alpha e_2\nu$,

$$\partial^2 : e_4\mu \mapsto \delta\varepsilon e_4\mu,$$
$$\partial^3 : e_4\mu \mapsto \zeta\varepsilon e_4\mu,$$

for $\nu, \mu \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : (e_1\eta, e_3\lambda) \mapsto \beta e_1\eta + \delta e_3\lambda,$$
$$\partial^2 : e_2\nu \mapsto (\alpha, -\gamma)e_2\nu,$$
$$\partial^3 : e_2\nu \mapsto \beta\alpha\beta\alpha e_2\nu,$$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\partial^1 : (e_2\nu, e_4\mu) \mapsto \gamma e_2\nu + \varepsilon e_4\mu,$$
$$\partial^2 : e_3\lambda \mapsto (\delta, -\zeta)e_3\lambda,$$
$$\partial^3 : e_3\lambda \mapsto \varepsilon\zeta\varepsilon\zeta e_3\lambda,$$

for $\eta, \mu, \lambda \in \Lambda$.

Moreover, the maps for S_4 are:

$$\partial^1 : e_3\lambda \mapsto \zeta e_3\lambda,$$
$$\partial^2 : e_1\eta \mapsto \gamma\beta e_1\eta,$$
$$\partial^3 : e_1\eta \mapsto \alpha\beta e_1\eta,$$

for $\eta, \lambda \in \Lambda$.

11.3. g^3 for S_1, S_2, S_3 and S_4.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_4 \xrightarrow{\partial^3} \zeta\varepsilon \xrightarrow{\partial^2} \delta\varepsilon\zeta\varepsilon \xrightarrow{\partial^1} \alpha\delta\varepsilon\zeta\varepsilon$, so $\alpha\delta\varepsilon\zeta\varepsilon \in g^3$.

For S_2

$e_2 \xrightarrow{\partial^3} \beta\alpha\beta\alpha \xrightarrow{\partial^2} (\alpha, -\gamma)\beta\alpha\beta\alpha \xrightarrow{\partial^1} \beta\alpha\beta\alpha\beta\alpha - \delta\gamma\beta\alpha\beta\alpha$,
so $\beta\alpha\beta\alpha\beta\alpha - \delta\gamma\beta\alpha\beta\alpha \in g^3$.

For S_3

$e_3 \xrightarrow{\partial^3} \varepsilon\zeta\varepsilon\zeta \xrightarrow{\partial^2} (\delta, -\zeta)\varepsilon\zeta\varepsilon\zeta \xrightarrow{\partial^1} \gamma\delta\varepsilon\zeta\varepsilon\zeta - \varepsilon\zeta\varepsilon\zeta\varepsilon\zeta$, so $\gamma\delta\varepsilon\zeta\varepsilon\zeta - \varepsilon\zeta\varepsilon\zeta\varepsilon\zeta \in g^3$.

For S_4

$e_1 \xrightarrow{\partial^3} \alpha\beta \xrightarrow{\partial^2} \gamma\beta\alpha\beta \xrightarrow{\partial^1} \zeta\gamma\beta\alpha\beta$, so $\zeta\gamma\beta\alpha\beta \in g^3$.

Let $g_1^3 = \alpha\delta\varepsilon\zeta\varepsilon$, $g_2^3 = \beta\alpha\beta\alpha\beta\alpha - \delta\gamma\beta\alpha\beta\alpha$, $g_3^3 = \gamma\delta\varepsilon\zeta\varepsilon\zeta - \varepsilon\zeta\varepsilon\zeta\varepsilon\zeta$, and $g_4^3 = \zeta\gamma\beta\alpha\beta$.

So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3\}$.

We know that $g^2 = \{\beta\alpha - \delta\gamma, \gamma\delta - \varepsilon\zeta, \alpha\delta\varepsilon, \zeta\gamma\beta\}$. Denote

$$g_1^2 = \beta\alpha - \delta\gamma,$$
$$g_2^2 = \gamma\delta - \varepsilon\zeta,$$
$g_3^2 = \alpha\delta\varepsilon$ and
$$g_4^2 = \zeta\gamma\beta.$$

So we have

$$g_1^3 = g_3^2\zeta\varepsilon = -\alpha g_1^2\delta\varepsilon - \alpha\delta g_2^2\varepsilon + \alpha\beta g_3^2,$$
$$g_2^3 = g_1^2\beta\alpha\beta\alpha = \beta g_3^2\zeta\gamma - \delta g_2^2\gamma\beta\alpha - \delta\gamma g_1^2\beta\alpha - \delta\varepsilon g_4^2\alpha + \beta\alpha g_1^2\delta\gamma + \beta\alpha\delta g_2^2\gamma + \beta\alpha\beta\alpha g_1^2,$$
$$g_3^3 = g_2^2\varepsilon\zeta\varepsilon\zeta = -\gamma g_1^2\delta\varepsilon\zeta - \varepsilon g_4^2\alpha\delta - \gamma\delta g_2^2\varepsilon\zeta + \gamma\beta g_3^2\zeta + \varepsilon\zeta g_2^2\gamma\delta + \varepsilon\zeta\gamma g_1^2\delta + \varepsilon\zeta\varepsilon\zeta g_2^2 \text{ and}$$
$$g_4^3 = g_4^2\alpha\beta = \zeta g_2^2\gamma\beta + \zeta\gamma g_1^2\beta + \zeta\varepsilon g_4^2.$$

11.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\,.$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots$$

11.4.1. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_2 \otimes_{g_1^2} e_2 \mapsto j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha\\
&e_3 \otimes_{g_2^2} e_3 \mapsto j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta\\
&e_1 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_4 \otimes_{g_4^2} e_1 \mapsto 0,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$e_1 \otimes_{g_1^3} e_4 \mapsto e_1 \otimes_{g_3^2} \zeta\varepsilon - [-\alpha \otimes_{g_1^2} \delta\varepsilon - \alpha\delta \otimes_{g_2^2} \varepsilon + \alpha\beta \otimes_{g_3^2} e_4],$$
$$e_2 \otimes_{g_2^3} e_2 \mapsto e_2 \otimes_{g_1^2} \beta\alpha\beta\alpha - [\beta \otimes_{g_3^2} \zeta\gamma - \delta \otimes_{g_2^2} \gamma\beta\alpha - \delta\gamma \otimes_{g_1^2} \beta\alpha - \delta\varepsilon \otimes_{g_4^2} \alpha + \beta\alpha \otimes_{g_1^2} \delta\gamma + \beta\alpha\delta \otimes_{g_2^2} \gamma + \beta\alpha\beta\alpha \otimes_{g_1^2} e_2],$$
$$e_3 \otimes_{g_3^3} e_3 \mapsto e_3 \otimes_{g_2^2} \varepsilon\zeta\varepsilon\zeta - [-\gamma \otimes_{g_1^2} \delta\varepsilon\zeta - \varepsilon \otimes_{g_4^2} \alpha\delta - \gamma\delta \otimes_{g_2^2} \varepsilon\zeta + \gamma\beta \otimes_{g_3^2} \zeta + \varepsilon\zeta \otimes_{g_2^2} \gamma\delta + \varepsilon\zeta\gamma \otimes_{g_1^2} \delta + \varepsilon\zeta\varepsilon\zeta \otimes_{g_2^2} e_3],$$
$$e_4 \otimes_{g_4^3} e_1 \mapsto e_4 \otimes_{g_4^2} \alpha\beta - [\zeta \otimes_{g_2^2} \gamma\beta + \zeta\gamma \otimes_{g_1^2} \beta + \zeta\varepsilon \otimes_{g_4^2} e_1].$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_4) = \alpha(j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)\delta\varepsilon + \alpha\delta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\varepsilon = 0$

for $\theta d^3(e_2 \otimes_{g_2^3} e_2) = (j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)\beta\alpha\beta\alpha - [-\delta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\gamma\beta\alpha - \delta\gamma(j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)\beta\alpha + \beta\alpha(j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)\delta\gamma + \beta\alpha\delta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\gamma + \beta\alpha\beta\alpha(j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)] = 0$

also for $\theta d^3(e_3 \otimes_{g_3^3} e_3) = (j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\varepsilon\zeta\varepsilon\zeta - [-\gamma(j_1e_2 + j_2\beta\alpha$
$+ j_3\beta\alpha\beta\alpha)\delta\varepsilon\zeta - \gamma\delta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\varepsilon\zeta + \varepsilon\zeta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\gamma\delta$
$+ \varepsilon\zeta\gamma(j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha)\delta + \varepsilon\zeta\varepsilon\zeta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)] = 0$
and for $\theta d^3(e_4 \otimes_{g_4^3} e_1) = -\zeta(j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta)\gamma\beta - \zeta\gamma(j_1e_2 + j_2\beta\alpha$
$+ j_3\beta\alpha\beta\alpha)\beta = 0.$

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_2 \otimes_{g_1^2} e_2 \mapsto j_1e_2 + j_2\beta\alpha + j_3\beta\alpha\beta\alpha\\
&e_3 \otimes_{g_2^2} e_3 \mapsto j_4e_3 + j_5\varepsilon\zeta + j_6\varepsilon\zeta\varepsilon\zeta\\
&e_1 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_4 \otimes_{g_4^2} e_1 \mapsto 0,
\end{aligned}$$

where $j_i \in K$. Hence $\dim \operatorname{Ker}\delta^2 = 6$ and then $\operatorname{Ker}\delta^2 = \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$.

11.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_2 \to z_1\alpha + z_2\alpha\beta\alpha$

$e_2 \otimes_\beta e_1 \to z_3\beta + z_4\beta\alpha\beta$

$e_2 \otimes_\delta e_3 \to z_5\delta + z_6\delta\varepsilon\zeta$

$e_3 \otimes_\gamma e_2 \to z_7\gamma + z_8\gamma\beta\alpha$

$e_3 \otimes_\varepsilon e_4 \to z_9\varepsilon + z_{10}\varepsilon\zeta\varepsilon$

$e_4 \otimes_\zeta e_3 \to z_{11}\zeta + z_{12}\zeta\varepsilon\zeta,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_2 \otimes_{g_1^2} e_2 \mapsto e_2 \otimes_\beta \alpha + \beta \otimes_\alpha e_2 - e_2 \otimes_\delta \gamma - \delta \otimes_\gamma e_2$

$e_3 \otimes_{g_2^2} e_3 \mapsto e_3 \otimes_\gamma \delta + \gamma \otimes_\delta e_3 - e_3 \otimes_\varepsilon \zeta - \varepsilon \otimes_\zeta e_3$

$e_1 \otimes_{g_3^2} e_4 \mapsto e_1 \otimes_\alpha \delta\varepsilon + \alpha \otimes_\delta \varepsilon + \alpha\delta \otimes_\varepsilon e_4$

$e_4 \otimes_{g_4^2} e_1 \mapsto e_4 \otimes_\zeta \gamma\beta + \zeta \otimes_\gamma \beta + \zeta\gamma \otimes_\beta e_1.$

Then the map φd^2 is given by

$\varphi d^2(e_2 \otimes_{g_1^2} e_2) = (z_3\beta + z_4\beta\alpha\beta)\alpha + \beta(z_1\alpha + z_2\alpha\beta\alpha) - (z_5\delta + z_6\delta\varepsilon\zeta)\gamma$
$- \delta(z_7\gamma + z_8\gamma\beta\alpha) = (z_1 + z_3 - z_5 - z_7)\beta\alpha + (z_2 + z_4 - z_6 - z_8)\beta\alpha\beta\alpha,$

$\varphi d^2(e_3 \otimes_{g_2^2} e_3) = (z_7\gamma + z_8\gamma\beta\alpha)\delta + \gamma(z_5\delta + z_6\delta\varepsilon\zeta) - (z_9\varepsilon + z_{10}\varepsilon\zeta\varepsilon)\zeta$
$- \varepsilon(z_{11}\zeta + z_{12}\zeta\varepsilon\zeta) = (z_5 + z_7 - z_9 - z_{11})\varepsilon\zeta + (z_6 + z_8 - z_{10} - z_{12})\varepsilon\zeta\varepsilon\zeta,$

$\varphi d^2(e_1 \otimes_{g_3^2} e_4) = (z_1\alpha + z_2\alpha\beta\alpha)\delta\varepsilon + \alpha(z_5\delta + z_6\delta\varepsilon\zeta)\varepsilon + \alpha\delta(z_9\varepsilon + z_{10}\varepsilon\zeta\varepsilon) = 0,$

$\varphi d^2(e_4 \otimes_{g_4^2} e_1) = (z_{11}\zeta + z_{12}\zeta\varepsilon\zeta)\gamma\beta + \zeta(z_7\gamma + z_8\gamma\beta\alpha)\beta + \zeta\gamma(z_3\beta + z_4\beta\alpha\beta) = 0.$

Thus φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_2 \otimes_{g_1^2} e_2 \mapsto (z_1 + z_3 - z_5 - z_7)\beta\alpha + (z_2 + z_4 - z_6 - z_8)\beta\alpha\beta\alpha\\
&e_3 \otimes_{g_2^2} e_3 \mapsto (z_5 + z_7 - z_9 - z_{11})\varepsilon\zeta + (z_6 + z_8 - z_{10} - z_{12})\varepsilon\zeta\varepsilon\zeta\\
&e_1 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_4 \otimes_{g_4^2} e_1 \mapsto 0,
\end{aligned}$$

where $z_i \in K$. Therefore $\dim \operatorname{Im} \delta^1 = 4$.

11.4.3. $\mathrm{HH}^2(\Lambda)$.

From 11.4.1 and 11.4.2 we have that $\dim \mathrm{HH}^2(\Lambda) = 2$ and then

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_2 \otimes_{g_1^2} e_2 \mapsto d_1 e_2 \\ e_3 \otimes_{g_2^2} e_3 \mapsto d_2 e_3 \\ e_1 \otimes_{g_3^2} e_4 \mapsto 0 \\ e_4 \otimes_{g_4^2} e_1 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_2 \otimes_{g_1^2} e_2 \mapsto e_2 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_3 \otimes_{g_2^2} e_3 \mapsto e_3 \\ \text{else} \mapsto 0. \end{array}$$

12. The algebra A_8

Definition 12.1. [5] *Let A_8 be the algebra KQ/I where Q is the quiver*

$$
\begin{tikzcd}
1 \arrow[d, "\sigma"'] \arrow[dr, shift left, "\alpha"] & 4 \arrow[l, "\delta"'] \\
2 \arrow[r, "\zeta"'] & 3 \arrow[u, "\gamma"'] \arrow[ul, shift left, "\beta"]
\end{tikzcd}
$$

and

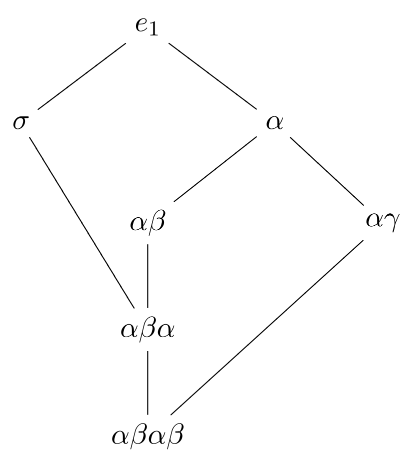
$$I = \langle \alpha\beta\alpha - \sigma\zeta, \beta\alpha\beta - \gamma\delta, \zeta\gamma, \delta\sigma, \beta\alpha\gamma, \alpha\beta\sigma, \zeta\beta\alpha, \delta\alpha\beta \rangle.$$

12.1. **The structure of the indecomposable projectives.**

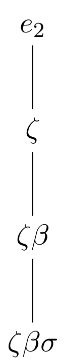
The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$$e_1\Lambda = sp\{e_1, \sigma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\alpha, \alpha\beta\alpha\beta\},$$
$$e_2\Lambda = sp\{e_2, \zeta, \zeta\beta, \zeta\beta\sigma\},$$
$$e_3\Lambda = sp\{e_3, \beta, \gamma, \beta\sigma, \beta\alpha, \beta\alpha\beta, \beta\alpha\beta\alpha\},$$
$$e_4\Lambda = sp\{e_4, \delta, \delta\alpha, \delta\alpha\gamma\}.$$

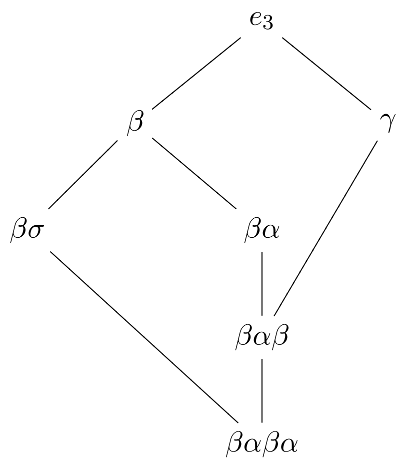
So we have for $e_1\Lambda$



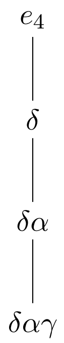
For $e_2\Lambda$



Also $e_3\Lambda$



And for $e_4\Lambda$



12.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

12.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto \sigma e_2\nu + \alpha e_3\lambda$, for $\nu, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

12.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$, let $e_2\nu = d_1e_2+d_2\zeta+d_3\zeta\beta+d_4\zeta\beta\sigma$ and $e_3\lambda = f_1e_3+f_2\beta+f_3\gamma+f_4\beta\sigma+f_5\beta\alpha+f_6\beta\alpha\beta+f_7\beta\alpha\beta\alpha$ with $d_i, f_i \in K$. Assume that $(e_2\nu, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\sigma e_2\nu+\alpha e_3\lambda = 0$, so $\sigma(d_1e_2+d_2\zeta+d_3\zeta\beta+d_4\zeta\beta\sigma)+\alpha(f_1e_3+f_2\beta+f_3\gamma+f_4\beta\sigma+f_5\beta\alpha+f_6\beta\alpha\beta+f_7\beta\alpha\beta\alpha) = d_1\sigma+d_2\sigma\zeta+d_3\sigma\zeta\beta+f_1\alpha+f_2\alpha\beta+f_3\alpha\gamma+f_5\alpha\beta\alpha+f_6\alpha\beta\alpha\beta = 0$, that is, $d_1\sigma + f_1\alpha + f_2\alpha\beta + f_3\alpha\gamma + (d_2+f_5)\alpha\beta\alpha + (d_3+f_6)\alpha\beta\alpha\beta = 0$ and then $d_1 = f_1 = f_2 = f_3 = 0, d_2+f_5 = 0, d_3+f_6 = 0$ so $f_5 = -d_2$ and $f_6 = -d_3$. Thus $e_2\nu = d_2\zeta+d_3\zeta\beta+d_4\zeta\beta\sigma$ and $e_3\lambda = f_4\beta\sigma - d_2\beta\alpha - d_3\beta\alpha\beta + f_7\beta\alpha\beta\alpha$.

Hence $\operatorname{Ker}\partial^1 = \{(d_2\zeta+d_3\zeta\beta+d_4\zeta\beta\sigma, f_4\beta\sigma-d_2\beta\alpha-d_3\beta\alpha\beta+f_7\beta\alpha\beta\alpha) : d_i, f_i \in K\}$.

Claim. $\operatorname{Ker} \partial^1 = (0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^1$; then $x = (d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma, f_4\beta\sigma - d_2\beta\alpha - d_3\beta\alpha\beta + f_7\beta\alpha\beta\alpha)$, that is, $x = (0, \beta\sigma)(f_4e_2) + (-\zeta, \beta\alpha)(-d_2e_3 - d_3\beta - d_4\beta\sigma + f_7\beta\alpha)$. Thus $x \in (0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda$ and therefore $\operatorname{Ker} \partial^1 \subseteq (0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda$.

On the other hand, let $y = (0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda \in (0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda) = \partial^1(-\zeta e_3\lambda, \beta\alpha e_3\lambda + \beta\sigma e_2\nu) = -\sigma\zeta e_3\lambda + \alpha\beta\alpha e_3\lambda + \alpha\beta\sigma e_2\nu = (\alpha\beta\alpha - \sigma\zeta)e_3\lambda + \alpha\beta\sigma e_2\nu = 0$. Therefore $y \in \operatorname{Ker} \partial^1$ and so $(0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda \subseteq \operatorname{Ker} \partial^1$.

Hence $\operatorname{Ker} \partial^1 = (0, \beta\sigma)e_2\Lambda + (-\zeta, \beta\alpha)e_3\Lambda$. $\square$

So $\partial^2 : e_2\Lambda \oplus e_3\Lambda \to e_2\Lambda \oplus e_3\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto (0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda$, for $\nu, \lambda \in \Lambda$.

12.2.3. $\operatorname{Ker} \partial^2$.

Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_1)$. Let $e_2\nu = d_1e_2 + d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma$ and $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$ with $d_i, f_i \in K$. Assume that $(e_2\nu, e_3\lambda) \in \operatorname{Ker} \partial^2$. Then $(0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda = (0, 0)$. So $(0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda = (0, \beta\sigma)(d_1e_2 + d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma) + (-\zeta, \beta\alpha)(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha) = (0, d_1\beta\sigma + d_2\beta\sigma\zeta) + (-f_1\zeta - f_2\zeta\beta - f_4\zeta\beta\sigma, f_1\beta\alpha + f_2\beta\alpha\beta + f_5\beta\alpha\beta\alpha) = (-f_1\zeta - f_2\zeta\beta - f_4\zeta\beta\sigma, d_1\beta\sigma + d_2\beta\sigma\zeta + f_1\beta\alpha + f_2\beta\alpha\beta + f_5\beta\alpha\beta\alpha) = (0, 0)$ so $-f_1\zeta - f_2\zeta\beta - f_4\zeta\beta\sigma = 0$, that is, $f_1 = f_2 = f_4 = 0$. Also $d_1\beta\sigma + f_1\beta\alpha + f_2\beta\alpha\beta + (d_2 + f_5)\beta\alpha\beta\alpha = 0$, that is, $d_1 = f_1 = f_2 = 0, d_2 + f_5 = 0$ so $d_1 = f_1 = f_2 = f_4 = 0$ and $f_5 = -d_2$. Thus $e_2\nu = d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma$ and $e_3\lambda = f_3\gamma - d_2\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$. Therefore $\operatorname{Ker} \partial^2 = \{(d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma, f_3\gamma - d_2\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha) : d_i, f_i \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = (-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = (d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma, f_3\gamma - d_2\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha)$ so $u = (-\zeta, \beta\alpha)(-d_2e_2 - d_3\beta - d_4\beta\sigma + f_7\beta\alpha) + (0, \gamma)(f_3e_4 + (d_3 + f_6)\delta)$. Hence $u \in (-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq (-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda$.

On the other hand, let $v = (-\zeta, \beta\alpha)e_3\lambda + (0, \gamma)e_4\mu \in (-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((-\zeta, \beta\alpha)e_3\lambda + (0, \gamma)e_4\mu) = \partial^2(-\zeta e_3\lambda, \beta\alpha e_3\lambda + \gamma e_4\mu) = (0, \beta\sigma)(-\zeta e_3\lambda) + (-\zeta, \beta\alpha)(\beta\alpha e_3\lambda + \gamma e_4\mu) = (0, -\beta\sigma\zeta e_3\lambda) + (-\zeta\beta\alpha e_3\lambda - \zeta\gamma e_4\mu, \beta\alpha\beta\alpha e_3\lambda + \beta\alpha\gamma e_4\mu) = (-\zeta\beta\alpha e_3\lambda - \zeta\gamma e_4\mu, (\beta\alpha\beta\alpha - \beta\sigma\zeta)e_3\lambda + \beta\alpha\gamma e_4\mu) = (0, 0)$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $(-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = (-\zeta, \beta\alpha)e_3\Lambda + (0, \gamma)e_4\Lambda$. $\square$

So the map $\partial^3 : e_3\Lambda \oplus e_4\Lambda \to e_2\Lambda \oplus e_3\Lambda$ is given by $(e_3\lambda, e_4\mu) \mapsto (-\zeta, \beta\alpha)e_3\lambda + (0, \gamma)e_4\mu$, for $\lambda, \mu \in \Lambda$.

12.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.

The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_2\Lambda$ is given by $e_3\lambda \to \zeta e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

12.2.5. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\zeta e_3\lambda = 0$ so $\zeta(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha) = f_1\zeta + f_2\zeta\beta + f_4\zeta\beta\sigma = 0$, that is, $f_1 = f_2 = f_4 = 0$. Thus $e_3\lambda = f_3\gamma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$.

Hence $\operatorname{Ker}\partial^1 = \{f_3\gamma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \beta\alpha e_3\Lambda + \gamma e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = f_3\gamma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$. So $x = \beta\alpha(f_5e_3 + f_6\beta + f_7\beta\alpha) + \gamma(f_3e_4)$. Thus $x \in \beta\alpha e_3\Lambda + \gamma e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \beta\alpha e_3\Lambda + \gamma e_4\Lambda$.

On the other hand, let $y = \beta\alpha e_3\lambda + \gamma e_4\mu \in \beta\alpha e_3\Lambda + \gamma e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\beta\alpha e_3\lambda + \gamma e_4\mu) = \zeta\beta\alpha e_3\lambda + \zeta\gamma e_4\mu = 0$. So $\beta\alpha e_3\Lambda + \gamma e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \beta\alpha e_3\Lambda + \gamma e_4\Lambda$. □

So $\partial^2 : e_3\Lambda \oplus e_4\Lambda \to e_3\Lambda$ is given by $(e_3\lambda, e_4\mu) \mapsto \beta\alpha e_3\lambda + \gamma e_4\mu$, for $\lambda, \mu \in \Lambda$.

12.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$ and $e_4\mu = t_1e_4 + t_2\delta + t_3\delta\alpha + t_4\delta\alpha\gamma$ with $f_i, t_i \in K$. Assume that $(e_3\lambda, e_4\mu) \in \operatorname{Ker}\partial^2$. Then $\beta\alpha e_3\lambda + \gamma e_4\mu = 0$. So $\beta\alpha e_3\lambda + \gamma e_4\mu = \beta\alpha(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha) + \gamma(t_1e_4 + t_2\delta + t_3\delta\alpha + t_4\delta\alpha\gamma) = f_1\beta\alpha + f_2\beta\alpha\beta + f_5\beta\alpha\beta\alpha + t_1\gamma + t_2\gamma\delta + t_3\gamma\delta\alpha = f_1\beta\alpha + (f_2 + t_2)\beta\alpha\beta + (f_5 + t_3)\beta\alpha\beta\alpha + t_1\gamma = 0$. So $f_1 = t_1 = 0, t_2 = -f_2, t_3 = -f_5$. Thus $e_3\lambda = f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha$ and $e_4\mu = -f_2\delta - f_5\delta\alpha + t_4\delta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{(f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha, -f_2\delta - f_5\delta\alpha + t_4\delta\alpha\gamma) : f_i, t_4 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\beta\alpha + f_6\beta\alpha\beta + f_7\beta\alpha\beta\alpha, -f_2\delta - f_5\delta\alpha + t_4\delta\alpha\gamma)$ so $u = (\beta, -\delta)(f_2e_1 + f_4\sigma + f_5\alpha + f_6\alpha\beta + f_7\alpha\beta\alpha - t_4\alpha\gamma) + (\gamma, 0)(f_3e_4)$. Hence $u \in (\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda$.

On the other hand, let $v = (\beta, -\delta)e_1\eta + (\gamma, 0)e_4\mu \in (\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\beta, -\delta)e_1\eta + (\gamma, 0)e_4\mu) = \partial^2(\beta e_1\eta + \gamma e_4\mu, -\delta e_1\eta) = (\beta\alpha\beta - \gamma\delta)e_1\eta + \beta\alpha\gamma e_4\mu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\beta, -\delta)e_1\Lambda + (\gamma, 0)e_4\Lambda$. □

So the map $\partial^3 : e_1\Lambda\oplus e_4\Lambda \to e_3\Lambda\oplus e_4\Lambda$ is given by $(e_1\eta, e_4\mu) \mapsto (\beta, -\delta)e_1\eta+(\gamma, 0)e_4\mu$, for $\eta, \mu \in \Lambda$.

12.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3.
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_4\Lambda \to e_3\Lambda$ is given by $(e_1\eta, e_4\mu) \mapsto \beta e_1\eta + \gamma e_4\mu$, for $\eta, \mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

12.2.8. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_1\eta = c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ with $c_i \in K$. Assume that $(e_1\eta, e_4\mu) \in \operatorname{Ker}\partial^1$. Then $\beta e_1\eta + \gamma e_4\mu = 0$ so $\beta(c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta) + \gamma(t_1e_4 + t_2\delta + t_3\delta\alpha + t_4\delta\alpha\gamma) = c_1\beta + c_2\beta\sigma + c_3\beta\alpha + c_5\beta\alpha\beta + c_6\beta\alpha\beta\alpha + t_1\gamma + t_2\gamma\delta + t_3\gamma\delta\alpha = c_1\beta + c_2\beta\sigma + c_3\beta\alpha + (c_5 + t_2)\beta\alpha\beta + (c_6 + t_3)\beta\alpha\beta\alpha + t_1\gamma = 0$, that is, $c_1 = c_2 = c_3 = t_1 = 0, t_2 = -c_5, t_3 = -c_6$. Thus $e_1\eta = c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ and $e_4\mu = -c_5\delta - c_6\delta\alpha + t_4\delta\alpha\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{(c_4\alpha\gamma+c_5\alpha\beta+c_6\alpha\beta\alpha+c_7\alpha\beta\alpha\beta, -c_5\delta-c_6\delta\alpha+t_4\delta\alpha\gamma) : c_i, t_4 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_4\alpha\gamma+c_5\alpha\beta+c_6\alpha\beta\alpha+c_7\alpha\beta\alpha\beta, -c_5\delta-c_6\delta\alpha+t_4\delta\alpha\gamma)$. So $x = (\alpha\beta, -\delta)(c_5e_1 + c_6\alpha + c_7\alpha\beta - t_4\alpha\gamma) + (\alpha\gamma, 0)(c_4e_4)$. Thus $x \in (\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda$.

On the other hand, let $y = (\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, 0)e_4\mu \in (\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, 0)e_4\mu) = \partial^1(\alpha\beta e_1\eta + \alpha\gamma e_4\mu, -\delta e_1\eta) = (\beta\alpha\beta - \gamma\delta)e_1\eta + \beta\alpha\gamma e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha\beta, -\delta)e_1\Lambda + (\alpha\gamma, 0)e_4\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_4\Lambda \to e_1\Lambda \oplus e_4\Lambda$ is given by $(e_1\eta, e_4\mu) \mapsto (\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, 0)e_4\mu$, for $\eta, \mu \in \Lambda$.

12.2.9. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_1\eta = c_1e_1+c_2\sigma+c_3\alpha+c_4\alpha\gamma+c_5\alpha\beta+c_6\alpha\beta\alpha+c_7\alpha\beta\alpha\beta$ and $e_4\mu = t_1e_4 + t_2\delta + t_3\delta\alpha + t_4\delta\alpha\gamma$ with $c_i, t_i \in K$. Assume that $(e_1\eta, e_4\mu) \in \operatorname{Ker}\partial^2$. Then $(\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, o)e_4\mu = (0, 0)$. So $(\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, o)e_4\mu = (\alpha\beta, - \delta)(c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta) + (\alpha\gamma, 0)(t_1e_4 + t_2\delta + t_3\delta\alpha + t_4\delta\alpha\gamma) = (c_1\alpha\beta + c_3\alpha\beta\alpha + c_5\alpha\beta\alpha\beta + t_1\alpha\gamma + t_2\alpha\gamma\delta, -c_1\delta - c_3\delta\alpha - c_4\delta\alpha\gamma) = (0, 0)$. So $c_1\alpha\beta+c_3\alpha\beta\alpha+c_5\alpha\beta\alpha\beta+t_1\alpha\gamma+t_2\alpha\gamma\delta = c_1\alpha\beta+c_3\alpha\beta\alpha+(c_5+t_2)\alpha\beta\alpha\beta+t_1\alpha\gamma = 0$, that is, $c_1 = c_3 = t_1 = 0$ and $c_5 = -t_2$. Also $-c_1\delta-c_3\delta\alpha-c_4\delta\alpha\gamma = 0$ and then $c_1 = c_3 = c_4 = 0$.

Thus $e_1\eta = c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ and $e_4\mu = -c_5\delta + t_3\delta\alpha + t_4\delta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta, -c_5\delta + t_3\delta\alpha + t_4\delta\alpha\gamma) : c_i, t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta, -c_5\delta + t_3\delta\alpha + t_4\delta\alpha\gamma)$, that is, $u = (\alpha\beta, -\delta)(c_5e_1 + c_6\alpha + c_7\alpha\beta - t_4\alpha\gamma) + (\sigma, 0)(c_2e_2) + (0, \delta\alpha)(t_3e_3 + c_6e_3)$. But we can show that $(0, \delta\alpha)(t_3e_3 + c_6e_3) \subseteq (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$, namely $(0, \delta\alpha)(t_3e_3 + c_6e_3) = (\alpha\beta, -\delta)e_1\eta + (\sigma, 0)e_2\nu$ where $e_1\eta = -(t_3 + c_6)\alpha$ and $e_2\nu = (t_3 + c_6)\zeta$. Hence $u \in (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$.

On the other hand, let $v = (\alpha\beta, -\delta)e_1\eta + (\sigma, 0)e_2\nu \in (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha\beta, -\delta)e_1\eta + (\sigma, 0)e_2\nu) = \partial^2(\alpha\beta e_1\eta + \sigma e_2\nu, -\delta e_1\eta) = (\alpha\beta, -\delta)(\alpha\beta e_1\eta + \sigma e_2\nu) + (\alpha\gamma, 0)(-\delta e_1\eta) = (\alpha\beta\alpha\beta e_1\eta + \alpha\beta\sigma e_2\nu, -\delta\alpha\beta e_1\eta - \delta\sigma e_2\nu) + (-\alpha\gamma\delta e_1\eta, 0) = ((\alpha\beta\alpha\beta - \alpha\gamma\delta)e_1\eta + \alpha\beta\sigma e_2\nu, -\delta\alpha\beta e_1\eta - \delta\sigma e_2\nu) = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha\beta, -\delta)e_1\Lambda + (\sigma, 0)e_2\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_4\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto (\alpha\beta, -\delta)e_1\eta + (\sigma, 0)e_2\nu$, for $\eta, \nu \in \Lambda$.

12.2.10. *The minimal projective resolution of the simple Λ-module S_4.*
The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_4\Lambda$ is given by $e_1\eta \mapsto \delta e_1\eta$, for $\eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

12.2.11. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$. Let $e_1\eta = c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^1$. Then $\delta e_1\eta = 0$ so $\delta e_1\eta = \delta(c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta) = c_1\delta + c_3\delta\alpha + c_4\delta\alpha\gamma = 0$, that is, Therefore $c_1 = c_3 = c_4 = 0$. Thus $e_1\eta = c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker}\partial^1 = \{c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \alpha\beta e_1\Lambda + \sigma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = c_2\sigma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$, that is, $x = \alpha\beta(c_5e_1 + c_6\alpha + c_7\alpha\beta) + \sigma(c_2e_2)$. Thus $x \in \alpha\beta e_1\Lambda + \sigma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \alpha\beta e_1\Lambda + \sigma e_2\Lambda$.

On the other hand, let $y = \alpha\beta e_1\eta + \sigma e_2\nu \in \alpha\beta e_1\Lambda + \sigma e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha\beta e_1\eta + \sigma e_2\nu) = \delta\alpha\beta e_1\eta + \delta\sigma e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\alpha\beta e_1\Lambda + \sigma e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \alpha\beta e_1\Lambda + \sigma e_2\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto \alpha\beta e_1\eta + \sigma e_2\nu$, for $\eta, \nu \in \Lambda$.

12.2.12. $\operatorname{Ker} \partial^2$.
Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_4)$. Let $e_1\eta = c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ and $e_2\nu = d_1e_2 + d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma$ where $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker} \partial^2$. Then $\alpha\beta e_1\eta + \sigma e_2\nu = 0$. So $\alpha\beta e_1\eta + \sigma e_2\nu = \alpha\beta(c_1e_1 + c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta) + \sigma(d_1e_2 + d_2\zeta + d_3\zeta\beta + d_4\zeta\beta\sigma) = c_1\alpha\beta + c_3\alpha\beta\alpha + c_5\alpha\beta\alpha\beta + d_1\sigma + d_2\sigma\zeta + d_3\sigma\zeta\beta = c_1\alpha\beta + (c_3 + d_2)\alpha\beta\alpha + (c_5 + d_3)\alpha\beta\alpha\beta + d_1\sigma = 0$. Thus $c_1 = d_1 = 0, d_2 = -c_3$ and $d_3 = -c_5$. So $e_1\eta = c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta$ and $e_2\nu = -c_3\zeta - c_5\zeta\beta + d_4\zeta\beta\sigma$. Therefore $\operatorname{Ker} \partial^2 = \{(c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta, -c_3\zeta - c_5\zeta\beta + d_4\zeta\beta\sigma) : c_i, d_4 \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = (\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = (c_2\sigma + c_3\alpha + c_4\alpha\gamma + c_5\alpha\beta + c_6\alpha\beta\alpha + c_7\alpha\beta\alpha\beta, -c_3\zeta - c_5\zeta\beta + d_4\zeta\beta\sigma)$, that is, $u = (\sigma, 0)(c_2e_2) + (\alpha, -\zeta)(c_3e_3 + c_4\gamma + c_5\beta + c_6\beta\alpha + c_7\beta\alpha\beta - d_4\beta\sigma)$. Hence $u \in (\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq (\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda$.

On the other hand, let $v = (\sigma, 0)e_2\nu + (\alpha, -\zeta)e_3\lambda \in (\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\sigma, 0)e_2\nu + (\alpha, -\zeta)e_3\lambda) = \partial^2(\sigma e_2\nu + \alpha e_3\lambda, -\zeta e_3\lambda) = (\alpha\beta\alpha - \sigma\zeta)e_3\lambda + \alpha\beta\sigma e_2\nu = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $(\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = (\sigma, 0)e_2\Lambda + (\alpha, -\zeta)e_3\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto (\sigma, 0)e_2\nu + (\alpha, -\zeta)e_3\lambda$, for $\nu, \lambda \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_2\nu, e_3\lambda) \mapsto \sigma e_2\nu + \alpha e_3\lambda,$$
$$\partial^2 : (e_2\nu, e_3\lambda) \mapsto (0, \beta\sigma)e_2\nu + (-\zeta, \beta\alpha)e_3\lambda,$$
$$\partial^3 : (e_3\lambda, e_4\mu) \mapsto (-\zeta, \beta\alpha)e_3\lambda + (0, \gamma)e_4\mu,$$

for $\nu, \lambda, \mu \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : e_3\lambda \to \zeta e_3\lambda,$$
$$\partial^2 : (e_3\lambda, e_4\mu) \mapsto \beta\alpha e_3\lambda + \gamma e_4\mu$$
$$\partial^3 : (e_1\eta, e_4\mu) \mapsto (\beta, -\delta)e_1\eta + (\gamma, 0)e_4\mu,$$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\partial^1 : (e_1\eta, e_4\mu) \mapsto \beta e_1\eta + \gamma e_4\mu,$$
$$\partial^2 : (e_1\eta, e_4\mu) \mapsto (\alpha\beta, -\delta)e_1\eta + (\alpha\gamma, 0)e_4\mu,$$
$$\partial^3 : (e_1\eta, e_2\nu) \mapsto (\alpha\beta, -\delta)e_1\eta + (\sigma, 0)e_2\nu,$$

for $\eta, \nu, \mu \in \Lambda$.

Moreover, the maps for S_4 are:

$$\partial^1 : e_1\eta \mapsto \delta e_1\eta,$$
$$\partial^2 : (e_1\eta, e_2\nu) \mapsto \alpha\beta e_1\eta + \sigma e_2\nu,$$
$$\partial^3 : (e_2\nu, e_3\lambda) \mapsto (\sigma, 0)e_2\nu + (\alpha, -\zeta)e_3\lambda,$$

for $\eta, \nu, \lambda \in \Lambda$.

12.3. g^3 **for** S_1, S_2, S_3 **and** S_4.

Now we want to find the elements of g^3; these are paths in $K\mathcal{Q}$.

For S_1

$(e_3, 0) \stackrel{\partial^3}{\to} (-\zeta, \beta\alpha) \stackrel{\partial^2}{\to} (0, \beta\sigma)(-\zeta) + (-\zeta, \beta\alpha)(\beta\alpha) = (0, -\beta\sigma\zeta) + (-\zeta\beta\alpha, \beta\alpha\beta\alpha) = (-\zeta\beta\alpha, \beta\alpha\beta\alpha - \beta\sigma\zeta) \stackrel{\partial^1}{\to} -\sigma\zeta\beta\alpha + \alpha\beta\alpha\beta\alpha - \alpha\beta\sigma\zeta$, so $-\sigma\zeta\beta\alpha + \alpha\beta\alpha\beta\alpha - \alpha\beta\sigma\zeta \in g^3$.

$(0, e_4) \stackrel{\partial^3}{\to} (0, \gamma) \stackrel{\partial^2}{\to} (0, \beta\sigma)(0) + (-\zeta, \beta\alpha)(\gamma) = (-\zeta\gamma, \beta\alpha\gamma) \stackrel{\partial^1}{\to} -\sigma\zeta\gamma + \alpha\beta\alpha\gamma$,
so $-\sigma\zeta\gamma + \alpha\beta\alpha\gamma \in g^3$.

For S_2

$(e_1, 0) \stackrel{\partial^3}{\to} (\beta, -\delta) \stackrel{\partial^2}{\to} \beta\alpha\beta - \gamma\delta \stackrel{\partial^1}{\to} \zeta\beta\alpha\beta - \zeta\gamma\delta$, so $\zeta\beta\alpha\beta - \zeta\gamma\delta \in g^3$.

$(0, e_4) \stackrel{\partial^3}{\to} (\gamma, 0) \stackrel{\partial^2}{\to} \beta\alpha\gamma \stackrel{\partial^1}{\to} \zeta\beta\alpha\gamma$, so $\zeta\beta\alpha\gamma \in g^3$.

For S_3

$(e_1, 0) \stackrel{\partial^3}{\to} (\alpha\beta, -\delta) \stackrel{\partial^2}{\to} (\alpha\beta, -\delta)\alpha\beta + (\alpha\gamma, 0)(-\delta) = (\alpha\beta\alpha\beta - \alpha\gamma\delta, -\delta\alpha\beta) \stackrel{\partial^1}{\to} \beta\alpha\beta\alpha\beta - \beta\alpha\gamma\delta - \gamma\delta\alpha\beta$, so $\beta\alpha\beta\alpha\beta - \beta\alpha\gamma\delta - \gamma\delta\alpha\beta \in g^3$.

$(0, e_2) \stackrel{\partial^3}{\to} (\sigma, 0) \stackrel{\partial^2}{\to} (\alpha\beta, -\delta)\sigma \stackrel{\partial^1}{\to} \beta\alpha\beta\sigma - \gamma\delta\sigma$, so $\beta\alpha\beta\sigma - \gamma\delta\sigma \in g^3$.

For S_4

$(e_2, 0) \stackrel{\partial^3}{\to} (\sigma, 0) \stackrel{\partial^2}{\to} \alpha\beta\sigma \stackrel{\partial^1}{\to} \delta\alpha\beta\sigma$, so $\delta\alpha\beta\sigma \in g^3$.

$(0, e_3) \stackrel{\partial^3}{\to} (\alpha, -\zeta) \stackrel{\partial^2}{\to} \alpha\beta\alpha - \sigma\zeta \stackrel{\partial^1}{\to} \delta\alpha\beta\alpha - \delta\sigma\zeta$, so $\delta\alpha\beta\alpha - \delta\sigma\zeta \in g^3$.

Let $g^3_1 = \alpha\beta\alpha\beta\alpha - \alpha\beta\sigma\zeta - \sigma\zeta\beta\alpha$, $g^3_2 = \alpha\beta\alpha\gamma - \sigma\zeta\gamma$, $g^3_3 = \zeta\beta\alpha\beta - \zeta\gamma\delta$, $g^3_4 = \zeta\beta\alpha\gamma$, $g^3_5 = \beta\alpha\beta\alpha\beta - \beta\alpha\gamma\delta - \gamma\delta\alpha\beta$, $g^3_6 = \beta\alpha\beta\sigma - \gamma\delta\sigma$, $g^3_7 = \delta\alpha\beta\sigma$ and $g^3_8 = \delta\alpha\beta\alpha - \delta\sigma\zeta$. So $g^3 = \{g^3_1, g^3_2, g^3_3, g^3_4, g^3_5, g^3_6, g^3_7, g^3_8\}$.

We know that $g^2 = \{\alpha\beta\alpha - \sigma\zeta, \beta\alpha\beta - \gamma\delta, \zeta\gamma, \delta\sigma, \beta\alpha\gamma, \alpha\beta\sigma, \zeta\beta\alpha, \delta\alpha\beta\}$. Denote

$g^2_1 = \alpha\beta\alpha - \sigma\zeta$,
$g^2_2 = \beta\alpha\beta - \gamma\delta$,
$g^2_3 = \zeta\gamma$,
$g^2_4 = \delta\sigma$,
$g^2_5 = \beta\alpha\gamma$,
$g^2_6 = \alpha\beta\sigma$,
$g^2_7 = \zeta\beta\alpha$ and
$g^2_8 = \delta\alpha\beta$.

So we have

$g^3_1 = g^2_1\beta\alpha - g^2_6\zeta = \alpha\beta g^2_1 - \sigma g^2_7$,
$g^3_2 = g^2_1\gamma = \alpha g^2_5 - \sigma g^2_3$,

$g_3^3 = g_7^2\beta - g_3^2\delta = \zeta g_2^2$,
$g_4^3 = g_7^2\gamma = \zeta g_5^2$,
$g_5^3 = g_2^2\alpha\beta - g_5^2\delta = \beta\alpha g_2^2 - \gamma g_8^2$,
$g_6^3 = g_2^2\sigma = \beta g_6^2 - \gamma g_4^2$,
$g_7^3 = g_8^2\sigma = \delta g_6^2$ and
$g_8^3 = g_8^2\alpha - g_4^2\zeta = \delta g_1^2$.

12.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots$$

12.4.1. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_3 \mapsto j_1\alpha + j_2\alpha\beta\alpha\\
&e_3 \otimes_{g_2^2} e_1 \mapsto j_3\beta + j_4\beta\alpha\beta\\
&e_2 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_4 \otimes_{g_4^2} e_2 \mapsto 0\\
&e_3 \otimes_{g_5^2} e_4 \mapsto j_5\gamma\\
&e_1 \otimes_{g_6^2} e_2 \mapsto j_6\sigma\\
&e_2 \otimes_{g_7^2} e_3 \mapsto j_7\zeta\\
&e_4 \otimes_{g_8^2} e_1 \mapsto j_8\delta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_3 \mapsto e_1 \otimes_{g_1^2} \beta\alpha - e_1 \otimes_{g_6^2} \zeta - \alpha\beta \otimes_{g_1^2} e_3 + \sigma \otimes_{g_7^2} e_3$,
$e_1 \otimes_{g_2^3} e_4 \mapsto e_1 \otimes_{g_1^2} \gamma - \alpha \otimes_{g_5^2} e_4 + \sigma \otimes_{g_3^2} e_4$,
$e_2 \otimes_{g_3^3} e_1 \mapsto e_2 \otimes_{g_7^2} \beta - e_2 \otimes_{g_3^2} \delta - \zeta \otimes_{g_2^2} e_1$,
$e_2 \otimes_{g_4^3} e_4 \mapsto e_2 \otimes_{g_7^2} \gamma - \zeta \otimes_{g_5^2} e_4$,
$e_3 \otimes_{g_5^3} e_1 \mapsto e_3 \otimes_{g_2^2} \alpha\beta - e_3 \otimes_{g_5^2} \delta - \beta\alpha \otimes_{g_2^2} e_1 + \gamma \otimes_{g_8^2} e_1$,
$e_3 \otimes_{g_6^3} e_2 \mapsto e_3 \otimes_{g_2^2} \sigma - \beta \otimes_{g_6^2} e_2 + \gamma \otimes_{g_4^2} e_2$,
$e_4 \otimes_{g_7^3} e_2 \mapsto e_4 \otimes_{g_8^2} \sigma - \delta \otimes_{g_6^2} e_2$,
$e_4 \otimes_{g_8^3} e_3 \mapsto e_4 \otimes_{g_8^2} \alpha - e_4 \otimes_{g_4^2} \zeta - \delta \otimes_{g_1^2} e_3$.

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_3) = (j_1\alpha + j_2\alpha\beta\alpha)\beta\alpha - (j_6\sigma)\zeta - \alpha\beta(j_1\alpha + j_2\alpha\beta\alpha) + \sigma(j_7\zeta) = (-j_6 + j_7)\alpha\beta\alpha = 0$ then $j_6 = j_7$.

For $\theta d^3(e_1 \otimes_{g_2^3} e_4) = (j_1\alpha + j_2\alpha\beta\alpha)\gamma - \alpha(j_5\gamma) = (j_1 - j_5)\alpha\gamma = 0$, that is, $j_1 = j_5$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_1) = (j_7\zeta)\beta - (j_3\zeta)\beta = (j_7 - j_3)\zeta\beta = 0$ and then $j_7 = j_3$.

And $\theta d^3(e_2 \otimes_{g_4^3} e_4) = 0$.

For $\theta d^3(e_3 \otimes_{g_5^3} e_1) = (j_3\beta + j_4\beta\alpha\beta)\alpha\beta - (j_5\gamma)\delta - \beta\alpha(j_3\beta + j_4\beta\alpha\beta) + \gamma(j_8\delta) = (j_8 - j_5)\beta\alpha\beta = 0$, that is, $j_8 = j_5$.

Also $\theta d^3(e_3 \otimes_{g_6^3} e_2) = (j_3\beta + j_4\beta\alpha\beta)\sigma - \beta(j_6\sigma) = (j_3 - j_6)\beta\sigma = 0$ and then $j_6 = j_3$.

And $\theta d^3(e_4 \otimes_{g_7^3} e_2) = 0$.

And $\theta d^3(e_4 \otimes_{g_8^3} e_3) = (j_8\delta)\alpha - \delta(j_1\alpha + j_2\alpha\beta\alpha) = (j_8 - j_1)\delta\alpha = 0$, so $j_8 = j_1$.

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$
\begin{aligned}
&\theta : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_3 \mapsto j_1\alpha + j_2\alpha\beta\alpha \\
&e_3 \otimes_{g_2^2} e_1 \mapsto j_3\beta + j_4\beta\alpha\beta \\
&e_2 \otimes_{g_3^2} e_4 \mapsto 0 \\
&e_4 \otimes_{g_4^2} e_2 \mapsto 0 \\
&e_3 \otimes_{g_5^2} e_4 \mapsto j_1\gamma \\
&e_1 \otimes_{g_6^2} e_2 \mapsto j_3\sigma \\
&e_2 \otimes_{g_7^2} e_3 \mapsto j_3\zeta \\
&e_4 \otimes_{g_8^2} e_1 \mapsto j_1\delta,
\end{aligned}
$$

where $j_i \in K$. Hence $\dim \operatorname{Ker}\delta^2 = 4$.

12.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\sigma e_2 \to z_1\sigma$

$e_2 \otimes_\zeta e_3 \to z_2\zeta$

$e_3 \otimes_\gamma e_4 \to z_3\gamma$

$e_4 \otimes_\delta e_1 \to z_4\delta$

$e_1 \otimes_\alpha e_3 \to z_5\alpha + z_6\alpha\beta\alpha$

$e_3 \otimes_\beta e_1 \to z_7\beta + z_8\beta\alpha\beta$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_3 \mapsto e_1 \otimes_\alpha \beta\alpha + \alpha \otimes_\beta \alpha + \alpha\beta \otimes_\alpha e_3 - e_1 \otimes_\sigma \zeta - \sigma \otimes_\zeta e_3$

$e_3 \otimes_{g_2^2} e_1 \mapsto e_3 \otimes_\beta \alpha\beta + \beta \otimes_\alpha \beta + \beta\alpha \otimes_\beta e_1 - e_3 \otimes_\gamma \delta - \gamma \otimes_\delta e_1$

$e_2 \otimes_{g_3^2} e_4 \mapsto e_2 \otimes_\zeta \gamma + \zeta \otimes_\gamma e_4$

$e_4 \otimes_{g_4^2} e_2 \mapsto e_4 \otimes_\delta \sigma + \delta \otimes_\sigma e_2$

$e_3 \otimes_{g_5^2} e_4 \mapsto e_3 \otimes_\beta \alpha\gamma + \beta \otimes_\alpha \gamma + \beta\alpha \otimes_\gamma e_4$

$e_1 \otimes_{g_6^2} e_2 \mapsto e_1 \otimes_\alpha \beta\sigma + \alpha \otimes_\beta \sigma + \alpha\beta \otimes_\sigma e_2$

$e_2 \otimes_{g_7^2} e_3 \mapsto e_2 \otimes_\zeta \beta\alpha + \zeta \otimes_\beta \alpha + \zeta\beta \otimes_\alpha e_3$

$e_4 \otimes_{g_8^2} e_1 \mapsto e_4 \otimes_\delta \alpha\beta + \delta \otimes_\alpha \beta + \delta\alpha \otimes_\beta e_1$.

Then the map φd^2 is given by

$$
\begin{aligned}
\varphi d^2(e_1 \otimes_{g_1^2} e_3) &= (z_5\alpha + z_6\alpha\beta\alpha)\beta\alpha + \alpha(z_7\beta + z_8\beta\alpha\beta)\alpha + \alpha\beta(z_5\alpha + z_6\alpha\beta\alpha) \\
&\quad - (z_1\sigma)\zeta - \sigma(z_2\zeta) = (-z_1 - z_2 + 2z_5 + z_7)\alpha\beta\alpha, \\
\varphi d^2(e_3 \otimes_{g_2^2} e_1) &= (z_7\beta + z_8\beta\alpha\beta)\alpha\beta + \beta(z_5\alpha + z_6\alpha\beta\alpha)\beta + \beta\alpha(z_7\beta + z_8\beta\alpha\beta) \\
&\quad - (z_3\gamma)\delta - \gamma(z_4\delta) = (-z_3 - z_4 + z_5 + 2z_7)\beta\alpha\beta,
\end{aligned}
$$

$\varphi d^2(e_2 \otimes_{g_3^2} e_4) = (z_2\zeta)\gamma + \zeta(z_3\gamma) = 0,$
$\varphi d^2(e_4 \otimes_{g_4^2} e_2) = (z_4\delta)\sigma + \delta(z_1\sigma) = 0,$
$\varphi d^2(e_3 \otimes_{g_5^2} e_4) = (z_7\beta + z_8\beta\alpha\beta)\alpha\gamma + \beta(z_5\alpha + z_6\alpha\beta\alpha)\gamma + \beta\alpha(z_3\gamma) = 0,$
$\varphi d^2(e_1 \otimes_{g_6^2} e_2) = (z_5\alpha + z_6\alpha\beta\alpha)\beta\sigma + \alpha(z_7\beta + z_8\beta\alpha\beta)\sigma + \alpha\beta(z_1\sigma) = 0,$
$\varphi d^2(e_2 \otimes_{g_7^2} e_3) = (z_2\zeta)\beta\alpha + \zeta(z_7\beta + z_8\beta\alpha\beta)\alpha + \zeta\beta(z_5\alpha + z_6\alpha\beta\alpha) = 0,$
$\varphi d^2(e_4 \otimes_{g_8^2} e_1) = (z_4\delta)\alpha\beta + \delta(z_5\alpha + z_6\alpha\beta\alpha)\beta + \delta\alpha(z_7\beta + z_8\beta\alpha\beta) = 0.$
We write $c = -z_1 - z_2 + 2z_5 + z_7$ and $c' = -z_3 - z_4 + z_5 + 2z_7$.

Thus φd^2 is given by

$$\begin{aligned}&P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_3 \mapsto c\alpha\beta\alpha\\ &e_3 \otimes_{g_2^2} e_1 \mapsto c'\beta\alpha\beta\\ &e_2 \otimes_{g_3^2} e_4 \mapsto 0\\ &e_4 \otimes_{g_4^2} e_2 \mapsto 0\\ &e_3 \otimes_{g_5^2} e_4 \mapsto 0\\ &e_1 \otimes_{g_6^2} e_2 \mapsto 0\\ &e_2 \otimes_{g_7^2} e_3 \mapsto 0\\ &e_4 \otimes_{g_8^2} e_1 \mapsto 0,\end{aligned}$$

where $c, c' \in K$.

So

$$\operatorname{Im}\delta^1 = \left\{\begin{array}{l} P^2 \to \Lambda\\ e_1 \otimes_{g_1^2} e_3 \mapsto c\alpha\beta\alpha\\ e_3 \otimes_{g_2^2} e_1 \mapsto c'\beta\alpha\beta\\ \text{else} \mapsto 0,\end{array}\right\}$$

with $c, c' \in K$. So $\dim \operatorname{Im}\delta^1 = 2$.

12.4.3. $\mathrm{HH}^2(\Lambda)$. From 12.4.1 and 12.4.2 we have that $\dim \mathrm{HH}^2(\Lambda) = 2$. Thus

$$\mathrm{HH}^2(\Lambda) = \left\{\begin{array}{l} P^2 \to \Lambda\\ e_1 \otimes_{g_1^2} e_3 \mapsto d_1\alpha\\ e_3 \otimes_{g_2^2} e_1 \mapsto d_2\beta\\ e_2 \otimes_{g_3^2} e_4 \mapsto 0\\ e_4 \otimes_{g_4^2} e_2 \mapsto 0\\ e_3 \otimes_{g_5^2} e_4 \mapsto d_1\gamma\\ e_1 \otimes_{g_6^2} e_2 \mapsto d_2\sigma\\ e_2 \otimes_{g_7^2} e_3 \mapsto d_2\zeta\\ e_4 \otimes_{g_8^2} e_1 \mapsto d_1\delta,\end{array}\right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{aligned}&x : P^2 \to \Lambda\\ &e_1 \otimes_{g_1^2} e_3 \mapsto \alpha\\ &e_3 \otimes_{g_5^2} e_4 \mapsto \gamma\\ &e_4 \otimes_{g_8^2} e_1 \mapsto \delta\\ &\text{else} \mapsto 0,\end{aligned}$$

$y : P^2 \to \Lambda$
$e_3 \otimes_{g_2^2} e_1 \mapsto \beta$
$e_1 \otimes_{g_6^2} e_2 \mapsto \sigma$
$e_2 \otimes_{g_7^2} e_3 \mapsto \zeta$
else $\mapsto 0.$

13. The algebra A_9

Definition 13.1. [5] *Let A_9 be the algebra KQ/I where Q is the quiver*

$$\begin{tikzcd}
1 \arrow[d, "\alpha"'] & 4 \arrow[l, "\delta"'] \arrow[d, shift left, "\varepsilon"] \\
2 \arrow[r, shift left, "\sigma"] & 3 \arrow[l, shift left, "\beta"] \arrow[u, shift left, "\gamma"]
\end{tikzcd}$$

and

$$I = \langle \delta\alpha - \varepsilon\beta, \gamma\varepsilon - \beta\sigma, \alpha\sigma\beta, \varepsilon\gamma\delta, \sigma\gamma\varepsilon\gamma \rangle.$$

13.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$$\begin{aligned}
e_1\Lambda &= sp\{e_1, \alpha, \alpha\sigma, \alpha\sigma\gamma, \alpha\sigma\gamma\delta\}, \\
e_2\Lambda &= sp\{e_2, \sigma, \sigma\beta, \sigma\gamma, \sigma\beta\sigma, \sigma\gamma\delta, \sigma\beta\sigma\beta\}, \\
e_3\Lambda &= sp\{e_3, \beta, \gamma, \beta\sigma, \gamma\delta, \beta\sigma\beta, \beta\sigma\gamma, \beta\sigma\beta\sigma\}, \\
e_4\Lambda &= sp\{e_4, \delta, \varepsilon, \delta\alpha, \varepsilon\gamma, \varepsilon\gamma\varepsilon, \varepsilon\gamma\varepsilon\gamma\}.
\end{aligned}$$

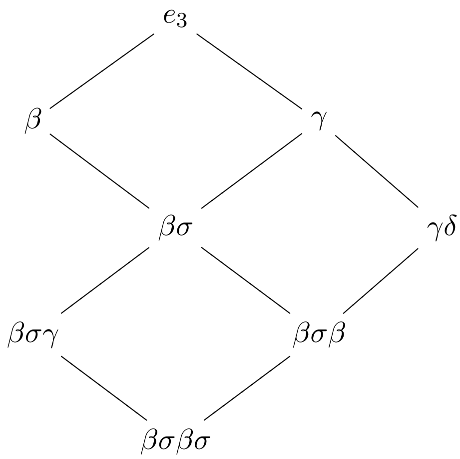
So we have for $e_1\Lambda$

$$\begin{tikzcd}
e_1 \arrow[d, dash] \\
\alpha \arrow[d, dash] \\
\alpha\sigma \arrow[d, dash] \\
\alpha\sigma\gamma \arrow[d, dash] \\
\alpha\sigma\gamma\delta
\end{tikzcd}$$

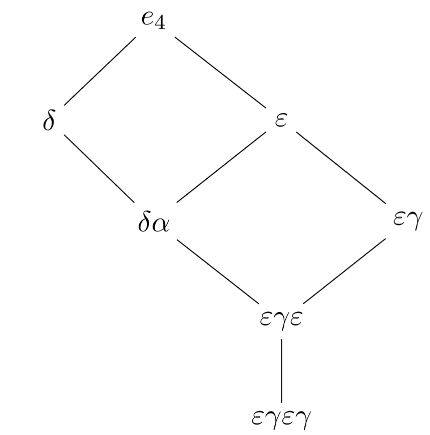
For $e_2\Lambda$

$$\begin{tikzcd}
& e_2 \arrow[d, dash] & & \\
& \sigma \arrow[dl, dash] \arrow[dr, dash] & & \\
\sigma\beta \arrow[dr, dash] & & \sigma\gamma \arrow[dl, dash] \arrow[dr, dash] & \\
& \sigma\beta\sigma \arrow[dr, dash] & & \sigma\gamma\delta \arrow[dl, dash] \\
& & \sigma\beta\sigma\beta &
\end{tikzcd}$$

Also $e_3\Lambda$



And for $e_4\Lambda$



13.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

13.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \alpha e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

13.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\alpha e_2\nu = 0$, so $\alpha(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta) = d_1\alpha + d_2\alpha\sigma + d_4\alpha\sigma\gamma + d_6\alpha\sigma\gamma\delta = 0$ and then $d_1 = d_2 = d_4 = d_6 = 0$. Thus $e_2\nu = d_3\sigma\beta + d_5\sigma\beta\sigma + d_7\sigma\beta\sigma\beta$.

Hence $\operatorname{Ker}\partial^1 = \{d_3\sigma\beta + d_5\sigma\beta\sigma + d_7\sigma\beta\sigma\beta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \sigma\beta e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^1$; then $x = d_3\sigma\beta + d_5\sigma\beta\sigma + d_7\sigma\beta\sigma\beta$, that is, $x = \sigma\beta(d_3e_2 + d_5\sigma + d_7\sigma\beta)$. Thus $x \in \sigma\beta e_2\Lambda$ and therefore $\operatorname{Ker} \partial^1 \subseteq \sigma\beta e_2\Lambda$.

On the other hand, let $y = \sigma\beta e_2\nu \in \sigma\beta e_2\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\sigma\beta e_2\nu) = \alpha\sigma\beta e_2\nu = 0$. Therefore $y \in \operatorname{Ker} \partial^1$ and so $\sigma\beta e_2\Lambda \subseteq \operatorname{Ker} \partial^1$.

Hence $\operatorname{Ker} \partial^1 = \sigma\beta e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\nu \mapsto \sigma\beta e_2\nu$, for $\nu \in \Lambda$.

13.2.3. $\operatorname{Ker} \partial^2$.
Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_1)$. Let $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker} \partial^2$. Then $\sigma\beta e_2\nu = 0$. So $\sigma\beta e_2\nu = \sigma\beta(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta) = d_1\sigma\beta + d_2\sigma\beta\sigma + d_3\sigma\beta\sigma\beta = 0$, that is, $d_1 = d_2 = d_3 = 0$. Thus $e_2\nu = d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$. Therefore $\operatorname{Ker} \partial^2 = \{d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta : d_i \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = \sigma\gamma e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = (d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta)$ so $u = \sigma\gamma(d_4e_4 + d_5\varepsilon + d_6\delta + d_7\varepsilon\beta)$. Hence $u \in \sigma\gamma e_4\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \sigma\gamma e_4\Lambda$.

On the other hand, let $v = \sigma\gamma e_4\mu \in \sigma\gamma e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\sigma\gamma e_4\mu) = \sigma\beta\sigma\gamma e_4\mu = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $\sigma\gamma e_4\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \sigma\gamma e_4\Lambda$. □

So the map $\partial^3 : e_4\Lambda \to e_2\Lambda$ is given by $e_4\mu \mapsto \sigma\gamma e_4\mu$, for $\mu \in \Lambda$.

13.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_2\Lambda$ is given by $e_3\lambda \to \sigma e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker} \partial^1$ and $\operatorname{Ker} \partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

13.2.5. $\operatorname{Ker} \partial^1$.
To find $\operatorname{Ker} \partial^1 = \Omega^2(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$. Assume that $e_3\lambda \in \operatorname{Ker} \partial^1$. Then $\sigma e_3\lambda = 0$ so $\sigma(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma) = f_1\sigma + f_2\sigma\beta + f_3\sigma\gamma + f_4\sigma\beta\sigma + f_5\sigma\gamma\delta + f_6\sigma\beta\sigma\beta = 0$, that is, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$. Thus $e_3\lambda = f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$.

Hence $\operatorname{Ker} \partial^1 = \{f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma : f_7, f_8 \in K\}$.

Claim. $\operatorname{Ker} \partial^1 = \gamma\varepsilon\gamma e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$. So $x = \gamma\varepsilon\gamma(f_7e_4 + f_8\varepsilon)$. Thus $x \in \gamma\varepsilon\gamma e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \gamma\varepsilon\gamma e_4\Lambda$.

On the other hand, let $y = \gamma\varepsilon\gamma e_4\mu \in \gamma\varepsilon\gamma e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\gamma\varepsilon\gamma e_4\mu) = \sigma\gamma\varepsilon\gamma e_4\mu = 0$. So $y \in \operatorname{Ker}\partial^1$ and $\gamma\varepsilon\gamma e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \gamma\varepsilon\gamma e_4\Lambda$. $\square$

So $\partial^2 : e_4\Lambda \to e_3\Lambda$ is given by $e_4\mu \mapsto \gamma\varepsilon\gamma e_4\mu$, for $\mu \in \Lambda$.

13.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_4\mu = t_1e_4 + t_2\delta + t_3\varepsilon + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma$. Assume that $e_4\mu \in \operatorname{Ker}\partial^2$. Then $\gamma\varepsilon\gamma e_4\mu = 0$. So $\gamma\varepsilon\gamma e_4\mu = \gamma\varepsilon\gamma(t_1e_4 + t_2\delta + t_3\varepsilon + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma) = t_1\gamma\varepsilon\gamma + t_3\gamma\varepsilon\gamma\varepsilon = 0$. So $t_1 = t_3 = 0$. Thus $e_4\mu = t_2\delta + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{t_2\delta + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \delta e_1\Lambda + \varepsilon\gamma e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = t_2\delta + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma$ so $u = \delta(t_2e_1 + t_4\alpha) + \varepsilon\gamma(t_5e_4 + t_6\varepsilon + t_7\varepsilon\gamma)$. Hence $u \in \delta e_1\Lambda + \varepsilon\gamma e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \delta e_1\Lambda + \varepsilon\gamma e_4\Lambda$.

On the other hand, let $v = \delta e_1\eta + \varepsilon\gamma e_4\mu \in \delta e_1\Lambda + \varepsilon\gamma e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\delta e_1\eta + \varepsilon\gamma e_4\mu) = \gamma\varepsilon\gamma\delta e_1\eta + \gamma\varepsilon\gamma\varepsilon\gamma e_4\mu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\delta e_1\Lambda + \varepsilon\gamma e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \delta e_1\Lambda + \varepsilon\gamma e_4\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \oplus e_4\Lambda \to e_4\Lambda$ is given by $(e_1\eta, e_4\mu) \mapsto \delta e_1\eta + \varepsilon\gamma e_4\mu$, for $\eta, \mu \in \Lambda$.

13.2.7. *The minimal projective resolution of the simple Λ-module S_3.*
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \oplus e_4\Lambda \to e_3\Lambda$ is given by $(e_2\nu, e_4\mu) \mapsto \beta e_2\nu + \gamma e_4\mu$, for $\nu, \mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

13.2.8. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$ and $e_4\mu = t_1e_4 + t_2\delta + t_3\varepsilon + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma$ with $d_i, t_i \in K$. Assume that $(e_2\nu, e_4\mu) \in \operatorname{Ker}\partial^1$. Then $\beta e_2\nu + \gamma e_4\mu = 0$ so $\beta(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta) + \gamma(t_1e_4 + t_2\delta + t_3\varepsilon + t_4\delta\alpha + t_5\varepsilon\gamma + t_6\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma) = d_1\beta + d_2\beta\sigma + d_3\beta\sigma\beta + d_4\beta\sigma\gamma + d_5\beta\sigma\beta\sigma + t_1\gamma + t_2\gamma\delta + t_3\gamma\varepsilon + t_4\gamma\delta\alpha + t_5\gamma\varepsilon\gamma + t_6\gamma\varepsilon\gamma\varepsilon = d_1\beta + t_1\gamma + t_2\gamma\delta + (d_2 + t_3)\beta\sigma + (d_3 + t_4)\beta\sigma\beta + (d_4 + t_5)\gamma\varepsilon\gamma + (d_5 + t_6)\beta\sigma\beta\sigma = 0$, that is, $d_1 = t_1 = t_2 = 0, t_3 = -d_2, t_4 = -d_3, t_5 = -d_4$ and $t_6 = -d_5$. Thus $e_2\nu = d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$ and $e_4\mu = -d_2\varepsilon - d_3\delta\alpha - d_4\varepsilon\gamma - d_5\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{(d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta, -d_2\varepsilon - d_3\delta\alpha - d_4\varepsilon\gamma - d_5\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma) : d_i, t_7 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (-\sigma, \varepsilon)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta, -d_2\varepsilon - d_3\delta\alpha - d_4\varepsilon\gamma - d_5\varepsilon\gamma\varepsilon + t_7\varepsilon\gamma\varepsilon\gamma)$. So $x = (-\sigma, \varepsilon)(-d_2e_3 - d_3\beta - d_4\gamma - d_5\beta\sigma - d_6\gamma\delta - d_7\beta\sigma\beta + t_7\gamma\varepsilon\gamma)$. Thus $x \in (-\sigma, \varepsilon)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (-\sigma, \varepsilon)e_3\Lambda$.

On the other hand, let $y = (-\sigma, \varepsilon)e_3\lambda \in (-\sigma, \varepsilon)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((-\sigma, \varepsilon)e_3\lambda) = (\gamma\varepsilon - \beta\sigma)e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(-\sigma, \varepsilon)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (-\sigma, \varepsilon)e_3\Lambda$. $\square$

So $\partial^2 : e_3\Lambda \to e_2\Lambda \oplus e_4\Lambda$ is given by: $e_3\lambda \mapsto (-\sigma, \varepsilon)e_3\lambda$, for $\lambda \in \Lambda$.

13.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^2$. Then $(-\sigma, \varepsilon)e_3\lambda = (0, 0)$. So $(-\sigma, \varepsilon)e_3\lambda = (-\sigma, \varepsilon)(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma) = (-f_1\sigma - f_2\sigma\beta - f_3\sigma\gamma - f_4\sigma\beta\sigma - f_5\sigma\gamma\delta - f_6\sigma\beta\sigma\beta, f_1\varepsilon + f_2\varepsilon\beta + f_3\varepsilon\gamma + f_4\varepsilon\beta\sigma + f_7\varepsilon\beta\sigma\gamma) = (0, 0)$. So $-f_1\sigma - f_2\sigma\beta - f_3\sigma\gamma - f_4\sigma\beta\sigma - f_5\sigma\gamma\delta - f_6\sigma\beta\sigma\beta = 0$, that is, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$. Also $f_1\varepsilon + f_2\varepsilon\beta + f_3\varepsilon\gamma + f_4\varepsilon\beta\sigma + f_7\varepsilon\beta\sigma\gamma = 0$ and then $f_1 = f_2 = f_3 = f_4 = f_7 = 0$. Thus $e_3\lambda = f_8\beta\sigma\beta\sigma$. Therefore $\operatorname{Ker}\partial^2 = \{f_8\beta\sigma\beta\sigma : f_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\sigma\beta\sigma e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_8\beta\sigma\beta\sigma$, that is, $u = \beta\sigma\beta\sigma(f_8e_3)$. Hence $u \in \beta\sigma\beta\sigma e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\sigma\beta\sigma e_3\Lambda$.

On the other hand, let $v = \beta\sigma\beta\sigma e_3\lambda \in \beta\sigma\beta\sigma e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\sigma\beta\sigma e_3\lambda) = (-\sigma\beta\sigma\beta\sigma, \varepsilon\beta\sigma\beta\sigma)e_3\lambda = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta\sigma\beta\sigma e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\sigma\beta\sigma e_3\Lambda$. $\square$

So the map $\partial^3 : e_3\Lambda \to e_3\Lambda$ is given by $e_3\lambda \mapsto \beta\sigma\beta\sigma e_3\lambda$, for $\lambda \in \Lambda$.

Note that $\operatorname{Ker}\partial^2 \cong S_3$ and so $\Omega^3(S_3) \cong S_3$.

13.2.10. *The minimal projective resolution of the simple* Λ*-module* S_4.

The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_4\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto \delta e_1\eta + \varepsilon e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

13.2.11. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta$ and $e_3\lambda = f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\delta e_1\eta + \varepsilon e_3\lambda = 0$ so $\delta e_1\eta + \varepsilon e_3\lambda = \delta(c_1e_1 + c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta) + \varepsilon(f_1e_3 + f_2\beta + f_3\gamma + f_4\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta + f_7\beta\sigma\gamma + f_8\beta\sigma\beta\sigma) = c_1\delta + c_2\delta\alpha + c_3\delta\alpha\sigma + c_4\delta\alpha\sigma\gamma + f_1\varepsilon + f_2\varepsilon\beta + f_3\varepsilon\gamma + f_4\varepsilon\beta\sigma + f_7\varepsilon\beta\sigma\gamma = c_1\delta + (c_2 + f_2)\delta\alpha + (c_3 + f_4)\varepsilon\beta\sigma + (c_4 + f_7)\varepsilon\beta\sigma\gamma + f_1\varepsilon + f_3\varepsilon\gamma = 0$, that is, $c_1 = f_1 = f_3 = 0, f_2 = -c_2, f_4 = -c_3$ and $f_7 = -c_4$. Thus $e_1\eta = c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta$ and $e_3\lambda = -c_2\beta - c_3\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta - c_4\beta\sigma\gamma + f_8\beta\sigma\beta\sigma$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta, -c_2\beta - c_3\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta - c_4\beta\sigma\gamma + f_8\beta\sigma\beta\sigma) : c_i, f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta, -c_2\beta - c_3\beta\sigma + f_5\gamma\delta + f_6\beta\sigma\beta - c_4\beta\sigma\gamma + f_8\beta\sigma\beta\sigma)$, that is, $x = (0, \gamma\delta)(f_5e_1) + (\alpha, -\beta)(c_2e_2 + c_3\sigma + c_4\sigma\gamma + c_5\sigma\gamma\delta - f_6\sigma\beta - f_8\sigma\beta\sigma)$. Thus $x \in (0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda$.

On the other hand, let $y = (0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu \in (0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu) = \partial^1(\alpha e_2\nu, \gamma\delta e_1\eta - \beta e_2\nu) = \delta\alpha e_2\nu + \varepsilon\gamma\delta e_1\eta - \varepsilon\beta e_2\nu = \varepsilon\gamma\delta e_1\eta + (\delta\alpha - \varepsilon\beta)e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (0, \gamma\delta)e_1\Lambda + (\alpha, -\beta)e_2\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto (0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu$, for $\eta, \nu \in \Lambda$.

13.2.12. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_4)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta$ and $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta$ with $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker}\partial^2$. Then $(0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu = (0, 0)$. So $(0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu = (0, \gamma\delta)(c_1e_1 + c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta) + (\alpha, -\beta)(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\gamma + d_5\sigma\beta\sigma + d_6\sigma\gamma\delta + d_7\sigma\beta\sigma\beta) = (d_1\alpha + d_2\alpha\sigma + d_4\alpha\sigma\gamma + d_6\alpha\sigma\gamma\delta, c_1\gamma\delta + c_2\gamma\delta\alpha + c_3\gamma\delta\alpha\sigma - d_1\beta - d_2\beta\sigma - d_3\beta\sigma\beta - d_4\beta\sigma\gamma - d_5\beta\sigma\beta\sigma) = (0, 0)$. So $d_1\alpha + d_2\alpha\sigma + d_4\alpha\sigma\gamma + d_6\alpha\sigma\gamma\delta = 0$, that is, $d_1 = d_2 = d_4 = d_6 = 0$. Also $c_1\gamma\delta + c_2\gamma\delta\alpha + c_3\gamma\delta\alpha\sigma - d_1\beta - d_2\beta\sigma - d_3\beta\sigma\beta - d_4\beta\sigma\gamma - d_5\beta\sigma\beta\sigma = c_1\gamma\delta - d_1\beta - d_2\beta\sigma + (c_2 - d_3)\beta\sigma\beta - d_4\beta\sigma\gamma + (c_3 - d_5)\beta\sigma\beta\sigma = 0$, that is, $c_1 = d_1 = d_2 = d_4 = 0, d_3 = c_2, d_5 = c_3$. Thus $e_1\eta = c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta$ and $e_2\nu = c_2\sigma\beta + c_3\sigma\beta\sigma + d_7\sigma\beta\sigma\beta$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta, c_2\sigma\beta + c_3\sigma\beta\sigma + d_7\sigma\beta\sigma\beta) : c_i, d_7 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha, \sigma\beta)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = (c_2\alpha + c_3\alpha\sigma + c_4\alpha\sigma\gamma + c_5\alpha\sigma\gamma\delta, c_2\sigma\beta + c_3\sigma\beta\sigma + d_7\sigma\beta\sigma\beta)$, that is, $u = (\alpha, \sigma\beta)(c_2e_2 + c_3\sigma + c_4\sigma\gamma + c_5\sigma\gamma\delta + d_7\sigma\beta)$. Hence $u \in (\alpha, \sigma\beta)e_2\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq (\alpha, \sigma\beta)e_2\Lambda$.

On the other hand, let $v = (\alpha, \sigma\beta)e_2\nu \in (\alpha, \sigma\beta)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha, \sigma\beta)e_2\nu) = (0, \gamma\delta)\alpha + (\alpha, -\beta)\sigma\beta = (\alpha\sigma\beta, \gamma\delta\alpha - \beta\sigma\beta)e_2\nu = (0, 0)$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $(\alpha, \sigma\beta)e_2\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = (\alpha, \sigma\beta)e_2\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_2\nu \mapsto (\alpha, \sigma\beta)e_2\nu$, for $\nu \in \Lambda$.

Thus the maps for S_1 are:

$\partial^1 : e_2\nu \mapsto \alpha e_2\nu,$

$\partial^2 : e_2\nu \mapsto \sigma\beta e_2\nu,$

$\partial^3 : e_4\mu \mapsto \sigma\gamma e_4\mu,$

for $\nu, \mu \in \Lambda$.

Also the maps for S_2 are:

$\partial^1 : e_3\lambda \to \sigma e_3\lambda,$

$\partial^2 : e_4\mu \mapsto \gamma\varepsilon\gamma e_4\mu$

$\partial^3 : (e_1\eta, e_4\mu) \mapsto \delta e_1\eta + \varepsilon\gamma e_4\mu,$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$\partial^1 : (e_2\nu, e_4\mu) \mapsto \beta e_2\nu + \gamma e_4\mu,$

$\partial^2 : e_3\lambda \mapsto (-\sigma, \varepsilon)e_3\lambda,$

$\partial^3 : e_3\lambda \mapsto \beta\sigma\beta\sigma e_3\lambda,$

for $\nu, \lambda, \mu \in \Lambda$.

Moreover, the maps for S_4 are:

$\partial^1 : (e_1\eta, e_3\lambda) \mapsto \delta e_1\eta + \varepsilon e_3\lambda,$

$\partial^2 : (e_1\eta, e_2\nu) \mapsto (0, \gamma\delta)e_1\eta + (\alpha, -\beta)e_2\nu,$

$\partial^3 : e_2\nu \mapsto (\alpha, \sigma\beta)e_2\nu,$

for $\eta, \nu, \lambda \in \Lambda$.

13.3. g^3 for S_1, S_2, S_3 and S_4.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_4 \xrightarrow{\partial^3} \sigma\gamma \xrightarrow{\partial^2} \sigma\beta\sigma\gamma \xrightarrow{\partial^1} \alpha\sigma\beta\sigma\gamma$, so $\alpha\sigma\beta\sigma\gamma \in g^3$.

For S_2

$(e_1, 0) \xrightarrow{\partial^3} \delta \xrightarrow{\partial^2} \gamma\varepsilon\gamma\delta \xrightarrow{\partial^1} \sigma\gamma\varepsilon\gamma\delta$, so $\sigma\gamma\varepsilon\gamma\delta \in g^3$.

$(0, e_4) \xrightarrow{\partial^3} \varepsilon\gamma \xrightarrow{\partial^2} \gamma\varepsilon\gamma\varepsilon\gamma \xrightarrow{\partial^1} \sigma\gamma\varepsilon\gamma\varepsilon\gamma$, so $\sigma\gamma\varepsilon\gamma\varepsilon\gamma \in g^3$.

For S_3

$e_3 \xrightarrow{\partial^3} \beta\sigma\beta\sigma \xrightarrow{\partial^2} (-\sigma\beta\sigma\beta\sigma, \varepsilon\beta\sigma\beta\sigma) \xrightarrow{\partial^1} -\beta\sigma\beta\sigma\beta\sigma + \gamma\varepsilon\beta\sigma\beta\sigma$, so $\gamma\varepsilon\beta\sigma\beta\sigma - \beta\sigma\beta\sigma\beta\sigma \in g^3$.

For S_4

$e_2 \xrightarrow{\partial^3} (\alpha, \sigma\beta) \xrightarrow{\partial^2} (0, \gamma\delta)\alpha + (\alpha, -\beta)\sigma\beta = (\alpha\sigma\beta, \gamma\delta\alpha - \beta\sigma\beta) \xrightarrow{\partial^1} \delta\alpha\sigma\beta + \varepsilon\gamma\delta\alpha - \varepsilon\beta\sigma\beta$, so $\delta\alpha\sigma\beta + \varepsilon\gamma\delta\alpha - \varepsilon\beta\sigma\beta \in g^3$.

Let $g_1^3 = \alpha\sigma\beta\sigma\gamma$, $g_2^3 = \sigma\gamma\varepsilon\gamma\delta$, $g_3^3 = \sigma\gamma\varepsilon\gamma\varepsilon\gamma$, $g_4^3 = \gamma\varepsilon\beta\sigma\beta\sigma - \beta\sigma\beta\sigma\beta\sigma$, $g_5^3 = \delta\alpha\sigma\beta + \varepsilon\gamma\delta\alpha - \varepsilon\beta\sigma\beta$.

So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3, g_5^3\}$.

We know that $g^2 = \{\delta\alpha - \varepsilon\beta, \gamma\varepsilon - \beta\sigma, \alpha\sigma\beta, \varepsilon\gamma\delta, \sigma\gamma\varepsilon\gamma\}$. Denote

$g_1^2 = \delta\alpha - \varepsilon\beta$,
$g_2^2 = \gamma\varepsilon - \beta\sigma$,
$g_3^2 = \alpha\sigma\beta$,
$g_4^2 = \varepsilon\gamma\delta$,
$g_5^2 = \sigma\gamma\varepsilon\gamma$.

So we have

$g_1^3 = g_3^2\sigma\gamma = -\alpha\sigma g_2^2\gamma + \alpha g_5^2$,
$g_2^3 = g_5^2\delta = \sigma\gamma g_4^2$,
$g_3^3 = g_5^2\varepsilon\gamma = \sigma g_2^2\gamma\varepsilon\gamma + \sigma\beta g_5^2$,
$g_4^3 = g_2^2\beta\sigma\beta\sigma = -\gamma g_1^2\sigma\beta\sigma - \beta g_5^2\varepsilon + \gamma\delta g_3^2\sigma + \beta\sigma g_2^2\gamma\varepsilon + \beta\sigma\beta\sigma g_2^2$,
$g_5^3 = g_1^2\sigma\beta + g_4^2\alpha = \delta g_3^2 + \varepsilon g_2^2\beta + \varepsilon\gamma g_1^2$.

13.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0 .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3, \Lambda) \to \dots$$

13.4.1. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$
\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_4 \otimes_{g_1^2} e_2 \mapsto j_1\delta\alpha\\
&e_3 \otimes_{g_2^2} e_3 \mapsto j_2 e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma\\
&e_1 \otimes_{g_3^2} e_2 \mapsto j_5\alpha\\
&e_4 \otimes_{g_4^2} e_1 \mapsto j_6\delta\\
&e_2 \otimes_{g_5^2} e_4 \mapsto j_7\sigma\gamma,
\end{aligned}
$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_4 \mapsto e_1 \otimes_{g_3^2} \sigma\gamma + \alpha\sigma \otimes_{g_2^2} \gamma - \alpha \otimes_{g_5^2} e_4,$

$e_2 \otimes_{g_2^3} e_1 \mapsto e_2 \otimes_{g_5^2} \delta - \sigma\gamma \otimes_{g_4^2} e_1,$

$e_2 \otimes_{g_3^3} e_4 \mapsto e_2 \otimes_{g_5^2} \varepsilon\gamma - \sigma \otimes_{g_2^2} \gamma\varepsilon\gamma - \sigma\beta \otimes_{g_5^2} e_4,$

$e_3 \otimes_{g_4^3} e_3 \mapsto e_3 \otimes_{g_2^2} \beta\sigma\beta\sigma + \gamma \otimes_{g_1^2} \sigma\beta\sigma + \beta \otimes_{g_5^2} \varepsilon - \gamma\delta \otimes_{g_3^2} \sigma - \beta\sigma \otimes_{g_2^2} \gamma\varepsilon - \beta\sigma\beta\sigma \otimes_{g_2^2} e_3,$

$e_4 \otimes_{g_5^3} e_2 \mapsto e_4 \otimes_{g_1^2} \sigma\beta + e_4 \otimes_{g_4^2} \alpha - \delta \otimes_{g_3^2} e_2 - \varepsilon \otimes_{g_2^2} \beta - \varepsilon\gamma \otimes_{g_1^2} e_2.$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_4) = j_5\alpha\sigma\gamma + j_2\alpha\sigma\gamma - j_7\alpha\sigma\gamma = (j_2 + j_5 - j_7)\alpha\sigma\gamma = 0$ then $j_7 = j_2 + j_5$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_1) = j_7\sigma\gamma\delta - j_6\sigma\gamma\delta = (-j_6 + j_7)\sigma\gamma\delta = 0$, that is, $j_6 = j_7$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_4) = 0$.

And $\theta d^3(e_3 \otimes_{g_4^3} e_3) = (j_2e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma)\beta\sigma\beta\sigma + j_1\gamma\delta\alpha\sigma\beta\sigma + \beta(j_7\sigma\gamma)\varepsilon - \gamma\delta(j_5\alpha)\sigma - \beta\sigma(j_2e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma)\gamma\varepsilon - \beta\sigma\beta\sigma(j_2e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma) = (-j_2 - j_5 + j_7)\beta\sigma\beta\sigma = 0$, that is, $j_7 = j_2 + j_5$.

Finally $\theta d^3(e_4 \otimes_{g_5^3} e_2) = j_1\delta\alpha\sigma\beta + j_6\delta\alpha - j_5\delta\alpha - \varepsilon(j_2e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma)\beta - j_1\varepsilon\gamma\delta\alpha = (-j_2 - j_5 + j_6)\delta\alpha = 0$, that is, $j_6 = j_2 + j_5$.

So $\theta \in \operatorname{Ker}\delta^2$ is given by

$$
\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_4 \otimes_{g_1^2} e_2 \mapsto j_1\delta\alpha\\
&e_3 \otimes_{g_2^2} e_3 \mapsto j_2 e_3 + j_3\beta\sigma + j_4\beta\sigma\beta\sigma\\
&e_1 \otimes_{g_3^2} e_2 \mapsto j_5\alpha\\
&e_4 \otimes_{g_4^2} e_1 \mapsto (j_2 + j_5)\delta\\
&e_2 \otimes_{g_5^2} e_4 \mapsto (j_2 + j_5)\sigma\gamma,
\end{aligned}
$$

where $j_i \in K$. Hence $\dim \operatorname{Ker}\delta^2 = 5$.

13.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_2 \to z_1\alpha$

$e_2 \otimes_\sigma e_3 \to z_2\sigma + z_3\sigma\beta\sigma$

$e_3 \otimes_\beta e_2 \to z_4\beta + z_5\beta\sigma\beta$

$e_3 \otimes_\gamma e_4 \to z_6\gamma + z_7\beta\sigma\gamma$

$e_4 \otimes_\varepsilon e_3 \to z_8\varepsilon + z_9\varepsilon\gamma\varepsilon$

$e_4 \otimes_\delta e_1 \to z_{10}\delta,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_4 \otimes_{g_1^2} e_2 \mapsto e_4 \otimes_\delta \alpha + \delta \otimes_\alpha e_2 - e_4 \otimes_\varepsilon \beta - \varepsilon \otimes_\beta e_2$
$e_3 \otimes_{g_2^2} e_3 \mapsto e_3 \otimes_\gamma \varepsilon + \gamma \otimes_\varepsilon e_3 - e_3 \otimes_\beta \sigma - \beta \otimes_\sigma e_3$
$e_1 \otimes_{g_3^2} e_2 \mapsto e_1 \otimes_\alpha \sigma\beta + \alpha \otimes_\sigma \beta + \alpha\sigma \otimes_\beta e_2$
$e_4 \otimes_{g_4^2} e_1 \mapsto e_4 \otimes_\varepsilon \gamma\delta + \varepsilon \otimes_\gamma \delta + \varepsilon\gamma \otimes_\delta e_1$
$e_2 \otimes_{g_5^2} e_4 \mapsto e_2 \otimes_\sigma \gamma\varepsilon\gamma + \sigma \otimes_\gamma \varepsilon\gamma + \sigma\gamma \otimes_\varepsilon \gamma + \sigma\gamma\varepsilon \otimes_\gamma e_4.$

Then the map φd^2 is given by

$\varphi d^2(e_4 \otimes_{g_1^2} e_2) = z_{10}\delta\alpha + z_1\delta\alpha - z_8\varepsilon\beta - z_9\varepsilon\gamma\varepsilon\beta - z_4\varepsilon\beta - z_5\varepsilon\beta\sigma\beta = (z_1 - z_4 - z_8 + z_{10})\delta\alpha,$

$\varphi d^2(e_3 \otimes_{g_2^2} e_3) = z_6\gamma\varepsilon + z_7\beta\sigma\gamma\varepsilon + z_8\gamma\varepsilon + z_9\gamma\varepsilon\gamma\varepsilon - z_4\beta\sigma - z_5\beta\sigma\beta\sigma - z_2\beta\sigma - z_3\beta\sigma\beta\sigma = (-z_2 - z_4 + z_6 + z_8)\beta\sigma + (-z_3 - z_5 + z_7 + z_9)\beta\sigma\beta\sigma,$

$\varphi d^2(e_1 \otimes_{g_3^2} e_2) = 0,$

$\varphi d^2(e_4 \otimes_{g_4^2} e_1) = 0,$

$\varphi d^2(e_2 \otimes_{g_5^2} e_4) = 0.$

Thus φd^2 is given by

$P^2 \to \Lambda$
$e_4 \otimes_{g_1^2} e_2 \mapsto (z_1 - z_4 - z_8 + z_{10})\delta\alpha$
$e_3 \otimes_{g_2^2} e_3 \mapsto (-z_2 - z_4 + z_6 + z_8)\beta\sigma + (-z_3 - z_5 + z_7 + z_9)\beta\sigma\beta\sigma$
$e_1 \otimes_{g_3^2} e_2 \mapsto 0$
$e_4 \otimes_{g_4^2} e_1 \mapsto 0$
$e_2 \otimes_{g_5^2} e_4 \mapsto 0,$

where $z_i \in K$. Therefore $\dim \operatorname{Im} \delta^1 = 3$.

13.4.3. $\mathrm{HH}^2(\Lambda)$.

From 13.4.1 and 13.4.2 we have that $\dim \mathrm{HH}^2(\Lambda) = 2$ and then

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_4 \otimes_{g_1^2} e_2 \mapsto 0 \\ e_3 \otimes_{g_2^2} e_3 \mapsto d_1 e_3 \\ e_1 \otimes_{g_3^2} e_2 \mapsto d_2\alpha \\ e_4 \otimes_{g_4^2} e_1 \mapsto (d_1 + d_2)\delta \\ e_2 \otimes_{g_5^2} e_4 \mapsto (d_1 + d_2)\sigma\gamma \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$x : P^2 \to \Lambda$
$e_3 \otimes_{g_2^2} e_3 \mapsto e_3$
$e_4 \otimes_{g_4^2} e_1 \mapsto \delta$
$e_2 \otimes_{g_5^2} e_4 \mapsto \sigma\gamma$
else $\mapsto 0,$

$y : P^2 \to \Lambda$
$e_1 \otimes_{g_3^2} e_2 \mapsto \alpha$
$e_4 \otimes_{g_4^2} e_1 \mapsto \delta$
$e_2 \otimes_{g_5^2} e_4 \mapsto \sigma\gamma$
else $\mapsto 0.$

14. The algebra A_{10}

Definition 14.1. [5] *Let A_{10} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd}
 & 1 \arrow[d, shift left, "\beta"] & \\
 & 2 \arrow[u, shift left, "\alpha"] \arrow[dr, "\delta"] & \\
4 \arrow[ur, "\zeta"] & & 3 \arrow[ll, "\gamma"]
\end{tikzcd}$$

and

$$I = \langle \zeta\alpha\beta - \zeta\delta\gamma\zeta, \alpha\beta\delta - \delta\gamma\zeta\delta, \beta\alpha, \gamma(\zeta\delta\gamma)^2 \rangle.$$

14.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$$\begin{aligned}
e_1\Lambda &= sp\{e_1, \beta, \beta\delta, \beta\delta\gamma, \beta\delta\gamma\zeta, \beta\delta\gamma\zeta\alpha\},\\
e_2\Lambda &= sp\{e_2, \alpha, \delta, \alpha\beta, \delta\gamma, \alpha\beta\delta, \delta\gamma\zeta, \delta\gamma\zeta\alpha, \alpha\beta\delta\gamma, \alpha\beta\delta\gamma\zeta\},\\
e_3\Lambda &= sp\{e_3, \gamma, \gamma\zeta, \gamma\zeta\alpha, \gamma\zeta\delta, \gamma\zeta\alpha\beta, \gamma\zeta\delta\gamma, \gamma\zeta\delta\gamma\zeta\delta\},\\
e_4\Lambda &= sp\{e_4, \zeta, \zeta\alpha, \zeta\delta, \zeta\delta\gamma, \zeta\delta\gamma\zeta, \zeta\alpha\beta\delta, \zeta\alpha\beta\delta\gamma\}.
\end{aligned}$$

So we have for $e_1\Lambda$

$$\begin{tikzcd}
e_1 \arrow[d, dash] \\
\beta \arrow[d, dash] \\
\beta\delta \arrow[d, dash] \\
\beta\delta\gamma \arrow[d, dash] \\
\beta\delta\gamma\zeta \arrow[d, dash] \\
\beta\delta\gamma\zeta\alpha
\end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd}
 & e_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha \arrow[d, dash] & & \delta \arrow[d, dash] \\
\alpha\beta \arrow[dd, dash] & & \delta\gamma \arrow[d, dash] \\
 & & \delta\gamma\zeta \arrow[dll, dash] \arrow[d, dash] \\
\alpha\beta\delta \arrow[d, dash] & & \delta\gamma\zeta\alpha \arrow[ddl, dash] \\
\alpha\beta\delta\gamma \arrow[dr, dash] & & \\
 & \alpha\beta\delta\gamma\zeta &
\end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{tikzcd}
 & e_3 \arrow[d, dash] & \\
 & \gamma \arrow[d, dash] & \\
 & \gamma\zeta \arrow[dl, dash] \arrow[dr, dash] & \\
\gamma\zeta\alpha \arrow[ddr, dash] & & \gamma\zeta\delta \arrow[d, dash] \\
 & & \gamma\zeta\delta\gamma \arrow[dl, dash] \\
 & \gamma\zeta\alpha\beta \arrow[d, dash] & \\
 & \gamma\zeta\delta\gamma\zeta\delta &
\end{tikzcd}$$

And for $e_4\Lambda$

$$
\begin{tikzcd}[arrows=dash]
 & e_4 \arrow[d] & \\
 & \zeta \arrow[dl] \arrow[dr] & \\
\zeta\alpha \arrow[ddr] & & \zeta\delta \arrow[d] \\
 & & \zeta\delta\gamma \arrow[dl] \\
 & \zeta\delta\gamma\zeta \arrow[d] & \\
 & \zeta\alpha\beta\delta \arrow[d] & \\
 & \zeta\alpha\beta\delta\gamma &
\end{tikzcd}
$$

14.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

14.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \beta e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

14.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\beta e_2\nu = 0$, so $\beta(d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta) = d_1\beta + d_3\beta\delta + d_5\beta\delta\gamma + d_7\beta\delta\gamma\zeta + d_8\beta\delta\gamma\zeta\alpha = 0$ and then $d_1 = d_3 = d_5 = d_7 = d_8 = 0$. Thus $e_2\nu = d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\delta + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\delta + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \alpha e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\delta + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$, that is, $x = \alpha(d_2e_1 + d_4\beta + d_6\beta\delta + d_9\beta\delta\gamma + d_{10}\beta\delta\gamma\zeta)$. Thus $x \in \alpha e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \alpha e_1\Lambda$.

On the other hand, let $y = \alpha e_1\eta \in \alpha e_1\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha e_1\eta) = \beta\alpha e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\alpha e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \alpha e_1\Lambda$. □

So $\partial^2 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \alpha e_1\eta$, for $\eta \in \Lambda$.

14.2.3. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^2$. Then $\alpha e_1\eta = 0$. So $\alpha e_1\eta = \alpha(c_1e_1 + c_2\beta + c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha) = c_1\alpha + c_2\alpha\beta + c_3\alpha\beta\delta + c_4\alpha\beta\delta\gamma + c_5\alpha\beta\delta\gamma\zeta = 0$, that is, $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Thus $e_1\eta = c_6\beta\delta\gamma\zeta\alpha$. Therefore $\operatorname{Ker}\partial^2 = \{c_6\beta\delta\gamma\zeta\alpha : c_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\delta\gamma\zeta\alpha e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = c_6\beta\delta\gamma\zeta\alpha$ so $u = \beta\delta\gamma\zeta\alpha(c_6e_1)$. Hence $u \in \beta\delta\gamma\zeta\alpha e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\delta\gamma\zeta\alpha e_1\Lambda$.

On the other hand, let $v = \beta\delta\gamma\zeta\alpha e_1\eta \in \beta\delta\gamma\zeta\alpha e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\delta\gamma\zeta\alpha e_1\eta) = \alpha\beta\delta\gamma\zeta\alpha e_1\eta = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta\delta\gamma\zeta\alpha e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\delta\gamma\zeta\alpha e_1\Lambda$. □

We remark that $\operatorname{Ker}\partial^2 \cong S_1$ and then $\Omega^3(S_1) \cong S_1$.

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \beta\delta\gamma\zeta\alpha e_1\eta$, for $\eta \in \Lambda$.

14.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \to \alpha e_1\eta + \delta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

14.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha$ and $e_3\lambda = f_1e_3 + f_2\gamma + f_3\gamma\zeta + f_4\gamma\zeta\alpha + f_5\gamma\zeta\delta + f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\alpha e_1\eta + \delta e_3\lambda = 0$ so $\alpha e_1\eta + \delta e_3\lambda = \alpha(c_1e_1 + c_2\beta + c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha) + \delta(f_1e_3 + f_2\gamma + f_3\gamma\zeta + f_4\gamma\zeta\alpha + f_5\gamma\zeta\delta + f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta) = c_1\alpha + c_2\alpha\beta + c_3\alpha\beta\delta + c_4\alpha\beta\delta\gamma + c_5\alpha\beta\delta\gamma\zeta + f_1\delta + f_2\delta\gamma + f_3\delta\gamma\zeta + f_4\delta\gamma\zeta\alpha + f_5\delta\gamma\zeta\delta + f_6\delta\gamma\zeta\alpha\beta + f_7\delta\gamma\zeta\delta\gamma = c_1\alpha + c_2\alpha\beta + f_1\delta + f_2\delta\gamma + f_3\delta\gamma\zeta + f_4\delta\gamma\zeta\alpha + (c_3 + f_5)\alpha\beta\delta + (c_4 + f_7)\alpha\beta\delta\gamma + (c_5 + f_6)\alpha\beta\delta\gamma\zeta = 0$, that is, $c_1 = c_2 = f_1 = f_2 = f_3 = f_4 = 0, f_5 = -c_3, f_7 = -c_4$ and $f_6 = -c_5$. Thus $e_1\eta = c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha$ and $e_3\lambda = -c_3\gamma\zeta\delta - c_4\gamma\zeta\delta\gamma - c_5\gamma\zeta\alpha\beta + f_8\gamma\zeta\delta\gamma\zeta\delta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha, -c_3\gamma\zeta\delta - c_4\gamma\zeta\delta\gamma - c_5\gamma\zeta\alpha\beta + f_8\gamma\zeta\delta\gamma\zeta\delta) : c_i, f_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\beta\delta, -\gamma\zeta\delta)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = (c_3\beta\delta + c_4\beta\delta\gamma + c_5\beta\delta\gamma\zeta + c_6\beta\delta\gamma\zeta\alpha, -c_3\gamma\zeta\delta - c_4\gamma\zeta\delta\gamma - c_5\gamma\zeta\alpha\beta + f_8\gamma\zeta\delta\gamma\zeta\delta)$. So $x = (\beta\delta, -\gamma\zeta\delta)(c_3e_3 + c_4\gamma + c_5\gamma\zeta + c_6\gamma\zeta\alpha - f_8\gamma\zeta\delta)$. Thus $x \in (\beta\delta, -\gamma\zeta\delta)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\beta\delta, -\gamma\zeta\delta)e_3\Lambda$.

On the other hand, let $y = (\beta\delta, -\gamma\zeta\delta)e_3\lambda \in (\beta\delta, -\gamma\zeta\delta)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\beta\delta, -\gamma\zeta\delta)e_3\lambda) = (\alpha\beta\delta - \delta\gamma\zeta\delta)e_3\lambda = 0$. So $(\beta\delta, -\gamma\zeta\delta)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\beta\delta, -\gamma\zeta\delta)e_3\Lambda$. □

So $\partial^2 : e_3\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_3\lambda \mapsto (\beta\delta, -\gamma\zeta\delta)e_3\lambda$, for $\lambda \in \Lambda$.

14.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\gamma + f_3\gamma\zeta + f_4\gamma\zeta\alpha + f_5\gamma\zeta\delta + f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^2$. Then $(\beta\delta, -\gamma\zeta\delta)e_3\lambda = (0,0)$. So $(\beta\delta, -\gamma\zeta\delta)e_3\lambda = (\beta\delta, -\gamma\zeta\delta)(f_1e_3 + f_2\gamma + f_3\gamma\zeta + f_4\gamma\zeta\alpha + f_5\gamma\zeta\delta + f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta) = (f_1\beta\delta + f_2\beta\delta\gamma + f_3\beta\delta\gamma\zeta + f_4\beta\delta\gamma\zeta\alpha, -f_1\gamma\zeta\delta - f_2\gamma\zeta\delta\gamma - f_3\gamma\zeta\delta\gamma\zeta - f_5\gamma\zeta\delta\gamma\zeta\delta) = (0,0)$. So $f_1\beta\delta + f_2\beta\delta\gamma + f_3\beta\delta\gamma\zeta + f_4\beta\delta\gamma\zeta\alpha = 0$, that is, $f_1 = f_2 = f_3 = f_4 = 0$. Also $-f_1\gamma\zeta\delta - f_2\gamma\zeta\delta\gamma - f_3\gamma\zeta\delta\gamma\zeta - f_5\gamma\zeta\delta\gamma\zeta\delta = 0$ and then $f_1 = f_2 = f_3 = f_5 = 0$. Thus $e_3\lambda = f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta$. Therefore $\operatorname{Ker}\partial^2 = \{f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \gamma\zeta\delta\gamma e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_6\gamma\zeta\alpha\beta + f_7\gamma\zeta\delta\gamma + f_8\gamma\zeta\delta\gamma\zeta\delta$ so $u = \gamma\zeta\delta\gamma(f_7e_4 + f_6\zeta + f_8\zeta\delta)$. Hence $u \in \gamma\zeta\delta\gamma e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \gamma\zeta\delta\gamma e_4\Lambda$.

On the other hand, let $v = \gamma\zeta\delta\gamma e_4\mu \in \gamma\zeta\delta\gamma e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\gamma\zeta\delta\gamma e_4\mu) = (\beta\delta\gamma\zeta\delta\gamma, -\gamma\zeta\delta\gamma\zeta\delta\gamma)e_4\mu = (0,0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\gamma\zeta\delta\gamma e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \gamma\zeta\delta\gamma e_4\Lambda$. □

So the map $\partial^3 : e_4\Lambda \to e_3\Lambda$ is given by $e_4\mu \mapsto \gamma\zeta\delta\gamma e_4\mu$, for $\mu \in \Lambda$.

14.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3.
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_4\Lambda \to e_3\Lambda$ is given by $e_4\mu \mapsto \gamma e_4\mu$, for $\mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

14.2.8. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_4\mu = t_1e_4 + t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^1$. Then $\gamma e_4\mu = 0$ so $\gamma(t_1e_4 + t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma) = t_1\gamma + t_2\gamma\zeta + t_3\gamma\zeta\alpha + t_4\gamma\zeta\delta + t_5\gamma\zeta\delta\gamma + t_6\gamma\zeta\delta\gamma\zeta + t_7\gamma\zeta\alpha\beta\delta = 0$, that is, $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = 0$. Thus $e_4\mu = t_8\zeta\alpha\beta\delta\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{t_8\zeta\alpha\beta\delta\gamma : t_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \zeta\delta\gamma\zeta\delta\gamma e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = t_8\zeta\alpha\beta\delta\gamma$. So $x = \zeta\alpha\beta\delta\gamma(t_8e_4)$. Thus $x \in \zeta\alpha\beta\delta\gamma e_4\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \zeta\alpha\beta\delta\gamma e_4\Lambda$.

On the other hand, let $y = \zeta\alpha\beta\delta\gamma e_4\mu \in \zeta\alpha\beta\delta\gamma e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\zeta\alpha\beta\delta\gamma e_4\mu) = \gamma\zeta\alpha\beta\delta\gamma e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\zeta\alpha\beta\delta\gamma e_4\Lambda \subseteq \operatorname{Ker}\partial^1$. Hence $\operatorname{Ker}\partial^1 = \zeta\alpha\beta\delta\gamma e_4\Lambda$. But $\zeta\alpha\beta\delta\gamma = \zeta\delta\gamma\zeta\delta\gamma$ and then $\operatorname{Ker}\partial^1 = \zeta\delta\gamma\zeta\delta\gamma e_4\Lambda$. □

Note that $\operatorname{Ker}\partial^1 \cong S_4$ and $\Omega^2(S_3) \cong S_4$.

So $\partial^2 : e_4\Lambda \to e_4\Lambda$ is given by $e_4\mu \mapsto \zeta\delta\gamma\zeta\delta\gamma e_4\mu$, for $\mu \in \Lambda$.

14.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_4\mu = t_1e_4 + t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^2$. Then $\zeta\delta\gamma\zeta\delta\gamma e_4\mu = 0$. So $\zeta\delta\gamma\zeta\delta\gamma e_4\mu = \zeta\delta\gamma\zeta\delta\gamma(t_1e_4 + t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma) = t_1\zeta\delta\gamma\zeta\delta\gamma = 0$, that is, $t_1 = 0$. Thus $e_4\mu = t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \zeta e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = t_2\zeta + t_3\zeta\alpha + t_4\zeta\delta + t_5\zeta\delta\gamma + t_6\zeta\delta\gamma\zeta + t_7\zeta\alpha\beta\delta + t_8\zeta\alpha\beta\delta\gamma$, that is, $u = \zeta(t_2e_2 + t_3\alpha + t_4\delta + t_5\delta\gamma + t_6\delta\gamma\zeta + t_7\alpha\beta\delta + t_8\alpha\beta\delta\gamma)$. Hence $u \in \zeta e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \zeta e_2\Lambda$.

On the other hand, let $v = \zeta e_2\nu \in \zeta e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\zeta e_2\nu) = \zeta\delta\gamma\zeta\delta\gamma\zeta e_2\nu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\zeta e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \zeta e_2\Lambda$. □

So the map $\partial^3 : e_2\Lambda \to e_4\Lambda$ is given by $e_2\nu \mapsto \zeta e_2\nu$, for $\nu \in \Lambda$.

14.2.10. *The minimal projective resolution of the simple Λ-module S_4.*

The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_4\Lambda$ is given by $e_2\nu \mapsto \zeta e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

14.2.11. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\zeta e_2\nu = 0$ so $\zeta e_2\nu = \zeta(d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta) = d_1\zeta + d_2\zeta\alpha + d_3\zeta\delta + d_4\zeta\alpha\beta + d_5\zeta\delta\gamma + d_6\zeta\alpha\beta\delta + d_7\zeta\delta\gamma\zeta + d_9\zeta\alpha\beta\delta\gamma = d_1\zeta + d_2\zeta\alpha + d_3\zeta\delta + (d_4 + d_7)\zeta\delta\gamma\zeta + d_5\zeta\delta\gamma + d_6\zeta\alpha\beta\delta + d_9\zeta\alpha\beta\delta\gamma = 0$, that is, $d_1 = d_2 = d_3 = d_5 = d_6 = d_9 = 0, d_7 = -d_4$. Thus $e_2\nu = d_4(\alpha\beta - \delta\gamma\zeta) + d_8\delta\gamma\zeta\alpha + d_{10}\alpha\beta\delta\gamma\zeta$.

Hence $\operatorname{Ker}\partial^1 = \{d_4(\alpha\beta - \delta\gamma\zeta) + d_8\delta\gamma\zeta\alpha + d_{10}\alpha\beta\delta\gamma\zeta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = d_4(\alpha\beta - \delta\gamma\zeta) + d_8\delta\gamma\zeta\alpha + d_{10}\alpha\beta\delta\gamma\zeta$, that is, $x = (\alpha\beta - \delta\gamma\zeta)(d_4e_2 - d_8\alpha) + \alpha\beta\delta\gamma\zeta(d_{10}e_2)$. However we can show that $\alpha\beta\delta\gamma\zeta(d_{10}e_2) \subseteq (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$ since $\alpha\beta\delta\gamma\zeta(d_{10}e_2) = (\alpha\beta - \delta\gamma\zeta)e_2\psi$ where $e_2\psi = -d_{10}\alpha\beta$. Thus $x \in (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$.

On the other hand, let $y = (\alpha\beta - \delta\gamma\zeta)e_2\nu \in (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha\beta - \delta\gamma\zeta)e_2\nu) = (\zeta\alpha\beta - \zeta\delta\gamma\zeta)e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha\beta - \delta\gamma\zeta)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha\beta - \delta\gamma\zeta)e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\nu \mapsto (\alpha\beta - \delta\gamma\zeta)e_2\nu$, for $\nu \in \Lambda$.

14.2.12. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_4)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^2$. Then $(\alpha\beta - \delta\gamma\zeta)e_2\nu = 0$. So $(\alpha\beta - \delta\gamma\zeta)e_2\nu = (\alpha\beta - \delta\gamma\zeta)(d_1e_2 + d_2\alpha + d_3\delta + d_4\alpha\beta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta) = 0$. So $d_1\alpha\beta + d_3\alpha\beta\delta + d_5\alpha\beta\delta\gamma + d_7\alpha\beta\delta\gamma\zeta - d_1\delta\gamma\zeta - d_2\delta\gamma\zeta\alpha - d_3\delta\gamma\zeta\delta - d_4\delta\gamma\zeta\alpha\beta - d_5\delta\gamma\zeta\delta\gamma - d_7\delta\gamma\zeta\delta\gamma\zeta = d_1(\alpha\beta - \delta\gamma\zeta) - d_2\delta\gamma\zeta\alpha - d_4\delta\gamma\zeta\alpha\beta = 0$, that is, $d_1 = d_2 = d_4 = 0$. Thus $e_2\nu = d_3\delta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$. Therefore $\operatorname{Ker}\partial^2 = \{d_3\delta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \delta e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = d_3\delta + d_5\delta\gamma + d_6\alpha\beta\delta + d_7\delta\gamma\zeta + d_8\delta\gamma\zeta\alpha + d_9\alpha\beta\delta\gamma + d_{10}\alpha\beta\delta\gamma\zeta$, that is, $u = \delta(d_3e_3 + d_5\gamma + d_6\gamma\zeta\delta + d_7\gamma\zeta + d_8\gamma\zeta\alpha + d_9\gamma\zeta\delta\gamma + d_{10}\gamma\zeta\delta\gamma\zeta)$. Hence $u \in \delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \delta e_3\Lambda$.

On the other hand, let $v = \delta e_3\lambda \in \delta e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\delta e_3\lambda) = (\alpha\beta\delta - \delta\gamma\zeta\delta)e_3\lambda = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \delta e_3\Lambda$. □

So the map $\partial^3 : e_3\Lambda \to e_2\Lambda$ is given by $e_3\lambda \mapsto \delta e_3\lambda$, for $\lambda \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : e_2\nu \mapsto \beta e_2\nu,$$
$$\partial^2 : e_1\eta \mapsto \alpha e_1\eta,$$
$$\partial^3 : e_1\eta \mapsto \beta\delta\gamma\zeta\alpha e_1\eta,$$

for $\eta, \nu \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : (e_1\eta, e_3\lambda) \to \alpha e_1\eta + \delta e_3\lambda,$$
$$\partial^2 : e_3\lambda \mapsto (\beta\delta, -\gamma\zeta\delta)e_3\lambda$$
$$\partial^3 : e_4\mu \mapsto \gamma\zeta\delta\gamma e_4\mu,$$

for $\eta, \mu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\partial^1 : e_4\mu \mapsto \gamma e_4\mu,$$
$$\partial^2 : e_4\mu \mapsto \zeta\delta\gamma\zeta\delta\gamma e_4\mu,$$
$$\partial^3 : e_2\nu \mapsto \zeta e_2\nu,$$

for $\nu, \mu \in \Lambda$.

Moreover, the maps for S_4 are:

$$\partial^1 : e_2\nu \mapsto \zeta e_2\nu,$$
$$\partial^2 : e_2\nu \mapsto (\alpha\beta - \delta\gamma\zeta)e_2\nu,$$
$$\partial^3 : e_3\lambda \mapsto \delta e_3\lambda,$$

for $\nu, \lambda \in \Lambda$.

14.3. g^3 **for** S_1, S_2, S_3 **and** S_4**.**

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_1 \xrightarrow{\partial^3} \beta\delta\gamma\zeta\alpha \xrightarrow{\partial^2} \alpha\beta\delta\gamma\zeta\alpha \xrightarrow{\partial^1} \beta\alpha\beta\delta\gamma\zeta\alpha$, so $\beta\alpha\beta\delta\gamma\zeta\alpha \in g^3$.

For S_2

$e_4 \xrightarrow{\partial^3} \gamma\zeta\delta\gamma \xrightarrow{\partial^2} (\beta\delta, -\gamma\zeta\delta)\gamma\zeta\delta\gamma = (\beta\delta\gamma\zeta\delta\gamma, -\gamma\zeta\delta\gamma\zeta\delta\gamma) \xrightarrow{\partial^1} \alpha\beta\delta\gamma\zeta\delta\gamma - \delta\gamma\zeta\delta\gamma\zeta\delta\gamma$,
so $\alpha\beta\delta\gamma\zeta\delta\gamma - \delta\gamma\zeta\delta\gamma\zeta\delta\gamma \in g^3$.

For S_3

$e_2 \xrightarrow{\partial^3} \zeta \xrightarrow{\partial^2} \zeta\delta\gamma\zeta\delta\gamma\zeta \xrightarrow{\partial^1} \gamma\zeta\delta\gamma\zeta\delta\gamma\zeta$, so $\gamma\zeta\delta\gamma\zeta\delta\gamma\zeta \in g^3$.

For S_4

$e_3 \xrightarrow{\partial^3} \delta \xrightarrow{\partial^2} \alpha\beta\delta - \delta\gamma\zeta\delta \xrightarrow{\partial^1} \zeta\alpha\beta\delta - \zeta\delta\gamma\zeta\delta$, so $\zeta\alpha\beta\delta - \zeta\delta\gamma\zeta\delta \in g^3$.

Let $g^3_1 = \beta\alpha\beta\delta\gamma\zeta\alpha$, $g^3_2 = \alpha\beta\delta\gamma\zeta\delta\gamma - \delta\gamma\zeta\delta\gamma\zeta\delta\gamma$, $g^3_3 = \gamma\zeta\delta\gamma\zeta\delta\gamma\zeta$, $g^3_4 = \zeta\alpha\beta\delta - \zeta\delta\gamma\zeta\delta$.

So $g^3 = \{g^3_1, g^3_2, g^3_3, g^3_4\}$.

We know that $g^2 = \{\zeta\alpha\beta - \zeta\delta\gamma\zeta, \alpha\beta\delta - \delta\gamma\zeta\delta, \beta\alpha, \gamma(\zeta\delta\gamma)^2\}$. Denote

$g_1^2 = \beta\alpha,$
$g_2^2 = \alpha\beta\delta - \delta\gamma\zeta\delta,$
$g_3^2 = \gamma(\zeta\delta\gamma)^2,$
$g_4^2 = \zeta\alpha\beta - \zeta\delta\gamma\zeta.$

So we have

$g_1^3 = g_1^2\beta\delta\gamma\zeta\alpha = \beta g_2^2\gamma\zeta\alpha - \beta\delta\gamma g_4^2\alpha + \beta\delta\gamma\zeta\alpha g_1^2,$
$g_2^3 = g_2^2\gamma\zeta\delta\gamma = \alpha g_1^2\beta\delta\gamma - \alpha\beta g_2^2\gamma + \delta\gamma g_4^2\delta\gamma - \delta\gamma\zeta g_2^2\gamma - \delta g_3^2,$
$g_3^3 = g_3^2\zeta = -\gamma g_4^2\delta\gamma\zeta - \gamma g_4^2\alpha\beta + \gamma\zeta g_2^2\gamma\zeta + \gamma\zeta\alpha g_1^2\beta - \gamma\zeta\delta\gamma g_4^2,$
$g_4^3 = g_4^2\delta = \zeta g_2^2.$

14.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \ldots$$

14.4.1. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\beta\delta\gamma\zeta\alpha\\
&e_2 \otimes_{g_2^2} e_3 \mapsto j_3\delta + j_4\alpha\beta\delta\\
&e_3 \otimes_{g_3^2} e_4 \mapsto j_5\gamma + j_6\gamma\zeta\delta\gamma\\
&e_4 \otimes_{g_4^2} e_2 \mapsto j_7\zeta + j_8\zeta\delta\gamma\zeta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \beta\delta\gamma\zeta\alpha - \beta \otimes_{g_2^2} \gamma\zeta\alpha + \beta\delta\gamma \otimes_{g_4^2} \alpha - \beta\delta\gamma\zeta\alpha \otimes_{g_1^2} e_1,$
$e_2 \otimes_{g_2^3} e_4 \mapsto e_2 \otimes_{g_2^2} \gamma\zeta\delta\gamma - \alpha \otimes_{g_1^2} \beta\delta\gamma + \alpha\beta \otimes_{g_2^2} \gamma - \delta\gamma \otimes_{g_4^2} \delta\gamma$
$\quad + \delta\gamma\zeta \otimes_{g_2^2} \gamma + \delta \otimes_{g_3^2} e_4,$
$e_3 \otimes_{g_3^3} e_2 \mapsto e_3 \otimes_{g_3^2} \zeta + \gamma \otimes_{g_4^2} \delta\gamma\zeta + \gamma \otimes_{g_4^2} \alpha\beta - \gamma\zeta \otimes_{g_2^2} \gamma\zeta - \gamma\zeta\alpha \otimes_{g_1^2} \beta$
$\quad + \gamma\zeta\delta\gamma \otimes_{g_4^2} e_2,$
$e_4 \otimes_{g_4^3} e_3 \mapsto e_4 \otimes_{g_4^2} \delta - \zeta \otimes_{g_2^2} e_3.$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\beta\delta\gamma\zeta\alpha)\beta\delta\gamma\zeta\alpha - \beta(j_3\delta + j_4\alpha\beta\delta)\gamma\zeta\alpha + \beta\delta\gamma(j_7\zeta + j_8\zeta\delta\gamma\zeta)\alpha - \beta\delta\gamma\zeta\alpha(j_1e_1 + j_2\beta\delta\gamma\zeta\alpha) = (-j_3 + j_7)\beta\delta\gamma\zeta\alpha = 0$ then $j_7 = j_3$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_4) = (j_3\delta + j_4\alpha\beta\delta)\gamma\zeta\delta\gamma - \alpha(j_1e_1 + j_2\beta\delta\gamma\zeta\alpha)\beta\delta\gamma + \alpha\beta(j_3\delta + j_4\alpha\beta\delta)\gamma - \delta\gamma(j_7\zeta + j_8\zeta\delta\gamma\zeta)\delta\gamma + \delta\gamma\zeta(j_3\delta + j_4\alpha\beta\delta)\gamma + \delta(j_5\gamma + j_6\gamma\zeta\delta\gamma) = (-j_1 + 3j_3 + j_6 - j_7)\alpha\beta\delta\gamma + j_5\delta\gamma = (-j_1 + 2j_3 + j_6)\alpha\beta\delta\gamma + j_5\delta\gamma = 0$, that is, $j_6 = j_1 - 2j_3, j_5 = 0$.

Also $\theta d^3(e_3 \otimes_{g_3^3} e_2) = (j_5\gamma + j_6\gamma\zeta\delta\gamma)\zeta + \gamma(j_7\zeta + j_8\zeta\delta\gamma\zeta)\delta\gamma\zeta + \gamma(j_7\zeta + j_8\zeta\delta\gamma\zeta)\alpha\beta - \gamma\zeta(j_3\delta + j_4\alpha\beta\delta)\gamma\zeta - \gamma\zeta\alpha(j_1e_1 + j_2\beta\delta\gamma\zeta\alpha)\beta + \gamma\zeta\delta\gamma(j_7\zeta + j_8\zeta\delta\gamma\zeta) = (-j_1 + 2j_7 + j_6)\gamma\zeta\delta\gamma\zeta + j_5\gamma\zeta = 0$, that is, $j_6 = j_1 - 2j_3, j_5 = 0$.

And $\theta d^3(e_4 \otimes_{g_4^3} e_3) = (j_7\zeta + j_8\zeta\delta\gamma\zeta)\delta - \zeta(j_3\delta + j_4\alpha\beta\delta) = (j_7 - j_3)\zeta\delta + (j_8 - j_4)\zeta\alpha\beta\delta = 0$, that is, $j_7 = j_3, j_8 = j_4$.

So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\beta\delta\gamma\zeta\alpha\\
&e_2 \otimes_{g_2^2} e_3 \mapsto j_3\delta + j_4\alpha\beta\delta\\
&e_3 \otimes_{g_3^2} e_4 \mapsto (j_1 - 2j_3)\gamma\zeta\delta\gamma\\
&e_4 \otimes_{g_4^2} e_2 \mapsto j_3\zeta + j_4\zeta\delta\gamma\zeta,
\end{aligned}$$

with $j_i \in K$. So $\dim \operatorname{Ker} \delta^2 = 4$. Note that if $\operatorname{char} K = 2$ or if $\operatorname{char} K \neq 2$ the $\dim \operatorname{Ker} \delta^2$ will remains the same.

14.4.2. $\operatorname{Im} \delta^1$.

Now to find $\operatorname{Im} \delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\beta e_2 \to z_1\beta + z_2\beta\delta\gamma\zeta$

$e_2 \otimes_\alpha e_1 \to z_3\alpha + z_4\delta\gamma\zeta\alpha$

$e_2 \otimes_\delta e_3 \to z_5\delta + z_6\alpha\beta\delta$

$e_3 \otimes_\gamma e_4 \to z_7\gamma + z_8\gamma\zeta\delta\gamma$

$e_4 \otimes_\zeta e_2 \to z_9\zeta + z_{10}\zeta\delta\gamma\zeta,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by:

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\beta \alpha + \beta \otimes_\alpha e_1$

$e_2 \otimes_{g_2^2} e_3 \mapsto e_2 \otimes_\alpha \beta\delta + \alpha \otimes_\beta \delta + \alpha\beta \otimes_\delta e_3 - e_2 \otimes_\delta \gamma\zeta\delta - \delta \otimes_\gamma \zeta\delta - \delta\gamma \otimes_\zeta \delta - \delta\gamma\zeta \otimes_\delta e_3$

$e_3 \otimes_{g_3^2} e_4 \mapsto e_3 \otimes_\gamma \zeta\delta\gamma\zeta\delta\gamma + \gamma \otimes_\zeta \delta\gamma\zeta\delta\gamma + \gamma\zeta \otimes_\delta \gamma\zeta\delta\gamma + \gamma\zeta\delta \otimes_\gamma \zeta\delta\gamma + \gamma\zeta\delta\gamma \otimes_\zeta \delta\gamma$
$\quad + \gamma\zeta\delta\gamma\zeta \otimes_\delta \gamma + \gamma\zeta\delta\gamma\zeta\delta \otimes_\gamma e_4$

$e_4 \otimes_{g_4^2} e_2 \mapsto e_4 \otimes_\zeta \alpha\beta + \zeta \otimes_\alpha \beta + \zeta\alpha \otimes_\beta e_2 - e_4 \otimes_\zeta \delta\gamma\zeta - \zeta \otimes_\delta \gamma\zeta - \zeta\delta \otimes_\gamma \zeta - \zeta\delta\gamma \otimes_\zeta e_2.$

Then the map φd^2 is given by:

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1\beta + z_2\beta\delta\gamma\zeta)\alpha + \beta(z_3\alpha + z_4\delta\gamma\zeta\alpha) = (z_2 + z_4)\beta\delta\gamma\zeta\alpha,$

$\varphi d^2(e_2 \otimes_{g_2^2} e_3) = (z_3\alpha + z_4\delta\gamma\zeta\alpha)\beta\delta + \alpha(z_1\beta + z_2\beta\delta\gamma\zeta)\delta + \alpha\beta(z_5\delta + z_6\alpha\beta\delta) - (z_5\delta + z_6\alpha\beta\delta)\gamma\zeta\delta - \delta(z_7\gamma + z_8\gamma\zeta\delta\gamma)\zeta\delta - \delta\gamma(z_9\zeta + z_{10}\zeta\delta\gamma\zeta)\delta - \delta\gamma\zeta(z_5\delta + z_6\alpha\beta\delta) = (z_1 + z_3 - z_5 - z_7 - z_9)\alpha\beta\delta,$

$\varphi d^2(e_3 \otimes_{g_4^2} e_4) = 0,$

$\varphi d^2(e_4 \otimes_{g_4^2} e_2) = (z_9\zeta + z_{10}\zeta\delta\gamma\zeta)\alpha\beta + \zeta(z_3\alpha + z_4\delta\gamma\zeta\alpha)\beta + \zeta\alpha(z_1\beta + z_2\beta\delta\gamma\zeta) - (z_9\zeta + z_{10}\zeta\delta\gamma\zeta)\delta\gamma\zeta - \zeta(z_5\delta + z_6\alpha\beta\delta)\gamma\zeta - \zeta\delta(z_7\gamma + z_8\gamma\zeta\delta\gamma)\zeta - \zeta\delta\gamma(z_9\zeta + z_{10}\zeta\delta\gamma\zeta) = (z_1 + z_3 - z_5 - z_7 - z_9)\zeta\delta\gamma\zeta$. We write $c = z_2 + z_4$ and $c' = z_1 + z_3 - z_5 - z_7 - z_9$.

Thus φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto c\beta\delta\gamma\zeta\alpha \\ e_2 \otimes_{g_2^2} e_3 \mapsto c'\alpha\beta\delta \\ e_3 \otimes_{g_3^2} e_4 \mapsto 0 \\ e_4 \otimes_{g_4^2} e_2 \mapsto c'\zeta\delta\gamma\zeta, \end{array}$$

where $c, c' \in K$. Therefore $\dim \operatorname{Im} \delta^1 = 2$.

14.4.3. $\mathrm{HH}^2(\Lambda)$.

From 14.4.1 and 14.4.2 we have that if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_2 \otimes_{g_2^2} e_3 \mapsto d_2\delta \\ e_3 \otimes_{g_3^2} e_4 \mapsto d_1\gamma\zeta\delta\gamma \\ e_4 \otimes_{g_4^2} e_2 \mapsto d_2\zeta, \end{array} \right\}$$

with $d_i \in K$.

And a basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_3^2} e_4 \mapsto \gamma\zeta\delta\gamma \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_3 \mapsto \delta \\ e_3 \otimes_{g_3^2} e_4 \mapsto 0 \\ e_4 \otimes_{g_4^2} e_2 \mapsto \zeta \\ \text{else} \mapsto 0. \end{array}$$

Now if $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_2 \otimes_{g_2^2} e_3 \mapsto d_2\delta \\ e_3 \otimes_{g_3^2} e_4 \mapsto (d_1 - 2d_2)\gamma\zeta\delta\gamma \\ e_4 \otimes_{g_4^2} e_2 \mapsto d_2\zeta \end{array} \right\}$$

with $d_i \in K$.

And a basis of $\mathrm{HH}^2(\Lambda) = sp\{u, w\}$ where

$$\begin{array}{l} u : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_3^2} e_4 \mapsto \gamma\zeta\delta\gamma \\ \text{else} \mapsto 0, \end{array}$$

and

$$\begin{array}{l} w : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_3 \mapsto \delta \\ e_3 \otimes_{g_3^2} e_4 \mapsto -\frac{1}{2}\gamma\zeta\delta\gamma \\ e_4 \otimes_{g_4^2} e_2 \mapsto \zeta \\ \text{else} \mapsto 0. \end{array}$$

15. The algebra A_{11}

Definition 15.1. [5] *Let A_{11} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\alpha"] \arrow[r, shift left, "\xi"] & 3 \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\zeta"] & 4 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

and

$$I = \langle \gamma\alpha\beta - \gamma\xi\gamma, \alpha\beta\xi - \xi\gamma\xi, \beta\alpha, \delta\gamma, \xi\zeta, (\gamma\xi)^2 - \zeta\delta \rangle.$$

15.1. The structure of the indecomposable projectives.

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda, e_3\Lambda$ and $e_4\Lambda$ where

$e_1\Lambda = sp\{e_1, \beta, \beta\xi, \beta\xi\gamma, \beta\xi\gamma\alpha\}$,
$e_2\Lambda = sp\{e_2, \alpha, \xi, \alpha\beta, \xi\gamma, \alpha\beta\xi, \xi\gamma\alpha, \alpha\beta\xi\gamma\}$,
$e_3\Lambda = sp\{e_3, \gamma, \zeta, \gamma\alpha, \gamma\xi, \zeta\delta, \gamma\alpha\beta\}$,
$e_4\Lambda = sp\{e_4, \delta, \delta\zeta\}$.

We have $e_1\Lambda$

$$\begin{tikzcd}[row sep=small] e_1 \arrow[d, dash] \\ \beta \arrow[d, dash] \\ \beta\xi \arrow[d, dash] \\ \beta\xi\gamma \arrow[d, dash] \\ \beta\xi\gamma\alpha \end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd}[column sep=small] & e_2 \arrow[dl, dash] \arrow[dr, dash] & & \\ \alpha \arrow[d, dash] & & \xi \arrow[d, dash] & \\ \alpha\beta \arrow[dr, dash] & & \xi\gamma \arrow[dl, dash] \arrow[dr, dash] & \\ & \alpha\beta\xi \arrow[dr, dash] & & \xi\gamma\alpha \arrow[dl, dash] \\ & & \alpha\beta\xi\gamma & \end{tikzcd}$$

Also $e_3\Lambda$

$$
\begin{tikzcd}
 & & e_3 \arrow[dl, dash] \arrow[drr, dash] & & \\
 & \gamma \arrow[dl, dash] \arrow[dr, dash] & & & \zeta \arrow[dddl, dash] \\
\gamma\alpha \arrow[dr, dash] & & \gamma\xi \arrow[dl, dash] & & \\
 & \gamma\alpha\beta \arrow[drr, dash] & & & \\
 & & & \zeta\delta &
\end{tikzcd}
$$

And for $e_4\Lambda$

$$
\begin{tikzcd}
e_4 \arrow[d, dash] \\
\delta \arrow[d, dash] \\
\delta\zeta
\end{tikzcd}
$$

15.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3 and S_4.**

15.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \beta e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

15.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\xi + d_4\alpha\beta + d_5\xi\gamma + d_6\alpha\beta\xi + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\beta e_2\nu = 0$, so $\beta(d_1e_2 + d_2\alpha + d_3\xi + d_4\alpha\beta + d_5\xi\gamma + d_6\alpha\beta\xi + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma) = d_1\beta + d_3\beta\xi + d_5\beta\xi\gamma + d_7\beta\xi\gamma\alpha = 0$ and then $d_1 = d_3 = d_5 = d_7 = 0$. Thus $e_2\nu = d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\xi + d_8\alpha\beta\xi\gamma$.

Hence $\operatorname{Ker}\partial^1 = \{d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\xi + d_8\alpha\beta\xi\gamma : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \alpha e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = d_2\alpha + d_4\alpha\beta + d_6\alpha\beta\xi + d_8\alpha\beta\xi\gamma$, that is, $x = \alpha(d_2e_1 + d_4\beta + d_6\beta\xi + d_8\beta\xi\gamma)$. Thus $x \in \alpha e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \alpha e_1\Lambda$.

On the other hand, let $y = \alpha e_1\eta \in \alpha e_1\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha e_1\eta) = \beta\alpha e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\alpha e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \alpha e_1\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \alpha e_1\eta$, for $\eta \in \Lambda$.

15.2.3. $\operatorname{Ker} \partial^2$.
Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker} \partial^2$. Then $\alpha e_1\eta = 0$. So $\alpha e_1\eta = \alpha(c_1e_1+c_2\beta+c_3\beta\xi+c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha) = c_1\alpha + c_2\alpha\beta + c_3\alpha\beta\xi + c_4\alpha\beta\xi\gamma = 0$, that is, $c_1 = c_2 = c_3 = c_4 = 0$. Thus $e_1\eta = c_5\beta\xi\gamma\alpha$. Therefore $\operatorname{Ker} \partial^2 = \{c_5\beta\xi\gamma\alpha : c_5 \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = \beta\xi\gamma\alpha e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = c_5\beta\xi\gamma\alpha$ so $u = \beta\xi\gamma\alpha(c_5e_1)$. Hence $u \in \beta\xi\gamma\alpha e_1\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \beta\xi\gamma\alpha e_1\Lambda$.

On the other hand, let $v = \beta\xi\gamma\alpha e_1\eta \in \beta\xi\gamma\alpha e_1\Lambda$. Then, from the definition of ∂^2, we have that$\partial^2(v) = \partial^2(\beta\xi\gamma\alpha e_1\eta) = \alpha\beta\xi\gamma\alpha e_1\eta = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $\beta\xi\gamma\alpha e_1\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \beta\xi\gamma\alpha e_1\Lambda$. □

We remark that $\operatorname{Ker} \partial^2 \cong S_1$ and then $\Omega^3(S_1) \cong S_1$.

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \beta\xi\gamma\alpha e_1\eta$, for $\eta \in \Lambda$.

15.2.4. *The minimal projective resolution of the simple Λ-module S_2.*
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \to \alpha e_1\eta + \xi e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker} \partial^1$ and $\operatorname{Ker} \partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

15.2.5. $\operatorname{Ker} \partial^1$.
To find $\operatorname{Ker} \partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha$ and $e_3\lambda = f_1e_3 + f_2\gamma + f_3\zeta + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker} \partial^1$. Then $\alpha e_1\eta + \xi e_3\lambda = 0$ so $\alpha e_1\eta + \xi e_3\lambda = \alpha(c_1e_1 + c_2\beta + c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha) + \xi(f_1e_3 + f_2\gamma + f_3\zeta + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta) = c_1\alpha + c_2\alpha\beta + c_3\alpha\beta\xi + c_4\alpha\beta\xi\gamma + f_1\xi + f_2\xi\gamma + f_4\xi\gamma\alpha + f_5\xi\gamma\xi + f_7\xi\gamma\alpha\beta = c_1\alpha + c_2\alpha\beta + (c_3 + f_5)\alpha\beta\xi + (c_4 + f_7)\alpha\beta\xi\gamma + f_1\xi + f_2\xi\gamma + f_4\xi\gamma\alpha = 0$, that is, $c_1 = c_2 = f_1 = f_2 = f_4 = 0, f_5 = -c_3, f_7 = -c_4$. Thus $e_1\eta = c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha$ and $e_3\lambda = f_3\zeta - c_3\gamma\xi + f_6\zeta\delta - c_4\gamma\alpha\beta$.

Hence $\operatorname{Ker} \partial^1 = \{(c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha, f_3\zeta - c_3\gamma\xi + f_6\zeta\delta - c_4\gamma\alpha\beta) : c_i, f_i \in K\}$.

Claim. $\operatorname{Ker} \partial^1 = (\beta\xi, -\gamma\xi)e_3\Lambda + (0, \zeta)e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^1$; then $x = (c_3\beta\xi + c_4\beta\xi\gamma + c_5\beta\xi\gamma\alpha, f_3\zeta - c_3\gamma\xi + f_6\zeta\delta - c_4\gamma\alpha\beta)$. So $x = (\beta\xi, -\gamma\xi)(c_3e_3 + c_4\gamma + c_5\gamma\alpha) + (0, \zeta)(f_3e_4 + f_6\delta)$. Thus $x \in (\beta\xi, -\gamma\xi)e_3\Lambda + (0, \zeta)e_4\Lambda$ and therefore $\operatorname{Ker} \partial^1 \subseteq (\beta\xi, -\gamma\xi)e_3\Lambda + (0, \zeta)e_4\Lambda$.

On the other hand, let $y = (\beta\xi, -\gamma\xi)e_3\lambda + (0, \zeta)e_4\mu \in (\beta\xi, -\gamma\xi)e_3\Lambda + (0, \zeta)e_4\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\beta\xi, -\gamma\xi)e_3\lambda + (0, \zeta)e_4\mu) =$

$\partial^1(\beta\xi e_3\lambda, -\gamma\xi e_3\lambda+\zeta e_4\mu) = (\alpha\beta\xi-\xi\gamma\xi)e_3\lambda+\xi\zeta e_4\mu = 0$. So $(\beta\xi, -\gamma\xi)e_3\Lambda+(0,\zeta)e_4\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\beta\xi, -\gamma\xi)e_3\Lambda + (0,\zeta)e_4\Lambda$. $\square$

So $\partial^2 : e_3\Lambda \oplus e_4\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $(e_3\lambda, e_4\mu) \mapsto (\beta\xi, -\gamma\xi)e_3\lambda + (0,\zeta)e_4\mu$, for $\lambda, \mu \in \Lambda$.

15.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_3\lambda = f_1e_3+f_2\gamma+f_3\zeta+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta$ and $e_4\mu = t_1e_4 + t_2\delta + t_3\delta\zeta$ with $f_i, t_i \in K$. Assume that $(e_3\lambda, e_4\mu) \in \operatorname{Ker}\partial^2$. Then $(\beta\xi, -\gamma\xi)e_3\lambda + (0,\zeta)e_4\mu = (0,0)$. So $(\beta\xi, -\gamma\xi)e_3\lambda + (0,\zeta)e_4\mu = (\beta\xi, -\gamma\xi)(f_1e_3 + f_2\gamma + f_3\zeta + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta) + (0,\zeta)(t_1e_4 + t_2\delta + t_3\delta\zeta) = (f_1\beta\xi + f_2\beta\xi\gamma + f_4\beta\xi\gamma\alpha, -f_1\gamma\xi - f_2\gamma\xi\gamma - f_5\gamma\xi\gamma\xi) + (0, t_1\zeta+t_2\zeta\delta) = (0,0)$, so $f_1\beta\xi+f_2\beta\xi\gamma+f_4\beta\xi\gamma\alpha = 0$, that is, $f_1 = f_2 = f_4 = 0$. Also $-f_1\gamma\xi - f_2\gamma\xi\gamma + t_1\zeta + (t_2 - f_5)\zeta\delta = 0$ and then $f_1 = f_2 = t_1 = 0$ and $f_5 = t_2$. Thus $e_3\lambda = f_3\zeta+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta$ and $e_4\mu = f_5\delta+t_3\delta\zeta$. Therefore $\operatorname{Ker}\partial^2 = \{(f_3\zeta + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta, f_5\delta + t_3\delta\zeta) : f_i, t_3 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (f_3\zeta + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta, f_5\delta + t_3\delta\zeta)$ so $u = (\gamma\xi, \delta)(f_5e_3 + f_7\gamma + t_3\zeta) + (\zeta, 0)(f_3e_4 + f_6\delta)$. Hence $u \in (\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda$.

On the other hand, let $v = (\gamma\xi, \delta)e_3\lambda + (\zeta, 0)e_4\mu \in (\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\gamma\xi, \delta)e_3\lambda + (\zeta, 0)e_4\mu) = (\gamma\xi e_3\lambda + \zeta e_4\mu, \delta e_3\lambda) = (\beta\xi, -\gamma\xi)(\gamma\xi e_3\lambda+\zeta e_4\mu)+(0,\zeta)(\delta e_3\lambda) = (\beta\xi\gamma\xi e_3\lambda+\beta\xi\zeta e_4\mu, -\gamma\xi\gamma\xi e_3\lambda - \gamma\xi\zeta e_4\mu)+(0,\zeta\delta e_3\lambda) = (\beta\xi\gamma\xi e_3\lambda+\beta\xi\zeta e_4\mu, -(\gamma\xi\gamma\xi-\zeta\delta)e_3\lambda-\gamma\xi\zeta e_4\mu) = (0,0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\gamma\xi, \delta)e_3\Lambda + (\zeta, 0)e_4\Lambda$. $\square$

So the map $\partial^3 : e_3\Lambda\oplus e_4\Lambda \to e_3\Lambda\oplus e_4\Lambda$ is given by $(e_3\lambda, e_4\mu) \mapsto (\gamma\xi, \delta)e_3\lambda+(\zeta, 0)e_4\mu$, for $\lambda, \mu \in \Lambda$.

15.2.7. *The minimal projective resolution of the simple Λ-module S_3.*
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \oplus e_4\Lambda \to e_3\Lambda$ is given by $(e_2\nu, e_4\mu) \mapsto \gamma e_2\nu + \zeta e_4\mu$, for $\nu, \mu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

15.2.8. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\xi + d_4\alpha\beta + d_5\xi\gamma + d_6\alpha\beta\xi + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma$ and $e_4\mu = t_1e_4 + t_2\delta + t_3\delta\zeta$ with $d_i, t_i \in K$. Assume that $(e_2\nu, e_4\mu) \in \operatorname{Ker}\partial^1$.

Then $\gamma e_2\nu+\zeta e_4\mu=0$ so $\gamma(d_1e_2+d_2\alpha+d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma)+\zeta(t_1e_4+t_2\delta+t_3\delta\zeta)=d_1\gamma+d_2\gamma\alpha+d_3\gamma\xi+d_4\gamma\alpha\beta+d_5\gamma\xi\gamma+d_6\gamma\alpha\beta\xi+t_1\zeta+t_2\zeta\delta=d_1\gamma+d_2\gamma\alpha+d_3\gamma\xi+(d_4+d_5)\gamma\alpha\beta+(d_6+t_2)\zeta\delta+t_1\zeta=0$, that is, $d_1=d_2=d_3=t_1=0, d_5=-d_4$ and $t_2=-d_6$. Thus $e_2\nu=d_4(\alpha\beta-\xi\gamma)+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma$ and $e_4\mu=-d_6\delta+t_3\delta\zeta$.

Hence $\operatorname{Ker}\partial^1=\{(d_4(\alpha\beta-\xi\gamma)+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma,-d_6\delta+t_3\delta\zeta):d_i,t_3\in K\}$.

Claim. $\operatorname{Ker}\partial^1=(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda$.

Proof. Let $x\in\operatorname{Ker}\partial^1$; then $x=(d_4(\alpha\beta-\xi\gamma)+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma,-d_6\delta+t_3\delta\zeta)$. So $x=(\alpha\beta-\xi\gamma,0)(d_4e_2+d_6\xi-d_7\alpha+d_8\xi\gamma)+(\xi\gamma\xi,-\delta)(d_6e_3+t_3\zeta+d_8\gamma)$. Thus $x\in(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1\subseteq(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda$.

On the other hand, let $y=(\alpha\beta-\xi\gamma,0)e_2\nu+(\xi\gamma\xi,-\delta)e_3\lambda\in(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y)=\partial^1((\alpha\beta-\xi\gamma,0)e_2\nu+(\xi\gamma\xi,-\delta)e_3\lambda)=\partial^1((\alpha\beta-\xi\gamma)e_2\nu+\xi\gamma\xi e_3\lambda,-\delta e_3\lambda)=(\gamma\alpha\beta-\gamma\xi\gamma)e_2\nu+(\gamma\xi\gamma\xi-\zeta\delta)e_3\lambda=0$. Therefore $y\in\operatorname{Ker}\partial^1$ and so $(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda\subseteq\operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1=(\alpha\beta-\xi\gamma,0)e_2\Lambda+(\xi\gamma\xi,-\delta)e_3\Lambda$. □

So $\partial^2:e_2\Lambda\oplus e_3\Lambda\to e_2\Lambda\oplus e_4\Lambda$ is given by $(e_2\nu,e_3\lambda)\mapsto(\alpha\beta-\xi\gamma,0)e_2\nu+(\xi\gamma\xi,-\delta)e_3\lambda$, for $\nu,\lambda\in\Lambda$.

15.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2=\Omega^3(S_3)$. Let $e_2\nu=d_1e_2+d_2\alpha+d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma$ and $e_3\lambda=f_1e_3+f_2\gamma+f_3\zeta+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta$ with $d_i,f_i\in K$. Assume that $(e_2\nu,e_3\lambda)\in\operatorname{Ker}\partial^2$. Then $(\alpha\beta-\xi\gamma,0)e_2\nu+(\xi\gamma\xi,-\delta)e_3\lambda=(0,0)$. So $(\alpha\beta-\xi\gamma,0)e_2\nu+(\xi\gamma\xi,-\delta)e_3\lambda=(\alpha\beta-\xi\gamma,0)(d_1e_2+d_2\alpha+d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma)+(\xi\gamma\xi,-\delta)(f_1e_3+f_2\gamma+f_3\zeta+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta)=(d_1\alpha\beta-d_1\xi\gamma-d_2\xi\gamma\alpha+d_3\alpha\beta\xi-d_3\xi\gamma\xi-d_4\xi\gamma\alpha\beta,0)+(f_1\xi\gamma\xi+f_2\xi\gamma\xi\gamma,-f_1\delta-f_3\delta\zeta)=(d_1\alpha\beta-d_1\xi\gamma-d_2\xi\gamma\alpha-d_4\xi\gamma\alpha\beta+f_1\xi\gamma\xi+f_2\xi\gamma\xi\gamma,-f_1\delta-f_3\delta\zeta)=(d_1\alpha\beta-d_1\xi\gamma-d_2\xi\gamma\alpha+(f_2-d_4)\xi\gamma\xi\gamma+f_1\xi\gamma\xi,-f_1\delta-f_3\delta\zeta)=(0,0)$. So $d_1\alpha\beta-d_1\xi\gamma-d_2\xi\gamma\alpha+(f_2-d_4)\xi\gamma\xi\gamma+f_1\xi\gamma\xi=0$, that is, $d_1=d_2=f_1=0$ and $f_2=d_4$. Also $-f_1\delta-f_3\delta\zeta=0$, that is, $f_1=f_3=0$. Thus $e_2\nu=d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma$ and $e_3\lambda=d_4\gamma+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta$. Therefore $\operatorname{Ker}\partial^2=\{(d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma,d_4\gamma+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta):d_i,f_i\in K\}$.

Claim. $\operatorname{Ker}\partial^2=(\alpha\beta,\gamma)e_2\Lambda+(\xi,0)e_3\Lambda$.

Proof. Let $u\in\operatorname{Ker}\partial^2$. Then $u=(d_3\xi+d_4\alpha\beta+d_5\xi\gamma+d_6\alpha\beta\xi+d_7\xi\gamma\alpha+d_8\alpha\beta\xi\gamma,d_4\gamma+f_4\gamma\alpha+f_5\gamma\xi+f_6\zeta\delta+f_7\gamma\alpha\beta)$, that is, $u=(\alpha\beta,\gamma)(d_4e_2+f_4\alpha+f_5\xi+f_6\xi\gamma\xi+f_7\alpha\beta)+(\xi,0)(d_3e_3+d_5\gamma-f_5\gamma\xi+d_6\gamma\xi+d_7\gamma\alpha+d_8\gamma\xi\gamma)$. Hence $u\in(\alpha\beta,\gamma)e_2\Lambda+(\xi,0)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2\subseteq(\alpha\beta,\gamma)e_2\Lambda+(\xi,0)e_3\Lambda$.

On the other hand, let $v=(\alpha\beta,\gamma)e_2\nu+(\xi,0)e_3\lambda\in(\alpha\beta,\gamma)e_2\Lambda+(\xi,0)e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v)=\partial^2((\alpha\beta,\gamma)e_2\nu+(\xi,0)e_3\lambda)=\partial^2(\alpha\beta e_2\nu+$

$\xi e_3\lambda, \gamma e_2\nu) = (\alpha\beta - \xi\gamma, 0)(\alpha\beta e_2\nu + \xi e_3\lambda) + (\xi\gamma\xi, -\delta)(\gamma e_2\nu) = (\alpha\beta\alpha\beta e_2\nu - \xi\gamma\alpha\beta e_2\nu + \alpha\beta\xi e_3\lambda - \xi\gamma\xi e_3\lambda, 0) + (\xi\gamma\xi\gamma e_2\nu, -\delta\gamma e_2\nu) = (\alpha\beta\alpha\beta e_2\nu - \xi(\gamma\alpha\beta - \gamma\xi\gamma)e_2\nu + (\alpha\beta\xi - \xi\gamma\xi)e_3\lambda, -\delta\gamma e_2\nu) = (0,0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha\beta,\gamma)e_2\Lambda + (\xi,0)e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha\beta,\gamma)e_2\Lambda + (\xi,0)e_3\Lambda$. □

So the map $\partial^3 : e_2\Lambda \oplus e_3\Lambda \to e_2\Lambda \oplus e_3\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto (\alpha\beta,\gamma)e_2\nu + (\xi,0)e_3\lambda$, for $\nu, \lambda \in \Lambda$.

15.2.10. *The minimal projective resolution of the simple* Λ*-module* S_4*.*
The minimal projective resolution of the simple Λ-module S_4 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_4\Lambda$ is given by $e_3\lambda \mapsto \delta e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_4.

15.2.11. $\operatorname{Ker}\partial^1$*.*
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_4)$. Let $e_3\lambda = f_1e_3 + f_2\gamma + f_3\zeta + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\delta e_3\lambda = 0$ so $\delta e_3\lambda = \delta(f_1e_3 + f_2\gamma + f_3\zeta + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta) = f_1\delta + f_3\delta\zeta = 0$, that is, $f_1 = f_3 = 0$. Thus $e_3\lambda = f_2\gamma + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta$.

Hence $\operatorname{Ker}\partial^1 = \{f_2\gamma + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \gamma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$; then $x = f_2\gamma + f_4\gamma\alpha + f_5\gamma\xi + f_6\zeta\delta + f_7\gamma\alpha\beta$, that is, $x = \gamma(f_2e_2 + f_4\alpha + f_5\xi + f_6\xi\gamma\xi + f_7\alpha\beta)$. Thus $x \in \gamma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \gamma e_2\Lambda$. On the other hand, let $y = \gamma e_2\nu \in \gamma e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\gamma e_2\nu) = \delta\gamma e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\gamma e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \gamma e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_3\Lambda$ is given by $e_2\nu \mapsto \gamma e_2\nu$, for $\nu \in \Lambda$.

15.2.12. $\operatorname{Ker}\partial^2$*.*
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_4)$. Let $e_2\nu = d_1e_2 + d_2\alpha + d_3\xi + d_4\alpha\beta + d_5\xi\gamma + d_6\alpha\beta\xi + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^2$. Then $\gamma e_2\nu = 0$. So $\gamma e_2\nu = \gamma(d_1e_2 + d_2\alpha + d_3\xi + d_4\alpha\beta + d_5\xi\gamma + d_6\alpha\beta\xi + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma) = 0$. So $d_1\gamma + d_2\gamma\alpha + d_3\gamma\xi + d_4\gamma\alpha\beta + d_5\gamma\xi\gamma + d_6\gamma\alpha\beta\xi = d_1\gamma + d_2\gamma\alpha + d_3\gamma\xi + (d_4 + d_5)\gamma\alpha\beta + d_6\gamma\alpha\beta\xi = 0$, that is, $d_1 = d_2 = d_3 = d_6 = 0$ and $d_5 = -d_4$. Thus $e_2\nu = d_4\alpha\beta - d_4\xi\gamma + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{d_4\alpha\beta - d_4\xi\gamma + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha\beta - \xi\gamma)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = d_4\alpha\beta - d_4\xi\gamma + d_7\xi\gamma\alpha + d_8\alpha\beta\xi\gamma$, that is, $u = (\alpha\beta - \xi\gamma)(d_4e_2 - d_7\alpha - d_8\alpha\beta)$. Hence $u \in (\alpha\beta - \xi\gamma)e_2\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq (\alpha\beta - \xi\gamma)e_2\Lambda$.

On the other hand, let $v = (\alpha\beta - \xi\gamma)e_2\nu \in (\alpha\beta - \xi\gamma)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha\beta - \xi\gamma)e_2\nu) = (\gamma\alpha\beta - \gamma\xi\gamma)e_2\nu = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $(\alpha\beta - \xi\gamma)e_2\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = (\alpha\beta - \xi\gamma)e_2\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\nu \mapsto (\alpha\beta - \xi\gamma)e_2\nu$, for $\nu \in \Lambda$.

Thus the maps for S_1 are:

$\partial^1 : e_2\nu \mapsto \beta e_2\nu$,
$\partial^2 : e_1\eta \mapsto \alpha e_1\eta$,
$\partial^3 : e_1\eta \mapsto \beta\xi\gamma\alpha e_1\eta$,

for $\eta, \nu \in \Lambda$.

Also the maps for S_2 are:

$\partial^1 : (e_1\eta, e_3\lambda) \to \alpha e_1\eta + \xi e_3\lambda$,
$\partial^2 : (e_3\lambda, e_4\mu) \mapsto (\beta\xi, -\gamma\xi)e_3\lambda + (0, \zeta)e_4\mu$
$\partial^3 : (e_3\lambda, e_4\mu) \mapsto (\gamma\xi, \delta)e_3\lambda + (\zeta, 0)e_4\mu$,

for $\eta, \mu, \lambda \in \Lambda$.

And the maps for S_3 are:

$\partial^1 : (e_2\nu, e_4\mu) \mapsto \gamma e_2\nu + \zeta e_4\mu$,
$\partial^2 : (e_2\nu, e_3\lambda) \mapsto (\alpha\beta - \xi\gamma, 0)e_2\nu + (\xi\gamma\xi, -\delta)e_3\lambda$,
$\partial^3 : (e_2\nu, e_3\lambda) \mapsto (\alpha\beta, \gamma)e_2\nu + (\xi, 0)e_3\lambda$,

for $\nu, \lambda, \mu \in \Lambda$.

Moreover, the maps for S_4 are:

$\partial^1 : e_3\lambda \mapsto \delta e_3\lambda$,
$\partial^2 : e_2\nu \mapsto \gamma e_2\nu$,
$\partial^3 : e_2\nu \mapsto (\alpha\beta - \xi\gamma)e_2\nu$,

for $\nu, \lambda \in \Lambda$.

15.3. g^3 **for** S_1, S_2, S_3 **and** S_4.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_1 \xrightarrow{\partial^3} \beta\xi\gamma\alpha \xrightarrow{\partial^2} \alpha\beta\xi\gamma\alpha \xrightarrow{\partial^1} \beta\alpha\beta\xi\gamma\alpha$, so $\beta\alpha\beta\xi\gamma\alpha \in g^3$.

For S_2

$(e_3, 0) \overset{\partial^3}{\to} (\gamma\xi, \delta) \overset{\partial^2}{\to} (\beta\xi, -\gamma\xi)\gamma\xi + (0, \zeta)\delta = (\beta\xi\gamma\xi, -\gamma\xi\gamma\xi) + (0, \zeta\delta) = (\beta\xi\gamma\xi, -\gamma\xi\gamma\xi + \zeta\delta) \overset{\partial^1}{\to} \alpha\beta\xi\gamma\xi - \xi\gamma\xi\gamma\xi + \xi\zeta\delta$, so $\alpha\beta\xi\gamma\xi - \xi\gamma\xi\gamma\xi + \xi\zeta\delta \in g^3$.

$(0, e_4) \xrightarrow{\partial^3} (\zeta, 0) \xrightarrow{\partial^2} (\beta\xi, -\gamma\xi)\zeta = (\beta\xi\zeta, -\gamma\xi\zeta) \xrightarrow{\partial^1} \alpha\beta\xi\zeta - \xi\gamma\xi\zeta$, so $\alpha\beta\xi\zeta - \xi\gamma\xi\zeta \in g^3$.

For S_3

$(e_2,0) \xrightarrow{\partial^3} (\alpha\beta,\gamma) \xrightarrow{\partial^2} (\alpha\beta-\xi\gamma,0)\alpha\beta+(\xi\gamma\xi,-\delta)\gamma = (\alpha\beta\alpha\beta-\xi\gamma\alpha\beta,0)+(\xi\gamma\xi\gamma,-\delta\gamma) = (\alpha\beta\alpha\beta-\xi\gamma\alpha\beta+\xi\gamma\xi\gamma,-\delta\gamma) \xrightarrow{\partial^1} (\gamma\alpha\beta\alpha\beta-\gamma\xi\gamma\alpha\beta+\gamma\xi\gamma\xi\gamma-\zeta\delta\gamma)$, so $\gamma\alpha\beta\alpha\beta-\gamma\xi\gamma\alpha\beta+\gamma\xi\gamma\xi\gamma-\zeta\delta\gamma \in g^3$.

$(0,e_3) \xrightarrow{\partial^3} (\xi,0) \xrightarrow{\partial^2} (\alpha\beta-\xi\gamma,0)\xi = (\alpha\beta\xi-\xi\gamma\xi,0) \xrightarrow{\partial^1} \gamma\alpha\beta\xi-\gamma\xi\gamma\xi$, so $\gamma\alpha\beta\xi-\gamma\xi\gamma\xi \in g^3$.

For S_4

$e_2 \xrightarrow{\partial^3} (\alpha\beta-\xi\gamma) \xrightarrow{\partial^2} \gamma\alpha\beta-\gamma\xi\gamma \xrightarrow{\partial^1} \delta\gamma\alpha\beta-\delta\gamma\xi\gamma$, so $\delta\gamma\alpha\beta-\delta\gamma\xi\gamma \in g^3$.

Let $g_1^3 = \beta\alpha\beta\xi\gamma\alpha$,
$g_2^3 = \alpha\beta\xi\gamma\xi-\xi\gamma\xi\gamma\xi+\xi\zeta\delta$,
$g_3^3 = \alpha\beta\xi\zeta-\xi\gamma\xi\zeta$,
$g_4^3 = \gamma\alpha\beta\alpha\beta-\gamma\xi\gamma\alpha\beta+\gamma\xi\gamma\xi\gamma-\zeta\delta\gamma$,
$g_5^3 = \gamma\alpha\beta\xi-\gamma\xi\gamma\xi$,
$g_6^3 = \delta\gamma\alpha\beta-\delta\gamma\xi\gamma$.
So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3, g_5^3, g_6^3\}$.

We know that $g^2 = \{\gamma\alpha\beta-\gamma\xi\gamma, \alpha\beta\xi-\xi\gamma\xi, \beta\alpha, \delta\gamma, \xi\zeta, (\gamma\xi)^2-\zeta\delta\}$. Denote

$g_1^2 = \beta\alpha$,
$g_2^2 = \alpha\beta\xi-\xi\gamma\xi$,
$g_3^2 = \xi\zeta$,
$g_4^2 = \gamma\alpha\beta-\gamma\xi\gamma$,
$g_5^2 = (\gamma\xi)^2-\zeta\delta$,
$g_6^2 = \delta\gamma$.

So we have

$g_1^3 = g_1^2\beta\xi\gamma\alpha = \beta g_2^2\gamma\alpha-\beta\xi g_4^2\alpha+\beta\xi\gamma\alpha g_1^2$,
$g_2^3 = g_2^2\gamma\xi+g_3^2\delta = \alpha g_1^2\beta\xi-\xi g_5^2-\alpha\beta g_2^2$,
$g_3^3 = g_2^2\zeta = \alpha\beta g_3^2-\xi\gamma g_3^2$,
$g_4^3 = g_4^2\alpha\beta+g_5^2\gamma = \gamma\alpha g_1^2\beta-\gamma\xi g_4^2-\zeta g_6^2$,
$g_5^3 = g_4^2\xi = \gamma g_2^2$,
$g_6^3 = g_6^2\alpha\beta-g_6^2\xi\gamma = \delta g_4^2$.

15.4. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0 .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \ldots$$

15.4.1. $\operatorname{Ker}\delta^2$.

To find $\operatorname{HH}^2(\Lambda)$ we need to find $\operatorname{Ker}\delta^2$ and $\operatorname{Im}\delta^1$. Let $\theta \in \operatorname{Ker}\delta^2$; then $\theta \in \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\beta\xi\gamma\alpha\\
&e_2 \otimes_{g_2^2} e_3 \mapsto j_3\xi + j_4\alpha\beta\xi\\
&e_2 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_3 \otimes_{g_4^2} e_2 \mapsto j_5\gamma + j_6\gamma\alpha\beta\\
&e_3 \otimes_{g_5^2} e_3 \mapsto j_7e_3 + j_8\gamma\xi + j_9\zeta\delta\\
&e_4 \otimes_{g_6^2} e_2 \mapsto 0,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \beta\xi\gamma\alpha - \beta \otimes_{g_2^2} \gamma\alpha + \beta\xi \otimes_{g_4^2} \alpha - \beta\xi\gamma\alpha \otimes_{g_1^2} e_1,$
$e_2 \otimes_{g_2^3} e_3 \mapsto e_2 \otimes_{g_2^2} \gamma\xi + e_2 \otimes_{g_3^2} \delta - \alpha \otimes_{g_1^2} \beta\xi + \xi \otimes_{g_5^2} e_3 + \alpha\beta \otimes_{g_2^2} e_3,$
$e_2 \otimes_{g_3^3} e_4 \mapsto e_2 \otimes_{g_2^2} \zeta - \alpha\beta \otimes_{g_3^2} e_4 + \xi\gamma \otimes_{g_3^2} e_4,$
$e_3 \otimes_{g_4^3} e_2 \mapsto e_3 \otimes_{g_4^2} \alpha\beta + e_2 \otimes_{g_5^2} \gamma - \gamma\alpha \otimes_{g_1^2} \beta + \gamma\xi \otimes_{g_4^2} e_2 + \zeta \otimes_{g_6^2} e_2,$
$e_3 \otimes_{g_5^3} e_3 \mapsto e_3 \otimes_{g_4^2} \xi - \gamma \otimes_{g_2^2} e_3,$
$e_4 \otimes_{g_6^3} e_2 \mapsto e_4 \otimes_{g_6^2} \alpha\beta - e_4 \otimes_{g_6^2} \xi\gamma - \delta \otimes_{g_4^2} e_2.$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\beta\xi\gamma\alpha)\beta\xi\gamma\alpha - \beta(j_3\xi + j_4\alpha\beta\xi)\gamma\alpha + \beta\xi(j_5\gamma + j_6\gamma\alpha\beta)\alpha - \beta\xi\gamma\alpha(j_1e_1 + j_2\beta\xi\gamma\alpha) = (-j_3 + j_5)\beta\xi\gamma\alpha = 0$ then $j_5 = j_3$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_3) = (j_3\xi + j_4\alpha\beta\xi)\gamma\xi - \alpha(j_1e_1 + j_2\beta\xi\gamma\alpha)\beta\xi + \xi(j_7e_3 + j_8\gamma\xi + j_9\zeta\delta) + \alpha\beta(j_3\xi + j_4\alpha\beta\xi) = (2j_3 - j_1 + j_8)\alpha\beta\xi + j_7\xi = 0$, that is, $j_8 = j_1 - 2j_3, j_7 = 0$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_4) = 0$.

And $\theta d^3(e_3 \otimes_{g_4^3} e_2) = (j_5\gamma + j_6\gamma\alpha\beta)\alpha\beta + (j_7e_3 + j_8\gamma\xi + j_9\zeta\delta)\gamma - \gamma\alpha(j_1e_1 + j_2\beta\xi\gamma\alpha)\beta + \gamma\xi(j_5\gamma + j_6\gamma\alpha\beta) = (2j_5 - j_1 + j_8)\gamma\alpha\beta + j_7\gamma = 0$, that is, $j_8 = j_1 - 2j_5, j_7 = 0$.

For $\theta d^3(e_3 \otimes_{g_5^3} e_3) = (j_5\gamma + j_6\gamma\alpha\beta)\xi - \gamma(j_3\xi + j_4\alpha\beta\xi) = (j_5 - j_3)\gamma\xi + (j_6 - j_4)\zeta\delta = 0$, that is, $j_5 = j_3, j_6 = j_4$.

Also $\theta d^3(e_4 \otimes_{g_6^3} e_2) = -\delta(j_5\gamma + j_6\gamma\alpha\beta) = 0$.

So So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\beta\xi\gamma\alpha\\
&e_2 \otimes_{g_2^2} e_3 \mapsto j_3\xi + j_4\alpha\beta\xi\\
&e_2 \otimes_{g_3^2} e_4 \mapsto 0\\
&e_3 \otimes_{g_4^2} e_2 \mapsto j_3\gamma + j_4\gamma\alpha\beta\\
&e_3 \otimes_{g_5^2} e_3 \mapsto (j_1 - 2j_3)\gamma\xi + j_9\zeta\delta\\
&e_4 \otimes_{g_6^2} e_2 \mapsto 0.
\end{aligned}$$

Hence $\dim \operatorname{Ker}\delta^2 = 5$.

15.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\beta e_2 \to z_1\beta + z_2\beta\xi\gamma$

$e_2 \otimes_\alpha e_1 \to z_3\alpha + z_4\xi\gamma\alpha$
$e_2 \otimes_\xi e_3 \to z_5\xi + z_6\alpha\beta\xi$
$e_3 \otimes_\gamma e_2 \to z_7\gamma + z_8\gamma\alpha\beta$
$e_3 \otimes_\zeta e_4 \to z_9\zeta$
$e_4 \otimes_\delta e_3 \to z_{10}\delta,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by:

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\beta \alpha + \beta \otimes_\alpha e_1$
$e_2 \otimes_{g_2^2} e_3 \mapsto e_2 \otimes_\alpha \beta\xi + \alpha \otimes_\beta \xi + \alpha\beta \otimes_\xi e_3 - e_2 \otimes_\xi \gamma\xi - \xi \otimes_\gamma \xi - \xi\gamma \otimes_\xi e_3$
$e_2 \otimes_{g_3^2} e_4 \mapsto e_2 \otimes_\xi \zeta + \xi \otimes_\zeta e_4$
$e_3 \otimes_{g_4^2} e_2 \mapsto e_3 \otimes_\gamma \alpha\beta + \gamma \otimes_\alpha \beta + \gamma\alpha \otimes_\beta e_2 - e_3 \otimes_\gamma \xi\gamma - \gamma \otimes_\xi \gamma - \gamma\xi \otimes_\gamma e_2$
$e_3 \otimes_{g_5^2} e_3 \mapsto e_3 \otimes_\gamma \xi\gamma\xi + \gamma \otimes_\xi \gamma\xi + \gamma\xi \otimes_\gamma \xi + \gamma\xi\gamma \otimes_\xi e_3 - e_3 \otimes_\zeta \delta - \zeta \otimes_\delta e_3$
$e_4 \otimes_{g_6^2} e_2 \mapsto e_4 \otimes_\delta \gamma + \delta \otimes_\gamma e_2.$

Then the map φd^2 is given by:

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1\beta + z_2\beta\xi\gamma)\alpha + \beta(z_3\alpha + z_4\xi\gamma\alpha) = (z_2 + z_4)\beta\xi\gamma\alpha,$

$\varphi d^2(e_2 \otimes_{g_2^2} e_3) = (z_3\alpha + z_4\xi\gamma\alpha)\beta\xi + \alpha(z_1\beta + z_2\beta\xi\gamma)\xi + \alpha\beta(z_5\xi + z_6\alpha\beta\xi) - (z_5\xi + z_6\alpha\beta\xi)\gamma\xi - \xi(z_7\gamma + z_8\gamma\alpha\beta)\xi - \xi\gamma(z_5\xi + z_6\alpha\beta\xi) = (z_1 + z_3 - z_5 - z_7)\alpha\beta\xi,$

$\varphi d^2(e_2 \otimes_{g_3^2} e_4) = (z_5\xi + z_6\alpha\beta\xi)\zeta + \xi(z_9\zeta) = 0,$

$\varphi d^2(e_3 \otimes_{g_4^2} e_2) = (z_7\gamma + z_8\gamma\alpha\beta)\alpha\beta + \gamma(z_3\alpha + z_4\xi\gamma\alpha)\beta + \gamma\alpha(z_1\beta + z_2\beta\xi\gamma) - (z_7\gamma + z_8\gamma\alpha\beta)\xi\gamma - \gamma(z_5\xi + z_6\alpha\beta\xi)\gamma - \gamma\xi(z_7\gamma + z_8\gamma\alpha\beta) = (z_1 + z_3 - z_5 - z_7)\gamma\alpha\beta,$

$\varphi d^2(e_3 \otimes_{g_5^2} e_3) = (z_7\gamma + z_8\gamma\alpha\beta)\xi\gamma\xi + \gamma(z_5\xi + z_6\alpha\beta\xi)\gamma\xi + \gamma\xi(z_7\gamma + z_8\gamma\alpha\beta)\xi + \gamma\xi\gamma(z_5\xi + z_6\alpha\beta\xi) - (z_9\zeta)\delta - \zeta(z_{10}\delta) = (2z_5 + 2z_7 - z_9 - z_{10})\zeta\delta.$

$\varphi d^2(e_4 \otimes_{g_6^2} e_2) = (z_{10}\delta)\gamma + \delta(z_7\gamma + z_8\gamma\alpha\beta) = 0.$

Thus φd^2 is given by

$$
\begin{aligned}
&P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\beta\xi\gamma\alpha \\
&e_2 \otimes_{g_2^2} e_3 \mapsto (z_1 + z_3 - z_5 - z_7)\alpha\beta\xi \\
&e_2 \otimes_{g_3^2} e_4 \mapsto 0 \\
&e_3 \otimes_{g_4^2} e_2 \mapsto (z_1 + z_3 - z_5 - z_7)\gamma\alpha\beta \\
&e_3 \otimes_{g_5^2} e_3 \mapsto (2z_5 + 2z_7 - z_9 - z_{10})\zeta\delta \\
&e_4 \otimes_{g_6^2} e_2 \mapsto 0,
\end{aligned}
$$

where $z_i \in K$. Note that if $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ the $\dim \operatorname{Im} \delta^1 = 3$. Therefore $\dim \operatorname{Im} \delta^1 = 3$.

15.4.3. $\mathrm{HH}^2(\Lambda)$.

From 15.4.1 and 15.4.2 we have that If $\mathrm{char}\, K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_2 \otimes_{g_2^2} e_3 \mapsto d_2 \xi \\ e_2 \otimes_{g_3^2} e_4 \mapsto 0 \\ e_3 \otimes_{g_4^2} e_2 \mapsto d_2 \gamma \\ e_3 \otimes_{g_5^2} e_3 \mapsto d_1 \gamma\xi \\ e_4 \otimes_{g_6^2} e_2 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{u, w\}$ where

$$\begin{array}{l} u : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto \gamma\xi \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} w : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_3 \mapsto \xi \\ e_3 \otimes_{g_4^2} e_2 \mapsto \gamma \\ \text{else} \mapsto 0. \end{array}$$

If $\mathrm{char}\, K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_2 \otimes_{g_2^2} e_3 \mapsto d_2 \xi \\ e_2 \otimes_{g_3^2} e_4 \mapsto 0 \\ e_3 \otimes_{g_4^2} e_2 \mapsto d_2 \gamma \\ e_3 \otimes_{g_5^2} e_3 \mapsto (d_1 - 2d_2)\gamma\xi \\ e_4 \otimes_{g_6^2} e_2 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto \gamma\xi \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_3 \mapsto \xi \\ e_3 \otimes_{g_4^2} e_2 \mapsto \gamma \\ e_3 \otimes_{g_5^2} e_3 \mapsto -\frac{1}{2}\gamma\xi \\ \text{else} \mapsto 0. \end{array}$$

16. The algebra A_{12}

Definition 16.1. [5] *Let A_{12} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, shift left, "\delta"] & & 3 \arrow[ll, shift left, "\beta"] \end{tikzcd}$$

and

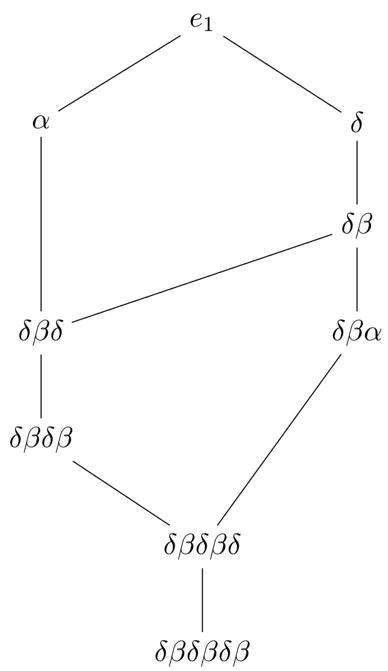
$$I = \langle \delta\beta\delta - \alpha\gamma, \gamma\beta\alpha, (\beta\delta)^3\beta \rangle.$$

16.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda$ and $e_3\Lambda$ where

$$e_1\Lambda = sp\{e_1, \alpha, \delta, \delta\beta, \delta\beta\delta, \delta\beta\delta\beta, \delta\beta\alpha, \delta\beta\delta\beta\delta, \delta\beta\delta\beta\delta\beta\},$$
$$e_2\Lambda = sp\{e_2, \gamma, \gamma\beta, \gamma\beta\delta, \gamma\beta\delta\beta, \gamma\beta\delta\beta\alpha\},$$
$$e_3\Lambda = sp\{e_3, \beta, \beta\alpha, \beta\delta, \beta\alpha\gamma, \beta\delta\beta, \beta\delta\beta\alpha, \beta\alpha\gamma\beta, \beta\alpha\gamma\beta\delta\}.$$

So we have for $e_1\Lambda$



For $e_2\Lambda$

$$
\begin{tikzcd}
e_2 \arrow[d, no head] \\
\gamma \arrow[d, no head] \\
\gamma\beta \arrow[d, no head] \\
\gamma\beta\delta \arrow[d, no head] \\
\gamma\beta\delta\beta \arrow[d, no head] \\
\gamma\beta\delta\beta\alpha
\end{tikzcd}
$$

Also $e_3\Lambda$

$$
\begin{tikzcd}
 & e_3 \arrow[d, no head] & \\
 & \beta \arrow[dl, no head] \arrow[dr, no head] & \\
\beta\alpha \arrow[dd, no head] & & \beta\delta \arrow[d, no head] \\
 & & \beta\delta\beta \arrow[dll, no head] \arrow[d, no head] \\
\beta\alpha\gamma \arrow[d, no head] & & \beta\delta\beta\alpha \arrow[ddl, no head] \\
\beta\alpha\gamma\beta \arrow[dr, no head] & & \\
 & \beta\alpha\gamma\beta\delta &
\end{tikzcd}
$$

16.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2 and S_3.**

16.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda$, for $\nu, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

16.2.2. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ and $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ with $d_i, f_i \in K$. Assume that $(e_2\nu, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\alpha e_2\nu + \delta e_3\lambda = 0$, so $\alpha(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) + \delta(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta) = d_1\alpha + d_2\alpha\gamma + d_3\alpha\gamma\beta + d_4\alpha\gamma\beta\delta + d_5\alpha\gamma\beta\delta\beta + f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha + f_4\delta\beta\delta + f_5\delta\beta\alpha\gamma + f_6\delta\beta\delta\beta + f_8\delta\beta\alpha\gamma\beta = d_1\alpha + (d_2 + f_4)\alpha\gamma + (d_3 + f_6)\alpha\gamma\beta + (d_4 + f_5)\alpha\gamma\beta\delta + (d_5 + f_8)\alpha\gamma\beta\delta\beta + f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha = 0$ and then $d_1 = f_1 = f_2 = f_3 = 0, d_2 + f_4 = 0, d_3 + f_6 = 0, d_4 + f_5 = 0$, and $d_5 + f_8 = 0$, so $f_4 = -d_2, f_6 = -d_3, f_5 = -d_4$, and $f_8 = -d_5$. Thus $e_2\nu = d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ and $e_3\lambda = -d_2\beta\delta - d_3\beta\delta\beta - d_4\beta\alpha\gamma - d_5\beta\alpha\gamma\beta + f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta$.

Hence $\operatorname{Ker}\partial^1 = \{(d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha, -d_2\beta\delta - d_3\beta\delta\beta - d_4\beta\alpha\gamma - d_5\beta\alpha\gamma\beta + f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta) : d_i, f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (-\gamma, \beta\delta)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha, -d_2\beta\delta - d_3\beta\delta\beta - d_4\beta\alpha\gamma - d_5\beta\alpha\gamma\beta + f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta)$, that is, $x = (-\gamma, \beta\delta)(-d_2e_3 - d_3\beta - d_4\beta\delta - d_5\beta\delta\beta - d_6\beta\delta\beta\alpha + f_7\beta\alpha + f_9\beta\delta\beta\delta)$. Thus $x \in (-\gamma, \beta\delta)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (-\gamma, \beta\delta)e_3\Lambda$.

On the other hand, let $y = (-\gamma, \beta\delta)e_3\lambda \in (-\gamma, \beta\delta)e_3\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((-\gamma, \beta\delta)e_3\lambda) = (-\alpha\gamma + \delta\beta\delta)e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(-\gamma, \beta\delta)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (-\gamma, \beta\delta)e_3\Lambda$. □

So $\partial^2 : e_3\Lambda \to e_2\Lambda \oplus e_3\Lambda$ is given by $e_3\lambda \mapsto (-\gamma, \beta\delta)e_3\lambda$, for $\lambda \in \Lambda$.

16.2.3. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^2$. Then $(-\gamma, \beta\delta)e_3\lambda = 0$. So $(-\gamma, \beta\delta)e_3\lambda = (-\gamma, \beta\delta)(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta) = (-f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\delta - f_6\gamma\beta\delta\beta - f_7\gamma\beta\delta\beta\alpha, f_1\beta\delta + f_2\beta\delta\beta + f_3\beta\delta\beta\alpha + f_4\beta\delta\beta\delta + f_5\beta\delta\beta\alpha\gamma + f_6\beta\delta\beta\delta\beta) = (0, 0)$, that is, $-f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\delta - f_6\gamma\beta\delta\beta - f_7\gamma\beta\delta\beta\alpha = 0$ and then $f_1 = f_2 = f_4 = f_6 = f_7 = 0$. Also $f_1\beta\delta + f_2\beta\delta\beta + f_3\beta\delta\beta\alpha + f_4\beta\delta\beta\delta + f_5\beta\delta\beta\alpha\gamma + f_6\beta\delta\beta\delta\beta = 0$ and then $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$. Thus $e_3\lambda = f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$. Therefore $\operatorname{Ker}\partial^2 = \{f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta : f_8, f_9 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\alpha\gamma\beta e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ so $u = \beta\alpha\gamma\beta(f_8e_1 + f_9\delta)$. Hence $u \in \beta\alpha\gamma\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\alpha\gamma\beta e_1\Lambda$.

On the other hand, let $v = \beta\alpha\gamma\beta e_1\eta \in \beta\alpha\gamma\beta e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\alpha\gamma\beta e_1\eta) = (\gamma, -\beta\delta)\beta\alpha\gamma\beta e_1\eta = (\gamma\beta\alpha\gamma\beta, -\beta\delta\beta\alpha\gamma\beta)e_1\eta = (0,0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta\alpha\gamma\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\alpha\gamma\beta e_1\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \to e_3\Lambda$ is given by $e_1\eta \mapsto \beta\alpha\gamma\beta e_1\eta$, for $\eta \in \Lambda$.

16.2.4. *The minimal projective resolution of the simple Λ-module S_2.*
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_2\Lambda$ is given by $e_3\lambda \to \gamma e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

16.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\gamma e_3\lambda = 0$ so $\gamma e_3\lambda = \gamma(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta) = f_1\gamma + f_2\gamma\beta + f_4\gamma\beta\delta + f_6\gamma\beta\delta\beta + f_7\gamma\beta\delta\beta\alpha = 0$, that is, $f_1 = f_2 = f_4 = f_6 = f_7 = 0$. Thus $e_3\lambda = f_3\beta\alpha + f_5\beta\alpha\gamma + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$.

Hence $\operatorname{Ker}\partial^1 = \{f_3\beta\alpha + f_5\beta\alpha\gamma + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \beta\alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = f_3\beta\alpha + f_5\beta\alpha\gamma + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$. So $x = \beta\alpha(f_3e_2 + f_5\gamma + f_8\gamma\beta + f_9\gamma\beta\delta)$. Thus $x \in \beta\alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \beta\alpha e_2\Lambda$.

On the other hand, let $y = \beta\alpha e_2\nu \in \beta\alpha e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\beta\alpha e_2\nu) = \gamma\beta\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and then $\beta\alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \beta\alpha e_2\Lambda$. $\square$

So $\partial^2 : e_2\Lambda \to e_3\Lambda$ is given by $e_2\nu \mapsto \beta\alpha e_2\nu$, for $\nu \in \Lambda$.

16.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^2$. Then $\beta\alpha e_2\nu = 0$. So $\beta\alpha e_2\nu = \beta\alpha(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) = d_1\beta\alpha + d_2\beta\alpha\gamma + d_3\beta\alpha\gamma\beta + d_4\beta\alpha\gamma\beta\delta = 0$, that is, $d_1 = d_2 = d_3 = d_4 = 0$. Thus $e_2\nu = d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$. Therefore $\operatorname{Ker}\partial^2 = \{d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \gamma\beta\delta\beta e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ so $u = \gamma\beta\delta\beta(d_5e_1 + d_6\alpha)$. Hence $u \in \gamma\beta\delta\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \gamma\beta\delta\beta e_1\Lambda$.

On the other hand, let $v = \gamma\beta\delta\beta e_1\eta \in \gamma\beta\delta\beta e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\gamma\beta\delta\beta e_1\eta) = \beta\alpha\gamma\beta\delta\beta e_1\eta = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\gamma\beta\delta\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \gamma\beta\delta\beta e_1\Lambda$. □

So the map $\partial^3 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \gamma\beta\delta\beta e_1\eta$, for $\eta \in \Lambda$.

16.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3*.*
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_3\Lambda$ is given by $e_1\eta \mapsto \beta e_1\eta$, for $\eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

16.2.8. $\operatorname{Ker}\partial^1$*.*
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^1$. Then $\beta e_1\eta = 0$ so $\beta(c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta) = c_1\beta + c_2\beta\alpha + c_3\beta\delta + c_4\beta\delta\beta + c_5\beta\delta\beta\delta + c_6\beta\delta\beta\delta\beta + c_7\beta\delta\beta\alpha + c_8\beta\delta\beta\delta\beta\delta = 0$, that is, $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$. Thus $e_1\eta = c_9\delta\beta\delta\beta\delta\beta$.

Hence $\operatorname{Ker}\partial^1 = \{c_9\delta\beta\delta\beta\delta\beta : c_9 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \delta\beta\delta\beta\delta\beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = c_9\delta\beta\delta\beta\delta\beta$. So $x = \delta\beta\delta\beta\delta\beta(c_9e_1)$. Thus $x \in \delta\beta\delta\beta\delta\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \delta\beta\delta\beta\delta\beta e_1\Lambda$.

On the other hand, let $y = \delta\beta\delta\beta\delta\beta e_1\eta \in \delta\beta\delta\beta\delta\beta e_1\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\delta\beta\delta\beta\delta\beta e_1\eta) = \beta\delta\beta\delta\beta\delta\beta e_1\eta = \beta\delta\beta\delta\beta\delta\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\delta\beta\delta\beta\delta\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \delta\beta\delta\beta\delta\beta e_1\Lambda$. □

So $\partial^2 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \delta\beta\delta\beta\delta\beta e_1\eta$, for $\eta \in \Lambda$.

Note that $\operatorname{Ker}\partial^1 \cong \Omega^2(S_3)$ and then $\Omega^2(S_3) \cong S_1$.

16.2.9. $\operatorname{Ker}\partial^2$*.*
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^2$. Then $\delta\beta\delta\beta\delta\beta e_1\eta = 0$. So $\delta\beta\delta\beta\delta\beta e_1\eta = \delta\beta\delta\beta\delta\beta(c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\alpha\gamma + c_6\alpha\gamma\beta + c_7\delta\beta\alpha + c_8\alpha\gamma\beta\delta + c_9\alpha\gamma\beta\delta\beta) = c_1\delta\beta\delta\beta\delta\beta = 0$, that is, $c_1 = 0$. Thus $e_1\eta = c_2\alpha + c_3\delta + c_4\delta\beta + c_5\alpha\gamma + c_6\alpha\gamma\beta + c_7\delta\beta\alpha + c_8\alpha\gamma\beta\delta + c_9\alpha\gamma\beta\delta\beta$. Therefore $\operatorname{Ker}\partial^2 = \{c_2\alpha + c_3\delta + c_4\delta\beta + c_5\alpha\gamma + c_6\alpha\gamma\beta + c_7\delta\beta\alpha + c_8\alpha\gamma\beta\delta + c_9\alpha\gamma\beta\delta\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha e_2\Lambda + \delta e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = c_2\alpha + c_3\delta + c_4\delta\beta + c_5\alpha\gamma + c_6\alpha\gamma\beta + c_7\delta\beta\alpha + c_8\alpha\gamma\beta\delta + c_9\alpha\gamma\beta\delta\beta$, that is, $u = \alpha(c_2e_2 + c_5\gamma + c_6\gamma\beta + c_8\gamma\beta\delta + c_9\gamma\beta\delta\beta) + \delta(c_3e_3 + c_4\beta + c_7\beta\alpha)$. Hence $u \in \alpha e_2\Lambda + \delta e_3\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \alpha e_2\Lambda + \delta e_3\Lambda$.

On the other hand, let $v = \alpha e_2\nu + \delta e_3\lambda \in \alpha e_2\Lambda + \delta e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha e_2\nu + \delta e_3\lambda) = \delta\beta\delta\beta\delta\beta\alpha e_2\nu + \delta\beta\delta\beta\delta\beta\delta e_3\lambda = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $\alpha e_2\Lambda + \delta e_3\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \alpha e_2\Lambda + \delta e_3\Lambda$. □

So the map $\partial^3 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda$, for $\nu, \lambda \in \Lambda$.

Thus the maps for S_1 are:

$$\begin{aligned}
&\partial^1 : (e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda,\\
&\partial^2 : e_3\lambda \mapsto (-\gamma, \beta\delta)e_3\lambda,\\
&\partial^3 : e_1\eta \mapsto \beta\alpha\gamma\beta e_1\eta,
\end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

Also the maps for S_2 are:

$$\begin{aligned}
&\partial^1 : e_3\lambda \to \gamma e_3\lambda,\\
&\partial^2 : e_2\nu \mapsto \beta\alpha e_2\nu\\
&\partial^3 : e_1\eta \mapsto \gamma\beta\delta\beta e_1\eta,
\end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\begin{aligned}
&\partial^1 : e_1\eta \mapsto \beta e_1\eta,\\
&\partial^2 : e_1\eta \mapsto \delta\beta\delta\beta\delta\beta e_1\eta,\\
&\partial^3 : (e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda,
\end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

16.3. g^3 **for** S_1, S_2 **and** S_3.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$$e_1 \xrightarrow{\partial^3} \beta\alpha\gamma\beta \xrightarrow{\partial^2} (-\gamma, \beta\delta)\beta\alpha\gamma\beta = (-\gamma\beta\alpha\gamma\beta, \beta\delta\beta\alpha\gamma\beta) \xrightarrow{\partial^1} -\alpha\gamma\beta\alpha\gamma\beta + \delta\beta\delta\beta\alpha\gamma\beta\ ,$$

so $-\alpha\gamma\beta\alpha\gamma\beta + \delta\beta\delta\beta\alpha\gamma\beta \in g^3$.

For S_2

$e_1 \xrightarrow{\partial^3} \gamma\beta\delta\beta \xrightarrow{\partial^2} \beta\alpha\gamma\beta\delta\beta \xrightarrow{\partial^1} \gamma\beta\alpha\gamma\beta\delta\beta$, so $\gamma\beta\alpha\gamma\beta\delta\beta \in g^3$.

For S_3

$(e_2, 0) \xrightarrow{\partial^3} \alpha \xrightarrow{\partial^2} \delta\beta\delta\beta\delta\beta\alpha \xrightarrow{\partial^1} \beta\delta\beta\delta\beta\delta\beta\alpha$, so $\beta\delta\beta\delta\beta\delta\beta\alpha \in g^3$.

$(0, e_3) \xrightarrow{\partial^3} \delta \xrightarrow{\partial^2} \delta\beta\delta\beta\delta\beta\delta \xrightarrow{\partial^1} \beta\delta\beta\delta\beta\delta\beta\delta$, so $\beta\delta\beta\delta\beta\delta\beta\delta \in g^3$.

Let $g_1^3 = -\alpha\gamma\beta\alpha\gamma\beta + \delta\beta\delta\beta\alpha\gamma\beta$,
$g_2^3 = \gamma\beta\alpha\gamma\beta\delta\beta$,

$g_3^3 = \beta\delta\beta\delta\beta\delta\beta\alpha,$

$g_4^3 = \beta\delta\beta\delta\beta\delta\beta\delta.$

So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3\}$.

We know that $g^2 = \{\delta\beta\delta - \alpha\gamma, \gamma\beta\alpha, (\beta\delta)^3\beta\}$. Denote

$g_1^2 = \delta\beta\delta - \alpha\gamma,$

$g_2^2 = \gamma\beta\alpha,$

$g_3^2 = (\beta\delta)^3\beta.$

So we have

$g_1^3 = g_1^2\beta\alpha\gamma\beta = -\alpha g_2^2\gamma\beta - \delta\beta\delta\beta g_1^2\beta + \delta g_3^2,$

$g_2^3 = g_2^2\gamma\beta\delta\beta = -\gamma\beta g_1^2\beta\delta\beta + \gamma g_3^2,$

$g_3^3 = g_3^2\alpha = \beta\delta\beta g_1^2\beta\alpha + \beta\delta\beta\alpha g_2^2,$

$g_4^3 = g_3^2\delta = \beta g_1^2\beta\alpha\gamma + \beta\alpha g_2^2\gamma + \beta\delta\beta\delta\beta g_1^2.$

16.3.1. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots$$

16.3.2. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_3 \mapsto j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta\\
&e_2 \otimes_{g_2^2} e_2 \mapsto j_4 e_2 + j_5\gamma\beta\delta\beta\alpha\\
&e_3 \otimes_{g_3^2} e_1 \mapsto j_6\beta + j_7\beta\delta\beta + j_8\beta\alpha\gamma\beta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \beta\alpha\gamma\beta + \alpha \otimes_{g_2^2} \gamma\beta + \delta\beta\delta\beta \otimes_{g_1^2} \beta - \delta \otimes_{g_3^2} e_1,$

$e_2 \otimes_{g_2^3} e_1 \mapsto e_2 \otimes_{g_2^2} \gamma\beta\delta\beta + \gamma\beta \otimes_{g_1^2} \beta\delta\beta - \gamma \otimes_{g_3^2} e_1,$

$e_3 \otimes_{g_3^3} e_2 \mapsto e_3 \otimes_{g_3^2} \alpha - \beta\delta\beta \otimes_{g_1^2} \beta\alpha - \beta\delta\beta\alpha \otimes_{g_2^2} e_2,$

$e_3 \otimes_{g_4^3} e_3 \mapsto e_3 \otimes_{g_3^2} \delta - \beta \otimes_{g_1^2} \beta\alpha\gamma - \beta\alpha \otimes_{g_2^2} \gamma - \beta\delta\beta\delta\beta \otimes_{g_1^2} e_3.$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta)\beta\alpha\gamma\beta + \alpha(j_4e_2 + j_5\gamma\beta\delta\beta\alpha)\gamma\beta + \delta\beta\delta\beta(j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta)\beta - \delta(j_6\beta + j_7\beta\delta\beta + j_8\beta\alpha\gamma\beta) = (2j_1 - j_8)\delta\beta\alpha\gamma\beta + (-j_7 + j_4)\delta\beta\delta\beta - j_6\delta\beta = 0$ then $j_8 = 2j_1, j_7 = j_4$ and $j_6 = 0$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_1) = (j_4e_2 + j_5\gamma\beta\delta\beta\alpha)\gamma\beta\delta\beta + \gamma\beta(j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta)\beta\delta\beta - \gamma(j_6\beta + j_7\beta\delta\beta + j_8\beta\alpha\gamma\beta) = (j_4 - j_7)\gamma\beta\delta\beta - j_6\gamma\beta = 0$, that is, $j_7 = j_4$ and $j_6 = 0$.

Also $\theta d^3(e_3 \otimes_{g_3^3} e_2) = (j_6\beta + j_7\beta\delta\beta + j_8\beta\alpha\gamma\beta)\alpha - \beta\delta\beta(j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta)\beta\alpha - \beta\delta\beta\alpha(j_4e_2 + j_5\gamma\beta\delta\beta\alpha) = j_6\beta\alpha + (-j_4 + j_7)\beta\delta\beta\alpha = 0$ then $j_7 = j_4$ and $j_6 = 0$.

And $\theta d^3(e_3 \otimes_{g_4^3} e_3) = (j_6\beta + j_7\beta\delta\beta + j_8\beta\alpha\gamma\beta)\delta - \beta(j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta)\beta\alpha\gamma - \beta\alpha(j_4e_2 + j_5\gamma\beta\delta\beta\alpha)\gamma - \beta\delta\beta\delta\beta(j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta) = (-2j_1 + j_8)\beta\alpha\gamma\beta\delta + (-j_4 + j_7)\beta\alpha\gamma + j_6\beta\delta = 0$, that is, $j_6 = 0, j_7 = j_4$ and $j_8 = 2j_1$.

So

$$\operatorname{Ker}\delta^2 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto j_4e_2 + j_5\gamma\beta\delta\beta\alpha \\ e_3 \otimes_{g_3^2} e_1 \mapsto j_4\beta\delta\beta + 2j_1\beta\alpha\gamma\beta \end{array} \right\}.$$

Now we will consider two cases if $\operatorname{char} K = 2$ and if $\operatorname{char} K \neq 2$.

If $\operatorname{char} K = 2$ then $\dim \operatorname{Ker}\delta^2 = 5$ and we have

$$\operatorname{Ker}\delta^2 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto j_4e_2 + j_5\gamma\beta\delta\beta\alpha \\ e_3 \otimes_{g_3^2} e_1 \mapsto j_4\beta\delta\beta \end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Ker}\delta^2 = 5$ and we have

$$\operatorname{Ker}\delta^2 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto j_1\delta + j_2\delta\beta\delta + j_3\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto j_4e_2 + j_5\gamma\beta\delta\beta\alpha \\ e_3 \otimes_{g_3^2} e_1 \mapsto j_4\beta\delta\beta + 2j_1\beta\alpha\gamma\beta \end{array} \right\}.$$

16.3.3. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_2 \to z_1\alpha + z_2\delta\beta\alpha$

$e_1 \otimes_\delta e_3 \to z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta$

$e_2 \otimes_\gamma e_3 \to z_6\gamma + z_7\gamma\beta\delta$

$e_3 \otimes_\beta e_1 \to z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_3 \mapsto e_1 \otimes_\delta \beta\delta + \delta \otimes_\beta \delta + \delta\beta \otimes_\delta e_3 - e_1 \otimes_\alpha \gamma - \alpha \otimes_\gamma e_3$

$e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \otimes_\gamma \beta\alpha + \gamma \otimes_\beta \alpha + \gamma\beta \otimes_\alpha e_2$

$e_3 \otimes_{g_3^2} e_1 \mapsto e_3 \otimes_\beta \delta\beta\delta\beta\delta\beta + \beta \otimes_\delta \beta\delta\beta\delta\beta + \beta\delta \otimes_\beta \delta\beta\delta\beta + \beta\delta\beta \otimes_\delta \beta\delta\beta + \beta\delta\beta\delta \otimes_\beta \delta\beta + \beta\delta\beta\delta\beta \otimes_\delta \beta + \beta\delta\beta\delta\beta\delta \otimes_\beta e_1$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_3) = (z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta)\beta\delta + \delta(z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta)\delta + \delta\beta(z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta) - (z_1\alpha + z_2\delta\beta\alpha)\gamma - \alpha(z_6\gamma + z_7\gamma\beta\delta) = (-z_1 + 2z_3 - z_6 + z_8)\delta\beta\delta + (-z_2 + 2z_4 - z_7 + z_9)\delta\beta\delta\beta\delta$,

$\varphi d^2(e_2 \otimes_{g_2^2} e_2) = (z_6\gamma + z_7\gamma\beta\delta)\beta\alpha + \gamma(z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta)\alpha + \gamma\beta(z_1\alpha + z_2\delta\beta\alpha) = (z_2 + z_7 + z_9)\gamma\beta\delta\beta\alpha$

$\varphi d^2(e_3 \otimes_{g_3^2} e_1) = (z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta)\delta\beta\delta\beta\delta\beta + \beta(z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta)\beta\delta\beta\delta\beta + \beta\delta(z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta)\delta\beta\delta\beta + \beta\delta\beta(z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta)\beta\delta\beta + \beta\delta\beta\delta(z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta)\delta\beta + \beta\delta\beta\delta\beta(z_3\delta + z_4\delta\beta\delta + z_5\delta\beta\delta\beta\delta)\beta + \beta\delta\beta\delta\beta\delta(z_8\beta + z_9\beta\delta\beta + z_{10}\beta\alpha\gamma\beta) = 0.$
Thus φd^2 is given by

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_3 \mapsto (-z_1 + 2z_3 - z_6 + z_8)\delta\beta\delta + (-z_2 + 2z_4 - z_7 + z_9)\delta\beta\delta\beta\delta \\
&e_2 \otimes_{g_2^2} e_2 \mapsto (z_2 + z_7 + z_9)\gamma\beta\delta\beta\alpha \\
&e_3 \otimes_{g_3^2} e_1 \mapsto 0,
\end{aligned}$$

where $z_i \in K$.

Note that if $\operatorname{char} K = 2$ then $\dim \operatorname{Im} \delta^1 = 2$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto (z_1 + z_6 + z_8)\delta\beta\delta + (z_2 + z_7 + z_9)\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto (z_2 + z_7 + z_9)\gamma\beta\delta\beta\alpha \\ e_3 \otimes_{g_3^2} e_1 \mapsto 0 \end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Im} \delta^1 = 3$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto (-z_1 + 2z_3 - z_6 + z_8)\delta\beta\delta + (-z_2 + 2z_4 - z_7 + z_9)\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto (z_2 + z_7 + z_9)\gamma\beta\delta\beta\alpha \\ e_3 \otimes_{g_3^2} e_1 \mapsto 0 \end{array} \right\}.$$

16.3.4. $\mathrm{HH}^2(\Lambda)$.

From 16.3.2 and 16.3.3 we have if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto d_1\delta + d_2\delta\beta\delta\beta\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto d_3 e_2 \\ e_3 \otimes_{g_3^2} e_1 \mapsto d_3\beta\delta\beta \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{aligned}
&x : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_3 \mapsto \delta \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&y : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_3 \mapsto \delta\beta\delta\beta\delta \\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&z : P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\
&e_3 \otimes_{g_3^2} e_1 \mapsto \beta\delta\beta \\
&\text{else} \mapsto 0.
\end{aligned}$$

Note that y represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}
&z : P^2 \to \Lambda \\
&e_2 \otimes_{g_2^2} e_2 \mapsto -\gamma\beta\delta\beta\alpha \\
&\text{else} \mapsto 0.
\end{aligned}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto d_1\delta \\ e_2 \otimes_{g_2^2} e_2 \mapsto d_2 e_2 \\ e_3 \otimes_{g_3^2} e_1 \mapsto d_2\beta\delta\beta + 2d_1\beta\alpha\gamma\beta \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_3 \mapsto \delta \\ e_3 \otimes_{g_3^2} e_1 \mapsto \frac{1}{2}\beta\alpha\gamma\beta \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_2 \otimes_{g_2^2} e_2 \mapsto e_2 \\ e_3 \otimes_{g_3^2} e_1 \mapsto \beta\delta\beta \\ \text{else} \mapsto 0. \end{array}$$

17. The algebra A_{13}

Definition 17.1. [5] *Let A_{13} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\gamma"] \arrow[loop above, "\alpha"] \arrow[r, shift left, "\delta"] & 3 \arrow[l, shift left, "\sigma"] \end{tikzcd}$$

and

$$I = \langle \alpha^2 - \gamma\beta, \alpha^3 - \delta\sigma, \beta\delta, \beta\gamma, \sigma\gamma, \alpha\delta, \sigma\alpha \rangle.$$

17.1. The structure of the indecomposable projectives.

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda$ and $e_3\Lambda$ where

$e_1\Lambda = sp\{e_1, \beta, \beta\alpha, \beta\alpha\gamma\}$,
$e_2\Lambda = sp\{e_2, \gamma, \delta, \alpha, \alpha\gamma, \alpha^2, \alpha^3\}$,
$e_3\Lambda = sp\{e_3, \sigma, \sigma\delta\}$.

So we have for $e_1\Lambda$

$$\begin{tikzcd} e_1 \arrow[d, dash] \\ \beta \arrow[d, dash] \\ \beta\alpha \arrow[d, dash] \\ \beta\alpha\gamma \end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd} & & e_2 \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \gamma \arrow[dr, dash] & & \alpha \arrow[dl, dash] \arrow[d, dash] & \delta \arrow[ddl, dash] \\ & \alpha^2 \arrow[dr, dash] & \alpha\gamma \arrow[d, dash] & \\ & & \alpha^3 & \end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{array}{c} e_3 \\ | \\ \sigma \\ | \\ \sigma\delta \end{array}$$

It can be seen that $\operatorname{rad} e_2\Lambda/\operatorname{soc} e_2\Lambda$ has a simple direct summand (isomorphic to S_3). This shows that the simple module S_3 does not lie at the end of a component of the stable Auslander-Reiten quiver.

17.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2 and S_3.**

17.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \beta e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

17.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\beta e_2\nu = 0$, so $\beta(d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3) = d_1\beta + d_4\beta\alpha + d_5\beta\alpha\gamma = 0$ and then $d_1 = d_4 = d_5 = 0$. Thus $e_2\nu = d_2\gamma + d_3\delta + d_6\alpha^2 + d_7\alpha^3$.

Hence $\operatorname{Ker}\partial^1 = \{d_2\gamma + d_3\delta + d_6\alpha^2 + d_7\alpha^3 : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \gamma e_1\Lambda + \delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = d_2\gamma + d_3\delta + d_6\alpha^2 + d_7\alpha^3$, that is, $x = \gamma(d_2e_1 + d_6\beta) + \delta(d_3e_3 + d_7\sigma)$. Thus $x \in \gamma e_1\Lambda + \delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \gamma e_1\Lambda + \delta e_3\Lambda$.

On the other hand, let $y = \gamma e_1\eta + \delta e_3\lambda \in \gamma e_1\Lambda + \delta e_3\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\gamma e_1\eta + \delta e_3\lambda) = \beta\gamma e_1\eta + \beta\delta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\gamma e_1\Lambda + \delta e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \gamma e_1\Lambda + \delta e_3\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto \gamma e_1\eta + \delta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

17.2.3. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma$ and $e_3\lambda = f_1e_3 + f_2\sigma + f_3\sigma\delta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^2$. Then $\gamma e_1\eta + \delta e_3\lambda = 0$. So $\gamma e_1\eta + \delta e_3\lambda = \gamma(c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma) + \delta(f_1e_3 + f_2\sigma + f_3\sigma\delta) = c_1\gamma + c_2\gamma\beta + c_3\gamma\beta\alpha + f_1\delta + f_2\delta\sigma = 0$, that is, $c_1 = c_2 = f_1 = 0, f_2 = -c_3$. Thus $e_1\eta = c_3\beta\alpha + c_4\beta\alpha\gamma$ and $e_3\lambda = -c_3\sigma + f_3\sigma\delta$. Therefore $\operatorname{Ker}\partial^2 = \{(c_3\beta\alpha + c_4\beta\alpha\gamma, -c_3\sigma + f_3\sigma\delta) : c_i, f_3 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\beta\alpha, -\sigma)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_3\beta\alpha + c_4\beta\alpha\gamma, -c_3\sigma + f_3\sigma\delta)$ so $u = (\beta\alpha, -\sigma)(c_3e_2 + c_4\gamma - f_3\delta)$. Hence $u \in (\beta\alpha, -\sigma)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\beta\alpha, -\sigma)e_2\Lambda$.

On the other hand, let $v = (\beta\alpha, -\sigma)e_2\nu \in (\beta\alpha, -\sigma)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\beta\alpha, -\sigma)e_2\nu) = (\gamma\beta\alpha - \delta\sigma)e_2\nu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\beta\alpha, -\sigma)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\beta\alpha, -\sigma)e_2\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_2\nu \mapsto (\beta\alpha, -\sigma)e_2\nu$, for $\nu \in \Lambda$.

17.2.4. *The minimal projective resolution of the simple* Λ*-module* S_2.

The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_2\nu, e_3\lambda) \to \gamma e_1\eta + \alpha e_2\nu + \delta e_3\lambda$, for $\eta, \nu, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

17.2.5. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, e_2\nu = d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$ and $e_3\lambda = f_1e_3 + f_2\sigma + f_3\sigma\delta$ with $c_i, d_i, f_i \in K$. Assume that $(e_1\eta, e_2\nu, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\gamma e_1\eta + \alpha e_2\nu + \delta e_3\lambda = 0$ so $\gamma e_1\eta + \alpha e_2\nu + \delta e_3\lambda = \gamma(c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma) + \alpha(d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3) + \delta(f_1e_3 + f_2\sigma + f_3\sigma\delta) = c_1\gamma + c_2\gamma\beta + c_3\gamma\beta\alpha + d_1\alpha + d_2\alpha\gamma + d_4\alpha^2 + d_6\alpha^3 + f_1\delta + f_2\delta\sigma = c_1\gamma + (c_2 + d_4)\alpha^2 + (c_3 + d_6 + f_2)\alpha^3 + d_1\alpha + d_2\alpha\gamma + f_1\delta = 0$, that is, $c_1 = d_1 = d_2 = f_1 = 0, d_4 = -c_2$ and $f_2 = -c_3 - d_6$. Thus $e_1\eta = c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, e_2\nu = d_3\delta - c_2\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$ and $e_3\lambda = -(c_3 + d_6)\sigma + f_3\sigma\delta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, d_3\delta - c_2\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3, -(c_3 + d_6)\sigma + f_3\sigma\delta) : c_i, d_i, f_3 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (-\beta, \alpha, 0)e_2\Lambda + (0, \alpha^2, -\sigma)e_2\Lambda + (0, \delta, 0)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, d_3\delta - c_2\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3, -(c_3 + d_6)\sigma + f_3\sigma\delta)$. So $x = (-\beta, \alpha, 0)(-c_2e_2 - c_3\alpha - c_4\alpha\gamma + d_5\gamma + d_7\alpha^2) +$

$(0,\alpha^2,-\sigma)((c_3+d_6)e_2-f_3\delta)+(0,\delta,0)(d_3e_3)$. Thus $x\in(-\beta,\alpha,0)e_2\Lambda+(0,\alpha^2,-\sigma)e_2\Lambda+(0,\delta,0)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1\subseteq(-\beta,\alpha,0)e_2\Lambda+(0,\alpha^2,-\sigma)e_2\Lambda+(0,\delta,0)e_3\Lambda$.

On the other hand, let $y=(-\beta,\alpha,0)e_2\nu+(0,\alpha^2,-\sigma)e_2\nu+(0,\delta,0)e_3\lambda\in(-\beta,\alpha,0)e_2\Lambda+(0,\alpha^2,-\sigma)e_2\Lambda+(0,\delta,0)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y)=\partial^1((-\beta,\alpha,0)e_2\nu+(0,\alpha^2,-\sigma)e_2\nu+(0,\delta,0)e_3\lambda)=(-\gamma\beta+\alpha^2)e_2\nu+(\alpha^3-\delta\sigma)e_2\nu+\alpha\delta e_3\lambda=0$. Thus $y\in\operatorname{Ker}\partial^1$ and then $(-\beta,\alpha,0)e_2\Lambda+(0,\alpha^2,-\sigma)e_2\Lambda+(0,\delta,0)e_3\Lambda\subseteq\operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1=(-\beta,\alpha,0)e_2\Lambda+(0,\alpha^2,-\sigma)e_2\Lambda+(0,\delta,0)e_3\Lambda$. □

So $\partial^2: e_2\Lambda\oplus e_2\Lambda\oplus e_3\Lambda\to e_1\Lambda\oplus e_2\Lambda\oplus e_3\Lambda$ is given by $(e_2\nu,e_2\nu,e_3\lambda)\mapsto(-\beta,\alpha,0)e_2\nu+(0,\alpha^2,-\sigma)e_2\nu+(0,\delta,0)e_3\lambda$, for $\nu,\lambda\in\Lambda$.

17.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2=\Omega^3(S_2)$. Let $e_2\nu=d_1e_2+d_2\gamma+d_3\delta+d_4\alpha+d_5\alpha\gamma+d_6\alpha^2+d_7\alpha^3$ and $e_3\lambda=f_1e_3+f_2\sigma+f_3\sigma\delta$ with $d_i,f_i\in K$. Assume that $(e_2\nu,e_2\nu,e_3\lambda)\in\operatorname{Ker}\partial^2$. Then $(-\beta,\alpha,0)e_2\nu+(0,\alpha^2,-\sigma)e_2\nu+(0,\delta,0)e_3\lambda=(0,0,0)$. So $(-\beta,\alpha,0)e_2\nu+(0,\alpha^2,-\sigma)e_2\nu+(0,\delta,0)e_3\lambda=(-\beta,\alpha,0)(d_1e_2+d_2\gamma+d_3\delta+d_4\alpha+d_5\alpha\gamma+d_6\alpha^2+d_7\alpha^3)+(0,\alpha^2,-\sigma)(d_1e_2+d_2\gamma+d_3\delta+d_4\alpha+d_5\alpha\gamma+d_6\alpha^2+d_7\alpha^3)+(0,\delta,0)(f_1e_3+f_2\sigma+f_3\sigma\delta)=(-d_1\beta-d_4\beta\alpha-d_5\beta\alpha\gamma,d_1\alpha+d_2\alpha\gamma+d_4\alpha^2+d_6\alpha^3,0)+(0,d_1\alpha^2+d_4\alpha^3,-d_1\sigma-d_3\sigma\delta)+(0,f_1\delta+f_2\delta\sigma,0)=(-d_1\beta-d_4\beta\alpha-d_5\beta\alpha\gamma,d_1\alpha+d_2\alpha\gamma+d_4\alpha^2+d_6\alpha^3+d_1\alpha^2+d_4\alpha^3+f_1\delta+f_2\delta\sigma,-d_1\sigma-d_3\sigma\delta)=(0,0,0)$. So $-d_1\beta-d_4\beta\alpha-d_5\beta\alpha\gamma=0$, that is, $d_1=d_4=d_5=0$. Also $d_1\alpha+d_2\alpha\gamma+d_4\alpha^2+d_6\alpha^3+d_1\alpha^2+d_4\alpha^3+f_1\delta+f_2\delta\sigma=0$, that is, $d_1=d_2=d_4=f_1=0,f_2=-d_6$. And for $-d_1\sigma-d_3\sigma\delta=0$, that is, $d_1=d_3=0$. Thus $e_2\nu=d_6\alpha^2+d_7\alpha^3$ and $e_3\lambda=-d_6\sigma+f_3\sigma\delta$. Therefore $\operatorname{Ker}\partial^2=\{(d_6\alpha^2+d_7\alpha^3,-d_6\sigma+f_3\sigma\delta):d_i,f_3\in K\}$.

Claim. $\operatorname{Ker}\partial^2=(\alpha^2,-\sigma)e_2\Lambda$.

Proof. Let $u\in\operatorname{Ker}\partial^2$. Then $u=(d_6\alpha^2+d_7\alpha^3,-d_6\sigma+f_3\sigma\delta)$ so $u=(\alpha^2,-\sigma)(d_6e_2+d_7\alpha-f_3\delta)$. Hence $u\in(\alpha^2,-\sigma)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2\subseteq(\alpha^2,-\sigma)e_2\Lambda$.

On the other hand, let $v=(\alpha^2,-\sigma)e_2\nu\in(\alpha^2,-\sigma)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v)=\partial^2((\alpha^2,-\sigma)e_2\nu)=(-\beta\alpha^2,\alpha^4+(\alpha^3-\delta\sigma),-\sigma\alpha^2)e_2\nu=(0,0,0)$. Therefore $v\in\operatorname{Ker}\partial^2$ and then $(\alpha^2,-\sigma)e_2\Lambda\subseteq\operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2=(\alpha^2,-\sigma)e_2\Lambda$. □

So the map $\partial^3: e_2\Lambda\to e_2\Lambda\oplus e_2\Lambda\oplus e_3\Lambda$ is given by $e_2\nu\mapsto(\alpha^2,-\sigma)e_2\nu$, for $\nu\in\Lambda$.

17.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3.

The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots\longrightarrow e_2\Lambda\xrightarrow{\partial^1}e_3\Lambda\longrightarrow S_3\longrightarrow 0$$

where $\partial^1: e_2\Lambda\to e_3\Lambda$ is given by $e_2\nu\mapsto\sigma e_2\nu$, for $\nu\in\Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2,∂^3 for the simple Λ-module S_3.

17.2.8. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_2\nu = d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\sigma e_2\nu = 0$ so $\sigma(d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3) = d_1\sigma + d_3\sigma\delta = 0$, that is, $d_1 = d_3 = 0$. Thus $e_2\nu = d_2\gamma + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$.

Hence $\operatorname{Ker}\partial^1 = \{d_2\gamma + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3 : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \gamma e_1\Lambda + \alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = d_2\gamma + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$. So $x = \gamma(d_2e_1) + \alpha(d_4e_2 + d_5\gamma + d_6\alpha + d_7\alpha^2)$. Thus $x \in \gamma e_1\Lambda + \alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \gamma e_1\Lambda + \alpha e_2\Lambda$.

On the other hand, let $y = \gamma e_1\eta + \alpha e_2\nu \in \gamma e_1\Lambda + \alpha e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\gamma e_1\eta + \alpha e_2\nu) = \sigma\gamma e_1\eta + \sigma\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\gamma e_1\Lambda + \alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \gamma e_1\Lambda + \alpha e_2\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_2\Lambda$ is given by: $(e_1\eta, e_2\nu) \mapsto \gamma e_1\eta + \alpha e_2\nu$, for $\eta, \nu \in \Lambda$.

17.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_1\eta = c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma$ and $e_2\nu = d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3$ with $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker}\partial^2$. Then $\gamma e_1\eta + \alpha e_2\nu = 0$. So $\gamma e_1\eta + \alpha e_2\nu = \gamma(c_1e_1 + c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma) + \alpha(d_1e_2 + d_2\gamma + d_3\delta + d_4\alpha + d_5\alpha\gamma + d_6\alpha^2 + d_7\alpha^3) = c_1\gamma + c_2\gamma\beta + c_3\gamma\beta\alpha + d_1\alpha + d_2\alpha\gamma + d_4\alpha^2 + d_6\alpha^3 = c_1\gamma + d_1\alpha + d_2\alpha\gamma + (c_2 + d_4)\alpha^2 + (c_3 + d_6)\alpha^3 = 0$, that is, $c_1 = d_1 = d_2 = 0, d_4 = -c_2$ and $d_6 = -c_3$. Thus $e_1\eta = c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma$ and $e_2\nu = d_3\delta - c_2\alpha + d_5\alpha\gamma - c_3\alpha^2 + d_7\alpha^3$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, -c_2\alpha + d_3\delta + d_5\alpha\gamma - c_3\alpha^2 + d_7\alpha^3) : c_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\beta + c_3\beta\alpha + c_4\beta\alpha\gamma, -c_2\alpha + d_3\delta + d_5\alpha\gamma - c_3\alpha^2 + d_7\alpha^3)$, that is, $u = (-\beta, \alpha)(-c_2e_2 - c_3\alpha - c_4\alpha\gamma + d_5\gamma + d_7\alpha^2) + (0, \delta)(d_3e_3)$. Hence $u \in (-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda$.

On the other hand, let $v = (-\beta, \alpha)e_2\nu + (0, \delta)e_3\lambda \in (-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((-\beta, \alpha)e_2\nu + (0, \delta)e_3\lambda) = \partial^2(-\beta e_2\nu, \alpha e_2\nu + \delta e_3\lambda) = \gamma(-\beta e_2\nu) + \alpha(\alpha e_2\nu + \delta e_3\lambda) = (\alpha^2 - \gamma\beta)e_2\nu + \alpha\delta e_3\lambda = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (-\beta, \alpha)e_2\Lambda + (0, \delta)e_3\Lambda$. □

So the map $\partial^3 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto (-\beta, \alpha)e_2\nu + (0, \delta)e_3\lambda$, for $\nu, \lambda \in \Lambda$.

Thus the maps for S_1 are:

$\partial^1 : e_2\nu \mapsto \beta e_2\nu,$

$\partial^2 : (e_1\eta, e_3\lambda) \mapsto \gamma e_1\eta + \delta e_3\lambda,$

$\partial^3 : e_2\nu \mapsto (\beta\alpha, -\sigma)e_2\nu,$

for $\eta, \nu, \lambda \in \Lambda$.

Also the maps for S_2 are:

$\partial^1 : (e_1\eta, e_2\nu, e_3\lambda) \to \gamma e_1\eta + \alpha e_2\nu + \delta e_3\lambda,$

$\partial^2 : (e_2\nu, e_2\nu, e_3\lambda) \mapsto (-\beta, \alpha, 0)e_2\nu + (0, \alpha^2, -\sigma)e_2\nu + (0, \delta, 0)e_3\lambda$

$\partial^3 : e_2\nu \mapsto (\alpha^2, -\sigma)e_2\nu,$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$\partial^1 : e_2\nu \mapsto \sigma e_2\nu,$

$\partial^2 : (e_1\eta, e_2\nu) \mapsto \gamma e_1\eta + \alpha e_2\nu,$

$\partial^3 : (e_2\nu, e_3\lambda) \mapsto (-\beta, \alpha)e_2\nu + (0, \delta)e_3\lambda,$

for $\eta, \nu, \lambda \in \Lambda$.

17.3. g^3 **for** S_1, S_2 **and** S_3**.**

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_2 \xrightarrow{\partial^3} (\beta\alpha, -\sigma) \xrightarrow{\partial^2} \gamma\beta\alpha - \delta\sigma \xrightarrow{\partial^1} \beta\gamma\beta\alpha - \beta\delta\sigma$, so $\beta\gamma\beta\alpha - \beta\delta\sigma \in g^3$.

For S_2

$e_2 \overset{\partial^3}{\to} (\alpha^2, -\sigma) \overset{\partial^2}{\to} ((-\beta, \alpha, 0) + (0, \alpha^2, -\sigma))\alpha^2 + (0, \delta, 0)(-\sigma) = (-\beta\alpha^2, \alpha^4 + \alpha^3 - \delta\sigma, -\sigma\alpha^2) \overset{\partial^1}{\to} -\gamma\beta\alpha^2 + \alpha^5 + \alpha^4 - \alpha\delta\sigma - \delta\sigma\alpha^2$, so $\alpha^5 + \alpha^4 - \alpha\delta\sigma - \gamma\beta\alpha^2 - \delta\sigma\alpha^2 \in g^3$.

For S_3

$(e_2, 0) \xrightarrow{\partial^3} (-\beta, \alpha) \xrightarrow{\partial^2} -\gamma\beta + \alpha^2 \xrightarrow{\partial^1} \sigma\alpha^2 - \sigma\gamma\beta$, so $\sigma\alpha^2 - \sigma\gamma\beta \in g^3$.

$(0, e_3) \xrightarrow{\partial^3} (0, \delta) \xrightarrow{\partial^2} \alpha\delta \xrightarrow{\partial^1} \sigma\alpha\delta$, so $\sigma\alpha\delta \in g^3$.

Let $g^3_1 = \beta\gamma\beta\alpha - \beta\delta\sigma,$

$g^3_2 = \alpha^5 + \alpha^4 - \alpha\delta\sigma - \gamma\beta\alpha^2 - \delta\sigma\alpha^2,$

$g^3_3 = \sigma\alpha^2 - \sigma\gamma\beta,$

$g^3_4 = \sigma\alpha\delta.$

So $g^3 = \{g^3_1, g^3_2, g^3_3, g^3_4\}$.

We know that $g^2 = \{\alpha^2 - \gamma\beta, \alpha^3 - \delta\sigma, \beta\delta, \beta\gamma, \sigma\gamma, \alpha\delta, \sigma\alpha\}$. Denote

$g^2_1 = \beta\gamma,$

$g^2_2 = \beta\delta,$

$g^2_3 = \alpha^2 - \gamma\beta,$

$g^2_4 = \alpha^3 - \delta\sigma,$

$g^2_5 = \alpha\delta,$

$g^2_6 = \sigma\gamma,$

$g^2_7 = \sigma\alpha.$

So we have

$g^3_1 = g^2_1\beta\alpha - g^2_2\sigma = -\beta g^2_3\alpha + \beta g^2_4,$

$$
\begin{aligned}
g_2^3 &= g_4^2\alpha^2 + g_3^2\alpha^2 - g_5^2\sigma = \alpha g_4^2 + \alpha^3 g_3^2 - \delta g_7^2\alpha - \gamma\beta g_3^2 - \gamma g_1^2\beta + \alpha g_3^2\gamma\beta \\
&\quad + \alpha\gamma g_1^2\beta, \\
g_3^3 &= g_7^2\alpha - g_6^2\beta = \sigma g_3^2, \\
g_4^3 &= g_7^2\delta = \sigma g_5^2.
\end{aligned}
$$

17.3.1. $\mathrm{HH}^2(\Lambda)$.

From [16] we have the projective resolution of Λ

$$
\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0 .
$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$
0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \dots
$$

17.3.2. $\mathrm{Ker}\,\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$
\begin{aligned}
&\theta : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\beta\alpha\gamma \\
&e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
&e_2 \otimes_{g_3^2} e_2 \mapsto j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3 \\
&e_2 \otimes_{g_4^2} e_2 \mapsto j_7 e_2 + j_8\alpha + j_9\alpha^2 + j_{10}\alpha^3 \\
&e_2 \otimes_{g_5^2} e_3 \mapsto j_{11}\delta \\
&e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
&e_3 \otimes_{g_7^2} e_2 \mapsto j_{12}\sigma,
\end{aligned}
$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$
\begin{aligned}
&e_1 \otimes_{g_1^3} e_2 \mapsto e_1 \otimes_{g_1^2} \beta\alpha - e_1 \otimes_{g_2^2} \sigma + \beta \otimes_{g_3^2} \alpha - \beta \otimes_{g_4^2} e_2, \\
&e_2 \otimes_{g_2^3} e_2 \mapsto e_2 \otimes_{g_4^2} \alpha^2 + e_2 \otimes_{g_3^2} \alpha^2 - e_2 \otimes_{g_5^2} \sigma + \delta \otimes_{g_7^2} \alpha + \gamma \otimes_{g_1^2} \beta \\
&\qquad - \alpha \otimes_{g_3^2} \gamma\beta - \alpha\gamma \otimes_{g_1^2} \beta - \alpha^3 \otimes_{g_3^2} e_2 + \gamma\beta \otimes_{g_3^2} e_2 - \alpha \otimes_{g_4^2} e_2, \\
&e_3 \otimes_{g_3^3} e_2 \mapsto e_3 \otimes_{g_7^2} \alpha - e_3 \otimes_{g_6^2} \beta - \sigma \otimes_{g_3^2} e_2, \\
&e_3 \otimes_{g_4^3} e_3 \mapsto e_3 \otimes_{g_7^2} \delta - \sigma \otimes_{g_5^2} e_3.
\end{aligned}
$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_2) = (j_1 e_1 + j_2\beta\alpha\gamma)\beta\alpha + \beta(j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3)\alpha - \beta(j_7 e_2 + j_8\alpha + j_9\alpha^2 + j_{10}\alpha^3) = 0$ then $j_7 = 0$ and $j_8 = j_1 + j_3$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_2) = (j_7 e_2 + j_8\alpha + j_9\alpha^2 + j_{10}\alpha^3)\alpha^2 + (j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3)\alpha^2 - (j_{11}\delta)\sigma + \delta(j_{12}\sigma)\alpha + \gamma(j_1 e_1 + j_2\beta\alpha\gamma)\beta - \alpha(j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3)\gamma\beta - \alpha\gamma(j_1 e_1 + j_2\beta\alpha\gamma)\beta - \alpha^3(j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3) + \gamma\beta(j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3) - \alpha(j_7 e_2 + j_8\alpha + j_9\alpha^2 + j_{10}\alpha^3) = (j_1 - j_8)\alpha^2 + (2j_4 - j_9 - j_{11})\alpha^3 = 0$, that is, $j_7 = j_3 = 0, j_8 = j_1$ and $j_{11} = 2j_4 - j_9$.

Also $\theta d^3(e_3 \otimes_{g_3^3} e_2) = (j_{12}\sigma)\alpha - \sigma(j_3 e_2 + j_4\alpha + j_5\alpha^2 + j_6\alpha^3) = 0$ then $j_3 = 0$.

And $\theta d^3(e_3 \otimes_{g_4^3} e_3) = (j_{12}\sigma)\delta - \sigma(j_{11}\delta)$, that is, $j_{12} = j_{11}$.

So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda \\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2 \beta\alpha\gamma \\
&e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
&e_2 \otimes_{g_3^2} e_2 \mapsto j_4\alpha + j_5\alpha^2 + j_6\alpha^3 \\
&e_2 \otimes_{g_4^2} e_2 \mapsto j_1\alpha + j_9\alpha^2 + j_{10}\alpha^3 \\
&e_2 \otimes_{g_5^2} e_3 \mapsto (2j_4 - j_9)\delta \\
&e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
&e_3 \otimes_{g_7^2} e_2 \mapsto (2j_4 - j_9)\sigma
\end{aligned}$$

Now if $\operatorname{char} K \neq 2$ then $\dim \operatorname{Ker} \delta^2 = 7$.

If $\operatorname{char} K = 2$ then $\dim \operatorname{Ker} \delta^2 = 7$ and

$$\operatorname{Ker} \delta^2 = \left\{ \begin{array}{l}
\theta : P^2 \to \Lambda \\
e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2 \beta\alpha\gamma \\
e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
e_2 \otimes_{g_3^2} e_2 \mapsto j_4\alpha + j_5\alpha^2 + j_6\alpha^3 \\
e_2 \otimes_{g_4^2} e_2 \mapsto j_1\alpha + j_9\alpha^2 + j_{10}\alpha^3 \\
e_2 \otimes_{g_5^2} e_3 \mapsto j_9\delta \\
e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
e_3 \otimes_{g_7^2} e_2 \mapsto j_9\sigma
\end{array} \right\}.$$

Now if $\operatorname{char} K \neq 2$ then $\dim \operatorname{Ker} \delta^2 = 7$ and

$$\operatorname{Ker} \delta^2 = \left\{ \begin{array}{l}
\theta : P^2 \to \Lambda \\
e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2 \beta\alpha\gamma \\
e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
e_2 \otimes_{g_3^2} e_2 \mapsto j_4\alpha + j_5\alpha^2 + j_6\alpha^3 \\
e_2 \otimes_{g_4^2} e_2 \mapsto j_1\alpha + j_9\alpha^2 + j_{10}\alpha^3 \\
e_2 \otimes_{g_5^2} e_3 \mapsto (2j_4 - j_9)\delta \\
e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
e_3 \otimes_{g_7^2} e_2 \mapsto (2j_4 - j_9)\sigma
\end{array} \right\}.$$

17.3.3. $\operatorname{Im} \delta^1$.

Now to find $\operatorname{Im} \delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\beta e_2 \to z_1\beta + z_2\beta\alpha$

$e_2 \otimes_\gamma e_1 \to z_3\gamma + z_4\alpha\gamma$

$e_2 \otimes_\alpha e_2 \to z_5 e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3$

$e_2 \otimes_\delta e_3 \to z_9\delta$

$e_3 \otimes_\sigma e_2 \to z_{10}\sigma,$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\beta \gamma + \beta \otimes_\gamma e_1$

$e_1 \otimes_{g_2^2} e_3 \mapsto e_1 \otimes_\beta \delta + \beta \otimes_\delta e_3$

$e_2 \otimes_{g_3^2} e_2 \mapsto e_2 \otimes_\alpha \alpha + \alpha \otimes_\alpha e_2 - e_2 \otimes_\gamma \beta - \gamma \otimes_\beta e_2$

$e_2 \otimes_{g_4^2} e_2 \mapsto e_2 \otimes_\alpha \alpha^2 + \alpha \otimes_\alpha \alpha + \alpha^2 \otimes_\alpha e_2 - e_2 \otimes_\delta \sigma - \delta \otimes_\sigma e_2$

$e_2 \otimes_{g_5^2} e_3 \mapsto e_2 \otimes_\alpha \delta + \alpha \otimes_\delta e_3$

$e_3 \otimes_{g_6^2} e_1 \mapsto e_3 \otimes_\sigma \gamma + \sigma \otimes_\gamma e_1$

$e_3 \otimes_{g_7^2} e_2 \mapsto e_3 \otimes_\sigma \alpha + \sigma \otimes_\alpha e_2.$

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1\beta + z_2\beta\alpha)\gamma + \beta(z_3\gamma + z_4\alpha\gamma) = (z_2 + z_4)\beta\alpha\gamma,$

$\varphi d^2(e_1 \otimes_{g_2^2} e_3) = (z_1\beta + z_2\beta\alpha)\delta + \beta(z_9\delta) = 0,$

$\varphi d^2(e_2 \otimes_{g_3^2} e_2) = (z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3)\alpha + \alpha(z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3) - (z_3\gamma + z_4\alpha\gamma)\beta - \gamma(z_1\beta + z_2\beta\alpha) = 2z_5\alpha + (-z_1 - z_3 + 2z_6)\alpha^2 + (-z_2 - z_4 + 2z_7)\alpha^3,$

$\varphi d^2(e_2 \otimes_{g_4^2} e_2) = (z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3)\alpha^2 + \alpha(z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3)\alpha + \alpha^2(z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3) - (z_9\delta)\sigma - \delta(z_{10}\sigma) = 3z_5\alpha^2 + (3z_6 - z_9 - z_{10})\alpha^3,$

$\varphi d^2(e_2 \otimes_{g_5^2} e_3) = (z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3)\delta + \alpha(z_9\delta) = z_5\delta,$

$\varphi d^2(e_3 \otimes_{g_6^2} e_1) = (z_{10}\sigma)\gamma + \sigma(z_3\gamma + z_4\alpha\gamma) = 0,$

$\varphi d^2(e_3 \otimes_{g_7^2} e_2) = (z_{10}\sigma)\alpha + \sigma(z_5e_2 + z_6\alpha + z_7\alpha^2 + z_8\alpha^3) = z_5\sigma.$

Thus φd^2 is given by

$$\begin{array}{l}
P^2 \to \Lambda \\
e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\beta\alpha\gamma \\
e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
e_2 \otimes_{g_3^2} e_2 \mapsto 2z_5\alpha + (-z_1 - z_3 + 2z_6)\alpha^2 + (-z_2 - z_4 + 2z_7)\alpha^3 \\
e_2 \otimes_{g_4^2} e_2 \mapsto 3z_5\alpha^2 + (3z_5 - z_9 - z_{10})\alpha^3 \\
e_2 \otimes_{g_5^2} e_3 \mapsto z_5\delta \\
e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
e_3 \otimes_{g_7^2} e_2 \mapsto z_5\sigma,
\end{array}$$

where $z_i \in K$.

If $\operatorname{char} K = 2$ then $\dim \operatorname{Im} \delta^1 = 4$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l}
P^2 \to \Lambda \\
e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\beta\alpha\gamma \\
e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
e_2 \otimes_{g_3^2} e_2 \mapsto (z_1 + z_3)\alpha^2 + (z_2 + z_4)\alpha^3 \\
e_2 \otimes_{g_4^2} e_2 \mapsto z_5\alpha^2 + (z_5 + z_9 + z_{10})\alpha^3 \\
e_2 \otimes_{g_5^2} e_3 \mapsto z_5\delta \\
e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
e_3 \otimes_{g_7^2} e_2 \mapsto z_5\sigma
\end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Im} \delta^1 = 5$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l}
\theta : P^2 \to \Lambda \\
e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\beta\alpha\gamma \\
e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\
e_2 \otimes_{g_3^2} e_2 \mapsto 2z_5\alpha + (-z_1 - z_3 + 2z_6)\alpha^2 + (-z_2 - z_4 + 2z_7)\alpha^3 \\
e_2 \otimes_{g_4^2} e_2 \mapsto 3z_5\alpha^2 + (3z_5 - z_9 - z_{10})\alpha^3 \\
e_2 \otimes_{g_5^2} e_3 \mapsto z_5\delta \\
e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\
e_3 \otimes_{g_7^2} e_2 \mapsto z_5\sigma
\end{array} \right\}.$$

17.3.4. $\mathrm{HH}^2(\Lambda)$.

From 17.3.2 and 17.3.3 we have if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_2 \mapsto d_2\alpha + d_2\alpha^3 \\ e_2 \otimes_{g_4^2} e_2 \mapsto d_1\alpha \\ e_2 \otimes_{g_5^2} e_3 \mapsto 0 \\ e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_7^2} e_2 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_2 \otimes_{g_4^2} e_2 \mapsto \alpha \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_2 \otimes_{g_3^2} e_2 \mapsto \alpha \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} z : P^2 \to \Lambda \\ e_2 \otimes_{g_3^2} e_2 \mapsto \alpha^3 \\ \text{else} \mapsto 0. \end{array}$$

Note that z represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto -\beta\alpha\gamma \\ \text{else} \mapsto 0. \end{array}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_2 \mapsto 0 \\ e_2 \otimes_{g_4^2} e_2 \mapsto d_1\alpha + d_2\alpha^2 \\ e_2 \otimes_{g_5^2} e_3 \mapsto 0 \\ e_3 \otimes_{g_6^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_7^2} e_2 \mapsto 0 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_2 \otimes_{g_4^2} e_2 \mapsto \alpha \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{aligned}
&y : P^2 \to \Lambda \\
&e_2 \otimes_{g_4^2} e_2 \mapsto \alpha^2 \\
&\text{else} \mapsto 0.
\end{aligned}$$

Note that y represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}
&P^2 \to \Lambda \\
&e_2 \otimes_{g_3^2} e_2 \mapsto -\tfrac{2}{3}\alpha \\
&e_2 \otimes_{g_5^2} e_3 \mapsto -\tfrac{1}{3}\delta \\
&e_3 \otimes_{g_7^2} e_2 \mapsto -\tfrac{1}{3}\sigma \\
&\text{else} \mapsto 0.
\end{aligned}$$

18. The algebra A_{14}

Definition 18.1. [5] *Let A_{14} be the algebra KQ/I where Q is the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\delta"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

and

$$I = \langle \beta\alpha - \delta\gamma\delta\gamma, \alpha\delta\gamma\delta, \gamma\delta\gamma\beta, \alpha\beta \rangle.$$

18.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda$ and $e_3\Lambda$ where

$$\begin{aligned} e_1\Lambda &= sp\{e_1, \alpha, \alpha\delta, \alpha\delta\gamma, \alpha\delta\gamma\beta\}, \\ e_2\Lambda &= sp\{e_2, \beta, \delta, \delta\gamma, \beta\alpha, \delta\gamma\beta, \delta\gamma\delta, \delta\gamma\beta\alpha, \delta\gamma\delta\gamma\delta\}, \\ e_3\Lambda &= sp\{e_3, \gamma, \gamma\beta, \gamma\delta, \gamma\delta\gamma, \gamma\delta\gamma\delta, \gamma\delta\gamma\delta\gamma, \gamma\delta\gamma\delta\gamma\delta\}. \end{aligned}$$

So we have $e_1\Lambda$

$$\begin{tikzcd} e_1 \arrow[d, dash] \\ \alpha \arrow[d, dash] \\ \alpha\delta \arrow[d, dash] \\ \alpha\delta\gamma \arrow[d, dash] \\ \alpha\delta\gamma\beta \end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd} & & e_2 \arrow[dll, dash] \arrow[dr, dash] & & \\ \beta \arrow[ddd, dash] & & & \delta \arrow[d, dash] & \\ & & & \delta\gamma \arrow[dl, dash] \arrow[dr, dash] & \\ & & \delta\gamma\delta \arrow[dll, dash] & & \delta\gamma\beta \arrow[dddl, dash] \\ \beta\alpha \arrow[d, dash] & & & & \\ \delta\gamma\delta\gamma\delta \arrow[drrr, dash] & & & & \\ & & & \delta\gamma\beta\alpha & \end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{tikzcd}[column sep=small]
 & & e_3 \arrow[d, dash] & & \\
 & & \gamma \arrow[dll, dash] \arrow[drr, dash] & & \\
\gamma\beta \arrow[dddrr, dash] & & & & \gamma\delta \arrow[d, dash] \\
 & & & & \gamma\delta\gamma \arrow[d, dash] \\
 & & & & \gamma\delta\gamma\delta \arrow[dll, dash] \\
 & & \gamma\delta\gamma\delta\gamma \arrow[d, dash] & & \\
 & & \gamma\delta\gamma\delta\gamma\delta & &
\end{tikzcd}$$

18.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2 and S_3.**

18.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \alpha e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

18.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\alpha e_2\nu = 0$, so $\alpha(d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta) = d_1\alpha + d_3\alpha\delta + d_4\alpha\delta\gamma + d_6\alpha\delta\gamma\beta = 0$ and then $d_1 = d_3 = d_4 = d_6 = 0$. Thus $e_2\nu = d_2\beta + d_5\beta\alpha + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$.

Hence $\operatorname{Ker}\partial^1 = \{d_2\beta + d_5\beta\alpha + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \beta e_1\Lambda + \delta\gamma\delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = d_2\beta + d_5\beta\alpha + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$, that is, $x = \beta(d_2e_1 + d_5\alpha + d_8\alpha\delta\gamma) + \delta\gamma\delta(d_7e_3 + d_9\gamma\delta)$. Thus $x \in \beta e_1\Lambda + \delta\gamma\delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \beta e_1\Lambda + \delta\gamma\delta e_3\Lambda$.

On the other hand, let $y = \beta e_1\eta + \delta\gamma\delta e_3\lambda \in \beta e_1\Lambda + \delta\gamma\delta e_3\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\beta e_1\eta + \delta\gamma\delta e_3\lambda) = \alpha\beta e_1\eta + \alpha\delta\gamma\delta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\beta e_1\Lambda + \delta\gamma\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \beta e_1\Lambda + \delta\gamma\delta e_3\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto \beta e_1\eta + \delta\gamma\delta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

18.2.3. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$ and $e_3\lambda = f_1e_3 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^2$. Then $\beta e_1\eta + \delta\gamma\delta e_3\lambda = 0$. So $\beta e_1\eta + \delta\gamma\delta e_3\lambda = \beta(c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta) + \delta\gamma\delta(f_1e_3 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta) = c_1\beta + c_2\beta\alpha + c_3\beta\alpha\delta + c_4\beta\alpha\delta\gamma + f_1\delta\gamma\delta + f_2\delta\gamma\delta\gamma + f_4\delta\gamma\delta\gamma\delta + f_5\delta\gamma\delta\gamma\delta\gamma = c_1\beta + f_1\delta\gamma\delta + (c_2 + f_2)\beta\alpha + (c_3 + f_4)\beta\alpha\delta + (c_4 + f_5)\beta\alpha\delta\gamma = 0$, that is, $c_1 = f_1 = 0, f_2 = -c_2, f_4 = -c_3$ and $f_5 = -c_4$. Thus $e_1\eta = c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$ and $e_3\lambda = -c_2\gamma + f_3\gamma\beta - c_3\gamma\delta - c_4\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta$. Therefore $\operatorname{Ker}\partial^2 = \{(c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta, -c_2\gamma + f_3\gamma\beta - c_3\gamma\delta - c_4\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta) : c_i, f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha, -\gamma)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta, -c_2\gamma + f_3\gamma\beta - c_3\gamma\delta - c_4\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta)$ so $u = (\alpha, -\gamma)(c_2e_2 + c_3\delta + c_4\delta\gamma + c_5\delta\gamma\beta - f_3\beta - f_6\delta\gamma - f_7\delta\gamma\delta - f_8\delta\gamma\delta\gamma\delta)$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha, -\gamma)e_2\Lambda$.

On the other hand, let $v = (\alpha, -\gamma)e_2\nu \in (\alpha, -\gamma)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha, -\gamma)e_2\nu) = (\beta\alpha - \delta\gamma\delta\gamma)e_2\nu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha, -\gamma)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha, -\gamma)e_2\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_2\nu \mapsto (\alpha, -\gamma)e_2\nu$, for $\nu \in \Lambda$.

18.2.4. *The minimal projective resolution of the simple Λ-module S_2.*
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \to \beta e_1\eta + \delta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

18.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$ and $e_3\lambda = f_1e_3 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\beta e_1\eta + \delta e_3\lambda = 0$ so $\beta e_1\eta + \delta e_3\lambda = \beta(c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta) + \delta(f_1e_3 + f_2\gamma + f_3\gamma\beta + f_4\gamma\delta + f_5\gamma\delta\gamma + f_6\gamma\delta\gamma\delta + f_7\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta) = c_1\beta + c_2\beta\alpha + c_3\beta\alpha\delta + c_4\beta\alpha\delta\gamma + f_1\delta + f_2\delta\gamma + f_3\delta\gamma\beta + f_4\delta\gamma\delta + f_5\delta\gamma\delta\gamma + f_6\delta\gamma\delta\gamma\delta + f_7\delta\gamma\delta\gamma\delta\gamma = c_1\beta + (c_2 + f_5)\beta\alpha + (c_3 + f_6)\beta\alpha\delta + (c_4 + f_7)\beta\alpha\delta\gamma + f_1\delta + f_2\delta\gamma + f_3\delta\gamma\beta + f_4\delta\gamma\delta = 0$, that is, $c_1 = f_1 = f_2 = f_3 = f_4 = 0, f_5 = -c_2, f_6 = -c_3$ and $f_7 = -c_4$. Thus $e_1\eta = c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$ and $e_3\lambda = -c_2\gamma\delta\gamma - c_3\gamma\delta\gamma\delta - c_4\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta, -c_2\gamma\delta\gamma - c_3\gamma\delta\gamma\delta - c_4\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta) : c_i, f_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, -\gamma\delta\gamma)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta, -c_2\gamma\delta\gamma - c_3\gamma\delta\gamma\delta - c_4\gamma\delta\gamma\delta\gamma + f_8\gamma\delta\gamma\delta\gamma\delta)$. So $x = (\alpha, -\gamma\delta\gamma)(c_2e_2 + c_3\delta + c_4\delta\gamma + c_5\delta\gamma\beta - f_8\delta\gamma\delta)$. Thus $x \in (\alpha, -\gamma\delta\gamma)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\gamma\delta\gamma)e_2\Lambda$.

On the other hand, let $y = (\alpha, -\gamma\delta\gamma)e_2\nu \in (\alpha, -\gamma\delta\gamma)e_2\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha, -\gamma\delta\gamma)e_2\nu) = (\beta\alpha - \delta\gamma\delta\gamma)e_2\nu = 0$. Thus $y \in \operatorname{Ker}\partial^1$ and then $(\alpha, -\gamma\delta\gamma)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\gamma\delta\gamma)e_2\Lambda$. □

So $\partial^2 : e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_2\nu \mapsto (\alpha, -\gamma\delta\gamma)e_2\nu$, for $\nu \in \Lambda$.

18.2.6. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^2$. Then $(\alpha, -\gamma\delta\gamma)e_2\nu = 0$. So $(\alpha, -\gamma\delta\gamma)(d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta) = (d_1\alpha + d_3\alpha\delta + d_4\alpha\delta\gamma + d_6\alpha\delta\gamma\beta, -d_1\gamma\delta\gamma - d_3\gamma\delta\gamma\delta - d_4\gamma\delta\gamma\delta\gamma - d_7\gamma\delta\gamma\delta\gamma\delta) = (0, 0)$. Then $d_1\alpha + d_3\alpha\delta + d_4\alpha\delta\gamma + d_6\alpha\delta\gamma\beta = 0$, that is, $d_1 = d_3 = d_4 = d_6 = 0$. Also $-d_1\gamma\delta\gamma - d_3\gamma\delta\gamma\delta - d_4\gamma\delta\gamma\delta\gamma - d_7\gamma\delta\gamma\delta\gamma\delta = 0$, that is, $d_1 = d_3 = d_4 = d_7 = 0$. Thus $e_2\nu = d_2\beta + d_5\beta\alpha + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$ Therefore $\operatorname{Ker}\partial^2 = \{d_2\beta + d_5\beta\alpha + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = d_2\beta + d_5\beta\alpha + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$ so $u = \beta(d_2e_1 + d_5\alpha + d_8\alpha\delta\gamma + d_9\alpha\delta)$. Hence $u \in \beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta e_1\Lambda$.

On the other hand, let $v = \beta e_1\eta \in \beta e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta e_1\eta) = (\alpha\beta, -\gamma\delta\gamma\beta)e_1\eta = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta e_1\Lambda$. □

So the map $\partial^3 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \beta e_1\eta$, for $\eta \in \Lambda$.

18.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3. The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_2\Lambda \to e_3\Lambda$ is given by $e_2\nu \mapsto \gamma e_2\nu$, for $\nu \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

18.2.8. $\operatorname{Ker}\partial^1$.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^1$. Then $\gamma e_2\nu = 0$ so $\gamma(d_1e_2 + d_2\beta + d_3\delta + d_4\delta\gamma + d_5\beta\alpha + d_6\delta\gamma\beta + d_7\delta\gamma\delta + d_8\delta\gamma\beta\alpha + d_9\delta\gamma\delta\gamma\delta) = d_1\gamma + d_2\gamma\beta + d_3\gamma\delta + d_4\gamma\delta\gamma + d_5\gamma\beta\alpha + d_7\gamma\delta\gamma\delta + d_9\gamma\delta\gamma\delta\gamma\delta = 0$, that is, $d_1 = d_2 = d_3 = d_4 = d_5 = d_7 = d_9 = 0$. Thus $e_2\nu = d_6\delta\gamma\beta + d_8\delta\gamma\beta\alpha$.

Hence $\operatorname{Ker}\partial^1 = \{d_6\delta\gamma\beta + d_8\delta\gamma\beta\alpha : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \delta\gamma\beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = d_6\delta\gamma\beta + d_8\delta\gamma\beta\alpha$. So $x = \delta\gamma\beta(d_6e_1 + d_8\alpha)$. Thus $x \in \delta\gamma\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \delta\gamma\beta e_1\Lambda$.

On the other hand, let $y = \delta\gamma\beta e_1\eta \in \delta\gamma\beta e_1\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\delta\gamma\beta e_1\eta) = \gamma\delta\gamma\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\delta\gamma\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \delta\gamma\beta e_1\Lambda$. □

So $\partial^2 : e_1\Lambda \to e_2\Lambda$ is given by: $e_1\eta \mapsto \delta\gamma\beta e_1\eta$, for $\eta \in \Lambda$.

18.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^2$. Then $\delta\gamma\beta e_1\eta = 0$. So $\delta\gamma\beta e_1\eta = \delta\gamma\beta(c_1e_1 + c_2\alpha + c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta) = c_1\delta\gamma\beta + c_2\delta\gamma\beta\alpha = 0$, that is, $c_1 = c_2 = 0$. Thus $e_1\eta = c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$. Therefore $\operatorname{Ker}\partial^2 = \{c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha\delta e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = c_3\alpha\delta + c_4\alpha\delta\gamma + c_5\alpha\delta\gamma\beta$, that is, $u = \alpha\delta(c_3e_3 + c_4\gamma + c_5\gamma\beta)$. Hence $u \in \alpha\delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \alpha\delta e_3\Lambda$.

On the other hand, let $v = \alpha\delta e_3\lambda \in \alpha\delta e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha\delta e_3\lambda) = \delta\gamma\beta\alpha\delta e_3\lambda = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\alpha\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \alpha\delta e_3\Lambda$. □

So the map $\partial^3 : e_3\Lambda \to e_1\Lambda$ is given by $e_3\lambda \mapsto \alpha\delta e_3\lambda$, for $\lambda \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : e_2\nu \mapsto \alpha e_2\nu,$$
$$\partial^2 : (e_1\eta, e_3\lambda) \mapsto \beta e_1\eta + \delta\gamma\delta e_3\lambda,$$
$$\partial^3 : e_2\nu \mapsto (\alpha, -\gamma)e_2\nu,$$

for $\eta, \nu, \lambda \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : (e_1\eta, e_3\lambda) \to \beta e_1\eta + \delta e_3\lambda,$$
$$\partial^2 : e_2\nu \mapsto (\alpha, -\gamma\delta\gamma)e_2\nu,$$

$$\partial^3 : e_1\eta \mapsto \beta e_1\eta,$$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\partial^1 : e_2\nu \mapsto \gamma e_2\nu,$$
$$\partial^2 : e_1\eta \mapsto \delta\gamma\beta e_1\eta,$$
$$\partial^3 : e_3\lambda \mapsto \alpha\delta e_3\lambda,$$

for $\eta, \nu, \lambda \in \Lambda$.

18.3. g^3 **for** S_1, S_2 **and** S_3**.**
Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$e_2 \xrightarrow{\partial^3} (\alpha, -\gamma) \xrightarrow{\partial^2} \beta\alpha - \delta\gamma\delta\gamma \xrightarrow{\partial^1} \alpha\beta\alpha - \alpha\delta\gamma\delta\gamma$, so $\alpha\beta\alpha - \alpha\delta\gamma\delta\gamma \in g^3$.

For S_2

$e_1 \xrightarrow{\partial^3} \beta \xrightarrow{\partial^2} (\alpha, -\gamma\delta\gamma)\beta = (\alpha\beta, -\gamma\delta\gamma\beta) \xrightarrow{\partial^1} \beta\alpha\beta - \delta\gamma\delta\gamma\beta$, so $\beta\alpha\beta - \delta\gamma\delta\gamma\beta \in g^3$.

For S_3

$e_3 \xrightarrow{\partial^3} \alpha\delta \xrightarrow{\partial^2} \delta\gamma\beta\alpha\delta \xrightarrow{\partial^1} \gamma\delta\gamma\beta\alpha\delta$, so $\gamma\delta\gamma\beta\alpha\delta \in g^3$.

Let $g^3_1 = \alpha\beta\alpha - \alpha\delta\gamma\delta\gamma$,
$g^3_2 = \beta\alpha\beta - \delta\gamma\delta\gamma\beta$,
$g^3_3 = \gamma\delta\gamma\beta\alpha\delta$.
So $g^3 = \{g^3_1, g^3_2, g^3_3\}$.

We know that $g^2 = \{\alpha\beta, \alpha\delta\gamma\delta, \beta\alpha - \delta\gamma\delta\gamma, \gamma\delta\gamma\beta\}$. Denote
$g^2_1 = \alpha\beta$,
$g^2_2 = \alpha\delta\gamma\delta$,
$g^2_3 = \beta\alpha - \delta\gamma\delta\gamma$,
$g^2_4 = \gamma\delta\gamma\beta$.
So we have
$g^3_1 = g^2_1\alpha - g^2_2\gamma = \alpha g^2_3$,
$g^3_2 = g^2_3\beta = \beta g^2_1 - \delta g^2_4$,
$g^3_3 = g^2_4\alpha\delta = -\gamma g^2_3\delta\gamma\delta + \gamma\delta\gamma g^2_3\delta + \gamma\beta g^2_2$.

18.4. $\mathrm{HH}^2(\Lambda)$**.**
From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3, \Lambda) \to \ldots$$

18.4.1. $\operatorname{Ker}\delta^2$.

To find $\mathrm{HH}^2(\Lambda)$ we need to find $\operatorname{Ker}\delta^2$ and $\operatorname{Im}\delta^1$. Let $\theta \in \operatorname{Ker}\delta^2$; then $\theta \in \operatorname{Hom}_{\Lambda^e}(P^2, \Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\alpha\delta\gamma\beta\\
&e_1 \otimes_{g_2^2} e_3 \mapsto j_3\alpha\delta\\
&e_2 \otimes_{g_3^2} e_2 \mapsto j_4 e_2 + j_5\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha\\
&e_3 \otimes_{g_4^2} e_1 \mapsto j_8\gamma\beta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$\begin{aligned}
&e_1 \otimes_{g_1^3} e_2 \mapsto e_1 \otimes_{g_1^2} \alpha - e_1 \otimes_{g_2^2} \gamma - \alpha g_3^2 e_2,\\
&e_2 \otimes_{g_2^3} e_1 \mapsto e_2 \otimes_{g_3^2} \beta - \beta \otimes_{g_1^2} e_1 + \delta \otimes_{g_4^2} e_1,\\
&e_3 \otimes_{g_3^3} e_3 \mapsto e_3 \otimes_{g_4^2} \alpha\delta + \gamma \otimes_{g_3^2} \delta\gamma\delta - \gamma\delta\gamma \otimes_{g_3^2} \delta - \gamma\beta \otimes_{g_2^2} e_3.
\end{aligned}$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_2) = (j_1 e_1 + j_2\alpha\delta\gamma\beta)\alpha - (j_3\alpha\delta)\gamma - \alpha(j_4 e_2 + j_5\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha) = (j_1 - j_4)\alpha - (j_3 + j_5)\alpha\delta\gamma = 0$ then $j_4 = j_1$ and $j_5 = -j_3$.

For $\theta d^3(e_2 \otimes_{g_2^3} e_1) = (j_4 e_2 + j_5\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha)\beta - \beta(j_1 e_1 + j_2\alpha\delta\gamma\beta) + \delta(j_8\gamma\beta) = (-j_1 + j_4)\beta + (j_5 + j_8)\delta\gamma\beta = 0$, that is, $j_4 = j_1$ and $j_8 = -j_5$.

Also $\theta d^3(e_3 \otimes_{g_3^3} e_3) = (j_8\gamma\beta)\alpha\delta + \gamma(j_4 e_2 + j_5\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha)\delta\gamma\delta - \gamma\delta\gamma(j_4 e_2 + j_5\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha)\delta - \gamma\beta(j_3\alpha\delta) = (j_8 - j_3)\gamma\beta\alpha\delta = 0$ then $j_8 = j_3$.

So the map θ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\alpha\delta\gamma\beta\\
&e_1 \otimes_{g_2^2} e_3 \mapsto j_3\alpha\delta\\
&e_2 \otimes_{g_3^2} e_2 \mapsto j_1 e_2 - j_3\delta\gamma + j_6\beta\alpha + j_7\delta\gamma\beta\alpha\\
&e_3 \otimes_{g_4^2} e_1 \mapsto j_3\gamma\beta.
\end{aligned}$$

Thus $\dim \operatorname{Ker}\delta^2 = 5$.

18.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$$\begin{aligned}
&e_1 \otimes_\alpha e_2 \to z_1\alpha + z_2\alpha\delta\gamma\\
&e_2 \otimes_\beta e_1 \to z_3\beta + z_4\delta\gamma\beta\\
&e_2 \otimes_\delta e_3 \to z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta\\
&e_3 \otimes_\gamma e_2 \to z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma,
\end{aligned}$$

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$$\begin{aligned}
&e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \beta + \alpha \otimes_\beta e_1\\
&e_1 \otimes_{g_2^2} e_3 \mapsto e_1 \otimes_\alpha \delta\gamma\delta + \alpha \otimes_\delta \gamma\delta + \alpha\delta \otimes_\gamma \delta + \alpha\delta\gamma \otimes_\delta e_3\\
&e_2 \otimes_{g_3^2} e_2 \mapsto e_2 \otimes_\beta \alpha + \beta \otimes_\alpha e_2 - e_2 \otimes_\delta \gamma\delta\gamma - \delta \otimes_\gamma \delta\gamma - \delta\gamma \otimes_\delta \gamma - \delta\gamma\delta \otimes_\gamma e_2\\
&e_3 \otimes_{g_4^2} e_1 \mapsto e_3 \otimes_\gamma \delta\gamma\beta + \gamma \otimes_\delta \gamma\beta + \gamma\delta \otimes_\gamma \beta + \gamma\delta\gamma \otimes_\beta e_1.
\end{aligned}$$

Then the map φd^2 is given by

$$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_1\alpha + z_2\alpha\delta\gamma)\beta + \alpha(z_3\beta + z_4\delta\gamma\beta) = (z_2 + z_4)\alpha\delta\gamma\beta,$$

$\varphi d^2(e_1 \otimes_{g_2^2} e_3) = (z_1\alpha + z_2\alpha\delta\gamma)\delta\gamma\delta + \alpha(z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta)\gamma\delta + \alpha\delta(z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma)\delta + \alpha\delta\gamma(z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta) = 0$,

$\varphi d^2(e_2 \otimes_{g_3^2} e_2) = (z_3\beta + z_4\delta\gamma\beta)\alpha + \beta(z_1\alpha + z_2\alpha\delta\gamma) - (z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta)\gamma\delta\gamma - \delta(z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma)\delta\gamma - \delta\gamma(z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta)\gamma - \delta\gamma\delta(z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma) = (z_1 + z_3 - 2z_5 - 2z_8)\beta\alpha + (z_2 + z_4 - 2z_6 - 2z_9)\delta\gamma\beta\alpha$,

$\varphi d^2(e_3 \otimes_{g_4^2} e_1) = (z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma)\delta\gamma\beta + \gamma(z_5\delta + z_6\delta\gamma\delta + z_7\delta\gamma\delta\gamma\delta)\gamma\beta + \gamma\delta(z_8\gamma + z_9\gamma\delta\gamma + z_{10}\gamma\delta\gamma\delta\gamma)\beta + \gamma\delta\gamma(z_3\beta + z_4\delta\gamma\beta) = 0$.

Thus φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\alpha\delta\gamma\beta \\ e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_2 \mapsto (z_1 + z_3 - 2z_5 - 2z_8)\beta\alpha + (z_2 + z_4 - 2z_6 - 2z_9)\delta\gamma\beta\alpha \\ e_3 \otimes_{g_4^2} e_1 \mapsto 0, \end{array}$$

where $z_i \in K$.

If $\operatorname{char} K = 2$ then $\dim \operatorname{Im} \delta^1 = 2$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\alpha\delta\gamma\beta \\ e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_2 \mapsto (z_1 + z_3)\beta\alpha + (z_2 + z_4)\delta\gamma\beta\alpha \\ e_3 \otimes_{g_4^2} e_1 \mapsto 0 \end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Im} \delta^1 = 3$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_2 + z_4)\alpha\delta\gamma\beta \\ e_1 \otimes_{g_2^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_2 \mapsto (z_1 + z_3 - 2z_5 - 2z_8)\beta\alpha + (z_2 + z_4 - 2z_6 - 2z_9)\delta\gamma\beta\alpha \\ e_3 \otimes_{g_4^2} e_1 \mapsto 0 \end{array} \right\}.$$

18.4.3. $\mathrm{HH}^2(\Lambda)$.

From 18.4.1 and 18.4.2 we have if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 \\ e_1 \otimes_{g_2^2} e_3 \mapsto d_2\alpha\delta \\ e_2 \otimes_{g_3^2} e_2 \mapsto d_1 e_2 - d_2\delta\gamma + d_3\delta\gamma\beta\alpha \\ e_3 \otimes_{g_4^2} e_1 \mapsto d_2\gamma\beta \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_2 \otimes_{g_3^2} e_2 \mapsto e_2 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{aligned}
&y : P^2 \to \Lambda\\
&e_1 \otimes_{g_2^2} e_3 \mapsto \alpha\delta\\
&e_2 \otimes_{g_3^2} e_2 \mapsto -\delta\gamma\\
&e_3 \otimes_{g_4^2} e_1 \mapsto \gamma\beta\\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&z : P^2 \to \Lambda\\
&e_2 \otimes_{g_3^2} e_2 \mapsto \delta\gamma\beta\alpha\\
&\text{else} \mapsto 0.
\end{aligned}$$

Note that z represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{aligned}
&P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto -\alpha\delta\gamma\beta\\
&\text{else} \mapsto 0.
\end{aligned}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l}
\theta : P^2 \to \Lambda\\
e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1\\
e_1 \otimes_{g_2^2} e_3 \mapsto d_2\alpha\delta\\
e_2 \otimes_{g_3^2} e_2 \mapsto d_1 e_2 - d_2\delta\gamma\\
e_3 \otimes_{g_4^2} e_1 \mapsto d_2\gamma\beta
\end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{aligned}
&x : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto e_1\\
&e_2 \otimes_{g_3^2} e_2 \mapsto e_2\\
&\text{else} \mapsto 0,
\end{aligned}$$

$$\begin{aligned}
&y : P^2 \to \Lambda\\
&e_1 \otimes_{g_2^2} e_3 \mapsto \alpha\delta\\
&e_2 \otimes_{g_3^2} e_2 \mapsto -\delta\gamma\\
&e_3 \otimes_{g_4^2} e_1 \mapsto \gamma\beta\\
&\text{else} \mapsto 0.
\end{aligned}$$

19. The algebra A_{15}

Definition 19.1. [5] *Let A_{15} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd}
 & 2 \arrow[dr, "\gamma"] & \\
1 \arrow[loop left, "\alpha"] \arrow[ur, "\sigma"] \arrow[rr, shift left, "\delta"] & & 3 \arrow[ll, shift left, "\beta"]
\end{tikzcd}$$

and

$$I = \langle \gamma\beta\alpha, \alpha^2 - \delta\beta, \beta\delta, \alpha\sigma, \alpha\delta - \sigma\gamma \rangle.$$

19.1. The structure of the indecomposable projectives.

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda$ and $e_3\Lambda$ where

$$\begin{aligned}
e_1\Lambda &= sp\{e_1, \alpha, \sigma, \delta, \alpha\delta, \alpha^2, \alpha^3\},\\
e_2\Lambda &= sp\{e_2, \gamma, \gamma\beta, \gamma\beta\sigma\},\\
e_3\Lambda &= sp\{e_3, \beta, \beta\alpha, \beta\sigma, \beta\alpha\delta\}.
\end{aligned}$$

So we have for $e_1\Lambda$

$$\begin{tikzcd}
 & & e_1 \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\sigma \arrow[dr, dash] & & \alpha \arrow[dl, dash] \arrow[ddr, dash] & & \delta \arrow[ddl, dash] \\
 & \alpha\delta & & & \\
 & & & \alpha^2 \arrow[d, dash] & \\
 & & & \alpha^3 &
\end{tikzcd}$$

For $e_2\Lambda$

$$\begin{tikzcd}
e_2 \arrow[d, dash] \\
\gamma \arrow[d, dash] \\
\gamma\beta \arrow[d, dash] \\
\gamma\beta\sigma
\end{tikzcd}$$

Also $e_3\Lambda$

$$\begin{tikzcd}[column sep=small]
 & e_3 \arrow[d, dash] & \\
 & \beta \arrow[dl, dash] \arrow[dr, dash] & \\
\beta\alpha \arrow[dr, dash] & & \beta\sigma \arrow[dl, dash] \\
 & \beta\alpha\delta &
\end{tikzcd}$$

19.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2 and S_3.**

19.2.1. *The minimal projective resolution of the simple Λ-module S_1.*

Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_1\eta, e_2\nu, e_3\lambda) \mapsto \alpha e_1\eta + \sigma e_2\nu + \delta e_3\lambda$, for $\eta, \nu, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

19.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$, $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma$ and $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta$ with $c_i, d_i, f_i \in K$. Assume that $(e_1\eta, e_2\nu, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\alpha e_1\eta + \sigma e_2\nu + \delta e_3\lambda = 0$, so $\alpha(c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3) + \sigma(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma) + \delta(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta) = c_1\alpha + c_2\alpha^2 + c_4\alpha\delta + c_6\alpha^3 + d_1\sigma + d_2\sigma\gamma + d_3\sigma\gamma\beta + f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha = c_1\alpha + (c_2 + f_2)\alpha^2 + (c_4 + d_2)\alpha\delta + (c_6 + d_3 + f_3)\alpha^3 + d_1\sigma + f_1\delta = 0$ and then $c_1 = d_1 = f_1 = 0, c_2 + f_2 = 0, c_4 + d_2 = 0$ and $c_6 + d_3 + f_3 = 0$, so $f_2 = -c_2, d_2 = -c_4, f_3 = -c_6 - d_3$. Thus $e_1\eta = c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$, $e_2\nu = -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma$ and $e_3\lambda = -c_2\beta - (c_6 + d_3)\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3, -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma, -c_2\beta - (c_6 + d_3)\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta) : c_i, d_i, f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3, -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma, -c_2\beta - (c_6 + d_3)\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta)$, that is, $x = (\alpha, 0, -\beta)(c_2e_1 + c_5\delta + c_7\alpha^2 + (c_6 + d_3)\alpha - f_4\sigma - f_5\alpha\delta) + (\sigma, 0, 0)(c_3e_2) + (\delta, -\gamma, 0)(c_4e_3 - d_3\beta - d_4\beta\sigma)$. Thus

$x \in (\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda$.

On the other hand, let $y = (\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda \in (\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda) = (\alpha^2 - \delta\beta)e_1\eta + \alpha\sigma e_2\nu + (\alpha\delta - \sigma\gamma)e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, 0, -\beta)e_1\Lambda + (\sigma, 0, 0)e_2\Lambda + (\delta, -\gamma, 0)e_3\Lambda$. □

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ is given by $(e_1\eta, e_2\nu, e_3\lambda) \mapsto (\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda$, for $\eta, \nu, \lambda \in \Lambda$.

19.2.3. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$, $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma$ and $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta$ with $c_i, d_i, f_i \in K$. Assume that $(e_1\eta, e_2\nu, e_3\lambda) \in \operatorname{Ker}\partial^2$. Then $(\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda = (0, 0, 0)$. So $(\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda = (\alpha, 0, -\beta)(c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3) + (\sigma, 0, 0)(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma) + (\delta, -\gamma, 0)(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta) = (c_1\alpha + c_2\alpha^2 + c_4\alpha\delta + c_6\alpha^3, 0, -c_1\beta - c_2\beta\alpha - c_3\beta\sigma - c_5\beta\alpha\delta) + (d_1\sigma + d_2\sigma\gamma + d_3\sigma\gamma\beta, 0, 0) + (f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha, -f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\sigma, 0) = (c_1\alpha + c_2\alpha^2 + c_4\alpha\delta + c_6\alpha^3 + d_1\sigma + d_2\sigma\gamma + d_3\sigma\gamma\beta + f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha, -f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\sigma, -c_1\beta - c_2\beta\alpha - c_3\beta\sigma - c_5\beta\alpha\delta) = (c_1\alpha + (c_2 + f_2)\alpha^2 + (c_4 + d_2)\alpha\delta + (c_6 + d_3 + f_3)\alpha^3 + d_1\sigma + f_1\delta, -f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\sigma, -c_1\beta - c_2\beta\alpha - c_3\beta\sigma - c_5\beta\alpha\delta) = (0, 0, 0)$. So $c_1\alpha + (c_2 + f_2)\alpha^2 + (c_4 + d_2)\alpha\delta + (c_6 + d_3 + f_3)\alpha^3 + d_1\sigma + f_1\delta = 0$ and then $c_1 = d_1 = f_1 = 0, f_2 = -c_2, d_2 = -c_4$ and $f_3 = -c_6 - d_3$. Also $-f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\sigma = 0$ and then $f_1 = f_2 = f_4 = 0$. And for $-c_1\beta - c_2\beta\alpha - c_3\beta\sigma - c_5\beta\alpha\delta = 0$, that is, $c_1 = c_2 = c_3 = c_5 = 0$. Thus $e_1\eta = c_4\delta + c_6\alpha^2 + c_7\alpha^3$, $e_2\nu = -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma$ and $e_3\lambda = -(c_6 + d_3)\beta\alpha + f_5\beta\alpha\delta$. Therefore $\operatorname{Ker}\partial^2 = \{(c_4\delta + c_6\alpha^2 + c_7\alpha^3, -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma, -(c_6 + d_3)\beta\alpha + f_5\beta\alpha\delta) : c_i, d_i, f_5 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_4\delta + c_6\alpha^2 + c_7\alpha^3, -c_4\gamma + d_3\gamma\beta + d_4\gamma\beta\sigma, -(c_6 + d_3)\beta\alpha + f_5\beta\alpha\delta)$ so $u = (\alpha^2, 0, -\beta\alpha)((c_6 + d_3)e_1 - f_5\delta) + (\delta, -\gamma, 0)(c_4e_3 - c_7\beta\alpha - d_3\beta - d_4\beta\sigma)$. Hence $u \in (\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda$.

On the other hand, let $v = (\alpha^2, 0, -\beta\alpha)e_1\eta + (\delta, -\gamma, 0)e_3\lambda \in (\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha^2, 0, -\beta\alpha)e_1\eta + (\delta, -\gamma, 0)e_3\lambda) = (\alpha^3 - \delta\beta\alpha, \gamma\beta\alpha, -\beta\alpha^2)e_1\eta + (\alpha\delta - \sigma\gamma, 0, -\beta\delta)e_3\lambda = (0, 0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha^2, 0, -\beta\alpha)e_1\Lambda + (\delta, -\gamma, 0)e_3\Lambda$. □

So the map $\partial^3 : e_1\Lambda \oplus e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto (\alpha^2, 0, -\beta\alpha)e_1\eta + (\delta, -\gamma, 0)e_3\lambda$, for $\eta, \lambda \in \Lambda$.

19.2.4. *The minimal projective resolution of the simple Λ-module S_2.*
The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_3\Lambda \to e_2\Lambda$ is given by $e_3\lambda \to \gamma e_3\lambda$, for $\lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

19.2.5. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\gamma e_3\lambda = 0$ so $\gamma e_3\lambda = \gamma(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta) = f_1\gamma + f_2\gamma\beta + f_4\gamma\beta\sigma = 0$, that is, $f_1 = f_2 = f_4 = 0$. Thus $e_3\lambda = f_3\beta\alpha + f_5\beta\alpha\delta$.

Hence $\operatorname{Ker}\partial^1 = \{f_3\beta\alpha + f_5\beta\alpha\delta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \beta\alpha e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = f_3\beta\alpha + f_5\beta\alpha\delta$. So $x = \beta\alpha(f_3e_1 + f_5\delta)$. Thus $x \in \beta\alpha e_1\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \beta\alpha e_1\Lambda$.

On the other hand, let $y = \beta\alpha e_1\eta \in \beta\alpha e_1\Lambda$. Then, from the of ∂^1, we have that $\partial^1(y) = \partial^1(\beta\alpha e_1\eta) = \gamma\beta\alpha e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and then $\beta\alpha e_1\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \beta\alpha e_1\Lambda$. ☐

So $\partial^2 : e_1\Lambda \to e_3\Lambda$ is given by $e_1\eta \mapsto \beta\alpha e_1\eta$, for $\eta \in \Lambda$.

19.2.6. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^2$. Then $\beta\alpha e_1\eta = 0$. So $\beta\alpha e_1\eta = \beta\alpha(c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3) = c_1\beta\alpha + c_4\beta\alpha\delta = 0$, that is, $c_1 = c_4 = 0$. Thus $e_1\eta = c_2\alpha + c_3\sigma + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$. Therefore $\operatorname{Ker}\partial^2 = \{c_2\alpha + c_3\sigma + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3 : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \alpha e_1\Lambda + \sigma e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = c_2\alpha + c_3\sigma + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$ so $u = \alpha(c_2e_1 + c_5\delta + c_6\alpha + c_7\alpha) + \sigma(c_3e_2)$. Hence $u \in \alpha e_1\Lambda + \sigma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \alpha e_1\Lambda + \sigma e_2\Lambda$.

On the other hand, let $v = \alpha e_1\eta + \sigma e_2\nu \in \alpha e_1\Lambda + \sigma e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\alpha e_1\eta + \sigma e_2\nu) = \beta\alpha^2 e_1\eta + \beta\alpha\sigma e_2\nu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\alpha e_1\Lambda + \sigma e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \alpha e_1\Lambda + \sigma e_2\Lambda$. ☐

So the map $\partial^3 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto \alpha e_1\eta + \sigma e_2\nu$, for $\eta, \nu \in \Lambda$.

19.2.7. *The minimal projective resolution of the simple Λ-module S_3.*
The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_3\Lambda$ is given by $e_1\eta \mapsto \beta e_1\eta$, for $\eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

19.2.8. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^1$. Then $\beta e_1\eta = 0$ so $\beta(c_1e_1 + c_2\alpha + c_3\sigma + c_4\delta + c_5\alpha\delta + c_6\alpha^2 + c_7\alpha^3) = c_1\beta + c_2\beta\alpha + c_3\beta\sigma + c_5\beta\alpha\delta = 0$, that is, $c_1 = c_2 = c_3 = c_5 = 0$. Thus $e_1\eta = c_4\delta + c_6\alpha^2 + c_7\alpha^3$.

Hence $\operatorname{Ker}\partial^1 = \{c_4\delta + c_6\alpha^2 + c_7\alpha^3 : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = c_4\delta + c_6\alpha^2 + c_7\alpha^3$. So $x = \delta(c_4e_3 + c_6\beta + c_7\beta\alpha)$. Thus $x \in \delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq \delta e_3\Lambda$.

On the other hand, let $y = \delta e_3\lambda \in \delta e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1(\delta e_3\lambda) = \beta\delta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = \delta e_3\Lambda$. ☐

So $\partial^2 : e_3\Lambda \to e_1\Lambda$ is given by $e_3\lambda \mapsto \delta e_3\lambda$, for $\lambda \in \Lambda$.

19.2.9. $\operatorname{Ker}\partial^2$.
Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^1$. Then $\delta e_3\lambda = 0$ so $\delta e_3\lambda = \delta(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\sigma + f_5\beta\alpha\delta) = f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha = 0$ then $f_1 = f_2 = f_3 = 0$. Thus $e_3\lambda = f_4\beta\sigma + f_5\beta\alpha\delta$. Therefore $\operatorname{Ker}\partial^2 = \{f_4\beta\sigma + f_5\beta\alpha\delta : f_4, f_5 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = \beta\sigma e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = f_4\beta\sigma + f_5\beta\alpha\delta$, that is, $u = \beta\sigma(f_4e_2 + f_5\gamma)$. Hence $u \in \beta\sigma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq \beta\sigma e_2\Lambda$.

On the other hand, let $v = \beta\sigma e_2\nu \in \beta\sigma e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\sigma e_2\nu) = \delta\beta\sigma e_2\nu = 0$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $\beta\sigma e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = \beta\sigma e_2\Lambda$. ☐

So the map $\partial^3 : e_2\Lambda \to e_3\Lambda$ is given by $e_2\nu \mapsto \beta\sigma e_2\nu$, for $\nu \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_1\eta, e_2\nu, e_3\lambda) \mapsto \alpha e_1\eta + \sigma e_2\nu + \delta e_3\lambda,$$
$$\partial^2 : (e_1\eta, e_2\nu, e_3\lambda) \mapsto (\alpha, 0, -\beta)e_1\eta + (\sigma, 0, 0)e_2\nu + (\delta, -\gamma, 0)e_3\lambda,$$

$$\partial^3 : (e_1\eta, e_3\lambda) \mapsto (\alpha^2, 0, -\beta\alpha)e_1\eta + (\delta, -\gamma, 0)e_3\lambda,$$

for $\eta, \nu, \lambda \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : e_3\lambda \to \gamma e_3\lambda,$$
$$\partial^2 : e_1\eta \mapsto \beta\alpha e_1\eta,$$
$$\partial^3 : (e_1\eta, e_2\nu) \mapsto \alpha e_1\eta + \sigma e_2\nu,$$

for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:

$$\partial^1 : e_1\eta \mapsto \beta e_1\eta,$$
$$\partial^2 : e_3\lambda \mapsto \delta e_3\lambda,$$
$$\partial^3 : e_2\nu \mapsto \beta\sigma e_2\nu,$$

for $\eta, \nu, \lambda \in \Lambda$.

19.3. g^3 **for** S_1, S_2 **and** S_3.

Now we want to find the elements of g^3; these are paths in KQ.

For S_1

$(e_1, 0) \overset{\partial^3}{\to} (\alpha^2, 0, -\beta\alpha) \overset{\partial^2}{\to} ((\alpha, 0, -\beta)\alpha^2 + (\sigma, 0, 0)(0) + (\delta, -\gamma, 0)(-\beta\alpha)) = (\alpha^3, 0, -\beta\alpha^2)$
$+ (-\delta\beta\alpha, \gamma\beta\alpha, 0) = (\alpha^3 - \delta\beta\alpha, \gamma\beta\alpha, -\beta\alpha^2) \overset{\partial^1}{\to} \alpha^4 - \alpha\delta\beta\alpha + \sigma\gamma\beta\alpha - \delta\beta\alpha^2$,
so $\alpha^4 - \alpha\delta\beta\alpha + \sigma\gamma\beta\alpha - \delta\beta\alpha^2 \in g^3$.

$(0, e_3) \overset{\partial^3}{\to} (\delta, -\gamma, 0) \overset{\partial^2}{\to} ((\alpha, 0, -\beta)\delta + (\sigma, 0, 0)(-\gamma) + (\delta, -\gamma, 0)(0)) = (\alpha\delta, 0,$
$-\beta\delta) + (-\sigma\gamma, 0, 0) = (\alpha\delta - \sigma\gamma, 0, -\beta\delta) \overset{\partial^1}{\to} \alpha^2\delta - \alpha\sigma\gamma - \delta\beta\delta$,
so $\alpha^2\delta - \alpha\sigma\gamma - \delta\beta\delta \in g^3$.

For S_2

$(e_1, 0) \xrightarrow{\partial^3} \alpha \xrightarrow{\partial^2} \beta\alpha^2 \xrightarrow{\partial^1} \gamma\beta\alpha^2$, so $\gamma\beta\alpha^2 \in g^3$.

$(0, e_2) \xrightarrow{\partial^3} \sigma \xrightarrow{\partial^2} \beta\alpha\sigma \xrightarrow{\partial^1} \gamma\beta\alpha\sigma$, so $\gamma\beta\alpha\sigma \in g^3$.

For S_3

$e_2 \xrightarrow{\partial^3} \beta\sigma \xrightarrow{\partial^2} \delta\beta\sigma \xrightarrow{\partial^1} \beta\delta\beta\sigma$, so $\beta\delta\beta\sigma \in g^3$.

Let $g^3_1 = \alpha^4 - \alpha\delta\beta\alpha + \sigma\gamma\beta\alpha - \delta\beta\alpha^2$,
$g^3_2 = \alpha^2\delta - \alpha\sigma\gamma - \delta\beta\delta$,
$g^3_3 = \gamma\beta\alpha^2$,
$g^3_4 = \gamma\beta\alpha\sigma$,
$g^3_5 = \beta\delta\beta\sigma$.

So $g^3 = \{g^3_1, g^3_2, g^3_3, g^3_4, g^3_5\}$.

We know that $g^2 = \{\gamma\beta\alpha, \alpha^2 - \delta\beta, \beta\delta, \alpha\sigma, \alpha\delta - \sigma\gamma\}$. Denote

$g^2_1 = \alpha^2 - \delta\beta$,
$g^2_2 = \alpha\sigma$,
$g^2_3 = \alpha\delta - \sigma\gamma$,

$g_4^2 = \gamma\beta\alpha,$
$g_5^2 = \beta\delta.$

So we have

$$\begin{aligned}
g_1^3 &= g_1^2\alpha^2 - g_3^2\beta\alpha = \alpha g_1^2\alpha - \delta g_5^2\beta + \sigma g_4^2 - \delta\beta g_1^2,\\
g_2^3 &= g_1^2\delta - g_2^2\gamma = \alpha g_3^2 - \delta g_5^2,\\
g_3^3 &= g_4^2\alpha = \gamma\beta g_1^2 + \gamma g_5^2\beta,\\
g_4^3 &= g_4^2\sigma = \gamma\beta g_2^2.\\
g_5^3 &= g_5^2\beta\sigma = -\beta g_1^2\sigma + \beta\alpha g_2^2.
\end{aligned}$$

19.4. $\mathrm{HH}^2(\Lambda)$. From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\ .$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \ldots$$

19.4.1. $\mathrm{Ker}\,\delta^2$. To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3\\
&e_1 \otimes_{g_2^2} e_2 \mapsto j_5\sigma\\
&e_1 \otimes_{g_3^2} e_3 \mapsto j_6\delta + j_7\alpha\delta\\
&e_2 \otimes_{g_4^2} e_1 \mapsto j_8\gamma\beta\\
&e_3 \otimes_{g_5^2} e_3 \mapsto j_9e_3 + j_{10}\beta\alpha\delta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$\begin{aligned}
e_1 \otimes_{g_1^3} e_1 &\mapsto e_1 \otimes_{g_1^2} \alpha^2 - e_1 \otimes_{g_3^2} \beta\alpha - \alpha \otimes_{g_1^2} \alpha + \delta \otimes_{g_5^2} \beta - \sigma \otimes_{g_4^3} e_1 + \delta\beta \otimes_{g_1^2} e_1,\\
e_1 \otimes_{g_2^3} e_3 &\mapsto e_1 \otimes_{g_1^2} \delta - e_1 \otimes_{g_2^2} \gamma - \alpha \otimes_{g_3^2} e_3 + \delta \otimes_{g_5^2} e_3,\\
e_2 \otimes_{g_3^3} e_1 &\mapsto e_2 \otimes_{g_4^2} \alpha - \gamma \otimes_{g_5^2} \beta - \gamma\beta \otimes_{g_1^2} e_1,\\
e_2 \otimes_{g_4^3} e_2 &\mapsto e_2 \otimes_{g_4^2} \sigma - \gamma\beta \otimes_{g_2^2} e_2,\\
e_3 \otimes_{g_5^3} e_2 &\mapsto e_3 \otimes_{g_5^2} \beta\sigma + \beta \otimes_{g_1^2} \sigma - \beta\alpha \otimes_{g_2^2} e_2.
\end{aligned}$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\alpha^2 - (j_6\delta + j_7\alpha\delta)\beta\alpha - \alpha(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\alpha + \delta(j_9e_3 + j_{10}\beta\alpha\delta)\beta - \sigma(j_8\gamma\beta) + \delta\beta(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3) = (j_1 + j_9)\alpha^2 + (j_2 - j_6 - j_8)\alpha^3 = 0$ then $j_9 = -j_1$ and $j_8 = j_6 - j_2$.

For $\theta d^3(e_1 \otimes_{g_2^3} e_3) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\delta - (j_5\sigma)\gamma - \alpha(j_6\delta + j_7\alpha\delta) + \delta(j_9e_3 + j_{10}\beta\alpha\delta) = (j_1 + j_9)\delta + (j_2 - j_5 - j_6)\alpha\delta = 0$, that is, $j_9 = -j_1$ and $j_5 = j_2 - j_6$, thus $j_8 = j_5$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_1) = (j_8\gamma\beta)\alpha - \gamma(j_9e_3 + j_{10}\beta\alpha\delta)\beta - \gamma\beta(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3) = -(j_1 + j_9)\gamma\beta = 0$ then $j_9 = -j_1$.

And $\theta d^3(e_2 \otimes_{g_4^3} e_2) = (j_8\gamma\beta)\sigma - \gamma\beta(j_5\sigma) = (j_8 - j_5)\gamma\beta\sigma = 0$, that is, $j_8 = j_5$.

For $\theta d^3(e_3 \otimes_{g_5^3} e_2) = (j_9 e_3 + j_{10}\beta\alpha\delta)\beta\sigma + \beta(j_1 e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\sigma - \beta\alpha(j_5\sigma) = (j_9 + j_1)\beta\sigma = 0$, that is, $j_9 = -j_1$.

So

$$\operatorname{Ker}\delta^2 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto j_1 e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto (-j_2 + j_6)\sigma \\ e_1 \otimes_{g_3^2} e_3 \mapsto j_6\delta + j_7\alpha\delta \\ e_2 \otimes_{g_4^2} e_1 \mapsto (-j_2 + j_6)\gamma\beta \\ e_3 \otimes_{g_5^2} e_3 \mapsto -j_1 e_3 + j_{10}\beta\alpha\delta \end{array} \right\}.$$

So $\dim \operatorname{Ker}\delta^2 = 7$.

19.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_1 \to z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3$

$e_1 \otimes_\sigma e_2 \to z_4\sigma$

$e_1 \otimes_\delta e_3 \to z_5\delta + z_6\alpha\delta$

$e_2 \otimes_\gamma e_3 \to z_7\gamma$

$e_3 \otimes_\beta e_1 \to z_8\beta + z_9\beta\alpha$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \alpha + \alpha \otimes_\alpha e_1 - e_1 \otimes_\delta \beta - \delta \otimes_\beta e_1$

$e_1 \otimes_{g_2^2} e_2 \mapsto e_1 \otimes_\alpha \sigma + \alpha \otimes_\sigma e_2$

$e_1 \otimes_{g_3^2} e_3 \mapsto e_1 \otimes_\alpha \delta + \alpha \otimes_\delta e_3 - e_1 \otimes_\sigma \gamma - \sigma \otimes_\gamma e_3$

$e_2 \otimes_{g_4^2} e_1 \mapsto e_2 \otimes_\gamma \beta\alpha + \gamma \otimes_\beta \alpha + \gamma\beta \otimes_\alpha e_1$

$e_3 \otimes_{g_5^2} e_3 \mapsto e_3 \otimes_\beta \delta + \beta \otimes_\delta e_3$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\alpha + \alpha(z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) - (z_5\delta + z_6\alpha\delta)\beta - \delta(z_8\beta + z_9\beta\alpha) = 2z_0\alpha + (2z_1 - z_5 - z_8)\alpha^2 + (2z_2 - z_6 - z_9)\alpha^3$,

$\varphi d^2(e_1 \otimes_{g_2^2} e_2) = (z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\sigma + \alpha(z_4\sigma) = z_0\sigma$,

$\varphi d^2(e_1 \otimes_{g_3^2} e_3) = (z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\delta + \alpha(z_5\delta + z_6\alpha\delta) - (z_4\sigma)\gamma - \sigma(z_7\gamma) = z_0\delta + (z_1 - z_4 + z_5 - z_7)\alpha\delta$,

$\varphi d^2(e_2 \otimes_{g_4^2} e_1) = (z_7\gamma)\beta\alpha + \gamma(z_8\beta + z_9\beta\alpha)\alpha + \gamma\beta(z_0 e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) = z_0\gamma\beta$,

$\varphi d^2(e_3 \otimes_{g_5^2} e_3) = (z_8\beta + z_9\beta\alpha)\delta + \beta(z_5\delta + z_6\alpha\delta) = (z_6 + z_9)\beta\alpha\delta$.

Thus φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_5 - z_8)\alpha^2 + (2z_2 - z_6 - z_9)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\sigma \\ e_1 \otimes_{g_3^2} e_3 \mapsto z_0\delta + (z_1 - z_4 + z_5 - z_7)\alpha\delta \\ e_2 \otimes_{g_4^2} e_1 \mapsto z_0\gamma\beta \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_6 + z_9)\beta\alpha\delta, \end{array}$$

where $z_i \in K$.

Note that if $\operatorname{char} K = 2$ then $\dim \operatorname{Im} \delta^1 = 4$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_5 + z_8)\alpha^2 + (z_6 + z_9)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\sigma \\ e_1 \otimes_{g_3^2} e_3 \mapsto z_0\delta + (z_1 - z_4 + z_5 - z_7)\alpha\delta \\ e_2 \otimes_{g_4^2} e_1 \mapsto z_0\gamma\beta \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_6 + z_9)\beta\alpha\delta \end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Im} \delta^1 = 5$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_5 - z_8)\alpha^2 + (2z_2 - z_6 - z_9)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\sigma \\ e_1 \otimes_{g_3^2} e_3 \mapsto z_0\delta + (z_1 - z_4 + z_5 - z_7)\alpha\delta \\ e_2 \otimes_{g_4^2} e_1 \mapsto z_0\gamma\beta \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_6 + z_9)\beta\alpha\delta \end{array} \right\}.$$

19.4.3. $\mathrm{HH}^2(\Lambda)$. From 19.4.1 and 19.4.2 we have if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha \\ e_1 \otimes_{g_2^2} e_2 \mapsto 0 \\ e_1 \otimes_{g_3^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_4^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -d_1e_3 + d_3\beta\alpha\delta \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, u\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -e_3 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} u : P^2 \to \Lambda \\ e_3 \otimes_{g_5^2} e_3 \mapsto \beta\alpha\delta \\ \text{else} \mapsto 0. \end{array}$$

Note that u represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto -\alpha^3 \\ \text{else} \mapsto 0. \end{array}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2 \alpha \\ e_1 \otimes_{g_2^2} e_2 \mapsto 0 \\ e_1 \otimes_{g_3^2} e_3 \mapsto 0 \\ e_2 \otimes_{g_4^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -d_1 e_3 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -e_3 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\ \text{else} \mapsto 0. \end{array}$$

Note that y represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_2^2} e_2 \mapsto -\frac{1}{2}\sigma \\ e_1 \otimes_{g_3^2} e_3 \mapsto -\frac{1}{2}\delta \\ e_2 \otimes_{g_4^2} e_1 \mapsto -\frac{1}{2}\gamma\beta \\ \text{else} \mapsto 0. \end{array}$$

20. The algebra A_{16}

Definition 20.1. [5] *Let A_{16} be the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver*

$$\begin{tikzcd} & 2 \arrow[dl, "\sigma"'] & \\ 1 \arrow[loop left, "\alpha"] \arrow[rr, shift left, "\beta"] & & 3 \arrow[ul, "\gamma"'] \arrow[ll, shift left, "\delta"] \end{tikzcd}$$

and

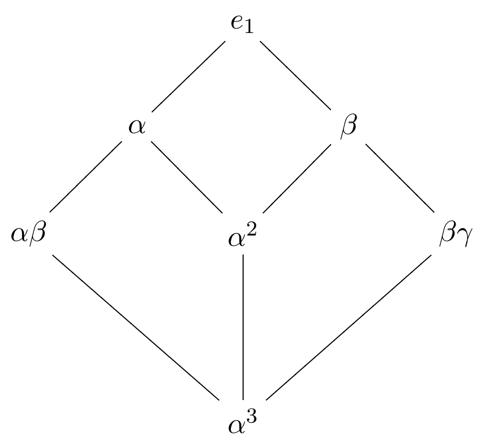
$$I = \langle \alpha\beta\gamma, \alpha^2 - \beta\delta, \delta\beta, \sigma\alpha, \delta\alpha - \gamma\sigma \rangle.$$

20.1. **The structure of the indecomposable projectives.**

The indecomposable projective Λ-modules are $e_1\Lambda, e_2\Lambda$ and $e_3\Lambda$ where

$$e_1\Lambda = sp\{e_1, \alpha, \beta, \alpha\beta, \alpha^2, \beta\gamma, \alpha^3\},$$
$$e_2\Lambda = sp\{e_2, \sigma, \sigma\beta, \sigma\beta\gamma\},$$
$$e_3\Lambda = sp\{e_3, \delta, \gamma, \delta\alpha, \gamma\sigma\beta\}.$$

So we have for $e_1\Lambda$



For $e_2\Lambda$

Also $e_3\Lambda$

$$\begin{tikzcd} & e_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \delta \arrow[dr, dash] & & \gamma \arrow[dl, dash] \\ & \delta\alpha \arrow[d, dash] & \\ & \gamma\sigma\beta & \end{tikzcd}$$

20.2. **The minimal projective resolutions of the simple Λ-modules S_1, S_2, S_3.**

20.2.1. *The minimal projective resolution of the simple Λ-module S_1.*
Now the minimal projective resolution of the simple Λ-module S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto \alpha e_1\eta + \beta e_3\lambda$, for $\eta, \lambda \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_1.

20.2.2. $\operatorname{Ker}\partial^1$.
To find $\operatorname{Ker}\partial^1 = \Omega^2(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_3\lambda = f_1e_3 + f_2\delta + f_3\gamma + f_4\delta\alpha + f_5\gamma\sigma\beta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $\alpha e_1\eta + \beta e_3\lambda = 0$, so $\alpha(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) + \beta(f_1e_3 + f_2\delta + f_3\gamma + f_4\delta\alpha + f_5\gamma\sigma\beta) = c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + c_5\alpha^3 + f_1\beta + f_2\beta\delta + f_3\beta\gamma + f_4\beta\delta\alpha = c_1\alpha + (c_2 + f_2)\alpha^2 + c_3\alpha\beta + (c_5 + f_4)\alpha^3 + f_1\beta + f_3\beta\gamma = 0$ and then $c_1 = c_3 = f_1 = f_3 = 0, f_2 = -c_2, f_4 = -c_5$. Thus $e_1\eta = c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_3\lambda = -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta$.

Hence $\operatorname{Ker}\partial^1 = \{(c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta) : c_i, f_5 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta)$, that is, $x = (\alpha, -\delta)(c_2e_1 + c_4\beta + c_5\alpha + c_7\alpha^2 - f_5\alpha\beta) + (\beta\gamma, 0)(c_6e_2)$. Thus $x \in (\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$.

On the other hand, let $y = (\alpha, -\delta)e_1\eta + (\beta\gamma, 0)e_2\nu \in (\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$. Then from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha, -\delta)e_1\eta + (\beta\gamma, 0)e_2\nu) = (\alpha^2 - \beta\delta)e_1\eta + \alpha\beta\gamma e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\delta)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto (\alpha, -\delta)e_1\eta + (\beta\gamma, 0)e_2\nu$, for $\eta, \nu \in \Lambda$.

20.2.3. Ker ∂^2.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker}\partial^2$. Then $(\alpha, -\delta)e_1\eta + (\beta\gamma, 0)e_2\nu = (\alpha, -\delta)(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) + (\beta\gamma, 0)(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + c_5\alpha^3, -c_1\delta - c_2\delta\alpha - c_4\delta\alpha\beta) + (d_1\beta\gamma + d_2\beta\gamma\sigma, 0) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + c_5\alpha^3 + d_1\beta\gamma + d_2\beta\gamma\sigma, -c_1\delta - c_2\delta\alpha - c_4\delta\alpha\beta) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + (c_5 + d_2)\alpha^3 + d_1\beta\gamma, -c_1\delta - c_2\delta\alpha - c_4\delta\alpha\beta) = (0, 0)$. So $c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + (c_5 + d_2)\alpha^3 + d_1\beta\gamma = 0$ and then $c_1 = c_2 = c_3 = d_1 = 0$ and $d_2 = -c_5$. Also $-c_1\delta - c_2\delta\alpha - c_4\delta\alpha\beta = 0$ and then $c_1 = c_2 = c_4 = 0$. Thus $e_1\eta = c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_2\nu = -c_5\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma$. Therefore $\operatorname{Ker}\partial^2 = \{(c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_5\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma) : c_i, d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_5\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma)$ so $u = (\alpha^2, -\sigma)(c_5e_1 + c_7\alpha - d_3\beta - d_4\beta\gamma) + (\beta\gamma, 0)(c_6e_2)$. Hence $u \in (\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$.

On the other hand, let $v = (\alpha^2, -\sigma)e_1\eta + (\beta\gamma, 0)e_2\nu \in (\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha^2, -\sigma)e_1\eta + (\beta\gamma, 0)e_2\nu) = ((\alpha^3 - \beta\gamma\sigma)e_1\eta + \alpha\beta\gamma e_2\nu, -\delta\alpha^2 e_1\eta - \delta\beta\gamma e_2\nu) = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha^2, -\sigma)e_1\Lambda + (\beta\gamma, 0)e_2\Lambda$. □

So the map $\partial^3 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto (\alpha^2, -\sigma)e_1\eta + (\beta\gamma, 0)e_2\nu$, for $\eta, \nu \in \Lambda$.

20.2.4. *The minimal projective resolution of the simple Λ-module S_2.*

The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \to \sigma e_1\eta$, for $\eta \in \Lambda$.

Now we need to find $\operatorname{Ker}\partial^1$ and $\operatorname{Ker}\partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_2.

20.2.5. Ker ∂^1.

To find $\operatorname{Ker}\partial^1 = \Omega^2(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^1$. Then $\sigma e_1\eta = 0$ so $\sigma e_1\eta = \sigma(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) = c_1\sigma + c_3\sigma\beta + c_6\sigma\beta\gamma = 0$, that is, $c_1 = c_3 = c_6 = 0$. Thus $e_1\eta = c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3$.

Hence $\operatorname{Ker}\partial^1 = \{c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3 : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = \alpha e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^1$. Then $x = c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3$. So $x = \alpha(c_2e_1 + c_4\beta + c_5\alpha + c_7\alpha^2)$. Thus $x \in \alpha e_1\Lambda$ and therefore $\operatorname{Ker} \partial^1 \subseteq \alpha e_1\Lambda$.

On the other hand, let $y = \alpha e_1\eta \in \alpha e_1\Lambda$. Then, from the of ∂^1, we have that $\partial^1(y) = \partial^1(\alpha e_1\eta) = \sigma\alpha e_1\eta = 0$. Therefore $y \in \operatorname{Ker} \partial^1$ and then $\alpha e_1\Lambda \subseteq \operatorname{Ker} \partial^1$.

Hence $\operatorname{Ker} \partial^1 = \alpha e_1\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \alpha e_1\eta$, for $\eta \in \Lambda$.

20.2.6. $\operatorname{Ker} \partial^2$.

Now we want to find $\operatorname{Ker} \partial^2 = \Omega^3(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker} \partial^2$. Then $\alpha e_1\eta = 0$. So $\alpha e_1\eta = \alpha(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) = c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + c_5\alpha^3 = 0$, that is, $c_1 = c_2 = c_3 = c_5 = 0$. Thus $e_1\eta = c_4\alpha\beta + c_6\beta\gamma + c_7\alpha^3$. Therefore $\operatorname{Ker} \partial^2 = \{c_4\alpha\beta + c_6\beta\gamma + c_7\alpha^3 : c_i \in K\}$.

Claim. $\operatorname{Ker} \partial^2 = \beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^2$. Then $u = c_4\alpha\beta + c_6\beta\gamma + c_7\alpha^3$ so $u = \beta\gamma(c_6e_2) + \alpha\beta(c_4e_3 + c_7\delta)$. Hence $u \in \beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda$ and therefore $\operatorname{Ker} \partial^2 \subseteq \beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda$.

On the other hand, let $v = \beta\gamma e_2\nu + \alpha\beta e_3\lambda \in \beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2(\beta\gamma e_2\nu + \alpha\beta e_3\lambda) = \alpha\beta\gamma e_2\nu + \alpha^2\beta e_3\lambda = 0$. Therefore $v \in \operatorname{Ker} \partial^2$ and then $\beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda \subseteq \operatorname{Ker} \partial^2$.

Hence $\operatorname{Ker} \partial^2 = \beta\gamma e_2\Lambda + \alpha\beta e_3\Lambda$. $\square$

So the map $\partial^3 : e_2\Lambda \oplus e_3\Lambda \to e_1\Lambda$ is given by $(e_2\nu, e_3\lambda) \mapsto \beta\gamma e_2\nu + \alpha\beta e_3\lambda$, for $\nu, \lambda \in \Lambda$.

20.2.7. *The minimal projective resolution of the simple* Λ*-module* S_3.

The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where $\partial^1 : e_1\Lambda \oplus e_2\Lambda \to e_3\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto \delta e_1\eta + \gamma e_2\nu$, for $\eta, \nu \in \Lambda$.

Now we need to find $\operatorname{Ker} \partial^1$ and $\operatorname{Ker} \partial^2$ in order to find ∂^2, ∂^3 for the simple Λ-module S_3.

20.2.8. $\operatorname{Ker} \partial^1$.

To find $\operatorname{Ker} \partial^1 = \Omega^2(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_2\nu = d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker} \partial^1$. Then $\delta e_1\eta + \gamma e_2\nu = 0$ so $\delta(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) + \gamma(d_1e_2 + d_2\sigma + d_3\sigma\beta + d_4\sigma\beta\gamma) = c_1\delta + c_2\delta\alpha + c_4\delta\alpha\beta + d_1\gamma + d_2\gamma\sigma + d_3\gamma\sigma\beta = c_1\delta + (c_2 + d_2)\delta\alpha + (c_4 + d_3)\delta\alpha\beta + d_1\gamma = 0$, that is, $c_1 = d_1 = 0, d_2 = -c_2$ and $d_3 = -c_4$. Thus $e_1\eta = c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_2\nu = -c_2\sigma - c_4\sigma\beta + d_4\sigma\beta\gamma$.

Hence $\operatorname{Ker} \partial^1 = \{(c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_2\sigma - c_4\sigma\beta + d_4\sigma\beta\gamma) : c_i, d_4 \in K\}$.

Claim. $\operatorname{Ker}\partial^1 = (\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^1$. Then $x = (c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3, -c_2\sigma - c_4\sigma\beta + d_4\sigma\beta\gamma)$. So $x = (\alpha, -\sigma)(c_2e_1 + c_4\beta + c_5\alpha + c_7\alpha^2 - d_4\beta\gamma) + (\beta, 0)(c_3e_3 + c_6\gamma)$. Thus $x \in (\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^1 \subseteq (\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda$.

On the other hand, let $y = (\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda \in (\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda$. Then, from the definition of ∂^1, we have that $\partial^1(y) = \partial^1((\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda) = (\delta\alpha - \gamma\sigma)e_1\eta + \delta\beta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^1$ and so $(\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda \subseteq \operatorname{Ker}\partial^1$.

Hence $\operatorname{Ker}\partial^1 = (\alpha, -\sigma)e_1\Lambda + (\beta, 0)e_3\Lambda$. $\square$

So $\partial^2 : e_1\Lambda \oplus e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\eta, e_3\lambda) \mapsto (\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda$, for $\eta, \lambda \in \Lambda$.

20.2.9. $\operatorname{Ker}\partial^2$.

Now we want to find $\operatorname{Ker}\partial^2 = \Omega^3(S_3)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3$ and $e_3\lambda = f_1e_3 + f_2\delta + f_3\gamma + f_4\delta\alpha + f_5\gamma\sigma\beta$ with $c_i, f_i \in K$. Assume that $(e_1\eta, e_3\lambda) \in \operatorname{Ker}\partial^1$. Then $(\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda = (0, 0)$ so $(\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda = (\alpha, -\sigma)(c_1e_1 + c_2\alpha + c_3\beta + c_4\alpha\beta + c_5\alpha^2 + c_6\beta\gamma + c_7\alpha^3) + (\beta, 0)(f_1e_3 + f_2\delta + f_3\gamma + f_4\delta\alpha + f_5\gamma\sigma\beta) = (c_1\alpha + c_2\alpha^2 + c_3\alpha\beta + c_5\alpha^3, -c_1\sigma - c_3\sigma\beta - c_6\sigma\beta\gamma) + (f_1\beta + f_2\beta\delta + f_3\beta\gamma + f_4\beta\delta\alpha, 0) = (c_1\alpha + f_1\beta + f_3\beta\gamma + (c_2 + f_2)\alpha^2 + c_3\alpha\beta + (c_5 + f_4)\alpha^3, -c_1\sigma - c_3\sigma\beta - c_6\sigma\beta\gamma) = (0, 0)$. So $c_1\alpha + f_1\beta + f_3\beta\gamma + (c_2 + f_2)\alpha^2 + c_3\alpha\beta + (c_5 + f_4)\alpha^3 = 0$ and then $c_1 = c_3 = f_1 = f_3 = 0, f_2 = -c_2$ and $f_4 = -c_5$. Also $-c_1\sigma - c_3\sigma\beta - c_6\sigma\beta\gamma = 0$, that is, $c_1 = c_3 = c_6 = 0$. Therefore $e_1\eta = c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3$ and $e_3\lambda = -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta$. Thus $\operatorname{Ker}\partial^2 = \{(c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3, -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta) : c_i, f_5 \in K\}$.

Claim. $\operatorname{Ker}\partial^2 = (\alpha, -\delta)e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^2$. Then $u = (c_2\alpha + c_4\alpha\beta + c_5\alpha^2 + c_7\alpha^3, -c_2\delta - c_5\delta\alpha + f_5\gamma\sigma\beta)$, that is, $u = (\alpha, -\delta)(c_2e_1 + c_4\beta + c_5\alpha + c_7\alpha^2 - f_5\alpha\beta)$. Hence $u \in (\alpha, -\delta)e_1\Lambda$ and therefore $\operatorname{Ker}\partial^2 \subseteq (\alpha, -\delta)e_1\Lambda$.

On the other hand, let $v = (\alpha, -\delta)e_1\eta \in (\alpha, -\delta)e_1\Lambda$. Then, from the definition of ∂^2, we have that $\partial^2(v) = \partial^2((\alpha, -\delta)e_1\eta) = ((\alpha, -\sigma)\alpha + (\beta, 0)(-\delta))e_1\eta = ((\alpha^2 - \beta\delta), -\sigma\alpha)e_1\eta = (0, 0)$. Therefore $v \in \operatorname{Ker}\partial^2$ and then $(\alpha, -\delta)e_1\Lambda \subseteq \operatorname{Ker}\partial^2$.

Hence $\operatorname{Ker}\partial^2 = (\alpha, -\delta)e_1\Lambda$. $\square$

So the map $\partial^3 : e_1\Lambda \to e_1\Lambda \oplus e_3\Lambda$ is given by $e_1\eta \mapsto (\alpha, -\delta)e_1\eta$, for $\eta \in \Lambda$.

Thus the maps for S_1 are:

$$\partial^1 : (e_1\eta, e_3\lambda) \mapsto \alpha e_1\eta + \beta e_3\lambda,$$
$$\partial^2 : (e_1\eta, e_2\nu) \mapsto (\alpha, -\delta)e_1\eta + (\beta\gamma, 0)e_2\nu,$$
$$\partial^3 : (e_1\eta, e_2\nu) \mapsto (\alpha^2, -\sigma)e_1\eta + (\beta\gamma, 0)e_2\nu,$$

for $\eta, \nu, \lambda \in \Lambda$.

Also the maps for S_2 are:

$$\partial^1 : e_1\eta \to \sigma e_1\eta,$$

$$\partial^2 : e_1\eta \mapsto \alpha e_1\eta,$$
$$\partial^3 : (e_2\nu, e_3\lambda) \mapsto \beta\gamma e_2\nu + \alpha\beta e_3\lambda,$$
for $\eta, \nu, \lambda \in \Lambda$.

And the maps for S_3 are:
$$\partial^1 : (e_1\eta, e_2\nu) \mapsto \delta e_1\eta + \gamma e_2\nu,$$
$$\partial^2 : (e_1\eta, e_3\lambda) \mapsto (\alpha, -\sigma)e_1\eta + (\beta, 0)e_3\lambda,$$
$$\partial^3 : e_1\eta \mapsto (\alpha, -\delta)e_1\eta,$$
for $\eta, \nu, \lambda \in \Lambda$.

20.3. g^3 **for** S_1, S_2 **and** S_3**.**
Now we want to find the elements of g^3; these are paths in $K\mathcal{Q}$.

For S_1
$(e_1, 0) \xrightarrow{\partial^3} (\alpha^2, -\sigma) \xrightarrow{\partial^2} (\alpha, -\delta)\alpha^2 + (\beta\gamma, 0)(-\sigma) = (\alpha^3 - \beta\gamma\sigma, -\delta\alpha^2) \xrightarrow{\partial^1} \alpha^4 - \alpha\beta\gamma\sigma - \beta\delta\alpha^2$,
so $\alpha^4 - \alpha\beta\gamma\sigma - \beta\delta\alpha^2 \in g^3$.
$(0, e_2) \xrightarrow{\partial^3} (\beta\gamma, 0) \xrightarrow{\partial^2} (\alpha, -\delta)\beta\gamma = (\alpha\beta\gamma, -\delta\beta\gamma) \xrightarrow{\partial^1} \alpha^2\beta\gamma - \beta\delta\beta\gamma$, so $\alpha^2\beta\gamma - \beta\delta\beta\gamma \in g^3$.

For S_2
$(e_2, 0) \xrightarrow{\partial^3} \beta\gamma \xrightarrow{\partial^2} \alpha\beta\gamma \xrightarrow{\partial^1} \sigma\alpha\beta\gamma$, so $\sigma\alpha\beta\gamma \in g^3$.
$(0, e_3) \xrightarrow{\partial^3} \alpha\beta \xrightarrow{\partial^2} \alpha^2\beta \xrightarrow{\partial^1} \sigma\alpha^2\beta$, so $\sigma\alpha^2\beta \in g^3$.

For S_3

$e_1 \xrightarrow{\partial^3} (\alpha, -\delta) \xrightarrow{\partial^2} (\alpha, -\sigma)\alpha + (\beta, 0)(-\delta) = (\alpha^2 - \beta\delta, -\sigma\alpha) \xrightarrow{\partial^1} \delta\alpha^2 - \delta\beta\delta - \gamma\sigma\alpha$,
so $\delta\alpha^2 - \delta\beta\delta - \gamma\sigma\alpha \in g^3$.

Let $g_1^3 = \alpha^4 - \alpha\beta\gamma\sigma - \beta\delta\alpha^2$,
$g_2^3 = \alpha^2\beta\gamma - \beta\delta\beta\gamma$,
$g_3^3 = \sigma\alpha\beta\gamma$,
$g_4^3 = \sigma\alpha^2\beta$,
$g_5^3 = \delta\alpha^2 - \delta\beta\delta - \gamma\sigma\alpha$.
So $g^3 = \{g_1^3, g_2^3, g_3^3, g_4^3, g_5^3\}$.

We know that $g^2 = \{\alpha\beta\gamma, \alpha^2 - \beta\delta, \delta\beta, \sigma\alpha, \delta\alpha - \gamma\sigma\}$. Denote
$g_1^2 = \alpha^2 - \beta\delta$,
$g_2^2 = \alpha\beta\gamma$,
$g_3^2 = \sigma\alpha$,
$g_4^2 = \delta\alpha - \gamma\sigma$,
$g_5^2 = \delta\beta$.
So we have
$g_1^3 = g_1^2\alpha^2 - g_2^2\sigma = \alpha g_1^2\alpha - \beta g_5^2\delta - \beta\delta g_1^2 + \alpha\beta g_4^2$,
$g_2^3 = g_1^2\beta\gamma = -\beta g_5^2\gamma + \alpha g_2^2$,
$g_3^3 = g_3^2\beta\gamma = \sigma g_2^2$,

$g_4^3 = g_3^2\alpha\beta = \sigma g_1^2\beta + \sigma\beta g_5^2,$
$g_5^3 = g_4^2\alpha - g_5^2\delta = \delta g_1^2 - \gamma g_3^2.$

20.4. $\mathrm{HH}^2(\Lambda)$.
From [16] we have the projective resolution of Λ

$$\cdots \longrightarrow P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0\,.$$

Apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to get

$$0 \to \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda) \xrightarrow{\delta^2} \mathrm{Hom}_{\Lambda^e}(P^3,\Lambda) \to \ldots$$

20.4.1. $\mathrm{Ker}\,\delta^2$.
To find $\mathrm{HH}^2(\Lambda)$ we need to find $\mathrm{Ker}\,\delta^2$ and $\mathrm{Im}\,\delta^1$. Let $\theta \in \mathrm{Ker}\,\delta^2$; then $\theta \in \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$. So the map $\theta : P^2 \to \Lambda$ is given by

$$\begin{aligned}
&\theta : P^2 \to \Lambda\\
&e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3\\
&e_1 \otimes_{g_2^2} e_2 \mapsto j_5\beta\gamma\\
&e_2 \otimes_{g_3^2} e_1 \mapsto j_6\sigma\\
&e_3 \otimes_{g_4^2} e_1 \mapsto j_7\delta + j_8\delta\alpha\\
&e_3 \otimes_{g_5^2} e_3 \mapsto j_9e_3 + j_{10}\gamma\sigma\beta,
\end{aligned}$$

for some $j_i \in K$. The map $d^3 : P^3 \to P^2$ is given by

$$\begin{aligned}
&e_1 \otimes_{g_1^3} e_1 \mapsto e_1 \otimes_{g_1^2} \alpha^2 - e_1 \otimes_{g_2^2} \sigma - \alpha \otimes_{g_1^2} \alpha + \beta \otimes_{g_5^2} \delta + \beta\delta \otimes_{g_1^2} e_1 - \alpha\beta \otimes_{g_4^2} e_1,\\
&e_1 \otimes_{g_2^3} e_2 \mapsto e_1 \otimes_{g_1^2} \beta\gamma + \beta \otimes_{g_5^2} \gamma - \alpha \otimes_{g_2^2} e_2,\\
&e_2 \otimes_{g_3^3} e_2 \mapsto e_2 \otimes_{g_3^2} \beta\gamma - \sigma \otimes_{g_2^2} e_2,\\
&e_2 \otimes_{g_4^3} e_3 \mapsto e_2 \otimes_{g_3^2} \alpha\beta - \sigma \otimes_{g_1^2} \beta - \sigma\beta \otimes_{g_5^2} e_3,\\
&e_3 \otimes_{g_5^3} e_1 \mapsto e_3 \otimes_{g_4^2} \alpha - e_3 \otimes_{g_5^2} \delta - \delta \otimes_{g_1^2} e_1 + \gamma \otimes_{g_3^2} e_1.
\end{aligned}$$

So the fact that $\theta d^3 = 0$ gives conditions on the coefficients $j_i \in K$.

We have $\theta d^3(e_1 \otimes_{g_1^3} e_1) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\alpha^2 - (j_5\beta\gamma)\sigma - \alpha(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\alpha + \beta(j_9e_3 + j_{10}\gamma\sigma\beta)\delta + \beta\delta(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3) - \alpha\beta(j_7\delta + j_8\delta\alpha) = (j_1 + j_9)\alpha^2 + (j_2 - j_5 - j_7)\alpha^3 = 0$ then $j_9 = -j_1$ and $j_7 = j_2 - j_5$.

For $\theta d^3(e_1 \otimes_{g_2^3} e_2) = (j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\beta\gamma + \beta(j_9e_3 + j_{10}\gamma\sigma\beta)\gamma - \alpha(j_5\beta\gamma) = (j_1 + j_9)\beta\gamma = 0$, that is, $j_9 = -j_1$.

Also $\theta d^3(e_2 \otimes_{g_3^3} e_2) = (j_6\sigma)\beta\gamma - \sigma(j_5\beta\gamma) = (-j_5 + j_6)\sigma\beta\gamma = 0$ then $j_6 = j_5$.

And $\theta d^3(e_2 \otimes_{g_4^3} e_3) = (j_6\sigma)\alpha\beta - \sigma(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3)\beta - \sigma\beta(j_9e_3 + j_{10}\gamma\sigma\beta) = (-j_1 - j_9)\sigma\beta = 0$, that is, $j_9 = -j_1$.

For $\theta d^3(e_3 \otimes_{g_5^3} e_1) = (j_7\delta + j_8\delta\alpha)\alpha - (j_9e_3 + j_{10}\gamma\sigma\beta)\delta - \delta(j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3) + \gamma(j_6\sigma) = -(j_1 + j_9)\delta + (-j_2 + j_6 + j_7)\delta\alpha = 0$, that is, $j_9 = -j_1$ and $j_7 = j_2 - j_6$. Hence $j_5 = j_6$.

So

$$\operatorname{Ker}\delta^2 = \left\{\begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto j_1e_1 + j_2\alpha + j_3\alpha^2 + j_4\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto j_5\beta\gamma \\ e_2 \otimes_{g_3^2} e_1 \mapsto j_5\sigma \\ e_3 \otimes_{g_4^2} e_1 \mapsto (j_2 - j_5)\delta + j_8\delta\alpha \\ e_3 \otimes_{g_5^2} e_3 \mapsto -j_1e_3 + j_{10}\gamma\sigma\beta \end{array}\right\}.$$

So $\dim\operatorname{Ker}\delta^2 = 7$.

20.4.2. $\operatorname{Im}\delta^1$.

Now to find $\operatorname{Im}\delta^1$. Let $\varphi \in \operatorname{Hom}_{\Lambda^e}(P^1, \Lambda)$. The map $\varphi : P^1 \to \Lambda$ is given by

$e_1 \otimes_\alpha e_1 \to z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3$

$e_1 \otimes_\beta e_3 \to z_4\beta + z_5\alpha\beta$

$e_2 \otimes_\sigma e_1 \to z_6\sigma$

$e_3 \otimes_\delta e_1 \to z_7\delta + z_8\delta\alpha$

$e_3 \otimes_\gamma e_2 \to z_9\gamma$,

for $z_i \in K$.

We have the map $d^2 : P^2 \to P^1$ given by

$e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \otimes_\alpha \alpha + \alpha \otimes_\alpha e_1 - e_1 \otimes_\beta \delta - \beta \otimes_\delta e_1$

$e_1 \otimes_{g_2^2} e_2 \mapsto e_1 \otimes_\alpha \beta\gamma + \alpha \otimes_\beta \gamma + \alpha\beta \otimes_\gamma e_2$

$e_2 \otimes_{g_3^2} e_1 \mapsto e_2 \otimes_\sigma \alpha + \sigma \otimes_\alpha e_1$

$e_3 \otimes_{g_4^2} e_1 \mapsto e_3 \otimes_\delta \alpha + \delta \otimes_\alpha e_1 - e_3 \otimes_\gamma \sigma - \gamma \otimes_\sigma e_1$

$e_3 \otimes_{g_5^2} e_3 \mapsto e_3 \otimes_\delta \beta + \delta \otimes_\beta e_3$.

Then the map φd^2 is given by

$\varphi d^2(e_1 \otimes_{g_1^2} e_1) = (z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\alpha + \alpha(z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) - (z_4\beta + z_5\alpha\beta)\delta - \beta(z_7\delta + z_8\delta\alpha) = 2z_0\alpha + (2z_1 - z_4 - z_7)\alpha^2 + (2z_2 - z_5 - z_8)\alpha^3$,

$\varphi d^2(e_1 \otimes_{g_2^2} e_2) = (z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3)\beta\gamma + \alpha(z_4\beta + z_5\alpha\beta)\gamma + \alpha\beta(z_9\gamma) = z_0\beta\gamma$,

$\varphi d^2(e_2 \otimes_{g_3^2} e_1) = (z_6\sigma)\alpha + \sigma(z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) = z_0\sigma$,

$\varphi d^2(e_3 \otimes_{g_4^2} e_1) = (z_7\delta + z_8\delta\alpha)\alpha + \delta(z_0e_1 + z_1\alpha + z_2\alpha^2 + z_3\alpha^3) - (z_9\gamma)\sigma - \gamma(z_6\sigma) = z_0\delta + (z_1 - z_6 + z_7 - z_9)\delta\alpha$,

$\varphi d^2(e_3 \otimes_{g_5^2} e_3) = (z_7\delta + z_8\delta\alpha)\beta + \delta(z_4\beta + z_5\alpha\beta) = (z_5 + z_8)\delta\alpha\beta = (z_5 + z_8)\gamma\sigma\beta$.

Thus φd^2 is given by

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_4 - z_7)\alpha^2 + (2z_2 - z_5 - z_8)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\beta\gamma \\ e_2 \otimes_{g_3^2} e_1 \mapsto z_0\sigma \\ e_3 \otimes_{g_4^2} e_1 \mapsto z_0\delta + (z_1 - z_6 + z_7 - z_9)\delta\alpha \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_5 + z_8)\gamma\sigma\beta, \end{array}$$

where $z_i \in K$.

Note that if $\operatorname{char} K = 2$ then $\dim \operatorname{Im} \delta^1 = 4$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto (z_4 + z_7)\alpha^2 + (z_5 + z_8)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\beta\gamma \\ e_2 \otimes_{g_3^2} e_1 \mapsto z_0\sigma \\ e_3 \otimes_{g_4^2} e_1 \mapsto z_0\delta + (z_1 - z_6 + z_7 - z_9)\delta\alpha \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_5 + z_8)\gamma\sigma\beta \end{array} \right\}.$$

If $\operatorname{char} K \neq 2$ then $\dim \operatorname{Im} \delta^1 = 5$ and

$$\operatorname{Im} \delta^1 = \left\{ \begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto 2z_0\alpha + (2z_1 - z_4 - z_7)\alpha^2 + (2z_2 - z_5 - z_8)\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto z_0\beta\gamma \\ e_2 \otimes_{g_3^2} e_1 \mapsto z_0\sigma \\ e_3 \otimes_{g_4^2} e_1 \mapsto z_0\delta + (z_1 - z_6 + z_7 - z_9)\delta\alpha \\ e_3 \otimes_{g_5^2} e_3 \mapsto (z_5 + z_8)\gamma\sigma\beta \end{array} \right\}.$$

20.4.3. $\mathrm{HH}^2(\Lambda)$.

From 20.4.1 and 20.4.2 we have if $\operatorname{char} K = 2$ then $\dim \mathrm{HH}^2(\Lambda) = 3$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1e_1 + d_2\alpha + d_3\alpha^3 \\ e_1 \otimes_{g_2^2} e_2 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_4^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -d_1e_3, \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y, z\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -e_3 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} z : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \alpha^3 \\ \text{else} \mapsto 0. \end{array}$$

Note that z represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_3 \otimes_{g_5^2} e_3 \mapsto -\gamma\sigma\beta \\ \text{else} \mapsto 0. \end{array}$$

If $\operatorname{char} K \neq 2$ then $\dim \mathrm{HH}^2(\Lambda) = 2$ and therefore

$$\mathrm{HH}^2(\Lambda) = \left\{ \begin{array}{l} \theta : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto d_1 e_1 + d_2 \alpha \\ e_1 \otimes_{g_2^2} e_2 \mapsto 0 \\ e_2 \otimes_{g_3^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_4^2} e_1 \mapsto 0 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -d_1 e_3 \end{array} \right\}$$

with $d_i \in K$.

A basis of $\mathrm{HH}^2(\Lambda) = sp\{x, y\}$ where

$$\begin{array}{l} x : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto e_1 \\ e_3 \otimes_{g_5^2} e_3 \mapsto -e_3 \\ \text{else} \mapsto 0, \end{array}$$

$$\begin{array}{l} y : P^2 \to \Lambda \\ e_1 \otimes_{g_1^2} e_1 \mapsto \alpha \\ \text{else} \mapsto 0. \end{array}$$

Note that y represents the same element of $\mathrm{HH}^2(\Lambda)$ as

$$\begin{array}{l} P^2 \to \Lambda \\ e_1 \otimes_{g_2^2} e_2 \mapsto -\frac{1}{2}\beta\gamma \\ e_2 \otimes_{g_3^2} e_1 \mapsto -\frac{1}{2}\sigma \\ e_3 \otimes_{g_4^2} e_1 \mapsto -\frac{1}{2}\delta \\ \text{else} \mapsto 0. \end{array}$$

So we have found $\mathrm{HH}^2(\Lambda)$ for all the algebras A_i where $i = 1, \ldots, 16$.

20.5. **Summary.**

Theorem 20.2. *Let $\lambda \in K \setminus \{0, 1\}$.*

For the algebra $\Lambda = A_1(\lambda)$, *we have* $\dim \mathrm{HH}^2(\Lambda) = 3$,

the algebra $\Lambda = A_2(\lambda)$ *has* $\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 6 \text{ if } \operatorname{char} K = 2 \\ 4 \text{ if } \operatorname{char} K \neq 2, \end{cases}$

the algebra $\Lambda = A_i$ *where* $i = 4, 7, 8, 9, 10, 11$ *has* $\dim \mathrm{HH}^2(\Lambda) = 2$,

the algebra $\Lambda = A_j$ *where* $j = 5, 6$ *has* $\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 3 \text{ if } \operatorname{char} K = 2 \\ 4 \text{ if } \operatorname{char} K = 3 \\ 3 \text{ if } \operatorname{char} K \neq 2, 3, \end{cases}$

and for the algebra $\Lambda = A_k$ *where* $k = 12, 13, 14, 15, 16$, *we have*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 3 \text{ if } \operatorname{char} K = 2 \\ 2 \text{ if } \operatorname{char} K \neq 2. \end{cases}$$

Recall from Chapter 4 that the derived equivalence classes are $\{A_1(\lambda)\}, \{A_2(\lambda)\}$, $\{A_4, A_7, A_8, A_9, A_{10}, A_{11}\}, \{A_5, A_6\}, \{A_{12}, A_{13}, A_{14}, A_{15}, A_{16}\}$.

In the remaining chapters we will find the periodicity of the simple modules of representatives of these derived equivalence classes of the algebras A_i. Recall that when we refer to periodicity we mean $\Omega-$periodic.

21. The periodicity and the bimodule resolution of the algebra $A_1(\lambda)$

We recall from Chapter 5 that $A_1(\lambda)$ is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$0 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\beta}{\overset{\sigma}{\rightleftarrows}} 2$$

and

$$I = \langle \alpha\gamma\alpha - \alpha\sigma\beta, \beta\gamma\alpha - \lambda\beta\sigma\beta, \gamma\alpha\gamma - \sigma\beta\gamma, \gamma\alpha\sigma - \lambda\sigma\beta\sigma \rangle$$

where $\lambda \in K \setminus \{0, 1\}$. Write Λ for $A_1(\lambda)$ and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are $e_0\Lambda, e_1\Lambda, e_2\Lambda$ with basis

$$e_0\Lambda = sp\{e_0, \alpha, \alpha\sigma, \alpha\gamma, \alpha\gamma\alpha, \alpha\gamma\alpha\gamma\},$$
$$e_1\Lambda = sp\{e_1, \sigma, \gamma, \sigma\beta, \gamma\alpha, \gamma\alpha\gamma, \sigma\beta\sigma, \gamma\alpha\gamma\alpha\},$$
$$e_2\Lambda = sp\{e_2, \beta, \beta\gamma, \beta\sigma, \beta\sigma\beta, \beta\sigma\beta\sigma\}.$$

21.1. The periodicity of the simple modules for $A_1(\lambda)$.

In this section we will look at the periodicity of the simple modules S_0, S_1 and S_2. From 5.2 and 5.3 we have the beginning of the projective resolution of the simple $A_1(\lambda)-$modules as follows.

The minimal projective resolution of S_0 starts with:

$$\cdots \longrightarrow e_0\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_0\Lambda \longrightarrow S_0 \longrightarrow 0$$

where

$$\partial^1 : e_1\zeta \mapsto \alpha e_1\zeta,$$
$$\partial^2 : e_1\zeta \mapsto (\gamma\alpha - \sigma\beta)e_1\zeta,$$
$$\partial^3 : e_0\nu \mapsto \gamma e_0\nu,$$

for $\zeta, \nu \in \Lambda$.

The minimal projective resolution of S_1 starts with

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_0\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_0\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$$\partial^1 : (e_0\nu, e_2\eta) \mapsto \gamma e_0\nu + \sigma e_2\eta,$$
$$\partial^2 : (e_0\nu, e_2\eta) \mapsto (\alpha\gamma, -\beta\gamma)e_0\nu + (\alpha\sigma, -\lambda\beta\sigma)e_2\eta,$$
$$\partial^3 : e_1\zeta \mapsto (\alpha, -\beta)e_1\zeta,$$

for $\nu, \eta, \zeta \in \Lambda$.

The minimal projective resolution of S_2 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$$\partial^1 : e_1\zeta \mapsto \beta e_1\zeta,$$
$$\partial^2 : e_1\zeta \mapsto (\gamma\alpha - \lambda\sigma\beta)e_1\zeta,$$

$$\partial^3 : e_2\eta \mapsto \sigma e_2\eta,$$

for $\zeta, \eta \in \Lambda$.

21.1.1. *The periodicity of S_0.*

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_0)$, let $e_0\nu = c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma$ with $c_i \in K$. Assume that $e_0\nu \in \operatorname{Ker}\partial^3$ then $\gamma e_0\nu = 0$ so $\gamma(c_0e_0 + c_1\alpha + c_2\alpha\sigma + c_3\alpha\gamma + c_4\alpha\gamma\alpha + c_5\alpha\gamma\alpha\gamma) = 0$, that is, $c_0\gamma + c_1\gamma\alpha + c_2\gamma\alpha\sigma + c_3\gamma\alpha\gamma + c_4\gamma\alpha\gamma\alpha = 0$, so that $c_0 = c_1 = c_2 = c_3 = c_4 = 0$. Thus $e_0\nu = c_5\alpha\gamma\alpha\gamma$.

Hence $\operatorname{Ker}\partial^3 = \{c_5\alpha\gamma\alpha\gamma : c_5 \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \alpha\gamma\alpha\gamma e_0\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_5\alpha\gamma\alpha\gamma$, that is, $x = \alpha\gamma\alpha\gamma(c_5e_0)$. Thus $x \in \alpha\gamma\alpha\gamma e_0\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \alpha\gamma\alpha\gamma e_0\Lambda$.

On the other hand, let $y = \alpha\gamma\alpha\gamma e_0\nu \in \alpha\gamma\alpha\gamma e_0\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \gamma(\alpha\gamma\alpha\gamma)e_0\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\alpha\gamma\alpha\gamma e_0\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \alpha\gamma\alpha\gamma e_0\Lambda$. □

Note that $\operatorname{Ker}\partial^3 \cong S_0$ and so $\Omega^4(S_0) \cong S_0$.

21.1.2. *The periodicity of S_1.*

Now we want to find $\operatorname{Ker}\partial^3 = \Omega^4(S_1)$. Let $e_1\zeta = d_1e_1 + d_2\sigma + d_3\gamma + d_4\sigma\beta + d_5\gamma\alpha + d_6\gamma\alpha\gamma + d_7\sigma\beta\sigma + d_8\gamma\alpha\gamma\alpha$ with $d_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker}\partial^3$. Then $(\alpha, -\beta)e_1\zeta = (0, 0)$. So $(\alpha, -\beta)e_1\zeta = (\alpha, -\beta)(d_1e_1 + d_2\sigma + d_3\gamma + d_4\sigma\beta + d_5\gamma\alpha + d_6\gamma\alpha\gamma + d_7\sigma\beta\sigma + d_8\gamma\alpha\gamma\alpha) = (0, 0)$. Thus $(d_1\alpha + d_2\alpha\sigma + d_3\alpha\gamma + d_4\alpha\sigma\beta + d_5\alpha\gamma\alpha + d_6\alpha\gamma\alpha\gamma, -d_1\beta - d_2\beta\sigma - d_3\beta\gamma - d_4\beta\sigma\beta - d_5\beta\gamma\alpha - d_7\beta\sigma\beta\sigma) = (0, 0)$. So $d_1\alpha + d_2\alpha\sigma + d_3\alpha\gamma + d_4\alpha\sigma\beta + d_5\alpha\gamma\alpha + d_6\alpha\gamma\alpha\gamma = 0$, that is, $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 0$. Also $-d_1\beta - d_2\beta\sigma - d_3\beta\gamma - d_4\beta\sigma\beta - d_5\beta\gamma\alpha - d_7\beta\sigma\beta\sigma = 0$ so $d_1 = d_2 = d_3 = d_4 = d_5 = d_7 = 0$. Thus $e_1\zeta = d_8\gamma\alpha\gamma\alpha$.

Hence $\operatorname{Ker}\partial^3 = \{d_8\gamma\alpha\gamma\alpha : d_8 \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \gamma\alpha\gamma\alpha e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^3$. Then $u = d_8\gamma\alpha\gamma\alpha$ so $u = \gamma\alpha\gamma\alpha(d_8e_1)$. Hence $u \in \gamma\alpha\gamma\alpha e_1\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \gamma\alpha\gamma\alpha e_1\Lambda$.

On the other hand, let $v = \gamma\alpha\gamma\alpha e_1\zeta \in \gamma\alpha\gamma\alpha e_1\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(v) = \partial^3(\gamma\alpha\gamma\alpha e_1\zeta) = (\alpha\gamma\alpha\gamma\alpha, -\beta\gamma\alpha\gamma\alpha)e_1\zeta = (0, 0)$. Therefore $\gamma\alpha\gamma\alpha e_1\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \gamma\alpha\gamma\alpha e_1\Lambda$. □

So the map $\partial^4 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\zeta \mapsto \gamma\alpha\gamma\alpha e_1\zeta$, for $\zeta \in \Lambda$.
Note that $\operatorname{Ker}\partial^3 \cong S_1$ and so $\Omega^4(S_1) \cong S_1$.

21.1.3. *The periodicity of S_2.*
To find $\operatorname{Ker}\partial^3 = \Omega^4(S_2)$, let $e_2\eta = c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma$ with $c_i \in K$. Assume that $e_2\eta \in \operatorname{Ker}\partial^3$ then $\sigma e_2\eta = 0$ so $\sigma(c_6e_2 + c_7\beta + c_8\beta\gamma + c_9\beta\sigma + c_{10}\beta\sigma\beta + c_{11}\beta\sigma\beta\sigma) = 0$, that is, $c_6\sigma + c_7\sigma\beta + c_8\sigma\beta\gamma + c_9\sigma\beta\sigma + c_{10}\sigma\beta\sigma\beta = 0$. So $c_6 = c_7 = c_8 = c_9 = c_{10} = 0$. Thus $e_2\eta = c_{11}\beta\sigma\beta\sigma$.

Hence $\operatorname{Ker}\partial^3 = \{c_{11}\beta\sigma\beta\sigma : c_{11} \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \beta\sigma\beta\sigma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_{11}\beta\sigma\beta\sigma$, that is, $x = (\beta\sigma\beta\sigma)(c_{11}e_2)$. Thus $x \in \beta\sigma\beta\sigma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \beta\sigma\beta\sigma e_2\Lambda$.

On the other hand, let $y = \beta\sigma\beta\sigma e_2\eta \in \beta\sigma\beta\sigma e_2\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \sigma\beta\sigma\beta\sigma e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\beta\sigma\beta\sigma e_2\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \beta\sigma\beta\sigma e_2\Lambda$. □

Note that $\operatorname{Ker}\partial^3 \cong S_2$ and so $\Omega^4(S_2) \cong S_2$.

21.2. **Summary.**

Theorem 21.1. *For the algebra $A_1(\lambda)$, we have $\Omega^4(S_0) \cong S_0, \Omega^4(S_1) \cong S_1$ and $\Omega^4(S_2) \cong S_2$. Hence $\Omega^4(S_i) \cong S_i$ for all $i = 0, 1, 2$.*

Theorem 21.2. *With the above notation, and for each simple module S_0, S_1, S_2 of the algebra $A_1(\lambda)$ we have that $\partial^m = \partial^{m+4}$ for all $m \geqslant 1$ where ∂^m is the mth map in the projective resolution of the simple $A_1(\lambda)$ module.*

Moreover let

$$\begin{aligned} g_1^0 &= e_0, \\ g_2^0 &= e_1, \\ g_3^0 &= e_2, \end{aligned}$$

and for $n \geqslant 0$, let

$$\begin{aligned}
&|g^{4n+1}| = 4 \textit{ and} \\
&\qquad g_1^{4n+1} = g_1^{4n}\alpha, \\
&\qquad g_2^{4n+1} = g_2^{4n}\gamma, \\
&\qquad g_3^{4n+1} = g_3^{4n}\sigma, \\
&\qquad g_4^{4n+1} = g_4^{4n}\beta; \\
&|g^{4n+2}| = 4 \textit{ and} \\
&\qquad g_1^{4n+2} = g_1^{4n+1}(\gamma\alpha - \sigma\beta), \\
&\qquad g_2^{4n+2} = g_2^{4n+1}\alpha\gamma - g_3^{4n+1}\beta\gamma, \\
&\qquad g_3^{4n+2} = g_2^{4n+1}\alpha\sigma - g_3^{4n}\lambda\beta\sigma, \\
&\qquad g_4^{4n+2} = g_4^{4n+1}(\gamma\alpha - \lambda\sigma\beta); \\
&|g^{4n+3}| = 3 \textit{ and} \\
&\qquad g_1^{4n+3} = g_1^{4n+2}\gamma, \\
&\qquad g_2^{4n+3} = g_2^{4n+2}\alpha - g_3^{4n+2}\beta,
\end{aligned}$$

$$
\begin{aligned}
&\qquad g_3^{4n+3} = g_4^{4n+2}\sigma;\\
&|g^{4n+4}| = 3 \textit{ and}\\
&\qquad g_1^{4n+4} = g_1^{4n+3}\alpha\gamma\alpha\gamma,\\
&\qquad g_2^{4n+4} = g_2^{4n+3}\gamma\alpha\gamma\alpha,\\
&\qquad g_3^{4n+4} = g_3^{4n+3}\beta\sigma\beta\sigma.
\end{aligned}
$$

Then $\Lambda/\mathfrak{r} = S_0 \oplus S_1 \oplus S_2$ has minimal projective resolution (Q^m, ε^m) as a right $\Lambda-$module where $Q^m = \oplus_{g_i^m \in g^m} \mathfrak{t}(g_i^m)\Lambda$ and, writing $g_i^m = \Sigma_j g_j^{m-1} p_j$ for appropriate elements $p_j \in \Lambda$ as above, the map $\varepsilon^m : Q^m \to Q^{m-1}$ is given by $\mathfrak{t}(g_i^m) \mapsto \Sigma_j \mathfrak{t}(g_j^{m-1})p_j$.

We remark that ε^m is the sum of the maps ∂^m for all simple modules S_0, S_1, S_2.

21.3. The bimodule resolution of $A_1(\lambda)$.

From Chapter 3 recall that, for a finite dimensional algebra Λ over a field K, we have $P^n = \oplus_{g_i^n \in g^n} \Lambda\mathfrak{o}(g_i^n) \otimes_K \mathfrak{t}(g_i^n)\Lambda$ and, for $n = 0, 1, 2, 3$, the map $d^n : P^n \to P^{n-1}$ is defined by how the elements of g^n relate to the elements in g^{n-1}. In this section, we use the ideas by Green and Snashall [16] to construct a minimal projective bimodule resolution (P^n, d^n) of $A_1(\lambda)$. We continue to write Λ for $A_1(\lambda)$, and use the sets g^n as defined in Theorem 21.2.

Definition 21.3. *Define $d^0 : P^0 \to \Lambda$ by*

$$
\begin{aligned}
&e_0 \otimes e_0 \mapsto e_0\\
&e_1 \otimes e_1 \mapsto e_1\\
&e_2 \otimes e_2 \mapsto e_2.
\end{aligned}
$$

For all $n \geqslant 0$, define $d^{4n+1} : P^{4n+1} \to P^{4n}$ by

$$
\begin{aligned}
&\mathfrak{o}(g_1^{4n+1}) \otimes \mathfrak{t}(g_1^{4n+1}) \mapsto \mathfrak{o}(g_1^{4n+1}) \otimes_{g_1^{4n}} \alpha - \alpha \otimes_{g_2^{4n}} \mathfrak{t}(g_1^{4n+1})\\
&\mathfrak{o}(g_2^{4n+1}) \otimes \mathfrak{t}(g_2^{4n+1}) \mapsto \mathfrak{o}(g_2^{4n+1}) \otimes_{g_2^{4n}} \gamma - \gamma \otimes_{g_1^{4n}} \mathfrak{t}(g_2^{4n+1})\\
&\mathfrak{o}(g_3^{4n+1}) \otimes \mathfrak{t}(g_3^{4n+1}) \mapsto \mathfrak{o}(g_3^{4n+1}) \otimes_{g_2^{4n}} \sigma - (-\lambda)^n \sigma \otimes_{g_3^{4n}} \mathfrak{t}(g_3^{4n+1})\\
&\mathfrak{o}(g_4^{4n+1}) \otimes \mathfrak{t}(g_4^{4n+1}) \mapsto \mathfrak{o}(g_4^{4n+1}) \otimes_{g_3^{4n}} \beta - (-\lambda)^{-n} \beta \otimes_{g_2^{4n}} \mathfrak{t}(g_4^{4n+1});
\end{aligned}
$$

define $d^{4n+2} : P^{4n+2} \to P^{4n+1}$ by

$$
\begin{aligned}
\mathfrak{o}(g_1^{4n+2}) \otimes \mathfrak{t}(g_1^{4n+2}) \mapsto{}& \mathfrak{o}(g_1^{4n+2}) \otimes_{g_1^{4n+1}} (\gamma\alpha - \sigma\beta) + \alpha \otimes_{g_2^{4n+1}} \alpha\\
&- \alpha \otimes_{g_3^{4n+1}} \beta + \alpha\gamma \otimes_{g_1^{4n+1}} \mathfrak{t}(g_1^{4n+2}) - (-\lambda)^n \alpha\sigma \otimes_{g_4^{4n+1}} \mathfrak{t}(g_1^{4n+2})\\
\mathfrak{o}(g_2^{4n+2}) \otimes \mathfrak{t}(g_2^{4n+2}) \mapsto{}& \mathfrak{o}(g_2^{4n+2}) \otimes_{g_2^{4n+1}} \alpha\gamma - \mathfrak{o}(g_2^{4n+2}) \otimes_{g_3^{4n+1}} \beta\gamma\\
&+ \gamma \otimes_{g_1^{4n+1}} \gamma - (-\lambda)^n \sigma \otimes_{g_4^{4n+1}} \gamma + (\gamma\alpha - \sigma\beta) \otimes_{g_2^{4n+1}} \mathfrak{t}(g_2^{4n+2})\\
\mathfrak{o}(g_3^{4n+2}) \otimes \mathfrak{t}(g_3^{4n+2}) \mapsto{}& \mathfrak{o}(g_3^{4n+2}) \otimes_{g_2^{4n+1}} \alpha\sigma - \mathfrak{o}(g_3^{4n+2}) \otimes_{g_3^{4n+1}} \lambda\beta\sigma\\
&+ \gamma \otimes_{g_1^{4n+1}} \sigma + (-\lambda)^{n+1} \sigma \otimes_{g_4^{4n+1}} \sigma + (\gamma\alpha - \lambda\sigma\beta) \otimes_{g_3^{4n+1}} \mathfrak{t}(g_3^{4n+2})\\
\mathfrak{o}(g_4^{4n+2}) \otimes \mathfrak{t}(g_4^{4n+2}) \mapsto{}& \mathfrak{o}(g_4^{4n+2}) \otimes_{g_4^{4n+1}} (\gamma\alpha - \lambda\sigma\beta) + (-\lambda)^{1-n} \beta \otimes_{g_3^{4n+1}} \beta\\
&+ (-\lambda)^{-n} \beta \otimes_{g_2^{4n+1}} \alpha + (-\lambda)^{-n} \beta\gamma \otimes_{g_1^{4n+1}} \mathfrak{t}(g_4^{4n+2})\\
&\quad - \lambda\beta\sigma \otimes_{g_4^{4n+1}} \mathfrak{t}(g_4^{4n+2});
\end{aligned}
$$

define $d^{4n+3} : P^{4n+3} \to P^{4n+2}$ *by*

$$\begin{aligned}
\mathfrak{o}(g_1^{4n+3}) \otimes \mathfrak{t}(g_1^{4n+3}) &\mapsto \mathfrak{o}(g_1^{4n+3}) \otimes_{g_1^{4n+2}} \gamma - \alpha \otimes_{g_2^{4n+2}} \mathfrak{t}(g_1^{4n+3}) \\
\mathfrak{o}(g_2^{4n+3}) \otimes \mathfrak{t}(g_2^{4n+3}) &\mapsto \mathfrak{o}(g_2^{4n+3}) \otimes_{g_2^{4n+2}} \alpha - \mathfrak{o}(g_2^{4n+3}) \otimes_{g_3^{4n+2}} \beta \\
&\quad - \gamma \otimes_{g_1^{4n+2}} \mathfrak{t}(g_2^{4n+3}) + (-\lambda)^n \sigma \otimes_{g_4^{4n+2}} \mathfrak{t}(g_2^{4n+3}) \\
\mathfrak{o}(g_3^{4n+3}) \otimes \mathfrak{t}(g_3^{4n+3}) &\mapsto \mathfrak{o}(g_3^{4n+3}) \otimes_{g_4^{4n+2}} \sigma - (-\lambda)^{-n} \beta \otimes_{g_3^{4n+2}} \mathfrak{t}(g_3^{4n+3});
\end{aligned}$$

and define $d^{4n+4} : P^{4n+4} \to P^{4n+3}$ *by*

$$\begin{aligned}
\mathfrak{o}(g_1^{4n+4}) \otimes \mathfrak{t}(g_1^{4n+4}) &\mapsto \mathfrak{o}(g_1^{4n+4}) \otimes_{g_1^{4n+3}} \alpha\gamma\alpha\gamma + \alpha \otimes_{g_2^{4n+3}} \gamma\alpha\gamma \\
&\quad + \alpha\gamma \otimes_{g_1^{4n+3}} \alpha\gamma + \alpha\gamma\alpha \otimes_{g_2^{4n+3}} \gamma + \alpha\gamma\alpha\gamma \otimes_{g_1^{4n+3}} \mathfrak{t}(g_1^{4n+4}) \\
&\quad - (-\lambda)^n \alpha\sigma \otimes_{g_3^{4n+3}} \beta\gamma \\
\mathfrak{o}(g_2^{4n+4}) \otimes \mathfrak{t}(g_2^{4n+4}) &\mapsto \mathfrak{o}(g_2^{4n+4}) \otimes_{g_2^{4n+3}} \gamma\alpha\gamma\alpha + \gamma \otimes_{g_1^{4n+3}} \alpha\gamma\alpha \\
&\quad + \gamma\alpha\gamma \otimes_{g_1^{4n+3}} \alpha + \gamma\alpha\gamma\alpha \otimes_{g_2^{4n+3}} \mathfrak{t}(g_2^{4n+4}) - (-\lambda)^n \gamma\alpha\sigma \otimes_{g_3^{4n+3}} \beta \\
&\quad - (-\lambda)^n \sigma \otimes_{g_3^{4n+3}} \beta\gamma\alpha + (1-\lambda)^{-1} \gamma\alpha \otimes_{g_2^{4n+3}} (\gamma\alpha - \lambda\sigma\beta) \\
&\quad + \lambda(1-\lambda)^{-1} \sigma\beta \otimes_{g_2^{4n+3}} (\sigma\beta - \gamma\alpha) \\
\mathfrak{o}(g_3^{4n+4}) \otimes \mathfrak{t}(g_3^{4n+4}) &\mapsto \mathfrak{o}(g_3^{4n+4}) \otimes_{g_3^{4n+3}} \beta\sigma\beta\sigma - (-\lambda)^{-n} \beta \otimes_{g_2^{4n+3}} \sigma\beta\sigma \\
&\quad + \beta\sigma \otimes_{g_3^{4n+3}} \beta\sigma - (-\lambda)^{-n} \beta\sigma\beta \otimes_{g_2^{4n+3}} \sigma + \beta\sigma\beta\sigma \otimes_{g_3^{4n+3}} \mathfrak{t}(g_3^{4n+4}) \\
&\quad + (-\lambda)^{-(n+1)} \beta\gamma \otimes_{g_1^{4n+3}} \alpha\sigma.
\end{aligned}$$

Remark. If $n = 0$ then these are the same as d^1, d^2, d^3 of Green and Snashall [16] as given in Chapter 3. The aim is to show that (P^n, d^n) is indeed a minimal projective resolution of Λ as a bimodule. We start by showing that (P^n, d^n) is a complex.

Theorem 21.4. *The composition map $d^m \cdot d^{m+1} = 0$ for all $m \geqslant 0$.*

Proof. This is true for $m = 0, 1, 2$ by [16]. So let $m \geqslant 3$. We consider four cases: $m = 4n+3, 4n+4, 4n+5, 4n+6$ where $n \geqslant 0$.

First we show $d^{4n+3} \cdot d^{4n+4} = 0$. We have

$$\begin{aligned}
&d^{4n+3} \cdot d^{4n+4}(\mathfrak{o}(g_1^{4n+4}) \otimes \mathfrak{t}(g_1^{4n+4})) \\
&\quad = [\mathfrak{o}(g_1^{4n+3}) \otimes_{g_1^{4n+2}} \gamma - \alpha \otimes_{g_2^{4n+2}} \mathfrak{t}(g_1^{4n+3})]\alpha\gamma\alpha\gamma + \alpha[\mathfrak{o}(g_2^{4n+3}) \otimes_{g_2^{4n+2}} \alpha - \mathfrak{o}(g_2^{4n+3}) \otimes_{g_3^{4n+2}} \beta \\
&\qquad -\gamma \otimes_{g_1^{4n+2}} \mathfrak{t}(g_2^{4n+3}) + (-\lambda)^n \sigma \otimes_{g_4^{4n+2}} \mathfrak{t}(g_2^{4n+3})]\gamma\alpha\gamma + \alpha\gamma[\mathfrak{o}(g_1^{4n+3}) \otimes_{g_1^{4n+2}} \gamma \\
&\qquad -\alpha \otimes_{g_2^{4n+2}} \mathfrak{t}(g_1^{4n+3})]\alpha\gamma + \alpha\gamma\alpha[\mathfrak{o}(g_2^{4n+3}) \otimes_{g_2^{4n+2}} \alpha - \mathfrak{o}(g_2^{4n+3}) \otimes_{g_3^{4n+2}} \beta - \gamma \otimes_{g_1^{4n+2}} \mathfrak{t}(g_2^{4n+3}) \\
&\qquad +(-\lambda)^n \sigma \otimes_{g_4^{4n+2}} \mathfrak{t}(g_2^{4n+3})]\gamma + \alpha\gamma\alpha\gamma[\mathfrak{o}(g_1^{4n+3}) \otimes_{g_1^{4n+2}} \gamma - \alpha \otimes_{g_2^{4n+2}} \mathfrak{t}(g_1^{4n+3})] \\
&\qquad -(-\lambda)^n \alpha\sigma[\mathfrak{o}(g_3^{4n+3}) \otimes_{g_4^{4n+2}} \sigma - (-\lambda)^{-n} \beta \otimes_{g_3^{4n+2}} \mathfrak{t}(g_3^{4n+3})]\beta\gamma \\
&\quad = -\alpha \otimes_{g_2^{4n+2}} \alpha\gamma\alpha\gamma + \alpha \otimes_{g_2^{4n+2}} \alpha\gamma\alpha\gamma - \alpha\gamma \otimes_{g_1^{4n+2}} \gamma\alpha\gamma + (-\lambda)^n \alpha\sigma \otimes_{g_4^{4n+2}} \gamma\alpha\gamma \\
&\qquad +\alpha\gamma \otimes_{g_1^{4n+2}} \gamma\alpha\gamma - \alpha\gamma\alpha \otimes_{g_2^{4n+2}} \alpha\gamma + \alpha\gamma\alpha \otimes_{g_2^{4n+2}} \alpha\gamma - \alpha\gamma\alpha \otimes_{g_3^{4n+2}} \beta\gamma - \alpha\gamma\alpha\gamma \otimes_{g_1^{4n+2}} \gamma \\
&\qquad +\alpha\gamma\alpha\gamma \otimes_{g_1^{4n+2}} \gamma - (-\lambda)^n \alpha\sigma \otimes_{g_4^{4n+2}} \sigma\beta\gamma + \alpha\sigma\beta \otimes_{g_3^{4n+2}} \beta\gamma \\
&\quad = 0,
\end{aligned}$$

$$
\begin{aligned}
&d^{4n+3}\cdot d^{4n+4}(\mathfrak{o}(g_2^{4n+4})\otimes\mathfrak{t}(g_2^{4n+4}))\\
&\quad=[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})]\gamma\alpha\gamma\alpha\\
&\qquad+\gamma[\mathfrak{o}(g_1^{4n+3})\otimes_{g_1^{4n+2}}\gamma-\alpha\otimes_{g_2^{4n+2}}\mathfrak{t}(g_1^{4n+3})]\alpha\gamma\alpha+\gamma\alpha\gamma[\mathfrak{o}(g_1^{4n+3})\otimes_{g_1^{4n+2}}\gamma-\alpha\otimes_{g_2^{4n+2}}\mathfrak{t}(g_1^{4n+3})]\alpha\\
&\qquad+\gamma\alpha\gamma\alpha[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})]\\
&\qquad-(-\lambda)^n\gamma\alpha\sigma[\mathfrak{o}(g_3^{4n+3})\otimes_{g_4^{4n+2}}\sigma-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\mathfrak{t}(g_3^{4n+3})]\beta-(-\lambda)^n\sigma[\mathfrak{o}(g_3^{4n+3})\otimes_{g_4^{4n+2}}\sigma\\
&\qquad-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\mathfrak{t}(g_3^{4n+3})]\beta\gamma\alpha+(1-\lambda)^{-1}\gamma\alpha[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta\\
&\qquad-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})](\gamma\alpha-\lambda\sigma\beta)+\lambda(1-\lambda)^{-1}\sigma\beta[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha\\
&\qquad-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})](\sigma\beta-\gamma\alpha)\\
&\quad=-\gamma\otimes_{g_1^{4n+2}}\gamma\alpha\gamma\alpha+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\gamma\alpha\gamma\alpha+\gamma\otimes_{g_1^{4n+2}}\gamma\alpha\gamma\alpha-\gamma\alpha\otimes_{g_2^{4n+2}}\alpha\gamma\alpha+\gamma\alpha\gamma\otimes_{g_1^{4n+2}}\gamma\alpha\\
&\qquad-\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+2}}\alpha+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+2}}\alpha-\gamma\alpha\gamma\alpha\otimes_{g_3^{4n+2}}\beta-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_4^{4n+2}}\sigma\beta+\gamma\alpha\sigma\beta\otimes_{g_3^{4n+2}}\beta\\
&\qquad-(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\sigma\beta\gamma\alpha+\sigma\beta\otimes_{g_3^{4n+2}}\beta\gamma\alpha+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+2}}\alpha\gamma\alpha-(1-\lambda)^{-1}\gamma\alpha\otimes_{g_3^{4n+2}}\beta\gamma\alpha\\
&\qquad-(1-\lambda)^{-1}\gamma\alpha\gamma\otimes_{g_1^{4n+2}}\gamma\alpha+(-\lambda)^n(1-\lambda)^{-1}\gamma\alpha\sigma\otimes_{g_4^{4n+2}}\gamma\alpha-\lambda(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+2}}\alpha\sigma\beta\\
&\qquad+\lambda(1-\lambda)^{-1}\gamma\alpha\otimes_{g_3^{4n+2}}\beta\sigma\beta+\lambda(1-\lambda)^{-1}\gamma\alpha\gamma\otimes_{g_1^{4n+2}}\sigma\beta+(-\lambda)^{n+1}(1-\lambda)^{-1}\gamma\alpha\sigma\otimes_{g_4^{4n+2}}\sigma\beta\\
&\qquad+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+2}}\alpha\sigma\beta-\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_3^{4n+2}}\beta\sigma\beta-\lambda(1-\lambda)^{-1}\sigma\beta\gamma\otimes_{g_1^{4n+2}}\sigma\beta\\
&\qquad-(-\lambda)^{n+1}(1-\lambda)^{-1}\sigma\beta\sigma\otimes_{g_4^{4n+2}}\sigma\beta-\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+2}}\alpha\gamma\alpha+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_3^{4n+2}}\beta\gamma\alpha\\
&\qquad+\lambda(1-\lambda)^{-1}\sigma\beta\gamma\otimes_{g_1^{4n+2}}\gamma\alpha+(-\lambda)^{n+1}(1-\lambda)^{-1}\sigma\beta\sigma\otimes_{g_4^{4n+2}}\gamma\alpha\\
&\quad=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&d^{4n+3}\cdot d^{4n+4}(\mathfrak{o}(g_3^{4n+4})\otimes\mathfrak{t}(g_3^{4n+4}))\\
&\quad=[\mathfrak{o}(g_3^{4n+3})\otimes_{g_4^{4n+2}}\sigma-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\mathfrak{t}(g_3^{4n+3})]\beta\sigma\beta\sigma-(-\lambda)^{-n}\beta[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha\\
&\qquad-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})]\sigma\beta\sigma\\
&\qquad+\beta\sigma[\mathfrak{o}(g_3^{4n+3})\otimes_{g_4^{4n+2}}\sigma-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\mathfrak{t}(g_3^{4n+3})]\beta\sigma-(-\lambda)^{-n}\beta\sigma\beta[\mathfrak{o}(g_2^{4n+3})\otimes_{g_2^{4n+2}}\alpha\\
&\qquad-\mathfrak{o}(g_2^{4n+3})\otimes_{g_3^{4n+2}}\beta-\gamma\otimes_{g_1^{4n+2}}\mathfrak{t}(g_2^{4n+3})+(-\lambda)^n\sigma\otimes_{g_4^{4n+2}}\mathfrak{t}(g_2^{4n+3})]\sigma+\beta\sigma\beta\sigma[\mathfrak{o}(g_3^{4n+3})\otimes_{g_4^{4n+2}}\sigma\\
&\qquad-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\mathfrak{t}(g_3^{4n+3})]+(-\lambda)^{-n-1}\beta\gamma[\mathfrak{o}(g_1^{4n+3})\otimes_{g_1^{4n+2}}\gamma-\alpha\otimes_{g_2^{4n+2}}\mathfrak{t}(g_1^{4n+3})]\alpha\sigma\\
&\quad=-(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\beta\sigma\beta\sigma+(-\lambda)^{-n}\beta\otimes_{g_3^{4n+2}}\beta\sigma\beta\sigma+(-\lambda)^{-n}\beta\gamma\otimes_{g_1^{4n+2}}\sigma\beta\sigma-\beta\sigma\otimes_{g_4^{4n+2}}\sigma\beta\sigma\\
&\qquad+\beta\sigma\otimes_{g_4^{4n+2}}\sigma\beta\sigma-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_3^{4n+2}}\beta\sigma-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_2^{4n+2}}\alpha\sigma+(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_3^{4n+2}}\beta\sigma\\
&\qquad-\beta\sigma\beta\sigma\otimes_{g_4^{4n+2}}\sigma+\beta\sigma\beta\sigma\otimes_{g_4^{4n+2}}\sigma+(-\lambda)^{-n-1}\beta\gamma\otimes_{g_1^{4n+2}}\gamma\alpha\sigma-(-\lambda)^{-n-1}\beta\gamma\alpha\otimes_{g_2^{4n+2}}\alpha\sigma\\
&\quad=0.
\end{aligned}
$$

Thus $d^{4n+3}\cdot d^{4n+4}=0$.

Next we show that $d^{4n+4}\cdot d^{4n+5}=0$. We have

$$
\begin{aligned}
&d^{4n+4}\cdot d^{4n+5}(\mathfrak{o}(g_1^{4n+5})\otimes\mathfrak{t}(g_1^{4n+5}))=d^{4n+4}\cdot d^{4(n+1)+1}(\mathfrak{o}(g_1^{4(n+1)+1})\otimes\mathfrak{t}(g_1^{4(n+1)+1}))\\
&\quad=[\mathfrak{o}(g_1^{4n+4})\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\gamma+\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma+\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma+\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\\
&\qquad+\alpha\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\mathfrak{t}(g_1^{4n+4})-(-\lambda)^n\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma]\alpha-\alpha[\mathfrak{o}(g_2^{4n+4})\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha+\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\\
&\qquad+\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\mathfrak{t}(g_2^{4n+4})-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta-(-\lambda)^n\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha\\
&\qquad+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}(\gamma\alpha-\lambda\sigma\beta)+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}(\sigma\beta-\gamma\alpha)]\\
&\quad=\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha+\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha+\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha+\alpha\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha-(-\lambda)^n\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha\\
&\qquad-\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha-\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha-\alpha\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha+(-\lambda)^n\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha\\
&\qquad-(1-\lambda)^{-1}\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha+\lambda(1-\lambda)^{-1}\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\sigma\beta-\lambda(1-\lambda)^{-1}\alpha\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta\\
&\qquad+\lambda(1-\lambda)^{-1}\alpha\sigma\beta\otimes_{g_2^{4n+3}}\gamma\alpha\\
&\quad=0,
\end{aligned}
$$

$$
\begin{aligned}
&d^{4n+4}\cdot d^{4n+5}(\mathfrak{o}(g_2^{4n+5})\otimes\mathfrak{t}(g_2^{4n+5}))\\
&\quad=[\mathfrak{o}(g_2^{4n+4})\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha+\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha+\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\mathfrak{t}(g_2^{4n+4})\\
&\qquad-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta-(-\lambda)^n\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}(\gamma\alpha-\lambda\sigma\beta)\\
&\qquad+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}(\sigma\beta-\gamma\alpha)]\gamma-\gamma[\mathfrak{o}(g_1^{4n+4})\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\gamma+\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\\
&\qquad+\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma+\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma+\alpha\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\mathfrak{t}(g_1^{4n+4})-(-\lambda)^n\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma]\\
&\quad=\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\gamma+\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma\\
&\qquad+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma-\lambda(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}\sigma\beta\gamma+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta\gamma\\
&\qquad-\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}\gamma\alpha\gamma-\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\gamma-\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\gamma-\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\\
&\qquad-\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\gamma+(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta\gamma\\
&\quad=0,
\end{aligned}
$$

$$
\begin{aligned}
&d^{4n+4}\cdot d^{4n+5}(\mathfrak{o}(g_3^{4n+5})\otimes\mathfrak{t}(g_3^{4n+5}))\\
&\quad=[\mathfrak{o}(g_2^{4n+4})\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha+\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha+\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\mathfrak{t}(g_2^{4n+4})\\
&\qquad-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta-(-\lambda)^n\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}(\gamma\alpha-\lambda\sigma\beta)\\
&\qquad+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}(\sigma\beta-\gamma\alpha)]\sigma-(-\lambda)^{n+1}\sigma[\mathfrak{o}(g_3^{4n+4})\otimes_{g_3^{4n+3}}\beta\sigma\beta\sigma\\
&\qquad-(-\lambda)^{-n}\beta\otimes_{g_2^{4n+3}}\sigma\beta\sigma+\beta\sigma\otimes_{g_3^{4n+3}}\beta\sigma-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_2^{4n+3}}\sigma+\beta\sigma\beta\sigma\otimes_{g_3^{4n+3}}\mathfrak{t}(g_3^{4n+4})\\
&\qquad+(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+3}}\alpha\sigma]\\
&\quad=\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha\sigma+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\sigma-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta\sigma-(-\lambda)^n\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha\sigma\\
&\qquad+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\sigma-\lambda(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}\sigma\beta\sigma+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta\sigma\\
&\qquad-\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}\gamma\alpha\sigma-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+3}}\beta\sigma\beta\sigma-\lambda\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta\sigma\\
&\qquad-(-\lambda)^{n+1}\sigma\beta\sigma\otimes_{g_3^{4n+3}}\beta\sigma-\lambda\sigma\beta\sigma\beta\otimes_{g_2^{4n+3}}\sigma-\sigma\beta\gamma\otimes_{g_1^{4n+3}}\alpha\sigma\\
&\quad=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&d^{4n+4}\cdot d^{4n+5}(\mathfrak{o}(g_4^{4n+5})\otimes\mathfrak{t}(g_4^{4n+5}))\\
&\quad=[\mathfrak{o}(g_3^{4n+4})\otimes_{g_3^{4n+3}}\beta\sigma\beta\sigma-(-\lambda)^{-n}\beta\otimes_{g_2^{4n+3}}\sigma\beta\sigma+\beta\sigma\otimes_{g_3^{4n+3}}\beta\sigma-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_2^{4n+3}}\sigma\\
&\qquad+\beta\sigma\beta\sigma\otimes_{g_3^{4n+3}}\mathfrak{t}(g_3^{4n+4})+(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+3}}\alpha\sigma]\beta-(-\lambda)^{-(n+1)}\beta[\mathfrak{o}(g_2^{4n+4})\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha\\
&\qquad+\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha+\gamma\alpha\gamma\otimes_{g_1^{4n+3}}\alpha+\gamma\alpha\gamma\alpha\otimes_{g_2^{4n+3}}\mathfrak{t}(g_2^{4n+4})-(-\lambda)^n\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta\\
&\qquad-(-\lambda)^n\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha+(1-\lambda)^{-1}\gamma\alpha\otimes_{g_2^{4n+3}}(\gamma\alpha-\lambda\sigma\beta)+\lambda(1-\lambda)^{-1}\sigma\beta\otimes_{g_2^{4n+3}}(\sigma\beta-\gamma\alpha)]\\
&\quad=-(-\lambda)^{-n}\beta\otimes_{g_2^{4n+3}}\sigma\beta\sigma\beta+\beta\sigma\otimes_{g_3^{4n+3}}\beta\sigma\beta-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta+\beta\sigma\beta\sigma\otimes_{g_3^{4n+3}}\beta\\
&\qquad+(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+3}}\alpha\sigma\beta-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+3}}\gamma\alpha\gamma\alpha-(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+3}}\alpha\gamma\alpha\\
&\qquad-\lambda^{-1}\beta\gamma\alpha\sigma\otimes_{g_3^{4n+3}}\beta-\lambda^{-1}\beta\sigma\otimes_{g_3^{4n+3}}\beta\gamma\alpha-(-\lambda)^{-(n+1)}(1-\lambda)^{-1}\beta\gamma\alpha\otimes_{g_2^{4n+3}}\gamma\alpha\\
&\qquad-(-\lambda)^{-n}(1-\lambda)^{-1}\beta\gamma\alpha\otimes_{g_2^{4n+3}}\sigma\beta+(-\lambda)^{-n}(1-\lambda)^{-1}\beta\sigma\beta\otimes_{g_2^{4n+3}}\sigma\beta\\
&\qquad-(-\lambda)^{-n}(1-\lambda)^{-1}\beta\sigma\beta\otimes_{g_2^{4n+3}}\gamma\alpha\\
&\quad=0.
\end{aligned}
$$

Hence $d^{4n+4}\cdot d^{4n+5}=0$.

Now we show that $d^{4n+5}\cdot d^{4n+6}=0$. We have

$$\begin{aligned}
&d^{4n+5}\cdot d^{4n+6}(\mathfrak{o}(g_1^{4n+6})\otimes \mathfrak{t}(g_1^{4n+6})) = d^{4(n+1)+1}\cdot d^{4(n+1)+2}(\mathfrak{o}(g_1^{4(n+1)+2})\otimes \mathfrak{t}(g_1^{4(n+1)+2}))\\
&\quad= [\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})](\gamma\alpha-\sigma\beta)+\alpha[\mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma\\
&\qquad-\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})]\alpha-\alpha[\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})]\beta\\
&\qquad+\alpha\gamma[\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})]-(-\lambda)^{n+1}\alpha\sigma[\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta\\
&\qquad-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})]\\
&\quad= \mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha\gamma\alpha-\alpha\otimes_{g_2^{4n+4}}\gamma\alpha-\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha\sigma\beta+\alpha\otimes_{g_2^{4n+4}}\sigma\beta+\alpha\otimes_{g_2^{4n+4}}\gamma\alpha\\
&\qquad-\alpha\gamma\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\sigma\beta+(-\lambda)^{n+1}\alpha\sigma\otimes_{g_3^{4n+4}}\beta+\alpha\gamma\otimes_{g_1^{4n+4}}\alpha-\alpha\gamma\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})\\
&\qquad-(-\lambda)^{n+1}\alpha\sigma\otimes_{g_3^{4n+4}}\beta+\alpha\sigma\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})\\
&\quad=0,
\end{aligned}$$

$$\begin{aligned}
&d^{4n+5}\cdot d^{4n+6}(\mathfrak{o}(g_2^{4n+6})\otimes \mathfrak{t}(g_2^{4n+6}))\\
&\quad= [\mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma-\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})]\alpha\gamma-[\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma\\
&\qquad-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})]\beta\gamma+\gamma[\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})]\gamma\\
&\qquad-(-\lambda)^{n+1}\sigma[\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})]\gamma\\
&\qquad+(\gamma\alpha-\sigma\beta)[\mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma-\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})]\\
&\quad= \mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma\alpha\gamma-\gamma\otimes_{g_1^{4n+4}}\alpha\gamma-\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma\beta\gamma+(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\beta\gamma\\
&\qquad+\gamma\otimes_{g_1^{4n+4}}\alpha\gamma-\gamma\alpha\otimes_{g_2^{4n+4}}\gamma-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\beta\gamma+\sigma\beta\otimes_{g_2^{4n+4}}\gamma+\gamma\alpha\otimes_{g_2^{4n+4}}\gamma\\
&\qquad-\gamma\alpha\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})-\sigma\beta\otimes_{g_2^{4n+4}}\gamma+\sigma\beta\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})\\
&\quad=0,
\end{aligned}$$

$$\begin{aligned}
&d^{4n+5}\cdot d^{4n+6}(\mathfrak{o}(g_3^{4n+6})\otimes \mathfrak{t}(g_3^{4n+6}))\\
&\quad= [\mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma-\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})]\alpha\sigma-[\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma\\
&\qquad-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})]\lambda\beta\sigma+\gamma[\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})]\sigma\\
&\qquad+(-\lambda)^{n+2}\sigma[\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})]\sigma+(\gamma\alpha\\
&\qquad-\lambda\sigma\beta)[\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})]\\
&\quad= \mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma\alpha\sigma-\gamma\otimes_{g_1^{4n+4}}\alpha\sigma-\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\lambda\sigma\beta\sigma+\lambda(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\beta\sigma\\
&\qquad+\gamma\otimes_{g_1^{4n+4}}\alpha\sigma-\gamma\alpha\otimes_{g_2^{4n+4}}\sigma-\lambda(-\lambda)^{n+1}\sigma\otimes_{g_2^{4n+4}}\beta\sigma+\lambda\sigma\beta\otimes_{g_3^{4n+4}}\sigma+\gamma\alpha\otimes_{g_2^{4n+4}}\sigma\\
&\qquad-\lambda\sigma\beta\otimes_{g_2^{4n+4}}\sigma-(-\lambda)^{n+1}\gamma\alpha\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})+\lambda(-\lambda)^{n+1}\sigma\beta\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})\\
&\quad=0,
\end{aligned}$$

and

$$\begin{aligned}
&d^{4n+5}\cdot d^{4n+6}(\mathfrak{o}(g_4^{4n+6})\otimes \mathfrak{t}(g_4^{4n+6}))\\
&\quad= [\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})](\gamma\alpha-\lambda\sigma\beta)\\
&\qquad+(-\lambda)^{-n}\beta[\mathfrak{o}(g_3^{4n+5})\otimes_{g_2^{4n+4}}\sigma-(-\lambda)^{n+1}\sigma\otimes_{g_3^{4n+4}}\mathfrak{t}(g_3^{4n+5})]\beta\\
&\qquad+(-\lambda)^{-(n+1)}\beta[\mathfrak{o}(g_2^{4n+5})\otimes_{g_2^{4n+4}}\gamma-\gamma\otimes_{g_1^{4n+4}}\mathfrak{t}(g_2^{4n+5})]\alpha\\
&\qquad+(-\lambda)^{-(n+1)}\beta\gamma[\mathfrak{o}(g_1^{4n+5})\otimes_{g_1^{4n+4}}\alpha-\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})]-\lambda\beta\sigma[\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta\\
&\qquad-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})]\\
&\quad= \mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\beta\gamma\alpha-(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\gamma\alpha-\mathfrak{o}(g_4^{4n+5})\otimes_{g_3^{4n+4}}\lambda\beta\sigma\beta\\
&\qquad-(-\lambda)^{-n}\beta\otimes_{g_2^{4n+4}}\sigma\beta+(-\lambda)^{-n}\beta\otimes_{g_2^{4n+4}}\sigma\beta+\lambda\beta\sigma\otimes_{g_3^{4n+4}}\beta\\
&\qquad+(-\lambda)^{-(n+1)}\beta\otimes_{g_2^{4n+4}}\gamma\alpha-(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+4}}\alpha+(-\lambda)^{-(n+1)}\beta\gamma\otimes_{g_1^{4n+4}}\alpha\\
&\qquad-(-\lambda)^{-(n+1)}\beta\gamma\alpha\otimes_{g_2^{4n+4}}\mathfrak{t}(g_1^{4n+5})-\lambda\beta\sigma\otimes_{g_3^{4n+4}}\beta-(-\lambda)^{-n}\beta\sigma\beta\otimes_{g_2^{4n+4}}\mathfrak{t}(g_4^{4n+5})\\
&\quad=0.
\end{aligned}$$

Thus $d^{4n+5}\cdot d^{4n+6}=0$.

Finally we show $d^{4n+6} \cdot d^{4n+7} = 0$. We have

$$\begin{aligned}
&d^{4n+6} \cdot d^{4n+7}(\mathfrak{o}(g_1^{4n+7}) \otimes \mathfrak{t}(g_1^{4n+7})) = d^{4(n+1)+2} \cdot d^{4(n+1)+3}(\mathfrak{o}(g_1^{4(n+1)+3}) \otimes \mathfrak{t}(g_1^{4(n+1)+3}))\\
&\quad = [\mathfrak{o}(g_1^{4n+6}) \otimes_{g_1^{4n+5}} (\gamma\alpha - \sigma\beta) + \alpha \otimes_{g_2^{4n+5}} \alpha - \alpha \otimes_{g_3^{4n+5}} \beta + \alpha\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_1^{4n+6})\\
&\qquad -(-\lambda)^{n+1}\alpha\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_1^{4n+6})]\gamma - \alpha[\mathfrak{o}(g_2^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\gamma - \mathfrak{o}(g_2^{4n+6}) \otimes_{g_3^{4n+5}} \beta\gamma\\
&\qquad +\gamma \otimes_{g_1^{4n+5}} \gamma - (-\lambda)^{n+1}\sigma \otimes_{g_4^{4n+5}} \gamma + (\gamma\alpha - \sigma\beta) \otimes_{g_2^{4n+5}} \mathfrak{t}(g_2^{4n+6})]\\
&\quad = \mathfrak{o}(g_1^{4n+6}) \otimes_{g_1^{4n+5}} \gamma\alpha\gamma - \mathfrak{o}(g_1^{4n+6}) \otimes_{g_1^{4n+5}} \sigma\beta\gamma + \alpha \otimes_{g_2^{4n+5}} \alpha\gamma - \alpha \otimes_{g_3^{4n+5}} \beta\gamma\\
&\qquad +\alpha\gamma \otimes_{g_1^{4n+5}} \gamma - (-\lambda)^{n+1}\alpha\sigma \otimes_{g_4^{4n+5}} \gamma - \alpha \otimes_{g_2^{4n+5}} \alpha\gamma + \alpha \otimes_{g_3^{4n+5}} \beta\gamma - \alpha\gamma \otimes_{g_1^{4n+5}} \gamma\\
&\qquad +(-\lambda)^{n+1}\alpha\sigma \otimes_{g_4^{4n+5}} \gamma - \alpha\gamma\alpha \otimes_{g_2^{4n+5}} \mathfrak{t}(g_2^{4n+6}) + \alpha\sigma\beta \otimes_{g_2^{4n+5}} \mathfrak{t}(g_2^{4n+6})\\
&\quad = 0,
\end{aligned}$$

$$\begin{aligned}
&d^{4n+6} \cdot d^{4n+7}(\mathfrak{o}(g_2^{4n+7}) \otimes \mathfrak{t}(g_2^{4n+7}))\\
&\quad = [\mathfrak{o}(g_2^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\gamma - \mathfrak{o}(g_2^{4n+6}) \otimes_{g_3^{4n+5}} \beta\gamma + \gamma \otimes_{g_1^{4n+5}} \gamma - (-\lambda)^{n+1}\sigma \otimes_{g_4^{4n+5}} \gamma\\
&\qquad +(\gamma\alpha - \sigma\beta) \otimes_{g_2^{4n+5}} \mathfrak{t}(g_2^{4n+6})]\alpha - [\mathfrak{o}(g_3^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\sigma - \mathfrak{o}(g_3^{4n+6}) \otimes_{g_3^{4n+5}} \lambda\beta\sigma\\
&\qquad +\gamma \otimes_{g_1^{4n+5}} \sigma + (-\lambda)^{n+2}\sigma \otimes_{g_4^{4n+5}} \sigma + (\gamma\alpha - \lambda\sigma\beta) \otimes_{g_3^{4n+5}} \mathfrak{t}(g_3^{4n+6})]\beta\\
&\qquad -\gamma[\mathfrak{o}(g_1^{4n+6}) \otimes_{g_1^{4n+5}} (\gamma\alpha - \sigma\beta) + \alpha \otimes_{g_2^{4n+5}} \alpha - \alpha \otimes_{g_3^{4n+5}} \beta + \alpha\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_1^{4n+6})\\
&\qquad -(-\lambda)^{n+1}\alpha\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_1^{4n+6})] + (-\lambda)^{n+1}\sigma[\mathfrak{o}(g_4^{4n+6}) \otimes_{g_4^{4n+5}} (\gamma\alpha - \lambda\sigma\beta)\\
&\qquad +(-\lambda)^{-n}\beta \otimes_{g_3^{4n+5}} \beta + (-\lambda)^{-(n+1)}\beta \otimes_{g_2^{4n+5}} \alpha + (-\lambda)^{-(n+1)}\beta\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_4^{4n+6})\\
&\qquad -\lambda\beta\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_4^{4n+6})]\\
&\quad = \mathfrak{o}(g_2^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\gamma\alpha - \mathfrak{o}(g_2^{4n+6}) \otimes_{g_3^{4n+5}} \beta\gamma\alpha + \gamma \otimes_{g_1^{4n+5}} \gamma\alpha - (-\lambda)^{n+1}\sigma \otimes_{g_4^{4n+5}} \gamma\alpha\\
&\qquad +\gamma\alpha \otimes_{g_2^{4n+5}} \alpha - \sigma\beta \otimes_{g_2^{4n+5}} \alpha - \mathfrak{o}(g_3^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\sigma\beta + \mathfrak{o}(g_3^{4n+6}) \otimes_{g_3^{4n+5}} \lambda\beta\sigma\beta\\
&\qquad -\gamma \otimes_{g_1^{4n+5}} \sigma\beta - (-\lambda)^{n+2}\sigma \otimes_{g_4^{4n+5}} \sigma\beta - \gamma\alpha \otimes_{g_3^{4n+5}} \beta + \lambda\sigma\beta \otimes_{g_3^{4n+5}} \beta - \gamma \otimes_{g_1^{4n+5}} \gamma\alpha\\
&\qquad +\gamma \otimes_{g_1^{4n+5}} \sigma\beta - \gamma\alpha \otimes_{g_2^{4n+5}} \alpha + \gamma\alpha \otimes_{g_3^{4n+5}} \beta - \gamma\alpha\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_1^{4n+6})\\
&\qquad +(-\lambda)^{n+1}\gamma\alpha\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_1^{4n+6}) + (-\lambda)^{n+1}\sigma \otimes_{g_4^{4n+5}} \gamma\alpha + (-\lambda)^{n+2}\sigma \otimes_{g_4^{4n+5}} \sigma\beta\\
&\qquad -\lambda\sigma\beta \otimes_{g_3^{4n+5}} \beta + \sigma\beta \otimes_{g_2^{4n+5}} \alpha + \sigma\beta\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_4^{4n+6}) + (-\lambda)^{n+2}\sigma\beta\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_4^{4n+6}))\\
&\quad = 0,
\end{aligned}$$

and

$$\begin{aligned}
&d^{4n+6} \cdot d^{4n+7}(\mathfrak{o}(g_3^{4n+7}) \otimes \mathfrak{t}(g_3^{4n+7}))\\
&\quad = [\mathfrak{o}(g_4^{4n+6}) \otimes_{g_4^{4n+5}} (\gamma\alpha - \lambda\sigma\beta) + (-\lambda)^{-n}\beta \otimes_{g_3^{4n+5}} \beta + (-\lambda)^{-(n+1)}\beta \otimes_{g_2^{4n+5}} \alpha\\
&\qquad +(-\lambda)^{-(n+1)}\beta\gamma \otimes_{g_1^{4n+5}} \mathfrak{t}(g_4^{4n+6}) - \lambda\beta\sigma \otimes_{g_4^{4n+5}} \mathfrak{t}(g_4^{4n+6})]\sigma\\
&\qquad -(-\lambda)^{-(n+1)}\beta[\mathfrak{o}(g_3^{4n+6}) \otimes_{g_2^{4n+5}} \alpha\sigma - \mathfrak{o}(g_3^{4n+6}) \otimes_{g_3^{4n+5}} \lambda\beta\sigma + \gamma \otimes_{g_1^{4n+5}} \sigma\\
&\qquad +(-\lambda)^{n+2}\sigma \otimes_{g_4^{4n+5}} \sigma + (\gamma\alpha - \lambda\sigma\beta) \otimes_{g_3^{4n+5}} \mathfrak{t}(g_3^{4n+6})]\\
&\quad = \mathfrak{o}(g_4^{4n+6}) \otimes_{g_4^{4n+5}} \gamma\alpha\sigma - \mathfrak{o}(g_4^{4n+6}) \otimes_{g_4^{4n+5}} \lambda\sigma\beta\sigma + (-\lambda)^{-n}\beta \otimes_{g_3^{4n+5}} \beta\sigma\\
&\qquad +(-\lambda)^{-(n+1)}\beta \otimes_{g_2^{4n+5}} \alpha\sigma + (-\lambda)^{-(n+1)}\beta\gamma \otimes_{g_1^{4n+5}} \sigma - \lambda\beta\sigma \otimes_{g_4^{4n+5}} \sigma\\
&\qquad -(-\lambda)^{-(n+1)}\beta \otimes_{g_2^{4n+5}} \alpha\sigma - (-\lambda)^{-n}\beta \otimes_{g_3^{4n+5}} \beta\sigma - (-\lambda)^{-(n+1)}\beta\gamma \otimes_{g_1^{4n+5}} \sigma\\
&\qquad +\lambda\beta\sigma \otimes_{g_4^{4n+5}} \sigma - (-\lambda)^{-(n+1)}\beta\gamma\alpha \otimes_{g_3^{4n+5}} \mathfrak{t}(g_3^{4n+6}) - (-\lambda)^{-n}\beta\sigma\beta \otimes_{g_3^{4n+5}} \mathfrak{t}(g_3^{4n+6})\\
&\quad = 0.
\end{aligned}$$

Thus $d^{4n+6} \cdot d^{4n+7} = 0$.

Hence $d^m \cdot d^{m+1} = 0$ for all $m \geqslant 0$. $\square$

Theorem 21.5. *With the above notation (P^m, d^m) is a minimal projective Λ−Λ−bimodule resolution of $\Lambda = A_1(\lambda)$.*

Proof. From Theorem 21.4, (P^m, d^m) is a complex. Now to prove that (P^m, d^m) is a minimal projective $\Lambda - \Lambda-$bimodule resolution of Λ, we use an argument which was given in [16]. Note that $\Lambda/\mathfrak{r} \otimes_\Lambda P^m \cong \oplus_{g_i^m \in g^m} \mathfrak{t}(g_i^m)\Lambda = Q^m$ as a right $\Lambda-$modules and that the map $\mathrm{id} \otimes d^m : \Lambda/\mathfrak{r} \otimes_\Lambda P^m \to \Lambda/\mathfrak{r} \otimes_\Lambda P^{m-1}$ is equivalent to the map $\varepsilon^m : Q^m \to Q^{m-1}$ given in Theorem 21.2, that is, for $m = 4n+1$ we have

$$\begin{aligned}
&\mathfrak{t}(g_1^{4n+1}) \mapsto \mathfrak{t}(g_1^{4n})\alpha\\
&\mathfrak{t}(g_2^{4n+1}) \mapsto \mathfrak{t}(g_2^{4n})\gamma\\
&\mathfrak{t}(g_3^{4n+1}) \mapsto \mathfrak{t}(g_3^{4n})\sigma\\
&\mathfrak{t}(g_4^{4n+1}) \mapsto \mathfrak{t}(g_4^{4n})\beta;
\end{aligned}$$

for $m = 4n+2$ we have:

$$\begin{aligned}
&\mathfrak{t}(g_1^{4n+2}) \mapsto \mathfrak{t}(g_1^{4n+1})(\gamma\alpha - \sigma\beta)\\
&\mathfrak{t}(g_2^{4n+2}) \mapsto \mathfrak{t}(g_2^{4n+1})\alpha\gamma - \mathfrak{t}(g_3^{4n+1})\beta\gamma\\
&\mathfrak{t}(g_3^{4n+2}) \mapsto \mathfrak{t}(g_2^{4n+1})\alpha\sigma - \mathfrak{t}(g_3^{4n+1})\lambda\beta\sigma\\
&\mathfrak{t}(g_4^{4n+2}) \mapsto \mathfrak{t}(g_4^{4n+1})(\gamma\alpha - \lambda\sigma\beta);
\end{aligned}$$

for $m = 4n+3$ we have:

$$\begin{aligned}
&\mathfrak{t}(g_1^{4n+3}) \mapsto \mathfrak{t}(g_1^{4n+2})\gamma\\
&\mathfrak{t}(g_2^{4n+3}) \mapsto \mathfrak{t}(g_2^{4n+2})\alpha - \mathfrak{t}(g_3^{4n+2})\beta\\
&\mathfrak{t}(g_3^{4n+3}) \mapsto \mathfrak{t}(g_4^{4n+2})\sigma;
\end{aligned}$$

and for $m = 4n+4$ we have:

$$\begin{aligned}
&\mathfrak{t}(g_1^{4n+4}) \mapsto \mathfrak{t}(g_1^{4n+3})\alpha\gamma\alpha\gamma\\
&\mathfrak{t}(g_2^{4n+4}) \mapsto \mathfrak{t}(g_2^{4n+3})\gamma\alpha\gamma\alpha\\
&\mathfrak{t}(g_3^{4n+4}) \mapsto \mathfrak{t}(g_3^{4n+3})\beta\sigma\beta\sigma.
\end{aligned}$$

From Theorem 21.2 $(\Lambda/\mathfrak{r} \otimes_\Lambda P^m, \mathrm{id} \otimes_\Lambda d^m)$ is a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right $\Lambda-$module. So from [16, Proposition 2.8] (and see [27, Theorem 1.6]) it follows that (P^m, d^m) is a minimal projective $\Lambda - \Lambda-$bimodule resolution of Λ. $\square$

We recall that $\lambda \in K \setminus \{0, 1\}$. Now for $n \geqslant 0$, if $(-\lambda)^n = 1$ then

$$d^{4n+1} = d^1, d^{4n+2} = d^2, d^{4n+3} = d^3, d^{4n+4} = d^4.$$

Theorem 21.6. *Let $\Lambda = A_1(\lambda)$ where $\lambda \in K \setminus \{0, 1\}$. If there exists some $n \geqslant 1$ such that $(-\lambda)^n = 1$ then $\Omega^{4n}_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules. Moreover Λ has a periodic projective $\Lambda - \Lambda-$bimodule resolution.*

Proof. The map d^{4n+1} is equal to d^1 so $\mathrm{Im}\, d^{4n+1} = \mathrm{Im}\, d^1$. Hence $\Omega^{4n+1}_{\Lambda^e}(\Lambda) \cong \Omega_{\Lambda^e}(\Lambda)$. But Λ is indecomposable and selfinjective so $\Lambda \cong \Omega^{4n}_{\Lambda^e}(\Lambda)$. Thus Λ is periodic of period dividing $4n$. $\square$

The Hochschild cohomology ring modulo nilpotence of a periodic algebra was determined by Green, Snashall and Solberg in [17]. Recall that the Hochschild cohomology ring is $\mathrm{HH}^*(\Lambda) = \oplus_{i \geqslant 0} \mathrm{HH}^i(\Lambda) = \oplus_{i \geqslant 0} \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$ with the Yoneda product. We let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(\Lambda)$ generated by the homogeneous nilpotent elements. The proof of the following theorem is immediate from [17, Theorem 1.6].

Theorem 21.7. *For $\Lambda = A_1(\lambda)$ we have that $\mathrm{HH}^*(\Lambda)/\mathcal{N} = K$ or $K[x]$. If there is $n \geqslant 1$ with $(-\lambda)^n = 1$ then $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x]$ where x is in degree m, and m is minimal such that $\Omega^m_{\Lambda^e}(\Lambda) \cong \Lambda$ as bimodules. In this case m divides $4n$.*

We remark that it is not yet known whether $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is isomorphic to K or $K[x]$ in the case where is no $n \geqslant 1$ with $(-\lambda)^n = 1$. It would be interesting to study this question further.

We recall from Chapter 6 that $A_2(\lambda)$ is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\sigma}{\rightleftarrows}} 2 \circlearrowleft \beta$$

and

$$I = \langle \alpha^2 - \sigma\gamma, \lambda\beta^2 - \gamma\sigma, \gamma\alpha - \beta\gamma, \sigma\beta - \alpha\sigma \rangle$$

where $\lambda \in K \setminus \{0, 1\}$. Write Λ for $A_2(\lambda)$ and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are:

$e_1\Lambda = sp\{e_1, \alpha, \sigma, \sigma\beta, \alpha^2, \alpha^3\}$ and $e_2\Lambda = sp\{e_2, \gamma, \beta, \gamma\alpha, \beta^2, \beta^3\}$.

From 6.2 we have the beginning of the projective resolution of the simple $A_2(\lambda)-$modules as follows.

The minimal projective resolution of S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$$\begin{aligned}
&\partial^1 : (e_1\zeta, e_2\eta) \mapsto \alpha e_1\zeta + \sigma e_2\eta,\\
&\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (-\sigma, \beta)e_2\eta,\\
&\partial^3 : e_1\zeta \mapsto (\alpha, \gamma)e_1\zeta,
\end{aligned}$$

for $\zeta, \eta \in \Lambda$.

The minimal projective resolution of S_2 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^3} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$$\begin{aligned}
&\partial^1 : (e_1\zeta, e_2\eta) \mapsto \gamma e_1\zeta + \beta e_2\eta,\\
&\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\gamma)e_1\zeta + (\sigma, -\lambda\beta)e_2\eta,\\
&\partial^3 : e_2\eta \mapsto (\sigma, -\beta)e_2\eta,
\end{aligned}$$

for $\zeta, \eta \in \Lambda$.

22.1. The periodicity of S_1.

Now we want to find $\operatorname{Ker}\partial^3 = \Omega^4(S_1)$. Let $e_1\zeta = c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker}\partial^3$ then $(\alpha, \gamma)e_1\zeta = (0, 0)$ so $(\alpha, \gamma)(c_1e_1 + c_2\alpha + c_3\sigma + c_4\sigma\beta + c_5\alpha^2 + c_6\alpha^3) = (0, 0)$. Then $(c_1\alpha + c_2\alpha^2 + c_3\alpha\sigma + c_5\alpha^3, c_1\gamma + c_2\gamma\alpha + c_3\gamma\sigma + c_4\gamma\sigma\beta) = (0, 0)$, that is, $c_1\alpha + c_2\alpha^2 + c_3\alpha\sigma + c_5\alpha^3 = 0$ so $c_1 = c_2 = c_3 = c_5 = 0$, and $c_1\gamma + c_2\gamma\alpha + c_3\gamma\sigma + c_4\gamma\sigma\beta = 0$ so $c_1 = c_2 = c_3 = c_4 = 0$. Thus $e_1\zeta = c_6\alpha^3$.

Hence $\operatorname{Ker}\partial^3 = \{c_6\alpha^3 : c_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \alpha^3 e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_6\alpha^3$, that is, $x = \alpha^3(c_6e_1)$. Thus $x \in \alpha^3e_1\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \alpha^3e_1\Lambda$.

On the other hand, let $y = \alpha^3 e_1\zeta \in \alpha^3 e_1\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = (\alpha, \gamma)(\alpha^3 e_1\zeta) = (\alpha^4, \gamma\alpha^3)e_1\zeta = (0, 0)$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\alpha^3 e_1\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \alpha^3 e_1\Lambda$. □

So the map $\partial^4 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\zeta \mapsto \alpha^3 e_1\zeta$, for $\zeta \in \Lambda$.
Note that $\operatorname{Ker}\partial^3 \cong S_1$ and so $\Omega^4(S_1) \cong S_1$.

22.2. **The periodicity of S_2.**
To find $\operatorname{Ker}\partial^3 = \Omega^4(S_2)$, let $e_2\eta = c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3$ with $c_i \in K$. Assume that $e_2\eta \in \operatorname{Ker}\partial^3$ then $(\sigma, -\beta)e_2\eta = (0, 0)$ so $(\sigma, -\beta)(c_7e_2 + c_8\gamma + c_9\beta + c_{10}\gamma\alpha + c_{11}\beta^2 + c_{12}\beta^3) = (c_7\sigma + c_8\sigma\gamma + c_9\sigma\beta + c_{10}\sigma\gamma\alpha, -c_7\beta - c_8\beta\gamma - c_9\beta^2 - c_{11}\beta^3) = (0, 0)$. Therefore $c_7\sigma + c_8\sigma\gamma + c_9\sigma\beta + c_{10}\sigma\gamma\alpha = 0$, that is, $c_7 = c_8 = c_9 = c_{10} = 0$. Also $-c_7\beta - c_8\beta\gamma - c_9\beta^2 - c_{11}\beta^3 = 0$, that is, $c_7 = c_8 = c_9 = c_{11} = 0$. Thus $e_2\eta = c_{12}\beta^3$.

Hence $\operatorname{Ker}\partial^3 = \{c_{12}\beta^3 : c_{12} \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \beta^3 e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_{12}\beta^3$, that is, $x = \beta^3(c_{12}e_2)$. Thus $x \in \beta^3 e_2\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \beta^3 e_2\Lambda$.

On the other hand, let $y = \beta^3 e_2\eta \in \beta^3 e_2\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = (\sigma, -\beta)\beta^3 e_2\eta = (\sigma\beta^3, -\beta^4)e_2\eta = (0, 0)$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\beta^3 e_2\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \beta^3 e_2\Lambda$. □

So $\partial^4 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\eta \mapsto \beta^3 e_2\eta$, for $\eta \in \Lambda$.
Note that $\operatorname{Ker}\partial^3 \cong S_2$ and so $\Omega^4(S_2) \cong S_2$.

We summarize this in the following Theorem.

Theorem 22.1. *For the algebra $A_2(\lambda)$, we have $\Omega^4(S_1) \cong S_1$ and $\Omega^4(S_2) \cong S_2$. Hence $\Omega^4(S_i) \cong S_i$ for all i.*

23. Periodicity of the simple A_5−modules

We recall from chapter 9 that A_5 is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\gamma"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\alpha"] \end{tikzcd}$$

and

$$I = \langle \gamma^2 - \beta\alpha, \alpha\gamma\beta \rangle.$$

Write Λ for A_5 and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are:

$e_1\Lambda = sp\{e_1, \gamma, \beta, \gamma\beta, \gamma^2, \gamma^2\beta, \gamma^3, \gamma^4\}$ and

$e_2\Lambda = sp\{e_2, \alpha, \alpha\gamma, \alpha\beta, \alpha\gamma^2, \alpha\beta\alpha\beta\}$.

Derived equivalence class of A_5 is $\{A_5, A_6\}$. We consider A_5 in this chapter and A_6 in the next chapter. Although it is known from [4, Lemma 2.1] that the simple modules are periodic and that they are derived equivalent, we show that the periodicity of the simples are not the same for these two algebras.

Recall from Section 9.2 the minimal projective resolution of S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \gamma e_1\zeta + \beta e_2\eta,$

$\partial^2 : e_1\zeta \mapsto (\gamma, -\alpha)e_1\zeta,$

$\partial^3 : e_1\zeta \mapsto \gamma^4 e_1\zeta,$

for $\zeta, \eta \in \Lambda$.

The minimal projective resolution of S_2 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^3} e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$\partial^1 : e_1\zeta \mapsto \alpha e_1\zeta,$

$\partial^2 : e_2\eta \mapsto \gamma\beta e_2\eta,$

$\partial^3 : e_2\eta \mapsto \alpha\beta e_2\eta,$

for $\zeta, \eta \in \Lambda$.

Also recall from 9.4.1 that $\operatorname{Ker}\partial^2 \cong S_1$ so that $\Omega^3(S_1) \cong S_1$.

23.1. The periodicity of S_2.

23.1.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_2)$, let $e_2\eta = d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$ with $d_i \in K$. Assume that $e_2\eta \in \operatorname{Ker}\partial^3$ then $\alpha\beta e_2\eta = 0$ so $\alpha\beta(d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta) = 0$, that is, $d_1\alpha\beta + d_2\alpha\beta\alpha + d_4\alpha\beta\alpha\beta = 0$, so that $d_1 = d_2 = d_4 = 0$. Thus $e_2\eta = d_3\alpha\gamma + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker}\partial^3 = \{d_3\alpha\gamma + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta : d_i \in K\}$.

Claim. $\operatorname{Ker} \partial^3 = \alpha\gamma e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^3$; then $x = d_3\alpha\gamma + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$, that is, $x = \alpha\gamma(d_3e_1 + d_5\gamma + d_6\gamma\beta)$. Thus $x \in \alpha\gamma e_1\Lambda$ and therefore $\operatorname{Ker} \partial^3 \subseteq \alpha\gamma e_1\Lambda$.

On the other hand, let $y = \alpha\gamma e_1\zeta \in \alpha\gamma e_1\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \alpha\beta(\alpha\gamma)e_1\zeta = \alpha\beta\alpha\gamma e_1\zeta = 0$. Therefore $y \in \operatorname{Ker} \partial^3$ and so $\alpha\gamma e_1\Lambda \subseteq \operatorname{Ker} \partial^3$.

Hence $\operatorname{Ker} \partial^3 = \alpha\gamma e_1\Lambda$. □

So $\partial^4 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\zeta \mapsto \alpha\gamma e_1\zeta$, for $\zeta \in \Lambda$.

23.1.2. $\operatorname{Ker} \partial^4$.

To find $\operatorname{Ker} \partial^4 = \Omega^5(S_2)$, let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker} \partial^4$. Then $\alpha\gamma e_1\zeta = 0$ so $\alpha\gamma(c_1e_1 + c_2\gamma + c_3\beta + c_4\gamma\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4) = c_1\alpha\gamma + c_2\alpha\gamma^2 + c_4\alpha\gamma^2\beta = 0$. So $c_1 = c_2 = c_4 = 0$. Thus $e_1\zeta = c_3\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$.

Hence $\operatorname{Ker} \partial^4 = \{c_3\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4 : c_i \in K\}$.

Claim. $\operatorname{Ker} \partial^4 = \beta e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^4$; then $x = c_3\beta + c_5\gamma^2 + c_6\gamma^2\beta + c_7\gamma^3 + c_8\gamma^4$, that is, $x = \beta(c_3e_2 + c_5\alpha + c_6\alpha\beta + c_7\alpha\gamma + c_8\alpha\gamma^2)$. Thus $x \in \beta e_2\Lambda$ and therefore $\operatorname{Ker} \partial^4 \subseteq \beta e_2\Lambda$.

On the other hand, let $y = \beta e_2\eta \in \beta e_2\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(y) = \alpha\gamma\beta e_2\eta = 0$. Therefore $y \in \operatorname{Ker} \partial^4$ and so $\beta e_2\Lambda \subseteq \operatorname{Ker} \partial^4$.

Hence $\operatorname{Ker} \partial^4 = \beta e_2\Lambda$. □

So $\partial^5 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\eta \mapsto \beta e_2\eta$, for $\eta \in \Lambda$.

23.1.3. $\operatorname{Ker} \partial^5$.

To find $\operatorname{Ker} \partial^5 = \Omega^6(S_2)$, let $e_2\eta = d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta$ with $d_i \in K$. Assume that $e_2\eta \in \operatorname{Ker} \partial^5$ then $\beta e_2\eta = 0$ so $\beta(d_1e_2 + d_2\alpha + d_3\alpha\gamma + d_4\alpha\beta + d_5\alpha\gamma^2 + d_6\alpha\beta\alpha\beta) = 0$, that is, $d_1\beta + d_2\beta\alpha + d_3\beta\alpha\gamma + d_4\beta\alpha\beta + d_5\beta\alpha\gamma^2 = 0$, so that $d_1 = d_2 = d_3 = d_4 = d_5 = 0$. Thus $e_2\eta = d_6\alpha\beta\alpha\beta$. Hence $\operatorname{Ker} \partial^5 = \{d_6\alpha\beta\alpha\beta : d_6 \in K\}$.

Claim. $\operatorname{Ker} \partial^5 = \alpha\beta\alpha\beta e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker} \partial^5$; then $x = d_6\alpha\beta\alpha\beta$, that is, $x = \alpha\beta\alpha\beta(d_6e_2)$. Thus $x \in \alpha\beta\alpha\beta e_2\Lambda$ and therefore $\operatorname{Ker} \partial^5 \subseteq \alpha\beta\alpha\beta e_2\Lambda$.

On the other hand, let $y = \alpha\beta\alpha\beta e_2\eta \in \alpha\beta\alpha\beta e_2\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = \beta(\alpha\beta\alpha\beta e_2\eta) = \beta\alpha\beta\alpha\beta e_2\eta = 0$. Therefore $y \in \operatorname{Ker} \partial^5$ and so $\alpha\beta\alpha\beta e_2\Lambda \subseteq \operatorname{Ker} \partial^5$. Hence $\operatorname{Ker} \partial^5 = \alpha\beta\alpha\beta e_2\Lambda$. □

So $\partial^6 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\eta \mapsto \alpha\beta\alpha\beta e_2\eta$, for $\eta \in \Lambda$. Note that $\operatorname{Ker} \partial^5 \cong S_2$ and so $\Omega^6(S_2) \cong S_2$.

We summarize this in the following Theorem.

Theorem 23.1. *For the algebra A_5, we have $\Omega^3(S_1) \cong S_1$ and $\Omega^6(S_2) \cong S_2$.*

24. Periodicity of the simple A_6-modules

We recall from Chapter 10 that A_6 is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\alpha \circlearrowright 1 \underset{\beta}{\overset{\gamma}{\rightleftarrows}} 2$$

and

$$I = \langle \alpha^3 - \gamma\beta, \beta\gamma, \beta\alpha^2, \alpha^2\gamma \rangle.$$

Write Λ for A_6 and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are:

$e_1\Lambda = sp\{e_1, \alpha, \gamma, \alpha\gamma, \alpha^2, \alpha^3, \alpha^4\}$ and

$e_2\Lambda = sp\{e_2, \beta, \beta\alpha, \beta\alpha\gamma\}$.

Recall from Section 10.2 the minimal projective resolution of S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^3} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$\partial^1 : (e_1\zeta, e_2\eta) \mapsto \alpha e_1\zeta + \gamma e_2\eta,$

$\partial^2 : (e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\alpha\gamma, 0)e_2\eta,$

$\partial^3 : (e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta,$

for $\zeta, \eta \in \Lambda$.

The minimal projective resolution of S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^3} e_1\Lambda \oplus e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$\partial^1 : e_1\zeta \mapsto \beta e_1\zeta,$

$\partial^2 : (e_1\zeta, e_2\eta) \mapsto \alpha^2 e_1\zeta + \gamma e_2\eta,$

$\partial^3 : (e_1\zeta, e_2\eta) \mapsto (\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta,$

for $\zeta, \eta \in \Lambda$.

24.1. **The periodicity of S_1.**

24.1.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_1)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^3$. Then $(\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta = (0, 0)$. So $(\alpha^2, -\beta)e_1\zeta + (\gamma, 0)e_2\eta = (\alpha^2, -\beta)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + (\gamma, 0)(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) + (d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, 0) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0, 0)$. So $(c_1\alpha^2 + (c_3 + d_2)\alpha^3 + (c_4 + d_3)\alpha^4 + d_1\gamma, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0, 0)$. Therefore $c_1\alpha^2 + (c_3 + d_2)\alpha^3 + (c_4 + d_3)\alpha^4 + d_1\gamma = 0$, that is, $c_1 = d_1 = 0, d_2 = -c_3, d_3 = -c_4$. Also $-c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma$, that is, $c_1 = c_3 = c_7 = 0$. Hence $e_1\zeta = c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4$ and $e_2\eta = -c_4\beta\alpha + d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^3 = \{(c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_4 \in K\}$.

Claim. $\operatorname{Ker} \partial^3 = (\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^3$. Then $u = (c_2\gamma + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta\alpha + d_4\beta\alpha\gamma)$ so $u = (\alpha^2, -\beta\alpha)(c_4e_1 + c_5\alpha + c_6\alpha^2 - d_4\gamma) + (\gamma, 0)(c_2e_2)$. So $u \in (\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda$ and therefore $\operatorname{Ker} \partial^3 \subseteq (\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda$.

On the other hand, let $v = (\alpha^2, -\beta\alpha)e_1\zeta + (\gamma, 0)e_2\eta \in (\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(v) = \partial^3((\alpha^2, -\beta\alpha)e_1\zeta + (\gamma, 0)e_2\eta) = (\alpha^2, -\beta)(\alpha^2 e_1\zeta + \gamma e_2\eta) + (\gamma, 0)(-\beta\alpha e_1\zeta) = (\alpha^4 e_1\zeta + \alpha^2\gamma e_2\eta - \gamma\beta\alpha e_1\zeta, -\beta\alpha^2 e_1\zeta - \beta\gamma e_2\eta) = (0, 0)$. Therefore $(\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda \subseteq \operatorname{Ker} \partial^3$.

Hence $\operatorname{Ker} \partial^3 = (\alpha^2, -\beta\alpha)e_1\Lambda + (\gamma, 0)e_2\Lambda$. □

So the map $\partial^4 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $(e_1\zeta, e_2\eta) \mapsto (\alpha^2, -\beta\alpha)e_1\zeta + (\gamma, 0)e_2\eta$, for $\zeta, \eta \in \Lambda$.

24.1.2. $\operatorname{Ker} \partial^4$.

To find $\operatorname{Ker} \partial^4 = \Omega^5(S_1)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker} \partial^4$. Then $(\alpha^2, -\beta\alpha)e_1\zeta + (\gamma, 0)e_2\eta = (0, 0)$. So $(\alpha^2, -\beta\alpha)e_1\zeta + (\gamma, 0)e_2\eta = (\alpha^2, -\beta\alpha)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + (\gamma, 0)(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4, -c_1\beta\alpha - c_2\beta\alpha\gamma) + (d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, 0) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, -c_1\beta\alpha - c_2\beta\alpha\gamma) = (0, 0)$. So $(c_1\alpha^2 + (c_3 + d_2)\alpha^3 + (c_4 + d_3)\alpha^4 + d_1\gamma, -c_1\beta\alpha - c_2\beta\alpha\gamma) = (0, 0)$. Thus $c_1\alpha^2 + (c_3 + d_2)\alpha^3 + (c_4 + d_3)\alpha^4 + d_1\gamma = 0$, that is, $c_1 = d_1 = 0, d_2 = -c_3$ and $d_3 = -c_4$. Also $-c_1\beta\alpha - c_2\beta\alpha\gamma = 0$, that is, $c_1 = c_2 = 0$. Hence $e_1\zeta = c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker} \partial^4 = \{(c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_4 \in K\}$.

Claim. $\operatorname{Ker} \partial^4 = (\alpha, -\beta)e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker} \partial^4$. Then $u = (c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma, -c_3\beta - c_4\beta\alpha + d_4\beta\alpha\gamma)$ so $u = (\alpha, -\beta)(c_3e_1 + c_4\alpha + c_5\alpha^2 + c_6\alpha^3 + c_7\gamma - d_4\alpha\gamma)$. So $u \in (\alpha, -\beta)e_1\Lambda$ and therefore $\operatorname{Ker} \partial^4 \subseteq (\alpha, -\beta)e_1\Lambda$.

On the other hand, let $v = (\alpha, -\beta)e_1\zeta \in (\alpha, -\beta)e_1\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(v) = \partial^4((\alpha, -\beta)e_1\zeta) = ((\alpha^2, -\beta\alpha)\alpha + (\gamma, 0)(-\beta))e_1\zeta = (\alpha^3, -\beta\alpha^2)e_1\zeta + (-\gamma\beta, 0)e_1\zeta = (\alpha^3 - \gamma\beta, -\beta\alpha^2)e_1\zeta = (0, 0)$. Therefore $(\alpha, -\beta)e_1\Lambda \subseteq \operatorname{Ker} \partial^4$.

Hence $\operatorname{Ker} \partial^4 = (\alpha, -\beta)e_1\Lambda$. □

So the map $\partial^5 : e_1\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_1\zeta \mapsto (\alpha, -\beta)e_1\zeta$, for $\zeta \in \Lambda$.

24.1.3. $\operatorname{Ker} \partial^5$.

To find $\operatorname{Ker} \partial^5 = \Omega^6(S_1)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker} \partial^5$ then $(\alpha, -\beta)e_1\zeta = (0, 0)$. So $(\alpha, -\beta)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) = (c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + c_4\alpha^3 + c_5\alpha^4, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0, 0)$. So $c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + c_4\alpha^3 + c_5\alpha^4 = 0$, that is, $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Also $-c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma = 0$, that is, $c_1 = c_3 = c_7 = 0$. Thus $e_1\zeta = c_6\alpha^4$.

Hence $\operatorname{Ker}\partial^5 = \{c_6\alpha^4 : c_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \alpha^4 e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = c_6\alpha^4$, that is, $x = \alpha^4(c_6e_1)$. Thus $x \in \alpha^4 e_1\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \alpha^4 e_1\Lambda$.

On the other hand, let $y = \alpha^4 e_1\zeta \in \alpha^4 e_1\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = (\alpha, -\beta)(\alpha^4 e_1\zeta) = (\alpha^5, -\beta\alpha^4)e_1\zeta = (0,0)$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\alpha^4 e_1\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \alpha^4 e_1\Lambda$. □

So $\partial^6 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\zeta \mapsto \alpha^4 e_1\zeta$, for $\zeta \in \Lambda$.

Hence $\operatorname{Ker}\partial^5 \cong S_1$ and so $\Omega^6(S_1) \cong S_1$.

24.2. **The periodicity of S_2.**

24.2.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_2)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ and $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $c_i, d_i \in K$. Assume that $(e_1\zeta, e_2\eta) \in \operatorname{Ker}\partial^3$. Then $(\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta = (0,0)$. So $(\alpha, -\beta)e_1\zeta + (\gamma, 0)e_2\eta = (\alpha, -\beta)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) + (\gamma, 0)(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = (c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + c_4\alpha^3 + c_5\alpha^4, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) + (d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, 0) = (c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + c_4\alpha^3 + c_5\alpha^4 + d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + (c_4+d_2)\alpha^3 + (c_5+d_3)\alpha^4 + d_1\gamma, -c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma) = (0,0)$. So $c_1\alpha + c_2\alpha\gamma + c_3\alpha^2 + (c_4+d_2)\alpha^3 + (c_5+d_3)\alpha^4 + d_1\gamma = 0$, that is, $c_1 = d_1 = c_2 = c_3 = 0, d_2 = -c_4$ and $d_3 = -c_5$. Also $-c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma = 0$, that is, $c_1 = c_3 = c_7 = 0$. Hence $e_1\zeta = c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4$ and $e_2\eta = -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^3 = \{(c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma) : c_i, d_4 \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = (\alpha^2, -\beta)e_1\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^3$. Then $u = (c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4, -c_4\beta - c_5\beta\alpha + d_4\beta\alpha\gamma)$ so $u = (\alpha^2, -\beta)(c_4e_1 + c_5\alpha + c_6\alpha^2 - d_4\alpha\gamma)$. So $u \in (\alpha^2, -\beta)e_1\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq (\alpha^2, -\beta)e_1\Lambda$.

On the other hand, let $v = (\alpha^2, -\beta)e_1\zeta \in (\alpha^2, -\beta)e_1\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(v) = \partial^3((\alpha^2, -\beta)e_1\zeta) = (((\alpha, -\beta)\alpha^2 + (\gamma, 0)(-\beta))e_1\zeta) = ((\alpha^3, -\beta\alpha^2) + (-\gamma\beta, 0))e_1\zeta = (\alpha^3 - \gamma\beta, -\beta\alpha^2)e_1\zeta = (0,0)$. Therefore $(\alpha^2, -\beta)e_1\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = (\alpha^2, -\beta)e_1\Lambda$. □

So the map $\partial^4 : e_1\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_1\zeta \mapsto (\alpha^2, -\beta)e_1\zeta$, for $\zeta \in \Lambda$.

24.2.2. $\operatorname{Ker}\partial^4$.

To find $\operatorname{Ker}\partial^4 = \Omega^5(S_2)$. Let $e_1\zeta = c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma$ with $c_i \in K$. Assume that $e_1\zeta \in \operatorname{Ker}\partial^4$. Then $(\alpha^2, -\beta)e_1\zeta = 0$. So $(\alpha^2, -\beta)e_1\zeta = (\alpha^2, -\beta)(c_1e_1 + c_2\gamma + c_3\alpha + c_4\alpha^2 + c_5\alpha^3 + c_6\alpha^4 + c_7\alpha\gamma) = (c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4, -c_1\beta -$

$c_3\beta\alpha - c_7\beta\alpha\gamma) = (0,0)$. So $c_1\alpha^2 + c_3\alpha^3 + c_4\alpha^4 = 0$. Thus $c_1 = c_3 = c_4 = 0$. Also $-c_1\beta - c_3\beta\alpha - c_7\beta\alpha\gamma = 0$, that is, $c_1 = c_3 = c_7 = 0$. Hence $e_1\zeta = c_2\gamma + c_5\alpha^3 + c_6\alpha^4$. Therefore $\operatorname{Ker}\partial^4 = \{c_2\gamma + c_5\alpha^3 + c_6\alpha^4 : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^4 = \gamma e_2\Lambda$.

Proof. Let $u \in \operatorname{Ker}\partial^4$. Then $u = c_2\gamma + c_5\alpha^3 + c_6\alpha^4$ so $u = \gamma(c_2e_2 + c_5\beta + c_6\beta\alpha)$. So $u \in \gamma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^4 \subseteq \gamma e_2\Lambda$.

On the other hand, let $v = \gamma e_2\eta \in \gamma e_2\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(v) = \partial^4(\gamma e_2\eta) = (\alpha^2, -\beta)\gamma e_2\eta = (\alpha^2\gamma, -\beta\gamma)e_2\eta = (0,0)$. Therefore $\gamma e_2\Lambda \subseteq \operatorname{Ker}\partial^4$.

Hence $\operatorname{Ker}\partial^4 = \gamma e_2\Lambda$. □

So the map $\partial^5 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\eta \mapsto \gamma e_2\eta$, for $\eta \in \Lambda$.

24.2.3. $\operatorname{Ker}\partial^5$.

To find $\operatorname{Ker}\partial^5 = \Omega^6(S_2)$. Let $e_2\eta = d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma$ with $d_i \in K$. Assume that $e_2\eta \in \operatorname{Ker}\partial^5$. Then $\gamma e_2\eta = 0$. So $\gamma e_2\eta = \gamma(d_1e_2 + d_2\beta + d_3\beta\alpha + d_4\beta\alpha\gamma) = d_1\gamma + d_2\gamma\beta + d_3\gamma\beta\alpha = 0$. Thus $d_1 = d_2 = d_3 = 0$. Hence $e_2\eta = d_4\beta\alpha\gamma$. Therefore $\operatorname{Ker}\partial^5 = \{d_4\beta\alpha\gamma : d_4 \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \beta\alpha\gamma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = d_4\beta\alpha\gamma$, that is, $x = \beta\alpha\gamma(d_4e_2)$. Thus $x \in \beta\alpha\gamma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \beta\alpha\gamma e_2\Lambda$.

On the other hand, let $y = \beta\alpha\gamma e_2\eta \in \beta\alpha\gamma e_2\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = \gamma(\beta\alpha\gamma e_2\eta) = \gamma\beta\alpha\gamma e_2\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\beta\alpha\gamma e_2\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \beta\alpha\gamma e_2\Lambda$. □

So $\partial^6 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\eta \mapsto \beta\alpha\gamma e_2\eta$, for $\eta \in \Lambda$.

Hence $\operatorname{Ker}\partial^5 \cong S_2$ and so $\Omega^6(S_2) \cong S_2$.

We summarize this in the following Theorem.

Theorem 24.1. *For the algebra A_6, we have $\Omega^6(S_1) \cong S_1$ and $\Omega^6(S_2) \cong S_2$. Hence $\Omega^6(S_i) \cong S_i$ for all $i = 1, 2$.*

We remark that the algebras A_5 and A_6 are derived equivalent, but nevertheless the simples do not have the same periodicity. Thus the period of a periodic simple module is not invariant under derived equivalence.

25. Periodicity of the simple A_7−modules

The algebras $A_4, A_7, A_8, A_9, A_{10}, A_{11}$ all lie in the same derived equivalence class. We give the periodicity of the simple A_7−modules, and do not discuss the periodicity for $A_4, A_8, A_9, A_{10}, A_{11}$.

We recall from Chapter 11 that A_7 is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} 3 \underset{\zeta}{\overset{\varepsilon}{\rightleftarrows}} 4$$

and

$$I = \langle \beta\alpha - \delta\gamma, \gamma\delta - \varepsilon\zeta, \alpha\delta\varepsilon, \zeta\gamma\beta \rangle.$$

Write Λ for A_7 and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are:

$e_1\Lambda = sp\{e_1, \alpha, \alpha\beta, \alpha\delta, \alpha\beta\alpha, \alpha\beta\alpha\beta\}$,
$e_2\Lambda = sp\{e_2, \beta, \delta, \beta\alpha, \delta\varepsilon, \beta\alpha\beta, \delta\varepsilon\zeta, \beta\alpha\beta\alpha\}$,
$e_3\Lambda = sp\{e_3, \gamma, \varepsilon, \gamma\beta, \varepsilon\zeta, \gamma\beta\alpha, \varepsilon\zeta\varepsilon, \varepsilon\zeta\varepsilon\zeta\}$,
$e_4\Lambda = sp\{e_4, \zeta, \zeta\gamma, \zeta\varepsilon, \zeta\varepsilon\zeta, \zeta\varepsilon\zeta\varepsilon\}$.

Recall from Section 11.2 the minimal projective resolution of S_1 starts with:

$$\cdots \longrightarrow e_4\Lambda \xrightarrow{\partial^3} e_4\Lambda \xrightarrow{\partial^2} e_2\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$\partial^1 : e_2\nu \mapsto \alpha e_2\nu$,
$\partial^2 : e_4\mu \mapsto \delta\varepsilon e_4\mu$,
$\partial^3 : e_4\mu \mapsto \zeta\varepsilon e_4\mu$,

for $\nu, \mu \in \Lambda$.

The minimal projective resolution of S_2 starts with:

$$\cdots \longrightarrow e_2\Lambda \xrightarrow{\partial^3} e_2\Lambda \xrightarrow{\partial^2} e_1\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$\partial^1 : (e_1\eta, e_3\lambda) \mapsto \beta e_1\eta + \delta e_3\lambda$,
$\partial^2 : e_2\nu \mapsto (\alpha, -\gamma)e_2\nu$,
$\partial^3 : e_2\nu \mapsto \beta\alpha\beta\alpha e_2\nu$,

for $\eta, \nu, \lambda \in \Lambda$.

The minimal projective resolution of S_3 starts with:

$$\cdots \longrightarrow e_3\Lambda \xrightarrow{\partial^3} e_3\Lambda \xrightarrow{\partial^2} e_2\Lambda \oplus e_4\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where

$\partial^1 : (e_2\nu, e_4\mu) \mapsto \gamma e_2\nu + \varepsilon e_4\mu$,
$\partial^2 : e_3\lambda \mapsto (\delta, -\zeta)e_3\lambda$,
$\partial^3 : e_3\lambda \mapsto \varepsilon\zeta\varepsilon\zeta e_3\lambda$,

for $\eta, \mu, \lambda \in \Lambda$.

The minimal projective resolution of S_4 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_3\Lambda \xrightarrow{\partial^1} e_4\Lambda \longrightarrow S_4 \longrightarrow 0$$

where

$$\begin{aligned}
&\partial^1 : e_3\lambda \mapsto \zeta e_3\lambda,\\
&\partial^2 : e_1\eta \mapsto \gamma\beta e_1\eta,\\
&\partial^3 : e_1\eta \mapsto \alpha\beta e_1\eta,
\end{aligned}$$

for $\eta, \lambda \in \Lambda$.

Also recall from 11.4.1 that $\operatorname{Ker}\partial^2 \cong S_2$ so that $\Omega^3(S_2) \cong S_2$ and from 11.4.2 that $\operatorname{Ker}\partial^2 \cong S_3$ so that $\Omega^3(S_3) \cong S_3$.

25.1. **The periodicity of S_1.**

25.1.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_1)$. Let $e_4\mu = t_1e_4+t_2\zeta+t_3\zeta\gamma+t_4\zeta\varepsilon+t_5\zeta\varepsilon\zeta+t_6\zeta\varepsilon\zeta\varepsilon$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^3$ then $\zeta\varepsilon e_4\mu = 0$ so $\zeta\varepsilon(t_1e_4+t_2\zeta+t_3\zeta\gamma+t_4\zeta\varepsilon+t_5\zeta\varepsilon\zeta+t_6\zeta\varepsilon\zeta\varepsilon) = t_1\zeta\varepsilon + t_2\zeta\varepsilon\zeta + t_4\zeta\varepsilon\zeta\varepsilon = 0$, so that $t_1 = t_2 = t_4 = 0$. Thus $e_4\mu = t_3\zeta\gamma + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$.

Hence $\operatorname{Ker}\partial^3 = \{t_3\zeta\gamma + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon : t_i \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \zeta\gamma e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = t_3\zeta\gamma+t_5\zeta\varepsilon\zeta+t_6\zeta\varepsilon\zeta\varepsilon$, that is, $x = \zeta\gamma(t_3e_2+t_5\delta+t_6\delta\varepsilon)$. Thus $x \in \zeta\gamma e_2\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \zeta\gamma e_2\Lambda$.

On the other hand, let $y = \zeta\gamma e_2\nu \in \zeta\gamma e_2\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \zeta\varepsilon(\zeta\gamma e_2\nu) = \zeta\varepsilon\zeta\gamma e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\zeta\gamma e_2\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \zeta\gamma e_2\Lambda$. □

So $\partial^4 : e_2\Lambda \to e_4\Lambda$ is given by $e_2\nu \mapsto \zeta\gamma e_2\nu$, for $\nu \in \Lambda$.

25.1.2. $\operatorname{Ker}\partial^4$.

To find $\operatorname{Ker}\partial^4 = \Omega^5(S_1)$. Let $e_2\nu = d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^4$. Then $\zeta\gamma e_2\nu = 0$ so $\zeta\gamma(d_1e_2 + d_2\beta + d_3\delta + d_4\beta\alpha + d_5\delta\varepsilon + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha) = d_1\zeta\gamma + d_3\zeta\gamma\delta + d_5\zeta\gamma\delta\varepsilon = 0$. So $d_1 = d_3 = d_5 = 0$. Thus $e_2\nu = d_2\beta + d_4\beta\alpha + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$.

Hence $\operatorname{Ker}\partial^4 = \{d_2\beta + d_4\beta\alpha + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^4 = \beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^4$; then $x = d_2\beta + d_4\beta\alpha + d_6\beta\alpha\beta + d_7\delta\varepsilon\zeta + d_8\beta\alpha\beta\alpha$, that is, $x = \beta(d_2e_1+d_4\alpha+d_6\alpha\beta+d_7\alpha\delta+d_8\alpha\beta\alpha)$. Thus $x \in \beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^4 \subseteq \beta e_1\Lambda$.

On the other hand, let $y = \beta e_1\eta \in \beta e_1\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(y) = \zeta\gamma\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^4$ and so $\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^4$.

Hence $\operatorname{Ker}\partial^4 = \beta e_1\Lambda$. □

So $\partial^5 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \beta e_1\eta$, for $\eta \in \Lambda$.

25.1.3. $\operatorname{Ker}\partial^5$.
To find $\operatorname{Ker}\partial^5 = \Omega^6(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^5$. Then $\beta e_1\eta = 0$ so $\beta(c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta) = c_1\beta + c_2\beta\alpha + c_3\beta\alpha\beta + c_4\beta\alpha\delta + c_5\beta\alpha\beta\alpha = 0$, so that $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Thus $e_1\eta = c_6\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker}\partial^5 = \{c_6\alpha\beta\alpha\beta : c_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \alpha\beta\alpha\beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = c_6\alpha\beta\alpha\beta$, that is, $x = \alpha\beta\alpha\beta(c_6e_1)$. Thus $x \in \alpha\beta\alpha\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \alpha\beta\alpha\beta e_1\Lambda$.

On the other hand, let $y = \alpha\beta\alpha\beta e_1\eta \in \alpha\beta\alpha\beta e_1\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = \beta(\alpha\beta\alpha\beta e_1\eta) = \beta\alpha\beta\alpha\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\alpha\beta\alpha\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \alpha\beta\alpha\beta e_1\Lambda$. □

So $\partial^6 : e_1\Lambda \to e_1\Lambda$ is given by $e_1\eta \mapsto \alpha\beta\alpha\beta e_1\eta$, for $\eta \in \Lambda$.

Thus $\operatorname{Ker}\partial^5 \cong S_1$ and so $\Omega^6(S_1) \cong S_1$.

25.2. **The periodicity of S_4.**

25.2.1. $\operatorname{Ker}\partial^3$.
To find $\operatorname{Ker}\partial^3 = \Omega^4(S_4)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^3$. Then $\alpha\beta e_1\eta = 0$ so $\alpha\beta(c_1e_1 + c_2\alpha + c_3\alpha\beta + c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta) = c_1\alpha\beta + c_2\alpha\beta\alpha + c_3\alpha\beta\alpha\beta = 0$. So $c_1 = c_2 = c_3 = 0$. Thus $e_1\eta = c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$.

Hence $\operatorname{Ker}\partial^3 = \{c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \alpha\delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_4\alpha\delta + c_5\alpha\beta\alpha + c_6\alpha\beta\alpha\beta$, that is, $x = \alpha\delta(c_4e_3 + c_5\gamma + c_6\gamma\beta)$. Thus $x \in \alpha\delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \alpha\delta e_3\Lambda$.

On the other hand, let $y = \alpha\delta e_3\lambda \in \alpha\delta e_3\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \alpha\beta(\alpha\delta e_3\lambda) = \alpha\beta\alpha\delta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\alpha\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \alpha\delta e_3\Lambda$. □

So $\partial^4 : e_3\Lambda \to e_1\Lambda$ is given by $e_3\lambda \mapsto \alpha\delta e_3\lambda$, for $\lambda \in \Lambda$.

25.2.2. $\operatorname{Ker}\partial^4$.
To find $\operatorname{Ker}\partial^4 = \Omega^5(S_4)$. Let $e_3\lambda = f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^4$. Then $\alpha\delta e_3\lambda = 0$ so $\alpha\delta(f_1e_3 + f_2\gamma + f_3\varepsilon + f_4\gamma\beta + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta) = f_1\alpha\delta + f_2\alpha\delta\gamma + f_4\alpha\delta\gamma\beta = 0$. So $f_1 = f_2 = f_4 = 0$. Thus $e_3\lambda = f_3\varepsilon + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$.

Hence $\operatorname{Ker}\partial^4 = \{f_3\varepsilon + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta : f_i \in K\}$.

Claim. $\operatorname{Ker}\partial^4 = \varepsilon e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^4$; then $x = f_3\varepsilon + f_5\varepsilon\zeta + f_6\gamma\beta\alpha + f_7\varepsilon\zeta\varepsilon + f_8\varepsilon\zeta\varepsilon\zeta$, that is, $x = \varepsilon(f_3e_4 + f_5\zeta + f_6\zeta\gamma + f_7\zeta\varepsilon + f_8\zeta\varepsilon\zeta)$. Thus $x \in \varepsilon e_4\Lambda$ and therefore $\operatorname{Ker}\partial^4 \subseteq \varepsilon e_4\Lambda$.

On the other hand, let $y = \varepsilon e_4\mu \in \varepsilon e_4\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(y) = \alpha\delta\varepsilon e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^4$ and so $\varepsilon e_4\Lambda \subseteq \operatorname{Ker}\partial^4$.

Hence $\operatorname{Ker}\partial^4 = \varepsilon e_4\Lambda$. □

So $\partial^5 : e_4\Lambda \to e_3\Lambda$ is given by $e_4\mu \mapsto \varepsilon e_4\mu$, for $\mu \in \Lambda$.

25.2.3. $\operatorname{Ker}\partial^5$.

To find $\operatorname{Ker}\partial^5 = \Omega^6(S_2)$. Let $e_4\mu = t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon$ with $t_i \in K$. Assume that $e_4\mu \in \operatorname{Ker}\partial^5$. Then $\varepsilon e_4\mu = 0$ so $\varepsilon(t_1e_4 + t_2\zeta + t_3\zeta\gamma + t_4\zeta\varepsilon + t_5\zeta\varepsilon\zeta + t_6\zeta\varepsilon\zeta\varepsilon) = t_1\varepsilon + t_2\varepsilon\zeta + t_3\varepsilon\zeta\gamma + t_4\varepsilon\zeta\varepsilon + t_5\varepsilon\zeta\varepsilon\zeta = 0$, so that $t_1 = t_2 = t_3 = t_4 = t_5 = 0$. Thus $e_4\mu = t_6\zeta\varepsilon\zeta\varepsilon$.

Hence $\operatorname{Ker}\partial^5 = \{t_6\zeta\varepsilon\zeta\varepsilon : t_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \zeta\varepsilon\zeta\varepsilon e_4\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = t_6\zeta\varepsilon\zeta\varepsilon$, that is, $x = \zeta\varepsilon\zeta\varepsilon(t_6e_4)$. Thus $x \in \zeta\varepsilon\zeta\varepsilon e_4\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \zeta\varepsilon\zeta\varepsilon e_4\Lambda$.

On the other hand, let $y = \zeta\varepsilon\zeta\varepsilon e_4\mu \in \zeta\varepsilon\zeta\varepsilon e_4\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = \varepsilon(\zeta\varepsilon\zeta\varepsilon e_4\mu) = \varepsilon\zeta\varepsilon\zeta\varepsilon e_4\mu = 0$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\zeta\varepsilon\zeta\varepsilon e_4\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \zeta\varepsilon\zeta\varepsilon e_4\Lambda$. □

So $\partial^6 : e_4\Lambda \to e_4\Lambda$ is given by $e_4\mu \mapsto \zeta\varepsilon\zeta\varepsilon e_4\mu$, for $\mu \in \Lambda$.

Hence $\operatorname{Ker}\partial^5 \cong S_4$ and so $\Omega^6(S_4) \cong S_4$.

We summarize this in the following Theorem.

Theorem 25.1. *For the algebra A_7, we have $\Omega^6(S_1) \cong S_1, \Omega^3(S_2) \cong S_2, \Omega^3(S_3) \cong S_3$ and $\Omega^6(S_4) \cong S_4$. Hence $\Omega^6(S_i) \cong S_i$ for all $i = 1, 2, 3, 4$.*

26. Periodicity of the simple A_{12}−modules

The algebras $A_{12}, A_{13}, A_{14}, A_{15}, A_{16}$ all lie in the same derived equivalence class. We give the periodicity of the simple A_{12}−modules, and do not discuss the periodicity for $A_{13}, A_{14}, A_{15}, A_{16}$.

We recall from Chapter 16 that A_{12} is the algebra $K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, shift left, "\delta"] & & 3 \arrow[ll, shift left, "\beta"] \end{tikzcd}$$

and

$$I = \langle \delta\beta\delta - \alpha\gamma, \gamma\beta\alpha, (\beta\delta)^3\beta \rangle.$$

Write Λ for A_{12} and S_i for the simple module at the vertex i.

The indecomposable projective Λ-modules are

$$\begin{aligned} e_1\Lambda &= sp\{e_1, \alpha, \delta, \delta\beta, \delta\beta\delta, \delta\beta\delta\beta, \delta\beta\alpha, \delta\beta\delta\beta\delta, \delta\beta\delta\beta\delta\beta\}, \\ e_2\Lambda &= sp\{e_2, \gamma, \gamma\beta, \gamma\beta\delta, \gamma\beta\delta\beta, \gamma\beta\delta\beta\alpha\}, \\ e_3\Lambda &= sp\{e_3, \beta, \beta\alpha, \beta\delta, \beta\alpha\gamma, \beta\delta\beta, \beta\delta\beta\alpha, \beta\alpha\gamma\beta, \beta\alpha\gamma\beta\delta\}. \end{aligned}$$

Recall from Section 16.2 the minimal projective resolution of S_1 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_3\Lambda \xrightarrow{\partial^2} e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^1} e_1\Lambda \longrightarrow S_1 \longrightarrow 0$$

where

$$\begin{aligned} &\partial^1 : (e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda, \\ &\partial^2 : e_3\lambda \mapsto (-\gamma, \beta\delta)e_3\lambda, \\ &\partial^3 : e_1\eta \mapsto \beta\alpha\gamma\beta e_1\eta, \end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

The minimal projective resolution of the simple Λ-module S_2 starts with:

$$\cdots \longrightarrow e_1\Lambda \xrightarrow{\partial^3} e_2\Lambda \xrightarrow{\partial^2} e_3\Lambda \xrightarrow{\partial^1} e_2\Lambda \longrightarrow S_2 \longrightarrow 0$$

where

$$\begin{aligned} &\partial^1 : e_3\lambda \to \gamma e_3\lambda, \\ &\partial^2 : e_2\nu \mapsto \beta\alpha e_2\nu \\ &\partial^3 : e_1\eta \mapsto \gamma\beta\delta\beta e_1\eta, \end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

The minimal projective resolution of the simple Λ-module S_3 starts with:

$$\cdots \longrightarrow e_2\Lambda \oplus e_3\Lambda \xrightarrow{\partial^3} e_1\Lambda \xrightarrow{\partial^2} e_1\Lambda \xrightarrow{\partial^1} e_3\Lambda \longrightarrow S_3 \longrightarrow 0$$

where

$$\begin{aligned} &\partial^1 : e_1\eta \mapsto \beta e_1\eta, \\ &\partial^2 : e_1\eta \mapsto \delta\beta\delta\beta\delta\beta e_1\eta, \\ &\partial^3 : (e_2\nu, e_3\lambda) \mapsto \alpha e_2\nu + \delta e_3\lambda, \end{aligned}$$

for $\eta, \nu, \lambda \in \Lambda$.

Note that $\Omega^2(S_3) \cong S_1$.

26.1. **The periodicity of S_1.**

26.1.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^3$ then $\beta\alpha\gamma\beta e_1\eta = 0$ so $\beta\alpha\gamma\beta e_1\eta = \beta\alpha\gamma\beta(c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta) = c_1\beta\alpha\gamma\beta + c_3\beta\alpha\gamma\beta\delta = 0$, so that $c_1 = c_3 = 0$. Thus $e_1\eta = c_2\alpha + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$.

Hence $\operatorname{Ker}\partial^3 = \{c_2\alpha + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \delta\beta e_1\Lambda + \alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_2\alpha + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$, that is, $x = \delta\beta(c_4e_1 + c_5\delta + c_6\delta\beta + c_7\alpha + c_8\delta\beta\delta + c_9\delta\beta\delta\beta) + \alpha(c_2e_2)$. Thus $x \in \delta\beta e_1\Lambda + \alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \delta\beta e_1\Lambda + \alpha e_2\Lambda$.

On the other hand, let $y = \delta\beta e_1\eta + \alpha e_2\nu \in \delta\beta e_1\Lambda + \alpha e_2\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \beta\alpha\gamma\beta(\delta\beta e_1\eta + \alpha e_2\nu) = \beta\alpha\gamma\beta\delta\beta e_1\eta + \beta\alpha\gamma\beta\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\delta\beta e_1\Lambda + \alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \delta\beta e_1\Lambda + \alpha e_2\Lambda$. $\square$

So $\partial^4 : e_1\Lambda \oplus e_2\Lambda \to e_1\Lambda$ is given by $(e_1\eta, e_2\nu) \mapsto \delta\beta e_1\eta + \alpha e_2\nu$, for $\eta, \nu \in \Lambda$.

26.1.2. $\operatorname{Ker}\partial^4$.

To find $\operatorname{Ker}\partial^4 = \Omega^5(S_1)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ and $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ with $c_i, d_i \in K$. Assume that $(e_1\eta, e_2\nu) \in \operatorname{Ker}\partial^4$. Then $\delta\beta e_1\eta + \alpha e_2\nu = 0$ so $\delta\beta e_1\eta + \alpha e_2\nu = \delta\beta(c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta) + \alpha(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) = c_1\delta\beta + c_2\delta\beta\alpha + c_3\delta\beta\delta + c_4\delta\beta\delta\beta + c_5\delta\beta\delta\beta\delta + c_6\delta\beta\delta\beta\delta\beta + d_1\alpha + d_2\alpha\gamma + d_3\alpha\gamma\beta + d_4\alpha\gamma\beta\delta + d_5\alpha\gamma\beta\delta\beta = c_1\delta\beta + c_2\delta\beta\alpha + d_1\alpha + (c_3 + d_2)\delta\beta\delta + (c_4 + d_3)\delta\beta\delta\beta + (c_5 + d_4)\delta\beta\delta\beta\delta + (c_6 + d_5)\delta\beta\delta\beta\delta\beta = 0$ so $c_1 = c_2 = d_1 = 0, d_2 = -c_3, d_3 = -c_4, d_4 = -c_5, d_5 = -c_6$. Thus $e_1\eta = c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ and $e_2\nu = -c_3\gamma - c_4\gamma\beta - c_5\gamma\beta\delta - c_6\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$.

Hence $\operatorname{Ker}\partial^4 = \{(c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta, -c_3\gamma - c_4\gamma\beta - c_5\gamma\beta\delta - c_6\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) : c_i, d_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^4 = (\delta, -\gamma)e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^4$; then $x = (c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta, -c_3\gamma - c_4\gamma\beta - c_5\gamma\beta\delta - c_6\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha)$, that is, $x = (\delta, -\gamma)(c_3e_3 + c_4\beta + c_5\beta\delta + c_6\beta\delta\beta + c_7\beta\alpha + c_8\beta\delta\beta\delta + c_9\beta\delta\beta\delta\beta - d_6\beta\delta\beta\alpha)$. Thus $x \in (\delta, -\gamma)e_3\Lambda$ and therefore $\operatorname{Ker}\partial^4 \subseteq (\delta, -\gamma)e_3\Lambda$.

On the other hand, let $y = (\delta, -\gamma)e_3\lambda \in (\delta, -\gamma)e_3\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(y) = \partial^4((\delta, -\gamma)e_3\lambda) = (\delta\beta\delta - \alpha\gamma)e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^4$ and so $(\delta, -\gamma)e_3\Lambda \subseteq \operatorname{Ker}\partial^4$.

Hence $\operatorname{Ker}\partial^4 = (\delta, -\gamma)e_3\Lambda$. □

So $\partial^5 : e_3\Lambda \to e_1\Lambda \oplus e_2\Lambda$ is given by $e_3\lambda \mapsto (\delta, -\gamma)e_3\lambda$, for $\lambda \in \Lambda$.

26.1.3. $\operatorname{Ker}\partial^5$.

To find $\operatorname{Ker}\partial^5 = \Omega^6(S_1)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^5$. Then $(\delta, -\gamma)e_3\lambda = (0, 0)$ so $(\delta, -\gamma)(f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta) = (f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha + f_4\delta\beta\delta + f_5\delta\beta\alpha\gamma + f_6\delta\beta\delta\beta + f_8\delta\beta\alpha\gamma\beta, -f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\delta - f_6\gamma\beta\delta\beta - f_7\gamma\beta\delta\beta\alpha) = (0, 0)$, so that $f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha + f_4\delta\beta\delta + f_5\delta\beta\alpha\gamma + f_6\delta\beta\delta\beta + f_8\delta\beta\alpha\gamma\beta = 0$, that is, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = f_8 = 0$ and $-f_1\gamma - f_2\gamma\beta - f_4\gamma\beta\delta - f_6\gamma\beta\delta\beta - f_7\gamma\beta\delta\beta\alpha = 0$, that is, $f_1 = f_2 = f_4 = f_6 = f_7 = 0$. Thus $e_3\lambda = f_9\beta\alpha\gamma\beta\delta$.

Hence $\operatorname{Ker}\partial^5 = \{f_9\beta\alpha\gamma\beta\delta : f_9 \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \beta\alpha\gamma\beta\delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = f_9\beta\alpha\gamma\beta\delta$, that is, $x = \beta\alpha\gamma\beta\delta(f_9e_3)$. Thus $x \in \beta\alpha\gamma\beta\delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \beta\alpha\gamma\beta\delta e_3\Lambda$.

On the other hand, let $y = \beta\alpha\gamma\beta\delta e_3\lambda \in \beta\alpha\gamma\beta\delta e_3\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = (\delta, -\gamma)(\beta\alpha\gamma\beta\delta e_3\lambda) = (\delta\beta\alpha\gamma\beta\delta, -\gamma\beta\alpha\gamma\beta\delta)e_3\lambda = (0, 0)$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\beta\alpha\gamma\beta\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \beta\alpha\gamma\beta\delta e_3\Lambda$. □

So $\partial^6 : e_3\Lambda \to e_3\Lambda$ is given by $e_3\lambda \mapsto \beta\alpha\gamma\beta\delta e_3\lambda$, for $\lambda \in \Lambda$.

Thus $\operatorname{Ker}\partial^5 \cong S_1$ and so $\Omega^6(S_1) \cong S_3$. And we have already seen that $\Omega^2(S_3) \cong S_1$. Thus $\Omega^8(S_1) \cong S_1$.

26.2. **The periodicity of S_2.**

26.2.1. $\operatorname{Ker}\partial^3$.

To find $\operatorname{Ker}\partial^3 = \Omega^4(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^3$ then $\gamma\beta\delta\beta e_1\eta = 0$ so $\gamma\beta\delta\beta e_1\eta = \gamma\beta\delta\beta(c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta) = c_1\gamma\beta\delta\beta + c_2\gamma\beta\delta\beta\alpha = 0$, so that $c_1 = c_2 = 0$. Thus $e_1\eta = c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$.

Hence $\operatorname{Ker}\partial^3 = \{c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^3 = \delta e_3\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^3$; then $x = c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$, that is, $x = \delta(c_3e_3 + c_4\beta + c_5\beta\delta + c_6\beta\delta\beta + c_7\beta\alpha + c_8\beta\delta\beta\delta + c_9\beta\delta\beta\delta\beta)$. Thus $x \in \delta e_3\Lambda$ and therefore $\operatorname{Ker}\partial^3 \subseteq \delta e_3\Lambda$.

On the other hand, let $y = \delta e_3\lambda \in \delta e_3\Lambda$. Then, from the definition of ∂^3, we have that $\partial^3(y) = \gamma\beta\delta\beta(\delta e_3\lambda) = \gamma\beta\delta\beta\delta e_3\lambda = 0$. Therefore $y \in \operatorname{Ker}\partial^3$ and so $\delta e_3\Lambda \subseteq \operatorname{Ker}\partial^3$.

Hence $\operatorname{Ker}\partial^3 = \delta e_3\Lambda$. □

So $\partial^4 : e_3\Lambda \to e_1\Lambda$ is given by $e_3\lambda \mapsto \delta e_3\lambda$, for $\lambda \in \Lambda$.

26.2.2. $\operatorname{Ker}\partial^4$.

To find $\operatorname{Ker}\partial^4 = \Omega^5(S_2)$. Let $e_3\lambda = f_1e_3 + f_2\beta + f_3\beta\alpha + f_4\beta\delta + f_5\beta\alpha\gamma + f_6\beta\delta\beta + f_7\beta\delta\beta\alpha + f_8\beta\alpha\gamma\beta + f_9\beta\alpha\gamma\beta\delta$ with $f_i \in K$. Assume that $e_3\lambda \in \operatorname{Ker}\partial^4$. Then $\delta e_3\lambda = 0$ so $\delta e_3\lambda = \delta(f_1e_3+f_2\beta+f_3\beta\alpha+f_4\beta\delta+f_5\beta\alpha\gamma+f_6\beta\delta\beta+f_7\beta\delta\beta\alpha+f_8\beta\alpha\gamma\beta+f_9\beta\alpha\gamma\beta\delta) = f_1\delta + f_2\delta\beta + f_3\delta\beta\alpha + f_4\delta\beta\delta + f_5\delta\beta\alpha\gamma + f_6\delta\beta\delta\beta + f_8\delta\beta\alpha\gamma\beta = 0$ so $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = f_8 = 0$. Thus $e_3\lambda = f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta$.

Hence $\operatorname{Ker}\partial^4 = \{f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta : f_7, f_9 \in K\}$.

Claim. $\operatorname{Ker}\partial^4 = \beta\delta\beta\alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^4$; then $x = f_7\beta\delta\beta\alpha + f_9\beta\alpha\gamma\beta\delta$, that is, $x = \beta\delta\beta\alpha(f_7e_2 + f_9\gamma)$. Thus $x \in \beta\delta\beta\alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^4 \subseteq \beta\delta\beta\alpha e_2\Lambda$.

On the other hand, let $y = \beta\delta\beta\alpha e_2\nu \in \beta\delta\beta\alpha e_2\Lambda$. Then, from the definition of ∂^4, we have that $\partial^4(y) = \partial^4(\beta\delta\beta\alpha e_2\nu) = \delta(\beta\delta\beta\alpha e_2\nu) = \delta\beta\delta\beta\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^4$ and so $\beta\delta\beta\alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^4$.

Hence $\operatorname{Ker}\partial^4 = \beta\delta\beta\alpha e_2\Lambda$. □

So $\partial^5 : e_2\Lambda \to e_3\Lambda$ is given by $e_2\nu \mapsto \beta\delta\beta\alpha e_2\nu$, for $\nu \in \Lambda$.

26.2.3. $\operatorname{Ker}\partial^5$. To find $\operatorname{Ker}\partial^5 = \Omega^6(S_2)$. Let $e_2\nu = d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^5$. Then $\beta\delta\beta\alpha e_2\nu = 0$ so $\beta\delta\beta\alpha(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) = d_1\beta\delta\beta\alpha + d_2\beta\delta\beta\alpha\gamma = 0$, so that, $d_1 = d_2 = 0$. Thus $e_2\nu = d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$.

Hence $\operatorname{Ker}\partial^5 = \{d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha : d_i \in K\}$.

Claim. $\operatorname{Ker}\partial^5 = \gamma\beta e_1\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^5$; then $x = d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha$, that is, $x = \gamma\beta(d_3e_1 + d_4\delta + d_5\delta\beta + d_6\delta\beta\alpha)$. Thus $x \in \gamma\beta e_1\Lambda$ and therefore $\operatorname{Ker}\partial^5 \subseteq \gamma\beta e_1\Lambda$.

On the other hand, let $y = \gamma\beta e_1\eta \in \gamma\beta e_1\Lambda$. Then, from the definition of ∂^5, we have that $\partial^5(y) = \beta\delta\beta\alpha(\gamma\beta e_1\eta) = \beta\delta\beta\alpha\gamma\beta e_1\eta = 0$. Therefore $y \in \operatorname{Ker}\partial^5$ and so $\gamma\beta e_1\Lambda \subseteq \operatorname{Ker}\partial^5$.

Hence $\operatorname{Ker}\partial^5 = \gamma\beta e_1\Lambda$. □

So $\partial^6 : e_1\Lambda \to e_2\Lambda$ is given by $e_1\eta \mapsto \gamma\beta e_1\eta$, for $\eta \in \Lambda$.

26.2.4. $\operatorname{Ker}\partial^6$. To find $\operatorname{Ker}\partial^6 = \Omega^7(S_2)$. Let $e_1\eta = c_1e_1 + c_2\alpha + c_3\delta + c_4\delta\beta + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_7\delta\beta\alpha + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$ with $c_i \in K$. Assume that $e_1\eta \in \operatorname{Ker}\partial^6$ then $\gamma\beta e_1\eta = 0$ so $\gamma\beta e_1\eta = \gamma\beta(c_1e_1+c_2\alpha+c_3\delta+c_4\delta\beta+c_5\delta\beta\delta+c_6\delta\beta\delta\beta+c_7\delta\beta\alpha+c_8\delta\beta\delta\beta\delta+c_9\delta\beta\delta\beta\delta\beta) = c_1\gamma\beta + c_3\gamma\beta\delta + c_4\gamma\beta\delta\beta + c_7\gamma\beta\delta\beta\alpha = 0$, so that $c_1 = c_3 = c_4 = c_7 = 0$. Thus $e_1\eta = c_2\alpha + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$.

Hence $\operatorname{Ker}\partial^6 = \{c_2\alpha + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta : c_i \in K\}$.

Claim. $\operatorname{Ker}\partial^6 = \alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^6$; then $x = c_2\alpha + c_5\delta\beta\delta + c_6\delta\beta\delta\beta + c_8\delta\beta\delta\beta\delta + c_9\delta\beta\delta\beta\delta\beta$, that is, $x = \alpha(c_2e_2+c_5\gamma+c_6\gamma\beta+c_8\gamma\beta\delta+c_9\gamma\beta\delta\beta)$. Thus $x \in \alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^6 \subseteq \alpha e_2\Lambda$.

On the other hand, let $y = \alpha e_2\nu \in \alpha e_2\Lambda$. Then, from the definition of ∂^6, we have that $\partial^6(y) = \gamma\beta(\alpha e_2\nu) = \gamma\beta\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^6$ and so $\alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^6$.

Hence $\operatorname{Ker}\partial^6 = \alpha e_2\Lambda$. □

So $\partial^7 : e_2\Lambda \to e_1\Lambda$ is given by $e_2\nu \mapsto \alpha e_2\nu$, for $\nu \in \Lambda$.

26.2.5. $\operatorname{Ker}\partial^7$. To find $\operatorname{Ker}\partial^7 = \Omega^8(S_2)$. Let $e_2\nu = d_1e_2+d_2\gamma+d_3\gamma\beta+d_4\gamma\beta\delta+d_5\gamma\beta\delta\beta+d_6\gamma\beta\delta\beta\alpha$ with $d_i \in K$. Assume that $e_2\nu \in \operatorname{Ker}\partial^7$. Then $\alpha e_2\nu = 0$ so $\alpha(d_1e_2 + d_2\gamma + d_3\gamma\beta + d_4\gamma\beta\delta + d_5\gamma\beta\delta\beta + d_6\gamma\beta\delta\beta\alpha) = d_1\alpha + d_2\alpha\gamma + d_3\alpha\gamma\beta + d_4\alpha\gamma\beta\delta + d_5\alpha\gamma\beta\delta\beta = 0$, so that $d_1 = d_2 = d_3 = d_4 = d_5 = 0$. Thus $e_2\nu = d_6\gamma\beta\delta\beta\alpha$.

Hence $\operatorname{Ker}\partial^7 = \{d_6\gamma\beta\delta\beta\alpha : d_6 \in K\}$.

Claim. $\operatorname{Ker}\partial^7 = \gamma\beta\delta\beta\alpha e_2\Lambda$.

Proof. Let $x \in \operatorname{Ker}\partial^7$; then $x = d_6\gamma\beta\delta\beta\alpha$, that is, $x = \gamma\beta\delta\beta\alpha(d_6e_2)$. Thus $x \in \gamma\beta\delta\beta\alpha e_2\Lambda$ and therefore $\operatorname{Ker}\partial^7 \subseteq \gamma\beta\delta\beta\alpha e_2\Lambda$.

On the other hand, let $y = \gamma\beta\delta\beta\alpha e_2\nu \in \gamma\beta\delta\beta\alpha e_2\Lambda$. Then, from the definition of ∂^7, we have that $\partial^7(y) = \partial^7(\gamma\beta\delta\beta\alpha e_2\nu) = \alpha\gamma\beta\delta\beta\alpha e_2\nu = 0$. Therefore $y \in \operatorname{Ker}\partial^7$ and so $\gamma\beta\delta\beta\alpha e_2\Lambda \subseteq \operatorname{Ker}\partial^7$.

Hence $\operatorname{Ker}\partial^7 = \gamma\beta\delta\beta\alpha e_2\Lambda$. □

So $\partial^8 : e_2\Lambda \to e_2\Lambda$ is given by $e_2\nu \mapsto \gamma\beta\delta\beta\alpha e_2\nu$, for $\nu \in \Lambda$.

Thus $\operatorname{Ker}\partial^7 \cong S_2$ and so $\Omega^8(S_2) \cong S_2$.

We summarize this in the following Theorem.

Theorem 26.1. *For the algebra A_{12}, we have $\Omega^8(S_1) \cong S_1, \Omega^8(S_2) \cong S_2$ and $\Omega^8(S_3) \cong S_3$. Hence $\Omega^8(S_i) \cong S_i$ for all $i = 1, 2, 3$.*

So we have found $\mathrm{HH}^2(\Lambda)$ and the $\Omega-$periodicity of each simple module for all tame weakly symmetric algebras Λ having only $\tau-$periodic modules.

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