Homotopy Types of Topological Groupoids and Lusternik-Schnirelmann Category of Topological Stacks

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Our greatest weakness lies in giving up. The most certain way to succeed is always to try just one more time.

Thomas Edison

Abstract

The concept of a groupoid was first introduced in 1926 by H. Brandt in his fundamental paper [7]. The idea behind it is a small category in which every arrow is invertible. This notion of groupoid can be thought of as a generalisation of the notion of a group. Namely, a group is a groupoid with only one object. After the introduction of topological and differentiable groupoids by Ehresmann in 1950 in his paper on connections [19], the concept has been widely studied by many mathematicians in many areas of topology, geometry and physics. In this thesis, we deal with topological groupoids as the main object of study. We first develop the main concepts of homotopy theory of topological groupoids. Also, we study general versions of Morita equivalence between topological groupoids, which lead us to discuss topological stacks. The main objective of this thesis is then to develop and analyse a notion of Lusternik-Schnirelmann category for general topological groupoids and topological stacks, generalising the classical work by Lusternik and Schnirelmann for topological spaces and manifolds [30] and for orbifolds and Lie groupoids as introduced by Colman [11]. Fundamental in the classical definition of the LS-category of a smooth manifold or topological space is the concept of a categorical set. A subset of a space is said to be categorical if it is contractible in the space. The Lusternik-Schnirelmann category $\mathrm{cat}(X)$ of a topological space X is defined to be the least number of categorical open sets required to cover X, if that number is finite. Otherwise the category $\mathrm{cat}(X)$ is said to be infinite. Here using a generalised notion of categorical subgroupoid and substack, we generalise the notion of the Lusternik-Schnirelmann category to topological groupoids and topological stacks with the intention of providing a new useful tool and invariant to study homotopy types of topological groupoids and topological stacks, which will be important also to understand the geometry and Morse theory of Lie groupoids and differentiable stacks from a purely homotopical viewpoint.

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Contents

Abbreviations

$\mathcal{G}$ = Groupoid

Top = Category of topological spaces

Grp = Category of groupoids

$Cov(\mathcal{G})$ = Category of coverings of a groupoid $\mathcal{G}$

Gpd = Bicategory in which the morphisms are generalized maps

Gpd' = The bicategory in which the morphisms are bibundles

CFG = Category fibered in groupoids

G = 2-category of topological groupoids, functors and natural transformations

H = 2-category of topological groupoids, functors and 1-homotopies

E = Set of essential equivalences in **G**

W = Set of essential 1-homotopy equivalences in **H**

St = 2-category of stacks

To my parents

Chapter 1

Introduction

The *Lusternik-Schnirelmann category* or (LS-category) of a topological space is a topological invariant introduced by Lusternik and Schnirelmann [30] in the early 1930s. This Lusternik-Schnirelmann category invariant became an important tool in algebraic topology and especially homotopy theory. For more details on the importance of the LS-category in topology and geometry see [27], [26] and [14]. The main concept in the definition of the Lusternik-Schnirelmann category is that of a categorical set. A subset of a space is said to be *categorical* if it is contractible in the space. The Lusternik-Schnirelmann category $\mathrm{cat}(X)$ of a topological space X is defined to be the least number of categorical open sets required to cover X, if that number is finite, otherwise the category $\mathrm{cat}(X)$ is said to be infinite.

Our aim in this thesis is to develop a Lusternik-Schnirelmann category theory invariant in the context of topological groupoids and generalise the notion of Lusternik-Schnirelmann category from topological spaces to topological stacks with the intention of providing a useful tool and invariant to study the homotopy theory of topological stacks. As we know, a groupoid is a common generalisation of the concepts of spaces and groups. A *groupoid* $\mathcal{G} = [G_1 \rightrightarrows G_0]$ consists of a set of objects G_0 and a set of arrows G_1 with source and target maps $s, t : G_1 \to G_0$ and there is a multiplication defined for composable pairs of morphisms. Finally there is also an inverse map. Specifically, *topological groupoids* are groupoids where both G_0 and G_1 in addition are topological spaces and for which all the structure maps (source, target, multiplication, unit and inversion) are continuous. In this thesis, we will study the homotopy theory of topological groupoids, and develop an analogue of the Lusternik-Schnirelmann category for topological groupoids. Furthermore, since topological stacks are defined as a Morita equivalence class of topological groupoids, [21], [15] and [43], we show that our notion of LS-category for topological groupoids is also invariant under Morita equivalence and therefore defines an invariant for topological stacks. In our analysis we have to study several notions of homotopy between topological groupoids. In particular, we discuss a notion of homotopy between generalised maps, which turns out to be Morita invariant. Roughly speaking, a*generalised map* from a topological groupoid $\mathcal{H}$ to a topological groupoid $\mathcal{G}$ is given by first replacing $\mathcal{H}$ by a Morita equivalent groupoid $\mathcal{H}'$ and then mapping $\mathcal{H}'$ into $\mathcal{G}$ by a homomorphism of topological groupoids. Topological groupoids, functors between them and continuous natural transformations between the functors defines a 2-category, which we will denote by $\mathbf{G}$. In general topological categories $\mathcal{J}$ and $\mathcal{J}'$, the usual notion of homotopy is called an *ordinary homotopy.* It is said that two continuous functors $f, g : \mathcal{J} \to \mathcal{J}'$ are homotopic if there is a continuous functor $H : \mathcal{J} \times \mathcal{I} \to \mathcal{J}'$ such that $H_0 = f$, $H_1 = g$ and $\mathcal{I}$ is the unit groupoid over the unit interval $[0, 1]$. But these two notions of natural transformation and ordinary homotopy are not invariant under Morita equivalence in general. So we have to look for an alternative notion of homotopy which is in fact invariant under Morita equivalence and generalises the notions of natural transformation and ordinary homotopy. To achieve this, we start to define strong and essential equivalence notions between general topological groupoids. We introduce the set of all essential equivalences between topological groupoids denoted by E and we obtain a bicategory of fractions of $\mathbf{G}$ when inverting all these essential equivalences E. In this way we get a new bicategory $\mathbf{Gpd} = \mathbf{G}(E^{-1})$. This bicategory is obtained by the collection of all topological groupoids as objects, generalised maps as morphisms and 2-morphisms between the generalised maps.

We then introduce a new notion of 1-homotopy between continuous functors which includes the notions of natural transformation and ordinary homotopy but this is not yet invariant under Morita equivalence. The resulting 2-category is denoted by $\mathbf{H}$. Afterwards we introduce a notion of essential 1-homotopy equivalence for the arrows in this 2-category $\mathbf{H}$. The class of essential 1-homotopy equivalences will be denoted by W and we prove that this class admits a bicalculus of fractions $\mathbf{H}(W^{-1})$ and will determine our correct 1-homotopy equivalences: a generalised map is a *1-homotopy equivalence* if it is an equivalence in $\mathbf{H}(W^{-1})$. In order to do so, we first recall the construction of Haefliger's $\mathcal{G}$-paths and the fundamental groupoid of a topological groupoid and prove that it admits a right calculus of fractions that inverts the essential equivalences W. We prove that the notion of 1-homotopy equivalence obtained in this bicategory is invariant under Morita equivalence and generalises the notions of natural transformation and ordinary homotopy.

When inverting the essential equivalences E in the 2-category $\mathbf{G}$ of topological groupoids, functors and continuous natural transformations, the following diagram of bicategories commutes

$$\begin{tikzcd} \mathbf{G} \arrow[r, "U|_{\mathbf{G}}"] \arrow[d] & \mathbf{G}(E^{-1}) \arrow[d, "i_{\mathbf{G}(E^{-1})}"] \\ \mathbf{H} \arrow[r, "U"] & \mathbf{H}(W^{-1}) \end{tikzcd}$$

where U and $U|_{\mathbf{G}}$ are the universal homomorphisms as defined in [42] and the arrows $i_{\mathbf{G}(E^{-1})}$ exists by the universal property of $U|_{\mathbf{G}}$.

We also develop a notion of homotopy between generalised maps. We start with a notion of strong homotopy between strict maps which is not Morita invariant, and introduce a related notion of essential homotopy equivalence that at the same time weakens strict homotopy and generalises an essential equivalence. We will say that two topological groupoids $\mathcal{G}$ and $\mathcal{K}$ have the same *Morita homotopy type* if there exists a third groupoid $\mathcal{J}$ and essential homotopy equivalences η and ν

$$\mathcal{K} \xleftarrow{\nu} \mathcal{J} \xrightarrow{\eta} \mathcal{G}$$

Our notion of Morita homotopy types corresponds to the 2-arrows in the bicategory $\mathbf{H}(W^{-1})$. This deformation within the groupoid is closely related to the notion of $\mathcal{G}$-path developed by Haefliger (see [24]and [23]). Based on this Morita homotopy type, we define the notion of a $\mathcal{G}$-categorical subgroupoid for the definition of the Lusternik-Schnirelmann category of topological groupoids. A subgroupoid $\mathcal{U}$ is $\mathcal{G}$-categorical if it can be deformed by a Morita homotopy into a transitive groupoid. The least number of $\mathcal{G}$-categorical subgroupoids that cover $\mathcal{G}$ is the *groupoid LS-category* of the topological groupoid $\mathcal{G}$. When $\mathcal{G}$ is the unit groupoid over a topological space X, this number specialises to the classical LS-category of the topological space X. We prove that $\mathrm{cat}(\mathcal{G})$ is an invariant of the Morita homotopy type. Therefore, $\mathrm{cat}(\mathcal{G})$ is invariant under Morita equivalence.

Finally, we introduce categories fibered in groupoids and define stacks over the category of topological spaces Top. Moreover, we prove how a category fibered in groupoids and stacks are associated to a given topological groupoid. That leads us to define topological stacks and also how to associate a topological stack to a topological groupoid. One of our aims is then to generalise the notion of Lusternik-Schnirelmann category from topological groupoids to topological stacks by showing that the groupoid LS-category is in fact a Morita invariant notion.

Classically, the Lusternik-Schnirelmann category (LS-category) of a topological space X is given as the least number n such that there is an open cover of X of $n + 1$ subsets contractible to a point in the space X. This concept was first introduced in 1930 by L. Lusternik and L. Schnirelmann [30] in the study of the geometry of differentiable manifolds via Morse theory. Later on, R. Fox [20] introduced the geometric category, another definition of the LS-category, where each subset of the cover is required to be contractible in itself, and he developed the LS-category in the field of algebraic topology further showing that it is indeed a homotopical invariant. More description of the two alternative definitions and a complete overview of topolgical and geometrical results about the LS-category for topological spaces and differentiable manifolds can be found in [14].
It can be expected that the stacky LS-category we discuss here for topological stacks will be also a very useful topological invariant for this kind of generalised spaces, including topological spaces, manifolds and orbifolds, which do appear naturally in geometry and physics. We aim to study these geometrical and topological aspects of the stacky LS-category also in further work.

The constructions of stack and groupoid LS-category here will be presented in a purely homotopical manner and employ the recent homotopy theory of topological groupoids and topological stacks (see [40] and [39]). The new notion of the stacky LS-category for topological stacks discussed here employs the new notion of categorical substacks and it turns out that this is again an invariant of the homotopy type of topological stacks. Therefore, this notion of stacky LS-category generalises the notions of LS-category of topological groupoids and is indeed an homotopical invariant of topological stacks.

Thesis outline

The content in this thesis is subdivided into five chapters and organised as follows:

Chapter 1 reviews the theory of categories and groupoids. In this chapter we introduce the basic concepts that will be used in the thesis. We begin with definitions from category theory and describe the groupoid notions, as well as giving some examples of groupoids, including the fundamental groupoid that will be relevant in the subsequent chapters. Then we introduce background material on covering groupoids and discuss the classification of coverings of groupoids. To end the chapter we present a classical description of topological groupoids that will be the base throughout this work and lay out some examples.

Chapter 2 defines morphisms and equivalences of topological groupoids. We first introduce the different notions of equivalence of topological groupoids, starting with essential equivalence and reaching Morita equivalence. Then we introduce some background on bicategories. To finalise the chapter we present the different notions of maps between topological groupoids that enlarge the usual definition of groupoid morphisms; these are the generalised maps and Hilsum-Skandalis maps. After that we will give an explicit construction of a bijective correspondence between generalized maps and bibundles which will allow us to switch from one to the other when needed.

Chapter 3 is concerned with the general homotopy theory of topological groupoids. It presents the notion of 1-homotopy for generalised maps. First we recall the construction of Haefliger $\mathcal{G}$-path and the fundamental groupoid of topological groupoids. Then we introduce the 1-homotopy bicategory H. We will show that the notion of 1-homotopy equivalence obtained in this bicategory is invariant under Morita equivalence and generalises the notions of natural transformation and ordinary homotopy for topological spaces. At the end of this chapter, we introduce the notion of homotopy of topological groupoids, essential equivalence homotopy and homotopy pullback of topological groupoids.

Chapter 4 is dedicated to stacks and topological stacks. We define categories fibered in groupoids over Top, the category of topological spaces and continuous maps, also introducing the language of 2-categories as categories fibered in groupoids over Top form a 2-category. Then we introduce the notion of stacks and see how stacks can be associated to topological groupoids. After that we define topological stacks which we will use in the final chapter.

Chapter 5 deals with Lusternik-Schnirelmann category theory. We start with recalling the notion of Lusternik-Schnirelmann category for topological spaces and give some examples. Then we introduce the notion of Lusternik-Schnirelmann category for topological groupoids and prove that it is invariant under Morita equivalence therefore giving a new topological invariant, the stacky Lusternik-Schnirelmann category for topological stacks. We will finish this final chapter by studying some basic but fundamental properties of these new notions of Lusternik-Schnirelmann category for topological groupoids and topological stacks.

Chapter 2

Categories and Groupoids

2.1 Categories

In this first chapter we introduce the basic concepts to be used in this thesis. We start to recall some of the basic definitions of categories, groupoids and topological groupoids. we follow here the book [8] regarding the notions for category theory.

Definition 2.1.1. *A category $\mathcal{C}$ consists of:*

- *A collection of objects of $\mathcal{C}$ denoted by $Ob(\mathcal{C})$.*
- *A set $\mathcal{C}(x, y)$ called the set of morphisms in $\mathcal{C}$ from x to y, for each pair of morphisms $x, y \in Ob(\mathcal{C})$.*
- *For each $x, y, z \in Ob(\mathcal{C})$ and for each pair of morphisms g in $\mathcal{C}(y, z)$ and f in $\mathcal{C}(x, y)$, there is a composition function $g \circ f$ in $\mathcal{C}(x, z)$, we will just write normally gf.*

These terms must satisfy the following axioms:

1. *(Associativity): If $h \in \mathcal{C}(z, w)$, $g \in \mathcal{C}(y, z)$, and $f \in \mathcal{C}(x, y)$, then $h(gf) = (hg)f$.*
2. *(Identity): For each x in $Ob(\mathcal{C})$ there is an element 1_x in $\mathcal{C}(x, x)$, such that if $g \in \mathcal{C}(w, x)$ and $f \in \mathcal{C}(x, y)$, then $1_x g = g$ and $f 1_x = f$.*

Definition 2.1.2. *A functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, consists of $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$, $\mathcal{C} \mapsto F\mathcal{C}$ and for any $c_1, c_2 \in Ob(\mathcal{C})$, $F : \mathcal{C}(c_1, c_2) \to \mathcal{D}(Fc_1, Fc_2)$ such that $F(id_{\mathcal{C}}) = id_{F\mathcal{C}}$ and $F(g \circ f) = F(g) \circ F(f)$. Here $\mathcal{C}(c_1, c_2)$ denotes the collection of morphisms in $\mathcal{C}$ from c_1 to c_2.*
A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an isomorphism if there exists a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$ such that $F^{-1}F$ is the identity functor $id_{\mathcal{C}}$ of $\mathcal{C}$ and FF^{-1} is the identity of functor $\mathcal{D}$ $id_{\mathcal{D}}$.

Definition 2.1.3. *A functor $F : \mathcal{C} \to \mathcal{D}$ is full if for any $x, y \in Ob(\mathcal{C})$ the map $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is onto. It is faithful if $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is injective. A functor that is full and faithful is fully faithful. A functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective if for any $y \in Ob(\mathcal{D})$ there is a $x \in Ob(\mathcal{C})$ and an invertible arrow $\gamma \in \mathcal{D}$ from $F(x)$ to y.*

Definition 2.1.4. *(Subcategory) If we have two categories $\mathcal{C}$ and $\mathcal{D}$, we say $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if the following conditions are satisfied:*

1. *$Ob(\mathcal{D}) \subseteq Ob(\mathcal{C})$, any object of $\mathcal{D}$ is an object of $\mathcal{C}$,*
2. *For each x, y in $Ob(\mathcal{D})$, we have $\mathcal{D}(x, y) \subseteq \mathcal{C}(x, y)$,*
3. *The composition of morphisms in $\mathcal{D}$ is the same as that for $\mathcal{C}$, and*
4. *The identity in $\mathcal{D}(x, x)$ is the identity in $\mathcal{C}(x, x)$, for each $x \in Ob(D)$.*

The subcategory $\mathcal{D}$ of $\mathcal{C}$ is called full if for each objects $x, y \in \mathcal{D}$, we have $\mathcal{D}(x, y) = \mathcal{C}(x, y)$.

Definition 2.1.5. *[16] A 2-category is a system of 2-cells or "maps" which can be compose in two different but commuting categorical ways, horizontal and vertical. A 2-category consists of*

1. *objects*
2. *1-morphisms between objects*
3. *2-morphisms between morphisms*

The morphisms can be composed along the objects, while the 2-morphisms can be composed in two different directions: along objects called horizontal composition and along morphisms called vertical composition.

Note that, this notion of 2-category is different from the more general notion of bicategory that we will talk more about it in the next section.

Definition 2.1.6. *[16] A 2-functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories is assignment*

$$Ob(\mathcal{C}) \ni x \mapsto F(x) \in Ob(\mathcal{D})$$

together with functors

$$\mathcal{C}(x,y) \xrightarrow{F_{x,y}} \mathcal{D}(F(x), F(y))$$

that preserve identity objects and intertwine the compositions of $\mathcal{C}$ and $\mathcal{D}$ up to coherent natural transformations.

2.2 Bicategories

The definition of a bicategory is used to extend the notion of a category to handle the cases where the composition of morphisms is not (strictly) associative, but only associative up to an isomorphism, we follow the work in [12].

A *bicategory* **B** consists of a class of objects, morphisms between objects and 2-morphisms between morphisms together with several ways of composing them.
We will picture the objects as points:

$$\bullet \mathcal{G}$$

the morphisms between objects as arrows:

$$\mathcal{K} \xrightarrow{\phi} \mathcal{G}$$

2-morphism as double arrows

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{G} \end{tikzcd}$$

Definition 2.2.1. *[12] A 2-morphism $a : \phi \Rightarrow \psi$ is a 2-isomorphism in* **B** *if it is invertible, i.e. if there exists a 2-morphism $b : \psi \Rightarrow \phi$ such that $ab = id_\psi$ and $ba = id_\phi$. In this case we say that the morphisms ϕ and ψ are equivalent and write $\phi \sim \psi$.*

Definition 2.2.2. *[12] A morphism $\varphi : \mathcal{K} \to \mathcal{G}$ is an equivalence in* **B** *if it is invertible up to a 2-isomorphism; i.e. if there exists a morphism $\xi : \mathcal{G} \to \mathcal{K}$ such* *that $\varphi\xi \sim id_{\mathcal{G}}$ and $\xi\varphi \sim id_{\mathcal{K}}$. In this case we will say that the objects $\mathcal{K}$ and $\mathcal{G}$ are equivalent and write $\mathcal{K} \sim \mathcal{G}$.*

Now we want to describe the composition $\phi\varphi$ of morphisms $\varphi : \mathcal{K} \to \mathcal{L}$ and $\phi : \mathcal{L} \to \mathcal{G}$ which is denoted by:

$$\mathcal{K} \xrightarrow{\varphi} \mathcal{L} \xrightarrow{\phi} \mathcal{G}\ .$$

We can compose 2-morphisms in two ways called *horizontal* and *vertical* composition. The horizontal composition ab of 2-morphisms $b : \varphi \Rightarrow \varphi'$ and $a : \phi \Rightarrow \phi'$ is denoted by

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\varphi"'] \arrow[r, phantom, "\Downarrow b"] & \mathcal{L} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{G}
\end{tikzcd}$$

The vertical composition $a \cdot b$ of 2-morphisms $b : \phi \Rightarrow \varphi$ and $a : \varphi \Rightarrow \psi$ is denoted by

$$\begin{tikzcd}
\mathcal{K}\bullet \arrow[r, bend left=60, "\phi"] \arrow[r] \arrow[r, bend right=60, "\psi"'] \arrow[r, phantom, shift left=2ex, "\Downarrow b"] \arrow[r, phantom, shift right=2ex, "\Downarrow a"] & \bullet\mathcal{G}
\end{tikzcd}$$

$(a \cdot b)(c \cdot d) = (ac) \cdot (bd)$ this law relates horizontal and vertical compositions together

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left=60, "\varphi"] \arrow[r] \arrow[r, bend right=60, "\varphi"'] \arrow[r, phantom, shift left=2ex, "\Downarrow d"] \arrow[r, phantom, shift right=2ex, "\Downarrow c"] & \mathcal{L} \arrow[r, bend left=60, "\phi"] \arrow[r] \arrow[r, bend right=60, "\psi"'] \arrow[r, phantom, shift left=2ex, "\Downarrow b"] \arrow[r, phantom, shift right=2ex, "\Downarrow a"] & \mathcal{G}
\end{tikzcd}$$

We can see that the vertical composition is strictly associative whereas horizontal composition is only associative up to 2-isomorphism (associator).
The unit laws for morphisms hold up to 2-isomorphisms (left and right unit constraints).
Associator and unit constraints are required to be natural with respect to their arguments.
The notion of a 2-category differs from the more general notion of a bicategory in which the horizontal composition is required to be strictly associative, whereas in

a bicategory it needs only be associative up to isomorphism. So a 2-category is a bicategory in which the natural 2-isomorphisms are identities.

2.3 Groupoids

Since 1926 [7] groupoids were first introduced by Heinrich Brandt, there are many works on groupoids and its algebraic structures. We follow in this section the works of Moerdijk [34] and Brown [9]. We introduce the notion $\mathcal{G}$ of a groupoid and provide some examples. In the general way of defining groupoids, we can see it as a certain generalization of a group that allows for individual elements to have "internal symmetries".

Definition 2.3.1. *A groupoid $\mathcal{G}$ is a small category in which every arrow is invertible. Small category means there is a set of objects $Ob(\mathcal{G})$ and a set of morphisms $Mor(\mathcal{G})$, and maps $s, t : Mor(\mathcal{G}) \to Ob(\mathcal{G})$, $u : Ob(\mathcal{G}) \to Mor(\mathcal{G})$ with appropriate properties.*
To clarify concepts and fix notation, a groupoid $\mathcal{G}$ in the category Sets of sets, denoted $[G_1 \rightrightarrows G_0]$, consists of a set of arrows G_1 and a set of objects G_0, together with five structure maps

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop above, "u"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, dashed] \end{tikzcd}$$

The maps s and t are called source and target. An element $g \in G_1$ with $s(g) = x$ and $t(g) = y$ is an arrow from x to y and will be denoted by $g : x \to y$ or $x \xrightarrow{g} y$
The set

$$G_1 \times_{G_0} G_1 = \{(h, g) \in G_1 \times G_1 | s(h) = t(g)\}$$

consists of composable arrows, and m is called the composition or multiplication map, where $G_1 \times_{G_0} G_1$ fits into the pullback square

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "pr"] \arrow[d, "pr"'] & G_1 \arrow[d, "s"] \\ G_1 \arrow[r, "t"'] & G_0 \end{tikzcd}$$

For a pair (h, g) of composable arrows, the composition is denoted by $m(g, h) = hg$. The map u is called the unit map and we write $u(x) = 1_x$, and the map i is called

the inversion map and we write $i(g) = g^{-1}$
The names of the maps become clear as they must satisfy the following conditions

- $s(h \cdot g) = s(g)$, $t(h \cdot g) = t(h)$
- *Associativity* $k \cdot (h \cdot g) = (k \cdot h) \cdot g$
- *Identity* $1_{t(g)}g = g = g1_{s(g)}$
- $s1_x = x = t1_x$
- *Inverse* $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$ *and* $g^{-1} \cdot g = 1_{s(g)}$ $g \cdot g^{-1} = 1_{t(g)}$

for any $k, h, g, \in G_1$ *with* $s(k) = t(h)$ *and* $s(h) = t(g)$.

Definition 2.3.2. *For any* $x, y \in G_0$ *and* $\mathcal{G}$ *a groupoid. The set* $\mathcal{O}_x := t(s^{-1}(x)) = \{y \in G_0 : \exists g : x \to y\} \subset X$ *is called the orbit of* x. *The orbits of* $\mathcal{G}$ *define an equivalence relation on* G_0

$$x \sim y \Leftrightarrow y \in \mathcal{O}_x$$

and the quotient space for this relation is called the orbit space of $\mathcal{G}$ *and denoted by* $G_0/\mathcal{G}$ *or* $|\mathcal{G}|$.

Definition 2.3.3. *If* $\mathcal{G} = [G_1 \rightrightarrows G_0]$ *is a groupoid, and* $x, y \in G_0$ *then*

1. *the source-fiber at* x *is the set of all arrows from* x*;* $G_x = G(x, \cdot) = s^{-1}(x) = \{g \in G_1 | s(g) = x\}$
2. *the target-fiber at* y *is the set of all arrows to* y $G^y = G(\cdot, y) = t^{-1}(y) = \{g \in G_1 | t(g) = y\}$
3. *the isotropy group at* x *is the set of self arrows at* x $G^x_x = s^{-1}(x) \cap t^{-1}(x) = \{g \in G_1 | x \to^g x\}$

Definition 2.3.4. *A groupoid* $\mathcal{G}$ *is connected if* $G(x, y)$ *is non-empty for all objects* $x, y \in \mathcal{G}$. *The components of* $\mathcal{G}$ *are the maximal connected subgroupoids of* $\mathcal{G}$. *If every arrow in* $\mathcal{G}$ *is an identity, we say that* $\mathcal{G}$ *is trivial or discrete.*

Definition 2.3.5. *A groupoid* $\mathcal{G} = [G_1 \rightrightarrows G_0]$ *is called transitive if for all* $x, y \in G_0$ *there is an* $g \in G_1$ *with* $s(g) = x$ *and* $t(g) = y$.

Theorem 2.3.6. *Let x, y, x', y' be objects of a connected groupoid $\mathcal{G}$. There is a bijection*

$$\varphi : Hom_{\mathcal{G}}(x, y) \longrightarrow Hom_{\mathcal{G}}(x', y')$$

which if $x = y$, $x' = y'$ can be chosen to be an isomorphism of groups.

Proof. Let's take $x \overset{\alpha}{\to} x'$ and $y \overset{\beta}{\to} y'$ in $\mathcal{G}$ which exist by connectedness. We define

$$\begin{gathered} \varphi : Hom_{\mathcal{G}}(x, y) \longrightarrow Hom_{\mathcal{G}}(x', y') \\ \gamma \mapsto \beta + \gamma - \alpha \\ \psi : Hom_{\mathcal{G}}(x', y') \longrightarrow Hom_{\mathcal{G}}(x, y) \\ \delta \mapsto -\beta + \delta + \alpha \end{gathered}$$

So we get $\varphi(\gamma) = \beta + \gamma - \alpha$ and $\psi(\gamma) = -\beta + \gamma + \alpha$

$$\begin{tikzcd} x \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\ x' \arrow[r, "\varphi(\gamma)"'] & y' \end{tikzcd}$$

It is clear that $\varphi\psi = 1_{Hom_{\mathcal{G}}(x',y')}$ i.e, $\varphi\psi = 1$ and $\psi\varphi = 1_{Hom_{\mathcal{G}}(x,y)}$ i.e, $\psi\varphi = 1$
So φ is a bijection.
Also, if $x = y$ and $x' = y'$ we have that $\alpha = \beta$ so that φ sends $\gamma \mapsto \alpha + \gamma - \alpha$. If $\gamma, \gamma' \in Hom_{\mathcal{G}}(x, x)$,then $\varphi\gamma + \varphi\gamma' = \alpha + \gamma - \alpha + \alpha - \gamma' - \alpha = \alpha + \gamma + \gamma' - \alpha = \varphi(\gamma + \gamma')$
Therefore, φ is an isomorphism. $\square$

We will now discuss some basic examples that will be playing a role in the rest of this work.

Example 2.3.7 (Unit groupoid). *Any set X can be viewed as a groupoid itself $\mathcal{G} = u(X)$, $\mathcal{G} = [X \rightrightarrows X]$, where the only arrows are the identities $1_x \in X$ and the object set is X, for any x in X. This is the trivial groupoid, or the unit groupoid, and is simply written as X. The source and target maps are the identity map 1_x, and multiplication is only defined between an element $x \in X$ and itself $xx = x$. The unit groupoid over the interval $I = [0, 1]$ with the space objects and space arrows are the same I and all the arrows are units is denoted by $\mathcal{I}$.*

Example 2.3.8 (Pair groupoid). *Any set X gives rise to the pair groupoid of X, denoted by $Pair(X)$. The objects set is $G_0 = X$, and the set of arrows is $G_1 = X \times X$, so we have $\mathcal{G} = [X \times X \rightrightarrows X]$. It has source map $s(y, x) := x$ and* *target map* $t(y, x) := y$ *for every pair* $(y, x) \in X \times X$ *(where* s *and* t *are the first and second projection maps). Multiplication is given by* $(z, y)(y, x) = (z, x)$*. The unit map is* $u(x) = (x, x)$*. The inversion is defined by* $(y, x)^{-1} = (x, y)$*.*

Example 2.3.9 (Equivalence relation)**.** *An equivalence relation* R *on* X *becomes a groupoid with* $s, t : R \to X$ *the two projections and product* $(x, y)(y, z) = (x, z)$ *whenever* $(x, y), (y, z) \in R$*. The identity for* $x \in X$ *is* (x, x)*. Thus, every equivalence relation* $R \subset X \times X$ *gives a groupoid* $[R \rightrightarrows X]$*.*

Example 2.3.10 (Group)**.** *Any group can be considered as a special case of a groupoid. If we have a group* G*, then we take* $G_0 = *$*,* $G_1 = G$*, the source (and target) map is the unique map* $G \to *$*, the unit map* $u(*) = 1_G$*, multiplication and inversion are the same as in the group* G*. So we have a groupoid* $[G \rightrightarrows *]$*.*
More generally, given a collection of points, a collection of groups over those points is a groupoid. So a group can be thought as a groupoid with only one object and a groupoid can be thought as a group with many objects.

2.4 The fundamental groupoid of a topological space

In this section we will introduce the fundamental groupoid of a topological space and in order to do that we shall first describe a path category PX of the topological space. Let X be a topological space. A *path* in X is just a continuous function $I := [0, 1] \to X$. But composition of paths might not be associative in general, so one needs to either use homotopy classes of paths or Moore paths as reparametrisation to repair it. *Moore paths* are parametrised paths $f\alpha : [0, r] \to X$ for real numbers $r \geq 0$ (see [9]). Though these do not have inverses and don't form a groupoid, Moore paths do form a category as we will now explain (see [9] for all details). The composition of Moore paths is associative as we will see below.
A *path component of* X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y. Reparametrization as used above is consistent with taking equivalence classes (see [9]).
So we define a category PX, *the path category of the topological space* X which has as objects the points of X, and for any $x, y \in X$, the set $PX(x; y)$ is the set of Moore paths from x to y .
Composition of paths is defined as follows and written additively: for $\alpha : [0, t_0] \to X$ and $\beta : [0, t_1] \to X$ we set: $\beta + \alpha : [0, t_0 + t_1] \to X$. The identity in $PX(x, x)$ is the zero path 0_x on the degenerate interval $[0, 0]$. Finally, addition of paths is

associative since if γ, β, α are paths of lengths r, q, p respectively, then $\gamma + (\beta + \alpha)$ is defined if and only if $(\gamma + \beta) + \alpha$ is defined, and both paths are given by

$$t \mapsto \begin{cases} \alpha(t), & 0 \leqslant t \leqslant p, \\ \beta(t-p), & p \leqslant t \leqslant p+q, \\ \gamma(t-p-q), & p+q \leqslant t \leqslant p+q+r \end{cases}$$

Thus, the category PX of Moore paths on X is not a groupoid since if α is a path in X of positive length then there is no path β such that $\beta + \alpha$ is a zero path.
We will work with Moore paths and call them simply paths here, but we will use reparametrisation to the unit interval in what follows to simplify notations if necessary (see also [9]).
Now we introduce an equivalence relation on $PX(x, y) = Hom_{PX}(x; y)$. We will say that two paths α, β are *homotopic rel endpoints* x, y if there exists a homotopy of paths between them (see [9]).
We use the notation $F : \alpha \sim \beta$ to mean that F is a homotopy rel endpoints from α to β. This is an equivalence relation because:

- (Reflexivity) There is a unique homotopy of length 0 from α to α.
- (Symmetry) If $F : \alpha \sim \beta$ is a homotopy of length q, then $-F$, defined by $(s,t) \mapsto F(s, q-t)$, is a homotopy $\beta \sim \alpha$ of length q.
- (Transitivity) If $F : \alpha \sim \beta$, $G : \beta \sim \gamma$ are homotopy of length q, q' respectively where α, β, γ are of length r, then the sum of F and G
  $G + F : [0, r] \times [0, q + q'] \longrightarrow X$

$$(s,t) \mapsto \begin{cases} F(s,t) & , 0 \leqslant t \leqslant q, \\ G(s, t-q) & , q \leqslant t \leqslant q + q' \end{cases}$$

is continuous by gluing continuous functions and is a homotopy $\alpha \sim \gamma$ of length $q + q'$.
So, $\alpha \sim \beta$ if we can construct a continuous family of paths from x to y such that the first of those paths is just α and the last is β. The equivalence classes of paths from x to y are called *homotopy classes of paths* from x to y.

Definition 2.4.1. *The fundamental groupoid* ΠX *of a topological space* X *is the groupoid whose objects are the points of* X *and whose morphisms* $x \to y$ *are* *homotopy classes of paths from x to y. Thus, the fundamental groupoid of a space X is a groupoid ΠX with $Ob(\Pi X) = X$ and $Hom_{\Pi X}(x,y) = Hom_{PX}(x,y)/\sim$.*

The set of homotopy classes of paths forms a groupoid, where the set of objects is the set X, $(\Pi X)_0 = X$ and the set of morphisms is the set of homotopy classes of paths in X denoted by $(\Pi X)_1$. This groupoid $[(\Pi X)_1 \rightrightarrows X]$ has a source map defined by $s([\alpha]) = \alpha(0)$ and a target map by $t([\alpha]) = \alpha(1)$. Composition is induced by concatenation of paths

$$\alpha' + \alpha(t) = \begin{cases} \alpha(2t) & ,t \leq \frac{1}{2} \\ \alpha'(2t-1) & ,t > \frac{1}{2} \end{cases}$$

The unit $u(x)$ at $x \in X$ is defined by the constant path $[x]$ and the inverse is defined by $[\alpha]^{-1} = [\alpha^{-1}]$ where $\alpha^{-1}(t) = \alpha(1-t)$. So the fundamental groupoid ΠX is a groupoid.
If we restrict to a single $x_0 \in X$, then the collection of homotopy classes of paths in X that start and end at x_0 is a group: the fundamental group $\pi_1(X,x)$, based at x_0. So the isotropy groups of the fundamental groupoid are the fundamental groups of the space, and are all isomorphic if the space is connected.

Theorem 2.4.2. *There is a functor $\Pi : \mathsf{Top} \longrightarrow Grpd$ from the category of topological spaces to the category of groupoids, which sends a topological space X to its fundamental groupoid ΠX.*

Proof. Let $f : X \to Y$ be a map of spaces and $Pf : PX \to PY$ the corresponding functor of paths categories. Suppose first of all that α, β are two homotopic paths in X of length r from x to x'. Then there is a map $F : [0,r] \times I \longrightarrow X$ such that $F_0 = \alpha$, $F_1 = \beta$ and $F(0,t) = x$, $F(r,t) = x'$ for all $t \in I$. The composite $f \circ F : [0,r] \times I \longrightarrow Y$ is a homotopy between $f \circ \alpha$ and $f \circ \beta$, $f \circ \alpha \sim f \circ \beta$.
If α, β are equivalent paths in X from x to x', then there are constant paths r, s such that $r + \alpha, s + \beta$ are homotopic $r + f\alpha = f(r+\alpha) \sim f(s+\beta) = s + f\beta$ and so $f\alpha$ is equivalent to $f\beta$. Thus we have a well defined function

$$\begin{gathered} \Pi f : \Pi X \longrightarrow \Pi Y \\ [\alpha] \mapsto [f\alpha] \\ \Pi f([\alpha]) = [f \circ \alpha], \end{gathered}$$

which is a morphism between the groupoids ΠX and ΠY. Now we have successfully created our functor $\Pi : \mathsf{Top} \longrightarrow Grpd$: given a space X it gives its fundamental groupoid ΠX and given a continuous $f : X \longrightarrow Y$ it gives the morphism $\Pi f : \Pi X \to \Pi Y$ as constructed above. $\square$

2.5 Covering groupoids

The theory of covering spaces is one of the most beautiful theories in classical algebraic topology. In this section we will extend the study of coverings of spaces to the case of general groupoids and give some definitions and theorems of coverings of groupoids. We will use the sources [9], [31].

Definition 2.5.1. *Given a topological space X, a covering space of X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ each of which maps homeomorphically onto U_α by p.*

Definition 2.5.2. *Let X and Y be two topological spaces with a covering space $p : \tilde{X} \to X$. Consider a map $f : Y \to X$. Then, the map $\tilde{f} : Y \to \tilde{X}$ is said to be a lift to f if $p \circ \tilde{f} = f$.*

Definition 2.5.3. *Let $\mathcal{G}$ be a groupoid and x be an object in $\mathcal{G}$. Define the star of x, denoted $St(x)$ or $St_{\mathcal{G}}(x)$, to be the set of morphisms of $\mathcal{G}$ with source x. Here we write $\pi(\mathcal{G}, x) = \mathcal{G}(x, x)$ for the group of automorphisms of the object x.*

Definition 2.5.4. *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be a morphism of groupoids. We say p is a covering morphism if for each object $\tilde{x}$ of $\tilde{\mathcal{G}}$ the restriction of $p : St_{\tilde{\mathcal{G}}}(\tilde{x}) \longrightarrow St_{\mathcal{G}}(p\tilde{x})$ is bijective. In such a case, we call $\tilde{\mathcal{G}}$ a covering groupoid of $\mathcal{G}$. The covering morphism p is called connected if both $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are connected.*

Remark 2.5.5. A covering morphism $p : \tilde{\mathcal{G}} \to \mathcal{G}$ of groupoids is called *transitive* if both groupoids $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are transitive.

Proposition 2.5.6. *[46] Let $p : (\tilde{\mathcal{G}}, e') \to (\mathcal{G}, e)$ be a covering of groupoids, then the induced map $p : \pi(\tilde{\mathcal{G}}, e') \to \pi(\mathcal{G}, e)$ is injective.*

Proposition 2.5.7. *[31] If $p : \tilde{X} \to X$ is a covering of spaces, then the induced functor $\Pi p : \Pi\tilde{X} \to \Pi X$ is a covering of groupoids.*

Example 2.5.8. *Let $p : \tilde{X} \longrightarrow X$ be a covering map, let A be a subset of X and let $\tilde{A} = p^{-1}(A)$. Then the induced morphism $\Pi p : \Pi\tilde{A} \longrightarrow \Pi A$ is a covering of groupoid.*

Proof. Let $\tilde{x} \in \tilde{A}$ and let $p(\tilde{x}) = x$. For each path α in X with initial point x, let $\tilde{\alpha}$ denote the unique covering path of $\tilde{X}$ with initial point $\tilde{x}$. If the final point of α is in A, then the final point of $\tilde{\alpha}$ is in $\tilde{A}$. Also, the equivalence class of $\tilde{\alpha}$ depends only on the equivalence class of α by path lifting property. So the function $[\alpha] \mapsto [\tilde{\alpha}]$ is inverse to the restriction of p which maps $St(\tilde{x}) \mapsto St(x)$. Thus we proved the bijectivity of the restriction map of p, so it is a covering groupoid. □

The next definition is from the source [31].

Definition 2.5.9. *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{B}$ be a covering of groupoids. For an object $x \in \mathcal{G}$, let F_x denote the set of objects of $\tilde{\mathcal{G}}$ such that $p(\tilde{x}) = x$, then $p^{-1}(St(x))$ is the disjoint union over $\tilde{x} \in F_x$ of $St(\tilde{x})$. Define the fiber translation functor $T = T(p) : \mathcal{B} \to \mathcal{G}$ as follows. For an object $x \in \mathcal{B}$, $T(x) = F_x$. For a morphism $f : x \to x'$ of $\mathcal{B}$, $T(f) : F_x \to F_{x'}$ is specified by $T(f)(e) = e'$, where e' is the target of the unique g in $St(e)$ such that $p(g) = f$.*

For a covering space $p : E \to B$ and a path $f : b \to b'$, $T(f) : F_b \to F_{b'}$ is given by $T(f)(e) = g(1)$ where g is the path in E that starts at e and covers f.

2.5.1 Group actions and groupoid actions

After having defined covering spaces and covering groupoids, we would like to classify the maps between theses objects . To do so, we need to give a brief detour by considering the theory of group actions and how we can generalise this to obtain the notion of groupoid actions. Let us first recall some basic definitions.

Definition 2.5.10. *Let G be a group, and X a set.*

1. *A (left) action of a group G on a set X is a function $\pi : G \times X \to X$ such that $e.x = x$ and $(g'g).x = g'(gx)$ for all $x \in X$.*

2. *The stabilizer or isotropy group G_x of a point x is the subgroup of G with elements $\{g|gx = x\}$.*

3. *An action is free or semiregular if all stabilizers are trivial.*

4. *An action is transitive if for every pair of elements x and $x' \in X$, there is a group element $g \in G$ such that $gx = x'$.*

5. *An action is regular if it is both free and transitive.*

The set X with a G-action is called a G-set.

Definition 2.5.11. *Let G be a group, and let X and Y be G-sets. A G-map $\phi : X \to Y$ is a set function that respects the G-set structure, i.e. such that $\phi(gx) = g\phi(x)$ for all $x \in X$ and $g \in G$. If ϕ is also a bijection, we call it an isomorphism of G-sets.*

Definition 2.5.12. *Let G be a group. The orbit category of G, denoted $\mathcal{O}(G)$ is defined to be the category with objects the G-sets G/H and morphisms the G-maps between them. These G-sets are called canonical orbits.*

It is possible to generalize the notion of group actions to groupoids. Since a group action is a functor from a group G to a category **Set**, where the objects are all small sets and arrows are functions between them, it is natural to define a groupoid action as a functor Ψ from a groupoid $\mathcal{G}$ to **Set**.

Definition 2.5.13. *[46] Form the category $\mathcal{A}$ with objects all triples $(G, X, \cdot)$, where G is a group, X a set, and $\cdot : G \times X \to X$ is a group action. Given two objects $(G, X, \cdot)$ and $(G', X', \cdot')$, we define a morphism $(\alpha, \phi) : (G, X, \cdot) \to (G', X', \cdot')$ whenever $\alpha : G \to G'$ is a homomorphism and $\phi : X \to X'$ is a G-map satisfying $\phi(g \cdot x) = \alpha(g) \cdot' \phi(x)$ for all $g \in G$ and $x \in X$. We call a morphism in this category a map of group actions, and an isomorphism, an isomorphism of group actions.*

Definition 2.5.14. *Let $\mathcal{G}$ be a groupoid. An action of $\mathcal{G}$ on a set S consists of a set S, a function $\mathsf{w} : S \longrightarrow Ob(\mathcal{G})$, and a partial function $\mathcal{G} \times S \rightarrowtail S$ defined as $(g, s) \mapsto g \cdot s$, which for each $x, y \in Ob(\mathcal{G})$, assigns to an element (g, s), where $g \in Hom_{\mathcal{G}}(x, y)$ and $s \in \mathsf{w}^{-1}(x)$, an element $g \cdot s \in \mathsf{w}^{-1}(y)$.*
The following rules are to be satisfied:

- *If $x \in Ob(\mathcal{G}), s \in \mathsf{w}^{-1}(x)$, then $1_x \cdot s = s$*
- *If $g \in Hom_{\mathcal{G}}(x, y), h \in Hom_{\mathcal{G}}(y, z), s \in \mathsf{w}^{-1}(x)$, then $(h \cdot g) \cdot s = h \cdot (g \cdot s)$*

We also say $\mathcal{G}$ acts on S via w, and that S is a $\mathcal{G}$-set.
The action of the groupoid $\mathcal{G}$ on the set S is said to be transitive if for all x, y in $Ob(\mathcal{G})$, $s \in \mathsf{w}^{-1}(x)$, $t \in \mathsf{w}^{-1}(y)$, there is a $g \in Hom_{\mathcal{G}}(x, y)$ such that $g \cdot s = t$.
If $s \in \mathsf{w}^{-1}(x)$, the group of stabilizes of S is the subgroup $\mathcal{G}_s$ of $\mathcal{G}$ of elements g such that $g \cdot s = s$.

An action is free (or semi-regular) if all stabilizers are trivial. So if $g \cdot s = s$ implies $g = e$, that is, if $\mathcal{G}_s = e$ for every $s \in S$.

Definition 2.5.15. *[9] Let G be a group and Γ be a groupoid with G acting on the left. The semidirect product groupoid $\Gamma \rtimes G$ has object set $Ob(\Gamma)$ and arrows $x \to y$ the set of pairs (γ, g) such that $g \in G$ and $\gamma \in \Gamma(g \cdot x, y)$. The sum of $(\gamma, g) : x \to y$ and $(\delta, h) : y \to z$ in $\Gamma \rtimes G$ is defined to be*

$$(\delta, h) + (\gamma, g) = (\delta + h \cdot \gamma, hg).$$

Definition 2.5.16. *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be a covering morphism of groupoids, and let $S = Ob(\tilde{\mathcal{G}})$, $\mathsf{w} = Ob(p)$, then we obtain an action of the groupoid $\mathcal{G}$ on the set S via a morphism $\mathsf{w} : S \longrightarrow Ob(\mathcal{G})$.*

Corollary 2.5.17. *The projection $p : \mathcal{G} \ltimes S \longrightarrow \mathcal{G}$, given on objects by $\mathsf{w} : S \to Ob(\mathcal{G})$ and on elements by $(s, g) \mapsto g$, is a covering morphism of groupoids.*

Proof. It is clear from the definition above that p is a morphism of groupoids. Also p is a covering morphism, because if $g \in Hom_{\mathcal{G}}(x, y)$, $s \in \mathsf{w}^{-1}(x)$, then (s, g) is the unique element of $\mathcal{G} \ltimes S$ which has source s and projects to g. $\square$

Proposition 2.5.18. *The groupoid $\mathcal{G} \ltimes S$ is connected if and only if the action is transitive.*

Proof. Here we need only to note that $(\mathcal{G} \ltimes S)(s, t)$ is non-empty if and only if there is a g in $\mathcal{G}$ such that $g \cdot s = t$. $\square$

2.5.2 The classification of coverings of groupoids

Given a groupoid $\mathcal{G}$, we want to classify all its coverings and the maps between them. In order to do this, we should clarify the notion of a map between coverings of groupoids. Since classification in mathematics is always done up to isomorphism type, we will show how two coverings of groupoids can be isomorphic. Now we shall start with the following theorem and in the whole section we will use May's book as the main source [31].

Definition 2.5.19. *A covering $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ of groupoids is regular if $p(\pi(\tilde{\mathcal{G}}, e))$ is a normal subgroup of $\pi(\mathcal{G}, b)$ and it is universal if $p(\pi(\tilde{\mathcal{G}}, e)) = \{e\}$, for each object $e \in \tilde{\mathcal{G}}$, where $b = p(e)$.*

Corollary 2.5.20. *A 1-connected covering groupoid of $\mathcal{G}$ covers every covering groupoid of $\mathcal{G}$. So a 1-connected covering groupoid of $\mathcal{G}$ is called a universal covering groupoid of $\mathcal{G}$.*

Proof. The proof is shown in [9]. □

Theorem 2.5.21 (The fundamental theorem of covering groupoid theory)**.** *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be a covering of groupoids, and $f : \mathcal{H} \longrightarrow \mathcal{G}$ be a functor. Choose $x_0 \in \mathcal{H}$ and let $b_0 = f(x_0)$ and $e_0 \in F_{b_0}$ in $\tilde{\mathcal{G}}$. Then there exists a functor $g : \mathcal{H} \longrightarrow \tilde{\mathcal{G}}$ such that $g(x_0) = e_0$ and $p \circ g = f$ if and only if $f(\pi(H, x_0)) \subset p(\pi(\tilde{G}, e_0))$ in $\pi(G, b_0)$. When this condition holds, there is a unique such functor g.*

Proof. It's shown in [31]. □

Definition 2.5.22. *let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and $f : \mathcal{H} \longrightarrow \mathcal{G}$ be coverings of a groupoid $\mathcal{G}$. A map $g : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ of coverings of $\mathcal{G}$ is a functor g such that the following diagram of functors is commutative:*

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[rr, "g"] \arrow[dr, "p"'] & & \mathcal{H} \arrow[dl, "f"] \\ & \mathcal{G} & \end{tikzcd}$$

Then we say that g is a map of coverings over $\mathcal{H}$.

Lemma 2.5.23. *A map $g : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ of coverings of $\mathcal{G}$ is itself a covering of $\mathcal{H}$.*

Proof. The proof is immediate from 2.5.22 of covering groupoids and it's shown in [31]. □

After we see that the map $g : \mathcal{H} \longrightarrow \tilde{\mathcal{G}}$ is a covering in the above lemma, we can rewrite the theorem in this way:

Theorem 2.5.24. *[31] Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and $f : \mathcal{H} \longrightarrow \mathcal{G}$ be coverings of $\mathcal{G}$ and choose base objects $x \in \mathcal{G}$, $\tilde{x} \in \tilde{\mathcal{G}}$, and $x' \in \mathcal{H}$ such that $p(\tilde{x}) = x = f(x')$. There exists a morphism $g : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ of coverings with $g(\tilde{x}) = x'$ if and only if $p(\pi(\tilde{G}, \tilde{x})) \subset f(\pi(H, x'))$. In particular, two maps of covers $g, g' : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ coincide if $g(\tilde{x}) = g'(\tilde{x})$ for any one object $\tilde{x} \in \tilde{\mathcal{G}}$.*

Proof. The proof is shown in [31]. □

Proposition 2.5.25. *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and $f : \mathcal{H} \longrightarrow \mathcal{G}$ be coverings, $g : \tilde{\mathcal{G}} \to \mathcal{H}$ is an isomorphism if and only if the displayed inclusion of subgroups of $\pi(\mathcal{G}, x)$ is an equality, for $x \in \mathcal{G}$. Therefore, $\tilde{G}$ and H are isomorphic if and only if $p(\pi(\tilde{G}, \tilde{x}))$ and $f(\pi(\mathcal{H}, x'))$ are conjugate whenever $p(\tilde{x}) = f(x')$. Furthermore, if the map $g : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ of coverings exists, the universal cover of $\mathcal{G}$ is unique up to isomorphism and covers any other cover.*

Definition 2.5.26. *Let $\mathcal{G}$ be a groupoid. We define $Cov(\mathcal{G})$ to be the category with objects the covering groupoids of $\mathcal{G}$, and morphisms the maps of coverings. So we write $Cov(\mathcal{G})$ to denote the category of coverings of $\mathcal{G}$; when $\mathcal{G}$ is understood, and we write $Cov(\tilde{\mathcal{G}}, \mathcal{H})$ for the set of maps $\tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ of coverings of $\mathcal{G}$.*

We have the following theorem

Theorem 2.5.27. *Let $p : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and $p' : \mathcal{H} \longrightarrow \mathcal{G}$ be coverings, choose a base object $x \in \mathcal{G}$, and let $A = \pi(\mathcal{G}, x)$. If $g : \tilde{\mathcal{G}} \longrightarrow \mathcal{H}$ is a map of coverings, then g restricts to a map $F_x \longrightarrow F'_x$ of A-sets, and restriction to fibers specifies a bijection between $Cov(\tilde{\mathcal{G}}, \mathcal{H})$ and the set of A-maps $F_x \longrightarrow F'_x$.*

Proof. Let $e \in F_x$ and $f \in \pi(\mathcal{G}, x)$. By definition, fe is the target of the map $\tilde{f} \in St_{\mathcal{G}}(e)$ such that $p(\tilde{f}) = f$. Clearly $g(fe)$ is the target of $g(\tilde{f}) \in St_{\mathcal{H}}(g(e))$ and $p'(g(\tilde{f})) = p(\tilde{f}) = f$. Again by definition, this gives $g(fe) = fg(e)$. The theorem above shows that restriction to fibers is an injection on $Cov(\tilde{\mathcal{G}}, \mathcal{H})$. To show surjectivity, let $\alpha : F_x \to F_{x'}$ be a A-map. Choose $e \in F_x$ and let $e' = \alpha(e)$. Since α is an A-map, the isotropy group $p(\pi(\tilde{\mathcal{G}}, e))$ of e is contained in the isotropy group $p'(\pi(\mathcal{G}, e'))$ of e'. Therefore the previous theorem ensures the existence of a covering map g that restricts to α on fibers. □

2.6 Topological groupoids

Until now, we have only considered groupoids $\mathcal{G} = [G_1 \rightrightarrows G_0]$, in the category Sets of sets, where G_0 and G_1 are sets. In most interesting cases, they could be topological spaces or smooth manifolds for example and its maps will have more structure. So in the first case when all the sets are topological spaces, the structure maps are continuous functions, it will be a groupoid in **Top** and it is called a *topological groupoid*, where **Top** is a topological category or a category of topological spaces and defined as the category whose objects are topological

spaces and whose morphisms are continuous maps. Also a *smooth groupoid* (or *Lie groupoid*) is a groupoid in **Diff** where all the objects are smooth i.e. C^∞-manifolds, all maps are smooth and, in addition, the source map is a smooth submersion map. So a groupoid with a topological structure (resp. a differential structure) is called a topological groupoid (resp. a Lie groupoid).
Here we will be concerned mainly with the case where the G_0 and G_1 are topological spaces, in which case $\mathcal{G} = [G_1 \rightrightarrows G_0]$ is a topological groupoid. At the beginning we define and describe the internal structure of topological groupoids. Then we will determine some examples of topological groupoids. (See also [34], [12], [36]) .

Definition 2.6.1. *A topological groupoid $\mathcal{G}$ is a small topological category such that all morphisms are invertible. Here small category means there is a set of objects $Ob(\mathcal{G})$ and a set of morphisms $Mor(\mathcal{G})$, and maps $s, t : Mor(\mathcal{G}) \to Ob(\mathcal{G})$, $u : Ob(\mathcal{G}) \to Mor(\mathcal{G})$ with appropriate properties. Topological means that all the maps s, t, u, m, i, are continuous functions.*
A topological groupoid is called open if the source map $s : Mor(\mathcal{G}) \to Ob(\mathcal{G})$ is open. Moreover, it is called an etale groupoid if in addition s (and so also t) is a local homeomorphism.

The internal categorical structure of a topological groupoid:
As we have seen in the definition above a topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ consists of the following:

1. the object space G_0 is a topological space,
2. the arrow space G_1 is a topological space,
3. (source)
   a source map which is a continuous map $s : G_1 \to G_0$,
4. (target)
   a target map which is a continuous map $t : G_1 \to G_0$,
5. (unit)
   a unit map which is a continuous map $u : G_0 \to G_1$,
6. (multiplication)
   a multiplication map which is a continuous map $m : G_1 \times_{s,t} G_1 \to G_1$, $(g, h) \mapsto g \cdot h$
7. (inverse) an inversion map which is a homeomorphism $i : G_1 \to G_1$.

These maps satisfy the following identities:

- $s(u(x)) = x = t(u(x))$
- $u(t(g)) \cdot g = g = g \cdot u(s(g))$
- $s(g \cdot h) = s(h)$, $t(g \cdot h) = t(g)$
- $(g \cdot h) \cdot k = g \cdot (h \cdot k)$
- $s(i(g)) = t(i(g)) = s(g)$, $g \cdot i(g) = u(t(g))$ and $i(g) \cdot g = u(s(g))$

$$\begin{tikzcd} s(g) \arrow[r, bend right, "g"'] & t(g) \arrow[l, bend right, "i(g)"'] \end{tikzcd} , \quad x \xrightarrow{u(x)} x \; , \quad \begin{tikzcd} z \arrow[r, "h"'] \arrow[rr, bend left, "h \cdot g"] & y \arrow[r, "g"'] & x \end{tikzcd}$$

Let us look at a few examples of topological groupoids related to the examples of groupoids in the last section.

Example 2.6.2 (Topological space)**.** *Let X be a topological space. Let $G_0 = G_1 = X$, all structure maps are the identity map. This turns any topological space into a topological groupoid $\mathcal{G} = [X \rightrightarrows X]$.*

Example 2.6.3 (Topological group)**.** *Let G be a topological group. Let $G_0 = *$, $G_1 = G$, s and t the unique maps $G \to *$, $u(*) = 1_G$, m and i the multiplication and inversion in G. This turns any topological group into a topological groupoid $\mathcal{G} = [G \rightrightarrows *]$.*
The disjoint union of topological groups is a topological groupoid which is not a topological group.

Example 2.6.4 (Pair groupoid)**.** *(see example 2.3.8) The pair groupoid $Pair(X)$ is viewed as topological groupoid with the set of objects $Pair(X)_0 = X$ and arrows $Pair(X)_1 = X \times X$. The source and target map are the first and second projection. The multiplication is unique, because any $x, x' \in X$ there is exactly one arrow from x to x'. Note that any continuous map $p : Y \to X$ induces a homomorphism of pair groupoids $p \times p : Pair(Y) \to Pair(X)$. Furthermore, if p is open surjection we may define the kernel groupoid $Ker(p)$ over Y, which is a topoligical subgroupoid of $Pair(Y)$, consisting of all pairs $(y, y') \in Y \times Y$ with $p(y) = p(y')$, i.e. $Ker(p)_1 = Y \times_X Y$.*

Example 2.6.5 (Fundamental groupoid of a topological space)**.** *Let X be a topological space. Consider the topological groupoid*

$$[(\Pi X)_1 \underset{t}{\overset{s}{\rightrightarrows}} X]$$

where $\Pi_1 = [\alpha] : \alpha : [0,1] \to X$ the set of homotopy classes of paths in X. For an arrow $[\alpha] \in \Pi_1(X)$ its source and target are given by $s([\alpha]) = \alpha(0)$, $t([\alpha]) = \alpha(1)$ while the composition of two arrows is just concatenation of paths $[\alpha][\beta] = [\alpha.\beta]$.

Example 2.6.6 (Covering groupoid)**.** *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of a locally compact space X. We use the notation $U_{ij} = U_i \cap U_j$ for $i, j \in I$. Consider the groupoid $[\bigsqcup_{i,j\in I} U_{i,j} \rightrightarrows \bigsqcup_{i\in I} U_i]$, whose object set is $\bigsqcup_i U_i$.*
Here the source map is the inclusion $U_{i,j} \to U_j$ and the target map is the inclusion $U_{i,j} \to U_i$. Composition $U_{i,j} \times_{s,t} U_{j,k} \to U_{i,k}$ is defined as $(x, y) \mapsto x(= y)$. The unit is the identity $U_i \mapsto U_{ii} = U_i$. The inverse map is the identity $U_{i,j} \to U_{j,i}$. This groupoid is called the covering groupoid associated to the cover $\mathcal{U}$ of X. Moreover this covering groupoid is an etale groupoid, as s and t are local homeomorphisms.

Example 2.6.7 (Action groupoid)**.** *Suppose a topological group G acts from the left on a topological space X. Then we construct the translation groupoid $G \ltimes X = [G \times X \rightrightarrows X]$, with the objects sets $(G \ltimes X)_0 = X$ and the arrows set $(G \ltimes X)_1 = G \times X$, $(g, x) \in G \times X$. The source map is defined by projection $s(g, x) := x$ and the target is given by the action map $t(g, x) := g \cdot x$. The multiplication is $(g', y)(g, x) = (g'g, x)$, if $y = g \cdot x$ or we can just say $(g', g \cdot x)(g, x) = (g'g, x)$. The unit map is defined by $u(x) := (e, x)$ and the inversion by $(g, x)^{-1} = (g^{-1}, g \cdot x)$. This groupoid is a topological groupoid if the action is continuous.*

Chapter 3

Morphisms and Equivalences of Groupoids

In this work, we deal with the homomorphisms of topological groupoids and we will give the definition of equivalences between them. Also, in this chapter we will present a broader concept of maps between topological groupoids, namely generalized maps and Hilsum-Skandalis maps, that will define equivalent bicategories.

3.1 Homomorphisms between groupoids

As the definition of a groupoid can be put in categorical terms, it is natural to define morphisms between groupoids in the following way by using [22] and [34] and provide a concept of natural transformation which gives a way of "moving between the images of two functors".

Definition 3.1.1. *A homomorphism between two groupoids $\mathcal{G}$ and $\mathcal{K}$ is a functor $\phi : \mathcal{G} \to \mathcal{K}$; it is given by a map on objects $G_0 \to K_0$ and a map on arrows $G_1 \to K_1$, which together preserve the groupoid structure, i.e. $\phi(s(g)) = s(\phi(g))$, $\phi(t(g)) = t(\phi(g))$, $\phi(1_x) = 1_{\phi(x)}$ and $\phi(gh) = \phi(g)\phi(h)$, for any $g, h \in G_1$ with $s(g) = t(h)$ and any $x \in G_0$.*

Definition 3.1.2. *A homomorphism between topological groupoids $\mathcal{G}$ and $\mathcal{K}$ is by definition a continuous functor $\phi : \mathcal{G} \to \mathcal{K}$ given by two continuous maps $\phi_0 : G_0 \to K_0$ and $\phi_1 : G_1 \to K_1$ that together commute with all the structure* *maps of the groupoids $\mathcal{G}$ and $\mathcal{K}$.*

Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. A homomorphism $\phi : \mathcal{G} \to \mathcal{K}$ is a continuous functor between topological gropoids. Specifically, ϕ is defined by two continuous maps, $\phi_0 : G_0 \to K_0$ and $\phi_1 : G_1 \to K_1$ which satisfy the functor relations:

- *If $x \in G_0$, then ϕ_1 takes the identity map on x in $\mathcal{G}$ to the identity map on $\phi_0(x)$ in $\mathcal{K}$; $\phi(1_x) = 1_\phi(x)$*
- *If $g \in G_1$, then ϕ preserves the source s and target t of g;*

$$\phi_0(s(g)) = s(\phi_1(g)), \qquad \phi_0(t(g)) = t(\phi_1(g))$$

- *If g_1 and g_2 are two arrows in G_1 such that $t(g_1) = s(g_2)$, then the arrows $\phi_1(g_1)$ and $\phi_1(g_2)$ can be composed in K_1 since ϕ preserves the source and target. Moreover, ϕ_1 respects the composition: $m(\phi_1(g_1), \phi_1(g_2)) = \phi_1(m(g_1, g_2))$. It follows from the above that ϕ preserves inverses as well.*

After we have defined the homomorphisms of topological groupoids, now we will define induced groupoids and transformations between these homomorphisms of topological groupoids. A natural transformation gives a way of "moving between the images of two functors".

Definition 3.1.3. *Let $\mathcal{G}$ be a topological groupoid and $\phi : X \to G_0$ a continuous map. Then we can define the induced groupoid $\phi^*(\mathcal{G})$ as the groupoid $\phi^*(\mathcal{G}) = [X \times_{G_0} G_1 \times_{G_0} X \rightrightarrows X]$ in which the arrows from x to y are the arrows in $\mathcal{G}$ from $\phi(x)$ to $\phi(y)$, i.e.*

$$\begin{aligned} \phi^*(\mathcal{G})_1 &= X \times_{G_0} G_1 \times_{G_0} X, \\ &= \{(x, g, y) | \phi(x) \xrightarrow{g} \phi(y)\} \end{aligned}$$

and the multiplication is given by the multiplication in $\mathcal{G}$, and defined as $(x, g, y)(m, h, n) = (x, gh, n)$ for any $x, y, m, n \in X$ and $g, h \in G_1$.

The space $\phi^*(\mathcal{G})_1$ can be constructed by two pull-backs as in the diagram

$$\begin{tikzcd}
\phi^*(\mathcal{G})_1 \arrow[rr] \arrow[d] & & X \arrow[d, "\phi"] \\
G_1 \times_{G_0} X \arrow[r, "pr_1"] \arrow[d] & G_1 \arrow[r, "t"] \arrow[d, "s"] & G_0 \\
X \arrow[r, "\phi"] & G_0 &
\end{tikzcd}$$

The lower pull-back has a natural continuous structure and if the composition $t \circ pr_1$ is an open surjection, then the upper pull-back has also a natural continuous structure. It follows that the square

$$\begin{tikzcd}
\phi^*(\mathcal{G})_1 \arrow[r] \arrow[d, "(s,t)"'] & G_1 \arrow[d, "(s,t)"] \\
X \times X \arrow[r, "\phi\times\phi"] & G_0 \times G_0
\end{tikzcd}$$

is a pull-back of topological spaces. Therefore $\phi^*(\mathcal{G})$ is a topological groupoid whenever the map $t \circ pr_1 : G_1 \times_{G_0} X \to G_0$ is an open surjection. The map ϕ induces a homomorphism of topological groupoids $\phi : \phi^*(\mathcal{G}) \to \mathcal{G}$.

Example 3.1.4 (Restricted)**.** *Let $\mathcal{G}$ be a groupoid and U open in G_0. Then the induced groupoid of $U \to G_0$ is the subgroupoid of arrows between elements of U.*

Definition 3.1.5. *A natural transformation T between two morphisms $\phi, \psi : \mathcal{G} \to \mathcal{K}$ of topological groupoids is a continuous map $T : G_0 \to K_1$ such that for each $x \in G_0$, $T(x)$ is an arrow from $\phi(x)$ to $\psi(x)$ in K_1, $T(x) : \phi(x) \to \psi(x)$ and for each arrow $g : x \to y$ in G_1 the square*

$$\begin{tikzcd}
\phi(x) \arrow[r, "T(x)"] \arrow[d, "\phi(g)"'] & \psi(x) \arrow[d, "\psi(g)"] \\
\phi(y) \arrow[r, "T(y)"] & \psi(y)
\end{tikzcd}$$

commutes and the identity $\psi(g)T(x) = T(y)\phi(g)$ holds. We write $\phi \sim_T \psi$ to indicate that T is such a transformation from ϕ to ψ.

We can compose two transformations as follows.

Definition 3.1.6. *Let $\phi, \psi, \rho : \mathcal{G} \to \mathcal{K}$ be topological groupoid homomorphism, T and S are two natural transformations between them as follows $T : \phi \to \psi$ and $S : \psi \to \rho$. We define $S \circ T : \phi \to \rho$ to be the transformation with $S \circ T : G_0 \to K_1$ given by*

$$(S \circ T)(x) = (\phi(x) \xrightarrow{T(x)} \psi(x) \xrightarrow{S(x)} \rho(x)) = S(x) \circ T(x)$$

Definition 3.1.7. *Given two functors $\phi, \psi : \mathcal{G} \to \mathcal{K}$, a natural isomorphism $T : \phi \Rightarrow \psi$ is a natural transformation that has an inverse i.e., a natural transformation $S : \psi \Rightarrow \phi$ such that $T \circ S = 1_\phi$ and $S \circ T = 1_\psi$.*

In particular, the homomorphisms from $\mathcal{G}$ to $\mathcal{K}$ are themselves the objects of a groupoid with transformations as arrows. In fact, topological groupoids, homomorphisms and natural transformations form a 2-category, denoted $\mathbf{G}$.

3.2 Equivalences of topological groupoids

Here we summarize some of the notions of equivalence of groupoids. These notions come directly from category theory, some amendments have to be made as we work in the continuous case [34]

Definition 3.2.1. *(Isomorphisms) Two topological groupoids $\mathcal{G}$ and $\mathcal{K}$ are said to be isomorphic if there are homomorphisms $\phi : \mathcal{G} \to \mathcal{K}$ and $\psi : \mathcal{K} \to \mathcal{G}$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity morphisms of $\mathcal{K}$ and $\mathcal{G}$ respectively.*

Now we give a definition of equivalence of categories and that will lead us to define equivalences of topological groupoids.

Definition 3.2.2. *(Equivalences of categories) Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\tau : F \circ G \to id_{\mathcal{D}}$ and $\sigma : G \circ F \to id_{\mathcal{C}}$.*
Alternatively, this notion of equivalence can be described as follows. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following two properties:

(i) F is essentially surjective; that is, for any object y of $\mathcal{D}$ there is an object x of $\mathcal{C}$ and an isomorphism $F(x) \to y$ in $\mathcal{D}$; and

(ii) F is full and faithful; that is, for any two objects x and x' in $\mathcal{C}$ the functor F induces a bijection

$$F : \mathcal{C}(x, x') \to \mathcal{D}(F(x), F(x'))$$

between the set of all arrows from x to x' in $\mathcal{C}$ and the set of all arrows from $F(x)$ to $F(x')$ in $\mathcal{D}$.

These two ways of describing equivalence of course apply to groupoids and they are coincide for general categories. However, for general categories in sets the notions of equivalence and essential equivalence are the same. This applies to particular case in which the categories are groupoids. But when some extra structure is involved like continuity or smoothness, these two notions are *not* the same any more. Now we want to define the morphisms of groupoids which induce an equivalence of topological groupoids. We follow the work in[34], [32] and [36].

Definition 3.2.3. *(Strong equivalence of topological groupoids) Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. A homomorphism $\phi : \mathcal{G} \to \mathcal{K}$ of topological groupoids is a strong equivalence of topological groupoids if there exists a morphism $\psi : \mathcal{K} \to \mathcal{G}$ of topological groupoids and natural transformations T and T' such that $T : \phi \circ \psi \to id_{\mathcal{K}}$ and $T' : \psi \circ \phi \to id_{\mathcal{G}}$.*

Definition 3.2.4. *(Weak equivalences of topological groupoids) Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. A homomorphism $\epsilon : \mathcal{G} \to \mathcal{K}$ is called a weak equivalence (or essential equivalence) of topological groupoids if it satisfies the following two modified conditions for being essentially surjective, and full and faithfull:*

(ES) the map $t \circ pr_1 : K_1 \times_{K_0} G_0 \to K_0$, sending a pair (h, x) with $s(h) = \epsilon(x)$ to $t(h)$, is an open surjection, where $K_1 \times_{K_0} G_0$ is the pullback along the source map $s : K_1 \to K_0$

$$K_1 \times_{K_0} G_0 \xrightarrow{pr_1} K_1 \xrightarrow{t} K_0$$

(FF) the following diagram is a pullback diagram

$$\begin{tikzcd} G_1 \arrow[r, "\epsilon"] \arrow[d, "{(s,t)}"'] & K_1 \arrow[d, "{(s,t)}"] \\ G_0 \times G_0 \arrow[r, "\epsilon \times \epsilon"'] & K_0 \times K_0 \end{tikzcd}$$

The first condition implies that for any object $y \in K_0$, there exists an object $x \in G_0$ whose image $\epsilon(x)$ can be connected to y by an arrow $h \in K_1$. The second condition implies that for all $x, z \in G_0$, the functor ϵ induces a homeomorphism $\epsilon : \mathcal{G}(x, z) \to \mathcal{K}(\epsilon(x), \epsilon(z))$ between the set of all arrows from x to z in $\mathcal{G}$ and the set of all arrows from $\epsilon(x)$ to $\epsilon(z)$ in $\mathcal{K}$.
Basically this means that ϵ is an equivalence of categories (where the first condition implies essential surjectivity and the second full and faithful).
However, for general categories the notions of equivalence and essential equivalence are the same. This applies to the particular case in which the categories are groupoids. Even so when some extra structure is involved like continuity or smoothness, these two notions are not the same anymore. An essential equivalence implies the existence of the inverse functor using the axiom of choice but not the

existence of continuous functors. To define generalised maps, we need to invert the essential equivalences.

Example 3.2.5 (see example 2.3.8). *Let X be a topological space and $Pair(X)$ its pair groupoid. The homomorphism $Pair(X) \to 1$, to the interval one point groupoid consisting of one object and one arrow, is a strong and essential equivalence.*

Example 3.2.6. *Let $p : X \to Y$ be an open surjective map between topological spaces. We view Y as the unit groupoid and consider the kernel groupoid $Ker(p)$, which is a topological subgroupoid of Pair(X) consisting of all Pairs $(y, y') \in X \times X$ with $p(y) = p(y')$, $X \times_Y X \rightrightarrows X$. The map p induces a weak equivalence $ker(p) \to Y$.*

Definition 3.2.7. *Let $\phi : \mathcal{G} \to \mathcal{K}$ be a homomorphism of topological groupoids such that $\phi_0 : G_0 \to K_0$ and $\phi_1 : G_1 \to K_1$*

(i) ϕ is called open if ϕ_1 and (hence) ϕ_0 are open maps.

(ii) ϕ is called essentially surjective if the map $spr_2 : G_0 \times_{K_0} K_1 \to K_0$ is an open surjection. (Here the pullback is along $t : K_1 \to K_0$; the condition is of course equivalent to the condition that $tpr_1 : K_1 \times_{K_0} G_0 \to K_0$ is an open surjection, where the pullback is along s.)

(iii) Consider the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & K_1 \arrow[d, "(s,t)"] \\
G_0 \times G_0 \arrow[r, "\phi_0 \times \phi_0"'] & K_0 \times K_0
\end{tikzcd}$$

ϕ is called faithful (resp. full, fully faithful if the map $((s,t), \phi_1) : G_1 \to P$ is an inclusion (resp. an open surjection, an homeomorphism) of spaces.

Proposition 3.2.8. *Every strong equivalence of topological groupoids is an essential equivalence.*

Proof. Let $\phi : \mathcal{G} \to \mathcal{K}$ be a strong equivalence, with $\psi : \mathcal{K} \to \mathcal{G}$ and T and T' as in the definition of strong equivalence above. We want to prove that the map

$$t \circ pr_1 : K_1 \times_{K_0} G_0 \longrightarrow K_0$$

in the definition of weak equivalence above is an open surjection. We can see this because any $y \in K_0$ is the image of $(T(y), \psi(y))$. To this end, consider the arrow $T(y_0)^{-1}h_0 : \phi(x_0) \to \phi(\psi(y_0))$ in $\mathcal{K}$. Since ϕ is an equivalence of categories, there is a unique arrow $g_0 : x_0 \to \psi(y_0)$ in $\mathcal{G}$ with $\phi(g_0) = T(y_0)^{-1}h_0$.
In particular, the fibred product $G_0 \times_{K_0} K_1 \times_{K_0} G_0$ of $t \circ pr_1$ along $\phi : G_0 \to K_0$ is a topological space, which fits into a pull-back diagram

$$\begin{tikzcd} G_0 \times_{K_0} K_1 \times_{K_0} G_0 \arrow[r, "pr_2"] \arrow[d, "(pr_3, pr_1)"'] & K_1 \arrow[d, "(s,t)"] \\ G_0 \times G_0 \arrow[r, "\phi\times\phi"'] & K_0 \times K_0 \end{tikzcd}$$

since ϕ is an equivalence of categories, the map $G_1 \longrightarrow G_0 \times_{K_0} K_1 \times_{K_0} G_0$, sending g to $(s(g), \phi(g), t(g))$, is a bijection. $\square$

The converse of 3.2.8 does not hold for topological groupoids.

Example 3.2.9 (see example 3.2.6)**.** *Let $p : X \to Y$ be an open surjective map. Consider the topological groupoid $X \times_Y X \rightrightarrows X$ defined by the pull-back of topological spaces*

$$\begin{tikzcd} X \times_Y X \arrow[r, "pr"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

and the essential equivalence $X \times_Y X \to Y$ induced by p, regarding Y as the unit topological groupoid. Any morphism $Y \to X \times_Y X$ amounts to choose a section of p. If X is non-trivial principal bundle over Y, then such sections do not exist and $X \times_Y X \to Y$ is not strong equivalence of topological groupoids.

3.3 Morita equivalence

Morita equivalence is the smallest equivalence relation between topological groupoids that they are equivalent whenever there exists an essential equivalence between them. Firstly, We will give a definition of a notion of weak pull-back and describe some properties of essential equivalences which will be necessary to define Morita equivalence of topological groupoids. Until the end of the chapter we follow the exposition in [12] and [36].

Definition 3.3.1. *Given morphisms of topological groupoids $\psi : \mathcal{K} \to \mathcal{G}$ and $\phi : \mathcal{L} \to \mathcal{G}$ the weak pullback or (fibered product) $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a groupoid whose space of objects is*

$$(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_0 = K_0 \times_{G_0} G_1 \times_{G_0} L_0$$

consisting of triples (x, g, y) with $x \in K_0$, $y \in L_0$ and g is an arrow in G_1 from $\psi(x)$ to $\phi(y)$. An arrow between (x, g, y) and (x', g', y') is a pair of arrows (k, l) with $k \in \mathcal{K}(x, x')$, $l \in \mathcal{L}(y, y')$ such that $g'\psi(k) = \phi(l)g$.

$$\begin{tikzcd}
\psi(x) \arrow[r, "g"] \arrow[d, "\psi(k)"'] & \phi(y) \arrow[d, "\phi(l)"] \\
\psi(x') \arrow[r, "g'"] & \phi(y')
\end{tikzcd}$$

The space of arrows can be identified with

$$(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_1 = K_1 \times_s^{\psi \circ s} G_1 \times_{\phi \circ s}^t L_1 = \{(k, g, l) | \psi \circ s(k) = s(g), \phi \circ s(l) = t(g)\}$$

which can be obtained by two fibered products

$$\begin{tikzcd}
K_1 \times_s^{\psi \circ s} G_1 \times_{\phi \circ s}^t L_1 \arrow[rr] \arrow[d] & & L_1 \arrow[d, "s"] \\
K_1 \times_s^{\psi \circ s} G_1 \times_{\phi}^t L_0 \arrow[r] \arrow[d, "pr_1"'] & K_0 \times_s^{\psi} G_1 \times_{\phi}^t L_0 \arrow[r, "pr_3"] \arrow[d, "pr_1"] & L_0 \\
K_1 \arrow[r, "s"] & K_0 &
\end{tikzcd}$$

If at least one of the two morphisms is a continuous map on objects, then the weak pullback $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a topological groupoid. In this case, the diagram of topological groupoids

$$\begin{tikzcd}
\mathcal{K} \times_{\mathcal{G}} \mathcal{L} \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & \mathcal{K} \arrow[d, "\psi"] \\
\mathcal{L} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

commutes up to a natural transformation and it is universal with this property.

The next proposition describes some properties of essential equivalences and the weak pull-back which will be necessary to define Morita equivalence of groupoids.

Proposition 3.3.2. *Let $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$ be topological groupoids.*

(i) *For two homomorphisms $\phi, \psi : \mathcal{L} \to \mathcal{G}$, if there is a transformation $T : \phi \to \psi$, then ϕ is an essential equivalence if and only if ψ is.*

(ii) *If for an essential equivalence $\phi : \mathcal{L} \to \mathcal{G}$ the map $\boldsymbol{t} \circ pr_1$ of the essentially surjective condition has a section, then ϕ is an equivalence.*

(iii) *The composition of two essential equivalences is an essential equivalence.*

(iv) *For any essential equivalence $\phi : \mathcal{L} \to \mathcal{G}$ and any homomorphism $\psi : \mathcal{K} \to \mathcal{G}$ the weak pull-back*

$$\begin{tikzcd} \mathcal{K} \times_{\mathcal{G}} \mathcal{L} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{L} \arrow[d, "\phi"] \\ \mathcal{K} \arrow[r, "\psi"] & \mathcal{G} \end{tikzcd}$$

exists and pr_2 is an essential equivalence for which $(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_0 \longrightarrow L_0$ is an open surjection.

Definition 3.3.3. *Two topological groupoids $\mathcal{G}$ and $\mathcal{K}$ are Morita equivalent if there exists a topological groupoid $\mathcal{J}$ and essential equivalences ϵ and σ*

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\sigma} \mathcal{G}$$

Remark 3.3.4. By using the definition of the weak pullback 3.3.1 and the property in Proposition 3.3.2 (iv), if there exist a Morita equivalence $\mathcal{G} \sim_M \mathcal{K}$ between two topological groupoids $\mathcal{G}$ and $\mathcal{K}$ that defines an equivalence relation between them. Indeed to check that we need to see the transitivity condition, suppose that we have another Morita equivalence $\mathcal{K} \sim_M \mathcal{L}$ we have a third topological groupoids between them and all the arrows are essential equivalences

$$\mathcal{G} \xleftarrow{\epsilon} \mathcal{H} \xrightarrow{\sigma} \mathcal{K} \qquad \mathcal{K} \xleftarrow{\epsilon'} \mathcal{H}' \xrightarrow{\sigma'} \mathcal{L}$$

We get the following diagram

$$\begin{tikzcd} & & \mathcal{H} \times_{\mathcal{K}} \mathcal{H}' \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\ & \mathcal{H} \arrow[dl, "\epsilon"'] \arrow[dr, "\sigma"] & & \mathcal{H}' \arrow[dl, "\epsilon'"'] \arrow[dr, "\sigma'"] & \\ \mathcal{G} & & \mathcal{K} & & \mathcal{L} \end{tikzcd}$$

were $\mathcal{H} \times \mathcal{H}'$ is the weak pullback of σ and ϵ', and observe that $\mathcal{G}$ and $\mathcal{L}$ are weakly equivalent via $\epsilon \circ pr_1$ and $\sigma' \circ pr_2$. Hence, the weak fibered product of $\mathcal{H}$ and $\mathcal{H}'$ over $\mathcal{K}$ provides a Morita equivalence between $\mathcal{G}$ and $\mathcal{L}$.

Example 3.3.5. *Let X be a connected topological space. Consider ΠX, the fundamental groupoid of X and $\pi_1(X,x)$ the fundamental group at x of X. There is a Morita equivalence $\Pi X \sim_M \pi_1(X,x)$ for $x \in X$, where we regard the fundamental group $\pi_1(X,x)$ as a groupoid over the singleton $\{x\}$. In this case, we can see this as the following*

$$\Pi X \hookleftarrow \pi_1(X,x) \xrightarrow{id} \pi_1(X,x).$$

Example 3.3.6. *Let G be a topological group acting freely and properly on a topological space X. Consider the action groupoid $G \ltimes X$ over X. There is a Morita equivalence $G \ltimes X \sim_M X/G$, where we regard the quotient space X/G as the unit groupoid. So,*

$$G \ltimes X \hookleftarrow X/G \xrightarrow{id} X/G.$$

3.4 Generalised maps

Generalized maps and Hilsum-Skandalis maps give different notions of morphisms between topological groupoids. The composition of these maps is not strictly associative. The category of topological groupoids and functors can be seen as a 2-category $\mathbf{G}$ with natural transformations as 2-morphisms. We will show a construction producing equivalent bicategories $\mathbf{Gpd}$ and $\mathbf{Gpd}'$ in which the morphisms are respectively generalized maps and bibundles. We describe in this section the generalised maps obtained by localisation of essential equivalences (see [21] and [42]). Considering the bicategory $\mathbf{G}$ of topological groupoids, functors and natural transformations, the bicategory $\mathbf{Gpd}$ is obtained as the bicategory of fractions of $\mathbf{G}$ when inverting the essential equivalences E, $\mathbf{Gpd} = \mathbf{G}(E^{-1})$.

Definition 3.4.1. *A generalised map from $\mathcal{K}$ to $\mathcal{G}$ is a pair of morphisms*

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

such that ϵ is an essential equivalence. We denote a generalised map by (ϵ, ϕ). So a generalised map from a topological groupoid $\mathcal{K}$ to a topological groupoid $\mathcal{G}$ is obtained by first replacing $\mathcal{K}$ by another groupoid $\mathcal{J}$ essentially equivalent to it and then mapping $\mathcal{J}$ into $\mathcal{G}$ by an ordinary morphism.

Definition 3.4.2. *Two generalised maps from $\mathcal{K}$ to $\mathcal{G}$, $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$, are isomorphic if there exists a groupoid $\mathcal{L}$ and essential equivalences $\mathcal{J} \xleftarrow{\alpha} \mathcal{L} \xrightarrow{\beta} \mathcal{J}'$*

$$
\begin{tikzcd}
 & & \mathcal{J} \arrow[dll, "\epsilon"'] \arrow[drr, "\phi"] & & \\
\mathcal{K} & \sim_T & \mathcal{L} \arrow[u, "\alpha"'] \arrow[d, "\beta"] & \sim_{T'} & \mathcal{G} \\
 & & \mathcal{J}' \arrow[ull, "\epsilon'"] \arrow[urr, "\phi'"'] & &
\end{tikzcd}
$$

where α and β are essential equivalences and the diagram commutes up to natural transformations. We write $(\epsilon, \phi) \sim (\epsilon', \phi')$.

In other words, there are natural transformations T and T' such that the generalised maps

$$\mathcal{K} \xleftarrow{\epsilon\alpha} \mathcal{L} \xrightarrow{\phi\alpha} \mathcal{G}$$

$$\mathcal{K} \xleftarrow{\epsilon'\beta} \mathcal{L} \xrightarrow{\phi'\beta} \mathcal{G}$$

satisfy $\epsilon\alpha \sim_T \epsilon'\beta$ and $\phi\alpha \sim_{T'} \phi'\beta$

Remark 3.4.3. Let $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{J}$ be topological groupoids

1. If $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi'} \mathcal{G}$ are two generalised maps with $\phi \sim_T \phi'$, then $(\epsilon, \phi) \sim (\epsilon, \phi')$.

2. If $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J} \xrightarrow{\phi'} \mathcal{G}$ are two generalised maps and $\delta : \mathcal{J}' \to \mathcal{J}$ an essential equivalence with $\phi' = \phi\delta$ and $\epsilon' = \epsilon\delta$, then $(\epsilon, \phi) \sim (\epsilon', \phi')$.

There is an equivalence relation between the diagrams above. A 2-isomorphism is an equivalence class of diagrams. We write $(\epsilon, \phi) \sim (\epsilon', \phi')$.

Proposition 3.4.4. *The collection of all topological groupoids as objects, generalised maps as morphisms and 2-isomorphisms is a bicategory. This bicategory will be denoted by* **Gpd**.

Proof. The proof of these facts can be found in [42] with more details and the explicit description of this bicategory **Gpd**:

- Objects are topological groupoids.

- A 1-morphism from $\mathcal{K}$ to $\mathcal{G}$ is a generalised map $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ such that ϵ is an essential equivalence.

- A 2-morphism from $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ to $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J} \xrightarrow{\phi'} \mathcal{G}$ is given by equivalence class diagrams

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\ \mathcal{K} \arrow[r, phantom, "\sim_T"] & \mathcal{L} \arrow[u, "\alpha"'] \arrow[d, "\beta"] \arrow[r, phantom, "\sim_{T'}"] & \mathcal{G} \\ & \mathcal{J}' \arrow[ul, "\epsilon'"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

where $\mathcal{L}$ is a topological groupoid and, α and β are essential equivalences.

□

All the 2-morphisms in **Gpd** are isomorphisms.
For each topological groupoid $\mathcal{G}$ the unit arrow (id, id) is defined as the generalised map $\mathcal{G} \xleftarrow{id} \mathcal{G} \xrightarrow{id} \mathcal{G}$. The composition of two arrows

$$(\mathcal{G} \xleftarrow{\delta} \mathcal{J}' \xrightarrow{\varphi} \mathcal{L}) \circ (\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$$

is given by the generalised map:

$$\mathcal{K} \xleftarrow{\epsilon pr_1} \mathcal{J} \times_{\mathcal{G}} \mathcal{J}' \xrightarrow{\varphi pr_3} \mathcal{L}$$

where pr_1 and pr_3 are the projections in the following weak pullback of topological groupoids:

$$\begin{tikzcd} \mathcal{J} \times_{\mathcal{G}} \mathcal{J}' \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{J}' \arrow[r, "\varphi"] \arrow[d, "\delta"] & \mathcal{L} \\ \mathcal{J} \arrow[r, "\phi"] \arrow[d, "\epsilon"] & \mathcal{G} & \\ \mathcal{K} & & \end{tikzcd}$$

The morphism pr_1 is an essential equivalence since it is the weak pullback of the essential equivalence δ. Then $\epsilon \circ pr_1$ is an essential equivalence. This composition is associative up to isomorphism.
The unit arrow is a left and right unit for this multiplication of arrows up to

isomorphism. The composition $(\mathcal{G} \xleftarrow{id} \mathcal{G} \xrightarrow{id} \mathcal{G}) \circ (\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$ is the generalised map $\mathcal{K} \xleftarrow{\epsilon pr_1} \mathcal{J} \times_{\mathcal{G}} \mathcal{G} \xrightarrow{pr_3} \mathcal{G}$. Since $\varphi = \delta = id$ implies $\varphi pr_3 = \phi pr_1$ and pr_1 is an essential equivalence. Then $pr_3 \sim_T \phi \circ pr_1$, where T is the natural transformation that makes the weak pull back square commute and pr_1 is an essential equivalence. We have that $(\epsilon \circ pr_1, \phi \circ pr_1) \sim (\epsilon, \phi)$ as it is easily seen by the diagram

$$\begin{tikzcd} & \mathcal{J} \times_{\mathcal{G}} \mathcal{G} \arrow[dl, "\epsilon \circ pr_1"'] \arrow[dr, "pr_3"] \arrow[dd, "pr_1"', "\sim_T"] & \\ \mathcal{K} & & \mathcal{G} \\ & \mathcal{J} \arrow[ul, "\epsilon"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

So there is an equivalence relation between the isomorphism diagrams of generalised maps. If we set a 2-isomorphism as an equivalence class of diagrams, then groupoids, generalised maps and isomorphisms form a bicategory. Vertical and horizontal composition of diagrams are defined in a natural way using weak pull-backs and they satisfy the coherence axioms for a bicategory. The proof of these facts can b found in [42] where one can find also the general construction of bicategory of fractions.

Remark 3.4.5. Note that the invertible generalized maps are exactly the Morita equivalences.

3.5 Hilsum-Skandalis maps

Another approach to generalised morphisms between topological groupoids is given by Hilsum-Skandalis maps. Although the relation with 2-morphisms of generalised maps is not immediate, the bicategory arising from this point of view is equivalent. In this setting morphisms are defined by right principle bibundles of topological groupoids. Such a map between topological groupoids $\mathcal{G}$ and $\mathcal{K}$ is an isomorphism class of principal $\mathcal{G}$-$\mathcal{K}$-bibundles. A Hilsum-Skandalis map between $\mathcal{G}$ and $\mathcal{H}$ is an isomorphism class of principle $\mathcal{G}$-$\mathcal{H}$-bibundles. These maps can be composed; they form a category in which two topological groupoids are isomorphic if and only if they are Morita equivalent. To give more details, we will define actions of groupoids on topological spaces and bibundles between two topological groupoids. Here we use [12], [36] and [42].

3.5.1 Actions of topological groupoids on topological spaces

Definition 3.5.1. *Let X be a topological space, $\mathcal{G}$ a topological groupoid and $\mu : X \to G_0$ a continuous map. A right action of $\mathcal{G}$ on X is a continuous map*

$$X \times^t_{G_0} G_1 \to X, \qquad (x,g) \mapsto xg$$

defined on $X \times^t_{G_0} G_1$ given by the following pullback of topological spaces along the target map:

$$\begin{tikzcd} X \times^t_{G_0} G_1 \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \arrow[d, "\mu"] \\ G_1 \arrow[r, "t"] & G_0 \end{tikzcd}$$

such that $\mu(xg) = t(g)$, $x1_{\mu(x)} = x$ and $(xg)h = x(gh)$, for any $x \in X$, $g, h \in G_1$ with $\mu(x) = t(g)$ and $s(g) = t(h)$.
Analogously, we have a left action by considering the pullback $G_1 \times^s_{G_0} X$ along the source map and demanding $\mu(xg) = s(g)$, $1_{\mu(x)}x = x$ and $h(gx) = (hg)x$, for $(x,g) \in G_1 \times^s_{G_0} X$ and $h \in G_1$ composable with g.

Definition 3.5.2. *Given two right $\mathcal{G}$-actions on X and Y with $\mu_1 : X \to G_0$ and $\mu_2 : Y \to G_0$, a map $f : X \to Y$ is equivariant if $\mu_1 = \mu_2 \circ f$ and $f(xg) = f(x)g$ for $(x,g) \in X \times^t_{G_0} G_1$.*
This map is invariant if $f(gx) = f(x)$.

Definition 3.5.3. *The translation groupoids $X \rtimes \mathcal{G}$ associated to a right action of $\mathcal{G}$ on X is given by $(X \rtimes \mathcal{G})_0 = X$ and $(X \rtimes \mathcal{G})_1 = X \times^t_{G_0} G_1$, where the source map is given by the action $s(x,g) = xg$ and the target map is just the projection $t(x,g) = x$.*

Example 3.5.4 (Unit groupoid)**.** *Consider the groupoid $e \ltimes X$ given by the action of the trivial group e on the topological space X. This is a topological groupoid whose arrows are all units. In this way, any topological space can be considered as a groupoid.*

Example 3.5.5 (Multiplication)**.** *Let H be a subgroup of topological group G. Then H acts by multiplication on G.*

Example 3.5.6 (Point groupoid)**.** *Let G be a topological group. Let $\bullet$ be a point. Consider the groupoid $G \ltimes \bullet$ where G acts trivially on the point. This is a topological groupoid with exactly one object $\bullet$ and G is the space of arrows in which the maps s and t coincide. We call $G \ltimes \bullet$ the point groupoid associated to G. In this way, any group can be considered as a groupoid.*

Definition 3.5.7. *The double translation groupoid $\mathcal{K} \ltimes X \rtimes \mathcal{G}$ associated to a left action of $\mathcal{K}$ on X and a right action of $\mathcal{G}$ on X which commute with each other is given by $(\mathcal{K} \ltimes X \rtimes \mathcal{G})_0 = X$ and $(\mathcal{K} \ltimes X \rtimes \mathcal{G})_1 = K_1 \times^s_{K_0} X \times^t_{G_0} G_1$ where the space of arrows is obtained by the following pullback of topological spaces:*

$$\begin{tikzcd}
K_1 \times^s_{K_0} X \times^t_{G_0} G_1 \arrow[rr] \arrow[d] & & K_1 \arrow[d, "s"] \\
X \times^t_{G_0} G_1 \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \arrow[r, "\tau"] \arrow[d, "\rho"] & K_0 \\
G_1 \arrow[r, "t"] & G_0 &
\end{tikzcd}$$

then $K_1 \times^s_{K_0} X \times^t_{G_0} G_1 = \{(h, x, g) | s(h) = \tau(x)$ and $t(g) = \rho(x)\}$ with $s(h, x, g) = x$ and $t(h, x, g) = hxg^{-1}$. The composition of arrows is given by $(h, x, g)(h', x', g') = (hh', x', gg')$.

3.5.2 Bibundles

Definition 3.5.8. *Let $\mathcal{G}$ be a topological groupoid. A right $\mathcal{G}$-bundle over a space B is a space X with a right $\mathcal{G}$-action corresponding to $\mu : X \to G_0$ and invariant map $\pi : X \to B$; that is $\pi(xg) = \pi(x)$ for $x \in X$ and $g \in G_1$ with $\mu(x) = t(g)$. Left bundles are defined similarly.*

$$\begin{tikzcd}
& X \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
B & & G_0
\end{tikzcd}$$

We say that a right $\mathcal{G}$-bundle is principal if :

(i) the map π is open surjective.

(ii) the map

$$X \times^t_{G_0} G_1 \xrightarrow{\alpha} X \times_B X, \qquad \alpha(x, g) = (xg, x)$$

is a homeomorphism.

Example 3.5.9 (Groupoid). *A groupoid $\mathcal{G}$ is itself a left (and a right) $\mathcal{G}$-bundle with $\mu = s : G_1 \to G_0$ and $\pi = t : G_1 \to G_0$ (and with $\mu = t$ and $\pi = s$). Both* *bundles are principal and actions commute. This bundle sometimes called the unit principal $\mathcal{G}$-bundle.*

We can define principal $\mathcal{G}$-bundles pull back as following

Definition 3.5.10. *If $\pi : X \to B$ is a principal $\mathcal{G}$-bundle and $f : N \to B$is a map then the pullback*

$$f^*X := N \times_B X \to N$$

is a principal $\mathcal{G}$-bundle as well

$$\begin{tikzcd}
N \times_B X \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & N \arrow[d, "f"] \\
X \arrow[r, "\pi"] & B
\end{tikzcd}$$

The action of $\mathcal{G}$ on f^*X is the restriction of the action of $\mathcal{G}$ on the product $N \times X$ to $N \times_B X \subset N \times X$.

Definition 3.5.11. *A morphism between principal $\mathcal{G}$-bundles $\pi : X \to B$ and $\pi' : Y \to B'$ is an equivariant map if $f : X \to Y$ is commutative with all structure maps.*

Definition 3.5.12. *Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. A $\mathcal{KG}$-bibundle is a space X that carries two bundle structures such that the corresponding two actions commute.*
There is a right $\mathcal{K}$-bundle structure over $\mu_l : X \to G_0$ with moment map $\mu_r : X \to K_0$, and a left $\mathcal{G}$-bundle structure over $\mu_r : X \to K_0$ with moment map $\mu_l : X \to G_0$. These two actions commute; that is
$(gx)k = g(xk)$ for $g \in G_1$, $x \in X$, $k \in K_1$ whenever either side is well defined

$$\begin{tikzcd}
G_1 \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] & & X \arrow[ddll, "\mu_l"] \arrow[ddrr, "\mu_r"'] & & K_1 \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] \\
& & & & \\
G_0 & & & & K_0
\end{tikzcd}$$

Corollary 3.5.13. *Let $\mathcal{G}$ be a topological groupoid, $\xi_1 \to N$, $\xi_2 \to N$ two principal $\mathcal{G}$-bundles with anchor maps μ_1 and μ_2 respectively. Any $\mathcal{G}$-equivariant map $\psi : \xi_1 \to \xi_2$ inducing the identity on N is homeomorphism.*

Definition 3.5.14. *A morphism $X \to Y$ of $\mathcal{KG}$-bibundles is a continuous map which is equivariant with respect to both actions.*

Definition 3.5.15. *A $\mathcal{K}\mathcal{G}$-bibundle X is right principal if the right $\mathcal{G}$-bundle $\tau : X \to K_0$ is principal. In this case, $X \times^t_{G_0} G_1$ is homeomorphic to $X \times_{K_0} X$ and $X/\mathcal{G}$ is homeomorphic to K_0 denoted by $(\mathcal{K}, X, \mathcal{G})$. Analogously, for left principal.*

Definition 3.5.16. *A $\mathcal{K}\mathcal{G}$-bibundle X is biprincipal if both left and right bundles ρ and τ are principal.*

Example 3.5.17 (see example 3.5.9)**.** *The groupoid $\mathcal{G}$ makes the biprincipal $\mathcal{G}\mathcal{G}$-bibundle.*

Definition 3.5.18. *Two $\mathcal{K}\mathcal{G}$-bibundles X and Y are isomorphic if there is a homeomorphism $f : X \to Y$ that intertwines the maps $X \to G_0$, $X \to K_0$ and also intertwines the $\mathcal{K}$ and $\mathcal{G}$ actions. In other words, $f(hxg) = hf(x)g$ and $\tau = \tau' f$, $\rho = \rho' f$. We write $(\mathcal{K}, X, \mathcal{G}) \sim (\mathcal{K}, Y, \mathcal{G})$.*

$$\begin{tikzcd} & X \arrow[dl, "\rho"'] \arrow[dr, "\tau"] \arrow[dd, "f"] & \\ G_0 & & K_0 \\ & Y \arrow[ul, "\rho'"] \arrow[ur, "\tau'"'] & \end{tikzcd}$$

Remark 3.5.19. Every morphism of right principal $\mathcal{K}\mathcal{G}$-bibundles $X \to Y$ is an isomorphism

Definition 3.5.20. *A Hilsum-Skandalis map $|(\mathcal{K}, X, \mathcal{G})|$ is an isomorphism class of right principal $\mathcal{K}\mathcal{G}$-bibundles.*

Definition 3.5.21. *Two topological groupoids are Morita equivalent if they are isomorphic in the localisation of the category of groupoids as equivalences. In particular, $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, if there is a bibundle $P : \mathcal{G} \to \mathcal{H}$ with the action of $\mathcal{G}$ being principal.*

Definition 3.5.22. *Let X be a two-sided principal $\mathcal{K}\mathcal{G}$-bibundles. We say that $\mathcal{G}$ and $\mathcal{K}$ are Morita equivalent and X is a Morita equivalence between $\mathcal{G}$ and $\mathcal{K}$.*

Proposition 3.5.23. *The collection of all topological groupoids as objects, right principal bibundles as morphisms and isomorphisms of bibundles as 2-morphisms forms a bicategory.*

Proof. The proof can be found in [28]. □

The bicategory will be denoted $\mathbf{Gpd'}$. All 2-morphisms in $\mathbf{Gpd'}$ are isomorphisms. For each groupoid $\mathcal{G}$ the unit arrow is defined as the $\mathcal{G}\mathcal{G}$-bibundle

$$\begin{tikzcd} & G_1 \arrow[dl, "t"'] \arrow[dr, "s"] & \\ G_0 & & G_0 \end{tikzcd}$$

The left and right actions of $\mathcal{G}$ on G_1 are given by the multiplication in the groupoid $\mathcal{G}$. The multiplication of arrows $(\mathcal{K}, X, \mathcal{G})$ and $(\mathcal{K}, Y, \mathcal{G})$ is given by the bibundle $(\mathcal{K}, (X \times_{G_0} Y)/\mathcal{G}, \mathcal{L})$ where $X \times_{G_0} Y$ is the pullback of topological spaces

$$\begin{tikzcd} X \times_{G_0} Y \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \arrow[d, "\rho X"] \\ Y \arrow[r, "\tau Y"] & G_0 \end{tikzcd}$$

and in addition, $\mathcal{G}$ acts on the topological space $X \times_{G_0} Y$ on the right $(x, y)g = (xg, g^{-1}y)$.
The orbit space is a $\mathcal{K}\mathcal{L}$-bibundle

$$\begin{tikzcd} & (X \times_{G_0} Y)/\mathcal{G} \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ L_0 & & K_0 \end{tikzcd}$$

where $\tau([x, y]) = \tau_X(x)$ and $\rho([x, y]) = \rho_Y(y)$ .The left $\mathcal{K}$-action is given by $k[x, y] = [kx, y]$ and the right $\mathcal{L}$-action by $[x, y]l = [x, yl]$. This bibundle is right principal. The multiplication is associative up to isomorphism.
The unit arrow $(\mathcal{G}, G_1, \mathcal{G})$ is a left and right unit for this multiplication of arrows up to isomorphism. We have that the bibundle $(\mathcal{K}, (X \times^t_{G_0} G_1)/\mathcal{G}, \mathcal{G})$ is isomorphic to $(\mathcal{K}, X, \mathcal{G})$ since the map

$$(X \times^t_{G_0} G_1)/\mathcal{G} \to^f X, \qquad f([x, y]) = xy$$

is a homeomorphism satisfying $f(h[x, y]g) = hf([x, y])g$. Hence there is a 2-morphism f from $(\mathcal{K}, X, \mathcal{G})$ to the composition

$$(\mathcal{G}, G_1, \mathcal{G}) \circ (\mathcal{K}, X, \mathcal{G}) = (\mathcal{K}, (X \times^t_{G_0} G_1)/\mathcal{G}, \mathcal{G})$$

.

3.6 A biequivalence Γ : $\mathbf{Gpd'} \to \mathbf{Gpd}$ of bicategories

We will present a process to get a bijection from the category of topological groupoids, generalized maps and isomorphisms to the category of topological groupoids, right principal bibundles and equivariant homeomorphisms. We will give an explicit construction of a bijective correspondence between generalised maps and bibundles. In addition, we conjecture that $\mathbf{Gpd'}$ is biequivalent to $\mathbf{Gpd}$. Recall that a homomorphism of bicategories is a generalisation of the notion of a functor sending objects, morphisms and 2-morphisms of one bicategory to items of the same types in the other one, preserving compositions and units up to 2-isomorphism.

Definition 3.6.1. *A homomorphism* Γ*:* $\mathbf{Gpd'} \to \mathbf{Gpd}$ *is a biequivalence if the functors* $\mathbf{Gpd'}$ $(\mathcal{K}, \mathcal{G}) \to \mathbf{Gpd}$ $(\Gamma\mathcal{K}, \Gamma\mathcal{G})$ *are equivalences for all objects* $\mathcal{K}$ *and* $\mathcal{G}$ *of* $\mathbf{Gpd'}$ *and if for every object* $\mathcal{L}$ *of Gpd there is an object* $\mathcal{K}$ *of* $\mathbf{Gpd'}$ *such that* $\Gamma\mathcal{K}$ *is equivalent to* $\mathcal{L}$ *in Gpd.*

This process is described more in [12] and [44].

Definition 3.6.2. *[12] Given a right principal* $\mathcal{K}\mathcal{G}$*-bibundle* X*:*

$$\begin{tikzcd} & X \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ G_0 & & K_0 \end{tikzcd}$$

where ρ *is a left* $\mathcal{K}$*-bundle and* τ *is a right principal* $\mathcal{G}$*-bundle, construct a generalized map* $\mathcal{K} \xleftarrow{\epsilon} \mathcal{K} \ltimes X \rtimes \mathcal{G} \xrightarrow{\phi} \mathcal{G}$ *where the double action groupoid* $\mathcal{K} \ltimes X \rtimes \mathcal{G}$ *is defined by* $(\mathcal{K} \ltimes X \rtimes \mathcal{G})_0 = X$ *and* $(\mathcal{K} \ltimes X \rtimes \mathcal{G})_1 = K_1 \times^s_\tau X \times^\rho_s G_1$*, this space* *of arrows is obtained by the following pull-backs of topological spaces*

$$\begin{tikzcd}
K_1 \times^s_\tau X \times^\rho_s G_1 \arrow[rr] \arrow[d] & & G_1 \arrow[d, "s"] \\
K_1 \times^s_\tau X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[r, "\rho"] \arrow[d, "\tau"] & G_0 \\
K_1 \arrow[r, "s"] & K_0 &
\end{tikzcd}$$

and it has the form $(\mathcal{K} \ltimes X \rtimes \mathcal{G})_1 = \{(k,x,g)|s(k) = \tau(x)$ *and* $s(g) = \rho(x)\}$ *with* $s(k,x,g) = x$ *and* $t(k,x,g) = kxg^{-1}$*. The composition of arrows is given by* $(k', kxg^{-1}, g')(k,x,g) = (k'k, x, g'g)$ *and the groupoid morphisms* ϕ, ϵ *are defined on arrows by the projections*

$$\phi(k,x,g) = k, \qquad \epsilon(k,x,g) = g$$

by taking the following morphisms ϵ *and* ϕ*:*

$$X \xrightarrow{\epsilon_0} K_0 \ , \qquad \epsilon_0 = \tau \text{ and } K_1 \times^s_{K_0} X \times^t_{G_0} G_1 \xrightarrow{\epsilon_1} K_1 \ , \qquad \epsilon_1 = pr_1$$

$$X \xrightarrow{\phi_0} G_0 \ , \qquad \phi_0 = \rho \text{ and } K_1 \times^s_{K_0} X \times^t_{G_0} G_1 \xrightarrow{\phi_1} G_1 \ , \qquad \phi_1 = pr_3$$

since τ *is a principal bundle,* ϵ *becomes an essential equivalence.*

We will show that if $(\mathcal{K}, X, \mathcal{G}) \sim (\mathcal{K}, Y, \mathcal{G})$ then the associated generalised maps (ϵ, ϕ) and (ϵ', ϕ') are isomorphic.

Let $f : X \to Y$ be the equivariant homeomorphism that intertwines the bundles. Define

$$\overline{f} : \mathcal{K} \ltimes X \rtimes \mathcal{G} \to \mathcal{K} \ltimes Y \rtimes \mathcal{G}$$

by $\overline{f}_0 = f$ on objects and $\overline{f}_1(h,x,g) = (h, f(x), g)$ on arrows. These maps commute with all the structure maps by the equivalence of f. Since $\overline{f}_0$ is a homeomorphism, it is in particular surjective and the space of arrows $K_1 \times^s_{K_0} X \times^t_{G_0} G_1$ is obtained from the following pullback of topological spaces:

$$\begin{tikzcd}
K_1 \times^s_{K_0} X \times^t_{G_0} G_1 \arrow[r, "\overline{f}_1"] \arrow[d, "(s,t)"'] & K_1 \times^s_{K_0} Y \times^t_{G_0} G_1 \arrow[d, "(s,t)"] \\
X \times X \arrow[r, "\overline{f}_0 \times \overline{f}_0"] & Y \times Y
\end{tikzcd}$$

Then $\overline{f}$ is an essential equivalence. Also, as f intertwines the bundles, we have that $\phi' = \phi\overline{f}$, $\epsilon' = \epsilon\overline{f}$ and follows that $(\epsilon, \phi) \sim (\epsilon', \phi')$ homomorphism of bicategories by

$$\Gamma : \mathbf{Gpd'} \to \mathbf{Gpd}, \qquad \Gamma((\mathcal{K}, X, \mathcal{G})) = (\epsilon, \phi)$$

as constructed above on morphisms and being the identity map on objects.
For 2-morphisms $f : X \to Y$, we define $\Gamma(f)$ as the following diagram:

$$\begin{tikzcd}
& \mathcal{K} \ltimes X \rtimes \mathcal{G} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\
\mathcal{K} & \mathcal{K} \ltimes X \rtimes \mathcal{G} \arrow[u, "id"] \arrow[d, "\overline{f}"] & \mathcal{G} \\
& \mathcal{K} \ltimes Y \rtimes \mathcal{G} \arrow[ul, "\epsilon'"] \arrow[ur, "\phi'"'] &
\end{tikzcd}$$

where $\overline{f} : \mathcal{K} \ltimes X \rtimes \mathcal{G} \to \mathcal{K} \ltimes Y \rtimes \mathcal{G}$ is defined by $\overline{f}(x) = f(x)$ on objects and $\overline{f}(h, x, g) = (h, f(x), g)$ on arrows. Since $\tau' f = \tau = \epsilon_0$ and $\rho' f = \rho = \phi_0$ we have that $s(h) = \tau'(f(x))$ and $t(g) = \rho'(f(x))$.

Conversely, given a generalized map from $\mathcal{K}$ to $\mathcal{G}$

$$\begin{tikzcd}
\mathcal{K} & \mathcal{J} \arrow[l, "\epsilon"'] \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

we construct an associated right principal $\mathcal{K}\mathcal{G}$-bibundle X

$$\begin{tikzcd}
& X \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
G_0 & & K_0
\end{tikzcd}$$

where X is the quotient by the action of $\mathcal{J}$ on $\tilde{X} = J_0 \times^t_{G_0} G_1 \times^t_{K_0} K_1$ given by the following pullbacks of topological spaces:

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] \arrow[rr, bend left, "pr_2"] \arrow[dd, bend right, "pr_4"'] & J_o \times_{G_o} G_1 \arrow[r] \arrow[d] & G_1 \arrow[r, "s"] \arrow[d, "t"] & G_o \\
J_o \times_{K_o} K_1 \arrow[r] \arrow[d] & J_o \arrow[r, "\epsilon"] \arrow[d, "\phi"] & G_o & \\
K_1 \arrow[r, "t"] \arrow[d, "s"] & K_o & & \\
K_o & & &
\end{tikzcd}$$

The maps ρ and τ are induced in the quotient by $\tilde{\rho} = s \circ pr_4$ and $\tilde{\tau} = s \circ pr_2$. The action of τ on $\tilde{X}$ is given by $((a,b,d),j) \mapsto (t(j), b\phi(j), d\epsilon(j))$. The left action of $\mathcal{K}$ on $X = \tilde{X}/\mathcal{J}$ is given by

$$K_1 \times^s_{K_0} X \to X, \qquad (k,[a,b,d]) \mapsto [a, bk^{-1}, c, d]$$

and the right action of $\mathcal{G}$ by

$$X \times^t_{G_0} G_1 \to X, \qquad ([a,b,d],g) \mapsto [a,b,dg].$$

If $(\epsilon,\phi) \sim (\epsilon',\phi')$ then the associated bibundles $(\mathcal{K}, X, \mathcal{G})$ and $(\mathcal{K}, Y, \mathcal{G})$ are isomorphic.

It can be proved that the correspondences give a weak equivalence of bicategories (composition is only preserved up to 2-morphisms).

3.7 Strict maps

This explicit construction can be used to characterize the generalised maps that come from a strict map. We follow the work in [12] and [36].

Any strict morphism $\phi : \mathcal{K} \to \mathcal{G}$ can be viewed as a generalised map by

$$\mathcal{K} \xleftarrow{id} \mathcal{K} \xrightarrow{\phi} \mathcal{G}$$

The corresponding bibundle is constructed by taking

$$X = (K_0 \times^t_{G_0} G_1 \times^t_{K_0} K_1)/\mathcal{K}$$

and considering the following identifications :

$$(K_0 \times^t_{G_0} G_1 \times^t_{K_0} K_1)/\mathcal{K} = K_0 \times^t_{G_0} G_1$$

$$[a,b,d] \mapsto (t(d), b\phi(d))$$

this bibundle is isomorphic to

$$\begin{tikzcd} & K_0 \times^t_{G_0} G_1 \arrow[dl, "spr_2"'] \arrow[dr, "pr_1"] & \\ G_0 & & K_0 \end{tikzcd}$$

Proposition 3.7.1. *[36] Let* $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ *be a generalised map and* $(\mathcal{K}, \rho, X, \mathcal{G}, \tau)$ *the associated right principal bibundle. Then* ϕ *is an essential equivalence iff* ρ *is principal.*

Remark 3.7.2. Essential equivalences correspond to *biprincipal* bibundles.

In both bicategories **Gpd** and **Gpd**$'$, Morita equivalences are the invertible morphisms up to a 2-isomorphism, i.e. the *equivalences* in **Gpd** (or **Gpd**$'$). If $\mathcal{K} \sim_M \mathcal{G}$, let $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\delta} \mathcal{G}$ be the associated generalised map in **Gpd** with ϵ and δ essential equivalences, then the inverse generalised map is $\mathcal{G} \xleftarrow{\delta} \mathcal{J} \xrightarrow{\epsilon} \mathcal{K}$.
In the category **Gpd**$'$, let

$$\begin{tikzcd} & X \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ G_0 & & K_0 \end{tikzcd}$$

be the biprincipal $\mathcal{K}\mathcal{G}$-bibundle representing the Morita equivalence, then the inverse biprincipal $\mathcal{G}\mathcal{K}$-bibundle is

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "\rho"] & \\ K_0 & & G_0 \end{tikzcd}$$

where the new actions are obtained from the original ones composing with the inverse: the left action of $\mathcal{G}$ on X is given by $g * x = xg^{-1}$ induced by the right action of $\mathcal{G}$ on the original bibundle. Similarly, the left action of $\mathcal{K}$ induces a right action in the inverse bundle.

Proposition 3.7.3. *[36] Let* $(\mathcal{K}, \rho, X, \mathcal{G}, \tau)$ *be a right principal* $\mathcal{K}\mathcal{G}$*-bibundle and* $(\epsilon, \varphi) = \Gamma((\mathcal{K}, X, \mathcal{G}))$*its associated generalised map. Then* $(\epsilon, \varphi) \sim (id, \phi)$ *if and only if* τ *has a section.*

In other words, a generalised map comes from a strict map iff when seen as a bibundle, the right principal $\mathcal{G}$-bundle has a section.
If a strict map $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, regarded as a generalised map it will be invertible. The inverse of a generalized (strict) map

$$\mathcal{K} \xleftarrow{id} \mathcal{J} \xrightarrow{\epsilon} \mathcal{G}$$

is the generalised map $\mathcal{G} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{id} \mathcal{K}$ which will not always come from a strict map.
The groupoids $\mathcal{G}$ and $\mathcal{K}$ are Morita equivalent but they are not equivalent by a natural transformation.

Example 3.7.4. *If $\mathcal{G}$ is a topological groupoid and X a topological space, the (principal) $\mathcal{G}$-X-bibundles are exactly the (principal) $\mathcal{G}$-bundles over X.*

Example 3.7.5. *Let X and Y be topological spaces. An action of Y on X is just a continuous map $X \to Y$. If $p : X \to Y$ is continuous map, then (X, p, id_X) is a principal Y-bundle over X. Conversely, let (E, p, w) be a principal Y-bundle over X. Then w is a homeomorphism. In particular, w is a Y-equivariant map between (E, p, w) and $(X, p \circ w^{-1}, id_X)$. We can thus identify the isomorphism classes of principal Y-bundles over X with the continuous maps from X to Y.*

Example 3.7.6. *If X and Y are topological spaces, then the Hilsum-Skandalis maps from X to Y are precisely the continuous maps from X to Y.*

Chapter 4

Homotopy Theory of Topological Groupoids

We look for notion of homotopy which is invariant under the Morita equivalences and generalizes the notion of natural transformation and ordinary homotopy. We will introduce a new notion of *1-homotopy* between continuous functors which includes the notions of natural transformation and ordinary homotopy and the resulting is 2-category denoted by $\mathbf{H}$. Then we introduce a notion of *essential 1-homotopy equivalence* for the arrows in this 2-category and we prove that the class W of essential 1-homotopy equivalences admits a bicalculus of fractions. The equivalences in this bicategory of fractions $\mathbf{H}(W^{-1})$ will determine our 1-homotopy equivalences: a generalized map is 1-homotopy equivalence if it is an equivalence in $\mathbf{H}(W^{-1})$.
Now we define a path in a topological groupoid $\mathcal{G}$ due to Haefliger and recall the notions of equivalence and homotopy of $\mathcal{G}$. In this chapter we follow some of the material in [12], [11] and [8].

4.1 Haefliger paths

Definition 4.1.1. *Given a topological groupoid $\mathcal{G}$, we define the $\mathcal{G}$-path (or a path in $\mathcal{G}$) from x to y over a subdivision of the unit interval $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ as a sequence*

$$(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n) \tag{4.1.1}$$

where

1. *$\alpha_i : [t_{i-1}, t_i] \to G_0$ are paths in G_0 , $1 \leqslant i \leqslant n$ and*

2. *$g_i \in G_1$, $0 \leqslant i \leqslant n$ are arrows in $\mathcal{G}$ such that;*
   $s(g_0) = x$ and $t(g_n) = y$
   $s(g_i) = \alpha_i(t_i)$ for all $0 < i \leqslant n$
   $t(g_i) = \alpha_{i+1}(t_i)$ for all $0 \leqslant i < n$

We call $(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ a $\mathcal{G}$-path c from $x = s(g_0)$ to $y = t(g_n)$.
Also we say that $\mathcal{G}$ is connected if and only if for any two points $x, y \in \mathcal{G}$ there exists a $\mathcal{G}$-path from x to y.
The inverse of a $\mathcal{G}$-path $c = (g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ over the subdivision $0 = t_0 \leq ... \leq t_k = 1$ is a $\mathcal{G}$-path

$$c^{-1} = (g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)$$

over the subdivision $0 = t'_0 \leq ... \leq t'_k = 1$, where $t'_i = 1 - t_i$, $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i+1}(1 - t)$.
So the terminal point of c is the initial point of c^{-1}.

Now we define an equivalence relation $\sim$ on the set of $\mathcal{G}$-paths, called *equivalence of $\mathcal{G}$-paths* by the following operations:

Definition 4.1.2. *Let $(g_0, \alpha_1, g_1, \cdots, \alpha_i, g_i, \cdots, \alpha_n, g_n)$ be a $\mathcal{G}$-path from x to y*

(i) If we add the identity arrow $1_{\alpha(s)}$ by taking a new point $s \in [t_{i-1}, t_i]$ in the subdivision, take the restrictions two paths α'_i and α''_i of the corresponding path α_i to the new intervals $[t_{i-1}, s]$ and $[s, t_i]$, then we will get that these two $\mathcal{G}$-paths

$$(g_0, \alpha_1, g_1, \cdots, \alpha_i, g_i, \cdots, \alpha_n, g_n) \tag{4.1.2}$$

and

$$(g_0, \alpha_1, g_1, \cdots, \alpha'_i, 1_{\alpha(s)}, \alpha''_i, \cdots, \alpha_n, g_n) \tag{4.1.3}$$

are equivalent.

(ii) If we give a map $h_i : [t_{i-1}, t_i] \to G_1$ with the source $s \circ h_i = \alpha_i$, and replace the following:
α_i by $t \circ h_i$, g_{i-1} by $h_i(t_{i-1})g_{i-1}$ and g_i by $g_i(h_i(t_i))^{-1}$
we will have the following $\mathcal{G}$-paths

$$(g_0, \alpha_1, g_1, \cdots, g_{i-1}, \alpha_i, g_i, \cdots, \alpha_n, g_n) \tag{4.1.4}$$

and

$$(g_0, \alpha_1, \cdots, h_i(t_{i-1}) \circ g_{i-1}, t \circ h_i, g_i \circ (h_i(t_i))^{-1}, \cdots, \alpha_n, g_n) \qquad (4.1.5)$$

are equivalent.

So if we have two $\mathcal{G}$-paths $(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ and $(g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)$ with $t(g_n) = s(g'_n)$, we can concatenate these two $\mathcal{G}$-paths into a new $\mathcal{G}$-path

$$g_0, \alpha_1, g_1, \cdots, \alpha_n, h, \alpha'_1 \cdots, \alpha'_n, g'_n,$$

where h is an arrow from $s(g_n)$ to $t(g'_n)$

Remark 4.1.3. [12] Note that the equivalence classes of $\mathcal{G}$-paths correspond to isomorphism classes of generalized maps from $\mathcal{I}$ to $\mathcal{G}$, where $\mathcal{I}$ is the unit groupoid associated to the interval $I = [0, 1]$.

Now we want to define the concept of *deformation* between two $\mathcal{G}$-paths of topological groupoids.

Definition 4.1.4. *A deformation between two $\mathcal{G}$-paths from x to y of the same order $(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ and $(g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)$ is given by:*

- *homotopies $H_i : [t_{i-1}, t_i] \times I \longrightarrow G_0$, with $(H_i)_0 = \alpha_i$ and $(H_i)_1 = \alpha'_i$ for $i = 1, \cdots, n$,and*
- *paths $\gamma_i : I \longrightarrow G_1$, with $g_i = (\gamma_i)_0$ to $g'_i = (\gamma_i)_1$ for $i = 1, \cdots, n-1$,*

such that $(g_0, (H_1)_s, \cdots, (\gamma_{n-1})_s, (H_n)_s, g_n)$ is a $\mathcal{G}$-path for each $s \in I$ which satisfy the following:

(a) $H_0(-, 0) = x$ *and* $H_n(-, 1) = y$

(b) $s \circ \gamma_i = H_i(-, 1)$ *and* $t \circ \gamma_i = H_i(-, 0)$ *for all* $i = 1, ..., n$

So we can see a deformation as a continuous family of $\mathcal{G}$-paths of order n from x to y, $t \in [0, 1]$.

Definition 4.1.5. *Two $\mathcal{G}$-paths between x and y are homotopic if one can be obtained from the other by a sequence of equivalences and deformations.*

Definition 4.1.6. *The multiplication of homotopy classes of $\mathcal{G}$-paths is defined by*

$$[(g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)][(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)] := [(g_0, \alpha_1, g_1, \cdots, \alpha_n, g'_0 g_n, \alpha'_1, \cdots, \alpha'_n, g'_n)],$$

where $g'_0 g_n$ is the multiplication of two composable arrows in $\mathcal{G}$ and the paths α_i are reparametrized to the new subdivision.
The inverse of the homotopy class of the $\mathcal{G}$-path $[(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)]$ from x to y is the class of the $\mathcal{G}$-path from y to x

$$[(g_n^{-1}, \alpha'_1, \cdots, g_1^{-1}, \alpha'_k, g_0^{-1})]$$

over the same subdivision, where $\alpha'_i : [t_{i-1}, t_i] \to G_0$ is given by

$$\alpha'_i(t) = \alpha_{k-i+1}(t_{k-i+1} + (\frac{t_{k-i} - t_{k-i+1}}{t_{i-1} - t_i})(t_{i-1} - t)).$$

Definition 4.1.7. *The fundamental groupoid $\Pi\mathcal{G}$ of a topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ is the groupoid whose objects are the objects of $\mathcal{G}$ and arrows are the homotopy classes of $\mathcal{G}$-paths with the multiplication as defined above in 4.1.6, such that*

$$[(g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)][(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)] := [(g_0, \alpha_1, g_1, \cdots, \alpha_n, g'_0 g_n, \alpha'_1, \cdots, \alpha'_n, g'_n)],$$

where $g'_0 g_n$ is the multiplication of two composable arrows in $\mathcal{G}$ and the paths α_i are reparametrized to the new subdivision.
If we restrict to a single point $x_0 \in G_0$, then the collection of homotopy classes of paths in $\mathcal{G}$ that start and end at x_0 is a group, the isotropy group at x_0, denoted by $\Pi(\mathcal{G})_{x_0}^{x_0} = \pi_1(\mathcal{G}, x_0)$. It consists of $\mathcal{G}$-homotopy classes of $(\mathcal{G}, x_0)$-loops (or loops in the pointed groupoid $(\mathcal{G}, x_0)$), which are by definition the $\mathcal{G}$-homotopy classes of $\mathcal{G}$-paths from x_0 to x_0.

Definition 4.1.8. *Let $\mathcal{G}$ be a topological groupoid and $x_0 \in G_0$ such that G_0 is locally path-connected. A $\mathcal{G}$-loop in G_0 with the base point x_0 thus consists of a sequence $(\alpha_i)_{i=1}^n$ of paths in G_0 and a sequence $(g_i)_{i=0}^n$ of arrows in G_1 with $s(g_0) = t(g_n) = x_0$ and $\alpha_i : [t_{i-1}, t_i] \to G_0$ such that any path α_i is from $t(g_{i-1})$ to $s(g_i)$ for all $i = 1, ..., n$. We denote this $\mathcal{G}$-loop by $(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ and denotes $\Omega(\mathcal{G}, x_0)$ the set of all $\mathcal{G}$-loops in G_0 with base point x_0.*
Now we will define an equivalence relation on $\Omega(\mathcal{G}, x_0)$ which we will call simply equivalence. It is the smallest equivalence relation such that

(i) a $\mathcal{G}$-loop

$$(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$$

is equivalent to $\mathcal{G}$-loop

$$(g_0, \alpha_1, g_1, \cdots, (g_{i+1} \circ g_i), \alpha_{i+1}, \cdots, \alpha_n, g_n)$$

if α_i is a constant path for some $1 \leq i \leq n$ and

(ii) a $\mathcal{G}$-loop

$$(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$$

is equivalent to the $\mathcal{G}$-loop

$$(g_0, \alpha_1, \cdots, g_{i-1}, (\alpha_{i-1}\alpha_i), \cdots g_1, \alpha_0)$$

if $g_i \in G_1$ for some $0 \leq i \leq n$.

Here $\alpha_{i-1}\alpha_i$ denotes the usual concatenation of α_{i-1} and α_i.

A deformation of a $\mathcal{G}$-loop $(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n)$ to $\mathcal{G}$-loop $(g'_0, \alpha'_1, g'_1, \cdots, \alpha'_n, g'_n)$ consists of homotopies

$$H_i : [0,1]^2 \to G_0$$

from $H_i(0, -) = \alpha_i$ to $H_i(1, -) = \alpha'_i$ $(i = 1, ..., n)$ and paths

$$h_i : [0,1] \to G_1$$

from g_i to g'_i $(i = 0, 1, ..., n)$ which satisfy

(a) $s \circ h_i = H_{i-1}(-, 1)$ and $t \circ h_i = H_i(-, 0)$ for all $i = 1, 2, ..., n$ and

(b) $H_0([0,1], 0) = H_n([0,1], 1) = \{x_0\}$.

Two $\mathcal{G}$-loops in $\Omega(\mathcal{G}, x_0)$ are homotopic if one can pass from one to another in a sequence of deformations and equivalences. With the multiplication induced by the concatenation, the homotopy classes of $\mathcal{G}$-loops in G_0 with the base point x_0 form a group $\pi_1(\mathcal{G}, x_0)$ called the fundamental group of a topological groupoid $\mathcal{G}$ with the base point x_0.

Let $\mathcal{K}$ be another topological groupoid with K_0 locally path-connected, and let $\phi : \mathcal{G} \to \mathcal{K}$ be a continuous functor. Let $y_0 = \phi_0(x_0)$. The functor ϕ gives a function

$$\phi_\sharp : \Omega(\mathcal{G}, x_0) \to \Omega(\mathcal{K}, y_0)$$

defined by

$$\phi_\sharp(g_0, \alpha_1, g_1, \cdots, \alpha_n, g_n) = (\phi(g_0)(\phi_0 \circ \alpha_1)...(\phi_0 \circ \alpha_n)\phi(g_n))$$

This function maps homotopic $\mathcal{G}$-loops to homotopic $\mathcal{K}$-loops, hence it induces a map

$$\phi_* : \pi_1(\mathcal{G}, x_0) \to \pi_1(\mathcal{K}, y_0)$$

which is clearly a homomorphism of groups.

Proposition 4.1.9. *The fundamental groupoid* $\Pi\mathcal{G}$ *of a topological groupoid* $\mathcal{G} = [G_1 \rightrightarrows G_0]$ *is a topological groupoid.*

Proof. As we have seen above the $\mathcal{G}$-paths in $\mathcal{G}$ give a sequence of paths and arrows over a subdivision $0 = t_0 \leq ... \leq t_n = 1$. Now we want to see that

$$\Pi\mathcal{G} = [(\Pi\mathcal{G})_1 \rightrightarrows (\Pi\mathcal{G})_0]$$

satisfies all the topological groupoid axioms. First, the set of objects $(\Pi\mathcal{G})_0 = G_0$ is the space G_0 of objects of the topological groupoid which is a topological space. Then we have the set of arrows $(\Pi\mathcal{G})_1$ which is the set of $\mathcal{G}$-paths denoted by $\mathcal{PG}$ and a sequence of deformations and equivalences of any two $\mathcal{G}$-paths in $\mathcal{PG}$, so it is the set of homotopy classes of $\mathcal{G}$-paths in $\mathcal{G}$ (with fixed end points), so $(\Pi\mathcal{G})_1 = \mathcal{PG}/\sim$. Thus this set is the quotient space of $\mathcal{G}$-paths on $\mathcal{G}$ given by the surjective map that sends each $\mathcal{G}$-path c to its homotopy class $[c]$. So it is clear that $(\Pi\mathcal{G})_1$ is a topological space.
For any $\mathcal{G}$-path $c = (g_0, \alpha_1, ..., \alpha_n, g_n)$ from x to y, we have that $\Pi\mathcal{G} = \{(x, [c], y); [c]$ is the homotopy class of $\mathcal{G}$-paths that start at x and end at $y\}$.
The source map is $s(x, [c], y) = x$ and the target map is $t(x, [c], y) = y$, therefore both of them are continuous maps, because they are projection maps. The unit map is defined as $u(x, [c]_x, x) = x$ and obviously continuous. Finally we see that the multiplication and the inversion maps, which are both defined above in 4.1.6, are continuous maps. So it is clear that $\Pi\mathcal{G}$ becomes a topological groupoid. $\square$

Note that, until the end of this chapter we will refer to the fundamental groupoid $\Pi\mathcal{G}$ of any topological groupoid $\mathcal{G}$ by $\mathcal{G}_*$, a morphism $\phi : \mathcal{K} \to \mathcal{G}$ of topological groupoids induces a morphism $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ between the fundamental groupoids given by $\phi_* = \phi$ on objects and $\phi_*([g_0, \alpha_1, g_1, ..., \alpha_n, g_n]) = [\phi(g_0), \phi \circ \alpha_1, \phi(g_1), ..., \phi \circ \alpha_n, \phi(g_n)]$ on arrows.

Proposition 4.1.10. *[33]*

(1) If $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, then $\epsilon_* : \mathcal{K}_* \to \mathcal{G}_*$ is an essential equivalence as well.

(2) If $\mathcal{K} \sim_M \mathcal{G}$ then $\mathcal{K}_* \sim_M \mathcal{G}_*$ and the fundamental groups are isomorphic.

(3) The fundamental groupoid $\mathcal{G}_{**}$ of $\mathcal{G}_*$ is isomorphic to $\mathcal{G}_*$.

4.2 The homotopy bicategories

Consider the category of topological groupoids and continuous functors. In the first section we will introduce a bicategory $\mathbf{H}$ and define W which is the set of essential 1-homotopy equivalences in $\mathbf{H}$, then we will define a bicategory $\mathbf{H}(W^{-1})$ having the same objects as $\mathbf{H}$ but inverse morphisms of morphisms in W have been added as well as more 2-morphisms.

4.2.1 The bicategory H

Now we introduce a notion of strict 1-homotopy between continuous functors. We follow the exposition of Hellen in [12] but in case of topological case.

Definition 4.2.1. *Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. The two continuous morphisms $\phi : \mathcal{K} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{G}$ are 1-homotopic if their induced morphisms ϕ_* and ψ_* between the fundamental groupoids are equivalent by a natural transformation. We write $\phi \simeq_H \psi$.*

Since a natural transformation from ϕ_* to ψ_* associates to each object x in $(G_*)_0 = G_0$ an arrow $g_x = [g_0, \alpha_1, g_1, ..., \alpha_n, g_n]$ in $(G_*)_1$ from $\phi(x)$ to $\psi(x)$, this notion of homotopy corresponds to the intuitive idea of continuously deforming ϕ

to ψ by morphisms from $\mathcal{K}$ to $\mathcal{G}$ along $\mathcal{G}$-paths.

We define a natural a 2-morphism $H_a : \phi \Rightarrow \psi$ as a natural transformation $a : \phi_* \to \psi_*$

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow H_a"] \arrow[d] & \mathcal{G} \arrow[d] \\
\mathcal{K}_* \arrow[r, bend left, "\phi_*"] \arrow[r, bend right, "\psi_*"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{G}_*
\end{tikzcd}$$

Topological groupoids, functors and 1-homotopies $H : \phi \Rightarrow \psi$ form a bicategory $\mathbf{H}$. All the 2-morphisms in $\mathbf{H}$ are isomorphisms.
Horizontal and vertical compositions of 2-morphisms are given by the horizontal and vertical compositions of natural transformations, $a_* b_*$ and $a_* \cdot b_*$ respectively. This notion of homotopy generalizes the concepts of natural transformation and ordinary homotopy, we have the following

Proposition 4.2.2. *Let $\phi, \psi : \mathcal{K} \to \mathcal{G}$ be morphisms of topological groupoids.*

(1) If $\phi \sim_T \psi$ where T is a natural transformation, then there is a 2-morphism $H : \phi \Rightarrow \psi$ in $\mathbf{H}$.

(2) If $\phi \simeq_F \psi$ where F is an ordinary homotopy, then there is a 2-morphism $H : \phi \Rightarrow \psi$ in $\mathbf{H}$.

Proof. Let $(g_0, \alpha_1, g_1, ..., \alpha_n, g_n)$ be a $\mathcal{K}$-path from x to y in $\mathcal{K}$. We will construct in each case a natural transformation $a : K_0 \to G_{1*}$ satisfying

$$\psi([g_0, \alpha_1, g_1, ..., \alpha_n, g_n])a(x) = a(y)\phi([g_0, \alpha_1, g_1, ..., \alpha_n, g_n].$$

(1) If $T : K_0 \to G_1$ is a natural transformation with $T(x) : \phi(x) \to \psi(x)$ an arrow in G_1, define a natural transformation $a : K_0 \to G_{1*}$ by $a(x) = [T(x)]$.

We have that $a(x)$ is an arrow in G_{1*} from $s(T(x)) = \phi(x)$ to $t(T(x)) = \psi(x)$ verifying the required equality.

(2) An ordinary homotopy $F : \mathcal{K} \times I \to \mathcal{G}$ with $F_0 = \phi$ and $F_1 = \psi$ determines for each $x \in K_0$ a path $F_x : I \to G_0$ from $\phi(x)$ to $\psi(x)$. Define $a : K_0 \to G_{1*}$ by $a(x) = [1_{\phi(x)}, F_x, 1_{\psi(x)}]$.

$$\begin{tikzcd} \bullet \arrow[loop above] \arrow[r, squiggly, "F_x"] & \bullet \arrow[loop above] \end{tikzcd}$$

$\square$

Definition 4.2.3. *Let $\mathcal{G}$ and $\mathcal{K}$ be topological groupoids. A strict 1-homotopy equivalence is a morphism $\phi : \mathcal{K} \to \mathcal{G}$ such that there exists another morphism $\varphi : \mathcal{G} \to \mathcal{K}$ and 2-isomorphisms $\phi\varphi \Rightarrow id_{\mathcal{G}}$ and $\varphi\phi \Rightarrow id_{\mathcal{K}}$ in $\mathbf{H}$. We will say that two groupoids $\mathcal{K}$ and $\mathcal{G}$ have the same strict 1-homotopy type if they are equivalent in the bicategory $\mathbf{H}$.*

However this notion of homotopy is not invariant under Morita equivalence. We need to add more morphisms and 2-morphisms to the bicategory $\mathbf{H}$ and define our notion of Morita homotopy in an extended bicategory.

We can characterize the strict 1-homotopy equivalences as the morphisms that induce an equivalence between the fundamental groupoids. Recall that $\mathbf{G}$ is the 2-category of topological groupoids, functors and natural transformations.

Proposition 4.2.4. *If $\phi : \mathcal{K} \to \mathcal{G}$ is a 1-homotopy equivalence in $\mathbf{H}$, then $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an equivalence in $\mathbf{G}$.*

Proof. If we have $\varphi : \mathcal{G} \to \mathcal{K}$, then we have the following diagram

$$\begin{tikzcd} & \mathcal{G} \arrow[dr, "\varphi"] \arrow[d, Rightarrow, "H"] & \\ \mathcal{K} \arrow[ur, "\phi"] \arrow[rr, "id_{\mathcal{K}}"'] & {} & \mathcal{K} \end{tikzcd}$$

then $(\varphi\phi)_* \sim_a (id_{\mathcal{K}})_*$ where $a : K_0 \to (K_*)_1$ is a natural transformation. Since $(\varphi\phi)_* = \varphi_*\phi_*$ and $(id_{\mathcal{K}})_* = id_{\mathcal{K}_*}$, we have that $\varphi_*\phi_* \sim_a id_{\mathcal{K}_*}$. In the same way $\phi_*\varphi_* \sim_b id_{\mathcal{G}_*}$ and ϕ_* is an equivalence. $\square$

Consider a 2-functor $\pi : \mathbf{G} \to \mathbf{G}$ between 2-categories given by $\pi(\mathcal{G}) = \mathcal{G}_*$, $\pi(\phi) = \phi_*$ and $\pi(T) = T_*$, where $T_* : \phi_* \Rightarrow \psi_*$ is a natural transformation defined in the following way. For each $x \in (\mathrm{K}_*)_0 = \mathrm{K}_0$, we define $T_*(x) : \phi(x) \to \psi(x)$ as the arrow in $\mathcal{G}_*$ given by $T_*(x) = [T(x)]$. This arrow satisfies the equality $\psi([g_0, \alpha_1, g_1, ..., \alpha_n, g_n])T_*(x) = T_*(y)\phi([g_0, \alpha_1, g_1, ..., \alpha_n, g_n])$. Then two morphisms ϕ and ψ are 1-homotopic if their images by π are equivalent. All the equivalences in $\mathbf{G}$ are 1-homotopy equivalences in $\mathbf{H}$, but there are 1-homotopy equivalences that do not come from equivalences in $\mathbf{G}$.
Now we introduce the *essential 1-homotopy equivalences* as the morphisms that induce an essential equivalence between the fundamental groupoids.

Definition 4.2.5. *A morphism $\phi : \mathcal{K} \to \mathcal{G}$ is an essential 1-homotopy equivalence if $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an essential equivalence.*

In this case, ϕ_* defines an isomorphism between fundamental groups.
Let E be the set of essential equivalences in $\mathbf{G}$ and W the set of essential 1-homotopy equivalences in $\mathbf{H}$. We have that every equivalence in $\mathbf{G}$ is an essential equivalence in $\mathbf{G}$ (see Proposition 3.2.8). We will see that it is also a 1-homotopy equivalence in $\mathbf{H}$.
Strong equivalences in $\mathbf{G}$ $\Rightarrow$ Essential equivalences in $\mathbf{G}$ $\Rightarrow$ Essential homotopy equivalence in $\mathbf{H}$
Strong equivalences in $\mathbf{G}$ $\Rightarrow$ 1-Homotopy equivalences in $\mathbf{H}$ $\Rightarrow$ Essential homotopy equivalence in $\mathbf{H}$

Proposition 4.2.6. *If $\phi : \mathcal{K} \to \mathcal{G}$ is an equivalence in $\mathbf{G}$, then ϕ is a 1-homotopy equivalence in $\mathbf{H}$.*

Proof. Let $\psi : \mathcal{G} \to \mathcal{K}$ be the inverse up to equivalence in $\mathbf{G}$. Then $\phi\psi \sim_T id_{\mathcal{G}}$ and $\psi\phi \sim_{T'} id_{\mathcal{K}}$. By Proposition 4.2.2(1) there are 2-morphisms $H : \phi\psi \Rightarrow id_{\mathcal{G}}$ and $H' : \psi\phi \Rightarrow id_{\mathcal{K}}$ in $\mathbf{H}$. □

Proposition 4.2.7. *For the bicategories $\mathbf{G}$ and $\mathbf{H}$ we have the following:*

(1) Every essential equivalence in $\mathbf{G}$ is an essential 1-homotopy equivalence in $\mathbf{H}$.

(2) In $\mathbf{H}$, all 1-homotopy equivalences are essential 1-homotopy equivalences

Proof. If $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, then ϵ is an essential 1-homotopy equivalence since it induces an essential equivalence between fundamental groupoids by Proposition 4.2.2(1).
If $\phi : \mathcal{K} \to \mathcal{G}$ is a 1-homotopy equivalence in $\mathbf{H}$, then $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an equivalence in G and ϕ is an essential 1-homotopy equivalence in $\mathbf{H}$. $\square$

Lemma 4.2.8. *For all $\epsilon : \mathcal{J} \to \mathcal{G}$ and $\phi : \mathcal{K} \to \mathcal{G}$ with ϵ being an essential 1-homotopy equivalence, there exists a groupoid $\mathcal{P}$ and morphisms $\delta : \mathcal{P} \to \mathcal{K}$ and $\psi : \mathcal{P} \to \mathcal{J}$ with δ an essential 1-homotopy equivalence such that the following square commutes up to a 2-isomorphism:*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\psi"] \arrow[d, "\delta"'] & \mathcal{J} \arrow[d, "\epsilon"] \\ \mathcal{K} \arrow[r, "\phi"] & \mathcal{G} \end{tikzcd}$$

Proof. We start by defining the *weak homotopy pullback* of $\mathcal{P}$ of the morphisms

$$\begin{tikzcd} & \mathcal{J} \arrow[d, "\epsilon"] \\ \mathcal{K} \arrow[r, "\phi"] & \mathcal{G} \end{tikzcd}$$

as follows. Objects are triples $(x, [g_0, \alpha_1, ..., \alpha_n, g_n], y)$ where $x \in J_0$, $y \in K_0$ and $[g_0, \alpha_1, ..., \alpha_n, g_n]$ is a $\mathcal{G}$-path from $\epsilon(x)$ to $\phi(y)$.
Arrows in $\mathcal{P}$ from $(x, [g_0, \alpha_1, ..., \alpha_n, g_n], y)$ to $(x^{'}, [g_0^{'}, \alpha_1^{'}, ..., \alpha_n^{'}, g_n^{'}], y^{'})$ are pairs (j, k) of arrows $j \in J_1$ and $k \in K_1$ such that

$$[g_0^{'}, \alpha_1^{'}, ..., \alpha_n^{'}, g_n^{'}][\epsilon(j)] = [\phi(k)][g_0, \alpha_1, ..., \alpha_n, g_n]$$

We observe that $\mathcal{P}$ is the ordinary weak pullback $\mathcal{J} \times_{\mathcal{G}_*} \mathcal{K}$ of the morphisms $\phi_* i_{\mathcal{K}}$ and $\epsilon_* i_{\mathcal{J}}$. Since ϵ_* is an essential equivalence and $i_{\mathcal{J}}$ is the identity on objects we can assume that $\epsilon_* i_{\mathcal{J}}$ is a continuous on objects and $\mathcal{P}$ is a topological groupoid. The square

$$\begin{tikzcd} \mathcal{P} \arrow[r, "p_1"] \arrow[d, "p_3"'] & \mathcal{J} \arrow[d, "\epsilon"] \\ \mathcal{K} \arrow[r, "\phi"] & \mathcal{G} \end{tikzcd}$$

does not necessarily commute up to a 2-cell but it does when taking the induced morphisms of the fundamental groupoids. Consider the weak pullback $\mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_*$

of groupoids

$$\begin{tikzcd} \mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_* \arrow[r, "\pi_1"] \arrow[d, "\pi_3"'] & \mathcal{J}_* \arrow[d, "\epsilon_*"] \\ \mathcal{K}_* \arrow[r, "\phi_*"] & \mathcal{G}_* \end{tikzcd}$$

where $\epsilon_* \pi_1 \sim \phi_* \pi_3$ and π_3 is an essential equivalence. By the definition of weak pullback and the explicit description of arrows in $\mathcal{P}$ and $\mathcal{P}^*$, we have that $\mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_* = \mathcal{P}_*$, $\pi_1 = p_{1*}$ and $\pi_3 = p_{3*}$, then $\mathcal{P}_*$ is the weak pullback of ϵ_* and ϕ_*. Since the weak pullback square commutes up to natural transformation, we have that there is a 2-morphism $H : \epsilon p_1 \Rightarrow \phi p_3$ with $p_3 \in W$. $\square$

Definition 4.2.9. *Two topological groupoids $\mathcal{K}$ and $\mathcal{G}$ are Morita 1-homotopy equivalent if there exist essential 1-homotopy equivalences:*

$$\mathcal{K} \xleftarrow{\omega} \mathcal{L} \xrightarrow{\theta} \mathcal{G}$$

for a third topological groupoid $\mathcal{L}$.

This defines an equivalence relation that we denote $\simeq_M$. The transitivity property follows from Lemma 4.2.8.

4.2.2 The bicategory $\mathbf{H}(W^{-1})$

The objects of $\mathbf{H}(W^{-1})$ are topological groupoids. The morphisms from $\mathcal{K}$ to $\mathcal{G}$ are formed by pairs (ω, ϕ)

$$\mathcal{K} \xleftarrow{\omega} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

such that ω is an essential 1-homotopy equivalence. The composition of morphisms $(\mathcal{G} \xleftarrow{\omega'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{L}) \circ (\mathcal{K} \xleftarrow{\omega} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$ is given by a morphism

$$\mathcal{K} \xleftarrow{\omega pr_2} \mathcal{P} \xrightarrow{\phi' pr_1} \mathcal{L}$$

where $\mathcal{P}$ is the weak homotopy pullback of ω' and ϕ.
A 2-morphism from (ω, ϕ) to (ω', ϕ') is given by a class of diagrams:

$$\begin{tikzcd}
 & & \mathcal{J} \arrow[lld, "\omega"'] \arrow[rrd, "\phi"] & & \\
\mathcal{K} \arrow[rr, phantom, "\Downarrow_H"] & & \mathcal{L} \arrow[u, "u"'] \arrow[d, "v"] \arrow[rr, phantom, "\Downarrow_{H'}"] & & \mathcal{G} \\
 & & \mathcal{J}' \arrow[llu, "\omega'"] \arrow[rru, "\phi'"'] & &
\end{tikzcd}$$

where $\mathcal{L}$ is a topological groupoid, u and v are essential 1-homotopy equivalences and $H : \omega u \Rightarrow \omega' v$ and $H' : \phi u \Rightarrow \phi' v$ are 2-isomorphisms in $\mathbf{H}$. The vertical composition is strictly associative, but horizontal composition is only associative up to the natural associativity isomorphism. The full details about the construction of bicategories of fractions and the description of the horizontal and vertical composition will be found in [42].
The notion of 1-homotopy we propose corresponds to 2-morphisms in the bicategory $\mathbf{H}(W^{-1})$. That is, we will say that two morphisms are 1-homotopic if there is a 2-morphism between them:

Definition 4.2.10. *Two morphisms (ω, ϕ) and (ω', ϕ') are 1-homotopic if there exists a diagram as above*

$$\begin{tikzcd}
 & & \mathcal{J} \arrow[lld, "\omega"'] \arrow[rrd, "\phi"] & & \\
\mathcal{K} \arrow[rr, phantom, "\Downarrow_H"] & & \mathcal{L} \arrow[u, "u"'] \arrow[d, "v"] \arrow[rr, phantom, "\Downarrow_{H'}"] & & \mathcal{G} \\
 & & \mathcal{J}' \arrow[llu, "\omega'"] \arrow[rru, "\phi'"'] & &
\end{tikzcd}$$

In this case, we write $(\omega, \phi) \simeq (\omega', \phi')$ and we say that there is a 1-homotopy between (ω, ϕ) and (ω', ϕ').
In particular, when ω and ω' are essential equivalences, we have a notion of 1-homotopy for generalised maps and when they are identities, we have a notion of 1-homotopy for strict maps.
Two objects $\mathcal{K}$ and $\mathcal{G}$ are equivalent in $\mathbf{H}(W^{-1})$ if there are morphisms (ω, ϕ) from $\mathcal{K}$ to $\mathcal{G}$ and (θ, ψ) from $\mathcal{G}$ to $\mathcal{K}$ such that $(\omega, \phi) \circ (\theta, \psi)$ is 1-homotopic to the identity $(id_{\mathcal{G}}, id_{\mathcal{G}})$ and $(\theta, \psi) \circ (\omega, \phi) \simeq (id_{\mathcal{K}}, id_{\mathcal{K}})$.

Proposition 4.2.11. *[12] A morphism (ω, ϕ) is invertible up to a 2-isomorphism in $\mathbf{H}(W^{-1})$ if and only if ϕ is an essential 1-homotopy equivalence. In this case, the inverse of (ω, ϕ) is the morphism (ϕ, ω).*

In other words, the definition of Morita 1-homotopy equivalence in above 4.2.9 amounts to *equivalence* in the bicategory $\mathbf{H}(W^{-1})$. So, we write $\mathcal{K} \simeq_M \mathcal{G}$ for equivalence of objects in $\mathbf{H}(W^{-1})$. The *1-homotopy type* of $\mathcal{G}$ is the class of $\mathcal{G}$ under the equivalence relation $\simeq_M$.
We show now that the 1-homotopy type is invariant under Morita equivalence.

Proposition 4.2.12. *If $\mathcal{K} \sim_M \mathcal{G}$, then $\mathcal{K} \simeq_M \mathcal{G}$.*

Proof. If $\mathcal{K}$ and $\mathcal{G}$ are Morita equivalent, then there is a topological groupoid $\mathcal{J}$ and essential equivalences:

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\delta} \mathcal{G}$$

The maps ϵ and δ are also essential 1-homotopy equivalences by Proposition 4.2.7(1). Then the morphisms (ϵ, δ) and (δ, ϵ) are inverse up to a 2-isomorphism in $\mathbf{H}$. Then $\mathcal{K}$ is equivalent to $\mathcal{G}$ in the bicategory $\mathbf{H}(W^{-1})$. $\square$

4.3 Homotopy of Topological Groupoids

We will now define two notions of homotopy that will generalize the notions of strong and Morita equivalence, respectively.

4.3.1 Strict homotopy

Here we will develop the notion of strict homotopy associated to the unit interval $I = [0, 1]$. We recall the notion of $\mathcal{G}$-path and at the end we will introduce the multiple $\mathcal{G}$-paths.

Definition 4.3.1. *Given a subdivision $S = \{0 = r_0 \leq r_1 \leq ... \leq r_n = 1\}$ of the interval $I = [0, 1]$, consider a topological groupoid $\mathcal{I}_s$ whose space of objects is given by the disjoint union of $\bigsqcup_{i=1}^{n} [r_{i-1}, r_i]$*
An element in the connected component $[r_{i-1}, r_i]$ will be denoted by (r, i). Then the topological space of objects $(\mathcal{I}_s)_0$ given by

$$(\mathcal{I}_s)_0 = \{(r, i) | r \in [r_{i-1}, r_i], i = 1, ..., n\}$$

The topological space of arrows $(\mathcal{I}_s)_1$ is given by the disjoint union

$$(\bigsqcup_{i=1}^{n}[r_{i-1}, r_i]) \sqcup \{r_1, ..., r_{n-1}, r'_1, ..., r'_{n-1}\}$$

where $\bigsqcup_{i=1}^{n}[r_{i-1}, r_i]$ is the set of unit arrows and for each point r_i in the subdivision S two arrows were added:
r_i and its inverse arrow r'_i such that the source of r_i is (r_i, i) and its target is $(r_i, i+1)$.
Let $\phi, \varphi : \mathcal{G} \to \mathcal{H}$ be morphisms of topological groupoids. We will say that ϕ is S-homotopic to φ if there exists a subdivision S and a morphism $H^S : \mathcal{G} \times \mathcal{I}_S \to \mathcal{K}$ such that $H^S_0 = \phi$ and $H^S_1 = \varphi$.

This defines an equivalence relation between morphisms that we will call *S-homotopy equivalence* and that sometimes we will refer to as *strong* or *strict* homotopy equivalence depending upon what feature we want to emphasize since we will introduce a weaker version (in the same sense that an essential equivalence weakens a strong equivalence) and a generalised version of this notion (in the same sense that a Morita equivalence generalises an essential equivalence).

Example 4.3.2. *A natural transformation $T : \phi \to \varphi$ determines a homotopy $H^S : \mathcal{G} \times \mathcal{I}_S \to \mathcal{K}$ over the subdivision $S = \{0 = r_0, r_1 = \frac{1}{2}, r_2 = 1\}$.*
The homotopy is given by

$$H^S(x, (r, i)) = \begin{cases} \phi(x) & , r \leq \frac{1}{2} \text{ and } i = 1, \\ \psi(x) & , r \geq \frac{1}{2} \text{ and } i = 2 \end{cases}$$

on objects;

$$H^S(k, u(r, i)) = \begin{cases} \phi(k) & , r \leq \frac{1}{2} \text{ and } i = 1, \\ \psi(k) & , r \geq \frac{1}{2} \text{ and } i = 2 \end{cases}$$

on unit arrows of $\mathcal{I}_S$ and $H^S(k, r_1) = T(y)\phi(k)$ for the arrow $r_1 : (\frac{1}{2}, 1) \to (\frac{1}{2}, 2)$. Therefore, there exists a subdivision S and a morphism H^S such that $H^S(x, 0) = \phi(x)$, $H^S(x, 1) = \psi(x)$, and $H^S(k, 0) = \phi(k)$, $H^S(k, 1) = \psi(k)$.

Example 4.3.3. *A morphism $H : \mathcal{G} \times \mathcal{I} \to \mathcal{K}$, with $\mathcal{I}$ being the unit groupoid over the interval $I = [0, 1]$, such that $H_0 \sim \phi$ and $H_1 \sim \psi$ determines a homotopy $H^S : \mathcal{G} \times \mathcal{I}_S \to \mathcal{K}$ over the subdivision $S = \{r_0 = 0, r_1 = 0, r_2 = 0, r_3 = 1, r_4 = 1, r_5 = 1\}$. The original morphism $H : \mathcal{G} \times \mathcal{I} \to \mathcal{K}$ does not define an equivalence relation, since transitivity fails.*

Example 4.3.4. *An ordinary homotopy* $H : \mathcal{G} \times \mathcal{I} \to \mathcal{K}$ *such that* $H_0 = \phi$ *and* $H_1 = \psi$ *determines a homotopy* $H^S : \mathcal{G} \times \mathcal{I}_S \to \mathcal{K}$ *over the subdivision* $S = \{0 = r_0, r_1 = 1\}$*. The morphism* $H : \mathcal{G} \times \mathcal{I} \to \mathcal{K}$ *defines an equivalence relation but fails to be invariant with respect to Morita equivalence (it is not even invariant of strong equivalence for groupoids).*

Proposition 4.3.5. *Let* $H : \mathcal{U} \times \mathcal{I}_S \to \mathcal{G}$ *be a homotopy of the inclusion,* $H_0 = i_u$*. Then, there is an injection* $j : G_x \to G_{y_r}$ *for all* $y_r = H^S(x, (r, i))$*.*

Proof. The homotopy H^S restricted to the connected component $U_x \times [0, r_1]$ determines an ordinary homotopy; then G_x injects into $G_{H(x,(r,1))}$ for all $r \in [0, r_1]$. The isotropy groups $G_{H(x,(r_1,1))}$ and $G_{H(x,(r_1,2))}$ coincide, since there is an arrow r_1 from $(r_1, 1)$ to $(r_1, 2)$. Then we have a finite number of injections:
$G_x m \rightarrowtail G_{H(x,(r,1))} \rightarrowtail G_{H(x,(r_1,1))} = G_{H(x,(r_1,2))} \rightarrowtail G_{H(x,(r,2))} ... \rightarrowtail G_{H(x,(r_n,n))}$ and G_x injects into $G_{H^S(x,(r,i))}$ for all $r \in [r_{i-1}, r_i]$ with $i = 1, ..., n$. □

A $\mathcal{G}$-path from x to y over the subdivision $S = \{0 = r_0 \leq r_1 \leq \cdots \leq r_n = 1\}$ of the interval $[0, 1]$ is a sequence: $(\alpha_1, g_1, \alpha_2, g_2, \cdots, \alpha_n)$, where

1. for all $1 \leq i \leq n$ the map $\alpha_i : [r_{i-1}, r_i] \to G_0$is a path with $\alpha_1(0) = x$ and $\alpha_n(1) = y$;
2. for all $1 \leq i \leq n - 1$, the arrow $g_i \in G_1$ satisfies:
   $s(g_i) = \alpha_i(r_i)$,
   $t(g_i) = \alpha_{i+1}(r_i)$,

Then our notion of homotopy $H^S : \mathcal{K} \times \mathcal{I}_S \to \mathcal{G}$ between $H_0^S = \phi$ and $H_1^S = \psi$ determines for each $x \in K_0$ a $\mathcal{G}$-path over the subdivision S between the objects $\phi(x)$ and $\psi(x)$ in G_0. In fact, it determines many more paths in G_1 other than the one defined by the unit arrows. If we think of a $\mathcal{G}$-path as a morphism from some version of the interval I to $\mathcal{G}$, we have that the Haefliger $\mathcal{G}$-paths correspond to morphisms $\sigma : \mathcal{I}_S \to \mathcal{G}$, where the groupoid $\mathcal{I}_S$ is the one constructed above.

Remark 4.3.6. A homotopy $H^S : \mathcal{K} \times \mathcal{I}_S \to \mathcal{G}$ determines a $\mathcal{G}$-path $H^S_{1_x} : \mathcal{I}_S \to \mathcal{G}$ where 1_x is the trivial groupoid over $x \in K_0$.

Note that if we thinking of a $\mathcal{G}$-path as a morphism σ from $\mathcal{I}_S$ to $\mathcal{G}$, since most of the arrows $\mathcal{I}_S$ are units so the image of the arrows in $\mathcal{I}_S$ by σ is almost entirely contained in $u(G_0)$, with the only exception being the arrows $\{r_1, \cdots, r_{n-1}\}$and

their inverse. In other words, we cannot fill the space G_1 with paths of arrows given by the $\mathcal{G}$-paths.
To define the several branches coming into a multiple $\mathcal{G}$-path we need to introduce several copies of each subinterval in the subdivision S. We define a groupoid $\mathcal{I}'_S$ associated to the interval I and the subdivision S in the following way. Given a subdivision $S = \{0 = r_0 \leq r_1 \leq \cdots \leq r_n = 1\}$ of the interval $I = [0, 1]$, consider the space of objects given by the disjoint union:

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m_i} [r_{i-1}, r_i]_j,$$

where the extra copies of each interval are indexed by j.
An element in the connected component $[r_{i-1}, r_i]_j$ will be denoted by (r, i, j). Then the space of objects is

$$(\mathcal{I}'_S)_0 = \{(r, i, j) | r \in [r_{i-1}, r_i], j = 1, ..., m, i = 1, ..., n\}.$$

The space of arrows of $\mathcal{I}'_S$ is generated by the disjoint union of:

1. $\bigsqcup_{i,j} [r_{i-1}, r_i]_j$, is the set of unit arrows;
2. $\bigsqcup_{i} r_i$, is the set of arrows connecting the jumps at the subdivision points r_i, i.e. the source and target of the arrow r_i are $s(r_i) = (r_i, i, 1)$ and $t(r_i) = (r_i, i + 1, 1)$ and
3. $\bigsqcup_{i,j} [r_{i-1}, r_i]_j$, the set of arrows between the different copies $[r_{i-1}, r_i]_j$ of each subinterval, .e. the source and target of the arrow $r_{ij} \in [r_{i-1}, r_i]_j$ are $s(r_{ij}) = (r, i, j)$ and $t(r_{ij}) = (r, i, j + 1)$.

Definition 4.3.7. *A multiple $\mathcal{G}$-path over a subdivision S is a morphism $\sigma : \mathcal{I}'_S \to \mathcal{G}$.*

Note that a $\mathcal{G}$-path in the sense of Haefliger is a multiple $\mathcal{G}$-path over the same subdivision by taking $j = 1$ for all subintervals and $\sigma((r, i)) = \alpha_i(r)$ on objects; $\sigma(r_i) = g_i$ on arrows r_i and $\sigma(u(r, i)) = u(\alpha_i(r)$for unit arrows.
We can think of a multiple $\mathcal{G}$-path between *orbits* or as a path between orbit subgroupoids. In this spirit, we will say that the initial subgroupoid of the path is $\sigma(0')$ and the end subgroupoids is $\sigma(1')$. Where $0'$ and $1'$ are the full subgroupoids over the orbits of 0 and 1, which in general will not be trivial groupoids.

Remark 4.3.8. A homotopy $H^S : \mathcal{K} \times \mathcal{I}_S \to \mathcal{G}$, when restricted to the full subgroupoid $\mathbf{X}$ over an orbit $\mathcal{O}(x)$, defines a multiple $\mathcal{G}$-path between the orbit subgroupoids $H(\mathbf{X}, 0)$ and $H(\mathbf{X}, 1)$. If H^S is a homotopy between ϕ and ψ, then for all $x \in K_0$, the S-homotopy defines a multiple $\mathcal{G}$-path between the orbits of $\phi(x)$ and $\psi(x)$.

Sometimes we will denote a multiple $\mathcal{G}$-path by

$$\sigma = (\alpha_1^{j_1}, g_1, \alpha_2^{j_2}, \cdots, \alpha_{n-1}^{j_{n-1}}, g_{n-1}, \alpha_n^j)$$

where $(\alpha_1^1, g_1, \alpha_2^1, \cdots, \alpha_{n-1}^1, g_{n-1}, \alpha_n^1)$ is a $\mathcal{G}$-path and

$$(\alpha_1^1, g_1, \alpha_2^1, \cdots, h_{ij} g_{i-1}, \alpha_i^j, g_i h_{ij}^{-1}, \cdots, \alpha_{n-1}^1, g_{n-1}, \alpha_n^1)$$

is a $\mathcal{G}$-path for each $j = 1, \cdots m_i$ and each $i = 1, \cdots, n$.

4.3.2 Essential homotopy equivalences of topological groupoids

Here we introduce the notion of essential homotopy equivalence and homotopy pullback of topological groupoids.

Definition 4.3.9. *A morphism $\eta : \mathcal{K} \to \mathcal{G}$ of topological groupoids is an essential homotopy equivalence of topological groupoids if there exists an S-homotopy equivalence $h : \mathcal{K} \to \mathcal{L}$ and an essential equivalence $\epsilon : \mathcal{L} \to \mathcal{G}$ such that $\eta \simeq_S \epsilon h$.*

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "\eta"] \arrow[dr, "h"'] & & \mathcal{G} \\ & \mathcal{L} \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

This implies that for any object $y \in G_0$, there exists an object $x \in K_0$ whose image $\eta(x)$ can be connected to y by a concatenation of paths and arrows. Also η induces a homotopy equivalence $|\eta|$ between the orbit spaces.

Remark 4.3.10. If $\eta : \mathcal{K} \to \mathcal{G}$ is an essential homotopy equivalence, then there is an injection between the corresponding isotropy groups K_x and $G_{\eta(x)}$.
It is clear that essential equivalences as well as (strong) homotopy equivalences are essential homotopy equivalences.

Proposition 4.3.11. *[11]*

1. *An essential equivalence is an essential homotopy equivalence.*

2. *A homotopy equivalence is an essential homotopy equivalence.*

Proposition 4.3.12. *If $\eta\phi \simeq_S \eta\psi$ and η is an essential homotopy equivalence, then $\phi \simeq_S \psi$.*

Proof. Since $\eta \simeq_S \epsilon h$, we have that $\epsilon h\phi \simeq_S \epsilon h\psi$. Then $h\phi \simeq_S h\psi$, because ϵ is an essential equivalence. If g is the homotopic inverse of h, we have that $gh\phi \simeq_S gh\psi$. Thus, $\phi \simeq_S \psi$. □

In the following we will define the groupoid homotopy pullback;

4.3.3 The groupoid homotopy pullback

Let $\phi : \mathcal{K} \to \mathcal{G}$ and $\psi : \mathcal{J} \to \mathcal{G}$ be morphisms of topological groupoids and S a subdivision of the interval $I = [0, 1]$. Let $P_S(\mathcal{G})$ be the space of $\mathcal{G}$-paths over the subdivision S.

Definition 4.3.13. *The groupoid homotopy pullback $P_S = \mathcal{K} \times^h_S \mathcal{J}$ is the topological groupoid whose space of objects is*
$(P_S)_0 = \{(x, \sigma, y)|x \in K_0, y \in J_0, \sigma \in P_S(\mathcal{G})$ *with* $\sigma(0) = \phi(x)$ *and* $\sigma(1) = \psi(y)\}$
whose space of arrows is
$(P_S)_1 = \{(k, \sigma, j)|k \in K_1, j \in J_1, \sigma \in P_S(\mathcal{G})$ *with* $\sigma(0) = \phi(s(k))$ *and* $\sigma(1) = \psi(t(j))\}$*, and whose source and target maps are given by:* $s(k, \sigma, j) = (s(k), \sigma, s(j))$ *and* $t(k, \sigma, j) = (t(k), \phi(j)\sigma\phi(k)^{-1}, t(j))$.

The groupoid homotopy pullback is well defined (up to homotopy) whenever one of the maps ϕ or ψ is homotopic to an open surjection on objects.
We will give the construction of the topological space of objects $(P_S)_0$ for a subdivision $S = \{t_0 = 0, t_1, t_2, t_3 = 1\}$ by a sequence of pullbacks and homotopy pullbacks of topological spaces.
We recall first the basic definition of a homotopy pullback of topological spaces. Here we use [11].

Definition 4.3.14. *Given two continuous maps $f : X \to Z$ and $g : Y \to Z$, the homotopy pullback is the topological space*
$X \times^h Y = \{(x, \alpha, y)|x \in X, y \in Y$ *and α is a path between $f(x)$ and $g(y)$*$\}$ *together*

with the projection maps making the following diagram commute up to homotopy and it is universal (up to homotopy) with respect to this property:

$$\begin{tikzcd} X \times^h Y \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Remark 4.3.15. If f is homotopic to a continuous map then the homotopic pullback $X \times^h Y$ is a space homotopic to the ordinary pullback.

Consider the following *ordinary* pullbacks of spaces:

$$\begin{tikzcd} K_0 \times_{G_0} G_1 \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & K_0 \arrow[d, "\phi"] \\ G_1 \arrow[r, "s"] & G_0 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} J_0 \times_{G_0} G_1 \arrow[r, "pr_1'"] \arrow[d, "pr_2'"'] & J_0 \arrow[d, "\psi"] \\ G_1 \arrow[r, "t"] & G_0 \end{tikzcd}$$

and the following *homotopy* pullback of spaces:

$$\begin{tikzcd} (P_S)_0 \arrow[rr] \arrow[dd] & & K_0 \times_{G_0} G_1 \arrow[d, "pr_2"] \\ & & G_1 \arrow[d, "t"] \\ J_0 \times_{G_0} G_1 \arrow[r, "pr_2'"] & G_1 \arrow[r, "s"] & G_0 \end{tikzcd}$$

Then we have that $(P_S)_0$ is the topological space given as:
$\{(x, g, \alpha, h, y) | \alpha$ is a path between $t(g)$ and $s(h)$ with $\phi(x) = s(g)$ and $\psi(y) = t(h)\}$.
Analogously, we can define the space of arrows $(P_S)_1$ as the homotopy pullback $(K_1 \times_{G_0} G_1) \times^h (J_1 \times_{G_0} G_1) = K_1 \times_{G_0} G_1 \times^h G_1 \times_{G_0} J_1$.
If (k, g, α, h, j) is an arrow from (x, g, α, h, y) to $(x', g\phi(k)^{-1}, \alpha, \psi(j)h, y')$ and (k', g', α, h', j') is an arrow from (x', g', α, h', y') to $(x'', g'\phi(k')^{-1}, \alpha, \psi(j')h', y'')$, the composition of arrows is given by

$$(k', g, \alpha, h, j') \circ (k, g', \alpha, h', j) = (k'k, g, \alpha, h, j', j)$$

We can generalise this construction to obtain the groupoid homotopy pullback $\mathcal{P}_S$ corresponding to the subdivision $S = \{0 = r_0 \leqslant r_1 \leqslant \cdots \leqslant r_n = 1\}$ by iterating n homotopy pullback.

Then we get

$$(P_S)_0 = K_0 \times^h_{G_0} G^{n-1} \times^h_{G_0} J_0$$

and

$$(P_S)_1 = K_1 \times^h_{G_0} G^{n-1} \times^h_{G_0} J_1.$$

We will show that the following diagram of groupoids commute up to strong homotopy:

$$\begin{tikzcd} \mathcal{P}_S \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & \mathcal{K} \arrow[d, "\phi"] \\ \mathcal{J} \arrow[r, "\psi"] & \mathcal{G} \end{tikzcd}$$

Consider the homotopy $H^S : \mathcal{P}_S \times \mathcal{I}_S \to \mathcal{G}$ given by $H^S((x,\sigma,y),(r,i,j)) = \sigma((r,i,j))$ on objects and $H^S((k,\sigma,j),r_i) = \sigma(r_i)$ on arrows (see remark 4.3.6) We have that $H^S_0 = \phi pr_1$ and $H^S_1 = \psi pr_2$.

The homotopy pullback of groupoids satisfies the following universal property. For any groupoid $\mathcal{A}$ and morphisms $\delta : \mathcal{A} \to \mathcal{K}$ and $\gamma : \mathcal{A} \to \mathcal{J}$ with $\phi\delta \simeq_{H^S} \psi\gamma$ there exists a morphisms $\xi : \mathcal{A} \to \mathcal{P}_S$ such that both triangles commute up to homotopy:

$$\begin{tikzcd} \mathcal{A} \arrow[drr, bend left, "\delta"] \arrow[dr, dotted, "\xi"] \arrow[ddr, bend right, "\gamma"'] & & \\ & \mathcal{P}_S \arrow[r, "pr_1"'] \arrow[d, "pr_2"] & \mathcal{K} \arrow[d, "\phi"'] \\ & \mathcal{J} \arrow[r, "\psi"] & \mathcal{G} \end{tikzcd}$$

We define $\xi : \mathcal{A} \to \mathcal{P}_S$ by $\xi(\omega) = (\delta(\omega), H^S_{1_W}, \gamma(\omega))$ on objects and $\xi(a) = (\delta(a), H^S_{1_W}, \gamma(a))$ on arrows, where 1_W is the trivial groupoid over ω.
Note that the groupoid homotopy pullback $\mathcal{P}_S$ is defined for each subdivision S of the interval $I = [0,1]$.

Remark 4.3.16. If $S = \{0 = r_0 \leq r_1 \leq ... \leq r_n = 1\}$ is a subdivision of $I = [0,1]$, then for all subdivisions $S' \supset S$ we have:

1. If $\phi \simeq_{H^S} \psi$, then $\phi \simeq_{H^{S'}} \psi$.

2. There exists a morphism $\xi : \mathcal{P}_S \to \mathcal{P}_{S'}$ such that $pr'_1\xi = pr_1$ and $pr'_2\xi = pr_2$. Since $\phi pr_1 \simeq_{H^S} \psi pr_2$, then $\phi pr_1 \simeq_{H^{S'}} \psi pr_2$ and the following commute up to S'-homotopy :

$$\begin{tikzcd}
\mathcal{P}_S \arrow[dr, dotted, "\xi"] \arrow[drr, bend left, "pr_1"] \arrow[ddr, bend right, "pr_2"'] & & \\
& \mathcal{P}'_S \arrow[r, "pr'_1"'] \arrow[d, "pr'_2"] & \mathcal{K} \arrow[d, "\phi"'] \\
& \mathcal{J} \arrow[r, "\psi"] & \mathcal{G}
\end{tikzcd}$$

Proposition 4.3.17. *If there is a subdivision S such that the following diagram commute up to S-homotopy:*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\eta"] \arrow[dr, "\epsilon"'] & \mathcal{G} \\
& \mathcal{L} \arrow[u, "h"']
\end{tikzcd}$$

where ϵ is an essential equivalence and h is a homotopy equivalence, then $\eta : \mathcal{K} \to \mathcal{G}$ is an essential homotopy equivalence.

Proof. Let g be the homotopic inverse of h. Then, the following triangle commutes up to S-homotopy:

$$\begin{tikzcd}
\mathcal{G} \arrow[d, "g"'] \arrow[dr, "id"] & \\
\mathcal{L} \arrow[r, "h"'] & \mathcal{G}
\end{tikzcd}$$

consider the following fibered product of groupoids:

$$\begin{tikzcd}
\mathcal{G} \times_{\mathcal{L}} \mathcal{K} \arrow[r, "\epsilon'"] \arrow[d, "g'"'] & \mathcal{G} \arrow[d, "g"] \\
\mathcal{K} \arrow[r, "\epsilon"] & \mathcal{L}
\end{tikzcd}$$

This square is commute up a natural transformation. Then it commutes up to S-homotopy and has the universal property. Since $\eta \simeq_S h\epsilon$, we have the following diagram up to S-homotopy:

$$\begin{tikzcd}
\mathcal{K} \arrow[dr, dotted, "\xi"] \arrow[drr, bend left, "\eta"] \arrow[ddr, bend right, "id"'] & & & \\
& \mathcal{G} \times_{\mathcal{L}} \mathcal{K} \arrow[r, "\epsilon'"'] \arrow[d, "g'"] & \mathcal{G} \arrow[d, "g"'] \arrow[dr, "id"] & \\
& \mathcal{K} \arrow[r, "\epsilon"] & \mathcal{L} \arrow[r, "h"'] & \mathcal{G}
\end{tikzcd}$$

Therefore, $\epsilon' \circ \xi \simeq_S \eta$ and $g' \circ \xi \simeq_S id$. We have that

$$\eta \circ g' \simeq_S h \circ \epsilon \circ g' \simeq_S h \circ g \circ \epsilon' \simeq_S id \circ \epsilon' \simeq_S \epsilon'$$

and also

$$\eta \circ g' \simeq_S \eta \circ id_{\mathcal{K}} \circ g' \simeq_S \eta \circ g' \circ \xi \circ g' \simeq_S \epsilon' \circ \xi \circ g'.$$

By Proposition 4.3.12 we have that $id \simeq_S \xi g'$ and ξ is the homotopic inverse of g'. Then $\eta \simeq_S \epsilon' h'$ with ϵ' and essential equivalence and $h' = \xi$ a homotopy equivalence.

$\square$

Lemma 4.3.18. *If ϵ is an essential equivalence and g is an S-homotopy equivalence, then the homotopic pullback*

$$\begin{tikzcd}
\mathcal{P}_S \arrow[r, "{}'\epsilon"] \arrow[d, "g'"'] & \mathcal{J} \arrow[d, "g"] \\
\mathcal{L} \arrow[r, "\epsilon"] & \mathcal{G}
\end{tikzcd}$$

exists and g' is an S-homotopy equivalence as well.

Proof. since ϵ is an essential equivalence, the pullback $\mathcal{P}_S$ exists. Let $f : \mathcal{G} \to \mathcal{J}$ be the homotopic inverse of g. Then $fg \simeq_S id_{\mathcal{J}}$ and $gf \simeq_S id_{\mathcal{G}}$. Consider the following pullbacks

$$\begin{tikzcd}
\mathcal{P}'_S \arrow[r, "\epsilon''"] \arrow[d, "f'"] & \mathcal{G} \arrow[d, "f"] \\
\mathcal{P}_S \arrow[r, "{}'\epsilon"] \arrow[d, "g'"] & \mathcal{J} \arrow[d, "g"] \\
\mathcal{L} \arrow[r, "\epsilon"] & \mathcal{G}
\end{tikzcd}$$

By the universal property of the large square, there exists $\xi : \mathcal{L} \to \mathcal{P}'_S$ such that the following diagram commutes up to S-homotopy:

$$\begin{tikzcd}
\mathcal{L} \arrow[dr, dotted, "\xi"] \arrow[drr, bend left, "\epsilon"] \arrow[ddr, bend right, "id"'] & & \\
& \mathcal{P}'_S \arrow[r, "\epsilon''"'] \arrow[d, "g'f'"] & \mathcal{G} \arrow[d, "id"'] \\
& \mathcal{L} \arrow[r, "\epsilon"] & \mathcal{G}
\end{tikzcd}$$

Then $g'f'\xi \simeq_S id$ and $\epsilon''\xi \simeq_S \epsilon$. We will see that $f'\xi$ is the homotopic inverse of g'. We have that

$$\epsilon' f'\xi g' \simeq_S f\epsilon''\xi g' \simeq_S fg\epsilon' \simeq_S \epsilon'.$$

By Proposition 4.3.12, $f'\xi g' \simeq_S id$ and g' is a homotopy equivalence.

□

Proposition 4.3.19. *If η and ν are essential homotopy equivalences, then $\nu\eta$ is an essential homotopy equivalence as well.*

Proof. We have that $\eta \simeq_{S'} \epsilon h$ and $\nu \simeq_{S''} \sigma f$ with ϵ and σ being essential equivalences and h and f homotopy equivalences. Consider the following diagram commute up to S-homotopy:

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "\eta"] \arrow[dr, "h"] & & \mathcal{G} \arrow[rr, "\nu"] & & \mathcal{L} \\
& \mathcal{A} \arrow[ur, "\epsilon"'] & & \mathcal{B} \arrow[ul, "g"'] \arrow[ur, "\sigma"'] & \\
& & \mathcal{P}_S \arrow[ul, "g'"] \arrow[ur, "\epsilon'"'] & &
\end{tikzcd}$$

Let h' be the homotopic inverse of g'. Then, $\nu\eta \simeq_S \sigma\epsilon' h' h$, where $\sigma\epsilon'$ is an essential equivalence and hh' is a homotopy equivalence. □

We can pull back the decomposition in Proposition 4.3.17 and we have the following.

Proposition 4.3.20. *If $\nu : \mathcal{K} \to \mathcal{G}$ is an essential homotopy equivalence, then there is a subdivision S such that the morphism $pr_2 : \mathcal{P}_S \to \mathcal{J}$ is an essential homotopy equivalence as well.*

Definition 4.3.21. *A topological groupoid $\mathcal{K}$ is Morita homotopy equivalent to $\mathcal{G}$ if there exists a topological groupoid $\mathcal{L}$ and essential homotopy equivalences*

$$\mathcal{K} \xleftarrow{\eta} \mathcal{L} \xrightarrow{\nu} \mathcal{G}\,.$$

Note that it is possible to choose these essential homotopy equivalences to be homotopic to open surjections on objects.

Chapter 5

Stacks and Topological Stacks

In this chapter we will start with the first categorical structures which play a role in the theory of stacks, the categories fibred in groupoids. The base category here is the category of topological spaces Top. We will be developing the conditions to be satisfied for a category fibred in groupoids to be a stack, and eventually for a stack to be a topological stack. As well our goal is to define the notion of topological stacks and explain the relation between topological stacks and topological groupoids. Topological stacks are those stacks which are isomorphic to quotient stacks of topological groupoids. For instance, if G group acts on a topological space X, then the canonical map $X \to [X/G]$ is an atlas. Roughly speaking, topological stacks are topological groupoids up to *Morita equivalence*. The base category Top will come with a Grothendieck topology being the usual open-cover topology. This means that we have a notion of coverings of a topological space X, which is a collection of morphisms $\{U \to X\}$, such that each point of X is in the image of at least one of these maps. A single map $U \to X$ is called a covering map if $\{U \to X\}$ is a covering. Any covering $\{U_\alpha \to X\}$ determines a covering map $U = \bigsqcup U_\alpha \to X$, which can often be used in place of the covering. A morphism $Y \to X$ admits local sections in the relevant topology if there exists a covering $\{U_\alpha \to X\}$ and a lift $U_\alpha \to Y$ for each α. A stack is not only a kind of space with some extra structure; rather it is a category. A stack (over a Top) is a category $\mathfrak{X}$, together with a functor $\pi : \mathfrak{X} \to \mathsf{Top}$, satisfying some properties. A category together with a functor to another category, with an appropriate notion of pullbacks, is known as a fibred category. Our fibred category will all be fibred over Top. We will use [5], [37], [38] and [13].

5.1 Categories fibred in groupoids

The first requirements for a category $\mathfrak{X}$, together with a functor $\pi : \mathfrak{X} \to \mathsf{Top}$ to be a stack is being a category fibred in groupoids over Top. This means that the following two axioms in the definition must be satisfied but before that we will start by recalling the definition of a fibre product of sets.

Definition 5.1.1. *Let $f : X \to Z$ and $g : Y \to Z$ be two maps of sets. The fibre product of f and g, or more precisely the fibre product of X and Y over Z is the set $X \times_{f,Z,g} Y = X \times_Z Y = \{(x, y)|f(x) = g(y)\}$.*

Remark 5.1.2. If $f : X \to Z$ and $g : Y \to Z$ are continuous maps between topological spaces then the fibre product $X \times_Z Y$ is a subset of $X \times Y$ and hence is naturally a topological space.

For any category $\mathcal{C}$ we want to define what it means to have a category over $\mathcal{C}$ and what it means to have a morphism between such objects:

Definition 5.1.3. *A morphism $\phi : x \to y$ in $\mathfrak{X}$ is said to be strongly Cartesian if it satisfies the following universal property. For every $f : z \to x$ and $g : p(z) \to p(y)$ such that $p(\phi)g = p(f)$ there is a unique $h : z \to y$ with $\phi h = f$ and $p(h) = g$.*

Definition 5.1.4. *Let $\mathcal{C}$ and $\mathfrak{X}$ be two categories, $p : \mathfrak{X} \to \mathcal{C}$ a functor between them. We say that $\mathfrak{X}$ is fibred over $\mathcal{C}$ if for every x in $\mathfrak{X}$ and $g : s \to p(x)$ in $\mathcal{C}$, there is a strongly Cartesian morphism $f : y \to x$ with $p(y) = s$ and $p(f) = g$.*

In this case, the fibre over an object s of $\mathcal{C}$ is the subcategory consisting of the objects x of $\mathfrak{X}$ with $p(x) = s$ and the morphisms lifting id_s.
A fibred category $P : \mathfrak{X} \to \mathcal{C}$ is called fibred in groupoids if all fibres are groupoids, (i.e, all morphisms in the fibre are isomorphisms).

We now define the notion of a category fibred in groupoids. As we will see in the next section, this will bring us very close to the definition of stacks. In fact, a stack is defined as a category fibred in groupoids satisfying a few more axioms.

Definition 5.1.5. *A category fibred in groupoids over* Top *is a category $\mathfrak{X}$, together with a functor $\pi : \mathfrak{X} \to \mathsf{Top}$ such that the following axioms hold:*

(i) For every morphism $f : V \to U$ in Top*, and every object x of $\mathfrak{X}$ lying over U with $\pi(x) = U$, there exists an object y and an arrow $\phi : y \to x$ in $\mathfrak{X}$ lying over f, with $\pi(\phi) = f$,*

(ii) *For every commutative triangle $W \to V \to U$ in* Top *and morphisms $z \to x$ lying over $W \to U$ and $y \to x$ lying over $V \to U$, there exists an isomorphism $z \to y$ lying over $W \to V$ such that the composition $z \to y \to x$ is the morphism $z \to x$.*

Any choice of such an object y is called a *pullback* of x via the morphism $f : V \to U$. We will write as usual $y = x|V$ or $y = f^*x$.
Given a category fibred in groupoids $\mathfrak{X}$ over Top and a topological space X of Top, the category of all objects of $\mathfrak{X}$ lying over a fixed object X of Top with all morphisms of $\mathfrak{X}$ lying over the identity morphism id_X is called the *fibre* or *category of sections* of $\mathfrak{X}$ over X and denoted by $\mathfrak{X}_X$ or $\mathfrak{X}(X)$. By the definition all fibres are groupoids.
Categories fibred in groupoids over Top form a 2-category. We will explain this in the next definition:

Definition 5.1.6. *Given two categories fibred in groupoids over* Top*, say $\mathfrak{X}$ and $\mathfrak{X}'$,*
A 1-morphism of categories fibred in groupoids over Top *is a functor $F : \mathfrak{X} \to \mathfrak{X}'$ such that $\pi' F = \pi$.*
A 2-morphism u from F to G, with $F, G : \mathfrak{X} \to \mathfrak{X}'$, is a natural transformation such that $\pi'(u(x)) = id_{\pi(x)}$ for all $x \in \mathfrak{X}_0$.
Categories fibred in groupoids with these notions of 1- and 2-morphisms form a 2-category. This 2-category denoted by CFG.

Definition 5.1.7. *Two categories fibred in groupoids $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic if there are* 1*-morphisms $\phi : \mathfrak{X} \to \mathfrak{Y}$ and $\psi : \mathfrak{Y} \to \mathfrak{X}$ and* 2*-morphisms T and T' such that $T : \phi \circ \psi \Rightarrow \mathrm{id}_{\mathfrak{Y}}$ and $T' : \psi \circ \phi \Rightarrow \mathrm{id}_{\mathfrak{X}}$.*

Example 5.1.8 (Identity)**.** *Let $\mathfrak{X}$ be the fixed category* Top*. Let $\pi = \mathrm{id}_{\mathsf{Top}} : \mathsf{Top} \to \mathsf{Top}$ be the projection functor. Then $\mathfrak{X} =$ Top together with the identity map is a category fibred in groupoids.*

Example 5.1.9 (Object)**.** *Given a fixed object $X \in$ Top, consider now the category $\underline{X}$ whose objects are (U, f) where $f : U \to X$ is a morphism in* Top *and U an object in* Top*, and whose arrows are diagrams*

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, "f"'] & & V \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

The projection functor is $\pi : \underline{X} \to \mathsf{Top}$ *with* $\pi((U,f)) = U$ *and* $\pi((U,f),\phi,(V,g)) = \phi$. *We have that* $\underline{X}$ *is a category fibred in groupoids. In particular, in the case that* X *is a point,* $X = *$, *we have that* $\underline{*} = \mathsf{Top}$.

Example 5.1.10. *Fix a topological groupoid* $\mathcal{G}$, *assign the category* $\mathcal{BG}$, *with objects principal* $\mathcal{G}$*-bundles and morphisms are* $\mathcal{G}$*-equivariant maps. The functor* $\pi : \mathcal{BG} \to \mathsf{Top}$ *that sends a principal* $\mathcal{G}$*-bundle to its base and a* $\mathcal{G}$*-equivariant map between two principal* $\mathcal{G}$*-bundles to the induced map between their bases makes the category* $\mathcal{BG}$ *into a category fibred groupoids over the category* Top *of topological spaces.*
To check the conditions in the Definition 5.1.5:
Given a map $f : N \to M$ *between two topological spaces and a principal* $\mathcal{G}$*-bundle* $\xi \to M$ *we have the pullback bundle* $f^*\xi \to N$ *and a* $\mathcal{G}$*-equivariant map* $\tilde{f} : f^*\xi \to \xi$ *inducing* f *on the bases of the bundles.*
Note that if $\pi' : \xi' \to N$ *is a principal* $\mathcal{G}$*-bundle and* $h : \xi' \to \xi$ *is a* $\mathcal{G}$*-equivariant map inducing* $f : N \to M$ *then there is a canonical* $\mathcal{G}$*-equivariant map* $\eta : \xi' \to f^*\xi$ *which is given by* $\eta(x) = (\pi'(x), h(x))$. *By Corollary 3.5.13, the map* η *is a homeomorphism. To check the second condition, suppose that we have three principal* $\mathcal{G}$*-bundles* $\xi'' \to M''$, $\xi' \to M'$ $\xi \to M$, *two* $\mathcal{G}$*-equivariant maps* $f : \xi'' \to \xi$, $h : \xi' \to \xi$ *inducing* $\overline{f} : M'' \to M$ *and* $\overline{h} : M' \to M$ *respectively and a map* $g : M'' \to M'$ *so that*

$$\begin{tikzcd} M'' \arrow[rr, "g"] \arrow[dr, "\overline{f}"'] & & M' \arrow[dl, "\overline{h}"] \\ & M & \end{tikzcd}$$

commutes. We want to construct a $\mathcal{G}$*-equivariant map* $\tilde{g} : \xi'' \to \xi'$ *with* $h \circ \tilde{g} = f$. *We assume that* $\xi'' = \overline{f}^*\xi = M'' \times_M \xi$ *and* $\xi' = \overline{h}^*\xi = M' \times_M \xi$. *Define* $\tilde{g} : M'' \times_M \xi \to M' \times_M \xi$ *by* $\tilde{g}(m,x) = (g(m),x)$. *Hence* $h \circ \tilde{g} = f$, *and we have verified that* $\pi : \mathcal{BG} \to \mathsf{Top}$ *is a category fibred in groupoids.*

Example 5.1.11 (Sheaves)**.** *Let* $F : \mathsf{Top} \to (Sets)$ *be a presheaf, i.e., a contravariant functor from a category of topological spaces to the category of sets. We get a category fibred in groupoids* $\mathfrak{X}$ *defined as follows: objects of are pairs* (U,x), *with* U *a topological space and* x *is an element of the set* $F(U)$, $x \in F(U)$. *A morphism* $(U,x) \to (V,y)$ *is a continuous map in* Top, $\alpha : U \to V$ *such that* $F(\alpha) : F(U) \to F(V)$ *maps* x *to* y, $x = F(\alpha)(y)$. *The functor is defined as a projection functor by*

$$\pi : \mathfrak{X} \to \mathsf{Top},\ (U,x) \mapsto U.$$

Especially any sheaf $F : \mathsf{Top} \to (Sets)$ gives therefore a category fibred in groupoids over Top *and in particular every topological space X gives a category fibred in groupoids $\underline{X}$ over* Top *as the sheaf represented by X, i.e., where*

$$\underline{X}(U) = Hom_{\mathsf{Top}}(U, X).$$

To simplify notation, we will freely identify $\underline{X}$ with the topological space X.

Definition 5.1.12. *A category $\mathfrak{X}$ fibred in groupoids over* Top *is representable if there exists a topological space X such that $\underline{X}$ is isomorphic to $\mathfrak{X}$ as categories fibred in groupoids over* Top*.*
We call a morphism of categories fibred in groupoids $\mathfrak{X} \to \mathfrak{Y}$ representable, if for every morphism $U \to \mathfrak{Y}$ from a topological space U, the fibred product $\mathfrak{X} \times_{\mathfrak{Y}} U$ is representable (or equivalent to) a topological spaces.

5.2 Stacks

Now let us recall the definition of a stack and describe it as category fibred in groupoid [6]. In the following, let Top always be the category of topological spaces.

Definition 5.2.1. *A category fibred in groupoids $\mathfrak{X}$ over* Top *is a stack over* Top *if the following gluing axioms hold:*

(i) For any topological space X in Top*, any two objects x, y in $\mathfrak{X}$ lying over X and any two isomorphisms $\phi, \psi : x \to y$ over X, such that $\phi|U_i = \psi|U_i$ for all U_i in a covering $\{U_i \to X\}$ it follows that $\phi = \psi$.*

(ii) For any topological space X in Top*, any two objects $x, y \in \mathfrak{X}$ lying over X, any covering $\{U_i \to X\}$ and, for every i, an isomorphism $\phi_i : x|U_i \to y|U_i$, such that $\phi_i|U_{ij} = \phi_j|U_{ij}$ for all i, j, there exists an isomorphism $\phi : x \to y$ with $\phi|U_i = \phi_i$ for all i.*

(iii) For any topological space X in Top*, any covering $\{U_i \to X\}$, any family $\{x_i\}$ of objects x_i in the fibre $\mathfrak{X}_{U_i}$ and any family of morphisms $\{\phi_{ij}\}$, where $\phi_{ij} : x_i|U_{ij} \to x_j|U_{ij}$ satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ in $\mathfrak{X}(U_{ijk})$ there exist an object x lying over X with isomorphisms $\phi_i : x|U_i \to x_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ in $\mathfrak{X}(U_{ij})$.*

The isomorphism ϕ in (ii) is unique by (i) and similar from (i) and (ii) it follows that the object x whose existence is asserted in (iii) is unique up to a unique isomorphism. All pullbacks mentioned in the definitions are also only unique up to isomorphism, but the properties do not depend on choices.
Stacks over Top form a full sub 2-category of CFG whose objects are stacks and denoted by St. Any fibred category over Top which is equivalent to a stack is itself a stack.

Definition 5.2.2. *A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks is called representable if for every map $T \to \mathfrak{Y}$ from a topological space T, the fibre product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is a topological space.*

Roughly speaking, this is saying that the fibres of f are topological spaces.
Any property $\mathbf{P}$ of morphisms of topological spaces which is invariant under base change can be defined for an arbitrary representable morphism of stacks. More precisely, we say that a representable morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is $\mathbf{P}$ if for every map $T \to \mathfrak{Y}$ from a topological space T, the base extension $f_T : T \times_{\mathfrak{Y}} \mathfrak{X}$ is $\mathbf{P}$ as a map of topological spaces; see [38].
In the next two sections, we follow [13] and recall how the category fibred in groupoids and stacks associated to a topological groupoid constructed.

5.3 The category fibred in groupoids associated to a topological groupoid

Definition 5.3.1. *Given a topological groupoid $\mathcal{G}$ and X a topological space in* Top *, we can construct a category fibred in groupoids $\pi : \mathfrak{X} \to \mathsf{Top}$ such that each fibre is given by the groupoid*

$$(\mathfrak{X}_X)_0 = \mathrm{Hom}(X, G_0)$$

$$(\mathfrak{X}_X)_1 = \mathrm{Hom}(X, G_1)$$

and the pullback functor is induced by composition. We will denote the category fibred in groupoids that was constructed by $\mathcal{G}(\mathfrak{X})$ and is *not* in general a stack as well. However, we can associate to each category fibred in groupoids a stack. To do this, we need first to introduce the notion of $\mathcal{G}$-torsor, which is a category of

principal $\mathcal{G}$-bundle that was defined in the Example 5.1.10, we can construct the fibred category in groupoids $\mathcal{BG}$ from a topological groupoid $\mathcal{G}$ where the objects of $\mathcal{BG}$ are principal $\mathcal{G}$-bundles and morphisms are $\mathcal{G}$-equivariant maps.
Now we recall the definition of $\mathcal{G}$-torsor:

5.3.1 Torsors

We provide the concept of action of groupoid on a topological space in the Definition 3.5.1, Now we define the groupoid torsor.

Definition 5.3.2. *Let $\mathcal{G}$ be a topological groupoid, X a topological space. A $\mathcal{G}$-torsor over X is an open surjection $p : E \to X$ equipped with an action $G_1 \times^t_{G_0} E \to E$ of $\mathcal{G}$ on the anchor map $a : E \to G_0$ such that*

$$G_1 \times^t_{G_0} E \to E \times_X E$$

is a homeomorphism.

A morphism (ϕ, f) from a $\mathcal{G}$-torsor over X to a $\mathcal{G}$-torsor over X' is given by a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

where ϕ is $\mathcal{G}$-invariant. Note that morphisms of $\mathcal{G}$-torsors over a fixed X are invertible.
Hence, we will denote the category whose objects are $\mathcal{G}$-torsors and whose arrows are morphisms of $\mathcal{G}$-torsors by $\mathcal{BG}$.
The diagram in the definition of a morphism of $\mathcal{G}$-torsors is a pullback diagram. In the special case, that the groupoid is defined as $\mathcal{G} = [G \rightrightarrows *]$ is a topological group, a G-torsor is simply a principal G-bundle.

Example 5.3.3. *(Trivial torsors) Let $f : S \to X$ be a continuous map. Given f, we can induce over S in a canonical way a $\mathcal{G}$-torsor, which we call the trivial $\mathcal{G}$-torsor given by f.*
Simply define P to be the fibred product $P = S \times_{f,X,s} \mathcal{G}$. The structure map $\pi : P \to S$ is the first projection. The anchor map of the $\mathcal{G}$-action is the second

projection followed by the target map t. The action is then defined by

$$(s, \gamma) \cdot \delta = (s, \gamma \cdot \delta).$$

One checks that this is, indeed, a $\mathcal{G}$-torsor over S.

5.4 The stack associated to a topological groupoid

We can associate to each groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ a stack, called *the stack completion* of $\mathcal{G}(\mathfrak{X})$ and we will denoted by $\mathcal{G}_{\mathfrak{X}}$, that is the same concept of quotient stacks that denoted by $[G_0/G_1]$. It is defined as the category fibred in groupoids $\pi : \mathcal{BG} \to \mathsf{Top}$ such that $\pi(E \to \mathcal{G}) = X$ on objects and $\pi((\phi, f)) = f : X \to X'$ on arrows. Each fibre is given by the groupoid
$(\mathcal{BG}_X)_0 = \mathcal{G}$-torsors over X
$(\mathcal{BG}_X)_1$ =morphisms of $\mathcal{G}$-torsor over X.
The category fibred in groupoids thus defined, is a stack. In the 2-category St we can say that two stacks are equal, isomorphic or equivalent.

Example 5.4.1. *The category fibred in groupoids over* Top *in the example 5.1.10 with canonical projection functor $\pi : \mathcal{BG} \to$* Top *, $(E, X) \mapsto X$ is stack.*

Example 5.4.2 (Quotient Stack)**.** *Let X be a topological space with a continuous (left) action $\rho : G \times X \to X$ by a topological group G. Let $[X/G]$ be the category which has as objects triples (P, S, μ), where S is a topological space of* Top*, P a principal G-bundle over S and $\mu : P \to X$ a G-equivariant continuous map. A morphism $(P, S, \mu) \to (Q, T, \nu)$ is a commutative diagram*

$$\begin{tikzcd} & X & \\ P \arrow[ur, "\mu"] \arrow[rr, "\varphi"] \arrow[d, "\pi"'] & & Q \arrow[ul, "\nu"'] \arrow[d, "\tau"] \\ S \arrow[rr, "\psi"] & & T \end{tikzcd}$$

where $\varphi : P \to Q$ is a G-equivariant map. Then $[X/G]$ together with the projection functor

$$\pi : [X/G] \to \mathsf{Top}, \ (P, S, \mu) \mapsto S$$

given by $\pi(P, S, \mu) = S$ *and* $\pi((\psi, \varphi)) = \varphi$ *is a category fibred in groupoids over* Top *and a stack.*

If two groupoids are Morita equivalent, their associated stacks are equivalent as stacks because the Morita equivalence between two groupoids defines an equivalence relation 3.3.3.

Corollary 5.4.3. *If* $\mathcal{G} \sim_M \mathcal{G}'$ *then* $\mathcal{G}_{\mathfrak{X}} \sim \mathcal{G}'_{\mathfrak{X}}$.

Proof. From Remark 3.3.4, it is clear that the Morita equivalence satisfied the equivalence relation. $\square$

For each topological groupoid $\mathcal{G}$ we can associate a category fibred in groupoids $\mathcal{G}(\mathfrak{X})$ and a stack $\mathcal{G}_{\mathfrak{X}}$. There is a canonical morphism of categories fibred in groupoids $F : \mathcal{G}(\mathfrak{X}) \to \mathcal{G}_{\mathfrak{X}}$. All groupoids in the same Morita equivalence class will have equivalent associated stacks. However, some of the groupoids in that equivalence class will have an associated category fibred in groupoids $\mathcal{G}(\mathfrak{X})$ that is not a stack, whereas some others in the same class will have an associated $\mathcal{G}'(\mathfrak{X})$ that is already a stack.
We will describe the groupoids $\mathcal{G}$ for which $F : \mathcal{G}(\mathfrak{X}) \to \mathcal{G}_{\mathfrak{X}}$ is an equivalence of categories fibred in groupoids, and therefore $\mathcal{G}(\mathfrak{X})$ is a stack.

5.5 Topological stacks

Topological stacks are defined over the category of topological spaces and continuous maps Top. We endow Top with a Grothendieck topology that defined by taking the open coverings to be the usual open coverings of topological spaces. This is a subcanonical topology. That means, the presheaf W_{pre} represented by any $W \in \mathsf{Top}$ is indeed a sheaf. Now we define some related concepts like sections and Grothendieck topology in order to define the topological stacks.

Definition 5.5.1. *A section of a fibre bundle* π *is a continuous right inverse of* π*,i.e if* $\pi : E \to B$ *then a section is* $\sigma : B \to E$ *s.t,* $\pi(\sigma(x)) = x\ \forall x \in B$.
A local section is a continuous map $s : U \to E$ *where* U *is an open set in* B *and* $\pi(s(x)) = x,\ \forall x \in U$.

Definition 5.5.2. *Let C be a category. A Grothendieck topology on C consists of, for each object X in C, a collection $Cov(X)$ of sets $\{X_i \to X\}$ of arrows, called coverings of X, such that*

1. *If $V \to X$ is an isomorphism, then $\{V \to X\} \in Cov(X)$.*
2. *If $\{X_i \to X\} \in Cov(X)$ and any arrow $Y \to X$, then the fibre products $X_i \times_X Y$ exist and $\{X_i \times_X Y\} \in Cov(Y)$.*
3. *If $\{X_i \to X\} \in Cov(X)$ and $\{V_{ij} \to X_i\} \in Cov(X_i)$ for each i, then $\{V_{ij} \to X_i \to X\} \in Cov(X)$.*

A site is a category C together with a Grothendieck topology.

Definition 5.5.3. *[40] A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks is called an epimorphism if it is an epimorphism in the sheaf theoretic sense (i.e., for every topological space W, every object in $\mathfrak{Y}(W)$ has a preimage in $\mathfrak{X}(W)$, possibly after passing to an open cover of W). In this case where $\mathfrak{X}$ and $\mathfrak{Y}$ are topological spaces, this is equivalent to saying that f admits local sections.*

Definition 5.5.4. *A topological stack is a stack $\mathfrak{X}$ over* Top *which admits a representable epimorphism $p : X \to \mathfrak{X}$ from a topological space X. Such a morphism is called atlas (or presentation) for $\mathfrak{X}$.*

It follows that every morphism $U \to \mathfrak{X}$ from a topological space U to a topological stack $\mathfrak{X}$ is representable.
Every quotient stack $[X_0/X_1]$ of a topological groupoid is a topological stack, and the quotient map $X_0 \to [X_0/X_1]$ is an atlas for it. Conversely, given an atlas $X \to \mathfrak{X}$ for a topological stack $\mathfrak{X}$, $\mathfrak{X}$ is equivalent to the quotient stack of the topological groupoid $[pr_1, pr_2 : X \times_{\mathfrak{X}} X \rightrightarrows X]$. [37]
More precisely we define a stack to be a topological stack in this way:

Definition 5.5.5. *A stack $\mathfrak{X}$ over* Top *is a topological stack if there exists a topological space X and a local section $x : X \to \mathfrak{X}$, i.e., there exists a topological space X together with a morphism of stacks $x : X \to \mathfrak{X}$ such that for every topological space U and every morphism of stacks $U \to \mathfrak{X}$ the fibre product $X \times_{\mathfrak{X}} U$ is representable and the induced morphism of topological spaces $X \times_{\mathfrak{X}} U$ is a local section.*

Remark 5.5.6. The atlas of a topological stack need not be unique, i.e. a topological stack can have different presentations.

Now we present some examples on topological stacks and note that all representable stacks are topological stacks.

Example 5.5.7. *Let X be a topological space. The category fibred in groupoids $\underline{X}$ is in fact a topological stack over* Top. *A presentation is given by the identity morphism id_X.*

Example 5.5.8 (Torsor)**.** *Let G be a topological group. Consider the category $\mathfrak{B}G$ which has as objects principal G-bundles (or G-torsors) P over S and as arrows commutative diagrams*

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\pi"'] & Q \arrow[d, "\tau"] \\ S \arrow[r, "\varphi"] & T \end{tikzcd}$$

where the map $\psi : P \to Q$ is equivariant.

The category $\mathfrak{B}G$ together with the projection functor $\pi : \mathfrak{B}G \to \mathsf{Top}$ given by $\pi(P \to S) = S$ and $\pi((\psi, \varphi)) = \varphi$ is a category fibred in groupoids, in fact a topological stack, the classifying stack of G whose atlas presentation is given by the representable open surjection $* \to \mathfrak{B}G$.

Example 5.5.9. *The quotient stack $[X/G]$ that constructed in the Example 5.4.2 is in fact a topological stack, the quotient stack of G. An atlas is given by the representable epimorphism $x : X \to [X/G]$. If $X = *$ is just a point, we get the topological stack $\mathfrak{B}G = [*/G]$, the classifying stack of G with atlas A presentation is given by the representable epimorphism $* \to \mathfrak{B}G$.*

Example 5.5.10 (Orbifolds)**.** *Every orbifold $\mathfrak{X}$ is a topological stack which is locally isomorphic to a stack of the form $[U/G]$, with G a finite group acting on a manifold U. The coarse moduli space of $\mathfrak{X}$ is the underlying topological space of $\mathfrak{X}$(so, locally it is U/G). The inertia group of a point x is the orbifold group (or the stabilizer group) of x. The inertia stack of $\mathfrak{X}$ is the stack of twisted sectors of $\mathfrak{X}$.*

Topological stacks are basically incarnations of topological groupoids. In order to do geometry on stacks, we have to compare them with topological spaces. Now we see how to associate topological stacks to topological groupoids and conversely.

5.6 The topological stack associated to a topological groupoid

Let $\mathfrak{X}$ be a topological stack with a given presentation $x : X \to \mathfrak{X}$. We can associate to $\mathfrak{X}$ a topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$. Let $G_0 = X$ and $G_1 = X \times_{\mathfrak{X}} X$. The source and target morphisms $s, t : X \times_{\mathfrak{X}} X \rightrightarrows X$ of $\mathcal{G}$ are given as the two canonical projection morphisms. The composition of morphisms m in $\mathcal{G}$ is given as projection to the first and third factor $X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} X \cong (X \times_{\mathfrak{X}} X) \times_X (X \times_{\mathfrak{X}} X) \to X \times_{\mathfrak{X}} X$. The morphism which interchanges factors $X \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} X$ gives the inverse morphism i and the unit morphism e is given by the diagonal morphism $X \to X \times_{\mathfrak{X}} X$. In other words, because the presentation $x : X \to \mathfrak{X}$ of a topological stack has a local section, it follows that the source and target morphisms $s, t : X \times_{\mathfrak{X}} X \rightrightarrows X$ have local sections as induced maps of the fibre product.
Given instead a topological groupoid we can associate a topological stack to it. Basically this is a generalisation of associating to a topological group G, i. e. a topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ with $G_0 = *$ and $G_1 = G$ the classifying stack $\mathcal{B}G$.

Example 5.6.1 ($\mathcal{G}$-torsors)**.** *The $\mathcal{B}\mathcal{G}$, the $\mathcal{G}$-torsor in the Example 5.5.8 is the topological stack that constructed over the category* Top*.*

We have the following fundamental property (see for example [6][Prop. 2.3]) of $\mathcal{B}\mathcal{G}$.

Theorem 5.6.2. *For every topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ the category fibred in groupoids $\mathcal{B}\mathcal{G}$ of $\mathcal{G}$-torsors is a topological stack with a presentation $\tau_0 : G_0 \to \mathcal{B}\mathcal{G}$.*

The stack $\mathcal{B}\mathcal{G}$ is also called the *classifying stack* of $\mathcal{G}$-torsors. It follows from this also that the topological groupoid $\mathcal{G} = [G_1 \rightrightarrows G_0]$ is isomorphic to the topological groupoid $\mathcal{G}(\tau_0)$ associated to the atlas $\tau_0 : G_0 \to \mathcal{B}\mathcal{G}$ of the stack $\mathcal{B}\mathcal{G}$.
As the presentations of a topological stack are not unique, the associated topological groupoids might be different. In order to define algebraic invariants, like cohomology or homotopy groups for topological stacks they should however not depend on a chosen presentation of the stack. Therefore it is important to know, when two different topological groupoids give rise to isomorphic stacks. This will

be the case when the topological groupoids are Morita equivalent.

We have the following main theorem concerning the relation of the various topological groupoids associated to various presentations of a topological stack (see [6], Theorem 2.24).

Theorem 5.6.3. *Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the associated topological stacks, i. e. $\mathfrak{X} = \mathcal{B}\mathcal{G}$ and $\mathfrak{Y} = \mathcal{B}\mathcal{H}$. Then the following are equivalent:*

(i) the topological stacks $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic,

(ii) the topological groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent.

As a special case we have the following fundamental property concerning different presentations of a topological stack $\mathfrak{X}$.

Proposition 5.6.4. *Let $\mathfrak{X}$ be a topological stack with two given presentations $x : X \to \mathfrak{X}$ and $x' : X' \to \mathfrak{X}$. Then the associated topological groupoids $\mathcal{G}(x)$ and $\mathcal{G}(x')$ are Morita equivalent.*

Therefore topological groupoids present isomorphic topological stacks if and only if they are Morita equivalent or in other words topological stacks correspond to Morita equivalence classes of topological groupoids.

We now recall the fundamental notion of a continuous map between topological stacks (see [25]).

Definition 5.6.5 (Continuous map)**.** *An arbitrary morphism $\mathfrak{X} \to \mathfrak{Y}$ of topological stacks is continuous, if there are atlases $X \to \mathfrak{X}$ and $Y \to \mathfrak{Y}$ such that the induced morphism from the fibred product $X \times_{\mathfrak{Y}} Y \to Y$ in the diagram below is a continuous map between spaces.*

$$\begin{tikzcd}
X \times_{\mathfrak{Y}} Y \arrow[r] \arrow[dd] & X \arrow[d] \\
 & \mathfrak{X} \arrow[d] \\
Y \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

Let $\mathfrak{U}$ be a subcategory of $\mathfrak{X}$. Recall that a subcategory is called *saturated* if whenever it contains an object x then it contains the entire isomorphism class $\bar{x}$ of that object and is called *full* if whenever it contains an arrow between x and y, it contains the entire set $\mathrm{Hom}(x,y)$ of arrows.

Let $\pi : \mathfrak{X} \to \mathsf{Top}$ be a topological stack and $x : X \to \mathfrak{X}$ be an atlas. Let $U \subset X$ be an open subset and consider the saturation U_0 of the image $x(U)$ in X_0, i.e.

$$U_0 = \{z \in X_0 |\ z \in \bar{x} \text{ for some } x \in U\}.$$

The full subcategory $\mathfrak{U}$ on U_0 is $U_1 \rightrightarrows U_0$ where $U_1 = \{g \in X_1 |\ s(g), t(g) \in U_0\}$.

Definition 5.6.6 (Restricted substack)**.** *Let $\pi : \mathfrak{X} \to \mathsf{Top}$ be a topological stack with atlas $x : X \to \mathfrak{X}$ and $U \subset X$ be an open set. Consider the full subcategory $\mathfrak{U}$ on U_0 and let $\pi' := \pi \circ i$, where $i : \mathfrak{U} \to \mathfrak{X}$ is the inclusion. We say that $\mathfrak{U}$ with the projection $\pi' : \mathfrak{U} \to \mathsf{Top}$ is the restricted substack of $\mathfrak{X}$ to $\mathfrak{U}$.*

Definition 5.6.7 (Constant morphism)**.** *Let $c : \mathfrak{X} \to \mathfrak{Y}$ be a continuous morphism between topological stacks. We say that c is a constant morphism if there are presentations $X \to \mathfrak{X}$ and $Y \to \mathfrak{Y}$ such that the induced morphism from the fibred product $X \times_{\mathfrak{Y}} Y \to Y$ is a constant map.*

For instance, any continuous map $\mathfrak{X} \to \mathfrak{Y}$ where $\mathfrak{Y}$ admits a presentation by a point $* \to \mathfrak{Y}$ is a constant morphism.

Example 5.6.8. *Let S^1 act on S^1 by rotation and consider the quotient stack $\mathfrak{X}$ associated to this action, $\mathfrak{X} = [S^1/S^1]$. We will show that the identity map $\mathrm{id}_{[S^1/S^1]} : [S^1/S^1] \to [S^1/S^1]$ is a constant map. The groupoid $[S^1 \times S^1 \rightrightarrows S^1]$ is Morita equivalent to a point groupoid $[* \rightrightarrows *]$, therefore the stacks $\mathfrak{X} = [S^1/S^1]$ and $\underline{*}$ are isomorphic. Since $* \to \underline{*}$ is a presentation for $\underline{*}$ it follows that $* \to [S^1/S^1]$ is a presentation for $\mathfrak{X}$. Hence any map with codomain $\mathfrak{X} = [S^1/S^1]$ is a constant morphism of stacks.*

Chapter 6

The Lusternik-Schnirelmann Category of a Topological Stack

The Lusternik-Schnirelmann category of a topological space is an invariant of the homotopy type of the space introduced in the early 1930s by Lusternik and Schnirelmann. Our aim is to develop a Lusternik-Schnirelmann theory in the context of topological groupoids. In this chapter, we will introduce the notion of LS-category and prove the basic results about it using [14], [11] and [3]. Also we will generalise this notion to the case of topological groupoids and show that LS-category is indeed an invariant of topological stacks.

6.1 Lusternik-Schnirelmann category of a space

In this section we present the classical notions of Lusternik-Schnirelmann category of a topological space and give some examples and results on that.

Definition 6.1.1.

- *The Lusternik-Schnirelmann or LS-category of a topological space X is the least integer n such that there exists an open covering $U_1, \cdots, U_{n+1}$ of X with each U_i contractible to a point in the space X. We denote this by* $\mathrm{cat}(X) = n$ *and we call such a covering* $\{U_i\}$ *categorical. If no such integer exists, we write* $\mathrm{cat}(X) = \infty$.

- *Let $A \subseteq X$, the subset category of A in X, denoted* $\mathrm{cat}_X(A)$ *is the least integer n such that there exist open subsets, $U_1, \cdots, U_{n+1}$ of X which cover A and which are contractible in X. If no such integer exists, we write* $\mathrm{cat}_X(A) = \infty$. *Note that* $\mathrm{cat}_X(X)$=$\mathrm{cat}(X)$ *and* $\mathrm{cat}_X(A) \leq \mathrm{cat}(X)$.

Example 6.1.2. *1. A contractible space X is a categorical cover of itself, so* $\mathrm{cat}(X) = 0$*.*

2. The sphere S^n may be covered by two hemispheres which have been extended slightly to make overlapping open sets. Each hemisphere is homeomorphic to a disk, so is contractible in the sphere. Hence we get $\mathrm{cat}(S^n) \leq 1$*.*

Definition 6.1.3. *If $f : X \to Y$ is a continuous map, a continuous map $g : Y \to X$ is a right homotopy inverse for f provided that fg is homotopic to the identity map on Y, left homotopy inverse is defined analogously.*

Lemma 6.1.4. *[14] If $f : X \to Y$ has a right homotopy inverse g (i.e. X dominates Y), then* $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$*.*

Proof. Let $g : Y \to X$ be the right homotopy inverse for f.
That is, $f \circ g \simeq id_Y$. Let $\mathrm{cat}(X) = n$ with categorical open cover $U_1, \cdots, U_{n+1}$. Define open sets $V_i = g^{-1}(U_i)$ for each i. For V_i, define a contracting homotopy K in Y by $K(v,t) = \begin{cases} G(v, 2t) & 0 \leqslant t \leqslant 1/2 \\ f(H(g(v), 2t-1)) & 1/2 \leqslant t \leqslant 1 \end{cases}$ where $G : Y \times I \to Y$ gives $f \circ g \simeq id_Y$ by $G(y, 0) = y$ and $G(y, 1) = f(g(y))$ and $H : U_i \times I \to X$ contracts U_i by $H(u, 0) = u$ and $H(u, 1) = x_0$. The homotopy K is well defined since $G(v, 1) = f(g(v)) = f(H(g(v), 0))$ and $K(v, 0) = G(v, 0) = v$, $K(v, 1) = f(H(g(v), 1)) = f(x_0) = y_0$. Hence, $\{V_i\}$ is a categorical cover for Y with $n + 1$ members and so $\mathrm{cat}(Y) \leqslant n = \mathrm{cat}(X)$. □

Theorem 6.1.5. *If $f : X \to Y$ is a homotopy equivalence, then* $\mathrm{cat}(X) = \mathrm{cat}(Y)$*,i.e, LS-category is a homotopy invariant.*

Proof. Because f is a homotopy equivalence, there exists a map $g : Y \to X$ with $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Now apply the lemma above to both f and g to get that the LS-category of X and Y is less than or equal to each other. □

6.2 Lusternik-Schnirelmann category of a topological groupoid

We recall first the definition and fundamental properties for the notion of Lusternik-Schnirelmann category of topological groupoids which is generalised from the

Lusternik-Schnirelmann category of Lie groupoids as defined in 2.6 and for more details see [11] and [1].

The most important property here is that the Lusternik-Schnirelmann category of topological groupoid is in fact Morita invariant, which means that it is in fact an invariant of the associated topological stack.

From Proposition 3.4.4 we consider our context to be the Morita bicategory of topological groupoids **Gpd** obtained from $\mathbf{Gpd} = \mathbf{G}(E^{-1})$ by formally inverting the essential equivalences E. Objects in this bicategory are topological groupoids, 1-morphisms are *generalised maps* $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ such that ϵ is an essential equivalence and 2-morphisms from $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ to $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ are given by classes of diagrams:

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\ \mathcal{K} \arrow[r, phantom, "\sim"] & \mathcal{L} \arrow[u, "u"] \arrow[d, "v"] \arrow[r, phantom, "\sim"] & \mathcal{G} \\ & \mathcal{J}' \arrow[ul, "\epsilon'"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

where $\mathcal{L}$ is a topological groupoid, and u and v are essential equivalences.

We define path groupoid as considered in [2] which $\mathcal{I}$ is the unit groupoid over the interval $I = [0, 1]$:

Definition 6.2.1. *The path groupoid of $\mathcal{G}$ is defined as the mapping groupoid in the bicategory* **Gpd**, $P\mathcal{G} = \mathcal{M}ap(\mathcal{I}, \mathcal{G})$, *which is a generalised map from the unit groupoid $\mathcal{I}$ to the topological groupoid $\mathcal{G}$. That is, a map $(\epsilon, \alpha) : \mathcal{I} \xleftarrow{\epsilon} \mathcal{I}' \xrightarrow{\alpha} \mathcal{G}$ where $\mathcal{I}'$ is a groupoid essentially equivalent to the unit groupoid $\mathcal{I}$.*

Let $(\sigma, f) : \mathcal{K} \xleftarrow{\sigma} \mathcal{K}' \xrightarrow{f} \mathcal{G}$ and $(\tau, g) : \mathcal{K} \xleftarrow{\tau} \mathcal{K}'' \xrightarrow{g} \mathcal{G}$ be generalised maps. The map (σ, f) is *groupoid homotopic* to (τ, g) if there exists $(\epsilon, H) : \mathcal{K} \xleftarrow{\epsilon} \tilde{\mathcal{K}} \xrightarrow{H} P\mathcal{G}$ and two commutative diagrams up to natural transformations:

$$\begin{tikzcd} & \tilde{\mathcal{K}} \arrow[dl, "\epsilon"'] \arrow[dr, "ev_0 H"] & \\ \mathcal{K} \arrow[r, phantom, "\sim"] & \mathcal{L}_0 \arrow[u, "u_0"] \arrow[d, "v_0"] \arrow[r, phantom, "\sim"] & \mathcal{G} \\ & \mathcal{K}' \arrow[ul, "\sigma"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} & \tilde{\mathcal{K}} \arrow[dl, "\epsilon"'] \arrow[dr, "ev_1 H"] & \\ \mathcal{K} \arrow[r, phantom, "\sim"] & \mathcal{L}_1 \arrow[u, "u_1"] \arrow[d, "v_1"] \arrow[r, phantom, "\sim"] & \mathcal{G} \\ & \mathcal{K}'' \arrow[ul, "\tau"] \arrow[ur, "g"'] & \end{tikzcd}$$

where $\mathcal{L}_i$ is a translation groupoid, coming from the action of topological group on a topological space which described in Example 2.6.7, and u_i and v_i are equivariant essential equivalences for $i = 0, 1$.

Similarly as for topological stacks we also have the concept of a restricted groupoid over a given invariant subset and that of a generalised constant map, which we will need to define the LS-category of topological groupoids.

Definition 6.2.2. *Let $\mathcal{G} = [G_1 \rightrightarrows G_0]$ be a topological groupoid. An open set $U \subset G_0$ is invariant if $t(s^{-1}(U)) \subset U$.*
The restricted groupoid $\mathcal{U}$ to an invariant set $U \subset G_0$ is the full groupoid over U. In other words, $U_0 = U$ and $U_1 = \{g \in G_1 : s(g), t(g) \in U\}$. We write $\mathcal{U} = \mathcal{G}|_U \subset \mathcal{G}$.

Now we define restricted groupoid over an orbit $\mathcal{O}$.

Definition 6.2.3. *A restricted groupoid $\mathcal{G}|_{\mathcal{O}}$ over an orbit $\mathcal{O}$ will be called an orbit groupoid and denoted by $\mathcal{O}^K$, where $K = G_x$ is the isotropy group of x for any $x \in \mathcal{O}$.*

Definition 6.2.4. *The map $(\epsilon, c) : \mathcal{K} \xleftarrow{\epsilon} \mathcal{K}' \xrightarrow{c} \mathcal{G}$ is a generalised constant map if for all $x \in K_0^{'}$ there exists $g \in G_1$ with $s(g) = x_0$ such that $c(x) = t(g)$ for a fixed $x_0 \in G_0$.*

In other words, the image of the generalised map (ϵ, c) is contained in a fixed orbit $\mathcal{O}$, the orbit of x_0.

Definition 6.2.5. *The restriction $(\epsilon, \phi)|_{\mathcal{V}}$ of a generalised map $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ to the restricted groupoid $\mathcal{V} \subset \mathcal{K}$ is the composition of the generalised map (ϵ, ϕ) and the inclusion functor $\mathcal{V} \xleftarrow{id} \mathcal{V} \xrightarrow{i_{\mathcal{V}}} \mathcal{K}$:*

$$\begin{tikzcd}
 & & \mathcal{J} \times_{\mathcal{K}} \mathcal{V} \arrow[dl, "pr_3"'] \arrow[dr, "pr_1"] & & \\
 & \mathcal{V} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{V}}"] & & \mathcal{J} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\
\mathcal{V} & & \mathcal{K} & & \mathcal{G}
\end{tikzcd}$$

where $\mathcal{J} \times_{\mathcal{K}} \mathcal{V}$ is the fibred product groupoid 3.3.1

Definition 6.2.6. *The product $(\epsilon, \phi) \times (\epsilon', \phi')$ of two generalised maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K}' \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}'$ is given by the generalised map $\mathcal{K} \times \mathcal{K}' \xleftarrow{\epsilon \times \epsilon'} \mathcal{J} \times \mathcal{J}' \xrightarrow{\phi \times \phi'} \mathcal{G} \times \mathcal{G}'$.*

We will now define a notion of $\mathcal{G}$-contraction within the topological groupoid $\mathcal{G}$. We keep the classical Lusternik-Schnirelmann terminology of *categorical* for contractible subspace in a given space.

Definition 6.2.7. *For an invariant open set $U \subset G_0$, we will say that the restricted groupoid $\mathcal{U}$ is $\mathcal{G}$-categorical if the inclusion functor $i_{\mathcal{U}} : \mathcal{U} \to \mathcal{G}$ is groupoid homotopic to a generalised constant map*

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{U}}"] & \\ \mathcal{U} & \mathcal{U}' \arrow[u, "u"'] \arrow[d, "v"] & \mathcal{G} \\ & \mathcal{U}' \arrow[ul, "\epsilon"] \arrow[ur, "c"'] & \end{tikzcd}$$

where u and v are essential homotopy equivalences.

Given a categorical subgroupoid $\mathcal{U}$, we have that the inclusion $i_{\mathcal{U}}$ composed with an essential homotopy equivalence u factors through an orbit groupoid up to homotopy:

$$\begin{tikzcd} \mathcal{U} \arrow[dr, "i_{\mathcal{U}}"] & \\ \mathcal{L} \arrow[u, "u"] \arrow[d, "v"] \arrow[r, phantom, "\Downarrow"] & \mathcal{G} \\ \mathcal{U}' \arrow[ur, "c"'] \arrow[r] & \mathcal{O}^K \arrow[u] \end{tikzcd}$$

In other words, if $\mathcal{U}$ is a $\mathcal{G}$-categorical, then there exists a groupoid $\mathcal{L}$ and a group K such that the diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, "c"] \arrow[d, "\epsilon"'] & \mathcal{O} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{G} \end{tikzcd}$$

is commutative up to groupoid homotopy where ϵ is an equivariant essential equivalence and $\mathcal{O}$ is an orbit..

Proposition 6.2.8. *A topological groupoid $\mathcal{G}$ is Morita equivalent to a point groupoid if and only if $\mathcal{G}$ is transitive.*

Proof. The map $t : G(x_0, -) \to G_0$ is the pullback of the open surjection (s, t) along the map $G_0 \to G_0 \times G_0$ which sends x to (x_0, x), hence it is itself open surjection.

Since any pointed topological groupoid is a topological group and it is clearly transitive, we only need to show that transitivity is stable under Morita equivalence. This is true because a pullback of an open surjection is open surjection, and because a map which pulls back along an open surjection to an open surjection is itself an open surjection. $\square$

Since an orbit groupoid is transitive, for $\mathcal{U}$ to be $\mathcal{G}$-categorical we can substitute the constant generalised map in 6.2.7 by a generalised map with image contained in a point groupoid.

Definition 6.2.9. *Let $\mathcal{G} = [G_1 \rightrightarrows G_0]$ be a topological groupoid. The groupoid LS-category,* $\mathrm{cat}(\mathcal{G})$*, is the least number of invariant open sets U needed to cover G_0 such that the restricted groupoid $\mathcal{U}$ is $\mathcal{G}$-categorical.*
If G_0 cannot be covered by a finite number of such open sets, we will say that $\mathrm{cat}(\mathcal{G}) = \infty$.

This number is invariant of Morita equivalence that generalises the ordinary LS-category of a topological space. If $\mathcal{G} = u(X)$ is the unit groupoid, then $\mathrm{cat}(\mathcal{G}) = \mathrm{cat}(X)$, where $\mathrm{cat}(X)$ means the ordinary LS-category.

6.2.1 Invariance of Morita homotopy type

We will show that the LS-category of groupoid is an invariant of Morita homotopy type, and then, invariant under Morita equivalence.

Proposition 6.2.10. *If $\mathcal{K} \simeq_M \mathcal{G}$, then* $\mathrm{cat}(\mathcal{K}) = \mathrm{cat}(\mathcal{G})$.

Proof. We will prove that if $\mathcal{K}$ dominates $\mathcal{G}$, then $\mathrm{cat}(\mathcal{K}) \geq \mathrm{cat}(\mathcal{G})$:
let $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{G} \xleftarrow{\delta} \mathcal{J}' \xrightarrow{\psi} \mathcal{K}$ be two generalised maps such that the composition $(\phi, \epsilon) \circ (\psi, \delta)$ is Morita homotopic to the identity in $\mathcal{G}$. We have the following diagram

$$
\begin{tikzcd}
 & & \mathcal{G} \arrow[dll, "id"'] \arrow[drr, "id"] & & \\
\mathcal{G} \arrow[rr, phantom, "\Downarrow S"] & & \mathcal{E} \arrow[u] \arrow[dd] \arrow[rr, phantom, "\Downarrow S"] & & \mathcal{G} \\
 & \mathcal{J}' \arrow[ul, "\delta"] & & \mathcal{J} \arrow[ur, "\phi"'] & \\
 & & \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \arrow[ul, "pr_2"] \arrow[ur, "p_1"'] & &
\end{tikzcd}
\tag{1}
$$

Let $\mathcal{U} \subset \mathcal{K}$ be a $\mathcal{K}$-categorical subgroupoid, the inclusion $i_{\mathcal{U}} : \mathcal{U} \to \mathcal{K}$ is homotopic to generalised constant map from the Definition 6.2.7 Then we have the following induced diagram:

$$
\begin{tikzcd}
 & \mathcal{U} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{U}}"] & \\
\mathcal{U} \arrow[r, phantom, "\Downarrow S'"] & \mathcal{L} \arrow[u] \arrow[dd] \arrow[r, phantom, "\Downarrow S'"] & \mathcal{K} \\
 & & \bullet^{K} \arrow[u] \\
 & \mathcal{U}' \arrow[uul, "z"] \arrow[ur] &
\end{tikzcd}
\tag{2}
$$

Let $\mathcal{V} \subset \mathcal{G}$ be the groupoid given by $\delta pr_2'(\mathcal{U} \times_{\mathcal{K}} \mathcal{J}')$ in the following pullback:

$$
\begin{tikzcd}
 & \mathcal{U} \times_{\mathcal{K}} \mathcal{J}' \arrow[r, "pr_1'"] \arrow[d, "pr_2'"'] & \mathcal{U} \arrow[d, "i_{\mathcal{U}}"] \\
\mathcal{G} & \mathcal{J}' \arrow[l, "\delta"'] \arrow[r, "\psi"] & \mathcal{K}
\end{tikzcd}
$$

In diagram (1), take the restriction to $\mathcal{V}$, i.e. compose both maps with the inclusion functor $\mathcal{V} \xleftarrow{id} \mathcal{V} \xrightarrow{i_{\mathcal{V}}} \mathcal{G}$:

$$
\begin{tikzcd}
 & & \mathcal{V} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{V}}"] & & \\
 & \mathcal{V} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{V}}"] \arrow[dd, equal] & & \mathcal{G} \arrow[dl, "id"'] \arrow[dr, "id"] & \\
\mathcal{V} & & \mathcal{G} \arrow[r, phantom, "\Downarrow S"] & \mathcal{E} \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow S"] & \mathcal{G} \\
 & \mathcal{V} \arrow[ul, "id"] \arrow[ur, "i_{\mathcal{V}}"] & & \mathcal{J}\times_{\mathcal{K}}\mathcal{J}' \arrow[ul, "\delta pr_2"'] \arrow[ur, "\phi pr_1"'] & \\
 & & \mathcal{J}\times_{\mathcal{K}}\mathcal{J}'\times_{\mathcal{G}}\mathcal{V} \arrow[ul] \arrow[ur] & &
\end{tikzcd}
\tag{3}
$$

Now take the restriction of $\mathcal{G} \xleftarrow{\delta} \mathcal{J}' \xrightarrow{\psi} \mathcal{K}$ to $\mathcal{V}$:

$$
\begin{tikzcd}
 & & \mathcal{J}'\times_{\mathcal{G}}\mathcal{V} \arrow[dl, "pr_3"] \arrow[dr, "pr_1"] & & \\
 & \mathcal{V} \arrow[dl, "id"'] \arrow[dr, "i\mathcal{V}"] & & \mathcal{J}' \arrow[dl, "\delta"'] \arrow[dr, "\psi"] & \\
\mathcal{V} & & \mathcal{G} & & \mathcal{K}
\end{tikzcd}
\tag{4}
$$

Lemma 6.2.11. *We have $\psi pr_1(\mathcal{J}' \times_{\mathcal{G}} \mathcal{V}) \subset \mathcal{U}$.*

Proof. Use that $\mathcal{U}$ is invariant and the essential equivalence δ includes a homomorphism between the isotropy groups. $\square$

Since $\psi pr_1(\mathcal{J}' \times_{\mathcal{G}} \mathcal{V}) \subset \mathcal{U}$ we obtain the generalised map $\mathcal{V} \longleftarrow \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \longrightarrow \mathcal{U}$ from diagram (4).

Now we compose this map with the inclusion functor $\mathcal{U} \xleftarrow{id} \mathcal{U} \longrightarrow \mathcal{K}$ and with the generalised map $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$. We have

$$\begin{tikzcd}
 & & & \mathcal{J} \times_{\mathcal{K}} \mathcal{U} \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[dll] \arrow[drr] & & & & \\
 & \mathcal{U} \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[dl] \arrow[dr] & & & & \mathcal{J} \arrow[ddll, "\epsilon"'] \arrow[ddrr, "\phi"] & & \\
\mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[d] \arrow[dr] & & \mathcal{U} \arrow[dl] \arrow[dr] & & & & & \\
\mathcal{V} & \mathcal{U} & & \mathcal{K} & & & & \mathcal{G}
\end{tikzcd} \tag{5}$$

We will show that the generalised map $\mathcal{V} \longleftarrow \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \xrightarrow{c'} \mathcal{G}$ obtained by this composition is homotopic to the map

$$\mathcal{V} \longleftarrow \mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \xrightarrow{c'} \mathcal{G}$$

obtained from the following compositions:

$$\begin{tikzcd}
 & & & \mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[dll] \arrow[drr, "pr_1"] & & & & \\
 & \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[dl] \arrow[dr] & & & & \mathcal{J} \arrow[ddll, "\epsilon"'] \arrow[ddrr, "\phi"] & & \\
\mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[d] \arrow[dr] & & \mathcal{U}' \arrow[dl, "z"] \arrow[dr, "c"] & & & & & \\
\mathcal{V} & \mathcal{U} & & \mathcal{K} & & & & \mathcal{G}
\end{tikzcd} \tag{6}$$

Letting $c' = \phi pr_1$, we have the following;

Lemma 6.2.12. *The image of $c' = \phi pr_1$ is contained in an orbit groupoid. Moreover, if $c(\mathcal{U}') \subset \mathcal{O}^K$, then $c'(\mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V}) \subset \mathcal{O}^{K'}$, where K is homeomorphic to K'.*

Proof. Use the fact that ϵ induces a homeomorphism between the orbit spaces and a homomorphism between the isotropy groups. □

Then $\mathcal{V} \longleftarrow \mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \xrightarrow{c'} \mathcal{G}$ is a generalised constant map. By using diagram (2) we have that

$$\mathcal{V} \longleftarrow \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \longrightarrow \mathcal{G} \quad \simeq_{S'} \quad \mathcal{V} \longleftarrow \mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \xrightarrow{c'} \mathcal{G} \tag{7}$$

Then we have a 2-morphism

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dl] \arrow[dr] & \\
\mathcal{V} \arrow[r, phantom, "\Downarrow S"] & \mathcal{L} \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow S"] & \mathcal{G} \\
 & \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{8}$$

given by diagram (3) and another 2-morphism

$$\begin{tikzcd}
 & \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[dl] \arrow[dr] & \\
\mathcal{V} \arrow[r, phantom, "\Downarrow S'"] & \mathcal{L}' \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow S'"] & \mathcal{G} \\
 & \mathcal{J} \times_{\mathcal{K}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{9}$$

given by equation (7).

The vertical composition of these 2-morphisms (8) and (9) gives a 2-morphism between the inclusion and a generalised constant map:

$$\begin{tikzcd}
 & & \mathcal{V} \arrow[ddll, "i_{\mathcal{V}}"'] \arrow[ddrr, "i_{\mathcal{V}}"] & & \\
 & & \mathcal{L} \arrow[u] \arrow[dr, "\mathcal{V}"] & & \\
\mathcal{K} \arrow[rr, phantom, "\Downarrow_{S''}"] & & \mathcal{P}_{S''} \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow_{S''}"] & \mathcal{D} \arrow[r, phantom, "\Downarrow_{S''}"] & \mathcal{G} \\
 & & \mathcal{L}' \arrow[ur, "\mathcal{U}'"'] \arrow[d] & & \\
 & & \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V} \arrow[uull] \arrow[uurr] & &
\end{tikzcd}$$

where $\mathcal{D} = \mathcal{J} \times_{\mathcal{K}} \mathcal{J}' \times_{\mathcal{G}} \mathcal{V}$. Therefore, $\mathcal{V}$ is categorical for $\mathcal{G}$. □

From the above we therefore immediately get the following

Theorem 6.2.13. *The Lusternik-Schnirelmann category of a topological groupoid is invariant under Morita equivalence of topological groupoids, i.e. if $\mathcal{G}$ is a topological groupoid which is Morita equivalent to a topological groupoid $\mathcal{G}'$, then we have*

$$\mathrm{cat}(\mathcal{G}) = \mathrm{cat}(\mathcal{G}').$$

The groupoid Lusternik-Schnirelmann category also generalises the ordinary Lusternik-Schnirelmann category of a topological space. In fact, if $\mathcal{G} = u(X)$ is the unit groupoid, then we have $\mathrm{cat}(\mathcal{G}) = \mathrm{cat}(X)$, where cat on the right hand side means the ordinary Lusternik-Schnirelmann category of a topological space X.

6.2.2 Homotopy restrictions

If $\mathcal{U}$ is $\mathcal{G}$-categorical, we have that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
 & \mathcal{U} \arrow[dl, "id"'] \arrow[dr, "i_{\mathcal{U}}"] & \\
\mathcal{U} \arrow[r, phantom, "\Downarrow_H"] & \mathcal{L} \arrow[u, "u"'] \arrow[d, "v"] \arrow[r, phantom, "\Downarrow_{H'}"] & \mathcal{G} \\
 & \mathcal{U}' \arrow[ul, "\epsilon"] \arrow[ur, "c"'] \arrow[r] & \bullet^{K} \arrow[u]
\end{tikzcd}$$

Then, for all $y \in L_0$, the isotropy group G_y injects into G_x for $x = u(y) \in U$ and into G_z for $z = v(y) \in U'$ by 4.3.5 and 4.3.10. We have that $G_x \rightarrowtail K$. In particular, if the isotropy groups are finite, we have that $|G_x|$ divides $|K|$ for all $x \in U$. For instance, a categorical subgroupoid $\mathcal{U}$ cannot factor through a trivial group K except in the case that all the points in $\mathcal{U}$ have trivial isotropy. Because of Proposition 5.6.4 and theorem 6.2.13 we can now define the Lusternik-Schnirelmann category of topological stack by using the groupoid Lusternik-Schnirelmann category.

6.3 Lusternik-Schnirelmann category of a topological stack

Now we will introduce the Lusternik-Schnirelmann category for topological stacks by using homotopical properties of topological stacks.

Definition 6.3.1. *Let $\mathfrak{X}$ be a topological stack. Consider the path stack $P\mathfrak{X} = \mathfrak{Map}([0,1], \mathfrak{X})$ of $\mathfrak{X}$ as defined by Noohi in [39] for general topological stacks. We will say that the morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{X} \to \mathfrak{Y}$ between topological stacks are* homotopic *if there exists a morphism of stacks $H : \mathfrak{X} \to P\mathfrak{Y}$ such that the following diagram of stack morphisms is commutative up to natural transformations:*

$$\begin{tikzcd} \mathfrak{Y} & P\mathfrak{Y} \arrow[l, "ev_0"'] \arrow[r, "ev_1"] & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ul, "f"] \arrow[u, "H"] \arrow[ur, "g"'] & \end{tikzcd}$$

Definition 6.3.2. *Let $\pi : \mathfrak{X} \to \mathsf{Top}$ be a topological stack with atlas $x : X \to \mathfrak{X}$ and $U \subset X$ be an open set. We will say that the restricted substack $\mathfrak{U}$ is $\mathfrak{X}$-*categorical *if the inclusion map $i_{\mathfrak{U}} \colon \mathfrak{U} \to \mathfrak{X}$ is homotopic to a constant morphism $c : \mathfrak{U} \to \mathfrak{X}$ between topological stacks.*

Example 6.3.3. *For instance, in Example 5.6.8 for the stack $\mathfrak{X} = [S^1/S^1]$, let U be the set of triples (P, S, μ), where S is a topological space, P a S^1-torsor over S and $\mu : P \to S^1$ a S^1-equivariant continuous map. That is, $\mathfrak{U} = [S^1/S^1]$. We have that the stack $\mathfrak{U}$ is $\mathfrak{X}$-categorical since the identity map* $\mathrm{id} : [S^1/S^1] \to [S^1/S^1]$ *is homotopic to a constant morphism of stacks. So we get* $\mathrm{cat}(\mathfrak{X}) = 1$.

Definition 6.3.4. *Let $\pi : \mathfrak{X} \to \mathsf{Top}$ be a topological stack with atlas $x : X \to \mathfrak{X}$. The stack LS-category,* $\mathrm{cat}(\mathfrak{X})$, *is the least number of open sets U needed to cover*

X such that the restricted substack $\mathcal{U}$ *is* $\mathfrak{X}$*-categorical. If* X *cannot be covered by a finite number of such open sets, we will say that* $\mathrm{cat}(\mathfrak{X}) = \infty$.

Example 6.3.5. *From Example 5.6.8 and the above definition, we get that*

$$\mathrm{cat}([S^1/S^1]) = 1.$$

The following theorems establish the relationship between Lusternik-Schnirelmann category of topological stacks and topological groupoids.

Theorem 6.3.6. *Let* $\mathfrak{X}$ *be a topological stack with a given presentation* $x : X_0 \to \mathfrak{X}$ *and associated topological groupoid* $\mathcal{G}(x) = [X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0]$*. Then*

$$\mathrm{cat}(\mathfrak{X}) = \mathrm{cat}(\mathcal{G}(x)).$$

Proof. This follows from the definition of LS-category for topological groupoids 6.2.9. The fact that LS-category of topological groupoids is Morita invariant, implies now by using Proposition 5.6.4 that it does not depend on the chosen presentation for the topological stack $\mathfrak{X}$, which gives the result. □

Theorem 6.3.7. *Let* $\mathcal{G}$ *be a topological groupoid and* $\mathfrak{B}\mathcal{G}$ *be the associated topological stack. Then the Lusternik-Schnirelmann category of* $\mathfrak{B}\mathcal{G}$*, the Lusternik-Schnirelmann category of the topological groupoid* $\mathcal{G}$ *is given by*

$$\mathrm{cat}(\mathcal{G}) = \mathrm{cat}(\mathfrak{B}\mathcal{G}).$$

Proof. This follows from the explicit construction of the classifying stack $\mathfrak{B}\mathcal{G}$ of $\mathcal{G}$-torsors for the given topological groupoid $\mathcal{G}$. Now the associated topological groupoid of the topological stack $\mathfrak{B}\mathcal{G}$ is Morita invariant to the given topological groupoid $\mathcal{G}$ following Theorem 5.6.3 above. □

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