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RELATED FIXED POINTS FOR SET-VALUED MAPPINGS ON TWO UNIFORM SPACES

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Some related fixed point theorems for set-valued mappings on two complete and compact uniform spaces are proved.

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1. Introduction. Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be uniform spaces. Families $\{d_1^i : i \in I$ being indexing set$\}$, $\{d_2^i : i \in I\}$ of pseudometrics on X, Y, respectively, are called associated families for uniformities $\mathcal{U}_1$, $\mathcal{U}_2$, respectively, if families

$$\begin{aligned}\beta_1 &= \{V_1(i,r) : i \in I,\ r > 0\},\\ \beta_2 &= \{V_2(i,r) : i \in I,\ r > 0\},\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}V_1(i,r) &= \{(x,x') : x,x' \in X,\ d_1^i(x,x') < r\},\\ V_2(i,r) &= \{(y,y') : y,y' \in Y,\ d_1^i(y,y') < r\},\end{aligned} \tag{1.2}$$

are subbases for the uniformities $\mathcal{U}_1$, $\mathcal{U}_2$, respectively. We may assume that β_1, β_2 themselves are a base by adjoining finite intersections of members of β_1, β_2, if necessary. The corresponding families of pseudometrics are called an augmented associated families for $\mathcal{U}_1$, $\mathcal{U}_2$. An associated family for $\mathcal{U}_1$, $\mathcal{U}_2$ will be denoted by $\mathcal{D}_1$, $\mathcal{D}_2$, respectively. For details, the reader is referred to [1, 4, 5, 6, 7, 8, 9, 10, 11].

Let A, B be a nonempty subset of a uniform space X, Y, respectively. Define

$$\begin{aligned}P_1^*(A) &= \sup\{d_1^i(x,x') : x,x' \in A,\ i \in I\},\\ P_2^*(B) &= \sup\{d_2^i(y,y') : y,y' \in B,\ i \in I\},\end{aligned} \tag{1.3}$$

where $\{d_1^i(x,x') : x,x' \in A,\ i \in I\} = P_1^*$, $\{d_2^i(y,y') : y,y' \in B,\ i \in I\} = P_2^*$. Then, $P_1^*(A)$, $P_2^*(B)$ are called an augmented diameter of A, B. Further, A, B are said to be $P_1^*(A) < \infty$, $P_2^*(B) < \infty$. Let

$$\begin{aligned}2^X &= \{A : A \text{ is a nonempty } P_1^*\text{-bounded subset of } X\},\\ 2^Y &= \{B : B \text{ is a nonempty } P_2^*\text{-bounded subset of } Y\}.\end{aligned} \tag{1.4}$$

For each $i \in I$ and $A_1, A_2 \in 2^X$, $B_1, B_2 \in 2^Y$, define

$$\begin{aligned} \delta_1^i(A_1, A_2) &= \sup\{d_1^i(x, x') : x \in A_1,\ x' \in A_2\}, \\ \delta_2^i(B_1, B_2) &= \sup\{d_2^i(y, y') : y \in B_1,\ y' \in B_2\}. \end{aligned} \tag{1.5}$$

Let $(X, \mathcal{U}_1)$ and $(X, \mathcal{U}_2)$ be uniform spaces and let $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ be arbitrary entourages. For each $A \in 2^X$, $B \in 2^Y$, define

$$\begin{aligned} U_1[A] &= \{x' \in X : (x, x') \in U_1 \text{ for some } x \in A\}, \\ U_2[B] &= \{y' \in Y : (y, y') \in U_2 \text{ for some } y \in B\}. \end{aligned} \tag{1.6}$$

The uniformities $2^{\mathcal{U}_1}$ on 2^X and $2^{\mathcal{U}_2}$ on 2^Y are defined by bases

$$2^{\beta_1} = \{\tilde{U}_1 : U_1 \in \mathcal{U}_1\}, \qquad 2^{\beta_2} = \{\tilde{U}_2 : U_2 \in \mathcal{U}_2\}, \tag{1.7}$$

where

$$\begin{aligned} \tilde{U}_1 &= \{(A_1, A_2) \in 2^X \times 2^X : A_1 \times A_2 \subset U_1\} \cup \Delta, \\ \tilde{U}_2 &= \{(B_1, B_2) \in 2^Y \times 2^Y : B_1 \times B_2 \subset U_2\} \cup \Delta, \end{aligned} \tag{1.8}$$

where Δ denotes the diagonal of $X \times X$ and $Y \times Y$.

The augmented associated families P_1^*, P_2^* also induce uniformities $\mathcal{U}_1^*$ on 2^X, $\mathcal{U}_2^*$ on 2^Y defined by bases

$$\begin{aligned} \beta_1^* &= \{V_1^*(i, r) : i \in I,\ r > 0\}, \\ \beta_2^* &= \{V_2^*(i, r) : i \in I,\ r > 0\}, \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} V_1^*(i, r) &= \{(A_1, A_2) : A_1, A_2 \in 2^X : \delta_1^i(A_1, A_2) < r\} \cup \Delta, \\ V_2^*(i, r) &= \{(B_1, B_2) : B_1, B_2 \in 2^Y : \delta_2^i(B_1, B_2) < r\} \cup \Delta. \end{aligned} \tag{1.10}$$

Uniformities $2^{\mathcal{U}_1}$ and $\mathcal{U}_1^*$ on 2^X are uniformly isomorphic and uniformities $2^{\mathcal{U}_2}$ and $\mathcal{U}_2^*$ on 2^Y are uniformly isomorphic. The space $(2^X, \mathcal{U}_1^*)$ is thus a uniform space called the hyperspace of $(X, \mathcal{U}_1)$. The $(2^Y, \mathcal{U}_2^*)$ is also a uniform space called the hyperspace of $(Y, \mathcal{U}_2)$.

Now, let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of nonempty subsets of uniform space $(X, \mathcal{U})$. We say that sequence $\{A_n\}$ converges to subset A of X if

(i) each point in a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n = 1, 2, \ldots$,

(ii) for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$, where

$$A_\varepsilon = \cup_{x \in A} U(x) = \{y \in X : d_i(x, y) < \varepsilon \text{ for some } x \text{ in } A,\ i \in I\}. \tag{1.11}$$

A is then said to be a limit of the sequence $\{A_n\}$.

It follows easily from the definition that if A is the limit of a sequence $\{A_n\}$, then A is closed.

Lemma 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded, nonempty subsets of a complete uniform space $(X,\mathcal{U})$ which converge to the bounded subsets A and B, respectively, then sequence $\{\delta_i(A_n,B_n)\}$ converges to $\delta_i(A,B)$.*

Proof. For arbitrary $\varepsilon > 0$, there exists an integer N such that

$$\delta_i(A_n,B_n) \le \delta_i(A_\varepsilon,B_\varepsilon) = \sup\{d_i(a',b') : a' \in A_\varepsilon,\ b' \in B_\varepsilon\} \tag{1.12}$$

for $n > N$. Now, for each a' in A_ε and b' in B_ε, we can find a in A and b in B with $d_i(a',a) < \varepsilon$, $d_i(b',b) < \varepsilon$, and so

$$\begin{aligned} d_i(a',b') &\le d_i(a',a) + d_i(a,b') \\ &\le d_i(a',a) + d_i(a,b) + d_i(b,b') \\ &\le d_i(a,b) + 2\varepsilon. \end{aligned} \tag{1.13}$$

It follows that

$$\delta_i(A_n,B_n) < \sup\{d_i(a,b) : a \in A,\ b \in B\} + 2\varepsilon = \delta_i(A,B) + 2\varepsilon \tag{1.14}$$

for $n > N$. Further, there exists an integer N' such that for each a in A and b in B we can find a_n in A_n and b_n in B_n with

$$d_i(a,a_n) < \varepsilon, \qquad d_i(b,b_n) < \varepsilon \tag{1.15}$$

for $n > N'$, and so

$$\begin{aligned} d_i(a,b) &\le d_i(a,a_n) + d_i(a_n,b) \\ &\le d_i(a,a_n) + d_i(a_n,b_n) + d_i(b_n,b) \\ &< d_i(a_n,b_n) + 2\varepsilon. \end{aligned} \tag{1.16}$$

It follows that

$$\begin{aligned} \delta_i(A,B) &= \sup\{d_i(a,b) : a \in A,\ b \in B\} \\ &\le \sup\{d_i(a_n,b_n) : a_n \in A_n,\ b_n \in B_n\} + 2\varepsilon \\ &= \delta_i(A_n,B_n) + 2\varepsilon \end{aligned} \tag{1.17}$$

for $n > N'$. The result of the lemma follows from inequalities (1.14) and (1.17). □

Remark 1.2. If we replace the uniform space $(X,\mathcal{U})$ in Lemma 1.1 by a metric space (i.e., a metrizable uniform space), then the result of the second author [2] will follow as special case of our result.

Theorem 1.3. *Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be complete Hausdorff uniform spaces defined by $\{d_1^i,\ i \in I\} = P_1^*$, $\{d_2^i,\ i \in I\} = P_2^*$, and $(2^X,\mathcal{U}_1^*)$, $(2^Y,\mathcal{U}_2^*)$ hyperspaces, let $F : X \to 2^Y$ and $G : Y \to 2^X$ satisfy inequalities*

$$\begin{aligned} &\delta_1^i(GFx,GFx') \le c_i \max\{d_1^i(x,x'),\delta_1^i(x,GFx),\delta_1^i(x',GFx'),\delta_2^i(Fx,Fx')\}, \\ &\delta_2^i(FGy,FGy') \le c_i \max\{d_2^i(y,y'),\delta_2^i(y,FGy),\delta_2^i(y',FGy'),\delta_1^i(Gy,Gy')\} \end{aligned} \tag{1.18}$$

for all $i \in I$ and $x, x' \in X$, $y, y' \in Y$, where $0 \le c_i < 1$. If F is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y. Further, $Fz = \{w\}$ and $Gw = \{z\}$.

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows. Choose a point y_1 in Fx_1 and then a point x_1 in Gy_1. In general, having chosen x_n in X and y_n in Y, choose x_{n+1} in Gy_n and then y_{n+1} in Fx_{n+1} for $n = 1, 2, \ldots$.

Let $U_1 \in \mathcal{U}_1$ be an arbitrary entourage. Since β_1 is a base for $\mathcal{U}_1$, there exists $V_1(i,r) \in \beta_1$ such that $V_1(i,r) \subseteq U_1$. We have

$$\begin{aligned} & d_1^i(x_{n+1}, x_{n+2}) \\ & \quad \le \delta_1^i(GFx_n, GFx_{n+1}) \\ & \quad \le c_i \max\{d_1^i(x_n, x_{n+1}), \delta_1^i(x_n, GFx_n), \delta_1^i(x_{n+1}, GFx_{n+1}), \delta_2^i(Fx_n, Fx_{n+1})\} \\ & \quad \le c_i \max\{\delta_1^i(GFx_{n-1}, GFx_n), \delta_1^i(GFx_n, GFx_{n+1}), \delta_2^i(Fx_n, Fx_{n+1})\} \\ & \quad = c_i \max\{\delta_1^i(GFx_{n-1}, GFx_n), \delta_2^i(Fx_n, Fx_{n+1})\} \end{aligned} \tag{1.19}$$

and, similarly let $U_2 \in \mathcal{U}_2$ be an arbitrary entourage. Since β_2 is a base for $\mathcal{U}_2$, there exists $V_2(i,r) \in \beta_2$ such that $V_2(i,r) \subseteq U_2$. We have

$$\begin{aligned} d_2^i(y_{n+1}, y_{n+2}) & \le \delta_2^i(FGy_n, FGy_{n+1}) \\ & \le c_i \max\{\delta_2^i(FGy_{n-1}, FGy_n), \delta_1^i(Gy_n, Gy_{n+1})\}. \end{aligned} \tag{1.20}$$

It follows that

$$\begin{aligned} d_1^i(x_n, x_{n+m}) & \le d_1^i(x_n, x_{n+1}) + d_1^i(x_{n+1}, x_{n+2}) + \cdots + d_1^i(x_{n+m-1}, x_{n+m}) \\ & \le \delta_1^i(GFx_{n-1}, GFx_n) + \cdots + \delta_1^i(GFx_{n+m-2}, GFx_{n+m-1}) \\ & \le c_i \max\{\delta_1^i(GFx_{n-2}, GFx_{n-1}), \delta_2^i(Fx_{n-1}, Fx_n)\} \\ & \quad + \cdots + c_i \max\{\delta_1^i(GFx_{n+m-3}, GFx_{n+m-2}), \delta_2^i(Fx_{n+m-2}, Fx_{n+m-1})\} \\ & \le (c_i^n + c_i^{n+1} + \cdots + c_i^{n+m-1})\delta_1^i(x_1, GFx_1) \end{aligned} \tag{1.21}$$

for n greater than some N. Since $c_i < 1$, it follows that there exists p such that $d_1^i(x_n, x_m) < r$ and hence $(x_n, x_m) \in U_1$ for all $n, m \ge p$. Therefore, sequence $\{x_n\}$ is Cauchy sequence in the d_1^i-uniformity on X.

Let $S_p = \{x_n : n \ge p\}$ for all positive integers p and let $\mathcal{B}_1$ be the filter basis $\{S_p : p = 1, 2, \ldots\}$. Then, since $\{x_n\}$ is a d_1^i-Cauchy sequence for each $i \in I$, it is easy to see that the filter basis $\mathcal{B}_1$ is a Cauchy filter in the uniform space $(X, \mathcal{U}_1)$. To see this, we first note that family $\{V_1(i,r) : i \in I,\ r > 0\}$ is a base for $\mathcal{U}_1$ as $P_1^* = \{d_1^i : i \in I\}$. Now, since $\{x_n\}$ is a d_1^i-Cauchy sequence in X, there exists a positive integer p such that $d_1^i(x_n, x_m) < r$ for $m \ge p$, $n \ge p$. This implies that $S_p \times S_p \subset V_1(i,r)$. Thus, given any $U_1 \in \mathcal{U}_1$, we can find an $S_p \in \mathcal{B}_1$ such that $S_p \times S_p \subset U_1$. Hence, $\mathcal{B}_1$ is a Cauchy filter in $(X, \mathcal{U}_1)$. Since $(X, \mathcal{U}_1)$ is a complete Hausdorff space, the Cauchy filter $\mathcal{B}_1 = \{S_p\}$

converges to a unique point $z \in X$. Similarly, the Cauchy filter $\mathcal{B}_2 = \{S_k\}$ converges to a unique point $w \in Y$.

Further,

$$\begin{aligned}\delta_1^i(z,GFS_p) &\le d_1^i(z,S_{m+1}) + \delta_1^i(S_{m+1},GFS_p)\\ &\le d_1^i(z,S_{m+1}) + \delta_1^i(GFS_m,GFS_p)\end{aligned} \tag{1.22}$$

since $S_{m+1} \subseteq GFS_m$. Thus, on using inequality (1.20), we have

$$\delta_1^i(z,GFS_p) \le d_1^i(z,S_{m+1}) + \varepsilon \tag{1.23}$$

for $n,m \ge p$. Letting m tend to infinity, it follows that

$$\delta_1^i(z,GFS_p) < \varepsilon \tag{1.24}$$

for $n > p$, and so

$$\lim_{n\to\infty} GFS_p = \{z\} \tag{1.25}$$

since ε is arbitrary. Similarly,

$$\lim_{n\to\infty} FGS_k = \{w\} = \lim_{n\to\infty} FS_p \tag{1.26}$$

since $S_{k+1} \in GS_k$. Using the continuity of F, we see that

$$\lim_{p\to\infty} FS_p = Fz = \{w\}. \tag{1.27}$$

Now, let $W \in \mathcal{U}_1$ be an arbitrary entourage. Since β_1 is a base for $\mathcal{U}_1$, there exists $V_1(j,t) \in \beta_1$ such that $V_1(j,t) \subseteq W$. Using inequality (1.14), we now have

$$\delta_1^i(GFS_p,GFz) \le c_i \max\{d_1^i(S_p,z),\delta_1^i(S_p,GFS_p),\delta_1^i(z,GFz),\delta_2^i(Fz,FS_p)\}. \tag{1.28}$$

Letting p tend to infinity and using (1.24) and (1.26), we have

$$\delta_1^i(z,GFz) \le c_i\delta_1^i(z,GFz). \tag{1.29}$$

Since $c_i < 1$, we have $\delta_1^i(z,GFz) = 0 < t$. Hence, $(z,GFz) \in V_1(j,t) \subseteq W$. Again, since W is arbitrary and X is Hausdorff, we must have $GFz = \{z\}$, proving that z is a fixed point of GF.

Further, using (1.26), we have

$$FGw = FGFz = w, \tag{1.30}$$

proving that w is a fixed point of FG.

Now, suppose that GF has a second fixed point z'. Then, using inequalities (1.18), we have

$$\begin{aligned}\delta_1^i(z',GFz') &\le \delta_1^i(GFz',GFz')\\ &\le c_i\max\{d_1^i(z',z'),\delta_1^i(z',GFz'),\delta_2^i(Fz',Fz')\}\\ &\le c_i\delta_2^i(Fz',Fz') \le c_i\delta_2^i(Fz',FGFz') \le c_i\delta_2^i(FGFz',FGFz')\\ &\le c_i^2\max\{\delta_2^i(Fz',FGFz'),\delta_2^i(Fz',FGFz'),\delta_1^i(GFz',GFz')\}\\ &\le c_i^2\delta_2^i(GFz',GFz'),\end{aligned}\tag{1.31}$$

and so Fz' is a singleton and $GFz' = \{z'\}$, since $c_i < 1$. Thus,

$$\begin{aligned}d_1^i(z,z') &\le \delta_1^i(GFz,GFz')\\ &\le c_i\max\{d_1^i(z,z'),\delta_1^i(z,GFz),\delta_1^i(z',GFz'),\delta_2^i(Fz,Fz')\}.\end{aligned}\tag{1.32}$$

But

$$\begin{aligned}d_2^i(Fz,Fz') &\le \delta_2^i(FGFz,FGFz')\\ &\le c_i\max\{\delta_2^i(Fz,Fz'),\delta_2^i(Fz,FGFz),\delta_2^i(Fz',FGFz'),\delta_1^i(GFz,GFz')\}\\ &= c_i\max\{d_2^i(Fz,Fz'),d_2^i(Fz,Fz),d_2^i(Fz',Fz'),d_1^i(z,z')\}\\ &= c_id_1^i(z,z'),\end{aligned}\tag{1.33}$$

and so

$$d_1^i(z,z') \le c_i^2d_1^i(z,z').\tag{1.34}$$

Since $c_i < 1$, the uniqueness of z follows.

Similarly, w is the unique fixed point of FG. This completes the proof of the theorem. □

If we let F be a single-valued mapping T of X into Y and G a single-valued mapping S of Y into X, we obtain the following result.

Corollary 1.4. *Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be complete Hausdorff uniform spaces. If T is a continuous mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$\begin{aligned}d_1^i(STx,STx') &\le c_i\max\{d_1^i(x,x'),d_1^i(x,STx),d_1^i(x',STx'),d_2^i(Tx,Tx')\},\\ d_2^i(TSy,TSy') &\le c_i\max\{d_2^i(y,y'),d_2^i(y,TSy),d_2^i(y',TSy'),d_1^i(Sy,Sy')\}\end{aligned}\tag{1.35}$$

for all $x,x' \in X$ and $y,y' \in Y$, $i \in I$ where $0 \le c_i < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Theorem 1.5. *Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be compact uniform spaces defined by $\{d_1^i : i \in I\} = P_1^*$ and $\{d_2^i : i \in I\} = P_2^*$, and, $(2^X,\mathcal{U}_1^*)$ and $(2^Y,\mathcal{U}_2^*)$ hyperspaces. If F is a continuous mapping of X into 2^Y and G is a continuous mapping of Y into 2^X satisfying the inequalities*

$$\begin{aligned}\delta_1^i(GFx,GFx') &< \max\{d_1^i(x,x'),\delta_1^i(x,GFx),\delta_1^i(x',GFx'),\delta_2^i(Fx,Fx')\},\\ \delta_2^i(FGy,FGy') &< \max\{d_2^i(y,y'),\delta_2^i(y,FGy),\delta_2^i(y',FGy'),\delta_1^i(Gy,Gy')\}\end{aligned} \tag{1.36}$$

for all $x,x' \in X$ and $y,y' \in Y$, $i \in I$ for which the right-hand sides of the inequalities are positive, then, FG has a unique fixed point $z \in X$ and GF has a unique fixed point $w \in Y$. Further, $FGz = \{z\}$ and $GFw = \{w\}$.

Proof. We denote the right-hand sides of inequalities (1.35) by $h(x,x')$ and $k(y,y')$, respectively. First of all, suppose that $h(x,x') \neq 0$ for all $x,x' \in X$ and $k(y,y') \neq 0$ for all $y,y' \in Y$. Define the real-valued function $f(x,x')$ on $X \times X$ by

$$f(x,x') = \frac{\delta_1^i(GFx,GFx')}{h(x,x')}. \tag{1.37}$$

Then, if $\{(x_n,x_n')\}$ is an arbitrary sequence in $X \times X$ converging to (x,x'), it follows from the lemma and the continuity of F and G that the sequence $\{f(x_n,x_n')\}$ converges to $f(x,x')$. The function f is therefore a continuous function defined on the compact uniform space $X \times X$ and so achieves its maximum value $c_1^i < 1$.

Thus,

$$\delta_1^i(GFx,GFx') \le c_1^i \max\{d_1^i(x,x'),\delta_1^i(x,GFx),\delta_1^i(x',GFx'),\delta_2^i(Fx,Fx')\} \tag{1.38}$$

for all x,x' in X, $i \in I$.

Similarly, there exists $c_2^i < 1$ such that

$$\delta_2^i(FGy,FGy') \le c_2^i \max\{d_2^i(y,y'),\delta_2^i(y,FGy),\delta_2^i(y',FGy'),\delta_1^i(Gy,Gy')\} \tag{1.39}$$

for all $y,y' \in Y$, $i \in I$. It follows that the conditions of Theorem 1.3 are satisfied with $c_i = \max\{c_1^i,c_2^i\}$ and so, once again there exists z in X and w in Y such that $GFz = \{z\}$ and $FGw = \{w\}$.

Now, suppose that $h(x,x') = 0$ for some x,x' in X. Then, $GFx = GFx' = \{x\} = \{x'\}$ is a singleton $\{w\}$. It follows that z is a fixed point of GF and $GFz = \{z\}$. Further,

$$FGw = FGFz = Fz = \{w\} \tag{1.40}$$

and so w is a fixed point of FG.

It follows similarly that if $k(y,y') = 0$ for some $y,y' \in Y$, then again GF has a fixed point z and FG has a fixed point w.

Now, we suppose that GF has a second fixed point z' in X so that z' is in GFz'. Then, on using inequalities (1.36), we have, on assuming that $\delta_2^i(Fz',Fz') \neq 0$ for each $i \in I$,

$$\begin{aligned}
\delta_1^i(z',GFz') &\leq \delta_1^i(GFz',GFz') \\
&< \max\{d_1^i(z',z'),\delta_1^i(z',GFz'),\delta_2^i(Fz',Fz')\} \\
&= \delta_2^i(Fz',Fz') \leq \delta_2^i(Fz',FGFz') \leq \delta_2^i(FGFz',FGFz') \\
&< \max\{\delta_2^i(Fz',Fz'),\delta_2^i(Fz',FGFz'),\delta_1^i(GFz',GFz')\} \\
&= \delta_2^i(GFz',GFz'),
\end{aligned} \tag{1.41}$$

a contradiction, and so Fz' is a singleton and $GFz' = \{z'\}$. Thus, if $z \neq z'$

$$\begin{aligned}
d_1^i(z,z') &= \delta_1^i(GFz,GFz') \\
&< \max\{d_1^i(z,z'),\delta_1^i(z,GFz),\delta_1^i(z',GFz'),\delta_2^i(Fz,Fz')\} \\
&= d_2^i(Fz,Fz').
\end{aligned} \tag{1.42}$$

But if $Fz \neq Fz'$, we have

$$\begin{aligned}
d_2^i(Fz,Fz') &\leq \delta_2^i(FGFz,FGFz') \\
&< \max\{\delta_2^i(Fz,Fz'),\delta_2^i(Fz,FGFz),\delta_2^i(Fz',FGFz'),\delta_1^i(GFz,GFz')\} \\
&= \max\{\delta_2^i(Fz,Fz'),d_2^i(Fz,Fz),d_2^i(Fz',Fz'),d_1^i(z,z')\} \\
&= d_i(z,z'),
\end{aligned} \tag{1.43}$$

and so

$$d_i(z,z') < d_i(z,z'), \tag{1.44}$$

a contradiction. The uniqueness of z follows.

Similarly, w is the unique fixed point of FG. This completes the proof of the theorem. □

If we let F be a single-valued mapping T of X into Y and G a single-valued mapping of Y into X, we obtain the following result.

Corollary 1.6. *Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be compact Hausdorff uniform spaces. If T is a continuous mapping of X into Y and S is a continuous mapping of Y into X satisfying the inequalities*

$$\begin{aligned}
d_1^i(STx,STx') &< \max\{d_1^i(x,x'),d_1^i(x,STx),d_1^i(x',STx'),d_2^i(Tx,Tx')\}, \\
d_2^i(TSy,TSy') &< \max\{d_2^i(y,y'),d_2^i(y,TSy),d_2^i(y',TSy'),d_1^i(Sy,Sy')\}
\end{aligned} \tag{1.45}$$

for all $x,x' \in X$ and $y,y' \in Y$, $i \in I$ for which the right-hand sides of the inequalities are positive, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Remark 1.7. If we replace the uniform spaces $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ in Theorems 1.3 and 1.5 and Corollaries 1.4 and 1.6, by a metric space (i.e., a metrizable uniform space), then the results of the authors [3] will follow as special cases of our results.

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