EULER CHARACTERISTICS AND COHOMOLOGY FOR QUASIPERIODIC PROJECTION PATTERNS

Thesis submitted for the degree of
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by

Claire Louise Irving MMath (Leicester)
Department of Mathematics
University of Leicester

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Abstract

Claire Irving

Euler characteristics and cohomology for quasiperiodic projection patterns

This thesis investigates quasiperiodic patterns and, in particular, polytopal projection patterns, which are produced using the projection method by choosing the acceptance domain to be a polytope. Cohomology theories applicable in this setting are defined, together with the Euler characteristic.

Formulae for the Čech cohomology $\check{H}^*(M\mathcal{P})$ and Euler characteristic $e_{\mathcal{P}}$ are determined for polytopal projection patterns of codimension 2 and calculations are carried out for several examples. The Euler characteristic is shown to be undefined for certain codimension 3 polytopal projection patterns. The Euler characteristic $e_{\mathcal{P}}$ is proved to be always defined for a particular class of codimension n polytopal projection patterns $\mathcal{P}$ and a formula for $e_{\mathcal{P}}$ for such patterns is given. The finiteness or otherwise of the rank of $\check{H}^m(M\mathcal{P}) \otimes \mathbb{Q}$ for $m \geqslant 0$ is also discussed for various classes of polytopal projection patterns. Lastly, a model for $M\mathcal{P}$ is considered which leads to an alternative method for computing the rank of $\check{H}^m(M\mathcal{P}) \otimes \mathbb{Q}$ for $\mathcal{P}$ a d-dimensional codimension n polytopal projection pattern with $d > n$.

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I am also grateful to my parents for their support and encouragement during all stages of my education, and to my friends for square meals, light relief and good company.

The Rhombus tiling in Figure 2.2 was drawn using Mathematica [38] using a modification of the package Octagonal.m by Uwe Grimm [17]

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Introduction

Patterns have fascinated people for thousands of years, both for their aesthetic and their mathematical properties [18]. The patterns we are most familiar with in everyday life are *periodic*. For example, a periodic pattern in the plane consists of a motif (fundamental domain) which repeats in a regular way (so the pattern is invariant under translations in two linearly independent directions).

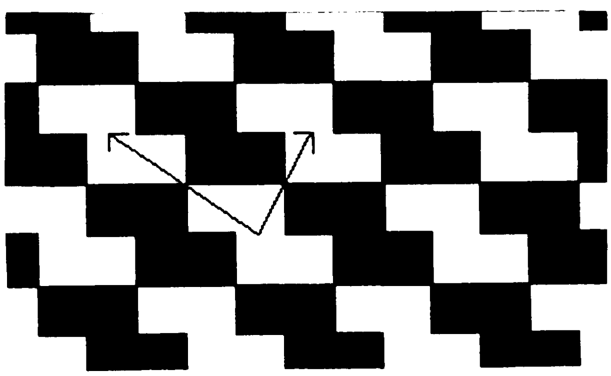


Figure 1: A periodic pattern, showing the translations under which it is invariant

Another type of pattern is repetitive, in the sense that any motif appears infinitely often throughout the whole plane, but the motifs do not repeat in a regular way (so the pattern is not periodic in any direction and hence not invariant under translation by any non-zero vector in the plane). These ideas are made more precise in Section 1.1 of Chapter 1.

Patterns with such properties are called *quasiperiodic* and they are the main kind of pattern considered in this document. An example can be seen in Figure 2 below.

Quasiperiodic patterns have been studied for many years. Perhaps the most well-known examples are the Penrose tilings of the plane, which were developed in the 1970s [29]. Note that a *tiling* of the plane is a division of the plane into regions called *tiles*. There are many other examples of quasiperiodic tilings of the plane, such as the Octagonal tiling below, which is discussed in more detail in Example 1.23.

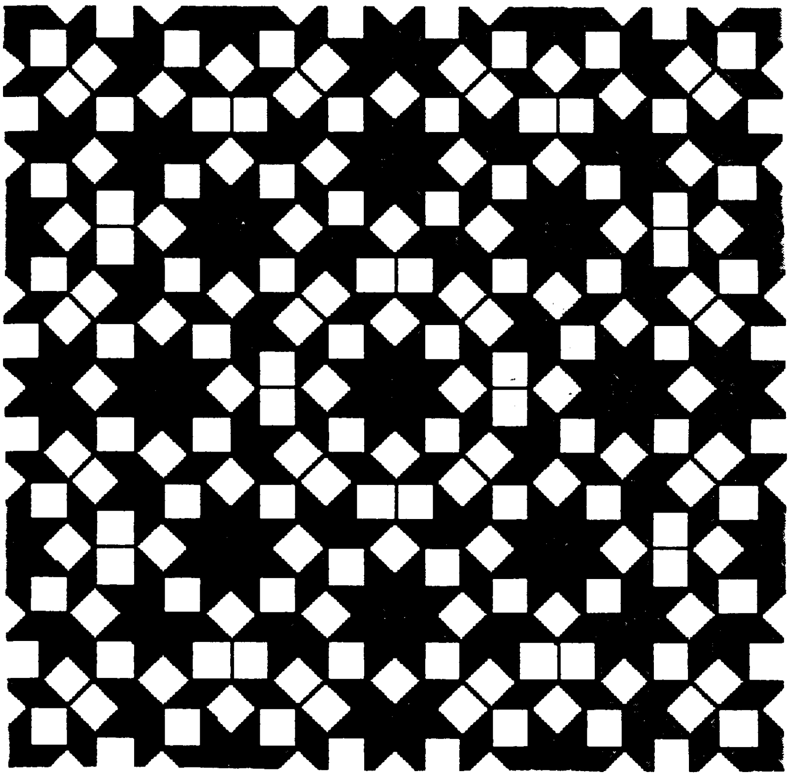


Figure 2: The Octagonal tiling

In recent years, interest in quasiperiodic patterns has increased following the publication in 1984 of a paper [33] identifying the existence of minerals with atoms arranged in patterns of this type. These minerals have come to be known as *quasicrystals*.

Theoretically, it is possible to construct quasiperiodic patterns in Euclidean space (or even hyperbolic space) in arbitrary dimensions. In this document, we consider tilings and patterns in d-dimensional Euclidean space $\mathbb{R}^d$. One-dimensional quasiperiodic tilings have been known for hundreds of years, such as [1] a pattern in $\mathbb{R}$ arising from the Fibonacci sequence. Such patterns have been extensively studied in the literature, so they are not considered in great detail in this document. Patterns in two- and three-dimensional Euclidean space will be considered in most detail, although some results are given which can be applied to patterns in higher dimensions. Developing an understanding of these patterns is of importance to physicists as well as mathematicians, as described in the following sections.

Physical motivation

From the perspective of physics, quasiperiodic patterns are of interest because [24] such patterns in two- and three-dimensional space can be used as models for quasicrystals. Thus an understanding of the mathematical properties of the quasiperiodic pattern associated to a quasicrystal may lead to an understanding of the physical properties of the quasicrystal.

Quasicrystals were first discovered through consideration of diffraction patterns, which are patterns produced on radiation-sensitive film by passing X-rays through thin slices of minerals. If the structure of the atomic arrangement within an object is sufficiently regular, the diffraction pattern will contain clear bright spots. The symmetries of the diffraction pattern are related to the symmetries of the arrangement of atoms in the object. *Crystals* are minerals which have periodic atomic arrangements and it was believed that only crystals had atomic structure regular enough to give rise to diffraction patterns with bright spots. A pattern arising from a periodic arrangement of atoms can [34, p6] only exhibit rotational symmetry of order 2, 3, 4 or 6 — any other orders are classically forbidden. The substances studied in the early 1980s, such as the one pictured in Figure 3 below, had diffraction patterns exhibiting such 'forbidden symmetry', so their atomic arrangements could not be periodic. This fact, together with the fact that sharp diffraction patterns were produced from these minerals, so the atomic arrangements were not totally random, indicated that

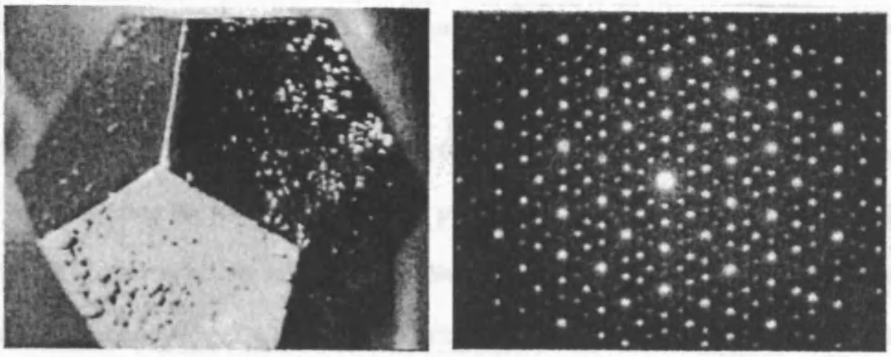


Figure 3: A quasicrystal and its diffraction pattern

the atoms must have been arranged in quasiperiodic patterns. Hence such minerals have come to be called quasicrystals.

Physicists study the properties of quasicrystals with the aim of finding uses for these substances in industry. One quasicrystal, an alloy of Aluminium, Iron and Copper, is currently [35] being manufactured into frying pans because it exhibits non-stick properties. Also [36] an alloy of Zirconium, Nickel and Titanium, another quasicrystal, is being studied with the aim of developing containers for the storage of hydrogen.

Mathematical methods

From the mathematical perspective, since a pattern is essentially a decoration of $\mathbb{R}^d$, a contractible space, its topology is uninteresting. An alternative way to study a pattern $\mathcal{P}$ is to form a topological space from $\mathcal{P}$ and then use tools from algebraic topology to investigate the (much more interesting) structure of this object.

DEFINITION 0.1 *The* continuous hull *$M\mathcal{P}$ for a pattern $\mathcal{P}$ is the set of all patterns $\mathcal{P}'$* locally congruent *to $\mathcal{P}$ (so any bounded region Π in $\mathcal{P}' \in M\mathcal{P}$ can be found somewhere in $\mathcal{P}$) with topology defined by a particular metric on patterns, such as Definition* 1.9 *ahead.*

There are several metrics which can be defined on the set of locally congruent patterns, but the metric of Definition 1.9 is chosen because the topology defined by this metric yields information about the properties of patterns in $M\mathcal{P}$. For more information about these ideas, see Section 1.1 in Chapter 1.

There are various tools in algebraic topology which can be applied to the continuous hull $M\mathcal{P}$ to investigate its structure. Since it provides a link between algebraic topology and physics, K-theory is briefly introduced in Section 1.6.1 in Chapter 1, but in this document we focus on cohomology, in particular Čech cohomology (defined in Section 1.6.2), group cohomology (defined in Section 1.6.3), and the Euler characteristic (considered in detail in Chapter 2) whose definition involves Čech cohomology in the first instance.

Apart from providing models for quasicrystals, quasiperiodic patterns have applications in other areas of mathematics, such as dynamical systems. A further aim of the study of quasiperiodic patterns is to find a classification of all such patterns up to some notion of equivalence.

We now consider a way of generating quasiperiodic patterns, called the *projection method*, which will be used in Chapters 2 and 3. The basic idea of this construction is that given a periodic pattern Λ in some large-dimensional Euclidean space $\mathbb{R}^N$, a pattern can be produced in a subspace E of $\mathbb{R}^N$ by projecting to E a subset of Λ. Careful choice of E and the points to be projected ensures that the resulting pattern in E is quasiperiodic. We now make these ideas more precise, although the projection method is considered in most detail in Chapter 1.

The standard projection method for producing a quasiperiodic point pattern $\mathcal{P}$ is set up by taking Λ an N-dimensional periodic point pattern (which will also be referred to as a *lattice*), a d-dimensional subspace E of $\mathbb{R}^N$ for $d < N$ called the *pattern space*, with orthogonal complement $E^\perp$, and K a compact subset of $E^\perp$ which is the closure of its interior, called the *acceptance domain* for $\mathcal{P}$. Those points of Λ which lie in the strip $K + E$ are projected orthogonally to E, creating a point pattern in E. An example of a projection scheme with $N = 2$ and $d = 1$ is shown below. The lattice is $\mathbb{Z}^2$ and the d-dimensional space is $E \cong \mathbb{R}$. Also $E^{\perp} \cong \mathbb{R}$ and the acceptance domain is a closed interval. To avoid cluttering the diagram, the orthogonal projections to E of only three points in the strip $K + E$ are shown. The point pattern in E generated by this projection scheme is produced by projecting all points in $K + E$ to E.

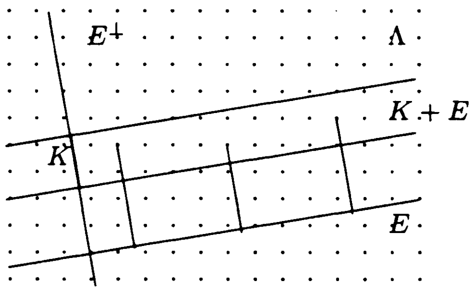


Figure 4: Diagram illustrating the projection method

To ensure that the pattern in E is quasiperiodic, we impose two conditions. The first is that $E \cap \Lambda = \{0\}$. This implies [34, Prop 2.17] that the resulting pattern is non-periodic. We also assume that no points of Λ lie on $\partial K + E$, the boundary of the strip $K + E$, since this ensures that the pattern is repetitive.

Call a projection pattern *canonical* if the acceptance domain is (a translate of) the subset of $E^{\perp}$ formed by projection of a unit cell in the lattice Λ to the space $E^{\perp}$. An example of a projection to $\mathbb{R}^2$ of a unit cell in $\mathbb{Z}^4$ is shown in Figure 5 below.

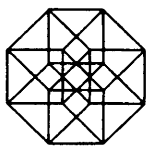


Figure 5: An example of a projection to $\mathbb{R}^2$ of a four-dimensional hypercube

In this document, we are interested in a class of quasiperiodic patterns whose acceptance domain is more general than those of canonical projection patterns. These will be referred to as *polytopal projection patterns* in this document (Def 1.33) and they are projection patterns with acceptance domains which are *polytopes* (Def 1.27). Examples include closed intervals in $\mathbb{R}$, polygons in $\mathbb{R}^2$ and polyhedra in $\mathbb{R}^3$.

Main Results and Document Layout

After conducting a survey of existing literature on quasiperiodic tilings and patterns, including [1], [12] and [20], it became apparent that topological invariants, such as cohomology, for *canonical* projection patterns have been extensively studied, but less is known about topological invariants for projection patterns which are not canonical. Also, the patterns which have been studied in most detail are those for which the cohomology groups, for example, are finitely generated. Different models for the continuous hull $M\mathcal{P}$ which enable the Čech cohomology groups $\check{H}^*(M\mathcal{P})$ to be described in terms of the structure of $M\mathcal{P}$ are of interest as well (see [12, Chapter III] and [20] for example). Thus in this thesis, we consider the following questions.

QUESTION 0.1 *Given a polytopal projection pattern $\mathcal{P}$, to what extent can the Čech cohomology $\check{H}^*(M\mathcal{P})$ of the continuous hull $M\mathcal{P}$ for $\mathcal{P}$ be determined, and can the Euler characteristic be computed?*

QUESTION 0.2 *If the Euler characteristic can be computed for a given polytopal projection pattern $\mathcal{P}$, what values can it take?*

QUESTION 0.3 *Given a polytopal projection pattern, under what circumstances is the Euler characteristic finite and for what values of m are the Čech cohomology groups $\check{H}^m(M\mathcal{P})$ finitely generated?*

QUESTION 0.4 *Can models for the continuous hull $M\mathcal{P}$ of a polytopal projection pattern $\mathcal{P}$ be produced analogous to those of* [20] *which enable the Čech cohomology $\check{H}^*(M\mathcal{P})$ groups to be computed in a more straightforward way?*

In tackling these questions, work has been divided into three main sections. Firstly, various constructions from [12] for canonical projection patterns have been generalised to polytopal projection patterns. Secondly, the Euler characteristic has been defined for general patterns and then computed where possible for several classes of polytopal projection patterns. Applications to the calculation of the cohomology of the continuous hull for such patterns have also been investigated, which lead to generalisations of other results from [12]. Lastly, models for the continuous hull $M\mathcal{P}$ of particular classes of polytopal projection patterns have been developed, which provide an alternative method for the calculation of the cohomology of $M\mathcal{P}$ and generalise results of [12, Chapter III] and [20].

The document is organised as follows. In Chapter 1, a more detailed discussion is given of quasiperiodic patterns, the projection method, polytopes and polytopal projection patterns, and the topological invariants which can be defined in this setting, in order to establish the context of this work and fix notation. Results extending some of the ideas in [12] (which were applicable only to canonical projection patterns) to the larger class of polytopal projection patterns are also provided. The Euler characteristic is introduced in Chapter 2 and calculated in various cases. The cohomology groups $\check{H}^*(M\mathcal{P})$ for polytopal projection patterns $\mathcal{P}$ are considered in Chapter 2 as well, and in particular some results are given indicating when these groups are finitely generated. Several examples of polytopal projection patterns are also presented. Chapter 3 presents an alternative model of the continuous hull $M\mathcal{P}$ for polytopal projection patterns which facilitates a more straightforward computation of the Čech cohomology $\check{H}^*(M\mathcal{P})$. A concluding chapter summarises the results obtained in this document and describes several open problems which were not satisfactorily dealt with during the period of study, and so present opportunities for further research. An appendix follows, containing the precise definitions and results relating to K-theory which are referred to in Chapter 1. Finally, lists of definitions and figures are provided, followed by the bibliography.

Chapter 1

General Setup

This chapter gives an overview of the main ideas in the study of patterns, and quasiperiodic patterns in particular. In the remainder of this document, we use the term *pattern* to refer to a set of points, but we will also make use of the term *tiling* to denote the division of a space into bounded regions called *tiles*. Tilings are more visually appealing than sets of points, so examples of quasiperiodic patterns, such as the Octagonal tiling in Figure 2 in the Introduction, will be drawn as tilings, but most results in this document will be stated for patterns only. The distinction between patterns and tilings is discussed further in Section 1.1.1 ahead. This chapter also contains a description of the projection method for generating quasiperiodic patterns. Polytopal projection patterns are introduced and properties of polytopes relevant to the calculations in Chapters 2 and 3 are investigated. Finally, various tools from algebraic topology which can be applied in this setting are discussed.

1.1 Patterns and tilings

In this section, definitions of patterns and tilings are given, which are not the most general (see [18] or [34], for example, for alternative descriptions) but they are suitable for the purposes of this document. Methods for constructing quasiperiodic patterns are introduced, and in particular the projection method is considered in greater detail than in the Introduction. Assumptions imposed on all patterns and tilings in later chapters are also stated here. We begin by giving definitions which enable the notion of a quasiperiodic pattern or tiling to be made precise.

DEFINITION 1.1 *A* (point) pattern $\mathcal{P}$ *in d-dimensional Euclidean space is a countable set of points.*

For a point pattern $\mathcal{P}$, a patch of radius r *is the set of points in $\mathcal{P}$ which are contained in a ball $B_r(x)$ of radius r with centre at some point $x \in \mathbb{R}^d$.*

DEFINITION 1.2 *A* tiling *of $\mathbb{R}^d$ consists of a countable family $\mathcal{T} = \{T_1, T_2, \ldots\}$ of closed sets called* tiles, *with the properties that $\bigcup_{i=1}^{\infty} T_i = \mathbb{R}^d$, each tile is homeomorphic to a d-dimensional disc, the interiors of any two distinct tiles are disjoint and the intersection of any two tiles is a connected set.*

Define a prototile *for a tiling $\mathcal{T}$ to be an equivalence class of tiles in $\mathcal{T}$ with respect to some notion of equivalence, such as congruence or translation.*

Define a patch *in a tiling $\mathcal{T}$ to be a subset Π of tiles in $\mathcal{T}$ whose union is homeomorphic to a d-dimensional disc.*

Say a patch Π in a tiling $\mathcal{T}$ has radius r *if there is $x \in \mathbb{R}^d$ such that $\Pi \subset B_r(x)$ and $r = \inf_{x \in \mathbb{R}^d} \{r' : \Pi \subset B_{r'}(x)\}$.*

Thus a patch Π in a tiling $\mathcal{T}$ has radius r if there is $x \in \mathbb{R}^d$ such that the ball $B_r(x)$ contains all the tiles in the patch, and any ball of smaller radius does not contain all the tiles.

1.1.1 Equivalences of tilings and patterns

Given a tiling $\mathcal{T}$, a point pattern $\mathcal{P}$ can be constructed from it in various ways. For example, each prototile for $\mathcal{T}$ could be 'punctured' (so a distinguished point on each tile is selected). The point pattern $\mathcal{P}$ is then the set of punctures in the tiling $\mathcal{T}$ as a subset of $\mathbb{R}^d$. Alternatively, a point pattern can be produced from a tiling by taking the vertices of each tile. Conversely, given a point pattern $\mathcal{P}$, one way of producing a tiling $\mathcal{T}$ from $\mathcal{P}$ is to define the faces of the tiles in $\mathcal{T}$ to be the perpendicular bisectors of the lines joining pairs of points (a process known as the *Voronoi construction* [12, I.4]). Another way to form a tiling is to join those pairs of points at a specified distance apart by line segments which form the 1-dimensional faces of tiles. Note that such processes are not necessarily inverse to one another, so if we take a tiling $\mathcal{T}$, produce a point pattern $\mathcal{P}$ and then construct a tiling $\mathcal{T}'$, the tiling $\mathcal{T}'$ need not look like $\mathcal{T}$. There could be some notion of equivalence between the two tilings, but in general they could be quite different.

For example, the Octagonal tiling (Fig 2 in the Introduction) is formed from the point pattern produced by the Projection method by joining pairs of points which lie at a certain distance from one another. However, it is not always the case that quasiperiodic patterns give rise to quasiperiodic tilings and vice-versa since for example [18] we could consider a quasiperiodic tiling consisting of a periodic tiling of the plane by unit squares which are then coloured to break the periodicity. Taking the vertices of these tiles and then joining points at a distance 1 from each other recovers the periodic tiling by squares.

There are several equivalences which can be defined on tilings and patterns. In this document, we will use *local congruence*, *mutual local derivability* and *topological conjugacy.*

DEFINITION 1.3 *The* local congruence class (LC class) *of a pattern $\mathcal{P}$ is the set of all patterns that look locally like (translates of) $\mathcal{P}$, so any patch in a pattern $\mathcal{P}'$ in the* LC *class of $\mathcal{P}$ is also a patch in $\mathcal{P}$.*

Note that in the literature, local congruence is also referred to as *local isomorphism.* The former term is used in this document since it fits with the notion that a pattern $\mathcal{P}'$ is locally congruent to $\mathcal{P}$ if bounded patches in $\mathcal{P}'$ are congruent up to translation to patches in $\mathcal{P}$.

DEFINITION 1.4 [8] *Given two d-dimensional patterns $\mathcal{P}$, $\mathcal{P}'$, and vectors $x_1, x_2 \in \mathbb{R}^d$, say $\mathcal{P}'$ is* locally derivable *from $\mathcal{P}$ if there is $R > 0$ such that if $\mathcal{P} - x_1$ agrees with $\mathcal{P} - x_2$*

on a ball of radius R about the origin, then $\mathcal{P}' - x_1$ agrees with $\mathcal{P}' - x_2$ on a ball of radius 1 about the origin.

If $\mathcal{P}'$ is locally derivable from $\mathcal{P}$ and vice versa, then $\mathcal{P}$ and $\mathcal{P}'$ are said to be mutually locally derivable (MLD).

Hence if two patterns $\mathcal{P}$ and $\mathcal{P}'$ are mutually locally derivable, then the positions of patches in $\mathcal{P}'$ are determined by the positions of patches in $\mathcal{P}$.

DEFINITION 1.5 [12, I.4.5] *Two patterns $\mathcal{P}$ and $\mathcal{P}'$ in $\mathbb{R}^d$, with $\mathbb{R}^d$-action by translation, are topologically conjugate if there is an $\mathbb{R}^d$-equivariant homeomorphism $M\mathcal{P} \cong M\mathcal{P}'$.*

For other types of equivalence, see [12, I.4], for example.

The following definitions are applicable to both tilings and patterns, but will be stated only for patterns.

DEFINITION 1.6 *A pattern $\mathcal{P}$ in $\mathbb{R}^d$ is* periodic *if it is invariant under translations by d linearly independent vectors $v_i \in \mathbb{R}^d$, so $\mathcal{P} + v_i = \mathcal{P}$ for $i = 1, \ldots, d$.*

Say $\mathcal{P}$ is subperiodic *if it is invariant under translations by k linearly independent vectors, for $0 < k < d$.*

Say $\mathcal{P}$ is non-periodic *if it is invariant under no translations by vectors in $\mathbb{R}^d$, so $\mathcal{P}+v \neq \mathcal{P}$ for all $0 \neq v \in \mathbb{R}^d$.*

DEFINITION 1.7 *A pattern $\mathcal{P}$ is* repetitive *if for every patch Π of finite radius in $\mathcal{P}$, there is an R satisfying $0 < R < \infty$ such that for every $x \in \mathbb{R}^d$, there is a translate of Π contained in $B_R(x)$.*

DEFINITION 1.8 *A pattern $\mathcal{P}$ is called* quasiperiodic *if it is non-periodic and repetitive.*

As in the Introduction, a topological space (in fact a metrisable space) can be constructed from a pattern $\mathcal{P}$ as follows. Take the set $\mathcal{P} + \mathbb{R}^d$ of all *distinct* patterns produced by translating the points of $\mathcal{P}$ by vectors in $\mathbb{R}^d$. Several metrics can be defined on this space, but the metric used in this document, taken from [12, I.3.1], is defined below.

DEFINITION 1.9 *For a pattern $\mathcal{P}$ in $\mathbb{R}^d$, define a metric on $\mathcal{P}+\mathbb{R}^d$ by*

$$\mu(\mathcal{P}_1,\mathcal{P}_2) = \inf\{1/(r+1) : d_H(B_r(\mathcal{P}_1), B_r(\mathcal{P}_2)) < 1/r\}$$

where $B_r(\mathcal{P}) = \mathcal{P} \cap (B_r(0) \cup \partial B_r(0))$ and d_H is the Hausdorff metric, which is defined on two non-empty closed subsets A and B of $\mathbb{R}^d$ by $d_H(A,B) := \inf\{\epsilon > 0 : N_\epsilon(A) \supset B \,\&\, N_\epsilon(B) \supset A\}$, where $N_\epsilon(A)$ denotes an epsilon neighbourhood of A.

Essentially, this metric says that patterns are close if they agree (up to a small translation) on a large patch around the origin. Note that $1/(r+1)$ is used to ensure that the value of μ is at most 1.

There is [13] a similarly-defined metric on the set $\mathcal{T}+\mathbb{R}^d$ of translates of a tiling which has the property that tilings are close if they agree up to a small translation on a large patch around the origin. Since we mainly consider point patterns in the remainder of this document, we do not state the precise definition of the metric for tilings here.

DEFINITION 1.10 *The* continuous hull *$M\mathcal{P}$ for a pattern $\mathcal{P}$ is the completion of $\mathcal{P}+\mathbb{R}^d$ with respect to the metric μ defined above.*

The action by translation of $\mathbb{R}^d$ on $\mathcal{P}$ is continuous with respect to the metric μ and so [22] can be extended to an action on $M\mathcal{P}$ which takes limit points to limit points.

Note that the continuous hull is defined with respect to the metric μ above rather than some other choice of metric since [13] sets $\{\mathcal{U}_\Pi + x : x \in B_\epsilon(0)\}$, for $\mathcal{U}_\Pi$ the set of patterns $\mathcal{P}' \in M\mathcal{P}$ which contain a given patch Π, are open in $M\mathcal{P}$ so the topology of $M\mathcal{P}$ encodes information about the patterns in $M\mathcal{P}$. Also as shown in Lemma 1.14 ahead, with this metric $M\mathcal{P}$ is a compact space if $\mathcal{P}$ satisfies a certain condition (Def 1.13).

For example, suppose $\mathcal{P}$ is a periodic pattern in $\mathbb{R}^2$, such as the one in Figure 1 in the Introduction, with a point at the origin of $\mathbb{R}^2$, so it is invariant under translations by two vectors v_1, v_2 in $\mathbb{R}^2$ and all integral linear combinations of these vectors. Distinct patterns in $\mathcal{P} + \mathbb{R}^2$ only arise when $\mathcal{P}$ is translated so that the origin of $\mathbb{R}^2$ lies within a half-open unit parallelogram with edges given by the vectors v_1, v_2, since $\mathcal{P}$ is invariant under unit translations by these vectors. As the origin approaches an open side of the parallelogram, by periodicity the pattern with the origin in this position gets closer (with respect to the metric μ) to the pattern with origin on the closed boundary of the parallelogram. Hence taking the closure of the half-open unit parallelogram under μ gives $M\mathcal{P}$ homeomorphic to a torus. Similarly for d-dimensional periodic patterns $\mathcal{P}$, the continuous hull is homeomorphic to a d-dimensional torus.

When $\mathcal{P}$ is quasiperiodic, so it is not invariant under translations by any vector in $\mathbb{R}^d$, new patterns are obtained when $\mathcal{P}$ is translated by any $0 \neq v \in \mathbb{R}^d$. However, since a quasiperiodic pattern $\mathcal{P}$ is repetitive, so any patch Π of finite radius R in $\mathcal{P}$ appears infinitely often throughout $\mathbb{R}^d$, translations of $\mathcal{P}$ can be chosen so that the new patterns $\mathcal{P} + x$ and $\mathcal{P} + y$ are closer than $1/(R+1)$ in the metric μ. Patches can be selected of any radius R, thus ensuring that two distinct translates of $\mathcal{P}$ can be arbitrarily close with respect to μ. Thus the continuous hull $M\mathcal{P}$ is similar to a torus when $\mathcal{P}$ is quasiperiodic, since large translates $\mathcal{P} + x$ of a pattern $\mathcal{P}$ can be found arbitrarily close to $\mathcal{P}$, but $M\mathcal{P}$ has richer topological structure than a torus because $\mathcal{P} + x \neq \mathcal{P}$ for $x \neq 0$. The torus-like nature of $M\mathcal{P}$ provides the focus for the work of Chapter 3.

The continuous hull $M\mathcal{P}$ of a pattern $\mathcal{P}$ encodes information about the local congruence (LC) class of $\mathcal{P}$ by the lemma below.

LEMMA 1.11 *A pattern $\mathcal{P}'$ is an element of the continuous hull $M\mathcal{P}$ for some other tiling or pattern $\mathcal{P}$ if and only if $\mathcal{P}'$ is in the* LC *class of $\mathcal{P}$.*

Proof First suppose $\mathcal{P}'$ is in the LC class of $\mathcal{P}$, so any patch Π in $\mathcal{P}'$ is found somewhere in $\mathcal{P}$. Take Π of radius $R > 0$ in $\mathcal{P}'$ about the origin in $\mathbb{R}^d$. Then there is $x \in \mathbb{R}^d$ with the property that the patch about the origin of radius R in $\mathcal{P} + x$ agrees with Π (up to some small translation). Thus $\mathcal{P}+x$ is within a distance $1/(R+1)$ of $\mathcal{P}'$ in the metric μ. Hence taking a sequence of patches Π_r of $\mathcal{P}'$ of increasing radii r produces a sequence of translates $\mathcal{P}+x_r$ of $\mathcal{P}$ which converges to $\mathcal{P}'$. Hence $\mathcal{P}' \in M\mathcal{P}$.

Now suppose $\mathcal{P}'$ is not in the LC class of $\mathcal{P}$ so there is some patch Π of radius R in $\mathcal{P}'$ which does not appear in $\mathcal{P}$. We can suppose (by taking a translate of $\mathcal{P}'$ if necessary, since if $\mathcal{P}' \in M\mathcal{P}$ then $\mathcal{P}'+x \in M\mathcal{P}$ for all $x \in \mathbb{R}^d$) that Π contains the origin. Then any translate $\mathcal{P}+x$ of $\mathcal{P}$ is at a distance greater than $1/(R+1) > 0$ from $\mathcal{P}'$ in the metric μ since the two patterns can only agree on a ball of radius less than R about the origin as $\mathcal{P}$ does not contain Π. Thus there is no sequence of translates of $\mathcal{P}$ which converges to $\mathcal{P}'$ and hence $\mathcal{P}' \notin M\mathcal{P}$. ∎

1.1.2 Standard assumptions

In order to produce quasiperiodic tilings and patterns which can be used as models for quasicrystals, certain assumptions are imposed. To simplify the model of a quasicrystal, we suppose that there is an infinite amount of the substance, so there is no boundary, and we also assume that atoms are distributed homogeneously within the substance, so there are no empty spaces of radius larger than some allowed amount. Note also that in any crystalline structure which gives rise to a sharp diffraction pattern, there is a minimum distance between atoms due to the action of interatomic forces. Hence we make the assumptions that there is a minimum distance between all points in a pattern and any chosen point in a pattern is surrounded by other points. These two concepts are formalised in the following definition.

DEFINITION 1.12 *Say a point pattern $\mathcal{P}$ is a* Delone set *if it is* uniformly discrete, *so there is an $\epsilon > 0$ such that for all $p \in \mathcal{P}$, the ball of radius ϵ centred at p satisfies $B_\epsilon(p) \cap \mathcal{P} = \{p\}$, and* relatively dense, *so there is a $\rho > 0$ such that any ball of radius greater than or equal to ρ contains at least one element of $\mathcal{P}$.*

It is shown in [34] that patterns produced by the projection method are Delone sets.

It is also known from physics that crystalline structures contain only a finite number of different atomic configurations, so in this document, we assume the set of prototiles for any tiling is finite, and we make the assumption that any tiling or pattern satisfies the following condition.

DEFINITION 1.13 *Let $\mathcal{P}$ be a point pattern. Then $\mathcal{P}$ satisfies the* Finite Local Complexity (FLC) *condition if for any $R_0 > 0$ there are, up to translation, only finitely many different (finite) subsets Π of $\mathcal{P}$ in the set $\{(\mathcal{P} - x) \cap B_{R_0}(0) \colon x \in \mathbb{R}^d\}$.*

For a tiling $\mathcal{T}$, we say that $\mathcal{T}$ has Finite Local Complexity *if for any $R_0 > 0$ there are, up to translation, only finitely many patches in $\mathcal{T}$ with radius R_0.*

Note that not all tilings satisfy FLC. The standard counterexample is the Pinwheel tiling. This is constructed using the substitution method (see Section 1.2.1 ahead). Its prototile is pictured below, showing the decomposition ω which leads to a tiling of the plane when the prototile is successively decomposed by ω and expanded by a factor of $\lambda = \sqrt{5}$.

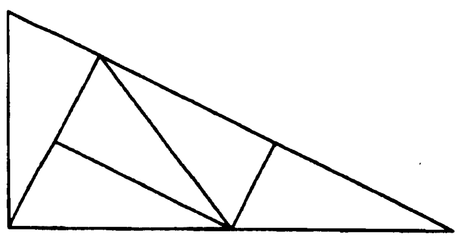


Figure 1.1: Prototile for the Pinwheel Tiling

This tiling contains an infinite number of configurations up to translation because the smallest angle in the prototile is irrational with respect to π, so the prototile appears in infinitely many orientations when infinitely many decompositions and expansions are carried out. Further details about this tiling can be found in [34].

Making the assumption that the tilings and patterns we consider have the FLC property means that the following result holds, which simplifies the study of the continuous hull $M\mathcal{P}$ (Def 1.10).

LEMMA 1.14 *If a pattern or tiling $\mathcal{P}$ satisfies the Finite Local Complexity condition, then the continuous hull $M\mathcal{P}$ is compact.*

Proof For metric spaces, compactness is [4, p25] equivalent to every sequence of points in the space having a convergent subsequence.

Thus we take $(\mathcal{P}_m)$ an infinite sequence of patterns in $M\mathcal{P}$. Also take an infinite unbounded monotone increasing sequence (R_k) of real numbers. By FLC, there are only finitely many patches in $\mathcal{P}$ (up to translation) of radius R_0, so at least one patch Π_0 must appear somewhere in infinitely many patterns $\mathcal{P}_m$. Similarly, there are only finitely many patches of radius R_1, so at least one patch must appear in infinitely many patterns $\mathcal{P}_m$ which also contain Π_0. Continuing in this way produces a subsequence $(\mathcal{P}_{m_i})$ of patterns for which $\mathcal{P}_{m_i}$ contains the patches Π_j of radius R_j for $j \leqslant i$. This subsequence converges in the metric μ (Def 1.9) to a limit pattern $\mathcal{P}_\infty$ which contains the patches Π_i for all i.

Therefore every sequence of patterns in $M\mathcal{P}$ contains a convergent subsequence and hence $M\mathcal{P}$ is compact. ∎

1.2 Methods for constructing quasiperiodic patterns

There are two main methods of generating quasiperiodic patterns or tilings, namely *substitution* and *projection*. In this document, we focus on the projection method, since the extra structure associated to patterns produced in this way will be used to compute topological invariants such as cohomology and the Euler characteristic. The projection method is also highly valuable to the study of quasicrystals, since it is known [24] that models for such minerals can be provided by projection schemes of some kind (possibly more general than those described here). The substitution method also gives rise to important examples of quasiperiodic tilings and patterns. Several papers have been published on such patterns and the topological invariants which can be associated to their continuous hulls, such as [1] and [22]. For completeness, the substitution method is described here, although it will not be used much in the remainder of this document.

1.2.1 The Substitution Method

One way of defining the substitution method for constructing quasiperiodic tilings is as follows.

DEFINITION 1.15 [22] *A* substitution tiling $\mathcal{T}$ *in* $\mathbb{R}^d$ *consists of a set* $\{T_1, \dots, T_m\}$ *of prototiles which are equivalence classes of tiles with respect to translation only, a scaling factor* $\lambda > 1$*, and a substitution rule* ω*. These satisfy the properties that* $\omega(T_i)$ *is a finite collection of tiles which overlap only on their boundaries, and the union of these tiles is exactly* $\lambda(T_i)$.

The substitution rule ω creates patches from prototiles, and the definition of ω can be extended to translates of the prototiles T_i by setting $\omega(T_i + x) = \omega(T_i) + \lambda x$ for $x \in \mathbb{R}^d$. Also, if P is a patch in the tiling $\mathcal{T}$, then we can define $\omega(P){:}\ = \{\omega(T) : T \in P\}$, which means that ω can be iterated, forming a sequence of patches $\omega^k(P)$ for $k = 1, 2, \dots$. As k increases, the radius of the patches increases, producing a tiling of the whole of $\mathbb{R}^d$ in the limit as $k \to \infty$. Finally note that if $\mathcal{T}$ is a tiling, then so is $\omega(\mathcal{T})$.

Examples of substitution tilings are the Pinwheel Tiling whose prototile is in Figure 1.1, and the Octagonal Tiling (Fig 2 in the Introduction) whose prototiles are pictured in Figure 1.3 ahead.

There are other ways of defining substitutions, which are slightly more general than the method above. In particular, we need not have $\omega(T_i) = \lambda T_i$ as ω may replace T_i by a collection of tiles which are not contained entirely within the boundary of λT_i. For example [1], the Penrose tiling can be described as a substitution tiling of this more general type. One example of a set of prototiles T_i for the Penrose tiling is shown below, together with the collections $\omega(T_i)$.

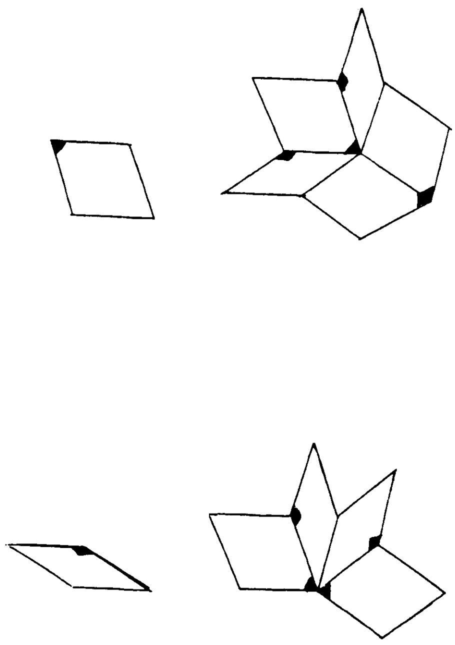


Figure 1.2: Prototiles and substitution for the Penrose tiling

1.2.2 The Projection Method

In this section, we give a more detailed construction of the projection method than that given in the Introduction.

As before, take an N-dimensional lattice Λ, a d-dimensional subspace E of $\mathbb{R}^N$ (the pattern space) which only intersects the lattice at the origin, and an acceptance domain K in the $(N-d)$-dimensional subspace $E^\perp$ of $\mathbb{R}^N$ which is the orthogonal complement of E. Denote by π and $\pi^\perp$ the orthogonal projections to E and $E^\perp$ respectively.

Note that the projection $\pi^\perp(\Lambda)$ of the lattice Λ to $E^\perp$ may not be dense in $E^\perp$. It is important for later work to consider a space V in which the group $\pi^\perp(\Lambda) \cap V$ is dense, so we make the following constructions.

DEFINITION 1.16 *Define Δ to be the real vector space generated by the discrete group $E^\perp \cap \Lambda$.*

LEMMA 1.17 [12, I.2.11] *The Euclidean closure $\overline{\pi^{\perp}(\Lambda)}$ of the projection of the lattice Λ to the subspace $E^{\perp}$ of $\mathbb{R}^N$ can be decomposed into a sum $V \oplus \Delta$, for V a linear subspace of $E^{\perp}$.* ∎

Note that Δ may be trivial. For example, it is trivial for the Octagonal tiling (Example 1.23) but [12, 1.2.7] for the Penrose tiling Δ is 1-dimensional.

Now recall that for a projection pattern to be quasiperiodic, we needed the space E to be ***totally irrational***, meaning that the only lattice point contained in E is the origin. We also required no lattice points to lie on the boundary of the strip $K + E$.

DEFINITION 1.18 *Points $v \in \mathbb{R}^N$ causing $\Lambda + v$ to intersect the boundary of $K + E$ are called* singular points. *If the boundary of $K + E$ does not intersect $\Lambda + v$ then v is called* non-singular, *or* regular. *Write NS to denote the set of all non-singular points in $\mathbb{R}^N$.*

DEFINITION 1.19 *The* projection pattern $\mathcal{P}_v$ *is the set of points $\{\pi(x) : x \in \Lambda + v,\ v \in \mathbb{R}^N$ non-singular, $\pi^{\perp}(x) \in K\}$. We say $\mathcal{P}_v$ is determined by the data (Λ, E, K, v).*

Note that in the following work, we will generally suppress the data and simply write $\mathcal{P}$ for polytopal projection patterns.

We next introduce an alternative description of the continuous hull $M\mathcal{P}$ (Def 1.10) for projection patterns.

LEMMA 1.20 [13, Cor 30] *The continuous hull $M\mathcal{P}$ for a projection pattern $\mathcal{P}$ is homeomorphic to Π/Λ, where Π is the completion of the set NS of non-singular points with respect to the metric $\tilde{\mu}$ defined by $\tilde{\mu}(u,v) = \mu(\mathcal{P}_u, \mathcal{P}_v) + \|u - v\|$ for $u, v \in NS$.* ∎

We can equivalently say that $v \in \mathbb{R}^N$ is non-singular if the boundary of $(K + E) + v$ does not intersect the lattice Λ. This means that, since the projection $\pi^{\perp}$ has kernel E, the singular points can alternatively be viewed in V as arising from translates by $\pi^{\perp}(\Lambda) \cap V$

of the component of the boundary of K which is in V. Hence the set NS of non-singular points can be considered in V via $\pi^{\perp}(NS) \cap V$. This perspective will be used in Section 1.4.

The results of Section 1.6.3 ahead show that we can restrict attention to V rather than $E^{\perp} \cong V \oplus \Delta$. We thus make the following definitions.

DEFINITION 1.21 *Given a projection pattern $\mathcal{P}$, the dimension n of V is called the* codimension *of $\mathcal{P}$ and the* dimension *of $\mathcal{P}$ is the dimension d of the pattern space E.*

DEFINITION 1.22 *Say a projection pattern is* canonical *if it can be produced from a projection scheme in which the acceptance domain $K \subset V$ is the component of $\pi^{\perp}(U)$ in V, for U a unit cell in the lattice Λ.*

To illustrate the above constructions, we now consider an example of a tiling which can be produced using both the substitution and projection methods.

EXAMPLE 1.23 *The Octagonal Tiling*

This tiling is pictured in Figure 2 in the Introduction. There are 12 prototiles up to translation for this tiling. The substitution rules for each congruence class of prototiles are shown below, and the substitution rules for the other prototiles can be obtained by rotations by $k\pi/4$ of the diagrams below, for $k = 0, \ldots 7$. The expansion factor is $\lambda = 1 + \sqrt{2}$.

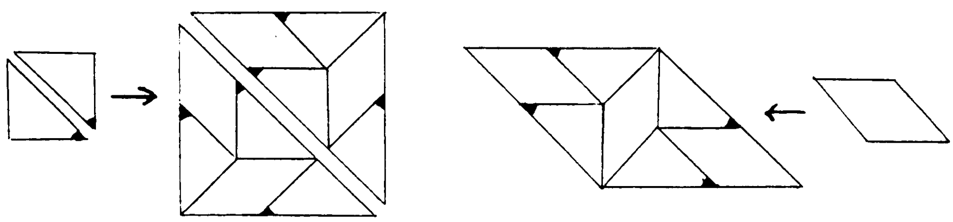


Figure 1.3: Substitution rules for the Octagonal Tiling

To generate the Octagonal Tiling using the projection method, the lattice Λ is chosen to be $\mathbb{Z}^4$ in standard position as a subspace of $\mathbb{R}^4$. Take

$E = Span\{(\sqrt{2}/2, 1/2, 0, -1/2), (0, 1/2, \sqrt{2}/2, 1/2)\}$

$E^{\perp} = Span\{(\sqrt{2}/2, -1/2, 0, 1/2), (0, -1/2, \sqrt{2}/2, -1/2)\}$.

The acceptance domain is the projection to $E^{\perp}$ of the four-dimensional unit hypercube in $\mathbb{Z}^4$ shown below. The projection of Λ to $E^{\perp}$ is dense in $E^{\perp}$, so the Octagonal Tiling is a 2-dimensional codimension 2 canonical projection pattern.

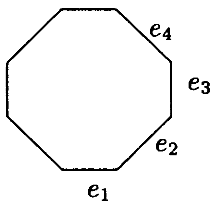


Figure 1.4: Acceptance domain for the Octagonal tiling

Although canonical projection patterns are of interest in this document, we also want to consider tilings and patterns whose acceptance domain is a *polytope*. The following section contains a definition of polytopes, together with a discussion of the properties which are applicable to the study of polytopal projection patterns.

1.3 Polytopes

The term *polytope* has been used in several different ways in the literature (see [7],[26] for example). A definition is given below which encapsulates the notions required for the purposes of this document. In particular, the polytopes considered here are compact subsets of $\mathbb{R}^n$, they have only a finite number of $(n-1)$-dimensional faces and they need not be convex. Several properties of these polytopes which will be used in later sections are also described below.

DEFINITION 1.24 *An m-*cell *c is a space which is homeomorphic to an m-dimensional closed ball B^m. Call a 0-cell a* vertex.

DEFINITION 1.25 *A* regular cell complex *is a space X and a collection of cells c_α with the following properties.*

1. *X is Hausdorff.*

2. $\bigcup_\alpha c_\alpha = X$.

3. *For each m-cell c_α, there is a homeomorphism $f_\alpha\colon B^m \to c_\alpha$ and ∂c_α is equal to a finite union of cells of X, each of dimension less than m.*

4. *A set A is closed in X if $A \cap c_\alpha$ is closed in c_α for each α.*

A regular cell complex is a locally finite CW complex with the added properties that cells are homeomorphic to balls and the boundary of a cell c_α is equal to a union of finitely many cells of X, whereas in a general CW complex, the map f_α need only be a homeomorphism between the interior of B^m and the interior of c_α, and in general f_α takes the boundary of B^m to a subset of a union of finitely many cells of X of dimension less than m (see Fig 1.5 below).

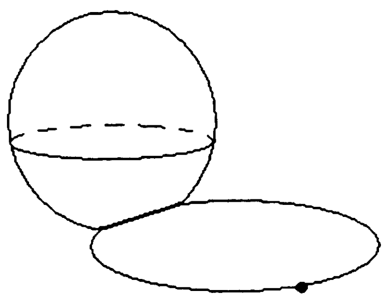


Figure 1.5: A locally finite CW complex which is not a regular cell complex

In the above diagram, the boundary of the 2-cell is contained in but not homeomorphic to the the union of the 1-cell and the 0-cell.

DEFINITION 1.26 *For a regular cell complex X, define an* edge-path *between vertices a and b to be a finite set $\{e_0, e_1, \dots, e_{k-1}\}$ of 1-dimensional cells of X such that for $i = 0, \dots, k-1$ each cell e_i has boundary vertices v_i and v_{i+1}, so $e_i \cap e_{i+1} = v_{i+1}$, and $a = v_0$, $b = v_k$.*

Call a maximal edge-connected subset of X an edge-path component.

Say a regular cell complex X is edge-connected *if there is an edge path between any pair a, b of vertices in X.*

Note that [4, IV.9] an orientation of a CW complex, and hence of a regular cell complex, can be determined.

We are now in a position to give the definition of a polytope.

DEFINITION 1.27 *An n-dimensional* polytope *$L \subset \mathbb{R}^n$ is the underlying space of a connected oriented regular cell complex consisting of finitely many cells c_k with the following properties.*

a) Cells have dimension at most n and there is at least one cell c of dimension n.

b) All m-cells are m-dimensional affine subspaces of $\mathbb{R}^n$ with boundary consisting of $(m-1)$-cells, for $1 \leqslant m \leqslant n$.

c) The interiors of any pair of cells $c_i \neq c_j$ are disjoint. The intersection of any two m-cells $c_i \neq c_j$ is either empty or a cell c_k of dimension less than m for $1 \leqslant m \leqslant n$.

d) Every m-cell is at the intersection of at least two $(m+1)$-cells for $0 \leqslant m \leqslant n-2$. All $(n-1)$-cells lie in the boundary of at least one n-cell.

A 1-dimensional polytope is a subset of $\mathbb{R}$ consisting of a finite union of closed intervals $[a_1, a_2] \cup [a_2, a_3] \cup \dots \cup [a_{k-2}, a_{k-1}] \cup [a_{k-1}, a_k]$, a 2-dimensional polytope is a collection of polygons intersecting in complete 1-cells, and a 3-dimensional polytope is a collection of polyhedra intersecting in complete 2-cells.

DEFINITION 1.28 *Say an n-dimensional polytope L has* inradius r *if the supremum of all radii of spheres S^{n-1} which can be inscribed in the n-dimensional cells of L is r.*

A useful tool in the study of polytopes is the notion of a *flag*.

DEFINITION 1.29 *A* flag $\mathcal{F}$ *in an n-dimensional polytope L is a set of cells $\{c_i : i = 0, \ldots, n\}$ of L, where c_i is i-dimensional and $c_i \subset c_{i+1}$ for each i.*

Note that given any cell c in a polytope L, a flag $\mathcal{F}$ (not unique in general) can be constructed which contains c.

LEMMA 1.30 [26] *Flags in a polytope satisfy the following.*

1. *For each i, there is a unique flag $\mathcal{F}'$ differing from a given flag $\mathcal{F}$ by exactly one element c_i.*

2. *Given a pair of flags $\mathcal{F}$ and $\mathcal{F}'$, there is a chain $(\theta_i)_{i\in\{0,\ldots,k\}}$ of flags $\mathcal{F} = \theta_0, \theta_1, \ldots, \theta_k = \mathcal{F}'$ in which θ_i differs by exactly one element from θ_{i+1} and $\mathcal{F} \cap \mathcal{F}' \subset \theta_i$ for each i.* ∎

We now describe two properties of polytopes which will be of use in subsequent sections.

LEMMA 1.31 *A polytope L is a compact subset of $\mathbb{R}^n$ which is the closure of its interior.*

Proof Since a polytope is the underlying space of a regular cell complex containing finitely many compact cells, it is compact.

As L is compact, it is closed in $\mathbb{R}^n$ and we write $L = \overline{L}$. Now the interior $Int(L)$ is a subset of L, so $\overline{Int(L)} \subset \overline{L} = L$. It remains to show that $L \subset \overline{Int(L)}$. We have $Int(L) = Int(\bigcup_{i\in N} c_i) \supset \bigcup_{i\in N} Int(c_i)$ so $\overline{Int(L)} \supset \overline{\bigcup_{i\in N} Int(c_i)} = \bigcup_{i\in N} \overline{Int(c_i)} = \bigcup_{i\in N} c_i$, where N enumerates the n-cells in L, as each n-cell c_i is homeomorphic to a closed ball and so is the closure of its interior in $\mathbb{R}^n$. Note that $\bigcup_{i\in N} c_i = L$ since the boundaries of the n-cells contain all other cells of lower dimension in L. Hence $\overline{Int(L)} = L$ as required. ∎

LEMMA 1.32 *A polytope L is edge-connected.*

Proof A 1-dimensional polytope is edge-connected by definition since it consists only of 1-dimensional cells intersecting at vertices.

Now consider a general n-dimensional polytope L and suppose there are two edge-path components. Take two 1-dimensional cells c, c', one in each component. Take two flags $\mathcal{F} \supset c$ and $\mathcal{F}' \supset c'$. By Lemma 1.30, since L is a polytope, statement 1 tells us there is a chain of flags between $\mathcal{F}$ and $\mathcal{F}'$, and statement 2 gives that any two 1-dimensional cells in the chain share a common vertex and hence form an edge-path. Thus there is an edge-path between c and c' and hence between vertices in both edge-path components, which is a contradiction. Therefore the polytope is edge-connected, as required. ∎

We now return to projection patterns. Lemma 1.31 shows that a polytope is a valid choice for the acceptance domain K. Thus we make the following definition.

DEFINITION 1.33 *Say a projection pattern is* polytopal *if it can be produced from a projection scheme in which the acceptance domain $K \subset V$ is a polytope.*

Note that canonical projection patterns are also polytopal since a unit cell in the lattice Λ is an N-dimensional polytope and its projection to V is again a polytope. However, not all polytopes arise as projections of hypercubes, so the class of polytopal projection patterns is larger than the class of canonical projection patterns.

1.4 Singular points

Given a polytopal projection pattern $\mathcal{P}$ with lattice Λ, acceptance domain K, and $\pi^{\perp}(\Lambda) \cap V$ dense in a space V, singular points in V are [13] those in the set $V \cap \bigcup_{p \in \partial K + \Lambda} \pi^{\perp}(p)$. Thus, since the acceptance domain K is a polytope, the singular points in K are arranged into lines, planes and so on. We call these objects singular spaces. Note that singular spaces are not just formed by the faces of K and its translates in V, but they also arise at the intersections of translates of faces of K. The notion of a singular space is made more precise in the definitions below.

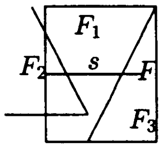
DEFINITION 1.34 *Write* $\Gamma := \pi^{\perp}(\Lambda) \cap V$ *and denote by* $S + \Gamma$ *the set* $\{S + \gamma : \gamma \in \Gamma\}$ *for* S *any subset of* V.

DEFINITION 1.35 *Define* $\mathcal{K}$ *to be the set of subsets of* $V \cong \mathbb{R}^n$ *consisting of the following.*

- $K + \Gamma := \{K + \gamma : \gamma \in \Gamma\}$.
- $F + \gamma$ *for all* $\gamma \in \Gamma$, *where* F *is a cell of any dimension in the boundary* ∂K *of* K.
- $\bigcap_{i=1}^{k}(F_i + \gamma_i)$ *finite intersections of translates of cells* F_i *in the boundary of* K.
- *All* i*-dimensional subsets* s *of* i*-dimensional elements* F *of* $\mathcal{K}$ *which are polytopes such that the boundary of* s *is a union of* $(i-1)$*-dimensional elements, for* $i = 1, \ldots n$.

An i-dimensional element of $\mathcal{K}$ is thus a subset of an i-dimensional face of some translate of K which is bounded by subsets of $(i-1)$-dimensional faces of other translates of K. The n-dimensional elements arise only from the translates of K itself. The intersection of singular n-spaces does not produce any singular $(n-1)$ spaces other than those contained in ∂K since the only way for two n-dimensional polytopes in n-dimensional space to have $(n-1)$-dimensional intersection is if they intersect only at their boundaries.

Compare this construction with the algebra $\mathcal{A}_u$ associated to a projection pattern $\mathcal{P}_u$, defined in [12, I.9.3], which consists of subsets of $NS \cap (\overline{E + \Lambda} + u) \cap V$ and is generated by $NS \cap (\overline{E + \Lambda} + u) \cap K + V \cap \pi^{\perp}(u)$ under finite unions, finite intersections and symmetric difference. The set $\mathcal{K}$ of subsets of the set of *singular points* is defined as above so as to rule out the difference $s \setminus s'$, for s' a singular space of dimension strictly less than the dimension of s, but to include subsets of elements which do not arise as intersections of other elements, such as s in the diagram below, which is a subset of the 1-dimensional element F but not in the intersection of F with the two-dimensional elements F_1, F_2, F_3.



DEFINITION 1.36 *Denote by I_K the set of $(n-1)$-dimensional boundary faces of the acceptance domain K which are distinct up to Γ-translation.*

Write $\mathcal{K}^i$ to denote the closure under finite union of the set of i-dimensional elements of $\mathcal{K}$ and call the elements of $\mathcal{K}^i$ singular i-spaces.

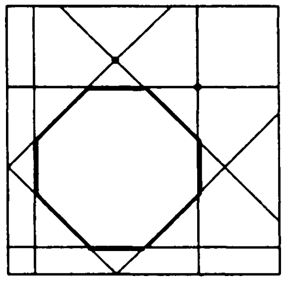
Write I_i for the set of Γ-orbits of singular i-spaces, which are the elements of $\mathcal{K}^i/\Gamma$. Define $L_i := |I_i|$ to be the cardinality of this set.

Note that since singular $(n-1)$-spaces arise only from Γ-translates of the $(n-1)$-dimensional faces of the acceptance domain K, and the fact that K is assumed to be a polytope implies that there are finitely many Γ-orbits of singular $(n-1)$-spaces, so I_K is always a set of finite cardinality.

Of particular note in this document is the set $\mathcal{K}^0$ of singular 0-spaces.

DEFINITION 1.37 *Say $\mathcal{K}^0$ is* finitely generated *if it is the union of finitely many Γ-orbits of singular 0-spaces, and say it is* infinitely generated *otherwise. Write L_0 for the number of Γ-orbits of singular 0-spaces, so $L_0 = |\mathcal{K}^0/\Gamma|$.*

As an example of how singular spaces arise from a given projection scheme, consider the Octagonal tiling from Example 1.23, a 2-dimensional codimension 2 canonical projection pattern. Its acceptance domain K is an octagon with vertices at points of Γ, so translating K by elements of Γ produces lines of infinite length in four distinct directions in V. These singular 1-spaces intersect at the vertices of K (elements of Γ) which form a Γ-orbit of singular 0-spaces but singular 0-spaces also arise at the intersection of pairs of orthogonal singular 1-spaces, and these are not vertices of K or its translates and hence are not elements of Γ.



Since any canonical projection pattern of arbitrary codimension $n \geqslant 2$ has acceptance domain K with vertices at points of Γ, the Γ-orbits of the $(n-1)$-dimensional faces of K consist of sets of $(n-1)$-dimensional hyperplanes. However, for a general polytopal projection pattern $\mathcal{P}$, the Γ-orbit of any face F of the acceptance domain K for $\mathcal{P}$ need not contain the hyperplane spanned by F. In addition, it is possible that two Γ-translates of parallel singular i-spaces S, S' for $i \leqslant n-1$ may have i-dimensional intersection $s \subsetneq S, S'$, even if there is no $\gamma \in \Gamma$ with $S' = S + \gamma$. This fact also contrasts with the canonical case, where parallel singular i-spaces are in the same Γ-orbit, which has the form of sets of disjoint hyperplanes. Thus we make the following definition.

DEFINITION 1.38 *Take the set consisting of translates under the action of* Γ *of all singular* i*-spaces parallel to a given* i*-dimensional face* θ *of* K*. Denote a typical element of the set* I_{ic}^{θ} *of connected components in the resulting space by* D*.*

Denote by I_{ic} *the set containing the elements of* I_{ic}^{θ} *for all faces* θ*.*

Write $\mathcal{D}$ *to denote the orbit class of* D *under the action of* Γ*.*

By definition, D is an element of $\mathcal{K}^i$ so is also referred to as a singular i-space.

We now describe some properties of singular spaces.

LEMMA 1.39 *There is a dense set of singular* j*-spaces contained in any singular* i*-space, for* $0 < i \leqslant n$ *and* $0 \leqslant j \leqslant i-1$*.*

Proof Note first that since Γ is dense in V, the translates of singular $(n-1)$-spaces (arising from the boundary faces of K) are dense in any translate $K+\gamma$ for $\gamma \in \Gamma$.

We now show that there is a dense set of singular $(n-2)$-spaces in any singular $(n-1)$-space. Take a singular $(n-1)$-space F and note that since the acceptance domain K is bounded in V, there is a singular $(n-1)$-space F' whose translates intersect F in singular $(n-2)$-spaces. The set of all translates of F' within an n-dimensional ball B containing F is dense in B by the density of Γ in V. Thus the translates of F' intersect F in a dense set of singular $(n-2)$-spaces since otherwise there would be an n-dimensional subset $F+F'$ of V which could contain no translates of F', contradicting the density of Γ in V.

The above two results imply that singular $(n-2)$-spaces are also dense in singular n-spaces $K+\gamma$. Thus the result follows if we show that singular $(i-1)$-spaces are dense in singular i-spaces for $0 < i < n-1$.

Take a singular i-space S and an n-dimensional ball B containing S. There is a singular $(n-1)$-space F which intersects S in a singular $(i-1)$-space since K is bounded in V, so has boundary faces with normals spanning V, and hence contains at least one face which does not have S as a subset. The Γ-translates of F are dense in the ball B by the density of Γ in V, and these translates intersect S in a dense set of singular i-spaces since if not, then the n-dimensional region in V, of the form $S+F$, would contain no translates of F, contradicting the density of Γ in V. ∎

LEMMA 1.40 *Every element of $\mathcal{K}^0$ lies in the intersection of at least two elements of $\mathcal{K}^1$.*

Proof If the acceptance domain K is 1-dimensional, then the lemma holds since elements of $\mathcal{K}^0$ are contained in $K+\Gamma$, which covers $V \cong \mathbb{R}$, a 1-dimensional space, so any singular 0-space is the end point of two singular 1-spaces. If K is 2-dimensional, then any singular 0-space arises at the intersection of boundary faces of K, which are elements of $\mathcal{K}^1$, so the result holds in this case also.

If K is 3-dimensional, at least three 2-dimensional faces of translates of K with normals linearly independent in V intersect to form any singular 0-space p. If p is a vertex of a face F, then the (1-dimensional) boundary edges of F intersecting at p are the required elements of $\mathcal{K}^1$. If p is not a vertex then it is either in the interior of all three faces, or it is in the boundary of at least one face F. In the former case, any two of the faces intersect in a line through p and there at least two distinct lines arising from the intersection of the three faces, giving the required elements of $\mathcal{K}^1$. In the latter case, the singular 1-space containing p in the boundary of F is one of the elements of $\mathcal{K}^1$. At least one other face F' of some translate of K intersects F to form p, so p is either in the (1-dimensional) boundary of F' as well, or F' intersects F in a line segment through p. Thus in all cases, there are at least two elements of $\mathcal{K}^1$ intersecting at p, as required.

Now if K is n-dimensional, then a singular 0-space p is either a vertex of a translate $K+\gamma$ of K or is in the interior of some i-dimensional face F of $K+\gamma$, for $1 \leqslant i \leqslant n-1$ and $\gamma \in \Gamma$. In the first case, the result holds since K is a polytope so by definition the vertex is at the intersection of two 1-dimensional faces of K. In the latter case, either p is at the intersection of two 1-dimensional boundary faces of Γ-translates of K, so the result holds immediately, or p is in the interior of a face F of dimension i greater than 1. Then since n faces of K of dimension $n-1$ with linearly independent normal vectors intersect to form p, and singular j-spaces arise at the intersection of $n-j$ of the translates of $(n-1)$-dimensional faces of K, there are at least two singular $(j = n-(i-1))$-spaces which intersect F in distinct singular 1-spaces containing p. ∎

LEMMA 1.41 *There is a finite path of singular* 1*-spaces from any singular* 0*-space* p *in* V *to a singular* 0*-space in a* 1*-dimensional boundary face of a translate of* K *containing* p.

Proof A path of singular 1-spaces can be constructed which starts at p since by Lemma 1.40 there is a singular 1-space containing p and by Lemma 1.39 this space also contains another singular 0-space which by Lemma 1.40 lies in another singular 1-space and so on. As singular 1-spaces l are contained in faces of K by definition, and by Lemma 1.39 there are infinitely many singular 0-spaces q in l, the point q can be chosen so that the second singular 1-space l' passing through $q \in l$ has length greater than or equal to r, the smallest inradius of all faces containing q. Hence the path can be constructed between p and a singular 0-space arbitrarily far from p and so the path will intersect the boundary of the translate of K containing p in finitely many steps since K is bounded in V.

Suppose the point of intersection q of the path with ∂K is contained in a face F of dimension i. By Lemma 1.40, there is a singular 1-space containing q and we can suppose the singular 1-space is contained in F. This is because if not, then q must be a vertex of some face F' of another translate of K and hence the result holds immediately because K is a polytope so its vertices are contained in 1-dimensional boundary faces. As above, a finite path of singular 1-spaces can be constructed from q to the boundary of F. Similarly construct paths in the lower dimensional faces of K until a singular 1-space intersects a 1-dimensional face of K in a point, as required to complete the path from p to the 1-dimensional boundary of K. ∎

LEMMA 1.42 *The set $\mathcal{K}^0$ is edge-connected, in the sense that for any pair v_0, v_k of singular 0-spaces in $\mathcal{K}^0$ there is a finite set $\{e_0, e_1, \ldots, e_{k-1}\}$ of singular 1-spaces in which e_i has boundary $\{v_i, v_{i+1}\}$.*

Proof Take two singular 0-spaces v and v' in $\mathcal{K}^0$ and the translates K_v, $K_{v'}$ of K which contain them. As K has positive inradius and $V \cong \mathbb{R}^n$ is locally compact, we can take finitely many translates $\{K_i : i = 1, \ldots m\}$ of K such that $K_i \cap K_{i+1} \neq \emptyset$, $K_0 = K_v$ and $K_m = K_{v'}$. By the above lemma there are paths from v to a 1-dimensional boundary face of K_v. By Lemma 1.32 there is a path of singular 1-spaces between any singular 0-spaces in the 1-dimensional boundary of K_v and hence to a singular 0-space in $K_v \cap K_1$. Lemma 1.41

then tells us that there is a path between singular 0-spaces in $K_v \cap K_1$ and singular 0-spaces in the 1-dimensional boundary of K_1. Applying Lemma 1.32 again links these points to points in $K_1 \cap K_2$. Lemma 1.41 gives paths between singular 0-spaces in $K_i \cap K_{i+1}$ and singular 0-spaces in the 1-dimensional boundary of K_{i+1} for each i. Hence there is a finite path of singular 1-spaces between v and v'. ∎

LEMMA 1.43 *Suppose the orbits of the* $(n-1)$*-dimensional faces of the acceptance domain* K *contain the hyperplanes spanned by the faces. Then the singular* i*-spaces are arranged into* i*-dimensional hyperplanes, for* $0 < i < n-1$.

Proof Take a singular $(n-2)$-space F at the intersection of two $(n-1)$-dimensional faces F_1, F_2 of translates of K. By assumption, each of these faces is contained in a hyperplane composed of singular $(n-1)$-spaces (arising from translates of $(n-1)$-dimensional faces of K). The hyperplanes associated to F_1 and F_2 intersect in a $(n-2)$-dimensional hyperplane H which contains F. The hyperplane H consists of a union of singular $(n-2)$-spaces, since singular $(n-2)$-spaces arise by definition at intersections of singular $(n-1)$-spaces. Hence if the orbits of the $(n-1)$-dimensional faces of K contain the hyperplanes spanned by the faces, the singular $(n-2)$-spaces form $(n-2)$-dimensional hyperplanes.

Similarly, given a singular $(n-3)$-space F', it is at the intersection of two singular $(n-2)$-spaces. The hyperplanes containing these singular $(n-2)$-spaces intersect to produce a $(n-3)$-dimensional hyperplane (composed of singular $(n-3)$-spaces) containing F'. Repeating this argument shows that given a singular 1-space l, it is at the intersection of two singular 2-spaces which are contained in planes. These planes intersect in an infinite line which contains l. Hence, if the orbits of the $(n-1)$-dimensional faces of K contain the hyperplanes spanned by the faces then the singular i-spaces are arranged into i-dimensional hyperplanes, for $0 < i < n-1$. ∎

DEFINITION 1.44 *Take a singular* i*-space* $D \in I_{ic}$ *with* Γ*-action* $\gamma(D) := D + \gamma$ *for* $\gamma \in \Gamma$. *Define the* stabiliser *of* D *to be the subset* $\{\gamma \in \Gamma : \gamma(D) = D\}$ *of* Γ.

Note that the stabiliser of an element D depends only on its orbit class, since the stabiliser of $D+\gamma'$ is $\{\gamma \in \Gamma : \gamma(D+\gamma') = D+\gamma'\} = \{\gamma \in \Gamma : \gamma(D)+\gamma' = D+\gamma'\} = \{\gamma \in \Gamma : \gamma(D) = D\}$, which is the stabiliser of D.

DEFINITION 1.45 *Denote the stabiliser of a connected component D in the orbit class $\mathcal{D}$ by $\Gamma^{\mathcal{D}}$.*

We now consider some general properties of the group $\Gamma\colon = \pi^{\perp}(\Lambda) \cap V$ and stabilisers of singular spaces.

LEMMA 1.46 *The group $\Gamma/\Gamma^{\mathcal{D}}$ is torsion free.*

Proof Suppose $\gamma \in \Gamma$ does not stabilise D but γ^m is an element of $\Gamma^{\mathcal{D}}$ for some $m > 0$. Then $\gamma^m(D) = D + m\gamma = D$ and $\gamma^{mp}(D) = D$ for $p \in \mathbb{Z}$ so, as D is connected, it contains a 1-dimensional affine subspace l. Since γ is a translation vector, if γ^m is a vector parallel to l, then so must $\gamma = \frac{1}{m}\gamma^m$ be. Thus γ is a translation in the direction of l and so takes points of D to points of D. Hence $\gamma(D) = D$. Therefore γ is an element of the stabiliser of D, which is a contradiction to the initial assumption. ∎

LEMMA 1.47 *The rank of Γ is $n+d$.*

Proof The lattice Λ has rank N by definition. Also, the dimension of the vector space $E + E^{\perp} \cong \mathbb{R}^N$ is N. Recall from Definition 1.16 that $E^{\perp} \cong V \oplus \Delta$ for V of dimension n and Δ the real vector space generated by $E^{\perp} \cap \Lambda$. Thus $N = d + n + \dim\Delta$. We can write $\Lambda = (E^{\perp} \cap \Lambda) \oplus \Lambda'$ where $\Lambda' = \{g \in \Lambda : \pi^{\perp}(g) \in V\}$, so rk $\Lambda' = N - \dim\Delta = n + d$. The definition of Λ' implies that $\Gamma = \pi^{\perp}(\Lambda')$. Since $\pi^{\perp}$ has kernel E but $E \cap \Lambda = \{0\}$, no two elements of Λ' project to the same vector in V so rk Γ = rk $\Lambda' = n + d$ as required. ∎

COROLLARY 1.48 *The group Γ splits as $\Gamma^{\mathcal{D}} \oplus \Gamma/\Gamma^{\mathcal{D}}$.*

Proof Since $\Gamma \cong \mathbb{Z}^{n+d}$ is torsion-free, the subgroup $\Gamma^{\mathcal{D}}$ is torsion-free. Also, by Lemma 1.46, $\Gamma/\Gamma^{\mathcal{D}}$ is torsion-free. Hence the result follows. ∎

Note that the rank of the stabiliser of any connected component is bounded above by the rank of Γ. The following lemma makes this bound more precise.

LEMMA 1.49 *For a connected component D in the orbit class $\mathcal{D}$ with stabiliser $\Gamma^{\mathcal{D}}$, we have*

$$rk\,\Gamma^{\mathcal{D}} \leqslant rk\,\Gamma - n + \dim D - 1 = \dim D + d - 1.$$

Proof Suppose D has codimension c in V, so $c = n - \dim D$. To ensure that Γ spans V, there must be c generators γ_i of Γ whose span is a hyperplane H complementary to D in V. To ensure that Γ is dense in V, there must be at least one additional generator γ, rationally independent of the c generators γ_i, in H with the property that integral linear combinations of the vectors γ, γ_i densely fill the fundamental domain for the action of the c generators of Γ on H. Hence the rank of the stabiliser $\Gamma^{\mathcal{D}}$ is at most $\mathrm{rk}\,\Gamma - (c+1) = n + d - n + \dim D - 1 = \dim D + d - 1$, as required. ∎

We now give the definitions of two special classes of polytopal projection patterns.

DEFINITION 1.50 *Call a polytopal projection pattern $\mathcal{P}$* hypergeneric *if all singular i-spaces D associated to $\mathcal{P}$ are i-dimensional hyperplanes with stabilisers $\Gamma^{\mathcal{D}}$ satisfying $rk\,\Gamma^{\mathcal{D}} = i$.*

Hypergeneric projection patterns have the property that the Γ-orbits of the $(n-1)$-dimensional boundary faces of the acceptance domain K contain the hyperplanes spanned by the faces. Such patterns were referred to as *generic* in [12] since they are the kind most likely to occur in a randomly selected canonical projection scheme. However, we reserve the term generic for the following class of projection patterns, which are the ones most likely to occur in a randomly selected polytopal projection scheme.

DEFINITION 1.51 *A polytopal projection pattern in which the stabilisers of all singular spaces are trivial is called* generic.

Lastly, to simplify terminology we make the following definition.

DEFINITION 1.52 *Suppose $\mathcal{P}$ is a codimension n polytopal projection pattern with acceptance domain K. Call $\mathcal{P}$ a* hyperplane polytopal projection pattern *if all connected components $D \in I_{n-1c}$ in the set of Γ-orbits of $(n-1)$-dimensional faces of K are $(n-1)$-dimensional hyperplanes.*

The following results yield information about the stabilisers of singular spaces when the number L_0 of Γ-orbits of singular 0-spaces is finite.

LEMMA 1.53 *If L_0 is finite, then the stabilisers of all i-dimensional connected components D are non-trivial, for $0 < i \leqslant n-1$.*

Proof Suppose there exists D of dimension i with trivial stabiliser.

By Lemma 1.39, there are infinitely many singular 0-spaces in D. Suppose two of these singular 0-spaces p_1 and p_2 are in the same orbit. Then there exists $\gamma \in \Gamma$ with $p_2 = p_1 + \gamma$. This means that $D \cap (D+\gamma) \supset \{p_2\} \neq \emptyset$ since $p_2 \in D$ and $p_1 + \gamma = p_2 \in D + \gamma$. Since D is a connected component in the set of Γ-orbits of singular spaces, $D \cap (D+\gamma) \neq \emptyset$, and γ acts by translation only, this implies that $D + \gamma = D$, so γ stabilises D, which is a contradiction. Hence the singular 0-spaces in D must be in distinct Γ-orbits. Thus we have shown that if the stabiliser of some $D \in I_{ic}$ is trivial then L_0 is infinite. ∎

LEMMA 1.54 *If L_0 is finite, then the singular i-spaces $D \in I_{ic}$ are i-dimensional hyperplanes for $i = 1, \ldots n-1$.*

Proof By Lemma 1.43 it suffices to prove this result for $i = n-1$.

First note that when L_0 is finite, singular 1-spaces have the form of lines of infinite length. This is because if there were a singular 1-space l which was only a line segment, then its stabiliser would be trivial and so any two singular 0-spaces in l would be in distinct

Γ-orbits. However, by Lemma 1.39 since there are infinitely many singular 0-spaces in any singular 1-space, this would imply that L_0 was infinite.

Lemma 1.39 also gives that singular 1-spaces are dense in D. Singular 1-spaces in $n-1$ linearly independent directions are contained in D because D contains translates of at least one $(n-1)$-dimensional face F of the acceptance domain K for the polytopal projection pattern under consideration and F is a polytope which is bounded in $\mathbb{R}^{n-1}$ and hence contains at least one set of singular 1-spaces linearly independent in $\mathbb{R}^{n-1}$. Thus D contains dense sets of lines of infinite length in $n-1$ linearly independent directions.

Now singular 0-spaces in D in the same Γ-orbit are translates of one another by elements in the stabiliser $\Gamma^{\mathcal{D}}$ of D since if they differed by some element not in the stabiliser of D then at least one of the points could not lie in D. As singular 0-spaces are dense in D by Lemma 1.39, but there are only finitely many Γ-orbits of singular 0-spaces by assumption, $\Gamma^{\mathcal{D}}$ is dense in D. Thus taking translates $F+\gamma$ of at least one of the faces F giving rise to D for $\gamma \in \Gamma^{\mathcal{D}}$, every point in the hyperplane H_D spanned by D is contained in the set $\{F+\gamma : \gamma \in \Gamma^{\mathcal{D}}\}$, which shows that $H_D \subset D$.

Hence since $D \subset H_D$ by definition of H_D, we have shown that D is an $(n-1)$-dimensional hyperplane and so all singular i-spaces in I_{ic} for $0 < i < n$ are i-dimensional hyperplanes, as required. ∎

COROLLARY 1.55 *If L_0 is finite then the dimension of the real vector space which is the span of the stabiliser of an $(n-1)$-dimensional singular space $D \in I_{n-1,c}$ is $n-1$.*

Proof As above, D contains singular 1-spaces in $n-1$ linearly independent directions. Since the singular 1-spaces are lines of infinite length, their stabilisers are non-trivial. Hence the stabiliser $\Gamma^{\mathcal{D}}$ of D contains $n-1$ linearly independent elements which span D (an $(n-1)$-dimensional affine subspace of V). ∎

LEMMA 1.56 *For a polytopal projection pattern, if L_0 is finite, then the rank of the stabiliser of an i-dimensional connected component D is rk $\Gamma^{\mathcal{D}} = \frac{n+d}{n} i$.*

Proof By the above results, when L_0 is finite, the connected components form hyperplanes so the argument of [12, IV.6.7] can be used to show that the above statement holds for polytopal projection patterns with $L_0 < \infty$. ∎

The results of this section imply that there are three distinct cases to consider.

- L_0 finite, so the orbits of singular i-spaces contain the hyperplanes spanned by the spaces for all i.
- At least one orbit of singular $(n-1)$-spaces does not contain the hyperplanes spanned by the spaces, so L_0 is infinite.
- L_0 is infinite but the orbits of singular spaces contain the hyperplanes spanned by the spaces.

1.5 Modules from singular spaces

In analogy with [12, V.3.1], given the set $\mathcal{K}^i$ of singular i-spaces, a Γ-module C^i can be defined, for $i = 1, \ldots, n$ as follows.

DEFINITION 1.57 *For $0 \leqslant i \leqslant n$, define C^i, to be the $\mathbb{Z}$-module of compactly-supported $\mathbb{Z}$-valued functions on singular i-spaces generated by indicator functions $[U]$ on i-dimensional elements U of $\mathcal{K}$, subject to the relation $[U_1]+[U_2] = [U_1 \cup U_2]+[U_1 \cap U_2]$, with zero element $0 = [W]$ for W the empty set or, if $i > 0$, a singular j-space for $j \leqslant i-1$.*

Note in particular that C^0 is a free $\mathbb{Z}$-module.

There is an action of Γ on C^i for each i given by $\gamma \cdot [U] = [U+\gamma]$ for $\gamma \in \Gamma$ and $[U] \in C^i$.

We now use these modules to define a complex analogous to that given in [12, V.3.2].

LEMMA 1.58 *For C^i as above, there is a complex of Γ-modules $0 \to C^n \xrightarrow{\delta} C^{n-1} \to \cdots \to C^1 \xrightarrow{\delta} C^0 \to 0$ for Γ-equivariant maps δ.*

Proof Let $\mathcal{R}$ be the set of all Delone subsets $R \subset \Gamma$ such that all connected components of $V_R\colon = V \setminus \{\partial K + \gamma : \gamma \in R\}$ are bounded and nonempty. Since K is polytopal with inradius r, the Delone set R which is such that any ball of radius $r/2$ contains a point of R is sufficient to ensure that the connected components of $V \setminus \{\partial K + \gamma\colon \gamma \in R\}$ are bounded and non-empty. Thus $\mathcal{R}$ is non-empty. Clearly, it is also closed under finite union. Hence $\mathcal{R}$ is a directed system under inclusion and $\bigcup_{R\in\mathcal{R}} R = \Gamma$.

For $R \in \mathcal{R}$ and $0 \leqslant i \leqslant n = \dim V$, define modules C_R^i in the same way as C^i above, but only for elements of $\mathcal{K}^i$ which arise from the translates $\{K + \gamma : \gamma \in R\}$ of the acceptance domain K. Define maps $\delta_R\colon C_R^i \to C_R^{i-1}$ on generators by $[U] \mapsto \sum_{j\in J} \pm[V_j]$, for J the set enumerating all singular $(i-1)$-spaces in the boundary of the singular i-spaces making up the space U and the sign of an indicator function $[V_j]$ is determined by the orientation of the singular $(i-1)$-space V_j. The orientation of singular i-spaces arising from $\{K + \gamma : \gamma \in R\}$ (a CW complex) can be chosen to ensure that $\delta_R\delta_R[U] = 0$ for $[U] \in C_R^i$ by orienting K and assigning the same orientations to singular i-spaces parallel to i-dimensional faces of K (note that the orientation of all singular n-spaces will be the same). Hence these modules and maps form a chain complex

$$0 \to C_R^n \xrightarrow{\delta_R} C_R^{n-1} \xrightarrow{\delta_R} \cdots \xrightarrow{\delta_R} C_R^0 \to 0. \tag{1.1}$$

For $R, R' \in \mathcal{R}$ with $R \subset R'$, the module C_R^i can be identified with a submodule of $C_{R'}^i$ and under this identification we have $\delta_R[U] = \delta_{R'}[U]$ for all $[U] \in C_R^i$. Thus, the direct limit of the modules C_R^i for $R \in \mathcal{R}$ is C^i for each i. Hence the complexes (1.1) for $R \in \mathcal{R}$ form a direct system with direct limit again a chain complex. Note also that $R \in \mathcal{R}$ implies that $R + \gamma \in \mathcal{R}$ for all $\gamma \in \Gamma$, and if $[U], [U+\gamma]$ are elements of C_R^i, then $\delta_R[U+\gamma] = \delta_R[U] + \gamma$, so the limit complex is a complex of Γ-modules and the maps δ are Γ-equivariant. ∎

LEMMA 1.59 *The sequence*

$$0 \to C^n \xrightarrow{\delta_{n-1}} C^{n-1} \to \cdots \to C^1 \xrightarrow{\delta_0} C^0 \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{1.2}$$

is exact at C^n, C^0 and $\mathbb{Z}$, where $\epsilon[U] = 1$ for all $[U] \in C^0$.

Proof Take an element $f = \sum n_i[U_i] \neq 0$ in C^n and suppose $\delta_{n-1}(f) = \sum\sum \pm n_i[V_{ij}] = 0$. The element f is a compactly supported function on singular n-spaces in $V \cong \mathbb{R}^n$. The compact subset of $\mathbb{R}^n$ formed by the singular n-spaces is bounded in $\mathbb{R}^n$ so it has non-empty boundary components V_{ij} and there are only finitely many such components since the singular n-spaces U_i are polytopes by definition. Now find a linearly independent set $[V'_{ik}]$ within the (finite) set of all elements $[V_{ij}]$ in $\delta_{n-1}(f)$ and write $[V_{ij}] = \sum m_k[V'_{ik}]$ so $\delta_{n-1}(f) = \sum_i \sum_k \pm n_i m_k[V'_{ik}]$. Since $\delta_{n-1}(f) = 0$, we must have $n_i m_k = 0$ for all i, k. As at least one of the $[V_{ij}]$ are non-zero for each i, there exists at least one k with $m_k[V'_{ik}] \neq 0$ so $m_k \neq 0$. This implies that $n_i = 0$ for all i, so the element $f = \sum n_i[U_i] = 0$, which is a contradiction. Hence if $f \in C^n$ is non-zero then $\delta_{n-1}(f) \neq 0$ in C^{n-1} and hence $Ker\delta_{n-1} = 0$ as required for the sequence to be exact at C^n.

To see exactness at C^0, note that any element in the image of δ_0 has the form $\sum a([p_i] - [p_j])$ for $a \in \mathbb{Z}$ and p_i, p_j singular 0-spaces. Thus $\epsilon(\sum a([p_i] - [p_j])) = \sum a\epsilon([p_i] - [p_j]) = \sum a(1 - 1) = 0$ and hence $Im\delta_0 \subset Ker\epsilon$. The fact that $Ker\epsilon \subset Im\delta_0$ is provided by Lemma 1.42.

Finally, the sequence is exact at $\mathbb{Z}$. This is because there is at least one singular point p in $\mathcal{K}^0$, so taking integer multiples of the indicator function $[p] \in C^0$ on p gives elements of C^0 which map to $a \in \mathbb{Z}$ for any a. Hence the map ϵ is surjective, as required for exactness at $\mathbb{Z}$. ∎

Unlike in the canonical case [12], for a polytopal projection pattern, it is not straightforward to show that the sequence is exact everywhere, and in fact it is possible that the sequence may not be exact in general. However, as shown in Section 2.2, the sequence is exact for all polytopal projection patterns of codimension 2 and for higher codimensions, there is one class of polytopal projection patterns for which the sequence is always exact.

LEMMA 1.60 *If the orbits of faces of K contain the hyperplanes spanned by the faces, then the sequence* (1.2) *is exact.*

Proof Exactness at $\mathbb{Z}, C^0$ and C^n is provided by Lemma 1.59. The removal of hyperplanes translated by elements of a Delone subset R of Γ decomposes the space $V \cong \mathbb{R}^n$ into convex (and hence contractible) sets. Thus the complex (1.1) of modules C^i_R is acyclic so the limit complex (1.2) of modules C^i is acyclic for $0 < i < n$. ∎

Thus, whether L_0 is finite or infinite, if the orbits of singular spaces contain the hyperplanes spanned by the spaces, then the sequence will again be exact.

Now note that the modules C^i can be decomposed into various submodules, as follows.

DEFINITION 1.61 *For $D \in I_{n-1,c}$ in the orbit class $\mathcal{D}$* (Def 1.38), *define C^{n-1}_D to be the submodule of C^{n-1} generated by indicator functions on all singular $(n-1)$-spaces D in the Γ-orbit $\mathcal{D}$. Also, define $C^{n-1}_{\mathcal{D}}$ to be the submodule of C^{n-1} generated by indicator functions on a single representative D of the Γ-orbit $\mathcal{D}$.*

Similarly for $0 \leqslant i < n-1$, define C^i_D for $D \in I_{n-1c}$ to be the submodule of C^i generated only by indicator functions on singular i-spaces which are contained in D and its Γ-translates. Lastly, define $C^i_{\mathcal{D}}$ as the submodule generated by singular i-spaces in one representative of the Γ-orbit $\mathcal{D}$.

Note in particular that C^0_D and $C^0_{\mathcal{D}}$ are free modules with generators consisting of indicator functions on singular points contained in $D + \Gamma$ and $D \in \mathcal{D}$ respectively.

For canonical projection patterns, there is exactly one Γ-orbit of singular $(n-1)$-spaces D parallel to θ for each $\theta \in I_K$ (Def 1.36), but in general, parallel singular $(n-1)$-spaces could lie in distinct orbits (although there will be only finitely many such orbits by the remark following Definition 1.36). Also note that for non-canonical projection patterns, the Γ-orbits of faces of K need not be disjoint, which means that the connected components D could be composed of translates of more than one distinct (up to Γ-translation) $(n-1)$-dimensional face of K, whereas for canonical projection patterns connected components D are formed from translates of only one face of K.

Since all parallel connected components $D \in I_{n-1c}$ are disjoint, and the intersection of two non-parallel components is a singular space of dimension $s < n-1$ which gives rise to the zero element in C^{n-1}, there is a decomposition $\bigoplus_{D \in I_{n-1c}} C_D^{n-1}$. A further decomposition of C^{n-1} is provided by the following lemma.

LEMMA 1.62 *The module C^{n-1} splits as $\bigoplus_{\mathcal{D} \in I_{n-1c}/\Gamma} C_{\mathcal{D}}^{n-1} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ for $\mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ the free $\mathbb{Z}$-module with basis consisting of elements of $\Gamma/\Gamma^{\mathcal{D}}$.*

Proof Given the above decomposition $C^{n-1} = \bigoplus_{D \in I_{n-1c}} C_D^{n-1}$, note that $C_D^{n-1} \cong C_{\mathcal{D}}^{n-1} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$. This is because $\mathcal{D}$ is the equivalence class of connected components $D \in I_{n-1c}$ under the relation $D \sim D'$ if $D = D' + \gamma$ for $\gamma \in \Gamma$, and since $D = D + \gamma$ for $\gamma \in \Gamma^{\mathcal{D}}$, distinct elements of C_D^{n-1} arise only from translations of D by $\Gamma/\Gamma^{\mathcal{D}}$. ∎

Lastly, note that the submodules defined above also fit into complexes analogous to the one in Lemma 1.58.

$$0 \to C_{\mathcal{D}}^{n-1} \to C_{\mathcal{D}}^{n-2} \to \cdots \to C_{\mathcal{D}}^{0} \to 0$$

The exactness and other properties of this sequence will be considered in greater detail in Chapter 2. However, we note here that the sequence is exact at $C_{\mathcal{D}}^{n-1}$, since the singular $(n-1)$-spaces underlying an element $\sum n_i[U_i]$ of C^{n-1} are bounded in the connected components D in the orbit $\mathcal{D}$ and hence the proof of Lemma 1.59 can be applied in this case.

1.6 Algebraic Topology and $M\mathcal{P}$

In this section, we introduce the main tools from algebraic topology which are used in the study of the continuous hull $M\mathcal{P}$ (Def 1.10) and hence, by Lemma 1.11, of a pattern $\mathcal{P}$. Relationships between the following topological invariants are stated in Theorem 1.76 ahead.

1.6.1 C^*-algebra K-theory

One way of investigating the structure of $M\mathcal{P}$ is provided by first obtaining a C^**-algebra* from $M\mathcal{P}$, written $C(M\mathcal{P}) \rtimes \mathbb{R}^d$, and then considering the K-theory of this object. Essentially, a C^*-algebra A is a complex vector space equipped with a product and another operation, $^*: A \to A$ called the ***adjoint***, together with a norm $\|\cdot\|$ such that A is complete with respect to this norm and elements $a \in A$ satisfy the C^****-algebra condition*** $\|a^*a\| = \|a\|^2$. For the precise definition of C^*-algebras, and the construction of the crossed-product $C(Y) \rtimes G$ for a group G acting on a compact Hausdorff topological space Y, see Appendix 1. To study such objects, C^*-algebra K-theory is used. This is defined using projections in A, which are elements $p \in A$ such that $p = p^* = p^2$. If A is unital, then $K_0(A)$ is defined to be the Grothendieck group of the semigroup of stable equivalence classes of projections in A. Appendix 1 contains the precise definition of the equivalence relation on projections and the definition of $K_0(A)$ when A is not unital. The group $K_1(A)$ can also be calculated, which is defined using suspensions of A (Def 4.4). By Bott periodicity, given in Appendix 1 as Theorem 4.5, these are the only distinct K-groups for C^*-algebras. Other properties of C^*-algebra K-theory which are utilised in the study of C^*-algebras arising from quasiperiodic patterns are also stated in Appendix 1. The simplest example of a C^*-algebra is the set $\mathbb{C}$ of complex numbers together with the usual addition, multiplication, complex conjugation and the modulus norm. The K-theory is $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$.

The C^*-algebra K-theory has applications to the study of particles moving within quasicrystals. A discussion of the link between K-theory and physics can be found in [22]. The main ideas giving rise to this connection are as follows.

Under certain simplifying assumptions, such as no external forces acting, the motion of a particle in Euclidean space can be modelled (in quantum mechanics) by its position and momentum operators. These operators generate an algebra, known as the algebra of observables of the particle, which in turn gives rise to the C^**-algebra of observables* $\mathcal{A}$.

The *tight-binding* model for a particle moving in a solid is used. In this model, a solid is represented by a tiling in which each tile corresponds to an atom. The motion of a particle in this structure is then discrete - the particle essentially jumps from tile to tile. The position operator of the particle is replaced by a tile in the tiling and the momentum operator is replaced by finite translations. Now consider the Hamiltonian operator $H \in \mathcal{A}$ (which encodes kinetic and potential energy) of a particle moving in a solid, in accordance with the tight-binding model. The Hamiltonian is a *bounded operator*, so its spectrum S of eigenvalues is a bounded subset of $\mathbb{R}$. Define a *gap* in the spectrum to be a maximal connected subset of the complement of S in $\mathbb{R}$. Note that since S is bounded in $\mathbb{R}$, there exist real numbers a and b such that the intervals $-\infty: = (-\infty, a)$ and $\infty: = (b, \infty)$ are gaps. The C^*-algebra K-theory labels the gaps, as follows.

Writing $Gap(H)$ for the set of all gaps in the spectrum of H, there is a map $Gap(H) \to K_0(A)$ which sends a gap g to the class $[P_g]$ of P_g, the spectral projection of the interval $(-\infty, g)$. Then $[P_g]$ is called a *label* for the gap g. It has the properties that $[P_{-\infty}] = 0$ and $[P_\infty] = 1$. If H is perturbed such that the gaps change in size but do not disappear, then the labels do not change, so the map $Gap(H) \to K_0(A)$ is injective. This result is useful if $K_0(A)$ can be computed, since the number of gaps in the spectrum of H is then known even if the precise nature of the spectrum is unknown. For more details about these ideas, see [22].

1.6.2 Čech cohomology

Given the topological space $M\mathcal{P}$, the *Čech cohomology* $\check{H}^*(M\mathcal{P};\mathbb{Z})$ can be defined. To do this [27], first take the set J of all open covers $\mathcal{U}$ of $M\mathcal{P}$, and say $\mathcal{U}_1 < \mathcal{U}_2$ if $\mathcal{U}_2$ is a *refinement* of $\mathcal{U}_1$, so for all $U \in \mathcal{U}_2$, there is at least one $U' \in \mathcal{U}_1$ containing U. Next define the *nerve* $N(\mathcal{U})$ of a cover $\mathcal{U}$ to be the simplicial complex with vertices the elements of $\mathcal{U}$ and n-simplices consisting of finite subsets $\{U_1, \ldots, U_n\}$ of elements of $\mathcal{U}$ with the property that $U_1 \cap \ldots \cap U_n \neq \emptyset$. Given two open covers $\mathcal{U}_1$,$\mathcal{U}_2$ of $M\mathcal{P}$ with $\mathcal{U}_1 < \mathcal{U}_2$, there is a map $f\colon \mathcal{U}_2 \to \mathcal{U}_1$ given by choosing $f(U)$ to be an element $U' \in \mathcal{U}_1$ which contains $U \in \mathcal{U}_2$. Note that if $U_1 \cap \cdots \cap U_n \neq \emptyset$, then $U' = f(U_1 \cap \cdots \cap U_n) \neq \emptyset$ because U' contains $U_1 \cap \cdots \cap U_n$ by definition. Thus f induces a map $f\colon N(\mathcal{U}_2) \to N(\mathcal{U}_1)$, which in turn induces a homomorphism $f^*\colon H^k(N(\mathcal{U}_1);\mathbb{Z}) \to H^k(N(\mathcal{U}_2);\mathbb{Z})$. Choosing a different map does not affect this construction since if we have two maps f and f' with the property that $f(U)$ and $f'(U)$ are elements of $\mathcal{U}_1$ containing U, then $\bigcap U_i \subset \bigcap_{i=1}^n (f(U_i) \cap f'(U_i))$, so the induced maps f^* and f'^* are the same.

DEFINITION 1.63 *The* kth Čech cohomology group of $M\mathcal{P}$ with coefficients in $\mathbb{Z}$ *is defined to be*

$$\check{H}^k(M\mathcal{P};\mathbb{Z}):= \varinjlim_{\mathcal{U}\in J} H^k(N(\mathcal{U});\mathbb{Z}).$$

Čech cohomology is used in Chapters 2 and 3. It is a good cohomology theory to use for the study of topological spaces like $M\mathcal{P}$, which is not a CW complex. Also, $M\mathcal{P}$ can be described [31] in terms of an inverse limit $\varprojlim K_n$, so the property [27] that $\check{H}^*(\varprojlim K_n) = \varinjlim \check{H}^*(K_n)$ means that Čech cohomology is suitable for use in this setting. Theorem 1.76 below provides further evidence that Čech cohomology is appropriate, since this theorem supplies isomorphisms linking the K-theory to the Čech cohomology of the topological space $M\mathcal{P}$. By Theorem 1.76, Čech cohomology is also isomorphic to group cohomology, which is defined in the following section. This invariant is more straightforward to compute and so will be used in Chapter 2.

1.6.3 Group (co)homology and dynamical systems

Later in this document, and in Chapter 2 in particular, we wish to compute the group homology $H_*(\Gamma;C)$ of $\Gamma := \pi^{\perp}(\Lambda) \cap V$ with coefficients in a Γ-module C. Following [12, V.4], this section gives the definition of group homology and states properties which will be used to facilitate calculations later in this document. Note that group cohomology can be obtained from group homology via Poincaré duality.

Group homology of a group Γ is defined to be the homology of a projective resolution of $\mathbb{Z}$ by modules over $\mathbb{Z}[\Gamma]$, the free $\mathbb{Z}$-module with basis consisting of all elements of Γ. The resolution we choose [12] is given below.

DEFINITION 1.64 *Denote by* $\{e_1, \ldots, e_{n+d}\}$ *a basis of* Γ. *Write* $\Lambda\Gamma$ *for the* exterior module *of the group* Γ, *which is* $\Lambda\Gamma := \bigoplus \Lambda_i\Gamma$, *and* $\Lambda_i\Gamma$ *is the* $\mathbb{Z}$*-module with basis denoted by* $\{e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_i} : j_k \in \{1, \ldots, n+d\}\}$ *for* $\wedge$ *subject to relations* $e_{j_1} \wedge \ldots \wedge e_{j_k} \wedge e_{j_{k+1}} \wedge \ldots \wedge e_{j_i} = -e_{j_1} \wedge \ldots \wedge e_{j_{k+1}} \wedge e_{j_k} \wedge \ldots \wedge e_{j_i}$, *for* $k = 1, \ldots n+d-1$.

Note that the relations imply that if $j_k = j_m$ for some $j_k, j_m \in \{1, \ldots, n+d\}$, then $e_{j_1} \wedge \ldots \wedge e_{j_i} = 0$.

Also $\mathbb{Z}[\Gamma]$ can be viewed as the ring of Laurent polynomials on $n+d$ variables $\{\gamma_1, \ldots, \gamma_{n+d}\}$ with integer coefficients, and hence is also referred to as the *integral group ring* of Γ.

A free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\Gamma]$-module is [14]

$$0 \to \Lambda_{n+d}\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda_0\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{1.3}$$

where $\epsilon[\gamma] = 1$ for all basis elements $[\gamma]$ of $\mathbb{Z}[\Gamma]$ and ∂ is determined uniquely as the $\mathbb{Z}[\Gamma]$-linear map of degree 1 satisfying $\partial(e_i) = t_i - 1$. The $\mathbb{Z}[\Gamma]$-action is trivial on the first factor and permutation on $\mathbb{Z}[\Gamma]$.

Applying the functor $- \otimes_\Gamma C$ gives the complex below.

$$0 \to \Lambda_{n+d}\Gamma \otimes \mathbb{Z}[\Gamma] \otimes_\Gamma C \xrightarrow{\partial\otimes 1} \Lambda_{n+d-1}\Gamma \otimes \mathbb{Z}[\Gamma] \otimes_\Gamma C \xrightarrow{\partial\otimes 1} \cdots$$

$$\xrightarrow{\partial\otimes 1} \Lambda_1\Gamma \otimes \mathbb{Z}[\Gamma] \otimes_\Gamma C \xrightarrow{\partial\otimes 1} \Lambda_0\Gamma \otimes \mathbb{Z}[\Gamma] \otimes C \to 0 \tag{1.4}$$

The boundary operator is given on basis elements by

$$\partial \otimes 1((e_{j_1} \wedge \ldots \wedge e_{j_i}) \otimes c) = \sum_{k=1}^{i} (-1)^k (e_{j_1} \wedge \ldots \hat{e}_{j_k} \ldots \wedge e_{j_i}) \otimes (e_{i_k} \cdot c - c)$$

where $\hat{e}_{j_k}$ signifies that e_{j_k} is omitted and $\gamma \cdot c$ denotes the action by translation of $\gamma \in \Gamma$ on $c \in C$, so $\gamma \cdot c = c + \gamma$. Group homology $H_*(\Gamma; C)$ is the homology of this complex.

We now list some results which will be useful for later calculations.

LEMMA 1.65 *With the above constructions, where Γ acts freely on $\mathbb{Z}[\Gamma]$, we have*

$$H_m(\Gamma;\mathbb{Z}[\Gamma]) \cong \begin{cases} \mathbb{Z} & m=0 \\ 0 & m>0. \end{cases}$$

Proof Since $\mathbb{Z}[\Gamma]$ is a free Γ-module, the functor $-\otimes_\Gamma \mathbb{Z}[\Gamma]$ is exact, meaning that tensoring over Γ by $\mathbb{Z}[\Gamma]$ does not alter the exactness of the sequence (1.3). Hence $H_k(\Gamma;\mathbb{Z}[\Gamma]) = 0$ for $k>0$.

The fact that the free resolution (1.3) of $\mathbb{Z}$ is exact after applying $-\otimes_\Gamma \mathbb{Z}[\Gamma]$ implies that $Ker(\partial\colon \Lambda_0\Gamma\otimes\mathbb{Z}[\Gamma]\to 0)/Im(\partial\colon \Lambda_1\Gamma\otimes\mathbb{Z}[\Gamma]\to\Lambda_0\Gamma\otimes\mathbb{Z}[\Gamma])$ is equal to $\mathbb{Z}$ since from (1.3) we have $\Lambda_0\Gamma\otimes\mathbb{Z}[\Gamma]\twoheadrightarrow\mathbb{Z}$. Thus $H_0(\Gamma;\mathbb{Z}[\Gamma])\cong\mathbb{Z}$. ∎

Now suppose Γ splits as $\Gamma_1 \bigoplus \Gamma_2$. Write $\mathbb{Z}[\Gamma_2]$ for the free $\mathbb{Z}$-module generated by Γ_2. This is also a Γ-module under the action $(\gamma_1\oplus\gamma_2)\cdot x = x+\gamma_2$. Then we have the following result.

LEMMA 1.66

$$H_m(\Gamma;\mathbb{Z}[\Gamma_2]) \cong \begin{cases} \mathbb{Z}^{\binom{rk\,\Gamma_1}{m}} & 0\leqslant m\leqslant rk\,\Gamma_1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof We have $\Lambda_k\Gamma\otimes\mathbb{Z}[\Gamma_2]\cong\bigoplus_{i+j=k}\Lambda_i\Gamma_1\otimes\Lambda_j\Gamma_2\otimes\mathbb{Z}[\Gamma_2]$ so, under the above action by Γ, the boundary operator ∂_k becomes $(-1)^k\otimes\partial'_k$ for ∂'_k the boundary of the complex $\Lambda\Gamma_2\otimes\mathbb{Z}[\Gamma_2]$. Hence $H_k(\Gamma;\mathbb{Z}[\Gamma_2])\cong\bigoplus_{i+j=k}\Lambda_i\Gamma_1\otimes H_j(\Gamma_2;\mathbb{Z}[\Gamma_2])\cong\Lambda_k\Gamma_1$. The fact that $\Lambda_k\Gamma_1\cong\mathbb{Z}^{\binom{rk\,\Gamma_1}{k}}$ gives the required result. ∎

COROLLARY 1.67 $H_k(\Gamma;\mathbb{Z})\cong\mathbb{Z}^{\binom{rk\,\Gamma}{k}}$. ∎

COROLLARY 1.68 $e_{\mathbb{Z}}\colon=\sum_{m=0}^{N}(-1)^m rk\, H_m(\Gamma;\mathbb{Z})\otimes\mathbb{Q}=0$. ∎

COROLLARY 1.69 *For C any Γ-module, $H_m(\Gamma_1\oplus\Gamma_2;C\otimes_\Gamma\mathbb{Z}[\Gamma_2])\cong H_m(\Gamma_1;C)$.* ∎

LEMMA 1.70 *If $C=\bigoplus_i C_i$ then $H_m(\Gamma;C)\cong\bigoplus_i H_m(\Gamma;C_i)$.* ∎

Note that since the complex used to define group homology is bounded below, we have $H_m(\Gamma; C) = 0$ for $m < 0$.

Now consider the modules C^i in Definition 1.57, which arise from consideration of singular i-spaces (Def 1.36). There are several results which we collect here for use in Chapter 2.

COROLLARY 1.71 *Group homology $H_m(\Gamma; C^{n-1})$ splits as a direct sum of groups*

$$H_m(\Gamma; C^{n-1}) \cong \bigoplus_{\mathcal{D} \in I_{n-1c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^{n-1}_{\mathcal{D}}).$$

Proof Lemmas 1.62, 1.70 and Corollary 1.66 give $H_m(\Gamma; C^{n-1}) \cong \bigoplus_{\mathcal{D} \in I_{n-1c}/\Gamma} H_m(\Gamma^{\mathcal{D}} \oplus \Gamma/\Gamma^{\mathcal{D}}; C^{n-1}_{\mathcal{D}} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]) \cong \bigoplus_{\mathcal{D} \in I_{n-1c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^{n-1}_{\mathcal{D}})$. ∎

LEMMA 1.72 *For a singular i-space $D \in I_{ic}$ in Γ-orbit $\mathcal{D}$ with stabiliser of rank $r \geqslant i$, the homology groups $H_m(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}})$ are non-trivial only for $m \leqslant rk\ \Gamma^{\mathcal{D}} - \dim D$.*

Proof If the rank of $\Gamma^{\mathcal{D}}$ is equal to i, then there is a fundamental domain Y for the action of $\Gamma^{\mathcal{D}}$ on elements D in $\mathcal{D}$, and $\Gamma^{\mathcal{D}}$ acts freely on Y, so $C^i_{\mathcal{D}}$ decomposes as $C^i_Y \otimes \mathbb{Z}[\Gamma^{\mathcal{D}}]$, for C^i_Y the submodule of $C^i_{\mathcal{D}}$ generated by singular i-spaces contained in Y. By Corollary 1.69 we have $H_*(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}}) \cong H_*(1; C^i_Y)$ so only $H_0(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}})$ is non-trivial. Similarly, if rk $\Gamma^{\mathcal{D}} > i$, then $H_m(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}}) \cong H_m(\Gamma_1 \oplus \Gamma_2; C^i_Y \otimes \mathbb{Z}[\Gamma_2])$, where Γ_2 acts freely on $Y \subset D$ and hence has rank i. By Corollary 1.69 we have $H_m(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}}) \cong H_m(\Gamma_1; C'^i_Y)$. Since Γ_1 has rank rk $\Gamma^{\mathcal{D}} - \dim D$, the result follows. ∎

COROLLARY 1.73 *If the rank of the stabiliser $\Gamma^{\mathcal{D}}$ of a singular i-space D is i, then $H_m(\Gamma^{\mathcal{D}}; C^i_{\mathcal{D}}) = 0$ for $m > 0$.* ∎

Spectral sequences

Now suppose we have a d-dimensional codimension n polytopal projection pattern with associated exact sequence (1.2) of Γ-modules $0 \to C^n \xrightarrow{\delta} C^{n-1} \to \cdots \to C^1 \xrightarrow{\delta} C^0 \to \mathbb{Z} \to 0$. Spectral sequences can be produced from the double complex $(\Lambda_p\Gamma \otimes C^q, \partial, (-1)^p \otimes \delta)$ as follows.

The first spectral sequence which can arise, denoted $\tilde{E}^r_{pq}$, is produced by taking the homology of the complex with respect to $(-1)^p \otimes \delta$. Since the sequence (1.2) above is exact, $\tilde{E}^1_{pq} = H_\delta(\Lambda_p\Gamma \otimes C^q) = Ker(\Lambda_p\Gamma \otimes C^q \to \Lambda_p\Gamma \otimes C^{q-1})/Im(\Lambda_p\Gamma \otimes C^{q+1} \to \Lambda_p\Gamma \otimes C^q)$ is trivial for $q > 0$. Hence the higher differentials arising from the map ∂ are trivial and $\tilde{E}^1_{pq} = \tilde{E}^\infty_{pq}$ is the limit of the spectral sequence. Thus [12] $\bigoplus_{p+q=k} \tilde{E}^\infty_{pq} = \Lambda_k\Gamma$ and $\tilde{E}^\infty_{pq} = 0$ for $q > 0$. Hence when $q = 0$, we have $\tilde{E}^\infty_{pq} \cong \Lambda_p\Gamma \cong H_p(\Gamma;\mathbb{Z})$. The spectral sequence is said to converge to $H_p(\Gamma;\mathbb{Z})$.

The second spectral sequence which can be produced from the above double complex is denoted E^r_{pq} and is derived by taking the homology of the complex $\Lambda_p\Gamma \otimes C^q$ with respect to ∂, so $E^1_{p,q} = H_p(\Gamma; C^q)$ and the first differential is $((-1)^p \otimes \delta)_*$. The first differential is a map from $E^1_{p,q}$ to $E^1_{p,q-1}$. There are also higher differentials from E^r_{pq} to $E^r_{p-r,q-r-1}$ Note that the limit term E^∞_{pq} is equal to the term E^r_{pq} if all higher differentials $E^t_{pq} \to E^t_{p-t,q-t-1}$ are trivial for $t \geqslant r$. Again, the spectral sequence is said to converge to $H_*(\Gamma;\mathbb{Z})$ and we write $E^r_{pq} \Rightarrow H_{p+q}(\Gamma;\mathbb{Z})$. This means that the vector spaces $E^\infty_{pq} \otimes \mathbb{Q}$ satisfy $\bigoplus_{p+q=k} E^\infty_{pq} \otimes \mathbb{Q} \cong H_k(\Gamma;\mathbb{Z}) \otimes \mathbb{Q}$ and hence the ranks of the terms on the diagonals $p + q = k$ of the table sum to the rank of $H_k(\Gamma;\mathbb{Z})$.

There are maps $H_{n+p}(\Gamma;\mathbb{Z}) \to H_p(\Gamma; C^n)$ for $0 \leqslant p \leqslant d$, which will be referred to as *edge homomorphisms* in this document. They are defined as the composition of the boundary homomorphisms in the long exact sequences in homology arising from the short exact seqences

$$\begin{gathered}
0 \to C^n \to C^{n-1} \to C_0^{n-2} \to 0 \\
0 \to C_0^{n-2} \to C^{n-2} \to C_0^{n-3} \to 0 \\
\vdots \\
0 \to C_0^0 \to C^0 \to \mathbb{Z} \to 0.
\end{gathered}$$

Spectral sequences associated to the complex (1.2) can be illustrated by drawing a table with n columns. For the terms E^1_{pq}, the ith column contains $H_*(\Gamma; C^i)$, with $H_0(\Gamma; C^i)$ in the 0th row, $H_1(\Gamma; C^i)$ in the first row and so on. Examples of tables for the E^1 and $E^3 = E^\infty$ terms in the spectral sequence of a 3-dimensional codimension 3 polytopal projection pattern with L_0 finite are shown below. The non-zero entries in the table are determined by Lemmas 1.56 and 1.72.

0	0	0	$H_3(\Gamma; C^3)$
0	0	$H_2(\Gamma; C^2) \xleftarrow{d_{32}}$	$H_2(\Gamma; C^3)$
0	$H_1(\Gamma; C^1) \xleftarrow{d_{21}}$	$H_1(\Gamma; C^2) \xleftarrow{d_{31}}$	$H_1(\Gamma; C^3)$
$H_0(\Gamma; C^0) \xleftarrow{d_{10}}$	$H_0(\Gamma; C^1) \xleftarrow{d_{20}}$	$H_0(\Gamma; C^2) \xleftarrow{d_{30}}$	$H_0(\Gamma; C^3)$

0	0	0	$H_3(\Gamma; C^3)$
0	0	$Kerd_{22}/Imd_{32}$	$Kerd_{32}$
0	$(H_1(\Gamma; C^1)/Imd_{21})/Im\partial_1$	$Kerd_{21}/Imd_{31}$	$Ker\partial_2$
$H_0(\Gamma; C^0)/Imd_{10}$	$Kerd_{10}/Imd_{20}$	$Kerd_{20}/Imd_{30}$	$Ker\partial_1$

In this case, there are short exact sequences $0 \to H_{m+1}(\Gamma; C^1)/Imd_{2,m+1} \to H_{m+2}(\Gamma; \mathbb{Z}) \to Kerd_{2m} \to 0$ and so on associated to the terms on the diagonals of the table, so

$$\text{rk } H_{m+1}(\Gamma; C^1)/Imd_{2,m+1} + \text{rk } Kerd_{2m} = \binom{d+2}{m+2}.$$

For more information about spectral sequences in general, see [25], and for more detail about this particular spectral sequence, see [12]. In Chapter 2 we study this spectral sequence in detail in order to determine the rational ranks of the groups $H_p(\Gamma; C^q)$, which are used in the calculation of the Euler characteristic.

Dynamical systems

We now consider an application of group homology, following [12]. Take a projection system (Λ, E, K, v). Decompose $\Lambda \cong \mathbb{Z}^N$ into $\Lambda_0 \oplus \Lambda_1$ in such a way that Λ_0 spans a space E' complementary to E. Recall the definition of the metric $\tilde{\mu}$ on the set NS of non-singular points given in Lemma 1.20 by $\tilde{\mu}(u, v) = \mu(\mathcal{P}_u, \mathcal{P}_v) + ||v - u||$.

DEFINITION 1.74 *Write $Q = \overline{E + \mathbb{Z}^N}$ for the Euclidean closure of $E + \mathbb{Z}^N$.*

Define $E'_u = \overline{E' \cap NS \cap (Q + u)}$, the completion of this space with respect to the metric $\tilde{\mu}$.

Define a map ϕ: $E'_u \to E' \cap (\overline{E + \mathbb{Z}^N} + u)$ which is such that $||\phi^{-1}(v)|| = 1$ for $v \in NS \cap E' \cap (Q + u)$.

The map ϕ 'closes' the 'gaps' made by removing singular points and completing with respect to $\tilde{\mu}$. It is one-to-one on non-singular points and m-to-one, for finite m, elsewhere.

Decompose $E' \cap Q$ into $V + \Delta$, where $\overline{\pi'(\Lambda)}$, the Euclidean closure of the projection of Λ to E', is dense in V.

DEFINITION 1.75 *Define* $V_u\colon = \{x \in E'_u : \phi(x) \in V + \pi'(u)\}$.

With this definition, there is [12, II.4.3] a decomposition $C_c(E'_u;\mathbb{Z}) \cong C_c(V_u;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[E' \cap \Lambda]$, where $C_c(Y;\mathbb{Z})$ denotes the continuous integer-valued functions on Y with compact support and $\mathbb{Z}[E' \cap \Lambda]$ denotes the free $\mathbb{Z}$-module on the discrete group $E' \cap \Lambda$ which gives rise to the vector space Δ in Definition 1.16.

Decompose Λ as $\Lambda' \oplus (\Lambda \cap E')$, for $\Lambda'\colon = \{g \in \Lambda : \pi'(g) \in V\}$. Note that $\Lambda' = (\Lambda' \cap \Lambda_0) \oplus \Lambda_1$ and $\pi'(\Lambda') = \Gamma = \pi'(\Lambda) \cap V$. Then [12, II.4.4] there is a decomposition $\mathbb{Z}[\Lambda] \cong \mathbb{Z}[\Lambda'] \otimes \mathbb{Z}[\Lambda \cap E']$ and each component of $\mathbb{Z}[\Lambda]$ acts on the corresponding component of $C_c(E'_u;\mathbb{Z})$. Thus by Corollary 1.69 there is an isomorphism $H_*(\Lambda; C_c(E'_u;\mathbb{Z})) \cong H_*(\Gamma; C_c(V_u;\mathbb{Z}))$. Hence when calculating the cohomology of projection patterns we can restrict attention to V rather than the whole of E' (which is isomorphic to $E^\perp$ via the projection $\pi^\perp$ with kernel E).

Now split Λ_0 into $\Lambda_a \oplus \Lambda_b$ so that Λ_a has rank equal to the dimension of V and V is the span of Λ_a. Take a fundamental domain X for the action of Λ_a on V_u. Then [12, I.10] there is a dynamical system consisting of the space $X\colon = V_u/\Lambda_a$ with action by $\Lambda_b \cong \mathbb{Z}^d$. Provided $E \cap \Lambda = \{0\}$, so the pattern produced by the projection system is non-periodic, the space X is a Cantor set and the $\mathbb{Z}^d$-action is *minimal*, meaning that every orbit is dense in X.

The continuous hull $M\mathcal{P}$ with action by $\mathbb{R}^d$ is the mapping torus of this dynamical system: $M\mathcal{P} \cong (X \times \mathbb{R}^d)/\sim$, where $(x, y + a) \sim (\alpha_a(x), y)$ for all $a \in \mathbb{Z}^d$, and α denotes the $\mathbb{Z}^d$-action on X.

For $C(X;\mathbb{Z})$ the continuous integer-valued functions on X, we have group cohomology $H_*(\mathbb{Z}^d; C(X;\mathbb{Z}))$, defined as in Section 1.6.3. The relationships between this invariant, those defined above, and the K-theory and Čech cohomology of the continuous hull $M\mathcal{P}$ is discussed in the following section.

1.6.4 Isomorphisms between invariants

This section contains the theorem providing links between the invariants defined above. Note $\check{H}^*(-)$ denotes Čech cohomology, $H^*(-)$ indicates group cohomology, $H_*(-)$ denotes group homology and $K_*(-)$ signifies C^*-algebra K-theory.

THEOREM 1.76 *For d-dimensional codimension n projection patterns, there are isomorphisms of groups for each m as follows.*

1. $\check{H}^m(M\mathcal{P}) \cong H^m(\Lambda; C(E'_u;\mathbb{Z})) \cong H_{d-m}(\Lambda; C_c(E'_u;\mathbb{Z}))$.

2. $H_{d-m}(\Lambda; C_c(E'_u;\mathbb{Z})) \cong H_{d-m}(\Gamma; C_c(V_u;\mathbb{Z})) \cong H_{d-m}(\mathbb{Z}^d; C(X;\mathbb{Z})) \cong H^m(\mathbb{Z}^d; C(X;\mathbb{Z}))$.

3. $K_m(C(X) \rtimes \mathbb{Z}^d) \otimes \mathbb{Q} \cong \bigoplus_{j=-\infty}^{\infty} H^{d+m+2j}(\mathbb{Z}^d; C(X;\mathbb{Z})) \otimes \mathbb{Q}$.

Proof Item 1 is proved in [12, II.4.2]. The second statement follows from the decomposition of E'_u, the definition of X, Lemma 1.69 and Poincaré duality. The final statement is proved in [11]. ∎

Note that for patterns of codimension $n \leqslant 3$ the result of item 3 has been strengthened [15, Thm 7.3] to an isomorphism $K_m(C(X) \rtimes \mathbb{Z}^d) \cong \bigoplus_{j=-\infty}^{\infty} H^{d+m+2j}(\mathbb{Z}^d; C(X;\mathbb{Z}))$. It is not yet known whether this result is true for general n, but for the purposes of this document, the result of the above theorem is enough.

COROLLARY 1.77 *The homology and cohomology groups described above are non-trivial only for* $0 \leqslant m \leqslant d$.

Proof The homology groups $H_m(\mathbb{Z}^d; C(X;\mathbb{Z}))$ are non-trivial only for m in this range. ∎

1.6.5 The Euler characteristic

The final tool which will be considered here and which can be used to investigate the continuous hull $M\mathcal{P}$ is the *Euler characteristic.*

As we will see in Chapter 2, there are several equivalent definitions of the Euler characteristic. For general patterns $\mathcal{P}$, one possible description is given in Definition 2.1 in Chapter 2. Alternative formulations of the Euler characteristic, for $\mathcal{P}$ a polytopal projection pattern, arise from the results of Theorem 1.76, and hence are equivalent to Definition 2.1. Corollaries 2.2 and 2.4 give two such alternatives. The Euler characteristic is a rational invariant, so it does not detect torsion, for example. Hence it encodes less information than Čech cohomology or C^*-algebra K-theory. However, the Euler characteristic is relatively straightforward to compute from the combinatorics of the acceptance domain and the singular spaces (Def 1.36) arising from it. In cases when quantities required for the computation of Čech cohomology or K-theory cannot be determined, the Euler characteristic can therefore often yield information.

Chapter 2

The Euler Characteristic

The main aim of this chapter is to define a topological invariant (the Euler characteristic) for polytopal projection patterns and carry out computations of this invariant in several cases. In order to achieve this aim, we first extend various ideas in [12] (which were applicable only to canonical projection patterns) to the class of polytopal projection patterns. We then derive several formulae for calculating the Euler characteristic and examine some consequences of the values obtained from such calculations.

Whereas cohomology is explicitly computed in [12] for *canonical* projection patterns, formulae for determining the cohomology of the continuous hull $M\mathcal{P}$ of polytopal projection patterns were not considered. For codimension 2 polytopal projection patterns, the method of calculation of the Euler characteristic in all possible cases can be adapted to enable a formula for the cohomology of such patterns to be determined, as shown in Section 2.2.2.

For polytopal projection patterns of higher codimension, it is shown in Section 2.3 that the Euler characteristic is not always defined. A discussion is given in Section 2.4 about how much can be said in such cases and formulae are provided for the calculation of the Euler characteristic in cases when it is defined. The chapter concludes with some applications of the results obtained.

We begin by defining the Euler characteristic for general point patterns before specialising to polytopal projection patterns for which a more tractable definition is available.

DEFINITION 2.1 *For a d-dimensional pattern $\mathcal{P}$, the* Euler characteristic *is defined to be*

$$e_{\mathcal{P}} := \sum_{i=0}^{d} (-1)^{d-i} rk\ \check{H}^i(M\mathcal{P}) \otimes \mathbb{Q},$$

where $\check{H}$ denotes Čech cohomology, provided no two terms are infinite and of opposite sign.

If any two terms are infinite and have opposite sign, then the Euler characteristic is said to be not defined.

Note that since some of the cohomology groups $\check{H}^i(M\mathcal{P})$ could have infinite rank, the Euler characteristic is allowed to take the value $\pm\infty$ if infinite terms in the expression for $e_{\mathcal{P}}$ have the same sign.

The sign $(-1)^{d-i}$ is chosen to ensure that the term involving rk $\check{H}^d(M\mathcal{P}) \otimes \mathbb{Q}$ is positive, so that the values of the Euler characteristic correspond to those given in [14] and [16] for various examples of patterns.

This definition is applicable to any point pattern $\mathcal{P}$. For the remainder of this chapter, we consider patterns which arise from a projection scheme with polytopal acceptance domain. By Theorem 1.76, in the case when $\mathcal{P}$ is a polytopal projection pattern, there is an alternative description of the Euler characteristic, as follows.

COROLLARY 2.2 *For a d-dimensional codimension n polytopal projection pattern $\mathcal{P}$*

$$e_{\mathcal{P}} = \sum_{i=0}^{d} (-1)^i rk\ H_i(\Gamma; C_c(V_u; \mathbb{Z})) \otimes \mathbb{Q},$$

when it is defined, where H denotes group homology and $C_c(V_u; \mathbb{Z})$ is the module of compactly supported continuous $\mathbb{Z}$-valued functions on V_u (Def 1.75). ∎

In order to produce yet another description of the Euler characteristic for a polytopal projection pattern $\mathcal{P}$, we first recall that there is a complex (1.2) $0 \to C^n \to C^{n-1} \to \cdots \to C^0 \to \mathbb{Z} \to 0$ of Γ-modules C^i (Def 1.57) arising from singular i-spaces associated to $\mathcal{P}$. The following lemmas show that this complex (1.2) is applicable to the calculation of the Euler characteristic.

LEMMA 2.3 *For a codimension n polytopal projection pattern $\mathcal{P}$, there is a group isomorphism $C^n \cong C_c(V_u; \mathbb{Z})$.*

Proof By the remark preceding [13, Prop 61], the isomorphism is given by sending an element $[U]$ to the indicator function on the closure of $U \setminus ((\partial K + \Gamma) \cap U)$. ∎

The following expression gives the definition of the Euler characteristic which will be most useful in the remainder of this chapter.

COROLLARY 2.4 *For a codimension n polytopal projection pattern $\mathcal{P}$, the Euler characteristic is*

$$e_{\mathcal{P}} = \sum_{i=0}^{d} (-1)^i rk\ H_i(\Gamma; C^n) \otimes \mathbb{Q}.$$

Proof An immediate corollary of the above lemma is that $H_*(\Gamma; C_c(V_\Gamma; \mathbb{Z})) \cong H_*(\Gamma; C^n)$. Substituting this into the expression for the Euler characteristic in Corollary 2.2 gives the result. ∎

If the sequence in Lemma 1.59 is exact everywhere, then the lemma below shows that the Euler characteristic can be computed by considering the modules C^i for $i < n$. We first define some notation.

DEFINITION 2.5 *For $\underline{Y}$: $= 0 \to C^{n-1} \to \ldots \to C^1 \to C^0 \to 0$, define $e_{\underline{Y}}$ to be $\sum_p \sum_q (-1)^{p+q} rk\ H_p(\Gamma; C^q) \otimes \mathbb{Q}$ and for a $\mathbb{Z}$-module C denote $\sum_p (-1)^p rk\ H_p(\Gamma; C) \otimes \mathbb{Q}$ by e_C.*

LEMMA 2.6 *For a codimension n polytopal projection pattern $\mathcal{P}$ such that the sequence $0 \to C^n \to \cdots \to C^0 \to \mathbb{Z} \to 0$ is exact everywhere, $e_{\mathcal{P}} = (-1)^{n+1} e_{\underline{Y}}$.*

Proof Take the exact sequence $0 \to C^n \to \cdots \to C^1 \to C^0 \to \mathbb{Z} \to 0$. This can be split up into a series of short exact sequences $0 \to C_0^i \to C^i \to C_0^{i-1} \to 0$, for $0 \leqslant i \leqslant n-1$, where

$C_0^i = Im(\delta\colon C^{i+1} \to C^i)$. Note $C_0^{n-1} = C^n$ and $C_0^{-1} = \mathbb{Z}$. These exact sequences give rise to long exact sequences in homology of the form shown below.

$$\begin{aligned}
\cdots &\xrightarrow{\beta_2} H_2(\Gamma; C_0^i) \longrightarrow H_2(\Gamma; C^i) \longrightarrow H_2(\Gamma; C_0^{i-1}) \\
&\xrightarrow{\beta_1} H_1(\Gamma; C_0^i) \longrightarrow H_1(\Gamma; C^i) \longrightarrow H_1(\Gamma; C_0^{i-1}) \\
&\xrightarrow{\beta_0} H_0(\Gamma; C_0^i) \longrightarrow H_0(\Gamma; C^i) \longrightarrow H_0(\Gamma; C_0^{i-1}) \longrightarrow 0
\end{aligned}$$

Thus we have

$$\text{rk } H_0(\Gamma; C^i) = \text{rk } H_0(\Gamma; C_0^i) + \text{rk } H_0(\Gamma; C_0^{i-1}) - \text{rk } \beta_0,$$

$$\text{rk } H_1(\Gamma; C^i) = \text{rk } H_1(\Gamma; C_0^i) + \text{rk } H_1(\Gamma; C_0^{i-1}) - \text{rk } \beta_0 - \text{rk } \beta_1,$$

$$\text{rk } H_2(\Gamma; C^i) = \text{rk } H_2(\Gamma; C_0^i) + \text{rk } H_2(\Gamma; C_0^{i-1}) - \text{rk } \beta_2 - \text{rk } \beta_1$$

and so on. This means that in the alternating sum $e_{C^i} = \sum_{q=0}^{d}(-1)^q \text{rk } H_q(\Gamma; C^i)$ the terms rk β_j cancel so that $e_{C^i} = e_{C_0^i} + e_{C_0^{i-1}}$. Similarly $e_{C_0^{i-1}} = e_{C^{i-1}} - e_{C_0^{i-2}}$. Thus $e_{C^i} = e_{C_0^i} + e_{C^{i-1}} - e_{C_0^{i-2}}$. Continuing in this way produces an expression $e_{C^n} = e_{C^{n-1}} - e_{C^{n-2}} + \ldots \pm e_{C^0} \mp e_{\mathbb{Z}}$. By Corollary 1.68, we have $e_{\mathbb{Z}} = 0$. Thus $\sum_{p=0}^{n}(-1)^p e_{C^p} = 0$, which means $\sum_{p=0}^{n}\sum_{q=0}^{d}(-1)^{p+q}\text{rk } H_q(\Gamma; C^p) \otimes \mathbb{Q} = 0$.

From this result, we have

$$\sum_{q=0}^{d}(-1)^{n+q}\text{rk } H_q(\Gamma; C^n) \otimes \mathbb{Q} \;=\; -\sum_{p=0}^{n-1}\sum_{q=0}^{d}(-1)^{p+q}\text{rk } H_q(\Gamma; C^p) \otimes \mathbb{Q}. \tag{2.1}$$

Since the Euler characteristic $e_{\mathcal{P}} = \sum_{q=0}^{d}(-1)^q \text{rk } H_q(\Gamma; C^n) \otimes \mathbb{Q}$ by Corollary 2.4, the left hand side of (2.1) gives $(-1)^n e_{\mathcal{P}}$. The right hand side (without the minus sign) is precisely $e_{\underline{Y}}$.

Hence $e_{\mathcal{P}} = (-1)^{n+1} e_{\underline{Y}}$ as required. ∎

Hence, when it can be shown to be exact, the sequence (1.2) in Lemma 1.59 is of use in the calculation of the Euler characteristic of a projection method pattern.

By the results of Chapter 1, there are three different cases to consider. In the case when L_0 is finite, by Lemma 1.54 the connected components in $\partial K + \Gamma$ are hyperplanes, so by Lemma 1.60 the sequence (1.2) is exact and, as shown in Section 2.4.1, a formula for the Euler characteristic of a polytopal projection pattern of any codimension can be determined in this case. In the second case, when L_0 is infinite and the connected components are hyperplanes, then the sequence (1.2) is again exact. However, it will be seen in Section 2.3 that the Euler characteristic cannot always be determined in this case. Lastly, in the case when connected components in $\partial K + \Gamma$ are not all hyperplanes, then exactness of the sequence (1.2) is not guaranteed. However, there are some results which can be deduced in this most general case, such as Theorems 2.28 and 2.60 ahead.

The following three sections consider projection patterns with low codimensions, for which calculations are easiest to visualise.

2.1 Codimension 1 projection patterns

This case was considered in full generality in [12, Chapter III]. We present the results here for completeness and note their applications to the Euler characteristic. Recall (Def 1.37) that L_0 denotes the number of Γ-orbits of singular 0-spaces in $\mathcal{K}^0$ and L_0 may be finite or infinite.

THEOREM 2.7 [12] *For a d-dimensional codimension* 1 *projection pattern* $\mathcal{P}$,

$$\check{H}^m(M\mathcal{P}) \cong \begin{cases} \mathbb{Z}^{\binom{d+1}{m}} & \text{for } 0 \leqslant m \leqslant d-1 \\ \mathbb{Z}^{L_0+d} & \text{for } m = d \end{cases}$$

and if $L_0 = \infty$ *then* $\mathbb{Z}^\infty$ *denotes the countably infinite direct sum of copies of* $\mathbb{Z}$.

Examples of calculations of the cohomology of codimension 1 projection patterns can be found in [22], for example.

COROLLARY 2.8 *For a d-dimensional codimension* 1 *projection pattern* $\mathcal{P}$, *the Euler characteristic is equal to* L_0.

Proof By the above theorem, $e_{\mathcal{P}} = d + L_0 + \sum_{m=0}^{d-1}(-1)^{d+m}\binom{d+1}{m}$ and since $\sum_{m=0}^{d+1}\binom{d+1}{m} = 0$ we have $e_{\mathcal{P}} = d + L_0 + \sum_{m=0}^{d+1}(-1)^{d+m}\binom{d+1}{m} - (d+1) + 1 = L_0$. ∎

COROLLARY 2.9 *For $\mathcal{P}$ a codimension 1 projection pattern, the Euler characteristic $e_{\mathcal{P}}$ is finite if and only if L_0 is finite.* ∎

COROLLARY 2.10 *For $\mathcal{P}$ a codimension 1 projection pattern, $e_{\mathcal{P}} \geqslant 1$.*

Proof There is at least one point in the boundary of a 1-dimensional polytope, so there is at least one Γ-orbit of singular 0-spaces. Hence $e_{\mathcal{P}} = L_0 \geqslant 1$. ∎

Finally, note that in the codimension 1 case, there is an immediate corollary of Lemma 1.59.

COROLLARY 2.11 *Associated to a codimension 1 projection pattern $\mathcal{P}$, there is an exact sequence $0 \to C^1 \to C^0 \to \mathbb{Z} \to 0$.* ∎

2.2 Codimension 2 projection patterns

In this section, the cohomology and Euler characteristics $e_{\mathcal{P}}$ for codimension 2 polytopal projection patterns $\mathcal{P}$ are investigated. For canonical projection patterns, this case was considered in detail in [12]. The results relevant to this work are stated below. We then consider the more general class of codimension 2 polytopal projection patterns and show that the complex (1.2) for these patterns is always exact. The two cases $L_0 < \infty$ and $L_0 = \infty$, for L_0 the number of orbit classes of singular 0-spaces are considered separately. A formula for the Euler characteristic is determined in Lemma 2.19 when L_0 is finite, which, when $\mathcal{P}$ is a canonical projection pattern, agrees with the formula from [12] given in the theorem below. We also show that the Euler characteristic is always defined for codimension 2 polytopal projection patterns. In the special case of a canonical projection pattern, the results obtained in this chapter yield an alternative proof of the theorem from [12, Chap IV,V] that the rational rank of the Čech cohomology group $\check{H}^d(M\mathcal{P})$ is finite if and only if

L_0 is finite. A stronger statement, that the Euler characteristic for a polytopal projection pattern is defined and is finite if and only if L_0 is finite, is also proved in Theorems 2.23 and 2.26. This section concludes by considering the other Čech cohomology groups of $M\mathcal{P}$ and investigating torsion in $\check{H}^*(M\mathcal{P})$.

THEOREM 2.12 [12, V.2.6] *For a d-dimensional codimension* 2 *canonical projection pattern* $\mathcal{P}$ *with* L_0 *finite,*

$$\begin{aligned} rk\ H_p(\Gamma; C^2) \otimes \mathbb{Q} &= \binom{d+2}{p+2} + L_1 \binom{\frac{d+2}{2}}{p+1} - r_{p+1} - r_p \ \text{for } p > 0 \\ rk\ H_0(\Gamma; C^2) \otimes \mathbb{Q} &= \binom{d+2}{2} - (d+2) + 1 + L_1\left(\frac{d+2}{2} - 1\right) + e_{\mathcal{P}} - r_1 \end{aligned}$$

where $L_1 = |I_1|$, *the number of* Γ*-orbits of singular* 1*-spaces,* $r_p = rk\ \langle \Lambda_{p+1}\Gamma^\xi : \xi \in I_1 \rangle$, *and the Euler characteristic is* $e_{\mathcal{P}} = -L_0 + \sum_{\xi \in I_1} L_0^\xi$, *where* L_0^ξ *is the number of* Γ*-orbits of singular* 1*-spaces contained in singular* 1*-spaces in the orbit* $\xi \in I_1$. ∎

THEOREM 2.13 [12, IV.2.9,V.2.4] *For a codimension* 2 *canonical projection pattern,* L_0 *is finite if and only if* $rk\ H_0(\Gamma; C^2) \otimes \mathbb{Q}$ *is finite.* ∎

EXAMPLE 2.14 *The Octagonal Tiling* (Example 1.23)

Consider again the Octagonal tiling. Its acceptance domain is the octagon in Figure 1.4. The Γ-orbits of the edges e_1, e_2, e_3, e_4 give rise to singular 0-spaces in three Γ-orbits — the vertices of the octagon, points at the intersection of e_1 and e_3 and points at the intersection of e_2 and e_4, as shown in the diagram below.

On e_1 there are singular 0-spaces in the first and second Γ-orbits described above. On e_2, singular 0-spaces are in the first and third Γ-orbits above. Similarly, there are two Γ-orbits of singular 0-spaces with representatives on e_3 and e_4. Hence for this pattern $\mathcal{P}$ we have $L_0 = 3$, and $L_0^\xi = 2$ for $\xi = e_i$, $i = 1, \ldots, 4$, and so the Euler characteristic is $e_{\mathcal{P}} = -3 + 4 \times 2 = 5$.

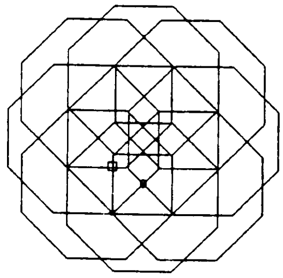


Figure 2.1: Translates of the Octagonal tiling acceptance domain

We also have $d = 2$, $L_1 = 4$, $r_p = 0$ for $p > 1$ and $r_1 = 3$ since $e_1 \wedge (e_2 - e_4) + e_2 \wedge (e_3 + e_1) + e_3 \wedge (e_2 + e_4) + e_4 \wedge (e_3 - e_1) = 0$. Thus the ranks of the cohomology groups for the Octagonal tiling are those stated below.

$$\begin{aligned} \text{rk } H_0(\Gamma; C^2) &= 9 \\ \text{rk } H_1(\Gamma; C^2) &= 5 \\ \text{rk } H_2(\Gamma; C^2) &= 1 \end{aligned}$$

Other examples of codimension 2 canonical projection patterns are considered in [16].

2.2.1 Polytopal projection patterns

We now consider codimension 2 polytopal projection patterns and first investigate the consequences of the results of Section 1.5 for this case.

LEMMA 2.15 *For modules* C^i (Def 1.57), *there is an exact sequence*

$$0 \to C^2 \xrightarrow{\delta_1} C^1 \xrightarrow{\delta_0} C^0 \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Proof By Lemma 1.59, the sequence is exact at C^2, C^0 and $\mathbb{Z}$ so to prove this result, it remains to show exactness at C^1.

Take a cycle z in C^1, which consists of compactly-supported integer-valued functions on a set of singular 1-spaces. Since C^1 is generated by indicator functions on singular 1-spaces,

z can be viewed as a collection of copies of singular 1-spaces. Now from the proof of Lemma 1.58, the module C^1 is a direct limit $\varinjlim C^1_R$ for R a Delone subset of Γ and so z can be considered to be a cycle in a locally finite CW decomposition of $V \cong \mathbb{R}^2$. Thus [27, §5] the cycle z has the form of a finite union of simple loops, each of which bounds a disc in $\mathbb{R}^2$ by the Jordan Curve theorem [27, §36] and hence is the boundary of an element of C^2. Hence any cycle in C^1 is the boundary of an element of C^2, so the sequence is exact at C^1, as required. ∎

Note also that the submodules $C^i_{\mathcal{D}}$ (Def 1.61) fit into an exact sequence, as described in the Lemma below.

LEMMA 2.16 *For $\mathcal{D} \in I_{1c}/\Gamma$, there is an exact sequence $0 \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0$.*

Proof By the remark at the end of Section 1.5, the sequence is exact at $C^1_{\mathcal{D}}$. Since representatives D of the Γ-orbit $\mathcal{D}$ are singular 1-spaces, any pair of singular 0-spaces in D is the set of end points of some singular 1-space in D, so $Ker(C^0_{\mathcal{D}} \to \mathbb{Z}) \subset Im(C^1_{\mathcal{D}} \to C^0_{\mathcal{D}})$. Also, $Im(\delta\colon C^1_{\mathcal{D}} \to C^0_{\mathcal{D}}) \subset Ker(\epsilon\colon C^0_{\mathcal{D}} \to \mathbb{Z})$ because $\delta(\sum n_i[U_{a_i b_i}]) = \sum n_i([a_i] - [b_i])$ and $\epsilon(\sum n_i([a_i] - [b_i])) = \sum n_i(1-1) = 0$. Thus the sequence is exact at $C^0_{\mathcal{D}}$. As in Lemma 1.59, the sequence is exact at $\mathbb{Z}$ because the map $C^0_{\mathcal{D}} \to \mathbb{Z}$ is a surjection. ∎

Finally, we state the results of Lemma 1.62 and Corollary 1.71 which are applicable in this situation.

LEMMA 2.17

$$C^1 \cong \bigoplus_{\mathcal{D} \in I_{1c}/\Gamma} C^1_{\mathcal{D}} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$$

∎

COROLLARY 2.18 *Group homology $H_m(\Gamma; C^1)$ splits as a direct sum of groups $\bigoplus_{\mathcal{D} \in I_{1c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$.* ∎

The Finite Case

Consider first the case when the number L_0 of Γ-orbits of singular 0-spaces is finite. By Lemma 1.53, we know that the stabilisers of the connected components in the Γ-orbits of faces of K are non-trivial, so the orbits contain the lines spanned by the faces and hence the singular 1-spaces $D \in I_{1c}$ are lines of infinite length. Note that in this case, the set I_{1c}/Γ of Γ-orbits $\mathcal{D}$ of singular 1-spaces D is equal to the set I_1 of Γ-orbits of 1-dimensional faces of the acceptance domain K since singular 0-spaces are dense in D and in finitely many Γ-orbits so the stabiliser $\Gamma^{\mathcal{D}}$ is dense in D. Thus the $\Gamma^{\mathcal{D}}$-orbit of one distinct (up to Γ-translation) face of K gives rise to D and hence the Γ-orbit of this face is the Γ-orbit $\mathcal{D}$ of D. We write ξ for a typical element of I_1.

LEMMA 2.19 *For a codimension* 2 *polytopal projection pattern* $\mathcal{P}$ *with* L_0 *finite,* $e_{\mathcal{P}} = -L_0 + \sum_{\xi \in I_1} L_0^{\xi}$, *where* L_0^{ξ} *is the number of distinct (up to* Γ*-translation) singular* 0*-spaces which are contained in any singular* 1*-space in the orbit* ξ.

Proof By Lemma 2.6, we compute $e_{\mathcal{P}} = -e_{\underline{Y}} = e_{C^1} - e_{C^0}$. Firstly, by the remark following Definition 1.57, C^0 is the free Γ-module with generators in one-to-one correspondence with the generators of $\mathcal{K}^0$, so $H_m(\Gamma; C^0) = 0$ for $m > 0$ and rk $H_0(\Gamma; C^0) \otimes \mathbb{Q} = L_0$, since L_0 is the number of Γ-orbits of singular 0-spaces in $\mathcal{K}^0$. Hence $e_{C^0} = L_0$.

To compute $e_{\mathcal{P}}$, it remains to determine e_{C^1}. Since in this case $I_{1c}/\Gamma = I_1$, we decompose C^1 as $\bigoplus_{\xi \in I_1} C^1_{\xi} \otimes \mathbb{Z}[\Gamma/\Gamma^{\xi}]$ as in Lemma 2.17 and consider C^1_{ξ} for $\xi \in I_1$. This module fits into an exact sequence $0 \to C^1_{\xi} \xrightarrow{\delta} C^0_{\xi} \to \mathbb{Z} \to 0$ by Lemma 2.16. Write $e_{C^i_{\xi}}$ to denote $\sum_{i=-\infty}^{\infty} (-1)^i \text{rk } H_i(\Gamma^{\xi}; C^i_{\xi}) \otimes \mathbb{Q}$ and set $e'_{\mathbb{Z}} = \sum_{i=-\infty}^{\infty} (-1)^i \text{rk } H_i(\Gamma^{\xi}; \mathbb{Z}) \otimes \mathbb{Q}$. We then have $e_{C^1_{\xi}} = e_{C^0_{\xi}} - e'_{\mathbb{Z}}$. As the stabilisers are non-trivial in this case, rk $H_m(\Gamma^{\xi}; \mathbb{Z}) \otimes \mathbb{Q} = \binom{rk\Gamma^{\xi}}{m}$ by Corollary 1.67 and hence $e'_{\mathbb{Z}} = 0$. Thus $e_{C^1_{\xi}} = e_{C^0_{\xi}}$ which is equal to the number L_0^{ξ} of Γ-orbits of singular 0-spaces contained in any singular 1-space in the orbit ξ. Note that L_0^{ξ} is well-defined since if there are distinct singular 0-spaces $\beta_1, \ldots, \beta_k$ in some singular 1-space

l, then the singular 0-spaces $\beta_1 + \gamma, \dots, \beta_k + \gamma$ are contained in $l + \gamma$ and if β is in $l + \gamma$ then $\beta - \gamma$ is in l.

Hence $e_{\mathcal{P}} = -e_{C^0} + e_{C^1} = -L_0 + \sum_{\xi \in I_1} L_0^{\xi}$, as required. ∎

When $\mathcal{P}$ is a canonical projection pattern, this clearly gives the same result for the Euler characteristic $e_{\mathcal{P}}$ as that given in [12]. We next provide an alternative description of the Euler characteristic in terms of *multiplicities* of singular 0-spaces.

DEFINITION 2.20 *The* multiplicity q_β *of a representative* β *of a* Γ*-orbit of singular* 0*-spaces is the number of distinct directions* $\theta \in I_K$ *of singular* 1*-spaces which intersect at* β*.*

The multiplicity is well-defined since if there were two representatives β, β' of an orbit of singular points with $\beta' = \beta + \gamma$, then taking $l + \gamma$ for all lines l intersecting at β produces lines which contribute to the multiplicity of β'. Similarly, those lines intersecting at β' give rise to lines contributing to the multiplicity of β. Hence $q_\beta = q_{\beta'}$.

THEOREM 2.21 *For a codimension* 2 *polytopal projection pattern* $\mathcal{P}$ *with* $L_0 < \infty$*, the Euler characteristic is* $e_{\mathcal{P}} = \sum_{\beta \in I_0}(q_\beta - 1)$*.*

Proof Given that $e_{\mathcal{P}} = -L_0 + \sum_{\xi \in I_1} L_0^{\xi}$, note that elements $\beta \in I_0$ contributing to L_0^{ξ} will be counted more than once as ξ varies. Each $\beta \in I_0$ will be counted q_β times in the sum over $\xi \in I_1$. Thus the sum can be rewritten as $-\sum_{\beta \in I_0} 1 + \sum_{\beta \in I_0} q_\beta = \sum_{\beta \in I_0}(q_\beta - 1)$ as required. ∎

The above results can be used to put bounds on the values that the Euler characteristic can take. First note the following points.

Any polytope has a vertex, and the orbit of this vertex under the action of Γ gives an element of I_0, so $L_0 \geqslant 1$. For t the number of distinct directions of faces of K, the multiplicity of a singular 0-space can be at most t (if all edges of K in distinct directions intersect at that point). We must also have $\sum_{\beta \in I_0} q_\beta \geqslant t$, since if this were not true then there would be faces in the boundary of K which did not contribute to the multiplicity of any singular point β, but the faces intersect at vertices of K by definition of K as a polytope.

LEMMA 2.22 *For a codimension 2 polytopal projection pattern $\mathcal{P}$ with acceptance domain K having t distinct directions of faces, the Euler characteristic $e_{\mathcal{P}}$ is bounded as follows.* $t-1 \leqslant e_{\mathcal{P}} \leqslant L_0(t-1)$

Proof To find an upper bound for $e_{\mathcal{P}}$, suppose every vertex has multiplicity t. Then $e_{\mathcal{P}} \leqslant L_0(t-1)$. For a lower bound, first suppose $L_0 = 1$, so $e_{\mathcal{P}} = q_\beta - 1$. Then using the property that $\sum_{\beta \in I_0} q_\beta \geqslant t$, we have $e_{\mathcal{P}} \geqslant t-1$ in this case. Now suppose $L_0 = 2$, so $e_{\mathcal{P}} = q_\beta + q_{\beta'} - 2$. As before, we know $q_\beta + q_{\beta'} \geqslant t$ but note that at least one singular 1-space must appear twice in $q_\beta + q_{\beta'}$. This is because if every singular 1-space arising from the faces of K contained singular 0-spaces in the Γ-orbits of either β or β' but not both, then K would not be edge-connected (Def 1.26) but K is a polytope so is edge-connected by Lemma 1.32. Hence $e_{\mathcal{P}} = q_\beta + q_{\beta'} - 2 \geqslant t+1-2 = t-1$. Similarly for $L_0 = 3$, two or more singular 1-spaces are counted at least twice each since K is edge-connected, again giving $e_{\mathcal{P}} \geqslant t-1$. Continuing in this way shows that for every extra singular 0-space β' arising, at least one singular 1-space containing some other singular 0-space β is counted in q_β and $q_{\beta'}$ and so $q_{\beta'} + \sum_{\beta \in I_0} q_\beta \geqslant t+1$. Hence $\sum_{\beta \in I_0}(q_\beta - 1) \geqslant t + (L_0 - 1) - L_0 = t-1$ as required. ∎

This bound implies the following results since in this document a polytope is assumed to have a finite number of faces, so $t < \infty$, but there must be at least $t = 2$ distinct directions in order for the polygon to be bounded in $V \cong \mathbb{R}^2$.

THEOREM 2.23 *For a codimension 2 polytopal projection pattern $\mathcal{P}$, if L_0 is finite, then the Euler characteristic $e_{\mathcal{P}}$ is finite.* ∎

COROLLARY 2.24 *For $\mathcal{P}$ a codimension 2 polytopal projection pattern, $e_{\mathcal{P}} \geqslant 1$.* ∎

Note that when $L_0 = 1$, the lower bound on $e_{\mathcal{P}}$ is attainable provided there is a pattern $\mathcal{P}$ with polytopal acceptance domain having faces in t distinct directions but with singular 0-spaces in one orbit class. Patterns with this property include the Penrose tiling [1], for which $t = 5$, and [16] the Heptagonal tiling of $\mathbb{R}^4$, which has $t = 7$.

Examples with $t = 2$ and hence $e_{\mathcal{P}} = 1$ also exist, since we can choose acceptance domains with $L_0 = 1$ and $t = 2$, giving $e_{\mathcal{P}} = t - 1 = 1$.

EXAMPLE 2.25 *The Rhombus tiling.*

This is a pattern with $t = 2$ which is polytopal and not canonical. It is formed by taking the data (Λ, E, K, v) which give rise to the Octagonal tiling (see Example 1.23), but instead of taking the acceptance domain K to be the projection of a four-dimensional hypercube, simply take the rhombus formed from the two non-orthogonal directions e_1 and e_2 in the projected hypercube of Figure 1.4. Since the positions of E and $E^{\perp}$ are the same as those for the Octagonal tiling, the rhombus acceptance domain is not the projection of any hypercube in $\mathbb{Z}^4$ to $E^{\perp}$ and hence the Rhombus tiling is not canonical.

Clearly $t = 2$ and, as we exclude the directions e_3 and e_4, only singular 0-spaces which are in the same Γ-orbit as the vertices of the Octagonal tiling's acceptance domain arise from Γ-translates of the above rhombus. Hence $L_0 = 1$ in this case. The tiling associated to this setup is pictured in Figure 2.2.

Similarly taking the setup (Λ, E, K, v) for the Penrose tiling but replacing K by the polygon formed by 3 or 4 of the distinct directions of faces of K yields tilings with Euler characteristic 2 or 3, respectively.

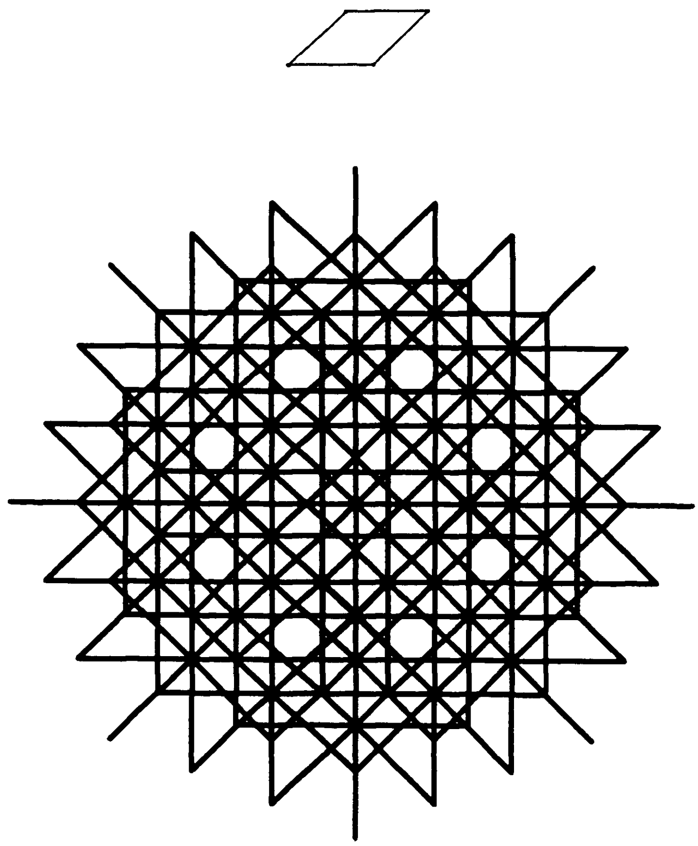


Figure 2.2: The Rhombus tiling and its acceptance domain

The Infinite Case

Now consider the case when L_0 is infinite. The converse of Theorem 2.23, stated in Theorem 2.26, is also true, and is proved below. The argument given covers both the case when the connected components D are lines of infinite length (with non-trivial stabiliser) and when the stabilisers of components D are trivial, since the exact sequence (2.15) exists in both cases. Note that if the connected components have trivial stabilisers then they may arise from parallel faces in distinct Γ-orbits whose translates have non-empty intersection. Connected components which are lines of infinite length may arise in the Γ-orbits of single faces of the acceptance domain K for the pattern under consideration but it is also possible that they arise only in the set of translates of all parallel faces of the acceptance domain, and the

Γ-orbits of individual faces are not connected. Hence I_{1c}/Γ need not be equal to I_1 in this case, so we use the notation $\mathcal{D} \in I_{1c}/\Gamma$ instead of $\xi \in I_1$ which was used in the L_0 finite case.

THEOREM 2.26 *For a codimension 2 projection pattern with polytopal acceptance domain such that L_0 is infinite, the Euler characteristic $e_{\mathcal{P}}$ is defined and is infinite.*

In order to prove this statement, and in particular demonstrate that $e_{\mathcal{P}}$ is defined, we make use of the following result. Note that the Euler characteristic is a rational invariant, so we need to compute the rank rk $(G \otimes \mathbb{Q})$ of various groups G as rational vector spaces. In the following work, we abbreviate rk $(G \otimes \mathbb{Q})$ by rk G.

THEOREM 2.27 *For a codimension 2 projection pattern with polytopal acceptance domain, rk $H_m(\Gamma; C^2)$ is finite for $m > 0$.*

Proof First note that the sequence $0 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ is exact by Lemma 2.15, so there is a spectral sequence $E^r_{pq} \Rightarrow H_{p+q}(\Gamma; \mathbb{Z})$ with $E^1_{pq} = H_p(\Gamma; C^q)$, as in Section 1.6.3. The E^1 terms are shown in the table below. Note that by Lemma 1.72 the groups $H_m(\Gamma; C^1)$ are non-zero only for $m < r = \max\{\text{rk } \Gamma^{\mathcal{D}} : \mathcal{D} \in I_{ic}/\Gamma\}$.

0	0	$H_d(\Gamma; C^2)$
0	0	$\vdots$
0	$H_{r-1}(\Gamma; C^1)$	$\xleftarrow{d_{2,r-1}}$ $H_{r-1}(\Gamma; C^2)$
$\vdots$	$\vdots$	$\vdots$
0	$H_1(\Gamma; C^1)$	$\xleftarrow{d_{21}}$ $H_1(\Gamma; C^2)$
$H_0(\Gamma; C^0)$	$\xleftarrow{d_{10}}$ $H_0(\Gamma; C^1)$	$\xleftarrow{d_{20}}$ $H_0(\Gamma; C^2)$

To prove the required result, we begin by showing that $H_m(\Gamma; C^1)$ is of finite rank for $m > 0$. By Lemma 2.17 and Corollary 2.18, there is a decomposition $C^1 \cong \bigoplus_{\mathcal{D} \in I_{1c}/\Gamma} C^1_{\mathcal{D}} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ and $H_m(\Gamma; C^1)$ splits as a direct sum of groups $H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$. By Lemma 2.16, there is also an exact sequence

$$0 \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0. \tag{2.2}$$

Applying the functor $H_*(\Gamma^{\mathcal{D}};-)$ to this short exact sequence gives the long exact sequence in homology below.

$$\cdots \longrightarrow H_2(\Gamma^{\mathcal{D}};\mathbb{Z}) \longrightarrow H_1(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}}) \longrightarrow H_1(\Gamma^{\mathcal{D}};C^0_{\mathcal{D}}) \longrightarrow H_1(\Gamma^{\mathcal{D}};\mathbb{Z})$$
$$\longrightarrow H_0(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}}) \longrightarrow H_0(\Gamma^{\mathcal{D}};C^0_{\mathcal{D}}) \longrightarrow H_0(\Gamma^{\mathcal{D}};\mathbb{Z}) \longrightarrow 0$$

Since $C^0_{\mathcal{D}}$ is a free Γ-module, $H_m(\Gamma;C^0_{\mathcal{D}}) = 0$ for $m > 0$ so there is an isomorphism $H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}}) \cong H_{m+1}(\Gamma^{\mathcal{D}};\mathbb{Z})$ for $m > 0$. Thus in particular rk $H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})$ is finite for positive m. Now $H_m(\Gamma;C^1) \cong \bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})$ and since K is a polytope, by the remark following Definition 1.36, the set I_{1c}/Γ is finite. Thus rk $H_m(\Gamma;C^1) =$ rk $[\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})]$ is finite for $m > 0$.

We now examine the rank of $H_m(\Gamma;C^2)$. The E^2 terms in the spectral sequence (which are the E^∞ terms since the second differentials $H_p(\Gamma;C^q) \to H_{p+1}(\Gamma;C^{q-2})$ are zero for $q \leqslant 2$) are shown in the table below.

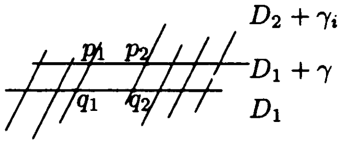
0	0	$H_d(\Gamma;C^2)$
0	0	$\vdots$
0	$H_{r-1}(\Gamma;C^1)/Imd_{2,r-1}$	$Kerd_{2,r-1}$
$\vdots$	$\vdots$	$\vdots$
0	$H_1(\Gamma;C^1)/Imd_{21}$	$Kerd_{21}$
$H_0(\Gamma;C^0)/Imd_{10}$	$Kerd_{10}/Imd_{20}$	$Kerd_{20}$

From this we can see that $H_m(\Gamma;C^2) \cong H_{m+2}(\Gamma;\mathbb{Z}) \cong \mathbb{Z}^{\binom{d+2}{m+2}}$ for $m \geqslant r$ since the edge homomorphisms are isomorphisms for these values of m. Also $Kerd_{2,r-1} \cong H_{r+1}(\Gamma;\mathbb{Z}) \cong \mathbb{Z}^{\binom{d+2}{r+1}}$. Thus these groups have finite rank over $\mathbb{Q}$. For $0 \leqslant m < r$, there are short exact sequences $0 \to H_{m+1}(\Gamma;C^1)/Imd_{2,m+1} \to H_{m+2}(\Gamma;\mathbb{Z}) \to Kerd_{2m} \to 0$, which give rk $(H_{m+1}(\Gamma;C^1)/Imd_{2,m+1}) +$ rk $Kerd_{2m} = \binom{d+2}{m+2}$. The finiteness of rk $H_p(\Gamma;C^1)$ implies finiteness of rk $Im(d_{2,p}\colon H_p(\Gamma;C^2) \to H_p(\Gamma;C^1))$ for $p > 0$, so rk $(H_{m+1}(\Gamma;C^1)/Imd_{2,m+1})$ is finite and hence the remaining term rk $Kerd_{2m}$ must be finite for $m > 0$. Therefore rk $H_m(\Gamma;C^2) =$ rk $Kerd_{2m} +$ rk Imd_{2m} is finite for $m > 0$. ∎

Proof of Theorem 2.26

It remains to determine rk $H_0(\Gamma; C^2)$ = rk $Kerd_{20}$ + rk Imd_{20}. Since the ranks of the terms on the diagonals of the above table sum to the rank of $H_k(\Gamma; \mathbb{Z})$, we have $H_0(\Gamma; C^0)/Imd_{10} \cong H_0(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$ so rk $Imd_{10} = L_0 - 1$, which is infinite since L_0 is infinite, and rk $Kerd_{10}$ − rk Imd_{20} = rk $H_1(\Gamma; \mathbb{Z})$. If we can show that rk $Kerd_{10}$ is infinite, then rk Imd_{20} must be infinite since rk $H_1(\Gamma; \mathbb{Z})$ is finite. Hence rk $H_0(\Gamma; C^2)$ = rk $Kerd_{20}$ + rk Imd_{20} will be infinite. Note that rk $Kerd_{20}$ is finite since rk $Kerd_{20}$ + rk $H_1(\Gamma; C^1)$ − rk Imd_{21} = rk $H_2(\Gamma; \mathbb{Z})$ and rk $H_1(\Gamma; C^1)$, rk Imd_{21} and $H_2(\Gamma; \mathbb{Z})$ are all finite quantities.

When L_0 is infinite, since there are only finitely many 1-dimensional faces of the acceptance domain K, and hence only finitely many Γ-orbits of singular 1-spaces $D \in I_{1c}$ there must be at least one orbit $\mathcal{D}_1$ whose representatives D_1 contain singular 0-spaces in infinitely many orbits. The finite number of edges in K also implies that there must be at least one D_2, transverse to D_1, with the property that infinitely many translates of D_2 intersect D_1 at singular 0-spaces which are in distinct Γ-orbits. Infinitely many of the translates of D_2 extend beyond at least one side of D_1 by some positive distance ϵ, since D_2 has positive length. Hence a translate $D_1 + \gamma$ can be found (by the density of Γ) which intersects infinitely many of the translates of D_2.



Consider the indicator function $f_{q_1 q_2}$ on the singular 1-space in D_1 which joins the singular 0-spaces q_1 and q_2 in the diagram above. Applying d_{10} to this function gives $f_{q_2} - f_{q_1}$, the difference of the indicator functions on the end points of the singular 1-space. Next consider the function $f = f_{q_1 p_1} + f_{p_1 p_2} + f_{p_2 q_2} \in C^1$, where p_i is the point of intersection of $D_1 + \gamma$ with the translate of D_2 passing through q_i. This function is distinct from $f_{q_1 q_2}$ since there is no translation of the line segment between q_1 and q_2 to the line segment between q_1 and p_1, for example, so $f \neq f_{q_1 q_2} + \gamma$ for $\gamma \in \Gamma$. Applying d_{10} to f gives $f_{q_2} - f_{q_1}$, the difference of indicator functions on the end points of the path q_1p_1, p_1p_2, p_2q_2 of singular 1-spaces. Thus we have found distinct elements of C^1 which map to the same element of C^0 under d_{10}, so the element $f_{q_1q_2} - f$ is in the kernel of d_{10}. Taking similar elements of C^1 corresponding to points q_1 and q_i for all points q_i in distinct Γ-orbits gives rise to infinitely many elements of $Kerd_{10}$. These elements are all distinct in $H_0(\Gamma; C^1) = C^1/Im(\delta\colon \Lambda_1\Gamma \otimes C^1 \to \Lambda_0\Gamma \otimes C^1)$ because of the fact that there is no element of Γ taking q_i to any other point q_j in D_1, so any two of the elements of the kernel constructed above do not differ by elements in $Im\delta$. Note that this infinite set of elements is also rationally independent. This is because if there were a finite set $\{l_i : i = 1, \ldots, m\}$ of elements such that any elements could be expressed as a linear combination $\sum_i a_i l_i$ for $a_i \in \mathbb{Q}$, then there would be a finite set $B = \{p_1, q_1, p_i, q_i : i = 2, \ldots, m\}$ of indicator functions on end points of the singular 1-spaces associated to each l_i. However, since L_0 is infinite, we can find an l with associated singular points p_1, q_1, p, q for p, q not in B which is not a linear combination of the elements l_i with coefficients in $\mathbb{Q}$ since the distance between q and any $q_i \in B$ is irrational.

Thus, we have shown that rk $Kerd_{10}$ is infinite, and hence Imd_{20} has infinite rank over $\mathbb{Q}$, which means that rk $H_0(\Gamma; C^2) = \infty$. However, rk $H_m(\Gamma; C^2)$ is finite for $m > 0$ by Theorem 2.27, so the Euler characteristic is defined and is infinite. ∎

Hence in this section, we have proved the following result.

THEOREM 2.28 *For a codimension* 2 *polytopal projection pattern* $\mathcal{P}$*, the Euler characteristic* $e_{\mathcal{P}}$ *is always defined, takes only positive values, and is finite if and only if* L_0 *is finite.* ∎

2.2.2 Cohomology formulae

For codimension 2 polytopal projection patterns, by Lemma 2.15 we have an exact sequence $0 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ in all cases. In this section, we use this sequence to determine formulae (analogous to those in Theorem 2.12) for the cohomology of the continuous hull for all polytopal projection patterns of codimension 2. These formulae agree with the previous results when the pattern under consideration is canonical.

THEOREM 2.29 *For a d-dimensional codimension 2 polytopal projection pattern $\mathcal{P}$, the torsion-free parts of the Čech cohomology groups of the continuous hull $M\mathcal{P}$ are given by*

$$\begin{aligned} rk\ \check{H}^d(M\mathcal{P}) \otimes \mathbb{Q} &= e_{\mathcal{P}} + \binom{d+2}{2} - (d+2) + 1 + \sum_{\mathcal{D} \in I_{1c}/\Gamma} (rk\Gamma^{\mathcal{D}} - 1) - r_1 \\ rk\ \check{H}^{d-m}(M\mathcal{P}) \otimes \mathbb{Q} &= \binom{d+2}{m+2} + \sum_{\mathcal{D} \in I_{1c}/\Gamma} \binom{rk\Gamma^{\mathcal{D}}}{m+1} - r_m - r_{m+1} \text{ for } 0 \leqslant m < d \end{aligned}$$

where $e_{\mathcal{P}}$ is the Euler characteristic and $r_m = rk\ \langle \Lambda_{m+1}\Gamma^{\mathcal{D}} : \mathcal{D} \in I_{1c}/\Gamma \rangle$.

The rank of $\check{H}^m(M\mathcal{P})$ is always finite for $m < d$ but $rk\ \check{H}^d(M\mathcal{P}) \otimes \mathbb{Q}$ is infinite if L_0, the number of Γ-orbits of singular 0-spaces, is infinite.

Proof Consider the exact sequence $0 \to C^2 \to C^1 \xrightarrow{\delta_0} C^0 \to \mathbb{Z} \to 0$. Writing $C^0_0 :\ = \delta_0(C^1)$ means that we can break the sequence into two short exact sequences, which is the technique used in [14].

$$0 \longrightarrow C^0_0 \longrightarrow C^0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow C^2 \longrightarrow C^1 \longrightarrow C^0_0 \longrightarrow 0$$

Note by Theorem 1.76 and Lemma 2.3 that $\check{H}^m(M\mathcal{P}) \cong H_{d-m}(\Gamma; C^2)$. Thus we apply the functor $H_*(\Gamma; -)$ to these sequences, to get long exact sequences of homology groups. The first sequence yields

$$\begin{aligned} \cdots \longrightarrow H_2(\Gamma;\mathbb{Z}) \longrightarrow H_1(\Gamma; C^0_0) \longrightarrow H_1(\Gamma; C^0) \longrightarrow H_1(\Gamma;\mathbb{Z}) \\ \longrightarrow H_0(\Gamma; C^0_0) \longrightarrow H_0(\Gamma; C^0) \longrightarrow H_0(\Gamma;\mathbb{Z}) \longrightarrow 0. \end{aligned}$$

Since C^0 is a free Γ-module, $H_m(\Gamma;C^0)=0$ for $m>0$ so $H_m(\Gamma;C_0^0)\cong H_{m+1}(\Gamma;\mathbb{Z})\cong \mathbb{Z}^{\binom{d+2}{m+1}}$. Also, rk $H_0(\Gamma;C^0)=L_0$, which means that rk $H_0(\Gamma;C_0^0)=(d+2)+L_0-1$.

Now $H_m(\Gamma;C^1)\cong\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma}H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})$ by Corollary 2.18. Applying the functor $H_*(\Gamma^{\mathcal{D}};-)$ to the short exact sequence $0\to C^1_{\mathcal{D}}\to C^0_{\mathcal{D}}\to\mathbb{Z}\to 0$ gives the long exact sequence in homology below.

$$\cdots\longrightarrow H_2(\Gamma^{\mathcal{D}};\mathbb{Z})\longrightarrow H_1(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})\longrightarrow H_1(\Gamma^{\mathcal{D}};C^0_{\mathcal{D}})\longrightarrow H_1(\Gamma^{\mathcal{D}};\mathbb{Z})$$
$$\longrightarrow H_0(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})\longrightarrow H_0(\Gamma^{\mathcal{D}};C^0_{\mathcal{D}})\longrightarrow H_0(\Gamma^{\mathcal{D}};\mathbb{Z})\longrightarrow 0 \qquad (2.3)$$

Since $C^0_{\mathcal{D}}$ is a free $\Gamma^{\mathcal{D}}$-module with rk $H_0(\Gamma^{\mathcal{D}};C^0_{\mathcal{D}})=L_0^{\mathcal{D}}$, for $L_0^{\mathcal{D}}$ the number of Γ-orbits of singular 0-spaces in representatives of $\mathcal{D}$, the above homology sequence yields rk $H_m(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})=$ rk $H_{m+1}(\Gamma^{\mathcal{D}};\mathbb{Z})=\binom{rk\Gamma^{\mathcal{D}}}{m+1}$ for $m>0$ and rk $H_0(\Gamma^{\mathcal{D}};C^1_{\mathcal{D}})=$ rk $\Gamma^{\mathcal{D}}+L_0^{\mathcal{D}}-1$. Hence

$$\text{rk } H_m(\Gamma;C^1)=\begin{cases}\sum_{\mathcal{D}\in I_{1c}/\Gamma}\binom{rk\Gamma^{\mathcal{D}}}{m+1} & \text{for } m>0\\ \sum_{\mathcal{D}\in I_{1c}/\Gamma}(\text{rk }\Gamma^{\mathcal{D}}+L_0^{\mathcal{D}}-1) & \text{for } m=0\end{cases}$$

Next examine the long exact sequence arising from the second sequence above.

$$\cdots\longrightarrow H_2(\Gamma;C_0^0)\longrightarrow H_1(\Gamma;C^2)\longrightarrow H_1(\Gamma;C^1)\xrightarrow{\beta_1}H_1(\Gamma;C_0^0)$$
$$\longrightarrow H_0(\Gamma;C^2)\longrightarrow H_0(\Gamma;C^1)\xrightarrow{\beta_0}H_0(\Gamma;C_0^0)\longrightarrow 0$$

This can be divided up into 5-term exact sequences

$$0\to Im\beta_{m+1}\to H_{m+1}(\Gamma;C_0^0)\to H_m(\Gamma;C^2)\to H_m(\Gamma;C^1)\to Im\beta_m\to 0.$$

Write r_m for the rank of the image of the map β_m, which is identified with $\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma}\Lambda_{m+1}\Gamma^{\mathcal{D}}\to\Lambda_{m+1}\Gamma$ for $m>0$, as in [15]. This gives rk $H_m(\Gamma;C^2)=\binom{d+2}{m+2}+\sum_{\mathcal{D}\in I_{1c}/\Gamma}\binom{rk\Gamma^{\mathcal{D}}}{m+1}-r_m-r_{m+1}$ for $m>0$ as required.

Lastly, consider $0\to Im\beta_1\to H_1(\Gamma;C_0^0)\to H_0(\Gamma;C^2)\to H_0(\Gamma;C^1)\to H_0(\Gamma;C_0^0)\to 0$. This yields

$$\text{rk } H_0(\Gamma;C^2) \;=\; \text{rk } H_0(\Gamma;C^1)-\text{rk } H_0(\Gamma;C^0)+1-(d+2)+\binom{d+2}{2}-r_1$$

If L_0 is finite, then all the above ranks are finite so the formulae are well-defined, so we can substitute $H_0(\Gamma; C^1)$ and $H_0(\Gamma; C^0)$ for the values obtained above to obtain

$$\begin{aligned}
\text{rk } H_0(\Gamma; C^2) &= \sum_{\mathcal{D}\in I_{1c}/\Gamma} (\text{rk } \Gamma^{\mathcal{D}} + L_0^{\mathcal{D}} - 1) - L_0 + 1 - (d+2) + \binom{d+2}{2} - r_1 \\
&= e_{\mathcal{P}} + \binom{d+2}{2} - (d+2) + 1 + \sum_{\mathcal{D}\in I_{1c}/\Gamma} (rk\Gamma^{\mathcal{D}} - 1) - r_1
\end{aligned}$$

as required, since by Lemma 2.19 the Euler characteristic is $e_{\mathcal{P}} = -L_0 + \sum_{\mathcal{D}\in I_{1c}/\Gamma} L_0^{\mathcal{D}}$ for $L_0^{\mathcal{D}}$ the number of Γ-orbits of singular 0-spaces in singular 1-spaces in the orbit $\mathcal{D}$. By Theorem 2.27, rk $\check{H}^m(M\mathcal{P}) \otimes \mathbb{Q}$ is finite for $m > 0$ so the Euler characteristic is always defined and by Theorem 2.26, if L_0 is infinite then $e_{\mathcal{P}}$ is infinite. Therefore $\check{H}^d(M\mathcal{P})$ is of infinite rank if L_0 is infinite and hence $e_{\mathcal{P}}$ is infinite. This is consistent with the formula given above. ∎

If L_0 is finite, then Lemma 1.56 gives rk $\Gamma^{\xi} = \frac{d+2}{2}$ for all $\xi \in I_1 = I_{1c}/\Gamma$, so the formula for rk $\check{H}^*(M\mathcal{P})$ reduces to that given in Theorem 2.12.

Torsion in $\check{H}^*(M\mathcal{P})$

Note that, as in [15], we can also investigate when torsion can arise in the cohomology of $M\mathcal{P}$. Given that $\check{H}^p(M\mathcal{P}) \cong H_{d-p}(\Gamma; C^2)$, we consider torsion in $H_m(\Gamma; C^2)$.

First consider the long exact sequence in homology (2.3) which is associated to the short exact sequence $0 \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0$. The fact that $C^0_{\mathcal{D}}$ is a free Γ-module implies that $H_m(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}})$ is torsion-free for all m and $H_m(\Gamma^{\mathcal{D}}; \mathbb{Z}) \cong \mathbb{Z}^{\binom{rk\Gamma^{\mathcal{D}}}{m}}$ is torsion-free for all m, so the same is true for $H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$. Hence $H_m(\Gamma; C^1) \cong \bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$ is torsion-free.

Next consider the long exact sequence in homology associated to the short exact sequence $0 \to C^2 \to C^1 \to C^0_0 \to 0$, described in the proof of the above theorem. Here we break the sequence in homology into short exact sequences $0 \to H_{m+1}(\Gamma; C^0_0)/Im\beta_{m+1} \to H_m(\Gamma; C^2) \to Ker\beta_m \to 0$ so $H_m(\Gamma; C^2) \cong Ker\beta_m \oplus (H_{m+1}(\Gamma; C^0_0)/Im\beta_{m+1})$. For $m > 0$

[15] we identify β_m with the homomorphism $\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} \Lambda_{m+1}\Gamma^{\mathcal{D}} \to \Lambda_{m+1}\Gamma$ and β_0 is identified with $\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma}(\Lambda_1\Gamma^{\mathcal{D}} \oplus Ker\epsilon^{\mathcal{D}}) \to \Lambda_1\Gamma \oplus Ker\epsilon$, for $\epsilon^{\mathcal{D}}: C^0_{\mathcal{D}} \to \mathbb{Z}$.

Since $Ker\beta_m$ is a subgroup of the free abelian group $H_m(\Gamma; C^1)$, and hence torsion-free, it remains to consider $H_{m+1}(\Gamma; C^0_0)/Im\beta_{m+1}$. Now $H_m(\Gamma; C^0_0)$ is of finite rank and torsion-free for $m > 0$ since it is isomorphic to the torsion-free group $H_{m+1}(\Gamma; \mathbb{Z})$, as shown in the proof of Theorem 2.29. This means that if $Im\beta_{m+1}$ is trivial then $H_m(\Gamma; C^2)$ is torsion-free.

The map β_{m+1} will be zero if $\Lambda_{m+2}\Gamma^{\mathcal{D}}$ is trivial for all $\mathcal{D} \in I_{ic}/\Gamma$, and this is the case if $m + 2 > \text{rk } \Gamma^{\mathcal{D}}$. By Lemma 1.49, for d-dimensional codimension 2 polytopal projection patterns the rank of the stabiliser $\Gamma^{\mathcal{D}}$ of a singular 1-space may have rank d, so the homology groups are always torsion-free for $m \geqslant d - 2$. Thus for a 2-dimensional codimension 2 polytopal projection pattern, torsion may appear in $H_0(\Gamma; C^2)$ only, for 4-dimensional patterns, torsion may appear in $H_0(\Gamma; C^2)$ and $H_1(\Gamma; C^2)$ and for 6-dimensional patterns, torsion may appear in $H_0(\Gamma; C^2)$ to $H_3(\Gamma; C^2) \cong \check{H}^3(M\mathcal{P})$. Compare this with the results in [15] for canonical projection patterns, which have stabilisers $\Gamma^{\mathcal{D}}$ of rank $\frac{d+2}{2}$. In this case, torsion can only arise in $H_m(\Gamma; C^2)$ for $m < d/2$. Thus for a 6-dimensional codimension 2 *canonical* projection pattern $H_3(\Gamma; C^2) \cong \check{H}^3(M\mathcal{P})$ is always torsion free.

Consequences

There are several corollaries of the above results.

COROLLARY 2.30 *For codimension* 2 *hypergeneric polytopal projection patterns* (Def 1.50), *rk* $H_m(\Gamma; C^2) = $ *rk* $H_{m+2}(\Gamma; \mathbb{Z})$ *for* $m > 0$ *and rk* $H_0(\Gamma; C^2) = \infty$.

Proof In this case, $L_0 = \infty$ so rk $H_0(\Gamma; C^2) = \infty$ and the ranks of the stabilisers of all singular 1-spaces are 1 so $r_m = 0$ and $H_m(\Gamma; C^1) = 0 = H_m(\Gamma; C^0)$ for all $m > 0$, which implies that the edge homomorphisms $H_{m+2}(\Gamma; \mathbb{Z}) \to H_m(\Gamma; C^2)$ in the spectral sequence (Page 48) are isomorphisms and hence rk $H_m(\Gamma; C^2) = \binom{d+2}{m+2} = $ rk $H_{m+2}(\Gamma; \mathbb{Z})$. ∎

COROLLARY 2.31 *For codimension* 2 *hypergeneric polytopal projection patterns the groups* $H_m(\Gamma; C^2)$ *are torsion-free for all* m.

Proof In this case, we have rk $\Gamma^{\mathcal{D}} = 1$ for all $\mathcal{D} \in I_{1c}/\Gamma$ so $\Lambda_{m+2}\Gamma^{\mathcal{D}}$ is trivial and hence $\beta_{m+1} : \bigoplus \Lambda_{m+2}\Gamma^{\mathcal{D}} \to \Lambda_{m+2}\Gamma$ is zero for all m, as required for $H_m(\Gamma; C^2)$ to be torsion-free by the results of the above section. ∎

COROLLARY 2.32 *For codimension* 2 *generic polytopal projection patterns* (Def 1.51), *we have rk* $H_m(\Gamma; C^2) =$ *rk* $H_{m+2}(\Gamma; \mathbb{Z})$ *for* $m > 0$ *and rk* $H_0(\Gamma; C^2)) = \infty$.

Proof Again $L_0 = \infty$ so rk $H_0(\Gamma; C^2) = \infty$ and since rk $\Gamma^{\mathcal{D}} = 0$ for all $\mathcal{D} \in I_{1c}/\Gamma$ we have $r_m = 0$ for all m and rk $H_m(\Gamma; C^2) = \binom{d+2}{m+2} =$ rk $H_{m+2}(\Gamma; \mathbb{Z})$ as required. ∎

As the stabilisers $\Gamma^{\mathcal{D}}$ are all trivial in this case, we have $\Lambda_{m+2}\Gamma^{\mathcal{D}}$ trivial for all m. Thus $H_m(\Gamma; C^2)$ is torsion-free for all m for generic polytopal projection patterns as well.

Note that it is shown in Section 2.4 that in fact for generic and hypergeneric polytopal projection patterns $H_m(\Gamma; C^2)$ is isomorphic to $H_{m+2}(\Gamma; \mathbb{Z})$ so these groups are not just of equal rank as rational vector spaces.

If the singular 1-spaces have stabilisers of different ranks, then L_0 will be infinite, so rk $H_0(\Gamma; C^2)$ will be infinite, and if the stabilisers of singular 1-spaces are all less than or equal to 1 then we will again have $H_m(\Gamma; C^2) \cong H_{m+2}(\Gamma; \mathbb{Z})$ for $m > 0$. If the ranks of some stabilisers are greater than 1 then the formula of Theorem 2.29 will give rk $H_m(\Gamma; C^2)$ for $m > 0$.

Note that for codimension 2 polytopal projection patterns which are not generic or hypergeneric, to compute the cohomology $H_m(\Gamma; C^2)$ we need to know the quantities r_m, whereas they do not appear in the Euler characteristic formula. Hence in general the Euler characteristic $e_{\mathcal{P}}$ is more straightforward to compute than the ranks of the Čech cohomology groups $\check{H}^*(M\mathcal{P})$ of the continuous hull $M\mathcal{P}$.

2.2.3 Examples

To show how the above ideas work in practice, we now consider some specific examples.

EXAMPLE 2.33 *The Rhombus tiling* (Example 2.25).

This is a tiling whose acceptance domain contains singular 0-spaces in a single Γ-orbit and gives rise to two distinct orbits of singular 1-spaces, each with stabiliser of rank 2. By the earlier calculations, we have $L_0 = 1$, $L_1 = 2$ (since $I_1 = \{e_1, e_2\}$ for e_1 and e_2 as in Example 2.25) and $e_{\mathcal{P}} = 1$. Lemma 1.56 gives that the ranks of the stabilisers of the Γ-orbits of e_1 and e_2 are 2 and in fact Γ^{e_1} is generated by e_1 and $e_2 - e_4$ and Γ^{e_2} is generated by e_2 and $e_1 + e_3$. This means that $r_1 = \mathrm{rk}\,\langle \Lambda_2\Gamma^{\xi} : \xi \in I_1 \rangle = 2$ since $\Lambda_2\Gamma^{e_1}$ is of rank 1, generated by $e_1 \wedge (e_2 - e_4)$, and $\Lambda_2\Gamma^{e_2}$ is generated by $e_2 \wedge (e_3 + e_1)$ and these two vectors are rationally independent. Thus for this tiling

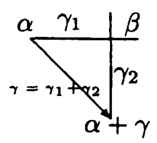
$$\begin{array}{rclcl}
\mathrm{rk}\,\check{H}^2(M\mathcal{P}) & = & e + 3 + L_1 - r_1 & = & 4 \\
\mathrm{rk}\,\check{H}^1(M\mathcal{P}) & = & 4 + L_1 - r_1 & = & 4 \\
\mathrm{rk}\,\check{H}^0(M\mathcal{P}) & = & 1 & = & 1.
\end{array}$$

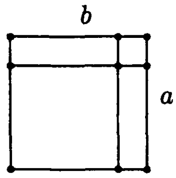
Compare this with the cohomology calculations for the Octagonal tiling in Example 2.14.

EXAMPLE 2.34 *Degenerate Octagonal tiling*

Consider again the setup for the Octagonal tiling (Example 1.23). Altering the position of E, and hence that of $E^{\perp}$, the unit hypercube can be projected to $E^{\perp}$ so that e_1 to e_4 are arranged into two pairs of parallel vectors with rationally independent lengths, as shown in Figure 2.3 below.

The singular 0-spaces marked on the acceptance domain K in Figure 2.3, and any others arising from translates of K are all in the same Γ-orbit.





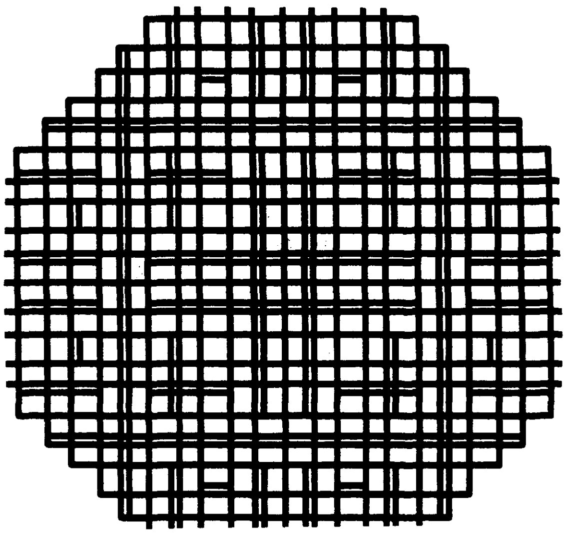


Figure 2.3: A degenerate canonical projection pattern and its acceptance domain

This is because any $\gamma \in \Gamma$ is a linear combination of elements in the stabilisers of the two singular 1-spaces a and b and so any point β at the intersection of translates of a and b is in the same Γ-orbit as the end points of a and b, which are lattice points (as shown in the diagram above). Thus $L_0 = 1$ and the Euler characteristic for patterns of this form also has the value $e_{\mathcal{P}} = 2 - 1 = 1$.

This tiling is a two-dimensional example of a class of patterns discussed in [21] and described briefly below; namely it is the product of two 1-dimensional codimension 1 canonical projection patterns with acceptance domains consisting of the non-parallel edges of the acceptance domain in Figure 2.3.

EXAMPLE 2.35 *Cartesian products of 1-dimensional tilings.*

Given a set of d tilings $\mathcal{P}_i$ of $\mathbb{R}$, form the d-dimensional tiling $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_d$. Tiles in this tiling are d-dimensional rhombs formed from the Cartesian product of intervals of

$\mathbb{R}$. The continuous hull of this tiling [21] can be expressed in terms of the hulls of the 1-dimensional tilings as $M\mathcal{P} = M\mathcal{P}_1 \times \ldots \times M\mathcal{P}_d$, with the product topology.

Using Definition 2.1 and Theorem 1.76, we have the following equivalent formulation of the Euler characteristic.

COROLLARY 2.36 *The Euler characteristic for a d-dimensional pattern $\mathcal{P}$ is*

$$e_{\mathcal{P}} :\ = rk\ K_0(C(M\mathcal{P}) \rtimes \mathbb{R}^d) \otimes \mathbb{Q} - rk\ K_1(C(M\mathcal{P}) \rtimes \mathbb{R}^d) \otimes \mathbb{Q}$$

when this makes sense. ∎

For a product $\mathcal{P}$ of two 1-dimensional tilings $\mathcal{P}_1$, $\mathcal{P}_2$, it is shown in [21] that the K-theory of the C^*-algebra $C(M\mathcal{P}) \rtimes \mathbb{R}^2$ decomposes as $K_0(C(M\mathcal{P}) \rtimes \mathbb{R}^2) \cong K_0(C(M\mathcal{P}_1) \rtimes \mathbb{R}) \otimes K_0(C(M\mathcal{P}_2) \rtimes \mathbb{R}) \oplus K_1(C(M\mathcal{P}_1) \rtimes \mathbb{R}) \otimes K_1(C(M\mathcal{P}_2) \rtimes \mathbb{R})$ and $K_1(C(M\mathcal{P}) \rtimes \mathbb{R}^2) \cong K_0(C(M\mathcal{P}_1) \rtimes \mathbb{R}) \otimes K_1(C(M\mathcal{P}_2) \rtimes \mathbb{R}) \oplus K_1(C(M\mathcal{P}_1) \rtimes \mathbb{R}) \otimes K_0(C(M\mathcal{P}_2) \rtimes \mathbb{R})$. There are similar (but more long-winded) formulae for products of three or more 1-dimensional tilings. Thus, if all quantities are finite, the Euler characteristic $e_{\mathcal{P}}$ is equal to $e_{\mathcal{P}_1} \times e_{\mathcal{P}_2}$, the product of the Euler characteristics of the 1-dimensional tilings. Given the formula for the K-theory of a d-dimensional product tiling, it can also be shown that $e_{\mathcal{P}} = \prod_{i=1}^{d} e_{\mathcal{P}_i}$ in general.

Summary

This section has provided a generalisation of Theorem 2.12 ([12, V.2.6]) to polytopal projection patterns and has also determined where torsion may arise, in analogy with [15]. For codimension 2 projection patterns, we have also given an alternative proof of the result from [12] that in the canonical case $\check{H}^d(M\mathcal{P})$ is of finite rank over $\mathbb{Q}$ if and only if L_0 is finite. In fact, we proved the stronger result that the Euler characteristic is defined and finite for codimension 2 polytopal projection patterns if and only if L_0 is finite. Formulae for the computation of the Euler characteristic of polytopal projection patterns were also derived and several examples were considered.

2.3 The codimension 3 case

In [15, Thm 5.1], formulae for the cohomology groups $\check{H}^*(M\mathcal{P})$ of a codimension 3 *canonical* projection pattern $\mathcal{P}$ with L_0 finite were given, and an expression for the Euler characteristic in this case was provided in [12, V.2.7]. The main results are stated below. In this section, we also obtain an expression for the Euler characteristic $e_{\mathcal{P}}$ for codimension 3 *polytopal* projection patterns $\mathcal{P}$ with L_0 finite, before considering $e_{\mathcal{P}}$ for patterns $\mathcal{P}$ with $L_0 = \infty$.

THEOREM 2.37 [15, Thm 5.1] *For a d-dimensional codimension* 3 *canonical projection pattern $\mathcal{P}$ with L_0 finite,*

$$\begin{aligned}
rk\ \check{H}^{d-p}(M\mathcal{P})\otimes\mathbb{Q} &= \binom{d+3}{p+3} + L_2\binom{2\frac{d+3}{3}}{p+2} + \sum_{\eta\in I_2} L_1^\eta\binom{\frac{d+3}{3}}{p+1} \\
&\quad + L_1\binom{\frac{d+3}{3}}{p+2} - R_p - R_{p+1} \text{ for } p>0 \\
rk\ \check{H}^{d}(M\mathcal{P})\otimes\mathbb{Q} &= \sum_{j=0}^{3}(-1)^j\binom{d+3}{3-j} + L_2\sum_{j=0}^{2}(-1)^j\binom{2\frac{d+3}{3}}{2-j} + \sum_{\eta\in I_2} L_1^\eta\sum_{j=0}^{1}(-1)^j\binom{\frac{d+3}{3}}{1-j} \\
&\quad + L_1\sum_{j=0}^{2}(-1)^j\binom{\frac{d+3}{3}}{2-j} + e_{\mathcal{P}} - R_1
\end{aligned}$$

where I_1^η is the set of Γ-orbits of singular 1*-spaces ξ contained in η a singular* 2*-space, $L_1^\eta = |I_1^\eta|$ and $R_p = rk\ \langle\Lambda_{p+2}\Gamma^\eta : \eta\in I_2\rangle + \sum_{\eta\in I_2} rk\ \langle\Lambda_{p-1}\Gamma^\xi : \xi\in I_1^\eta\rangle + rk\ \langle(\bigoplus_{\xi\in I_1^\eta}\Lambda_{p+1}\Gamma^\xi)\cap(Ker\beta_p^\eta\colon \bigoplus_{\xi\in I_1^\eta}\Lambda_{p+1}\Gamma^\xi\to\Lambda_{p+1}\Gamma) : \eta\in I_2\rangle$. The Euler characteristic is given by $e_{\mathcal{P}} = L_0 - \sum_{\eta\in I_2} L_0^\eta + \sum_{\eta\in I_2}\sum_{\xi\in I_1^\eta} L_0^\xi - \sum_{\xi\in I_1} L_0^\xi$.* ∎

THEOREM 2.38 [12, Thm IV.2.9,Thm V.2.4] *The quantity L_0 is finite if and only if $rk\ H_0(\Gamma; C^3)\otimes\mathbb{Q}$ is finite.* ∎

Several examples of cohomology calculations for codimension 3 canonical projection patterns are provided in [15].

2.3.1 Polytopal projection patterns

We now turn to polytopal projection patterns of codimension 3. To simplify calculations, we consider separately the cases introduced at the end of Section 1.4.

Case 1: L_0 finite

When $L_0 < \infty$, we know by Lemma 1.54 that the Γ-orbits of the 2-dimensional faces of K contain the planes spanned by the faces and the orbits of singular 1-spaces consist of lines of infinite length. Also, the set I_{ic}/Γ of Γ-orbits of connected components in the orbits of singular i-spaces is equal to the set I_i (Def 1.36) of Γ-orbits of faces of K in this case since by Lemma 1.39 singular 0-spaces are dense in a connected component D and as there are only finitely many Γ-orbits of singular 0-spaces, the stabiliser of D is dense in D. Hence the orbit of a single face of K gives rise to D. By Lemma 1.60, the sequence $0 \to C^3 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ of Γ-modules is exact. Using this sequence, and group homology $H_*(\Gamma; C^i)$ as defined in Section 1.6.3, a formula for the Euler characteristic can be determined in this case, as follows. Note that I_i^η denotes the set enumerating distinct orbit classes of singular i-spaces contained in the singular 2-space $\eta \in I_2$.

THEOREM 2.39 *For a codimension 3 polytopal projection pattern $\mathcal{P}$ with L_0 finite, the Euler characteristic is given by*

$$e_{\mathcal{P}} = L_0 - \sum_{\xi \in I_1} L_0^\xi + \sum_{\eta \in I_2} \sum_{\xi \in I_1^\eta} L_0^\xi - \sum_{\eta \in I_2} L_0^\eta.$$

Proof By Lemma 2.6, we need to compute $e_{\underline{Y}} = e_{C^2} - e_{C^1} + e_{C^0}$. By Lemma 2.19 we have $e_{C^0} = L_0$ and $e_{C^1} = \sum_{\xi \in I_1} L_0^\xi$ since C^1 can again be decomposed as $\bigoplus_{\xi \in I_1} C_\xi^1 \otimes \mathbb{Z}[\Gamma/\Gamma^\xi]$. It thus remains to compute e_{C^2}. There is a decomposition of C^2 by Lemma 1.62 as $C^2 = \bigoplus_{\eta \in I_2} C_\eta^2 \otimes \mathbb{Z}[\Gamma/\Gamma^\eta]$. For each Γ-orbit of singular 2-spaces $\eta \in I_2$, there is a sequence $0 \to C_\eta^2 \to C_\eta^1 \to C_\eta^0 \to \mathbb{Z} \to 0$, where C_η^i is the module generated as in Definition 1.57 but only by singular i-spaces contained in singular 2-spaces in the Γ-orbit $\eta \in I_2$. This sequence is exact by Lemma 2.15 since we consider singular 0- and 1-spaces in a plane $D \cong \mathbb{R}^2$ in the orbit η. Exactness of the sequence implies $e_{C_\eta^2} = e_{C_\eta^1} - e_{C_\eta^0} + e'_{\mathbb{Z}}$, using the notation of Definition 2.5 and Lemma 2.19. Since the orbits $\eta \in I_2$ are sets of infinite planes, the stabilisers Γ^η are non-trivial by Lemma 1.53 and hence Corollary 1.68 gives

$e'_{\mathbb{Z}} = \sum(-1)^j H_j(\Gamma^\eta;\mathbb{Z}) = 0$. Thus, by Lemma 2.19, we have $e_{C^2_\eta} = -L_0^\eta + \sum_{\xi\in I_1^\eta} L_0^\xi$, for I_1^η as above. Hence $e_{C^2} = \sum_{\eta\in I_2}(L_0^\eta - \sum_{\xi\in I_1^\eta} L_0^\xi)$ and so $e_{C^3} = L_0 - \sum_{\xi\in I_1} L_0^\xi + \sum_{\eta\in I_2}\sum_{\xi\in I_1^\eta} L_0^\xi - \sum_{\eta\in I_2} L_0^\eta$. ∎

As in the codimension 2 case, there is an alternative expression of this formula in terms of the number of distinct directions of singular 1-spaces intersecting at each singular 0-space.

DEFINITION 2.40 *Define the* multiplicity q_β *of a singular* 0*-space* β *to be the number of distinct directions of singular* 1*-spaces in the set* $\mathcal{K}^1$ *intersecting at* β*. Write* q_β^η *for the number of such singular* 1*-spaces intersecting at* β *but lying in any plane in the* Γ*-orbit* $\eta \in I_2$*.*

As in the remark following Definition 2.20, these quantities are well-defined for any singular 0-space in the same Γ-orbit as β.

LEMMA 2.41 *For a codimension* 3 *polytopal projection pattern* $\mathcal{P}$ *with* L_0 *finite*

$$e_{\mathcal{P}} = \sum_{\eta\in I_2}\sum_{\beta\in I_0^\eta}(q_\beta^\eta - 1) - \sum_{\beta\in I_0}(q_\beta - 1).$$

Proof As in Lemma 2.21, we can write $-L_0 + \sum_{\eta\in I_2} L_0^\eta = \sum_{\beta\in I_0}(q_\beta - 1)$ and similarly $-L_0^\eta + \sum_{\xi\in I_1^\eta} L_0^\xi = \sum_{\beta\in I_0^\eta}(q_\beta^\eta - 1)$ so $\sum_{\eta\in I_2}(L_0^\eta - \sum_{\xi\in I_1^\eta} L_0^\xi) = \sum_{\eta\in I_2}\sum_{\beta\in I_0^\eta}(q_\beta^\eta - 1)$. Substituting these quantities into the formula of Theorem 2.39 gives the result. ∎

Of most use later in this section is a formula for the Euler characteristic in which the calculation is carried out by taking each singular 0-space $\beta \in I_0$ in turn and examining the singular 1- and 2-spaces which intersect at β. With the aim of producing such a formula, we first define some notation.

DEFINITION 2.42 *Write* p_β *to denote the number of singular* 2*-spaces* $\eta \in I_2$ *with distinct normal vectors intersecting to form a given singular* 0*-space* β*. Write* I_β *for the set enumerating all distinct directions of singular* 1*-spaces intersecting at* β*, so* $|I_\beta| = q_\beta$*. Finally, write* q_l^β *for the number of distinct singular* 2*-spaces intersecting to form the singular* 1*-space* $l \in I_\beta$*.*

Note that p_β is well-defined since if we take two singular 0-spaces β_1, β_2 in the same Γ-orbit, so $\beta_2 = \beta_1 + \gamma$ then any singular 2-space D which passes through β_1 and hence contributes to p_{β_1} will be such that $D + \gamma$ contributes to p_{β_2}. Similarly any D passing through β_2 has the property that $D - \gamma$ passes through β_1. Hence $p_{\beta_1} = p_{\beta_2}$. Also note that $L_0 < \infty$ implies that the Γ-orbits of singular 1- and 2-spaces are lines and planes respectively, so if a singular 2-space D contains a singular 1-space l, and hence contributes to q_l^β, then $D + \gamma$ will pass through another line $l' = l + \gamma$ in the same orbit as l, so $q_l^\beta = q_{l'}^\beta$ which means that q_l^β is well-defined.

THEOREM 2.43 *For a codimension* 3 *polytopal projection pattern with* L_0 *finite,*

$$e_{\mathcal{P}} = \sum_{\beta \in I_0} \left[-(p_\beta - 1) + \sum_{l \in I_\beta} (q_l^\beta - 1) \right].$$

Proof From the formula in Lemma 2.41, we have

$$e_{\mathcal{P}} = \sum_{\beta \in I_0} 1 - \sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} 1 + \sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} q_\beta^\eta - \sum_{\beta \in I_0} q_\beta = L_0 - \sum_{\eta \in I_2} L_0^\eta + \sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} q_\beta^\eta - \sum_{\beta \in I_0} q_\beta.$$

Now in the term $\sum_{\eta \in I_2} L_0^\eta$ singular 0-spaces β are counted p_β times in the sum over all $\eta \in I_2$ since if β has multiplicity p_β then by definition it lies in p_β non-parallel planes Π, each in some orbit η. Thus $\sum_{\eta \in I_2} L_0^\eta = \sum_{\beta \in I_0} p_\beta$. Hence $-L_0 + \sum_{\eta \in I_2} L_0^\eta = -\sum_{\beta \in I_0} 1 + \sum_{\eta \in I_2} L_0^\eta = \sum_{\beta \in I_0} (p_\beta - 1)$.

Next consider $\sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} q_\beta^\eta$. The quantity q_β^η counts the number of lines passing through β but which lie in a given $\eta \in I_2$. Thus in the sum over all η, each line l passing through $\beta \in I_0$ will be counted q_l^β times. Hence $\sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} q_\beta^\eta = \sum_{\beta \in I_0} \sum_{l \in I_\beta} q_l^\beta$. For any given $\beta \in I_0$, singular 1-spaces passing through β are counted exactly once by q_β, so $\sum_{\beta \in I_0} q_\beta = \sum_{\beta \in I_0} \sum_{l \in I_\beta} 1$. Thus $\sum_{\eta \in I_2} \sum_{\beta \in I_0^\eta} q_\beta^\eta - \sum_{\beta \in I_0} q_\beta = \sum_{\beta \in I_0} \sum_{l \in I_\beta} (q_l^\beta - 1)$. Combining these results gives $e_{\mathcal{P}} = \sum_{\beta \in I_0} \left[-(p_\beta - 1) + \sum_{l \in I_\beta} (q_l^\beta - 1) \right]$ as required. ∎

As for polytopal projection patterns of codimension 2, we can consider bounds on the Euler characteristic in terms of the number t of 2-dimensional faces of the acceptance domain K with distinct normal vectors.

LEMMA 2.44 *For a codimension 3 polytopal projection pattern $\mathcal{P}$ with L_0 finite and acceptance domain K having t faces with distinct normal vectors, the Euler characteristic is bounded and $L_0 \leqslant e_{\mathcal{P}} \leqslant \frac{L_0}{2}[t^3 - 2t^2 + t - 4]$.*

Proof Given the formula in the above theorem, note that for a given $\beta \in I_0$, we have $p_\beta \leqslant \sum_{l \in I_\beta}(q_l^\beta - 1)$. This is because firstly $q_l^\beta \geqslant 2$, as any singular 1-space lies at the intersection of two or more faces of K, so $\sum_{l \in I_\beta}(q_l^\beta - 1) \geqslant \sum_{l \in I_\beta} 1 = q_\beta$. Secondly, if p_β faces intersect at a singular point β, then the number of distinct lines which intersect at β (the multiplicity q_β) is at least p_β. To see this, note that each face of K could intersect only two others, or $p_\beta - 1$ faces could intersect in a single line and the remaining face, which must be transverse to the others as K is a polytope, intersects all the other faces, creating $1 + p_\beta - 1 = p_\beta$ lines through β. However, if neither of these cases occur, then $q_\beta > p_\beta$ since, for example, one plane D could intersect three others, D_1, D_2, D_3, producing three lines in D plus the three or more lines arising at the intersection of D_1, D_2 and D_3 with one another.

Hence $\sum_{l \in I_\beta}(q_l^\beta - 1) - (p_\beta - 1) \geqslant 1$ so $e_{\mathcal{P}} = \sum_{\beta \in I_0}\left[-(p_\beta - 1) + \sum_{l \in I_\beta}(q_l^\beta - 1)\right] \geqslant \sum_{\beta \in I_0} 1 = L_0$.

To determine the upper bound, note the following points. Firstly, $q_l \leqslant t - 1$ for any $l \in I_\beta$, since at most $t - 1$ distinct faces of K can intersect in any one line, as there must be at least one face transverse to l to ensure that K is bounded. Also, the number $q_\beta = |I_\beta|$ of lines through β satisfies $|I_\beta| \leqslant \binom{t}{2}$ since the greatest number of lines through β is produced when all faces of K intersect at β and each pair of faces intersects in a distinct line. Lastly, $p_\beta \geqslant 3$ since singular points β arise at the intersection of three or more planes in $V \cong \mathbb{R}^3$.

These results give $e_{\mathcal{P}} \leqslant L_0[-2 + \binom{t}{2}(t-1-1)] = \frac{L_0}{2}[t(t-1)(t-2) - 4] = \frac{L_0}{2}[t^3 - 3t^2 + 2t - 4]$. Note that since $t \geqslant 3$, we have $\frac{L_0}{2}[t^3 - 3t^2 + 2t - 4] \geqslant L_0$. ∎

This result shows that for codimension 3 polytopal projection patterns with L_0 finite, $e_{\mathcal{P}} \geqslant 1$. This is because the acceptance domain K is a polytope and so has at least one vertex, which gives rise to a Γ-orbit of singular 0-spaces, so $L_0 \geqslant 1$. Thus $e_{\mathcal{P}} \geqslant L_0 \geqslant 1$, as required.

The lower bound $e_{\mathcal{P}} = 1$ is attainable for some codimension 3 polytopal projection pattern, since we can take a pattern consisting of the product of three 1-dimensional codimension 1 patterns (as described in Example 2.35), each with Euler characteristic $e = 1$ to give a 3-dimensional codimension 3 pattern $\mathcal{P}$ with $e_{\mathcal{P}} = 1$.

As t is assumed to be finite by definition of a polytope, there is another corollary of the above result.

COROLLARY 2.45 *For a codimension 3 polytopal projection pattern $\mathcal{P}$, if L_0 is finite, then the Euler characteristic $e_{\mathcal{P}}$ is defined and is finite.* ∎

Unlike for codimension 2 polytopal projection patterns, we will see later in this section that if L_0 is infinite then it is not always the case that the Euler characteristic is defined and infinite, since for some codimension 3 polytopal projection patterns the Euler characteristic is not defined. However, note that for a codimension 3 polytopal projection pattern with L_0 finite, so the Euler characteristic $e_{\mathcal{P}}$ is defined, it is more straightforward to compute this invariant than the Čech cohomology groups $H^*(M\mathcal{P})$ for $\mathcal{P}$. This is because the quantities R_p defined in Theorem 2.37 arising from ranks of maps in the spectral sequence (Section 1.6.3) are [12, V.6] hard to compute in general. However, they are not required for the Euler characteristic calculations.

The Euler characteristic has a particularly neat formulation in the case when there is one Γ-orbit of singular 0-spaces and exactly two singular 2-spaces intersect at any singular 1-space.

LEMMA 2.46 *For a codimension 3 polytopal projection pattern $\mathcal{P}$ with $L_0 = 1$ and such that exactly two singular 2-spaces intersect at any singular 1-space, $e_{\mathcal{P}} = \frac{1}{2}(t-1)(t-2)$.*

Proof In this case, $q_\beta = \binom{t}{2}$ since a line through β is produced at the intersection of any pair of singular 2-spaces. Also, for each η, in this situation we have $q_\beta^\eta = (t-1)$ since the $t-1$ planes transverse to a plane D in the orbit η intersect D in distinct lines through β. Substituting these values into the formula in Lemma 2.41 gives $e_{\mathcal{P}} = \sum_{\eta \in I_2}[(t-1)-1] - \binom{t}{2} + 1] = t(t-1) - t - \frac{t(t-1)}{2} + 1 = \frac{t(t-1)}{2} - 2t + 2 = \frac{1}{2}(t^2 - 3t + 2) = \frac{1}{2}(t-1)(t-2)$ ∎

An example of a pattern satisfying these conditions is the Danzer tiling, described in [14]. This pattern is canonical, $L_0 = 1$ and its acceptance domain has 6 faces in distinct orbit classes, each pair of which has 1-dimensional intersection, so its Euler characteristic is $\frac{1}{2}(5 \times 4) = 10$.

Case 2: L_0 infinite but orbits of faces of K contain planes

Unlike codimension 3 polytopal projection patterns $\mathcal{P}$ with a finite number of Γ-orbits of singular 0-spaces, for which the Γ-orbits of the faces of the acceptance K for $\mathcal{P}$ automatically have the form of planes by Lemma 1.54, when L_0 is infinite and singular 2-spaces $D \in I_{2c}$ are planes, there are two possibilities. Firstly, the Γ-orbit of any face of K may consist of planes. Secondly, the Γ-orbit of a single 2-dimensional face of the acceptance domain K may not consist of planes, but in the set of Γ-orbits of all faces of K the connected components D are planes. Recall (Def 1.52) that we refer to patterns $\mathcal{P}$ in this case as hyperplane polytopal projection patterns. By Lemma 1.60 the sequence $0 \to C^3 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ is exact for hyperplane polytopal projection patterns and so a spectral sequence can be set up as in Section 1.6.3. However, as we shall see in Theorem 2.50 below, even in the simplest case of a three-dimensional codimension 3 hyperplane polytopal projection pattern, the Euler characteristic (Def 2.1) is not always defined. There are cases in which the Euler characteristic of such a codimension 3 pattern is defined, namely under the conditions of Theorem 2.53 or Corollaries 2.55 and 2.56. To state and prove these results, we show in Lemma 2.49 that for codimension 3 hyperplane polytopal patterns, the finiteness of the rational rank of $H_*(\Gamma; C^3) \cong \check{H}^{d-*}(M\mathcal{P})$ depends in part on the rank of $H_*(\Gamma; C^1)$. We begin with a result about rk $H_*(\Gamma; C^1)$.

LEMMA 2.47 *For a codimension 3 hyperplane polytopal projection pattern, if the number L_{1c} of Γ-orbits of singular 1-spaces $\mathcal{D} \in I_{1c}/\Gamma$ is finite, then rk $H_m(\Gamma; C^1) \otimes \mathbb{Q}$ is finite for $m > 0$.*

Proof First decompose the module C^1 as $\bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} C^1_{\mathcal{D}} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ by Lemma 1.62, where $C^1_{\mathcal{D}}$ is the submodule of C^1 arising from singular 1-spaces contained in a single representative of the Γ-orbit $\mathcal{D}$. Corollary 1.71 then gives $H_m(\Gamma; C^1) \cong \bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$. Now we can use Lemma 2.16 to show that for each $\mathcal{D} \in I_{1c}/\Gamma$, the module $C^1_{\mathcal{D}}$ fits into a sequence $0 \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0$ which is exact. Applying the functor $H_*(\Gamma^{\mathcal{D}}; -)$ to this short exact sequence yields a long exact sequence in homology

$$\begin{aligned}\cdots \longrightarrow H_2(\Gamma^{\mathcal{D}}; \mathbb{Z}) \longrightarrow H_1(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}}) \longrightarrow H_1(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}}) \longrightarrow H_1(\Gamma^{\mathcal{D}}; \mathbb{Z})\\ \longrightarrow H_0(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}}) \longrightarrow H_0(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}}) \longrightarrow H_0(\Gamma^{\mathcal{D}}; \mathbb{Z}) \longrightarrow 0\end{aligned}$$

and since $C^0_{\mathcal{D}}$ is a free $\Gamma^{\mathcal{D}}$-module, $H_m(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}}) = 0$ for $m > 0$. This implies that $H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}}) \cong H_{m+1}(\Gamma^{\mathcal{D}}; \mathbb{Z})$ for $m > 0$. In particular, rk $H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}}) < \infty$.

Returning to $H_1(\Gamma; C^1) \cong \bigoplus_{\mathcal{D}\in I_{1c}/\Gamma} H_1(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$, since $L_{1c} = |I_{1c}/\Gamma|$ is finite by assumption, $H_1(\Gamma; C^1)$ is a finite sum of groups of finite rank over $\mathbb{Q}$ and hence rk $H_1(\Gamma; C^1) \otimes \mathbb{Q} < \infty$, as required. ∎

As in the codimension 2 case (Section 2.2), for a hyperplane polytopal projection pattern $\mathcal{P}$ with exact sequence $0 \to C^3 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$, we can obtain other exact sequences, as shown in the following lemma.

LEMMA 2.48 *Given a codimension 3 hyperplane polytopal projection pattern, and Γ-modules C^i_D for $i = 0, 1, 2$ (Def 1.61), there is an exact sequence*

$$0 \longrightarrow C^2_D \longrightarrow C^1_D \longrightarrow C^0_D \longrightarrow \mathbb{Z}^s \longrightarrow 0. \tag{2.4}$$

Proof In this case, singular 2-spaces are planes. Since parallel planes are disjoint, and non-parallel planes in distinct Γ-orbits intersect in at most singular 1-spaces, which give rise to the zero element in C^2, we can decompose C^2 as $\bigoplus_{D \in I_{2c}} C^2_{\mathcal{D}}$. Note that, as in Lemma 1.62, we could write $C^2_{\mathcal{D}} \cong C^2_{D} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$, for C^2_{D} the module defined on singular 2-spaces in a single representative of the orbit $\mathcal{D}$ with action by $\Gamma^{\mathcal{D}}$, but in this section we generally want to consider the module $C^2_{\mathcal{D}}$ with action by the whole of Γ.

Note that the sequence $0 \to C^2_{D} \to C^1_{D} \to C^0_{D} \to \mathbb{Z} \to 0$ is exact by Lemma 2.15, as we are restricting to an orbit $\mathcal{D}$ of a (two-dimensional) plane D in which the singular 1-spaces are lines of infinite length by Lemma 1.43. Thus the sequence $0 \to C^2_{\mathcal{D}} \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z}^s \to 0$ is exact, because the functor $- \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ is exact. Note that $\mathbb{Z} \otimes_{\Gamma} \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}] \cong \mathbb{Z}^s$ is of infinite rank. ∎

Tables for the E^1 and E^3 terms of the spectral sequence described in Section 1.6.3 are given below for a 3-dimensional codimension 3 hyperplane polytopal projection pattern $\mathcal{P}$. Note that $E^3 = E^{\infty}$ since only the first and second differentials are non-zero in this case. The groups in the table which are possibly non-trivial are determined by Lemma 1.72. As we will always be working rationally in the work which follows, we again suppress $\mathbb{Q}$ by writing E^r_{pq} for $E^r_{pq} \otimes \mathbb{Q}$ and rk G for rk $(G \otimes \mathbb{Q})$.

0	0	0	$H_3(\Gamma; C^3)$
0	$H_2(\Gamma; C^1) \xleftarrow{d_{22}}$	$H_2(\Gamma; C^2) \xleftarrow{d_{32}}$	$H_2(\Gamma; C^3)$
0	$H_1(\Gamma; C^1) \xleftarrow{d_{21}}$	$H_1(\Gamma; C^2) \xleftarrow{d_{31}}$	$H_1(\Gamma; C^3)$
$H_0(\Gamma; C^0) \xleftarrow{d_{10}}$	$H_0(\Gamma; C^1) \xleftarrow{d_{20}}$	$H_0(\Gamma; C^2) \xleftarrow{d_{30}}$	$H_0(\Gamma; C^3)$

0	0	0	$H_3(\Gamma; C^3)$
0	$(H_2(\Gamma; C^1)/Imd_{22})/Im\partial_2$	$Kerd_{22}/Imd_{32}$	$Kerd_{32}$
0	$(H_1(\Gamma; C^1)/Imd_{21})/Im\partial_1$	$Kerd_{21}/Imd_{31}$	$Ker\partial_2$
$H_0(\Gamma; C^0)/Imd_{10}$	$Kerd_{10}/Imd_{20}$	$Kerd_{20}/Imd_{30}$	$Ker\partial_1$

LEMMA 2.49 *For a codimension 3 hyperplane polytopal projection pattern $\mathcal{P}$, if the rank of $H_m(\Gamma; C^1)$ over $\mathbb{Q}$ is infinite, then the rank over $\mathbb{Q}$ of $H_m(\Gamma; C^3)$ is infinite for $m > 0$.*

Proof To prove that rk $H_m(\Gamma; C^3)$ is infinite, consideration of the spectral sequence tables above indicates that it suffices to show $Ker(d_{2m}\colon H_m(\Gamma; C^2) \to H_m(\Gamma; C^1))$ is of infinite rank over $\mathbb{Q}$. This is because of the fact that rk $(Kerd_{2m}/Imd_{3m})$ is a summand of rk $H_{m+2}(\Gamma; \mathbb{Z})$, which is finite by Lemma 1.66, so if rk $Kerd_{2m} = \infty$ then rk $Imd_{3m} = \infty$ and hence rk $H_m(\Gamma; C^3) =$ rk $Kerd_{3m} +$ rk $Imd_{3m} = \infty$.

In order to prove rk $Kerd_{2m} = \infty$, in analogy with the proof of Theorem 2.26 we essentially find (infinitely many) linearly independent pairs of elements $[x], [y] \in H_m(\Gamma; C^2)$ with the property that $d_{2m}[x] = d_{2m}[y]$ in $H_m(\Gamma; C^1)$ so their difference $[x] - [y]$ is in the kernel of d_{2m}. In fact, we establish the following diagram, in which $H_m(\Gamma; C^1_{DD'})$ is an infinite rank subgroup of $H_m(\Gamma; C^1)$ with the property that an infinite rank subspace $A \cap A'$ lifts to distinct summands $H_m(\Gamma; C^2_D)$ and $H_m(\Gamma; C^2_{D'})$ of $H_m(\Gamma; C^2)$, thus producing the required (infinite) set of pairs of elements of $H_m(\Gamma; C^2)$ whose images under d_{2m} are equal.

$$\begin{tikzcd}
A \cap A' \arrow[rrr, dashed] \arrow[dr, hook] \arrow[ddd, dashed] & & & H_m(\Gamma; C^2_{D'}) \arrow[dl, "d_{2m}"'] \arrow[ddd, hook] \\
& H_m(\Gamma; C^1_{DD'}) \arrow[r, hook] \arrow[d, hook] & H_m(\Gamma; C^1_{D'}) \arrow[d, hook] & \\
& H_m(\Gamma; C^1_D) \arrow[r, hook] & H_m(\Gamma; C^1) & \\
H_m(\Gamma; C^2_D) \arrow[ur, "d_{2m}"] \arrow[rrr, hook] & & & H_m(\Gamma; C^2) \arrow[ul, "d_{2m}"']
\end{tikzcd}$$

We first produce the group $H_m(\Gamma; C^1_{DD'})$. By Lemma 2.47, there are an infinite number L_{1c} of orbits of singular 1-spaces since rk $H_m(\Gamma; C^1)$ is infinite. Since the number of Γ-orbits of singular 2-spaces is finite for a codimension 3 polytopal projection pattern, by the remark following Definition 1.36, at least one pair D, D' of singular 2-spaces has the property that the translates of D and D' under Γ intersect in singular 1-spaces in infinitely many Γ-orbits. We can also suppose that these singular 1-spaces have stabilisers of rank $m + 1$ or greater, and hence cause $H_m(\Gamma; C^1)$ to be non-trivial and of infinite rank.

Consider the exact sequence (2.4) in Lemma 2.48. In C^1_D, there is a submodule $C^1_{DD'}$ of functions on singular 1-spaces at the intersection of singular 2-spaces D and D' and their Γ-translates. As the singular 1-spaces $(D+\gamma)\cap(D'+\gamma')$ as γ and γ' vary are all parallel, there is a decomposition $C^1_{DD'} = \bigoplus_{\xi\in I_1^{DD'}} C^1_\xi \otimes \mathbb{Z}[\Gamma/\Gamma^\xi]$, where C^1_ξ is the Γ-module generated by indicator functions on a single representative of a Γ-orbit of singular 1-spaces $(D+\gamma)\cap(D'+\gamma')$ and $I_1^{DD'}$ is the set of all such orbits. By Corollary 1.71 we have $H_m(\Gamma; C^1_{DD'}) \cong \bigoplus_{\xi\in I_1^{DD'}} H_m(\Gamma^\xi; C^1_\xi)$. Now we assumed that the singular 1-spaces at the intersection of D and D' had stabilisers of rank strictly greater than m, so $H_m(\Gamma^\xi; C^1_\xi)$ is non-trivial, and we assumed the singular 1-spaces were in infinitely many Γ-orbits, so $I_1^{DD'}$ is an infinite set. Hence we have produced a group $H_m(\Gamma; C^1_{DD'})$ of infinite rank over $\mathbb{Q}$.

Note that there is an inclusion $H_m(\Gamma; C^1_{DD'}) \hookrightarrow H_m(\Gamma; C^1_D)$ induced by the inclusion $C^1_{DD'} \subset C^1_D$. Thus rk $H_m(\Gamma; C^1_D)$ is also infinite and the inclusion $C^1_D \subset C^1$ induces a map $H_m(\Gamma; C^1_D) \hookrightarrow H_m(\Gamma; C^1)$.

We now show that $H_m(\Gamma; C^2)$ is of infinite rank over $\mathbb{Q}$. Consider again the sequence (2.4). We can break this sequence into two short exact sequences

$$0 \to C^2_D \to C^1_D \to C^{00}_D \to 0 \tag{2.5}$$

$$0 \to C^{00}_D \to C^0_D \to \mathbb{Z}^s \to 0 \tag{2.6}$$

where C^{00}_D is the image of the map $\delta_0\colon C^1_D \to C^0_D$. Note that Γ acts on these modules, as mentioned in Lemma 2.48, so we apply the functor $H_*(\Gamma; -)$ to both sequences.

The long exact sequence in homology associated to (2.6) is

$$\cdots \to H_m(\Gamma; C^{00}_D) \to H_m(\Gamma; C^0_D) \to H_m(\Gamma; \mathbb{Z}^s) \to H_{m-1}(\Gamma; C^{00}_D) \to \cdots. \tag{2.7}$$

Now $H_m(\Gamma; C^0_D)$ is trivial for $m > 0$ since C^0_D is a free Γ-module, so we have $H_m(\Gamma; C^{00}_D) \cong H_{m+1}(\Gamma; \mathbb{Z}^s)$ for $m > 0$. By definition of s and Corollary 1.71, we have $H_m(\Gamma; \mathbb{Z}^s) =$

$H_m(\Gamma^{\mathcal{D}} \oplus \Gamma/\Gamma^{\mathcal{D}}; \mathbb{Z} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]) \cong H_m(\Gamma^{\mathcal{D}}; \mathbb{Z})$, which is of finite rank over $\mathbb{Q}$. Thus $H_m(\Gamma; C_D^{00})$ has finite rank over $\mathbb{Q}$ for $m > 0$.

The sequence in homology associated to (2.5) is as follows.

$$\cdots \to H_m(\Gamma; C_D^2) \stackrel{d_{2*}}{\to} H_m(\Gamma; C_D^1) \stackrel{\delta_0^m}{\to} H_m(\Gamma; C_D^{00}) \to H_{m-1}(\Gamma; C_D^2) \to \cdots \qquad (2.8)$$

As shown above, rk $H_m(\Gamma; C_D^1) = \infty$ but rk $H_m(\Gamma; C_D^{00}) < \infty$ so $Ker\delta_0^m$ must be of infinite rank over $\mathbb{Q}$. The exactness of the sequence (2.8) implies that there is a surjection $H_m(\Gamma; C_D^2) \twoheadrightarrow Ker\delta_0^m$ and so $H_m(\Gamma; C_D^2)$ is of infinite rank over $\mathbb{Q}$. The inclusion of the summand C_D^2 into C^2 induces an inclusion $H_m(\Gamma; C_D^2) \hookrightarrow H_m(\Gamma; C^2)$ so $H_m(\Gamma; C^2)$ is of infinite rank over $\mathbb{Q}$ for $m > 0$.

We could equivalently have made the above constructions using the summand $C_{D'}^2$ of C^2, so we also have $H_m(\Gamma; C_{D'}^2)$ an infinite rank summand of $H_m(\Gamma; C^2)$ and $H_m(\Gamma; C_{D'}^1)$ of infinite rank, with inclusions $H_m(\Gamma; C_{DD'}^1) \hookrightarrow H_m(\Gamma; C_{D'}^1) \hookrightarrow H_m(\Gamma; C^1)$.

Given the diagrams

$$\begin{array}{ccc} H_m(\Gamma; C_{DD'}^1) & \hookrightarrow & H_m(\Gamma; C_D^1) \\ & & \downarrow \delta_0^m \\ & & H_m(\Gamma; C_D^{00}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H_m(\Gamma; C_{DD'}^1) & \hookrightarrow & H_m(\Gamma; C_{D'}^1) \\ & & \downarrow \delta_0'^m \\ & & H_m(\Gamma; C_{D'}^{00}) \end{array}$$

we can decompose $H_m(\Gamma; C_{DD'}^1)$ as a direct sum $A \oplus B$ for $A \subset H_m(\Gamma; C_{DD'}^1) \subset H_m(\Gamma; C_D^1)$ in the kernel of δ_0^m and B such that the restriction $\delta_0^m|_B$ of δ_0^m to B is an inclusion. As $H_m(\Gamma; C_D^{00})$ is of finite rank, B is of finite rank, so A is of infinite rank since rk $H_m(\Gamma; C_{DD'}^1) = \infty$. Similarly there is a decomposition of $H_m(\Gamma; C_{DD'}^1)$ as $A' \oplus B'$ for $A' \subset Ker\delta_0'^m$ of infinite rank and B' of finite rank.

The intersection $A \cap A'$ is of infinite rank. To see this, consider $\pi\colon A \oplus B \to B$ and $\pi'\colon A' \oplus B' \to B'$ projection maps with kernel A and A' respectively. The kernel of $H_m(\Gamma; C_{DD'}^1) \stackrel{\pi \oplus \pi'}{\longrightarrow} B \oplus B'$ is precisely $A \cap A'$ but this is of infinite rank since rk $H_m(\Gamma; C_{DD'}^1) = \infty$ and $B \oplus B'$ is of finite rank.

Now since $H_m(\Gamma; C_D^2)$ surjects onto $Ker\delta_0^m$ by exactness of the long exact sequence in homology above and $A \cap A'$ is contained in $Ker\delta_0^m$, the elements of $A \cap A'$ can be lifted to elements of $H_m(\Gamma; C^2_D)$. Similarly, the elements of $A \cap A'$ can be lifted to $H_m(\Gamma; C^2_{D'})$. Thus we can find two non-zero elements $[x] \in H_m(\Gamma; C^2_D)$ and $[y] \in H_m(\Gamma; C^2_{D'})$ which both map to some element $[z] \in A \cap A' \subset H_m(\Gamma; C^1_{DD'})$ under $d_{2m}\colon H_m(\Gamma; C^2) \to H_m(\Gamma; C^1)$. Since $[x]$ and $[y]$ are in different direct summands of $H_m(\Gamma; C^2)$ their difference $[x] - [y]$ is non-zero in $H_m(\Gamma; C^2)$ but $d_{2m}([x] - [y]) = [z] - [z] = 0$. As $A \cap A'$ is of infinite rank, there exist infinitely many such elements $[z]$ associated to linearly independent pairs $[x], [y]$ of elements of $H_m(\Gamma; C^2)$. Thus $Kerd_{2m}$ is of infinite rank, so rk $H_m(\Gamma; C^3)$ is infinite. ∎

We can now formulate a theorem giving conditions under which the Euler characteristic is not defined.

THEOREM 2.50 *Given a codimension* 3 *hyperplane polytopal projection pattern for which the rank of* $H_1(\Gamma; C^1)$ *is infinite, the Euler characteristic is not defined.*

Proof Since rk $H_1(\Gamma; C^1) = \infty$ by assumption, setting $m = 1$ in Lemma 2.49 gives that $H_1(\Gamma; C^3)$ is of infinite rank over $\mathbb{Q}$. If we can show that rk $H_0(\Gamma; C^3)$ is also infinite, then since it has a sign opposite to that of rk $H_1(\Gamma; C^3)$ in the expression for the Euler characteristic in Corollary 2.4, the Euler characteristic is not defined in this case.

First note that under the above assumptions, L_0 is infinite since rk $H_1(\Gamma; C^1) = \infty$ implies that $L_{1c} = |I_{1c}/\Gamma|$ is infinite by Lemma 2.47 and this in turn gives that L_0 is infinite, since each Γ-orbit of singular 1-spaces in I_{1c}/Γ contains at least one Γ-orbit of singular 0-spaces by Lemma 1.39. Hence to show that rk $H_0(\Gamma; C^3) = \infty$, we could use Theorem 2.60 ahead, but in this relatively low dimensional case for which we know by Lemma 1.60 the sequence $0 \to C^3 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ is exact, we prove the result as follows. The argument used is similar to that of Lemma 2.49 — we show that the kernel of the map $d_{20}\colon H_0(\Gamma; C^2) \to H_0(\Gamma; C^1)$ is of infinite rank by finding an infinite set of elements in $H_0(\Gamma; C^1)$ which lift to two distinct summands of $H_0(\Gamma; C^2)$ so their difference is non-trivial in $H_0(\Gamma; C^2)$ but zero in $H_0(\Gamma; C^1)$.

For a codimension 3 polytopal projection pattern, there are only finitely many Γ-orbits of singular 2-spaces and hence at least one, denoted D, must contain singular 0-spaces in infinitely many Γ-orbits. As in Lemma 2.49, we consider the exact sequence $0 \to C_D^2 \to C_D^1 \to C_D^0 \to \mathbb{Z}^s \to 0$ and the short exact sequences (2.5) and (2.6).

The long exact sequence arising from (2.6) yields rk $H_0(\Gamma; C_D^{00}) =$ rk $H_0(\Gamma; C_D^0) -$ rk $H_0(\Gamma; \mathbb{Z}^s)+$rk $H_1(\Gamma; \mathbb{Z}^s)$ and this is equal to rk $H_0(\Gamma^{\mathcal{D}}; C_D^0)-$rk $H_0(\Gamma^{\mathcal{D}}; \mathbb{Z})+$rk $H_1(\Gamma^{\mathcal{D}}; \mathbb{Z})$ by Corollary 1.69, for $\Gamma^{\mathcal{D}}$ the stabiliser of all singular 2-spaces D in the orbit $\mathcal{D}$. Hence, writing $L_0^{\mathcal{D}}$ for the number of Γ-orbits of singular 0-spaces in D (or any other singular 2-spaces in the Γ-orbit $\mathcal{D}$ of D) we have rk $H_0(\Gamma; C_D^{00}) = L_0^{\mathcal{D}} +$ rk $\Gamma^{\mathcal{D}} - 1$ which is infinite since $L_0^{\mathcal{D}}$ is infinite by the choice of D.

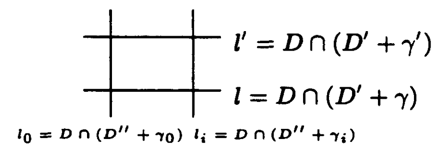
Next considering the long exact sequence (2.7) arising from (2.6), the map $H_0(\Gamma; C_D^1) \to H_0(\Gamma; C_D^{00})$ is a surjection so rk $H_0(\Gamma; C_D^1)$ is infinite and hence $H_0(\Gamma; C^1) \supset H_0(\Gamma; C_D^1)$ is of infinite rank.

Now there is a singular 1-space in D which contains singular 0-spaces in an infinite number of Γ-orbits. To see this, note first that there are infinitely many Γ-orbits of singular 0-spaces in D but any singular 0-space arises at the intersection of three or more singular 2-spaces and since there are only finitely many Γ-orbits of singular 2-spaces for a codimension 3 polytopal projection pattern, there is at least one set D', D'' of singular 2-spaces whose translates intersect D in singular 0-spaces in infinitely many Γ-orbits. If the intersection of translates of D'' with the singular 1-space $D \cap (D' + \gamma)$ and the intersection of translates of D' with $D \cap (D'' + \gamma')$ were in finitely many Γ-orbits for some $\gamma, \gamma' \in \Gamma$ then there could be only finitely many Γ-orbits of singular 0-spaces arising at the intersection of translates of D, D' and D'', which is a contradiction to the choice of D' and D''. Thus at least one of the lines $D \cap (D' + \gamma)$ or $D \cap (D'' + \gamma')$ contain infinitely many Γ-orbits of singular 0-spaces. Denote this singular 1-space by l and write l_i for the set of parallel singular 1-spaces which intersect l at the singular 0-spaces in infinitely many Γ-orbits and which are formed at the intersection of D with the singular 2-space D' or D'' not used to form l.

The above construction enables us to be able to apply the argument in the proof of Theorem 2.26. Namely we produce infinitely many elements ℓ_i of C^1_D giving rise to distinct elements of $Kerd_{10}\colon H_0(\Gamma; C^1_D) \to H_0(\Gamma; C^{00}_D)$, which is thus of infinite rank. Hence, since $H_0(\Gamma; C^2_D)$ surjects onto $Kerd_{10}\colon H_0(\Gamma; C^1_D) \to H_0(\Gamma; C^{00}_D)$ by exactness of (2.8) at $H_0(\Gamma; C^1_D)$ we have rk $H_0(\Gamma; C^2_D)$ infinite.

We now proceed by constructing an infinite set of elements in the kernel of the map $\delta^0_0\colon H_0(\Gamma; C^1_D) \to H_0(\Gamma; C^{00}_D)$ which lifts to two distinct summands of $H_0(\Gamma; C^2)$, thus giving rise to pairs of elements whose difference in $H_0(\Gamma; C^2)$ is non-trivial, but which are in the kernel of $d_{20}\colon H_0(\Gamma; C^2) \to H_0(\Gamma; C^1)$.

Take the set S_i of singular 2-spaces in D which are bounded by the singular 1-spaces associated to the elements $\ell_i \in Ker\delta^0_0$, where ℓ_i is the indicator function on the loop formed by the singular 1-spaces l, l_0, l_i and l' for l' parallel to l and l_0 parallel to l_i formed by the same pairs of singular 2-spaces as l and l_i respectively, as shown in the diagram below. These singular 2-spaces then give rise to elements S_i of C^2_D whose images under $\delta_1\colon C^2_D \to C^1_D$ are ℓ_i which in turn yield distinct elements $[\ell_i]$ in $Ker(H_0(\Gamma; C^1_D) \to H_0(\Gamma; C^{00}_D))$ as in the proof of Theorem 2.26. Hence $\{\ell_i\}$ is a set of elements in $H_0(\Gamma; C^1_D)$ which lifts to $H_0(\Gamma; C^2_D)$.



Now translates of D, D' and D'' can be found which form parallelepipeds Π_i having base S_i since each singular 1-space in the boundary of S_i is at the intersection of D with some translate of D' or D'' and since D, D' and D'' are planes of infinite extent by assumption a translate $D + \gamma$ of D can be taken which intersects the translates of D' and D'' in singular 1-spaces forming the boundary of $S_i + \gamma$. Thus as above the element $[\ell_i] + [\ell_i + \gamma] \in Ker(H_0(\Gamma; C^1_D) \to H_0(\Gamma; C^{00}_D))$ lifts to $H_0(\Gamma; C^2_D) \subset H_0(\Gamma; C^2)$, but it lifts to $H_0(\Gamma; C^2_{D'}) \oplus H_0(\Gamma; C^2_{D''}) \subset H_0(\Gamma; C^2)$ as well, since the singular 1-spaces associated to ℓ_i also form the boundary of the cylinder in Π_i arising from D', D'' and their translates.

Hence we have found elements $[x] \in H_0(\Gamma; C^2_D)$ and $[y] \in H_0(\Gamma; C^2_{D'}) \oplus H_0(\Gamma; C^2_{D''})$ such that $[x] - [y]$ is non-trivial but $d_{20}([x] - [y]) = [\ell_i] + [\ell_i + \gamma] - ([\ell_i] + [\ell_i + \gamma]) = 0$ and since there are infinitely many elements $[\ell_i]$, we have rk $Ker(d_{20}\colon H_0(\Gamma; C^2) \to H_0(\Gamma; C^1)) = \infty$, which means that rk $H_0(\Gamma; C^3) = \infty$ as required. ∎

The above theorem established criteria under which the Euler characteristic of a codimension 3 polytopal projection pattern is not defined. We now consider an example satisfying these conditions.

EXAMPLE 2.51 *A 3-dimensional codimension 3 degenerate canonical projection pattern.*

To produce such a pattern, we take a canonical 3-dimensional codimension 3 projection pattern, so the acceptance domain K is the projection to V of a six-dimensional hypercube. The projection is chosen so that three of the 1-dimensional faces of K coincide but with lengths that are mutually irrational. Any singular 1-space l in the direction of these three faces thus has stabiliser of rank 3 since there are three elements of Γ in the direction of l and they are rationally independent. Note by Lemma 1.49 that the stabiliser of any singular 1-space associated to a 3-dimensional codimension 3 projection pattern cannot be of rank larger than 3. To show that a pattern with this property satisfies the criteria above, note the following lemma.

LEMMA 2.52 *For a 3-dimensional codimension 3 hyperplane polytopal projection pattern such that at least one singular 1-space has stabiliser Γ^l of rank 3, then the number of orbit classes of singular 1-spaces with rk $\Gamma^l = 3$ is infinite.*

Proof Under these assumptions, and using Lemma 1.49, the rank of the stabiliser of any singular 2-space D is at most 4, so if there is a singular 1-space l with stabiliser $\Gamma^l = \langle e_1, e_2, e_3 \rangle$ of rank 3 in some singular 2-space D, then there must be another singular 1-space l' in D with stabiliser of rank 1. This is because D is a plane by assumption, but there are only finitely many Γ-orbits of singular 2-spaces for a codimension 3 polytopal projection pattern so the vector space span of the stabiliser of D must be two-dimensional. Hence all stabiliser vectors cannot be parallel to the singular 1-space l.

Now suppose the singular 1-space l is formed at the intersection of singular 2-spaces D and D'. Then as in Lemma 1.39 the lines formed at the intersection of D and D' are dense in D by the density of Γ in V (which implies that the projection of Γ to D along D' is dense in D). Similarly, the singular 0-spaces at the intersection of l' and the singular 1-spaces parallel to l are dense in l'. The singular 0-spaces in l' are thus not all in the same Γ-orbit since the stabiliser of l' is of rank 1 so singular 0-spaces closer than the length of the generator of $\Gamma^{l'}$ are in distinct Γ-orbits.

Note that the elements of the stabiliser $\Gamma^{\mathcal{D}}$ of D are linear combinations of the elements of Γ^{l} and $\Gamma^{l'}$. This means that the orbit of l under $\Gamma^{\mathcal{D}}$ is not dense in D because in the direction of l' (not parallel to l) the only translates are of length $\|v\|$ for v the generator of $\Gamma^{l'}$. Since singular 1-spaces parallel to l are dense in D, there are infinitely many which are closer than $\|v\|$ to l and to each other and hence must be in distinct Γ-orbits. ∎

The above lemma shows that $H_1(\Gamma; C^1)$ is infinite for this pattern so Lemma 2.49 tells us that $H_1(\Gamma; C^3)$ is of infinite rank over $\mathbb{Q}$. Lemma 2.47 also gives L_0 infinite for this pattern so rk $H_0(\Gamma; C^3)$ is infinite by the result in the proof of Theorem 2.50 or by Theorem 2.60. Hence the Euler characteristic of this pattern is not defined.

We now give three results considering situations when the Euler characteristic is defined for a codimension 3 hyperplane polytopal projection pattern with L_0 infinite.

THEOREM 2.53 *For a d-dimensional codimension 3 hyperplane polytopal projection pattern $\mathcal{P}$ with L_0 infinite, if $H_m(\Gamma; C^1)$ is of finite rank over $\mathbb{Q}$ for all $m > 0$ then the Euler characteristic $e_{\mathcal{P}}$ is defined and is infinite.*

Proof To prove this result, we begin by showing that rk $H_m(\Gamma; C^3)$ is finite for $m > 0$ by showing that the kernel and image of the map $d_{3m}\colon H_m(\Gamma; C^3) \to H_m(\Gamma; C^2)$ have finite rank over $\mathbb{Q}$.

By Lemma 1.62, we can decompose C^2 as $\bigoplus_{\mathcal{D}\in I_{2c}/\Gamma} C^2_{\mathcal{D}} \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$ for $C^2_{\mathcal{D}}$ the Γ-module in Definition 1.61. From the proof of Lemma 2.48 there is an associated exact sequence $0 \to C^2_{\mathcal{D}} \to C^1_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0$. Break this sequence into two short exact sequences $0 \to C^2_{\mathcal{D}} \to C^1_{\mathcal{D}} \to C^{00}_{\mathcal{D}} \to 0$ and $0 \to C^{00}_{\mathcal{D}} \to C^0_{\mathcal{D}} \to \mathbb{Z} \to 0$, where $C^{00}_{\mathcal{D}} = Im\delta\colon C^1_{\mathcal{D}} \to C^0_{\mathcal{D}}$ and apply the functor $H_*(\Gamma^{\mathcal{D}}; -)$ to obtain long exact sequences in homology. From the sequence $\ldots \to H_m(\Gamma^{\mathcal{D}}; C^{00}_{\mathcal{D}}) \to H_m(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}}) \to H_m(\Gamma^{\mathcal{D}}; \mathbb{Z}) \to \cdots$ we see that $H_m(\Gamma^{\mathcal{D}}; C^{00}_{\mathcal{D}}) \cong H_{m+1}(\Gamma^{\mathcal{D}}; \mathbb{Z})$ for $m > 0$ since $C^0_{\mathcal{D}}$ is a free Γ-module so $H_m(\Gamma^{\mathcal{D}}; C^0_{\mathcal{D}}) = 0$ for $m > 0$. Now in the long exact sequence $\cdots \to H_m(\Gamma^{\mathcal{D}}; C^2_{\mathcal{D}}) \to H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}}) \to H_m(\Gamma^{\mathcal{D}}; C^{00}_{\mathcal{D}}) \to \cdots$, for $m > 0$ we have $H_m(\Gamma^{\mathcal{D}}; C^{00}_{\mathcal{D}})$ of finite rank over $\mathbb{Q}$ and $H_m(\Gamma^{\mathcal{D}}; C^1_{\mathcal{D}})$ of finite rank by assumption, so $H_m(\Gamma^{\mathcal{D}}; C^2_{\mathcal{D}})$ is of finite rank. Hence, as the set I_{2c}/Γ of Γ-orbits of singular 2-spaces is finite for codimension 3 polytopal projection patterns, $H_m(\Gamma; C^2) \cong \bigoplus_{\mathcal{D}\in I_{2c}/\Gamma} H_m(\Gamma^{\mathcal{D}}; C^2_{\mathcal{D}})$ is of finite rank. Thus the map $d_{3m}\colon H_m(\Gamma; C^3) \to H_m(\Gamma; C^2)$ has image of finite rank.

Note that $Kerd_{3m}$ is also of finite rank, since from the $E^3 = E^\infty$ terms in the spectral sequence (in the table preceding Lemma 2.49) rk $Ker\partial_{m+1}$ is a summand of $H_{m+3}(\Gamma; \mathbb{Z})$, which is of finite rank. This implies that rk $Kerd_{3m}$ is finite since if $Kerd_{3m}$ were of infinite rank, then the image of the second differential $\partial_{m+1}\colon H_m(\Gamma; C^3) \to H_{m+1}(\Gamma; C^1)$ would have infinite rank. However, $H_m(\Gamma; C^1)$ is of finite rank by assumption, so rk $Kerd_{3m}$ cannot be infinite. Hence rk $H_m(\Gamma; C^3) =$ rk $Kerd_{3m} +$ rk Imd_{3m} is finite for $m > 0$. Thus the Euler characteristic is defined in this case. The proof of Theorem 2.50 or Theorem 2.60 ahead imply that $H_0(\Gamma; C^3)$ is of infinite rank over $\mathbb{Q}$ and hence the Euler characteristic is infinite. ∎

COROLLARY 2.54 *The Euler characteristic for a codimension 3 polytopal projection pattern is defined if and only if rk $H_m(\Gamma; C^1)$ is finite for all $m > 0$.* ∎

COROLLARY 2.55 *For a codimension* 3 *hyperplane polytopal projection pattern with* L_0 *infinite and such that the stabilisers of all singular* 1*-spaces have rank* 1*, the Euler characteristic is defined and is infinite.*

Proof By Lemma 1.72, in this situation the homology groups $H_m(\Gamma^\xi; C^1_\xi)$ are trivial for $m > 0$, where $\xi \in I_{1c}/\Gamma$ are Γ-orbits of singular 1-spaces and C^1_ξ is the Γ-module (Def 1.57) defined only for one singular 1-space in the Γ-orbit ξ. Note that $C^1 \cong \bigoplus_{\xi \in I_{1c}/\Gamma} C^1_\xi \otimes \mathbb{Z}[\Gamma/\Gamma^\xi]$ since by assumption the singular 1-spaces are lines of infinite length, so parallel singular 1-spaces are disjoint and non-parallel lines intersect in at most a point, which gives the zero element in C^1. Thus $H_m(\Gamma; C^1) \cong \bigoplus_{\xi \in I_{1c}/\Gamma} H_m(\Gamma^\xi; C^1_\xi)$ is trivial. In particular rk $H_m(\Gamma; C^1)$ is finite, so Theorem 2.53 tells us that $H_m(\Gamma; C^3)$ is of finite rank over $\mathbb{Q}$ for $m > 0$, which means that the Euler characteristic is defined. Again, by the proof of Theorem 2.50 or Theorem 2.60, the group $H_0(\Gamma; C^3)$ is of infinite rank. Thus the Euler characteristic is infinite. ∎

COROLLARY 2.56 *For a hypergeneric codimension* 3 *polytopal projection pattern of arbitrary dimension, the Euler characteristic is defined and is infinite.*

Proof If the stabilisers of singular i-spaces are i-dimensional, for $i = 1, 2$, then by Lemma 1.72 the groups $H_m(\Gamma; C^i) \cong \bigoplus_{\theta \in I_{ic}/\Gamma} H_m(\Gamma^\theta; C^i_\theta)$ are trivial for $m > 0$. Thus in particular $H_m(\Gamma; C^1)$ is of finite rank over $\mathbb{Q}$ for $m > 0$ so Theorem 2.53 gives that rk $H_m(\Gamma; C^3)$ is finite for m in this range, and so the Euler characteristic is defined.

By Theorem 2.50 or Theorem 2.60, we know that $H_0(\Gamma; C^3)$ is of infinite rank, so the Euler characteristic is infinite. ∎

Note that something more can be said about the (co)homology of hypergeneric polytopal projection patterns than is yielded by the Euler characteristic. Considering the spectral sequence tables on Page 88, if $H_m(\Gamma; C^1) = 0 = H_m(\Gamma; C^2)$ for $m > 0$, then the edge homomorphisms $H_{m+3}(\Gamma; \mathbb{Z}) \to H_m(\Gamma; C^3)$ are isomorphisms, so we have the following result.

THEOREM 2.57 *The rational ranks of the Čech cohomology groups of the continuous hull $M\mathcal{P}$ for a codimension 3 hypergeneric polytopal projection pattern are as follows.*

$$rk\ \check{H}^p(M\mathcal{P}) = \begin{cases} \binom{rk\Gamma}{p+3} & \text{for } 0 \leqslant p \leqslant d-1 \\ \infty & \text{for } p = d \end{cases}$$

∎

We conclude consideration of this case with a general result giving a necessary condition for $H_m(\Gamma; C^3)$ to be of finite rank.

THEOREM 2.58 *For a d-dimensional codimension 3 hyperplane polytopal projection pattern, the homology groups $H_m(\Gamma; C^3)$ have finite rank over $\mathbb{Q}$ for $m > r-1$, where $r = \max\{rk\,\Gamma^{\xi} : \xi \in I_1\}$.*

Proof By Lemma 1.72, the group $H_m(\Gamma; C^1)$ is trivial for $m > r-1$, where r is the maximum of the ranks of stabilisers of singular 1-spaces (which are lines of infinite length in this case). Thus the second differentials $\partial\colon H_{m-1}(\Gamma; C^3) \to H_m(\Gamma; C^1)$ will be zero maps for such values of m and so $Kerd_{3m}$ is of finite rank over $\mathbb{Q}$ since $Ker\partial$ is of finite rank, being a summand of $H_{m+3}(\Gamma; \mathbb{Z})$, and $Im\partial$ is of finite (zero) rank. Also, the images of the first differentials $d_{3m}\colon H_m(\Gamma; C^3) \to H_m(\Gamma; C^2)$ have finite rank for $m > r-1$ by Theorem 2.53, since rk $H_m(\Gamma; C^1) = 0 < \infty$ for $m > r-1$. Thus the groups $H_m(\Gamma; C^3)$ are of finite rank for $m > r-1$. ∎

Case 3: L_0 infinite and orbits not hyperplanes

For a codimension 3 polytopal projection pattern $\mathcal{P}$, if the Γ-orbits of singular 2-spaces do not contain the planes spanned by the spaces, then it is possible that the sequence (1.2) of modules C^i (Def 1.57) may not be exact. However, if there is such an exact sequence, then we may proceed in the same way as in the previous cases to obtain various results about the Euler characteristic in this situation.

Note that when the Γ-orbits of singular 2-spaces do not consist of planes, then there are three possibilities. Firstly, the connected components may be homeomorphic to 2-dimensional balls and hence have trivial stabilisers. Secondly, the connected components could have the form of strips of infinite length, with stabiliser having real span of dimension 1. Thirdly, the connected component may have the form of a plane with holes in, so the stabiliser has real span of dimension 2 but there are some points in the span of a 2-dimensional face F of the acceptance domain K for $\mathcal{P}$ which are not in the orbit $F + \Gamma$ of F. In the latter two cases, singular 1-spaces $l \in I_{1c}$ could have the form of lines of infinite length.

The result below considers polytopal projection patterns with connected components $D \in I_{2c}$ having trivial stabiliser (referred to as *generic* polytopal projection patterns in Definition 1.51).

THEOREM 2.59 *For a codimension* 3 *generic polytopal projection pattern for which the exact sequence* (1.2) *exists, then the Euler characteristic is defined and is infinite.*

Proof Given the exact sequence $0 \to C^3 \to C^2 \to C^1 \to C^0 \to \mathbb{Z} \to 0$, a spectral sequence can be set up as in Section 1.6.3. In this case, the stabilisers Γ^l of singular 1-spaces l are trivial so $H_m(\Gamma; C^1) \cong \bigoplus_{\xi \in I_{1c}/\Gamma} H_m(1; C^1_\xi) = 0$ for $m > 0$ and similarly $H_m(\Gamma; C^2) = 0$ for $m > 0$. This means that the edge homomorphisms $H_{m+3}(\Gamma; \mathbb{Z}) \to H_m(\Gamma; C^3)$ in the spectral sequence are isomorphisms for $m > 0$ so rk $H_m(\Gamma; C^3)$ is finite for $m > 0$ and hence the Euler characteristic is defined since at most one group in the expression given in Corollary 2.2, namely $H_0(\Gamma; C^3)$, can be of infinite rank.

Since L_0 is infinite for generic patterns (because the stabiliser of any singular 2-space D is trivial and so any two singular 0-spaces in D must be in different Γ-orbits and there are infinitely many singular 0-spaces in D by Lemma 1.39), by Theorem 2.60 ahead or Theorem 2.50 we have $H_0(\Gamma; C^3)$ of infinite rank. Hence the Euler characteristic is infinite. ∎

When the stabilisers of singular 2-spaces are non-trivial, but the singular spaces are not hyperplanes, then subject to the modification of the exact sequence associated to $C_{\mathcal{D}}^2$ described below, the results of Case 2 can be applied here to yield similar results.

There is an exact sequence $0 \to C_{\mathcal{D}}^2 \to C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^0 \to \mathbb{Z}^s \to 0$, for $C_{\mathcal{D}}^2$ the Γ-module in Definition 1.61. This is because within the 2-dimensional affine subspace D of $V \cong \mathbb{R}^3$, we can apply Lemma 2.15 to give exactness at $C_{\mathcal{D}}^2$ and $C_{\mathcal{D}}^1$. However, since singular 2-spaces in I_{2c} are not hyperplanes, it is possible that there are singular 0-spaces in D which are not contained in singular 1-spaces. These can only arise from the intersection with D of vertices of translates of the acceptance domain K, since if p is a singular 0-space at the intersection of singular 2-spaces D and D' which is in the interior of D and not a vertex of D', then $D \cap D'$ consists of a line segment containing p. Thus there are singular 0-spaces for which the difference of their indicator functions is in the kernel of $C_{\mathcal{D}}^0 \to \mathbb{Z}$ but not in the image of $C_{\mathcal{D}}^1 \to C_{\mathcal{D}}^0$ since there is no path of singular 1-spaces between the singular 0-spaces. Hence for exactness at $C_{\mathcal{D}}^0$, we require a map $C_{\mathcal{D}}^0 \to \mathbb{Z}^s$ for s the number of edge-path components (Def 1.26) in D. There can only be finitely many singular 0-spaces in D not contained in any singular 1-space since there are only finitely many vertices of K and as the stabiliser of D is trivial, no more than one singular 0-space in the same Γ-orbit can lie in D. All other singular 0-spaces in D are in the same edge-path component by Lemma 1.42 so s is finite.

Summary

This section has shown that for codimension 3 polytopal projection patterns when L_0 is finite we can produce formulae for the calculation of the Euler characteristic, as we did in the codimension 2 case. However, when L_0 is infinite, codimension 3 polytopal projection patterns differ from patterns of codimension 2 since we have seen that the Euler characteristic is not always defined in this case. Lastly, we showed that the Euler characteristic is always defined and infinite for generic and hypergeneric polytopal projection patterns of codimension 3.

Having considered low-codimension patterns, the next section gives a discussion of polytopal projection patterns of arbitrary codimension.

2.4 Higher codimensions

This section produces results generalising the statements made above for polytopal projection patterns of low codimensions to patterns of any codimension. We begin with the proof of the result referred to in previous sections, namely that (no matter whether the sequence (1.2) of modules C^i is exact or not) if L_0 is infinite for a polytopal projection pattern $\mathcal{P}$ then $\check{H}^d(M\mathcal{P}) \cong H_0(\Gamma; C^n)$ is of infinite rank over $\mathbb{Q}$. The result generalises [12, IV.2.9], which was only applicable to canonical projection patterns.

THEOREM 2.60 *For a codimension n polytopal projection pattern with L_0 infinite, we have rk $H_0(\Gamma; C^n) \otimes \mathbb{Q} = \infty$.*

Proof To prove this result, we restrict to a region small with respect to the inradii of faces of K so that there are infinitely many translates of singular $(n-1)$-spaces with the property that the region intersects the spaces in their interior only. Thus we recover the setup for [12, IV.2.9], namely a region containing a dense set of hyperplanes whose normals span the region, and so the method of proof of [12, IV.2.9] can be applied here.

Since L_0 is infinite, but the number of Γ-orbits of $(n-1)$-dimensional faces of the acceptance domain K is finite, there is an $(n-1)$-dimensional face F containing representatives of infinitely many orbit classes of singular 0-spaces. Take a singular 0-space β in F. This point is formed at the intersection of at least n faces of K, including F, and there must be a subset of n faces with linearly independent normal vectors for the intersection of these faces to be a single point.

Since there are infinitely many orbit classes of singular points, and K is a polytope so has a finite number of vertices by definition, without loss of generality we can take a singular 0-space β which is not a vertex of K. Thus it lies in the interior of F and all the other faces F_i intersecting F to form β. Since β is in the interior of these faces, there is some $\epsilon > 0$ such that $(n-1)$-dimensional balls $B_\epsilon(\beta)$ of radius ϵ centred at β can be contained in each face intersecting at β. We can suppose that there is another point β' in F which is not in the same orbit as β and is formed at the intersection of translates $F_i + \gamma_i$ of the same faces that form β (but note that given translates $F_i + \gamma_i$, $F_j + \gamma_j$ of the faces forming β, the elements γ_i and γ_j need not be equal). This is because if all singular 0-spaces in F were formed at the intersection of different faces, then there would have to be an infinite number of distinct faces in K, which is not true since K is a polytope. We can also suppose that β' lies in the ball $B_\epsilon(\beta)$ in F since if there were no translates of the faces F_i intersecting F at a point within this ball, then the n-dimensional region $v \times B_\epsilon(\beta)$ of the n-dimensional space V could contain no translates of at least one of the faces F' of K, for v a vector in F' but not F. This contradicts the density of Γ in V, so β' is within ϵ of β, as stated. Thus, the translates of faces forming β' intersect the faces forming β. Taking a translate of F within ϵ of β then produces a bounded region Π, which is an n-dimensional generalisation of a parallelepiped. Since Π encloses an n-dimensional subset of V, it is also a singular n-space.

Thus we have constructed the required region and so the proof of [12, IV.2.9] can be applied to give the result. The basic idea behind the proof is to construct a map $\phi\colon C^n \to \bigoplus_{\mathcal{J}_0} \mathbb{Z}/2$ for $\mathcal{J}_0$ the set of flags (Def 1.29) on all singular n-spaces in Π such that the image of ϕ is an infinitely generated subgroup of $\bigoplus_{\mathcal{J}_0} \mathbb{Z}/2$, so then taking quotients by Γ and writing $\mathcal{J} = \mathcal{J}_0/\Gamma$ for the set of Γ-orbits of flags associated to singular n-spaces, there is a homomorphism $H_0(\Gamma; C^n) \to \bigoplus_{\mathcal{J}} \mathbb{Z}/2$ whose image is infinitely generated. By [12, IV.2.10], this implies that $H_0(\Gamma; C^n) \otimes \mathbb{Q}$ is infinite-dimensional as a $\mathbb{Q}$-vector space as required. ∎

In the next section, a formula for the calculation of the Euler characteristic is presented, which generalises the results of the low codimension cases. This formula only applies to polytopal projection patterns for which L_0 is finite, so the result below shows that the Euler characteristic is defined and finite in this case.

THEOREM 2.61 *For a d-dimensional codimension n projection pattern $\mathcal{P}$, if L_0 is finite, then the Euler characteristic is defined and is finite.*

Proof To prove that the Euler characteristic is defined and finite, we need to show that rk $H_m(\Gamma; C^n)$ is finite for all m. We proceed as in [12, V.2.4] by induction on the codimension n of the pattern $\mathcal{P}$. The result is true for $n = 1$ since rk $H_0(\Gamma; C^0) = L_0$ is finite by assumption so using the formulae in Theorem 2.7 rk $H_0(\Gamma; C^1) = L_0 + d$ is finite and rk $H_m(\Gamma; C^1) =$ rk $H_m(\Gamma; \mathbb{Z})$ is also finite for $m > 0$.

Now suppose the result is true for patterns of codimension $q < n$. As L_0 is finite, Lemma 1.54 gives that the orbits of the faces of the acceptance domain K contain the hyperplanes spanned by the faces and by Lemma 1.60 the sequence $0 \to C^n \to \cdots \to C^0 \to \mathbb{Z} \to 0$ is exact. From Section 1.6.3, there is also a spectral sequence with $E^1_{pq} = H_p(\Gamma; C^q)$. Note that the module C^{n-1} decomposes as $\bigoplus_{\theta \in I_{n-1}} C^{n-1}_\theta \otimes \mathbb{Z}[\Gamma/\Gamma^\theta]$, for Γ-modules C^{n-1}_θ as in Definition 1.57 but only for singular $(n-1)$-spaces in the Γ-orbit θ. This is because L_0 is finite so any singular $(n-1)$-space is a hyperplane arising from the Γ-orbit of one face of K (so $I_{n-1c}/\Gamma = I_{n-1}$) and non-parallel hyperplanes intersect in singular spaces of dimension less than $n-1$ which give the zero element in C^{n-1}. Hence by Corollary 1.71 there is a decomposition $\bigoplus_{\theta \in I_{n-1}} H_p(\Gamma^\theta; C^{n-1}_\theta)$ of $H_p(\Gamma; C^{n-1})$. Also note that by Lemma 1.60 applied to θ (since the singular spaces in θ are hyperplanes by Lemma 1.43) there is an exact sequence $0 \to C^{n-1}_\theta \to C^{n-2}_\theta \to \cdots \to C^0_\theta \to \mathbb{Z} \to 0$ and we can view this as arising from a codimension $n-1$ projection scheme. Hence by the induction hypothesis, $H_p(\Gamma^\theta; C^{n-1}_\theta)$ is of finite rank for all p. Note that the number $L_{n-1} = |I_{n-1}|$ of Γ-orbits of singular $(n-1)$-spaces is finite, since singular $(n-1)$-spaces arise only from Γ-translates of faces of K, and the fact that K is a polytope implies that there are only finitely many Γ-orbits of faces. Thus $H_p(\Gamma; C^{n-1}) \cong \bigoplus_{\theta \in I_{n-1}} H_p(\Gamma^\theta; C^{n-1}_\theta)$ is of finite rank for all p for a codimension n pattern.

When L_0 is finite [12, V.2.3], the number L_q of Γ-orbits of singular q-spaces is also finite, for $q < n-1$. Thus a similar argument to the one given above shows that $H_p(\Gamma; C^q)$ is of finite rank for all p.

Using the notation of spectral sequences (Page 48), we have shown that E^1_{pq} has finite rational rank for $q < n$. Since the terms E^k_{pq} for $k > 1$ are produced as quotients of kernels by images of the differential maps, E^1_{pq} of finite rank implies that rk E^k_{pq} is finite. Now E^∞_{pn} is produced by successively taking kernels of higher differentials, so if rk $E^1_{pn} = \infty$, then the rank of the image of some differential ∂ must be infinite. However, the fact that rk $E^k_{pq} < \infty$ for all k and $q < n$ implies that the images of the higher differentials must have finite rank. Hence rk $H_p(\Gamma; C^n) \otimes \mathbb{Q} < \infty$ for all p.

Thus the Euler characteristic $e_{\mathcal{P}} = \sum_{i=0}^{d}(-1)^i \text{rk } H_i(\Gamma; C^n) \otimes \mathbb{Q}$ is defined and it is finite as it is a finite sum of finite quantities. ∎

2.4.1 General formula for the Euler characteristic

The aim of this section is to produce a formula for the Euler characteristic for codimension n polytopal projection patterns when L_0 is finite, generalising the formulae found for polytopal projection patterns of codimension 1, 2 and 3. We begin by producing a formula describing the Euler characteristic in codimension n in terms of the Euler characteristic for codimension $n-1$ patterns.

For a codimension n polytopal projection pattern with L_0 finite and acceptance domain K, recall (Lemma 1.54) that the Γ-orbits of $(n-1)$-dimensional faces of K consist of sets of $(n-1)$-dimensional hyperplanes. We thus make the following definition.

DEFINITION 2.62 *Denote by θ_i a Γ-orbit of singular i-spaces and write I_i for the set of all such orbits. Write $I_i^{\theta_j}$ to denote the set of orbit classes of i-dimensional hyperplanes which are contained in singular j-spaces in the Γ-orbit θ_j.*

In the notation of previous sections, we have $\beta = \theta_0$, $\xi = \theta_1$ and $\eta = \theta_2$.

Now from Lemma 1.60, for a codimension n polytopal projection pattern $\mathcal{P}$ with L_0 finite, the sequence (1.2) $0 \to C^n \to \cdots \to C^0 \to \mathbb{Z} \to 0$ is exact and hence by Lemma 2.6, we have $e_{\mathcal{P}} = (-1)^{n+1} e_{\underline{Y}} = e_{C^{n-1}} - e_{C^{n-2}} + \cdots + (-1)^{n+1} e_{C^0}$. This expression arises by breaking the exact sequence (1.2) into short exact sequences $0 \to C^n \to C^{n-1} \to C_0^{n-2} \to 0$, $0 \to C_0^{n-2} \to C^{n-2} \to C_0^{n-3} \to 0$ and so on. This means that $e_{\mathcal{P}} = e_{C^n} = e_{C^{n-1}} + e_{C_0^{n-2}}$ for $e_{C_0^{n-2}} = \pm e_{C^{n-2}} \mp e_{C^{n-1}} \pm \cdots \pm e_{C^0}$.

DEFINITION 2.63 *For a codimension n polytopal projection pattern $\mathcal{P}$ with L_0 finite, write e_n to denote the Euler characteristic $e_{\mathcal{P}} = \sum_{i=0}^{d}(-1)^i rk\ H_i(\Gamma; C^n) \otimes \mathbb{Q}$ and let e_q^θ denote the formula for the Euler characteristic $\sum_{i=0}^{d}(-1)^i rk\ H_i(\Gamma^\theta; C_\theta^q)$. Write e_q^0 for the Euler characteristic $e_{C_0^{q-1}} = e_{C^{q-1}} + e_{C^{q-2}} + \cdots \pm e_{C^0}$.*

LEMMA 2.64 *The Euler characteristic e_n for a codimension n polytopal projection pattern with L_0 finite can be expressed as*

$$e_n = -e_{n-1}^0 + \sum_{\theta_{n-1} \in I_{n-1}} e_{n-1}^{\theta_{n-1}}$$

for $n \geqslant 2$ and $e_1 = L_0$.

Proof Use proof by induction. By the results of Section 2.1, $e_1 = e_{C^0} = L_0$. Also, Lemma 2.19 gives $e_2 = -L_0 + \sum_{\theta_1 \in I_1} L_0^{\theta_0} = -e_1^0 + \sum_{\theta_1 \in I_1} e_1^{\theta_1}$ since $e_{C_0^0} = e_{C^0} = L_0$ as this module fits into a short exact sequence $0 \to C_0^0 \to C^0 \to \mathbb{Z} \to 0$ and $e_{\mathbb{Z}} = 0$ by Corollary 1.68.

Now suppose the result is true for k. Thus we suppose that we have a formula of the above form for the Euler characteristic e_k. Since L_0 is finite, there is an exact sequence $0 \to C^k \to C^{k-1} \to \cdots \to C^0 \to \mathbb{Z} \to 0$ by Lemma 1.60. Using the notation of Lemma 2.6, by Corollary 1.68 we have $e_{\mathbb{Z}} = 0$ so $e_k = e_{C^k} = e_{C^{k-1}} - e_{C^{k-2}} + \cdots \pm e_{C^0}$.

For $e_{k+1} = e_{C^{k+1}}$, we have an exact sequence $0 \to C^{k+1} \to C^k \to \cdots \to C^1 \to C^0 \to \mathbb{Z} \to 0$, and we need to compute $e_{C^k} - e_{C^{k-1}} + \cdots \pm e_{C^0}$. Since we assume a formula for e_k has already been determined, we have an expression for $e_{C_0^{k-1}} = e_{C^{k-1}} - \cdots \pm e_{C^0}$. Thus $e_{C^{k+1}} = e_{C^k} - e_k^0$.

It remains to compute e_{C^k}. Decompose C^k as $\bigoplus_{\theta_k \in I_k} C_{\theta_k}^k \otimes \mathbb{Z}[\Gamma/\Gamma^{\theta_k}]$ as in Lemma 1.62. Since L_0 is finite, by Lemma 1.60 there is also an exact sequence $0 \to C_{\theta_k}^k \to \cdots \to$

$C^0_{\theta_k} \to \mathbb{Z} \to 0$. By the induction hypothesis, the formula for the Euler characteristic $e_{C^k_{\theta_k}} = \sum(-1)^i \mathrm{rk}\, H_i(\Gamma^{\theta_k}; C^k_{\theta_k}) = e^{\theta_k}_k$ is known. Hence $e_{C^k} = \sum_{\theta_k \in I_k} e^{\theta_k}_k$ and $e_{k+1} = -e^0_k + \sum_{\theta_k \in I_k} e^{\theta_k}_k$ as required. ∎

Note that in the same way that e_2 can be equivalently written as $\sum_{\beta \in I_0}(q_\beta - 1)$ (Theorem 2.21) for q_β the multiplicity of a singular 0-space β, there is a similar formulation for e_n. In order to produce this equivalent formula, we begin with some definitions and notation.

DEFINITION 2.65 *Write N_n for the set $\{2, 3, \ldots, n-2, n-1\}$ and set $N_1 = \emptyset = N_2$.*

Define a k-multi-index *to be $\underline{i}\colon = \{i_1, \ldots i_k\}$ for $k \geqslant 1$, where $i_k > i_{k-1} > \ldots > i_1$ and $i_j \in N_n$. Set $\underline{i}\colon = \emptyset$ for $k = 0$.*

DEFINITION 2.66 *Define q_{θ_0} for $\theta_0 \in I_0$ to be the* multiplicity *of a singular 0-space θ_0, that is the number of non-parallel singular 1-spaces intersecting at θ_0. If $\theta_i \in I^{\theta_j}_i$, then the multiplicity of θ_0 is counted only over those singular 1-spaces contained in singular j-spaces in the Γ-orbit θ_j.*

DEFINITION 2.67 *We define the following notation.*

$$\sum_{\theta_{\underline{j}} \in I_{\underline{j}}} (q_{\theta_0} - 1)\colon = \sum_{\theta_{j_k} \in I_{j_k}} \sum_{\theta_{j_{k-1}} \in I^{\theta_{j_k}}_{j_{k-1}}} \cdots \sum_{\theta_{j_1} \in I^{\theta_{j_2}}_{j_1}} \sum_{\theta_0 \in I^{\theta_{j_1}}_0} (q_{\theta_0} - 1)$$

If $\underline{j} = \emptyset$ then $\sum_{\theta_{\underline{j}} \in I_{\underline{j}}}(q_{\theta_0} - 1) = \sum_{\theta_0 \in I_0}(q_{\theta_0} - 1)$.

THEOREM 2.68 *With the above constructions, the Euler characteristic associated to a codimension n polytopal projection pattern with L_0 finite is*

$$e_n = (-1)^{n-1} \sum_{k=0}^{n-2} (-1)^{k+1} \sum_{\underline{j} \text{ a } k\text{-multi-index}} \sum_{\theta_{\underline{j}} \in I_{\underline{j}}} (q_{\theta_0} - 1) \text{ for } n \geqslant 2$$

and $e_1 = L_0$.

Proof Using Lemma 2.64, we prove this result by induction on the codimension n.

For $n = 1$, by Section 2.1, $e_1 = L_0$ as required.

For $n = 2$, we have $N_2 = \emptyset$ and $k = 0$, so the expression reduces to

$$e_2 = (-1)^1(-1)^1 \sum_{\theta_0 \in I_0} (q_{\theta_0} - 1) = \sum_{\theta_0 \in I_0} (q_{\theta_0} - 1)$$

which is equivalent to the result of Theorem 2.21.

Now suppose the result is true for m, so

$$e_m = (-1)^{m-1} \sum_{k=0}^{m-2} (-1)^{k+1} \sum_{\underline{j} \text{ a } k\text{-multi-index}} \sum_{\theta_{\underline{j}} \in I_{\underline{j}}} (q_{\theta_0} - 1).$$

By the recursion formula for the Euler characteristic (Lemma 2.64), we have $e_{m+1} = -e_m^0 + \sum_{\theta_m \in I_m} e_m^{\theta_m}$. Thus

$$\begin{aligned} e_{m+1} &= (-1)^{m-1} \left(-\sum_{k=0}^{m-2} (-1)^{k+1} \sum_{\underline{j} \text{ a } k\text{-multi-index}} \sum_{\theta_{\underline{j}} \in I_{\underline{j}}} (q_{\theta_0} - 1) \right. \\ &\quad \left. + \sum_{\theta_m \in I_m} \sum_{k=0}^{m-2} (-1)^{k+1} \sum_{\underline{j} \text{ a } k\text{-multi-index}} \sum_{\theta_{\underline{j}} \in I_{\underline{j}}} (q_{\theta_0} - 1) \right). \end{aligned}$$

Now the second term in the expression has a sum indexed by $(k+1)$-multi-indices, where $i_{k+1} = m$, for $k = 0, \ldots, m-2$. Equivalently, we can think of this as a sum indexed by k-multi-indices with $i_k = m$ for $k = 1, \ldots, m-1$. In the first term, there will also be sums indexed by k-multi-indices, for $k = 0, \ldots m-2$. In particular, there are no multi-indices with $i_k = m$. Thus no multi-indices appear in both the first and second terms. However, all possible k-multi-indices on $\{2, \ldots, m-1, m\}$ for $k = 0, \ldots, m-1$ appear in either one term or the other since the only $(m-1)$-multi-indices are those with $i_{m-1} = m$, the only 0-multi-index is $\emptyset$, and a k-multi-index $\underline{j}$ for $k = 1, \ldots, m-2$ either has $j_k = m$ and the other entries in $\underline{j}$ have the form of a $(k-1)$-multi-index on $\{2, \ldots, m-1\}$ (since $j_k = m > j_l$ for $l < k$ by definition) which appears in the second term, or $j_k \neq m$ in which case $\underline{j}$ is a multi-index on $\{2, \ldots, m-1\}$, appearing in the first term. Thus we can combine terms to give

$$\begin{aligned}
e_{m+1} &= (-1)^{m-1}\left(\sum_{k=0}^{m-2}(-1)^k \sum_{\underline{j}\text{ a }k\text{-multi-index}} \sum_{\theta_{\underline{j}}\in I_{\underline{j}}}(q_{\theta_0}-1)\right. \\
&\quad \left. + \sum_{k=1}^{m-1}(-1)^k \sum_{\substack{\underline{j}\text{ a }k\text{-multi-index}\\ \text{with } i_k=m}} \sum_{\theta_{\underline{j}}\in I_{\underline{j}}}(q_{\theta_0}-1)\right) \\
&= (-1)^{m-1}\sum_{k=0}^{m-1}(-1)^k \sum_{\underline{j}\text{ a }k\text{-multi-index}} \sum_{\theta_{\underline{j}}\in I_{\underline{j}}}(q_{\theta_0}-1) \\
&= (-1)^{(m+1)-1}\sum_{k=0}^{(m+1)-2}(-1)^{k+1} \sum_{\underline{j}\text{ a }k\text{-multi-index}} \sum_{\theta_{\underline{j}}\in I_{\underline{j}}}(q_{\theta_0}-1)
\end{aligned}$$

which is the required result for a codimension $(m+1)$ projection pattern. Hence by induction, the result holds for all m. ∎

Given the results of the sections investigating polytopal projection patterns with codimensions $n \leqslant 3$, the above results enable us in particular to determine a result about the Euler characteristic for a codimension 4 polytopal projection pattern, as shown below.

THEOREM 2.69 *For a codimension n polytopal projection pattern, if L_0 is finite, then the Euler characteristic satisfies $e_n \geqslant L_0 > 0$ for $n \leqslant 4$.*

Proof For $n = 1$, we have $e_1 = L_0 \geqslant 1$ by Corollary 2.8 and the fact that the polytope K (the acceptance domain for the pattern) has at least one vertex, which gives rise to a Γ-orbit of singular 0-spaces.

For $n = 2$, we have $e_2 = -L_0 + \sum_{\theta_1\in I_1} L_0^{\theta_1}$. Now every singular 0-space is at the intersection of at least two singular 1-spaces θ_1 and θ_1', so a given singular 0-space θ_0 appears in both $L_0^{\theta_1}$ and $L_0^{\theta_1'}$. Thus $\sum_{\theta_1\in I_1} L_0^{\theta_1} \geqslant 2L_0$ as every singular 0-space appears at least twice in the sum. Hence $e_2 \geqslant -L_0 + 2L_0 = L_0 > 0$.

When $n = 3$, by Lemma 2.44 we have $e_3 \geqslant L_0 \geqslant 1$. This result was obtained by first writing e_3 as a sum over $\theta_0 \in I_0$ and then showing that the terms in the sum had value at least L_0.

For $n = 4$, we use $e_4 = -e_3^0 + \sum_{\theta_3 \in I_3} e_3^{\theta_3}$ and the formula $e_3 = \sum_{\theta_0 \in I_0}[-(p_{\theta_0} - 1) + \sum_{l \in I_{\theta_0}}(q_l^{\theta_0} - 1)]$ of Lemma 2.44, where the notation is as in Definition 2.42. Given an element $\theta_0 \in I_0$, in the sum $\sum_{\theta_3 \in I_3} \sum_{\theta_0 \in I_0^{\theta_3}} p_{\theta_0}$, each plane passing through θ_0 is counted at least twice since planes θ_2 are at the intersection of at least two elements of I_3. Also every plane passing through θ_0 contributes at least two to the sum $\sum_{\theta_3 \in I_3} \sum_{\theta_0 \in I_0^{\theta_3}} \sum_{l \in I_{\theta_0}} q_l^{\theta_0}$ since the plane passes through at least one singular 1-space l through θ_0 and is counted at least twice. Thus if $\sum_{\theta_3 \in I_3} \sum_{\theta_0 \in I_0^{\theta_3}} p_\beta$ increases by some amount, then $\sum_{\theta_3 \in I_3} \sum_{\theta_0 \in I_0^{\theta_3}} \sum_{l \in I_{\theta_0}} q_l^{\theta_0}$ increases by at least the same amount. Also note that $p_{\theta_0} > 1$ and $q_l^{\theta_0} > 1$ since singular 0-spaces and singular 1-spaces are at the intersection of at least two singular 2-spaces in θ_3. Hence since $\sum_{\theta_3 \in I_3} L_0^{\theta_3} \geqslant 2L_0$ we have $\sum_{\theta_3 \in I_3} \sum_{\theta_0 \in I_0^{\theta_3}}[-(p_{\theta_0} - 1) + \sum_{l \in I_{\theta_0}}(q_l - 1)] \geqslant 2\sum_{\theta_0 \in I_0}[-(p_{\theta_0} - 1) + \sum_{l \in I_{\theta_0}}(q_l - 1)]$ which means that $e_4 = -e_3^0 + \sum_{\theta_3 \in I_3} e_3^{\theta_3} \geqslant e_3 \geqslant L_0$, as required. ∎

Potentially, Lemma 2.64 and the methods used in the proof of the above result lead to an induction argument on the codimension n which could be used to show that $e_n \geqslant L_0$ for polytopal projection patterns of arbitrary codimension n. However, the notation which would be required in order to do this becomes rather more complicated, so e_5, e_6 and so on are not considered here.

COROLLARY 2.70 *For a codimension n polytopal projection pattern with $n \leqslant 4$, if e_n is defined and finite then L_0 is finite.* ∎

Finally note that by Theorem 2.61, if L_0 is finite then e_n is finite.

We now move on to consideration of some codimension n polytopal projection patterns for which L_0 is infinite.

2.4.2 The hypergeneric case

For hypergeneric polytopal projection patterns (Def 1.50) we have L_0 infinite, but the stabilisers of all singular spaces have rank equal to the dimensions of the spaces they stabilise, and the singular spaces are arranged into hyperplanes. This means that the sequence (1.2) is exact everywhere and a spectral sequence exists for computing the group homology $H_*(\Gamma; C^n)$. The argument of the proof of Corollary 2.56 immediately generalises to verify the following result.

THEOREM 2.71 *For a hypergeneric codimension n polytopal projection pattern $\mathcal{P}$, the Euler characteristic is defined and is infinite.* ∎

Also, the ranks of the cohomology groups $\check{H}^*(M\mathcal{P})$ can be determined in this case, because the methods of Corollary 2.56 can again be used to show that $H_m(\Gamma; C^n) \cong H_{m+n}(\Gamma; \mathbb{Z})$ for $m > 0$ and rk $H_0(\Gamma; C^n) = \infty$.

As in the codimension 2 case, we can also consider whether torsion can arise in the cohomology of hypergeneric projection patterns of codimension n. By the above result, since $H_m(\Gamma; C^n) \cong H_{m+n}(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n+d}{m+n}}$ for $m > 0$, these groups are torsion-free, so it remains to check $H_0(\Gamma; C^n)$. Now we can break the sequence $0 \to C^n \xrightarrow{\delta_{n-1}} C^{n-1} \to \cdots \to C^1 \xrightarrow{\delta_0} C^0 \to \mathbb{Z} \to 0$ (1.2), which is exact in the hypergeneric case, into short exact sequences

$$0 \longrightarrow C^n \longrightarrow C^{n-1} \longrightarrow C_0^{n-2} \longrightarrow 0$$

$$0 \longrightarrow C_0^{n-2} \longrightarrow C^{n-2} \longrightarrow C_0^{n-3} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow C_0^1 \longrightarrow C^1 \longrightarrow C_0^0 \longrightarrow 0$$

$$0 \longrightarrow C_0^0 \longrightarrow C^0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where C_0^q is the image of $\delta_q \colon C^{q+1} \to C^q$. Note that C^0 is free, so $H_0(\Gamma; C^0)$ is torsion-free, and so is $H_0(\Gamma; \mathbb{Z})$. Thus $H_0(\Gamma; C_0^0)$ is torsion-free. Applying the techniques described for the codimension 2 case on Page 74 to the sequence $0 \to C_0^1 \to C^1 \to C^0 \to \mathbb{Z} \to 0$ gives that $H_0(\Gamma; C_0^1)$ is torsion-free since singular 1-spaces have stabilisers Γ^ξ of rank 1 in this case so $\beta_1 \colon \bigoplus_{\xi \in I_{1c}/\Gamma} \colon \Lambda_2\Gamma^\xi \to \Lambda_2\Gamma$ is the zero map. Similarly, considering $C^2 \cong \bigoplus_{\mathcal{D} \in I_{2c}/\Gamma} C_{\mathcal{D}}^2 \otimes \mathbb{Z}[\Gamma/\Gamma^{\mathcal{D}}]$, the methods for codimension 2 patterns yield $H_0(\Gamma^{\mathcal{D}}; C_{\mathcal{D}}^2)$ torsion-free in this case, so $H_0(\Gamma; C^2)$ is torsion-free. Thus from the long exact sequence in homology associated to the short exact sequence $0 \to C_0^2 \to C^2 \to C_0^1 \to 0$, since $H_0(\Gamma; C^2)$ and $H_0(\Gamma; C_0^1)$ are torsion-free, we see that $H_0(\Gamma; C_0^2)$ is torsion-free as well. Similarly considering the sequence $0 \to C_{\mathcal{D}}^3 \to C_{\mathcal{D}}^2 \to C_{0\mathcal{D}}^1 \to 0$ for $\mathcal{D} \in I_{3c}/\Gamma$ gives $H_0(\Gamma; C^3) \cong \bigoplus_{\mathcal{D} \in I_{3c}/\Gamma} H_0(\Gamma^{\mathcal{D}}; C_{\mathcal{D}}^3)$ torsion-free. Proceeding in the same way for C_0^3 and C^4 and so on shows that $H_0(\Gamma; C^n)$ is torsion-free.

Note also that $H_0(\Gamma; C^n) \cong \mathbb{Z}^s$ for $s = \infty$ since $H_p(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^{\binom{\mathrm{rk}\,\Gamma}{p}}$ and $H_0(\Gamma; C^0) \cong \mathbb{Z}^{L_0}$ so the spectral sequences of Section 1.6.3 and induction on n give the result. Hence we have the following.

THEOREM 2.72 *For a hypergeneric codimension n polytopal projection pattern, we have $\check{H}^m(M\mathcal{P})$ torsion-free for all m and*

$$\check{H}^m(M\mathcal{P}) \cong \begin{cases} \mathbb{Z}^{\binom{n+d}{m+n}} & \text{for } m < d \\ \mathbb{Z}^\infty & \text{for } m = d \end{cases}$$

where $\mathbb{Z}^\infty$ denotes a countable direct sum of copies of $\mathbb{Z}$. ∎

2.4.3 Infinite generation of cohomology groups

In the low codimension cases, we saw that there are circumstances under which the cohomology groups $\check{H}^*(M\mathcal{P}) \cong H_{d-*}(\Gamma; C^n)$ for polytopal projection patterns $\mathcal{P}$ have infinite rank over $\mathbb{Q}$. This section contains some general results determining when rk $H_m(\Gamma; C^n)$ will be finite or infinite. We have already seen in Theorem 2.60 that if L_0 is infinite, then $H_0(\Gamma; C^n)$ is of infinite rank, so we next give a corollary of Lemma 2.49. Note that we always assume in this section that the sequence (1.2) of Γ-modules C^i is exact.

THEOREM 2.73 *If rk $H_m(\Gamma; C^{n-2})$ is infinite and the stabilisers of singular $(n-3)$-spaces have rank less than $m+n-3$ for some $m > 0$, then rk $H_m(\Gamma; C^n)$ is infinite.*

Proof With these assumptions, by Lemma 1.72 we know $H_m(\Gamma; C^{n-3}) = 0$. Also $H_p(\Gamma, C^q) = 0$ for $q < n-3$ and $p \geqslant m$ since the stabilisers of singular spaces contained in a singular $(n-3)$-space D have ranks which are not greater than the rank of the stabiliser of D. This means that in the spectral sequence (Section 1.6.3) the higher differentials with domain $H_m(\Gamma; C^{n-1})$ and $H_m(\Gamma; C^{n-2})$ are trivial so we can use the proof of Theorem 2.50 with $n = 3$ replaced by n and C^{00}_η replaced by $C^{(n-3)0}_{\theta_{n-1}}$, to give rk $Ker(d_{n-1,m}\colon H_m(\Gamma; C^{n-1}) \to H_m(\Gamma; C^{n-2}))$ infinite, and hence rk Imd_{nm} infinite. Thus rk $H_m(\Gamma; C^n) =$ rk $Kerd_{nm} +$ rk Imd_{nm} is infinite, as required. ∎

There are also cases when the ranks of groups $H_m(\Gamma; C^n)$ are known to be finite, as shown by the next two results.

THEOREM 2.74 *For a codimension n polytopal projection pattern $\mathcal{P}$, if the stabilisers of singular $(n-1)$-spaces are of rank less than $m+n-1$ for some $m > 0$ then $H_m(\Gamma; C^n) \cong \check{H}^{d-m}(M\mathcal{P})$ is of finite rank over $\mathbb{Q}$.*

Proof With these assumptions, $H_m(\Gamma; C^{n-1}) = 0$ and, as in the proof of the above theorem, $H_m(\Gamma; C^q) = 0$ for all $q < n$. This means that the edge homomorphisms $H_{m+n}(\Gamma; \mathbb{Z}) \to H_m(\Gamma; C^n)$ are isomorphisms, so $H_m(\Gamma; C^n) \cong H^{d-m}(M\mathcal{P}) \cong \mathbb{Z}^{\binom{n+d}{n+m}}$ is of finite rank. ∎

THEOREM 2.75 *For a codimension n polytopal projection pattern $\mathcal{P}$, if only finitely many Γ-orbits of singular q-spaces have stabilisers of rank $r \geqslant m+q$ for all $q < n$ then $H_p(\Gamma; C^n)$ is of finite rank for $p \geqslant m$.*

Proof With these assumptions, $H_p(\Gamma; C^q)$ is finite for $p \geqslant m$ by Lemma 1.72. Hence in the spectral sequence with $E^1_{pn} = H_p(\Gamma; C^n)$, the images of all differentials with domain E^k_{pn} for any k will be of finite rank. In particular $Im(d_{pn}\colon H_p(\Gamma; C^n) \to H_p(\Gamma; C^{n-1}))$ is of finite rank for $p \geqslant m$.

The kernels of the differentials with domain $E^k_{pn} = E^{\infty}_{pn}$ are also of finite rank because rk $Ker\partial$ is a summand of rk $H_{p+n}(\Gamma; \mathbb{Z})$ for some differential ∂ with domain E^k_{pn}, so is finite. If any other differential with domain E^r_{pn} for $r < k$ had kernel of infinite rank, then the image of a higher differential would thus have to be of infinite rank, which is impossible by the first paragraph. In particular $Ker(d_{pn}\colon H_p(\Gamma; C^n) \to H_p(\Gamma; C^{n-1})$ is of finite rank for $p \geqslant m$.

Hence rk $H_p(\Gamma; C^n) =$ rk $Kerd_{pn} +$ rk Imd_{pn} is finite for $p \geqslant m$. ∎

The above discussion does not enable the finiteness or otherwise of all groups $H_m(\Gamma; C^n)$ to be determined. For example, given a codimension 4 polytopal projection pattern with $H_m(\Gamma; C^1)$ of infinite rank, the methods used in Lemma 2.49 are not strong enough to tell us whether $H_m(\Gamma; C^4)$ is of infinite rank. We suspect that rk $H_m(\Gamma; C^4) = \infty$ if rk $H_m(\Gamma; C^1) = \infty$, since expressions like those for the Euler characteristics in Lemma 2.19 or Theorem 2.39 arise when computing cohomology for higher-codimension projection patterns, but with L_0 replaced by L_1, and if the Euler characteristic is defined and infinite, then so should these quantities be. For example, for a codimension 3 canonical projection pattern $\mathcal{P}$, in the formula of Theorem 2.37 for the rank of $H^*(M\mathcal{P}) \otimes \mathbb{Q}$, terms of the form $-L_1 + \sum_{\eta \in I_2} L_1^{\eta}$ appear, which are analogous to the formula for the Euler characteristic for a codimension 2 canonical projection pattern. Thus if $e_{\mathcal{P}}$ is infinite, then such expressions should also be infinite and hence the homology groups $H_m(\Gamma; C^n)$ are likely to have infinite rank. However, in order to prove such results, more work is needed.

2.5 Applications

In this section, we consider several applications of the calculations and results obtained in this chapter.

2.5.1 Euler characteristic 0

Note that the Euler characteristic $e_{\mathcal{P}}$ is zero for a two-dimensional periodic point pattern $\mathcal{P}$, since the continuous hull $M\mathcal{P}$ is a torus in this case. Thus the work of this chapter, and in particular corollary 2.24, shows that periodic patterns do not arise as projection patterns as described above. In fact, periodic patterns have the form of codimension 0 projection patterns, since they can be produced by taking a lattice Λ of any dimension and choosing the space E so that it intersects Λ at more than one point. Then [34] a periodic pattern consisting of points from Λ automatically appears in E, without the need for any projection.

There are two-dimensional quasiperiodic patterns which have $e_{\mathcal{P}} = 0$. One example is the Pinwheel tiling in Figure 1.1. Since this tiling does not satisfy the Finite Local Complexity condition (Def 1.13), the above theory cannot be used to compute its Euler characteristic $e_{\mathcal{P}}$, but [28] the continuous hull $M\mathcal{P}$ of this tiling can be viewed as an S^1-bundle over a certain simplicial complex and its Čech cohomology (with rational coefficients) is related to the cohomology of the circle S^1, for which the Euler characteristic is 0. Since $e_{\mathcal{P}} = 0$, Corollary 2.24 tells us that the Pinwheel tiling cannot be produced from the projection of points in a lattice selected by a polytopal acceptance domain.

2.5.2 Canonical projection patterns

Since canonical projection patterns have acceptance domains which are polytopal, the above work is applicable in this more specific case. In particular, we have shown that for canonical projection patterns, the group $\check{H}^d(M\mathcal{P})$ is of infinite rank over $\mathbb{Q}$ if and only if the number L_0 of Γ-orbits of singular 0-spaces is infinite, a result which appears in [12]. We have also extended the work of [12] to consideration of higher homology groups $H_m(\Gamma; C^n)$ and what can be said about their ranks over $\mathbb{Q}$. In the codimension 2 case, we obtained the result that the Euler characteristic is always defined, which is stronger than the above result for $\check{H}^d(M\mathcal{P})$ only, since its proof entailed showing that the groups $\check{H}^m(M\mathcal{P})$ are always of finite rank for $m < d$.

2.5.3 K-theory

Consider the isomorphism $K^{i-d}(M\mathcal{P}) \otimes \mathbb{Q} \cong \bigoplus_{j=-\infty}^{\infty} \check{H}^{2j+i}(M\mathcal{P}) \otimes \mathbb{Q}$ from Theorem 1.76 for $\mathcal{P}$ a codimension 2 polytopal projection pattern. In this case we have seen that if $L_0 = \infty$ then $\check{H}^d(M\mathcal{P})$ has infinite rank over $\mathbb{Q}$, but all other cohomology groups have finite rational rank, so rk $K^0(M\mathcal{P}) \otimes \mathbb{Q} =$ rk $\check{H}^d(M\mathcal{P}) \otimes \mathbb{Q} +$ rk $\check{H}^{d-2}(M\mathcal{P}) \otimes \mathbb{Q} + \cdots$ could be infinite, but rk $K^1(M\mathcal{P}) \otimes \mathbb{Q} =$ rk $\check{H}^{d-1}(M\mathcal{P}) \otimes \mathbb{Q} +$ rk $\check{H}^{d-3}(M\mathcal{P}) \otimes \mathbb{Q} + \cdots$ will always be finite. Additionally, for 3-dimensional codimension 3 patterns, since rk $K^0(M\mathcal{P}) =$ rk $\check{H}^3(M\mathcal{P})+$rk $\check{H}^1(M\mathcal{P})$ and rk $K^1(M\mathcal{P}) \otimes \mathbb{Q} =$ rk $\check{H}^2(M\mathcal{P}) \otimes \mathbb{Q}+$rk $\check{H}^0(M\mathcal{P}) \otimes \mathbb{Q}$, both $K^0(M\mathcal{P})$ and $K^1(M\mathcal{P})$ could be of infinite rank over $\mathbb{Q}$. For polytopal projection patterns of higher dimension and codimension, it is again possible that both rk $K^0(M\mathcal{P}) \otimes \mathbb{Q}$ and rk $K^1(M\mathcal{P}) \otimes \mathbb{Q}$ could be infinite.

This has applications to ideas related to those in Section 1.6.1. In Section 1.6.1, we saw the gap labelling map $Gap(H) \to K_0(A)$ for $Gap(H)$ a set of subsets of $\mathbb{R}$ and A the C^*-algebra associated to tilings $\mathcal{P}$ in $M\mathcal{P}$. A related concept is to consider the topological K-theory of $M\mathcal{P}$, for which there is a trace map $K^0(M\mathcal{P}) \to \mathbb{R}$ sending projections to intervals of $\mathbb{R}$ which has image consisting of finitely many intervals since the Hamiltonian H has bounded spectrum in $\mathbb{R}$. Thus if rk $K^0(M\mathcal{P})$ is infinite, then the kernel of this map is of infinite rank over $\mathbb{Q}$. Elements in the kernel are called infinitesimals, and are of interest as knowledge of these elements should yield more information about the structure of the space $M\mathcal{P}$ than is detected by the trace, especially since the group of infinitesimals has been shown by the results of this chapter to be of infinite rank in many cases. Work in this area is being carried out by Bellissard, Bendetti and Gambaudo [2] amongst others.

2.5.4 Deformations

In [8], *deformations* of tilings are considered. Given a d-dimensional tiling $\mathcal{T}$ (Def 1.2), a deformation is a tiling $\mathcal{T}'$ which is combinatorially identical to $\mathcal{T}$, so the tiles in $\mathcal{T}$ and $\mathcal{T}'$

can be labelled in such a way that if tiles meet in $\mathcal{T}$ then they correspond to tiles which meet in $\mathcal{T}'$, but the tiling $\mathcal{T}'$ differs geometrically from $\mathcal{T}$, so the shapes and positions of tiles in $\mathcal{T}'$ could be different from the shapes and positions of tiles in $\mathcal{T}$. A deformation f is [8] given by a set of vectors specifying the one-dimensional edges of each prototile in a tiling so that if two prototiles meet at an edge in a tiling, then the vectors associated to the edge are the same. The sum of all vectors associated to a prototile should be zero. Such deformations [32] do not alter the topology of the continuous hull — given two deformations f, g, and the associated tilings $\mathcal{T}_f$ and $\mathcal{T}_g$ then $M\mathcal{T}_f$ is homeomorphic to $M\mathcal{T}_g$. However, the relationship between the tilings in the spaces $M\mathcal{T}_f$ and $M\mathcal{T}_g$ is more interesting. In [8], a map $\mathcal{I}$ is defined from the space Ξ of deformations to $\check{H}^1(M\mathcal{P};\mathbb{R}^d) = \check{H}^1(M\mathcal{P}) \otimes \mathbb{R}^d$. The image of this map is then investigated to produce conditions for when tilings in $M\mathcal{T}_f$ and $M\mathcal{T}_g$ are related, and how strong the relationship is. For example [8], if $f, g \in \Xi$, and $\mathcal{I}(f) = \mathcal{I}(g)$, then given an $\mathbb{R}^d$-equivariant homeomorphism ϕ: $M\mathcal{T}_f \to M\mathcal{T}_g$ the patterns $\mathcal{T}' \in M\mathcal{T}_f$ and $\phi(\mathcal{T}') \in M\mathcal{T}_g$ are mutually locally derivable.

Since we have shown that for codimension 2 polytopal projection patterns the group $H^1(M\mathcal{P})$ is of finite rank, this means that for associated tilings, deformations only produce finitely many MLD classes (Def 1.4). For higher codimensions, there are possibly infinitely many MLD classes, but Chapter 2 gives conditions when this is not the case. In particular, for tilings associated to hypergeneric (Def 1.50) or generic (Def 1.51) polytopal projection patterns for which the cohomology can be computed, there are only finitely many possible MLD classes produced by deformations.

We have seen in this chapter a consideration of one particular topological invariant for polytopal projection patterns $\mathcal{P}$ — the Euler characteristic. Since this invariant was defined using the Čech cohomology $\check{H}^*(M\mathcal{P})$ of the continuous hull $M\mathcal{P}$ for $\mathcal{P}$, some results about these cohomology groups were also obtained. The next chapter contains further discussion of Čech cohomology $\check{H}^*(M\mathcal{P})$ for polytopal projection patterns and models for the continuous hull $M\mathcal{P}$ which simplify the computation of $\check{H}^*(M\mathcal{P})$.

Chapter 3

Cohomology and models for $M\mathcal{P}$

The cohomology $\check{H}^*(M\mathcal{P})$ of the continuous hull $M\mathcal{P}$ for a canonical projection pattern $\mathcal{P}$ can be calculated as in [12], but recent work in this area aims to produce alternative models for $M\mathcal{P}$ whose cohomology is more straightforward to compute and gives a more intuitive idea of the structure of $M\mathcal{P}$ which is detected by the cohomology. It was shown in [12, Chapter III] that for a d-dimensional codimension 1 projection pattern (which need not be canonical), the continuous hull in that case is homeomorphic to a punctured $(d+1)$-dimensional torus, where the number of punctures depends on the number of Γ-orbit classes of points in the boundary of the acceptance domain K. These ideas were extended in a paper by Pavel Kalugin [20] to provide a geometric interpretation of the continuous hull and its cohomology, for certain projection patterns of codimension $n > 1$.

More specifically, in [20], a d-dimensional codimension n canonical projection pattern $\mathcal{P}$ is considered, for which $n = d$ and which satisfies the following (referred to in [20] as the *rationality condition*).

Firstly, there should be two vectors, $k_i \in V$ and $n_i \in E$ associated to each face F_i in the boundary of the acceptance domain K of $\mathcal{P}$, with the properties that the hyperplane H_i spanned by F_i has k_i as its normal vector and n_i is normal to some hyperplane H_i' in E which intersects F_i at a point. Secondly, the space $H_i + H_i'$ should be an affine torus in $\mathbb{T}^{n+d}$ of dimension $n + d - 2 = 2(d-1)$ orthogonal to n_i and k_i.

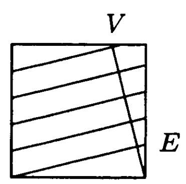
It is shown in [20] that the cohomology of $M\mathcal{P}$ is isomorphic to the cohomology of $\mathbb{T}^{n+d} \setminus A$, where A is an arrangement of $(2(d-1))$-dimensional tori, thickened so that A is a $2d$-dimensional submanifold of $\mathbb{T}^{2d}$. Note that the tori in A correspond to the tori $H_i + H_i'$ defined above.

Further work in this area is being carried out by Pavel Kalugin, Franz Gähler and others. However, on studying [20] a generalisation readily suggested itself, so the following results have been obtained independently. It is shown in this chapter that results analogous to those of [12] hold for the larger class of d-dimensional codimension n polytopal projection patterns with the number L_0 of Γ-orbits of singular 0-spaces finite (for which n divides d by Lemma 1.56 but n need not be equal to d). Also in this chapter, we consider a further generalisation which potentially yields results for polytopal projection patterns of codimension 2 with L_0 infinite.

We begin by setting out the methodology and main results of [20].

3.1 Setup and existing result

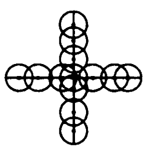
In [20], projection patterns are viewed in a different way from those described in Section 1.1. As before, we have a lattice Λ in $\mathbb{R}^N$, a subspace E of $\mathbb{R}^N$, a space V orthogonal to E with the property that $\pi^\perp(\Lambda) \cap V$ is dense in V for $\pi^\perp$ the projection map with kernel E, and an acceptance domain K in V. However, in this setting, the projection scheme is considered to be within the torus $\mathbb{T}^{n+d}$, formed as the quotient $(V+E)/\Lambda'$, where $V + E \cong \mathbb{R}^{n+d}$ and $\Lambda' \cong \mathbb{Z}^{n+d}$ is the subset of Λ with the property that $\pi^\perp(\Lambda') \cap V = \Gamma$ for Γ as in Definition 1.34. If the tiling space E intersects Λ' only at the origin, then E is dense in $\mathbb{T}^{n+d}$.



The paper [20] proceeds by noting that for a canonical projection pattern $\mathcal{P}$ in the space E with L_0 finite, if E intersects the hyperplane H_i which is the span of an $(n-1)$-dimensional boundary face F_i of the acceptance domain K at some point, then in the torus $\mathbb{T}^{n+d}$ it intersects H_i in a dense set of points. Viewed in E, this set of points is a Delone set in a hyperplane subspace of E.

Since H_i is of dimension $n-1$, it has codimension 1 in V and hence the vector k_i in the rationality condition is a normal to H_i. When $n = d$, if the pattern $\mathcal{P}$ satisfies the rationality condition, then [20] the hyperplane H_i' in E containing the Delone set of points of intersection with H_i is of codimension 1 in E with normal n_i. The rationality condition also gives that the space $H_i + H_i'$ is an affine torus of codimension 2 in $\mathbb{T}^{n+d}$ orthogonal to n_i and k_i. Note that if hyperplanes associated to faces of K are in the same Γ-orbit, then their associated vectors will be of equal magnitude and direction, so we need only consider distinct Γ-orbits of faces of K.

The fact that the set of points $E \cap F_i$ is a Delone set, and in particular is relatively dense so there exists $R > 0$ such that any ball in H_i of radius R contains a point of $E \cap F_i$, implies that a (thickened) hyperplane containing these points can be produced by taking balls in E of radius R about each point, as shown in the diagram below.



This leads to consideration in [20] of the set $Y_R = \bigcup_{i \in I_K} F_i + B_R^E$, whose intersection with E is the set of hyperplanes $\{H_i'\}_{i \in I_k}$ thickened by some amount in the direction n_i, for I_K the set enumerating the Γ-orbits of faces of K. Note that more than one translate of the acceptance domain K may be needed in order for all possible Γ-orbits of singular spaces to appear in Y_R, but since L_0 is finite by assumption, there are only a finite number of Γ-orbits of singular spaces of any dimension, so finitely many translates of K are sufficient. In fact, since the intention below is to excise Y_R from $\mathbb{T}^{n+d}$ and then complete the resulting space, to ensure that the completion of $\mathbb{T}^{n+d} \setminus Y_R$ is not $\mathbb{T}^{n+d}$ the space Y_R must be of dimension $n + d$. This means that we instead consider $Y_R = \bigcup_{i \in I_K}((H_i + B_\epsilon^V) + B_R^E)$ where B_ϵ^V is a one-dimensional space in V of fixed length ϵ containing the vector k_i which 'thickens' the hyperplane H_i, for ϵ small with respect to the inradius of the smallest n-cell in the set $\bigcup_\gamma (K + \gamma)$ of translates of K in which representatives of all Γ-orbits of singular spaces appear. Thus $(H_i + B_\epsilon^V) + B_R^E$ is an $(n+d)$-dimensional object and Y_R is $(n+d)$-dimensional as well.

The next step in [20] is to show that the continuous hull $M\mathcal{P}$ for $\mathcal{P}$ is homeomorphic to the inverse limit $\varprojlim X_R$ as R increases, where X_R is the completion of $\mathbb{T}^{n+d} \setminus Y_R$ with respect to the metric ρ defined below. First note that there is a metric in $\mathbb{T}^{n+d} \cong \mathbb{R}^{n+d}/\mathbb{Z}^{n+d}$ induced from the Euclidean metric on $\mathbb{R}^{n+d}$. This in turn induces the following metric on $\mathbb{T}^{n+d} \setminus Y_R$.

DEFINITION 3.1 *Define ρ to be the metric on $\mathbb{T}^{n+d} \setminus Y_R$ which gives $\rho(a,b)$ as the infimum of the lengths of all paths connecting the points a and b in $\mathbb{T}^{n+d}$ but avoiding Y_R.*

DEFINITION 3.2 *Define $X := \varprojlim X_{r_k}$ for (r_k) a monotone increasing sequence of real numbers $r_k \geqslant R$, where the maps $\imath_k : X_{r_{k+1}} \to X_{r_k}$ are the extensions of the maps $i_k : \mathbb{T}^{n+d} \setminus Y_{r_{k+1}} \to \mathbb{T}^{n+d} \setminus Y_{r_k}$ (which are continuous with respect to ρ). Write $\pi_k : X \to X_{r_k}$ for the projection maps.*

It is then shown in [20] that there is an isomorphism $\check{H}^*(M\mathcal{P}) \cong H^*(\mathbb{T}^{n+d} \setminus Y_{r_0})$ for some r_0. Denoting Y_{r_0} by A, by definition A is an arrangement of thickened tori of the form $(H_i + B_\epsilon^V) + B_R^E$. A formula for the rank of $H^*(\mathbb{T}^{n+d} \setminus A)$ over $\mathbb{Q}$ is then determined in [20] as follows, using the long exact sequence in relative cohomology for the pair $(\mathbb{T}^{n+d}, \mathbb{T}^{n+d} \setminus A)$.

$$\cdots H^{m-1}(\mathbb{T}^{n+d} \setminus A) \xrightarrow{d^m} H^m(\mathbb{T}^{n+d}, \mathbb{T}^{n+d} \setminus A) \xrightarrow{\alpha^m} H^m(\mathbb{T}^{n+d}) \xrightarrow{\beta^m} H^m(\mathbb{T}^{n+d} \setminus A) \cdots$$

Since $\mathbb{T}^{n+d}$ is a compact orientable manifold containing the compact subset $\mathbb{T}^{n+d} \setminus A$, by Poincaré-Alexander-Lefschetz duality [4, Thm 8.3], we have $H^m(\mathbb{T}^{n+d}, \mathbb{T}^{n+d} \setminus A) \cong H_{n+d-m}(A)$. Hence the long exact sequence can be broken into five-term exact sequences of the form

$$0 \to Im(\beta^{m-1}) \to H^{m-1}(\mathbb{T}^{n+d} \setminus A) \overset{d^m}{\to} H_{n+d-m}(A) \overset{\alpha^m}{\to} H^m(\mathbb{T}^{n+d}) \to Im(\beta^m) \to 0. \quad (3.1)$$

From this, we obtain one of the main results of [20], stated in the following theorem. Recall that rk G denotes rk $(G \otimes \mathbb{Q})$.

THEOREM 3.3 *For a d-dimensional codimension n canonical projection pattern $\mathcal{P}$ which satisfies the rationality condition, the torsion-free part of $\check{H}^{m-1}(M\mathcal{P})) \cong H^{m-1}(\mathbb{T}^{n+d} \setminus A)$ is given by the formula*

$$rk\ H^{m-1}(\mathbb{T}^{n+d} \setminus A) = rk\ H_{n+d-m}(A) + rk\ Im(\beta^{m-1}) + rk\ Im(\beta^m) - \binom{n+d}{m}. \quad (3.2)$$

∎

3.2 Extension of existing result

For canonical projection patterns with $d > n$, the rationality condition fails to hold as stated, because the hyperplane in E defined by the points of intersection with a face F_i of the acceptance domain K will be of codimension greater than one. To see this, we begin by assuming that L_0, the number of orbit classes of singular 0-spaces, is finite. This then implies that the cohomology $\check{H}^*(M\mathcal{P})$ is finitely generated over $\mathbb{Q}$ (see Theorem 2.61) and also that the orbits of the faces of K contain the hyperplanes spanned by the faces by Lemma 1.54. By Lemma 1.56 the rank of the stabiliser of a hyperplane H_i associated to an $(n-1)$-dimensional face of K is $\frac{n+d}{n}(n-1)$. Hence, in the quotient with respect to Λ', the real span of those vectors $\Lambda'^{\mathcal{D}}$ in Λ' which project to the stabiliser $\Gamma^{\mathcal{D}}$ of an $(n-1)$-dimensional hyperplane D forms a $(\frac{n+d}{n}(n-1))$-dimensional torus of codimension

$n+d-\frac{n+d}{n}(n-1) = \frac{n+d}{n} =: \nu$ in $\mathbb{T}^{n+d} \cong (V+E)/\Lambda'$. If $d > n$ then $\nu > 2$ so the subtori of $\mathbb{T}^{n+d}$ have dimension $\nu(n-1) = n+d-\nu < n+d-2$ and hence the rationality condition does not hold.

However, the theory of [20] does not break down if we extend to the more general setting of d-dimensional codimension n polytopal projection patterns with L_0 finite and $d \geqslant n$. The proof of [20, Corollary 1], which asserts that $M\mathcal{P}$ is homeomorphic to the space X, is still valid since it does not use the assumption that H_i is of codimension 1 in E. The vectors k_i and n_i are used explicitly in the proof of [20, Corollary 2], which gives that the cohomology of $M\mathcal{P}$ is isomorphic to that of $\mathbb{T}^{n+d} \setminus A$. They are used to define a local coordinate system on $\mathbb{T}^{n+d} \setminus A$ and then to define hyperplanes $\{x \in V : x \cdot k_i = 0\}$ and $\{x \in E : x \cdot n_i = 0\}$. An amendment can be made to this result to ensure that the conclusion of the therorem still holds for projection patterns with $n \neq d$. When $d > n$, the hyperplane H_i' is of codimension greater than 1 in E, so there is a hyperplane of vectors normal to H_i' in E. Associate to F_i the vector k_i as before, but now also associate to F_i a set of vectors $\{n_i^j\}$ which forms a basis of the hyperplane normal to H_i'. Irrespective of the choice of basis $\{n_i^j\}$, the torus $H_i + H_i'$ is orthogonal to all the vectors n_i^j. Thus if we replace the vector n_i by the set $\{n_i^j\}_{j=1}^{\nu-1}$ and consider the hyperplane $\{x \in E : x \cdot n_i^j = 0 \text{ for all } j\}$ instead of $\{x \in E : x \cdot n_i = 0\}$, the result from [20] holds in this setting.

The cohomology of $\mathbb{T}^{n+d} \setminus A$ is computed in exactly the same way as before, using relative cohomology for the pair $(\mathbb{T}^{n+d}, \mathbb{T}^{n+d} \setminus A)$, so we have the following result.

THEOREM 3.4 *For a d-dimensional codimension n polytopal projection pattern $\mathcal{P}$ with L_0 finite, the torsion-free part of $\check{H}^{m-1}(M\mathcal{P})) \cong H^{m-1}(\mathbb{T}^{n+d} \setminus A)$ is given by the formula*

$$rk\, H^{m-1}(\mathbb{T}^{n+d} \setminus A) \;=\; rk\, H_{n+d-m}(A) + rk\, Im(\beta^{m-1}) + rk\, Im(\beta^m) - \binom{n+d}{m}. \quad (3.3)$$

∎

3.2.1 The codimension 2 case

We now consider the codimension 2 case and investigate the consequences of Theorem 3.4 in more detail. In particular, we show that the formula (3.3) is equivalent to the formula stated in Theorem 2.12 in Chapter 2 (taken from [12]). A specific example of a 4-dimensional codimension 2 polytopal projection pattern is considered, which does not satisfy the rationality condition of [20] but whose cohomology is isomorphic to that of $\mathbb{T}^6 \setminus A$, a six-dimensional torus with an arrangement A of three-dimensional tori removed from it. Several examples of 2-dimensional codimension 2 polytopal projection patterns which do satisfy the rationality condition can be found in [20].

Note that for a codimension 2 pattern with L_0 finite, we have $\nu = \frac{d+2}{2} \in \mathbb{Z}$ so in particular the dimension of the pattern must be even. Also in this case, the faces of the acceptance domain K give rise to lines of infinite length which intersect in points. This means that the quantities in the formula are straightforward to compute. We do not discuss codimension 3 patterns here, for which the faces are planes intersecting in lines, giving rise to tori which intersect in circles, since the calculations are not so straightforward. One example of a 3-dimensional codimension 3 pattern is discussed in [20].

It is shown in [20] that the arrangement A of tori arising from the $F_1, \ldots, F_t$ of K in distinct Γ-orbits have 0-dimensional intersection only, since the faces of K (which are 1-dimensional) intersect at points. Note that, for codimension $n = 2$ polytopal projection patterns, these codimension ν tori in $\mathbb{T}^{n+d}$ also have dimension $n + d - \nu = n + d - (n + d)/2 = \nu$. The following result gives the homology groups for the arrangement A, in which $e_A :\, = \sum_{i=0}^{\infty}(-1)^i \mathrm{rk}\; H_i(A) \otimes \mathbb{Q}$.

LEMMA 3.5 *The homology groups $H_*(A)$ of an arrangement A of t tori of dimension ν, intersecting only at points, with p connected components are the following.*

$$
\begin{aligned}
H_0(A) &= \mathbb{Z}^p \\
H_1(A) &= \mathbb{Z}^{(\nu-1)t-e_A+p} \\
H_2(A) &= \mathbb{Z}^{\binom{\nu}{2}t} \\
H_3(A) &= \mathbb{Z}^{\binom{\nu}{3}t} \\
\vdots \;\; &\;\; \vdots \;\;\; \vdots \\
H_{\nu-1}(A) &= \mathbb{Z}^{\binom{\nu}{\nu-1}t} \\
H_\nu(A) &= \mathbb{Z}^t
\end{aligned}
$$

Proof Consider the spectral sequence with $\bigoplus_{i=1}^m H_*(\mathbb{T}^\nu)$ in the E^1_{0p} column and $\bigoplus H_*(\mathbb{T}^\nu_{i_1} \cap \ldots \cap \mathbb{T}^\nu_{i_j})$ in the $E^1_{j-1,p}$ column, for $j = 2, \ldots, t$.

$\bigoplus_{i=1}^t H_\nu(\mathbb{T}_i^\nu)$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bigoplus_{i=1}^t H_1(\mathbb{T}_i^\nu)$ $\xleftarrow{d^1_{11}}$	0	0	0
$\bigoplus_{i=1}^t H_0(\mathbb{T}_i^\nu)$ $\xleftarrow{d^1_{01}}$	$\bigoplus H_0(\mathbb{T}^\nu_{i_1} \cap \mathbb{T}^\nu_{i_2})$ $\xleftarrow{d^1_{02}}$	$\cdots$ $\xleftarrow{d^1_{03}}$	$H_0(\bigcap_{i=0}^t \mathbb{T}_i^\nu)$

Since the tori in the arrangement A intersect only in points, we have $H_*(\mathbb{T}^\nu_{i_1} \cap \ldots \cap \mathbb{T}^\nu_{i_j}) = 0$ for $* > 0$ and $j = 2, \ldots, t$. On the level of chain complexes, we have an exact sequence $0 \to C_*(\bigcap_{i=1}^t \mathbb{T}_i^\nu) \to \bigoplus C_*(\bigcap_{j=1}^{t-1} \mathbb{T}^\nu_{i_j}) \xrightarrow{\delta_{t-1}} \bigoplus C_*(\bigcap_{j=1}^{t-2} \mathbb{T}^\nu_{i_j}) \to \cdots \to \bigoplus C_*(\mathbb{T}^\nu_{i_1} \cap \mathbb{T}^\nu_{i_2}) \to \bigoplus C_*(\mathbb{T}_i^\nu) \to C_*(A) \to 0$. This can be split into a series of short exact sequences $0 \to C_*(\bigcap_{i=1}^t \mathbb{T}_i^\nu) \to \bigoplus C_*(\bigcap_{j=1}^{t-1} \mathbb{T}^\nu_{i_j}) \to Im\delta_{t-1} \to 0$, then $0 \to Im\delta_{t-1} \to \bigoplus C_*(\bigcap_{j=1}^{t-2} \mathbb{T}^\nu_{i_j}) \to Im\delta_{t-2} \to 0$ and so on, until $0 \to Im\delta_3 \to \bigoplus C_*(\bigcap_{j=1}^{3} \mathbb{T}^\nu_{i_j}) \to Im\delta_2 \to 0$ and finally $0 \to Im\delta_2 \to \bigoplus C_*(\mathbb{T}^\nu_{i_1} \cap \mathbb{T}^\nu_{i_2}) \to \bigoplus C_*(\mathbb{T}_i^\nu) \to C_*(A) \to 0$. In homology, we then have short exact sequences $0 \to H_0(Im\delta_i) \to \bigoplus H_0(\bigcap_{j=1}^{i-1} \mathbb{T}^\nu_{i_j}) \to H_0(Im\delta_{i-1}) \to 0$ since $\bigcap_{j=1}^{i-1} \mathbb{T}^\nu_{i_j}$ consists only of points for $i = 2, \ldots, t-1$ so $H_m(\bigcap_{j=1}^{i-2} \mathbb{T}^\nu_{i_j}) = 0$ and hence $H_m(\delta_{i-1}) = 0$ for $m > 0$. The spectral sequence splices all such short exact sequences in homology together [4] so the zeroth row of the spectral sequence table is exact up to $\bigoplus H_0(\bigcap_{j=1}^{3} \mathbb{T}^\nu_{i_j})$. Passing to the E^2 page thus gives the following result. There are only two non-zero columns, so $E^2 = E^\infty$.

$\bigoplus_{i=1}^{t} H_\nu(\mathbb{T}_i^\nu)$	0	0
$\vdots$	$\vdots$	$\vdots$
$\bigoplus_{i=1}^{t} H_1(\mathbb{T}_i^\nu)$	0	0
$\bigoplus_{i=1}^{t} H_0(\mathbb{T}_i^\nu)/Imd_{01}^1$	$Kerd_{01}^1/Imd_{02}^1$	0

The alternating sum of all the entries in the spectral sequence table gives [6] the Euler characteristic e_A. However, the Euler characteristic of a torus $\mathbb{T}^\nu$ is zero, so the alternating sum of the groups $H_*(\mathbb{T}_i^\nu)$ is zero. Hence $e_A = 0 - \text{rk } Imd_{01}^1 - \text{rk } Kerd_{01}^1 + \text{rk } Imd_{02}^1$. As the arrangement A has p connected components, $H_0(A) = \mathbb{Z}^p$, so the rank of the image of d_{01}^1 must be $t-p$. Hence $\text{rk } Kerd_{01}^1 - \text{rk } Imd_{02}^1 = -e_A - (t-p)$.

Therefore, $H_0(A) = \mathbb{Z}^p$, $H_1(A) = \mathbb{Z}^{\nu t - e_A - (t-p)}$ and $H_m(A) = \mathbb{Z}^{\binom{\nu}{m} t}$ for $2 \leqslant m \leqslant \nu$, as required. ∎

Some results about the Euler characteristic of codimension 2 polytopal projection patterns (see Chapter 2) can also be deduced in this setting.

LEMMA 3.6 *The Euler characteristic $e_A = -e$, where e is* (Def 2.1) *the Euler characteristic of $M\mathcal{P}$.*

Proof The isomorphism $\check{H}^*(M\mathcal{P}) \cong H^*(\mathbb{T}^{d+2} \setminus A)$ implies that the Euler characteristic of $H^*(\mathbb{T}^{d+2} \setminus A)$ is e. The isomorphism $H^m(\mathbb{T}^{d+2}, \mathbb{T}^{d+2} \setminus A) \cong H_{d+2-m}(A)$, together with the fact that $n = 2$ is even and d is even since L_0 is finite so the parity of m is equal to the parity of $d+2-m$, implies that the Euler characteristic of $H^*(\mathbb{T}^{d+2}, \mathbb{T}^{d+2} \setminus A)$ is e_A. Also Lemma 1.68 gives that the Euler characteristic of the torus $\mathbb{T}^{n+d}$ is zero.

From the sequence (3.1) we thus see that $0 = e_A + e$, that is the Euler characteristic of $H^*(\mathbb{T}^{n+d})$ is equal to the sum of the Euler characteristic of $H^*(\mathbb{T}^{n+d}, \mathbb{T}^{n+d} \setminus A)$ and that of $H^*(\mathbb{T}^{n+d} \setminus A)$. Hence $e = -e_A$, as required. ∎

COROLLARY 3.7 *For a codimension 2 polytopal projection pattern with L_0 finite, its Euler characteristic e is positive.*

Proof The Euler characteristic of a one-point union A of t tori of dimension ν is $e_A = 0-(1-t) = 1-t$ as the Euler characteristic of a single torus is 0 and to create the one-point union, we add in $t-1$ 1-simplices (homotopy equivalent to points) which each contribute -1 to e_A. Since there are at least two tori in the arrangement A arising from a codimension 2 polytopal projection pattern (because the acceptance domains for such patterns have at least two faces) we have $e_A < 0$ and hence by the above lemma $e = -e_A > 0$. ∎

This provides an alternative proof of Corollary 2.24 in Chapter 2.

Next consider the maps in the relative cohomology sequence (3.1). Since $H_i(A) = 0$ for $i > \nu$, the map β^m is an isomorphism for $m < \nu$ so rk $Im(\beta^m) = \binom{n+d}{m}$ in these cases.

The map $\alpha^{d+1}\colon H_1(A) \to H^{d+1}(\mathbb{T}^{d+2}) \cong H_1(\mathbb{T}^{d+2})$ is surjective. To see this, note that $e \geqslant t-1$ by Lemma 2.22 and the number t of distinct faces of the acceptance domain K is at least 2 since K is bounded in the 2-dimensional space V. This then gives rk $H_1(A) = \frac{d+2}{2}t - t + e + 1 \geqslant \frac{d+2}{2}2 = d+2 = \text{rk } H^{d+1}(\mathbb{T}^{2+d})$. It thus remains to show that the $d+2$ generators of $H^{d+1}(\mathbb{T}^{d+2}) \cong H_1(\mathbb{T}^{d+2})$ correspond to elements of $H_1(A)$ under α^{d+1}.

Note the following points. Every $\gamma \in \Gamma$ is in at least one stabiliser when L_0 is finite since the stabilisers of singular 1-spaces have rank $\frac{n+d}{n}$ and there are at least n such spaces because K is of positive volume, so if there were fewer than n linearly independent lines v_i intersecting at any vertex of K then the parallelepiped formed by these lines and some of their translates would have volume $v_1 \wedge \ldots \wedge v_n = 0$ which is a contradiction. Also, the tori in A are formed from the real span of the vectors in Λ' projecting to the stabilisers of hyperplanes. Thus in the quotient $\mathbb{T}^{n+d} \cong (V+E)/\Lambda'$, the tori in A are embedded in $\mathbb{T}^{n+d}$ (as a simplicial complex) so that they contain cycles which generate $H_1(\mathbb{T}^{n+d})$ and hence $H_1(A) \twoheadrightarrow H_1(\mathbb{T}^{n+d})$ as required.

The above results give that β^{d+1} and β^{d+2} are zero and hence $H^m(\mathbb{T}^{d+2} \setminus A) = 0$ for $m > d$.

Now rk $Im(\beta^m) = \binom{d+2}{m} - r_{d+2-(m+1)}$ for $\nu \leqslant m \leqslant d$, where r_m (which arises in the calculations of [12]) is the rank of the map δ in the sequence $\cdots \to H_m(\Gamma; C^1) \xrightarrow{\delta} H_m(\Gamma; C_0^0) \to H_{m-1}(\Gamma; C^2) \to \cdots$, where $H_m(\Gamma; C_0^0) \cong H_{m+1}(\Gamma; \mathbb{Z}) \cong H_{m+1}(\mathbb{T}^{d+2})$ for $m > 0$ and rk $H_0(\Gamma; C_0^0) = \text{rk } H_1(\mathbb{T}^{d+2}) + \text{rk } H_0(\Gamma; C^0) - \text{rk } H_0(\mathbb{T}^{d+2})$.

Substituting these results into the formula (3.3) gives the following, in which we write $\nu = \frac{d+2}{2}$.

$$\begin{aligned}
\text{rk } H^{d-p}(\mathbb{T}^{2+d} \setminus A) &= \text{rk } H_{p+1}(A) + \text{rk } Im(\beta^{d-p}) + \text{rk } Im(\beta^{d-p+1}) - \binom{d+2}{d-p+1} \\
&= \binom{\frac{d+2}{2}}{p+1} t + \binom{d+2}{d-p} - r_{d+2-(d-p+1)} \\
&\quad + \binom{d+2}{d-p+1} - r_{d+2-(d-p+1+1)} - \binom{d+2}{d-p+1} \\
&= \binom{\nu}{p+1} t + \binom{2\nu}{p+2} - r_p - r_{p+1} \text{ for } p > 0 \\
\text{rk } H^{d}(\mathbb{T}^{2+d} \setminus A) &= (\nu - 1)t + e + 1 + \binom{2\nu}{2} - (d+2) - r_1
\end{aligned}$$

This is in accordance with the formulae given in [12], which were only applicable to canonical projection patterns.

Hence we have proved the following result.

THEOREM 3.8 *For a d-dimensional codimension* 2 *polytopal projection pattern with* L_0 *finite* (*so* d *is even*)

$$\begin{aligned}
rk\ \check{H}^{d-p}(\mathcal{MP}) \cong rk\ H^{d-p}(\mathbb{T}^{2+d} \setminus A) &= \binom{\nu}{p+1} t + \binom{2\nu}{p+2} - r_p - r_{p+1} \text{ for } p > 0 \\
rk\ \check{H}^{d}(\mathcal{MP}) \cong rk\ H^{d}(\mathbb{T}^{2+d} \setminus A) &= (\nu - 1)t + e + 1 + \binom{2\nu}{2} - (d+2) - r_1. \quad \blacksquare
\end{aligned}$$

Compare this with the results of Section 2.2.2.

EXAMPLE 3.9 *The Heptagonal tiling*

This tiling was described in [16]. It is a codimension 2 tiling of four-dimensional space whose acceptance domain is a polygon with 14 sides. The sides are arranged into seven

Γ-orbits. There is only one Γ-orbit of singular 0-spaces, corresponding to the projection of the lattice points in $\mathbb{Z}^6$ to the two-dimensional space V. Thus, the arrangement A consists of seven tori of dimension $\nu = \frac{2+4}{2} = 3$, all intersecting at a single point.

By Lemma 3.5, the homology of A is as follows. Note that the Euler characteristic for a one-point union of seven 3-tori (as cellular complexes) is $e_A = 1 - 3\times 7 + 3\times 7 - 1\times 7 = -6$.

$$H_0(A) \cong \mathbb{Z},\ H_1(A) \cong \mathbb{Z}^{2\times 7-(-6)+1} = \mathbb{Z}^{21},\ H_2(A) \cong \mathbb{Z}^{21},\ H_3(A) \cong \mathbb{Z}^7$$

Now rk $Im\beta^0 = \binom{6}{0} = 1$, rk $Im\beta^1 = 6$ and rk $Im\beta^2 = 15$. Also, we know $Im\beta^5 = Im\beta^6 = 0$. Lastly, rk $Im\beta^m = \binom{6}{m} - r_{5-m}$ for $m = 3, 4$. Note [16] that $r_1 = 12$ and $r_2 = 4$ for this tiling. The formula (3.3) then tells us that the rational ranks of the cohomology of $\mathbb{T}^6 \setminus A$, and hence of the Čech cohomology of the continuous hull $M\mathcal{P}$ for $\mathcal{P}$ the Heptagonal tiling, are the following.

$$\begin{array}{lcl cl}
\text{rk } \check{H}^0(M\mathcal{P}) & = & 0+1+6-6 & = & 1 \\
\text{rk } \check{H}^1(M\mathcal{P}) & = & 0+6+15-15 & = & 6 \\
\text{rk } \check{H}^2(M\mathcal{P}) & = & 7+15+20-r_2-20 & = & 18 \\
\text{rk } \check{H}^3(M\mathcal{P}) & = & 21+20-r_2+15-r_1-15 & = & 25 \\
\text{rk } \check{H}^4(M\mathcal{P}) & = & 21+15-r_1+0-6 & = & 18 \\
\text{rk } \check{H}^5(M\mathcal{P}) & = & 1+0+0-1 & = & 0
\end{array}$$

These results agree with those calculated in [16].

3.3 Possible further extensions

If we weaken the hypotheses of the constructions from [20] by supposing that there are an infinite number of Γ-orbits of singular 0-spaces, so $L_0 = \infty$, then it is likely that similar results hold, which are analogous to those obtained in Chapter 2 for polytopal projection patterns with L_0 infinite. As before, we need to describe the continuous hull $M\mathcal{P}$ as an inverse limit space, but in the cases below, the description X' we produce has to be rather different from that given in Section 3.1 so the methods of [20] cannot be used to show that $M\mathcal{P}$ is homeomorphic to X'. We first restrict to consideration of hypergeneric patterns (Def 1.50).

3.3.1 Hypergeneric polytopal projection patterns

Hypergeneric polytopal projection patterns $\mathcal{P}$ are such that the Γ-orbits of the $(n-1)$-dimensional faces of the acceptance domain K consist of hyperplanes, but the number L_0 of Γ-orbits of singular 0-spaces is infinite. Thus any finite set of translates of K will not contain representatives of all Γ-orbits of singular spaces. To describe the continuous hull $M\mathcal{P}$ for such patterns $\mathcal{P}$, the construction of Section 3.1 is therefore not good enough. However, if an inverse limit space can be produced which is homeomorphic to $M\mathcal{P}$ then results can be obtained in a similar way as before.

Note also that for hypergeneric polytopal projection patterns, the stabilisers of $(n-1)$-dimensional hyperplanes are of rank $n-1$ and there is at least one set of n hyperplanes with linearly independent normals since the acceptance domain is bounded in the n-dimensional space V. As in Section 3.2, the stabilisers of hyperplanes are projections to V of elements of Λ' and each set of these elements defines an $(n-1)$-dimensional torus in the quotient $(V+E)/\Lambda'$. Define $T_i := (H_i+E)/\Lambda' \subset (V+E)/\Lambda' \cong \mathbb{T}^{n+d}$, which is homotopy equivalent to an $(n-1)$-dimensional torus since E is totally irrational with respect to Λ' and hence is contractible in $(V+E)/\Lambda'$.

Again, in order to be able to remove the objects T_i from $\mathbb{T}^{n+d}$ and obtain a non-trivial space, the T_i need to be thickened to ensure they are $(n+d)$-dimensional. This leads to problems when trying to find an inverse limit homeomorphic to $M\mathcal{P}$ since there are infinitely many T_i which need to be considered in order to include all the singular 0-spaces which arise at the intersection of the hyperplanes H_i. If k objects T_i are each thickened by an amount ϵ, then $k+1$ objects T_i must each be thickened by less than ϵ to ensure that the limit space is non-trivial. In the remainder of this section, we proceed as if we had such an inverse limit construction for $M\mathcal{P}$ and investigate the consequences for the cohomology of $M\mathcal{P}$.

The codimension 2 case

We consider codimension $n = 2$ hypergeneric polytopal projection patterns, so that we are again dealing with sets of lines intersecting in points in V. The hyperplanes H_i are lines of infinite length with stabilisers $\Gamma^{H_i} \subset \Gamma = \pi^{\perp}(\Lambda')$ of rank 1 in this case which thus give rise to circular cuts in the quotient $(V + E)/\Lambda'$.

A collection of $t = |I_K|$ circles S^1 intersecting in various points is homotopy equivalent to a one-point union of κ circles, for $\kappa \geqslant t$, by a homotopy which collapses the arcs of circles between points. The homology of such an object is straightforward to compute, as follows.

$$H_m(\bigvee_{i=1}^{\kappa} S^1) \cong \begin{cases} \mathbb{Z} & m = 0 \\ \mathbb{Z}^{\kappa} & m = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that the objects $T_i = (H_i + E)/\Lambda'$ are cylinders and hence are homotopy equivalent to circles S^1.

To compute the cohomology of $\mathbb{T}^{d+2} \setminus Y_k \colon = \mathbb{T}^{n+d} \setminus \{T_i + \gamma_j : \gamma_j \in \Gamma, j = 1, \dots k, i \in I\}$, we again consider the relative cohomology of the pair $(\mathbb{T}^{d+2}, \mathbb{T}^{d+2} \setminus Y_k)$, so as before we need to determine rk $H^{m-1}(\mathbb{T}^{d+2} \setminus Y_k) = \text{rk } H_{d+2-m}(Y_k) + \text{rk } Im(\beta^{m-1}) + \text{rk } Im(\beta^m) - \binom{d+2}{m}$.

Now $H_{d+2-m}(Y_k) \cong H_{d+2-m}(\bigvee_{i=1}^{\kappa} S^1) = 0$ for $m \neq d+1, d+2$. From the long exact sequence (3.1) in relative cohomology, this implies that $H^{m-1}(\mathbb{T}^{d+2} \setminus Y_k) \cong H^{m-1}(\mathbb{T}^{d+2})$ for such m.

It remains to consider $H^d(\mathbb{T}^{d+2} \setminus Y_k)$ and $H^{d+1}(\mathbb{T}^{d+2} \setminus Y_k)$. For $m = d+1$, we have rk $H^d(\mathbb{T}^{d+2} \setminus Y_k) = \text{rk } H_1(Y_k) + \text{rk } Im\beta^d + \text{rk } Im\beta^{d+1} - (d+2) = \kappa + \binom{d+2}{d} + \text{rk } Im\beta^{d+1} - (d+2)$. For $m = d+2$, we have rk $H^{d+1}(\mathbb{T}^{d+2} \setminus Y_k) = \text{rk } H_0(Y_k) + \text{rk } Im\beta^{d+1} + \text{rk } Im\beta^{d+2} - 1 = \text{rk } Im\beta^{d+1} + \text{rk } Im\beta^{d+2}$. We thus need to determine the ranks of β^{d+1} and β^{d+2}.

As in the case for patterns with L_0 finite, we again have $H_0(Y_k) \cong \mathbb{Z} \cong H_0(\mathbb{T}^{d+2}) \cong H^{d+2}(\mathbb{T}^{d+2})$ and the generators of $H_1(Y_k) \cong H_1(\bigvee_{i=1}^{\kappa} S^1)$ map to generators of $H_1(\mathbb{T}^{d+2}) \cong H^{d+1}(\mathbb{T}^{d+2})$ under α^{d+1}, so $\beta^{d+2} = 0$ and if $\kappa \geqslant d+2$ then $\alpha \colon \mathbb{Z}^{\kappa} \to \mathbb{Z}^{d+2}$ will be surjective and $\beta^{d+1} = 0$.

LEMMA 3.10 *For $Y_k \simeq \bigvee_{i=1}^{\kappa} S^1$ a subset of $\mathbb{T}^{d+2}$ arising from a d-dimensional codimension 2 hyperplane polytopal projection pattern as above, $\kappa \geqslant d+2$.*

Proof If there are $t \geqslant d+2$ distinct Γ-orbits of faces of K, then the result holds immediately, as in the above case of patterns with L_0 finite, since Y_k contains lines H_i for $i \in I_K$ and $|I_K| = t$, so $\kappa \geqslant t \geqslant d+2$ as required.

Now suppose that there are only $t <$ rk Γ orbit classes of singular 1-spaces, so the vectors giving the directions of the corresponding lines in $\mathbb{R}^{d+2}$ do not span this space. We still have $\kappa \geqslant d+2$ in this case as well, for the number k of translates of the acceptance domain K large enough. This is because by Lemma 1.42 any pair of points in $\partial K + \Gamma$ can be connected by a path of singular 1-spaces. Hence any two points in the same Γ-orbit differing by an element $\gamma \in \Gamma = \pi^{\perp}(\Lambda')$ which is not a linear combination of the stabilisers of singular 1-spaces can be connected by a path of singular 1-spaces and so these singular 1-spaces with end points in the same Γ-orbit form circles in the quotient $(V+E)/\Lambda' \cong \mathbb{T}^{d+2}$ which are embedded non-trivially and correspond to cycles in $\mathbb{T}^{d+2}$. Only finitely many translates of lines H_i are needed to create circular cuts in the compact space $\mathbb{T}^{d+2}$ because the lines are of positive length, not of irrational slope with respect to Γ (since their stabilisers are non-trivial, so they each contain two points in the same Γ-orbit) and two or more lines contain a given singular 0-space in any Γ-orbit, so the required paths can be formed which give rise to the circles in $\mathbb{T}^{d+2}$.

Hence in Y_k for k large enough we have $d+2$ circles which are embedded non-trivially in $\mathbb{T}^{d+2}$ so $\kappa \geqslant d+2$ in all cases. ∎

THEOREM 3.11 *A $(d+2)$-dimensional torus with a one-point union of infinitely many circles removed (denoted M) has Čech cohomology groups with ranks*

$$\begin{aligned} rk\ \check{H}^d(M) &= \infty \\ rk\ \check{H}^m(M) &= \binom{d+2}{m} \text{ for } 0 \leqslant m < d. \end{aligned}$$

Proof The above calculations show that $\check{H}^m(M) \cong H^m(\mathbb{T}^{d+2})$ for $m < d$. Given that rk $\check{H}^d(\mathbb{T}^{d+2} \setminus Y_k) = \binom{d+2}{2} + \kappa - (d+2)$, in the limit as the number of Γ-orbits of singular 0-spaces arising at intersections of elements in the set Y_k increases with increasing k, and hence the number κ of circles in the one-point union $\bigvee_{i=1}^{\kappa} S^1$ increases, rk $\check{H}^d(M) = \infty$. ∎

These results are in accordance with the results for hypergeneric projection patterns computed in Theorem 2.71, giving evidence for a link between the continuous hull $M\mathcal{P}$ and the cut torus M.

3.3.2 Non-hyperplane polytopal projection patterns

In this section, we weaken the assumptions still further and suppose that the patterns under consideration are not hypergeneric so the Γ-orbits of singular $(n-1)$-spaces do not all contain the hyperplanes spanned by the spaces. Note that $L_0 = \infty$ in this case as well.

We again need to describe the continuous hull as an inverse limit, but similar problems to those discussed in the previous section arise here, since L_0 is infinite so infinitely many translates of the acceptance domain for the pattern under consideration need to be taken in order for representatives of all Γ-orbits of singular 0-spaces to appear. We consider the space $\mathbb{T}^{n+d} \setminus Y_k$ formed by removing finitely many translates of the acceptance domain from the torus. More precisely, we define $Y_k = \{\bigcup_{i=1}^{k}((\partial K + E)/\Lambda' + \gamma_i) : \gamma_i \in \Gamma\}$.

Suppose that $\mathbb{T}^{n+d} \setminus Y_k$ can be used to describe $M\mathcal{P}$ as an inverse limit and consider codimension 2 generic polytopal projection patterns only. Some consequences of this are given below.

Codimension 2 generic patterns

In this case, by Definition 1.51 the stabilisers of the faces of the acceptance domain K for any generic polytopal projection pattern $\mathcal{P}$ are trivial, so no two singular 0-spaces in the same orbit appear on any singular 1-space. However, as in the case of hypergeneric patterns, since the singular 0-spaces can be connected by paths of singular 1-spaces in $\partial K + \Gamma$, if enough translates of K are taken, then there will be paths between two points in the same Γ-orbit, which become circles in the quotient $(V+E)/\Lambda'$ as $\Gamma = \pi^{\perp}(\Lambda')$. Note that only finitely many translates are required to form the paths since $\mathbb{T}^{d+2}$ is compact and the faces of K have positive length. Hence, as before, for A_κ the $(d+2)$-dimensional submanifold of $\mathbb{T}^{d+2}$ consisting of the one-point union of κ circles $\bigvee_{i=1}^{\kappa} S^1$, which is homotopy equivalent to Y_k, we have $H^m(\mathbb{T}^{d+2} \setminus A_\kappa) \cong H^m(\mathbb{T}^{d+2})$ for $m < d$ and $H^{d+1}(\mathbb{T}^{d+2} \setminus A_\kappa) = 0$. Now as the number of translates of ∂K in Y_k increases, the number of singular 0-spaces arising increases. Since paths can be constructed between representatives of each orbit appearing, as k increases, the number κ of circles which arise increases. Thus we again have $H^d(\mathbb{T}^{d+2} \setminus A_\kappa) = \binom{d+2}{d} + \kappa - (d+2)$, giving the following result.

THEOREM 3.12 *For $M := \mathbb{T}^{n+d} \setminus A$, where A is a one-point union of a countably infinite number of circles S^1 we have*

$$\begin{aligned} rk\, \check{H}^d(M) &= \infty \\ rk\, \check{H}^m(M) &= \binom{d+2}{m} \text{ for } 0 \leqslant m < d. \end{aligned}$$

Proof The above calculations show that $\check{H}^m(M) \cong H^m(\mathbb{T}^{d+2})$ for $m < d$ and, given that rk $H^d(\mathbb{T}^{d+2} \setminus A_\kappa) = \binom{d+2}{2} + \kappa - (d+2)$, in the limit as the number of Γ-orbits of singular 0-spaces arising at intersections of elements in the set A_κ increases with increasing κ, we have rk $\check{H}^d(M) = \infty$. ∎

Again, this is consistent with the results for generic polytopal projection patterns obtained in Chapter 2.

Non-hyperplane polytopal projection patterns with non-trivial stabilisers

Lastly, consider a general codimension 2 polytopal projection pattern $\mathcal{P}$ with L_0 infinite. In this situation, problems with describing the continuous hull as an inverse limit still arise but the constructions of all the cases discussed above can be combined to yield similar results, where the dimensions of the tori which are removed from $\mathbb{T}^{n+d}$ depend on the ranks of the stabilisers of the $(n-1)$-dimensional faces of the acceptance domain K for $\mathcal{P}$. If the stabilisers of all $(n-1)$-dimensional faces of K are less than or equal to $(n-1)$, then the space $\mathbb{T}^{n+d} \setminus A$ which arises has Čech cohomology as given in Theorem 3.12 above, but if a face has stabiliser of rank greater than $(n-1)$ then $\check{H}^m(\mathbb{T}^{n+d} \setminus A)$ will not be isomorphic to $H^{m+2}(\mathbb{T}^{d+2})$ for $0 < m < d$ and will have rank greater than rk $H^{m+2}(\mathbb{T}^{d+2})$.

Chapter 4

Conclusion

This document has established two sets of results. Firstly, a theory has been developed in analogy with work in [12] for investigating projection patterns with polytopal acceptance domains, a class which contains the set of canonical projection patterns. Explicit calculations of the Euler characteristic were carried out for such patterns in codimension 2 and 3. In particular, we showed for a polytopal projection pattern that $\check{H}^d(M\mathcal{P})$ is finite if and only if L_0, the number of orbit classes of singular 0-spaces, is finite. As a corollary, this reproves a result of [12] which gives that $\check{H}^d(M\mathcal{P})$ has finite rank if and only if L_0 is finite, for $\mathcal{P}$ a *canonical* projection pattern. A formula for computing the Euler characteristic of a polytopal projection pattern of any codimension, provided L_0 is finite, was produced, and patterns with L_0 infinite were also considered. Secondly, it has shown that for codimension 2 polytopal projection patterns of arbitrary dimension with L_0 finite, the Čech cohomology of the continuous hull can be calculated and also can be expressed as the cohomology of a $(d+2)$-dimensional torus with an arrangement of lower-dimensional tori removed. This extends ideas presented in [12] and [20].

Thus, in answer to Question 0.1 we have shown that for codimension 2 polytopal projection patterns, the Euler characteristic, as defined in Definition 2.1, always exists, but under certain circumstances it takes the value infinity. Also, the rational ranks of the Čech cohomology groups $\check{H}^*(M\mathcal{P})$ can be computed, although rk $\check{H}^d(M\mathcal{P})$ is infinite if L_0 is infinite. However, in general, the Euler characteristic of a codimension n polytopal projection pattern for $n \geqslant 3$ is not defined according to Definition 2.1. There are classes of codimension n polytopal projection patterns for which the Euler characteristic can be computed: in particular, for a codimension n polytopal projection pattern with L_0 finite, a formula yielding the Euler characteristic of a pattern of any codimension was produced in Chapter 2. Also the Euler characteristic is defined for a given polytopal projection pattern if the stabilisers $\Gamma^{\mathcal{D}}$ of singular spaces have sufficiently small ranks. With regard to Question 0.2, in Chapter 2, when the Euler characteristic is finite, bounds on the values it could take were found, in particular for patterns of codimension 2 and 3. Such calculations, plus consideration of patterns with L_0 infinite enabled Question 0.3 to be addressed, since necessary conditions for certain cohomology groups of polytopal projection patterns to be of infinite rank were determined. However, the question was not answered in complete generality since there are some cohomology groups, for polytopal projection patterns of codimension 4 or more, to which the methods developed in this document cannot be applied. Finally, Question 0.4, which asked about alternative models of the continuous hull, was answered positively for polytopal projection patterns with L_0 finite, and calculations were carried out for patterns of codimension 2. Thus the work of [20] was generalised to a larger class of projection patterns than previously considered.

There are various questions which have arisen during the period of study but which are still unresolved due to the time constraints associated to this project, and so provide scope for future research.

The continuous hull $M\mathcal{P}$ for polytopal projection patterns $\mathcal{P}$ with L_0 infinite could be investigated further, in particular to find models which yield the results described at the end of Chapter 3. Since [15] the cohomology of $M\mathcal{P}$ is not always torsion-free, the work of Chapter 3 could be re-evaluated with the aim of determining $\check{H}^*(M\mathcal{P})$ with coefficients in $\mathbb{Z}$ rather than $\mathbb{Q}$. Also, the tools of Chapter 3 were not used to investigate the structure of the continuous hull of codimension 3 polytopal projection patterns, even though an example of a canonical projection pattern was considered in [20], because the geometry of the singular spaces in this higher-dimensional situation is more complicated. Thus the methods of Chapter 3 and [20] could also be considered in more depth with respect to polytopal projection patterns of codimension 3 in order to produce further examples of computations of the Čech cohomology of the continuous hull in this case.

A method for determining when the sequence (1.2) of Γ-modules C^i is exact could be developed, so more specific conditions on the theorems of Chapter 2 could be obtained. If the sequence (1.2) is not always exact, then the spectral sequences described in Section 1.6.3 could be considered more carefully with the aim of producing similar results. A full characterisation of when the ranks of the cohomology groups $\check{H}^*(M\mathcal{P})$ are infinite could be produced. Alternative definitions of the Euler characteristic could also be considered, which may be applicable in more cases where Definition 2.1 is not always suitable, such as for 3-dimensional codimension 3 patterns with $H_0(\Gamma; C^0)$ and $H_1(\Gamma; C^1)$ of infinite rank. Lastly, geometric interpretations of the higher cohomology groups $\check{H}^m(M\mathcal{P})$ for $m > 1$ could be considered which would also extend the results of [8].

List of Definitions

Page 27 Group Γ

Page 27 $\mathcal{K}$

Page 28 Set $\mathcal{K}^i$ of singular i-spaces, I_i the set of Γ-orbits of singualr i-spaces, $L_i = |I_1|$

Page 28 L_0, the number of Γ-orbits of singular 0-spaces in $\mathcal{K}^0$

Page 29 Set I_{ic} of connected components D in Γ-orbits of singular spaces, $\mathcal{D}$ Γ-orbit of D

Page 34 Stabiliser $\Gamma^{\mathcal{D}}$ of $D \in I_{ic}$ in orbit $\mathcal{D}$

Page 35 Hypergeneric polytopal projection pattern

Page 35 Generic polytopal projection pattern

Page 36 Hyperplane polytopal projection pattern

Page 38 Module C^i arising from singular i-spaces

Page 41 Γ-modules C^i_D, $C^i_{\mathcal{D}}$ for $0 \leqslant i \leqslant n-1$

Page 45 kth Čech cohomology group $\check{H}^k(M\mathcal{P})$

Page 46 Exterior module $\Lambda\Gamma$

Page 50 E'_u

Page 51 V_u

Page 55 Euler characteristic $e_{\mathcal{P}}$

Page 56 Euler characteristics $e_{\underline{Y}}$, e_C

Page 64 Codimension 2 multiplicity q_β of singular 0-spaces $\beta \in I_0$

Page 82 Codimension 3 multiplicities q_β, q^η_β of singular 0-spaces

Page 82 Multiplicities p_β, I_β, q^β_l

Page 105 Orbits of singular spaces $\theta_i \in I_i$, $I^{\theta_j}_i$

Page 106 Euler characteristics e_n, e'_n, e^θ_n

Page 107 Codimension n multiplicities $q^{\theta_i}_{\theta_0}$

Page 121 Metric ρ on $\mathbb{T}^{n+d} \setminus Y_r$

Page 121 $X_r = \mathbb{T}^{n+d} \setminus Y_r$, $X = \varprojlim X_r$

Page 142 C^*-algebra A

Page 144 K-theory $K_0(A)$, $K_1(A)$

Page 145 Suspension SA of a C^*-algebra A

List of Figures

Appendix 1: C^*-algebras and K-theory

This appendix consists of the definition of C^*-algebras and the definition of C^*-algebra K-theory. Crossed-product C^*-algebras are also constructed here. The main references are [3] and [37].

C^*-algebras

DEFINITION 4.1 *A* C^*-algebra *is an algebra* A *over* $\mathbb{C}$ *together with an operation, called the* adjoint *and an algebra norm* $\|\cdot\|$, *where*

1. *A is complete with respect to the norm $\|\cdot\|$.*

2. *for all $a, b \in A$ and $\lambda \in \mathbb{C}$ the adjoint satisfies*

   - $(a+b)^* = a^* + b^*$.

   - $(\lambda a)^* = \overline{\lambda} a^*$.

   - $a^{**} = a$.

   - $(ab)^* = b^* a^*$.

3. $\|a^* a\| = \|a\|^2$.

A C^*-algebra A does not necessarily have to contain a unit, which is an element $1 \in A$ with $1.a = a = a.1$ for all $a \in A$. However, we can always adjoin a unit to A by embedding A into a larger, unital, C^*-algebra. This C^*-algebra, $A^\sim$, contains A as an ideal and is such that if A is unital, then $A^\sim = A$ and if A is not unital, then $A^\sim/A \simeq \mathbb{C}$. Further details about this construction can be found in [37, p16], for example.

Given two C^*-algebras A and B, say a map $A \to B$ is a *$*$-homomorphism* if it preserves the addition, scalar multiplication, product and adjoint operations. In general, $*$-homomorphisms are norm-decreasing. Lastly, define a *$*$-representation* of a C^*-algebra A to be a $*$-homomorphism $\pi\colon A \to B(\mathcal{H})$ from A to the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$.

Examples of C^*-algebras are the set $C(Y)$ of continuous complex-valued functions on a compact Hausdorff space Y, together with the operations pointwise addition, pointwise multiplication and complex conjugation, and the supremum norm, and, for a C^*-algebra A, the matrix algebra $M_n(A)$ of all $n \times n$ matrices with entries in A, with the usual matrix operations and norm.

The definition of the *crossed-product* C^*-algebra $A \rtimes_\alpha G$ is given in full generality in [10]. For tiling theory, we use $A = C(Y)$ for Y either $M\mathcal{P}$ or X; the group G is $\mathbb{R}^d$ or $\mathbb{Z}^d$ respectively and α is a homomorphism of G into the group of automorphisms of $C(Y)$ (the action of G on $C(Y)$). First take the space $C_c(G, C(Y), \alpha)$ of compactly supported $C(Y)$-valued functions on G. Write $f(g, y)$ to denote the element of $\mathbb{C}$ which is the value of the function $f(g) \in C(Y)$ on the element $y \in Y$. Define the product of two such elements f_1 and f_2 by

$$f_1 * f_2(g, y) = \int_{h \in G} f_1(h, y) f_2(g - h, y - h)\, dh$$

and define the adjoint operation by

$$f_1^*(g, y) = \overline{f(g, y - g)}$$

for $g \in G$ and $y \in C(Y)$. Define a norm on this algebra by $\|f\| = \sup \|\sigma(f)\|$, where σ runs over all $*$-representations of $C_c(G, C(Y), \alpha)$ on a Hilbert space $\mathcal{H}$. The completion of $C_c(G, C(Y), \alpha)$ with respect to this norm is the C^*-algebra $C(Y) \rtimes_\alpha G$, the crossed-product of $C(Y)$ by G.

K-theory

Given the C^*-algebra $C(Y) \rtimes G$, we can define its *C^*-algebra K-theory* $K_*(C(Y) \rtimes G)$, as described below.

For a C^*-algebra A, say an element $p \in M_n(A)$ is a *projection* if it satisfies $p = p^* = p^2$. Write $P(A) = \bigcup_{n=1}^{\infty} \{\text{projections in } M_n(A)\}$. Say projections $p \in M_n(A)$ and $q \in M_m(A)$, for $m \geqslant n$ are *$*$-equivalent* if there exists ω in $M_m(A)$ with the property that $q = \omega^*\omega$ and $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \omega\omega^*$, where $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ denotes the matrix p with sufficiently many zero entries added to give an $m \times m$ matrix. Say $p, q \in P(A)$ are *stably equivalent*, and write $p \overset{*}{\sim} q$, if there exists $e \in P(A)$ such that the block diagonal projection matrix $p \oplus e\colon = \begin{pmatrix} p & 0 \\ 0 & e \end{pmatrix}$ is $*$-equivalent to $q \oplus e\colon = \begin{pmatrix} q & 0 \\ 0 & e \end{pmatrix}$. Denote by $[p]$ the stable equivalence class of p. The operation $\oplus$ induces a commutative associative semigroup operation $[p] + [q] = [p \oplus q]$ on the set $P(A)/\overset{*}{\sim}$ of stable equivalence classes of projections. Define A^+ to be $A^\sim$ if A is non-unital, or $A \oplus \mathbb{C}$ if A is unital. Writing $K_0(A^+)$ to denote the Grothendieck group of $P(A^+)/\overset{*}{\sim}$, we can then make the following definition.

DEFINITION 4.2 *The zeroth K-group of the C^*-algebra A is*

$$K_0(A)\colon = Ker(\pi_*\colon K_0(A^+) \to \mathbb{Z}),$$

where π_* is induced from the map $\pi\colon \mathbb{C} \to A^+$ which takes the unit in $\mathbb{C}$ to the adjoined unit in $A^\sim$ or to the second factor in $A \oplus \mathbb{C}$.

To define $K_1(A)$, first write $GL(A) = \bigcup_{n=1}^{\infty} GL_n(A)$ the nested union of all $n \times n$ invertible matrices with entries in A. Inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$ is given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Finally, denote by $GL(A^+)_0$ the connected component of $GL(A^+)$ which contains the unit, i.e. it is the set of elements which are homotopic to the unit in $GL(A^+)$. We then have the following.

DEFINITION 4.3 $K_1(A) = GL(A^+)/GL(A^+)_0$

An alternative way of defining $K_1(A)$, in analogy with the construction of $K^1(Y)$ in topological K-theory, is to use suspensions.

DEFINITION 4.4 [37] *The* suspension *of a C^*-algebra A is the C^*-algebra $SA\colon = A \otimes C_0(\mathbb{R})$ for $C_0(\mathbb{R})$ the algebra of continuous functions on $\mathbb{R}$ with compact support.*

It is shown in [37, Thm 7.2.5] that there is an isomorphism $\theta_A\colon K_1(A) \to K_0(SA)$.

The process of suspension can be iterated, giving $S^n A = A \otimes C_0(\mathbb{R}^n)$, and hence higher K-groups can be defined as $K_n(A)\colon = K_0(S^n A)$. However, [37, Chap 9] for a C^*-algebra A there is also an isomorphism $\beta_A\colon K_0(A) \to K_1(SA)$. This result gives rise to *Bott periodicity* for C^*-algebra K-theory.

THEOREM 4.5 [37] The Bott Periodicity Theorem

For a C^*-algebra A, there is an isomorphism $K_i(A) \cong K_{i+2}(A)$. ∎

Note that the most useful notion of equivalence of C^*-algebras is *Morita equivalence*. A precise definition is given in [30]. The main point to note is that Morita equivalent C^*-algebras have isomorphic K-theory. Finally, two other properties of C^*-algebra K-theory relevant to the study of C^*-algebras arising from quasiperiodic patterns are listed below.

THEOREM 4.6 [5] The Serre-Swan Theorem

For a compact Hausdorff topological space Y, there is an isomorphism $K_*(C(Y)) \cong K^*(Y)$ of the C^*-algebra K-theory of the algebra of continuous functions on Y with the ordinary topological K-theory of the space Y. ∎

THEOREM 4.7 [9] Connes' Generalised Thom Isomorphism Theorem

For a C^*-algebra A, there is an isomorphism

$$K_i(A \rtimes \mathbb{R}^n) \cong K_{i-n}(A),$$

where $i-n$ denotes $(i-n)$ mod 2. ∎

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