Self-injective algebras and the second Hochschild cohomology group

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

by

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2005

Abstract

In this thesis we study the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ of a finite dimensional algebra Λ. In particular, we determine $\mathrm{HH}^2(\Lambda)$ where Λ is a finite dimensional self-injective algebra of finite representation type over an algebraically closed field K and show that this group is zero for most such Λ; we give a basis for $\mathrm{HH}^2(\Lambda)$ in the few cases where it is not zero.

Then we consider algebras of tame representation type; more specifically, we study finite dimensional self-injective one parametric tame algebras which are not weakly symmetric. Here we show that $\mathrm{HH}^2(\Lambda)$ is non-zero and find a non-zero element η in $\mathrm{HH}^2(\Lambda)$ and an associative deformation Λ_η of Λ.

ACKNOWLEDGEMENTS

I thank Prof. Amin Kashmeery for suggesting the University of Leicester to study at and for supporting us the first time we came to England, Taif University in Saudi Arabia for funding my PhD. research, Dr. Nicole Snashall for her helpful supervision and valuable suggestions. Also I would like to extend my thanks to my parents (Sobhi and Nafisah) for supporting me. Many thanks to them all and special thanks to my husband Jehad for his encouragement and to my daughters Lma and Lmees for being patient.

Contents

INTRODUCTION

In this thesis we study the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ of all finite dimensional self-injective algebras Λ of finite representation type over an algebraically closed field K. The second Hochschild cohomology group is linked to deformations of an algebra, that is, if $\mathrm{HH}^2(\Lambda) = 0$ then all deformations of Λ are trivial. The converse of this result is false in general and was shown in [13]. However, for monomial algebras with directed quiver the converse holds and was shown in [8]. We then study certain finite dimensional algebras Λ of tame representation type and find a non-zero element η in $\mathrm{HH}^2(\Lambda)$ and an associative deformation Λ_η of Λ.

In general, finite dimensional self-injective algebras of finite representation type over an algebraically closed field K fall into type A, type D and type E. Riedtmann in her paper [23] classified the stable equivalence representatives of these algebras and Asashiba then showed that stable equivalence classes are exactly the derived equivalence classes in [2, Theorem 1.2]. In [2], the derived equivalence class representatives are given explicitly by quiver and relations.

Happel in [17] showed that Hochschild cohomology is invariant under derived equivalence. So if A and B are derived equivalent then $\mathrm{HH}^2(A) \cong \mathrm{HH}^2(B)$. Hence to study $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras of finite representation type over an algebraically closed field K, it is enough to study $\mathrm{HH}^2(\Lambda)$ for the representatives of the derived equivalence classes. The algebras of type A fall into two types: Nakayama algebras and Möbius algebras and the Hochschild cohomology of these algebras has been studied in the literature. In [9], Erdmann and Holm give the dimension of the second Hochschild cohomology group of a Nakayama algebra. In [15], Green and Snashall find the second Hochschild cohomology group for the Möbius algebras.

The main work in the thesis is in determining $\mathrm{HH}^2(\Lambda)$ for the finite dimensional self-injective algebras of finite representation type D and E. The algebras of type D fall into 5 classes, and the algebras of type E fall into 2 classes. In Chapter 5 we give a general theorem, Theorem 5.11, which we use to show that $\mathrm{HH}^2(\Lambda) = 0$ for most of these algebras. This is motivated by work in [15]. The strategy of the theorem is to show that every element in $\mathrm{Hom}(Q^2, \Lambda)$ is a coboundary so that $\mathrm{HH}^2(\Lambda) = 0$, where Q^2 is the second projective in a minimal projective resolution of Λ as a Λ,Λ-bimodule. For all other cases which are not covered by Theorem 5.11, we determine $\mathrm{HH}^2(\Lambda)$ by direct calculation, and find a basis for $\mathrm{HH}^2(\Lambda)$ in the instances where $\mathrm{HH}^2(\Lambda) \neq 0$. Chapters 4, 6, 7, 8 deal with the satandard algebra of type D, Chapters 9, 10 deal with standard algebra of type E and Chapter 11 deals with the nonstandard algebra of type D.

In [2, Proposition 1.1], Asashiba gives a classification of finite dimensional self-injective algebras of finite representation type over an algebraically closed field according to the type of the algebra. I give more details in Chapter 2; however it is sufficient here to note that the following theorem is the main result of this thesis which is given in Theorem 11.10 in Chapter 11, and deals with every finite dimensional self-injective algebra of finite representation type over an algebraically closed field K.

Theorem. *Let Λ be a finite dimensional self-injective algebra of finite representation type over an algebraically closed field K. If Λ is the standard algebra of type $\Lambda(A_{2p+1}, s, 2)$ with $s, p > 1$, $\Lambda(D_n, s, 1), \Lambda(D_4, s, 3)$ with $n \geq 4, s \geq 1$, $\Lambda(D_n, s, 2), \Lambda(D_{3m}, s/3, 1)$ with $n \geq 4, m \geq 2, s \geq 2$ or $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ with $n \in \{6, 7, 8\}, s \geq 1$; then $\mathrm{HH}^2(\Lambda) = 0$.*

If Λ is of type $\Lambda(A_n, s/n, 1)$ then $\dim \mathrm{HH}^2(\Lambda) = m$ where $n + 1 = ms + r$ and $0 \leq r < s$.

For $\Lambda(A_{2p+1}, s, 2)$ with $s = p = 1$, $\dim \mathrm{HH}^2(\Lambda) = 1$.

Let Λ be $\Lambda(D_n, 1, 2)$; then $\dim \mathrm{HH}^2(\Lambda) = 1$.

Let Λ be the standard algebra $\Lambda(D_{3m}, 1/3, 1)$; then

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } \operatorname{char} K \neq 2, \\ 4 & \text{if } \operatorname{char} K = 2. \end{cases}$$

If Λ is the nonstandard algebra $\Lambda(m)$ of type $(D_{3m}, 1/3, 1)$ where $m \geq 2$ we have

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } \operatorname{char} K \neq 2, \\ 3 & \text{if } \operatorname{char} K = 2. \end{cases}$$

Thus we have determined $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras Λ of finite representation type over an algebraically closed field.

In the final chapter of the thesis we consider certain finite dimensional self-injective one parametric but not weakly symmetric tame algebras. The classification of these algebras is given in [6]. The algebras in [6] are divided into two types, and in this thesis we study some of one type. For our algebras Λ we show in Theorem 12.15 that $\mathrm{HH}^2(\Lambda)$ is non-zero and find a non-zero element η in $\mathrm{HH}^2(\Lambda)$ and an associative deformation Λ_η of Λ. This illustrates the connection between the second Hochschild cohomology group and deformation theory.

This leaves infinitely many algebras in the classification of [6] of the first type and the algebras of the second type to look at for future work. We also intend to look at the classification in [7] of finite dimensional self-injective one parametric of finite representation type tame weakly symmetric algebras.

1. Hochschild cohomology and deformations

The aim of the thesis is to study the second Hochschild cohomology group and to determine the second Hochschild cohomology group of all finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

We start by introducing Hochschild cohomology and explain how the second Hochschild cohomology group controls the deformations of an algebra, thus providing an important link between algebra and algebraic geometry.

Since we are interested in finite dimensional algebras over an algebraically closed field, we assume throughout that Λ is a finite dimensional algebra over an algebraically closed field K. Let $\Lambda^e = \Lambda \otimes_K \Lambda^{op}$ be the enveloping algebra of Λ. For ease of notation we write $\otimes$ for $\otimes_K$.

Now we will define the acyclic Hochschild complex of Λ.

Definition 1.1. [5, p75] Let $S_n(\Lambda) = \Lambda^{\otimes n+2}$. Then $S_n(\Lambda)$ is a Λ,Λ-bimodule via

$$(a \otimes b)(\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n+1}) = a\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n+1} b.$$

Let $\tilde{S}_n(\Lambda) = \Lambda^{\otimes n}$. Then $S_n(\Lambda) = \Lambda \otimes \tilde{S}_n(\Lambda) \otimes \Lambda \cong (\Lambda \otimes \Lambda^{op}) \otimes \tilde{S}_n(\Lambda) \cong \Lambda^e \otimes \tilde{S}_n(\Lambda)$. The acyclic Hochschild complex is the projective resolution of Λ over Λ^e

$$S_*(\Lambda) : \cdots \to S_n(\Lambda) \xrightarrow{d_n} S_{n-1}(\Lambda) \xrightarrow{d_{n-1}} \cdots \to S_1(\Lambda) \xrightarrow{d_1} S_0(\Lambda) \to \Lambda \to 0.$$

The map $d_n : S_n(\Lambda) \to S_{n-1}(\Lambda)$ is given by

$$a_0 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{j=0}^{n} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}$$

and it is a Λ,Λ-homomorphism.

Note that the acyclic Hochschild complex is known also as the bar resolution of Λ as in [20] or the standard resolution as in [17].

Let X be a Λ,Λ-bimodule. Applying $\mathrm{Hom}_{\Lambda^e}(-, X)$ to the acyclic Hochschild complex gives the complex

$S^*(\Lambda, X) : 0 \to \mathrm{Hom}_{\Lambda^e}(S_0(\Lambda), X) \to \mathrm{Hom}_{\Lambda^e}(S_1(\Lambda), X) \to \ldots \to \mathrm{Hom}_{\Lambda^e}(S_n(\Lambda), X) \to \mathrm{Hom}_{\Lambda^e}(S_{n+1}(\Lambda), X) \to \ldots.$

We have $\mathrm{Hom}_{\Lambda^e}(S_n(\Lambda), X) \cong \mathrm{Hom}_{\Lambda^e}(\Lambda^e \otimes \tilde{S}_n(\Lambda), X) \cong \mathrm{Hom}_K(\tilde{S}_n(\Lambda), X)$. So the map

$$\mathrm{Hom}_{\Lambda^e}(S_n(\Lambda), X) \to \mathrm{Hom}_{\Lambda^e}(S_{n+1}(\Lambda), X)$$

is the map

$$b^n : \mathrm{Hom}_K(\tilde{S}_n(\Lambda), X) \to \mathrm{Hom}_K(\tilde{S}_{n+1}(\Lambda), X)$$

given by

$$\begin{aligned} b^n(f) : a_1 \otimes \cdots \otimes a_{n+1} \mapsto & \; a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, \end{aligned}$$

where $f \in \mathrm{Hom}_K(\tilde{S}_n(\Lambda), X)$.

Note that $\tilde{S}_0(\Lambda) = K$ so $\mathrm{Hom}_K(\tilde{S}_0(\Lambda), X) \cong X$. Therefore the map

$$b^0 : X \to \mathrm{Hom}_K(\Lambda, X) \text{ is given by}$$

$$b^0(x) : \Lambda \to X, a \mapsto ax - xa \text{ for } x \in X, a \in \Lambda.$$

In [19], Hochschild started with this complex, that is, with

$$C^* : 0 \to X \xrightarrow{b^0} \mathrm{Hom}_K(\Lambda, X) \xrightarrow{b^1} \mathrm{Hom}_K(\Lambda^{\otimes 2}, X) \xrightarrow{b^2} \ldots.$$

Definition 1.2. The n-th Hochschild cohomology group of Λ with coefficients in the bimodule X is denoted $\mathrm{H}^n(\Lambda, X)$ and is the n-th cohomology group of the complex

$$C^* : 0 \to X \xrightarrow{b^0} \mathrm{Hom}_K(\Lambda, X) \xrightarrow{b^1} \mathrm{Hom}_K(\Lambda^{\otimes 2}, X) \xrightarrow{b^2} \ldots.$$

Thus $\mathrm{H}^n(\Lambda, X) = \mathrm{Ker}\, b^n / \mathrm{Im}\, b^{n-1}$.

The group $\mathrm{H}^n(\Lambda, X)$ is also the n-th cohomology group of the complex $S^*(\Lambda, X)$. Since $S_*(\Lambda)$ is a projective resolution of Λ as a Λ,Λ-bimodule, this gives that $\mathrm{H}^n(\Lambda, X) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, X)$. Note with $X = \Lambda$, then $\mathrm{H}^n(\Lambda, \Lambda)$ is denoted by $\mathrm{HH}^n(\Lambda)$, and is called the n-th Hochschild cohomology group of Λ.

The low dimensional groups $\mathrm{HH}^0(\Lambda), \mathrm{HH}^1(\Lambda)$ and $\mathrm{HH}^2(\Lambda)$ have important interpretations. Though this thesis is interested in $\mathrm{HH}^2(\Lambda)$, we consider briefly the descriptions of $\mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^1(\Lambda)$ in this chapter as well. Let us take the Hochschild complex C^* with $X = \Lambda$ and write δ^n for b^n:

$$C^* : 0 \to \Lambda \xrightarrow{\delta^0} \mathrm{Hom}_K(\Lambda, \Lambda) \xrightarrow{\delta^1} \mathrm{Hom}_K(\Lambda^{\otimes 2}, \Lambda) \xrightarrow{\delta^2} \dots .$$

Then we have $\delta^0 : \Lambda \to \mathrm{Hom}_K(\Lambda, \Lambda)$ where $\delta^0(\lambda)$ is the element in $\mathrm{Hom}_K(\Lambda, \Lambda)$ which is given by $a \mapsto a\lambda - \lambda a$. Since $\mathrm{HH}^0(\Lambda) = \mathrm{Ker}\,\delta^0$, then

$$\begin{aligned} \mathrm{HH}^0(\Lambda) &= \{\lambda \in \Lambda \,|\, a \mapsto a\lambda - \lambda a \text{ is the zero map}\} \\ &= \{\lambda \in \Lambda \,|\, a\lambda - \lambda a = 0 \ \forall a \in \Lambda\} \\ &= Z(\Lambda), \text{ the centre of } \Lambda. \end{aligned}$$

Now we look at $\mathrm{HH}^1(\Lambda)$ and again use the complex C^*. The group $\mathrm{HH}^1(\Lambda)$ is related to the derivations of Λ.

Definition 1.3. [17, 1.2] Let X be a Λ,Λ-bimodule. The set of derivations of Λ on X is the set

$$\mathrm{Der}_K(\Lambda, X) := \{f \in \mathrm{Hom}_K(\Lambda, X) | f(a_1a_2) = a_1f(a_2) + f(a_1)a_2, \forall a_1, a_2 \in \Lambda\}.$$

The set of inner derivations of Λ on X is the set

$$\mathrm{Inn}_K(\Lambda, X) := \{f_x \in \mathrm{Hom}_K(\Lambda, X) | f_x(a) = ax - xa, \text{ with } x \in X\}.$$

We know that $\mathrm{H}^1(\Lambda, X) = \mathrm{Ker}\,\delta^1/\mathrm{Im}\,\delta^0$. Let $f \in \mathrm{Ker}\,\delta^1$; then $\delta^1(f) = 0$, where $\delta^1(f) : \Lambda^{\otimes 2} \to X$ is defined by:

$$a_1 \otimes a_2 \mapsto a_1f(a_2) - f(a_1a_2) + f(a_1)a_2.$$

So $a_1f(a_2) - f(a_1a_2) + f(a_1)a_2 = 0, \forall a_1, a_2 \in \Lambda$. Hence $f(a_1a_2) = a_1f(a_2) + f(a_1)a_2$ and so $f \in \mathrm{Der}_K(\Lambda, X)$. And $\mathrm{Im}\,\delta^0 = \{f_x \in \mathrm{Hom}_K(\Lambda, X) | f_x(a) = ax - xa, \text{ with } x \in X\} = \mathrm{Inn}_K(\Lambda, X)$. Therefore,

$$\mathrm{H}^1(\Lambda, X) = \mathrm{Der}_K(\Lambda, X)/\mathrm{Inn}_K(\Lambda, X).$$

So $\mathrm{HH}^1(\Lambda) = \mathrm{Der}_K(\Lambda, \Lambda)/\mathrm{Inn}_K(\Lambda, \Lambda)$.

Definition 1.4. [17, 1.2] If f is a derivation of Λ on X which represents a non-zero element of $\mathrm{H}^1(\Lambda, X)$ then f is called an outer derivation of Λ on X.

Now let us look at $\mathrm{HH}^2(\Lambda)$. This group is related to the deformations of Λ. The deformation theory of an associative algebra was introduced by Gerstenhaber in [12]. We take the results here from [11] which provides a useful summary and introduction to this theory.

A one-parameter algebraic deformation of Λ may be considered as a family of algebras $\{\Lambda_t\}$ such that $\Lambda_0 \cong \Lambda$ and the multiplicative structure of Λ_t varies in some "nice" algebraic way with t. We will look at the next example before giving a formal definition.

Example 1.5. Let $\Lambda = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
& \circ \arrow[dr, "c"] & \\
\circ \arrow[ur, "b"] \arrow[rr, "a"'] & & \circ
\end{tikzcd}$$

and $I = \langle bc \rangle$. Let $\Lambda_t = KQ/I_t$ where $I_t = \langle bc - at \rangle$. Notice that $bc = 0$ in Λ but $bc = at$ in Λ_t so the product varies with t.

We will return to this example in 1.19, but first we give the formal definition of a deformation.

Definition 1.6. [11, Definition 1.2] A one-parameter deformation of Λ is the power series ring $\Lambda[[t]]$ together with a multiplication F which is a formal power series $F = \sum_{n=0}^{\infty} f_n t^n$ with $f_n \in \mathrm{Hom}_K(\Lambda^{\otimes 2}, \Lambda)$ and f_0 is multiplication in Λ. We write (a, b) for $a \otimes b$. Then F gives $\Lambda[[t]]$ a $K[[t]]$-algebra structure with

$$\begin{aligned}
F(a,b) &= f_0(a,b) + f_1(a,b)t + f_2(a,b)t^2 + \dots \\
&= ab + f_1(a,b)t + f_2(a,b)t^2 + \dots
\end{aligned}$$

for $a, b \in \Lambda$. The deformation $\Lambda[[t]]$ with multiplication F is written Λ_F.

If $t = 0$ then $\Lambda[[t]] \cong \Lambda$ with multiplication $F(a, b) = ab$ so we get back to Λ as required.

Note that [11] uses $\Lambda \otimes_K K[[t]]$ instead of $\Lambda[[t]]$ but claims that $\Lambda \otimes_K K[[t]] \cong \Lambda[[t]]$. However, this isomorphism does not always hold. For, as Buchweitz remarked, if $\Lambda = K[a]$ then $\sum_{i=0}^{\infty} a_i t^i \in \Lambda[[t]]$ but $\sum_{i=0}^{\infty} a_i t^i \notin \Lambda \otimes_K K[[t]]$. So we must use $\Lambda[[t]]$ throughout.

Now, we started with an associative algebra Λ over K so we want the deformation Λ_F to be associative.

Definition 1.7. [11, Definition 1.2] The deformation Λ_F is associative if

$$F(F(a,b),c) = F(a,F(b,c)),\ \forall a,b,c \in \Lambda.$$

By expanding both sides we have:

$F(F(a,b),c) = F(\sum_{n=0}^{\infty} f_n(a,b)t^n, c)$
$= \sum_{n=0}^{\infty} F(f_n(a,b),c)t^n$
$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_m(f_n(a,b),c)t^{n+m}$,
and
$F(a,F(b,c)) = F(a, \sum_{n=0}^{\infty} f_n(b,c)t^n)$
$= \sum_{n=0}^{\infty} F(a, f_n(b,c))t^n$
$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_m(a, f_n(b,c))t^{n+m}$.

Now if we collect the coefficients of t^r we get

$$\sum_{i=0}^{r} f_i(f_{r-i}(a,b),c) = \sum_{i=0}^{r} f_i(a, f_{r-i}(b,c)) \qquad (1)$$

Definition 1.8. [11, p22] Let f_m be the first non-zero coefficient where $m \geq 1$ in the power series for F. Then f_m is called the infinitesimal of F.

Now let f_m be the infinitesimal of F and put $r = m$ in (1) to give:

$$f_0(f_m(a,b),c) + f_m(f_0(a,b),c) = f_0(a, f_m(b,c)) + f_m(a, f_0(b,c)).$$

Since f_0 is multiplication in Λ we have:

$$f_m(a,b)c + f_m(ab,c) = af_m(b,c) + f_m(a,bc).$$

Hence $af_m(b,c) - f_m(ab,c) + f_m(a,bc) - f_m(a,b)c = 0$. Therefore, $f_m \in \operatorname{Ker} \delta^2$.

This leads to the first theorem which connects deformation theory and cohomology theory.

Theorem 1.9. [11, Theorem 2.1] *If F is an associative deformation of Λ then the infinitesimal f_m of F is in* $\operatorname{Ker}\delta^2$*, that is, f_m is a 2-cocycle.*

Now we consider when any 2-cocycle may be extended to give an associative deformation of Λ.

For arbitrary n, we may write (1) as:

$$f_0(f_n(a,b),c) + \textstyle\sum_{i=1}^{n-1} f_i(f_{n-i}(a,b),c) + f_n(f_0(a,b),c)$$
$$= f_0(a, f_n(b,c) + \textstyle\sum_{i=1}^{n-1} f_i(a, f_{n-i}(b,c)) + f_n(a, f_0(b,c)).$$

Then

$$(\delta^2 f_n)(a,b,c) = \sum_{i=1}^{n-1} f_i(f_{n-i}(a,b),c) - \sum_{i=1}^{n-1} f_i(a, f_{n-i}(b,c)). \qquad (2)$$

If $f_1, \ldots, f_{m-1}$ satisfy (2) then let

$$g = \sum_{i=1}^{m-1} [f_i(f_{m-i}(a,b),c) - f_i(a, f_{m-i}(b,c))],$$

that is, the right hand side of (2). Then the cohomology class of g may be viewed as an obstruction to the construction of f_m which extends the deformation.

We now have the following important theorem proved by Gerstenhaber; a proof may also be found in [11, p33]

Theorem 1.10. [12, §5 Proposition 3] *The obstruction g is a 3-cocycle, that is, $\delta^3 g = 0$.*

Corollary 1.11. [11, Corollary 2.3] *If* $\mathrm{HH}^3(\Lambda) = 0$ *then every 2-cocycle of Λ may be extended to an associative deformation of Λ.*

Proof. Let f be a 2-cocycle of Λ. We construct an associative deformation F of Λ with infinitesimal f, thus showing that every 2-cocycle of Λ may be extended to an associative deformation of Λ.

Let f_0 be the usual multiplication in Λ and let $f_1 = f$. Then, from Theorem 1.9, f_1 satisfies (2). Let $g_2 = f_1(f_1(a,b),c) - f_1(a, f_1(b,c))$, that is, the right hand side of (2) with $n = 2$. From Theorem 1.10 we have $\delta^3 g_2 = 0$ so $g_2 \in \operatorname{Ker}\delta^3$. Since $\mathrm{HH}^3(\Lambda) = 0$ then $\operatorname{Ker}\delta^3 = \operatorname{Im}\delta^2$. Hence $g_2 \in \operatorname{Im}\delta^2$. Therefore $g_2 = \delta^2 f_2$ for some f_2.

Now let $g_3 = \sum_{i=1}^{2}[f_i(f_{3-i}(a,b),c) - f_i(a, f_{3-i}(b,c))]$, that is, the right hand side of (2) with $n = 3$. Again from Theorem 1.10 we have $\delta^3 g_3 = 0$ so that $g_3 = \delta^2 f_3$ for some f_3.

Continuing in this way gives f_n for $n \geq 2$. Let $F = \sum_{n\geq 0}^{\infty} f_n t^n$. Then F is an associative deformation that extends the cocycle $f = f_1$. □

Now we want to describe whether or not two deformations are significantly different from one another. Given associative deformations Λ_F and Λ_G of Λ we want to know when there is an isomorphism $\Psi : \Lambda_F \to \Lambda_G$ which keeps Λ fixed.

Definition 1.12. [11, p23] A formal isomorphism $\Psi : \Lambda_F \to \Lambda_G$ is a $K[[t]]$-linear map that may be written in the form

$$\Psi(a) = a + \psi_1(a)t + \psi_2(a)t^2 + \cdots, \text{ for } a \in \Lambda.$$

We remark that it is enough to consider $a \in \Lambda$ since Ψ is $K[[t]]$-linear, and we also assume that each $\psi_i \in \mathrm{Hom}_K(\Lambda, \Lambda)$.

The formal isomorphism Ψ is an algebraic isomorphism if Ψ is multiplication preserving, that is, if

$$G(\Psi(a), \Psi(b)) = \Psi(F(a,b)) \text{ for all } a \text{ and } b \text{ in } \Lambda.$$

The deformations Λ_F and Λ_G are said to be equivalent if there is an algebraic isomorphism $\Psi : \Lambda_F \to \Lambda_G$. In this case we write $\Lambda_F \cong \Lambda_G$.

If Λ_F and Λ_G are equivalent, then from [11, p23], we have $\delta^1 \psi_1 = f_1 - g_1$. So f_1 and g_1 represent the same element of $\mathrm{HH}^2(\Lambda)$.

Now suppose that we have constructed $\psi_1, \psi_2, \ldots, \psi_{m-1}$, from Λ_F to Λ_G. Then we have the following theorem and corollary.

Theorem 1.13. [12, §5 Proposition 2] *The obstruction to finding ψ_m which extends the isomorphism is a 2-cocycle.*

Thus if $\mathrm{HH}^2(\Lambda) = 0$, then there is no obstruction to finding ψ_m so all such obstructions vanish and $\Lambda_F \cong \Lambda_G$.

Corollary 1.14. [12] *If* $\mathrm{HH}^2(\Lambda) = 0$ *then all deformations of* Λ *are isomorphic.*

Definition 1.15. [11, p23] The trivial deformation of Λ is the deformation Λ_F with $F = f_0$. We say that a deformation is trivial if it is isomorphic to the trivial deformation. An algebra Λ is rigid if it has no non-trivial deformations.

Theorem 1.14 implies that if $\mathrm{HH}^2(\Lambda) = 0$ then all deformations of Λ are trivial. Hence we have the following result.

Theorem 1.16. [12, §3 Corollary] *If* $\mathrm{HH}^2(\Lambda) = 0$ *then* Λ *is rigid.*

However, the converse to Theorem 1.16 is not true in general as was shown in [13]. However, it was shown in [8] that the converse does hold for monomial algebras KQ/I where Q has no oriented cycles. We briefly discuss [8] in relation to Example 1.5.

Definition 1.17. A finite dimensional algebra KQ/I over a field K is a monomial algebra if I is a two sided ideal of KQ generated by a set of paths each of length at least 2.

Theorem 1.18. [8, Theorem 3.12, Theorem 4.2] *Let* Q *be a quiver with no oriented cycles and* I *an ideal of* KQ *generated by a set of paths each of length at least 2. Then* $\Lambda = KQ/I$ *is rigid if and only if* $\mathrm{HH}^2(\Lambda) = 0$.

Example 1.19. Let $\Lambda = KQ/I$ as in Example 1.5 so that Q is the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "c"] & \\ 1 \arrow[ur, "b"] \arrow[rr, "a"'] & & 3 \end{tikzcd}$$

and $I = \langle bc \rangle$. Thus Λ is a monomial algebra.

It is straightforward to show that there is a non-zero element of $\mathrm{HH}^2(\Lambda)$ given by

$$P^2 = \Lambda e_1 \otimes e_3 \Lambda \to \Lambda \text{ such that } e_1 \otimes e_3 \mapsto a.$$

(Indeed, in [4] a minimal projective resolution of a monomial algebra is given.)

Thus from [8], Λ is not rigid, that is, Λ has a non-trivial deformation. The construction of a deformation is given in [8], and for this example, $\Lambda_t = KQ/I_t$ where $I_t = \langle bc - at \rangle$ is a non-trivial deformation of Λ.

Thus we have shown the link between $\mathrm{HH}^2(\Lambda)$ and deformations. Our aim is to compute $\mathrm{HH}^2(\Lambda)$ for a specific class of finite dimensional algebras.

In the next chapter we introduce our algebras together with the notion of derived equivalence.

2. Derived equivalence and self-injective algebras of finite representation type

In the following chapters we will look at $\mathrm{HH}^2(\Lambda)$ for all finite dimensional self-injective algebras of finite representation type over an algebraically closed field. Recall that a finite dimensional algebra is of finite representation type if there are only finitely many isomorphism classes of indecomposable finitely generated Λ-modules. These algebras were classified in [2] by quiver and relations up to derived equivalence. It is known from Happel's result in [17] that Hochschild cohomology is invariant under derived equivalence. We start with a brief description of some category theory to enable us to introduce the concept of derived equivalence. This is mainly taken from [24]; details can also be found in [21].

Definition 2.1. [24, Definition A.1.1] A category $\mathcal{C}$ consists of a class of objects $\mathcal{O}bj(\mathcal{C})$, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ for every ordered pair (A,B) of objects, an identity morphism $id_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ for each object A, and a composition function

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$$

for every ordered triple (A,B,C) of objects. In addition, a category needs to satisfy two axioms:

(i) Associativity Axiom, that is, $(hg)f = h(gf)$ for $f : A \to B, g : B \to C, h : C \to D$.

(ii) Unit axiom, that is, $id_B f = f = f id_A$ for $f : A \to B$.

Example 2.2. For Λ a finite dimensional algebra, $\operatorname{mod}\Lambda$ is the category whose objects are finitely generated Λ-modules and morphisms are Λ-module homomorphisms.

Definition 2.3. [24, p421] A functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule that associates an object $F(C)$ of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphism $f : C_1 \to C_2$ in $\mathcal{C}$. In addition, F is required to preserve identity morphisms, that is, $F(id_C) = id_{F(C)}$ and composition, that is, $F(gf) = F(g)F(f)$.

Definition 2.4. [24, p422] A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is a collection of some objects and some morphisms of $\mathcal{C}$, such that the morphisms of $\mathcal{B}$ are closed under composition and include id_B for every object $B \in \mathcal{B}$. A subcategory is itself a category.

A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ in which $\mathrm{Hom}_{\mathcal{B}}(B_1, B_2) = \mathrm{Hom}_{\mathcal{C}}(B_1, B_2)$, for every B_1, B_2 in $\mathcal{B}$ is called a full subcategory.

To define the stable category we need to specify the objects and morphisms. For an algebra Λ, let $\mathrm{mod}\,\Lambda$ denote the category of finite dimensional Λ-modules and $pro\,\Lambda$ denote the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of finite dimensional projective Λ-modules.

Definition 2.5. Let Λ be a self-injective algebra. Then the category of finitely generated Λ-modules modulo projectives is the stable category $\underline{\mathrm{mod}}\,\Lambda$, that is, $\underline{\mathrm{mod}}\,\Lambda := \mathrm{mod}\,\Lambda/\langle pro\,\Lambda\rangle$. So the objects of $\underline{\mathrm{mod}}\,\Lambda$ are finitely generated Λ-modules. If M, N are in $\mathrm{mod}\,\Lambda$, let $P\mathrm{Hom}_{\Lambda}(M, N)$ be the subspace of $\mathrm{Hom}_{\Lambda}(M, N)$ consisting of all those Λ-morphisms which factor through projective modules. Note that $f : M \to N$ factors through a projective if there exists a projective module P and two morphisms $g : M \to P$ and $h : P \to N$ such that $f = hg$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g"'] & N \\ & P \arrow[u, "h"'] \end{tikzcd}$$

Now we define the morphisms in the stable category by

$$\underline{\mathrm{Hom}}(M, N) := \mathrm{Hom}_{\Lambda}(M, N)/P\mathrm{Hom}_{\Lambda}(M, N).$$

Remark. If M is projective module and $f : M \to N$, then f factors through a projective, that is, the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "id"'] & N \\ & M \arrow[u, "f"'] \end{tikzcd}$$

So in $\underline{\mathrm{mod}}\,\Lambda, f : M \to N$ is the same as the zero map $0 : M \to N$ and, in particular, $id_M : M \to M$ is the zero map $0 : M \to M$. Therefore, every projective module is isomorphic to zero in the stable category.

Definition 2.6. [24, A.3] Suppose that F and G are two functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\eta : F \to G$ is a rule that associates a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$ to every object C of $\mathcal{C}$ in such a way that for every morphism $f : C \to C'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F(C) \arrow[r, "F_f"] \arrow[d, "\eta_C"'] & F(C') \arrow[d, "\eta_{C'}"] \\ G(C) \arrow[r, "G_f"] & G(C') \end{tikzcd}$$

If each η_C is an isomorphism, we say η is natural equivalence or natural isomorphism and write $F \cong G$.

An equivalence of categories is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are naturally isomorphic to the appropriate identity functor, that is, $FG \cong id_{\mathcal{D}}$ and $GF \cong id_{\mathcal{C}}$.

Now the following definition tells us when two algebras are stably equivalent.

Definition 2.7. [2, p1] For two algebras Λ, Π, an equivalence $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Pi$ is called a stable equivalence from Λ to Π. The algebras Λ, Π are said to be stably equivalent if there exists a stable equivalence between them.

In fact, for a self-injective algebra Λ, the stable category $\underline{\mathrm{mod}}\,\Lambda$ is a triangulated category. We see this in Theorem 2.13 but first we will define a triangulated category using [18] and [24].

Definition 2.8. [24, A.4.1] A category $\mathcal{C}$ is called an **Ab**-category if the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group such that composition distributes over addition, that is, for appropriate morphisms f, g, g' and h, we have $f(g + g')h = fgh + fg'h$.

Definition 2.9. [24, p425] An **Ab**-category $\mathcal{C}$ is called an additive category if $\mathcal{C}$ satisfies the following:

(1) There is a zero object $0 \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(0, A) \cong e \cong \mathrm{Hom}_{\mathcal{C}}(A, 0)$, where e is the trivial abelian group consisting of one element.

(2) For all $X, Y \in Obj(\mathcal{C})$, there exists $X \oplus Y \in Obj(\mathcal{C})$ with maps:

$$\begin{tikzcd} & X \oplus Y \arrow[dl, shift left, "p_X"] \arrow[dr, shift left, "p_Y"] & \\ X \arrow[ur, shift left, "i_X"] & & Y \arrow[ul, shift left, "i_Y"] \end{tikzcd}$$

that satisfies the following:

$$p_X i_X = id_X; \quad p_Y i_Y = id_Y;$$

$$p_X i_Y = 0; \quad p_Y i_X = 0;$$

$$i_X p_X + i_Y p_Y = id_{X \oplus Y}.$$

Definition 2.10. [18, 1.1] Let $\mathcal{C}$ be an additive category and T an automorphism of $\mathcal{C}$. A sextuple (X, Y, Z, u, v, w) in $\mathcal{C}$ is given by objects $X, Y, Z \in Obj(\mathcal{C})$ and morphisms $u : X \to Y, v : Y \to Z$ and $w : Z \to TX$. The sextuple can be written as follows:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX.$$

The automorphism T is usually called the translation functor, and its inverse is denoted by T^{-1}. A morphism of sextuples from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple (f, g, h) forming a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

If f, g and h are isomorphisms in $\mathcal{C}$ the morphism is then called an isomorphism. Then we say that the two sextuples are isomorphic.

Definition 2.11. [18, p2] An additive category $\mathcal{C}$ is called a triangulated category if there is an automorphism $T : \mathcal{C} \to \mathcal{C}$, a set of sextuples in $\mathcal{C}$ and a fixed set of sextuples which we call triangles such that the following conditions hold.

(TR1) Every sextuple isomorphic to a triangle is a triangle.

Every morphism $u : X \to Y$ can be embedded into a triangle.

The sextuple $(A, A, 0, id_A, 0, 0)$ is a triangle.

(TR2) (Rotation).

If (X, Y, Z, u, v, w) is a triangle then $(Y, Z, TX, v, w, -Tu)$ is a triangle.

(TR3) (Morphisms).

Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and morphisms $f : X \to X', g : Y \to Y'$ such that $u'f = gu$, there exists a morphism (f, g, h) of triangles.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "\exists h"'] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

(TR4) (The octahedral axiom).

Consider three triangles $(X, Y, Z', u, i, i'), (Y, Z, X', v, j, j')$ and (X, Z, Y', vu, k, k'). Then there is a triangle $(Z', Y', X', TZ', f, g, Tij')$ such that $gk = j, k'f = i', fi = kv$ and $Tuk' = j'g$.

Definition 2.12. [18, p4] For additive categories $\mathcal{C}, \mathcal{D}$ an additive functor $F : \mathcal{C} \to \mathcal{D}$ is a functor such that each $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is a group homomorphism.

An additive functor F between two triangulated categories $\mathcal{C}, \mathcal{C}'$ is called exact if for T and T' automorphisms on $\mathcal{C}$ and $\mathcal{C}'$ respectively, there exists an invertible natural transformation $\alpha : FT \to T'F$ such that $(FX, FY, FZ, Fu, Fv, Fw\alpha_X)$ is a triangulation of $\mathcal{C}$ whenever (X, Y, Z, u, v, w) is a triangulation of $\mathcal{C}$.

If an exact functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories, then we call it a triangle equivalence. Then $\mathcal{C}$ and $\mathcal{C}'$ are called triangle equivalent.

For $M \in \underline{\mathrm{mod}}\,\Lambda$, if $0 \to M \to I_1 \to N_1 \to 0$ and $0 \to M \to I_2 \to N_2 \to 0$ are short exact sequences in $\mathrm{mod}\,\Lambda$ with I_1, I_2 injective Λ-modules then from [18, Lemma 2.2] we know that $N_1 \cong N_2$ in $\underline{\mathrm{mod}}\,\Lambda$. We let $\Omega^{-1} = N_1$ which is unique up to isomorphism in $\underline{\mathrm{mod}}\,\Lambda$.

Theorem 2.13. [18, Theorem 2.6] *If Λ is self-injective algebra the stable category $\underline{\mathrm{mod}}\,\Lambda$ is a triangulated category with $T = \Omega^{-1}$ and the triangles are the sextuples isomorphic to a standard triangle.*

In the next part using [24] we will define the derived category $D(\mathcal{A})$ of an abelian category. It is obtained from the category $Ch(\mathcal{A})$ of cochain complexes in two stages; the first step is to construct a quotient category of $Ch(\mathcal{A})$ and the next step is to localize this quotient category.

Definition 2.14. [24, A.4.2] An abelian category is an additive category $\mathcal{A}$ such that:

(1) Every map in $\mathcal{A}$ has a kernel and cokernel,
(2) Every monic in $\mathcal{A}$ is the kernel of its cokernel, and
(3) Every epi in $\mathcal{A}$ is the cokernel of its kernel.

Definition 2.15. [24, p2] Let $\mathcal{A}$ be an abelian category. The category of cochain complexes $Ch(\mathcal{A})$ is the category where the objects are bi-infinite cochain complexes in $\mathcal{A}$ and the morphisms are cochain maps. A cochain map $f : A \to B$ in $Ch(\mathcal{A})$ is a commutative diagram:

$$\begin{tikzcd}
A \arrow[d, "f"'] & = \cdots \arrow[r] & \cdots \arrow[r] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d, "f^{-1}"'] & A^0 \arrow[r, "d_A^0"] \arrow[d, "f^0"'] & A^1 \arrow[r, "d_A^1"] \arrow[d, "f^1"'] & A^2 \arrow[r] \arrow[d, "f^2"'] & \cdots \\
B & = \cdots \arrow[r] & \cdots \arrow[r] & B^{-1} \arrow[r, "d_B^{-1}"] & B^0 \arrow[r, "d_B^0"] & B^1 \arrow[r, "d_B^1"] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

where $d_A^{i+1}d_A^i = 0, d_B^{i+1}d_B^i = 0, A^i, B^i \in \mathcal{A}, \forall i \in \mathbb{Z}$.

Let f and g be two cochain maps from A to B. We say that f and g are (cochain) homotopic if there are maps $s_n : A^n \to B^{n-1}$ such that $f - g = sd + ds$. The maps $\{s_n\}$ are called a cochain homotopy from f to g.

The objects of the homotopy category $\mathcal{H}(\mathcal{A})$ are cochains in $\mathcal{A}$ and the morphisms are homotopy equivalence classes of cochain maps. Thus

$$\mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(A, B) \cong \mathrm{Hom}_{Ch(\mathcal{A})}(A, B)/\sim,$$

where $f \sim g$ if and only if f is homotopic to g. Note that $\mathcal{H}(\mathcal{A})$ is a triangulated category and it is a quotient category of $Ch(\mathcal{A})$.

To define the derived category we now need to define a quasi-isomorphism and a localisation.

Definition 2.16. [24, Definition 1.1.2] A cochain map $f : A \to B$ in $\mathcal{H}(\mathcal{A})$ is a quasi-isomorphism if the induced cohomology maps $\tilde{f}^n : H^n(A) \to H^n(B)$ are all isomorphisms.

Definition 2.17. [24, Definition 10.3.1] Let S be a collection of morphisms in a category $\mathcal{C}$. A localisation of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$, together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that:

i) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.

ii) Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors in a unique way through q.

We now come to the definition of the derived category.

Definition 2.18. [24, p379] The derived category of an abelian category $\mathcal{A}$ is defined as the localisation $D(\mathcal{A}) := Q^{-1}\mathcal{H}(\mathcal{A})$, where Q is the collection of quasi-isomorphisms in $\mathcal{H}(\mathcal{A})$.

Theorem 2.19. [24, Corollary 10.4.3] *Let $\mathcal{A}$ be an abelian category. Then $D(\mathcal{A}), D^b(\mathcal{A}), D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ are all triangulated categories where $D^b(\mathcal{A})$, $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ are the full subcategories of $D(\mathcal{A})$ whose objects are the cochain complexes which are bounded, bounded below and bounded above, respectively.*

Definition 2.20. [2, p2] A triangle equivalence $D(\mathcal{A}) \to D(\mathcal{B})$ between two derived categories is called a derived equivalence.

If $\mathcal{A} = \underline{\text{mod}}\,\Lambda$ and $\mathcal{B} = \underline{\text{mod}}\,\Pi$ for algebras Λ, Π and if there is a triangle equivalence $D(\mathcal{A}) \to D(\mathcal{B})$ then we say Λ, Π are derived equivalent.

Now we describe Asashiba's work of [2]. The main result for us is [2, Theorem 1.2] in the paper which we state in Theorem 2.24. In brief, Asashiba gives the derived equivalence class representatives of all self-injective finite dimensional algebras of finite representation type over a field K. His description is given in terms of the type of Λ. Throughout the paper K denotes an algebraically closed field, all algebras are assumed to be basic, connected, finite dimensional algebras with identity.

The type $\text{typ}(\Lambda)$ was defined in [3]. We recall the definition here. Some definitions are needed first and they are taken from [1].

Definition 2.21. [1, p166] A morphism $g : B \to C$ in $\text{mod}\,\Lambda$ is called irreducible if g is neither a split monomorphism nor a split epimorphism and if $g = ts$ for some $s : B \to X$ and $t : X \to C$ then s is a split monomorphism or t is a split epimorphism.

Definition 2.22. [1, p225] For any algebra Λ of finite representation type the Auslander-Reiten quiver of Λ (AR-quiver) is the quiver where the vertices are the indecomposable finitely generated Λ-modules and the arrows are the irreducible morphisms between indecomposable finitely generated modules together with the Auslander-Reiten translate τ.

Now we define the type typ(Λ). From [22], the stable AR-quiver of a self-injective algebra Λ of finite representation type has the form $\mathbb{Z}\Delta/\langle g\rangle$, where Δ is a Dynkin graph, $g = \zeta\tau^{-r}$ such that r is a natural number, ζ is an automorphism of the quiver $\mathbb{Z}\Delta$ with a fixed vertex, τ is the AR-translate.

Then typ(Λ) $:= (\Delta, f, t)$, where t is the order of ζ and $f := r/m_\Delta$ such that $m_\Delta = n, 2n-3, 11, 17$ or 29 as $\Delta = A_n, D_n, E_6, E_7$ or E_8, respectively.

Remark. The type of Λ is uniquely determined by the stable Auslander-Reiten quiver.

Proposition 2.23. [2, Proposition 1.1] *Given Λ a self-injective algebra of finite representation type then the type typ(Λ) is one of the following:*

$\{(A_n, s/n, 1)|n, s \in \mathbb{N}\}$;

$\{(A_{2p+1}, s, 2)|p, s \in \mathbb{N}\}$;

$\{(D_n, s, 1)|n, s \in \mathbb{N}, n \geq 4\}$;

$\{(D_n, s, 2)|n, s \in \mathbb{N}, n \geq 4\}$;

$\{(D_4, s, 3)|s \in \mathbb{N}\}$;

$\{(D_{3m}, s/3, 1)|m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$;

$\{(E_n, s, 1)|n = 6, 7, 8, s \in \mathbb{N}\}$; *and*

$\{(E_6, s, 2)|s \in \mathbb{N}\}$.

Now we state the main theorem for us.

Theorem 2.24. [3, Theorem] *Let Λ and Π be representation-finite self-injective algebras.*

(i) If Λ is standard and Π is non-standard then Λ and Π are not derived equivalent.

(ii)If Λ and Π are either both standard or both non-standard then the following are equivalent:

1) Λ and Π are derived equivalent;

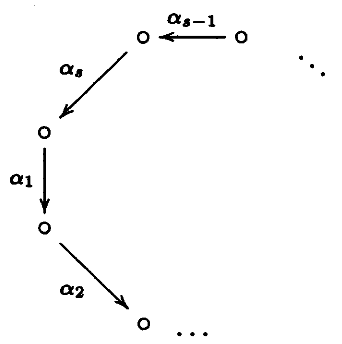
2) Λ and Π are stably equivalent;

3) typ(Λ) = typ(Π).

Thus we may list the derived equivalence class representatives according to their type. Asashiba gives a precise description of these algebras by quiver and relations in [2]. We give here the full classification of [2] of derived equivalence class representatives of the finite dimensional self-injective algebras of finite representation type over an algebraically closed field. Using Proposition 2.23 and Theorem 2.24, the derived equivalence representatives are given in 2.25-2.33. Note that $[j]$ denotes the residue of j modulo s where $s \geq 1$ and we write paths from left to right (whereas paths are written from right to left in [2]).

2.25. $\Lambda(A_n, s/n, 1)$ *with* $s, n \geq 1$.

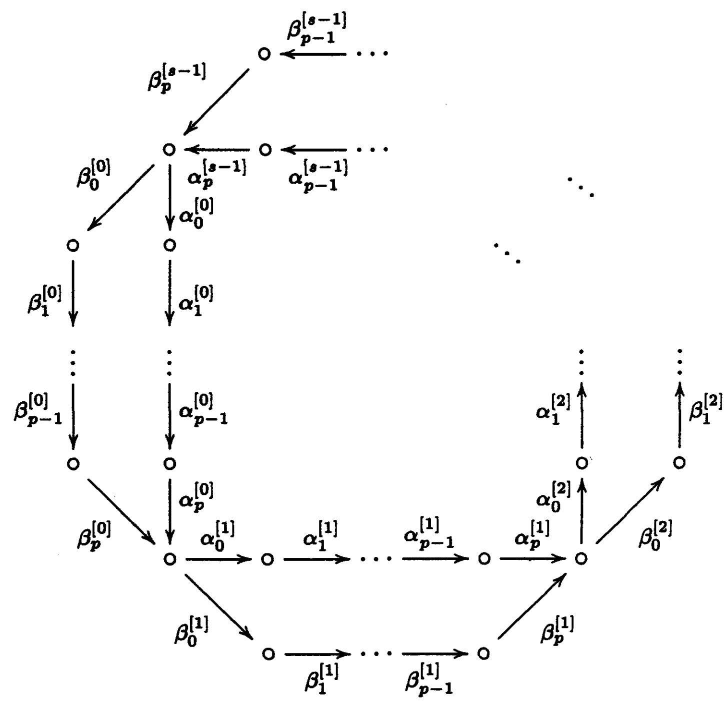
$\Lambda(A_n, s/n, 1)$ with $s, n \geq 1$ is the Nakayama algebra $N_{s,n}$ and it is given by the quiver $Q(N_{s,n})$:



with relations $R(N_{s,n})$:
$\alpha_i\alpha_{i+1}\cdots\alpha_{i+n} = 0$, for all $i \in \{1, 2, \ldots, s\} = \mathbb{Z}/\langle s\rangle$.

2.26. $\Lambda(A_{2p+1}, s, 2)$ *with* $s, p \geq 1$.

$\Lambda(A_{2p+1}, s, 2)$ with $s, p \geq 1$ is the Möbius algebra $M_{p,s}$ and it is given by the quiver $Q(M_{p,s})$:



with relations $R(M_{p,s})$:

(i) $\alpha_0^{[i]} \cdots \alpha_p^{[i]} = \beta_0^{[i]} \cdots \beta_p^{[i]}$, for all $i \in \{0, \ldots, s-1\}$,

(ii) for all $i \in \{0, \ldots, s-2\}$,

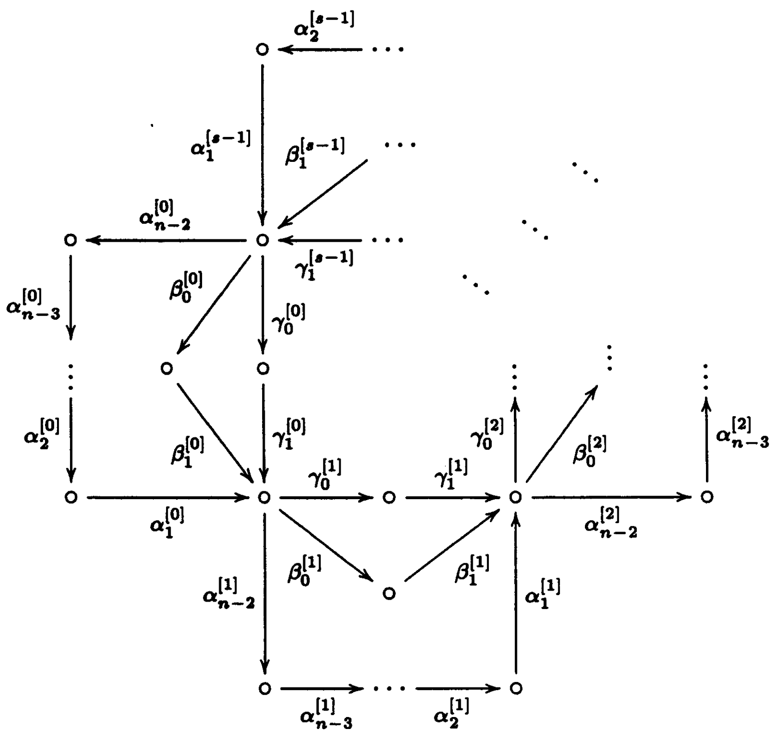
$$\alpha_p^{[i]}\beta_0^{[i+1]} = 0, \qquad \beta_p^{[i]}\alpha_0^{[i+1]} = 0,$$

$$\alpha_p^{[s-1]}\alpha_0^{[0]} = 0, \qquad \beta_p^{[s-1]}\beta_0^{[0]} = 0,$$

(iii) paths of length $p+2$ are equal to 0.

2.27. $\Lambda(D_n, s, 1)$ *with* $n \geq 4, s \geq 1$.

$\Lambda(D_n, s, 1)$ with $n \geq 4, s \geq 1$ is given by the quiver $Q(D_n, s)$:



with relations $R(D_n, s, 1)$:

(i) $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$

$$\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0,$$

$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$

(iii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$ and for all $j \in \{1, \ldots, n-2\} = \mathbb{Z}/\langle n-2\rangle$,

$$\alpha_j^{[i]} \ldots \alpha_{j-n+2}^{[i+1]} = 0,$$
$$\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]} = 0, \qquad \gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]} = 0.$$

The set of relations (iii) means that "α-paths" of length $n-1$ are equal to 0, "β-paths" of length 3 are equal to 0 and "γ-paths" of length 3 are equal to 0.

2.28. $\Lambda(D_n, s, 2)$ *with* $n \geq 4, s \geq 1$.

$\Lambda(D_n, s, 2)$ with $n \geq 4, s \geq 1$ is given by the quiver $Q(D_n, s)$ above with relations $R(D_n, s, 2)$:

(i) $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0,$$

and for all $i \in \{0, \ldots, s-2\}$,

$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$
$$\beta_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0,$$

(iii) "α-paths" of length $n-1$ are equal to 0, and for all $i \in \{0, \ldots, s-2\}$,

$$\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]} = 0, \qquad \gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]} = 0 \text{ and}$$
$$\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]} = 0, \qquad \gamma_0^{[s-1]}\gamma_1^{[s-1]}\beta_0^{[0]} = 0,$$
$$\beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]} = 0, \qquad \gamma_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} = 0.$$

2.29. $\Lambda(D_4, s, 3)$ *with* $s \geq 1$.

$\Lambda(D_4, s, 3)$ with $s \geq 1$ is given by the quiver $Q(D_4, s)$ above with relations $R(D_4, s, 3)$:

(i) $\alpha_0^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

(ii) for all $i \in \{0, \ldots, s-2\}$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$

$$\beta_1^{[i]}\alpha_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_0^{[i+1]} = 0,$$

$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$

and

$$\alpha_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \alpha_1^{[s-1]}\gamma_0^{[0]} = 0,$$

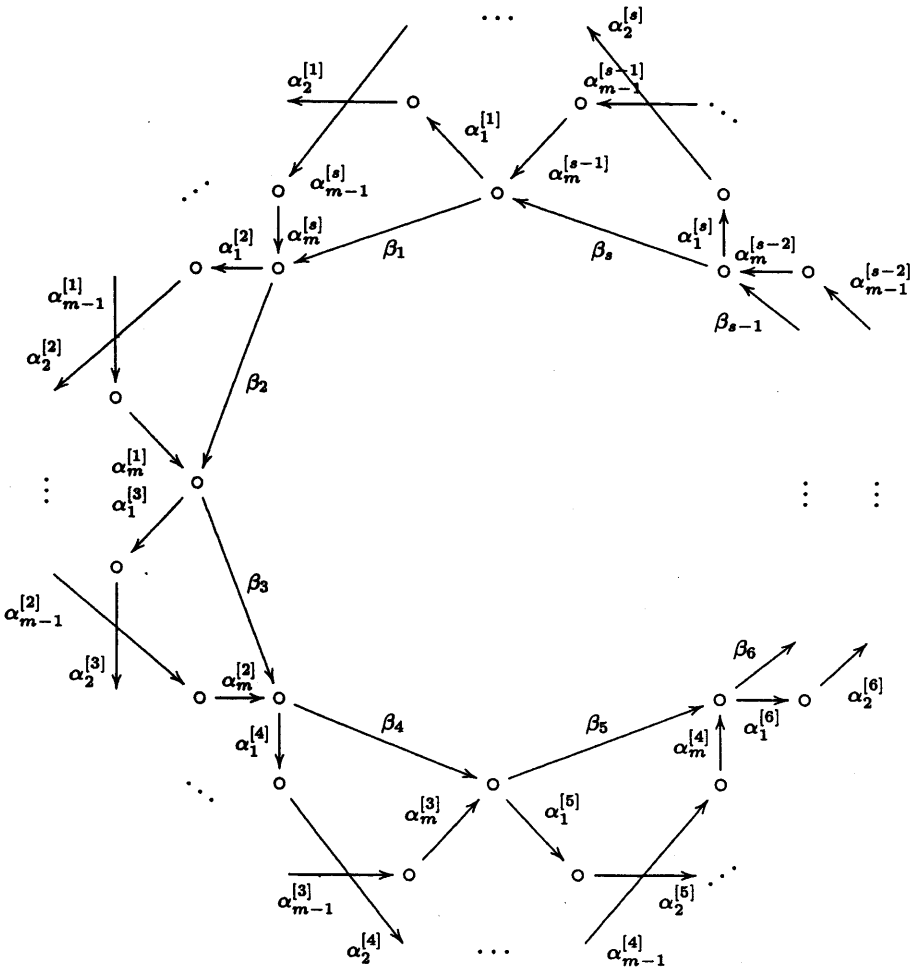
$$\beta_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_0^{[0]} = 0,$$

$$\gamma_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0,$$

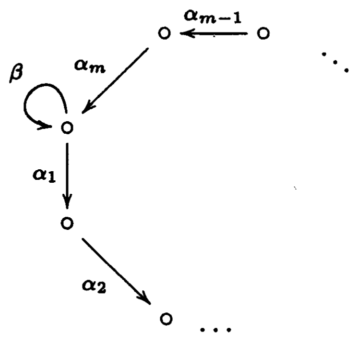
(iii) paths of length 3 are equal to 0.

2.30. $\Lambda(D_{3m}, s/3, 1)$ *with* $m \geq 2$ *and* $3 \nmid s \geq 1$.

$\Lambda(D_{3m}, s/3, 1)$ with $m \geq 2$ and $3 \nmid s \geq 1$ is given by the quiver $Q(D_{3m}, s/3)$:



and for $s = 1$, $Q(D_{3m}, 1/3)$:



with relations $R(D_{3m}, s/3, 1)$:

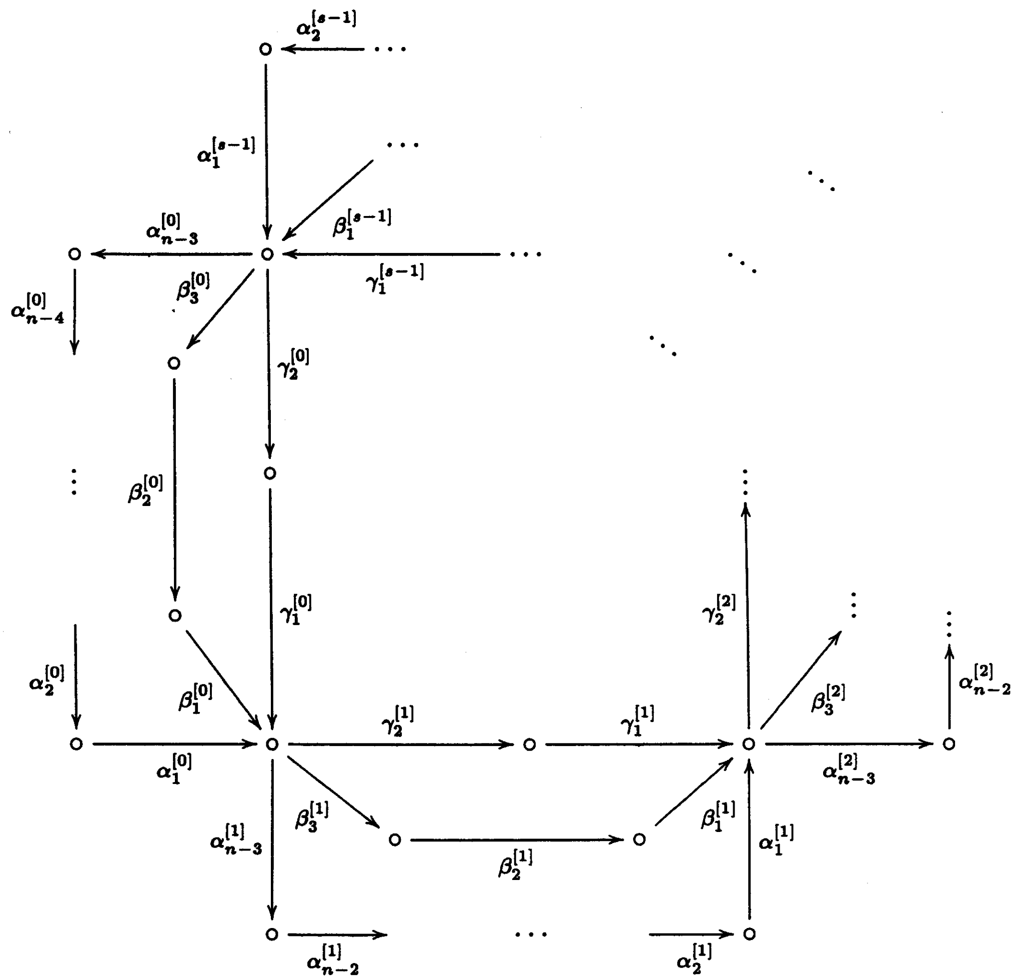
(i) $\alpha_1^{[i]}\alpha_2^{[i]}\cdots\alpha_m^{[i]} = \beta_i\beta_{i+1}$, for all $i \in \{1,\ldots,s\} = \mathbb{Z}/\langle s\rangle$,

(ii) $\alpha_m^{[i]}\alpha_1^{[i+2]} = 0$, for all $i \in \{1,\dots,s\} = \mathbb{Z}/\langle s\rangle$,

(iii) $\alpha_j^{[i]}\cdots\alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]}\cdots\alpha_j^{[i+3]} = 0$, for all $i \in \{1,\dots,s\} = \mathbb{Z}/\langle s\rangle$ and for all $j \in \{1,\dots,m\}$ (i.e. paths of length $m+2$ are equal to 0).

2.31. $\Lambda(E_n, s, 1)$ *with* $n \in \{6,7,8\}$ *and* $s \geq 1$.

$\Lambda(E_n, s, 1)$ is given by the quiver $Q(E_n, s)$:



with relations $R(E_n, s, 1)$:

(i) $\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} = \gamma_2^{[i]}\gamma_1^{[i]}$, for all $i \in \{0,\dots,s-1\}$,

(ii) for all $i \in \{0,\dots,s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\alpha_1^{[i]}\beta_3^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_2^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_{n-3}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-3}^{[i+1]} = 0,$$

$$\beta_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$

(iii) "α-paths" of length $n-2$ are equal to 0, "β-paths" of length 4 are equal to 0 and "γ-paths" of length 3 are equal to 0.

2.32. $\Lambda(E_6, s, 2)$ *with* $s \geq 1$.

$\Lambda(E_6, s, 2)$ is given by the quiver $Q(E_6, s)$ above with relations $R(E_6, s, 2)$:
(i) $\alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} = \gamma_2^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \dots, s-1\}$,
(ii) for all $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\gamma_1^{[i]}\alpha_3^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$

$$\alpha_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \beta_1^{[i]}\gamma_2^{[i+1]} = 0,$$

and for all $i \in \{0, \dots, s-2\}$,

$$\alpha_1^{[i]}\beta_3^{[i+1]} = 0, \qquad \beta_1^{[i]}\alpha_3^{[i+1]} = 0,$$

$$\alpha_1^{[s-1]}\alpha_3^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_3^{[0]} = 0,$$

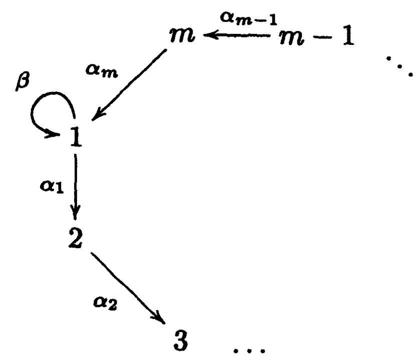
(iii) "γ-paths" of length 3 are equal to 0 and for all $i \in \{0, \dots, s-2\}$ and for all $j \in \{1, 2, 3\} = \mathbb{Z}/\langle 3 \rangle$,

$$\alpha_j^{[i]} \cdots \alpha_{j-3}^{[i+1]} = 0, \qquad \beta_j^{[i]} \cdots \beta_{j-3}^{[i+1]} = 0,$$

$$\alpha_j^{[s-1]} \cdots \alpha_1^{[s-1]}\beta_3^{[0]} \cdots \beta_{j-3}^{[0]} = 0, \qquad \beta_j^{[s-1]} \cdots \beta_1^{[s-1]}\alpha_3^{[0]} \cdots \alpha_{j-3}^{[0]} = 0.$$

2.33. *Nonstandard algebras* $\Lambda(m)$ *with* $m \geq 2$.

From [2] the derived equivalence representatives of the nonstandard self-injective algebras of finite representation type over an algebraically closed field K are the algebras $\Lambda(m)$ for each $m \geq 2$, where $\Lambda(m)$ is given by the quiver $Q(D_{3m}, 1/3)$:



with relations $R(m)$:

(i) $\alpha_1\alpha_2\cdots\alpha_m = \beta^2$,

(ii) $\alpha_m\alpha_1 = \alpha_m\beta\alpha_1$,

(iii) $\alpha_i\alpha_{i+1}\cdots\alpha_i = 0$, for all $i \in \{1,\ldots,m\} = \mathbb{Z}/\langle m\rangle$ (i.e. "α" paths of length $m+1$ are equal to 0).

3. Projective resolutions

To find the Hochschild cohomology groups for any finite dimensional algebra Λ, a projective resolution of Λ as a Λ^e-module is needed. So in this chapter we will look at the projective resolutions of [15] and [16] in order to describe the second Hochschild cohomology group. In [15, Theorem 2.9], for $\Lambda = KQ/I$ where Q is a quiver, I is an admissible ideal of KQ and the set f^2 is a minimal set of generators for the ideal I, a minimal projective resolution of Λ as a Λ,Λ-bimodule is given which begins:

$$\cdots \to Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0,$$

where the projective Λ,Λ-bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v,vertex} \Lambda v \otimes v\Lambda,$$

$$Q^1 = \bigoplus_{a,arrow} \Lambda \mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda,$$

where $\mathfrak{o}(a)$ is the origin of the arrow a and $\mathfrak{t}(a)$ is the end of a,

$$Q^2 = \bigoplus_{x \in f^2} \Lambda \mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda.$$

We now explain the notation of [15] (including the notation $\mathfrak{o}(x)$ and $\mathfrak{t}(x)$ for $x \in f^2$) and start by defining the Λ,Λ-bimodule homomorphisms g, A_1 and A_2.

Definition 3.1. The map $g : Q^0 \to \Lambda$, is the multiplication map so is given by $v \otimes v \mapsto v$. The map $A_1 : Q^1 = \bigoplus_a \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda \to Q^0$, is given by $\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$ for each arrow a.

To define the map $A_2 : Q^2 \to Q^1$, we fix the set f^2 (which is a minimal set of generators of I) and let x be one of the minimal relations. Then $x = \sum_{j=1}^r c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j}$, that is, x is a linear combination of paths $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ for $j = 1, \ldots, r$ and $c_j \in K$ and there are unique vertices v and w such that each path $a_{ij} \cdots a_{kj} \cdots a_{s_j j}$ starts at v and ends at w for all j. We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Then $A_2 : Q^2 = \bigoplus_{x \in f^2} \Lambda\mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda \to Q^1$ is given by $\mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^r c_j (\sum_{k=1}^{s_j} a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j})$, where $a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j} \in \Lambda\mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$.

In order to find the projective Q^3 and the map A_3 in the Λ^e-resolution of Λ, Green and Snashall in [15] start by finding a projective resolution of $\Lambda/\mathfrak{r}$ as a Λ-module, where $\Lambda = KQ/I$, Q is a quiver, I is an admissible ideal and $\mathfrak{r} = J(\Lambda)$ the Jacobson radical of Λ. The paper [16] by Green, Solberg and Zacharia provides a procedure to find such a resolution, in the graded and finite dimensional cases. In the first part of this section we describe the work of [16]. Then we will relate this to the Λ^e-resolution of [15].

We are interested in the finite dimensional case. Given $\Lambda = KQ/I$ finite dimensional with I an admissible ideal, with the notation of [16], consider the right Λ-module $\Lambda/\mathfrak{r}$. Let the vertices of Q be labelled $1, \ldots, n$ and let $F = \coprod_{i=1}^n e_i KQ$. Then

$$F \to \Lambda/\mathfrak{r} \to 0$$

is an exact sequence of KQ-modules. Note that F is a projective KQ-module (since it is the sum of projective KQ-modules).

Now [16] constructs a filtration of F by KQ-submodules that contains all the information needed to construct the Λ-projective resolution of $\Lambda/\mathfrak{r}$. This filtration is

$$\cdots \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F,$$

such that the F^i's are projective KQ-modules (since the F^i's are submodules of F and KQ is a hereditary algebra), and

$$\cdots \to F^n/F^nI \to F^{n-1}/F^{n-1}I \to \cdots \to F^1/F^1I \to F/FI \to \Lambda/\mathfrak{r} \to 0$$

is a Λ-projective resolution of $\Lambda/\mathfrak{r}$ with the maps induced by the inclusions of the filtration.

We introduce the notation needed to define these submodules F^i. Let $R = KQ$, $f_i^0 = e_i$ for $i = 1, \ldots, n$ so that the projective Λ-module

$$\coprod_{i=1}^n f_i^0 R / \coprod_{i=1}^n f_i^0 I \text{ maps onto } \Lambda/\mathfrak{r}.$$

Let $f^0 = \{f_i^0\}_{i=1}^n$. We have the exact sequence of R-modules

$$0 \to \Omega_R^1(\Lambda/\mathfrak{r}) \to \coprod_{i=1}^n e_i KQ \xrightarrow{f} \Lambda/\mathfrak{r} \to 0.$$

Now choose a set $\{f_i^{1*}\}$ of elements of $\coprod_{i=1}^n f_i^0 R = \coprod_{i=1}^n e_i KQ$ such that $\Omega_R^1(\Lambda/\mathfrak{r}) = \coprod_{i\in A} f_i^{1*}R$. We will find this set by finding $\Omega_R^1(\Lambda/\mathfrak{r})$ which is $\operatorname{Ker} f$. The map f is given by: $e_1r_1 + \cdots + e_nr_n \mapsto e_1r_1 + \cdots + e_nr_n + \mathfrak{r}$, with $r_i \in KQ$. Then $\Omega_R^1(\Lambda/\mathfrak{r}) = \{e_1r_1 + \cdots + e_nr_n | e_1r_1 + \cdots + e_nr_n \in \mathfrak{r}\} = \coprod_{a,arrow} aKQ$. Therefore we may take the set $\{f_i^{1*}\}$ to be the set of arrows of the quiver Q.

Next we discard all the elements f_i^{1*} that are in $\coprod_{i=1}^n f_i^0 I$; note that $\coprod_{i=1}^n f_i^0 I$ is equal to $\coprod_{i=1}^n e_i I = I$. Denote the remaining elements of the set $\{f_i^{1*}\}$ by $\{f_i^1\}$. However, there are no elements f_i^{1*} in I since I is admissible. Hence $\{f_i^1\} = \{a, \text{arrows}\}$. Let $f^1 = \{f_i^1\}$.

The algorithm now proceeds as follows: firstly, assume we have constructed $f_i^0, f_i^1, \ldots, f_i^n$'s. Then construct the set f^{n+1} by considering

$$(\coprod_i f_i^n R)\bigcap(\coprod_j f_j^{n-1} I) := \coprod_k f_k^{n+1*}R.$$

Note that we may have $\{f_k^{n+1*}\} = \emptyset$. Next discard all the elements f_k^{n+1*} that are in $\coprod_i f_i^n I$ and denote the remaining elements by f_k^{n+1}. Let $f^{n+1} = \{f_k^{n+1}\}$. If $\{f_k^{n+1}\} = \emptyset$ then we stop at this point. Note that the set f^2 is a minimal set of generators for the ideal I.

Now [16] gives the following definitions and results for the resolution of $\Lambda/\mathfrak{r}$.

Definition 3.2. [16, Definition 1.1]. For each $n \geq 0$, let $P_n = \coprod_i f_i^n R / f_i^n I$ and let $\delta^n : P_n \to P_{n-1}$ be the homomorphism induced by the inclusion $\coprod_i f_i^n R \subset \coprod_j f_j^{n-1} R$.

Note that if $\{f_i^n\} = \emptyset$ then $P_n = 0$. For this reason if $\{f_i^n\} = \emptyset$ then we can stop at this stage of the construction.

Theorem 3.3. [16, Theorem 1.2]. *With the above notation,*

$$(\mathcal{P}, \delta) : \cdots \to P_n \xrightarrow{\delta^n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta^1} P_0 \to \Lambda/\mathfrak{r} \to 0$$

is a projective resolution of $\Lambda/\mathfrak{r}$ *over* Λ.

It was then shown in [16] that, when Λ is a finite dimensional algebra, we can always choose the elements $\{f_j^n\}$ in such a way as to obtain a minimal projective resolution when I is an admissible ideal.

Theorem 3.4. [16, Theorem 2.2] *In the resolution $(\mathcal{P},\delta)$ the elements $\{f_j^n\}$ can be chosen in such a way that, for each n, no proper K-linear combination of a subset of $\{f_j^n\}$ is in $\coprod_j f_j^{n-1}I + \coprod_k f_k^{n*}\mathfrak{r}$.*

Moreover, there is a decomposition

$$\coprod_k f_k^{n*}R = (\coprod_i f_i^n R)\coprod(\coprod_i f_i^{n'} R),$$

where the elements $f_i^{n'}$ can be chosen to be in $\coprod_j f_j^{n-1}I$.

Theorem 3.5. [16, Theorem 2.4] *Let $(\mathcal{P},\delta)$ be the projective resolution of $\Lambda/\mathfrak{r}$ as in Theorem 3.3, where $\{f_j^n\}$ are chosen as in Theorem 3.4. Then, the resolution $(\mathcal{P},\delta)$ is minimal.*

We are interested in the minimal projective resolution of Λ as a Λ^e-module, and in particular in the part $Q^3 \xrightarrow{A_3} Q^2$ of this resolution. We keep the notation of [16]. Then [15, 2.5] describes Q^3 in the following way. Suppose that the elements of f^2 are $\{f_1^2,\dots,f_m^2\}$, where f^2 is our fixed minimal set of elements in the generating set of I. Each element of f^3 is in $(\coprod f^2R)\bigcap(\coprod f^1I) = (\coprod f^2R)\bigcap(\coprod_a aI)$. Let y denote an arbitrary element of f^3, so $y \in \coprod f^2R$ and $y \in \coprod_a aI$. Therefore, $y = \Sigma f_i^2 p_i$, with $p_i \in R$. Also $y = \Sigma_a a(\Sigma_i \alpha_i f_i^2 \beta_i) = \Sigma_{a,i} a\alpha_i f_i^2\beta_i$ for elements $\alpha_i, \beta_i \in R$, arrows a, so we may write $y = \Sigma q_i f_i^2 r_i$, where q_i is in the ideal of R generated by the arrows and $r_i \in R$.

Then [15] gives that $Q^3 = \coprod_{y\in f^3} \Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda$ and describes the map A_3. For $y \in f^3$ the component of $A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2)\otimes\mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\Sigma(\mathfrak{o}(y)\otimes p_i - q_i\otimes r_i)$.

Thus we can describe the part of the minimal projective Λ^e-resolution of Λ

$$Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0$$

since we have determined the set f^3 satisfying the conditions of Theorem 3.4. Applying $\mathrm{Hom}(-,\Lambda)$ to this resolution gives us the complex

$$0 \to \mathrm{Hom}(Q^0,\Lambda) \xrightarrow{d_1} \mathrm{Hom}(Q^1,\Lambda) \xrightarrow{d_2} \mathrm{Hom}(Q^2,\Lambda) \xrightarrow{d_3} \mathrm{Hom}(Q^3,\Lambda)$$

where d_i is the map induced from A_i for $i = 1, 2, 3$. Then $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. We keep this notation for the rest of the thesis.

The next proposition tells us how many elements there are in the set f^3.

Proposition 3.6. [17, Lemma 1.5] *Let A be a finite dimensional algebra over a field K. Let $P(i)$ denote the indecomposable projective A-module e_iA. Let $S_i = P(i)/\mathrm{rad}P(i)$ be the corresponding simple A-module. Denote by $P(i,j)$, the indecomposable projective A^e-module $A(e_i \otimes e_j)A$.*

Then if

$$\cdots R_n \to R_{n-1} \to \cdots R_1 \to R_0 \to A \to 0$$

is a minimal projective resolution of A over A^e, then

$$R_n = \bigoplus_{i,j} P(i,j)^{\dim \mathrm{Ext}^n_A(S_i,S_j)}.$$

In order to find $\mathrm{Ext}^n(S_i, S_j)$ we use the well-known result, (see [5, Corollary 2.5.4]), that for S_i, S_j simple right Λ-modules, we have

$$\mathrm{Ext}^n_\Lambda(S_i, S_j) \cong \mathrm{Hom}_\Lambda(\Omega^n S_i, S_j).$$

So Proposition 3.6 says that $\Lambda e_i \otimes e_j\Lambda$ occurs $\dim \mathrm{Ext}^n_\Lambda(S_i, S_j)$ times in the n-th projective Λ^e-module of the minimal projective resolution of Λ. This provides a check that we have found all elements of f^3.

The paper [16] and subsequent work by Green and Solberg (unpublished) provides us with an explicit description of the set f^3. Throughout we fix f^0 as the set of vertices of the quiver $\mathcal{Q}$, f^1 as the set of arrows of $\mathcal{Q}$ and f^2 as our chosen minimal set of generators of the ideal I.

The maps induced by the filtration

$$\cdots \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F,$$

in Theorem 3.5 give the beginning of the minimal projective resolution of $\Lambda/\mathfrak{r}$ as

$$\coprod_{x\in f^2} \mathfrak{t}(x)\Lambda \xrightarrow{\delta^2} \coprod_{a\in f^1} \mathfrak{t}(a)\Lambda \xrightarrow{\delta^1} \coprod_{v\in f^0} v\Lambda \to \Lambda/\mathfrak{r} \to 0$$

with maps δ^2 and δ^1 given as follows. For $a \in f^1$, $\delta^1(\mathfrak{t}(a)\lambda) = a\mathfrak{t}(a)\lambda$ in the summand $\mathfrak{t}(a)\Lambda$. For $x \in f^2$ where $x = \sum_{j=1}^r c_j a_{1j} \cdots a_{kj} \cdots a_{s_jj}$ with the notation of Definition 3.1, $\delta^2(\mathfrak{t}(x)\lambda)$ has component $c_j a_{2j} \cdots a_{kj} \cdots a_{s_jj}\lambda$ in the summand $\mathfrak{t}(a_{1j})\Lambda$.

Moreover, as $\Lambda/\mathfrak{r}$ is the direct sum of the simple right Λ-modules, for each simple $S = v(\Lambda/\mathfrak{r})$, we have the beginning of a minimal projective resolution of S given by

$$\coprod_{x\in f^2, o(x)=v} \mathfrak{t}(x)\Lambda \xrightarrow{\delta^2} \coprod_{a\in f^1, o(a)=v} \mathfrak{t}(a)\Lambda \xrightarrow{\delta^1} v\Lambda \to S \to 0.$$

Now using δ^2 to compute $\operatorname{Ker}\delta^2 = \Omega^3_\Lambda(S)$ and P^3, the third projective in the minimal projective resolution of S, we may write $P^3 = \bigoplus_{i\in\mathcal{I}_s} e_i\Lambda$ for some index set $\mathcal{I}_s$ where $|\mathcal{I}_s| = dim\,\mathrm{Hom}(\Omega^3(S), \Lambda/\mathfrak{r}) = dim\,\mathrm{Ext}^3_\Lambda(S, \Lambda/\mathfrak{r})$.

Moreover this explicit calculation of $\operatorname{Ker}\delta^2$ gives us the map

$$\delta^3 : P^3 \to \coprod_{x\in f^2, o(x)=v} \mathfrak{t}(x)\Lambda$$

$$e_i \mapsto \Sigma_{x\in f^2, o(x)=v}\mathfrak{t}(x)p_{i,x}e_i, \text{ for each } i \in \mathcal{I}_s.$$

For each $i \in \mathcal{I}_s$, we set $f^3_i := \Sigma_{x\in f^2, o(x)=v}\, x p_{i,x} \in KQ$. Taking the union over all simple right Λ-modules S gives us our set $f^3 = \bigcup_{S,simple}\{f^3_i | i \in \mathcal{I}_s\}$. It is easy to verify for each of our algebras that we consider that this set f^3 does have the required properties and that $|f^3| = dim\,\mathrm{Ext}^3_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$.

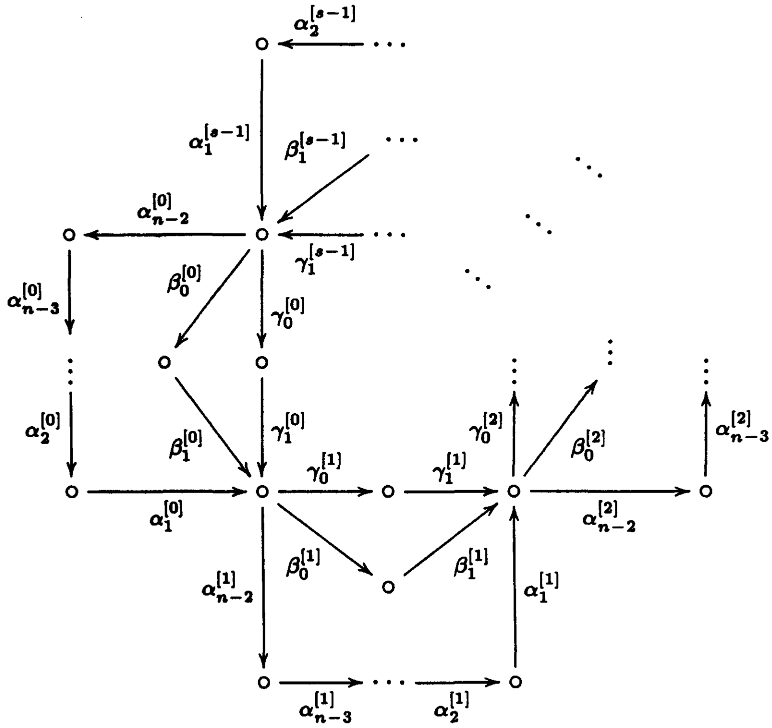
Now we are ready to compute $\mathrm{HH}^2(\Lambda)$ for the derived equivalence representatives of the self-injective finite dimensional algebras of finite representation type over an algebraically closed field.

First we note that the algebras of type $(A_n, s/n, 1)$ and $(A_{2p+1}, s, 2)$ have been considered in [9] and [15] respectively. We will come back to these results later in Chapter 5, but we start by considering type D_n, beginning with $(D_n, s, 1)$ in the next chapter.

Throughout, all tensor products are tensor products over K, and we write $\otimes$ for $\otimes_K$. When considering an element of the projective Λ^e-module $Q^1 = \bigoplus_{a,arrow} \Lambda o(a) \otimes \mathfrak{t}(a)\Lambda$ it is important to keep track of the individual summands of Q^1. So to avoid confusion we usually denote an element in the summand $\Lambda o(a) \otimes \mathfrak{t}(a)\Lambda$ by $\lambda \otimes_a \lambda'$ using the subscript 'a' to remind us in which summand this element lies. Similarly, an element $\lambda \otimes_{f_i^2} \lambda'$ lies in the summand $\Lambda \mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda \otimes_{f_i^3} \lambda'$ lies in the summand $\Lambda \mathfrak{o}(f_i^3) \otimes \mathfrak{t}(f_i^3)\Lambda$ of Q^3.

4. $\Lambda(D_n, s, 1)$

In this chapter we will determine $dim\,\mathrm{HH}^2(\Lambda)$ for $\Lambda(D_n, s, 1)$ with $n \geq 4, s \geq 1$. The algebra $\Lambda(D_n, s, 1)$ is given by the quiver $\mathcal{Q}(D_n, s)$:



with relations $R(D_n, s, 1)$:

(i) $\alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1 = \beta^{[i]}_0\beta^{[i]}_1 = \gamma^{[i]}_0\gamma^{[i]}_1$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\alpha^{[i]}_1\beta^{[i+1]}_0 = 0, \qquad \alpha^{[i]}_1\gamma^{[i+1]}_0 = 0,$$

$$\beta^{[i]}_1\alpha^{[i+1]}_{n-2} = 0, \qquad \gamma^{[i]}_1\alpha^{[i+1]}_{n-2} = 0,$$

$$\beta^{[i]}_1\gamma^{[i+1]}_0 = 0, \qquad \gamma^{[i]}_1\beta^{[i+1]}_0 = 0,$$

(iii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$ and for all $j \in \{1, \ldots, n-2\} = \mathbb{Z}/\langle n-2\rangle$,

$$\alpha^{[i]}_j \ldots \alpha^{[i+1]}_{j-n+2} = 0,$$

$$\beta^{[i]}_0\beta^{[i]}_1\beta^{[i+1]}_0 = 0, \qquad \gamma^{[i]}_0\gamma^{[i]}_1\gamma^{[i+1]}_0 = 0,$$

$$\beta^{[i]}_1\beta^{[i+1]}_0\beta^{[i+1]}_1 = 0, \qquad \gamma^{[i]}_1\gamma^{[i+1]}_0\gamma^{[i+1]}_1 = 0.$$

The set of relations (iii) means that "α-paths" of length $n-1$ are equal to 0, "β-paths" of length 3 are equal to 0 and "γ-paths" of length 3 are equal to 0.

We label the vertices of the quiver $\mathcal{Q}(D_n, s)$ as follows. For $i = 0, \dots, s-1$, $\mathfrak{o}(\alpha_{n-2}^{[i]}) = \mathfrak{o}(\beta_0^{[i]}) = \mathfrak{o}(\gamma_0^{[i]}) = e_{1,i}$, $\mathfrak{o}(\alpha_j^{[i]}) = e_{j+1,i}$, for $j = 1, \dots, n-3$, $\mathfrak{o}(\beta_1^{[i]}) = e_{n-1,i}$ and $\mathfrak{o}(\gamma_1^{[i]}) = e_{n,i}$.

We need a set f^2 of minimal relations but note that $R(D_n, s, 1)$ for $s \geq 1$ is not minimal. So now we will discard some of the relations of $R(D_n, s, 1)$ to give a minimal set.

All relations of type (ii) are in f^2. For relations of type (i), choose $\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]} \in f^2$ and $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. With these choices, we now consider the relations of type (iii). So $(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]})\gamma_0^{[i+1]} = (\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} - \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]}) \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} \in I$. Therefore $\gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and is not in our minimal set of relations f^2. Also $\gamma_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}) = (\gamma_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} - \gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]}) \in I$ and $\gamma_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$. So $\gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ and is not in f^2. Similarly we can show that neither $\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]}$ nor $\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]}$ are in f^2.

Now consider "α-paths". We have $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. So $(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]})\alpha_{n-2}^{[i+1]} \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\alpha_{n-2}^{[i+1]} \in I$. Therefore $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \in I$ and is not in f^2. Also $\alpha_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]}) \in I$ and $\alpha_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$. So $\alpha_1^{[i-1]}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} \in I$ and not in f^2.

However, the path $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_2^{[i+1]}$ cannot be obtained from any other paths, so $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_2^{[i+1]} \in f^2$. In general, $\alpha_k^{[i]}\alpha_{k-1}^{[i]} \cdots \alpha_{k+1}^{[i+1]}\alpha_k^{[i+1]} \in f^2$ for $k = \{2, \dots, n-3\}$. Now let us label the elements of f^2 as follows.

$$f_{1,1,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f_{1,2,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]},$$

$$f_{2,1,i}^2 = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f_{2,2,i}^2 = \alpha_1^{[i]}\gamma_0^{[i+1]},$$

$$f_{2,3,i}^2 = \beta_1^{[i]}\alpha_{n-2}^{[i+1]}, \qquad f_{2,4,i}^2 = \gamma_1^{[i]}\alpha_{n-2}^{[i+1]},$$

$$f_{2,5,i}^2 = \beta_1^{[i]}\gamma_0^{[i+1]}, \qquad f_{2,6,i}^2 = \gamma_1^{[i]}\beta_0^{[i+1]} \text{ and}$$

$$f_{3,k,i}^2 = \alpha_k^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]}, \text{ for } k = \{2, \dots, n-3\}.$$

Hence $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}, f^2_{2,5,i}, f^2_{2,6,i}, f^2_{3,k,i}\}$ for $i = 0,\ldots,s-1$ and $k = 2,\ldots,n-3$.

Next we need to find f^3.

For $i \in \{0,\ldots,s-1\}$ the indecomposable projective modules are:

$e_{1,i}\Lambda$

$$
\begin{tikzcd}[arrows=dash]
 & 1,i \arrow[dl] \arrow[d] \arrow[dr] & \\
n-1,i \arrow[dddr] & n-2,i \arrow[d] & n,i \arrow[dddl] \\
 & n-3,i \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & 1,i+1 &
\end{tikzcd}
$$

For $2 \leq m \leq n-2$

$e_{m,i}\Lambda$

$$
\begin{tikzcd}[arrows=dash]
m,i \arrow[d] \\
m-1,i \arrow[d] \\
\vdots \arrow[d] \\
2,i \arrow[d] \\
1,i+1 \arrow[d] \\
n-2,i+1 \arrow[d] \\
n-3,i+1 \arrow[d] \\
\vdots \arrow[d] \\
m,i+1
\end{tikzcd}
$$

$$
\begin{array}{ccc}
e_{n-1,i}\Lambda & \qquad & e_{n,i}\Lambda \\
 & & \\
n-1,i & & n,i \\
| & & | \\
1,i+1 & & 1,i+1 \\
| & & | \\
n-1,i+1 & & n,i+1
\end{array}
$$

From the minimal projective resolutions of each simple Λ-module we easily see that:

$$
\Omega^3(S_{2,i}) \quad \cong \quad
\begin{array}{ccc}
 & 1,i+2 & \\
\swarrow & & \searrow \\
n,i+2 & & n-1,i+2
\end{array}
$$

for $3 \leq m \leq n-2$ we have,

$$
\Omega^3(S_{m,i}) \quad \cong \quad \begin{array}{c}
m-2, i+1 \\
| \\
m-3, i+1 \\
| \\
\vdots \\
| \\
1, i+2 \\
| \\
n-2, i+2 \\
| \\
\vdots \\
| \\
m-1, i+2
\end{array}
$$

$$
\Omega^3(S_{n-1,i}) \quad \cong \quad \begin{array}{ccccc}
 & & 1, i+2 & & \\
 & \diagup & & \diagdown & \\
n, i+2 & & & & n-2, i+2
\end{array}
$$

$$
\Omega^3(S_{n,i}) \quad \cong \quad \begin{array}{ccccc}
 & & 1, i+2 & & \\
 & \diagup & & \diagdown & \\
n-2, i+2 & & & & n-1, i+2
\end{array}
$$

For $\Omega^3(S_{1,i})$ we need more details. We have the map

$$
\psi : e_{n-2,i}\Lambda \oplus e_{n-1,i}\Lambda \oplus e_{n,i}\Lambda \to \Omega(S_{1,i})
$$

given by:

$$e_{n-2,i}\lambda \mapsto \alpha_{n-2}^{[i]} e_{n-2,i}\lambda,$$

$$e_{n-1,i}\mu \mapsto \beta_0^{[i]} e_{n-1,i}\mu,$$

$$e_{n,i}\xi \mapsto \gamma_0^{[i]} e_{n,i}\xi$$

where $\lambda, \mu, \xi \in \Lambda$. Note that $\Omega^2(S_{1,i}) = \operatorname{Ker}\psi$.

Proposition 4.1. $\Omega^2(S_{1,i}) = (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\Lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})\Lambda$.

Proof. On one hand, let $x \in \Omega^2(S_{1,i})$. Then $x = (e_{n-2,i}\lambda, e_{n-1,i}\mu, e_{n,i}\xi)$. Write $e_{n-2,i}\lambda = c_{0,i}e_{n-2,i} + c_{1,i}\alpha_{n-3}^{[i]} + c_{2,i}\alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]} + \cdots + c_{n-2,i}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]}$, $e_{n-1,i}\mu = c'_{0,i}e_{n-1,i} + c'_{1,i}\beta_1^{[i]} + c'_{2,i}\beta_1^{[i]}\beta_0^{[i+1]}$ and $e_{n,i}\xi = d_{0,i}e_{n,i} + d_{1,i}\gamma_1^{[i]} + d_{2,i}\gamma_1^{[i]}\gamma_0^{[i+1]}$ with all coefficients $c_{j,i}, c'_{l,i}, d_{l,i} \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$. So $\alpha_{n-2}^{[i]}(c_{0,i}e_{n-2,i} + c_{1,i}\alpha_{n-3}^{[i]} + c_{2,i}\alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]} + \cdots + c_{n-2,i}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]}) + \beta_0^{[i]}(c'_{0,i}e_{n-1,i} + c'_{1,i}\beta_1^{[i]} + c'_{2,i}\beta_1^{[i]}\beta_0^{[i+1]}) + \gamma_0^{[i]}(d_{0,i}e_{n,i} + d_{1,i}\gamma_1^{[i]} + d_{2,i}\gamma_1^{[i]}\gamma_0^{[i+1]}) = c_{0,i}\alpha_{n-2}^{[i]} + c_{1,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} + c_{2,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]} + \cdots + c_{n-4,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]} + c_{n-3,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]} + c'_{0,i}\beta_0^{[i]} + c'_{1,i}\beta_0^{[i]}\beta_1^{[i]} + d_{0,i}\gamma_0^{[i]} + d_{1,i}\gamma_0^{[i]}\gamma_1^{[i]} = c_{0,i}\alpha_{n-2}^{[i]} + c_{1,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} + c_{2,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]} + \cdots + c_{n-4,i}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]} + c'_{0,i}\beta_0^{[i]} + d_{0,i}\gamma_0^{[i]} + (c_{n-3,i} + c'_{1,i} + d_{1,i})\beta_0^{[i]}\beta_1^{[i]} = 0$. Thus $c_{0,i} = \cdots = c_{n-4,i} = c'_{0,i} = d_{0,i} = 0$ and $c_{n-3,i} + c'_{1,i} + d_{1,i} = 0$ so let $c'_{1,i} = -(c_{n-3,i} + d_{1,i})$. Therefore,
$x = (c_{n-3,i}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]} + c_{n-2,i}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]}, -(c_{n-3,i} + d_{1,i})\beta_1^{[i]} + c'_{2,i}\beta_1^{[i]}\beta_0^{[i+1]}, d_{1,i}\gamma_1^{[i]} + d_{2,i}\gamma_1^{[i]}\gamma_0^{[i+1]}) = (-\alpha_{n-3,i}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)(-c_{n-3,i}e_{1,i+1} + c_{n-2,i}\alpha_{n-2}^{[i+1]}) + (0, \beta_1^{[i]}, -\gamma_1^{[i]})(-d_{1,i}e_{1,i+1} + c'_{2,i}\beta_0^{[i+1]} - d_{2,i}\gamma_0^{[i+1]})$. So $x \in (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\Lambda$
$+ (0, \beta_1^{[i]}, -\gamma_1^{[i]})\Lambda$.

Thus $\Omega^2(S_{1,i}) \subseteq (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\Lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})\Lambda$.

On the other hand, let $x \in (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\Lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})\Lambda$. So $x = (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})\mu = (-e_{n-2,i}\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}\lambda, e_{n-1,i}\beta_1^{[i]}\lambda + e_{n-1,i}\beta_1^{[i]}\mu, -e_{n,i}\gamma_1^{[i]}\mu)$ where $\lambda, \mu \in \Lambda$. It is immediate from the definition of ψ that $\psi(x) = 0$ and so $x \in \Omega^2(S_{1,i})$.

Thus $(-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)\Lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})\Lambda \subseteq \Omega^2(S_{1,i})$.

Therefore, $\Omega^2(S_{1,i}) = (-\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]}, \beta_1^{[i]}, 0)e_{1,i+1}\Lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})e_{1,i+1}\Lambda$.

$\square$

To find $\Omega^3(S_{1,i})$ the second projective in the resolution of $S_{1,i}$ is $e_{1,i+1}\Lambda \oplus e_{1,i+1}\Lambda$. With the map:

$$\theta : e_{1,i+1}\Lambda \oplus e_{1,i+1}\Lambda \to \Omega^2(S_{1,i})$$

given by:

$$(e_{1,i+1}\lambda, e_{1,i+1}\mu) \mapsto (-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}, \beta_1^{[i]}, 0)e_{1,i+1}\lambda + (0, \beta_1^{[i]}, -\gamma_1^{[i]})e_{1,i+1}\mu$$

where $\lambda, \mu \in \Lambda$, we have $\Omega^3(S_{1,i}) = \operatorname{Ker}\theta$.

Proposition 4.2.

$\Omega^3(S_{1,i}) = (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}, 0)\Lambda + (0, \alpha_{n-2}^{[i+1]})\Lambda + (-\beta_0^{[i+1]}, \beta_0^{[i+1]})\Lambda + (\gamma_0^{[i+1]}, 0)\Lambda.$

Proof. On one hand, let $y \in \Omega^3(S_{1,i})$. Then $y = (e_{1,i+1}\lambda, e_{1,i+1}\mu) \in e_{1,i+1}\Lambda \oplus e_{1,i+1}\Lambda$ where $\lambda, \mu \in \Lambda$. Let $\lambda = c_{0,i+1}e_{1,i+1} + c_{1,i+1}\alpha_{n-2}^{[i+1]} + c_{2,i+1}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} + \cdots + c_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} + c_{n-1,i+1}\beta_0^{[i+1]} + c_{n,i+1}\gamma_0^{[i+1]}$ and $\mu = c'_{0,i+1}e_{1,i+1} + c'_{1,i+1}\alpha_{n-2}^{[i+1]} + c'_{2,i+1}\alpha_{n-2,i+1}\alpha_{n-3}^{[i+1]} + \cdots + c'_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} + c'_{n-1,i+1}\beta_0^{[i+1]} + c'_{n,i+1}\gamma_0^{[i+1]}$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus, $(-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}, \beta_1^{[i]}, 0)$ $(c_{0,i+1}e_{1,i+1} + c_{1,i+1}\alpha_{n-2}^{[i+1]} + c_{2,i+1}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} + \cdots + c_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} + c_{n-1,i+1}\beta_0^{[i+1]} + c_{n,i+1}\gamma_0^{[i+1]}) + (0, \beta_1^{[i]}, -\gamma_1^{[i]})(c'_{0,i+1}e_{1,i+1} + c'_{1,i+1}\alpha_{n-2}^{[i+1]} + c'_{2,i+1}\alpha_{n-2,i+1}\alpha_{n-3}^{[i+1]} + \cdots + c'_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} + c'_{n-1,i+1}\beta_0^{[i+1]} + c'_{n,i+1}\gamma_0^{[i+1]})$ $= c_{0,i+1}(-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}, \beta_1^{[i]}, 0) + (c_{1,i+1}\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}, c_{n-1,i+1}\beta_1^{[i]}\beta_0^{[i+1]}, 0) + c'_{0,i+1}(0, \beta_1^{[i]}, -\gamma_1^{[i]}) + (0, c'_{n-1,i+1}\beta_1^{[i]}\beta_0^{[i+1]}, c'_{n,i+1}\gamma_1^{[i]}\gamma_0^{[i+1]}) = c_{0,i+1}(-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}, \beta_1^{[i]}, 0) + (c_{1,i+1}\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}, (c_{n-1,i+1} + c'_{n-1,i+1})\beta_1^{[i]}\beta_0^{[i+1]}, c'_{n,i+1}\gamma_1^{[i]}\gamma_0^{[i+1]}) + c'_{0,i+1}(0, \beta_1^{[i]}, -\gamma_1^{[i]}) = 0$. Thus $c_{0,i+1} = c_{1,i+1} = c'_{0,i+1} = c'_{n,i+1} = 0$ and $c'_{n-1,i+1} + c_{n-1,i+1} = 0$. Replace $c'_{n-1,i+1}$ by $-c_{n-1,i+1}$. Therefore, $y = (c_{2,i+1}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} + \cdots + c_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} + c_{n-1,i+1}\beta_0^{[i+1]} + c_{n,i+1}\gamma_0^{[i+1]}, c'_{1,i+1}\alpha_{n-2}^{i+1} + c'_{2,i+1}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} + \cdots + c'_{n-2,i+1}\alpha_{n-2}^{[i+1]}\cdots\alpha_1^{[i+1]} - c_{n-1,i+1}\beta_0^{[i+1]}) =$ $\Big(\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}(c_{2,i+1}e_{n-3,i+1} + \cdots + c_{n-2,i+1}\alpha_{n-4}^{[i+1]}\cdots\alpha_1^{[i+1]})$ $+\beta_0^{[i+1]}c_{n-1,i+1}e_{n-1,i+1} + \gamma_0^{[i+1]}c_{n,i+1}e_{n,i+1}, \alpha_{n-2}^{[i+1]}(c'_{1,i+1}e_{n-2,i+1} + c'_{2,i+1}\alpha_{n-3}^{[i+1]}$ $+ \cdots + c'_{n-2,i+1}\alpha_{n-3}^{[i+1]}\cdots\alpha_1^{[i+1]}) - \beta_0^{[i+1]}c_{n-1,i+1}e_{n-1,i+1}\Big)$. Thus

$$y = (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}\lambda + \beta_0^{[i+1]}\xi + \gamma_0^{[i+1]}\nu, \alpha_{n-2}^{[i+1]}\mu - \beta_0^{[i+1]}\xi)$$

$$= (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}, 0)\lambda + (0, \alpha_{n-2}^{[i+1]})\mu + (-\beta_0^{[i+1]}, \beta_0^{[i+1]})(-\xi) + (\gamma_0^{[i+1]}, 0)\nu,$$

where $\lambda, \mu, \xi, \nu \in \Lambda$.

So $y \in (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}, 0)\Lambda + (0, \alpha_{n-2}^{[i+1]})\Lambda + (-\beta_0^{[i+1]}, \beta_0^{[i+1]})\Lambda + (\gamma_0^{[i+1]}, 0)\Lambda$.

Thus $\Omega^3(S_{1,i}) \subseteq (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)\Lambda+(0,\alpha_{n-2}^{[i+1]})\Lambda+(-\beta_0^{[i+1]},\beta_0^{[i+1]})\Lambda+(\gamma_0^{[i+1]},0)\Lambda$.

On the other hand, let $y \in (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)\Lambda+(0,\alpha_{n-2}^{[i+1]})\Lambda+(-\beta_0^{[i+1]},\beta_0^{[i+1]})\Lambda$ $+(\gamma_0^{[i+1]},0)\Lambda$. So write $y = (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)\lambda+(0,\alpha_{n-2}^{[i+1]})\mu+(-\beta_0^{[i+1]},\beta_0^{[i+1]})\xi'+$ $(\gamma_0^{[i+1]},0)\nu$, where $\lambda,\mu,\xi',\nu \in \Lambda$. Then $\theta(y) = (-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]},\beta_1^{[i]},0)\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}\lambda$ $+(0,\beta_1^{[i]},-\gamma_1^{[i]})\alpha_{n-2}^{[i+1]}\mu-(-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]},\beta_1^{[i]},0)\beta_0^{[i+1]}\xi'+(0,\beta_1^{[i]},-\gamma_1^{[i]})\beta_0^{[i+1]}\xi'+$ $(-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]},\beta_1^{[i]},0)\gamma_0^{[i+1]}\nu = 0$. So $y \in \Omega^3(S_{1,i})$.

Thus $(\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)\Lambda + (0,\alpha_{n-2}^{[i+1]})\Lambda + (-\beta_0^{[i+1]},\beta_0^{[i+1]})\Lambda + (\gamma_0^{[i+1]},0)\Lambda \subseteq$ $\Omega^3(S_{1,i})$.

Therefore, $\Omega^3(S_{1,i}) = (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)e_{n-3,i+1}\Lambda + (0,\alpha_{n-2}^{[i+1]})e_{n-2,i+1}\Lambda +$ $(-\beta_0^{[i+1]},\beta_0^{[i+1]})e_{n-1,i+1}\Lambda + (\gamma_0^{[i+1]},0)e_{n,i+1}\Lambda$. Let ϕ be the the map:

$$\phi : e_{n-3,i+1}\Lambda \oplus e_{n-2,i+1}\Lambda \oplus e_{n-1,i+1}\Lambda \oplus e_{n,i+1}\Lambda \to \Omega^3(S_{1,i})$$

given by:

$$\begin{array}{lcl} e_{n-3,i+1}\lambda & \mapsto & (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0)e_{n-3,i+1}\lambda, \\ e_{n-2,i+1}\mu & \mapsto & (0,\alpha_{n-2}^{[i+1]})e_{n-2,i+1}\mu, \\ e_{n-1,i+1}\xi & \mapsto & (-\beta_0^{[i+1]},\beta_0^{[i+1]})e_{n-1,i+1}\xi, \\ e_{n,i+1}\nu & \mapsto & (\gamma_0^{[i+1]},0)e_{n,i+1}\nu \end{array}$$

where $\lambda,\mu,\xi,\nu \in \Lambda$. □

From the projective resolution for simples we now know that the 3rd projective in the Λ^e resolution of Λ is:

$Q^3 = \bigoplus_{i=0}^{s-1}[(\Lambda e_{1,i}\otimes e_{n-3,i+1}\Lambda)\oplus(\Lambda e_{1,i}\otimes e_{n-2,i+1}\Lambda)\oplus(\Lambda e_{1,i}\otimes e_{n-1,i+1}\Lambda)\oplus$ $(\Lambda e_{1,i}\otimes e_{n,i+1}\Lambda)\oplus(\Lambda e_{2,i}\otimes e_{1,i+2}\Lambda)\oplus(\Lambda e_{n-1,i}\otimes e_{1,i+2}\Lambda)\oplus(\Lambda e_{n,i}\otimes e_{1,i+2}\Lambda)\oplus$ $\bigoplus_{m=3}^{n-2}(\Lambda e_{m,i}\otimes e_{m-2,i+1}\Lambda)]$.

We now give a specific illustration of finding an element of f^3. From above we know that Q^3 has the term $\Lambda e_{1,i}\otimes e_{n-3,i+1}\Lambda$ which corresponds to some element of f^3. Using the filtration in KQ we have $e_{n-3,i+1} \xrightarrow{\phi} (\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},0) \xrightarrow{\theta} (\beta_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]},-\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}) \xrightarrow{\psi}$ $\beta_0^{[i]}\beta_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}$. So we may choose $f_{1,1,i}^3 =$ $\beta_0^{[i]}\beta_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]} = f_{1,2,i}^2\alpha_{n-2}^{[i+1]}\alpha_{n-3}^{[i+1]}$. Similarly the other summands of $\Omega^3(S_{1,i})$ give us the terms $f_{1,2,i}^3, f_{1,3,i}^3, f_{1,4,i}^3$ which correspond to the summand $\Lambda e_{1,i}\otimes e_{n-2,i+1}\Lambda$, $\Lambda e_{1,i}\otimes e_{n-1,i+1}\Lambda$ and

$\Lambda e_{1,i} \otimes e_{n,i+1}\Lambda$ of Q^3 respectively. Then we find by direct calculation the expressions of each f^3 in the form $f_i^3 = \sum_j q_j f_j^2 r_j$ with $q_j, r_j \in KQ$.

From the previous comments the set f^3 for all $s \geq 1$ consists of the following elements:

$$\{f^3_{1,1,i}, f^3_{1,2,i}, f^3_{1,3,i}, f^3_{1,4,i}, f^3_{1,5,i}, f^3_{1,6,i}, f^3_{1,7,i}, f^3_{2,3,i}, f^3_{2,m,i}\}, \text{ with } m \in \{4, \ldots, n-2\}$$

where

$$\begin{array}{lllll}
f^3_{1,1,i} & = & f^2_{1,2,i}\alpha^{[i+1]}_{n-2}\alpha^{[i+1]}_{n-3} & = & \beta^{[i]}_0 f^2_{2,3,i}\alpha^{[i+1]}_{n-3} - \alpha^{[i]}_{n-2} f^2_{3,n-3,i}, \\
f^3_{1,2,i} & = & f^2_{1,1,i}\alpha^{[i+1]}_{n-2} & = & \beta^{[i]}_0 f^2_{2,3,i} - \gamma^{[i]}_0 f^2_{2,4,i}, \\
f^3_{1,3,i} & = & f^2_{1,1,i}\beta^{[i+1]}_0 - f^2_{1,2,i}\beta^{[i+1]}_0 & = & \alpha^{[i]}_{n-2}\cdots\alpha^{[i]}_2 f^2_{2,1,i} - \gamma^{[i]}_0 f^2_{2,6,i}, \\
f^3_{1,4,i} & = & f^2_{1,2,i}\gamma^{[i+1]}_0 & = & \beta^{[i]}_0 f^2_{2,5,i} - \alpha^{[i]}_{n-2}\cdots\alpha^{[i]}_2 f^2_{2,2,i}, \\
f^3_{1,5,i} & = & f^2_{2,1,i}\beta^{[i+1]}_1 - f^2_{2,2,i}\gamma^{[i+1]}_1 & = & \alpha^{[i]}_1 f^2_{1,1,i+1}, \\
f^3_{1,6,i} & = & f^2_{2,3,i}\alpha^{[i+1]}_{n-3}\cdots\alpha^{[i+1]}_1 - f^2_{2,5,i}\gamma^{[i+1]}_1 & = & \beta^{[i]}_1 f^2_{1,1,i+1} - \beta^{[i]}_1 f^2_{1,2,i+1}, \\
f^3_{1,7,i} & = & f^2_{2,6,i}\beta^{[i+1]}_1 - f^2_{2,4,i}\alpha^{[i+1]}_{n-3}\cdots\alpha^{[i+1]}_1 & = & \gamma^{[i]}_1 f^2_{1,2,i+1}, \\
f^3_{2,3,i} & = & f^2_{3,2,i}\alpha^{[i+1]}_1 & = & \alpha^{[i]}_2 f^2_{2,1,i}\beta^{[i+1]}_1 - \alpha^{[i]}_2\alpha^{[i]}_1 f^2_{1,2,i+1}, \\
f^3_{2,m,i} & = & f^2_{3,m-1,i}\alpha^{[i+1]}_{m-2} & = & \alpha^{[i]}_{m-1} f^2_{3,m-2,i} \text{ for } m \in \{4,\ldots,n-2\}.
\end{array}$$

To find $\mathrm{HH}^2(\Lambda)$, we know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$ (from the notation of Chapter 3). First we will find $\mathrm{Im}\, d_2$.

$\mathbf{Im}\, \mathbf{d_2}$ for $\mathbf{s \geq 1}$.

Since $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$, then $d_2 f \in \mathrm{Im}\, d_2$, where $f \in \mathrm{Hom}(Q^1, \Lambda)$ and $d_2 f = f A_2$. Here $Q^1 = \bigoplus_{i=0}^{s-1}[(\Lambda e_{1,i} \otimes_{\beta_0^{[i]}} e_{n-1,i}\Lambda) \oplus (\Lambda e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1}\Lambda) \oplus (\Lambda e_{1,i} \otimes_{\gamma_0^{[i]}} e_{n,i}\Lambda) \oplus (\Lambda e_{n,i} \otimes_{\gamma_1^{[i]}} e_{1,i+1}\Lambda) \oplus (\Lambda e_{2,i} \otimes_{\alpha_1^{[i]}} e_{1,i+1}\Lambda) \oplus \bigoplus_{l=2}^{n-2}(\Lambda e_{l+1,i} \otimes_{\alpha_l^{[i]}} e_{l,i}\Lambda)]$. The module Q^1 has $5s + s(n-3)$ summands, that is, $2s + sn$ summands. Let $f \in \mathrm{Hom}(Q^1, \Lambda)$ and write

$$f(e_{1,i} \otimes_{\beta_0^{[i]}} e_{n-1,i}) = c_{1,i}\beta_0^{[i]}, \qquad f(e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1}) = c_{2,i}\beta_1^{[i]},$$

$$f(e_{1,i} \otimes_{\gamma_0^{[i]}} e_{n,i}) = c_{3,i}\gamma_0^{[i]}, \qquad f(e_{n,i} \otimes_{\gamma_1^{[i]}} e_{1,i+1}) = c_{4,i}\gamma_1^{[i]},$$

$$f(e_{2,i} \otimes_{\alpha_1^{[i]}} e_{1,i+1}) = d_{1,i}\alpha_1^{[i]}$$

and

$$f(e_{l+1,i} \otimes_{\alpha_l^{[i]}} e_{l,i}) = d_{l,i}\alpha_l^{[i]} \text{ for } l \in \{2, \ldots, n-2\},$$

where $i \in \{0, \ldots, s-1\}$, and $c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i}, d_{1,i}, d_{l,i} \in K$.

We have $Q^2 = \bigoplus_{i=0}^{s-1}[(\Lambda e_{1,i} \otimes_{f^2_{1,1,i}} e_{1,i+1}\Lambda) \oplus (\Lambda e_{1,i} \otimes_{f^2_{1,2,i}} e_{1,i+1}\Lambda) \oplus (\Lambda e_{2,i} \otimes_{f^2_{2,1,i}} e_{n-1,i+1}\Lambda) \oplus (\Lambda e_{2,i} \otimes_{f^2_{2,2,i}} e_{n,i+1}\Lambda) \oplus (\Lambda e_{n-1,i} \otimes_{f^2_{2,3,i}} e_{n-2,i+1}\Lambda) \oplus (\Lambda e_{n,i} \otimes_{f^2_{2,4,i}} e_{n-2,i+1}\Lambda) \oplus (\Lambda e_{n-1,i} \otimes_{f^2_{2,5,i}} e_{n,i+1}\Lambda) \oplus (\Lambda e_{n,i} \otimes_{f^2_{2,6,i}} e_{n-1,i+1}\Lambda) \oplus \bigoplus_{k=2}^{n-3}(\Lambda e_{k+1,i} \otimes_{f^2_{3,k,i}} e_{k,i+1}\Lambda)]$. Here Q^2 has $4s + sn$ summands. Now we find fA_2. Recall the definition of the map A_2 from Definition 3.1.

We have $fA_2(e_{1,i} \otimes_{f^2_{1,1,i}} e_{1,i+1}) = f(e_{1,i} \otimes_{\beta_0^{[i]}} \beta_1^{[i]} - e_{1,i} \otimes_{\gamma_0^{[i]}} \gamma_1^{[i]} + \beta_0^{[i]} \otimes_{\beta_1^{[i]}} e_{1,i+1} - \gamma_0^{[i]} \otimes_{\gamma_1^{[i]}} e_{1,i+1}) = f(e_{1,i} \otimes_{\beta_0^{[i]}} e_{n-1,i})\beta_1^{[i]} - f(e_{1,i} \otimes_{\gamma_0^{[i]}} e_{n,i})\gamma_1^{[i]} + \beta_0^{[i]} f(e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1}) - \gamma_0^{[i]} f(e_{n,i} \otimes_{\gamma_1^{[i]}} e_{1,i+1}) = c_{1,i}\beta_0^{[i]}\beta_1^{[i]} - c_{3,i}\gamma_0^{[i]}\gamma_1^{[i]} + c_{2,i}\beta_0^{[i]}\beta_1^{[i]} - c_{4,i}\gamma_0^{[i]}\gamma_1^{[i]} = (c_{1,i} - c_{3,i} + c_{2,i} - c_{4,i})\beta_0^{[i]}\beta_1^{[i]}$.

Also $fA_2(e_{1,i} \otimes_{f^2_{1,2,i}} e_{1,i+1}) = f(e_{1,i} \otimes_{\beta_0^{[i]}} e_{n-1,i})\beta_1^{[i]} + \beta_0^{[i]} f(e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1}) - f(e_{1,i} \otimes_{\alpha_{n-2}^{[i]}} e_{n-2})\alpha_{n-3}^{[i]} \cdots \alpha_1^{[i]} - \alpha_{n-2}^{[i]} f(e_{n-2,i} \otimes_{\alpha_{n-3}^{[i]}} e_{n-3,i})\alpha_{n-4}^{[i]} \cdots \alpha_1^{[i]} - \ldots - \alpha_{n-2}^{[i]} \cdots \alpha_2^{[i]} f(e_{2,i} \otimes_{\alpha_1^{[i]}} e_{1,i+1}) = c_{1,i}\beta_0^{[i]}\beta_1^{[i]} + c_{2,i}\beta_0^{[i]}\beta_1^{[i]} - d_{n-2,i}\alpha_{n-2}^{[i]} \cdots \alpha_1^{[i]} - \ldots - d_{1,i}\alpha_{n-2}^{[i]} \cdots \alpha_2^{[i]}\alpha_1^{[i]} = (c_{1,i} + c_{2,i} - d_{n-2,i} - \ldots - d_{1,i})\beta_0^{[i]}\beta_1^{[i]}$.

And $fA_2(e_{2,i} \otimes_{f^2_{2,1,i}} e_{n-1,i+1}) = f(e_{2,i} \otimes_{\alpha_1^{[i]}} e_{1,i+1})\beta_0^{[i+1]} + \alpha_1^{[i]} f(e_{1,i+1} \otimes_{\beta_0^{[i+1]}} e_{n-1,i+1}) = d_{1,i}\alpha_1^{[i]}\beta_0^{[i+1]} + c_{1,i+1}\alpha_1^{[i]}\beta_0^{[i+1]} = (d_{1,i} + c_{1,i+1})\alpha_1^{[i]}\beta_0^{[i+1]} = 0$,

$fA_2(e_{2,i} \otimes_{f^2_{2,2,i}} e_{n,i+1}) = f(e_{2,i} \otimes_{\alpha_1^{[i]}} e_{1,i+1})\gamma_0^{[i+1]} + \alpha_1^{[i]} f(e_{1,i+1} \otimes_{\gamma_0^{[i+1]}} e_{n,i+1}) = d_{1,i}\alpha_1^{[i]}\gamma_0^{[i+1]} + c_{3,i+1}\alpha_1^{[i]}\gamma_0^{[i+1]} = (d_{1,i} + c_{3,i+1})\alpha_1^{[i]}\gamma_0^{[i+1]} = 0$,

$fA_2(e_{n-1,i} \otimes_{f^2_{2,3,i}} e_{n-2,i+1}) = f(e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1})\alpha_{n-2}^{[i+1]} + \beta_1^{[i]} f(e_{1,i+1} \otimes_{\alpha_{n-2}^{[i+1]}} e_{n-2,i+1}) = c_{2,i}\beta_1^{[i]}\alpha_{n-2}^{[i+1]} + d_{n-2,i+1}\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0$,

$fA_2(e_{n,i} \otimes_{f^2_{2,4,i}} e_{n-2,i+1}) = f(e_{n,i} \otimes_{\gamma_1^{[i]}} e_{1,i+1})\alpha_{n-2}^{[i+1]} + \gamma_1^{[i]} f(e_{1,i+1} \otimes_{\alpha_{n-2}^{[i+1]}} e_{n-2,i+1}) = c_{4,i}\gamma_1^{[i]}\alpha_{n-2}^{[i+1]} + d_{n-2,i+1}\gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0$,

$fA_2(e_{n-1,i} \otimes_{f^2_{2,5,i}} e_{n,i+1}) = f(e_{n-1,i} \otimes_{\beta_1^{[i]}} e_{1,i+1})\gamma_0^{[i+1]} + \beta_1^{[i]} f(e_{1,i+1} \otimes_{\gamma_0^{[i+1]}} e_{n,i+1}) = c_{2,i}\beta_1^{[i]}\gamma_0^{[i+1]} + c_{3,i+1}\beta_1^{[i]}\gamma_0^{[i+1]} = 0$,

$fA_2(e_{n,i} \otimes_{f^2_{2,6,i}} e_{n-1,i+1}) = f(e_{n,i} \otimes_{\gamma_1^{[i]}} e_{1,i+1})\beta_0^{[i+1]} + \gamma_1^{[i]} f(e_{1,i+1} \otimes_{\beta_0^{[i+1]}} e_{n-1,i+1}) = c_{4,i}\gamma_1^{[i]}\beta_0^{[i+1]} + c_{1,i+1}\gamma_1^{[i]}\beta_0^{[i+1]} = 0$.

For $k \in \{2, \ldots, n-3\}$, $fA_2(e_{k+1,i} \otimes_{f^2_{3,k,i}} e_{k,i+1}) = f(e_{k+1,i} \otimes_{\alpha_k^{[i]}} e_{k,i})\alpha_{k-1}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]} + \alpha_k^{[i]} f(e_{k,i} \otimes_{\alpha_{k-1}^{[i]}} e_{k-1,i})\alpha_{k-2}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]} + \cdots + \alpha_k^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_{k-1}^{[i+1]} f(e_{k+1,i+1} \otimes_{\alpha_k^{[i+1]}} e_{k,i+1}) = d_{k,i}\alpha_k^{[i]}\alpha_{k-1}^{[i]} \cdots \alpha_1^{[i]} \alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]} + d_{k-1,i}\alpha_k^{[i]}\alpha_{k-1}^{[i]}\alpha_{k-2}^{[i]} \cdots \alpha_1^{[i]}\alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]} + \cdots + d_{k,i+1}\alpha_k^{[i]} \cdots \alpha_1^{[i]}$

$\alpha_{n-2}^{[i+1]} \cdots \alpha_{k-1}^{[i+1]} \alpha_k^{[i+1]} = (d_{k,i} + d_{k-1,i} + \cdots + d_{k,i+1}) \alpha_k^{[i]} \alpha_{k-1}^{[i]} \cdots \alpha_1^{[i]} \alpha_{n-2}^{[i+1]} \cdots \alpha_k^{[i+1]}$
$= 0$.

Hence $fA_2(e_{1,i} \otimes_{f^2_{1,1,i}} e_{1,i+1}) = (c_{1,i} - c_{3,i} + c_{2,i} - c_{4,i})\beta_0^{[i]}\beta_1^{[i]} = c'\beta_0^{[i]}\beta_1^{[i]}$ and $fA_2(e_{1,i} \otimes_{f^2_{1,2,i}} e_{1,i+1}) = (c_{1,i} + c_{2,i} - d_{n-2,i} - \ldots - d_{1,i})\beta_0^{[i]}\beta_1^{[i]} = c''\beta_0^{[i]}\beta_1^{[i]}$ for some $c', c'' \in K$. Moreover fA_2 takes every other idempotent $\mathfrak{o}(f^2_{j,l,i}) \otimes \mathfrak{t}(f^2_{j,l,i})$ with $j \neq 1$ to zero. So $dim\,\mathrm{Im}\,d_2 = 2s$.

To find $\mathrm{Ker}\,d_3$ we consider separately the cases $s = 1$ and $s \geq 2$.

Ker $\mathbf{d_3}$ for $\mathbf{s \geq 2}$.

To find $\mathrm{Ker}\,d_3$, we have $d_3 : \mathrm{Hom}(Q^2, \Lambda) \to \mathrm{Hom}(Q^3, \Lambda)$. Let $h \in \mathrm{Ker}\,d_3$, so $h \in \mathrm{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then, as $s \geq 2$, the map $h : Q^2 \to \Lambda$ is given by

$$h(e_{1,i} \otimes_{f^2_{1,1,i}} e_{1,i+1}) = c_{1,i}\beta_0^{[i]}\beta_1^{[i]},$$

$$h(e_{1,i} \otimes_{f^2_{1,2,i}} e_{1,i+1}) = c_{2,i}\beta_0^{[i]}\beta_1^{[i]},$$

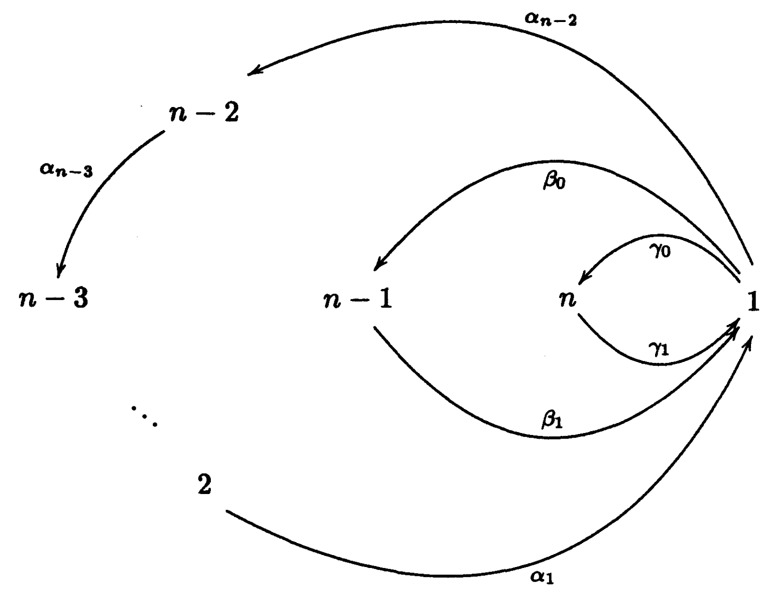
$$h(\mathfrak{o}(f^2_{2,j,i}) \otimes_{f^2_{2,j,i}} \mathfrak{t}(f^2_{2,j,i})) = 0, \text{ for } j \in \{1, \ldots, 6\} \text{ and}$$

$$h(\mathfrak{o}(f^2_{3,k,i}) \otimes_{f^2_{3,k,i}} \mathfrak{t}(f^2_{3,k,i})) = 0, \text{ for } k \in \{2, \ldots, n-3\}$$

for some $c_{1,i}, c_{2,i} \in K$, where $i \in \{0, \ldots, s-1\}$. Hence $dim\,\mathrm{Hom}(Q^2, \Lambda) = 2s$.

Note that $\mathrm{Im}\,d_2 \subseteq \mathrm{Ker}\,d_3 \subseteq \mathrm{Hom}(Q^2, \Lambda)$. So since $dim\,\mathrm{Im}\,d_2 = dim\,\mathrm{Hom}(Q^2, \Lambda) = 2s$ then $dim\,\mathrm{Ker}\,d_3 = 2s$ so $\mathrm{Im}\,d_2 = \mathrm{Ker}\,d_3$. Therefore $\mathrm{HH}^2(\Lambda) = 0$.

Ker $\mathbf{d_3}$ for $\mathbf{s = 1}$. We write δ and $f^r_{a,b}$ to indicate $\delta^{[0]}$ and $f^r_{a,b,0}$ respectively for an arrow δ in $Q(D_n, s)$ since there is no confusion here. The algebra $(D_n, 1, 1)$ is given by the quiver $Q(D_n, 1)$:



Recall from the beginning of the chapter that for $s = 1$ the minimal relations are

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1,$$

$$f^2_{2,1} = \alpha_1\beta_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$

$$f^2_{2,3} = \beta_1\alpha_{n-2}, \qquad f^2_{2,4} = \gamma_1\alpha_{n-2},$$

$$f^2_{2,5} = \beta_1\gamma_0, \qquad f^2_{2,6} = \gamma_1\beta_0 \text{ and}$$

$$f^2_{3,k} = \alpha_k\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k, \text{ for } k = \{2,\ldots,n-3\}.$$

and the set f^3 is

$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_{1,5}, f^3_{1,6}, f^3_{1,7}, f^3_{2,3}, f^3_{2,m}\}$, with $m \in \{4,\ldots,n-2\}$ where

$$\begin{array}{lllll}
f^3_{1,1} & = & f^2_{1,2}\alpha_{n-2}\alpha_{n-3} & = & \beta_0 f^2_{2,3}\alpha_{n-3} - \alpha_{n-2} f^2_{3,n-3}, \\
f^3_{1,2} & = & f^2_{1,1}\alpha_{n-2} & = & \beta_0 f^2_{2,3} - \gamma_0 f^2_{2,4}, \\
f^3_{1,3} & = & f^2_{1,1}\beta_0 - f^2_{1,2}\beta_0 & = & \alpha_{n-2}\cdots\alpha_2 f^2_{2,1} - \gamma_0 f^2_{2,6}, \\
f^3_{1,4} & = & f^2_{1,2}\gamma_0 & = & \beta_0 f^2_{2,5} - \alpha_{n-2}\cdots\alpha_2 f^2_{2,2}, \\
f^3_{1,5} & = & f^2_{2,1}\beta_1 - f^2_{2,2}\gamma_1 & = & \alpha_1 f^2_{1,1}, \\
f^3_{1,6} & = & f^2_{2,3}\alpha_{n-3}\cdots\alpha_1 - f^2_{2,5}\gamma_1 & = & \beta_1 f^2_{1,1} - \beta_1 f^2_{1,2}, \\
f^3_{1,7} & = & f^2_{2,6}\beta_1 - f^2_{2,4}\alpha_{n-3}\cdots\alpha_1 & = & \gamma_1 f^2_{1,2}, \\
f^3_{2,3} & = & f^2_{3,2}\alpha_1 & = & \alpha_2 f^2_{2,1}\beta_1 - \alpha_2\alpha_1 f^2_{1,2}, \\
f^3_{2,m} & = & f^2_{3,m-1}\alpha_{m-2} & = & \alpha_{m-1} f^2_{3,m-2} \text{ for } m \in \{4,\ldots,n-2\}.
\end{array}$$

We have $d_3 : \mathrm{Hom}(Q^2,\Lambda) \to \mathrm{Hom}(Q^3,\Lambda)$. Let $h \in \mathrm{Ker}\, d_3$, so $h \in \mathrm{Hom}(Q^2,\Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_1 e_1 + c_2\beta_0\beta_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4\beta_0\beta_1,$$

$$h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j \in \{1,\ldots,6\} \text{ and}$$

$$h(\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k})) = d_k\alpha_k, \text{ for } k \in \{2,\ldots,n-3\}$$

for some $c_1, c_2, c_3, c_4, d_k \in K$. Hence $dim\, \mathrm{Hom}(Q^2,\Lambda) = n$.

Recall the definition of the map A_3 from Definition 3.1. Then $hA_3(e_1 \otimes_{f^3_{1,1}} e_{n-3}) = h(e_1 \otimes_{f^2_{1,2}} \alpha_{n-2}\alpha_{n-3} - [\beta_0 \otimes_{f^2_{2,3}} \alpha_{n-3} - \alpha_{n-2} \otimes_{f^2_{3,n-3}} e_{n-3}]) = h(e_1 \otimes_{f^2_{1,2}} e_1)\alpha_{n-2}\alpha_{n-3} - \beta_0 h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2})\alpha_{n-3} + \alpha_{n-2} h(e_{n-2} \otimes_{f^2_{3,n-3}} e_{n-3}) = (c_3 e_1 + c_4\beta_0\beta_1)\alpha_{n-2}\alpha_{n-3} - 0 + d_{n-3}\alpha_{n-2}\alpha_{n-3} = (c_3 + d_{n-3})\alpha_{n-2}\alpha_{n-3}$. As $h \in \mathrm{Ker}\, d_3, c_3 + d_{n-3} = 0$.

$hA_3(e_1 \otimes_{f^3_{1,2}} e_{n-2}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\alpha_{n-2} - \beta_0 h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2}) + \gamma_0 h(e_n \otimes_{f^2_{2,4}} e_{n-2}) = (c_1 e_1 + c_2\beta_0\beta_1)\alpha_{n-2} = c_1\alpha_{n-2}$, so $c_1 = 0$.

$hA_3(e_1 \otimes_{f^3_{1,3}} e_{n-1}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\beta_0 - h(e_1 \otimes_{f^2_{1,2}} e_1)\beta_0 - \alpha_{n-2}\cdots\alpha_2 h(e_2 \otimes_{f^2_{2,1}} e_{n-1}) + \gamma_0 h(e_n \otimes_{f^2_{2,6}} e_{n-1}) = (c_1 e_1 + c_2\beta_0\beta_1)\beta_0 - (c_3 e_1 + c_4 e_1\beta_0\beta_1)\beta_0 = (c_1 - c_3)\beta_0$. So we have $c_1 - c_3 = 0$ and $c_1 = c_3$. As $c_1 = 0$ it follows that $c_3 = 0$ and therefore $d_{n-3} = 0$.

$hA_3(e_1 \otimes_{f^3_{1,4}} e_n) = h(e_1 \otimes_{f^2_{1,2}} e_1)\gamma_0 - \beta_0 h(e_{n-1} \otimes_{f^2_{2,5}} e_n) + \alpha_{n-2}\cdots\alpha_2 h(e_2 \otimes_{f^2_{2,2}} e_n) = (c_3 e_1 + c_4\beta_0\beta_1)\gamma_0 = c_3\gamma_0$. We already know $c_3 = 0$, so this gives no new information.

Similarly, $hA_3(e_2 \otimes_{f^3_{1,5}} e_1) = h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_1 - h(e_2 \otimes_{f^2_{2,2}} e_n)\gamma_1 - \alpha_1 h(e_1 \otimes_{f^2_{1,1}} e_1) = -c_1\alpha_1$,

$hA_3(e_{n-1} \otimes_{f^3_{1,6}} e_1) = h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2})\alpha_{n-3}\alpha_{n-4}\cdots\alpha_1 - h(e_{n-1} \otimes_{f^2_{2,5}} e_n)\gamma_1 - \beta_1 h(e_1 \otimes_{f^2_{1,1}} e_1) + \beta_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -c_1\beta_1 + c_3\beta_1 = (c_3 - c_1)\beta_1$,

$hA_3(e_n \otimes_{f^3_{1,7}} e_1) = h(e_n \otimes_{f^2_{2,6}} e_{n-1})\beta_1 - h(e_n \otimes_{f^2_{2,4}} e_{n-2})\alpha_{n-3}\cdots\alpha_1 - \gamma_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -\gamma_1(c_3e_1 + c_4\beta_0\beta_1) = -c_3\gamma_1$, all give no new information.

$hA_3(e_3 \otimes_{f^3_{2,3}} e_1) = h(e_3 \otimes_{f^2_{3,2}} e_2)\alpha_1 - \alpha_2 h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_1 - \alpha_2\alpha_2 h(e_1 \otimes_{f^2_{1,2}} e_1) = d_2\alpha_2\alpha_1 - 0 - \alpha_2\alpha_1(c_3e_1 + c_4\beta_0\beta_1) = (d_2 - c_3)\alpha_2\alpha_1$. Thus we have $d_2 - c_3 = 0$. As $c_3 = 0$, it follows that $d_2 = 0$.

Finally for $m \in \{4, \ldots, n-2\}$, we have $hA_3(e_m \otimes_{f^3_{2,m}} e_{m-2}) = h(e_m \otimes_{f^2_{3,m-1}} e_{m-1})\alpha_{m-2} - \alpha_{m-1}h(e_{m-1} \otimes_{f^2_{3,m-2}} e_{m-2}) = d_{m-1}\alpha_{m-1}\alpha_{m-2} - d_{m-2}\alpha_{m-1}\alpha_{m-2} = (d_{m-1} - d_{m-2})\alpha_{m-1}\alpha_{m-2}$. Then $d_{m-1} - d_{m-2} = 0$ and so $d_{m-1} = d_{m-2}$. Hence $d_{n-3} = d_{n-4} = \ldots = d_3 = d_2$. We already have $d_{n-3} = 0$ and $d_2 = 0$ so $d_k = 0$ for $k = 2, \ldots, n-3$.

Thus h is given by

$$
\begin{aligned}
h(e_1 \otimes_{f^2_{1,1}} e_1) &= c_2\beta_0\beta_1,\\
h(e_1 \otimes_{f^2_{1,2}} e_1) &= c_4\beta_0\beta_1,\\
h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) &= 0, \text{ for } j \in \{1, \ldots, 6\} \text{ and}\\
h(\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k})) &= 0, \text{ for } k \in \{2, \ldots, n-3\}
\end{aligned}
$$

for some $c_2, c_4 \in K$ and so $dim\,\mathrm{Ker}\, d_3 = 2$.

We recall that $dim\,\mathrm{Im}\, d_2 = 2s = 2$. Therefore $dim\,\mathrm{HH}^2(\Lambda) = 2 - 2 = 0$.

Theorem 4.3. *For $\Lambda = \Lambda(D_n, s, 1)$ with $n \geq 4, s \geq 1$ we have $\mathrm{HH}^2(\Lambda) = 0$.*

5. Vanishing of $\mathrm{HH}^2(\Lambda)$

In chapter 4 we showed that $\mathrm{HH}^2(\Lambda) = 0$ for Λ of type $(D_n, s, 1)$ with $s \geq 1$. Together with Section 3 of [15], which gives some sufficient conditions for the vanishing of $\mathrm{HH}^2(\Lambda)$, this motivates new results on the vanishing of $\mathrm{HH}^2(\Lambda)$ which may be applied to some of the self-injective algebras of finite representation type. Throughout $\Lambda = KQ/I$ is a finite dimensional algebra over the algebraically closed field K where I is an admissible ideal with a minimal set of uniform relations $f^2 = \{f_1^2, \dots, f_m^2\}$. Note that we may choose each relation to be uniform, that is, for each relation f_i^2 there are some vertices v, w such that $f_i^2 = vf_i^2$ and $f_i^2 = f_i^2 w$.

We start by recalling some definitions from Section 3 of [15] and from the theory of Gröbner bases (see [15] and [14]).

A length-lexicographic order $>$ on the paths of Q is an arbitrary linear order of both the vertices and the arrows of Q, so that any vertex is smaller than any path of length at least one. For paths p and q, both not vertices, we define $p > q$ if the length of p is greater than the length of q. If the lengths are equal, say $p = a_1 \cdots a_t$ and $q = b_1 \cdots b_t$ where the a_i and b_i are arrows, then we say $p > q$ if there is an $i, 0 \leq i \leq t-1$, such that $a_j = b_j$ for $j \leq i$ but $a_{i+1} > b_{i+1}$.

Let f be an element in KQ written as a linear combination of paths $\sum_{j=1}^{s} c_j \rho_j$ with $c_j \in K\backslash\{0\}$ and paths ρ_j. Following [15], we say a path ρ occurs in f if $\rho = \rho_j$ for some j.

Definition 5.1. Fix a length-lexicographic order on a quiver Q. Let f be a non-zero element of KQ. Let $tip(f)$ denote the largest path occuring in f. Then

$$Tip(I) = \{tip(f) | f \in I\backslash\{0\}\}.$$

Define $\mathrm{NonTip}(I)$ to be the set of paths in KQ that are not in Tip(I). Note that for vertices v and w, $v\mathrm{NonTip}(I)w$ is a K-basis of paths for $v\Lambda w$.

Definition 5.2. [15, Definition 3.1] The boundary of f^2, denoted by $Bdy(f^2)$, is defined to be the set

$$Bdy(f^2) = \{(\mathfrak{o}(f_1^2), \mathfrak{t}(f_1^2)), \dots, (\mathfrak{o}(f_m^2), \mathfrak{t}(f_m^2))\} = \{(\mathfrak{o}(x), \mathfrak{t}(x)) | x \in f^2\}.$$

Definition 5.3. [15, Definition 3.3] Let $\mathcal{G}^2 = \bigcup v\mathrm{NonTip}(I)w$, where the union is taken over all (v, w) in $Bdy(f^2)$.

We consider now elements of $\mathrm{Hom}(Q^2, \Lambda)$.

Definition 5.4. [15, Definition 3.4] For p in $\mathcal{G}^2$ and $x \in f^2$ with $\mathfrak{o}(x) = \mathfrak{o}(p)$ and $\mathfrak{t}(x) = \mathfrak{t}(p)$, define $\phi_{p,x} : Q^2 \to \Lambda$ to be the Λ^e-homomorphism given by

$$\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2) \mapsto \begin{cases} p & \text{if } f_i^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$ be the map induced by A_2. Each element of $\mathrm{HH}^2(\Lambda)$ may be represented by a map in $\mathrm{Hom}(Q^2, \Lambda)$ and so is represented by a linear combination over K of maps $\phi_{p,x}$. If every $\phi_{p,x}$ is in $\mathrm{Im}\, d_2$ then $\mathrm{Hom}(Q^2, \Lambda) = \mathrm{Im}\, d_2$ and hence $\mathrm{HH}^2(\Lambda) = 0$. Our strategy in Theorem 5.11 is to show that $\mathrm{HH}^2(\Lambda) = 0$ for certain self-injective algebras of finite representation type by showing that every $\phi_{p,x}$ is in $\mathrm{Im}\, d_2$.

First we return to [15] and modify [15, Definition 3.6].

Definition 5.5. Let X be a set of paths in KQ. Define

$L_0(X) = \{p \in X | \exists$ some arrow a which occurs in p and
which does not occur in any element of $X \backslash \{p\}\}$.

For $p \in L_0(X)$, we call such an a an arrow associated to p.

Define $L_i(X)$ for $i \in \mathbb{N}$ by

$$L_i(X) = L_0(X \backslash \bigcup_{j=0}^{i-1} L_j(X)).$$

Definition 5.6. [15, Definition 3.9] Let X be a set of paths in $\mathrm{NonTip}(I)$. The arrows are said to separate X if $X = \bigcup_{i \geq 0} L_i(X)$.

Now we will state the main Theorem of [15].

Theorem 5.7. [15, Theorem 3.10] *Let $X = \mathcal{G}^2$ and suppose that the arrows separate X. Suppose further that for all $(v, w) \in Bdy(f^2)$ there is some $x = vxw \in f^2$ and constants $c_{x,i} \in K \backslash \{0\}$ such that $vf^2w = \{tip(x) +$*

$\sum_{p_i \in v\mathrm{NonTip}(I)w} c_{x,i} p_i\}$. *For each path p in X, let α_p be an arrow associated to p and let s_p be the number of occurrences of α_p in p. Suppose that char K does not divide s_p for all $p \in X$. Then every element of* $\mathrm{Hom}(Q^2, \Lambda)$ *is a coboundary, that is, $\phi_{p,x} \in \mathrm{Im}\, d_2$ for all $p \in \mathcal{G}^2$ and $x \in f^2$. Thus* $\mathrm{HH}^2(\Lambda) = 0$.

Theorem 5.7 was used in [15] to determine $\mathrm{HH}^2(\Lambda)$ for the self-injective Möbius algebras of finite representation type A_n. We state Theorem 4.2 of [15] with the notation of [2] which we have given in chapter 3.

Theorem 5.8. [15, Theorem 4.2] *For the Möbius algebra $M_{p,s}$ we have* $\mathrm{HH}^2(M_{p,s}) = 0$ *except when $p = 1$ and $s = 1$.*

From [15], if $p = 1$ and $s = 1$ then $M_{p,s}$ is the preprojective algebra of type A_3. In [10, 7.2.1], a basis for the Hochschild cohomology groups of the algebras of type A_n is given. Using [10, 7.2.1] with $n = 3$ we have the following proposition.

Proposition 5.9. *For the Möbius algebra $M_{p,s}$ with $p = 1$ and $s = 1$ we have dim* $\mathrm{HH}^2(M_{p,s}) = 1$.

Self-injective algebras of finite representation type A_n fall into two types. They are the Möbius algebras $M_{p,s}$ above, and the self-injective Nakayama algebras. In [9], the dimension of $\mathrm{HH}^{2j}(\Lambda)$ is given for a self-injective Nakayama algebra for all $j \geq 1$. In particular this gives us $\mathrm{HH}^2(\Lambda)$ when $j = 1$. The self-injective Nakayama algebra $\Lambda(A_n, s/n, 1)$ of [2] is the algebra B_s^{n+1} of [9]. Write $n + 1 = ms + r$ where $0 \leq r < s$. From [9], with $j = 1$, we have the following result.

Proposition 5.10. [9, Proposition 4.4] *For $\Lambda = \Lambda(A_n, s/n, 1)$, we have dim* $\mathrm{HH}^2(\Lambda) = m$.

Thus we need now to determine $\mathrm{HH}^2(\Lambda)$ for the algebras of type D_n and $E_{6,7,8}$.

For the algebras $\Lambda(D_n, s, 1), \Lambda(D_n, s, 2)$ with $n \geq 4, \Lambda(D_4, s, 3), \Lambda(D_{3m}, s/3, 1)$ with $m \geq 2$ and $3 \nmid s$, $\Lambda(E_n, s, 1)$ with $n \in \{6, 7, 8\}$ and $\Lambda(E_6, s, 2)$ all for

$s \geq 1$, Theorem 5.7 does not apply since there is some $x = vxw \in f^2$ such that $|vf^2w| > 1$.

Motivated by Theorem 5.7 we give a new theorem on the vanishing of $\mathrm{HH}^2(\Lambda)$ which we will show applies to all these algebras when $s \geq 2$. (We will consider the case $s = 1$ later.)

Theorem 5.11. *Suppose that for all $(v, w) \in Bdy(f^2)$ either $v\Lambda w = \{0\}$ or there is some path p such that $v\mathrm{NonTip}(I)w = \{p\}$. If $v\Lambda w \neq 0$ suppose further that $vf^2w = \{p - q_1, \ldots, p - q_t\}$ for paths $q_1, \ldots, q_t$. Thus we may write $\mathcal{G}^2 = \{p_1, \ldots, p_r\}$, where for each $i = 1, \ldots, r$, we have non-zero paths $q_{i1}, \ldots, q_{it_i}$ with $\mathfrak{o}(p_i)f^2\mathfrak{t}(p_i) = \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$.*

Let $Y = \{p_1, \ldots, p_r, q_{ij} | 1 \leq i \leq r, 1 \leq j \leq t_i\}$. Suppose that $L_O(Y) = Y$. Let a_{ij} be an arrow associated to q_{ij} and assume that a_{ij} occurs only once in the path q_{ij}. Then every element of $\mathrm{Hom}(Q^2, \Lambda)$ is a coboundary, that is, $\phi_{p,x} \in \mathrm{Im}\, d_2$ for all $p \in \mathcal{G}^2$ and $x \in f^2$, and thus $\mathrm{HH}^2(\Lambda) = 0$.

Proof. It is enough to show that each element $\phi_{p,x}$ of $\mathrm{Hom}(Q^2, \Lambda)$ where p is a path in $\mathcal{G}^2$ and $x \in f^2$ with $\mathfrak{o}(x) = \mathfrak{o}(p)$ and $\mathfrak{t}(x) = \mathfrak{t}(p)$, is a coboundary. By hypothesis $\mathcal{G}^2 = \{p_1, \ldots, p_r\}$. Note that the paths $p_1, \ldots, p_r$ are distinct. Consider the path p_i where $i \in \{1, \ldots, r\}$. Then by hypothesis there are vertices v_i, w_i with $v_i\mathrm{NonTip}(I)w_i = \{p_i\}$ and $v_if^2w_i = \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$. Thus if $x \in f^2$ and $\mathfrak{o}(x) = \mathfrak{o}(p_i)$ and $\mathfrak{t}(x) = \mathfrak{t}(p_i)$ then $x \in v_if^2w_i$. Thus $x \in \{p_i - q_{i1}, \ldots, p_i - q_{it_i}\}$. Consider $x = p_i - q_{ij}$ where $j \in \{1, \ldots, t_i\}$.

The map $\phi_{p_i,x} : Q^2 \to \Lambda$ is given by

$$\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2) \mapsto \begin{cases} p_i & \text{if } f_k^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

We have $Y = \{p_1, \ldots, p_r, q_{ij} | 1 \leq i \leq r, 1 \leq j \leq t_i\}$ and $Y = L_O(Y)$ so $q_{ij} \in L_O(Y)$. Therefore there exists some arrow a_{ij} which occurs in q_{ij} and does not occur in any element of $Y \backslash \{q_{ij}\}$.

Define $\psi : Q^1 \to \Lambda$ by

$$\mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) \mapsto \begin{cases} -a_{ij} & \text{if } \alpha = a_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we want to show that $\psi A_2 = \phi_{p_i,x}$. Take $\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2) \in Q^2$. We start by finding $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2))$ by considering two cases.

Case $f_k^2 = x$.

Here, we have $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = \psi A_2(\mathfrak{o}(x) \otimes \mathfrak{t}(x))$, where $x = p_i - q_{ij}$ and $q_{ij} = \rho_1 a_{ij} \rho_2$ for paths ρ_1, ρ_2 such that a_{ij} does not occur in ρ_1 or ρ_2 since a_{ij} occurs only once in q_{ij} by hypothesis. Let $p_i = \sigma_1 \cdots \sigma_l, \rho_1 = \epsilon_1 \cdots \epsilon_n, \rho_2 = b_1 \cdots b_m$, where σ's, ϵ's, b's are arrows. Then $\psi A_2(\mathfrak{o}(x) \otimes \mathfrak{t}(x)) =$

$\psi[(\mathfrak{o}(x) \otimes_{\sigma_1} (\sigma_2 \cdots \sigma_l) + \sigma_1 \otimes_{\sigma_2} (\sigma_3 \cdots \sigma_l) + \cdots + (\sigma_1\sigma_2 \cdots \sigma_{l-1} \otimes_{\sigma_l} \mathfrak{t}(x)) - (\mathfrak{o}(x) \otimes_{\epsilon_1} (\epsilon_2 \cdots \epsilon_n) a_{ij}\rho_2 + \epsilon_1 \otimes_{\epsilon_2} (\epsilon_3 \cdots \epsilon_n) a_{ij}\rho_2 + \cdots + (\epsilon_1\epsilon_2 \cdots \epsilon_{n-1}) \otimes_{\epsilon_n} a_{ij}\rho_2 + \rho_1 \otimes_{a_{ij}} \rho_2 + \rho_1 a_{ij} \otimes_{b_1} (b_2 \cdots b_m) + \rho_1 a_{ij} b_1 \otimes_{b_2} (b_3 \cdots b_m) + \cdots + \rho_1 a_{ij}(b_1 b_2 \cdots b_{m-1}) \otimes_{b_m} \mathfrak{t}(x))]$.

As $q_{ij}, p_i \in Y = L_0(Y)$ and a_{ij} occurs in q_{ij}, we have that a_{ij} does not occur in p_i. So a_{ij} is not equal to any of the σ's, ϵ's or b's. Therefore

$$\begin{aligned}
&\psi A_2(\mathfrak{o}(x) \otimes \mathfrak{t}(x))\\
&= -\psi(\rho_1 \otimes_{a_{ij}} \rho_2)\\
&= -\rho_1 \psi(\mathfrak{t}(\rho_1) \otimes_{a_{ij}} \mathfrak{o}(\rho_2))\rho_2\\
&= -\rho_1 \psi(\mathfrak{o}(a_{ij}) \otimes_{a_{ij}} \mathfrak{t}(a_{ij}))\rho_2\\
&= \rho_1 a_{ij} \rho_2 = q_{ij}.
\end{aligned}$$

Case $f_k^2 \neq x$.

We consider the cases $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$ and $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) \neq 0$.

a) If $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$ then $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = \mathfrak{o}(f_k^2)\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2))\mathfrak{t}(f_k^2) = 0$ as $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) \in \Lambda$ and $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = 0$.

b) If $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) \neq 0$ then $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = Sp\{p_u\}$, the vector space spanned by p_u, for some $1 \leq u \leq r$. Hence $f_k^2 = p_u - q_{ul}$ for some $1 \leq l \leq t_u$.

We have $L_0(Y) = Y$ so a_{ij} does not occur in any element of $Y \backslash \{q_{ij}\}$. Suppose for contradiction that a_{ij} occurs in q_{ul}, so that $q_{ul} = q_{ij}$ as paths in KQ. Then

$$\mathfrak{o}(f_k^2) = \mathfrak{o}(q_{ul}) = \mathfrak{o}(q_{ij}) = \mathfrak{o}(x)$$

and

$$\mathfrak{t}(f_k^2) = \mathfrak{t}(q_{ul}) = \mathfrak{t}(q_{ij}) = \mathfrak{t}(x).$$

Therefore, $\mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = \mathfrak{o}(x)\Lambda\mathfrak{t}(x) = Sp\{p_i\}$. Hence, $p_u = p_i$ by the choice of $\mathcal{G}^2$. Therefore, $f_k^2 = p_u - q_{ul} = p_i - q_{ij} = x$. This gives a contradiction since we assumed $f_k^2 \neq x$. Hence a_{ij} does not occur in q_{ul}.

Now suppose for contradiction that a_{ij} occurs in p_u so that $p_u = q_{ij}$ as paths in KQ. Then

$$\mathfrak{o}(f_k^2) = \mathfrak{o}(p_u) = \mathfrak{o}(q_{ij}) = \mathfrak{o}(x)$$

and

$$\mathfrak{t}(f_k^2) = \mathfrak{t}(p_u) = \mathfrak{t}(q_{ij}) = \mathfrak{t}(x).$$

Therefore, $Sp\{p_u\} = \mathfrak{o}(f_k^2)\Lambda\mathfrak{t}(f_k^2) = \mathfrak{o}(x)\Lambda\mathfrak{t}(x) = Sp\{p_i\}$. Therefore, $p_u = p_i$ by the choice of $\mathcal{G}^2$. Hence $p_i = p_u = q_{ij}$ in KQ. So $p_i - q_{ij} = 0$ in KQ. This contradicts $p_i - q_{ij}$ being a minimal generator of I. Therefore, a_{ij} does not occur in p_u.

Thus a_{ij} does not occur in f_k^2. So $\psi A_2(\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2)) = 0$.

Hence ψA_2 is the map

$$\mathfrak{o}(f_k^2) \otimes \mathfrak{t}(f_k^2) \mapsto \begin{cases} q_{ij} & \text{if } f_k^2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

As $p_i - q_{ij} \in f^2$, we know that $p_i = q_{ij}$ in Λ. Hence $\psi A_2 = \phi_{p_i,x}$. Thus $\phi_{p_i,x}$, and hence each element of $\mathrm{Hom}(Q^2, \Lambda)$, is a coboundary. Hence $\mathrm{HH}^2(\Lambda) = 0$. □

We now want to apply Theorem 5.11. So we need minimal relations for each algebra in Asashiba's list. We consider the standard algebras in this chapter. Theorem 5.11 does not apply to the nonstandard algebras and so we consider the nonstandard case separately in Chapter 11.

5.12. $\Lambda(D_n, s, 2)$.

Recall the quiver and relations from 2.28. We need a set f^2 of minimal relations but note that $R(D_n, s, 2)$ is not minimal. So now we will discard some of the relations of $R(D_n, s, 2)$ to give a minimal set.

All relations of type (ii) are in f^2. For relations of type (i), let $\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]} \in f^2$ and $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} \in f^2$ for $i \in \{0,\ldots,s-1\}$.

We now consider the relations of type (iii). Now $(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]})\gamma_0^{[i+1]} = (\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} - \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]}) \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} \in I$ for all $i \in \{0,\ldots,s-2\}$. Therefore $\gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and so is not in f^2 for all $i \in \{0,\ldots,s-2\}$. Similarly we can show that $\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]}, \beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]}$ and $\gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]}$ are not in f^2 for $i \in \{0,\ldots,s-2\}$.

Now consider the path $\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]}$. Here we have $(\beta_0^{[s-1]}\beta_1^{[s-1]} - \gamma_0^{[s-1]}\gamma_1^{[s-1]})\gamma_0^{[0]} = (\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]} - \gamma_0^{[s-1]}\gamma_1^{[s-1]}\gamma_0^{[0]}) \in I$ and $\gamma_0^{[s-1]}\gamma_1^{[s-1]}\gamma_0^{[0]} \in I$. Therefore $\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]} \in I$ and so is not in f^2. Similarly we can show that $\gamma_0^{[s-1]}\gamma_1^{[s-1]}\beta_0^{[0]}, \beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]}$ and $\gamma_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]}$ are not in f^2.

Now consider "α-paths". We have $\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. So $(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]})\alpha_{n-2}^{[i+1]} \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\alpha_{n-2}^{[i+1]} \in I$. Therefore $(\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]})\alpha_{n-2}^{[i+1]} \in I$ and is not in f^2. Also $\alpha_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]}) \in I$ and $\alpha_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$. So $\alpha_1^{[i-1]}\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} \in I$ and is not in f^2 for $i \in \{0,\ldots,s-1\}$.

However, the path $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\cdots\alpha_2^{[i+1]}$ for $i \in \{0,\ldots,s-1\}$ cannot be obtained from any other relations, so $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\cdots\alpha_2^{[i+1]} \in f^2$. More generally, $\alpha_k^{[i]}\alpha_{k-1}^{[i]}\cdots\alpha_{k+1}^{[i+1]}\alpha_k^{[i+1]} \in f^2$ for $i \in \{0,\ldots,s-1\}$ and for $k = \{2,\ldots,n-3\}$. So we have the following proposition.

Proposition 5.13. *For $\Lambda = \Lambda(D_n, s, 2)$ with $s \geq 2$, let*

for all $i \in \{0,\ldots,s-1\}$,

$$f_{1,1,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f_{1,2,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]},$$

$$f_{2,1,i}^2 = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f_{2,2,i}^2 = \alpha_1^{[i]}\gamma_0^{[i+1]},$$

$$f_{2,3,i}^2 = \beta_1^{[i]}\alpha_{n-2}^{[i+1]}, \qquad f_{2,4,i}^2 = \gamma_1^{[i]}\alpha_{n-2}^{[i+1]},$$

for all $i \in \{0,\ldots,s-2\}$,

$$f_{2,5,i}^2 = \beta_1^{[i]}\gamma_0^{[i+1]}, \qquad f_{2,6,i}^2 = \gamma_1^{[i]}\beta_0^{[i+1]},$$

$$f_{2,7,s-1}^2 = \beta_1^{[s-1]}\beta_0^{[0]}, \qquad f_{2,8,s-1}^2 = \gamma_1^{[s-1]}\gamma_0^{[0]},$$

for $i \in \{0,\ldots,s-1\}$,

$$f_{3,k,i}^2 = \alpha_k^{[i]}\cdots\alpha_1^{[i]}\alpha_{n-2}^{[i+1]}\cdots\alpha_k^{[i+1]}, \text{ for } k = \{2,\ldots,n-3\}.$$

Then $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i}$ *for* $i = 0, \ldots, s-1\} \cup \{f^2_{2,5,i}, f^2_{2,6,i}$ *for* $i = 0, \ldots, s-2\} \cup \{f^2_{2,7,s-1}, f^2_{2,8,s-1}\} \cup \{f^2_{3,k,i}$ *for* $i = 0, \ldots, s-1$ *and* $k = 2, \ldots, n-3\}$ *is a minimal set of relations.*

In the case $s = 1$, we write δ and $f^r_{a,b}$ to indicate $\delta^{[0]}$ and $f^r_{a,b,0}$ respectively for an arrow δ in $\mathcal{Q}(D_n, s)$ since there is no confusion here. The relations for the algebra $\Lambda(D_n, 1, 2)$ are $R(D_n, 1, 2)$ as follows.

(i) $\alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1 = \beta_0\beta_1 = \gamma_0\gamma_1$,

(ii)

$$\alpha_1\beta_0 = 0, \qquad \alpha_1\gamma_0 = 0,$$

$$\beta_1\alpha_{n-2} = 0, \qquad \gamma_1\alpha_{n-2} = 0,$$

$$\beta_1\beta_0 = 0, \qquad \gamma_1\gamma_0 = 0,$$

(iii) "α-paths" of length $n-1$ are equal to 0,

$$\beta_0\beta_1\gamma_0 = 0, \qquad \gamma_0\gamma_1\beta_0 = 0,$$

$$\beta_1\gamma_0\gamma_1 = 0, \qquad \gamma_1\beta_0\beta_1 = 0.$$

Note that $R(D_n, 1, 2)$ is not minimal. So now we will discard some of the relations of $R(D_n, 1, 2)$ to give a minimal set f^2.

All relations of type (ii) are in f^2. For relations of type (i), let $\beta_0\beta_1 - \gamma_0\gamma_1 \in f^2$ and $\beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1 \in f^2$.

We now consider the relations of type (iii). First consider "α-paths". We have $\beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1 \in f^2$. So $(\beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1)\alpha_{n-2} \in I$ and $\beta_0\beta_1\alpha_{n-2} \in I$. Therefore $\alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1\alpha_{n-2} \in I$ and is not in f^2. Also $\alpha_1(\beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1) \in I$ and $\alpha_1\beta_0\beta_1 \in I$. So $\alpha_1\alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1 \in I$ and is not in f^2.

However, the path $\alpha_2\alpha_1\alpha_{n-2}\cdots\alpha_2$ cannot be obtained from any other relations, so $\alpha_2\alpha_1\alpha_{n-2}\cdots\alpha_2 \in f^2$. More generally, $\alpha_k\alpha_{k-1}\cdots\alpha_{k+1}\alpha_k \in f^2$ for $k \in \{2, \ldots, n-3\}$. It is easy to see that the other relations of type (iii) are not in f^2. So we have the following proposition for $s = 1$.

Proposition 5.14. *For $\Lambda = \Lambda(D_n, 1, 2)$, let*

$$f_{1,1}^2 = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f_{1,2}^2 = \beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1,$$
$$f_{2,1}^2 = \alpha_1\beta_0, \qquad f_{2,2}^2 = \alpha_1\gamma_0,$$
$$f_{2,3}^2 = \beta_1\alpha_{n-2}, \qquad f_{2,4}^2 = \gamma_1\alpha_{n-2},$$
$$f_{2,5}^2 = \beta_1\beta_0, \qquad f_{2,6}^2 = \gamma_1\gamma_0 \text{ and}$$
$$f_{3,k}^2 = \alpha_k\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k, \text{ for } k \in \{2, \ldots, n-3\}.$$

Then $f^2 = \{f_{1,1}^2, f_{1,2}^2, f_{2,1}^2, f_{2,2}^2, f_{2,3}^2, f_{2,4}^2, f_{2,5}^2, f_{2,6}^2, f_{3,k}^2 \text{ for } k = 2, \ldots, n-3\}$ is a minimal set of relations.

5.15. $\Lambda(D_4, s, 3)$.

Recall the quiver and relations from 2.29. We need a set f^2 of minimal relations but note that $R(D_4, s, 3)$ is not minimal. So now we will discard some of the relations of $R(D_4, s, 3)$ to give a minimal set. All relations of type (ii) are in f^2. For relations of type (i) let $\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]} \in f^2$ and $\beta_0^{[i]}\beta_1^{[i]} - \alpha_0^{[i]}\alpha_1^{[i]} \in f^2$ for all $i \in \{0, \ldots, s-1\}$.

We now consider the relations of type (iii). Consider first "β-paths". So $(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]})\beta_0^{[i+1]} = \beta_0^{[i]}\beta_0^{[i]}\beta_0^{[i+1]} - \gamma_0^{[i]}\gamma_1^{[i]}\beta_0^{[i+1]} \in I$ and $\gamma_0^{[i]}\gamma_1^{[i]}\beta_0^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-1\}$. Therefore $\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} \in I$ and is not in f^2 for all $i \in \{0, \ldots, s-1\}$. We also have $\beta_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}) = \beta_1^{[i-1]}\beta_0^{[i]}\beta_0^{[i]} - \beta_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ and $\beta_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-2\}$. Therefore $\beta_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-2\}$ and is not in f^2 for all $i \in \{0, \ldots, s-2\}$. So $\beta_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} \in f^2$.

Now consider "γ-paths". We have $(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]})\gamma_0^{[i+1]} = \beta_0^{[i]}\beta_0^{[i]}\gamma_0^{[i+1]} - \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and $\beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-2\}$. Therefore $\gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and is not in f^2 for all $i \in \{0, \ldots, s-2\}$ but $\gamma_1^{[s-1]}\gamma_0^{[0]} \in f^2$. Therefore $\gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} \in I$ and is not in f^2 for all $i \in \{0, \ldots, s-1\}$. We also have $\gamma_1^{[i-1]}(\beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}) = \gamma_1^{[i-1]}\beta_0^{[i]}\beta_0^{[i]} - \gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ and $\gamma_1^{[i-1]}\beta_0^{[i]}\beta_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-1\}$. Therefore $\gamma_1^{[i-1]}\gamma_0^{[i]}\gamma_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-1\}$ and is not in f^2 for all $i \in \{0, \ldots, s-1\}$. Similarly we can show that for "α-paths" neither $\alpha_0^{[i]}\alpha_1^{[i]}\alpha_0^{[i+1]}$ nor $\alpha_1^{[s-1]}\alpha_0^{[0]}\alpha_1^{[0]}$ are in f^2.

Now we consider the other possible paths of length 3. We have $\gamma_0^{[i]}\gamma_1^{[i]}\beta_0^{[i+1]}$ in I and is not in f^2 since $\gamma_1^{[i]}\beta_0^{[i+1]} \in f^2$ for all $i \in \{0,\ldots,s-1\}$. Similarly we can easily see that non of $\alpha_0^{[i]}\alpha_1^{[i]}\gamma_0^{[i+1]}, \beta_0^{[i]}\beta_1^{[i]}\alpha_0^{[i+1]}, \alpha_1^{[i]}\gamma_1^{[i+1]}\gamma_1^{[i+1]}, \beta_1^{[i]}\alpha_0^{[i+1]}\alpha_1^{[i+1]}$ or $\gamma_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]}$ are in f^2 for all $i \in \{0,\ldots,s-1\}$.

We also have $\gamma_0^{[i]}\gamma_1^{[i]}\alpha_0^{[i+1]} \in I$ and is not in f^2 for all $i \in \{0,\ldots,s-2\}$. Therefore, $\gamma_0^{[s-1]}\gamma_1^{[s-1]}\alpha_0^{[0]} \in f^2$. Similarly we can easily see that non of $\alpha_0^{[i]}\alpha_1^{[i]}\beta_0^{[i+1]}, \beta_0^{[i]}\beta_1^{[i]}\gamma_0^{[i+1]}, \alpha_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]}, \beta_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]}$ or $\gamma_1^{[i]}\alpha_0^{[i+1]}\alpha_1^{[i+1]}$ are in f^2 for all $i \in \{0,\ldots,s-2\}$. Hence $\alpha_0^{[s-1]}\alpha_1^{[s-1]}\beta_0^{[0]}$, $\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]}$, $\alpha_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]}$, $\beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]}$ and $\gamma_1^{[s-1]}\alpha_0^{[0]}\alpha_1^{[0]}$ are in f^2. So we have the following proposition.

Proposition 5.16. *For $\Lambda = \Lambda(D_4, s, 3)$ with $s \geq 2$, let*

for all $i \in \{0,\ldots,s-1\}$:

$$f_{1,1,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \gamma_0^{[i]}\gamma_1^{[i]}, \qquad f_{1,2,i}^2 = \beta_0^{[i]}\beta_1^{[i]} - \alpha_0^{[i]}\alpha_1^{[i]},$$

$$f_{2,1,i}^2 = \beta_1^{[i]}\alpha_0^{[i+1]}, \qquad f_{2,2,i}^2 = \alpha_1^{[i]}\gamma_0^{[i+1]},$$

$$f_{2,3,i}^2 = \gamma_1^{[i]}\beta_0^{[i+1]},$$

for all $i \in \{0,\ldots,s-2\}$:

$$f_{2,4,i}^2 = \alpha_1^{[i]}\beta_0^{[i+1]}, \qquad f_{2,5,i}^2 = \beta_1^{[i]}\gamma_0^{[i+1]},$$

$$f_{2,6,i}^2 = \gamma_1^{[i]}\alpha_0^{[i+1]},$$

$$f_{2,7,s-1}^2 = \gamma_1^{[s-1]}\gamma_0^{[0]}, \qquad f_{2,8,s-1}^2 = \beta_1^{[s-1]}\beta_0^{[0]},$$

$$f_{2,9,s-1}^2 = \alpha_1^{[s-1]}\alpha_0^{[0]};$$

$$f_{3,1,s-1}^2 = \beta_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]}, \qquad f_{3,2,s-1}^2 = \alpha_0^{[s-1]}\alpha_1^{[s-1]}\beta_0^{[0]},$$

$$f_{3,4,s-1}^2 = \beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]}, \qquad f_{3,5,s-1}^2 = \alpha_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} \text{ and}$$

$$f_{3,6,s-1}^2 = \beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]}, \qquad f_{3,7,s-1}^2 = \gamma_1^{[s-1]}\alpha_0^{[0]}\alpha_1^{[0]}.$$

Then $f^2 = \{f_{1,1,i}^2, f_{1,2,i}^2, f_{2,1,i}^2, f_{2,2,i}^2, f_{2,3,i}^2,$ for $i = 0,\ldots,s-1\} \cup \{f_{2,4,i}^2, f_{2,5,i}^2, f_{2,6,i}^2$ for $i = 0,\ldots,s-2\} \cup \{f_{2,7,s-1}^2, f_{2,8,s-1}^2, f_{2,9,s-1}, f_{3,1,s-1}^2, f_{3,2,s-1}^2, f_{3,3,s-1}^2, f_{3,4,s-1}^2, f_{3,5,s-1}^2, f_{3,6,s-1}^2\}$ is a minimal set of relations.

Consider $s = 1$. We write δ and $f_{a,b}^r$ to indicate $\delta^{[0]}$ and $f_{a,b,0}^r$ respectively since there is no confusion here. The relations $R(D_4, 1, 3)$ is given as follows:

(i)

$$\beta_0\beta_1 = \gamma_0\gamma_1 = \alpha_0\alpha_1,$$

(ii)

$$\beta_1\alpha_0 = 0, \qquad \alpha_1\gamma_0 = 0,$$

$$\gamma_1\beta_0 = 0, \qquad \gamma_1\gamma_0 = 0,$$

$$\beta_1\beta_0 = 0, \qquad \alpha_1\alpha_0 = 0,$$

(iii) paths of length 3 are equal to 0.

It is easy to see from Proposition 5.16 that the set of minimal relations f^2 when $s = 1$ is different. It is given in the next proposition.

Proposition 5.17. *For* $\Lambda = \Lambda(D_4, 1, 3)$, *let*

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_0\alpha_1,$$

$$f^2_{2,1} = \beta_1\alpha_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$

$$f^2_{2,3} = \gamma_1\beta_0,$$

$$f^2_{2,4} = \gamma_1\gamma_0, \qquad f^2_{2,5} = \beta_1\beta_0 \text{ and}$$

$$f^2_{2,6} = \alpha_1\alpha_0.$$

Then $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}\}$ *is a minimal set of relations.*

5.18. $\Lambda(D_{3m}, s/3, 1)$.

We recall the quiver and relations from 2.30. We need a set f^2 of minimal relations but note that $R(D_{3m}, s/3, 1)$ is not minimal. So we discard some of the relations of $R(D_{3m}, s/3, 1)$ to give a minimal set.

Relations of type (i) and (ii) are in f^2.

We now consider the relations of type (iii). If $j = 1$ we have $\alpha_1^{[i]} \cdots \alpha_m^{[i]}\beta_{i+2}$ $\alpha_1^{[i+3]} = \beta_i\beta_{i+1}\beta_{i+2}\alpha_1^{[i+3]} = \beta_i\alpha_1^{[i+1]} \cdots \alpha_m^{[i+1]}\alpha_1^{[i+3]} \in I$. So $\alpha_1^{[i]} \cdots \alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]}$ $\in I$ and is not in f^2. Similarly if $j = m$ then it can be shown that $\alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]} \cdots \alpha_m^{[i+3]} \in I$ and is not in f^2.

However, the paths $\alpha_j^{[i]} \cdots \alpha_m^{[i]} \beta_{i+2} \alpha_1^{[i+3]} \cdots \alpha_j^{[i+3]} \forall j \in \{2, \dots, m-1\}$ cannot be obtained from any other relations, so $\alpha_j^{[i]} \cdots \alpha_m^{[i]} \beta_{i+2} \alpha_1^{[i+3]} \cdots \alpha_j^{[i+3]} \in f^2 \forall j \in \{2, \dots, m-1\}$. So we have the following proposition.

Proposition 5.19. *For $\Lambda = \Lambda(D_{3m}, s/3, 1)$ with $s \geq 1$, let for all $i \in \{1, \dots, s\}$*

$$f_{1,i}^2 = \beta_i \beta_{i+1} - \alpha_1^{[i]} \cdots \alpha_m^{[i]}, \qquad f_{2,i}^2 = \alpha_m^{[i]} \alpha_1^{[i+2]},$$

$$f_{3,i,j}^2 = \alpha_j^{[i]} \cdots \alpha_m^{[i]} \beta_{i+2} \alpha_1^{[i+3]} \cdots \alpha_j^{[i+3]} \text{ for all } j \in \{2, \dots, m-1\}.$$

Then $f^2 = \{f_{1,i}^2, f_{2,i}^2, f_{3,i,j}^2$ for $j = 2, \dots, m-1$ and for $i = 1, \dots, s\}$ is a minimal set of relations.

5.20. $\Lambda(E_n, s, 1)$.

Recall the quiver and relation from 2.31. We need a set f^2 of minimal relations but note that $R(E_n, s, 1)$ is not minimal. So we discard some of the relations of $R(E_n, s, 1)$ to give a minimal set. Throughout this part $i \in \{0, \dots, s-1\}$.All relations of type (ii) are in f^2. For relations of type (i) let $\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \gamma_2^{[i]} \gamma_1^{[i]} \in f^2$ and $\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_1^{[i]} \in f^2$ for all $i \in \{0, \dots, s-1\}$.

We now consider the relations of type (iii) and start by considering "α-paths". We have $\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]} \in f^2$. So $(\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]}) \alpha_{n-3}^{[i+1]} \in I$ and $\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} \alpha_{n-3}^{[i+1]} \in I$. Therefore the relation $\alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]} \alpha_{n-3}^{[i+1]} \in I$ and is not in f^2. Also $\alpha_1^{[i-1]} (\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]}) \in I$ and $\alpha_1^{[i-1]} \beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} \in I$. So $\alpha_1^{[i-1]} \alpha_{n-3}^{[i]} \alpha_{n-4}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]} \in I$ and not in f^2.

However, the path $\alpha_2^{[i]} \alpha_1^{[i]} \alpha_{n-3}^{[i+1]} \cdots \alpha_2^{[i+1]}$ cannot be obtained from any other relations, so $\alpha_2^{[i]} \alpha_1^{[i]} \alpha_{n-3}^{[i+1]} \cdots \alpha_2^{[i+1]} \in f^2$. More generally, the relation $\alpha_k^{[i]} \alpha_{k-1}^{[i]} \cdots \alpha_{k+1}^{[i+1]} \alpha_k^{[i+1]} \in f^2$ for $k \in \{2, \dots, n-4\}$.

Now consider "β-paths" of length 4. We have $(\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \gamma_2^{[i]} \gamma_1^{[i]}) \beta_3^{[i+1]} = \beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} \beta_3^{[i+1]} - \gamma_2^{[i]} \gamma_1^{[i]} \beta_3^{[i+1]} \in I$ and $\gamma_2^{[i]} \gamma_1^{[i]} \beta_3^{[i+1]} \in I$. Therefore $\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} \beta_3^{[i+1]} \in I$ and is not in f^2. We also have $\beta_1^{[i-1]} (\beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \gamma_2^{[i]} \gamma_1^{[i]}) = \beta_1^{[i-1]} \beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} - \beta_1^{[i-1]} \gamma_2^{[i]} \gamma_1^{[i]} \in I$ and $\beta_1^{[i-1]} \gamma_2^{[i]} \gamma_1^{[i]} \in I$. Therefore $\beta_1^{[i-1]} \beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} \in I$ and is not in f^2.

However, the path $\beta_2^{[i-1]}\beta_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}$ cannot be obtained from any other relations, so $\beta_2^{[i-1]}\beta_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]} \in f^2$.

Now consider "γ-paths" of length 3. We have $(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]})\gamma_2^{[i+1]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\gamma_2^{[i+1]} - \gamma_2^{[i]}\gamma_1^{[i]}\gamma_2^{[i+1]} \in I$ and $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\gamma_2^{[i+1]} \in I$. Therefore, $\gamma_2^{[i]}\gamma_1^{[i]}\gamma_2^{[i+1]} \in I$ and is not in f^2. We also have $\gamma_1^{[i-1]}(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}) = \gamma_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ and $\gamma_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} \in I$. Therfore, $\gamma_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ and is not in f^2.

So the elements of f^2 is given in the following proposition.

Proposition 5.21. *For $\Lambda = \Lambda(E_n, s, 1)$ with $s \geq 1$ and for all $i \in \{0, \ldots, s-1\}$, let*

$$f_{1,1,i}^2 = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}, \qquad f_{1,2,i}^2 = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_{n-3}^{[i]}\alpha_{n-4}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]},$$

$$f_{2,1,i}^2 = \alpha_1^{[i]}\beta_3^{[i+1]}, \qquad f_{2,2,i}^2 = \alpha_1^{[i]}\gamma_2^{[i+1]},$$

$$f_{2,3,i}^2 = \beta_1^{[i]}\alpha_{n-3}^{[i+1]}, \qquad f_{2,4,i}^2 = \beta_1^{[i]}\gamma_2^{[i+1]},$$

$$f_{2,5,i}^2 = \gamma_1^{[i]}\alpha_{n-3}^{[i+1]}, \qquad f_{2,6,i}^2 = \gamma_1^{[i]}\beta_3^{[i+1]},$$

$$f_{3,k,i}^2 = \alpha_k^{[i]}\alpha_{k-1}^{[i]}\cdots\alpha_{k+1}^{[i+1]}\alpha_k^{[i+1]} \text{ for } k \in \{2, \ldots, n-4\} \text{ and}$$

$$f_{4,i}^2 = \beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i]}\beta_2^{[i+1]}.$$

Then $f^2 = \{f_{1,1,i}^2, f_{1,2,i}^2, f_{2,1,i}^2, f_{2,2,i}^2, f_{2,3,i}^2, f_{2,4,i}^2, f_{2,5,i}^2, f_{2,6,i}^2, f_{3,k,i}^2$ for $k \in \{2, \ldots, n-4\}, f_{4,i}^2\}$ is a minimal set of relations.

5.22. $\Lambda(E_6, s, 2)$.

We recall the quiver and relations from 2.32. We need a set f^2 of minimal relations but note that $R(E_6, s, 2)$ is not minimal. So now we will discard some of the relations of $R(E_6, s, 2)$ to give a minimal set. All relations of type (ii) are in f^2. For relations of type (i) let $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]} \in f^2$ and $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} \in f^2$ for all $i \in \{0, \ldots, s-1\}$.

We now consider the relations of type (iii) and start with "γ-paths" of length 3. We have $(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]})\gamma_2^{[i+1]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\gamma_2^{[i+1]} - \gamma_2^{[i]}\gamma_1^{[i]}\gamma_2^{[i+1]} \in I$ and $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\gamma_2^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-1\}$. Therefore, $\gamma_2^{[i]}\gamma_1^{[i]}\gamma_2^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-1\}$ and is not in f^2. We also have $\gamma_1^{[i-1]}(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}) = \gamma_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ and $\gamma_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} \in I$ for all

$i \in \{0, \ldots, s-1\}$. Therefore, $\gamma_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-1\}$ and is not in f^2.

Now consider "α-paths". When $j = 1$ we have $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} \in f^2$. So $\alpha_1^{[i-1]}(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]}) \in I$ and $\alpha_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-2\}$. So $\alpha_1^{[i-1]}\alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-2\}$ and is not in f^2. Similarly if $j = 3$ it can be shown that $\alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]}\alpha_3^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-2\}$ and is not in f^2. However, if $j = 2$ and $s \geq 2$ then the path $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_3^{[i+1]}\alpha_2^{[i+1]}$ cannot be obtained from any other paths, so $\alpha_2^{[i]}\alpha_1^{[i]}\alpha_3^{[i+1]}\alpha_2^{[i+1]} \in f^2$ for all $i \in \{0, \ldots, s-2\}$.

Now consider "β-paths". When $j = 1$ we have $\beta_1^{[i-1]}(\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}) = \beta_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \beta_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ and $\beta_1^{[i-1]}\gamma_2^{[i]}\gamma_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-1\}$. Therefore $\beta_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} \in I$ for all $i \in \{0, \ldots, s-1\}$ and is not in f^2. So in particular $\beta_1^{[i-1]}\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}$ is not in f^2 for all $i \in \{0, \ldots, s-2\}$. Similarly when $j = 3$ it can be shown that $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]} \in I$ for all $i \in \{0, \ldots, s-1\}$ and is not in f^2. So in particular $\beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]} \in I$ is not in f^2 for all $i \in \{0, \ldots, s-2\}$. However, if $j = 2$ and only if $s \geq 2$ then the path $\beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]}\beta_2^{[i+1]}$ cannot be obtained from any other paths, so $\beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]}\beta_2^{[i+1]} \in f^2$ for all $i \in \{0, \ldots, s-2\}$.

This leaves two relations of type (iii) to consider. Firstly consider the relation $\alpha_j^{[s-1]} \cdots \alpha_1^{[s-1]}\beta_3^{[0]} \cdots \beta_{j-3}^{[0]} = 0$. If $j = 1$ we have $\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} - \gamma_2^{[0]}\gamma_1^{[0]} \in f^2$. So $\alpha_1^{[s-1]}(\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} - \gamma_2^{[0]}\gamma_1^{[0]}) \in I$ and $\alpha_1^{[s-1]}\gamma_2^{[0]}\gamma_1^{[0]} \in I$. Therefore, $\alpha_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} \in I$ and is not in f^2. Similarly, when $j = 3$ we can show that $\alpha_3^{[s-1]}\alpha_2^{[s-1]}\alpha_1^{[s-1]}\beta_3^{[0]} \in I$ and is not in f^2. However when $j = 2$ then the path $\alpha_2^{[s-1]}\alpha_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]}$ cannot be obtained from any other paths, so $\alpha_2^{[s-1]}\alpha_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]} \in f^2$.

Finally consider the relation $\beta_j^{[s-1]} \cdots \beta_1^{[s-1]}\alpha_3^{[0]} \cdots \alpha_{j-3}^{[0]} = 0$. If $j = 1$ we have $\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} - \alpha_3^{[0]}\alpha_2^{[0]}\alpha_1^{[0]} \in f^2$. So $\beta_1^{[s-1]}(\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} - \alpha_3^{[0]}\alpha_2^{[0]}\alpha_1^{[0]}) \in I$ and $\beta_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]}\beta_1^{[0]} \in I$. Therefore, $\beta_1^{[s-1]}\alpha_3^{[0]}\alpha_2^{[0]}\alpha_1^{[0]} \in I$ and is not in f^2. Similarly, when $j = 3$ we can show that $\beta_3^{[s-1]}\beta_2^{[s-1]}\beta_1^{[s-1]}\alpha_3^{[0]} \in I$ and is not in f^2. However when $j = 2$ then the path $\beta_2^{[s-1]}\beta_1^{[s-1]}\alpha_3^{[0]}\alpha_2^{[0]}$ cannot be obtained from any other paths, so $\beta_2^{[s-1]}\beta_1^{[s-1]}\alpha_3^{[0]}\alpha_2^{[0]} \in f^2$. So the elements of f^2 is given in the following proposition.

Proposition 5.23. *For $\Lambda = \Lambda(E_6, s, 2)$ with $s \geq 2$, let for all $i \in \{0, \ldots, s-1\}$:*

$$f^2_{1,1,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \gamma_2^{[i]}\gamma_1^{[i]}, \qquad f^2_{1,2,i} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} - \alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]},$$

$$f^2_{2,1,i} = \gamma_1^{[i]}\alpha_3^{[i+1]}, \qquad f^2_{2,2,i} = \alpha_1^{[i]}\beta_3^{[i+1]},$$

$$f^2_{2,3,i} = \alpha_1^{[i]}\gamma_2^{[i+1]}, \qquad f^2_{2,4,i} = \beta_1^{[i]}\gamma_2^{[i+1]},$$

and for all $i \in \{0, \ldots, s-2\}$

$$f^2_{2,5,i} = \alpha_1^{[i]}\beta_3^{[i+1]}, \qquad f^2_{2,6,i} = \beta_1^{[i]}\alpha_3^{[i+1]},$$

$$f^2_{2,7,s-1} = \alpha_1^{[s-1]}\alpha_3^{[0]}, \qquad f^2_{2,8,s-1} = \beta_1^{[s-1]}\beta_3^{[0]}$$

$$f^2_{3,1,i} = \alpha_2^{[i]}\alpha_1^{[i]}\alpha_3^{[i+1]}\alpha_2^{[i+1]}, \qquad f^2_{3,2,i} = \beta_2^{[i]}\beta_1^{[i]}\beta_3^{[i+1]}\beta_2^{[i+1]},$$

$$f^2_{3,3,s-1} = \alpha_2^{[s-1]}\alpha_1^{[s-1]}\beta_3^{[0]}\beta_2^{[0]}, \qquad f^2_{3,4,s-1} = \beta_2^{[s-1]}\beta_1^{[s-1]}\alpha_3^{[0]}\alpha_2^{[0]}.$$

Hence $f^2 = \{f^2_{1,1,i}, f^2_{1,2,i}, f^2_{2,1,i}, f^2_{2,2,i}, f^2_{2,3,i}, f^2_{2,4,i},$ for $i \in \{0, \ldots s-1\}\} \cup \{f^2_{2,5,i}, f^2_{2,6,i},$ for $i \in \{0, \ldots, s-2\}\} \cup \{f^2_{2,7,s-1}, f^2_{2,8,s-1}\} \cup \{f^2_{3,1,i}, f^2_{3,2,i},$ for $i \in \{0, \ldots, s-2\}\} \cup \{f^2_{3,3,s-1}, f_{3,4,s-1}\}$ is a minimal set of relations.

Consider $s = 1$. We write δ and $f^r_{a,b}$ to indicate $\delta^{[0]}$ and $f^r_{a,b,0}$ respectively since there is no confusion here. The set of minimal relations f^2 when $s = 1$ is different from the case $s \geq 2$ above. The minimal relations for $s = 1$ is given in the next proposition.

Proposition 5.24. *Let $\Lambda = \Lambda(E_6, 1, 2)$, let*

$$f^2_{1,1} = \beta_3\beta_2\beta_1 - \gamma_2\gamma_1, \qquad f^2_{1,2} = \beta_3\beta_2\beta_1 - \alpha_3\alpha_2\alpha_1,$$

$$f^2_{2,1} = \gamma_1\alpha_3, \qquad f^2_{2,2} = \gamma_1\beta_3,$$

$$f^2_{2,3} = \alpha_1\gamma_2, \qquad f^2_{2,4} = \beta_1\gamma_2,$$

$$f^2_{2,5} = \alpha_1\alpha_3, \qquad f^2_{2,6} = \beta_1\beta_3,$$

$$f^2_{3,1} = \alpha_2\alpha_1\beta_3\beta_2, \qquad f^2_{3,2} = \beta_2\beta_1\alpha_3\alpha_2.$$

Hence $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,1}, f^2_{3,2}\}$ *is a minimal set of relations.*

We now apply Theorem 5.11 to the self-injective algebras of type D_n and $E_{6,7,8}$ using Propositions 5.13, 5.14, 5.16, 5.17, 5.19, 5.21, 5.23, 5.24.

For example consider the algebra $\Lambda(D_n, s, 2)$ for $s \geq 2$. Fix an order on the vertices and the arrows:
$\alpha^{[0]}_{n-2} > \alpha^{[0]}_{n-3} > \cdots > \alpha^{[0]}_1 > \gamma^{[0]}_0 > \gamma^{[0]}_1 > \beta^{[0]}_0 > \beta^{[0]}_1 > \alpha^{[1]}_{n-2} > \cdots > \beta^{[1]}_1 > \cdots > \alpha^{[s-1]}_{n-2} > \cdots > \beta^{[s-1]}_1$
and $\beta^{[s-1]}_1 > e_{1,0} > e_{n-2,0} > \cdots > e_{1,1} > e_{n,0} > e_{n-1,0} > \cdots > e_{1,s-1} > e_{n-2,s-1} > \cdots > e_{n,s-1} > e_{n-1,s-1}$.

Then $tip(f^2_{1,1,i}) = tip(\beta^{[i]}_0\beta^{[i]}_1 - \gamma^{[i]}_0\gamma^{[i]}_1) = \gamma^{[i]}_0\gamma^{[i]}_1$ and $tip(f^2_{1,2,i}) = tip(\beta^{[i]}_0\beta^{[i]}_1 - \alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1) = \alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1$ for $i = 0, \ldots, s-1$. For all other $f^2_j \in f^2$ with $f^2_j \neq f^2_{1,1,i}, f^2_{1,2,i}$ we know that f^2_j is a path in KQ so $tip(f^2_j) = f^2_j$. In these cases $\mathfrak{o}(f^2_j)\mathrm{NonTip}(I)\mathfrak{t}(f^2_j) = \{0\}$. Let $v_i = \mathfrak{o}(f^2_{1,1,i}) = \mathfrak{o}(f^2_{1,2,i})$ and let $w_i = \mathfrak{t}(f^2_{1,1,i}) = \mathfrak{t}(f^2_{1,2,i})$ for $i = 0, \ldots, s-1$. Then $(v_i, w_i) \in Bdy(f^2)$ and $v_i\mathrm{NonTip}(I)w_i = \{\beta^{[i]}_0\beta^{[i]}_1\}$ for all $i = 0, \ldots, s-1$. So let $p^{[i]} = \beta^{[i]}_0\beta^{[i]}_1$ for $i = 0, \ldots, s-1$. Then $v_i f^2 w_i = \{\beta^{[i]}_0\beta^{[i]}_1 - \gamma^{[i]}_0\gamma^{[i]}_1, \beta^{[i]}_0\beta^{[i]}_1 - \alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1\} = \{p^{[i]} - q^{[i]}_1, p^{[i]} - q^{[i]}_2\}$, where $q^{[i]}_1 = \gamma^{[i]}_0\gamma^{[i]}_1$, $q^{[i]}_2 = \alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1$. With the notation of Theorem 5.11, $\mathcal{G}^2 = \{\beta^{[i]}_0\beta^{[i]}_1 \mid i = 0, \ldots, s-1\}$ and $Y = \{\beta^{[i]}_0\beta^{[i]}_1, \gamma^{[i]}_0\gamma^{[i]}_1, \alpha^{[i]}_{n-2}\alpha^{[i]}_{n-3}\cdots\alpha^{[i]}_2\alpha^{[i]}_1 \mid i = 0, \ldots, s-1\} = L_0(Y)$. Choose $a^{[i]}_1 = \gamma^{[i]}_0$ and $a^{[i]}_2 = \alpha^{[i]}_{n-2}$ so that $a^{[i]}_1$ and $a^{[i]}_2$ are arrows associated to $q^{[i]}_1$ and $q^{[i]}_2$ respectively, and $a^{[i]}_j$ occurs once in $q^{[i]}_j$ for $j = 1, 2$. Then by applying Theorem 5.11, every element of $\mathrm{Hom}(Q^2, \Lambda)$ is a coboundary and so $\mathrm{HH}^2(\Lambda) = 0$.

Similar arguments give the following corollary.

Corollary 5.25. *Let* Λ *be one of the standard algebras* $\Lambda(D_n, s, 1), \Lambda(D_n, s, 2)$ *for* $n \geq 4$, $\Lambda(D_4, s, 3)$, $\Lambda(D_{3m}, s/3, 1)$ *with* $m \geq 2, 3 \nmid s$, $\Lambda(E_n, s, 1)$ *with* $n \in \{6, 7, 8\}$ *and* $\Lambda(E_6, s, 2)$ *and* $s \geq 2$. *Then* $\mathrm{HH}^2(\Lambda) = 0$.

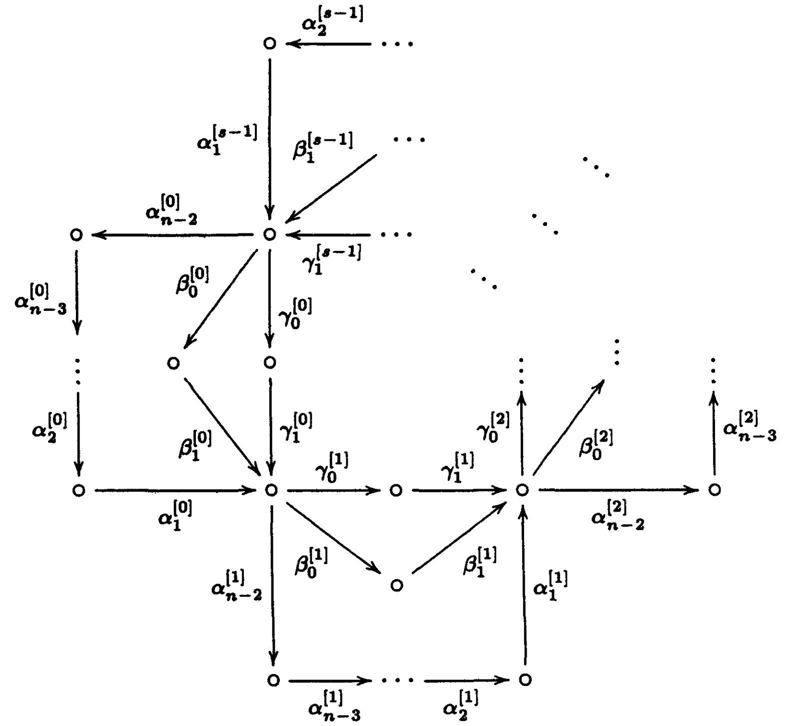
Remark. i) Theorem 5.11 does not apply if $s = 1$.

ii) Corollary 5.25 provides an alternative proof of our result (see Theorem 4.3) in Chapter 4 that $\mathrm{HH}^2(\Lambda) = 0$ for $\Lambda = \Lambda(D_n, s, 1)$ with $s \geq 2$.

In the next chapters we will determine $\mathrm{HH}^2(\Lambda)$ for the algebras $\Lambda(D_n, s, 2)$, $\Lambda(D_4, s, 3)$, $\Lambda(D_{3m}, s/3, 1)$, $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 1)$ when $s = 1$.

6. $\Lambda(D_n, s, 2)$

In this chapter we calculate $\mathrm{HH}^2(\Lambda)$ for $\Lambda(D_n, s, 2)$ with $n \geq 4, s = 1$. It is known that $\mathrm{HH}^2(\Lambda) = 0$ for $s \geq 2$ from Theorem 5.11. We start by recalling the algebra $\Lambda(D_n, s, 2)$ for all $s \geq 1$. From [2] and 2.4 in Chapter 2, the algebra $\Lambda(D_n, s, 2)$ is given by the quiver $\mathcal{Q}(D_n, s)$:



with relations $R(D_n, s, 2)$:

(i) $\alpha_{n-2}^{[i]}\alpha_{n-3}^{[i]}\cdots\alpha_2^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$

$$\beta_1^{[i]}\alpha_{n-2}^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_{n-2}^{[i+1]} = 0,$$

and for all $i \in \{0, \ldots, s-2\}$,

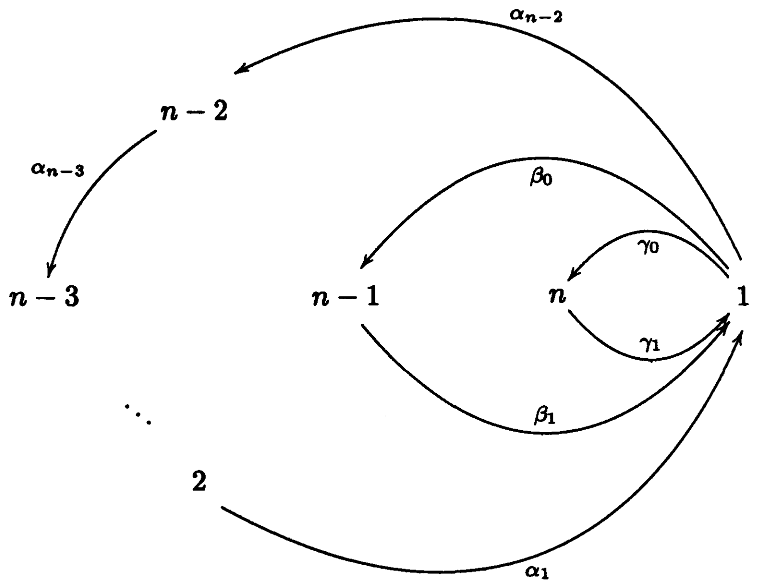
$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0,$$

$$\beta_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0,$$

(iii) "α-paths" of length $n-1$ are equal to 0, and for all $i \in \{0, \dots, s-2\}$,

$$\beta_0^{[i]}\beta_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \gamma_0^{[i]}\gamma_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\beta_0^{[i+1]}\beta_1^{[i+1]} = 0, \qquad \gamma_1^{[i]}\gamma_0^{[i+1]}\gamma_1^{[i+1]} = 0 \text{ and}$$
$$\beta_0^{[s-1]}\beta_1^{[s-1]}\gamma_0^{[0]} = 0, \qquad \gamma_0^{[s-1]}\gamma_1^{[s-1]}\beta_0^{[0]} = 0,$$
$$\beta_1^{[s-1]}\gamma_0^{[0]}\gamma_1^{[0]} = 0, \qquad \gamma_1^{[s-1]}\beta_0^{[0]}\beta_1^{[0]} = 0.$$

In the case $s = 1$, the algebra $\Lambda(D_n, 1, 2)$ is given by the quiver $\mathcal{Q}(D_n, 1)$:



with the minimal set of relations given in Proposition 5.14 as follows.

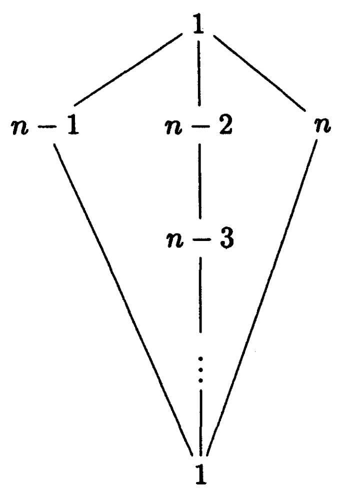
$f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,k}$ for $k = 2, \dots, n-3\}$ where

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_{n-2}\alpha_{n-3}\cdots\alpha_2\alpha_1,$$
$$f^2_{2,1} = \alpha_1\beta_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$
$$f^2_{2,3} = \beta_1\alpha_{n-2}, \qquad f^2_{2,4} = \gamma_1\alpha_{n-2},$$
$$f^2_{2,5} = \beta_1\beta_0, \qquad f^2_{2,6} = \gamma_1\gamma_0 \text{ and}$$

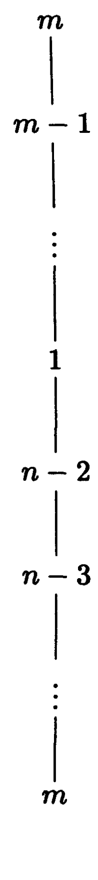
$$f^2_{3,k} = \alpha_k \cdots \alpha_1 \alpha_{n-2} \cdots \alpha_k, \text{ for } k \in \{2, \ldots, n-3\}.$$

Next we need to find f^3. This again can be done by looking at the 3rd projective in each of the minimal projective resolution of the simple Λ-modules.

$e_1\Lambda$



$e_m\Lambda$, for $2 \leq m \leq n-2$



$$\begin{array}{ccc} e_{n-1}\Lambda & \qquad\qquad & e_n\Lambda \\ & & \\ n-1 & & n \\ | & & | \\ 1 & & 1 \\ | & & | \\ n & & n-1 \end{array}$$

From the minimal projective resolution of each simple Λ-module we easily see that:

$$\Omega^3(S_2) \quad \cong \quad \begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ n & & n-1 \end{array}$$

For $3 \leq m \leq n-2$ we have,

$$\Omega^3(S_m) \cong \begin{array}{c} m-2 \\ | \\ m-3 \\ | \\ \vdots \\ | \\ 1 \\ | \\ n-2 \\ | \\ \vdots \\ | \\ m-1 \end{array}$$

$$\Omega^3(S_{n-1}) \cong \begin{array}{ccc} & 1 & \\ / & & \backslash \\ n & & n-2 \end{array}$$

$$\Omega^3(S_n) \cong \begin{array}{ccc} & 1 & \\ / & & \backslash \\ n-2 & & n-1 \end{array}$$

For $\Omega^3(S_1)$ we need more details. We have the map

$$\psi : e_{n-2}\Lambda \oplus e_{n-1}\Lambda \oplus e_n\Lambda \to \Omega(S_1)$$

given by:

$$e_{n-2}\lambda \mapsto \alpha_{n-2}e_{n-2}\lambda,$$

$$e_{n-1}\mu \mapsto \beta_0 e_{n-1}\mu,$$

$$e_n\xi \mapsto \gamma_0 e_n\xi$$

where $\lambda, \mu, \xi \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 6.1. $\Omega^2(S_1) = (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \gamma_1)\Lambda$.

Proof. On one hand, let $x \in \Omega^2(S_1)$. Then $x = (e_{n-2}\lambda, e_{n-1}\mu, e_n\xi)$ with $\lambda, \mu, \xi \in \Lambda$. Write $e_{n-2}\lambda = c_0e_{n-2} + c_1\alpha_{n-3} + c_2\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-2}\alpha_{n-3}\cdots\alpha_1\alpha_{n-2}$, $e_{n-1}\mu = c_0'e_{n-1} + c_1'\beta_1 + c_2'\beta_1\gamma_0$ and $e_n\xi = d_0e_n + d_1\gamma_1 + d_2\gamma_1\beta_0$ with $c_j, c_i', d_i \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$. Thus $\alpha_{n-2}(c_0e_{n-2} + c_1\alpha_{n-3} + c_2\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-2}\alpha_{n-3}\cdots\alpha_1\alpha_{n-2}) + \beta_0(c_0'e_{n-1}+c_1'\beta_1+c_2'\beta_1\gamma_0)+\gamma_0(d_0e_n+d_1\gamma_1+d_2\gamma_1\beta_0) = c_0\alpha_{n-2}+c_1\alpha_{n-2}\alpha_{n-3}+c_2\alpha_{n-2}\alpha_{n-3}\alpha_{n-4}+\cdots+c_{n-4}\alpha_{n-2}\alpha_{n-3}\cdots\alpha_2+c_{n-3}\alpha_{n-2}\alpha_{n-3}\cdots\alpha_1+c_0'\beta_0+c_1'\beta_0\beta_1 + d_0\gamma_0 + d_1\gamma_0\gamma_1 = c_0\alpha_{n-2} + c_1\alpha_{n-2}\alpha_{n-3} + c_2\alpha_{n-2}\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-4}\alpha_{n-2}\alpha_{n-3}\cdots\alpha_2+c_0'\beta_0+d_0\gamma_0+(c_{n-3}+c_1'+d_1)\beta_0\beta_1 = 0$. Hence $c_0 = c_1 = c_2 = \ldots = c_{n-4} = c_0' = d_0 = 0$ and $c_{n-3} + c_1' + d_1 = 0$. Write $c_1' = -(c_{n-3} + d_1)$. Therefore, $x = (c_{n-3}\alpha_{n-3}\cdots\alpha_1+c_{n-2}\alpha_{n-3}\cdots\alpha_1\alpha_{n-2}, -(c_{n-3}+d_1)\beta_1+c_2'\beta_1\gamma_0, d_1\gamma_1+d_2\gamma_1\beta_0) = (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)(c_{n-3}e_1+c_{n-2}\alpha_{n-2})+(0, -\beta_1, \gamma_1)(d_1e_1 - c_2'\gamma_0 + d_2\beta_0)$. So $x \in (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \gamma_1)\Lambda$.

Thus $\Omega^2(S_1) \subseteq (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \gamma_1)\Lambda$.

On the other hand, let $x \in (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \gamma_1)\Lambda$. So $x = (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\lambda + (0, -\beta_1, \gamma_1)\mu = (e_{n-2}\alpha_{n-3}\cdots\alpha_1\lambda, -e_{n-1}\beta_1\lambda - e_{n-1}\beta_1\mu, e_n\gamma_1\mu)$. By definition of ψ it is easy to see that $\psi(x) = 0$.

Thus $(\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \gamma_1)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)e_1\Lambda + (0, -\beta_1, \gamma_1)e_1\Lambda$. □

To find $\Omega^3(S_1)$, we need to find the kernel of the map:

$$\theta : e_1\Lambda \oplus e_1\Lambda \to \Omega^2(S_1)$$

given by

$$(e_1\lambda, e_1\mu) \mapsto (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)e_1\lambda + (0, -\beta_1, \gamma_1)e_1\mu$$

where $\lambda, \mu \in \Lambda$. Note that $\Omega^3(S_1) = \operatorname{Ker}\theta$.

Proposition 6.2. $\Omega^3(S_1) = (\alpha_{n-2}\alpha_{n-3}, 0)\Lambda+(0, \alpha_{n-2})\Lambda+(\beta_0, 0)\Lambda+(\gamma_0, -\gamma_0)\Lambda$.

Proof. On one hand, let $y \in \Omega^3(S_1)$. Then $y = (e_1\lambda, e_1\mu)$. Write $y = (c_0e_1 + c_1\alpha_{n-2} + c_2\alpha_{n-2}\alpha_{n-3} + \cdots + c_{n-2}\alpha_{n-2}\cdots\alpha_1 + c_{n-1}\beta_0 + c_n\gamma_0, c'_0e_1 + c'_1\alpha_{n-2} + c'_2\alpha_{n-2}\alpha_{n-3} + \cdots + c'_{n-2}\alpha_{n-2}\cdots\alpha_1 + c'_{n-1}\beta_0 + c'_n\gamma_0)$ with $c_j, c'_j \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus $(\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)(c_0e_1 + c_1\alpha_{n-2} + c_2\alpha_{n-2}\alpha_{n-3} + \cdots + c_{n-2}\alpha_{n-2}\cdots\alpha_1 + c_{n-1}\beta_0 + c_n\gamma_0) + (0, -\beta_1, \gamma_1)(c'_0e_1 + c'_1\alpha_{n-2} + c'_2\alpha_{n-2}\alpha_{n-3} + \cdots + c'_{n-2}\alpha_{n-2}\cdots\alpha_1 + c'_{n-1}\beta_0 + c'_n\gamma_0) = (c_0\alpha_{n-3}\cdots\alpha_1 + c_1\alpha_{n-3}\cdots\alpha_1\alpha_{n-2}, -c_0\beta_1 - c_n\beta_1\gamma_0, 0) + (0, -c'_0\beta_1 - c'_n\beta_1\gamma_0, c'_0\gamma_1 + c'_{n-1}\gamma_1\beta_0) = (c_0\alpha_{n-3}\cdots\alpha_1 + c_1\alpha_{n-3}\cdots\alpha_1\alpha_{n-2}, -(c_0+c'_0)\beta_1 - (c_n+c'_n)\beta_1\gamma_0, c'_0\gamma_1 + c'_{n-1}\gamma_1\beta_0)$. Thus $c_0 = c_1 = c'_{n-1} = c'_0 = 0, c_n + c'_n = 0$. Let $c'_n = -c_n$. Therefore, $y = (c_2\alpha_{n-2}\alpha_{n-3} + \cdots + c_{n-2}\alpha_{n-2}\cdots\alpha_1 + c_{n-1}\beta_0 + c_n\gamma_0, c'_1\alpha_{n-2} + c'_2\alpha_{n-2}\alpha_{n-3} + \cdots + c'_{n-2}\alpha_{n-2}\cdots\alpha_1 - c_n\gamma_0) = (\alpha_{n-2}\alpha_{n-3}(c_2e_{n-3} + \cdots + c_{n-2}\alpha_{n-4}\cdots\alpha_1) + \beta_0c_{n-1}e_{n-1} + \gamma_0c_ne_n, \alpha_{n-2}(c'_1e_{n-2} + c'_2\alpha_{n-3} + \cdots + c'_{n-2}\alpha_{n-3}\cdots\alpha_1) - \gamma_0c_ne_n)$. So

$$y = (\alpha_{n-2}\alpha_{n-3}\lambda + \beta_0\xi + \gamma_0\nu, \alpha_{n-2}\mu - \gamma_0\nu)$$

$$= (\alpha_{n-2}\alpha_{n-3}, 0)\lambda + (0, \alpha_{n-2})\mu + (\beta_0, 0)\xi + (\gamma_0, -\gamma_0)\nu),$$

where $\lambda, \mu, \xi, \nu \in \Lambda$. Hence $y \in (\alpha_{n-2}\alpha_{n-3}, 0)\Lambda + (0, \alpha_{n-2})\Lambda + (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda$.

Thus

$$\Omega^3(S_1) \subseteq (\alpha_{n-2}\alpha_{n-3}, 0)\Lambda + (0, \alpha_{n-2})\Lambda + (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda.$$

On the other hand, let $y \in (\alpha_{n-2}\alpha_{n-3}, 0)\Lambda + (0, \alpha_{n-2})\Lambda + (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda$. So $y = (\alpha_{n-2}\alpha_{n-3}, 0)\lambda + (0, \alpha_{n-2})\mu + (\beta_0, 0)\xi + (\gamma_0, -\gamma_0)\nu$, where $\lambda, \mu, \xi, \nu \in \Lambda$. Then $\theta(y) = (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\alpha_{n-2}\alpha_{n-3}\lambda + (0, -\beta_1, \gamma_1)\alpha_{n-2}\mu + (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\beta_0\xi + (\alpha_{n-3}\cdots\alpha_1, -\beta_1, 0)\gamma_0\nu - (0, -\beta_1, \gamma_1)\gamma_0\nu = 0$. So $y \in \Omega^3(S_1)$.

Thus $(\alpha_{n-2}\alpha_{n-3}, 0)\Lambda + (0, \alpha_{n-2})\Lambda + (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda \subseteq \Omega^3(S_1)$. Therefore, $\Omega^3(S_1) = (\alpha_{n-2}\alpha_{n-3}, 0)e_{n-3}\Lambda + (0, \alpha_{n-2})e_{n-2}\Lambda + (\beta_0, 0)e_{n-1}\Lambda + (\gamma_0, -\gamma_0)e_n\Lambda$. □

From the projective resolutions for simples we now know that the 3rd projective $Q^3 = (\Lambda e_1 \otimes e_{n-3}\Lambda) \oplus (\Lambda e_1 \otimes e_{n-2}\Lambda) \oplus (\Lambda e_1 \otimes e_{n-1}\Lambda) \oplus (\Lambda e_1 \otimes e_n\Lambda) \oplus (\Lambda e_2 \otimes e_1\Lambda) \oplus (\Lambda e_{n-1} \otimes e_1\Lambda) \oplus (\Lambda e_n \otimes e_1\Lambda) \oplus \bigoplus_{m=3}^{n-2}(\Lambda e_m \otimes e_{m-2}\Lambda)$.

We choose the set f^3 to consist of the following elements:

$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_{1,5}, f^3_{1,6}, f^3_{1,7}, f^3_{2,3}, f^3_{2,m}\}$, with $m \in \{4,\dots,n-2\}$ where

$$\begin{array}{lcllcl}
f^3_{1,1} & = & f^2_{1,2}\alpha_{n-2}\alpha_{n-3} & & = & \beta_0 f^2_{2,3}\alpha_{n-3} - \alpha_{n-2} f^2_{3,n-3}, \\
f^3_{1,2} & = & f^2_{1,1}\alpha_{n-2} & & = & \beta_0 f^2_{2,3} - \gamma_0 f^2_{2,4}, \\
f^3_{1,3} & = & f^2_{1,2}\beta_0 & & = & \beta_0 f^2_{2,5} - \alpha_{n-2}\cdots\alpha_2 f^2_{2,1}, \\
f^3_{1,4} & = & f^2_{1,1}\gamma_0 - f^2_{1,2}\gamma_0 & & = & \alpha_{n-2}\cdots\alpha_2 f^2_{2,2} - \gamma_0 f^2_{2,6}, \\
f^3_{1,5} & = & f^2_{2,1}\beta_1 - f^2_{2,2}\gamma_1 & & = & \alpha_1 f^2_{1,1}, \\
f^3_{1,6} & = & f^2_{2,5}\beta_1 - f^2_{2,3}\alpha_{n-3}\cdots\alpha_1 & & = & \beta_1 f^2_{1,2}, \\
f^3_{1,7} & = & f^2_{2,4}\alpha_{n-3}\cdots\alpha_1 - f^2_{2,6}\gamma_1 & & = & \gamma_1 f^2_{1,1} - \gamma_1 f^2_{1,2}, \\
f^3_{2,3} & = & f^2_{3,2}\alpha_1 & & = & \alpha_2 f^2_{2,1}\beta_1 - \alpha_2\alpha_1 f^2_{1,2}, \\
f^3_{2,m} & = & f^2_{3,m-1}\alpha_{m-2} & & = & \alpha_{m-1} f^2_{3,m-2} \text{ for } m \in \{4,\dots,n-2\}.
\end{array}$$

We know that $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$.

Find $\mathbf{Im\, d_2}$.

Since $d_2 : \mathrm{Hom}(Q^1,\Lambda) \to \mathrm{Hom}(Q^2,\Lambda)$, then $d_2 f \in \operatorname{Im} d_2$, where $f \in \mathrm{Hom}(Q^1,\Lambda)$ and $d_2 f = fA_2$. Here $Q^1 = (\Lambda e_1 \otimes_{\beta_0} e_{n-1}\Lambda) \oplus (\Lambda e_{n-1} \otimes_{\beta_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\gamma_0} e_n\Lambda) \oplus (\Lambda e_n \otimes_{\gamma_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\alpha_{n-2}} e_{n-2}\Lambda) \oplus \bigoplus_{l=1}^{n-3}(\Lambda e_{l+1} \otimes_{\alpha_l} e_l\Lambda)$. Let $f \in \mathrm{Hom}(Q^1,\Lambda)$ and so write

$$f(e_1 \otimes_{\beta_0} e_{n-1}) = c_1\beta_0, \qquad f(e_{n-1} \otimes_{\beta_1} e_1) = c_2\beta_1,$$

$$f(e_1 \otimes_{\gamma_0} e_n) = c_3\gamma_0, \qquad f(e_n \otimes_{\gamma_1} e_1) = c_4\gamma_1,$$

$$f(e_1 \otimes_{\alpha_{n-2}} e_{n-2}) = d_{n-2}\alpha_{n-2}$$

and

$$f(e_{l+1} \otimes_{\alpha_l} e_l) = d_l\alpha_l \text{ for } l \in \{1,\dots,n-3\},$$

where $c_1, c_2, c_3, c_4, d_{n-2}, d_l \in K$ for $l \in \{1,\dots,n-3\}$.

We have $Q^2 = (\Lambda e_1 \otimes_{f^2_{1,1}} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{f^2_{1,2}} e_1\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,1}} e_{n-1}\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,2}} e_n\Lambda) \oplus (\Lambda e_{n-1} \otimes_{f^2_{2,3}} e_{n-2}\Lambda) \oplus (\Lambda e_n \otimes_{f^2_{2,4}} e_{n-2}\Lambda) \oplus (\Lambda e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}\Lambda) \oplus (\Lambda e_n \otimes_{f^2_{2,6}} e_n\Lambda) \oplus \bigoplus_{k=2}^{n-3}(\Lambda e_{k+1} \otimes_{f^2_{3,k}} e_k\Lambda)$.

Now we find fA_2. We have

$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = f(e_1 \otimes_{\beta_0} e_{n-1})\beta_1 - f(e_1 \otimes_{\gamma_0} e_n)\gamma_1 + \beta_0 f(e_{n-1} \otimes_{\beta_1} e_1) - \gamma_0 f(e_n \otimes_{\gamma_1} e_1) = c_1\beta_0\beta_1 - c_3\gamma_0\gamma_1 + c_2\beta_0\beta_1 - c_4\gamma_0\gamma_1 = (c_1 - c_3 + c_2 - c_4)\beta_0\beta_1$.

Also $fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = f(e_1 \otimes_{\beta_0} e_{n-1})\beta_1 + \beta_0 f(e_{n-1} \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\alpha_{n-2}} e_{n-2})\alpha_{n-3}\cdots\alpha_1 - \alpha_{n-2} f(e_{n-2} \otimes_{\alpha_{n-3}} e_{n-3})\alpha_{n-4}\cdots\alpha_1 - \dots - \alpha_{n-2}\cdots\alpha_2 f(e_2 \otimes_{\alpha_1}$

$e_1) = c_1\beta_0\beta_1 + c_2\beta_0\beta_1 - d_{n-2}\alpha_{n-2}\cdots\alpha_1 - \ldots - d_1\alpha_{n-2}\cdots\alpha_2\alpha_1 = (c_1 + c_2 - d_{n-2} - \ldots - d_1)\beta_0\beta_1$.

And $fA_2(e_2 \otimes_{f^2_{2,1}} e_{n-1}) = f(e_2 \otimes_{\alpha_1} e_1)\beta_0 + \alpha_1 f(e_1 \otimes_{\beta_0} e_{n-1}) = d_1\alpha_1\beta_0 + c_1\alpha_1\beta_0 = (d_1 + c_1)\alpha_1\beta_0 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,2}} e_n) = f(e_2 \otimes_{\alpha_1} e_1)\gamma_0 + \alpha_1 f(e_1 \otimes_{\gamma_0} e_n) = d_1\alpha_1\gamma_0 + c_3\alpha_1\gamma_0 = (d_1 + c_3)\alpha_1\gamma_0 = 0$,

$fA_2(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2}) = f(e_{n-1} \otimes_{\beta_1} e_1)\alpha_{n-2} + \beta_1 f(e_1 \otimes_{\alpha_{n-2}} e_{n-2}) = c_2\beta_1\alpha_{n-2} + d_{n-2}\beta_1\alpha_{n-2} = 0$,

$fA_2(e_n \otimes_{f^2_{2,4}} e_{n-2}) = f(e_n \otimes_{\gamma_1} e_1\alpha_{n-2} + \gamma_1 f(e_1 \otimes_{\alpha_{n-2}} e_{n-2}) = c_4\gamma_1\alpha_{n-2} + d_{n-2}\gamma_1\alpha_{n-2} = 0$,

$fA_2(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) = f(e_{n-1} \otimes_{\beta_1} e_1)\beta_0 + \beta_1 f(e_1 \otimes_{\beta_0} e_{n-1}) = c_2\beta_1\beta_0 + c_1\beta_1\beta_0 = (c_2 + c_1)\beta_1\beta_0 = 0$,

$fA_2(e_n \otimes_{f^2_{2,6}} e_n) = f(e_n \otimes_{\gamma_1} e_1)\gamma_0 + \gamma_1 f(e_1 \otimes_{\gamma_0} e_n) = c_4\gamma_1\gamma_0 + c_3\gamma_1\gamma_0 = (c_4 + c_3)\gamma_1\gamma_0 = 0$.

Finally, for $k = 2, \ldots, n-3$, we have $fA_2(e_{k+1} \otimes_{f^2_{3,k}} e_k) = f(e_{k+1} \otimes_{\alpha_k} e_k)\alpha_{k-1}\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k + \alpha_k f(e_k \otimes_{\alpha_{k-1}} e_{k-1})\alpha_{k-2}\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k + \ldots + \alpha_k\cdots\alpha_1\alpha_{n-2}\cdots\alpha_{k-1} f(e_{k+1} \otimes_{\alpha_k} e_k) = d_k\alpha_k\alpha_{k-1}\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k + d_{k-1}\alpha_k\alpha_{k-1}\alpha_{k-2}\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k + \ldots + d_k\alpha_k\cdots\alpha_1\alpha_{n-2}\cdots\alpha_{k-1}\alpha_k = (d_k + d_{k-1} + \cdots + d_k)\alpha_k\alpha_{k-1}\cdots\alpha_1\alpha_{n-2}\cdots\alpha_k = 0$.

Thus f is given by

$$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = (c_1 - c_3 + c_2 - c_4)\beta_0\beta_1 = c'\beta_0\beta_1 \text{ and}$$

$$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = (c_1 + c_2 - d_{n-2} - \ldots - d_1)\beta_0\beta_1 = c''\beta_0\beta_1$$

for some $c', c'' \in K$

$$fA_2(\mathfrak{o}(f^2_j) \otimes \mathfrak{t}(f^2_j)) = 0$$

for all $f^2_j \neq f^2_{1,1}, f^2_{1,2}$. So $dim\,\mathrm{Im}\,d_2 = 2$.

Find $\mathbf{Ker}\,\mathbf{d_3}$.

We have $d_3 : \mathrm{Hom}(Q^2, \Lambda) \to \mathrm{Hom}(Q^3, \Lambda)$. Let $h \in \mathrm{Ker}\,d_3$, so $h \in \mathrm{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_1 e_1 + c_2\beta_0\beta_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4\beta_0\beta_1,$$

$$h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j \in \{1, \dots, 4\},$$

$$h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) = c_5 e_{n-1},$$

$$h(e_n \otimes_{f^2_{2,6}} e_n) = c_6 e_n \text{ and}$$

$$h(\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k})) = d_k \alpha_k, \text{ for } k \in \{2, \dots, n-3\}$$

for some $c_1, \dots, c_6, d_k \in K$. Hence $dim\,\mathrm{Hom}(Q^2, \Lambda) = n - 4 + 6 = n + 2$.

Then $hA_3(e_1 \otimes_{f^3_{1,1}} e_{n-3}) = h(e_1 \otimes_{f^2_{1,2}} e_1)\alpha_{n-2}\alpha_{n-3} - \beta_0 h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2})\alpha_{n-3}$ $+\alpha_{n-2}h(e_{n-2} \otimes_{f^2_{3,n-3}} e_{n-3}) = (c_3 e_1 + c_4\beta_0\beta_1)\alpha_{n-2}\alpha_{n-3} - 0 + d_{n-3}\alpha_{n-2}\alpha_{n-3} =$ $(c_3 + d_{n-3})\alpha_{n-2}\alpha_{n-3}$. As $h \in \mathrm{Ker}\, d_3$ we have $c_3 + d_{n-3} = 0$.

$hA_3(e_1 \otimes_{f^3_{1,2}} e_{n-2}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\alpha_{n-2} - \beta_0 h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2}) + \gamma_0 h(e_n \otimes_{f^2_{2,4}} e_{n-2}) = (c_1 e_1 + c_2\beta_0\beta_1)\alpha_{n-2} = c_1\alpha_{n-2}$. As $h \in \mathrm{Ker}\, d_3$, $c_1 = 0$.

Next, $hA_3(e_1 \otimes_{f^3_{1,3}} e_{n-1}) = h(e_1 \otimes_{f^2_{1,2}} e_1)\beta_0 - \beta_0 h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) + \alpha_{n-2}\cdots\alpha_2 h(e_1 \otimes_{f^2_{2,1}} e_{n-1}) = (c_3 e_1 + c_4\beta_0\beta_1)\beta_0 - c_5\beta_0 e_{n-1} = (c_3 - c_5)\beta_0$. So we have $c_3 - c_5 = 0$ and hence $c_3 = c_5$.

$hA_3(e_1 \otimes_{f^3_{1,4}} e_n) = h(e_1 \otimes_{f^2_{1,1}} e_1)\gamma_0 - h(e_1 \otimes_{f^2_{1,2}} e_1)\gamma_0 - \alpha_{n-2}\cdots\alpha_2 h(e_2 \otimes_{f^2_{2,2}} e_n) + \gamma_0 h(e_n \otimes_{f^2_{2,6}} e_n) = (c_1 e_1 + c_2\beta_0\beta_1)\gamma_0 - (c_3 e_1 + c_4\beta_0\beta_1)\gamma_0 - 0 + c_6\gamma_0 =$ $(c_1 - c_3 + c_6)\gamma_0$. Therefore $c_3 = c_6$ as $c_1 = 0$.

$hA_3(e_2 \otimes_{f^3_{1,5}} e_1) = h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_1 - h(e_2 \otimes_{f^2_{2,2}} e_n)\gamma_1 - \alpha_1 h(e_1 \otimes_{f^2_{1,1}} e_1) =$ $-c_1\alpha_1 = 0$. This gives no new information since we have already $c_1 = 0$.

$hA_3(e_{n-1} \otimes_{f^3_{1,6}} e_1) = h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1})\beta_1 - h(e_{n-1} \otimes_{f^2_{2,3}} e_{n-2})\alpha_{n-3}\cdots\alpha_1 -$ $\beta_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = c_5 e_{n-1}\beta_1 - \beta_1(c_3 e_1 + c_4\beta_0\beta_1) = c_5\beta_1 - c_3\beta_1 = (c_5 - c_3)\beta_1$. Thus again we have $c_5 = c_3$.

$hA_3(e_n \otimes_{f^3_{1,7}} e_1) = h(e_n \otimes_{f^2_{2,4}} e_{n-2})\alpha_{n-3}\cdots\alpha_1 - h(e_1 \otimes_{f^2_{2,6}} e_n)\gamma_1 - \gamma_1 h(e_1 \otimes_{f^2_{1,1}} e_1) + \gamma_1(e_1 \otimes_{f^2_{1,2}} e_1) = -c_6 e_n\gamma_1 - \gamma_1(c_1 e_1 + c_2\beta_0\beta_1) + \gamma_1(c_3 e_1 + c_4\beta_0\beta_1) =$ $-c_6\gamma_1 - c_1\gamma_1 + c_3\gamma_1 = (c_3 - c_1 - c_6)\gamma_1$. So again this gives no new information.

$hA_3(e_3 \otimes_{f^3_{2,3}} e_1) = h(e_3 \otimes_{f^2_{3,2}} e_2)\alpha_1 - \alpha_2 h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_1 + \alpha_2\alpha_2 h(e_1 \otimes_{f^2_{1,2}} e_1) = d_2\alpha_2\alpha_1 - 0 + \alpha_2\alpha_1(c_3 e_1 + c_4\beta_0\beta_1) = (d_2 + c_3)\alpha_2\alpha_1$. So we have $d_2 + c_3 =$ 0.

Finally for $m \in \{4, \dots, n-2\}$, we have $hA_3(e_m \otimes_{f^3_{2,m,}} e_{m-2}) = h(e_m \otimes_{f^2_{3,m-1}} e_{m-1})\alpha_{m-2} - \alpha_{m-1}h(e_{m-1} \otimes_{f^2_{3,m-2}} e_{m-2}) = d_{m-1}\alpha_{m-1}\alpha_{m-2} - d_{m-2}\alpha_{m-1}\alpha_{m-2}$ $= (d_{m-1} - d_{m-2})\alpha_{m-1}\alpha_{m-2}$. Then $d_{m-1} - d_{m-2} = 0$ and so $d_{m-1} = d_{m-2}$. Hence $d_{n-3} = d_{n-4} = \ldots = d_3 = d_2$.

Thus h is given by

$$\begin{aligned}
h(e_1 \otimes_{f^2_{1,1}} e_1) &= c_2\beta_0\beta_1,\\
h(e_1 \otimes_{f^2_{1,2}} e_1) &= c_3 e_1 + c_4\beta_0\beta_1,\\
h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) &= 0, \text{ for } j \in \{1,\dots,4\},\\
h(e_{n-1} \otimes_{f^2_{2,5}} e_{n-1}) &= c_3 e_{n-1}\\
h(e_n \otimes_{f^2_{2,6}} e_n) &= c_3 e_n \text{ and}\\
h(\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k})) &= -c_3\alpha_k, \text{ for } k \in \{2,\dots,n-3\}
\end{aligned}$$

for some $c_2, c_3, c_4 \in K$. So we have $\dim \operatorname{Ker} d_3 = 3$.

Therefore $\dim \mathrm{HH}^2(\Lambda) = 3 - 2 = 1$. Hence $\dim \mathrm{HH}^2(\Lambda) = 1$.

Theorem 6.3. *For $\Lambda = \Lambda(D_n, 1, 2)$ we have $\dim \mathrm{HH}^2(\Lambda) = 1$.*

6.4. *A basis for* $\mathrm{HH}^2(\Lambda)$.

Now we will find a non-zero element of $\mathrm{HH}^2(\Lambda)$. An element of $Ker\, d_3$ represent a non-zero element of $\mathrm{HH}^2(\Lambda)$ if it is not an element of $\operatorname{Im} d_2$. Let η be the map in $\operatorname{Ker} d_3$ given by

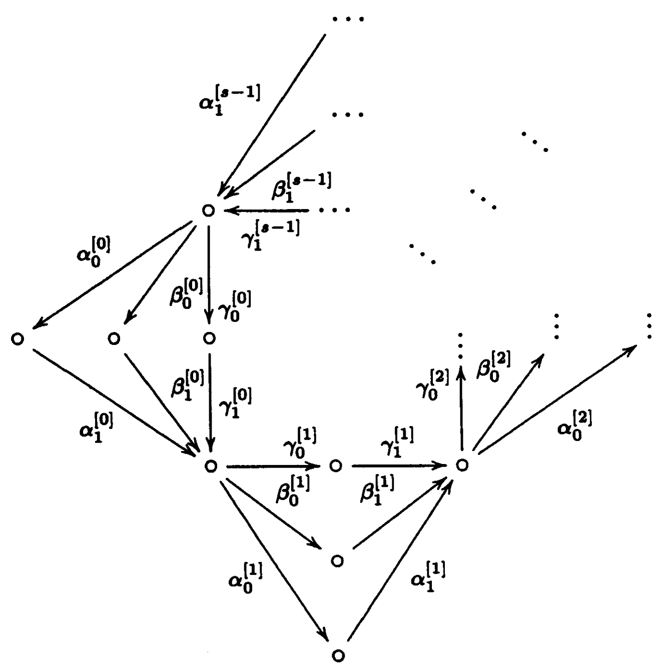
$$\begin{array}{rcl}
e_1 \otimes_{f^2_{1,2}} e_1 & \mapsto & e_1,\\
e_{n-1} \otimes_{f^2_{2,5}} e_{n-1} & \mapsto & e_{n-1},\\
e_n \otimes_{f^2_{2,6}} e_n & \mapsto & e_n,\\
\mathfrak{o}(f^2_{3,k}) \otimes_{f^2_{3,k}} \mathfrak{t}(f^2_{3,k}) & \mapsto & -\alpha_k, \text{ for } k \in \{2,\dots,n-3\},\\
\text{else} & \mapsto & 0.
\end{array}$$

Clearly, η is a non-zero map. Suppose for contradiction that $\eta \in \operatorname{Im} d_2$. Then by the definition of η, we have $\eta(e_n \otimes_{f^2_{2,6}} e_n) = e_n$. On the other hand, $\eta(e_n \otimes_{f^2_{2,6}} e_n) = fA_2(e_n \otimes_{f^2_{2,6}} e_n)$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. So $\eta(e_n \otimes_{f^2_{2,6}} e_n) = 0$. So we have a contradiction as we get $e_n = 0$. Therefore $\eta \notin \operatorname{Im} d_2$.

Thus $\eta + \operatorname{Im} d_2$ is a non-zero element of $\mathrm{HH}^2(\Lambda)$ and so the set $\{\eta + \operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$.

7. $\Lambda(D_4, s, 3)$

It is known for $\Lambda = \Lambda(D_4, s, 3)$ with $s \geq 2$ that $\mathrm{HH}^2(\Lambda) = 0$ from Theorem 5.11. The aim of this chapter is to find $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \Lambda(D_4, s, 3)$ with $s = 1$. Following Asashiba in [2], we write α_0 for α_2. We start by recapping the definition of $\Lambda(D_4, s, 3)$ with $s \geq 1$. From [2] and 2.5 in Chapter 2 the algebra $\Lambda(D_4, s, 3)$ is given by the quiver $\mathcal{Q}(D_4, s)$:



with relations $R(D_4, s, 3)$:
(i) $\alpha_0^{[i]}\alpha_1^{[i]} = \beta_0^{[i]}\beta_1^{[i]} = \gamma_0^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,
(ii) for all $i \in \{0, \ldots, s-2\}$,

$$\alpha_1^{[i]}\beta_0^{[i+1]} = 0, \qquad \alpha_1^{[i]}\gamma_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\alpha_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\alpha_0^{[i+1]} = 0,$$
$$\beta_1^{[i]}\gamma_0^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_0^{[i+1]} = 0;$$

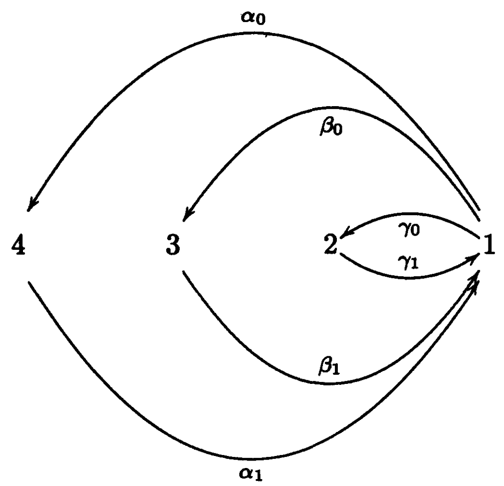
and

$$\alpha_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \alpha_1^{[s-1]}\gamma_0^{[0]} = 0,$$
$$\beta_1^{[s-1]}\alpha_0^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_0^{[0]} = 0,$$

$$\gamma_1^{[s-1]}\beta_0^{[0]} = 0, \qquad \gamma_1^{[s-1]}\gamma_0^{[0]} = 0;$$

(iii) paths of length 3 are equal to 0.

Consider $s = 1$. The algebra $\Lambda(D_4, 1, 3)$ is given by the quiver $\mathcal{Q}(D_4, 1)$. Note that we changed the notation of the vertices from that in Chapter 4.



with the minimal set of relations given in 5.17 as follows.
$f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}\}$ where

$$f^2_{1,1} = \beta_0\beta_1 - \gamma_0\gamma_1, \qquad f^2_{1,2} = \beta_0\beta_1 - \alpha_0\alpha_1,$$

$$f^2_{2,1} = \beta_1\alpha_0, \qquad f^2_{2,2} = \alpha_1\gamma_0,$$

$$f^2_{2,3} = \gamma_1\beta_0,$$

$$f^2_{2,4} = \gamma_1\gamma_0, \qquad f^2_{2,5} = \beta_1\beta_0 \text{ and}$$

$$f^2_{2,6} = \alpha_1\alpha_0.$$

Next we need to find f^3. The indecomposable projective Λ-modules are:

$$e_1\Lambda \qquad\qquad\qquad e_2\Lambda \qquad\qquad e_3\Lambda \qquad\qquad e_4\Lambda$$

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
4 \arrow[dr, dash] & 3 \arrow[d, dash] & 2 \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 4
\end{tikzcd}
\qquad
\begin{tikzcd}
3 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 2
\end{tikzcd}
\qquad
\begin{tikzcd}
4 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 3
\end{tikzcd}$$

From the minimal projective resolution of each simple Λ-module we have

$$\Omega^3(S_2) \cong \begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
2 & & 4
\end{tikzcd}$$

$$\Omega^3(S_3) \cong \begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
2 & & 3
\end{tikzcd}$$

$$\Omega^3(S_4) \cong \begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
3 & & 4
\end{tikzcd}$$

For $\Omega^3(S_1)$ we need more details. We have the map

$$\psi : e_2\Lambda \oplus e_3\Lambda \oplus e_4\Lambda \to \Omega(S_1)$$

given by:

$$e_2\lambda \mapsto \gamma_0 e_2\lambda,$$

$$e_3\mu \mapsto \beta_0 e_3\mu,$$

$$e_4\xi \mapsto \alpha_0 e_4\xi$$

where $\lambda, \mu, \xi \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 7.1. $\Omega^2(S_1) = (\gamma_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \alpha_1)\Lambda$.

Proof. On one hand, let $x \in \Omega^2(S_1)$. Then $x = (e_2\lambda, e_3\mu, e_4\xi)$ where $\lambda, \mu, \xi \in \Lambda$. Write $e_2\lambda = c_0e_2 + c_1\gamma_1 + c_2\gamma_1\alpha_0$, $e_3\mu = c_0'e_3 + c_1'\beta_1 + c_2'\beta_1\gamma_0$ and $e_4\xi = d_0e_4 + d_1\alpha_1 + d_2\alpha_1\beta_0$ with $c_i, c_i', d_i \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$. Thus, $\gamma_0(c_0e_2 + c_1\gamma_1 + c_2\gamma_1\alpha_0) + \beta_0(c_0'e_3 + c_1'\beta_1 + c_2'\beta_1\gamma_0) + \alpha_0(d_0e_4 + d_1\alpha_1 + d_2\alpha_1\beta_0) = c_0\gamma_0 + c_1\gamma_0\gamma_1 + c_0'\beta_0 + c_1'\beta_0\beta_1 + d_0\alpha_0 + d_1\alpha_0\alpha_1 = c_0\gamma_0 + c_0'\beta_0 + d_0\alpha_0 + (c_1 + c_1' + d_1)\beta_0\beta_1 = 0$. Thus $c_0 = c_0' = d_0 = 0$ and $c_1 + c_1' + d_1 = 0$. Let $c_1' = -(c_1 + d_1)$. Therefore, $x = (c_1\gamma_1 + c_2\gamma_1\alpha_0, -c_1\beta_1 - d_1\beta_1 + c_2'\beta_1\gamma_0, d_1\alpha_1 + d_2\alpha_1\beta_0) = c_1(\gamma_1, -\beta_1, 0) + d_1(0, -\beta_1, \alpha_1) + (c_2\gamma_1\alpha_0, c_2'\beta_1\gamma_0, d_2\alpha_1\beta_0) = (\gamma_1, -\beta_1, 0)(c_1e_1 + c_2\alpha_0 - c_2'\gamma_0) + (0, -\beta_1, \alpha_1)(d_1e_1 + d_2\beta_0) = (\gamma_1, -\beta_1, 0)\lambda + (0, -\beta_1, \alpha_1)\mu$, where $\lambda, \mu \in \Lambda$. So $x \in (\gamma_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \alpha_1)\Lambda$.

Thus $\Omega^2(S_1) \subseteq (\gamma_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \alpha_1)\Lambda$.

On the other hand, let $x \in (\gamma_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \alpha_1)\Lambda$. So $x = (\gamma_1, -\beta_1, 0)\lambda + (0, -\beta_1, \alpha_1)\mu = (e_2\gamma_1\lambda, -e_3\beta_1\lambda - e_3\beta_1\mu, e_4\alpha_1\mu)$. Then it is easy to see from the definition of ψ that $\psi(x) = 0$.

Thus $(\gamma_1, -\beta_1, 0)\Lambda + (0, -\beta_1, \alpha_1)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\gamma_1, -\beta_1, 0)e_1\Lambda + (0, -\beta_1, \alpha_1)e_1\Lambda$. □

Next $\Omega^3(S_1)$ is the kernel of the map

$$\theta : e_1\Lambda \oplus e_1\Lambda \to \Omega^2(S_1)$$

given by

$$(e_1\lambda, e_1\mu) \mapsto (\gamma_1, -\beta_1, 0)e_1\lambda + (0, -\beta_1, \alpha_1)e_1\mu$$

where $\lambda, \mu \in \Lambda$.

Proposition 7.2. $\Omega^3(S_1) = (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda + (0, \alpha_0)\Lambda$.

Proof. Firstly, let $y \in \Omega^3(S_1)$. Then $y = (e_1\lambda, e_1\mu) \in e_1\Lambda \oplus e_1\Lambda$ with $\lambda, \mu \in \Lambda$. Write $y = (c_0e_1+c_1\alpha_0+c_2\alpha_0\alpha_1+c_3\beta_0+c_4\gamma_0, c'_0e_1+c'_1\alpha_0+c'_2\alpha_0\alpha_1+c'_3\beta_0+c'_4\gamma_0)$ with $c_i, c'_i \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus, $(\gamma_1, -\beta_1, 0)(c_0e_1 + c_1\alpha_0 + c_2\alpha_0\alpha_1 + c_3\beta_0 + c_4\gamma_0) + (0, -\beta_1, \alpha_1)(c'_0e_1 + c'_1\alpha_0 + c'_2\alpha_0\alpha_1+c'_3\beta_0+c'_4\gamma_0) = c_0(\gamma_1, -\beta_1, 0)+(c_1\gamma_1\alpha_0, -c_4\beta_1\gamma_0, 0)+c'_0(0, -\beta_1, \alpha_1)+(0, -c'_4\beta_1\gamma_0, c'_3\alpha_1\beta_0) = (c_0\gamma_1 + c_1\gamma_1\alpha_0, -(c_4 + c'_4)\beta_1\gamma_0 - (c_0 + c'_0)\beta_1, c'_0\alpha_1 + c'_3\alpha_1\beta_0) = 0$. Thus $c_0 = c_1 = c'_0 = c'_3 = 0, c_4 + c'_4 = 0$ and $c_0 + c'_0 = 0$. Let $c'_4 = -c_4$. Therefore, $y = (c_2\alpha_0\alpha_1 + c_3\beta_0 + c_4\gamma_0, c'_1\alpha_0 + c'_2\alpha_0\alpha_1 - c_4\gamma_0) = (\beta_0(c_3e_3 + c_2\beta_1) + c_4\gamma_0, \alpha_0(c'_1e_4 + c'_2\alpha_1) - c_4\gamma_0) = (\beta_0\lambda+\gamma_0\mu, \alpha_0\xi-\gamma_0\mu) = (\beta_0, 0)\lambda + (\gamma_0, -\gamma_0)\mu + (0, \alpha_0)\xi$, where $\lambda, \mu, \xi \in \Lambda$. So $y \in (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda + (0, \alpha_0)\Lambda$.

Thus $\Omega^3(S_1) \subseteq (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda + (0, \alpha_0)\Lambda$.

Conversely, let $y \in (\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda + (0, \alpha_0)\Lambda$. So $y = (\beta_0, 0)\lambda + (\gamma_0, -\gamma_0)\mu+(0, \alpha_0)\xi = (e_1\beta_0, 0)\lambda+(e_1\gamma_0, -e_1\gamma_0)\mu+(0, e_1\alpha_0)\xi$, where $\lambda, \mu, \xi \in \Lambda$. Then

$\theta(y) = (\gamma_1, -\beta_1, 0)\beta_0\lambda+(\gamma_1, -\beta_1, 0)\gamma_0\mu-(0, -\beta_1, \alpha_1)\gamma_0\mu+(0, -\beta_1, \alpha_1)\alpha_0\xi = 0$. So $y \in \Omega^3(S_1)$.

Thus $(\beta_0, 0)\Lambda + (\gamma_0, -\gamma_0)\Lambda + (0, \alpha_0)\Lambda \subseteq \Omega^3(S_1)$.

Therefore, $\Omega^3(S_1) = (\beta_0, 0)e_3\Lambda + (\gamma_0, -\gamma_0)e_2\Lambda + (0, \alpha_0)e_4\Lambda$. □

From the projective resolution for simples we now know that the 3rd projective $Q^3 = (\Lambda e_1 \otimes e_2\Lambda) \oplus (\Lambda e_1 \otimes e_3\Lambda) \oplus (\Lambda e_1 \otimes e_4\Lambda) \oplus (\Lambda e_2 \otimes e_1\Lambda) \oplus (\Lambda e_3 \otimes e_1\Lambda) \oplus (\Lambda e_4 \otimes e_1\Lambda)$.

We choose the set f^3 to consist of the following elements:

$$\begin{array}{lllll}
f^3_{1,1} & = & f^2_{1,1}\gamma_0 - f^2_{1,2}\gamma_0 & = & \alpha_0 f^2_{2,2} - \gamma_0 f^2_{2,4}, \\
f^3_{1,2} & = & f^2_{1,1}\beta_0 & = & \beta_0 f^2_{2,5} - \gamma_0 f^2_{2,3}, \\
f^3_{1,3} & = & f^2_{1,2}\alpha_0 & = & \beta_0 f^2_{2,1} - \alpha_0 f^2_{2,6}, \\
f^3_{1,4} & = & f^2_{2,3}\beta_1 - f^2_{2,4}\gamma_1 & = & \gamma_1 f^2_{1,1}, \\
f^3_{1,5} & = & f^2_{2,5}\beta_1 - f^2_{2,1}\alpha_1 & = & \beta_1 f^2_{1,2}, \\
f^3_{1,6} & = & f^2_{2,6}\alpha_1 - f^2_{2,2}\gamma_1 & = & \alpha_1 f^2_{1,1} - \alpha_1 f^2_{1,2},
\end{array}$$

Find $\mathbf{Im\, d_2}$.

We know that $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Since $d_2 : \operatorname{Hom}(Q^1, \Lambda) \to \operatorname{Hom}(Q^2, \Lambda)$, then $d_2 f \in \operatorname{Im} d_2$, where $f \in \operatorname{Hom}(Q^1, \Lambda)$ and $d_2 f = fA_2$. Here $Q^1 = (\Lambda e_1 \otimes_{\gamma_0} e_2\Lambda) \oplus (\Lambda e_2 \otimes_{\gamma_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\beta_0} e_3\Lambda) \oplus (\Lambda e_3 \otimes_{\beta_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\alpha_0} e_4\Lambda) \oplus (\Lambda e_4 \otimes_{\alpha_1} e_l\Lambda)$. Let $f \in \operatorname{Hom}(Q_1, \Lambda)$ and write

$$f(e_1 \otimes_{\gamma_0} e_2) = c_1\gamma_0, \qquad f(e_2 \otimes_{\gamma_1} e_1) = c_2\gamma_1,$$
$$f(e_1 \otimes_{\beta_0} e_3) = c_3\beta_0, \qquad f(e_3 \otimes_{\beta_1} e_1) = c_4\beta_1,$$
$$f(e_1 \otimes_{\alpha_0} e_4) = c_5\alpha_0$$

and

$$f(e_4 \otimes_{\alpha_1} e_1) = c_6\alpha_1,$$

where $c_1, \ldots, c_6 \in K$.

We have $Q^2 = (\Lambda e_1 \otimes_{f^2_{1,1}} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{f^2_{1,2}} e_1\Lambda) \oplus (\Lambda e_3 \otimes_{f^2_{2,1}} e_4\Lambda) \oplus (\Lambda e_4 \otimes_{f^2_{2,2}} e_2\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,3}} e_3\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,4}} e_2\Lambda) \oplus (\Lambda e_3 \otimes_{f^2_{2,5}} e_3\Lambda) \oplus (\Lambda e_4 \otimes_{f^2_{2,6}} e_4\Lambda)$.

Now we find fA_2. We have

$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = f(e_1 \otimes_{\beta_0} e_3)\beta_1 - f(e_1 \otimes_{\gamma_0} e_2)\gamma_1 + \beta_0 f(e_3 \otimes_{\beta_1} e_1) - \gamma_0 f(e_2 \otimes_{\gamma_1} e_1) = c_3\beta_0\beta_1 - c_1\gamma_0\gamma_1 + c_4\beta_0\beta_1 - c_2\gamma_0\gamma_1 = (c_3 - c_1 + c_4 - c_2)\beta_0\beta_1$.

Also $fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = f(e_1 \otimes_{\beta_0} e_3)\beta_1 + \beta_0 f(e_3 \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\alpha_0} e_4) - \alpha_0 f(e_4 \otimes_{\alpha_1} e_1) = c_3\beta_0\beta_1 + c_4\beta_0\beta_1 - c_5\alpha_0\alpha_1 - c_6\alpha_0\alpha_1 = (c_3 + c_4 - c_5 - c_6)\beta_0\beta_1$.

$fA_2(e_3 \otimes_{f^2_{2,1}} e_4) = f(e_3 \otimes_{\beta_1} e_1)\alpha_0 + \beta_1 f(e_1 \otimes_{\alpha_0} e_4) = c_4\beta_1\alpha_0 + c_5\beta_1\alpha_0 = 0$,

$fA_2(e_4 \otimes_{f^2_{2,2}} e_2) = f(e_4 \otimes_{\alpha_1} e_1)\gamma_0 + \alpha_1 f(e_1 \otimes_{\gamma_0} e_2) = c_6\alpha_1\gamma_0 + c_1\alpha_1\gamma_0 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,3}} e_3) = f(e_2 \otimes_{\gamma_1} e_1)\beta_0 + \gamma_1 f(e_1 \otimes_{\beta_0} e_3) = c_2\gamma_1\beta_0 + c_3\gamma_1\beta_0 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,4}} e_2) = f(e_2 \otimes_{\gamma_1} e_1)\gamma_0 + \gamma_1 f(e_1 \otimes_{\gamma_0} e_2) = c_2\gamma_1\gamma_0 + c_1\gamma_1\gamma_0 = 0$,

$fA_2(e_3 \otimes_{f^2_{2,5}} e_3) = f(e_3 \otimes_{\beta_1} e_1)\beta_0 + \beta_1 f(e_1 \otimes_{\beta_0} e_3) = c_4\beta_1\beta_0 + c_3\beta_1\beta_0 = 0$,

$fA_2(e_4 \otimes_{f^2_{2,6}} e_4) = f(e_4 \otimes_{\alpha_1} e_1)\alpha_0 + \alpha_1 f(e_1 \otimes_{\alpha_0} e_4) = c_6\alpha_1\alpha_0 + c_5\alpha_1\alpha_0 = 0$,

Hence f is given by

$$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = (c_3 - c_1 + c_4 - c_2)\beta_0\beta_1 = c'\beta_0\beta_1,$$
$$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = (c_3 + c_4 - c_5 - c_6)\beta_0\beta_1 = c''\beta_0\beta_1 \text{ and}$$
$$fA_2(\mathfrak{o}(f^2_{2,j}) \otimes \mathfrak{t}(f^2_{2,j})) = 0, \text{where } j \in \{1, 2, 3, 4, 5, 6\}.$$

So $dim\, \operatorname{Im} d_2 = 2$.

Find $\operatorname{Ker} d_3$. We have $d_3 : \operatorname{Hom}(Q^2, \Lambda) \to \operatorname{Hom}(Q^3, \Lambda)$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f_{1,1}^2} e_1) = c_1e_1 + c_2\beta_0\beta_1,$$
$$h(e_1 \otimes_{f_{1,2}^2} e_1) = c_3e_1 + c_4\beta_0\beta_1,$$
$$h(\mathfrak{o}(f_{2,j}^2) \otimes_{f_{2,j}^2} \mathfrak{t}(f_{2,j}^2)) = 0, \text{ for } j \in \{1,2,3\},$$
$$h(e_2 \otimes_{f_{2,4}^2} e_2) = c_5e_2,$$
$$h(e_3 \otimes_{f_{2,5}^2} e_3) = c_6e_3 \text{ and}$$
$$h(e_4 \otimes_{f_{2,6}^2} e_4) = c_7e_4,$$

for some $c_1, \ldots, c_7 \in K$. Hence $dim\,\mathrm{Hom}(Q^2, \Lambda) = 7$.

Then $hA_3(e_1 \otimes_{f_{1,1}^3} e_2) = h(e_1 \otimes_{f_{1,1}^2} e_1)\gamma_0 - h(e_1 \otimes_{f_{1,2}^2} e_1)\gamma_0 - \alpha_0 h(e_4 \otimes_{f_{2,2}^2} e_2) + \gamma_0 h(e_2 \otimes_{f_{2,4}^2} e_2) = (c_1e_1 + c_2\beta_0\beta_1)\gamma_0 - (c_3e_1 + c_4\beta_0\beta_1)\gamma_0 + c_5\gamma_0 e_2 = c_1\gamma_0 - c_3\gamma_0 + c_5\gamma_0 = (c_1 - c_3 + c_5)\gamma_0$. As $h \in \mathrm{Ker}\, d_3$, $c_1 - c_3 + c_5 = 0$.

$hA_3(e_1 \otimes_{f_{1,2}^3} e_3) = h(e_1 \otimes_{f_{1,1}^2} e_1)\beta_0 - \beta_0 h(e_3 \otimes_{f_{2,5}^2} e_3) + \gamma_0 h(e_2 \otimes_{f_{2,3}^2} e_3) = (c_1e_1 + c_2\beta_0\beta_1)\beta_0 - c_6\beta_0 e_3 = (c_1 - c_6)\beta_0$. Thus we have $c_1 - c_6 = 0$ and so $c_1 = c_6$.

$hA_3(e_1 \otimes_{f_{1,3}^3} e_4) = h(e_1 \otimes_{f_{1,2}^2} e_1)\alpha_0 - \beta_0 h(e_3 \otimes_{f_{2,1}^2} e_4) + \alpha_0 h(e_4 \otimes_{f_{2,6}^2} e_4) = (c_3e_1 + c_4\beta_0\beta_1)\alpha_0 + c_7\alpha_0 e_4 = c_3\alpha_0 + c_7\alpha_0 = (c_3 + c_7)\alpha_0$. So we have $c_3 = -c_7$.

$hA_3(e_2 \otimes_{f_{1,4}^3} e_1) = h(e_2 \otimes_{f_{2,3}^2} e_3)\beta_1 - h(e_2 \otimes_{f_{2,4}^2} e_2)\gamma_1 - \gamma_1 h(e_1 \otimes_{f_{1,1}^2} e_1) = -c_5e_2\gamma_1 - \gamma_1(c_1e_1 + c_2\beta_0\beta_1) = -c_5\gamma_1 - c_1\gamma_1 = (-c_5 - c_1)\gamma_1$. Therefore $c_1 = -c_5$. Thus $c_3 = 0$ as we had $c_1 - c_3 + c_5 = 0$ above. Also $c_7 = 0$ since we already had $c_3 = -c_7$.

$hA_3(e_3 \otimes_{f_{1,5}^3} e_1) = h(e_3 \otimes_{f_{2,5}^2} e_3)\beta_1 - h(e_3 \otimes_{f_{2,1}^2} e_4)\alpha_1 - \beta_1 h(e_1 \otimes_{f_{1,2}^2} e_1) = c_6e_3\beta_1 - \beta_1(c_3e_1 + c_4\beta_0\beta_1) = (c_6 - c_3)\beta_1$. Hence $c_6 = 0$ since we had $c_3 = 0$ already. Moreover, since $c_1 = c_6$, we know $c_1 = 0$.

$hA_3(e_4 \otimes_{f_{1,6}^3} e_1) = h(e_4 \otimes_{f_{2,6}^2} e_4)\alpha_1 - h(e_4 \otimes_{f_{2,2}^2} e_2)\gamma_1 - \alpha_1 h(e_1 \otimes_{f_{1,1}^2} e_1) + \alpha_1 h(e_1 \otimes_{f_{1,2}^2} e_1) = c_7e_4\alpha_1 - \alpha_1(c_1e_1 + c_2\beta_0\beta_1) + \alpha_1(c_3e_1 + c_4\beta_0\beta_1) = c_7\alpha_1 - c_1\alpha_1 + c_3\alpha_1 = (c_7 + c_3 - c_1)\alpha_1$. This gives no new information.

Thus h is given by

$$h(e_1 \otimes_{f_{1,1}^2} e_1) = c_2\beta_0\beta_1,$$
$$h(e_1 \otimes_{f_{1,2}^2} e_1) = c_4\beta_0\beta_1,$$
$$h(\mathfrak{o}(f_{2,j}^2) \otimes_{f_{2,j}^2} \mathfrak{t}(f_{2,j}^2)) = 0, \text{ for } j = \{1, \ldots, 6\},$$

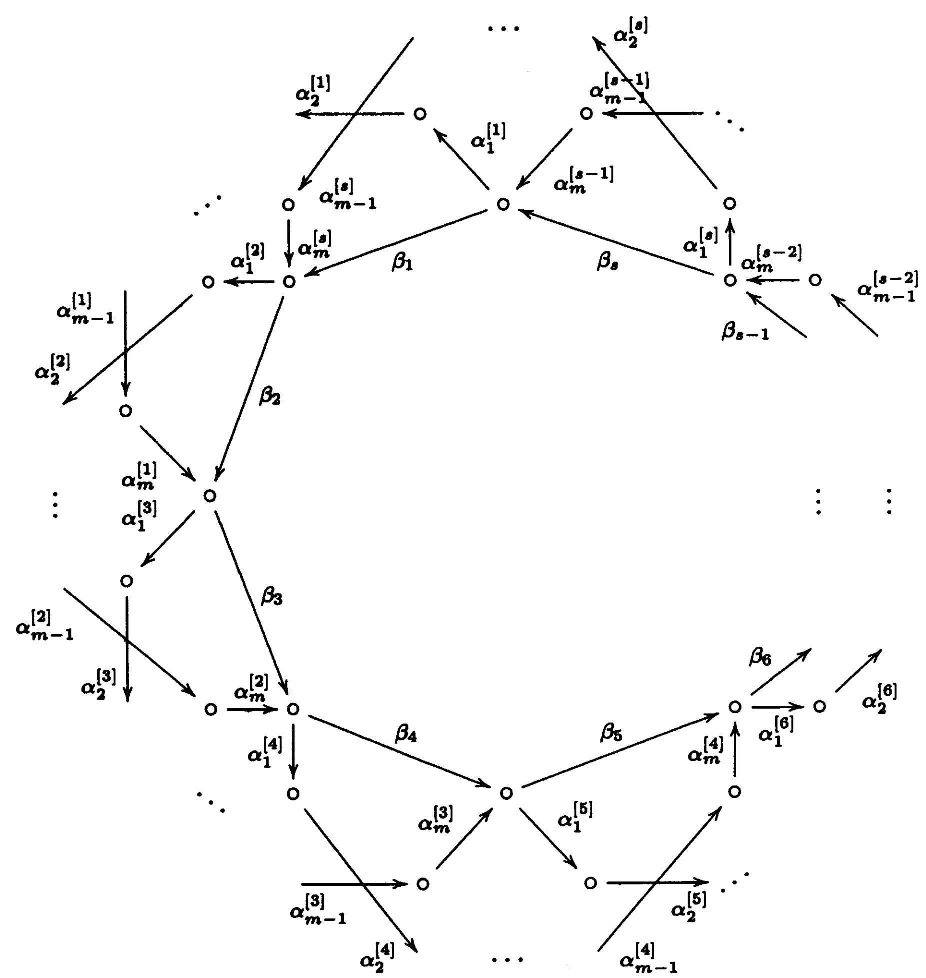
for some $c_2, c_4 \in K$ and so $dim\,\mathrm{Ker}\, d_3 = 2$.

Therefore $dim\, \mathrm{HH}^2(\Lambda) = 2 - 2 = 0$.

Theorem 7.3. *For* $\Lambda = \Lambda(D_4, 1, 3)$ *we have* $\mathrm{HH}^2(\Lambda) = 0$.

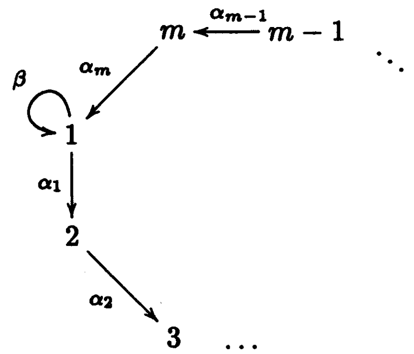
8. The standard algebra $\Lambda(D_{3m}, s/3, 1)$

It is known for $\Lambda = \Lambda(D_{3m}, s/3, 1)$ with $m \geq 2$ and $3 \nmid s \geq 2$ that $\mathrm{HH}^2(\Lambda) = 0$ from Theorem 5.11. The aim of this chapter is to find $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \Lambda(D_{3m}, s/3, 1)$ with $s = 1$. We start by recapping the definition of $\Lambda(D_{3m}, s/3, 1)$ with $s \geq 1$. From [2] and 2.30, the algebra $\Lambda(D_{3m}, s/3, 1)$ with $m \geq 2$ and $3 \nmid s \geq 1$ is given by the quiver $\mathcal{Q}(D_{3m}, s/3)$:



with relations $R(D_{3m}, s/3, 1)$:
(i) $\alpha_1^{[i]}\alpha_2^{[i]}\cdots\alpha_m^{[i]} = \beta_i\beta_{i+1}$, for all $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$,
(ii) $\alpha_m^{[i]}\alpha_1^{[i+2]} = 0$, for all $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$,
(iii) $\alpha_j^{[i]}\cdots\alpha_m^{[i]}\beta_{i+2}\alpha_1^{[i+3]}\cdots\alpha_j^{[i+3]} = 0$, for all $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$ and for all $j \in \{1, \ldots, m\}$ (i.e. paths of length $m+2$ are equal to 0).

Consider $s = 1$. We write δ and $f^r_{a,b}$ to indicate $\delta^{[1]}$ and $f^r_{a,1,b}$ respectively for an arrow δ in $(\mathcal{Q}, s)$ since there is no confusion here. The algebra $\Lambda(D_{3m}, 1/3, 1)$ is given in [2] and has quiver $\mathcal{Q}(D_{3m}, 1/3)$:



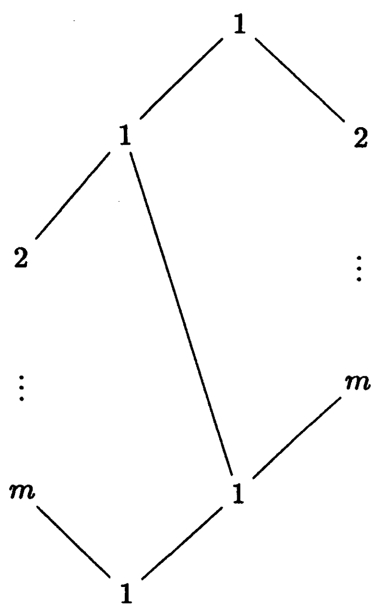
From Proposition 5.19 in Chapter 5 the set of minimal relations f^2 for $\Lambda(D_{3m}, 1/3, 1)$ is as follows:

$$f_1^2 = \beta^2 - \alpha_1\cdots\alpha_m, \qquad f_2^2 = \alpha_m\alpha_1,$$

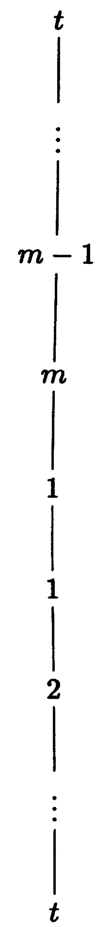
$$f_{3,j}^2 = \alpha_j\cdots\alpha_m\beta\alpha_1\cdots\alpha_j \text{ for all } j \in \{2, \ldots, m-1\}.$$

Next we need to find f^3. The indecomposable projective right Λ-modules are:

$e_1\Lambda$



$e_t\Lambda$, for $2 \leq t \leq m$



From the minimal projective resolutions of each simple Λ-module we see that: for $2 \leq t \leq m-2$ we have,

$$\Omega^3(S_t) \quad \cong \quad \begin{array}{c} t+2 \\ | \\ t+3 \\ | \\ \vdots \\ | \\ m-1 \\ | \\ m \\ | \\ 1 \\ | \\ 1 \\ | \\ 2 \\ | \\ \vdots \\ | \\ t \\ | \\ t+1 \end{array}$$

$$\Omega^3(S_{m-1}) \quad \cong \quad \begin{array}{c} 1 \\ | \\ 1 \\ | \\ \vdots \\ | \\ m-1 \\ | \\ m \end{array}$$

and

$$\Omega^3(S_m) \cong S_2.$$

For $\Omega^3(S_1)$ we need more details. We have the map

$$\psi : e_1\Lambda \oplus e_2\Lambda \to \Omega(S_1)$$

given by:

$$e_1\lambda \mapsto \beta e_1\lambda,$$

$$e_2\mu \mapsto \alpha_1 e_2\mu,$$

where $\lambda, \mu \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 8.1. $\Omega^2(S_1) = (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$.

Proof. First let $x \in \Omega^2(S_1)$. Then $x = (e_1\lambda, e_2\mu)$ with $\lambda, \mu \in \Lambda$. Write $e_1\lambda = c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}$ and $e_2\mu = d_0e_2 + d_1\alpha_2 + d_2\alpha_2\alpha_3 + \cdots + d_{m-1}\alpha_2 \cdots \alpha_m + d_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1$ where $c_i, d_i, c'_j \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$, that is, $\beta(c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}) + \alpha_1(d_0e_2 + d_1\alpha_2 + d_2\alpha_2\alpha_3 + \cdots + d_{m-1}\alpha_2 \cdots \alpha_m + d_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = 0$. So $c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + c_m\beta^3 + c_{m+1}\beta^2 + d_0\alpha_1 + d_1\alpha_1\alpha_2 + d_2\alpha_1\alpha_2\alpha_3 + \cdots + d_{m-2}\alpha_1 \cdots \alpha_{m-1} + d_{m-1}\alpha_1 \cdots \alpha_m + d_m\alpha_1 \cdots \alpha_m\beta = c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + d_0\alpha_1 + d_1\alpha_1\alpha_2 + d_2\alpha_1\alpha_2\alpha_3 + \cdots + d_{m-2}\alpha_1 \cdots \alpha_{m-1} + (d_{m-1} + c_{m+1})\beta^2 + (d_m + c_m)\beta^3 = 0$. Thus $c_0 = c_1 = \cdots = c_{m-1} = d_0 = \cdots = d_{m-2} = 0, d_{m-1} + c_{m+1} = 0$ and $d_m + c_m = 0$. Therefore, $x = (c_m\beta^2 + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}, -c_{m+1}\alpha_2 \cdots \alpha_m - c_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = c_{m+1}(\beta, -\alpha_2 \cdots \alpha_m) + c_m(\beta, -\alpha_2 \cdots \alpha_m)\beta + (c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}, d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = (\beta, -\alpha_2 \cdots \alpha_m)(c_{m+1}e_1 + c_m\beta + c'_0\beta^2 + c'_1\alpha_1 + c'_2\alpha_1\alpha_2 + \cdots + c'_{m-1}\alpha_1 \cdots \alpha_{m-1} + d_{m+1}\beta\alpha_1) = (\beta, -\alpha_2 \cdots \alpha_m)\nu$, where $\nu \in \Lambda$. So $x \in (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$. Thus $\Omega^2(S_1) \subseteq (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$.

On the other hand, let $x \in (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$. So $x = (\beta, -\alpha_2 \cdots \alpha_m)\lambda = (e_1\beta\lambda, -e_2\alpha_2 \cdots \alpha_m\lambda)$, where $\lambda \in \Lambda$. Then $\psi(x) = (\beta^2 - \alpha_1 \cdots \alpha_m)\lambda = 0$. Thus $(\beta, -\alpha_2 \cdots \alpha_m)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\beta, -\alpha_2 \cdots \alpha_m)e_1\Lambda$. □

To find $\Omega^3(S_1)$ we have that $\Omega^3(S_1) = \operatorname{Ker}\theta$ where θ is the map:

$$\theta : e_1\Lambda \to \Omega^2(S_1)$$

given by:

$$e_1\lambda \mapsto (\beta, -\alpha_2 \cdots \alpha_m)e_1\lambda$$

where $\lambda \in \Lambda$.

Proposition 8.2. $\Omega^3(S_1) = \beta\alpha_1\alpha_2\Lambda$.

Proof. On one hand, let $y \in \Omega^3(S_1)$. Then $y = e_1\lambda$ where $\lambda \in \Lambda$. Write $y = c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}$ where $c_i, c'_j \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus $0 = (\beta, -\alpha_2 \cdots \alpha_m)(c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}) = (c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + c_m\beta^3 + c_{m+1}\beta^2, -c_0\alpha_2 \cdots \alpha_m - c_{m+1}\alpha_2 \cdots \alpha_m\beta - c'_1\alpha_2 \cdots \alpha_m\beta\alpha_1) = c_0(\beta, -\alpha_2 \cdots \alpha_m) + (c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + c_m\beta^3, 0) + c_{m+1}(\beta, -\alpha_2 \cdots \alpha_m)\beta + (0, -c'_1\alpha_2 \cdots \alpha_m\beta\alpha_1)$. Thus $c_0 = \cdots = c_{m+1} = c'_1 = 0$. Therefore, $y = c'_0\beta^3 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} = \beta\alpha_1\alpha_2(c'_0\alpha_3 \cdots \alpha_m + c'_2e_3 + \cdots + c'_{m-1}\alpha_3 \cdots \alpha_{m-1}) = \beta\alpha_1\alpha_2\mu$, where $\mu \in \Lambda$. So $y \in \beta\alpha_1\alpha_2\Lambda$. Thus $\Omega^3(S_1) \subseteq \beta\alpha_1\alpha_2\Lambda$.

On the other hand, let $y \in \beta\alpha_1\alpha_2\Lambda$. So $y = \beta\alpha_1\alpha_2\lambda$ where $\lambda \in \Lambda$. Then $\theta(y) = (\beta, -\alpha_2 \cdots \alpha_m)\beta\alpha_1\alpha_2\lambda = 0$. So $y \in \Omega^3(S_1)$. Thus $\beta\alpha_1\alpha_2\Lambda \subseteq \Omega^3(S_1)$.

Therefore, $\Omega^3(S_1) = \beta\alpha_1\alpha_2e_3\Lambda$. □

From the projective resolutions for simples we now know that the 3rd projective $Q^3 = (\Lambda e_1 \otimes e_3\Lambda) \oplus \bigoplus_{t=2}^{m-2}(\Lambda e_t \otimes e_{t+2}\Lambda) \oplus (\Lambda e_{m-1} \otimes e_1\Lambda) \oplus (\Lambda e_m \otimes e_2\Lambda)$.

We choose the set f^3 to consist of the following elements:

$$\{f_1^3, f_{2,t}^3, f_3^3, f_4^3\} \text{ with } t \in \{2,\ldots,m-2\} \text{ where}$$

$$\begin{array}{lllll}
f_1^3 & = & f_1^2\beta\alpha_1\alpha_2 & = & \beta f_1^2\alpha_1\alpha_2 + \beta\alpha_1\cdots\alpha_{m-1}f_2^2\alpha_2 - \alpha_1 f_{3,2}^2, \\
f_{2,t}^3 & = & f_{3,t}^2\alpha_{t+1} & = & \alpha_t f_{3,t+1}^2, \\
f_3^3 & = & f_{3,m-1}^2\alpha_m & = & \alpha_{m-1}f_2^2\alpha_2\cdots\alpha_m\beta + \alpha_{m-1}\alpha_m f_1^2\beta - \alpha_{m-1}\alpha_m\beta f_1^2, \\
f_4^3 & = & f_2^2\alpha_2\cdots\alpha_m\beta\alpha_1 & = & -\alpha_m f_1^2\beta\alpha_1 + \alpha_m\beta f_1^2\alpha_1 + \alpha_m\beta\alpha_1\cdots\alpha_{m-1}f_2^2.
\end{array}$$

Find $\mathbf{Im}\,\mathbf{d_2}$.

We know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$. First we will find $\mathrm{Im}\, d_2$. Since $d_2 : \mathrm{Hom}(Q^1,\Lambda) \to \mathrm{Hom}(Q^2,\Lambda)$, then $d_2 f \in \mathrm{Im}\, d_2$, where $f \in \mathrm{Hom}(Q^1,\Lambda)$ and $d_2 f = fA_2$. Here $Q^1 = (\Lambda e_1 \otimes_\beta e_1\Lambda) \oplus \bigoplus_{l=1}^{m-1}(\Lambda e_l \otimes_{\alpha_l} e_{l+1}\Lambda) \oplus (\Lambda e_m \otimes_{\alpha_m} e_1\Lambda)$. Let $f \in \mathrm{Hom}(Q^1,\Lambda)$ and so

$$f(e_1 \otimes_\beta e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$f(e_l \otimes_{\alpha_l} e_{l+1}) = d_l\alpha_l, \text{ for } l \in \{1,\ldots,m-1\},$$

$$f(e_m \otimes_{\alpha_m} e_1) = d_m\alpha_m,$$

where $c_1, c_2, c_3, c_4, d_l, d_m \in K$, for $l \in \{1,\ldots,m-1\}$.

We have $Q^2 = (\Lambda e_1 \otimes_{f_1^2} e_1\Lambda) \oplus (\Lambda e_m \otimes_{f_2^2} e_2\Lambda) \oplus \bigoplus_{j=2}^{m-1}(\Lambda e_j \otimes_{f_{3,j}^2} e_{j+1}\Lambda)$.

Now we find fA_2. We have

$fA_2(e_1 \otimes_{f_1^2} e_1) = f(e_1 \otimes_\beta e_1)\beta + \beta f(e_1 \otimes_\beta e_1) - f(e_1 \otimes_{\alpha_1} e_2)\alpha_2\cdots\alpha_m - \alpha_1 f(e_2 \otimes_{\alpha_2} e_3)\alpha_3\cdots\alpha_m - \cdots - \alpha_1\alpha_2\cdots\alpha_{m-1}f(e_m \otimes_{\alpha_m} e_1) = (c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta + \beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3) - d_1\alpha_1\cdots\alpha_m - d_2\alpha_1\cdots\alpha_m - \ldots - d_m\alpha_1\cdots\alpha_m = 2c_1\beta - (d_1 + d_2 + \cdots + d_m - 2c_2)\beta^2 + 2c_3\beta^3$.

Also $fA_2(e_m \otimes_{f_2^2} e_2) = f(e_m \otimes_{\alpha_m} e_1)\alpha_1 + \alpha_m f(e_1 \otimes_{\alpha_1} e_2) = d_m\alpha_m\alpha_1 + d_1\alpha_m\alpha_1 = (d_m + d_1)\alpha_m\alpha_1 = 0$.

Finally, for $j = 2,\ldots,m-1$ we have $fA_2(e_j \otimes_{f_{3,j}^2} e_{j+1}) = f(e_j \otimes_{\alpha_j} e_{j+1})\alpha_{j+1}\cdots\alpha_m\beta\alpha_1\cdots\alpha_j + \alpha_j f(e_{j+1} \otimes_{\alpha_{j+1}} e_{j+2})\alpha_{j+2}\cdots\alpha_m\beta\alpha_1\cdots\alpha_j + \cdots + \alpha_j\cdots\alpha_{m-1}f(e_m \otimes_{\alpha_m} e_1)\beta\alpha_1\cdots\alpha_j + \alpha_j\cdots\alpha_m f(e_1 \otimes_\beta e_1)\alpha_1\cdots\alpha_j + \alpha_j\cdots\alpha_m\beta f(e_1 \otimes_{\alpha_1} e_2)\alpha_2\cdots\alpha_j + \cdots + \alpha_j\cdots\alpha_m\beta\alpha_1\cdots\alpha_{j-1}f(e_j \otimes_{\alpha_j} e_{j+1})$

$= \quad d_j\alpha_j\alpha_{j+1}\cdots\alpha_m\beta\alpha_1\cdots\alpha_j \;+\; d_{j+1}\alpha_j\alpha_{j+1}\alpha_{j+2}\cdots\alpha_m\beta\alpha_1\cdots\alpha_j \;+\; \ldots$

$+d_m\alpha_j\cdots\alpha_{m-1}\alpha_m\beta\alpha_1\cdots\alpha_j+\alpha_j\cdots\alpha_m(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\alpha_1\cdots\alpha_j+d_1\alpha_j\cdots\alpha_m\beta\alpha_1\alpha_2\cdots\alpha_j+\cdots+d_j\alpha_j\cdots\alpha_m\beta\alpha_1\cdots\alpha_{j-1}\alpha_j$. So $fA_2(e_j\otimes_{f^2_{3,j}}e_{j+1})=(d_j+d_{j+1}+\cdots+d_m+d_1+\cdots+d_j)\alpha_j\cdots\alpha_m\beta\alpha_1\cdots\alpha_j=0$. Thus f is given by

$$fA_2(e_1\otimes_{f_1^2}e_1)=2c_1\beta-(d_1+d_2+\cdots+d_m-2c_2)\beta^2+2c_3\beta^3=2c'\beta+c''\beta^2+2c'''\beta^3,$$

$$fA_2(e_m\otimes_{f_2^2}e_2)=0,\qquad f(e_j\otimes_{f^2_{3,j}}e_{j+1})=0,\text{ for all } j\in\{2,\dots,m-1\}$$

for some $c',c'',c'''\in K$. So

$$dim\,\mathrm{Im}\,d_2=\begin{cases}3 & \text{if } char\,K\neq 2,\\ 1 & \text{if } char\,K=2.\end{cases}$$

Find Ker $\mathbf{d_3}$.

We have $d_3:\mathrm{Hom}(Q^2,\Lambda)\to\mathrm{Hom}(Q^3,\Lambda)$. Let $h\in\mathrm{Ker}\,d_3$, so $h\in\mathrm{Hom}(Q^2,\Lambda)$ and $d_3h=0$. Then $h:Q^2\to\Lambda$ is given by

$$h(e_1\otimes_{f_1^2}e_1)=c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3,$$

$$h(e_m\otimes_{f_2^2}e_2)=c_5\alpha_m\beta\alpha_1 \text{ and}$$

$$h(e_j\otimes_{f^2_{3,j}}e_{j+1})=d_j\alpha_j,\text{ for } j\in\{2,\dots,m-1\},$$

for some $c_1,\dots,c_5,d_j\in K$ where $j=2,\dots,m-1$. Hence $dim\,\mathrm{Hom}(Q^2,\Lambda)=m+3$.

Then $hA_3(e_1\otimes_{f_1^3}e_3)=h(e_1\otimes_{f_1^2}e_1)\beta\alpha_1\alpha_2-\beta h(e_1\otimes_{f_1^2}e_1)\alpha_1\alpha_2-\beta\alpha_1\cdots\alpha_{m-1}h(e_m\otimes_{f_2^2}e_2)\alpha_2+\alpha_1h(e_2\otimes_{f^2_{3,2}}e_3)=(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\beta\alpha_1\alpha_2-\beta(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\alpha_1\alpha_2+d_2\alpha_1\alpha_2=c_1\beta\alpha_1\alpha_2-c_1\beta\alpha_1\alpha_2+d_2\alpha_1\alpha_2=d_2\alpha_1\alpha_2$. As $h\in\mathrm{Ker}\,d_3$ we have $d_2=0$.

For $t\in\{2,\dots,m-2\}$, we have $hA_3(e_t\otimes_{f^3_{2,t}}e_{t+2})=h(e_t\otimes_{f^2_{3,t}}e_{t+1})\alpha_{t+1}-\alpha_th(e_{t+1}\otimes_{f^2_{3,t+1}}e_{t+2})=d_t\alpha_t\alpha_{t+1}-d_{t+1}\alpha_t\alpha_{t+1}=(d_t-d_{t+1})\alpha_t\alpha_{t+1}$. Then $d_t-d_{t+1}=0$ and so $d_t=d_{t+1}$ for $t=2,\dots,m-2$. Hence $d_2=d_3=\ldots=d_{m-2}=d_{m-1}$. We already have $d_2=0$ so $d_j=0$ for $j=2,\dots,m-1$.

Now $hA_3(e_{m-1}\otimes_{f_3^3}e_1)=h(e_{m-1}\otimes_{f^2_{3,m-1}}e_m)\alpha_m-\alpha_{m-1}h(e_m\otimes_{f_2^2}e_2)\alpha_2\cdots\alpha_m\beta-\alpha_{m-1}\alpha_mh(e_1\otimes_{f_1^2}e_1)\beta+\alpha_{m-1}\alpha_m\beta h(e_1\otimes_{f_1^2}e_1)$
$=d_{m-1}\alpha_{m-1}\alpha_m-c_5\alpha_{m-1}\alpha_m\beta\alpha_1\alpha_2\cdots\alpha_m\beta-\alpha_{m-1}\alpha_m(c_1e_1+c_2\beta+c_3\beta^2+$

$c_4\beta^3)\beta+\alpha_{m-1}\alpha_m\beta(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3) = d_{m-1}\alpha_{m-1}\alpha_m - c_1\alpha_{m-1}\alpha_m\beta +$ $c_1\alpha_{m-1}\alpha_m\beta = 0$, and

$hA_3(e_m \otimes_{f_4^3} e_2) = h(e_m \otimes_{f_2^2} e_2)\alpha_2 \cdots \alpha_m\beta\alpha_1 + \alpha_m h(e_1 \otimes_{f_1^2} e_1)\beta\alpha_1 - \alpha_m\beta$ $h(e_1 \otimes_{f_1^2} e_1)\alpha_1 - \alpha_m\beta\alpha_1 \cdots \alpha_{m-1}h(e_m \otimes_{f_2^2} e_2) = c_5\alpha_m\beta\alpha_1\alpha_2 \cdots \alpha_m\beta\alpha_1 +$ $\alpha_m(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta\alpha_1 - \alpha_m\beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\alpha_1 -$ $c_5\alpha_m\beta\alpha_1 \cdots \alpha_{m-1}\alpha_m\beta\alpha_1 = 0$, and so these give no information on the constants occuring in h.

Thus h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$
$$h(e_m \otimes_{f_2^2} e_2) = c_5\alpha_m\beta\alpha_1 \text{ and}$$
$$h(e_j \otimes_{f_{3,j}^2} e_{j+1}) = 0, \text{ for } j \in \{2, \dots, m-1\}$$

for some $c_1, \dots, c_5 \in K$ and so $\dim \operatorname{Ker} d_3 = 5$.

Therefore,

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 5-3=2 & \text{if } char\, K \neq 2, \\ 5-1=4 & \text{if } char\, K = 2. \end{cases}$$

Hence,

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } char\, K \neq 2, \\ 4 & \text{if } char\, K = 2. \end{cases}$$

Theorem 8.3. *For $\Lambda = \Lambda(D_{3m}, 1/3, 1)$ we have*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } char\, K \neq 2, \\ 4 & \text{if } char\, K = 2. \end{cases}$$

8.4. *A basis for* $\mathrm{HH}^2(\Lambda)$.

char K $\neq$ 2.

We know that $\dim \mathrm{HH}^2(\Lambda) = 2$. So we need to find two non-zero linearly independent elements in $\mathrm{HH}^2(\Lambda)$. We start by defining two non-zero maps h_1, h_2 in $\operatorname{Ker} d_3$. Let h_1 be the map given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0, \end{array}$$

and the map h_2 given by

$$\begin{array}{rcl} e_m \otimes_{f_2^2} e_2 & \mapsto & \alpha_m\beta\alpha_1, \\ \text{else} & \mapsto & 0. \end{array}$$

Now suppose for contradiction that $h_1 \in \operatorname{Im} d_2$. Then $h_1(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. So we have $e_1 = 2c'\beta + c''\beta^2 + 2c'''\beta^3$ for some $c', c'', c''' \in K$. This gives a contradiction. Therefore $h_1 \notin \operatorname{Im} d_2$, that is, $h_1 + \operatorname{Im} d_2 \neq 0 + \operatorname{Im} d_2$ is a non-zero element in $\mathrm{HH}^2(\Lambda)$. Similarly, if we suppose that $h_2 \in \operatorname{Im} d_2$, then $h_2(e_m \otimes_{f_2^2} e_2) = fA_2(e_m \otimes_{f_2^2} e_2)$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. So we have $\alpha_m\beta\alpha_1 = 0$. But this contradicts having $\alpha_m\beta\alpha_1$ a non-zero path in Λ. Therefore $h_2 \notin \operatorname{Im} d_2$, that is, $h_2 + \operatorname{Im} d_2 \neq 0 + \operatorname{Im} d_2$ is a non-zero element in $\mathrm{HH}^2(\Lambda)$.

Next we need to show that $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2\}$ is a basis for $\operatorname{Ker} d_3 / \operatorname{Im} d_2 = \mathrm{HH}^2(\Lambda)$. Suppose $c(h_1 + \operatorname{Im} d_2) + d(h_2 + \operatorname{Im} d_2) = 0 + \operatorname{Im} d_2$ for some $c, d \in K$. So $ch_1 + dh_2 \in \operatorname{Im} d_2$. Then $(ch_1 + dh_2)(e_m \otimes_{f_2^2} e_2) = fA_2(e_m \otimes_{f_2^2} e_2)$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. So we have $d\alpha_m\beta\alpha_1 = 0$. Hence $d = 0$. Also $(ch_1 + dh_2)(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$. So $ce_1 = 2c'\beta + c''\beta^2 + 2c'''\beta^3$. Therefore $ce_1 - 2c'\beta + c''\beta^2 + 2c'''\beta^3 = 0$. Since $\{e_1, \beta, \beta^2, \beta^3\}$ is a linearly independent set in Λ, we have $c = c' = c'' = c''' = 0$. Hence $h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2$ are linearly independent.

So $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$ when *char* $K \neq 2$.

char K = 2.

Here *dim* $\mathrm{HH}^2(\Lambda) = 4$. So we need to find four non-zero linearly independent elements in $\mathrm{HH}^2(\Lambda)$. We start by defining non-zero maps h_1, h_2, h_3, h_4 in $\operatorname{Ker} d_3$.

Let h_1 be the map given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0, \end{array}$$

h_2 be given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta, \\ \text{else} & \mapsto & 0, \end{array}$$

h_3 be given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta^3, \\ \text{else} & \mapsto & 0, \end{array}$$

and h_4 be given by

$$\begin{array}{rcl} e_m \otimes_{f_2^2} e_2 & \mapsto & \alpha_m\beta\alpha_1, \\ \text{else} & \mapsto & 0. \end{array}$$

It can be shown as before that these maps are not in $\operatorname{Im} d_2$. Now we will show that $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2, h_4 + \operatorname{Im} d_2\}$ is a linearly independent set in $\operatorname{Ker} d_3/\operatorname{Im} d_2 = \mathrm{HH}^2(\Lambda)$.

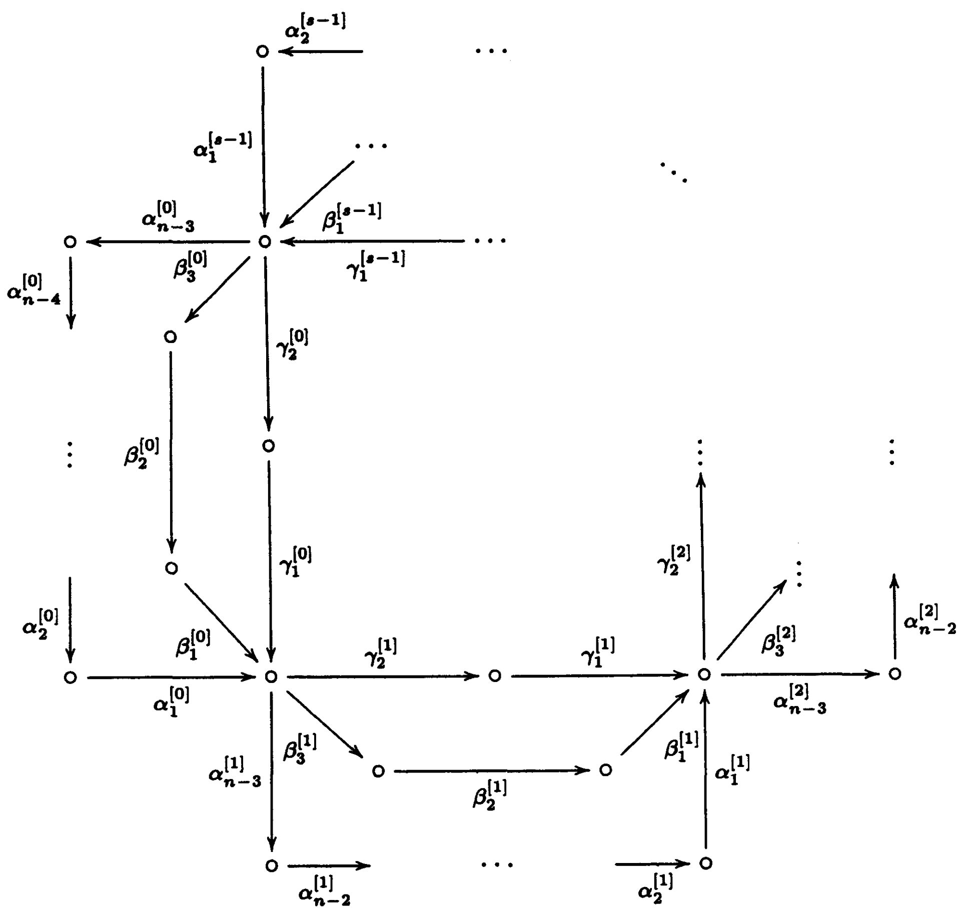
Suppose $a(h_1 + \operatorname{Im} d_2) + b(h_2 + \operatorname{Im} d_2) + c(h_3 + \operatorname{Im} d_2) + d(h_4 + \operatorname{Im} d_2) = 0 + \operatorname{Im} d_2$ for some $a, b, c, d \in K$. So $ah_1 + bh_2 + ch_3 + dh_4 \in \operatorname{Im} d_2$. Hence $ah_1 + bh_2 + ch_3 + dh_4 = fA_2$ for some $f \in \operatorname{Hom}(Q^1, \Lambda)$. Then $(ah_1 + bh_2 + ch_3 + dh_4)(e_m \otimes_{f_2^2} e_2) = fA_2(e_m \otimes_{f_2^2} e_2)$. So we have $d\alpha_m\beta\alpha_1 = 0$. Hence $d = 0$.

Also $(ah_1+bh_2+ch_3+dh_4)(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$. So $ae_1+b\beta+c\beta^3 = c''\beta^2$. Therefore $ae_1 + b\beta + c\beta^3 - c''\beta^2 = 0$. Since $\{e_1, \beta, \beta^2, \beta^3\}$ is linearly independent in Λ, we have $a = b = c = c'' = 0$. Hence $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2, h_4 + \operatorname{Im} d_2\}$ is linearly independent in $\mathrm{HH}^2(\Lambda)$ and forms a basis of $\mathrm{HH}^2(\Lambda)$ when *char* $K = 2$.

This completes the discussion of the standard self-injective algebras of finite representation type D_n. In the next chapter we start to look at the algebras of type E_n.

9. $\Lambda(E_n, s, 1)$

It is known for $\Lambda = \Lambda(E_n, s, 1)$ with $s \geq 2$ that $\mathrm{HH}^2(\Lambda) = 0$ from Theorem 5.11. The aim of this chapter is to find $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \Lambda(E_n, s, 1)$ with $s = 1$. We start by recapping the definition of $\Lambda(E_n, s, 1)$ with $s \geq 1$. From [2] and 2.31, the algebra $\Lambda(E_n, s, 1)$ with $n \in \{6, 7, 8\}$ is given by the quiver $\mathcal{Q}(E_n, s)$:



with relations $R(E_n, s, 1)$:

(i) $\alpha_{n-3}^{[i]} \cdots \alpha_2^{[i]} \alpha_1^{[i]} = \beta_3^{[i]} \beta_2^{[i]} \beta_1^{[i]} = \gamma_2^{[i]} \gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\}$,

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

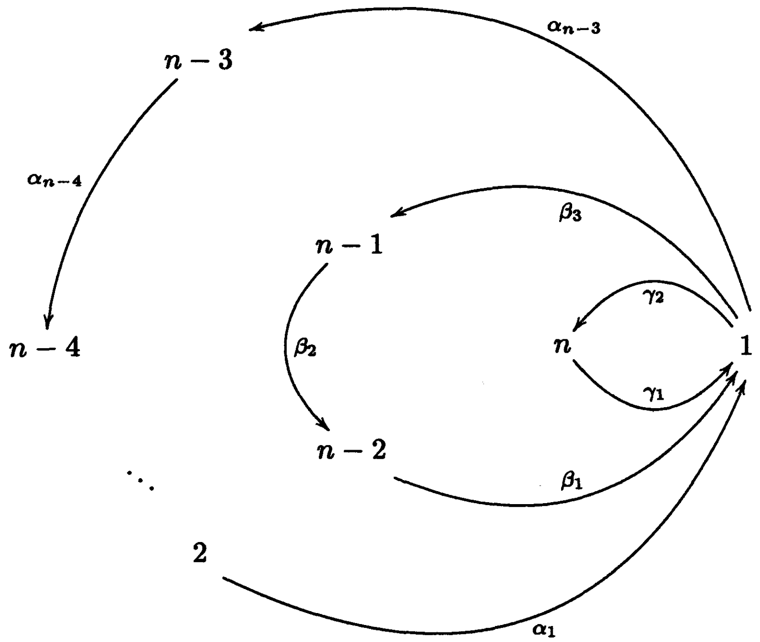
$$\alpha_1^{[i]} \beta_3^{[i+1]} = 0, \qquad \alpha_1^{[i]} \gamma_2^{[i+1]} = 0,$$

$$\beta_1^{[i]} \alpha_{n-3}^{[i+1]} = 0, \qquad \gamma_1^{[i]} \alpha_{n-3}^{[i+1]} = 0,$$

$$\beta_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$

(iii) "α-paths" of length $n-2$ are equal to 0, "β-paths" of length 4 are equal to 0 and "γ-paths" of length 3 are equal to 0.

In order to complete our investigation of $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \Lambda(E_n, s, 1)$ we consider $s = 1$. We write δ and $f^r_{a,b}$ to indicate $\delta^{[0]}$ and $f^r_{a,b,0}$ respectively for δ an arrow in $Q(E_n, s)$ since there is no confusion here. The algebra $\Lambda(E_n, 1, 1)$ is given by the quiver $\mathcal{Q}(E_n, 1)$:



with the set of minimal relations f^2 given in Proposition 5.21 as follows:

$$f^2_{1,1} = \beta_3\beta_2\beta_1 - \gamma_2\gamma_1, \qquad f^2_{1,2} = \beta_3\beta_2\beta_1 - \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2\alpha_1,$$

$$f^2_{2,1} = \alpha_1\beta_3, \qquad f^2_{2,2} = \alpha_1\gamma_2,$$

$$f^2_{2,3} = \beta_1\alpha_{n-3}, \qquad f^2_{2,4} = \beta_1\gamma_2,$$

$$f^2_{2,5} = \gamma_1\alpha_{n-3}, \qquad f^2_{2,6} = \gamma_1\beta_3,$$

$$f^2_{3,k} = \alpha_k\alpha_{k-1}\cdots\alpha_{k+1}\alpha_k \text{ for } k \in \{2,\ldots,n-4\} \text{ and}$$

$$f^2_4 = \beta_2\beta_1\beta_3\beta_2.$$

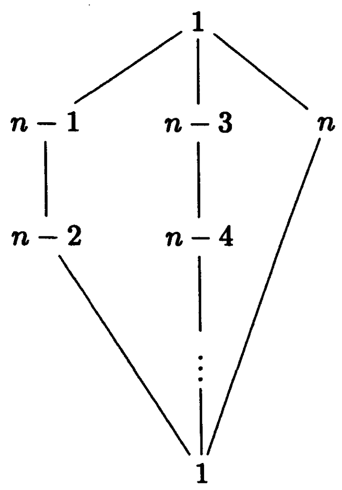
Hence $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,k}$ for $k = 2,\ldots,n-4, f^2_4\}$.

Note that f^2 is a subset of $R(E_n, 1, 1)$ since the set $R(E_n, 1, 1)$ is not minimal.

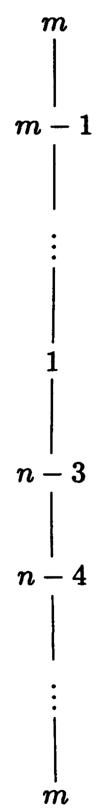
Next we need to find f^3.

The indecomposable projective right Λ-modules are:

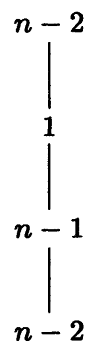
$e_1\Lambda$



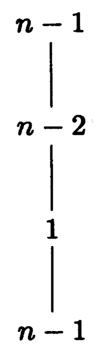
$e_m\Lambda$, for $2 \leq m \leq n-3$



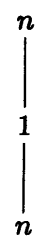
$e_{n-2}\Lambda$



$e_{n-1}\Lambda$



$e_n\Lambda$



From the minimal projective resolutions of each simple Λ-module we see that:

$$\Omega^3(S_2) \quad \cong \quad \begin{array}{ccc} & 1 & \\ / & & \backslash \\ n & & n-1 \end{array}$$

For $3 \leq m \leq n-3$ we have,

$$\Omega^3(S_m) \quad \cong \quad \begin{array}{c} m-1 \\ | \\ m-2 \\ | \\ \vdots \\ | \\ 1 \\ | \\ n-3 \\ | \\ \vdots \\ | \\ m \end{array}$$

$$\Omega^3(S_{n-2}) \quad \cong \quad \begin{array}{ccc} & 1 & \\ / & & \backslash \\ n-3 & & n \end{array}$$

$$\Omega^3(S_{n-1}) \quad \cong \quad \begin{array}{c} 1 \\ | \\ n-1 \\ | \\ n-2 \end{array}$$

$$\Omega^3(S_n) \quad \cong \quad \begin{array}{ccc} & 1 & \\ \diagup & & \diagdown \\ n-3 & & n-1 \end{array}$$

For $\Omega^3(S_1)$ we need more details. We have the map

$$\psi : e_{n-3}\Lambda \oplus e_{n-1}\Lambda \oplus e_n\Lambda \to \Omega(S_1)$$

given by:

$$e_{n-3}\lambda \mapsto \alpha_{n-3}e_{n-3}\lambda,$$

$$e_{n-1}\mu \mapsto \beta_3 e_{n-1}\mu,$$

$$e_n\xi \mapsto \gamma_2 e_n\xi$$

where $\lambda, \mu, \xi \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 9.1. $\Omega^2(S_1) = (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$.

Proof. On one hand, let $x \in \Omega^2(S_1)$. Then $x = (e_{n-3}\lambda, e_{n-1}\mu, e_n\xi)$. Write $e_{n-3}\lambda = c_0e_{n-3} + c_1\alpha_{n-4} + c_2\alpha_{n-4}\alpha_{n-5} + \cdots + c_{n-3}\alpha_{n-4}\cdots\alpha_1\alpha_{n-3}$, $e_{n-1}\mu = c'_0e_{n-1} + c'_1\beta_2 + c'_2\beta_2\beta_1 + c'_3\beta_2\beta_1\beta_3$ and $e_n\xi = d_0e_n + d_1\gamma_1 + d_2\gamma_1\gamma_2$ where $c_i, c'_j, d_l \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$. Thus $\alpha_{n-3}(c_0e_{n-3} + c_1\alpha_{n-4} + c_2\alpha_{n-4}\alpha_{n-5} + \cdots + c_{n-3}\alpha_{n-4}\cdots\alpha_1\alpha_{n-3}) + \beta_3(c'_0e_{n-1} + c'_1\beta_2 + c'_2\beta_2\beta_1 + c'_3\beta_2\beta_1\beta_3) + \gamma_2(d_0e_n + d_1\gamma_1 + d_2\gamma_1\gamma_2) = c_0\alpha_{n-3} + c_1\alpha_{n-3}\alpha_{n-4} + c_2\alpha_{n-3}\alpha_{n-4}\alpha_{n-5} + \cdots + c_{n-4}\alpha_{n-3}\alpha_{n-4}\cdots\alpha_1 + c'_0\beta_3 + c'_1\beta_3\beta_2 + c'_2\beta_3\beta_2\beta_1 + d_0\gamma_2 + d_1\gamma_2\gamma_1 = 0$. Hence $c_0\alpha_{n-3} + c_1\alpha_{n-3}\alpha_{n-4} + c_2\alpha_{n-3}\alpha_{n-4}\alpha_{n-5} + \cdots + c_{n-5}\alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 + c'_0\beta_3 + c'_1\beta_3\beta_2 + d_0\gamma_2 + (c_{n-4} + c'_2 + d_1)\beta_3\beta_2\beta_1 = 0$. Thus $c_0 = c_1 = c_2 = \ldots = c_{n-5} = c'_0 = c'_1 = d_0 = 0$ and $c_{n-4} + c'_2 +$

$d_1 = 0$. So let $c'_2 = -(c_{n-4} + d_1)$. Therefore, $x = (c_{n-4}\alpha_{n-4}\cdots\alpha_1 + c_{n-3}\alpha_{n-4}\cdots\alpha_1\alpha_{n-3}, -(c_{n-4} + d_1)\beta_2\beta_1 + c'_3\beta_2\beta_1\beta_3, d_1\gamma_1 + d_2\gamma_1\gamma_2)$ $= (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)(c_{n-4}e_1 + c_{n-3}\alpha_{n-3}) + (0, \beta_2\beta_1, -\gamma_1)(c'_2e_1 + c'_3\beta_3 + d_2\gamma_2) = (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\lambda + (0, \beta_2\beta_1, -\gamma_1)\mu$, where $\lambda, \mu \in \Lambda$. So $x \in (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$. Thus

$$\Omega^2(S_1) \subseteq (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda.$$

On the other hand, let $x \in (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$. So $x = (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\lambda + (0, \beta_2\beta_1, -\gamma_1)\mu$ with $\lambda, \mu \in \Lambda$. From the definition of ψ, it follows that $\psi(x) = 0$. Thus $(\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)e_1\Lambda + (0, \beta_2\beta_1, -\gamma_1)e_1\Lambda$. □

To find $\Omega^3(S_1)$, we use the map:

$$\theta : e_1\Lambda \oplus e_1\Lambda \to \Omega^2(S_1)$$

given by:

$$(e_1\lambda, e_1\mu) \mapsto (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)e_1\lambda + (0, \beta_2\beta_1, -\gamma_1)e_1\mu$$

where $\lambda, \mu \in \Lambda$, since we know that $\Omega^3(S_1) = \operatorname{Ker}\theta$.

Proposition 9.2.

$\Omega^3(S_1) = (\alpha_{n-3}\alpha_{n-4}, 0)\Lambda + (\gamma_2, 0)\Lambda + (\beta_3, -\beta_3)\Lambda + (0, \alpha_{n-3})\Lambda + (0, \beta_3\beta_2)\Lambda$.

Proof. On one hand, let $y \in \Omega^3(S_1)$. Then $y = (e_1\lambda, e_1\mu)$. Write $e_1\lambda = c_0e_1 + c_1\alpha_{n-3} + c_2\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-3}\alpha_{n-3}\cdots\alpha_1 + c_{n-2}\gamma_2 + c_{n-1}\beta_3 + c_n\beta_3\beta_2$ and $e_1\mu = c'_0e_1 + c'_1\alpha_{n-3} + c'_2\alpha_{n-3}\alpha_{n-4} + \cdots + c'_{n-3}\alpha_{n-3}\cdots\alpha_1 + c'_{n-2}\gamma_2 + c'_{n-1}\beta_3 + c'_n\beta_3\beta_2$ with $c_i, c'_i \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus $(\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)(c_0e_1 + c_1\alpha_{n-3} + c_2\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-3}\alpha_{n-3}\cdots\alpha_1 + c_{n-2}\gamma_2 + c_{n-1}\beta_3 + c_n\beta_3\beta_2) + (0, \beta_2\beta_1, -\gamma_1)(c'_0e_1 + c'_1\alpha_{n-3} + c'_2\alpha_{n-3}\alpha_{n-4} + \cdots + c'_{n-3}\alpha_{n-3}\cdots\alpha_1 + c'_{n-2}\gamma_2 + c'_{n-1}\beta_3 + c'_n\beta_3\beta_2) = (c_0\alpha_{n-4}\cdots\alpha_1 + c_1\alpha_{n-4}\cdots\alpha_1\alpha_{n-3}, -c_0\beta_2\beta_1 - c_{n-1}\beta_2\beta_1\beta_3, 0) + (0, c'_0\beta_2\beta_1 - c'_{n-1}\beta_2\beta_1\beta_3, -c'_0\gamma_1 - c'_{n-2}\gamma_1\gamma_2) = (c_0\alpha_{n-4}\cdots\alpha_1 + c_1\alpha_{n-4}\cdots\alpha_1\alpha_{n-3}, -(c_0 - c'_0)\beta_2\beta_1 - (c_{n-1} + c'_{n-1})\beta_2\beta_1\beta_3, -c'_0\gamma_1 - c'_{n-2}\gamma_1\gamma_2) = 0$. Thus $c_0 = c_1 = c'_0 = c'_{n-2} = 0, c_{n-1} + c'_{n-1} = 0$. Let $c'_{n-1} = -c_{n-1}$. Therefore $y = (c_2\alpha_{n-3}\alpha_{n-4} + \cdots + c_{n-3}\alpha_{n-3}\cdots\alpha_1 + c_{n-2}\gamma_2 + c_{n-1}\beta_3 + c_n\beta_3\beta_2, c'_1\alpha_{n-3} + c'_2\alpha_{n-3}\alpha_{n-4} + \cdots + c'_{n-3}\alpha_{n-3}\cdots\alpha_1 - c_{n-1}\beta_3 + c'_n\beta_3\beta_2)$. So

$y = (\alpha_{n-3}\alpha_{n-4}(c_2e_{n-4} + \cdots + c_{n-3}\alpha_{n-5}\cdots\alpha_1) + c_{n-2}\gamma_2 + \beta_3(c_{n-1}e_{n-1} + c_n\beta_2), \alpha_{n-3}(c'_1e_{n-3} + c'_2\alpha_{n-4} + \cdots + c'_{n-3}\alpha_{n-4}\cdots\alpha_1) - c_{n-1}\beta_3 + c'_n\beta_3\beta_2)$. Then $y = (\alpha_{n-3}\alpha_{n-4}(c_2e_{n-4}+\cdots+c_{n-3}\alpha_{n-5}\cdots\alpha_1)+c_{n-2}\gamma_2+\beta_3(c_{n-1}e_{n-1}+c_n\beta_2), \alpha_{n-3}(c'_1e_{n-3}+c'_2\alpha_{n-4}+\cdots+c'_{n-3}\alpha_{n-4}\cdots\alpha_1)-\beta_3(c_{n-1}e_{n-1}+c_n\beta_2)+(c_n+c'_n)\beta_3\beta_2)$. Hence

$$y = (\alpha_{n-3}\alpha_{n-4}\zeta + \gamma_2\nu + \beta_3\xi, \alpha_{n-3}\kappa - \beta_3\xi + \beta_3\beta_2\eta)$$

$$= (\alpha_{n-3}\alpha_{n-4}, 0)\zeta + (\gamma_2, 0)\nu + (\beta_3, -\beta_3)\xi + (0, \alpha_{n-3})\kappa + (0, \beta_3\beta_2)\eta,$$

where $\zeta, \nu, \xi, \kappa, \eta \in \Lambda$. So $y \in (\alpha_{n-3}\alpha_{n-4}, 0)\Lambda + (\gamma_2, 0)\Lambda + (\beta_3, -\beta_3)\Lambda + (0, \alpha_{n-3})\Lambda + (0, \beta_3\beta_2)\Lambda$. Thus

$$\Omega^3(S_1) \subseteq (\alpha_{n-3}\alpha_{n-4}, 0)\Lambda + (\gamma_2, 0)\Lambda + (\beta_3, -\beta_3)\Lambda + (0, \alpha_{n-3})\Lambda + (0, \beta_3\beta_2)\Lambda.$$

On the other hand, let $y \in (\alpha_{n-3}\alpha_{n-4}, 0)\Lambda + (\gamma_2, 0)\Lambda + (\beta_3, -\beta_3)\Lambda + (0, \alpha_{n-3})\Lambda+(0, \beta_3\beta_2)\Lambda$. So $y = (e_1\alpha_{n-3}\alpha_{n-4}, 0)\zeta+(e_1\gamma_2, 0)\nu+(e_1\beta_3, -e_1\beta_3)\xi+(0, e_1\alpha_{n-3})\kappa+(0, e_1\beta_3\beta_2)\eta$, where $\zeta, \nu, \xi, \kappa, \eta \in \Lambda$. Then $\theta(y) = (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\alpha_{n-3}\alpha_{n-4}\zeta+(0, \beta_2\beta_1, -\gamma_1)\alpha_{n-3}\kappa+(0, \beta_2\beta_1, -\gamma_1)\beta_3\beta_2\eta+(\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\beta_3\xi + (\alpha_{n-4}\cdots\alpha_1, -\beta_2\beta_1, 0)\gamma_2\nu - (0, \beta_2\beta_1, -\gamma_1)\gamma_2\nu = 0$. So $y \in \Omega^3(S_1)$. Thus $(\alpha_{n-3}\alpha_{n-4}, 0)\Lambda+(\gamma_2, 0)\Lambda+(\beta_3, -\beta_3)\Lambda+(0, \alpha_{n-3})\Lambda+(0, \beta_3\beta_2)\Lambda \subseteq \Omega^3(S_1)$.

Therefore, $\Omega^3(S_1) = (\alpha_{n-3}\alpha_{n-4}, 0)e_{n-4}\Lambda + (\gamma_2, 0)e_n\Lambda + (\beta_3, -\beta_3)e_{n-1}\Lambda + (0, \alpha_{n-3})e_{n-3}\Lambda + (0, \beta_3\beta_2)e_{n-2}\Lambda$. □

From the projective resolution for simples we now know that the 3rd projective $Q^3 = (\Lambda e_1 \otimes e_{n-4}\Lambda) \oplus (\Lambda e_1 \otimes e_{n-3}\Lambda) \oplus (\Lambda e_1 \otimes e_{n-2}\Lambda) \oplus (\Lambda e_1 \otimes e_{n-1}\Lambda) \oplus (\Lambda e_1 \otimes e_n\Lambda) \oplus (\Lambda e_2 \otimes e_1\Lambda) \oplus (\Lambda e_{n-1} \otimes e_1\Lambda) \oplus (\Lambda e_{n-2} \otimes e_1\Lambda) \oplus (\Lambda e_n \otimes e_1\Lambda) \oplus \bigoplus_{m=3}^{n-3}(\Lambda e_m \otimes e_{m-2}\Lambda)$.

We choose the set f^3 to consist of the following elements:

$$\{f^3_{1,1}, f^3_{1,2}, f^3_{1,3}, f^3_{1,4}, f^3_{1,5}, f^3_{1,6}, f^3_{1,7}, f^3_{1,8}, f^3_{1,9}, f^3_{2,3}, f^3_{2,m}\}$$

$$\begin{array}{lcll}
f^3_{1,1} & = & f^2_{1,2}\alpha_{n-3}\alpha_{n-4} & = \beta_3\beta_2 f^2_{2,3}\alpha_{n-4} - \alpha_{n-3}f^2_{3,n-4}, \\
f^3_{1,2} & = & f^2_{1,1}\alpha_{n-3} & = \beta_3\beta_2 f^2_{2,3} - \gamma_2 f^2_{2,5}, \\
f^3_{1,3} & = & f^2_{1,1}\beta_3\beta_2 & = \beta_3 f^2_4 - \gamma_2 f^2_{2,6}\beta_2, \\
f^3_{1,4} & = & f^2_{1,1}\beta_3 - f^2_{1,2}\beta_3 & = \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 f^2_{2,1} - \gamma_2 f^2_{2,6}, \\
f^3_{1,5} & = & f^2_{1,2}\gamma_2 & = \beta_3\beta_2 f^2_{2,4} - \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 f^2_{2,2}, \\
f^3_{1,6} & = & f^2_{2,1}\beta_2\beta_1 - f^2_{2,2}\gamma_1 & = \alpha_1 f^2_{1,1}, \\
f^3_{1,7} & = & f^2_4\beta_1 & = \beta_2\beta_1 f^2_{1,1} + \beta_2 f^2_{2,4}\gamma_1, \\
f^3_{1,8} & = & f^2_{2,3}\alpha_{n-4}\cdots\alpha_2\alpha_1 - f^2_{2,4}\gamma_1 & = \beta_1 f^2_{1,1} - \beta_1 f^2_{1,2}, \\
f^3_{1,9} & = & f^2_{2,6}\beta_2\beta_1 - f^2_{2,5}\alpha_{n-4}\cdots\alpha_2\alpha_1 & = \gamma_1 f^2_{1,2}, \\
f^3_{2,3} & = & f^2_{3,2}\alpha_1 & = \alpha_2 f^2_{2,1}\beta_2\beta_1 - \alpha_2\alpha_1 f^2_{1,2}, \\
f^3_{2,m} & = & f^2_{3,m-1}\alpha_{m-2} & = \alpha_{m-1} f^2_{3,m-2} \text{ for } m = 4,\ldots n-3.
\end{array}$$

Find $\mathrm{Im}\, d_2$.

We know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$. First we will find $\mathrm{Im}\, d_2$. Since $d_2 : \mathrm{Hom}(Q^1,\Lambda) \to \mathrm{Hom}(Q^2,\Lambda)$, then $d_2 f \in \mathrm{Im}\, d_2$, where $f \in \mathrm{Hom}(Q^1,\Lambda)$ and $d_2 f = fA_2$. Here $Q^1 = (\Lambda e_1 \otimes_{\beta_3} e_{n-1}\Lambda) \oplus (\Lambda e_{n-1} \otimes_{\beta_2} e_{n-2}\Lambda) \oplus (\Lambda e_{n-2} \otimes_{\beta_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\gamma_2} e_n\Lambda) \oplus (\Lambda e_n \otimes_{\gamma_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\alpha_{n-3}} e_{n-3}\Lambda) \oplus \bigoplus_{l=1}^{n-4}(\Lambda e_{l+1} \otimes_{\alpha_l} e_l\Lambda)$. Let $f \in \mathrm{Hom}(Q^2,\Lambda)$ so

$$f(e_1 \otimes_{\beta_3} e_{n-1}) = c_1\beta_3, \qquad f(e_{n-1} \otimes_{\beta_2} e_{n-2}) = c_2\beta_2,$$

$$f(e_{n-2} \otimes_{\beta_1} e_1) = c_3\beta_1$$

$$f(e_1 \otimes_{\gamma_2} e_n) = c_4\gamma_2, \qquad f(e_n \otimes_{\gamma_1} e_1) = c_5\gamma_1,$$

$$f(e_1 \otimes_{\alpha_{n-3}} e_{n-3}) = d_{n-3}\alpha_{n-3}$$

and

$$f(e_{l+1} \otimes_{\alpha_l} e_l) = d_l\alpha_l, \text{ for } l \in \{1,\ldots,n-4\},$$

where $c_1,\ldots,c_5,d_1,d_l \in K$ for $l = 1,\ldots,n-4$.

We have $Q^2 = (\Lambda e_1 \otimes_{f^2_{1,1}} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{f^2_{1,2}} e_1\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,1}} e_{n-1}\Lambda) \oplus (\Lambda e_2 \otimes_{f^2_{2,2}} e_n\Lambda) \oplus (\Lambda e_{n-2} \otimes_{f^2_{2,3}} e_{n-3}\Lambda) \oplus (\Lambda e_{n-2} \otimes_{f^2_{2,4}} e_n\Lambda) \oplus (\Lambda e_n \otimes_{f^2_{2,5}} e_{n-3}\Lambda) \oplus (\Lambda e_n \otimes_{f^2_{2,6}} e_{n-1}\Lambda) \oplus \bigoplus_{k=2}^{n-4}(\Lambda e_{k+1} \otimes_{f^2_{3,k}} e_k\Lambda) \oplus (\Lambda e_{n-1} \otimes_{f^2_4} e_{n-2}\Lambda)$.

Now we find fA_2.

We have $fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = f(e_1 \otimes_{\beta_3} e_{n-1})\beta_2\beta_1 + \beta_3 f(e_{n-1} \otimes_{\beta_2} e_{n-2})\beta_1 + \beta_3\beta_2 f(e_{n-2} \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\gamma_2} e_n)\gamma_1 - \gamma_2 f(e_n \otimes_{\gamma_1} e_1) = c_1\beta_3\beta_2\beta_1 + c_2\beta_3\beta_2\beta_1 + c_3\beta_3\beta_2\beta_1 - c_4\gamma_2\gamma_1 - c_5\gamma_2\gamma_1 = (c_1 + c_2 + c_3 - c_4 - c_5)\beta_3\beta_2\beta_1,$

$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = f(e_1 \otimes_{\beta_3} e_{n-1})\beta_2\beta_1 + \beta_3 f(e_{n-1} \otimes_{\beta_2} e_{n-2})\beta_1 + \beta_3\beta_2 f(e_{n-2} \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\alpha_{n-3}} e_{n-3})\alpha_{n-4} \dots \alpha_1 - \dots - \alpha_{n-3} \dots \alpha_2 f(e_2 \otimes_{\alpha_1} e_1) = c_1\beta_3\beta_2\beta_1 + c_2\beta_3\beta_2\beta_1 + c_3\beta_3\beta_2\beta_1 - d_{n-3}\alpha_{n-3}\alpha_{n-4} \dots \alpha_1 - \dots - d_1\alpha_{n-3}\alpha_{n-4} \dots \alpha_1 = (c_1 + c_2 + c_3)\beta_3\beta_2\beta_1 - (d_{n-3} + \dots + d_1)\beta_3\beta_2\beta_1 = (c_1 + c_2 + c_3 - d_{n-3} - \dots - d_1)\beta_3\beta_2\beta_1$.

Also $fA_2(e_2 \otimes_{f^2_{2,1}} e_{n-1}) = f(e_2 \otimes_{\alpha_1} e_1)\beta_3 + \alpha_1 f(e_1 \otimes_{\beta_3} e_{n-1}) = d_1\alpha_1\beta_3 + c_1\alpha_1\beta_3 = (d_1 + c_1)\alpha_1\beta_3 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,2}} e_n) = f(e_2 \otimes_{\alpha_1} e_1)\gamma_2 + \alpha_1 f(e_1 \otimes_{\gamma_2} e_n) = d_1\alpha_1\gamma_2 + c_4\alpha_1\gamma_2 = (d_1 + c_4)\alpha_1\gamma_2 = 0$,

$fA_2(e_{n-2} \otimes_{f^2_{2,3}} e_{n-3}) = f(e_{n-2} \otimes_{\beta_1} e_1)\alpha_{n-3} + \beta_1 f(e_1 \otimes_{\alpha_{n-3}} e_{n-3}) = c_3\beta_1\alpha_{n-3} + d_{n-3}\beta_1\alpha_{n-3} = (c_3 + d_{n-3})\beta_1\alpha_{n-3} = 0$,

$fA_2(e_{n-2} \otimes_{f^2_{2,4}} e_n) = f(e_{n-2} \otimes_{\beta_1} e_1)\gamma_2 + \beta_1 f(e_1 \otimes_{\gamma_2} e_n) = c_3\beta_1\gamma_2 + c_4\beta_1\gamma_2 = (c_3 + c_4)\beta_1\gamma_2 = 0$,

$fA_2(e_n \otimes_{f^2_{2,5}} e_{n-3}) = f(e_n \otimes_{\gamma_1} e_1)\alpha_{n-3} + \gamma_1 f(e_1 \otimes_{\alpha_{n-3}} e_{n-3}) = c_5\gamma_1\alpha_{n-3} + d_{n-3}\gamma_1\alpha_{n-3} = (c_5 + d_{n-3})\gamma_1\alpha_{n-3} = 0$,

$fA_2(e_n \otimes_{f^2_{2,6}} e_{n-1}) = f(e_n \otimes_{\gamma_1} e_1)\beta_3 + \gamma_1 f(e_1 \otimes_{\beta_3} e_{n-1}) = c_5\gamma_1\beta_3 + c_1\gamma_1\beta_3 = (c_5 + c_1)\gamma_1\beta_3 = 0$.

For $k = 2, \dots, n-4$, $fA_2(e_{k+1} \otimes_{f^2_{3,k}} e_k) = f(e_{k+1} \otimes_{\alpha_k} e_k)\alpha_{k-1} \cdots \alpha_1 \alpha_{n-3} \cdots \alpha_k + \alpha_k f(e_k \otimes_{\alpha_{k-1}} e_{k-1})\alpha_{k-2} \cdots \alpha_1\alpha_{n-3} \cdots \alpha_k + \dots + \alpha_k \cdots \alpha_1\alpha_{n-3} \cdots \alpha_{k+1} f(e_{k+1} \otimes_{\alpha_k} e_k) = (d_k + d_{k-1} + \dots + d_k)\alpha_k\alpha_{k-1} \cdots \alpha_1\alpha_{n-3} \cdots \alpha_k = 0$.

Finally, $fA_2(e_{n-1} \otimes_{f^2_4} e_{n-2}) = f(e_{n-1} \otimes_{\beta_2} e_{n-2})\beta_1\beta_3\beta_2 + \beta_2 f(e_{n-2} \otimes_{\beta_1} e_1)\beta_3\beta_2 + \beta_2\beta_1 f(e_1 \otimes_{\beta_3} e_{n-1})\beta_2 + \beta_2\beta_1\beta_3 f(e_{n-1} \otimes_{\beta_2} e_{n-2}) = c_2\beta_2\beta_1\beta_3\beta_2 + c_3\beta_2\beta_1\beta_3\beta_2 + c_1\beta_2\beta_1\beta_3\beta_2 + c_2\beta_2\beta_1\beta_3\beta_2 = (2c_2 + c_3 + c_1)\beta_2\beta_1\beta_3\beta_2 = 0$.

Hence f is given by

$$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = (c_1 + c_2 + c_3 - c_4 - c_5)\beta_3\beta_2\beta_1 = c'\beta_3\beta_2\beta_1,$$

$$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = (c_1 + c_2 + c_3 - d_{n-3} - \dots - d_1)\beta_3\beta_2\beta_1 = c''\beta_3\beta_2\beta_1,$$

$$fA_2(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j = 1, \dots, 6,$$

$$fA_2(e_{k+1} \otimes_{f^2_{3,k}} e_k) = 0 \text{ for } k = 2, \dots, n-4,$$

$$fA_2(e_{n-1} \otimes_{f^2_4} e_{n-2}) = 0,$$

for some $c', c'' \in K$. Hence $dim \operatorname{Im} d_2 = 2$.

Find Ker $\mathbf{d_3}$.

We have $d_3 : \mathrm{Hom}(Q^2, \Lambda) \to \mathrm{Hom}(Q^3, \Lambda)$. Let $h \in \mathrm{Ker}\, d_3$, so $h \in \mathrm{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_1 e_1 + c_2\gamma_2\gamma_1,$$
$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4\gamma_2\gamma_1,$$
$$h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j \in \{1, \dots, 6\},$$
$$h(e_{k+1} \otimes_{f^2_{3,k}} e_k) = d_k\alpha_k \text{ for } k = 2, \dots, n-4,$$
$$h(e_{n-1} \otimes_{f^2_4} e_{n-2}) = c_5\beta_2$$

for some $c_1, \dots, c_5, d_k \in K$. Hence $dim\, \mathrm{Hom}(Q^2, \Lambda) = n$.

Then $hA_3(e_1 \otimes_{f^3_{1,1}} e_{n-4}) = h(e_1 \otimes_{f^2_{1,2}} e_1)\alpha_{n-3}\alpha_{n-4} - \beta_3\beta_2 h(e_{n-2} \otimes_{f^2_{2,3}} e_{n-3})\alpha_{n-4} + \alpha_{n-3} h(e_{n-3} \otimes_{f^2_{3,n-4}} e_{n-4}) = (c_3 e_1 + c_4\gamma_2\gamma_1)\alpha_{n-3}\alpha_{n-4} + d_{n-4}\alpha_{n-3}\alpha_{n-4} = c_3\alpha_{n-3}\alpha_{n-4} + d_{n-4}\alpha_{n-3}\alpha_{n-4} = (c_3 + d_{n-4})\alpha_{n-3}\alpha_{n-4}$. As $h \in \mathrm{Ker}\, d_3$, we have $c_3 + d_{n-4} = 0$ and so $c_3 = -d_{n-4}$.

$hA_3(e_1 \otimes_{f^3_{1,2}} e_{n-3}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\alpha_{n-3} - \beta_3\beta_2 h(e_{n-2} \otimes_{f^2_{2,3}} e_{n-3}) + \gamma_2 h(e_n \otimes_{f^2_{2,5}} e_{n-3}) = (c_1 e_1 + c_2\gamma_2\gamma_1)\alpha_{n-3} = c_1\alpha_{n-3}$. As $h \in \mathrm{Ker}\, d_3$, we have $c_1 = 0$.

$hA_3(e_1 \otimes_{f^3_{1,3}} e_{n-2}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\beta_3\beta_2 - \beta_3 h(e_{n-1} \otimes_{f^2_4} e_{n-2}) + \gamma_2 h(e_n \otimes_{f^2_{2,6}} e_{n-1})\beta_2 = (c_1 e_1 + c_2\gamma_2\gamma_1)\beta_3\beta_2 - c_5\beta_3\beta_2 = (c_1 - c_5)\beta_3\beta_2$. Thus, since $h \in \mathrm{Ker}\, d_3$, we have $c_5 = 0$ as we have already $c_1 = 0$.

Now, $hA_3(e_1 \otimes_{f^3_{1,4}} e_{n-1}) = h(e_1 \otimes_{f^2_{1,1}} e_1)\beta_3 - h(e_1 \otimes_{f^2_{1,2}} e_1)\beta_3 - \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 h(e_2 \otimes_{f^2_{2,1}} e_{n-1}) + \gamma_2 h(e_n \otimes_{f^2_{2,6}} e_{n-1}) = (c_1 e_1 + c_2\gamma_2\gamma_1)\beta_3 - (c_3 e_1 + c_4\gamma_2\gamma_1)\beta_3 = (c_1 - c_3)\beta_3$. So we have $c_1 - c_3 = 0$ and so $c_3 = 0$ as $c_1 = 0$. It then follows that $d_{n-4} = 0$ as $c_3 = -d_{n-4}$ above.

Next, $hA_3(e_1 \otimes_{f^3_{1,5}} e_n) = h(e_1 \otimes_{f^2_{1,2}} e_1)\gamma_2 - \beta_3\beta_2 h(e_{n-2} \otimes_{f^2_{2,4}} e_n) + \alpha_{n-3}\alpha_{n-4}\cdots\alpha_2 h(e_2 \otimes_{f^2_{2,2}} e_n) = (c_3 e_1 + c_4\gamma_2\gamma_1)\gamma_2 = c_3\gamma_2 = 0$. This gives no new information.

Similarly, $hA_3(e_2 \otimes_{f^3_{1,6}} e_1) = h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_2\beta_1 - h(e_2 \otimes_{f^2_{2,2}} e_n)\gamma_1 - \alpha_1 h(e_1 \otimes_{f^2_{1,1}} e_1) = -\alpha_1(c_1 e_1 + c_2\gamma_2\gamma_1) = -c_1\alpha_1 = 0$,

$hA_3(e_{n-1} \otimes_{f^3_{1,7}} e_1) = h(e_{n-1} \otimes_{f^2_4} e_{n-2})\beta_1 - \beta_2\beta_1 h(e_1 \otimes_{f^2_{1,1}} e_1) - \beta_2 h(e_{n-2} \otimes_{f^2_{2,4}} e_n)\gamma_1 = c_5\beta_2\beta_1 - \beta_2\beta_1(c_1 e_1 + c_2\gamma_2\gamma_1) = (c_5 - c_1)\beta_2\beta_1 = 0$,

$hA_3(e_{n-2} \otimes_{f^3_{1,8}} e_1) = h(e_{n-2} \otimes_{f^2_{2,3}} e_{n-3})\alpha_{n-4}\cdots\alpha_2\alpha_1 - h(e_{n-2} \otimes_{f^2_{2,4}} e_n)\gamma_1 - \beta_1 h(e_1 \otimes_{f^2_{1,1}} e_1) + \beta_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -\beta_1(c_1 e_1 + c_2\gamma_2\gamma_1) + \beta_1(c_3 e_1 + c_4\gamma_2\gamma_1) = (c_3 - c_1)\beta_1 = 0$, and

$hA_3(e_n \otimes_{f^3_{1,9}} e_1) = h(e_n \otimes_{f^2_{2,6}} e_{n-1})\beta_2\beta_1 - h(e_n \otimes_{f^2_{2,5}} e_{n-3})\alpha_{n-4}\cdots\alpha_2\alpha_1 - \gamma_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -\gamma_1(c_3e_1 + c_4\gamma_2\gamma_1) = -c_3\gamma_1 = 0$, all give no new information.

Next, $hA_3(e_3 \otimes_{f^3_{2,3}} e_1) = h(e_3 \otimes_{f^2_{3,2}} e_2)\alpha_1 - \alpha_2 h(e_2 \otimes_{f^2_{2,1}} e_{n-1})\beta_2\beta_1 + \alpha_2\alpha_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = d_2\alpha_2\alpha_1 + \alpha_2\alpha_1(c_3e_1 + c_4\gamma_2\gamma_1) = (d_2 + c_3)\alpha_2\alpha_1$. Since $h \in \operatorname{Ker} d_3$, we have $d_2 + c_3 = 0$. So $d_2 = 0$ as $c_3 = 0$ from above.

Finally, for $m \in \{4, \ldots, n-3\}$, we have $hA_3(e_m \otimes_{f^3_{2,m}} e_{m-2}) = h(e_m \otimes_{f^2_{3,m-1}} e_{m-1})\alpha_{m-2} - \alpha_{m-1}h(e_{m-1} \otimes_{f^2_{3,m-2}} e_{m-2}) = d_{m-1}\alpha_{m-1}\alpha_{m-2} - d_{m-2}\alpha_{m-1}\alpha_{m-2} = (d_{m-1} - d_{m-2})\alpha_{m-1}\alpha_{m-2}$. Then $h \in \operatorname{Ker} d_3$ gives $d_{m-1} - d_{m-2} = 0$ and so $d_{m-1} = d_{m-2}$. Hence $d_{n-4} = d_{n-5} = \ldots = d_3 = d_2$. We already have $d_2 = 0$ so $d_k = 0$ for $k = 2, \ldots, n-4$.

Thus h is given by

$$\begin{gathered} h(e_1 \otimes_{f^2_{1,1}} e_1) = c_2\gamma_2\gamma_1, \\ h(e_1 \otimes_{f^2_{1,2}} e_1) = c_4\gamma_1\gamma_1, \\ h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j \in \{1, \ldots, 6\}, \\ h(e_{k+1} \otimes_{f^2_{3,k}} e_k) = 0, \text{ for } k \in \{2, \ldots, n-4\} \text{ and} \\ h(e_{n-1} \otimes_{f^2_4} e_{n-2}) = 0, \end{gathered}$$

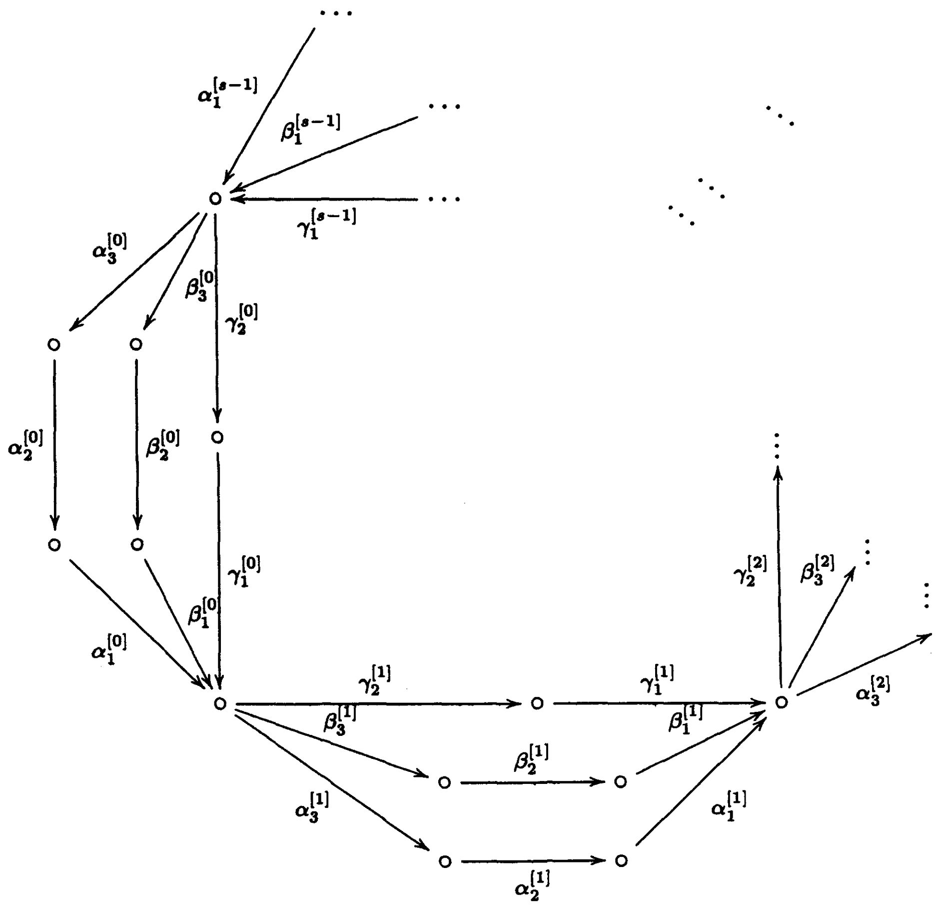
for some $c_2, c_4 \in K$ and so $dim \operatorname{Ker} d_3 = 2$.

Therefore $dim\, \mathrm{HH}^2(\Lambda) = 2 - 2 = 0$. Hence $\mathrm{HH}^2(\Lambda) = 0$.

Theorem 9.3. *For $\Lambda = \Lambda(E_n, 1, 1)$ with $n = 6, 7, 8$, we have $\mathrm{HH}^2(\Lambda) = 0$.*

10. $\Lambda(E_6, s, 2)$

It is known for $\Lambda = \Lambda(E_6, s, 1)$ with $s \geq 2$ that $\mathrm{HH}^2(\Lambda) = 0$ from Theorem 5.11. But we recap all $s \geq 1$ first. The algebra $\Lambda(E_6, s, 2)$ is given by the quiver $\mathcal{Q}(E_6, s)$:



with relations $R(E_6, s, 2)$:

(i) $\alpha_3^{[i]}\alpha_2^{[i]}\alpha_1^{[i]} = \beta_3^{[i]}\beta_2^{[i]}\beta_1^{[i]} = \gamma_2^{[i]}\gamma_1^{[i]}$, for all $i \in \{0, \ldots, s-1\}$;

(ii) for all $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\gamma_1^{[i]}\alpha_3^{[i+1]} = 0, \qquad \gamma_1^{[i]}\beta_3^{[i+1]} = 0,$$
$$\alpha_1^{[i]}\gamma_2^{[i+1]} = 0, \qquad \beta_1^{[i]}\gamma_2^{[i+1]} = 0;$$

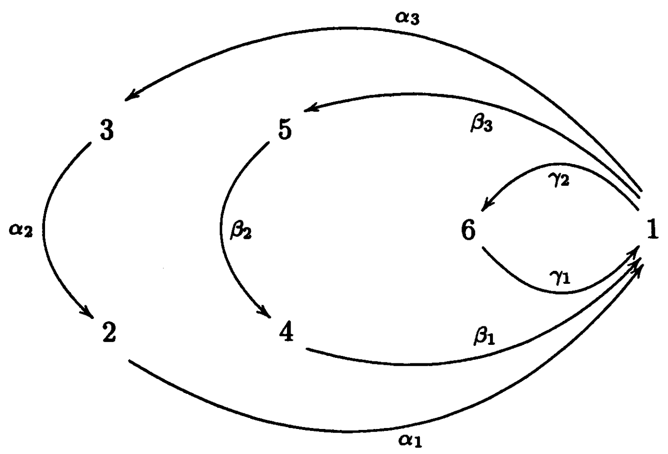
and for all $i \in \{0, \ldots, s-2\}$,

$$\alpha_1^{[i]}\beta_3^{[i+1]} = 0, \qquad \beta_1^{[i]}\alpha_3^{[i+1]} = 0,$$
$$\alpha_1^{[s-1]}\alpha_3^{[0]} = 0, \qquad \beta_1^{[s-1]}\beta_3^{[0]} = 0;$$

(iii) "γ-paths" of length 3 are equal to 0 and for all $i \in \{0, \dots, s-2\}$ and for all $j \in \{1,2,3\} = \mathbb{Z}/\langle 3 \rangle$,

$$\alpha_j^{[i]} \cdots \alpha_1^{[i]} \alpha_3^{[i+1]} \cdots \alpha_{j-3}^{[i+1]} = 0, \qquad \beta_j^{[i]} \cdots \beta_1^{[i]} \beta_3^{[i+1]} \cdots \beta_{j-3}^{[i+1]} = 0,$$
$$\alpha_j^{[s-1]} \cdots \alpha_1^{[s-1]} \beta_3^{[0]} \cdots \beta_{j-3}^{[0]} = 0, \qquad \beta_j^{[s-1]} \cdots \beta_1^{[s-1]} \alpha_3^{[0]} \cdots \alpha_{j-3}^{[0]} = 0.$$

Consider $s = 1$. Recall that we write δ and $f^r_{a,b}$ to indicate $\delta^{[0]}$ and $f^r_{a,b,0}$ respectively since there is no confusion here. The algebra $\Lambda(E_6, 1, 2)$ is given by the quiver $\mathcal{Q}(E_6, 1)$.



with the set of minimal relations f^2 given in 5.24 as follows.

$$f^2_{1,1} = \beta_3\beta_2\beta_1 - \gamma_2\gamma_1, \qquad f^2_{1,2} = \beta_3\beta_2\beta_1 - \alpha_3\alpha_2\alpha_1,$$
$$f^2_{2,1} = \gamma_1\alpha_3, \qquad f^2_{2,2} = \gamma_1\beta_3,$$
$$f^2_{2,3} = \alpha_1\gamma_2, \qquad f^2_{2,4} = \beta_1\gamma_2,$$
$$f^2_{2,5} = \alpha_1\alpha_3, \qquad f^2_{2,6} = \beta_1\beta_3,$$
$$f^2_{3,1} = \alpha_2\alpha_1\beta_3\beta_2, \qquad f^2_{3,2} = \beta_2\beta_1\alpha_3\alpha_2.$$

Hence $f^2 = \{f^2_{1,1}, f^2_{1,2}, f^2_{2,1}, f^2_{2,2}, f^2_{2,3}, f^2_{2,4}, f^2_{2,5}, f^2_{2,6}, f^2_{3,1}, f^2_{3,2}\}$.

Next we need to find f^3. The projective indecomposable Λ-modules are:

$e_1\Lambda$

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
5 \arrow[dd, dash] & & 3 \arrow[dd, dash] \\
 & 6 \arrow[dd, dash] & \\
4 \arrow[dr, dash] & & 2 \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}$$

$e_2\Lambda$

$$\begin{tikzcd}
2 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 5 \arrow[d, dash] \\ 4
\end{tikzcd}$$

$e_3\Lambda$

$$\begin{tikzcd}
3 \arrow[d, dash] \\ 2 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 5
\end{tikzcd}$$

$e_4\Lambda$

$$\begin{tikzcd}
4 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 2
\end{tikzcd}$$

$e_5\Lambda$

$$\begin{tikzcd}
5 \arrow[d, dash] \\ 4 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 3
\end{tikzcd}$$

$e_6\Lambda$

$$\begin{tikzcd}
6 \arrow[d, dash] \\ 1 \arrow[d, dash] \\ 6
\end{tikzcd}$$

From the minimal projective resolutions of each simple Λ-module we have:

$$\Omega^3(S_2) \quad \cong \quad \begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
5 & & 6
\end{tikzcd}$$

$$\Omega^3(S_3) \quad\cong\quad \begin{tikzcd} 1 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 2 \end{tikzcd}$$

$$\Omega^3(S_3) \quad\cong\quad \begin{tikzcd} 1 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 2 \end{tikzcd}$$

$$\Omega^3(S_4) \quad\cong\quad \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 3 & & 6 \end{tikzcd}$$

$$\Omega^3(S_5) \quad\cong\quad \begin{tikzcd} 1 \arrow[d, dash] \\ 5 \arrow[d, dash] \\ 4 \end{tikzcd}$$

$$\Omega^3(S_6) \quad\cong\quad \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 3 & & 5 \end{tikzcd}$$

For $\Omega^3(S_1)$ we need more details. We have the map

$$\psi : e_3\Lambda \oplus e_5\Lambda \oplus e_6\Lambda \to \Omega(S_1)$$

given by:

$$e_3\lambda \mapsto \alpha_3 e_3\lambda,$$

$$e_5\mu \mapsto \beta_3 e_5\mu,$$

$$e_6\xi \mapsto \gamma_2 e_6\xi$$

where $\lambda, \mu, \xi \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 10.1. $\Omega^2(S_1) = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$.

Proof. On one hand, let $x \in \Omega^2(S_1)$. Then $x = (e_3\lambda, e_5\mu, e_6\xi)$. Write $e_3\lambda = c_0e_3 + c_1\alpha_2 + c_2\alpha_2\alpha_1 + c_3\alpha_2\alpha_1\beta_3$, $e_5\mu = c_0'e_5 + c_1'\beta_2 + c_2'\beta_2\beta_1 + c_3'\beta_2\beta_1\alpha_3$ and $e_6\xi = d_0e_6 + d_1\gamma_1 + d_2\gamma_1\gamma_2$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$. Thus $\alpha_3(c_0e_3 + c_1\alpha_2 + c_2\alpha_2\alpha_1 + c_3\alpha_2\alpha_1\beta_3) + \beta_3(c_0'e_5 + c_1'\beta_2 + c_2'\beta_2\beta_1 + c_3'\beta_2\beta_1\alpha_3) + \gamma_2(d_0e_6 + d_1\gamma_1 + d_2\gamma_1\gamma_2) = c_0\alpha_3 + c_1\alpha_3\alpha_2 + c_2\alpha_3\alpha_2\alpha_1 + c_0'\beta_3 + c_1'\beta_3\beta_2 + c_2'\beta_3\beta_2\beta_1 + d_0\gamma_2 + d_1\gamma_2\gamma_1 = c_0\alpha_3 + c_1\alpha_3\alpha_2 + c_0'\beta_3 + c_1'\beta_3\beta_2 + d_0\gamma_2 + (c_2 + c_2' + d_1)\gamma_2\gamma_1 = 0$. Thus $c_0 = c_1 = c_0' = c_1' = d_0 = 0$ and $c_2 + c_2' + d_1 = 0$. Let $c_2' = -(c_2 + d_1)$. Therefore, $x = (c_2\alpha_2\alpha_1 + c_3\alpha_2\alpha_1\beta_3, -(c_2 + d_1)\beta_2\beta_1 + c_3'\beta_2\beta_1\alpha_3, d_1\gamma_1 + d_2\gamma_1\gamma_2) = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)(c_2e_1 + c_3\beta_3) + (0, \beta_2\beta_1, -\gamma_1)(-d_1e_1 + c_3'\alpha_3 - d_2\gamma_2) = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\lambda + (0, \beta_2\beta_1, -\gamma_1)\mu$, where $\lambda, \mu \in \Lambda$. So $x \in (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$. Thus $\Omega^2(S_1) \subseteq (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$.

On the other hand, let $x \in (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda$. So $x = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\lambda + (0, \beta_2\beta_1, -\gamma_1)\mu$, where $\lambda, \mu \in \Lambda$. From the definition of ψ, it follows that $\psi(x) = 0$. Thus $(\alpha_2\alpha_1, -\beta_2\beta_1, 0)\Lambda + (0, \beta_2\beta_1, -\gamma_1)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)e_1\Lambda + (0, \beta_2\beta_1, -\gamma_1)e_1\Lambda$. □

To find $\Omega^3(S_1)$. We have the map:

$$\theta : e_1\Lambda \oplus e_1\Lambda \to \Omega^2(S_1)$$

given by:

$$(e_1\lambda, e_1\mu) \mapsto (\alpha_2\alpha_1, -\beta_2\beta_1, 0)e_1\lambda + (0, \beta_2\beta_1, -\gamma_1)e_1\mu$$

where $\lambda, \mu \in \Lambda$.

Note that $\Omega^3(S_1) = \operatorname{Ker}\theta$.

Proposition 10.2. $\Omega^3(S_1) = (\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda$.

Proof. On one hand, let $y \in \Omega^3(S_1)$. Then $y = (e_1\lambda, e_1\mu)$ with $\lambda, \mu \in \Lambda$. Write $e_1\lambda = c_0e_1 + c_1\alpha_3 + c_2\alpha_3\alpha_2 + c_3\alpha_3\alpha_2\alpha_1 + c_4\gamma_2 + c_5\beta_3 + c_6\beta_3\beta_2$ and $e_1\mu = c_0'e_1 + c_1'\alpha_3 + c_2'\alpha_3\alpha_2 + c_3'\alpha_3\alpha_2\alpha_1 + c_4'\gamma_2 + c_5'\beta_3 + c_6'\beta_3\beta_2$ with all coefficients $c_i, c_i' \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus $(\alpha_2\alpha_1, -\beta_2\beta_1, 0)(c_0e_1 + c_1\alpha_3 + c_2\alpha_3\alpha_2 + c_3\alpha_3\alpha_2\alpha_1 + c_4\gamma_2 + c_5\beta_3 + c_6\beta_3\beta_2) + (0, \beta_2\beta_1, -\gamma_1)(c_0'e_1 + c_1'\alpha_3 + c_2'\alpha_3\alpha_2 + c_3'\alpha_3\alpha_2\alpha_1 + c_4'\gamma_2 + c_5'\beta_3 + c_6'\beta_3\beta_2) = (c_0\alpha_2\alpha_1 + c_5\alpha_2\alpha_1\beta_3, -c_0\beta_2\beta_1 - c_1\beta_2\beta_1\alpha_3, 0) + (0, c_0'\beta_2\beta_1 + c_1'\beta_2\beta_1\alpha_3, -c_0'\gamma_1 - c_4\gamma_1\gamma_2) = (c_0\alpha_2\alpha_1 + c_5\alpha_2\alpha_1\beta_3, (c_0' - c_0)\beta_2\beta_1 + (c_1' - c_1)\beta_2\beta_1\alpha_3, -c_0'\gamma_1 - c_4\gamma_1\gamma_2) = 0$. Thus $c_0 = c_5 = c_0' = c_4' = 0, c_1' = c_1$. Therefore, $y = (c_1\alpha_3 + c_2\alpha_3\alpha_2 + c_3\alpha_3\alpha_2\alpha_1 + c_4\gamma_2 + c_6\beta_3\beta_2, c_1\alpha_3 + c_2'\alpha_3\alpha_2\alpha_1 + c_3'\alpha_3\alpha_2\alpha_1 + c_5'\beta_3 + c_6'\beta_3\beta_2)$
$= (\alpha_3(c_1e_3 + c_2\alpha_2 + c_3\alpha_2\alpha_1) + c_4\gamma_2 + c_6\beta_3\beta_2, \alpha_3(c_1e_3 + c_2'\alpha_2 + c_3'\alpha_2\alpha_1) + \beta_3(c_5'e_5 + c_6'\beta_2))$
$= (\alpha_3(c_1e_3 + c_2\alpha_2 + c_3\alpha_2\alpha_1) + c_4\gamma_2 + c_6\beta_3\beta_2, \alpha_3(c_1e_3 + c_2\alpha_2 + (c_2' - c_2)\alpha_2 + c_3\alpha_2\alpha_1 + (c_3' - c_3)\alpha_2\alpha_1) + \beta_3(c_5'e_5 + c_6'\beta_2))$
$= (\alpha_3(c_1e_3 + c_2\alpha_2 + c_3\alpha_2\alpha_1), \alpha_3(c_1e_3 + c_2\alpha_2 + c_3\alpha_2\alpha_1) + (0, \alpha_3\alpha_2((c_2' - c_2)e_2 + (c_3' - c_3)\alpha_1)) + (c_4\gamma_2, 0) + (0, \beta_3(c_5'e_5 + c_6'\beta_2) + (c_6\beta_3\beta_2, 0)$. Hence $y = (\alpha_3, \alpha_3)\eta + (0, \alpha_3\alpha_2)\zeta + (\gamma_2, 0)\xi + (0, \beta_3)\kappa + (\beta_3\beta_2, 0)\nu$ where $\eta, \zeta, \xi, \kappa, \nu \in \Lambda$. So $y \in (\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda$ Thus $\Omega^3(S_1) \subseteq (\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda$.

On the other hand, let $y \in (\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda$. So $y = (e_1\alpha_3, e_1\alpha_3)\eta + (0, e_1\alpha_3\alpha_2)\zeta + (e_1\gamma_2, 0)\xi + (0, e_1\beta_3)\kappa + (e_1\beta_3\beta_2, 0)\nu$ where $\eta, \zeta, \xi, \kappa, \nu \in \Lambda$. Then $\theta(y) = (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\alpha_3\eta + (0, \beta_2\beta_1, -\gamma_1)\alpha_3\eta + (0, \beta_2\beta_1, -\gamma_1)\alpha_3\alpha_2\zeta + (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\gamma_2\xi + (0, \beta_2\beta_1, -\gamma_1)\beta_3\kappa + (\alpha_2\alpha_1, -\beta_2\beta_1, 0)\beta_3\beta_2\nu = 0$. So $y \in \Omega^3(S_1)$. Thus

$$(\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda \subseteq \Omega^3(S_1).$$

Therefore, $\Omega^3(S_1) = (\alpha_3, \alpha_3)\Lambda + (0, \alpha_3\alpha_2)\Lambda + (\gamma_2, 0)\Lambda + (0, \beta_3)\Lambda + (\beta_3\beta_2, 0)\Lambda$.

□

From the projective resolution for simples we now know that the 3rd projective $Q^3 = (\Lambda e_1 \otimes e_6\Lambda) \oplus (\Lambda e_1 \otimes e_5\Lambda) \oplus (\Lambda e_1 \otimes e_4\Lambda) \oplus (\Lambda e_1 \otimes e_3\Lambda) \oplus (\Lambda e_1 \otimes e_2\Lambda) \oplus (\Lambda e_2 \otimes e_1\Lambda) \oplus (\Lambda e_3 \otimes e_1\Lambda) \oplus (\Lambda e_4 \otimes e_1\Lambda) \oplus (\Lambda e_5 \otimes e_1\Lambda) \oplus (\Lambda e_6 \otimes e_1\Lambda)$.

The set f^3 consists of the following elements:

$$\{f_1^3, f_2^3, f_3^3, f_4^3, f_5^3, f_6^3, f_7^3, f_8^3, f_9^3, f_{10}^3\} \text{ where}$$

$$\begin{array}{lllll}
f_1^3 & = & f_{1,2}^2\gamma_2 & = & \beta_3\beta_2 f_{2,4}^2 - \alpha_3\alpha_2 f_{2,3}^2, \\
f_2^3 & = & f_{1,1}^2\beta_3 & = & \beta_3\beta_2 f_{2,6}^2 - \gamma_2 f_{2,2}^2, \\
f_3^3 & = & f_{1,2}^2\beta_3\beta_2 - \beta_3\beta_2 f_{2,6}^2\beta_2 & = & -\alpha_3 f_{3,1}^2, \\
f_4^3 & = & f_{1,1}^2\alpha_3 - f_{1,2}^2\alpha_3 & = & \alpha_3\alpha_2 f_{2,5}^2 - \gamma_2 f_{2,1}^2, \\
f_5^3 & = & f_{1,1}^2\alpha_3\alpha_2 + \gamma_2 f_{2,1}^2\alpha_2 & = & \beta_3 f_{3,2}^2, \\
f_6^3 & = & f_{2,5}^2\alpha_2\alpha_1 - f_{2,3}^2\gamma_1 & = & \alpha_1 f_{1,1}^2 - \alpha_1 f_{1,2}^2, \\
f_7^3 & = & f_{3,1}^2\beta_1 - \alpha_2 f_{2,3}^2\gamma_1 & = & \alpha_2\alpha_1 f_{1,1}^2, \\
f_8^3 & = & f_{2,6}^2\beta_2\beta_1 - f_{2,4}^2\gamma_1 & = & \beta_1 f_{1,1}^2, \\
f_9^3 & = & f_{3,2}^2\alpha_1 - \beta_2 f_{2,6}^2\beta_2\beta_1 & = & -\beta_2\beta_1 f_{1,2}^2, \\
f_{10}^3 & = & f_{2,2}^2\beta_2\beta_1 - f_{2,1}^2\alpha_2\alpha_1 & = & \gamma_1 f_{1,2}^2.
\end{array}$$

Find $\mathbf{Im}\, \mathbf{d_2}$.

We know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$. First we will find $\mathrm{Im}\, d_2$. Since $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$, then $d_2 f \in \mathrm{Im}\, d_2$, where $f \in \mathrm{Hom}(Q^1, \Lambda)$ and $d_2 f = f A_2$. Here $Q^1 = (\Lambda e_1 \otimes_{\beta_3} e_5\Lambda) \oplus (\Lambda e_5 \otimes_{\beta_2} e_4\Lambda) \oplus (\Lambda e_4 \otimes_{\beta_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\gamma_2} e_6\Lambda) \oplus (\Lambda e_6 \otimes_{\gamma_1} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{\alpha_3} e_3\Lambda) \oplus (\Lambda e_3 \otimes_{\alpha_2} e_2\Lambda) \oplus (\Lambda e_2 \otimes_{\alpha_1} e_1\Lambda)$.
Let

$$f(e_1 \otimes_{\beta_3} e_5) = c_1\beta_3, \qquad f(e_5 \otimes_{\beta_2} e_5) = c_2\beta_2,$$

$$f(e_4 \otimes_{\beta_1} e_1) = c_3\beta_1$$

$$f(e_1 \otimes_{\gamma_2} e_6) = c_4\gamma_2, \qquad f(e_6 \otimes_{\gamma_1} e_1) = c_5\gamma_1,$$

$$f(e_1 \otimes_{\alpha_3} e_3) = c_6\alpha_3, \qquad f(e_3 \otimes_{\alpha_2} e_2) = c_7\alpha_2 \text{ and}$$

$$f(e_2 \otimes_{\alpha_1} e_1) = c_8\alpha_1$$

where $c_1, \ldots, c_8 \in K$.

We have $Q^2 = (\Lambda e_1 \otimes_{f_{1,1}^2} e_1\Lambda) \oplus (\Lambda e_1 \otimes_{f_{1,2}^2} e_1\Lambda) \oplus (\Lambda e_6 \otimes_{f_{2,1}^2} e_3\Lambda) \oplus (\Lambda e_6 \otimes_{f_{2,2}^2} e_5\Lambda) \oplus (\Lambda e_2 \otimes_{f_{2,3}^2} e_6\Lambda) \oplus (\Lambda e_4 \otimes_{f_{2,4}^2} e_6\Lambda) \oplus (\Lambda e_2 \otimes_{f_{2,5}^2} e_3\Lambda) \oplus (\Lambda e_4 \otimes_{f_{2,6}^2} e_5\Lambda)$.

Now we find $f A_2$.

We have $f A_2(e_1 \otimes_{f_{1,1}^2} e_1) = f(e_1 \otimes_{\beta_3} e_5)\beta_2\beta_1 + \beta_3 f(e_5 \otimes_{\beta_2} e_4)\beta_1 + \beta_3\beta_2 f(e_4 \otimes_{\beta_1}$

$e_1) - f(e_1 \otimes_{\gamma_2} e_6)\gamma_1 - \gamma_2 f(e_6 \otimes_{\gamma_1} e_1) = c_1\beta_3\beta_2\beta_1 + c_2\beta_3\beta_2\beta_1 + c_3\beta_3\beta_2\beta_1 - c_4\gamma_2\gamma_1 - c_5\gamma_2\gamma_1 = (c_1 + c_2 + c_3 - c_4 - c_5)\beta_3\beta_2\beta_1$,

Also $fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = f(e_1 \otimes_{\beta_3} e_5)\beta_2\beta_1 + \beta_3 f(e_5 \otimes_{\beta_2} e_4)\beta_1 + \beta_3\beta_2 f(e_4 \otimes_{\beta_1} e_1) - f(e_1 \otimes_{\alpha_3} e_3)\alpha_2\alpha_1 - \alpha_3 f(e_3 \otimes_{\alpha_2} e_2)\alpha_1 - \alpha_3\alpha_2 f(e_2 \otimes_{\alpha_1} e_1) = c_1\beta_3\beta_2\beta_1 + c_2\beta_3\beta_2\beta_1 + c_3\beta_3\beta_2\beta_1 - c_6\alpha_3\alpha_2\alpha_1 - c_7\alpha_3\alpha_2\alpha_1 - c_8\alpha_3\alpha_2\alpha_1 = (c_1 + c_2 + c_3 - c_6 - c_7 - c_8)\beta_3\beta_2\beta_1$,

$fA_2(e_6 \otimes_{f^2_{2,1}} e_3) = f(e_6 \otimes_{\gamma_1} e_1)\alpha_3 + \gamma_1 f(e_1 \otimes_{\alpha_3} e_3) = c_5\gamma_1\alpha_3 + c_6\gamma_1\alpha_3 = (c_5 + c_6)\gamma_1\alpha_3 = 0$,

$fA_2(e_6 \otimes_{f^2_{2,2}} e_5) = f(e_6 \otimes_{\gamma_1} e_1)\beta_3 + \gamma_1 f(e_1 \otimes_{\beta_3} e_5) = c_5\gamma_1\beta_3 + c_1\gamma_1\beta_3 = (c_5 + c_1)\gamma_1\beta_3 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,3}} e_6) = f(e_2 \otimes_{\alpha_1} e_1)\gamma_2 + \alpha_1 f(e_1 \otimes_{\gamma_2} e_6) = c_8\alpha_1\gamma_2 + c_4\alpha_1\gamma_2 = (c_8 + c_4)\alpha_1\gamma_2 = 0$,

$fA_2(e_4 \otimes_{f^2_{2,4}} e_6) = f(e_4 \otimes_{\beta_1} e_1)\gamma_2 + \beta_1 f(e_1 \otimes_{\gamma_2} e_6) = c_3\beta_1\gamma_2 + c_4\beta_1\gamma_2 = (c_3 + c_4)\beta_1\gamma_2 = 0$,

$fA_2(e_2 \otimes_{f^2_{2,5}} e_3) = f(e_2 \otimes_{\alpha_1} e_1)\alpha_3 + \alpha_1 f(e_1 \otimes_{\alpha_3} e_3) = c_8\alpha_1\alpha_3 + c_6\alpha_1\alpha_3 = (c_8 + c_6)\alpha_1\alpha_3 = 0$,

$fA_2(e_4 \otimes_{f^2_{2,6}} e_5) = f(e_4 \otimes_{\beta_1} e_1)\beta_3 + \beta_1 f(e_1 \otimes_{\beta_3} e_5) = c_3\beta_1\beta_3 + c_1\beta_1\beta_3 = (c_3 + c_1)\beta_1\beta_3 = 0$,

$fA_2(3_3 \otimes_{f^2_{3,1}} e_4) = f(e_3 \otimes_{\alpha_2} e_2)\alpha_1\beta_3\beta_2 + \alpha_2 f(e_2 \otimes_{\alpha_1} e_1)\beta_3\beta_2 + \alpha_2\alpha_1 f(e_1 \otimes_{\beta_3} e_5)\beta_2 + \alpha_2\alpha_1\beta_3 f(e_5 \otimes_{\beta_2} e_4) = c_7\alpha_2\alpha_1\beta_3\beta_2 + c_8\alpha_2\alpha_1\beta_3\beta_2 + c_1\alpha_2\alpha_1\beta_3\beta_2 + c_2\alpha_2\alpha_1\beta_3\beta_2 = (c_7 + c_8 + c_1 + c_2)\alpha_2\alpha_1\beta_3\beta_2 = 0$,

Finally, $fA_2(e_5 \otimes_{f^2_{3,2}} e_2) = f(e_5 \otimes_{\beta_2} e_4)\beta_1\alpha_3\alpha_2 + \beta_2 f(e_4 \otimes_{\beta_1} e_1)\alpha_3\alpha_2 + \beta_2\beta_1 f(e_1 \otimes_{\alpha_3} e_3)\alpha_2 + \beta_2\beta_1\alpha_3 f(e_3 \otimes_{\alpha_2} e_2) = c_2\beta_2\beta_1\alpha_3\alpha_2 + c_3\beta_2\beta_1\alpha_3\alpha_2 + c_6\beta_2\beta_1\alpha_3\alpha_2 + c_7\beta_2\beta_1\alpha_3\alpha_2 = (c_2 + c_3 + c_6 + c_7)\beta_2\beta_1\alpha_3\alpha_2$.

Hence f is given by

$$fA_2(e_1 \otimes_{f^2_{1,1}} e_1) = (c_1 + c_2 + c_3 - c_4 - c_5)\beta_3\beta_2\beta_1 = c'\beta_3\beta_2\beta_1,$$

$$fA_2(e_1 \otimes_{f^2_{1,2}} e_1) = (c_1 + c_2 + c_3 - c_6 - c_7 - c_8)\beta_3\beta_2\beta_1 = c''\beta_3\beta_2\beta_1,$$

$$fA_2(\mathfrak{o}(f^2_j) \otimes_{f^2_j} \mathfrak{t}(f^2_j)) = 0 \text{ for all } f^2_j \neq f^2_{1,1}, f^2_{1,2}.$$

So $dim\, \mathrm{Im}\, d_2 = 2$.

Find Ker $\mathbf{d_3}$.

We have $d_3 : \mathrm{Hom}(Q^2, \Lambda) \to \mathrm{Hom}(Q^3, \Lambda)$. Let $h \in \mathrm{Ker}\, d_3$, so $h \in \mathrm{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Then $h : Q^2 \to \Lambda$ is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_1 e_1 + c_2 \gamma_2\gamma_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_3 e_1 + c_4 \gamma_2\gamma_1,$$

$$h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j \in \{1, \dots, 6\},$$

for some $c_1, c_2, c_3, c_4 \in K$. Hence $dim\, \mathrm{Hom}(Q^2, \Lambda) = 4$.

$$h(\mathfrak{o}(f^2_{2,l}) \otimes_{f^2_{2,l}} \mathfrak{t}(f^2_{2,l})) = 0, \text{ for } l = 1, 2,$$

Then $hA_3(e_1 \otimes_{f^3_1} e_n) = h(e_1 \otimes_{f^2_{1,2}} e_1)\gamma_2 - \beta_3\beta_2 h(e_4 \otimes_{f^2_{2,4}} e_6) + \alpha_3\alpha_2 h(e_2 \otimes_{f^2_{2,3}} e_6) = (c_3 e_1 + c_4\gamma_2\gamma_1)\gamma_2 = c_3\gamma_2 + c_4\gamma_2\gamma_1\gamma_2 = c_3\gamma_2$. As $h \in \mathrm{Ker}\, d_3$ we have $c_3 = 0$.

$hA_3(e_1 \otimes_{f^3_2} e_5) = h(e_1 \otimes_{f^2_{1,1}} e_1)\beta_3 - \beta_3\beta_2 h(e_4 \otimes_{f^2_{2,6}} e_5) + \gamma_2 h(e_6 \otimes_{f^2_{2,2}} e_5) = (c_1 e_1 + c_2\gamma_2\gamma_1)\beta_3 = c_1\beta_3 + c_2\gamma_2\gamma_1\beta_3 = c_1\beta_3$. As $h \in \mathrm{Ker}\, d_3$, $c_1 = 0$.

Now, $hA_3(e_1 \otimes_{f^3_3} e_4) = h(e_1 \otimes_{f^2_{1,2}} e_1)\beta_3\beta_2 - \beta_3\beta_2 h(e_4 \otimes_{f^2_{2,6}} e_5)\beta_2 + \alpha_3 h(e_3 \otimes_{f^2_{3,1}} e_4) = (c_3 e_1 + c_4\gamma_2\gamma_1)\beta_3\beta_2 = c_3\beta_3\beta_2 + c_4\gamma_2\gamma_1\beta_3\beta_2 = c_3\beta_3 = 0$. This gives no new information.

Next, $hA_3(e_1 \otimes_{f^3_4} e_3) = h(e_1 \otimes_{f^2_{1,1}} e_1)\alpha_3 - h(e_1 \otimes_{f^2_{1,2}} e_1)\alpha_3 - \alpha_3\alpha_2 h(e_2 \otimes_{f^2_{2,5}} e_3) + \gamma_2 h(e_6 \otimes_{f^2_{2,1}} e_3) = (c_1 e_1 + c_2\gamma_2\gamma_1)\alpha_3 - (c_3 e_1 + c_4\alpha_3\gamma_1)\alpha_3 = (c_1 - c_3)\alpha_3 + (c_2 - c_4)\gamma_2\gamma_1\alpha_3 = (c_1 - c_3)\alpha_3$, so we have $c_3 = 0$ as we already have $c_1 = 0$.

$hA_3(e_1 \otimes_{f^3_5} e_2) = h(e_1 \otimes_{f^2_{1,1}} e_1)\alpha_3\alpha_2 + \gamma_2 h(e_6 \otimes_{f^2_{2,1}} e_3)\alpha_2 - \beta_3 h(e_5 \otimes_{f^2_{3,2}} e_2) = (c_1 e_1 + c_2\gamma_2\gamma_1)\alpha_3\alpha_2 = c_1\alpha_3\alpha_2 + c_2\gamma_2\gamma_1\alpha_3\alpha_2 = c_1\alpha_3\alpha_2 = 0$,

$hA_3(e_2 \otimes_{f^3_6} e_1) = h(e_2 \otimes_{f^2_{2,5}} e_3)\alpha_2\alpha_1 - h(e_2 \otimes_{f^2_{2,3}} e_6)\gamma_1 - \alpha_1 h(e_1 \otimes_{f^2_{1,1}} e_1) + \alpha_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -\alpha_1(c_1 e_1 + c_2\gamma_2\gamma_1) + \alpha_1(c_3 e_1 + c_4\gamma_2\gamma_1) = -c_1\alpha_1 - c_2\alpha_1\gamma_2\gamma_1 + c_3\alpha_1 + c_4\alpha_1\gamma_2\gamma_1 = (c_3 - c_1)\alpha_1 = 0$, since we have already $c_3 = c_1 = 0$.

Similarly, $hA_3(e_3 \otimes_{f^3_7} e_1) = h(e_3 \otimes_{f^2_{3,1}} e_4)\beta_1 - \alpha_2 h(e_2 \otimes_{f^2_{2,3}} e_6)\gamma_1 - \alpha_2\alpha_1 h(e_1 \otimes_{f^2_{1,1}} e_1) = -\alpha_2\alpha_1(c_1 e_1 + c_2\gamma_2\gamma_1) = -c_1\alpha_2\alpha_1 - c_2\alpha_2\alpha_1\gamma_2\gamma_1 = 0$, and

$hA_3(e_4 \otimes_{f^3_8} e_1) = h(e_4 \otimes_{f^2_6} e_5)\beta_2\beta_1 - h(e_4 \otimes_{f^2_{2,4}} e_6)\gamma_1 - \beta_1 h(e_1 \otimes_{f^2_{1,1}} e_1) = -\beta_1(c_1 e_1 + c_2\gamma_2\gamma_1) = -c_1\beta_1 - c_2\beta_1\gamma_2\gamma_1 = -c_1\beta_1 = 0$, as we have $c_1 = 0$.

Next, $hA_3(e_5 \otimes_{f^3_9} e_1) = h(e_5 \otimes_{f^2_{3,2}} e_2)\alpha_1 - \beta_2 h(e_4 \otimes_{f^2_{2,6}} e_5)\beta_2\beta_1 + \beta_2\beta_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = \beta_2\beta_1(c_3 e_1 + c_4\gamma_2\gamma_1) = c_3\beta_2\beta_1 + c_4\beta_2\beta_1\gamma_2\gamma_1 = c_3\beta_2\beta_1 = 0$, Finally, $hA_3(e_6 \otimes_{f^3_{10}} e_1) = h(e_6 \otimes_{f^2_{2,3}} e_5)\beta_2\beta_1 - h(e_6 \otimes_{f^2_{2,1}} e_3)\alpha_2\alpha_1 - \gamma_1 h(e_1 \otimes_{f^2_{1,2}} e_1) = -\gamma_1(c_3e_1 + c_4\gamma_2\gamma_1) = -c_3\gamma_1 - c_4\gamma_1\gamma_2\gamma_1 = -c_3\gamma_1 = 0$, as we have $c_3 = 0$.

Thus h is given by

$$h(e_1 \otimes_{f^2_{1,1}} e_1) = c_2\gamma_2\gamma_1,$$

$$h(e_1 \otimes_{f^2_{1,2}} e_1) = c_4\gamma_1\gamma_1,$$

$$h(\mathfrak{o}(f^2_{2,j}) \otimes_{f^2_{2,j}} \mathfrak{t}(f^2_{2,j})) = 0, \text{ for } j = \{1, \ldots, 6\},$$

$$h(\mathfrak{o}(f^2_{2,l}) \otimes_{f^2_{2,l}} \mathfrak{t}(f^2_{2,l})) = 0, \text{ for } l = 1, 2,$$

for some $c_2, c_4 \in K$ and so $dim \operatorname{Ker} d_3 = 2$.

Therefore $dim \, \mathrm{HH}^2(\Lambda) = 2 - 2 = 0$.

Theorem 10.3. *For $\Lambda = \Lambda(E_6, 1, 2)$ we have $\mathrm{HH}^2(\Lambda) = 0$.*

Now we summarise the results of chapters 4, 5, 6, 7, 8, 9, 10 using Theorem 5.11, as we have already pointed out, we have the following theorem.

Theorem 10.4. *Let Λ be the standard algebras of type $\Lambda(D_n, s, 1), \Lambda(D_n, s, 2)$, with $n \geq 4$, $\Lambda(D_4, s, 3)$, $\Lambda(D_{3m}, s/3, 1)$ with $m \geq 2, 3 \nmid s$, $\Lambda(E_n, s, 1)$ with $n \in \{6, 7, 8\}$ or $\Lambda(E_6, s, 2)$. If $s \geq 2$ then $\mathrm{HH}^2(\Lambda) = 0$.*

The case $s = 1$ has been dealt with in Theorems 4.3, 6.3, 7.3, 8.3, 9.3 and 10.3. Combining these with Theorem 10.4 we complete our description of the second Hochschild cohomology for all finite dimensional standard self-injective algebras over an algebraically closed field of type D and E, thus summarising Chapters 4 and 6-10.

Theorem 10.5. *Let Λ be a standard algebra of type $\Lambda(D_n, s, 1), \Lambda(D_4, s, 3)$ with $n \geq 4, s \geq 1$, $\Lambda(D_n, s, 2), \Lambda(D_{3m}, s/3, 1)$ with $n \geq 4, m \geq 2, s \geq 2$ or $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ with $n \in \{6, 7, 8\}, s \geq 1$. Then $\mathrm{HH}^2(\Lambda) = 0$.*

Let Λ be $\Lambda(D_n, 1, 2)$; then $dim \, \mathrm{HH}^2(\Lambda) = 1$.

Let Λ be $\Lambda(D_{3m}, 1/3, 1)$; then

$$dim \, \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } char\, K \neq 2, \\ 4 & \text{if } char\, K = 2. \end{cases}$$

To conclude we now know the second Hochschild cohomology group for all finite dimensional standard self-injective algebras over an algebraically closed field K.

Theorem 10.6. *Let* Λ *be a finite dimensional standard self-injective algebras over an algebraically closed field* K*. If* Λ *is of type* $\Lambda(A_{2p+1}, s, 2)$ *with* $s, p > 1$, $\Lambda(D_n, s, 1), \Lambda(D_4, s, 3)$ *with* $n \geq 4, s \geq 1$, $\Lambda(D_n, s, 2), \Lambda(D_{3m}, s/3, 1)$ *with* $n \geq 4, m \geq 2, s \geq 2$ *or* $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ *with* $n \in \{6, 7, 8\}, s \geq 1$*; then* $\mathrm{HH}^2(\Lambda) = 0$.

If Λ *is of type* $\Lambda(A_n, s/n, 1)$ *then* $\dim \mathrm{HH}^2(\Lambda) = m$.

For $\Lambda(A_{2p+1}, s, 2)$ *with* $s = p = 1$, $\dim \mathrm{HH}^2(\Lambda) = 1$.

Let Λ *be* $\Lambda(D_n, 1, 2)$*; then* $\dim \mathrm{HH}^2(\Lambda) = 1$.

Let Λ *be* $\Lambda(D_{3m}, 1/3, 1)$*; then*

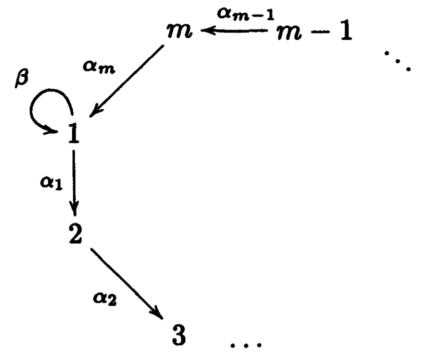
$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if char } K \neq 2, \\ 4 & \text{if char } K = 2. \end{cases}$$

Further a basis of $\mathrm{HH}^2(\Lambda)$ where $\dim HH^2(\Lambda) \neq 0$ for $\Lambda(A_n, s/n, 1)$, $\Lambda(A_{2p+1}, s, 2)$, $\Lambda(D_n, 1, 2)$ and $\Lambda(D_{3m}, 1/3, 1)$ can be found in [9], [10], 6.4 and 8.4 respectively.

We consider the nonstandard algebra in Chapter 11.

11. Nonstandard algebras

From [2] the derived equivalence representatives of the nonstandard self-injective algebras of finite representation type over an algebraically closed field K are the algebras $\Lambda(m)$ of type $(D_{3m}, 1/3, 1)$ for each $m \geq 2$, where $\Lambda(m)$ is given by the quiver $\mathcal{Q}(D_{3m}, 1/3)$:



with relations $R(m)$:

(i) $\alpha_1\alpha_2\cdots\alpha_m = \beta^2$,

(ii) $\alpha_m\alpha_1 = \alpha_m\beta\alpha_1$,

(iii) $\alpha_i\alpha_{i+1}\cdots\alpha_i = 0$, for all $i \in \{1,\ldots,m\} = \mathbb{Z}/\langle m\rangle$ (i.e. "α" paths of length $m+1$ are equal to 0).

We need a set f^2 of minimal relations. Relations of type (i) and (ii) are in f^2. So let $\beta^2 - \alpha_1\cdots\alpha_m =: f_1^2$ and $\alpha_m\alpha_1 - \alpha_m\beta\alpha_1 =: f_2^2$. Now consider the relations of type (iii). Let $f_{3,2}^2 := \alpha_2\alpha_3\cdots\alpha_2$ if $m \geq 3$. If $i = 1$ then the path $\alpha_1\cdots\alpha_m\alpha_1$ is one of the elements of I. It can be shown, for $m \geq 3$, that $\alpha_1\cdots\alpha_m\alpha_1 = \alpha_1\cdots\alpha_{m-1}f_2^2 - f_1^2\beta\alpha_1 + \beta f_1^2\alpha_1 + \beta\alpha_1\cdots\alpha_{m-1}f_2^2 - \beta f_1^2\beta\alpha_1 + f_1^2 f_1^2\alpha_1 + \alpha_1\cdots\alpha_m f_1^2\alpha_1 + f_1^2\alpha_1\cdots\alpha_m\alpha_1 + \alpha_1 f_{3,2}^2\alpha_3\cdots\alpha_m\alpha_1$. So $\alpha_1\cdots\alpha_m\alpha_1 \in I$ and not in f^2. Also it can be shown, for $m \geq 3$, that

$\alpha_m\alpha_1\cdots\alpha_m = f_2^2\alpha_2\cdots\alpha_m - \alpha_m\beta f_1^2 + \alpha_m f_1^2\beta + f_2^2\alpha_2\cdots\alpha_m\beta - \alpha_m\beta f_1^2\beta + \alpha_m f_1^2 f_1^2 + \alpha_m\cdots\alpha_m f_1^2 + \alpha_m f_1^2\alpha_1\cdots\alpha_m + \alpha_m\alpha_1 f_{3,2}^2\alpha_3\cdots\alpha_m$. So, if $m \geq 3$, then $\alpha_m\alpha_1\cdots\alpha_m \in I$ and not in f^2. If $m = 2$ then $\alpha_2\alpha_1\alpha_2 \in f^2$ and we let $f_{3,2}^2 := \alpha_2\alpha_1\alpha_2$. In this case $\alpha_1\alpha_2\alpha_1 = \alpha_1 f_2^2 - f_1^2\beta\alpha_1 + \beta f_1^2\alpha_1 + \beta\alpha_1 f_2^2 - \beta f_1^2\beta\alpha_1 + f_1^2 f_1^2\alpha_1 + \alpha_1\alpha_2 f_1^2\alpha_1 + f_1^2\alpha_1\alpha_2\alpha_1 + \alpha_1 f_{3,2}^2\alpha_1$ so is not in f^2. Let

$$f_{3,j}^2 := \alpha_j\alpha_{j+1}\cdots\alpha_j \text{ for } \begin{cases} j = 2,\ldots,m-1 & \text{if } m \geq 3, \\ j = 2 & \text{if } m = 2. \end{cases}$$

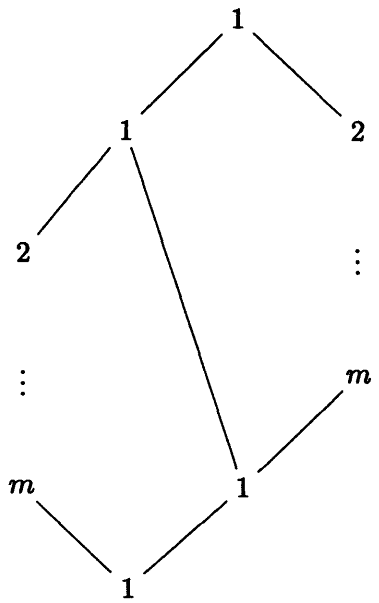
These paths cannot be obtained from any other relations. Now we have the elements of f^2 as follows.

$$f_1^2 = \beta^2 - \alpha_1\cdots\alpha_m, \qquad f_2^2 = \alpha_m\alpha_1 - \alpha_m\beta\alpha_1,$$

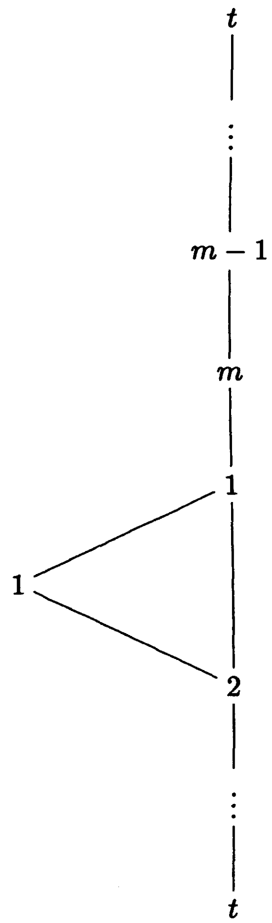
$$f_{3,j}^2 = \alpha_j\alpha_{j+1}\cdots\alpha_j \text{ for } \begin{cases} j = 2,\ldots,m-1 & \text{if } m \geq 3, \\ j = 2 & \text{if } m = 2. \end{cases}$$

Next we need to find f^3. The indecomposable projective right Λ-modules are:

$e_1\Lambda$



$e_t\Lambda$, for $2 \leq t \leq m$



From the minimal projective resolution of each simple Λ-module we see that: for $2 \leq t \leq m-2$ we have,

$$\Omega^3(S_t) \quad \cong \quad \begin{tikzcd}[row sep=small]
 & t+2 \arrow[d, dash] \\
 & t+3 \arrow[d, dash] \\
 & \vdots \arrow[d, dash] \\
 & m-1 \arrow[d, dash] \\
 & m \arrow[d, dash] \\
 & 1 \arrow[dd, dash] \\
1 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 2 \arrow[d, dash] \\
 & \vdots \arrow[d, dash] \\
 & t \arrow[d, dash] \\
 & t+1
\end{tikzcd}$$

and that

$$\Omega^3(S_{m-1}) \quad \cong \quad \begin{tikzcd}[row sep=small]
 & 1 \arrow[d, dash] \\
 & 1 \arrow[dd, dash] \\
1 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 2 \arrow[d, dash] \\
 & \vdots \arrow[d, dash] \\
 & m-1 \arrow[d, dash] \\
 & m
\end{tikzcd}$$

For $\Omega^3(S_1)$ and $\Omega^3(S_m)$ we need more details. We will find $\Omega^3(S_1)$ first. We have the map

$$\psi : e_1\Lambda \oplus e_2\Lambda \to \Omega(S_1)$$

given by:

$$e_1\lambda \mapsto \beta e_1\lambda,$$

$$e_2\mu \mapsto \alpha_1 e_2\mu,$$

where $\lambda, \mu \in \Lambda$. Note that $\Omega^2(S_1) = \operatorname{Ker}\psi$.

Proposition 11.1. $\Omega^2(S_1) = (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$.

Proof. First let $x \in \Omega^2(S_1)$. Then $x = (e_1\lambda, e_2\mu)$ with $\lambda, \mu \in \Lambda$. Write $e_1\lambda = c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}$ and $e_2\mu = d_0e_2 + d_1\alpha_2 + d_2\alpha_2\alpha_3 + \cdots + d_{m-1}\alpha_2 \cdots \alpha_m + d_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1$ where $c_i, d_i, c'_j \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x) = 0$, that is, $\beta(c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots + c_{m-1}\alpha_1 \cdots \alpha_{m-1} + c_m\alpha_1 \cdots \alpha_m + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}) + \alpha_1(d_0e_2 + d_1\alpha_2 + d_2\alpha_2\alpha_3 + \cdots + d_{m-1}\alpha_2 \cdots \alpha_m + d_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = 0$. So $c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + c_m\beta^3 + c_{m+1}\beta^2 + d_0\alpha_1 + d_1\alpha_1\alpha_2 + d_2\alpha_1\alpha_2\alpha_3 + \cdots + d_{m-2}\alpha_1 \cdots \alpha_{m-1} + d_{m-1}\alpha_1 \cdots \alpha_m + d_m\alpha_1 \cdots \alpha_m\beta = c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1 \cdots \alpha_{m-1} + d_0\alpha_1 + d_1\alpha_1\alpha_2 + d_2\alpha_1\alpha_2\alpha_3 + \cdots + d_{m-2}\alpha_1 \cdots \alpha_{m-1} + (d_{m-1} + c_{m+1})\beta^2 + (d_m + c_m)\beta^3 = 0$. Thus $c_0 = c_1 = \cdots = c_{m-1} = d_0 = \cdots = d_{m-2} = 0, d_{m-1} + c_{m+1} = 0$ and $d_m + c_m = 0$. Therefore, $x = (c_m\beta^2 + c_{m+1}\beta + c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}, -c_{m+1}\alpha_2 \cdots \alpha_m - c_m\alpha_2 \cdots \alpha_m\beta + d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = c_{m+1}(\beta, -\alpha_2 \cdots \alpha_m) + c_m(\beta, -\alpha_2 \cdots \alpha_m)\beta + (c'_0\beta^3 + c'_1\beta\alpha_1 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1 \cdots \alpha_{m-1}, d_{m+1}\alpha_2 \cdots \alpha_m\beta\alpha_1) = (\beta, -\alpha_2 \cdots \alpha_m)(c_{m+1}e_1 + c_m\beta + c'_0\beta^2 + c'_1\alpha_1 + c'_2\alpha_1\alpha_2 + \cdots + c'_{m-1}\alpha_1 \cdots \alpha_{m-1} - (c'_1 + d_{m+1})\beta\alpha_1) = (\beta, -\alpha_2 \cdots \alpha_m)\nu$, where $\nu \in \Lambda$. So $x \in (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$. Thus $\Omega^2(S_1) \subseteq (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$.

On the other hand, let $x \in (\beta, -\alpha_2 \cdots \alpha_m)\Lambda$. So $x = (\beta, -\alpha_2 \cdots \alpha_m)\lambda = (e_1\beta\lambda, -e_2\alpha_2 \cdots \alpha_m\lambda)$, where $\lambda \in \Lambda$. Then $\psi(x) = (\beta^2 - \alpha_1 \cdots \alpha_m)\lambda = 0$. Thus $(\beta, -\alpha_2 \cdots \alpha_m)\Lambda \subseteq \Omega^2(S_1)$.

Therefore, $\Omega^2(S_1) = (\beta, -\alpha_2 \cdots \alpha_m)e_1\Lambda$. ◻

To find $\Omega^3(S_1)$ we have that $\Omega^3(S_1) = \operatorname{Ker}\theta$ where θ is the map:

$$\theta : e_1\Lambda \to \Omega^2(S_1)$$

given by:

$$e_1\lambda \mapsto (\beta, -\alpha_2\cdots\alpha_m)e_1\lambda$$

where $\lambda \in \Lambda$.

Proposition 11.2. $\Omega^3(S_1) = \beta\alpha_1\alpha_2 e_3\Lambda$.

Proof. On one hand, let $y \in \Omega^3(S_1)$. Then $y = e_1\lambda$ where $\lambda \in \Lambda$. Write $y = c_0e_1+c_1\alpha_1+c_2\alpha_1\alpha_2+\cdots+c_{m-1}\alpha_1\cdots\alpha_{m-1}+c_m\alpha_1\cdots\alpha_m+c_{m+1}\beta+c'_0\beta^3+c'_1\beta\alpha_1+c'_2\beta\alpha_1\alpha_2+\cdots+c'_{m-1}\beta\alpha_1\cdots\alpha_{m-1}$ where $c_i, c'_j \in K$. Since $y \in \operatorname{Ker}\theta$ we know that $\theta(y) = 0$. Thus $0 = (\beta, -\alpha_2\cdots\alpha_m)(c_0e_1 + c_1\alpha_1 + c_2\alpha_1\alpha_2 + \cdots+c_{m-1}\alpha_1\cdots\alpha_{m-1}+c_m\alpha_1\cdots\alpha_m+c_{m+1}\beta+c'_0\beta^3+c'_1\beta\alpha_1+c'_2\beta\alpha_1\alpha_2+\cdots+c'_{m-1}\beta\alpha_1\cdots\alpha_{m-1}) = (c_0\beta + c_1\beta\alpha_1 + c_2\beta\alpha_1\alpha_2 + \cdots + c_{m-1}\beta\alpha_1\cdots\alpha_{m-1} + c_m\beta\alpha_1\cdots\alpha_m + c_{m+1}\beta^2, -c_0\alpha_2\cdots\alpha_m - (c'_1 + c_1)\alpha_2\cdots\alpha_m\beta\alpha_1)$. Thus $c_0 = \cdots = c_{m+1} = 0$ and $c'_1 + c_1 = 0$ so that $c'_1 = 0$. Therefore, $y = c'_0\beta^3 + c'_2\beta\alpha_1\alpha_2 + \cdots + c'_{m-1}\beta\alpha_1\cdots\alpha_{m-1} = \beta\alpha_1\alpha_2(c'_0\alpha_3\cdots\alpha_m + c'_2e_3 + \cdots + c'_{m-1}\alpha_3\cdots\alpha_{m-1}) = \beta\alpha_1\alpha_2\mu$, where $\mu \in \Lambda$. So $y \in \beta\alpha_1\alpha_2\Lambda$. Thus $\Omega^3(S_1) \subseteq \beta\alpha_1\alpha_2\Lambda$.

On the other hand, let $y \in \beta\alpha_1\alpha_2\Lambda$. So $y = \beta\alpha_1\alpha_2\lambda$ where $\lambda \in \Lambda$. Then $\theta(y) = (\beta, -\alpha_2\cdots\alpha_m)\beta\alpha_1\alpha_2\lambda = 0$. So $y \in \Omega^3(S_1)$. Thus $\beta\alpha_1\alpha_2\Lambda \subseteq \Omega^3(S_1)$.

Therefore, $\Omega^3(S_1) = \beta\alpha_1\alpha_2e_3\Lambda$. □

To find $\Omega^3(S_m)$ we have the map

$$\psi : e_1\Lambda \to \Omega(S_m)$$

given by:

$$e_1\lambda \mapsto \alpha_m e_1\lambda, \text{ where } \lambda \in \Lambda.$$

Note that $\Omega^2(S_m) = \operatorname{Ker}\psi$.

Proposition 11.3. $\Omega^2(S_m) = (\alpha_1 - \beta\alpha_1)\Lambda$.

Proof. First let $x \in \Omega^2(S_m)$. Then $x = e_1\lambda$ with $\lambda \in \Lambda$. Write $x = c_0e_1 + c_1\alpha_1+c_2\alpha_1\alpha_2+\cdots+c_{m-1}\alpha_1\cdots\alpha_{m-1}+c_m\alpha_1\cdots\alpha_m+c_{m+1}\beta+c'_0\beta^3+c'_1\beta\alpha_1+c'_2\beta\alpha_1\alpha_2+\cdots+c'_{m-1}\beta\alpha_1\cdots\alpha_{m-1}$ where $c_i, c'_j \in K$. Since $x \in \operatorname{Ker}\psi$ we know that $\psi(x)=0$, that is, $\alpha_m(c_0e_1+c_1\alpha_1+c_2\alpha_1\alpha_2+\cdots+c_{m-1}\alpha_1\cdots\alpha_{m-1}+c_m\alpha_1\cdots\alpha_m+c_{m+1}\beta+c'_0\beta^3+c'_1\beta\alpha_1+c'_2\beta\alpha_1\alpha_2+\cdots+c'_{m-1}\beta\alpha_1\cdots\alpha_{m-1})=0$.

So $c_0\alpha_m+c_1\alpha_m\alpha_1+c_2\alpha_m\alpha_1\alpha_2+\cdots+c_{m-1}\alpha_m\alpha_1\cdots\alpha_{m-1}+c_{m+1}\alpha_m\beta+c'_1\alpha_m\beta\alpha_1+c'_2\alpha_m\beta\alpha_1\alpha_2+\cdots+c'_{m-1}\alpha_m\beta\alpha_1\cdots\alpha_{m-1}=c_0\alpha_m+(c_1+c'_1)\alpha_m\alpha_1+(c_2+c'_2)\alpha_m\alpha_1\alpha_2+\cdots+(c_{m-1}+c'_{m-1})\alpha_m\alpha_1\cdots\alpha_{m-1}+c_{m+1}\alpha_m\beta=0$. Thus $c_0=c_{m+1}=0$ and $c_1+c'_1=c_2+c'_2=\cdots=c_{m-1}+c'_{m-1}=0$. Therefore, $x=c_1\alpha_1+c_2\alpha_1\alpha_2+\cdots+c_{m-1}\alpha_1\cdots\alpha_{m-1}+c_m\alpha_1\cdots\alpha_m+c'_0\beta^3-c_1\beta\alpha_1-\ldots-c_{m-1}\beta\alpha_1\cdots\alpha_{m-1}=c_1(\alpha_1-\beta\alpha_1)+c_2(\alpha_1-\beta\alpha_1)\alpha_2+\cdots+c_{m-1}(\alpha_1-\beta\alpha_1)\alpha_2\cdots\alpha_m+c_m\alpha_1\cdots\alpha_m+c'_0\beta^3=(\alpha_1-\beta\alpha_1)(c_1e_2+c_2\alpha_2+\cdots+c_{m-2}\alpha_2\cdots\alpha_{m-1}+(c_{m-1}+c_m)\alpha_2\cdots\alpha_m)+(c_m+c'_0)\beta^3=(\alpha_1-\beta\alpha_1)(c_1e_2+c_2\alpha_2+\cdots+c_{m-2}\alpha_2\cdots\alpha_{m-1}+(c_{m-1}+c_m)\alpha_2\cdots\alpha_m+(c_m+c'_0)(\alpha_2\cdots\alpha_m\beta))=(\alpha_1-\beta\alpha_1)\nu$, where $\nu\in\Lambda$. So $x\in(\alpha_1-\beta\alpha_1)\Lambda$. Thus $\Omega^2(S_m)\subseteq(\alpha_1-\beta\alpha_1)\Lambda$.

On the other hand, let $x\in(\alpha_1-\beta\alpha_1)\Lambda$. So $x=(\alpha_1-\beta\alpha_1)\lambda$, where $\lambda\in\Lambda$. Then $\psi(x)=(\alpha_m\alpha_1-\alpha_m\beta\alpha_1)\lambda=0$. Thus $(\alpha_1-\beta\alpha_1)\Lambda\subseteq\Omega^2(S_m)$.

Therefore, $\Omega^2(S_m)=(\alpha_1-\beta\alpha_1)e_2\Lambda$. □

To find $\Omega^3(S_m)$ we have that $\Omega^3(S_m)=\operatorname{Ker}\theta$ where θ is the map:

$$\theta: e_2\Lambda\to\Omega^2(S_m)$$

given by:

$$e_2\lambda\mapsto(\alpha_1-\beta\alpha_1)e_2\lambda$$

where $\lambda\in\Lambda$.

Proposition 11.4. $\Omega^3(S_m)=\alpha_2\cdots\alpha_m\alpha_1e_2\Lambda$.

Proof. On one hand, let $y\in\Omega^3(S_m)$. Then $y=e_2\lambda$ where $\lambda\in\Lambda$. Write $y=d_0e_2+d_1\alpha_2+d_2\alpha_2\alpha_3+\cdots+d_{m-1}\alpha_2\cdots\alpha_m+d_m\alpha_2\cdots\alpha_m\beta+d_{m+1}\alpha_2\cdots\alpha_m\alpha_1$ where $d_i\in K$. Since $y\in\operatorname{Ker}\theta$ we know that $\theta(y)=0$. Thus $0=(\alpha_1-\beta\alpha_1)(d_0e_2+d_1\alpha_2+d_2\alpha_2\alpha_3+\cdots+d_{m-1}\alpha_2\cdots\alpha_m+d_m\alpha_2\cdots\alpha_m\beta+d_{m+1}\alpha_2\cdots\alpha_m\alpha_1)=d_0(\alpha_1-\beta\alpha_1)+d_1(\alpha_1-\beta\alpha_1)\alpha_2+d_2(\alpha_1-\beta\alpha_1)\alpha_2\alpha_3+\cdots+d_{m-2}(\alpha_1-\beta\alpha_1)\alpha_2\cdots\alpha_{m-1}+d_{m-1}\alpha_1\cdots\alpha_m+(d_m-d_{m-1})\beta^3$. Thus $d_0=\cdots=d_{m-1}=0$ and $d_m-d_{m-1}=0$ so that $d_m=0$. Therefore, $y=d_{m+1}\alpha_2\cdots\alpha_m\alpha_1\mu$, where $\mu\in\Lambda$. So $y\in\alpha_2\cdots\alpha_m\alpha_1\Lambda$. Thus $\Omega^3(S_m)\subseteq\alpha_2\cdots\alpha_m\alpha_1\Lambda$.

On the other hand, let $y \in \alpha_2 \cdots \alpha_m \alpha_1 \Lambda$. So $y = \alpha_2 \cdots \alpha_m \alpha_1 \lambda$ where $\lambda \in \Lambda$. Then $\theta(y) = (\alpha_1 - \beta\alpha_1)\alpha_2 \cdots \alpha_m \alpha_1 \lambda = 0$. So $y \in \Omega^3(S_m)$. Thus $\alpha_2 \cdots \alpha_m \alpha_1 \Lambda \subseteq \Omega^3(S_m)$.

Therefore, $\Omega^3(S_m) = \alpha_2 \cdots \alpha_m \alpha_1 e_2 \Lambda$. □

To determine $\mathrm{HH}^2(\Lambda)$ we start by finding $\mathrm{Im}\, d_2$.

Find $\mathbf{Im}\, \mathbf{d_2}$.

We know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$. First we will find $\mathrm{Im}\, d_2$. Since $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$, then $d_2 f \in \mathrm{Im}\, d_2$, where $f \in \mathrm{Hom}(Q^1, \Lambda)$ and $d_2 f = f A_2$. Here $Q^1 = (\Lambda e_1 \otimes_\beta e_1 \Lambda) \oplus \bigoplus_{l=1}^{m-1} (\Lambda e_l \otimes_{\alpha_l} e_{l+1} \Lambda) \oplus (\Lambda e_m \otimes_{\alpha_m} e_1 \Lambda)$. Let $f \in \mathrm{Hom}(Q^1, \Lambda)$ and so

$$f(e_1 \otimes_\beta e_1) = c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3,$$

$$f(e_l \otimes_{\alpha_l} e_{l+1}) = d_l \alpha_l, \text{ for } l \in \{1, \ldots, m-1\},$$

$$f(e_m \otimes_{\alpha_m} e_1) = d_m \alpha_m,$$

where $c_1, c_2, c_3, c_4, d_l, d_m \in K$.

We have $Q^2 = (\Lambda e_1 \otimes_{f_1^2} e_1 \Lambda) \oplus (\Lambda e_m \otimes_{f_2^2} e_2 \Lambda) \oplus \bigoplus_{j=2}^{m-1} (\Lambda e_j \otimes_{f_{3,j}^2} e_{j+1} \Lambda)$ if $m \geq 3$ and $Q^2 = (\Lambda e_1 \otimes_{f_1^2} e_1 \Lambda) \oplus (\Lambda e_2 \otimes_{f_2^2} e_2 \Lambda) \oplus (\Lambda e_2 \otimes_{f_{3,2}^2} e_3 \Lambda)$ if $m = 2$.

Now we find $f A_2$. We have

$f A_2(e_1 \otimes_{f_1^2} e_1) = f(e_1 \otimes_\beta e_1)\beta + \beta f(e_1 \otimes_\beta e_1) - f(e_1 \otimes_{\alpha_1} e_2)\alpha_2 \cdots \alpha_m - \alpha_1 f(e_2 \otimes_{\alpha_2} e_3)\alpha_3 \cdots \alpha_m - \cdots - \alpha_1 \alpha_2 \cdots \alpha_{m-1} f(e_m \otimes_{\alpha_m} e_1) = (c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3)\beta + \beta(c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3) - d_1 \alpha_1 \cdots \alpha_m - d_2 \alpha_1 \cdots \alpha_m - \ldots - d_m \alpha_1 \cdots \alpha_m = 2c_1 \beta - (d_1 + d_2 + \cdots + d_m - 2c_2)\beta^2 + 2c_3 \beta^3$.

Also $f A_2(e_m \otimes_{f_2^2} e_2) = f(e_m \otimes_{\alpha_m} e_1)\alpha_1 + \alpha_m f(e_1 \otimes_{\alpha_1} e_2) - f(e_m \otimes_{\alpha_m} e_1)\beta\alpha_1 - \alpha_m f(e_1 \otimes_\beta e_1)\alpha_1 - \alpha_m \beta f(e_1 \otimes_{\alpha_1} e_2) = d_m \alpha_m \alpha_1 + d_1 \alpha_m \alpha_1 - d_m \alpha_m \beta \alpha_1 - \alpha_m (c_1 e_1 + c_2 \beta + c_3 \beta^2 + c_4 \beta^3)\alpha_1 - d_1 \alpha_m \beta \alpha_1 = (d_m + d_1 - c_1)\alpha_m \alpha_1 - (d_m + c_2 + d_1)\alpha_m \beta \alpha_1 = (d_m + d_1 - c_1 - d_m - c_2 - d_1)\alpha_m \alpha_1 = -(c_1 + c_2)\alpha_m \alpha_1$.

Finally, for $m \geq 3$ and $j = 2, \ldots, m-1$ or for $m = 2$ and $j = 2$, we have $f A_2(e_j \otimes_{f_{3,j}^2} e_{j+1}) = f(e_j \otimes_{\alpha_j} e_{j+1})\alpha_{j+1} \cdots \alpha_j + \alpha_j f(e_{j+1} \otimes_{\alpha_{j+1}} e_{j+2})\alpha_{j+2} \cdots \alpha_j + \cdots + \alpha_j \cdots \alpha_{j-1} f(e_j \otimes_{\alpha_j} e_{j+1}) = (d_j + d_{j+1} + \cdots + d_j)\alpha_j \cdots \alpha_j = 0$.

Thus fA_2 is given by

$$fA_2(e_1\otimes_{f_1^2}e_1) = 2c_1\beta-(d_1+d_2+\cdots+d_m-2c_2)\beta^2+2c_3\beta^3 = 2c'\beta+c''\beta^2+2c'''\beta^3,$$

$$fA_2(e_m\otimes_{f_2^2}e_2) = -(c_1+c_2)\alpha_m\alpha_1,$$

$$f(e_j\otimes_{f_{3,j}^2}e_{j+1}) = 0, \text{ for all } j\in\{2,\dots,m-1\} \text{ if } m\geq 3 \text{ or } j=2 \text{ if } m=2$$

for some $c', c'', c''', c_1, c_2 \in K$. So

$$dim\,\mathrm{Im}\,d_2 = \begin{cases} 4 & \text{if } char\,K \neq 2, \\ 2 & \text{if } char\,K = 2. \end{cases}$$

Next we determine $\mathrm{Ker}\,d_3$ in the case where $m \geq 4$. We will consider the cases $m=2$ and $m=3$ later.

Find Ker $\mathbf{d_3}$ for $m \geq 4$.

From the projective resolutions for simples we know,, for $m\geq 4$, that the third projective $Q^3 = (\Lambda e_1\otimes e_3\Lambda)\oplus\bigoplus_{t=2}^{m-2}(\Lambda e_t\otimes e_{t+2}\Lambda)\oplus(\Lambda e_{m-1}\otimes e_1\Lambda)\oplus(\Lambda e_m\otimes e_2\Lambda)$.

For $m\geq 4$, the set f^3 consists of the following elements:

$$\{f_1^3, f_{2,t}^3, f_3^3, f_4^3\} \text{ with } t\in\{2,\dots,m-2\} \text{ where}$$

$$\begin{array}{lclcl}
f_1^3 &=& f_1^2\beta\alpha_1\alpha_2 &=& \beta f_1^2\alpha_1\alpha_2+\alpha_1\cdots\alpha_{m-1}f_2^2\alpha_2+(\beta\alpha_1-\alpha_1)f_{3,2}^2,\\
f_{2,t}^3 &=& f_{3,t}^2\alpha_{t+1} &=& \alpha_t f_{3,t+1}^2,\\
f_3^3 &=& f_{3,m-1}^2(\alpha_m-\alpha_m\beta) &=& \alpha_{m-1}f_2^2\alpha_2\cdots\alpha_m+\alpha_{m-1}\alpha_m f_1^2\beta-\alpha_{m-1}\alpha_m\beta f_1^2,\\
f_4^3 &=& f_2^2\alpha_2\cdots\alpha_m\alpha_1 &=& -\alpha_m f_1^2\beta\alpha_1+\alpha_m\beta f_1^2\alpha_1+\alpha_m\alpha_1\cdots\alpha_{m-1}f_2^2.
\end{array}$$

We have $d_3 : \mathrm{Hom}(Q^2,\Lambda)\to\mathrm{Hom}(Q^3,\Lambda)$. Let $h\in\mathrm{Ker}\,d_3$, so $h\in\mathrm{Hom}(Q^2,\Lambda)$ and $d_3h=0$. Then $h : Q^2\to\Lambda$ is given by

$$h(e_1\otimes_{f_1^2}e_1) = c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3,$$

$$h(e_m\otimes_{f_2^2}e_2) = c_5\alpha_m\alpha_1 \text{ and}$$

$$h(e_j\otimes_{f_{3,j}^2}e_{j+1}) = d_j\alpha_j, \text{ for } j\in\{2,\dots,m-1\},$$

for some $c_1,\dots,c_5, d_j\in K$ where $j=2,\dots,m-1$.

Then $hA_3(e_1\otimes_{f_1^3}e_3) = h(e_1\otimes_{f_1^2}e_1)\beta\alpha_1\alpha_2-\beta h(e_1\otimes_{f_1^2}e_1)\alpha_1\alpha_2-\alpha_1\cdots\alpha_{m-1}h(e_m\otimes_{f_2^2}e_2)\alpha_2-(\beta\alpha_1-\alpha_1)h(e_2\otimes_{f_{3,2}^2}e_3) = (c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\beta\alpha_1\alpha_2 -$

$\beta(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\alpha_1\alpha_2-c_5\alpha_1\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2-d_2\beta\alpha_1\alpha_2+d_2\alpha_1\alpha_2 = d_2(\alpha_1\alpha_2-\beta\alpha_1\alpha_2)$. As $h \in \operatorname{Ker} d_3$ we have $d_2 = 0$.

For $t \in \{2,\ldots,m-2\}$, we have $hA_3(e_t \otimes_{f^3_{2,t}} e_{t+2}) = h(e_t \otimes_{f^2_{3,t}} e_{t+1})\alpha_{t+1} - \alpha_t h(e_{t+1} \otimes_{f^2_{3,t+1}} e_{t+2}) = d_t\alpha_t\alpha_{t+1} - d_{t+1}\alpha_t\alpha_{t+1} = (d_t - d_{t+1})\alpha_t\alpha_{t+1}$. Then $d_t - d_{t+1} = 0$ and so $d_t = d_{t+1}$ for $t = 2,\ldots,m-2$. Hence $d_2 = d_3 = \ldots = d_{m-2} = d_{m-1}$. We already have $d_2 = 0$ so $d_j = 0$ for $j = 2,\ldots,m-1$.

Now $hA_3(e_{m-1}\otimes_{f^3_3} e_1) = h(e_{m-1}\otimes_{f^2_{3,m-1}} e_m)(\alpha_m - \alpha_m\beta) - \alpha_{m-1}h(e_m\otimes_{f^2_2} e_2)\alpha_2\cdots\alpha_m - \alpha_{m-1}\alpha_m h(e_1\otimes_{f^2_1} e_1)\beta + \alpha_{m-1}\alpha_m\beta h(e_1\otimes_{f^2_1} e_1) = d_{m-1}\alpha_{m-1}\alpha_m - d_{m-1}\alpha_{m-1}\alpha_m\beta - c_5\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_m - \alpha_{m-1}\alpha_m(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\beta + \alpha_{m-1}\alpha_m\beta(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3) = d_{m-1}(\alpha_{m-1}\alpha_m - \alpha_{m-1}\alpha_m\beta) = 0$, as $d_{m-1} = 0$ above.

Finally, $hA_3(e_m\otimes_{f^3_4} e_2) = h(e_m\otimes_{f^2_2} e_2)\alpha_2\cdots\alpha_m\alpha_1 + \alpha_m h(e_1\otimes_{f^2_1} e_1)\beta\alpha_1 - \alpha_m\beta h(e_1\otimes_{f^2_1} e_1)\alpha_1 - \alpha_m\alpha_1\cdots\alpha_{m-1}h(e_m\otimes_{f^2_2} e_2) = c_5\alpha_m\alpha_1\alpha_2\cdots\alpha_m\alpha_1 + \alpha_m(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\beta\alpha_1 - \alpha_m\beta(c_1e_1+c_2\beta+c_3\beta^2+c_4\beta^3)\alpha_1 - c_5\alpha_m\alpha_1\cdots\alpha_{m-1}\alpha_m\alpha_1 = -c_1\alpha_m\beta\alpha_1 + c_1\alpha_m\beta\alpha_1 = 0$, and so this gives no information on the constants occuring in h.

Thus h is given by

$$h(e_1\otimes_{f^2_1} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$h(e_m\otimes_{f^2_2} e_2) = c_5\alpha_m\alpha_1 \text{ and}$$

$$h(e_j\otimes_{f^2_{3,j}} e_{j+1}) = 0, \text{ for } j \in \{2,\ldots,m-1\}$$

for some $c_1,\ldots,c_5 \in K$ and so $\dim \operatorname{Ker} d_3 = 5$.

Therefore, for $m \geq 4$ we have

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 5-4=1 & \text{if } char\, K \neq 2, \\ 5-2=3 & \text{if } char\, K = 2. \end{cases}$$

Theorem 11.5. *For $\Lambda = \Lambda(m)$ and $m \geq 4$ we have*

$$\dim \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } char\, K \neq 2, \\ 3 & \text{if } char\, K = 2. \end{cases}$$

11.6. *A basis for* $\mathrm{HH}^2(\Lambda)$ *for* $\Lambda = \Lambda(m)$ *and* $m \geq 4$.

char K $\neq$ **2.**

Suppose that $m \geq 4$. We know that $dim\,\mathrm{HH}^2(\Lambda) = 1$. So we need to find one non-zero element in $\mathrm{HH}^2(\Lambda)$. Define the non-zero map h_1 in $\mathrm{Ker}\, d_3$ to be given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0. \end{array}$$

Now suppose for contradiction that $h_1 \in \mathrm{Im}\, d_2$. Then $h_1(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$ for some $f \in \mathrm{Hom}(Q^1, \Lambda)$. So we have $e_1 = 2c'\beta + c''\beta^2 + 2c'''\beta^3$ for some $c', c'', c''' \in K$. This gives a contradiction. Therefore $h_1 \notin \mathrm{Im}\, d_2$. Hence $h_1 + \mathrm{Im}\, d_2$ is a non-zero element in $\mathrm{HH}^2(\Lambda)$.

So $\{h_1 + \mathrm{Im}\, d_2\}$ is a basis of $\mathrm{HH}^2(\Lambda)$ when $char\, K \neq 2$.

char K $=$ **2.**

Suppose that $m \geq 4$. Here $dim\,\mathrm{HH}^2(\Lambda) = 3$. So we need to find three non-zero linearly independent elements in $\mathrm{HH}^2(\Lambda)$. We start by defining non-zero maps h_1, h_2, h_3 in $\mathrm{Ker}\, d_3$.

Let h_1 be the map given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0, \end{array}$$

h_2 be given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta^3, \\ \text{else} & \mapsto & 0, \end{array}$$

and h_3 be given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta, \\ \text{else} & \mapsto & 0. \end{array}$$

It can be shown as before that these maps are not in $\mathrm{Im}\, d_2$. Now we will show that $\{h_1 + \mathrm{Im}\, d_2, h_2 + \mathrm{Im}\, d_2, h_3 + \mathrm{Im}\, d_2\}$ is a linearly independent set in $\mathrm{Ker}\, d_3/\mathrm{Im}\, d_2 = \mathrm{HH}^2(\Lambda)$.

Suppose $a(h_1 + \mathrm{Im}\, d_2) + b(h_2 + \mathrm{Im}\, d_2) + c(h_3 + \mathrm{Im}\, d_2) = 0 + \mathrm{Im}\, d_2$ for some $a, b, c \in K$. So $ah_1 + bh_2 + ch_3 \in \mathrm{Im}\, d_2$. Hence $ah_1 + bh_2 + ch_3 = fA_2$ for some $f \in \mathrm{Hom}(Q^1, \Lambda)$. Then $(ah_1 + bh_2 + ch_3)(e_1 \otimes_{f_1^2} e_1) = fA_2(e_1 \otimes_{f_1^2} e_1)$. So $ae_1 + b\beta^3 + c\beta = c''\beta^2$ for some $c'' \in K$. Since $\{e_1, \beta, \beta^2, \beta^3\}$ is linearly independent in Λ, we have $a = b = c = c'' = 0$. Hence $\{h_1 + \mathrm{Im}\, d_2, h_2 +$

$\operatorname{Im} d_2, h_3 + \operatorname{Im} d_2\}$ is linearly independent in $\mathrm{HH}^2(\Lambda)$ and forms a basis of $\mathrm{HH}^2(\Lambda)$ when *char* $K = 2$.

11.7. $\mathrm{HH}^2(\Lambda)$ *for* $\Lambda = \Lambda(m)$ *and* $m = 3$.

Note that if $m = 3$ then the set f^3 is $\{f_1^3, f_3^3, f_4^3\}$ where

$$\begin{array}{lllll} f_1^3 & = & f_1^2\beta\alpha_1\alpha_2 & = & \beta f_1^2\alpha_1\alpha_2 + \alpha_1\alpha_2 f_2^2\alpha_2 + (\beta\alpha_1 - \alpha_1)f_{3,2}^2, \\ f_3^3 & = & f_{3,2}^2(\alpha_3 - \alpha_3\beta) & = & \alpha_2 f_2^2\alpha_2\alpha_3 + \alpha_2\alpha_3 f_1^2\beta - \alpha_2\alpha_3\beta f_1^2, \\ f_4^3 & = & f_2^2\alpha_2\alpha_3\alpha_1 & = & -\alpha_3 f_1^2\beta\alpha_1 + \alpha_3\beta f_1^2\alpha_1 + \alpha_3\alpha_1\alpha_2 f_2^2. \end{array}$$

To find $\operatorname{Ker} d_3$. It can be seen easily from the case where $m \geq 4$ that a typical element $h \in \operatorname{Ker} d_3$ is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$h(e_3 \otimes_{f_2^2} e_2) = c_5\alpha_3\alpha_1 \text{ and}$$

$$h(e_2 \otimes_{f_{3,2}^2} e_3) = 0,$$

for some $c_1, \ldots, c_5 \in K$ and so *dim* $\operatorname{Ker} d_3 = 5$. Recall that

$$dim \operatorname{Im} d_2 = \begin{cases} 4 & \text{if } char\ K \neq 2, \\ 2 & \text{if } char\ K = 2. \end{cases}$$

Therefore,

$$dim\, \mathrm{HH}^2(\Lambda) = \begin{cases} 5 - 4 = 1 & \text{if } char\ K \neq 2, \\ 5 - 2 = 3 & \text{if } char\ K = 2. \end{cases}$$

A basis for $\Lambda = \Lambda(m)$ where $m = 3$ is the same basis as for $m \geq 4$ above.

11.8. $\mathrm{HH}^2(\Lambda)$ *for* $\Lambda = \Lambda(m)$ *and* $m = 2$.

The set f^3 is the set $\{f_1^3, f_4^3\}$ where

$$\begin{array}{lllll} f_1^3 & = & f_1^2\beta\alpha_1\alpha_2 & = & \beta f_1^2\alpha_1\alpha_2 + \alpha_1 f_2^2\alpha_2 + (\beta\alpha_1 - \alpha_1)f_{3,2}^2, \\ f_4^3 & = & f_2^2\alpha_2\alpha_1 & = & -\alpha_2 f_1^2\beta\alpha_1 + \alpha_2\beta f_1^2\alpha_1 + \alpha_2\alpha_1 f_2^2. \end{array}$$

Let $h \in \operatorname{Ker} d_3$. So $h \in \operatorname{Hom}(Q^2, \Lambda)$ and

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$h(e_2 \otimes_{f_2^2} e_2) = c_5\alpha_2\alpha_1 \text{ and}$$

$$h(e_2 \otimes_{f_{3,2}^2} e_1) = d_2\alpha_2$$

for some $c_1, \ldots, c_5, d_2 \in K$.

Then $hA_3(e_1 \otimes_{f_1^3} e_1) = h(e_1 \otimes_{f_1^2} e_1)\beta\alpha_1\alpha_2 - \beta h(e_1 \otimes_{f_1^2} e_1)\alpha_1\alpha_2 - \alpha_1 h(e_2 \otimes_{f_2^2} e_2)\alpha_2 - (\beta\alpha_1 - \alpha_1)h(e_2 \otimes_{f_{3,2}^2} e_1) = (c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta\alpha_1\alpha_2 - \beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\alpha_1\alpha_2 - c_5\alpha_1\alpha_2\alpha_1\alpha_2 - d_2\beta\alpha_1\alpha_2 + d_2\alpha_1\alpha_2 = d_2(\alpha_1\alpha_2 - \beta\alpha_1\alpha_2)$. As $h \in \operatorname{Ker} d_3$ we have $d_2 = 0$.

Also $hA_3(e_2 \otimes_{f_4^3} e_2) = h(e_2 \otimes_{f_2^2} e_2)\alpha_2\alpha_1 + \alpha_2 h(e_1 \otimes_{f_1^2} e_1)\beta\alpha_1 - \alpha_2\beta h(e_1 \otimes_{f_1^2} e_1)\alpha_1 - \alpha_2\alpha_1 h(e_2 \otimes_{f_2^2} e_2) = c_5\alpha_2\alpha_1\alpha_2\alpha_1 + \alpha_2(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\beta\alpha_1 - \alpha_2\beta(c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3)\alpha_1 - c_5\alpha_2\alpha_1\alpha_2\alpha_1 = -c_1\alpha_2\beta\alpha_1 + c_1\alpha_2\beta\alpha_1 = 0$, and so this gives no information on the constants occuring in h.

So h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\beta + c_3\beta^2 + c_4\beta^3,$$

$$h(e_2 \otimes_{f_2^2} e_2) = c_5\alpha_2\alpha_1 \text{ and}$$

$$h(e_2 \otimes_{f_{3,2}^2} e_1) = 0,$$

for some $c_1, \ldots, c_5 \in K$.

So $dim \operatorname{Ker} d_3 = 5$. Recalling that $dim \operatorname{Im} d_2$ was found earlier, we have

$$dim\, \mathrm{HH}^2(\Lambda) = \begin{cases} 5 - 4 = 1 & \text{if } char\, K \neq 2, \\ 5 - 2 = 3 & \text{if } char\, K = 2. \end{cases}$$

The basis for $\Lambda = \Lambda(m)$ where $m = 2$ is again the same as above.

We summarise the results in the following theorem.

Theorem 11.9. *For $\Lambda = \Lambda(m)$ where $m \geq 2$ we have*

$$dim\, \mathrm{HH}^2(\Lambda) = \begin{cases} 5 - 4 = 1 & \textit{if char } K \neq 2, \\ 5 - 2 = 3 & \textit{if char } K = 2. \end{cases}$$

If $char\, K \neq 2$, then $\{h_1 + \operatorname{Im} d_2\}$ is a basis for $\mathrm{HH}^2(\Lambda)$ where h_1 is the map given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0. \end{array}$$

If $char\, K = 2$, then $\{h_1 + \operatorname{Im} d_2, h_2 + \operatorname{Im} d_2, h_3 + \operatorname{Im} d_2\}$ is a basis for $\mathrm{HH}^2(\Lambda)$ where h_1 is the map given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & e_1, \\ \text{else} & \mapsto & 0, \end{array}$$

h_2 is given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta^3, \\ \text{else} & \mapsto & 0, \end{array}$$

and h_3 is given by

$$\begin{array}{rcl} e_1 \otimes_{f_1^2} e_1 & \mapsto & \beta, \\ \text{else} & \mapsto & 0. \end{array}$$

This completes the discussion of $\mathrm{HH}^2(\Lambda)$ for the non-standard self-injective algebras of finite representation type over an algebraically closed field.

Theorem 11.10. *Let Λ be a finite dimensional self-injective algebra of finite representation type over an algebraically closed field K. If Λ is the standard algebra of type $\Lambda(A_{2p+1}, s, 2)$ with $s, p > 1$, $\Lambda(D_n, s, 1), \Lambda(D_4, s, 3)$ with $n \geq 4, s \geq 1$, $\Lambda(D_n, s, 2), \Lambda(D_{3m}, s/3, 1)$ with $n \geq 4, m \geq 2, s \geq 2$ or $\Lambda(E_n, s, 1)$, $\Lambda(E_6, s, 2)$ with $n \in \{6, 7, 8\}, s \geq 1$; then $\mathrm{HH}^2(\Lambda) = 0$.*

If Λ is of type $\Lambda(A_n, s/n, 1)$ then $dim\, \mathrm{HH}^2(\Lambda) = m$ where $n + 1 = ms + r$ and $0 \leq r < s$.

For $\Lambda(A_{2p+1}, s, 2)$ with $s = p = 1$, $dim\, \mathrm{HH}^2(\Lambda) = 1$.

Let Λ be $\Lambda(D_n, 1, 2)$; then $dim\, \mathrm{HH}^2(\Lambda) = 1$.

Let Λ be the standard algebra $\Lambda(D_{3m}, 1/3, 1)$; then

$$dim\, \mathrm{HH}^2(\Lambda) = \begin{cases} 2 & \text{if } char\, K \neq 2, \\ 4 & \text{if } char\, K = 2. \end{cases}$$

If Λ is the nonstandard algebra $\Lambda(m)$ of type $(D_{3m}, 1/3, 1)$ where $m \geq 2$ we have

$$dim\, \mathrm{HH}^2(\Lambda) = \begin{cases} 1 & \text{if } char\, K \neq 2, \\ 3 & \text{if } char\, K = 2. \end{cases}$$

12. DERIVED EQUIVALENCE AND ONE-PARAMETRIC SELF-INJECTIVE ALGEBRAS

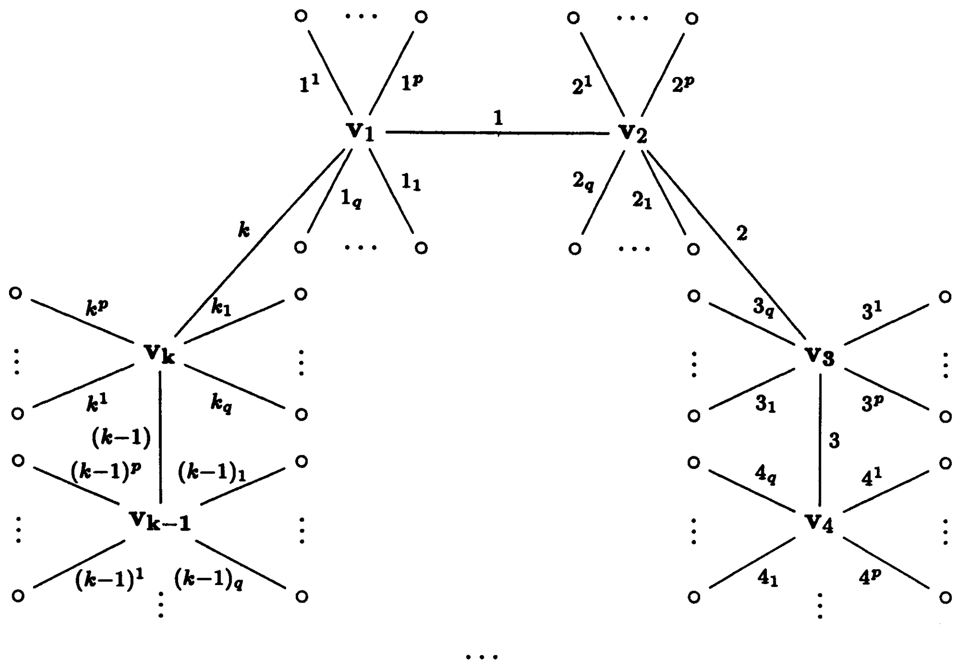
In this chapter we look at self-injective algebras of tame representation type (not finite representation type). We start by describing the work of Bocian, Holm and Skowroński in [6] which gives a classification of all standard one-parametric but not weakly symmetric self-injective algebras. We consider the algebras of [6], but remark that there are other classes of self-injective tame representation type algebras that we could also have considered. In particular, the classification of all standard one-parametric weakly symmetric algebras has been given in [7]. The algebras in [6] are divided into two types, and here we study $\mathrm{HH}^2(\Lambda)$ for some of one type Λ. The main result here is Theorem 12.13, and this finishes the thesis. Future work will continue the computation for the second type.

We start by describing this first type. The classification in [6] gives each algebra by quiver and relations and is constructed from Brauer graphs. The definitions are all taken from [6].

Definition 12.1. A Brauer graph T is a finite connected undirected graph, where for each vertex there is a fixed circular order on the edges adjacent to it.

In the context of the paper [6], T has at most one cycle. Moreover, the edges adjacent to a given vertex are clockwise ordered.

Let T be a Brauer graph with exactly one cycle R_k, having $k \geq 2$ edges. Let $v_1, \ldots, v_k$ be the vertices of R_k and $n(v_i) = \{v_i, v_{i+1}\}$ the edges of R_k where $i = 1, \ldots, k$ and $v_{k+1} = v_1$. In that case the Brauer graph T is of the form $(*)$:



As can be seen from the diagram, the main edges are labelled $1, \ldots, k$. Sometimes we will use the notation $n(v_i)$ for the main edge i where $1 \leq i \leq k$. The outer edges are those labelled by i^m where $1 \leq m \leq p$ and the inner edges are those labelled by i_m where $1 \leq m \leq q$.

The outer vertices are labelled $w_1, \ldots, w_{kp}$ where $w_{(i-1)p+m}$ and v_i are the vertices of the edge i^m, for $1 \leq m \leq p$ and $1 \leq i \leq k$. The inner vertices are labelled $y_1, \ldots, y_{qk}$ where $y_{(i-1)q+m}$ and v_i are the vertices of the edge i_m, for $1 \leq m \leq q$ and $1 \leq i \leq k$.

For ease of following the edges, we will label the inner edges by i^t where $p + 2 \leq t \leq p + q + 1$ and $1 \leq i \leq k$. So $i^t = i_{t-p-1}$.

As in [6], the edges of T are then rotated by an automorphism of T.

Definition 12.2. An automorphism of the Brauer graph T is an automorphism which preserves the fixed circular order on the edges adjacent to any vertex.

A rotation of T is an automorphism σ_s of T, for some integer s with $1 \leq s \leq k-1$, such that $\sigma_s(v_i) = v_{i+s}$ for all $i = 1, 2, \ldots, k$ (where $k + r = r$ for $r \geq 1$).

As in [6], we assume $\gcd(s+2, k) = 1$. With the following definitions we will define the generalized Brauer quiver $\mathcal{Q}(T, \sigma_s)$.

Definition 12.3. Let T be a Brauer graph. A σ_s-orbit of a vertex v of T is the orbit of v with respect to the action of the cyclic group generated by σ_s on the vertices of T. Moreover, all σ_s-orbits of vertices of T have k/d elements where $d = \gcd(s, k)$.

Definition 12.4. Let T be a Brauer graph. An order $P(T, \sigma_s)$ of the edges of T is defined as

$$P(T, \sigma_s) = \bigcup_v p(T, \sigma_s, v)$$

where v are the representatives of all pairwise different σ_s-orbits of vertices of T. A cyclic order $p(T, \sigma_s, v)$ is defined using the cyclic orders of edges around the vertices $v, \sigma_s(v), \ldots, \sigma_s^{(k/d)-1}(v)$ in the Brauer graph T.

Specifically, let $r \in \{0, 1, \ldots, (k/d) - 1\}$, let i be an edge of T adjacent to the vertex $\sigma_s^r(v)$ and let j be the direct successor of i in the cyclic order in T around $\sigma_s^r(v)$. If $j \neq n(\sigma_s^r(v))$, then j is defined to be the direct successor of i in the cyclic order $p(T, \sigma_s, v)$. If $j = n(\sigma_s^r(v))$ then $n(\sigma_s^{r+1}(v)) = \sigma_s(n(\sigma_s^r(v)))$ is said to be the direct successor of i in the cyclic order $P(T, \sigma_s, v)$.

Now we are ready to define the generalized Brauer quiver $\mathcal{Q}(T, \sigma_s)$.

Definition 12.5. Given a Brauer graph T, we define a generalized Brauer quiver $\mathcal{Q}(T, \sigma_s)$ where the vertices and arrows are given as follows. The vertices of $\mathcal{Q}(T, \sigma_s)$ are the edges of T. An arrow $i \to j$ exists if and only if j is the direct successor of i in the order $p(T, \sigma_s)$.

Definition 12.6. The algebra $\Omega^{(1)}(T, \sigma_s)$ is the algebra

$$K\mathcal{Q}(T, \sigma_s)/\bar{I}^{(1)}(T, \sigma_s),$$

where $KQ(T, \sigma_s)$ is the path algebra of the quiver $Q(T, \sigma_s)$ and $\bar{I}^{(1)}(T, \sigma_s)$ is the ideal in $KQ(T, \sigma_s)$ generated by the elements

(1) ab where $a : i_1 \to i_2, b : i_2 \to i_3$ and i_1, i_2, i_3 are not consecutive elements in the cyclic order $P(T, \sigma_s)$.

(2) $C(i, p(T, \sigma_s, v)) - C(i, p(T, \sigma_s, w))$, where $i = \{v, w\}$ is an edge of T and $C(i, p(T, \sigma_s, v)), C(i, p(T, \sigma_s, w))$ are the paths from i to $\sigma_s(i)$ in the quiver $Q(T, \sigma_s)$ corresponding to the consecutive elements $i, \ldots, \sigma_s(i)$ of the cyclic orders $p(T, \sigma_s, v)$ and $p(T, \sigma_s, w)$, respectively.

Definition 12.7. If the Brauer graph T is of the form $(*)$ and $\gcd(s, k) = 1$ then $\Omega^{(1)}(T, \sigma_s)$ is denoted by $\Lambda(p, q, k, s)$.

As [6, Theorem 1] shows, these algebras $\Lambda(p, q, k, s)$ are one of the types of standard one-parametric but not weakly symmetric algebras. The other type is labelled $\Gamma^*(n)$ and the definition of $\Gamma^*(n)$ is in [6].

Theorem 12.8. [6, Theorem 1] *For a standard self-injective algebra Λ the following are equivalent:*

i) Λ is one-parametric but not weakly symmetric;

ii) Λ is derived equivalent to an algebra of the form $\Lambda(p, q, k, s)$ or $\Gamma^*(n)$.

Remark. Two algebras of the form $\Lambda(p, q, k, s)$ and $\Gamma^*(n)$ are derived equivalent if and only if they are isomorphic. Note that an algebra of the form $\Lambda(p, q, k, s)$ is never isomorphic to an algebra of the form $\Gamma^*(n)$ as their stable Auslander-Reiten quivers are not isomorphic. Precise details of when two algebras either both of type Λ or both of type Γ are given in [6, Proposition 7.6] and [6, §0] respectively.

The overall aim is to find $\mathrm{HH}^2(\Lambda)$ for all standard one-parametric but not weakly symmetric algebras and in this thesis we consider here some of the derived equivalent representatives of these algebras $\Lambda(p, q, k, s)$, namely the algebras $\Lambda(p, q, k, k-1)$.

We fix $s = k - 1$ and will show that the second Hochschild cohomology group is non-zero, for the algebras of the form $\Lambda(p, q, k, k-1)$ where p, q, k are arbitrary. With $s = k - 1$ it is true that $\gcd(s + 2, k) = 1$ and $\gcd(s, k) = 1$. As Hochschild cohomology is invariant under derived equivalence, the second

Hochschild cohomology group is non-zero for any algebra derived equivalent to one of these representatives $\Lambda(p,q,k,k-1)$.

For an arbitrary p,q,k and $s=k-1$, let $d=\gcd(s,k)=\gcd(k-1,k)=1$ and let σ_s be the rotation:

for the main vertices v_i,

$$\sigma_s : v_i \mapsto v_{i+k-1} \text{ where } i+k-1 = i-1, i \geq 2.$$

for the outer vertices w_i,

$$\sigma_s : w_i \mapsto \begin{cases} w_{i+(k-1)p} & \text{if } w_i \text{ is a vertex around } v_1, \\ w_{i-p} & \text{otherwise,} \end{cases}$$

and for the inner vertices y_i,

$$\sigma_s : y_i \mapsto \begin{cases} y_{i+(k-1)q} & \text{if } y_i \text{ is a vertex around } v_1, \\ y_{i-q} & \text{otherwise.} \end{cases}$$

For ease of notation we write σ for σ_s.

The orbit of a vertex v with respect to the action of the cyclic group generated by σ on the vertices of T is:

$$\{id(v), \sigma(v), \sigma^2(v), \ldots\}.$$

Recall from Definition 12.3 that all σ-orbits of vertices of T have the same number of elements, namely k. Now we find the orbit $\mathcal{O}(v)$ of each vertex v of T.

$$\begin{aligned}
\mathcal{O}(v_1) &= \{v_1, v_k, v_{k-1}, \ldots, v_3, v_2\} \\
\mathcal{O}(w_1) &= \{w_1, w_{(k-1)p+1}, w_{(k-2)p+1}, \ldots, w_{2p+1}, w_{p+1}\} \\
&\vdots \\
\mathcal{O}(w_p) &= \{w_p, w_{kp}, w_{(k-1)p}, \ldots, w_{3p}, w_{2p}\} \\
\mathcal{O}(y_1) &= \{y_1, w_{(k-1)q+1}, w_{(k-2)q+1}, \ldots, w_{2q+1}, w_{q+1}\} \\
&\vdots \\
\mathcal{O}(y_q) &= \{y_q, y_{kq}, y_{(k-1)q}, \ldots, y_{3q}, y_{2q}\}.
\end{aligned}$$

Now we introduce an order $P(T,\sigma)$ of the edges of the Brauer graph T. We have $p+q+1$ distinct representatives of the orbits of the vertices of T, they are $v_1, w_1, \ldots, w_p, y_1, \ldots, y_q$. So we have $p+q+1$ cyclic orders. Hence, from Definition 12.4 we know that

$$P(T,\sigma) = p(T,\sigma,v_1) \cup \bigcup_{i=1}^{p} p(T,\sigma,w_i) \cup \bigcup_{j=1}^{q} p(T,\sigma,y_j).$$

Using Definition 12.4 we define the cyclic order around each vertex. For the cyclic order $p(T,\sigma,v_1)$, starting from any edge, say i, and moving clockwise around the vertex v_1, if the direct successor of i is not a main edge then it is the direct successor of i in the cyclic order. If the direct successor is a main edge $n(\sigma^r(v_1))$ then the direct successor of the edge i is $n(\sigma^{r+1}(v_1))$ in the cyclic order. So we have,

$p(T,\sigma,v_1) = \{1, 1^{p+2}, \ldots, 1^{p+q+1}, k, 1^1, \ldots, 1^p, k, k^{p+2}, \ldots, k^{p+q+1}, k-1, k^1, \ldots, k^p, k-1, (k-1)^{p+2}, \ldots, 2, 2^{p+2}, \ldots, 2^{p+q+1}, 1, 2^1, \ldots, 2^p\}$.

The other cyclic orders are found more easily, and can be seen to be:

$p(T,\sigma,w_1) = \{1^1, k^1, (k-1)^1, \ldots, 3^1, 2^1\}$,

$\vdots$

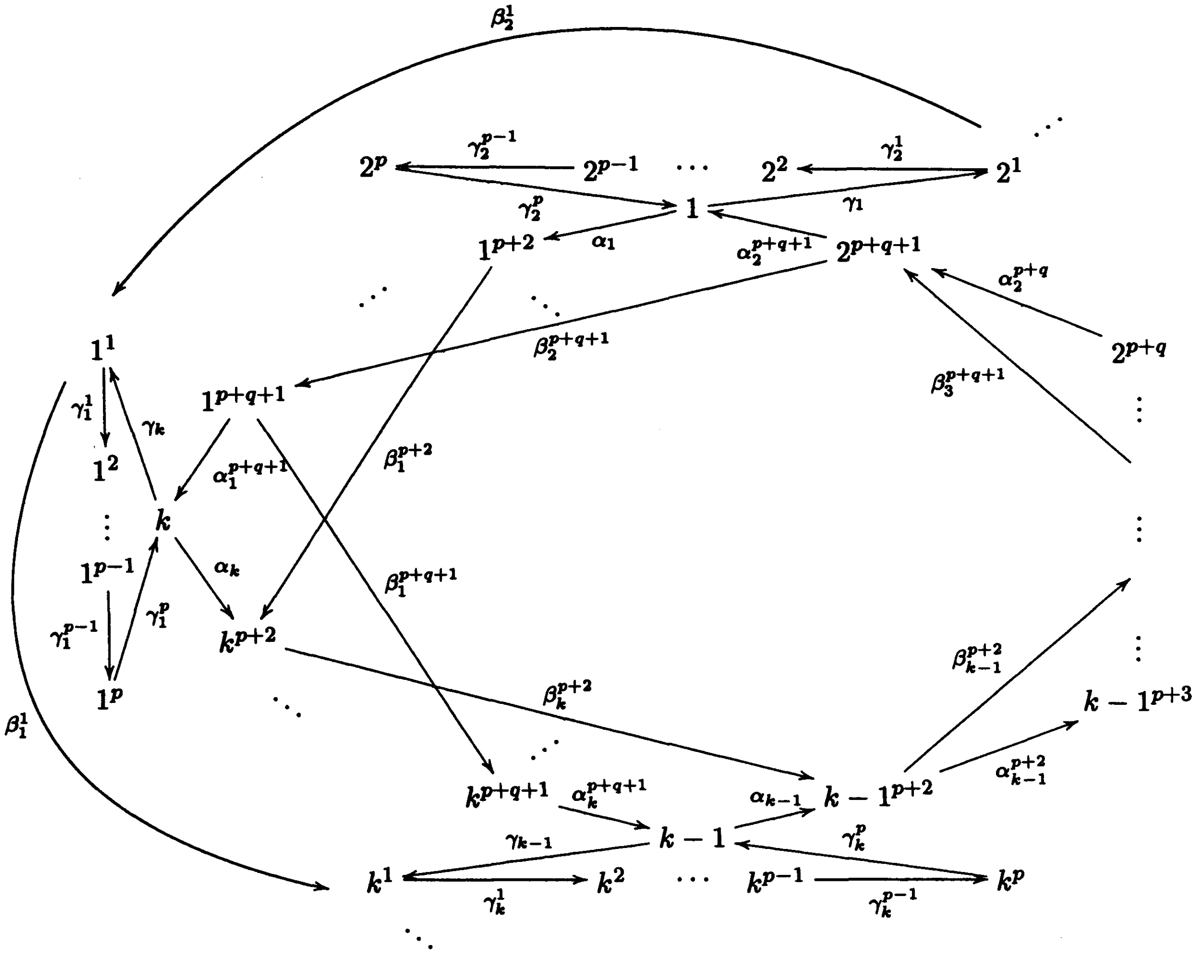
$p(T,\sigma,w_p) = \{1^p, k^p, (k-1)^p, \ldots, 3^p, 2^p\}$,

$p(T,\sigma,y_1) = \{1^{p+2}, k^{p+2}, (k-1)^{p+2}, \ldots, 3^{p+2}, 2^{p+2}\}$,

$\vdots$

$p(T,\sigma,y_q) = \{1^{p+q+1}, k^{p+q+1}, (k-1)^{p+q+1}, \ldots, 3^{p+q+1}, 2^{p+q+1}\}$.

Then the generalized Brauer quiver $Q(T,\sigma)$ using Definition 12.5 is of the form:



Hence, by Definition 12.6, the algebra $\Lambda(p, q, k, k-1)$ is given by the above quiver $Q(T, \sigma)$ with the relations:

(1) for all $i \in \{1, \dots, k\}$, we have relations

$\alpha_i \beta_i^{p+2}$,

$\gamma_i \beta_{i+1}^1$,

$\gamma_i^j \beta_i^{j+1}$ where $j \in \{1, \dots, p-1\}$,

$\gamma_i^p \gamma_{\sigma(i)}$,

$\beta_i^m \gamma_{\sigma(i)}^m$ where $m \in \{1, \dots, p\}$,

$\alpha_i^{l-1} \beta_i^l$ where $l \in \{p+3, \dots, p+q+1\}$,

$\alpha_i^{p+q+1} \alpha_{\sigma(i)}$ and

$\beta_i^t \alpha_{\sigma(i)}^t$ where $t \in \{p+2, \ldots, p+q+1\}$.

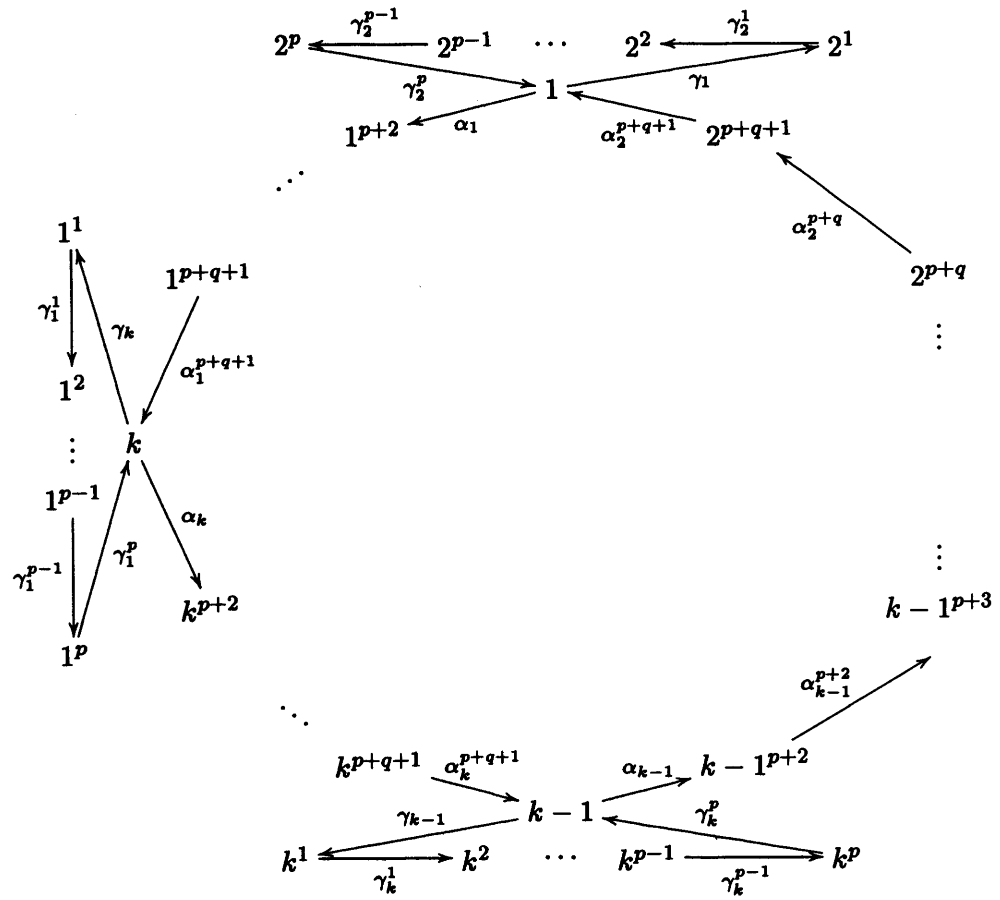
(2) for all $i \in \{1, \ldots, k\}$, we have relations

$\alpha_i \alpha_i^{p+2} \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p - \gamma_i \gamma_{i+1}^1 \cdots \gamma_{i+1}^p \alpha_i \alpha_i^{p+2} \cdots \alpha_i^{p+q+1}$,

$\beta_i^m - \gamma_i^m \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{p+q+1} \gamma_{\sigma^2(i)} \gamma_{\sigma(i)}^1 \cdots \gamma_{\sigma(i)}^{m-1}$, for $m \in \{1, \ldots, p\}$,

$\beta_i^t - \alpha_i^t \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{t-1}$,

Now we want to find an algebra $\Lambda'(p, q, k, k-1)$ isomorphic to $\Lambda(p, q, k, k-1)$ and so that $\Lambda' = KQ'/I'$ and the ideal I' is contained in the square of the arrow ideal of KQ'. So we are going to replace β's in the relations where it is possible. Let $Q'(p, q, k, k-1)$ be the quiver:



Then $\Lambda(p,q,k,k-1)$ is isomorphic to the algebra given by the quiver $Q'(p,q,k,k-1)$ with relations, for all $i \in \{1,\ldots,k\}$:

(1) $\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)}$,

$\gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}$,

$\gamma_i^j\gamma_i^{j+1}\cdots\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{p+q+1}\gamma_{i+1}\gamma_{\sigma(i)}^1\cdots\gamma_{\sigma(i)}^j$ where $j \in \{1,\ldots,p-1\}$,

$\gamma_i^p\gamma_{\sigma(i)}$,

$\gamma_i^m\cdots\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{p+q+1}\gamma_{\sigma^2(i)}\gamma_{\sigma(i)}^1\cdots\gamma_{\sigma(i)}^{m-1}\gamma_{\sigma(i)}^m$

where $m \in \{1,\ldots,p\}$,

$\alpha_i^{l-1}\alpha_i^l\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{l-1}$

where $l \in \{p+3,\ldots,p+q+1\}$,

$\alpha_i^{p+q+1}\alpha_{\sigma(i)}$,

$\alpha_i^t\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{t-1}\alpha_{\sigma(i)}^t$

where $t \in \{p+2,\ldots,p+q+1\}$

and finally

(2) $\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p - \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}$.

These relations are not minimal. So next we will find a minimal set of relations f^2 for this algebra.

Let $f_{1,i}^2 = \alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p - \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}$, $f_{2,i}^2 = \gamma_i^p\gamma_{\sigma(i)}$ and $f_{3,i}^2 = \alpha_i^{p+q+1}\alpha_{\sigma(i)}$ so $f_{1,i}^2, f_{2,i}^2, f_{3,i}^2$ are in f^2 for $i \in \{1,\ldots,k\}$.

Now consider the other relations. We have $(\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p - \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1})\alpha_{\sigma(i)}$ and $\gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\alpha_{\sigma(i)}$ are both in I. Therefore $\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)} \in I$ and is not in f^2.

Also we have $f_{1,i}^2\gamma_{\sigma(i)} \in I$, so that $(\gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1})\gamma_{\sigma(i)}$ is in I and not in f^2.

Now consider the relation $\gamma_i^m\cdots\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{p+q+1}\gamma_{\sigma^2(i)}\gamma_{\sigma(i)}^1\cdots\gamma_{\sigma(i)}^{m-1}\gamma_{\sigma(i)}^m$ where $m \in \{1,\ldots,p\}$.

If $r \in I$. So $\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{p+q+1}\gamma_{\sigma^2(i)}\gamma_{\sigma(i)}^1\cdots\gamma_{\sigma(i)}^{p-1}\gamma_{\sigma(i)}^p \in I$ and not in f^2 $\in I$. So $\gamma_i^p\alpha_{\sigma(i)}\alpha_{\sigma(i)}^{p+2}\cdots\alpha_{\sigma(i)}^{p+q+1}\gamma_{\sigma^2(i)}\gamma_{\sigma(i)}^1\cdots\gamma_{\sigma(i)}^{p-1}\gamma_{\sigma(i)}^p \in I$ and not in f^2.

If $m \in \{1, \ldots, p-1\}$, then
$f^2_{4,i,m} = \gamma_i^m \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{p+q+1} \gamma_{\sigma^2(i)} \gamma_{\sigma(i)}^1 \cdots \gamma_{\sigma(i)}^{m-1} \gamma_{\sigma(i)}^m$
cannot be obtained from any other elements of I. So $f^2_{4,i,m} = \gamma_i^m \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{p+q+1} \gamma_{\sigma^2(i)} \gamma_{\sigma(i)}^1 \cdots \gamma_{\sigma(i)}^{m-1} \gamma_{\sigma(i)}^m$ is in f^2 where $m \in \{1, \ldots, p-1\}$.

Now consider the relation $\gamma_i^j \gamma_i^{j+1} \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{p+q+1} \gamma_{i+1} \gamma_{\sigma(i)}^1 \cdots \gamma_{\sigma(i)}^j$ where $j \in \{1, \ldots, p-1\}$. Note that this relation is precisely $f^2_{4,i,m}$ above with $j = m$, which is in f^2.

Next consider the relation $\alpha_i^{l-1} \alpha_i^l \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{l-1}$ where $l \in \{p+3, \ldots, p+q+1\}$. It cannot be obtained from any other relations in I. Therefore $f^2_{5,i,l-1} = \alpha_i^{l-1} \alpha_i^l \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{l-1}$ where $l \in \{p+3, \ldots, p+q+1\}$ is in f^2.

Finally consider $\alpha_i^t \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{t-1} \alpha_{\sigma(i)}^t$ where $t \in \{p+2, \ldots, p+q+1\}$. This is precisely the element $f^2_{5,i,l-1}$ with $t = l-1$ above.

So we now have the minimal set of relations.

Proposition 12.9. *For $\Lambda = \Lambda(p, q, k, k-1)$, the minimal set of relations, for all $i \in \{1, \ldots, k\}$, is*

$$f^2 = \{f^2_{1,i}, f^2_{2,i}, f^2_{3,i}, f^2_{4,i,j}, f^2_{5,i,l-1}\}$$

where

$$\begin{aligned}
f^2_{1,i} &= \alpha_i \alpha_i^{p+2} \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p - \gamma_i \gamma_{i+1}^1 \cdots \gamma_{i+1}^p \alpha_i \alpha_i^{p+2} \cdots \alpha_i^{p+q+1}, \\
f^2_{2,i} &= \gamma_i^p \gamma_{\sigma(i)}, \\
f^2_{3,i} &= \alpha_i^{p+q+1} \alpha_{\sigma(i)}, \\
f^2_{4,i,j} &= \gamma_i^j \gamma_i^{j+1} \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{p+q+1} \gamma_{i+1} \gamma_{\sigma(i)}^1 \cdots \gamma_{\sigma(i)}^j \\
&\quad \textit{where } j \in \{1, \ldots, p-1\}, \\
f^2_{5,i,l-1} &= \alpha_i^{l-1} \alpha_i^l \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p \alpha_{\sigma(i)} \alpha_{\sigma(i)}^{p+2} \cdots \alpha_{\sigma(i)}^{l-1} \\
&\quad \textit{where } l \in \{p+3, \ldots, p+q+1\}.
\end{aligned}$$

In contrast to the majority of the self-injective algebras of finite representation type, we will show that the algebra $\Lambda(p, q, k, k-1)$ has $\mathrm{HH}^2(\Lambda) \neq 0$.

Specifically, we will find a non-zero element in $\mathrm{HH}^2(\Lambda)$. As $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$, the first step is to find a non-zero map in $\operatorname{Ker} d_3$. We have $d_3 : \operatorname{Hom}(Q^2, \Lambda) \to \operatorname{Hom}(Q^3, \Lambda)$.

12.10. *Define* $h \in \operatorname{Hom}(Q^2, \Lambda)$ *by*

$$\begin{array}{rcl} \mathfrak{o}(f_{1,1}^2) \otimes \mathfrak{t}(f_{1,1}^2) = e_1 \otimes e_k & \mapsto & \alpha_1 \alpha_1^{p+2} \cdots \alpha_1^{p+q+1} \gamma_k \gamma_1^1 \cdots \gamma_1^p =: \rho, \\ \textit{else} & \mapsto & 0. \end{array}$$

We note that $\rho \neq 0$ so h is a non-zero map. But $\rho\gamma_k = 0$ and $\rho\alpha_k = 0$. Hence $\rho\mathfrak{r} = 0$ where $\mathfrak{r} = \operatorname{rad}\Lambda$. Similarly we have $\mathfrak{r}\rho = 0$.

To show that $h \in \operatorname{Ker} d_3$ we show that $hA_3 = 0$.

Recall from Chapter 3 that the 3rd projective in a minimal Λ^e-resolution of Λ is $Q^3 = \coprod_{y \in f^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda$ where $y = \sum_u f_u^2 p_u = \sum_u q_u f_u^2 r_u$, p_u is a path of length ≥ 1 and q_u is in the ideal generated by the arrows. For $y \in f^3$ the component of $A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y))$ in $\Lambda\mathfrak{o}(f_u^2) \otimes \mathfrak{t}(f_u^2)\Lambda$ is

$$\Sigma(\mathfrak{o}(y) \otimes_{f_u^2} p_u - q_u \otimes_{f_u^2} r_u).$$

Then

$$hA_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y)) = \Sigma_u(h(\mathfrak{o}(y) \otimes_{f_u^2} p_u) - q_u h(\mathfrak{o}(f_u^2) \otimes_{f_u^2} \mathfrak{t}(f_u^2)) r_u).$$

Thus $h(\mathfrak{o}(y) \otimes_{f_u^2} p_u) = \begin{cases} \rho p_u & \text{if } f_u^2 = f_{1,1}^2 \\ 0 & \text{otherwise.} \end{cases}$

As p_u is in the arrow ideal of KQ, $\rho p_u \in \rho\mathfrak{r} = 0$. So we have $h(o(y) \otimes p_u) = 0$. Similarly $h(q_u \otimes_{f_u^2} r_u) = 0$ as $q_u \rho r_u \in \mathfrak{r}\rho r_u = 0$.

Therefore $hA_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y)) = 0$ forall $y \in f^3$ so $hA_3 = 0$. Thus $h \in \operatorname{Ker} d_3$. Thus we have a non-zero map h in $\operatorname{Ker} d_3$.

Proposition 12.11. *For h given in 12.10, h is a non-zero map in* $\operatorname{Ker} d_3$.

Next we find $\operatorname{Im} d_2$.

Since $d_2 : \operatorname{Hom}(Q^1, \Lambda) \to \operatorname{Hom}(Q^2, \Lambda)$, let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and consider $d_2 f = f A_2$. Here

$Q^1 = \bigoplus_{i=1}^k [(\Lambda e_i \otimes_{\gamma_i} e_{(i+1)^1}\Lambda) \oplus \bigoplus_{j=1}^{p-1} (\Lambda e_{i^j} \otimes_{\gamma_i^j} e_{i^{j+1}}\Lambda) \oplus (\Lambda e_{i^p} \otimes_{\gamma_i^p} e_{\sigma(i)}\Lambda) \oplus (\Lambda e_i \otimes_{\alpha_i} e_{i^{p+2}}\Lambda) \oplus \bigoplus_{t'=p+2}^{p+q} (\Lambda e_{i^{t'}} \otimes e_{i^{t'+1}}\Lambda) \oplus (\Lambda e_{i^{p+q+1}} \otimes_{\alpha_i^{p+q+1}} e_{\sigma(i)}\Lambda)]$.

Let

$f(e_i \otimes_{\gamma_i} e_{(i+1)^1}) = c_{1,i}\gamma_i,$
$f(e_{i^j} \otimes_{\gamma_i^j} e_{i^{j+1}}) = c_{2,i,j}\gamma_i^j$ for $j \in \{1,\dots,p-1\}$,
$f(e_{i^p} \otimes_{\gamma_i^p} e_{\sigma(i)}) = c_{2,i,p}\gamma_i^p$,
$f(e_i \otimes_{\alpha_i} e_{i^{p+2}}) = c_{3,i}\alpha_i$,
$f(e_{i^{t'}} \otimes e_{i^{t'+1}}) = c_{4,i,t'}\alpha_i^{t'}$ for $t' \in \{p+2,\dots,p+q\}$ and
$f(e_{i^{p+q+1}} \otimes_{\alpha_i^{p+q+1}} e_{\sigma(i)}) = c_{4,i,p+q+1}\alpha_i^{p+q+1} + d_{1,i}\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p$,
where all coefficients $c_{1,i}, c_{2,i,j}$ for $j \in \{1,\dots p-1\}, c_{2,i,p}, c_{3,i}, c_{4,i,t'}$ for $t' \in \{p+2,\dots,p+q\}, c_{4,i,p+q+1}, d_{1,i} \in K$.

We have $Q^2 = \bigoplus_{i=1}^{k}[(\Lambda e_i \otimes_{f^2_{1,i}} e_{\sigma(i)}\Lambda) \oplus (\Lambda e_{i^p} \otimes_{f^2_{2,i}} e_{i^1}\Lambda) \oplus (\Lambda e_{i^{p+q+1}} \otimes_{f^2_{3,i}} e_{(\sigma(i))^{p+2}}\Lambda) \oplus \bigoplus_{j=1}^{p-1}(\Lambda e_{i^j} \otimes_{f^2_{4,i,j}} e_{(\sigma(i))^{j+1}}\Lambda) \oplus \bigoplus_{l=p+3}^{p+q+1}(\Lambda e_{i^{l-1}} \otimes_{f^2_{5,i,l-1}} e_{(\sigma(i))^l}\Lambda)]$.

Now we find fA_2.

We have, $fA_2(e_i \otimes_{f^2_{1,i}} e_{\sigma(i)}) = f(e_i \otimes_{\alpha_i} e_{i^{p+2}})\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p + \alpha_i f(e_{i^{p+2}} \otimes_{\alpha_i^{p+2}} e_{i^{p+3}})\alpha_i^{p+3}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p + \cdots + \alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q}$ $f(e_{i^{p+q+1}} \otimes_{\alpha_i^{p+q+1}} e_{\sigma(i)})\,\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p + \alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1} f(e_{\sigma(i)} \otimes_{\gamma_{\sigma(i)}} e_{i^1})\gamma_i^1\cdots\gamma_i^p + \alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\, f(e_{i^1} \otimes_{\gamma_i^1} e_{i^2})\gamma_i^2\cdots\gamma_i^p + \cdots + \alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}$ $\gamma_i^1\cdots\gamma_i^{p-1} f(e_{i^p} \otimes_{\gamma_i^p} e_{\sigma(i)})$
$-[\gamma_i f(e_{(i+1)^1} \otimes_{\gamma_{i+1}^1} e_{(i+1)^2})\gamma_{i+1}^2\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1} + \cdots + \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^{p-1}$ $f(e_{(i+1)^p} \otimes_{\gamma_{i+1}^p} e_i)\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1} + \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p f(e_i \otimes_{\alpha_i} e_{i^{p+2}})\alpha_i^{p+2}\cdots\alpha_i^{p+q+1} + \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i f(e_{i^{p+2}} \otimes_{\alpha_i^{p+2}} e_{i^{p+3}})\alpha_i^{p+3}\cdots\alpha_i^{p+q+1} + \cdots + \gamma_i\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q}\ f(e_{i^{p+q+1}} \otimes_{\alpha_i^{p+q+1}} e_{\sigma(i)})]$
$= (c_{3,i} + c_{4,i,p+2} + \cdots + c_{4,i,p+q+1} + c_{1,\sigma(i)} + c_{2,i,1} + \cdots + c_{2,i,p})\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}$ $\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p - [c_{1,i} + c_{2,i+1,1} + \cdots + c_{2,i+1,p} + c_{3,i} + c_{4,i,p+2} + \cdots + c_{4,i,p+q+1}]\gamma_i$ $\gamma_{i+1}^1\cdots\gamma_{i+1}^p\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}$.
So $fA_2(e_i \otimes_{f^2_{1,i}} e_{\sigma(i)}) = (c_{1,\sigma(i)} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \cdots - c_{2,i+1,p})\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p$.

Next, $fA_2(e_{i^p} \otimes_{f^2_{2,i}} e_{i^1}) = f(e_{i^p} \otimes_{\gamma_i^p} e_{\sigma(i)})\gamma_{\sigma(i)} + \gamma_i^p f(e_{\sigma(i)} \otimes_{\gamma_{\sigma(i)}} e_{i^1}) = (c_{2,i,p} + c_{1,\sigma(i)})\gamma_i^p\gamma_{\sigma(i)} = 0$.

Now, $fA_2(e_{i^{p+q+1}} \otimes_{f^2_{3,i}} e_{(\sigma(i))^{p+2}}) = f(e_{i^{p+q+1}} \otimes_{\alpha_i^{p+q+1}} e_{\sigma(i)})\alpha_{\sigma(i)} + \alpha_i^{p+q+1}$ $f(e_{\sigma(i)} \otimes_{\alpha_{\sigma(i)}} e_{(\sigma(i))^{p+2}}) = c_{4,i,p+q+1}\alpha_i^{p+q+1}\alpha_{\sigma(i)} + d_{1,i}\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)} + c_{3,\sigma(i)}\alpha_i^{p+q+1}\alpha_{\sigma(i)} = d_{1,i}\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)}$.

For the remaining terms, we note that $fA_2(e_{i^j}\otimes_{f^2_{4,i,j}} e_{(\sigma(i))^{j+1}}) \in e_{i^j}\Lambda e_{(\sigma(i))^{j+1}}$ $\forall j \in \{1,\dots,p-1\}$ and $e_{i^j}\Lambda e_{(\sigma(i))^{j+1}} = 0$, so $fA_2(e_{i^j} \otimes_{f^2_{4,i,j}} e_{(\sigma(i))^{j+1}}) = 0$. Similarly $fA_2(e_{i^{l-1}} \otimes_{f^2_{5,i,l-1}} e_{(\sigma(i))^l}) \in e_{i^{l-1}}\Lambda e_{(\sigma(i))^l}$ $\forall l \in \{p+3,\dots,p+q+1\}$ and $e_{i^{l-1}}\Lambda e_{(\sigma(i))^l} = 0$, so $fA_2(e_{i^{l-1}} \otimes_{f^2_{5,i,l-1}} e_{(\sigma(i))^l}) = 0$.

Let $c'_i = c_{1,\sigma(i)} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \dots - c_{2,i+1,p}$ for $i = 1,\dots,k$.
Thus for $i \in \{1,\dots,k\}$, f is given by
$fA_2(e_i \otimes_{f^2_{1,i}} e_{\sigma(i)}) = c'_i\alpha_i\alpha_i^{p+2}\cdots\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p$,
$fA_2(e_{i^p} \otimes_{f^2_{2,i}} e_{i^1}) = 0$,
$fA_2(e_{i^{p+q+1}} \otimes_{f^2_{3,i}} e_{(\sigma(i))^{p+2}}) = d_{1,i}\alpha_i^{p+q+1}\gamma_{\sigma(i)}\gamma_i^1\cdots\gamma_i^p\alpha_{\sigma(i)}$,
$fA_2(e_{i^j} \otimes_{f^2_{4,i,j}} e_{(\sigma(i))^{j+1}}) = 0$ where $j \in \{1,\dots,p-1\}$ and
$fA_2(e_{i^{l-1}} \otimes_{f^2_{5,i,l-1}} e_{(\sigma(i))^l}) = 0$ where $l \in \{p+3,\dots,p+q+1\}$,
where $c'_1,\dots,c'_k \in K$ with $\Sigma_{i=1}^k c'_i = 0$ and $d_{1,1},\dots,d_{1,k} \in K$.

To find $dim\mathrm{Im}\, d_2$, let $c_{1,\sigma(i)} + c_{2,i,1} + \cdots + c_{2,i,p} =: b_{\sigma(i)}$ and $c_{1,i} + c_{2,i+1,1} + \cdots + c_{2,i+1,p} =: b_i$. Therefore we have the following k equations

$$c'_1 = b_k - b_1$$

$$c'_2 = b_1 - b_2$$

$$\vdots$$

$$c'_{k-1} = b_{k-2} - b_{k-1}$$

$$c'_k = b_{k-1} - b_k.$$

Thus we can form the matrix A of coefficients so that

$$\begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_{k-1} \\ c'_k \end{bmatrix} = A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} -1 & 0 & \cdots & & 0 & 1 \\ 1 & -1 & 0 & \cdots & & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & & 0 & 1 & -1 \end{bmatrix}.$$

Then it is easy to show that $rank(A) = k - 1$.

Hence $dim \mathrm{Im}\, d_2 = k - 1 + k = 2k - 1$.

Proposition 12.12. $dim \mathrm{Im}\, d_2 = 2k - 1$.

We now come to the main result of this chapter.

Theorem 12.13. *For* $\Lambda(p, q, k, k-1)$, $\mathrm{HH}^2(\Lambda) \neq 0$ *with* $h + \mathrm{Im}\, d_2$ *being a non-zero element.*

Proof. Consider the element $h + \mathrm{Im}\, d_2$ of $\mathrm{HH}^2(\Lambda)$ where h is given as in 12.10 by

$$\begin{array}{rcl} \mathfrak{o}(f^2_{1,1}) \otimes \mathfrak{t}(f^2_{1,1}) = e_1 \otimes e_k & \mapsto & \rho, \\ \text{else} & \mapsto & 0. \end{array}$$

Recall $\rho = \alpha_1 \alpha_1^{p+2} \cdots \alpha_1^{p+q+1} \gamma_k \gamma_1^1 \cdots \gamma_1^p$.

We have from Proposition 12.11 that $0 \neq h \in \mathrm{Ker}\, d_3$. Now suppose for contradiction that $h \in \mathrm{Im}\, d_2$. Then $h(e_1 \otimes e_k) = f A_2(e_1 \otimes e_k)$. So $\rho = c'_1 \rho$ and so $c'_1 = 1$. Also $h(e_i \otimes e_{\sigma(i)}) = f A_2(e_i \otimes e_{\sigma(i)})$ where $i \in \{2, \ldots, k\}$. Then $0 = c'_i \alpha_i \alpha_i^{p+2} \cdots \alpha_i^{p+q+1} \gamma_{\sigma(i)} \gamma_i^1 \cdots \gamma_i^p$, where $i \in \{2, \ldots, k\}$. But this contradicts having $\Sigma_{i=1}^k c'_i = 0$. Therefore $h \notin \mathrm{Im}\, d_2$, that is, $h + \mathrm{Im}\, d_2 \neq 0 + \mathrm{Im}\, d_2$. So $h + \mathrm{Im}\, d_2$ is a non-zero element in $\mathrm{HH}^2(\Lambda)$. $\square$

It is then immediate from Proposition 12.12 and Theorem 12.13 that we have the following result.

Corollary 12.14. $dim \mathrm{Ker}\, d_3 \geq 2k$.

Remark. There are other non-zero maps h_i in $\mathrm{Ker}\, d_3$ defined similarly, for each i, such as h_2 given by

$$\begin{array}{rcl} \mathfrak{o}(f^2_{1,2}) \otimes \mathfrak{t}(f^2_{1,2}) = e_2 \otimes e_1 & \mapsto & \alpha_2 \alpha_2^{p+2} \cdots \alpha_2^{p+q+1} \gamma_1 \gamma_2^1 \cdots \gamma_2^p, \\ \text{else} & \mapsto & 0. \end{array}$$

However they all represent the same element of $\mathrm{HH}^2(\Lambda)$, since, for example, $h_2 - h \in \mathrm{Im}\, d_2$.

Let $h + \operatorname{Im} d_2 =: \eta$. Since we have $\mathrm{HH}^2(\Lambda) \neq 0$ we might have a deformation related to that element η in $\mathrm{HH}^2(\Lambda)$. We introduce a new parameter t to get a deformed algebra Λ_η such that if $t = 0$ we get back to Λ. The deformed algebra Λ_η is the algebra KQ/I_η where I_η is the ideal generated by the following elements:

(1) $f_{1,1}^2 - t\rho, f_{1,j}^2$ where $j \in \{2, \dots, k\}$,

(2) for all $i \in \{1, \dots, k\}$, $f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,l-1}^2$, where $j \in \{1, \dots, p-1\}$, $l \in \{p+3, \dots, p+q+1\}$,

(3) ρa for all arrows a with $\mathrm{t}(\rho) = o(a)$,

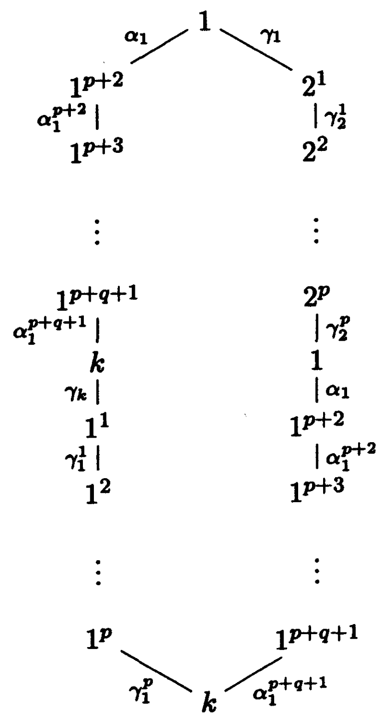
(4) $a\rho$ for all arrows a with $\mathrm{t}(a) = o(\rho)$.

We now need to show that $dim\, \Lambda_\eta = dim\, \Lambda$ to make sure that Λ_η is indeed a deformation. I thank Dr. Karin Erdmann for her helpful comments when considering Λ_η.

It is clear that $dim\, e_j\Lambda_\eta = dim\, e_j\Lambda$ for all t and for all vertices e_j with $e_j \neq e_1$ because the relation $f_{1,1}^2 - \rho t$ does not affect this.

Now we consider $e_1\Lambda$, $e_1\Lambda_\eta$ with $t \neq 1$, $e_1\Lambda_\eta$ with $t = 1$.

$e_1\Lambda$



$e_1\Lambda_\eta$ with $t \neq 1$

$$
\begin{tikzcd}
& 1 \arrow[dl, dash, "\alpha_1"'] \arrow[dr, dash, "\gamma_1"] & \\
1^{p+2} \arrow[d, dash, "\alpha_1^{p+2}"'] & & 2^1 \arrow[d, dash, "\gamma_2^1"] \\
1^{p+3} & & 2^2 \\
\vdots & & \vdots \\
1^{p+q+1} \arrow[d, dash, "\alpha_1^{p+q+1}"'] & & 2^p \arrow[d, dash, "\gamma_2^p"] \\
k \arrow[d, dash, "\gamma_k"'] & & 1 \arrow[d, dash, "\alpha_1"] \\
1^1 \arrow[d, dash, "\gamma_1^1"'] & & 1^{p+2} \arrow[d, dash, "\alpha_1^{p+2}"] \\
1^2 & & 1^{p+3} \\
\vdots & & \vdots \\
1^p \arrow[dr, dash, "\gamma_1^p"'] & & 1^{p+q+1} \arrow[dl, dash, "\alpha_1^{p+q+1}"] \\
& k &
\end{tikzcd}
$$

$e_1\Lambda_\eta$ with $t = 1$

$$
\begin{tikzcd}
& 1 \arrow[dl, dash, "\alpha_1"'] \arrow[dr, dash, "\gamma_1"] & \\
1^{p+2} \arrow[d, dash, "\alpha_1^{p+2}"'] & & 2^1 \arrow[d, dash, "\gamma_2^1"] \\
1^{p+3} & & 2^2 \\
\vdots & & \vdots \\
1^{p+q+1} \arrow[d, dash, "\alpha_1^{p+q+1}"'] & & 2^p \arrow[d, dash, "\gamma_2^p"] \\
k \arrow[d, dash, "\gamma_k"'] & & 1 \arrow[d, dash, "\alpha_1"] \\
1^1 \arrow[d, dash, "\gamma_1^1"'] & & 1^{p+2} \arrow[d, dash, "\alpha_1^{p+2}"] \\
1^2 & & 1^{p+3} \\
\vdots & & \vdots \\
1^p \arrow[dr, dash, "\gamma_1^p"'] & & 1^{p+q+1} \\
& k &
\end{tikzcd}
$$

In each case we see that $dim\, e_1\Lambda = dim\, e_1\Lambda_\eta = 2p+2q+4$ for all t. Hence

$$dim\, \Lambda_\eta = dim\, \Lambda.$$

Theorem 12.15. *With Λ, η, and Λ_η as defined above, then Λ_η is a deformation of Λ.*

So we have found a non-zero element in $\mathrm{HH}^2(\Lambda)$ and a corresponding associative deformation of Λ. Thus we can see the connection between the second Hochschild cohomology group and deformation theory.

This concludes the chapter and the thesis. In future work we will consider the other derived equivalent representatives of the algebras of the first type

$\Lambda(p, q, k, s)$. After that, it would be interesting to consider $\mathrm{HH}^2(\Lambda)$ for the second type $\Gamma^*(n)$ and to look at the classification in [7] of tame weakly symmetric finite dimensional self-injective algebras.

References

[1] Auslander, M., Reiten, I. and Smalø, S., *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics, **36**, CUP, 1995.

[2] Asashiba, H., *On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras*, Algebras and Representation Theory, **6** (2003), 427–447.

[3] Asashiba, H., *The derived equivalence classification of representation-finite self-injective algebras*, J. Algebra, **214** (1999), 182–221.

[4] Bardzell, M., *The Alternating Syzygy Behavior of Monomial Algebras*, J. Algebra, **188** (1997), 69–89.

[5] Benson, D., *Representations and Cohomology: vols I, II*, Cambridge studies in advanced mathematics **30**, **31**, CUP, 1995.

[6] Bocian, R., Holm, T. and Skowroński, A., *Derived equivalence classification of one-parametric self-injective algebras*, preprint (2004).

[7] Bocian, R., Holm, T. and Skowroński, A., *Derived equivalence classification of weakly symmetric algebras of Euclidean type*, J. Pure Appl. Alg., **191** (2004), 43–74.

[8] Cibils, C., *Rigid monomial algebras*, Math. Annalen, **289** (1991), 95–109.

[9] Erdmann, K. and Holm, T., *Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n*, Forum Math., **11** (1999), 177–201.

[10] Erdmann, K. and Snashall, N., *On Hochschild Cohomology of Preprojective Algebras, I*, J. Algebra, **205** (1998), 391–412.

[11] Fox, T., *An introduction to algebraic deformation theory*, J. Pure Appl. Alg., **84** (1993), 17–41.

[12] Gerstenhaber, M., *On the deformations of rings and algebras*, Annals of Math., **79** (1964), 59–103.

[13] Gerstenhaber, M. and Schack, S.D., *Relative Hochschild cohomology, rigid algebras and the Bockstein*, J. Pure Appl. Alg., **43** (1986), 53–74.

[14] Green, E. L., *Noncommutative Gröbner bases and projective resolutions*, In: Computational methods for representations of groups and algebras (Essen 1997), Progr. Math., **173** (1999), 29–60, Birkhäuser, Basel.

[15] Green, E. L. and Snashall, N., *Projective Bimodule Resolutions of an Algebra and Vanishing of the Second Hochschild Cohomology Group*, Forum Math., **16** (2004), 17–36.

[16] Green, E. L., Solberg, Ø. and Zacharia, D., *Minimal Projective Resolutions*, Trans. Amer. Math. Soc., **353** no. 7 (2001), 2915–2939.

[17] Happel, D., *Hochschild cohomology of finite-dimensional algebras*, Lecture Notes in Mathematics, Springer-Verlag, **1404** (1989), 108–126.

[18] Happel, D., *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Mathematical Society Lecture Note Series, **119**, CUP, 1988.

[19] Hochschild, G., *On the cohomology groups of an associative algebra*, Annals of Math., **46** (1945), 58–67.

[20] Maclane, S., *Homology*, Classics in Mathematics, Springer, 1995.

[21] Maclane, S., *Categories for the Working Mathematician*, Graduate Texts in Mathematics, **5**, Springer-Verlag, 1971.

[22] Riedtmann, C., *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. **55** (1980), 199–224.

[23] Riedtmann, C., *Representation-finite selfinjective algebras of class A_n*, Representation Theory II, Lecture Notes in Mathematics, Springer-Verlag, **832** (1979), 449–520.

[24] Weibel, C., *An introduction to homological algebra*, Cambridge studies in advanced mathematics, **38**, CUP, 1995.