

Robust Sliding Mode Control Using Output Information

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by

Sujit Kumar Bag, B. Tech.(Hons.) (India)
Department of Engineering
University of Leicester
Leicester LE1 7RH
United Kingdom

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*To my family
who has prayed for me
one thousand and ninety five days and nights,
and would never be forgotten, this work is dedicated*

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S. K. Bag
Leicester

September 5, 1997

Statement of Originality

The accompanying thesis submitted for the degree of Doctor of Philosophy entitled 'Robust Sliding Mode Control using Output Information' is based on research work carried out by the author in the Department of Engineering of the University of Leicester mainly during the period between September 1994 and May 1997. All the results reported in this thesis are original unless otherwise acknowledged in the text or references. No part of this work has been submitted for any degree in this or any other University.

S. K. Bag

Leicester

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Robust Sliding Mode Control

Using Output Information

Sujit Kumar Bag

Abstract

The thesis considers the development of robust output feedback sliding mode controllers for linear time invariant uncertain systems where output information alone is available to the controller. Two approaches to controller design are discussed. The first uses only the plant dynamics and is called static output feedback sliding mode control. It is shown that a quadratically stable sliding motion may be attained for a bounded uncertain system if and only if the system is minimum phase and a particular subsystem triple satisfies the output feedback design criteria. Sliding mode controllers are sensitive to unmatched uncertainty. Hence a robust design is considered which minimises the effect of uncertainty. The second approach is developed for systems which have design difficulties when only the plant dynamics are considered. Extra dynamics are added and the method is called dynamic output feedback sliding mode control. Closed-loop analysis is carried out and stability of the augmented system is observed. Both controllers guarantee a stable sliding motion despite the presence of bounded uncertainty.

Finally, two practical uncertain multivariable industrial examples demonstrate the theoretical developments. The first application is a helicopter model. The open loop dynamics have unstable poles with two stable invariant zeros, variations in model parameters and exhibit high levels of cross coupling. A model following sliding mode controller is used to force the plant outputs to track the outputs of an ideal model. Nonlinear simulation results show the practicality of the method. The second application considers the dynamic output feedback sliding mode control of an aircraft model. The system possesses unstable invariant zeros and requires a dynamic output feedback technique. Simulation results are obtained at different operating points to show the effect of unstable invariant zeros. The examples illustrate the benefits of these robust output feedback based sliding mode control developments.

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Chapter 1

Introduction

1.1 An Overview

Much of the research in the area of control systems theory during the seventies and eighties has focused towards the issues of nonlinear control and robustness of dynamical systems, i.e. designing controllers with the ability to maintain desired performance and stability in the presence of discrepancies between the plant and model due to the changes of parameters and disturbances. The subject of nonlinear control increases in importance as modern technology develops and produces more and more complex plants. For such systems linear control theory may not be sufficient to solve the associated problems although linear control theory is a mature subject with a variety of powerful methods and a long history of industrial application [67, 72, 86]. However, it is limited to a relatively small range of operations. When the operation range is large, a linear controller is likely to perform poorly or to be unstable, because the nonlinearities in the system and changes in the dynamics cannot be properly compensated. A nonlinear controller, on the other hand, may handle the nonlinearities over a large range of operation directly [94, 105]. For the application of linear control, the system model must be linearizable. However, in control systems there are many nonlinearities whose discontinuous nature does not allow linear approximation. These are so called hard nonlinearities [94, 104] such as coulomb friction, saturation, dead zones, backlash, and hysteresis. A nonlinear controller may be able to handle these phenomena more easily. For controller design, it is necessarily assumed that the system model is reasonably well

known. However, in practice it is impossible to find out an exact model. Moreover, while the process is operating and measurements are carried out, the instruments add some noise, which gives perturbation to the system parameters. The uncertainties and perturbations are thus very common in control systems and a linear controller may be unable to tolerate these. A nonlinear controller, on the other hand, may introduce some bounded nonlinearities into the controller so that it can tolerate the uncertainties. Two major and complementary approaches to dealing with such model uncertainties are *robust control* and *adaptive control*. One such robust technique is Variable Structure Systems (VSS) with a sliding mode which is the main issue of this thesis. The method of adaptive control is not discussed in this thesis. The structure of such a robust controller is composed of a nominal part, similar to a feedback linearizing control law and of additional terms aimed at dealing with model uncertainty or nonlinearity, although there are some robust controllers whose structure does not comprise with two parts such as H_∞ controllers.

There are many other reasons to use a nonlinear controller such as design simplicity, cost and performance optimisation. A good nonlinear control design may be simpler and more intuitive than its linear counterparts. In the industrial setting, an ad-hoc extension of linear techniques to control advanced processes with significant nonlinearities may result in undue cost and a lengthy development process. Linear control may require high quality actuators and sensors to produce linear behaviour in the specified range of operation, while nonlinear control may permit the use of less expensive components with nonlinear characteristics [94].

It is thus seen that the subject of nonlinear control is an important area of control engineering. In the past, the application of nonlinear control methods had been limited due to computational difficulties, but in recent years, advancement of computational power has increased interest in nonlinear control research. While the analysis of a nonlinear controller may be difficult, serious efforts have been made to develop appropriate theoretical tools. Presently, there are various tools available for designing and analysing nonlinear systems.

One of the most important steps in the development of nonlinear control systems is Lyapunov stability analysis. Basic Lyapunov theory comprises two methods, introduced

by the Russian Mathematician Alexander Mikhailovich Lyapunov [77] in the late 19th century, and called the indirect method and the direct method respectively [20, 52, 69, 94, 105]. The indirect method or linearisation method is often used to analyse nonlinear control systems. It states that the stability properties of the nonlinear system in the close vicinity of an equilibrium point are essentially the same as those of its linearised approximation [94, 105]. The method has provided the theoretical justification for the use of linear control systems. The direct method is most important in analysing nonlinear systems. It is based on energy concepts associated with mechanical systems [94, 105]. The motion of a mechanical system is stable if its total energy decreases all the time. In using the direct method to analyse the stability of a nonlinear system, the idea is to construct a scalar energy like function, called a *Lyapunov Function* for the system, and a control action so that the total energy decreases all the time. The big advantage of this method is that it avoids the solution of differential equations. It allows engineers to solve many complex design problems such as robotics, spacecraft and many adaptive control engineering problems. However, sometimes effort is required to find the appropriate Lyapunov function.

Another important method to analyse nonlinear systems is the describing function method [20, 52, 94, 97]. The basic idea of this method is to approximate the nonlinear components in a nonlinear control system by linear ‘equivalents’ and then use frequency domain techniques to design and analyse the resulting systems. Unlike Lyapunov methods, whose applicability to a specific system hinges on the success of a trial and error search for a proper Lyapunov function, its application is straightforward.

Phase plane analysis [4, 20, 52, 94] is another method used to analyse nonlinear systems. In this method, the system differential equation is solved graphically, instead of seeking an analytical solution. The main disadvantage of this method is that it is restricted to two dimensional problems because of its graphical nature. However, the ideas are useful to illustrate the concepts of the Variable Structure Systems (VSS) which will be discussed in Chapter §2.

The Variable Structure Systems (VSS) [122, 123] methodology is another nonlinear approach to robust controller design. Although the approach is applicable to systems of nonlinear differential equations, much of the research has been directed towards de-

velopments based on linear uncertain systems. However, now Symbolic Toolboxes are available for nonlinear controller design from nonlinear models. Most of the early work in the field of variable structure systems has been published in Russian and very few publications are available in the open literature describing the methods [102]. One such method is described in Chapter §2. Much of the work published in English in the literature in the area of variable structure systems considers sliding modes which is one of the modern developing robust methods for controlling dynamical systems in the presence of nonlinearities and uncertainties. Another important area involves variable structure systems with simplex control. However, in this thesis only the ‘variable structure systems with sliding mode control’ will be considered.

In general, variable structure control systems comprise a feedback control law which employs a discontinuous control action and a decision rule [32, 104]. The decision rule is often based upon the behaviour of a *switching function* and determines which of the control laws is activated at any instant in time. The control objective is to force the system states to reach and subsequently remain on a pre-determined surface, called the *switching surface* or *hyperplane*. The dynamical behaviour of the system when it reaches and remains on the surface is defined as a *sliding motion*. The system exhibits two important properties during sliding motion. Firstly, there is a reduction in order of the system. In the case of nonlinear systems, an appropriate selection of the switching surface may render the reduced order motion to be linear or almost linear. Secondly the sliding motion is insensitive to the parameter variations implicit in the input channels, the so called *matched uncertainty* [104, 122]. These invariance properties of the sliding mode are attractive to researchers seeking to design robust controllers for uncertain systems. The design approach comprises two components: the design of a suitable switching surface or hyperplane in the state space, so that the reduced order sliding motion satisfies the performance specifications imposed by the designer; and synthesis of a control law, discontinuous on the sliding surface, such that the trajectories of the closed-loop system are directed towards the surface. In other words, the second property ensures the discontinuous control action renders the sliding surface invariant, attractive and locally stable.

For a variable structure control with a sliding mode, the dynamical behaviour is divided

into two stages. The first stage or initial phase is to drive the states to the surface. During this motion the system is in general affected by any matched and unmatched disturbances present. In practice this initial phase should be kept as small as possible to minimise the effect of uncertainty; this causes high control action and some times the system may go unstable. The second stage seeks to maintain the states on the switching surface for the remaining period where it is insensitive to all matched uncertainty. This inherent property is well known in sliding mode control. However, the dynamics are continuously affected by unmatched uncertainty. It is shown that the effect is unavoidable but can be minimised using a robust design technique which leads to the development of a ‘robust sliding mode controller’. Hence, a robust switching surface design may be preferable to minimise this effect. Much of the work in this field considers the use of a state feedback control law and assumes that all states are available to feed through to the controller. In practice only a few states or measurable outputs may be available. This leads to the design of an observer system or an output strategy for such systems. The former will be discussed briefly in Chapter §6 and the latter will be extensively discussed in Chapter §4. First consider a simple example which shows how a multivariable system design is performed with a discontinuous function.

In this thesis, linear uncertain systems of the following form are considered;

$$\dot{x}(t) = Ax(t) + Bu(t) + h(x, u, t) \quad (1.1a)$$

$$y(t) = Cx(t) \quad (1.1b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are known plant model matrices where n , m and p are the number of states x , inputs u and outputs y respectively of the system and the function $h(t, u, x)$ is considered to represent any nonlinearities and model uncertainties in the system. Its contribution is unknown but assumed bounded. A mathematical model as in equation (1.1a) with a discontinuous function on the right hand side is feasible as a number of processes in mechanical, chemical and electrical systems and other areas have a discontinuous function on the right hand side [104]. A typical example of such a system is the dry (Coulomb) friction of the mechanical system shown in Figure 1.1. The equation of this system can be written as

$$M\ddot{x}(t) + P(\dot{x}) + Kx(t) = 0 \quad (1.2)$$

where x is displacement, M is mass, K is the spring rigidity, and

$$P(\dot{x}) = \begin{cases} P_o & \text{if } \dot{x}(t) > 0 \\ -P_o & \text{if } \dot{x}(t) < 0 \end{cases} \quad (1.3)$$

where P_o is a positive constant. It is quite obvious that the discontinuous surface in

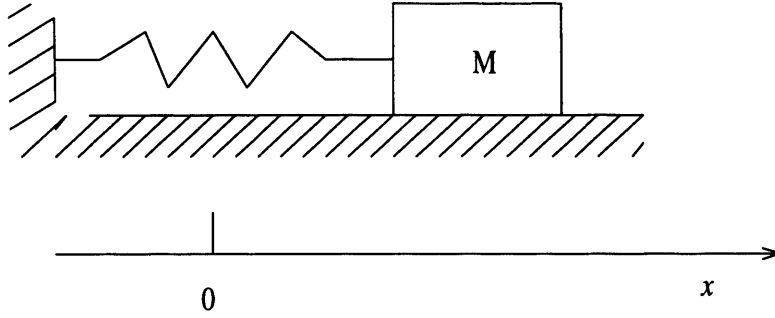


Figure 1.1: Coulomb friction in mechanical mass-spring system

this case is the x axis; i.e. $s(x) = \dot{x}(t) = 0$. Here the resistance force may take either positive or negative sign depending on the direction of the motion [104]. This situation may arise in many control application. Another such example is electric motors and power converters where the control action is naturally discontinuous [90]. In some cases, especially in mechanical systems, the introduction of a discontinuous control action will not introduce an ideal sliding motion due to imperfections in the process such as delays and hysteresis which create high frequency motion known as *chattering*. This idea is characterised by the states repeatedly crossing rather than remaining on the surface. Such a motion is undesirable in the control system and will result in high wear and tear which reduces the life of the actuator components. In this case the discontinuous control action is modified such that the states lie in a small boundary of the switching surface and subsequently remain within the boundary. This type of sliding motion is referred to as *pseudo sliding motion* of the control system [32]. The total invariance properties associated with the ideal sliding motion will be lost. However, an arbitrarily close approximation to ideal sliding motion can usually be attained by appropriate modification. These results are described in Ryan and Corless [88] in terms of *practical stability* and implementation issues are discussed in Davies and Spurgeon [21].

From the above discussion, the sliding mode approach is embedded within the state-space framework. Most of the published work relies on the assumptions that all the

internal states are measurable and available to feed through to the controller. Under this assumption, provided the linear uncertain system described in equation (1.1a) is completely controllable and the uncertainty is matched into the input channel and bounded, a regulatory controller may always be designed which guarantees asymptotic convergence to the zero state [88]. In practice, such an idealised situation is quite rare. Usually certain states will be either impossible or expensive to measure. Another factor is that the linear models may represent an approximation of a distributed parameter system or may be obtained using system identification. As a result, all the internal states are not possible to define and they do not have any physical meaning. In these situations much of the sliding mode control methods are no longer directly applicable. An alternative framework is essential for this purpose which is the main issue of this thesis. Before going through further discussions, a brief discussion of the contents of the thesis is presented.

1.2 Structure of the Thesis

The underlying assumption of this thesis is that *only measurable outputs* are available to feed through to the controller and the system is at least completely controllable and the uncertainty is bounded; any additional assumptions will be stated in due course as necessary. This will be named Output Feedback Variable Structure Control (VSCOF) or Output Feedback Sliding Mode Control. The early chapters of the thesis describe important mathematical background for analysis and design of sliding mode controllers. A brief description of state and output feedback sliding mode techniques and the invariance properties are described. The existence methods and its limitations, the need for further developments including the output feedback sliding mode controller design and necessary conditions for the existence of a sliding mode are discussed. The existence method is essentially a static output feedback method. Many systems do not fall into the static output feedback classes. These problems can be solved using dynamic output feedback sliding mode control giving extra freedom in design. Two different approaches are given for classes of systems which are difficult to control using static output feedback sliding mode control or may require greater freedom in design for better performance. The remaining two chapters consider the use of these new developments for practical

problems. Two different realistic problems involving the development of helicopter and aircraft controllers are emphasised here. In short it can be said that throughout the thesis, emphasis is made towards:

- the development of robust hyperplane design techniques and a suitable controller law formulation, using output feedback information only, for stabilisation and induction of sliding motion.
- the implementation of robust sliding mode control using output information for practical examples is also considered.

Finally conclusions are made based on the above work. Possible future investigations and the extension of the work are described in the final chapter.

In structure, the thesis may be divided into two parts. The early chapters develop new theoretical results whilst the later chapters apply these developments to helicopter and aircraft systems. Details of mathematical preliminaries and some important results pertaining to the development and design of sliding mode controllers are presented as Appendices. A brief summary of the chapters is given below.

Chapter (2) presents the background in variable structure systems. The motivation for robust multivariable controller design is briefly described. An introduction to variable structure systems is demonstrated with different examples. These examples are given to describe the sliding mode behaviour, attainment of sliding motion and the effect of chattering etc. The condition to ensure the hyperplane or switching surface is reached, known as the *reaching condition*, is also described. The system of equation (1.1a) with a discontinuous right hand side is most straightforward to analyse using the concept of the *equivalent control*. This enables the reduced order sliding dynamics to be formulated. This is emphasised for the case of output measurements since the switching surface design is associated with only the outputs of the system. The invariance properties of the reduced order sliding motion to disturbances occurring in the input distribution matrix is examined. The sliding mode technique uses a discontinuous control action. A common discontinuous control vector used in output feedback sliding mode control is presented. Finally, some continuous approximations to the variable structure controllers

are described. These are commonly used at the implementation stage to avoid the problem of chattering. Details of the existing developments and associated problems in output feedback sliding mode design are given in the following chapter.

Chapter (3) considers a brief review of the recent developments in output feedback sliding mode control and the limitations of these methods are presented. These may be classified in two categories. The class of system which do not require any additional dynamics for improvements in the controller performance, or to circumvent design difficulties associated with dimensionality requirements is called ‘static output feedback sliding mode control’. Necessary assumptions and the major problems are pointed out. If the system does not satisfy the necessary assumptions, an alternative approach is to add extra dynamics to the system, hence giving extra degrees of freedom in design. This is known as ‘dynamic output feedback sliding mode control’. Finally, keeping in mind existing methods and their limitations, further developments are made in subsequent chapters of this thesis.

Chapter (4) introduces a new framework for output feedback sliding mode control. It reviews the mathematical description of the system and the necessary assumptions for developing static output feedback sliding mode controllers. The uncertainty on the right hand side of the system is given a more general structure which is decomposed into unmatched and matched uncertainty, and the regular form of the system is defined. This structure is used throughout the thesis unless mentioned. It enables the class of systems considered by other researchers in this field to be extended and provides an explicitly realisable controller structure without any additional assumptions. Like state feedback sliding mode schemes, the two main design issues associated with the output feedback sliding mode schemes, are the switching surface or hyperplane design and realisation of a controller. It is shown that there is no freedom of choice in the switching surface design for square plant. The switching surface design for non-square plant is an output feedback problem which is affected by unmatched uncertainty. The conditions for existence of a stable reduced order sliding motion are presented. The effects of the invariant zeros and the ‘Kimura-Davison’ condition are described. Two major criteria for developing the static output feedback sliding mode control are; the invariant zeros must be stable and a particular closed-loop design triple must satisfy the

‘Kimura-Davison’ condition. To minimise the effect of unmatched uncertainty, a robust switching surface design is considered based on a normal matrix design technique. The controller formulation, its freedom in design and reaching conditions are examined. Finally, two numerical examples demonstrate the methods and simulation results show the tractability of application. Further details of the application of this theory to a nonlinear helicopter system are available in a later chapter and the associated problems are solved in Chapters §5 & §6. The theoretical results obtained and its application to uncertain systems are published in the proceedings of the IFAC and IEE conferences.

Chapter (5) develops the ‘dynamic output feedback sliding mode control strategy’ for systems which do not fall into the static output feedback sliding mode control framework or may require greater freedom in design to meet the performance requirements. This chapter emphasizes problems which do not satisfy the required ‘Kimura-Davison’ condition or may require extra degree of freedom in design and a subsequent chapter is presented to develop another type of dynamic output feedback sliding mode control for systems with unstable invariant zeros. The plant is augmented with a compensator and the augmented system is considered for static output feedback sliding mode design, giving more freedom in calculating the switching surface and controller parameters. Hence, the same robust output feedback technique of Chapter §4 is applicable for the design of the switching surface and the compensator parameterisation together. The controller structure is similar to that described in Chapter §4. However it is used with the augmented system parameters. Numerical examples are given to demonstrate the effect of dynamic compensator to solve problems with the ‘Kimura-Davison’ condition in the static output feedback problem and also to improve the performance. This theoretical result is published in the proceedings of an IEEE conference. A article involving the work from Chapters §4 & §5 has been accepted for publication in the IEE Proc. Part-D.

Chapter (6) examines the case of dynamic output feedback sliding mode control for systems with unstable invariant zeros. An additional assumption is imposed. Previous developments in this area are described in Chapter §3. The idea used here is similar to previous work but it extends the contribution to the case of nonlinear systems and the effect of uncertainty in the closed-loop dynamics is properly examined. The dynamic

compensator is parameterised independently and it is nonlinear in nature. The controller formulation and the reaching condition are examined. The closed-loop analysis and switching surface design are described together with limitations in the uncertainty. The issue of robustness is also applicable during the design of both the compensator and controller parameters. Further examples are given to demonstrate the numerical tractability and simulations show the output regulation, the reaching phase and the control action etc. More about its application is presented in Chapter §8.

Chapter (7) presents the design of a controller for a helicopter system. This is an application of static output feedback sliding mode control to a fully nonlinear industrial system. The reduced order realisable mathematical model of the helicopter and its uncertainties are described. The objectives of the design are achieved using a model following technique. An ideal model is designed based on the H_∞ 1-DOF technique. The H_∞ control produces a higher order controller due to the parameters of the shaped plant. The discrepancy between the higher order H_∞ controller and the minimum order sliding mode controller is solved. The switching surface and controller parameters are calculated using the robust output feedback technique. Fully nonlinear helicopter simulation results are presented for different flight conditions. These nonlinear simulation results justify the theoretical work presented in Chapter §4.

Chapter (8) considers controller design for an aircraft with unstable transmission zeros. The dynamic output feedback sliding mode control technique developed in Chapter §6 is applied for controller design. The mathematical model is derived from a GARTEUR benchmark problem called the High Incidence Research Model (HIRM). The control problem is divided into longitudinal and lateral channels. Each channel is controlled separately. The dynamic compensator and controller are robustly designed for each channel as presented in Chapter §6. The simulation results are obtained for linearly perturbed systems at different operating conditions. These results illustrate the application of dynamic output feedback sliding mode control for systems with unstable transmission zeros and the use of robust techniques for uncertain systems.

Chapter (9) summarises the contributions of the thesis highlighting the use of ‘robust sliding mode control using output information’ and recommends future work.

1.3 Notation

The system triple (A, B, C) is considered as a linear, time invariant, finite dimensional representation and the bounded uncertainty $h(t, u, x)$ comprises of all model uncertainties plus any other nonlinearities present in the system as represented in equation (1.1a). The linear system is said to be *asymptotically stable* if and only if each of the eigenvalues of the matrix A has a strictly negative real part.

1.3.1 Mathematical Notation

A^T	the transpose of the matrix A
$A^\dagger$	the Moore Penrose inverse of the matrix A
A^{-1}	the inverse of the square matrix A
$\det(A)$	the determinant of the square matrix A
$\lambda_{\max}(A)$	the largest eigenvalue of the square matrix A
$\lambda_{\min}(A)$	the smallest eigenvalue of the square matrix A
I_n	the $n \times n$ identity matrix
M	lump mass
$\mathcal{M}$	mach number
$N(S)$	the null space of S
$P > 0$	the matrix P has positive eigenvalues
$Q = Q^T \geq 0$	symmetric matrix Q is positive semi-definite
$\text{Re}(A)$	the real part
$\mathbb{R}$	the field of real numbers
$\mathbb{R}_+$	the field of scalar time
$\mathbb{R}^{n \times m}$	the set of real matrices with n rows and m columns
$R(B)$	the range space of B
$\mathcal{E}$	the bounded set
$ a $	the absolute value of the real number a
a_{ij}	the (i, j) element of the matrix A
$\text{diag}\{\sigma_j\}$	a diagonal matrix with σ_j on the main diagonal
h	vertical distance

i	$\sqrt{-1}$; sometimes an index, as in a_{ij}
$\rho(t, u, x)$	the scalar function of t, u, x
$\text{sgn}(\cdot)$	the signum function
t	time in seconds
$\ x\ $	Euclidean norm of vector
$\dot{x}$	the derivative of x w.r.t time
$\ddot{x}$	the double derivative of x w.r.t time
$\cup$	‘the union of element of’
$\subset$	‘the subset’
$\cap$	‘the intersection’
$\exists$	‘there exists’
$\in$	‘an element of’
$\notin$	‘not an element of’
$\forall$	‘for all’
$\neq$	‘not equal to’

1.3.2 Abbreviations

DOF	degrees-of-freedom
DRA	Defence Research Agency
GARTEUR	Group for Aeronautical Research and Technology in Europe
LHP	left half-plane
LQG	linear quadratic Gaussian
LSDP	loop-shaping design procedure
HIRM	High Incidence Research Model
MIMO	multi-input multi-output
RHP	right half-plane
SISO	single-input single-output
VSS	Variable Structure System
VSCS	Variable Structure Control System
VSCOF	Output Feedback Variable Structure Control
sec.	time in second

Chapter 2

Background and Scope in Variable Structure Systems

2.1 Introduction

In this chapter the concepts and ideas of variable structure systems are briefly reviewed. Section §2.2 motivates the use of robust control for tackling uncertain multivariable design problems. Section §2.3 introduces variable structure systems. The method of equivalent control and the invariance properties of variable structure systems to a certain class of uncertainty are discussed in subsequent subsections. The existence of a sliding mode, the nature of the control law and the effect of chattering are described. Section §2.4 summarises the main points of the chapter.

2.2 Motivation for Robust Multivariable Control

A mathematical model of any physical system is always an approximation of the true plant system dynamics. The difference between the model and the actual plant dynamics, i.e. the *plant uncertainty*, depends on various factors. Typical sources of uncertainties include unmodelled dynamics (high frequency), neglected nonlinearities, effects of model reduction, and the plant parameter changes due to environmental factors and variations in operating condition with time. These uncertainties enter into the plant dynamics either through the input or output channel. In addition the system may have unmatched nonlinearities, parameter changes etc. A feedback control system (in

which the controller design is based on an imperfect plant model) is required to be *robust* to such perturbations; it must maintain stability and some level of performance in the presence of uncertainty. In classical frequency response analysis for single-input single-output (SISO) systems, gain and phase margins have been used as measures of robustness. It is now well known that these measures (taken one loop at a time) are not good indicators of robustness for multivariable feedback systems [53].

Many SISO design techniques have been used to tackle multivariable problems. Examples are the Characteristic Root Locus and Nyquist Array design methods. These techniques have also been applied to some design problems. One example is the aircraft control problem discussed by Maciejowski [78]. The main drawback of these design methods is that they rely mainly on the notion of gain and phase margins to address robustness, and these measures, as indicated above, can be poor indicators of robust stability. Moreover if designs obtained through such methods do not yield satisfactory closed-loop behaviour, it is often not obvious what can be done to improve the performance. The question of optimisation is also important and one does not know if the design can be improved or not. Hence, when faced with the design of controllers for complex multivariable systems, one is motivated to look towards techniques which are inherently multivariable and which provide a degree of robustness to modelling errors and uncertainties. Variable structure systems are one such method which is inherently robust in nature and can easily handle multivariable problems with matched uncertainty. However, the effect of unmatched uncertainty can not be tolerated. Hence, a robust variable structure control is appreciated to minimise this class of uncertainty. Since most of the analysis in variable structure systems is based on state-space methods, one robustness measure for this case is the position of the eigenvalues and eigenvectors and the norm of the closed-loop system for both MIMO and SISO systems (as shown, for example, in [6]). One such approach is considered in this thesis.

2.3 Variable Structure Systems

The concept of variable structure systems first originated in the Soviet literature in 1955. The pioneer authors are Tsypkin [101], Emel'yanov [41], Aizerman and Gant-

macher [2], Filippov [46]. Most of this work considered variable structure regulation of linear systems. Most of the ideas were not published outside Russia until the end of 1964. Later, results were more broadly published in the English literature when a book edited by Itkis [66] appeared in 1976 and a survey paper was published by Utkin [102] in 1977. The application of variable structure system theory was first illustrated by Young [116] in 1978 for a manipulator controller design. The combination of sliding controllers with state observers was discussed by Bondarev *et al.*[9] in the linear case, and Hedrick and Gopalswamy [58] in the nonlinear case. Observers based on sliding surfaces were discussed in [30, 93, 107]. Variable structure schemes have subsequently been utilised and developed for the design of underwater vehicles [115], robust regulators [19, 54, 88, 92], model reference systems [99, 100], adaptive control schemes [61], tracking problems [55] and state observers [104, 107]. The method has successfully been applied to problems in control of electrical motors, chemical process plant and robot manipulators and in simulation of automatic flight control, helicopter stability and space systems etc.

Variable structure systems as the name reflects, are a class of systems whereby the ‘control law’ is changed during the control process according to predefined rules which depend on the system states. The main ideas can be described using phase plane analysis for second order systems. For the purpose of illustration consider the example of a satellite control system in Figure 2.1. The satellite, depicted in Figure 2.2(a), is

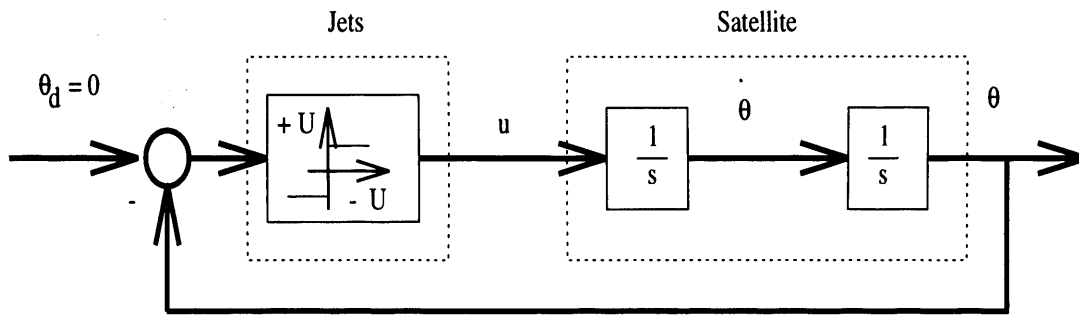


Figure 2.1: Satellite control system

simply a rotational unit inertia controlled by a pair of thrusters, which can provide either a positive constant torque U (positive firing) or a negative torque $-U$ (negative firing). The purpose of the control system is to maintain the satellite antenna at zero angle to the reference axis by appropriately firing the thrusters. The mathematical

model of the satellite is

$$\ddot{\theta}(t) = u(t) \quad (2.1)$$

where $u(t)$ is the torque provided by the thrusters and $\theta(t)$ is the satellite angle. Consider the phase plane behaviour of the control system when the thrusters are fired according to the control law defined by

$$u(t) = \begin{cases} -U & \text{if } \theta(t) > 0 \\ U & \text{if } \theta(t) < 0 \end{cases} \quad (2.2)$$

which means that the thrusters push the satellite in a counter clockwise direction if θ is positive, and vice versa. To generate the complete phase portrait, first consider the phase portrait when the thruster provides a positive torque U . The dynamics of the system are

$$\ddot{\theta}(t) = U \quad (2.3)$$

which implies that the phase trajectories are a family of parabolas defined by

$$\dot{\theta}^2(t) = 2U\theta(t) + c_1 \quad (2.4)$$

where c_1 is a constant. The corresponding phase portrait of the system is shown in

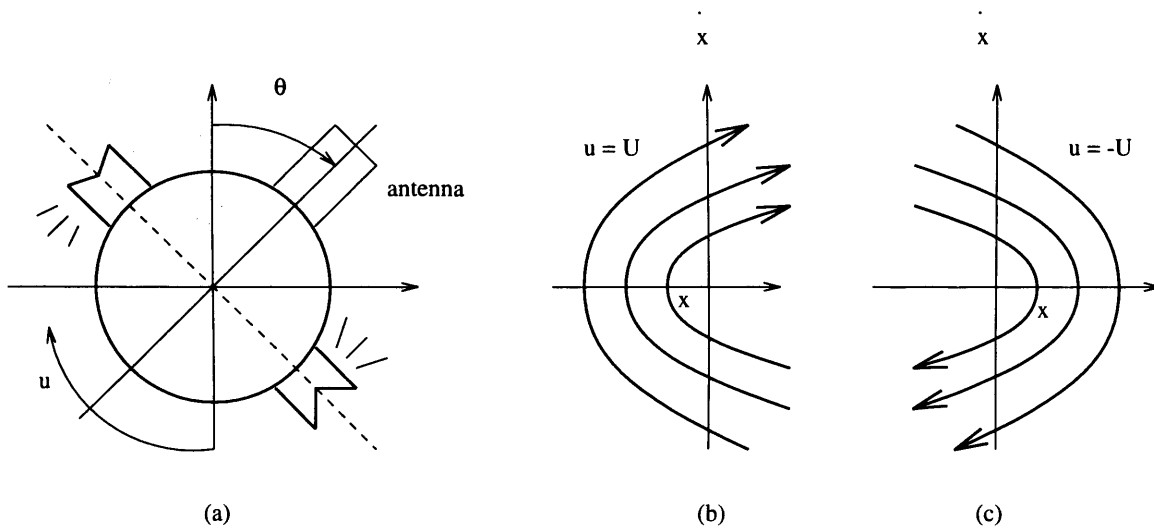


Figure 2.2: Satellite control using on - off thrusters

Figure 2.2(b). When the thrusters provide a negative torque $-U$, the phase trajectories are similarly found with the corresponding phase portrait shown in Figure 2.2(c). The complete phase portrait of the closed-loop system can be obtained simply by connecting

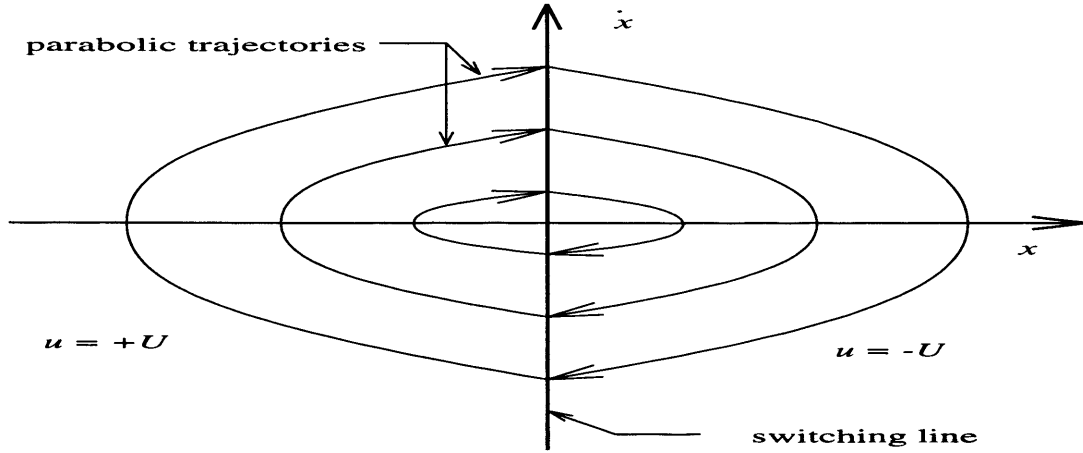


Figure 2.3: Complete phase portrait of the control system

the above two trajectories as shown in Figure 2.3. The resulting response shows that the system is stable while the control actions U and $-U$ simultaneously act on the system and the trajectory directed to the vertical axis $\dot{x}(t)$. It is concluded from the above discussion that the control law changes on the vertical axis which may be called the *switching line*.

Consider now the general form of a double integrator [32, 102] given by

$$\ddot{y}(t) = u(t) \quad (2.5)$$

Using the control law

$$u(t) = -ky(t) \quad k > 0 \quad (2.6)$$

results in simple harmonic motion characterised by an elliptical phase portrait which may be considered to be marginally stable. Consider instead the control law

$$u^*(t) = \begin{cases} -k_2 y(t) & \text{if } y\dot{y} > 0 \\ -k_1 y(t) & \text{otherwise} \end{cases} \quad (2.7)$$

where $k_2 > 1 > k_1 > 0$. The phase plane $(y, \dot{y})$ is partitioned by the switching rule into four quadrants separated by the Cartesian coordinate axis as shown in Figure 2.4. These examples clearly fit the description of the name of ‘Variable Structure Control Systems’. The control $u(t) = -k_2 y(t)$ is effected in the quadrants of the phase plane labelled (a). In this region it can be verified that the distance from the origin of the points in the phase portrait decreases along the trajectory. Similarly, in the region (b) when the control law $u(t) = -k_1 y(t)$ is activated, the distance from the origin of the points in the

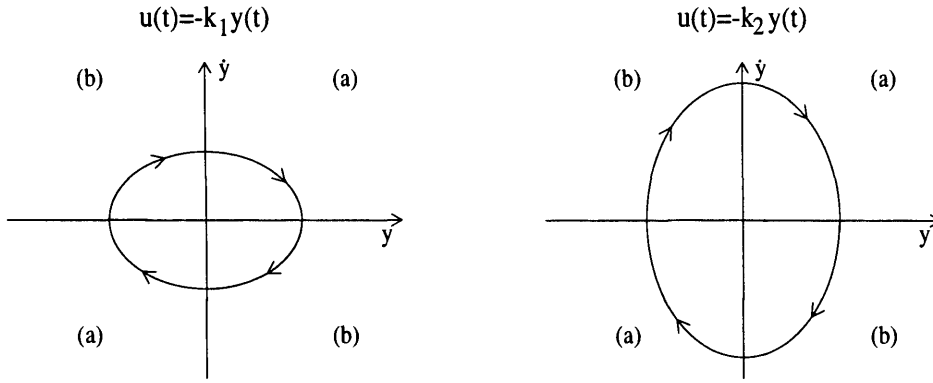


Figure 2.4: Phase portraits of simple harmonic motion

phase plane portrait also decreases. The phase portrait of the closed loop system under the variable structure control law u^* is obtained by placing together the appropriate regions from the two phase portraits. An asymptotically stable motion results as shown in Figure 2.5. By introducing a rule for switching between two control structures,

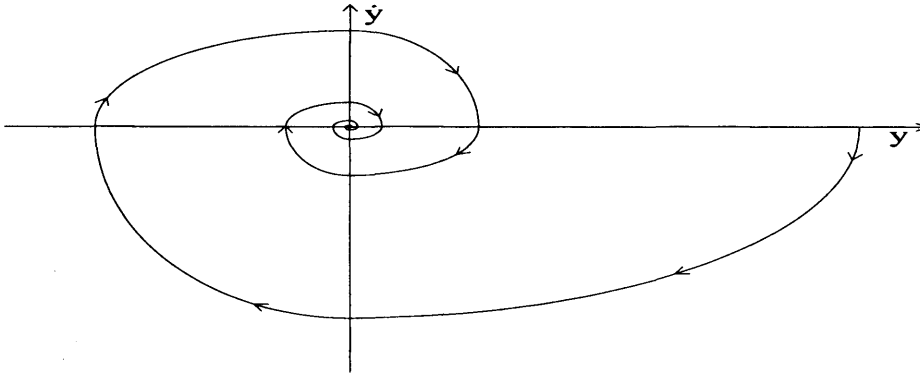


Figure 2.5: Phase portraits of the system under VSC

which independently do not provide stability, an asymptotically stable system has been obtained.

A more significant example results from using the control law

$$u^*(t) = \begin{cases} -1 & \text{if } s(y, \dot{y}) > 0 \\ 1 & \text{if } s(y, \dot{y}) < 0 \end{cases} \quad (2.8)$$

where the switching function is defined by

$$s(y, \dot{y}) = my(t) + \dot{y}(t) \quad m > 0 \quad (2.9)$$

Now, the term switching function is clear as $s(\cdot)$ is used to decide which control structure is in use at the point $(y, \dot{y})$ in the phase plane. For all values of $\dot{y}(t)$ which satisfy the inequality $m|\dot{y}| < 1$, the systems total energy will decrease for all time. This can be expressed mathematically as

$$s\dot{s} = s(m\dot{y}(t) - \text{sgn}(s)) < -|s|(1 - m|\dot{y}|) < 0 \quad (2.10)$$

where $\text{sgn}(\cdot)$ is the signum function. Equivalently it can be written [104] as

$$\lim_{s \rightarrow 0^+} \dot{s} < 0 \quad \text{and} \quad \lim_{s \rightarrow 0^-} \dot{s} > 0$$

This is called the ‘reachability condition’ of the system. Consequently when $m|\dot{y}| < 1$ the system trajectories lie on either side of the line

$$\mathcal{L}_s = \{(y, \dot{y}) : s(y, \dot{y}) = 0\} \quad (2.11)$$

In this case high frequency switching between the two control structures will take place as the system trajectories repeatedly cross the line $\mathcal{L}_s$. This type of high frequency motion is called *chattering*. If infinite frequency switching were possible, intuitively at least, the motion would remain on the line $\mathcal{L}_s$. The motion satisfies the condition $s(y, \dot{y}) = 0$, which generates the first order differential equation obtained from rearranging equation (2.9).

$$\dot{y}(t) = -my(t) \quad (2.12)$$

This represents a first order decay and the trajectories will slide along the line $\mathcal{L}_s$ to the origin as in Figure 2.6. Such dynamical behaviour is called a *sliding mode* and the line $\mathcal{L}_s$ is called a *sliding surface*. During the sliding motion, the system behaves as a free system with all control action expended in ensuring that the system maintains the reduced order motion, i.e. $s(y, \dot{y}) = 0$. The reduced order dynamics is solely dependent on the choice of the gradient of the line $\mathcal{L}_s$. The above discussion can easily be extended to higher order systems using a state space representation of variable structure control. Define a state vector

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \quad (2.13)$$

Equation (2.5) can then be rewritten in state space form as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (2.14)$$

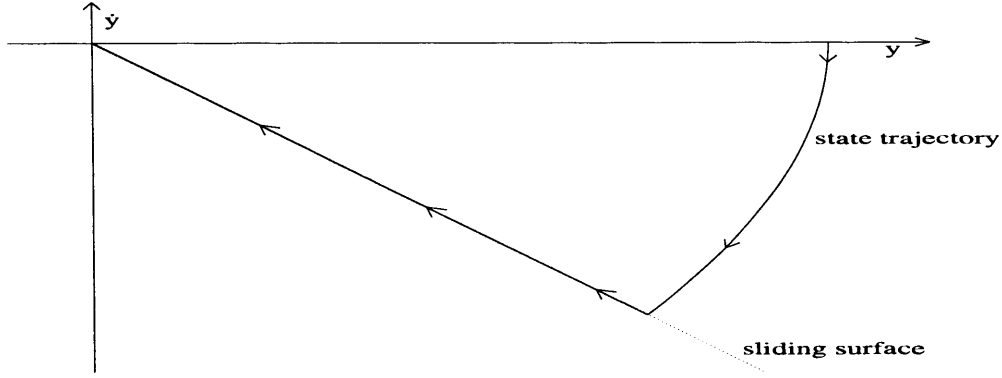


Figure 2.6: Sliding motion under VSC

and the switching function can also be expressed in the matrix form as

$$s(y, \dot{y}) = [m \quad 1] x(t) \quad (2.15)$$

This suggests that the state space representation is more convenient for describing multi-input and multi-output variable structure control systems. Now, consider the n^{th} order linear time invariant system with m inputs given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.16)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ with $1 \leq m \leq n$. Without loss of generality it can be assumed that the input matrix B has full rank.

Define a switching function $s(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be the linear map $s(x) = Sx(t)$ where $S \in \mathbb{R}^{m \times n}$ is of full rank and let $\mathcal{S}$ be the hyperplane defined by

$$\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = Sx(t) = 0\} \quad (2.17)$$

Suppose a control law exists such that u^* is a function of $s(x)$ and $x(t)$ where the changes in control strategy are dependent on the value of the switching function $s(x)$. The control law can then be considered as a map $u^* : x \rightarrow u^*(x)$ which is discontinuous. It is natural to choose a control action and the switching strategy such that sliding takes place on the hyperplane $\mathcal{S}$, i.e. the system states are driven onto and then subsequently forced to remain on the surface $\mathcal{S}$. Intuitively, all the system trajectories in a neighbourhood of $\mathcal{S}$ must be directed to the switching surface. Graphically, it can be represented as in

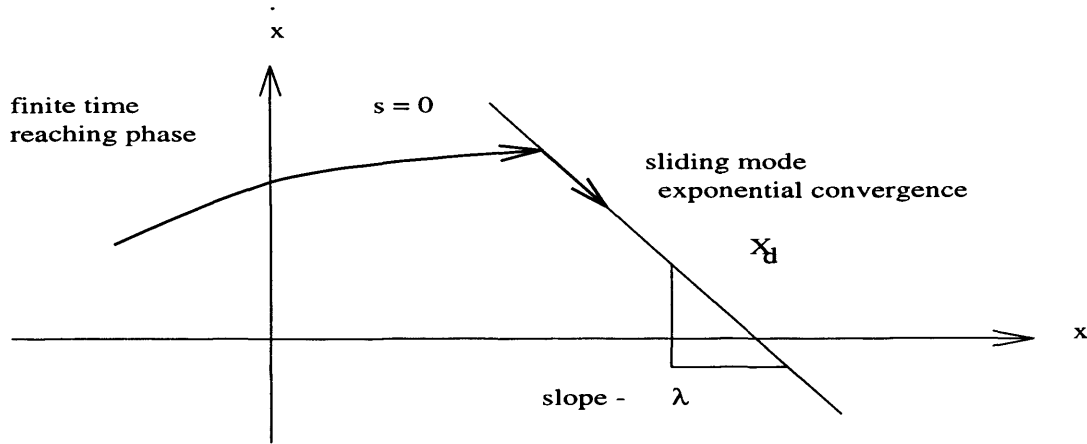


Figure 2.7: Graphical interpretation of Sliding motion

Figure 2.7. In practice an ideal sliding motion may not be attained due to the presence of imperfections such as delays, hysteresis and unmodelled dynamics which will result in a chattering motion in the boundary of the sliding surface as shown in Figure 2.8. Such

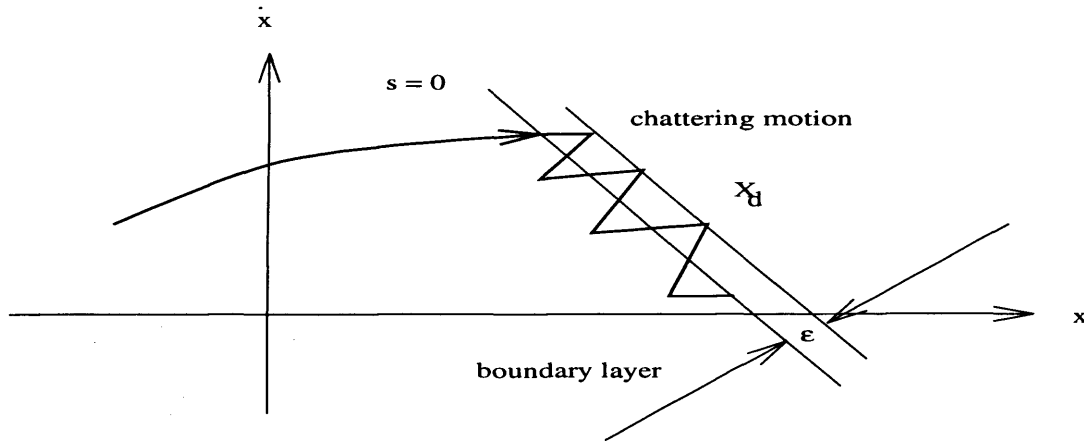


Figure 2.8: Chattering and boundary layer on the control switching line

a system does not fall within the scope of classical differential equation theory. Ideal sliding motion can be thought of as the limiting solution obtained as the imperfections diminish. A formal discussion along these lines appears in [104]. The solution concept proposed by Filippov [46] for differential equations with discontinuous right hand sides is also described in [104]. A rigorous approach for a certain class of variable structure controllers is given by Leitmann [74] and Ryan [87]. The chattering can be eliminated by replacing the discontinuity in the control law. Slotine and Sastry [95] and many other researchers utilise the *boundary layer* approach, where the discontinuous component is

replaced by a continuous approximation. This is discussed further in a later section.

Variable structure systems theory has a long history of development. Like other control methods, it has many different classifications based upon the particular nature of control law used; ‘full state or observer based controller’, ‘output feedback based controller’, ‘simplex type controller’, ‘adaptive type controller’ etc. Most of the work available in the literature on theoretical and practical developments in the area of sliding mode control systems are based on the assumption that the system state vector is available for use by the controller. In practice, it may not be possible or may be impractical to measure all of the states. However, researchers have developed methods to estimate the unmeasurable states in order to use the control strategies. The mathematical model used to generate the states is termed an *observer* and the idea was first developed by Luenberger [76]. He proposed a method to observe the states either completely, called a *full order state observer*, or partially called a *minimal order state observer* for linear systems. Despite fruitful research and development in the theory of variable structure control, there are very few authors who have applied the underlying principles to the problem of observer design. The work of Utkin [103, 104] is fundamental. He has described a discontinuous observer strategy where the error between the estimated and measured output vector is forced to exhibit a sliding motion. Dorling and Zinober [29] have applied this observer to an uncertain system and observe the difficulties in the selection of a proper switching gain which ensures sliding motion without excessive chattering. Walcott and Żak [107, 108] studied the sliding observer from a Lyapunov viewpoint. Under appropriate assumptions the observer exhibits asymptotic state error decay in the presence of bounded matched nonlinearities/uncertainties. Walcott *et. al.* [106] present a direct approach to nonlinear observer design when the nonlinearities present in the system are perfectly known. Recently Edwards and Spurgeon [32, 33] have developed a methodology for determining the magnitude of the discontinuous gain required by the Utkin observer to ensure the existence of a sliding mode despite the presence of a class of bounded plant uncertainty. This method explicitly solves the constrained Lyapunov problem appearing in the work of Walcott and Żak. An alternative approach to the VSS control of systems where the states are not all available for measurement is an *output feedback sliding mode control* which is the main topic of this thesis. The controller will

be designed based upon output information only.

Many practical systems have limited numbers of measured outputs available where either all or few of them are to be controlled. Extra measurements increase the complexity in implementation and the cost of actuator and sensors is also important from the economic point of view. Hence an alternative framework for state feedback sliding mode control is necessarily important for economy and easy implementation in industry. An ‘output feedback sliding mode control’ can easily circumvent these issue. Before discussion it is first necessary to establish the most common approach for system analysis.

2.3.1 Method of Equivalent Control

This section describes a method of establishing the nominal control action required to maintain a sliding motion on the switching surface $\mathcal{S}$ and the equations representing the dynamical behaviour of the states while constrained to the surface. These dynamics are called the ‘equivalent dynamics’. At this point it is stressed that the method about to be described is not confined to linear systems. In this regards, more general nonlinear systems and sliding surfaces are considered in DeCarlo *et. al.* [25] and Utkin [104] for state feedback systems.

Consider the linear system described in equation (2.16) which has outputs $y(t)$ represented by the equation

$$y(t) = Cx(t) \quad (2.18)$$

where $y \in \mathbb{R}^p$ and the output matrix C is known and full rank. The switching surface equation in (2.17) may be defined in terms of outputs by

$$\mathcal{S} = \{y \in \mathbb{R}^p : s(y) = Fy(t) = 0\} \quad (2.19)$$

for some matrix $F \in \mathbb{R}^{m \times p}$ called the ‘switching surface matrix’.

Suppose at time $t = t_s$ the system output reaches the switching surface $\mathcal{S}$ and an ideal sliding motion is attained. This can be expressed mathematically as $s(y) = FCx(t) = 0$ and $\dot{s}(y) = FC\dot{x}(t) = 0$ for all time $t \geq t_s$. Substituting for $\dot{x}(t)$ from equation (2.16) gives

$$FCAx(t) + FCBu(t) = 0 \quad \forall t \geq t_s \quad (2.20a)$$

$$u_{eq}(t) = -(FCB)^{-1}FCAx(t) \quad (2.20b)$$

This shows that the matrix FCB is required to be invertible if an ‘equivalent control’, $u_{eq}(t)$ is to exist. The singular case may be considered in the way described in Utkin [104] for state feedback control. The ideal sliding mode dynamics may be given by substituting the equivalent control $u_{eq}(t)$ into equation (2.16). This gives the free motion

$$\dot{x}(t) = (I_n - B(FCB)^{-1}FC)Ax(t) \quad \forall t \geq t_s \text{ and } s(x) = 0 \quad (2.21)$$

This shows that if the matrix $S = FC$ for a state feedback control defined in [32, 121] is considered then the properties of a *linear projection operator* are satisfied. Define the linear projection operator $P_s = (I_n - B(FCB)^{-1}FC) = (I_n - B(SB)^{-1}S)$ which satisfies

$$SP_s = 0 \quad \text{and} \quad P_sB = 0 \quad (2.22)$$

The system matrix governing the sliding motion P_sA therefore belongs to $N(S)$ and consequently the sliding motion is a reduced order dynamic. Hence the system can be decomposed into two sub-dynamics. One is called the *range space dynamics* $R(B)$ and the other is called the *null space dynamics* $N(S)$. The stability and the ‘invariance properties’ of the ‘range space dynamics’ is discussed in [111]. More about the reduced order dynamic behaviour and the design of the switching surface are discussed in later chapters. The next section considers the issue of invariance conditions which attracts the researcher for further development in this field.

2.3.2 Invariance Property of VSS

The invariance conditions for sliding systems were originally formulated by Draženović [31] for state feedback sliding mode control. The invariance conditions for output feedback sliding systems are similar to those described by Draženović for the choice of $S = FC$. This was also noted by Diong [26] in sliding mode control based on dynamic output feedback. Consider the uncertain linear system defined by

$$\dot{x}(t) = Ax(t) + Bu(t) + G\xi(t, x) \quad (2.23)$$

where the matrix $G \in \mathbb{R}^{n \times l}$ and the function $\xi(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^l$ represents any uncertainty or nonlinearity in the system. The outputs are represented by equation

(2.18). Suppose an output feedback control law exists which induces a sliding motion on the plane $\mathcal{S}$, then the equivalent control $u_{eq}(t)$ represented in Subsection §2.3.1 is written as

$$u_{eq}(t) = -(FCB)^{-1}FC \{Ax(t) + G\xi(t, x)\} \quad \forall t \geq t_s \quad (2.24)$$

The equivalent control law is never realised in practice due to the presence of the unknown function $\xi(t, x)$. However this does not present any difficulties in analysis. Substituting equation (2.24) into the uncertain system in equation (2.23) yields the sliding motion

$$\dot{x}(t) = P_s Ax(t) + P_s G \xi(t, x) \quad \forall t \geq t_s \text{ and } s(x) = 0 \quad (2.25)$$

where P_s is defined in Subsection §2.3.1. Suppose $\mathbb{R}(G) \subset R(B)$ then there exists a matrix $R \in \mathbb{R}^{m \times l}$ such that $G = BR$. This leads to $P_s G = 0$ since $P_s B = 0$ as described previously. The sliding mode dynamics may be written as

$$\dot{x}(t) = P_s Ax(t) \quad \forall t \geq t_s \text{ and } s(x) = 0 \quad (2.26)$$

where the function $\xi(t, x)$ does not affect the motion. The reduced order dynamics are thus insensitive to any disturbances occurring in the range space $R(B)$. This class of uncertainty is called *matched uncertainty*. The invariance property with respect to the matched uncertainty makes the ‘variable structure systems’ a powerful tool for controlling uncertain systems and motivates continuing research in this area.

The remaining uncertainty which does not fall in the range space $R(B)$ is described as *unmatched uncertainty* and it will appear in the null space dynamics. Its affect on the sliding motion is thus unavoidable. However, this effect may be reduced by using an appropriate robust design technique to prescribe the sliding function and thus the reduced order dynamics. It is also one of the main contributions of this thesis. Before going into further details, two important topics must be discussed relating to the robustness property outlined above. The robustness property exists only if the system attains and maintains a sliding motion. The choice of control to ensure this is thus fundamental.

2.3.3 Conditions for Existence of Sliding Mode

From the definition of a sliding mode, it is clear that motion in the neighbourhood of the switching surface $\mathcal{S}$ must be directed towards the surface. The surface must be locally stable. Using a Lyapunov¹ approach, a sufficient condition for existence of a sliding motion is described in DeCarlo *et al.*[25], Utkin [102], and also Edwards [32].

Theorem 2.1 *For a domain $D \subset \mathcal{S}$ to be the domain of a sliding mode it is sufficient that in some region $\Omega \subset \mathbb{R}^n$ where $D \subset \Omega$ there exists a continuously differentiable scalar function $V : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ satisfying*

i) $V(t, x, s)$ is positive definite with respect to s , i.e. $V(t, x, s) > 0$ if $s \neq 0$, for all $x \in \Omega$, and on the spheres $\|s\| = r$, for all $x \in \Omega$ and any time t the following relations hold

- $\inf V(t, x, s) = h_r$ and $h_r > 0$ for $r \neq 0$
- $\sup V(t, x, s) = H_r > 0$

where h_r and H_r depend on r .

ii) The total time derivative of $V(t, x, s)$ has a negative supremum for every $x \in \Omega$ except for x on the switching surface where the control inputs are not defined and hence the derivative of $V(t, x, s)$ does not exist.

Proof: The simplest Lyapunov function for a multivariable switching system is the quadratic form

$$V(t, x, s) = \frac{1}{2} s^T s \quad (2.27)$$

which is always a positive function with respect to s and satisfies the supremum and infimum conditions. The total derivative is given by

$$\dot{V}(t, x, s) = s^T \dot{s} \quad (2.28)$$

a sufficient condition for a sliding mode to exist is that the time derivative of the Lyapunov function is strictly decreasing, i.e.

$$s^T \dot{s} \leq 0 \quad \forall x \text{ and } t \quad (2.29)$$

¹A brief review of Lyapunov stability ideas is given in Appendix B

This existence condition is the one most often cited in the literature. In practice, certain controller designs are based upon this method which will be discussed in due course. ■

2.3.4 Structure of Control Law

Most of the control laws used in VSC have two parts; one has a fixed gain, mostly related to controlling the linear part of the system, and the other is switched or nonlinear in nature and is used to control the uncertain part of the system. Mathematically this is given as

$$u(t) = G_l y(t) + \nu(t, s) \quad (2.30)$$

where G_l is linear gain matrix and $\nu(t, s)$ is called the discontinuity vector. Its definition is available in the literature in terms of state feedback control in Slotine and Sastry [95], Ryan and Corless [88], de Jager [24], Burton and Zinober [14] and Spurgeon and Davies [98] and in terms of output feedback in Heck and Ferri [56], El-Khazali and Decarlo [38, 39] and also in Žak and Hui [118]. Edwards and Spurgeon [34] define it as

$$\nu(t, s) = \begin{cases} \rho(t, u, y) \frac{s(y)}{\|s(y)\|} & \text{if } s(y) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.31)$$

where $\rho(t, u, y)$ is a positive scalar quantity defined by the uncertainty bound parameters. It will be defined precisely in the controller design sections. As discussed earlier, chattering is common with this sort of control structure. A continuous approximation is used to avoid chattering. This is discussed in the next section

2.3.5 Continuous Approximations of the Discontinuous Control Law

The most common approach is to soften the discontinuous vector $\nu(t, s)$ in the control law. Slotine and Sastry [95] and many other workers utilise the boundary layer theory where the discontinuous component is replaced by a continuous nonlinear approximation. Ryan and Corless [88] use a variant on this boundary layer approach and advocate the power law interpolation structure. A detailed description of this is available in Edwards [32]. An alternative differential approximation is given by Burton and Zinober [14] and Spurgeon and Davies [98] who consider

$$u_n = -\rho(t, x) B_2^{-1} \frac{P_2 \phi(t)}{\|P_2 \phi(t)\| + \delta} \quad (2.32)$$

where δ is a small positive constant. As shown in Figure 2.9 the scalar δ does not define

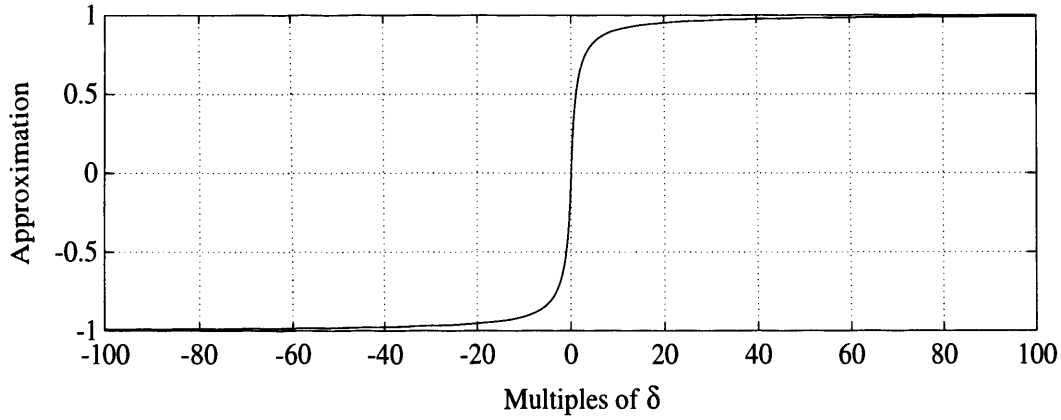


Figure 2.9: A differentiable approximation of the *signum* function

the boundary layer; at $\|s\| = \delta$ the nonlinear control action is half of its maximum value. However, ultimate boundedness results can be demonstrated. Other types of approximations are proposed in [80, 119]. A comparison of these different approaches to eliminate chattering is made by de Jager [24]. It is suggested that there is no significant difference between the various methods. Based on the above study of approximation methods, the output feedback control discontinuity part is approximated as

$$\nu(t, s) = \begin{cases} \rho(t, u, y) \frac{s(y)}{\|s(y)\| + \delta} & \text{if } s(y) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.33)$$

The size of δ and its effect on the robustness is studied by Davies and Spurgeon [21]. Some steady state errors result from this approximation. The introduction of integral control action may remove these effect. This approximation is mainly used during the simulation of numerical examples in this thesis.

2.4 Summary

A brief discussion of nonlinear multivariable control systems has been presented. The basic ideas associated with the use of variable structure systems for controlling multivariable systems are discussed. The sufficient conditions for the existence of a sliding mode, the control structure and its effects are also reviewed. These ideas are fundamental to the developments of the proposed approaches which follow in the remaining

chapters. The exposition has been restricted to linear uncertain multivariable systems. In the remaining chapters the ideas of variable structure control will be examined for uncertain linear multivariable systems based upon output information. In this situation, either the class of hyperplanes or the controllers considered must be restricted to those requiring only system output information. A robust technique for switching surface design will be investigated.

Chapter 3

Developments in Output Feedback Variable Structure Control

3.1 Introduction

Most of the techniques for the design of variable structure control systems that are available in the literature assume that either the state vector is directly measurable or that an observer is used to reconstruct the states. Even if the states are all measurable, implementation with state feedback is necessarily complex, requiring many feedback loops. An observer may be used to overcome this problem. Details about observer design is available in the literature such as Slotine *et al.* [93], and Walcott and Zak [108]. Young [116] has first applied this control structure to industrial problems. These added dynamics increase the integration complexity and cost of implementation. In fact, an observer may not be reliable for large order systems. An alternative approach to avoid state measurement or observer design is to use an output feedback control. Output feedback for linear systems has generated much interest in the last 20 years. The main benefits are that there are no extra dynamics involved in controller implementation and there are thus fewer computational loops producing less complexity. There are however, inherent problems with output feedback in linear systems that are also present in Output Feedback Variable Structure Control (VSCOF) design. For example, it may not be possible to stabilise a system with a given set of plant outputs called ‘static output feedback control’. A static output feedback controller is a controller where there are no additional compensator or observer dynamics involved. The only dynamics are associ-

ated with the plant dynamics. The minimum requirement to use such output feedback sliding mode control is that the number of outputs must be either equal or greater than the number of inputs in addition the system must satisfy the general output feedback criteria such as controllability, observability etc. [22, 23] and the ‘Kimura-Davison’ conditions [70, 71]. In the case of output feedback sliding mode control, a particular sliding mode design triple must satisfy these conditions. Static output feedback sliding mode control was first investigated by White [110, 111] for a linear system without uncertainty and more recently by El-Khazali and DeCarlo [37, 38] and in application the works of Heck and Ferri [56] are important. Later on Hui and Zak [63] investigated uncertainty models and proposed an algorithm for output feedback dependent hyperplane design which is based upon eigenvector methods. The problem has also been addressed by Edwards and Spurgeon [34] and this work is fundamental to the developments in this thesis.

If the system does not satisfy the static output feedback conditions then the most common way to overcome this is to add some additional dynamics so that the augmented system satisfies the conditions. This is called ‘dynamic output feedback control’ or ‘compensation technique’. If the sliding mode design triple does not satisfy these conditions then a dynamic compensator is added to the system. This type of output feedback is called ‘dynamic output feedback sliding mode control’ as described by El-Khazali and DeCarlo [40], Diong and Medanic [27, 28], and also Edwards and Spurgeon [35].

This chapter describes the existing contributions on the developments of output feedback sliding mode controller design as outlined above. Section §3.2 describes the general form of the system and the developments of static and dynamic output feedback sliding mode controllers are described in Sections §3.3 and §3.4 respectively.

3.2 System Definition

Different authors invoke different assumptions and also consider different forms for the uncertainty and nonlinearity. However, the most common way to define an uncertain plant is to consider a linear time invariant state space model with some uncertainties in

the system

$$\dot{x}(t) = Ax(t) + Bu(t) + h(t, u, x) \quad (3.1a)$$

$$y(t) = Cx(t) \quad (3.1b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ with $m \leq p \leq n$. Assume that the nominal linear system triple (A, B, C) is known and the input and output matrices B and C are both of full rank. The unknown function $h(t, u, x) : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the system nonlinearities plus any model uncertainties present in the system. Different authors have imposed different restrictions on the unknown function $h(t, u, x)$, but the uncertainty is largely assumed to be bounded by known constants. The aim of this chapter is to study the existing methods and their associated problems in sliding mode output feedback control and then develop a framework for output feedback sliding mode control which circumvents the difficulties with the existing methods.

3.3 Developments in Static Output Feedback VSC Design

This section discusses the developments in output feedback variable structure control. The approaches of White [110, 111], Heck and Ferri [56] and also El-Khazali and DeCarlo [38, 37] use a linear model without nonlinearity and parameter uncertainty. The approaches of Žak and Hui [118] and also Edwards and Spurgeon [32, 34] use a linear model with bounded nonlinearity and parameter uncertainty. The approaches of all of the above authors are based on static output feedback sliding mode control. Another recent development by El-Khazali and DeCarlo [40] and Diong and Medanic [27, 28] in the field of output feedback variable structure control uses dynamic output feedback. The latter discusses the special case where the system cannot be stabilised using only output information. Such problems may be solved using a dynamic compensator which gives some extra degrees of freedom in output feedback sliding mode design.

3.3.1 The Approach of White

White [110, 111] developed preliminary results in the field of output dependent variable structure control. He used a linear model without uncertainty and the system triple

(A, B, C) must be both controllable and observable. He did not consider any particular regular form in his design. His approach is quite straight forward and very similar to the state feedback approach. The controller $u(t)$ is chosen as

$$u(t) = -(K + \Delta K)y(t) \quad (3.2)$$

with the switching surface

$$s(y) = Fy(t) \quad (3.3)$$

where K and F are design matrices. The switching gain component $\Delta Ky(t)$ will be realised by a relay type action operating on the range space outputs. The closed-loop system description is given by

$$\dot{x}(t) = (A - BKC)x(t) - B\Delta Ky(t) \quad (3.4)$$

and the range space dynamics are described by

$$\dot{x}(t) = FC(A - BKC)x(t) - FCB\Delta Ky(t) \quad (3.5)$$

He has pointed out that the range space dynamics theorem [111] when applied to the output feedback sliding mode case is not generally possible as the fixed gain matrix K must be chosen as

$$K = (FCB)^{-1}FCA + QFC \quad (3.6)$$

where $Q \in \mathbb{R}^{m \times m}$ is an arbitrary matrix. He has not defined any methods for the design of the switching surface matrix F and the matrix Q in equation (3.6). The above expression shows that the selection of F to make FCB full rank is not always possible because he has not mentioned the rank of the output matrix C . This work motivated researchers to further develop output feedback variable structure control.

3.3.2 The Approach of Heck and Ferri

Heck and Ferri [56] describe a simplified linear model without uncertainty and point out that the hyperplane design in output feedback VSC can be considered as a linear output feedback dynamic problem. Their approach has considered the system triple (A, B, C) to be controllable and observable and in the regular form with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad C = [C_1 \quad C_2] \quad (3.7)$$

where $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$, $B_2 \in \mathbb{R}^{m \times m}$ and $C_2 \in \mathbb{R}^{p \times m}$. The switching surface is defined by

$$s(y) = Fy(t)$$

The reduced order dynamics can be expressed as

$$\begin{aligned} \dot{x}_1(t) &= [A_{11} - A_{12}(FC_2)^{-1}FC_1] x_1(t) \\ &= [A_{11} - A_{12}KC_1] x_1(t) \end{aligned} \quad (3.8)$$

where the matrix $K = (FC_2)^{-1}F$ can be calculated using pole placement algorithms and the switching surface parameter F can be derived from K using a pseudo inverse. They consider a control law of the form

$$u = -(FCB)^{-1}FCANy(t) - (FCB)^{-1} \frac{s(y)}{\|s(y)\|^2} \quad (3.9)$$

which is similar to that defined by DeCarlo *et al.*[25] in the state feedback case. The design matrix N is used to ensure the reaching condition, i.e. $s^T(y)\dot{s}(y) < 0$ is satisfied for all $y(t)$. This gives the condition that

$$\mathcal{L}(N) = (FC)^T FCA(I - NC) \leq 0$$

i.e. the matrix $\mathcal{L}(N)$ must be negative semi-definite. It is concluded that the selection of the matrix F to make FCB full rank is not always possible with this regular form as in the case of White's approach. The work of Yallapragada and Heck [113] and Heck *et al.*[57] relating to the design of the reaching phase in the above control law is important in this respect. The rate of convergence of the trajectories to the switching surface is solely dependent on the freedom of design of the matrix N . The asymptotic reaching phase shows difficulties in attaining a sliding mode. The work does not consider the effect of invariant zeros and also the dimensionality of the sliding mode design triple.

3.3.3 The Approach of El-Khazali and DeCarlo

El-Khazali and DeCarlo [37, 38, 39] consider only nominal linear systems and do not consider the effect of uncertainty. They assume that the linear plant triple (A, B, C) is both controllable and observable, that the rank of (CB) is equal to the number of

inputs and that the system exists in the regular form as in the case of Heck *et al.*[56]. Their work requires that the pair $(\hat{A}_{11}, \hat{C}_1)$ is completely observable where

$$\hat{A}_{11} = A_{11} - A_{12}C_2^\dagger C_1, \quad \hat{C}_1 = M^T C_1$$

where $M \in \mathbb{R}^{p \times (p-m)}$ is a full rank left annihilator of C_2 and $C_2^\dagger \in \mathbb{R}^{m \times p}$ is the left pseudo inverse of C_2 . If the matrix A_{12} has either full row or column rank, and the ‘Kimura-Davison’ conditions [22, 23, 70, 71], written as

$$m + p - 1 > n \quad \text{and} \quad p > m \geq 2$$

are satisfied, then there exists a matrix $\Gamma \in \mathbb{R}^{m \times (p-m)}$ such that $\lambda(\hat{A}_{11} - A_{12}\Gamma\hat{C}_1)$ can be assigned arbitrarily close to any set of $n - m$ poles. They have shown that one choice for the sliding surface matrix F is

$$F = C_2^\dagger + \Gamma M^T$$

In [39] the authors describe an alternative approach for design of the hyperplane based on an eigenstructure method. Essentially, they show that the eigenvectors associated with the eigenvalues of the reduced order system must lie in the null space $\mathcal{N}(C)$. The use of the pseudo inverse in the design of switching surface matrix F is limiting in their work. Also their work does not consider the effects of invariant zeros in the reduced order sliding dynamics.

3.3.4 The Approach of Žak and Hui

The approach of Žak and Hui [63, 118] is similar to that of El-Khazali and DeCarlo [37, 38]. They have established an appropriate eigenstructure method to define the reduced order sliding motion. It is assumed that the $\text{rank}(CB) = m$ and the system is controllable and observable. It is proved that the problem of designing the switching surface matrix is equal to finding the matrix $W \in \mathbb{R}^{n \times (n-m)}$ of full rank such that

1. $R(W) \cap R(B) = \{0\}$
2. $AW - W\Lambda \subset R(B)$
3. $\text{rank}(CW) = p - m$

where Λ is the diagonal matrix formed from a self conjugate set of complex numbers $\{\lambda_1, \dots, \lambda_{n-m}\}$ which represent the desired eigenvalues of the sliding motion. For a given matrix W satisfying the above conditions, then the hyperplane matrix $F \in \mathbb{R}^{m \times p}$ exists such that $FCW = 0$. The main problem arises in designing the matrix W which is not formally addressed in the paper. An interesting feature of the work is the consideration of bounded uncertainty and the role of invariant zeroes. They assume the uncertainty is bounded by a known parameter and matched to the input channel, i.e. the uncertainty in equation (3.1a) can be expressed as $h(t, u, x) = B\xi(t, u)$, for some bounded function $\xi(t, u) : \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ where $\|\xi(t, u)\| < K_g\|u\| + K_\alpha$ for some known positive constants $0 \leq K_g < 1$ and K_α . They identify that the invariant zeros in the system play an important component of the spectrum of the reduced order model and must be included in the proposed set of sliding mode eigenvalues. A necessary condition for the existence of a hyperplane providing a stable reduced order dynamic is therefore that the nominal system is minimum phase. In addition the structural constraint $FCA = MC$ where $M \in \mathbb{R}^{m \times p}$ must be satisfied for the control law. Robustness of the reduced order sliding motion is explored in [63].

3.3.5 The Approach of Edwards and Spurgeon

Edwards and Spurgeon [33] have given a new framework for the design of output feedback sliding mode control. Their approach is quite different to that of Žak and Hui [63, 118] and is applicable to a wider class of systems than those proposed by Heck and Ferri [56] and El-Khazali and DeCarlo [38, 39]. They assume that the nominal system triple (A, B, C) has the pair (A, B) controllable and that the $\text{rank}(CB) = m$. However, the pair (A, C) is not necessarily observable as required by Žak and Hui [118]. In addition, they consider bounded and matched nonlinearity and parameter uncertainty. They also propose a method to find the hyperplane matrix in the presence of stable invariant zeros. The method presents a useful framework for output feedback sliding mode control. However, some practical issues such as the effect of unmatched uncertainty, robust switching surface design and tailored controller construction remain to be addressed.

The second type of output feedback sliding mode control is ‘dynamic output feedback

sliding mode control'. Its use and development is discussed in the section below.

3.4 Developments in Dynamic Output Feedback VSC Design

The developments in Section §3.3 are based on the assumptions that the system satisfies the static output feedback sliding mode conditions. If the system does not satisfy them or if the performance is unsatisfactory then usually some extra dynamics, called a 'dynamic compensator', are used to give some extra degrees of freedom in the design process. Its use is also essential if the sliding mode dynamics is not stabilisable or does not meet the performance requirements using only measured variables. Two such cases are that the reduced order dynamics cannot be stabilised if the system does not satisfy the 'Kimura-Davison' conditions and / or if the system possesses any unstable invariant zeros. One way to solve those problems is to use dynamic output feedback sliding mode control. Some recent developments in this area are described below.

3.4.1 The Approach of El-Khazali and DeCarlo

This work [40] extends the work in [37, 38, 39] to the case of dynamic output feedback sliding mode control. Again they have considered linear systems without uncertainty. The system is considered completely controllable and observable, does not satisfy the 'Kimura-Davison' conditions and does not possess any invariant zeros. It is mentioned that the observability of the matrix pair (A, C) does not guarantee the observability of the reduced order sliding dynamics [40]. However no observability condition is presented for the reduced order sliding dynamics. A compensator of appropriate size which is driven by the output of the plant is used to augment the system. The sliding manifold is designed for the augmented system and the compensator parameters are chosen during switching surface design. Graphically this type of augmented system is represented in Figure 3.1. Mathematically the compensator is defined as

$$\dot{x}_c(t) = Hx_c(t) + Dy(t) \quad (3.10)$$

where the compensator parameters $H \in \mathbb{R}^{q \times q}$ and $D \in \mathbb{R}^{q \times p}$ are real matrices to be designed. The order of the compensator is chosen to ensure the augmented system

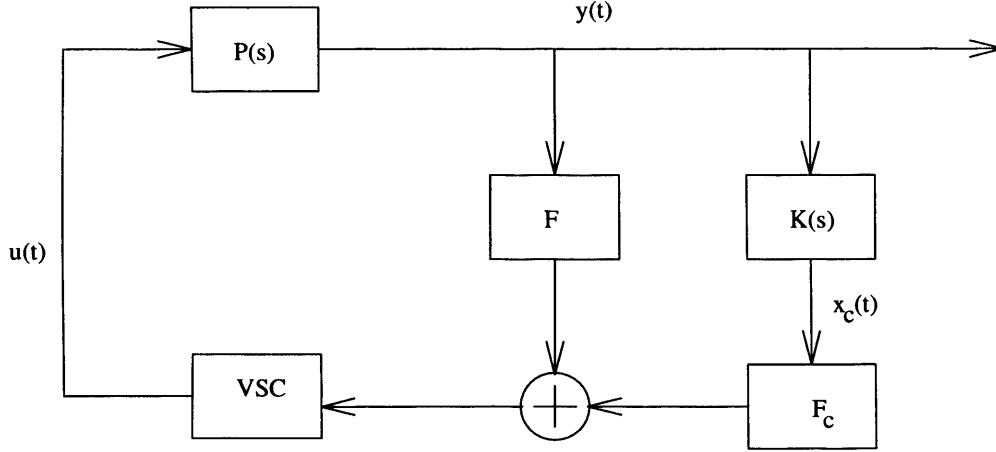


Figure 3.1: Compensator type dynamic output feedback

satisfies the ‘Kimura-Davison’ conditions where in practice only the output feedback design triple should satisfy the ‘Kimura-Davison’ condition. The plant in equations (3.1a)-(3.1b) is combined with the compensator dynamics in equation (3.10). This is given as

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t) \quad (3.11a)$$

$$\tilde{y}(t) = \tilde{C}z(t) \quad (3.11b)$$

with augmented switching surface

$$\mathcal{S}_a = \{z \in \mathbb{R}^{q+n} : s(\tilde{y}) = \tilde{F}\tilde{y}(t) = 0\} \quad (3.12)$$

where the augmented states $z = [x_c^T \ x^T]^T \in \mathbb{R}^{(q+n)}$, the outputs $\tilde{y} = [x_c^T \ y^T]^T \in \mathbb{R}^{(q+p)}$ and the matrix $\tilde{F} = [F_c \ F]$ where the matrices F_c and F are the switching surface matrices associated with the compensator and plant dynamics respectively and the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is given as

$$\tilde{A} = \begin{bmatrix} H & DC \\ 0 & A \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I_q & 0 \\ 0 & C \end{bmatrix}$$

The augmented reduced order closed-loop sliding mode dynamics yield a static output feedback stabilisation problem. As before, a pseudo inverse is used for designing the switching surface. The controller has no flexibility in its reaching time and the method is applicable only to minimal phase systems satisfying the controllability and observability conditions. Diong and Medanic [28] have proposed a dynamic output feedback sliding

mode control for systems with unstable invariant zeros. Their work is described briefly below.

3.4.2 The Approach of Diong and Medanic

Diong and Medanic [28] consider dynamic output feedback variable structure control for linear systems where a simplex method is used to solve the reachability problem. The dynamic output feedback VSC design in [40] is similar to the static output feedback design case. Diong and Medanic have proposed an observer based dynamic output feedback VSC, where the control vector $u(t)$ is considered as an additional input to the linear compensator dynamics defined in equation (3.10). The nominal plant may thus not necessarily be minimum phase. The block diagram in Figure 3.2 describes the augmented system for this type of dynamic output feedback sliding mode control. Mathematically this type of compensator is given by

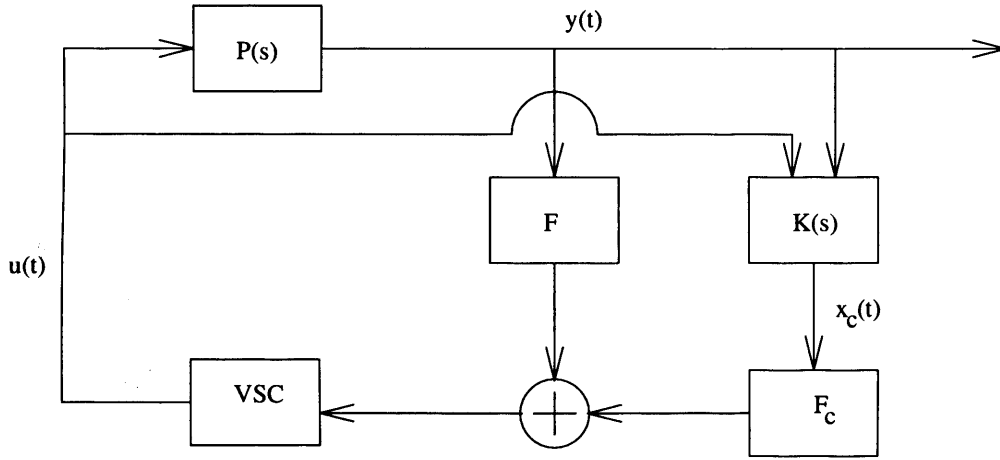


Figure 3.2: Observer type dynamic output feedback

$$\dot{x}_c(t) = Hx_c(t) + Dy(t) + Eu(t) \quad (3.13)$$

where the matrix $E \in \mathbb{R}^{q \times m}$. In this case the index q is dependent on the observer structure and the method adopted for the design of the compensator parameters. The augmented system defined in equations (3.11a)-(3.11b) has input matrix

$$\tilde{B} = \begin{bmatrix} E \\ B \end{bmatrix}$$

The choice of sliding surface is

$$\mathcal{S}_a = \{z \in \mathbb{R}^{q+n} : s(\tilde{y}) = [F_c x_c(t) + FCx(t)] = 0\} \quad (3.14)$$

Using the method of equivalent control and the simplex control method, gives

$$\dot{S}_a = F_c H x_c(t) + (F_c DC + FCA)x(t) + (F_c E + FCB)u(t) = 0 \quad (3.15)$$

in the sliding regime $\mathcal{S}_a$ and $F_c E + FCB = I$ respectively. So the equivalent control is

$$u_{eq}(t) = -F_c H x_c(t) - (F_c DC + FCA)x(t) \quad (3.16)$$

which results in the equivalent system dynamics being described by

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} H - EF_c H & DC - E[F_c DC + FCA] \\ -BF_c H & A - B[F_c DC + FCA] \end{bmatrix} \begin{bmatrix} x_c(t) \\ x(t) \end{bmatrix} \quad (3.17)$$

which is required to be stable for a stable sliding motion. Different methods are described to solve this problem. The main difficulty is that the method is applicable to square systems only. The great advantage is that the system does not have to be minimum phase. The simplex control is defined as a function of the switching surface $s(\tilde{y})$. This may have difficulty in achieving a reasonable reaching time. A further publication by the same author [27] considers the effect of nonlinearity in the above formulation. A H_∞ technique is used to minimise the effect of uncertainty on the augmented closed-loop system.

Further work relating to static and dynamic output feedback sliding mode control appears in [15, 18, 42, 43, 89, 109, 114]. Most of this work does not describe new theoretical contributions or is irrelevant to the work of this thesis and is thus omitted from the discussion.

Based on the developments of the static and dynamic output feedback VSC design, a framework is proposed in this thesis for output feedback sliding mode control of MIMO systems which may be non-square. The cases of stable and unstable invariant zeros, the matched and unmatched uncertainties together with robust switching surface design techniques are considered. A controller is formulated which shows low control effort whilst allowing the reaching time to be monitored.

3.5 Summary

A full state feedback control or reconstruction of unmeasurable states using an observer is a relatively straightforward procedure for variable structure control. However, it essentially increases the computational effort required to implement the control system. An alternative approach is to use an output feedback control strategy and there is thus less complexity in computation and implementation. The literature relating to static and dynamic output feedback sliding mode control is described. The major difficulties with existing approaches are pointed out. The dynamic output feedback system may be considered as additional dynamics. However its use is essential in many practical applications for satisfactory performance and stability. Keeping in mind problems associated in the existing methods such as invariant zeros, 'Kimura-Davison' condition, observability of the pair (A, C) , rank deficiency of the matrix FCB , robustness considerations, multivariable non-square systems and low control effort etc., the next chapter develops a static output feedback sliding mode control design method. Later, where necessary a dynamic output feedback sliding mode control design method is used. The application of both methods to wide classes of industrial systems is presented.

Chapter 4

Static Output Feedback Sliding Mode Control

4.1 Introduction

Based on the developments and the difficulties associated with static output feedback sliding mode control in the work of previous authors, this chapter describes a practical procedure for variable structure control which is based upon output feedback for a class of multivariable uncertain systems. It is shown that the problem of designing a suitable hyperplane, and hence ensuring a reduced order stable sliding mode dynamic, is equivalent to an output feedback design problem for a particular triple [33, 38, 56]. The reduced order motion is affected by the unmatched uncertainty in the system and it is thus necessary to consider methods to produce a sliding mode dynamic which is robust to this uncertainty. A robust output feedback algorithm is adapted to design the hyperplane. The method is based on normal matrix design as the eigenvalues of a normal matrix are most insensitive to perturbations of the matrix parameters [75, 112]. This result has been used to design a robust controller in the frequency domain by Hung and MacFarlane [64]. In the time domain, Davison and Wang [23] solved the problem of output feedback pole assignment and Porter [85], based on the Lyapunov direct method, solved the closed-loop pole assignment problem with prescribed stability by state feedback using these ideas. On the basis of their work, Changsheng [17] has worked on the Lyapunov direct method and exploited the design freedom in output feedback. Using this work, the hyperplane will be designed to give robustness to the reduced order sliding mode dynamics. In this way, when the working triple (A, B, C) of

the system varies, the assigned pole positions of the reduced order system will be minimally sensitive. A controller is formulated based on plant outputs only. The controller guarantees the sliding mode is attained and maintains stability. The controller has a similar structure to that described in DeCarlo *et al.*[25] in state feedback control and Heck *et al.*[56, 57] in output feedback control problems. However, the discontinuous vector is different and the controller has extra freedom to change its rate of convergence to the sliding mode.

The outline of the chapter is as follows. The system description and necessary assumptions, the uncertainty structure and special regular form are presented in Section §4.2. The main objective of the design is given in Section §4.3. Sections §4.4 & §4.5 describe the switching surface design for non-square and square plant respectively. A robust switching surface design procedure and its measurements are presented in Section §4.6. Section §4.7 gives the controller formulation and the necessary conditions for reachability. The design procedure and examples are presented to illustrate the design method in Sections §4.9 and §4.8.

4.2 System Description

Recall the linear time invariant state space model defined in equations (3.1a) and (3.1b) with some uncertainties in the system

$$\dot{x}(t) = Ax(t) + Bu(t) + h(t, u, x) \quad (4.1a)$$

$$y(t) = Cx(t) \quad (4.1b)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ with $m \leq p < n$ and the nominal linear system triple (A, B, C) is known and the input and output matrices B and C respectively are both of full rank. In addition, it is assumed that

A1) the matrix pair (A, B) is completely controllable;

A2) the matrix CB has rank m ;

The unknown function $h(t, u, x) : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the system nonlinearities plus any model uncertainties present in the system. It is assumed bounded as

described below.

4.2.1 Classification of Uncertainty

The uncertainty/nonlinearity in the problem is assumed to be decomposed into unmatched and matched contributions. Consider the decomposition as follows

$$h(t, u, x) = f(t, x) + g(t, u, x) \quad (4.2)$$

where

$$\begin{aligned} f(t, x) &: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow I_m(B)^\perp \\ g(t, u, x) &: \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow I_m(B) \end{aligned}$$

and $I_m(\cdot)$ denotes the range of the matrix $(\cdot)$, and the operation $(\cdot)^\perp$ refers to the orthogonal component of $(\cdot)$. It is assumed that the unknown function $f(t, x)$ can be expressed in the form

$$f(t, x) = F_1(t, y)y(t) + F_2(t, x) \quad (4.3)$$

where the matrix $F_1(t, y) \in \mathbb{R}^{n \times p}$ represents that unmatched uncertainty entering through the output channels and $F_2(t, x) \in \mathbb{R}^n$ denotes that unmatched uncertainty which is not implicit in the output channels. The unknown function $g(t, u, x)$ is written as

$$g(t, u, x) = G_1(t, u)u(t) + G_2(t, x) \quad (4.4)$$

where $G_1(t, u) \in \mathbb{R}^{n \times m}$ is matched components that multiply with the input $u(t)$ and $G_2(t, x)$ is that matched uncertainty which does not multiply with the input $u(t)$ but both are implicit in the input channels. Defining appropriate bounds on the uncertainty

$$\begin{aligned} \|F_1(t, y)\| &< K_f; \quad \|F_2(t, x)\| < K_d \\ \|G_1(t, u, x)\| &< K_g; \quad \|G_2(t, x)\| < K_\alpha \end{aligned}$$

this implies that

$$\|f(t, x)\| \leq K_f\|y\| + K_d \quad (4.5a)$$

$$\|g(t, u, x)\| \leq K_g\|u\| + K_\alpha \quad (4.5b)$$

Therefore, from equations (4.5a) and (4.5b), the uncertainty function $h(t, u, x)$ in equation (4.1a) is bounded.

4.2.2 Regular Form

For variable structure control design problems, a particular canonical form is most useful for analysis. With the underlying assumptions, a nonsingular transformation T (in practice a set of transformations) exists [33] such that in the new coordinates, the plant matrices (A, B, C) have the following form

1. The system state matrix can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (4.6)$$

where $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$. If there are any invariant zeros present in the system then they must appear in the matrix A_{11} [63]. It can then be partitioned accordingly, so that the sub-block A_{11} has the structure

$$A_{11} = \left[\begin{array}{cc|c} A_{11}^o & A_{12}^o & A_{12}^m \\ 0 & A_{22}^o & \\ \hline 0 & A_{21}^o & A_{22}^m \end{array} \right] \quad (4.7)$$

where $A_{11}^o \in \mathbb{R}^{r \times r}$, $A_{22}^o \in \mathbb{R}^{(n-p-r) \times (n-p-r)}$ and $A_{21}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$ for some $r \geq 0$ where r is number of invariant zeros in the system and the pair (A_{22}^o, A_{21}^o) is completely observable.

2. The input matrix has the form

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (4.8)$$

where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular

3. The output matrix has the structure

$$C = [0 \quad T_o] \quad (4.9)$$

where $T_o \in \mathbb{R}^{p \times p}$ and is orthogonal.

Proof: With the help of successive linear transformations of the system triple (A, B, C) , it is possible to express the system in the required canonical form. Define a nonsingular

transformation matrix T_c as

$$T_c = \begin{bmatrix} N_c \\ C \end{bmatrix} \quad (4.10)$$

where $N_c \in \mathbb{R}^{(n-p) \times n}$ is any matrix whose rows span the null space of C . The new state space triple (A_c, B_c, C_c) has the form

$$\begin{aligned} A_c &= T_c A T_c^{-1} \\ C_c &= C T_c^{-1} = [0 \quad I_p] \\ B_c &= T_c B = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \end{aligned}$$

where $B_{c1} \in \mathbb{R}^{(n-p) \times m}$ and $B_{c2} \in \mathbb{R}^{p \times m}$. Then $CB = C_c B_c = B_{c2}$ and so the rank of $B_{c2} = m$. Let $T_{22} \in \mathbb{R}^{p \times p}$ be any orthogonal transformation matrix so that

$$T_{22} B_{c2} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (4.11)$$

where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular. Consequently the transformation T_b can be set such that

$$T_b = \begin{bmatrix} I_{n-p} & T_{12} \\ 0 & T_{22} \end{bmatrix} \quad (4.12)$$

where $T_{12} = -B_{c1}(B_{c2}^T B_{c2})^{-1} B_{c2}^T$ and is nonsingular. So the new transformed matrices (A_b, B_b, C_b) are in the form

$$\begin{aligned} A_b = T_b A_c T_b^{-1} &= \left[\begin{array}{c|c} A_{11}^b & A_{12}^b \\ \hline A_{211}^b & A_{22}^b \\ A_{212}^b & \end{array} \right] \\ B_b = T_b B_c &= \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ C_b = C_c T_b^{-1} &= [0 \quad T_{22}^T] \end{aligned}$$

where $A_{11}^b \in \mathbb{R}^{(n-p) \times (n-p)}$, $A_{211}^b \in \mathbb{R}^{(p-m) \times (n-p)}$ and $A_{22}^b \in \mathbb{R}^{p \times p}$. In order to examine the observability of the pair (A_{11}^b, A_{211}^b) , define a transformation matrix $T_{obs} \in \mathbb{R}^{(n-p) \times (n-p)}$ such that the pair (A_{11}^b, A_{211}^b) has the following observability canonical form

$$T_{obs} A_{11}^b T_{obs}^{-1} = \begin{bmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22}^o \end{bmatrix}$$

and

$$A_{211}^b T_{obs}^{-1} = [0 \quad A_{21}^o]$$

where $A_{11}^o \in \mathbb{R}^{r \times r}$, $A_{21}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$ and the pair (A_{22}^o, A_{21}^o) is observable and $r \geq 0$ where r is number of unobservable states of the pair (A_{11}^b, A_{211}^b) .

Finally, consider T_a be the transformation matrix which gives the required system canonical form; T_a has the structure

$$T_a = \begin{bmatrix} T_{obs} & 0 \\ 0 & I_p \end{bmatrix} \quad (4.13)$$

and the transformed matrices are given by

$$B = T_a B_b = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

and

$$C = C_b T_a^{-1} = [0 \quad T_o]$$

where $T_o = T_{22}^T$, which is the required canonical form. Repartitioning the system matrix, gives

$$A = T_a A_b T_a^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

In equation (4.7), the invariant zeros are the r eigenvalues of matrix A_{11}^o [33], where the triple (A, B, C) is in canonical form and the nonsingular transformation $T = T_a T_b T_c$. The regular form in this section is a special case of the regular form defined in Section §3.3.2 of Chapter §3. Throughout the thesis this regular form is used for the analysis and design of variable structure output feedback controllers unless it is mentioned otherwise.

4.3 Design Objectives

As in state feedback based variable structure control, the output feedback sliding mode design problem can be decomposed into two stages. First construct a switching surface so that a stable motion can occur on the surface

$$\mathcal{S} = \{y \in \mathbb{R}^p : s(y) = Fy(t) = 0\} \quad (4.14)$$

for some selected matrix $F \in \mathbb{R}^{m \times p}$. Secondly, develop a control law which can induce a sliding motion on the switching surface $\mathcal{S}$. The controller has the proposed form

$$u(t) = Gy(t) - \nu(y) \quad (4.15)$$

where G is a fixed gain matrix which will be described later in the section and the discontinuity vector $\nu(y)$ is dependent on outputs only. The key problem is to design the switching surface matrix F and the gain matrix G , so that the system is stable and robust.

Later sections describe the general procedure for switching surface design. The case where the number of outputs is greater than the number of inputs exhibits a richer mathematical structure than the case of equal number of outputs and inputs. Each case is described separately. First consider the case of non-square plant.

4.4 Switching Surface Design for Non-square Plant

Consider the equations (4.1a) and (4.1b) to represent a non-square plant satisfying the assumptions (A1) - (A2) and the uncertainty decompositions in Subsection §4.2.1. Then the regular form in equations (4.6) - (4.9) exists for the non-square plant. Assume a controller defined in equation (4.15) exists such that it induces and maintains a sliding motion on the surface $\mathcal{S}$ defined in equation (4.14), then the derivative of the switching surface is given as

$$\dot{\mathcal{S}} = FC\dot{x}(t) = 0 \quad (4.16)$$

An equivalent control can be derived using equations (4.16) and (4.1a) as

$$u_{eq}(t) = -(FCB)^{-1}FC \{Ax(t) + h(t, u, x)\} \quad (4.17)$$

This shows that for a unique equivalent control law [104] to exist the matrix $FCB \in \mathbb{R}^{m \times m}$ must be of full rank for the matrix $F \in \mathbb{R}^{m \times p}$. This gives that the matrix FCB must be invertible. This implies that $\text{rank}(CB) = m$. To demonstrate this, consider the system in canonical form and define

$$\begin{array}{c} \overset{p-m}{\leftrightarrow} \quad \overset{m}{\leftrightarrow} \\ \left[\begin{array}{cc} F_1 & F_2 \end{array} \right] = FT_o \end{array} \quad (4.18)$$

From the above definition it follows that

$$FC = [F_1C_1 \quad F_2] \quad (4.19)$$

where C_1 is defined as

$$C_1 = [0_{(p-m) \times (n-p)} \quad I_{(p-m)}] \quad (4.20)$$

Then the matrix $FCB = F_2B_2$ which implies that if $\det(F_2) \neq 0$ then the matrix FCB is invertible. It will be shown later that the matrix F_2 can be chosen arbitrarily and it does not affect the switching surface performance as well as the control action defined in a later section. The equivalent dynamics can be written, using the equivalent control law in equation (4.17), as follows

$$\begin{aligned} \dot{x}(t) &= [I_n - B(FCB)^{-1}FC] Ax(t) + [I_n - B(FCB)^{-1}FC] h(t, u, x) \\ &= [I_n - B(FCB)^{-1}FC] [Ax(t) + f(t, x) + g(t, u, x)] \end{aligned} \quad (4.21)$$

Consider the state partitions as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} T_1 x(t) \\ T_2 x(t) \end{bmatrix} \quad (4.22)$$

where $x_1 \in \mathbb{R}^{n-m}$ and $T_1 \in \mathbb{R}^{(n-m) \times n}$ are components of $x(t)$ and T respectively. Substituting the regular form of equations (4.6) and (4.8)-(4.9) into equation (4.21) and using partitioned states in equation(4.22), the non-square system representation becomes

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + \tilde{f}(t, x_1, x_2) \quad (4.23a)$$

$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + \tilde{g}(t, u, x_1, x_2) \quad (4.23b)$$

where the uncertainty functions

$$\tilde{f}(t, x_1, x_2) = T_1 f(t, x) = T_1 f \left(t, T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \quad (4.24)$$

$$\tilde{g}(t, u, x_1, x_2) = T_2 g(t, u, x) = T_2 g \left(t, u, T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \quad (4.25)$$

If it is considered that the uncertainty is only matched, i.e. if the unmatched uncertainty function

$$\tilde{f}(t, x_1, x_2) = 0 \quad (4.26)$$

then the design procedure turns to a matched uncertainty problem. In this case the switching surface may be written as

$$x_2(t) = -F_2^{-1}F_1C_1x_1(t) \quad (4.27)$$

and the reduced order sliding dynamics can be written as

$$\dot{x}_1(t) = (A_{11} - A_{12}F_2^{-1}F_1C_1)x_1(t) \quad (4.28)$$

Hence, for a stable sliding motion

$$A_{11}^s = A_{11} - A_{12}F_2^{-1}F_1C_1 \quad (4.29)$$

must be stable. Let the matrix $K \in \mathbb{R}^{m \times (p-m)}$ be defined as $K = F_2^{-1}F_1$, then equation (4.28) becomes

$$\dot{x}_1(t) = (A_{11} - A_{12}KC_1)x_1(t) \quad (4.30)$$

Thus the design of the switching surface is an output feedback stabilisation problem for the triple (A_{11}, A_{12}, C_1) . The switching surface matrix F is finally expressed in terms of the output gain matrix K as follows

$$F = F_2 [K \quad I_m] T_o^T \quad (4.31)$$

where the matrix F_2 is merely a scaling of the matrix F . It is also possible to model the sliding mode dynamics if the system has both the unmatched and matched uncertainties. In this case the switching surface design also becomes an output feedback stabilisation problem and the sliding motion will be constrained to the surface $\mathcal{S}$. The effect of unmatched uncertainty upon the sliding mode dynamics will now be explored. In particular the conditions for stability of the sliding system will be developed. In the presence of unmatched uncertainty, define a second transformation

$$\hat{T} = \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \quad (4.32)$$

where K is a design matrix and C_1 is defined in equation (4.20) then in the new coordinates

$$\tilde{x}(t) = \begin{bmatrix} x_1(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ KC_1x_1(t) + x_2(t) \end{bmatrix} = \hat{T}x(t) \quad (4.33)$$

in which the constraints (4.26) and (4.27) are not necessary to be imposed. The system equations become

$$\dot{x}_1(t) = \Sigma x_1(t) + A_{12}\phi(t) + \hat{f}(t, x_1, \phi) \quad (4.34a)$$

$$\dot{\phi}(t) = \Theta x_1(t) + \Omega\phi(t) + B_2u(t) + \hat{g}(t, u, x_1, \phi) \quad (4.34b)$$

where the system matrices and uncertainties are given as

$$\Sigma = A_{11} - A_{12}KC_1 \quad (4.35a)$$

$$\Theta = A_{21} - A_{22}KC_1 + KC_1\Sigma \quad (4.35b)$$

$$\Omega = A_{22} + KC_1A_{12} \quad (4.35c)$$

$$\hat{f}(t, x_1, \phi) = \tilde{f}(t, x_1, (\phi - KC_1x_1)) \quad (4.35d)$$

$$\hat{g}(t, u, x_1, \phi) = KC_1\tilde{f}(t, x_1, (\phi - KC_1x_1)) + \tilde{g}(t, u, x_1, (\phi - KC_1x_1)) \quad (4.35e)$$

Assume a control authority exists as in equation (4.15) such that the system trajectories are constrained to $\mathcal{S}$ as defined in equation (4.14). In this situation, the states $\phi(t) = 0$ must exist. Then the reduced order sliding mode dynamics in equation (4.34a) become

$$\dot{x}_1(t) = \Sigma x_1(t) + \hat{f}(t, x_1, 0) \quad (4.36)$$

It can be proved that if the uncertainty function $\hat{f}(t, x_1, 0)$ in equation (4.36) satisfies the structural condition

$$\hat{f}(t, x_1, 0) = \Delta\Sigma x_1(t) + \Delta\hat{f}(t, x_1, 0) \quad (4.37)$$

where $\Delta\Sigma x_1(t)$ corresponds to the uncertainty function $F_1(t, y)y(t)$ and $\Delta\hat{f}(t, x_1, 0)$ represents the uncertainty corresponding to the remaining states plus any other uncertainties, then the equation (4.36) can be written as

$$\dot{x}_1(t) = \hat{\Sigma}x_1(t) + \Delta\hat{f}(t, x_1, 0) \quad (4.38)$$

where $\hat{\Sigma}, \Delta\Sigma, \Delta\hat{f}(t, x_1, 0)$ are represented by

$$\hat{\Sigma} = \Sigma + \Delta\Sigma \quad (4.39a)$$

$$\Delta\Sigma = T_1 \left(F_1(t, y)CT^{-1} \begin{bmatrix} I \\ -KC_1 \end{bmatrix} \right) \quad (4.39b)$$

$$\Delta\hat{f}(t, x_1, 0) = T_1 F_2 \left(t, T^{-1} \begin{bmatrix} I \\ -KC_1 \end{bmatrix} x_1 \right) \quad (4.39c)$$

Then the motion is globally uniformly ultimately bounded¹ [88, 98] to the switching surface $\mathcal{S}$. The proof of this is discussed in Lemma 4.1 below.

¹The definition of globally uniformly ultimately bounded systems is given in Appendix B.

Lemma 4.1 *If the uncertainty function $\hat{f}(t, x_1, 0)$ in equation (4.36) satisfies the structural constraint in equation (4.37) such that a Lyapunov equation*

$$P_1 \hat{\Sigma} + \hat{\Sigma}^T P_1 \leq -v I_{(n-m)}$$

for $v > 0$ exists where the matrix P_1 is symmetric positive definite then

a) *the system in equation (4.36) is globally uniformly ultimately bounded with respect to the ellipsoid*

$$\mathcal{E}(r_1) = \left\{ x_1 \in \mathbb{R}^{n-m} : \frac{1}{2} x_1^T P_1 x_1 \leq r_1 \right\} \quad (4.40)$$

where

$$r_1 = \varepsilon + 2 \frac{K_{r1}^2}{v^2} \|P_1\|^2 \quad (4.41)$$

with $\varepsilon > 0$ defined to be small constant and

$$K_{r1} = \sup_{F_2} \|P_1^{\frac{1}{2}} T_1 F_2\| \quad (4.42)$$

b) *if $\Delta x_1 = x_1 - x_1^m$, where $x_1^m = \exp[\Sigma(t - t_o)] x_1(t_o)$ defines the corresponding ideal sliding mode dynamics at time t from an initial condition t_o and $\Delta x_1(t_o) = 0$, then the deviation from ideal sliding motion Δx_1 is bounded with respect to the ellipsoid $\mathcal{E}(r_2)$ where*

$$r_2 = \begin{cases} 2\|P_1\|^2 \left(K_{r2} \|P_1^{\frac{1}{2}} x_1(t_o)\| + K_{r1} \right)^2 & \text{if } x_1(t_o) \notin \mathcal{E}(r_1) \\ 2\|P_1\|^2 \left(K_{r2} \sqrt{2r_1} + K_{r1} \right)^2 & \text{if } x_1(t_o) \in \mathcal{E}(r_1) \end{cases} \quad (4.43)$$

with

$$K_{r2} = \sup_{F_1} \|P_1^{\frac{1}{2}} \Delta \Sigma P_1^{-\frac{1}{2}}\| \quad (4.44)$$

i.e. $\Delta x_1(t) \in \mathcal{E}(r_2)$ for all $t \geq t_o$

Proof: The proof largely follows the work in [88, 98].

a) Consider the Lyapunov candidate

$$V_1(x_1) = \frac{1}{2} x_1^T P_1 x_1 \quad (4.45)$$

where P_1 is the unique solution of

$$P_1 \Sigma + \Sigma^T P_1 + I_{(n-m)} = 0 \quad (4.46)$$

Taking derivatives along the state trajectory and substituting the values of $\dot{x}_1(t)$ from equation (4.36) and $\hat{f}(t, x_1, 0)$ from equation (4.37), gives

$$\begin{aligned} \dot{V}_1(x_1) &= x_1^T P_1 (\Sigma x_1 + \hat{f}(t, x_1, 0)) \\ &= x_1^T P_1 (\Sigma x_1 + \Delta \Sigma x_1 + \Delta \hat{f}(t, x_1, 0)) \end{aligned} \quad (4.47)$$

Substitute the values from equations (4.39a) and (4.39c) into equation (4.47) and simplifying, gives

$$\begin{aligned} \dot{V}_1(x_1) &= x_1^T P_1 (\hat{\Sigma} x_1 + \Delta \hat{f}(t, x_1, 0)) \\ &= \frac{1}{2} x_1^T (P_1 \hat{\Sigma} + \hat{\Sigma}^T P_1) x_1 + x_1^T P_1 T_1 F_2(\cdot) \\ &\leq -\frac{1}{2} v \|x_1\|^2 + x_1^T P_1 T_1 F_2(\cdot) \end{aligned} \quad (4.48)$$

This implies that

$$\dot{V}_1(x_1) \leq -v V_1(x_1) \|P_1\|^{-1} + [2V_1(x_1)]^{\frac{1}{2}} \|P_1^{\frac{1}{2}} T_1 F_2\| \quad (4.49)$$

The equation (4.49) is negative scalar, i.e. $\dot{V}_1(x_1) \leq 0$ if and only if $V_1(x_1) > r_1 - \varepsilon$ where r_1 is defined in equation (4.41). Hence the motion in equation (4.36) is bounded.

b) Ideal sliding motion is defined by the dynamics of the system

$$\dot{x}_1^m(t) = \Sigma x_1^m(t) \quad (4.50)$$

The dynamics of the error system therefore satisfy

$$\Delta \dot{x}_1 = \Sigma \Delta x_1 + \hat{f}(t, x_1, 0) \quad (4.51)$$

Consider the Lyapunov function in equation (4.45) evaluated with the state deviation Δx_1 as

$$V_1(\Delta x_1) = \frac{1}{2} \Delta x_1^T P_1 \Delta x_1 \quad (4.52)$$

with

$$\begin{aligned}
\dot{V}_1(\Delta x_1) &= \frac{1}{2} \left(\Delta \dot{x}_1^T P_1 \Delta x_1 + \Delta x_1^T P_1 \Delta \dot{x}_1 \right) \\
&= \frac{1}{2} \Delta x_1^T \left(P_1 \Sigma + \Sigma^T P_1 \right) \Delta x_1 + \Delta x_1^T P_1 \Delta \Sigma x_1 + \Delta x_1^T P_1 T_1 F_2 \\
&\leq -\frac{1}{2} \|\Delta x_1\|^2 + \|\Delta x_1^T P_1 \Delta \Sigma x_1\| + \|\Delta x_1^T P_1 T_1 F_2\| \\
&\leq -V_1(\Delta x_1) \|P_1\|^{-1} + [2V_1(\Delta x_1)]^{\frac{1}{2}} \left\{ \|P_1^{\frac{1}{2}} \Delta \Sigma P_1^{-\frac{1}{2}}\| \right. \\
&\quad \left. \|P_1^{\frac{1}{2}} x_1\| + \|P_1^{\frac{1}{2}} T_1 F_2\| \right\}
\end{aligned} \tag{4.53}$$

Also from part (a)

$$\|P_1^{\frac{1}{2}} x_1\| \leq \begin{cases} P_1^{\frac{1}{2}} x_1(t_o) & \text{if } x_1(t_o) \notin \mathcal{E}(r_1) \\ \sqrt{2r_1} & \text{if } x_1(t_o) \in \mathcal{E}(r_1) \end{cases} \tag{4.54}$$

It follows directly from equations (4.53) and (4.54) that $\dot{V}_1(\Delta x_1) < 0$ if $V_1(\Delta x_1) > r_2$ where r_2 is defined as in equation (4.43). The boundedness of the deviation from the ideal sliding mode dynamics is thus proved. Hence once the motion attains the sliding mode it is constrained to the switching surface $\mathcal{S}$.

■

The structure of Σ is similar to that of A_{11}^s defined in equation (4.29). This has the form of a dynamic output feedback problem so the gain matrix K can be determined using any output feedback technique. The switching surface matrix F is similarly expressed with the output feedback gain matrix K as defined in equation (4.31). The effects of the matrix F_2 is already discussed; thus its choice is any invertible arbitrary matrix.

However, the presence of invariant zeros and dimensional constraints affect the design process in this output feedback problem. In addition the triple (A_{11}, A_{12}, C_1) must be completely controllable and observable. The reduced order sliding mode dynamics have the same requirements. If there are any invariant zeros in the original system, then the invariant zeros will appear in the spectrum of A_{11}^s and Σ [63]. As a consequence it follows that the poles of A_{11}^s or Σ cannot be assigned arbitrarily by the choice of the gain matrix K . However if the matrices A_{12} and A_{12}^m are partitioned as

$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix} \tag{4.55}$$

where $A_{122} \in \mathbb{R}^{(n-m-r) \times m}$, $A_{122}^m \in \mathbb{R}^{(n-m-r) \times (p-m)}$ and a new sub-system is formed from the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ where

$$\tilde{A}_{11} \triangleq \begin{bmatrix} A_{22}^o & A_{122}^m \\ A_{21}^o & A_{22}^m \end{bmatrix} \quad (4.56)$$

and

$$\tilde{C}_1 \triangleq [0_{(p-m) \times (n-p-r)} \quad I_{(p-m)}] \quad (4.57)$$

then it can be shown that

$$\lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1) \quad (4.58)$$

If the invariant zeros are unstable then an unstable sliding motion results. The invariant zeros of the system must thus be stable in this approach. The case of unstable invariant zeros will be considered in a later chapter.

Note that the matrix A_{122} is not necessarily of rank m . Let the rank of the matrix A_{122} be m' where $0 < m' \leq m$. If m' is zero then the pair $(\tilde{A}_{11}, A_{122})$ is not completely controllable which implies (A, B) is not completely controllable which violates the assumption (A1). If $m' < m$ then it is possible to construct a matrix $T_{m'} \in \mathbb{R}^{m \times m}$ by elementary column operations such that

$$A_{122}T_{m'} = [\tilde{B}_1 \quad 0] \quad (4.59)$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$ and is of full rank. If the matrix K is transformed and partitioned as

$$T_{m'}^{-1}K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{matrix} \mathbb{I}_{m'} \\ \mathbb{I}_{m-m'} \end{matrix} \quad (4.60)$$

it follows that

$$\begin{aligned} \tilde{A}_{11}^s &= \tilde{A}_{11} - A_{122}K\tilde{C}_1 \\ &= \tilde{A}_{11} - \tilde{B}_1K_1\tilde{C}_1 \end{aligned} \quad (4.61)$$

therefore $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ is stabilisable by output feedback if and only if $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilisable by output feedback. If the triple (A_{11}, A_{12}, C_1) , or where the invariant zeros are present the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$, is completely controllable and observable respectively [34] and if it satisfies the condition $p + m - 1 \geq n$ or $p + m' + r - 1 \geq n$ respectively

then the well known ‘Kimura-Davison’ condition is satisfied for the appropriate design triple. Under these circumstances all the poles of the appropriate closed loop subsystem can be assigned at the specified desired locations. If these conditions are not satisfied then a compensator of appropriate size may be used to give extra freedom in the design process. The stable invariant zeros and the dimensionality requirements are discussed in Bag *et al.*[5]. The use of such a dynamical compensator is considered in the next chapter. The problem with unstable invariant zeros is also addressed in a subsequent chapter.

Note that the variable structure control provides insensitivity to matched uncertainty only. It could be verified from the reduced order dynamics in equation (4.36) which is insensitive to the matched uncertainty $\hat{g}(t, u, x_1, 0)$ but sensitive to the unmatched uncertainty $\hat{f}(t, x_1, 0)$. If the unmatched uncertainty $f(t, x) = 0$ and the matched uncertainty $g(t, u, x) = B\xi(t, u)$ is considered as a bounded function $\|\xi(t, u)\| \leq K_g\|u\| + K_\alpha$ where K_g and K_α are two positive constant then from the equivalent dynamics in equation (4.21) it is easily concluded from equation (2.22) that the dynamics is invariant to this matched uncertainty. This is similar to the case described by Edwards [32]. However, in practice the matched uncertainty may not enter through the input distribution matrix B but it could be expressed as defined in equation (2.23) and the unmatched uncertainty is not always zero. Hence, the uncertainty decomposition in the Subsection §4.2.1 is useful. Before going further it is necessary to discuss the switching surface design for square plant.

4.5 Switching Surface Design for Square Plant

Consider the equations (4.1a) and (4.1b) represent a square plant satisfying the assumptions (A1) - (A2) and the uncertainty $h(t, u, x)$ is decomposed as defined in Subsection §4.2.1. In this case the system regular form in equations (4.6) - (4.9) may be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = [0 \quad I_m] \quad (4.62)$$

where the partition matrix $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$, $B_2 \in \mathbb{R}^{m \times m}$ and if there are any invariant zeros present in the system then they will appear amongst the eigenvalues of

the matrix A_{11} . Assume the system is in the above canonical form and a control law, as defined in equation (4.15), exists to induce and maintain a sliding motion on the surface $\mathcal{S}$ as given in equation (4.14). Then the equivalent control and its dynamics as defined in equations (4.17) and (4.21) respectively are satisfied if the matrix FCB has full rank which further implies that $\det(CB) \neq 0$. As for the non-square system, the states of the square system are also partitioned as in equation (4.22). Substituting the above canonical form into the equation (4.21) and using the state partition in equation (4.22), the reduced order sliding dynamics for the square system can be written as

$$\dot{x}_1(t) = A_{11}x_1(t) + \tilde{f}(t, x_1, 0) \quad \text{and} \quad x_2(t) = 0 \quad (4.63)$$

where $\tilde{f}(t, x_1, 0) = T_1 f(t, x)$. Subsequently, the transformation $\hat{T}$ in equation (4.32) is the identity. This shows that the reduced order sliding motion is independent of the matrix F . Consequently the sliding motion should be stable. Hence the necessary conditions for the existence of a sliding motion are

1. the determinant of the matrix (CB) must be non zero;
2. the nominal linear system (A, B, C) must be minimum phase, so that the matrix A_{11} has stable eigenvalues.

If the system satisfies the conditions (i) and (ii), without loss of generality it can be assumed that the matrix A_{11} will have stable eigenvalues. Hence the reduced order system is stable in the absence of unmatched uncertainty $\tilde{f}(t, x_1, 0)$. In the presence of unmatched uncertainty $\tilde{f}(t, x_1, 0)$ the sliding dynamics may exhibit a stable sliding motion in the sense of Lyapunov. This will further lead to a constrained motion to the switching surface $\mathcal{S}$ and it will be globally uniformly ultimately bounded. Its proof is similar to that given in Lemma 4.1 for the non-square system. Therefore, the switching surface matrix F has no effect on the dynamics of the sliding motion for square plant, and this gives limitation in the closed loop design freedom.

In Lemma 4.1, it is shown that the reduced order dynamics produces a bounded stability for a certain class of unmatched uncertainty. However, it should be noted that in many situations the unmatched uncertainty may not be in this form and it could give higher control action. This will be further discussed in the example section. The effect of

unmatched uncertainty is unavoidable but it may be minimised using a number of robust design techniques for non-square systems. The poles of the designed system matrix must be robust to any unmatched variations in the system parameters which is discussed in next section. For the square system this effect may not be minimised. However, the matrix F may be chosen in the null space of C . This may give maximum insensitivity to uncertainty. The section below will describe the robust switching surface design procedure for a non-square plant.

4.6 Robust Switching Surface Design Technique

The eigenvalues of a normal matrix are most insensitive to perturbations in the matrix parameters [75, 112]. This result is used here to design the hyperplane in output feedback variable structure control design. The method is based on the Lyapunov theory used in the robust pole placement technique of Changsheng [17]. Some non-normal measurements will be described based on the description given by Changsheng [17]. First note the following important Theorem.

Theorem 4.1 *For any square matrices $P > 0$ and $Q = Q^T > 0$, of dimension $n \times n$, there must exist a unique symmetric negative definite solution A_c for the matrix equation below.*

$$P^T A_c + A_c P = -Q \quad (4.64)$$

and the matrix A_c can be expressed as

$$A_c = \int_0^{-\infty} e^{P^T t} Q e^{P t} dt$$

Proof: By the use of Kronecker product, equation (4.64) can be rewritten as

$$(P^T \otimes I + I \otimes P^T) rsA_c = -rsQ \quad (4.65)$$

where rsA_c and rsQ represent the column vectors spanned by the rows of the matrices A_c and Q , respectively. Since $\lambda_k(I \otimes P^T + P^T \otimes I) rsA_c = \lambda_i(P^T) + \lambda_j(P) \neq 0$; ($k = 1, \dots, nn; j = 1, \dots, n$) then equation (4.65) has a unique solution and so the equation

(4.64) has a negative definite solution. Next, deriving $(e^{P^T t} Q e^{P t})$ and differentiating it yields

$$\frac{d}{dt} (e^{P^T t} Q e^{P t}) = P^T (e^{P^T t} Q e^{P t}) + (e^{P^T t} Q e^{P t}) P$$

Integrating the above expression over the interval from 0 to $-\infty$,

$$\int_0^{-\infty} d(e^{P^T t} Q e^{P t}) = P^T \int_0^{-\infty} (e^{P^T t} Q e^{P t}) + \int_0^{-\infty} (e^{P^T t} Q e^{P t}) dt P$$

Considering $\text{Re}[\lambda(P)] > 0$, then the above representation becomes

$$P^T \left(\int_0^{-\infty} e^{P^T t} Q e^{P t} dt \right) + \left(\int_0^{-\infty} e^{P^T t} Q e^{P t} dt \right) P = -Q$$

It is seen from the above that

$$A_c = \int_0^{-\infty} e^{P^T t} Q e^{P t} dt$$

is the unique symmetrical solution of equation (4.64). As $Q = Q^T > 0$, it is known that $A_c < 0$, i.e. $\text{Re}[\lambda(A_c)] < 0$. ■

The design approach is based upon minimisation of the norm of the closed-loop system, i.e. the designed closed-loop matrix behaves as a normal matrix and hence is insensitive to the matrix perturbations.

4.6.1 Normal Matrix Design Approach

For simplicity consider the triple (A_{11}, A_{12}, C_1) , A_{11}^s and K , or in case where the invariant zeros are present in the original system, the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$, $\tilde{A}_{11}^s$ and K_1 , correspond to the triple (A_q, B_q, C_q) , A_c and K_q respectively where it is assumed that the matrices have the following dimension: $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$ and $C_q \in \mathbb{R}^{p \times n}$. Then from Theorem 4.1, let $A_c = A_q - B_q K_q C_q$ and equation (4.64) can be rewritten as

$$[P^T B_q K_q C_q + B_q K_q C_q P] = [Q + P^T A_q + A_q P] = Q_q \quad (4.66)$$

Using the Kronecker product, the above equation becomes

$$(P^T B_q \otimes C_q^T + B_q \otimes P^T C_q^T) r s K_q = r s Q_q \quad (4.67)$$

For simplicity, this can be written down as

$$Y r s K_q = r s Q_q \quad (4.68)$$

where $Y = (P^T B_q \otimes C_q^T + B_q \otimes P^T C_q^T) \in \mathbb{R}^{nn \times mp}$ is a tall matrix, and $rsK_q \in \mathbb{R}^{mp \times 1}$ and $rsQ_q \in \mathbb{R}^{nn \times 1}$ represent the column vectors spanned by the rows of the matrices K_q and Q_q respectively. The solution of equation (4.68) can be divided into three cases.

- a) *The unique solution:* By assumption the matrices $P > 0$ and $Q = Q^T > 0$ and if the matrices are optimised then the following condition holds:

$$\text{rank}(Y) = \text{rank}[Y, rsQ_q] = mp \quad (4.69)$$

In this case the equation (4.68) has unique solution.

- b) *The linear least-square solution:* If the conditions in equation (4.69) do not hold, the solution of equation (4.68) turns into a least-squares problem, i.e. given $P > 0$ and $Q = Q^T > 0$, that is to say, given $Y \in \mathbb{R}^{nn \times mp}$ and $rsQ_q \in \mathbb{R}^{nn \times 1}$, the matrix $rsK_q \in \mathbb{R}^{mp \times 1}$ can be solved by considering the following minimal problem.

$$\min \|Y rsK_q - rsQ_q\|^2 \quad (4.70)$$

The solution can be written as

$$(Y^T Y) rsK_q = Y^T rsQ_q \quad (4.71)$$

and

$$rsK_q = (Y^T Y)^{-1} Y^T rsQ_q = Y^\dagger rsQ_q \quad (4.72)$$

where $Y^\dagger = (Y^T Y)^{-1} Y^T$ is the Moore Penrose generalised inverse of the matrix Y . If the column rank of Y is full, the linear least-squares solution is the unique solution.

- c) *The nonlinear least-square solution:* If the assumptions of Theorem 4.1 hold, the above minimal problem becomes a nonlinear least-squares problem. It must be minimised first with respect to the matrices P and Q , and next with respect to the vector rsK_q . This can be expressed as

$$\min \|Y(P) rsK_q - rsQ_q(P, Q)\|^2 \quad (4.73)$$

where $P > 0, Q = Q^T > 0 \in \mathbb{R}^{n \times n}$ and $rsK_q \in \mathbb{R}^{mp \times 1}$. Then the solution of rsK_q is given as

$$rsK_q = Y^\dagger(P) rsQ_q(P, Q) \quad (4.74)$$

The equation (4.73) can be solved with the help of any standard optimisation technique. One such technique is presented in Appendix C. The following theorem may be easily concluded from above discussion.

Theorem 4.2 *If the linear time invariant multivariable system (A_q, B_q, C_q) is controllable and observable and satisfies the ‘Kimura-Davison’ condition, then the matrices $P > 0$ and $Q = Q^T > 0$ can always be chosen to minimise the equation (4.73). The equation (4.68) may thus have either a unique or a least-squares solution for rsK_q . The former will make the system normal and the latter will make it nearly normal. The eigenvalues $Re[\lambda(A_c)] < 0$, for either solution of K_q .*

It is essential to verify that the designed closed-loop matrix attains the necessary normal properties so that the robustness of the system is confirmed. Hence it is necessary to consider some norm measurements for the normal matrix.

4.6.2 Measurements of Robustness

In Chapter §2, it is mentioned that the robustness measurement for classical frequency response analysis is based on the measure of gain and phase margins and it is limited to SISO systems [53]. The robustness for MIMO systems may be defined based on the following non-normal measurements [17]. If the matrix A_c is non zero then for the evaluation of the normal measure of the matrix A_c consider

1. The non-symmetrical measurement of matrix

$$\Delta_{2,F}(A_c) = \frac{\|A_c A_c^T - A_c^T A_c\|_{2,F}^{\frac{1}{2}}}{\|A_c\|_{2,F}}$$

2. The condition number of the frame

$$K(W_{A_c}) = \|W\|_2 \cdot \|W^{-1}\|_2$$

3. The measure of skewness

$$MS(A_c) = \frac{\|T\|_2}{\|A_c\|_2}$$

4. The non-symmetrical measure of the polar decomposition

$$\delta(A_c) = \frac{\|\phi M_r - M_r \phi\|_2}{\|A_c\|_2}$$

or

$$\delta(A_c) = \frac{\|\phi M_l - M_l \phi\|_2}{\|A_c\|_2}$$

5. The measure of robustness [68]

$$\nu(A_c) = n^{\frac{1}{2}} \|Q_r^{-1}\|_F$$

The meaning of the symbols in the above items depends on the following decompositions of the matrix A_c

$$\begin{aligned} A_c &= W \Lambda W^{-1} \text{ (Characteristic decomposition)} \\ &= S(D + T)S^* \text{ (Schur triangular decomposition)} \\ &= Y \Sigma U^* \text{ (Singular value decomposition)} \\ &= \phi M_r = P \Theta_d P^* \cdot U \Sigma U^* = M_l \phi = Y \Sigma Y^* \cdot P \Theta_d P^* \text{ (Polar decomposition)} \\ &= Q_r \Lambda Q_r^{-1} \end{aligned}$$

where $\Theta_d = \text{diag}(e^{j\theta_i})$ and $D = \text{diag}(d_i)$. $Q_r = (q_1, q_2, \dots, q_n)$, $\|q_i\|_2 = 1$, $(i = 1, 2, \dots, n)$. If the matrix A_c is a normal matrix then $K(W_{A_c}) = \nu(A_c) = 1$ and $\Delta(A_c) = MS(A_c) = \delta(A_c) = 0$. Therefore, the greater the value of the above items, the worse is the normal measure of the matrix A_c .

Once the robust switching surface is designed, the next step is to design a control law which can induce a sliding motion and maintain the system outputs on the surface thereafter.

4.7 Controller Formulation

The controller structure is similar to that described by Ryan and Corless [88] and Edwards and Spurgeon [33]. It is modified from the controller presented by El-Khazali and DeCarlo [39] and also Yallapragada and Heck [113]. The controller guarantees the reachability condition is satisfied and has the power to modify the convergence rate to

the switching surface $\mathcal{S}$. An arbitrary choice of invertible matrix F_2 as described earlier does not effect the control authority with this controller formulation. The use of the pseudo inverse in Yallapragada and Heck [113] is also eliminated in this formulation. Without loss of generality it can be assumed that the nominal system triple (A, B, C) is known. The controller design is based on the Lyapunov stability criteria. Assume a Lyapunov function of the form

$$V(s) = \frac{1}{2} s^T(y) s(y) \quad (4.75)$$

Here $s(y) = Fy(t)$ where F defines the switching surface matrix. The reachability condition is satisfied if $\dot{V}(s) < 0$, i.e. if $s^T(y)\dot{s}(y) < 0$ for all $y(t)$. Consider a control law of the form

$$u(t) = -(FCB)^{-1} [G_l y(t) + \nu(y)] \quad (4.76)$$

The gain matrix G_l is defined as

$$G_l = \left[FCAN + \frac{\alpha}{2} F \right]$$

where $N \in \mathbb{R}^{n \times p}$ is a design matrix and α is some positive constant to be chosen accordingly. It can be shown that the reachability condition can be attained by appropriate choice of N for a certain value of α , where α relates to the rate of convergence of the Lyapunov Convergence Lemma² as suggested in Yallapragada and Heck [113]. The aim is to demonstrate that

$$\dot{V}(s) + \alpha V(s) \leq 0 \quad \text{where} \quad \alpha > 0 \quad (4.77)$$

Integrating equation (4.77) over the time limit 0 to t_s , implies

$$V[s(t_s)] \leq V[s(0)]e^{-\alpha t_s} \quad (4.78)$$

Thus the parameter α corresponds to the rate of convergence; the larger the value of α , the smaller will be the reaching time. The component $\nu(y)$ is defined by

$$\nu(y) = \begin{cases} \rho(t, u, y) \frac{s(y)}{\|s(y)\|} & \text{if } s(y) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.79)$$

²For details the Lyapunov Convergence Lemma see for example [94]

where $\rho(t, u, y)$ is a positive scalar quantity defined by the uncertainty bound parameters K_f , K_d and K_g , K_α such that

$$\rho(t, u, y) = \frac{kK_g\|G_ly\| + K_f\|FC\|\|y\| + K_\alpha\|FC\|}{(1 - kK_g)} \quad (4.80)$$

where the denominator is considered to be always positive and the parameter $K_a = K_d + K_\alpha$. It follows that K_g is limited by $0 \leq K_g < \frac{1}{k}$ and k is defined as $k = \|(FC)\|\|(FCB)^{-1}\|$. Clearly lower values of k increase the limiting parameter K_g , which gives a higher bound on the *matched uncertainty*. The reaching condition and time to attain the sliding mode will be shown to be prescribed by the design of the matrix N . The following Lemma may be concluded for attainment of the reaching phase.

Lemma 4.2 *It can be shown that the control law in equation (4.76) induces a sliding mode on the relevant switching surface $\mathcal{S}$ if the matrix N is chosen so that*

$$\mathcal{L}(N) = (FC)^T \left[FCA(I - NC) - \frac{\alpha}{2}FC \right] \leq 0 \quad (4.81)$$

for a certain value of α , i.e. the matrix $\mathcal{L}(N)$ is required negative semi-definite³.

Proof: The proof of this Lemma can be demonstrated using the Lyapunov function in equation (4.75). Taking time derivatives along the system trajectory

$$\dot{V}(s) = s^T(y)\dot{s}(y) \quad (4.82)$$

which is required to be negative scalar quantity. Substituting equation (4.1a) into the reaching condition in equation (4.82), gives

$$s^T(y)\dot{s}(y) = x^T(FC)^T(FCA)x + x^T(FC)^T(FCB)u + s^T(y)FCh(t, u, x) \quad (4.83)$$

Inserting the control law from (4.76) into the equation (4.83), gives

$$s^T(y)\dot{s}(y) = x^T\mathcal{L}(N)x - s^T(y)[\nu(y) - FCh(t, u, x)] \quad (4.84)$$

The equation above can be rewritten as

$$s^T(y)\dot{s}(y) \leq x^T\mathcal{L}(N)x - \|s(y)\|[\rho(t, u, y) - \|FC\|\|h(t, u, x)\|] \quad (4.85)$$

³The method for symmetric negative definite matrix design is presented in Appendix C.

Using the structural constraint in equations (4.5a) and (4.5b), gives

$$\begin{aligned} s^T(y)\dot{s}(y) &\leq x^T \mathcal{L}(N)x - \|s(y)\| [\rho(t, u, y) - \|FC\| \{K_g\|u\| + K_f\|y\| + K_a\}] \\ &\leq x^T \mathcal{L}(N)x - \|s(y)\| \Psi(t, u, y) \end{aligned} \quad (4.86)$$

where

$$\Psi(t, u, y) \triangleq [\rho(t, u, y) - \|FC\| \{K_g\|u\| + K_f\|y\| + K_a\}] \quad (4.87)$$

It will be shown that $\Psi(t, u, y)$ is always a positive scalar quantity for all values of $y(t)$.

Hence, equation (4.76) will meet the reaching condition $s^T(y)\dot{s}(y) < 0$ if and only if

$$x^T \left\{ (FC)^T \left[FCA(I - NC) - \frac{\alpha}{2} FC \right] \right\} x \leq 0$$

It is thus necessary to select N such that the matrix

$$\mathcal{L}(N) = (FC)^T \left[FCA(I - NC) - \frac{\alpha}{2} FC \right] \quad (4.88)$$

is negative semi-definite in order to satisfy the reachability condition $s^T(y)\dot{s}(y) \leq 0$ for some value of α . ■

Proposition 4.1 *The function $\Psi(t, u, y)$ in equation (4.87) is always a positive scalar for all values of $u(t)$ and $y(t)$.*

Proof: Taking the norm of $u(t)$ in equation (4.76) and post multiplying on both sides by $\|FC\|$, equation (4.76) becomes

$$\|FC\| \cdot \|u\| \leq k \{ \rho(y) + \|G_l y\| \} \quad (4.89)$$

where $k = \|FC\| \cdot \|(FCB)^{-1}\|$. Rearranging expression (4.80) and substituting from equation (4.89), gives

$$\begin{aligned} \rho(t, u, y) &= kK_g\rho(t, u, y) + kK_g\|G_l y\| + K_f\|FC\| \cdot \|y\| + K_a\|FC\| \\ &\geq \|FC\| \{K_g\|u\| + K_f\|y\| + K_a\} \end{aligned} \quad (4.90)$$

It follows that $\Psi(t, u, y) \geq 0$ which is justified from the equation (4.90). ■

Numerical methods to design the reaching phase of the output feedback VSC control as discussed in Heck *et al.*[57] can be used to find an appropriate N .

Remark For the case where $(FC)^T FCA \leq 0$, then $N = 0$ and $\alpha = 0$ is sufficient. Note that N and α influence the time taken to reach the sliding surface and may be chosen to meet this particular objective. Recall that the inherent robustness properties of sliding mode systems are only exhibited once the sliding mode is reached. For real applications, it is necessary to limit the control action while achieving a sufficiently fast response to the sliding mode. The sections below demonstrate the design approach and the effectiveness of the proposed method described in this chapter.

4.8 Design Procedure

In this section a step by step design procedure for designing robust static output feedback sliding mode controller is presented. The design procedure consists of the following main steps.

1. *Compute the regular form and identify the reduced order sliding dynamics.* In this step the linear nominal model matrix triple (A, B, C) is first checked to see if it satisfies the static output feedback sliding mode controller design requirements. If the system triple satisfies the assumptions presented in Section §4.2, the triple is transformed into regular form and then the system dimension (n, m, p) and the presence of invariant zeros are identified. Finally the working triple for the reduced order sliding dynamics is obtained. If the system triple (A, B, C) does not satisfy the dimensionability requirements then it is possible to use dynamic output feedback sliding mode control design techniques as illustrated in subsequent chapters.
2. *Normal closed-loop matrix design for robust sliding dynamics.* The robust design procedure adopted is based on the normal matrix design approach as described by Changsheng [17]. An optimisation toolbox is used to place the eigenvalues of the closed-loop working triple in robust positions. The user is required to supply the range of eigenvalues where the closed-loop poles are to be assigned. The computational algorithm which assigns the robust poles is given in Appendix C. The gain matrix K_q is then obtained from an optimal solution of P and Q of equation (4.74).

3. *Verification of robust properties of the designed closed-loop matrix.* Once the reduced order closed-loop system is designed, it is necessary to verify the measurements of robustness. This is illustrated explicitly in the design examples below. There may be differences in the measurements for the reduced order system and the working system when the system possesses any invariant zeros.
4. *Calculation of switching surface matrix F .* The matrix F_2 does not affect the sliding dynamics. Hence it may be chosen arbitrarily. An identity matrix may be a good choice. Then the equation (4.31) is used to calculate the matrix F where the matrix K is obtained from step 2.
5. *Computation of controller parameter N .* The controller design involves mainly the selection of parameters N and α . The remaining parameters are associated with the uncertainty bounds. These are obtained during simulation. An optimisation algorithm is used to find an N which satisfies the condition given in equation (4.81) for an initial value of α . The user is required to supply the initial values N and α . The computational algorithm is presented in Appendix C.
6. *Selection of simulation parameters.* The uncertainty bound parameters are chosen to control a certain amount of uncertainty. These are obtained in this thesis by a trial and error basis at the time of simulation. It is possible to perform analytical computations on perturbed system representations. However, these may lead to a conservative controller. The value of α may be adjusted to enhance the reaching time. Some knowledge of the uncertainty and the operation range may help the user to tune these parameters.

A more general form of the algorithm is presented using the flow chart in Figure 4.1. This procedure will be illustrated on the numerical design examples, presented in the next section.

4.9 Numerical Design Examples

The following non-trivial examples substantiate the practicality of the proposed robust switching surface and controller design method. Application to a fully nonlinear indus-

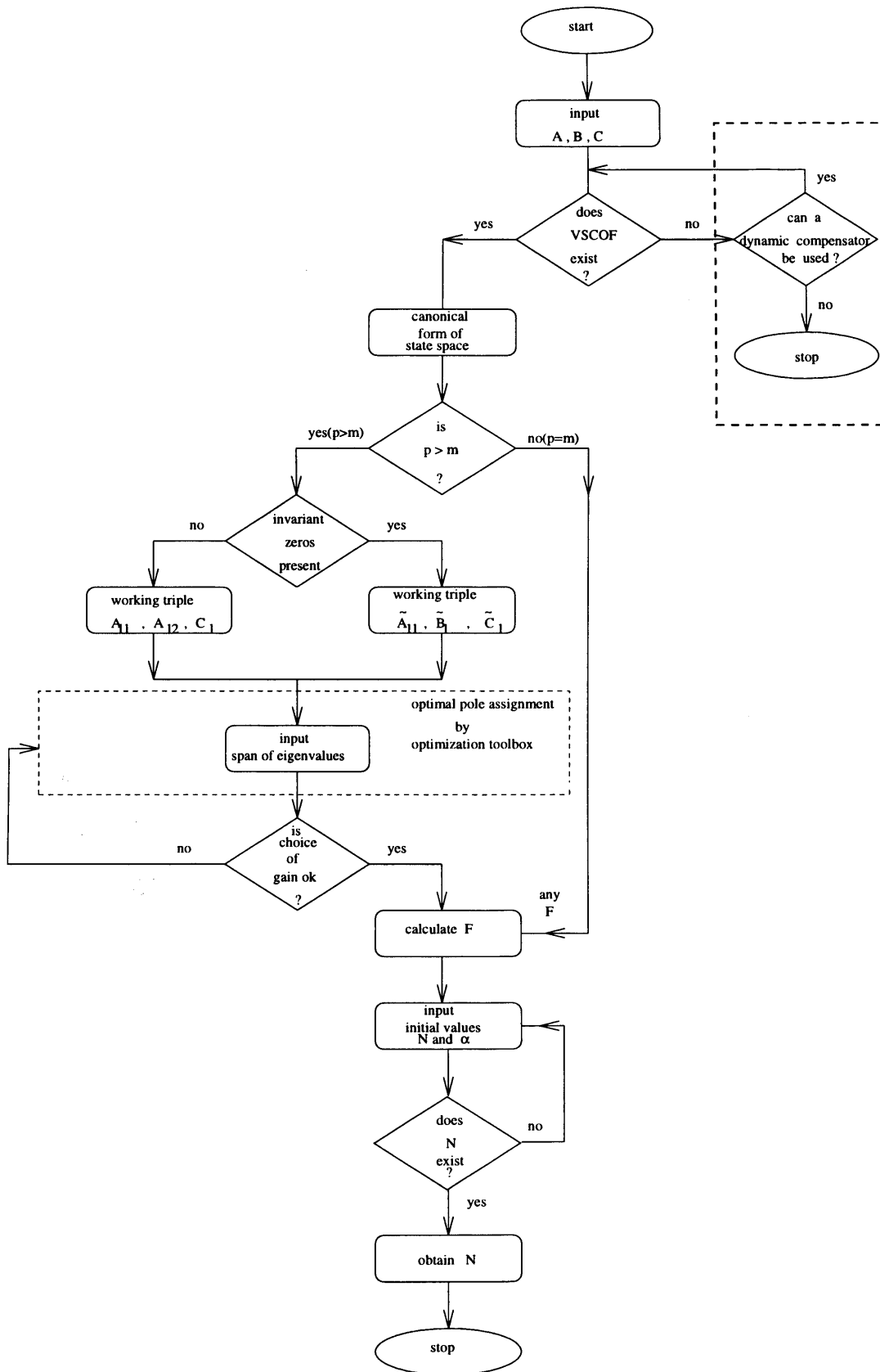


Figure 4.1: Algorithm for output feedback sliding mode design

trial system is described in a later chapter.

4.9.1 Example 1

Consider the example of a fixed wing aircraft (L-1011) from Sobel and Shapiro [96] (which also appeared in Edwards [32]) to demonstrate the design technique presented in this chapter. The design procedure is illustrated in following steps below.

Step 1: Compute the regular form and identify the reduced order sliding dynamics.

The nominal plant triple (A, B, C) is presented in the canonical form as described in Subsection 4.2.2.

$$\begin{aligned}
 A &= \begin{bmatrix} -0.4631 & 0.0007 & 0.0087 & -0.0214 & -0.0473 \\ -0.0000 & -0.0000 & -0.0008 & -0.9110 & 0.4125 \\ -0.6986 & 0.0386 & -0.0856 & 0.4118 & 0.9102 \\ -0.2539 & 0.0004 & 4.0994 & -0.7263 & 0.6161 \\ -0.1352 & 0.0009 & -3.5390 & 0.2270 & -0.4960 \end{bmatrix} \\
 B &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.8170 \\ 0 & 0 & 0 & 1.0335 & -0.4328 \end{bmatrix}^T \\
 C &= \begin{bmatrix} 0 & 0 & 0.0265 & -0.4124 & -0.9106 \\ 0 & 0 & -0.0008 & -0.9110 & 0.4125 \\ 0 & -0.0001 & 0.9996 & 0.0103 & 0.0245 \\ 0 & 1.0002 & 0 & 0 & 0.0001 \end{bmatrix}
 \end{aligned}$$

The reduced order system matrices are identified as

$$\begin{aligned}
 A_{11} &= \begin{bmatrix} -0.4631 & 0.0007 & 0.0087 \\ -0.0000 & -0.0000 & -0.0008 \\ -0.6986 & 0.0386 & -0.0856 \end{bmatrix} \\
 A_{12} &= \begin{bmatrix} -0.0214 & -0.0473 \\ -0.9110 & 0.4125 \\ 0.4118 & 0.9102 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

It is clear that there is no invariant zero in A_{11} and it is possible to design a stable reduced order closed-loop using pole placement techniques since the sliding mode design triple

satisfies the necessary output feedback design conditions.

Step 2: Normal closed-loop matrix design for robust sliding dynamics. Based on the robust design procedure presented in Section §4.6 the optimal values of the matrices P and Q of Theorem 4.1 are computed as

$$P = \begin{bmatrix} -0.4631 & 0.0007 & 0.0087 \\ -0.0000 & -0.0000 & -0.0008 \\ -0.6986 & 0.0386 & -0.0856 \end{bmatrix}, \quad Q = \begin{bmatrix} 4.1037 & 4.3692 & 3.2605 \\ 4.3692 & 4.6612 & 3.4706 \\ 3.2605 & 3.4706 & 2.5906 \end{bmatrix}$$

This gives a gain matrix

$$K = \begin{bmatrix} -0.3949 & 0.6366 \\ 0.4411 & 0.1800 \end{bmatrix}$$

which gives closed-loop poles at $(-0.5147, -0.5008 \pm 0.3481i)$. Hence the matrix K is designed to perform robustly. It is noted that computational algorithm has flexibility to choose any range of eigenvalues and it will compute the robust positions of the eigenvalues.

Step 3: Verification of robust properties of the designed closed-loop matrix. Consider the robustness measurements as described in Subsection §4.6.2. The parameters are given in Table 4.1. The robustness of the closed-loop matrix A_{11}^s is compared with the open-loop system matrix A and A_{11} . This shows that the measure parameters in the

No.	Measure of Robustness	Optimal Value	Open-loop Matrix A	Open-loop Matrix A_{11}	Closed-loop Matrix A_{11}^s
1	$\Delta_2(A_c)$	0	0.9736	0.8876	0.7268
2	$\Delta_F(A_c)$	0	1.1162	1.0528	0.6349
3	$K(W_{A_c})$	1	8.3848	6.6048	5.9942
4	$MS(A_c)$	0	0.9770	0.8416	0.7182
5	$\delta(A_c)$	0	0.9085	0.7873	0.4600
6	$\nu(A_c)$	1	14.6761	7.2736	5.9466

Table 4.1: : Comparison of Robustness Measures of Example 1

closed-loop system matrix A_{11}^s are reduced when compared to the open-loop system

matrix A and A_{11} . The behaviour of the switching surface is completely defined by the design of the reduced order closed-loop dynamic. Thus the greater the value of the resulting robustness parameters the worse the normal measure of the system. It is concluded that the closed-loop matrix does not reach the optimal values. It may be possible to reduce the normal measurements further using a different set of desired closed-loop poles.

Step 4: Calculation of switching surface matrix F . The matrix F_2 has no effect on the switching surface and serves merely to scale the matrix F ; let $F_2 = I_2$. Then using equation (4.31) the matrix

$$F = \begin{bmatrix} -0.3955 & -0.9114 & 0.6466 & -0.3949 \\ -0.9059 & 0.4123 & 0.2044 & 0.4411 \end{bmatrix}$$

is calculated which is also robustly designed.

Step 5: Computation of controller parameter N . The linear gain matrix G_l is designed to satisfy the reachability criteria. Hence the matrix N is obtained to satisfy the equation (4.81) of Lemma 4.2 as

$$N = \begin{bmatrix} -0.7002 & -0.0786 & 0.5516 & 0.3857 \\ 12.2155 & 3.8089 & -7.5342 & -0.8980 \\ -9.2473 & -2.7829 & 5.8624 & 0.9053 \\ -1.9513 & -7.8676 & 4.9498 & -3.7824 \\ 12.7980 & 2.6199 & -7.0324 & -1.5237 \end{bmatrix}$$

with an initial value of $\alpha = 0.0$.

Step 6: Selection of simulation parameters. To show the effect of uncertainty and robust behaviour of the controller, the plant system matrices (A, B) are perturbed arbitrarily and the output matrix C is unperturbed. These are given by

$$A_p = \begin{bmatrix} 0 & 0 & 2.0000 & 0.0004 & 0 \\ 0 & -0.3095 & -0.0059 & 3.0700 & 0.0005 \\ 0.0100 & 0.4980 & -2.0100 & -10.4000 & 0 \\ 0.0772 & -1.9766 & -0.0526 & -0.2352 & 0 \\ 0 & 0.1000 & 0 & 0 & -1.0050 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 & -1.4068 & 0.6372 & 0.0378 & 0 \\ 0 & -0.0605 & -2.1177 & 0 & 0 \end{bmatrix}^T$$

From the perturbed plant the effect of structural uncertainty can be demonstrated. The matched uncertainty can be tolerated by inherent properties of sliding systems. Only the effect of unmatched uncertainty is considered in this discussion. Assume that the uncertainty $\Delta \hat{f}(t, \tilde{x}_1, 0)$ in equation (4.37) equals zero. If the perturbed plant is partitioned accordingly then the amount of unmatched uncertainty entering the reduced order system may be written as

$$\Delta \Sigma = \Delta A_{11} - \Delta A_{12} K C_1 = \begin{bmatrix} 0 & -0.0002 & -0.9998 \\ 0 & -0.4484 & 0.9759 \\ -0.0100 & 1.8045 & -2.3004 \end{bmatrix}$$

Using the norm bound condition

$$\|\Delta \Sigma\| \leq \|T_1 \left(F_1(t, y) C T^{-1} \begin{bmatrix} I \\ -K C_1 \end{bmatrix} \right)\|$$

This gives an approximately $\|F_1(t, x)\| \leq K_f = 1.8119$. If the value of K_f is chosen

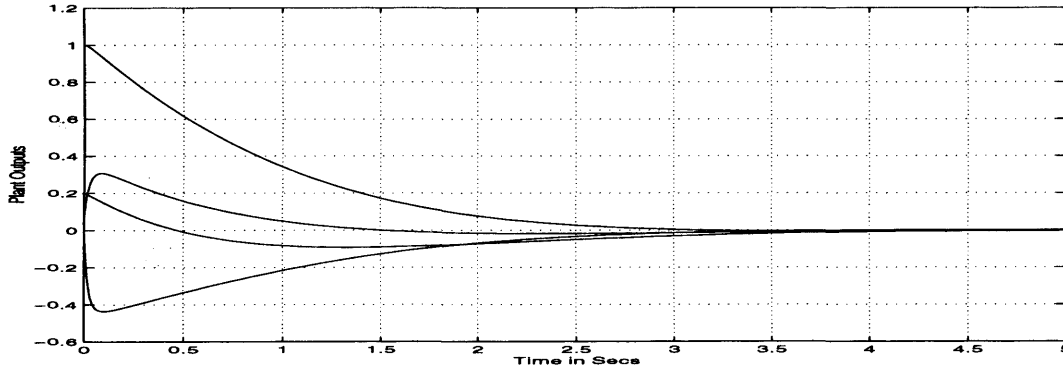


Figure 4.2: Output trajectory of perturbed plant in Example 1

higher than 1.8119, then the system will be stabilised which also increases the control action. However in practice the value of $K_f = 0.0001$ is used which is very small. This may be due to the analysis being very conservative. The robustness is important since it makes the eigenvalues insensitive to its parameter variations. Therefore the system is stable in the worst case. The other uncertainty bound parameters are chosen as $K_g = 0.0002$, $K_\alpha = K_d = 0.0$, and $\alpha = 0.005$, the initial values of the states at simulation are chosen as $[1.0 \ 0.05 \ 0.01 \ 0.2 \ 0.0]$. A small value of $\delta = 0.05$ is added to the denominator of the discontinuous part of the control law which avoids chattering as discussed in Sub-section §2.3.5. It is noted that there is no straightcut way to obtain

the uncertainty bound parameters. It is based on trial and error method, however some knowledge of uncertainty may help to guess these parameters. The simulation results

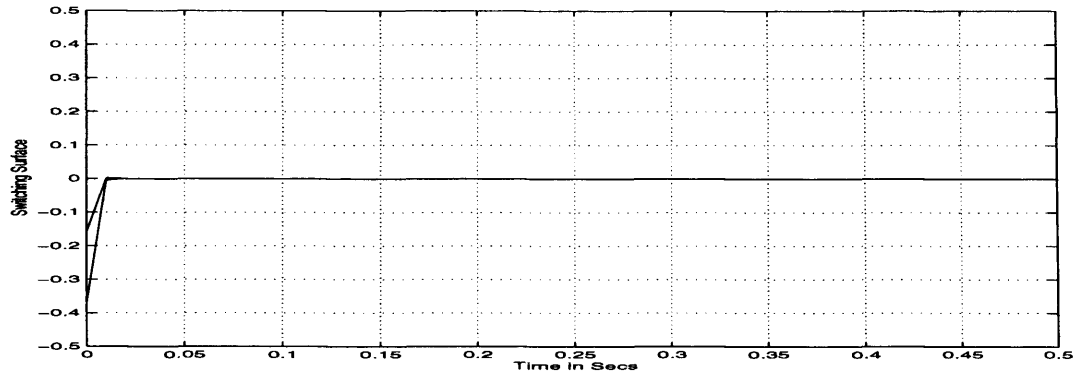


Figure 4.3: Switching surface of perturbed plant in Example 1

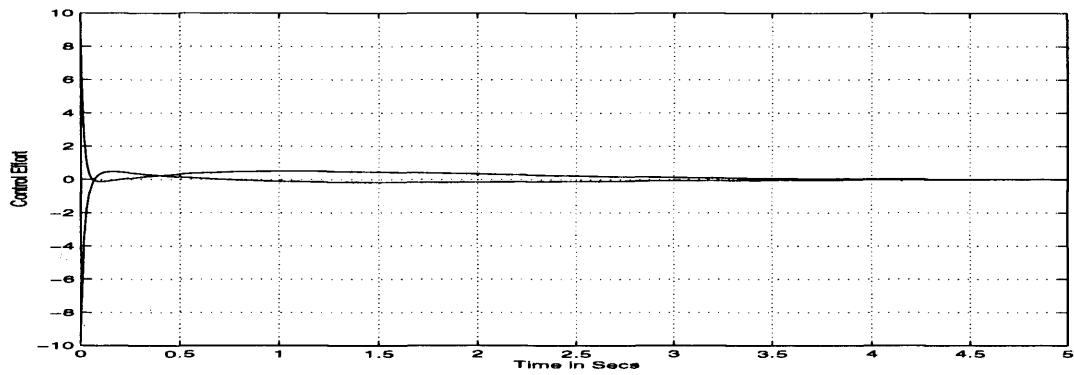


Figure 4.4: Control effort of perturbed plant in Example 1

are presented showing the behaviour of the plant outputs in Figure 4.2, switching surface in Figure 4.3 and Figure 4.4 shows the controller effort. These show good robustness of the design approach. The control action is reasonable.

4.9.2 Example 2

Consider the example presented by El-Khazali and DeCarlo in the paper [38].

Step 1: Compute the regular form and identify the reduced order sliding dynamics. The

regular form of the system triple (A, B, C) is

$$A = \begin{bmatrix} -1.0000 & -1.4142 & 1.4142 & 0 \\ 0 & 0 & 0 & -1.0000 \\ -0.7071 & -1.0000 & 1.0000 & 0 \\ -0.7071 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Using the canonical form the reduced order system is represented by

$$A_{11} = \begin{bmatrix} -1.0000 & -1.4142 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1.4142 & 0 \\ 0 & -1.0000 \end{bmatrix}, \quad C_1 = [0 \quad 1]$$

It is shown that the matrix A_{11} has an invariant zero at -1 , hence it is clear that an arbitrary pole placement is not possible. Eliminating the appropriate row and column a new reduced order sub-system $\tilde{A}_{11} = 0, \tilde{B}_1 = -1, \tilde{C}_1 = 1$ is formed. This is a system with $n = m = p = 1$; i.e. the number of inputs is equal to the number of states and outputs and the pole and the associated eigenvector can be assigned arbitrarily with maximum insensitivity to parameter variation. This also demonstrates the results obtained in this design.

Step 2: Normal closed-loop matrix design for robust sliding dynamics. Using the robust switching surface design method presented in Section §4.6, the gain matrix $K_1 = -3.0028$ is found which places the pole at -3.0082 . Using the transformation

$$T_{m'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in equation (4.60), the gain matrix $K = [0 \quad -3.0028]^T$ is obtained, where $K_2 = 0$ is considered.

Step 3: Verification of robust properties of the designed closed-loop matrix. Consider the robustness property of the reduced order closed-loop system A_{11}^s and the sub-system $\tilde{A}_{11}^s$. The Table 4.2 shows the measurements of robustness as in Subsection §4.6.2. These show

No.	Measure of Robustness	Optimal Value	Open-loop Matrix A	Closed-loop Matrix $\tilde{A}_{11}^s$	Closed-loop Matrix A_{11}^s
1	$\Delta_2(A_c)$	0	0.9603	0	0.5561
2	$\Delta_F(A_c)$	0	0.9603	0	0.6388
3	$K(W_{A_c})$	1	1.0476×10^{53}	1	1.9303
4	$MS(A_c)$	0	1	0	0.4223
5	$\delta(A_c)$	0	1	0	0.2439
6	$\nu(A_c)$	1	1.0476×10^{53}	1	2.4483

Table 4.2: : Comparison of Robustness Measures of Example 2

that the closed-loop sub-system $\tilde{A}_{11}^s$ has attained its optimal values of the measurements. The measurements also show some variations between the reduced order systems A_{11}^s and the sub-system $\tilde{A}_{11}^s$ where the robust design is performed. This variation is due to the addition of the invariant zero into the new sub-system and cannot be avoided. The normal measures of the open-loop matrix $\tilde{A}_{11}$ are not given because it is a null matrix and its comparison is not important since the robustness property of the controller and the switching surface is solely dependent on the reduced order closed-loop dynamics.

Step 4: Calculation of switching surface matrix F . The choice of F_2 does not affect the robustness of the reduced order system, hence it can be chosen arbitrarily. Letting $F_2 = I_2$ the switching surface matrix

$$F = \begin{bmatrix} 0 & 0 & -1.0000 \\ -3.0028 & -1.0000 & 0 \end{bmatrix}$$

is calculated.

Step 5: Computation of controller parameter N . The next step is to design the linear gain matrix G_l such that equation (4.81) is satisfied so that the control law (4.76) guarantees the reachability condition holds. The matrix

$$N = \begin{bmatrix} 0.1000 & 0 & 0 \\ 0.4848 & 0.2018 & 13.3649 \\ 18.6242 & 6.3225 & 5.6101 \\ 6.6810 & 2.1858 & 15.2017 \end{bmatrix}$$

is chosen to satisfy the condition in equation (4.81) and the value of $\alpha = 0.0$ is considered.

Step 6: Selection of simulation parameters. To demonstrate the robustness of the design procedure further choose a perturbed system defining the triple

$$A_p = \begin{bmatrix} 0.0050 & 1.0020 & 0 & 0.8500 \\ 0 & 0 & 1.0250 & 0 \\ -1.4000 & 0 & 0 & 2.0000 \\ 0.0005 & 0 & 0 & 0 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0.0010 & 0 \\ 0 & 0 \\ 1.0100 & 0 \\ 0 & -1.2000 \end{bmatrix}$$

The amount of unmatched structural uncertainty in this example is given by

$$\Delta\Sigma = \Delta A_{11} - \Delta A_{12}KC_1 = \begin{bmatrix} -0.0050 & 0.4484 \\ 0 & 0 \end{bmatrix}$$

Using the norm bound condition the $\|F_1(t, x)\| \leq 0.1002$. However the worst case parameter of $K_f = 0.0001$ is sufficient to stabilise the perturbed plant. The affect can be similarly explained and it is demonstrated that the value of $K_g = 0.0002$, $K_a = 0.0$ and the value of $\alpha = 0.0$ are sufficient since the rate of convergence is good from the design of matrix N . Figure 4.5 represents the perturbed plant output responses. The

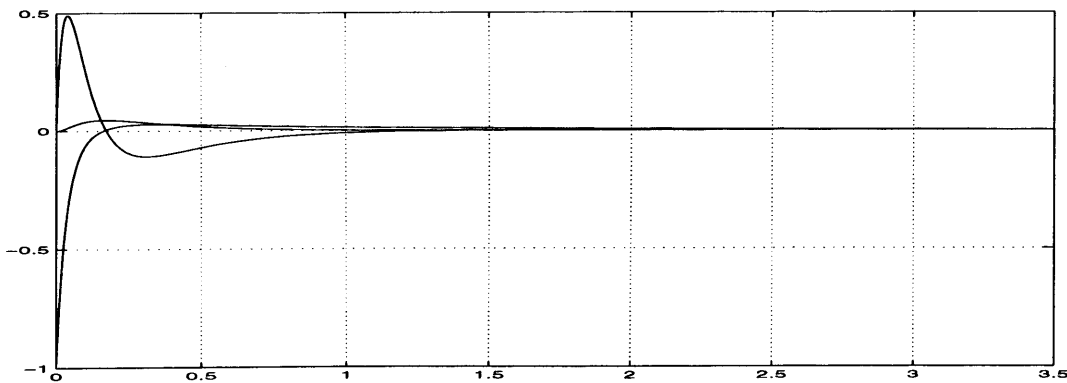


Figure 4.5: Output trajectory of perturbed plant in Example 2

switching surface response of the perturbed plant is shown in Figure 4.6. A sliding

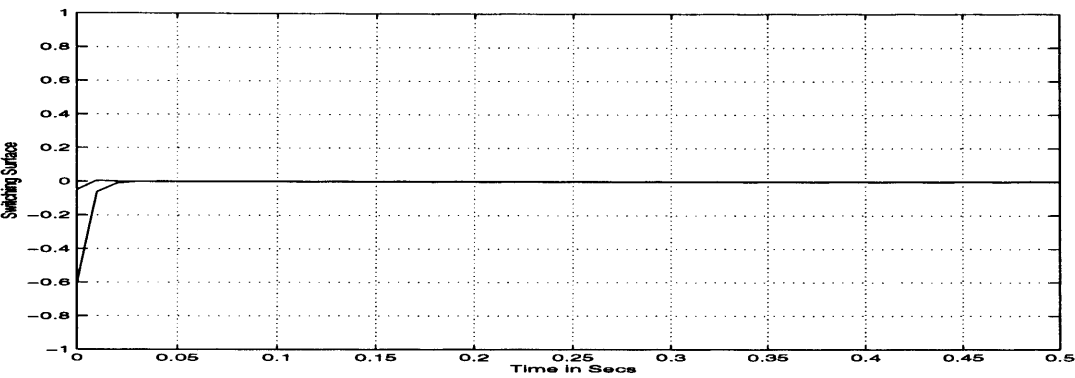


Figure 4.6: Switching surface of perturbed plant in Example 2

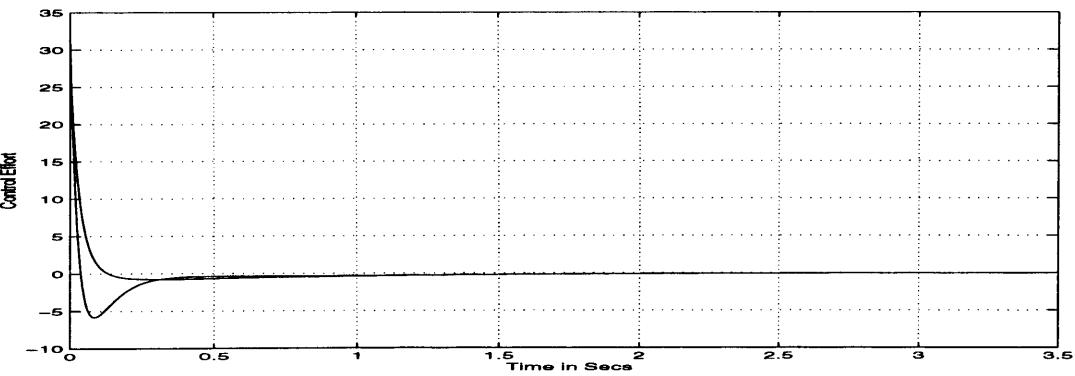


Figure 4.7: Control effort of perturbed plant in Example 2

mode is attained within 0.005 seconds approximately. These show good robustness to the design approach. The control action is given in Figure 4.7; this shows reasonable control effort is applied. It is further possible to reduce the control action by increasing the reaching time to the switching surface.

4.10 Summary

A unified framework for variable structure output feedback control is presented for the case of multivariable linear time invariant uncertain systems which may possess stable invariant zeros. The switching surface design in static output feedback sliding mode control is equivalent to an output feedback problem for the reduced order system. It follows that problems arise when the reduced order system triple does not satisfy the ‘Kimura-Davison’ condition and/or invariant zeros are present. Here, a method is described which tolerates the presence of stable invariant zeros. The problem of the ‘Kimura-Davison’ condition and unstable transmission zeros will be discussed in later chapters. A robust approach to output feedback stabilisation is used to determine the switching surface. There is no need for extra dynamics to be added if the reduced order subsystem triple satisfies the output feedback pole placement criteria. The design procedure is straightforward and the controller is easy to implement. The measurements of robustness are considered.

The controller design is straightforward. The controller gain can be varied to influence the reaching time to the switching surface. The closed-loop configuration is shown to eliminate the effects of certain model uncertainty and nonlinearity in the system. Numerical examples show the effectiveness of the technique. The simulation results demonstrate the applicability of the method; the proposed controllers guarantee the attainment of a sliding mode despite the presence of uncertainty. Some computational problems exist for high dimensional systems. As one would expect, the greater the design freedom in calculating the output feedback gain, the better the attainable normal measure of the closed loop system. The robustness measurements of the two examples justify the properties. If the ‘Kimura-Davison’ condition does not hold for the reduced order system and stable invariant zeros are present, a compensator may be added to

the VSCOF framework. This is discussed in the next chapter and the case of unstable invariant zeros is considered in the following chapter.

Chapter 5

Dynamic Output Feedback VSC with Stable Transmission Zeros

5.1 Introduction

This chapter considers the development of dynamic output feedback sliding mode controllers for a class of uncertain linear systems. The existence problem in variable structure control by output feedback, where no compensator dynamics are employed, is essentially a static output feedback control problem. In order to carry out this procedure the system is usually assumed to satisfy the pole placement conditions relating to controllability and observability and the appropriate design subsystem must satisfy the well known ‘Kimura-Davison’ condition pertaining to that subsystem dimensions. An appropriate switching surface matrix which assigns arbitrary eigenvalues to the sliding mode subsystem may not be chosen if that appropriate subsystem does not satisfy the latter condition. The sufficient conditions for developing static output feedback sliding mode controllers were discussed in the previous chapter. If the so-called ‘Kimura-Davison’ condition is not satisfied for that design triple, it has been shown [70, 71] that it may not be possible to determine a static output feedback sliding mode controller. A dynamic compensator of appropriate size may be used in the design procedure to satisfy this condition. This approach has been investigated in the sliding mode context by El-Khazali and DeCarlo [40] for linear systems. It is known from previous discussions that the invariant zeros affect the sliding mode controller design [32, 63]. The work of El-Khazali and DeCarlo [40] does not consider the implications of invariant

zeros or model uncertainty in the system. Recently Diong and Medanic [27, 28] studied the dynamic output feedback problem for linear systems based upon a simplex control law. A comparison of different classes of compensator and their effect on minimum and non-minimum phase systems was considered.

In this work, a dynamic output feedback strategy is proposed which circumvents problems caused in developing static output feedback sliding mode controllers for linear uncertain systems which do not satisfy the ‘Kimura-Davison’ condition for the design subsystem triple and may possess stable invariant zeros. In addition, it is shown that if the system does not satisfy the required performance specifications then this type of compensator may also be used to give some extra freedom in design. This is explained with an example. The system must be controllable but it is not necessary for the system to be observable as the reduced order sliding mode dynamics do not depend on the observability of the system. Here the parameters of the compensator are determined explicitly during the switching surface design procedure. As noted in the static case, during sliding the system is sensitive to any unmatched uncertainty. Again these effects can be minimised by ensuring that the reduced order sliding dynamics are maximally robust. The unstable invariant zero case will be considered in Chapter §6 with another type of dynamic output feedback sliding mode control.

The outline of the chapter is as follows: The compensator and system definition are introduced in Section §5.2. Section §5.3 describes a parameterisation of dynamic output feedback VSC. The control authority required for the augmented system and its reachability criteria are discussed in Section §5.6. Numerical examples illustrate the technique in Section §5.5.

5.2 Compensator and System Description

Recall the linear time invariant state space model of the plant defined in Section §4.2.

$$\dot{x}(t) = Ax(t) + Bu(t) + h(t, u, x) \quad (5.1a)$$

$$y(t) = Cx(t) \quad (5.1b)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ with $m \leq p < n$. Let the assumptions in Chapter §4 hold. In this case, a sufficient condition to solve a static output feedback problem was seen to depend on the relative dimensions of a particular subsystem; i.e. the design subsystem triple must satisfy the ‘Kimura-Davison’ condition. If this condition is not found to hold for this design subsystem triple, then it is natural to explore the effect of adding a *compensator* – i.e a dynamical system driven by the output of the plant – to introduce extra dynamics to provide additional degrees of freedom. Consider a dynamic compensator given by

$$\dot{x}_c(t) = Hx_c(t) + Dy(t) \quad (5.2)$$

where the matrices $H \in \mathbb{R}^{q \times q}$ and $D \in \mathbb{R}^{q \times p}$ are to be determined. The uncertain system of equation (5.1a) together with the compensator (5.2) will be referred to as the augmented uncertain system and the index q is chosen so that the augmented reduced order sliding dynamics satisfy the ‘Kimura-Davison’ condition. The augmented system may be written as

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t) + \tilde{h}(t, u, x) \quad (5.3a)$$

$$\tilde{y}(t) = \tilde{C}z(t) \quad (5.3b)$$

where the augmented state $z = [x_c^T \ x^T]^T \in \mathbb{R}^{(q+n)}$ and output $\tilde{y} = [x_c^T \ y^T]^T \in \mathbb{R}^{(q+p)}$ and the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is given

$$\tilde{A} = \begin{bmatrix} H & DC \\ 0 & A \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I_q & 0 \\ 0 & C \end{bmatrix}, \quad \tilde{h}(t, u, x) = \begin{bmatrix} 0 \\ h(t, u, x) \end{bmatrix} \quad (5.4)$$

Note that the augmented system in equation (5.3a) still satisfies the unmatched and matched uncertainty decompositions in Chapter §4. It was demonstrated that the reduced order dynamics are unavoidably affected by unmatched uncertainty, but it is possible to reduce the effects by appropriate robust design of the reduced order closed-loop matrix. Hence, for simplicity it is worthwhile to investigate only the case of matched uncertainty with a robust closed-loop design for the reduced order sliding dynamics. For the purpose of investigation of the effect of a dynamic compensator consider the unmatched uncertainty

$$f(t, x) = 0$$

and the matched uncertainty

$$g(t, u, x) = G_1(t, u)u(t) + G_2(t, y) \quad (5.5)$$

where $G_1(t, u) \in \mathbb{R}^{n \times m}$, $G_2(t, y) \in \mathbb{R}^{n \times 1}$ and its boundedness is defined in Subsection §4.2.1. It is required that the augmented system must exhibit a stable sliding motion in the augmented state space, formed from the plant and compensator state spaces. Define a new hyperplane in the augmented state space, as

$$\mathcal{S}_a = \{z \in \mathbb{R}^{q+n} : s(\tilde{y}) = F_c x_c(t) + FCx(t) = 0\} \quad (5.6)$$

where $F_c \in \mathbb{R}^{m \times q}$ and F define the relevant switching surface matrices. Once the switching surface is designed, the second stage is then to develop a control law based upon plant and compensator outputs which can induce a sliding motion on the switching surface $\mathcal{S}_a$ of the augmented system. The proposed control law is similar to that used for static output feedback sliding mode control but depends on augmented outputs, giving

$$u(t) = G\tilde{y}(t) - \nu(\tilde{y}) \quad (5.7)$$

where $\tilde{y}(t)$ represents the output of the augmented system. The parameter G is defined as a gain matrix and the discontinuity vector $\nu(\tilde{y})$ depends on the augmented outputs $\tilde{y}(t)$ only. The key problem is to design the switching surface matrices F and F_c , the compensator parameters H and D , and the gain matrix G , so that the closed loop system is both stable and robust.

5.3 Dynamic Output Feedback Parameterisation

To carry out the design and analysis for dynamic output feedback sliding mode control the canonical form defined in Subsection §4.2.2 is also useful. It is considered that the plant triple (A, B, C) is in the canonical form defined in equations (4.6) - (4.9) with $p \geq m$ and $\text{rank}(CB) = m$. Having obtained the necessary canonical form, the development of a framework for dynamic output feedback sliding mode controller design will be considered. It has been seen that the square plant case has no freedom to design the switching surface for static output feedback sliding mode control. It is seen that in this case the augmented reduced order dynamics become a static output feedback stabilisation problem and the square system essentially does not have any design freedom. This chapter is largely devoted to the case of more outputs than inputs and the effect of the dynamics on a square plant is mentioned. It is noted that in many cases a stabilising

controller gain may be obtained but this might not meet the performance requirements due to limited design freedom available. In this case this type of compensator dynamics may be used with the plant. One such example is considered.

5.3.1 Non-square Plant

The design subsystem triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ in equation (4.61) is stabilisable by output feedback if the so-called ‘Kimura-Davison’ condition holds or it may require more design freedom. If it is not possible to synthesis a K_1 to stabilise the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ then the plant in equation (5.1a) is augmented with the dynamical compensator defined in equation (5.2) to form the augmented uncertain system as in equation (5.3a). As in Section §4.4 the matrix FC is partitioned as in equation (4.19). In an analogous way define $D_1 \in \mathbb{R}^{q \times (p-m)}$ and $D_2 \in \mathbb{R}^{q \times m}$ as

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} = DT \quad (5.8)$$

If the states of the uncertain system are partitioned as in equation (4.22) then the compensator dynamics can be written as

$$\dot{x}_c(t) = Hx_c(t) + D_1C_1x_1(t) + D_2x_2(t) \quad (5.9)$$

where C_1 is defined in equation (4.20). Assume that a control action exists which forces and maintains motion on the hyperplane $\mathcal{S}_a$ given in (5.6). As in Section §4.4 in order for a unique equivalent control to exist the square matrix F_2 must be invertible. By writing $K = F_2^{-1}F_1$, defining $K_c \triangleq F_2^{-1}F_c$ and using the equation (5.6), the augmented system matrix governing the reduced order sliding motion, obtained by eliminating the coordinates x_2 , can be written as

$$\dot{x}_1(t) = (A_{11} - A_{12}KC_1)x_1(t) - A_{12}K_cx_c(t) \quad (5.10a)$$

$$\dot{x}_c(t) = (D_1 - D_2K)C_1x_1(t) + (H - D_2K_c)x_c(t) \quad (5.10b)$$

If the unmatched uncertainty $\tilde{f}(t, x_1, x_2)$ is non zero then it will appear in equation (5.10a) as defined in equation (4.24). From the above equations it is clear that the introduction of the compensator has introduced more design freedom than was available

in Section §4.4. It is considered that the unmatched uncertainty $f(t, x) = 0$ hence the closed-loop analysis is straightforward. Defining

$$\tilde{D}_1 \triangleq D_1 - D_2 K \quad (5.11a)$$

$$\tilde{H}_1 \triangleq H - D_2 K_c \quad (5.11b)$$

then the following exposition will determine an optimal choice of $\tilde{D}_1, \tilde{H}_1, K$ and K_c and thus the compensator will be parameterised by D_2 . Some remarks regarding its selection will be made later. Unfortunately the invariant zeros of the uncertain system are still embedded in the dynamics. From the definition of the partition of A_{11} given in (4.7), it can be shown that

$$A_{11}^d \triangleq \begin{bmatrix} A_{11} - A_{12} K C_1 & -A_{12} K_c \\ (D_1 - D_2 K) C_1 & H - D_2 K_c \end{bmatrix} = \begin{bmatrix} A_{11}^o & [A_{12}^o \mid A_{121}^m - A_{121} K] & -A_{121} K_c \\ 0 & \tilde{A}_{11} - A_{122} K \tilde{C}_1 & -A_{122} K_c \\ 0 & \tilde{D}_1 \tilde{C}_1 & \tilde{H}_1 \end{bmatrix}$$

As in the uncompensated case, it is necessary for the eigenvalues of A_{11}^o to have negative real parts. The design problem becomes one of selecting the matrices $\tilde{D}_1$ and $\tilde{H}_1$, and a hyperplane represented by the matrices K and K_c so that the matrix

$$\tilde{A}_{11}^d \triangleq \begin{bmatrix} \tilde{A}_{11} - A_{122} K \tilde{C}_1 & -A_{122} K_c \\ \tilde{D}_1 \tilde{C}_1 & \tilde{H}_1 \end{bmatrix}$$

is stable. Again if there is rank deficiency in the matrix A_{122} the problem is over-parameterised. As in Section §4.4 suppose $\text{rank}(A_{122}) = m' < m$ and let $T_{m'} \in \mathbb{R}^{m \times m}$ be a matrix of elementary column operations such that equation (4.59) holds, then $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$ is a full rank matrix. Define partitions of the transformed hyperplane matrices as in equation (4.60)

$$T_{m'}^{-1} K_c = \begin{bmatrix} K_{c1} \\ K_{c2} \end{bmatrix} \begin{matrix} \mathbb{I}_{m'} \\ \mathbb{I}_{m-m'} \end{matrix}$$

then it follows that

$$\tilde{A}_{11}^d = \begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1 & -\tilde{B}_1 K_{c1} \\ \tilde{D}_1 \tilde{C}_1 & \tilde{H}_1 \end{bmatrix} \quad (5.12)$$

As before, the unknowns present in the matrix given in (5.12) will be expressed as the result of an output feedback stabilisation problem for a certain known system triple.

This is comparable to the situation which occurred in the uncompensated case. Note that K_2 in equation (4.60) was found to have no effect on $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$. Similarly K_2 and K_{c2} have no effect on the equation (5.12). The key observation is that equation (5.12) can be written as

$$\begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1 & -\tilde{B}_1 K_{c1} \\ \tilde{D}_1 \tilde{C}_1 & \tilde{H}_1 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} K_1 & K_{c1} \\ \tilde{D}_1 & \tilde{H}_1 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & I_q \end{bmatrix} \quad (5.13)$$

Thus by defining

$$\hat{A}_1 \triangleq \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0_{q \times q} \end{bmatrix} \quad \hat{B}_1 \triangleq \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \quad \hat{C}_1 \triangleq \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & I_q \end{bmatrix} \quad \hat{K}_1 \triangleq \begin{bmatrix} K_1 & K_{c1} \\ \tilde{D}_1 & \tilde{H}_1 \end{bmatrix}$$

then the parameters $K_1, K_{c1}, \tilde{D}_1$ and $\tilde{H}_1$ can be obtained from an output feedback pole placement controller design on the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$. In order to use standard output feedback results it is necessary for the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ to be both controllable and observable:

Lemma 5.1 *The matrix pairs $(\hat{A}_1, \hat{B}_1)$ and $(\hat{A}_1, \hat{C}_1)$ are completely controllable and observable respectively if the matrix pair (A, B) is completely controllable.*

Proof: From the definition of $(\hat{A}_1, \hat{B}_1)$ it follows that

$$\text{rank} \begin{bmatrix} sI - \hat{A}_1 & \hat{B}_1 \end{bmatrix} = \text{rank} \begin{bmatrix} sI - \tilde{A}_{11} & \tilde{B}_1 \end{bmatrix} + q \quad \text{for all } s \in \mathbb{C}_-$$

It is possible to prove that the rank $\begin{bmatrix} sI - \tilde{A}_{11} & \tilde{B}_1 \end{bmatrix} = n - m - r$ for all $s \in \mathbb{C}_-$, i.e. the matrix pair $(\tilde{A}_{11}, \tilde{B}_1)$ is required to be completely controllable. Because the matrix pair (A, B) is in the canonical form of Section §4.2.2, which is a special case of the regular form used in sliding mode controller design, it is well known that the pair (A, B) completely controllable if and only if the pair (A_{11}, A_{12}) is completely controllable. Therefore from the PHB rank test

$$\text{rank} \begin{bmatrix} sI - A_{11} & A_{12} \end{bmatrix} = n - m \quad \text{for all } s \in \mathbb{C}_-$$

Substituting for A_{11} from equation (4.7) and A_{12} from equation (4.55), gives

$$\text{rank} \begin{bmatrix} sI - A_{11}^o & [A_{12}^o & A_{121}^m] & A_{121} \\ 0 & sI - \tilde{A}_{11} & A_{122} \end{bmatrix} = n - m \quad \text{for all } s \in \mathbb{C}_-$$

This implies that

$$\text{rank} \begin{bmatrix} sI - \tilde{A}_{11} & A_{122} \end{bmatrix} = n - m - r \quad \text{for all } s \in \mathbb{C}_-$$

and therefore $(\tilde{A}_{11}, A_{122})$ is completely controllable by PHB rank test. By construction $(\tilde{A}_{11}, A_{122})$ is completely controllable if and only if the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable and so the first part of the Lemma 5.1 is established, hence the pair $(\hat{A}_1, \hat{B}_1)$ is controllable.

Applying the PHB observability rank test to the pair $(\hat{A}_1, \hat{C}_1)$, gives

$$\text{rank} \begin{bmatrix} sI - \hat{A}_1 \\ \hat{C}_1 \end{bmatrix} = \text{rank} \begin{bmatrix} sI - \tilde{A}_{11} \\ \tilde{C}_1 \end{bmatrix} + q \quad \text{for all } s \in \mathbb{C}_-$$

It follows that if the pair $(\tilde{A}_{11}, \tilde{C}_1)$ is observable then the pair $(\hat{A}_1, \hat{C}_1)$ is observable. Therefore

$$\begin{aligned} \text{rank} \begin{bmatrix} sI - \tilde{A}_{11} \\ \tilde{C}_1 \end{bmatrix} &= \text{rank} \begin{bmatrix} sI - A_{22}^o & -A_{122}^m \\ A_{21}^o & sI - A_{22}^m \\ 0 & I_{(p-m)} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} sI - A_{22}^o \\ A_{21}^o \end{bmatrix} + (p - m) \quad \text{for all } s \in \mathbb{C}_- \end{aligned} \quad (5.14)$$

By construction the pair (A_{22}^o, A_{21}^o) is observable and has

$$\text{rank} \begin{bmatrix} sI - A_{22}^o \\ -A_{21}^o \end{bmatrix} = n - p - r \quad \text{for all } s \in \mathbb{C}_-$$

therefore

$$\text{rank} \begin{bmatrix} sI - \tilde{A}_{11} \\ \tilde{C}_1 \end{bmatrix} = n - m - r \quad \text{for all } s \in \mathbb{C}_-$$

hence the pair $(\hat{A}_1, \hat{C}_1)$ is completely observable. ■

Obtaining the matrices K and K_c , the switching surface matrix sub-block relating to the plant states is given as in equation (4.31) and that relating to the compensator states is given by

$$F_c = F_2 K_c \quad (5.15)$$

Note the matrix D_2 is not yet defined. It is observed that the matrix D_2 has no role in the augmented reduced order sliding dynamics, so an arbitrary choice of D_2 is possible.

However, in many situations, especially where a robust output feedback approach is used, the design matrix $\tilde{H}_1$ becomes very large value and may be unstable. This may cause very high control authority. An appropriate choice of D_2 can circumvent this problem. If the pair $(\tilde{H}_1, K_c)$ resulting from the output feedback design in equation (5.13) is observable, then rearranging the equation (5.11b), an observer gain approach may be utilised to evaluate the matrix D_2 to ensure that H is stable. From a practical implementation view point, a stable compensator is preferable. Similarly, rearranging the equation (5.11a) gives the matrix D_1 .

5.3.2 Square Plant

The square plant has no design freedom to select a switching surface for the static output feedback case and the addition of this type compensator does not improve the situation. Although it is possible to form the reduced order dynamics into an output feedback dynamic problem, stability depends on the eigenvalues of the matrix A_{11} which must be stable. This can be demonstrated from equations (5.8) and (5.9). If the square plant satisfies the sliding conditions in equation (5.6) and the canonical form in equation (4.62) exists, then eliminating the states $x_2(t)$, the reduced order sliding dynamics can be written as follows

$$\dot{x}_1(t) = A_{11}x_1(t) - A_{12}K_c x_c(t) \quad (5.16a)$$

$$\dot{x}_c(t) = (H - DK_c)x_c(t) \quad (5.16b)$$

where $K_c = F^{-1}F_c$. Using the structure of A_{11} in equation (4.7) and eliminating the invariant zeros from the dynamics in equations (5.16a) and (5.16b), the reduced order closed loop matrix may be written as

$$\tilde{A}_{11}^d = \begin{bmatrix} \tilde{A}_{11} & -\tilde{B}_1 K_{c1} \\ 0 & \tilde{H}_1 \end{bmatrix} \quad (5.17)$$

where $\tilde{H}_1 = H - DK_c$. The eigenvalues of the matrix $\tilde{A}_{11}^d$ are given as

$$\lambda(\tilde{A}_{11}^d) = \lambda(\tilde{A}_{11}) \cup \lambda(\tilde{H}_1)$$

Hence it is clear that addition of this compensator does not benefit the switching surface design and the eigenvalues of the matrix A_{11} must be stable, i.e. the square plant must

be minimum phase. This can be further written as an output feedback problem as follows

$$\begin{bmatrix} \tilde{A}_{11} & -\tilde{B}_1 K_{c1} \\ 0 & \tilde{H}_1 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} K_{c1} \\ \tilde{H}_1 \end{bmatrix} \begin{bmatrix} 0_{q \times n-m-r} & I_q \end{bmatrix} \quad (5.18)$$

Thus by defining

$$\hat{A}_1 \triangleq \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0_{q \times q} \end{bmatrix} \quad \hat{B}_1 \triangleq \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \quad \hat{C}_1 \triangleq \begin{bmatrix} 0_{q \times n-m-r} & I_q \end{bmatrix} \quad \hat{K}_1 \triangleq \begin{bmatrix} K_{c1} \\ \tilde{H}_1 \end{bmatrix}$$

The matrix pairs $(\hat{A}_1, \hat{B}_1)$ and $(\hat{A}_1, \hat{C}_1)$ are completely controllable and observable respectively and further discussion is irrelevant to this chapter.

The effects of unmatched uncertainty upon the sliding mode for this case can similarly be explored as discussed in Chapter §4 and a constrained sliding motion can be obtained. The closed-loop design for this purpose is similar to that described for the case of matched uncertainty. From the design point of view, all the effort is required to find the stabilising gain matrix $\hat{K}_1$ which defines the switching surface parameters and compensator parameters. From previous discussion, it is concluded that the closed-loop sliding dynamic must be robust to the unmatched uncertainty. As in Chapter §4, of Section §4.6 the normal matrix design approach discussed is also applicable here to design the robust switching surface and the compensator parameters since the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ resulting from the use of dynamic compensation satisfies the assumptions of the ‘Kimura-Davison’ condition in order to use an output feedback approach. The output feedback design will seek to maximise the robustness of these reduced order sliding mode dynamics. This will guarantee the stability and robustness of the ideal sliding dynamics. The second stage of the design is to find an appropriate controller which will bring the augmented system states onto the augmented switching surface. Further it is possible that the addition of compensator dynamics, although not necessary, can improve the robustness where the system does not attain the required robustness and performance specifications. This is demonstrated with an example later. The next section will discuss the design of a controller which will force the augmented outputs on to the augmented switching surface.

5.4 Controller Design

Without loss of generality it can be assumed that the plant and the augmented system triples (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ respectively are known. The controller formulation is similar that of described in case of static output feedback sliding mode controller and based on the Lyapunov stability criteria, however the augmented parameters are applicable in this case. Consider a Lyapunov function

$$V(s) = \frac{1}{2} s^T(\tilde{y}) s(\tilde{y}) \quad (5.19)$$

Here $s(\tilde{y}) = \tilde{F}\tilde{y}(t)$ where $\tilde{F} = [F_c \quad F]$ defines the augmented switching surface matrix. The reachability condition as defined in equation (4.86) must satisfy $\dot{V}(s) < 0$, i.e. if $s^T(\tilde{y})\dot{s}(\tilde{y}) < 0$ for all $\tilde{y}(t)$. Hence the control law of equation (4.76) becomes

$$u(t) = -(\tilde{F}\tilde{C}\tilde{B})^{-1} \left[\tilde{F}\tilde{C}\tilde{A}N + \frac{\alpha}{2}\tilde{F} \right] \tilde{y}(t) - (\tilde{F}\tilde{C}\tilde{B})^{-1} \nu(\tilde{y}) \quad (5.20)$$

where $N \in \mathbb{R}^{(q+n) \times (q+p)}$ is a design matrix and α is some positive constant to be chosen accordingly. Subsequently the component $\nu(\tilde{y})$ is defined by

$$\nu(\tilde{y}) = \begin{cases} \rho(t, u, \tilde{y}) \frac{s(\tilde{y})}{\|s(\tilde{y})\|} & \text{if } s(\tilde{y}) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.21)$$

and the positive scalar quantity $\rho(t, u, \tilde{y})$ defined by the matched uncertainty bound parameters K_g and K_α such that

$$\rho(t, u, \tilde{y}) = \frac{kK_g \left\| \left[\tilde{F}\tilde{C}\tilde{A}N + \frac{\alpha}{2}\tilde{F} \right] \tilde{y} \right\| + K_\alpha \|\tilde{F}\tilde{C}\|}{(1 - kK_g)} \quad (5.22)$$

where the denominator is again considered to be a positive scalar quantity.

It follows that K_g is limited by $0 \leq K_g < \frac{1}{k}$ where the scalar k is defined as

$$k = \|(\tilde{F}\tilde{C})\| \cdot \|(\tilde{F}\tilde{C}\tilde{B})^{-1}\|$$

and define K_α as a positive scalar. Clearly lower values of k increase the limiting parameter K_g , which gives a higher bound on the *matched uncertainty*. The reaching condition and time to attain the sliding mode will be shown to be prescribed by the design of the matrix N . The proof of this follows the Lemma 4.2. The next section demonstrates the technique using numerical examples.

5.5 Numerical Design Examples

The examples below describe the design procedure in this chapter. The first example demonstrates the applicability of the proposed theory using a non-square system. The second example demonstrates the possibility of increasing the robustness using a dynamic compensator, although the compensator is not essential for the design purposes.

5.5.1 Example 1

A system configuration which does not satisfy the ‘Kimura-Davison’ conditions and contains a stable invariant zero is considered in this example. The reduced order dynamics cannot be stabilised by static output feedback. The plant triple is presented in the canonical form of equations (4.6)-(4.9) for ease of exposition

$$A = \begin{bmatrix} -0.0800 & 0 & 0 & 1.0000 & 1.0000 \\ 0 & 0 & 25.0000 & -1.0000 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 25.0074 & 4.9093 & -0.1971 & -0.0007 \\ 0 & -24.9958 & 0.8948 & -0.0391 & -0.0033 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The triple (A, B, C) contains a stable invariant zero at -0.08 . The triple (A_{11}, A_{12}, C_1) is identified as

$$A_{11} = \begin{bmatrix} -0.08 & 0 & 0 \\ 0 & 0 & 25.00 \\ 0 & 1.00 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the matrix $C_1 = [0 \ 0 \ 1]$. Since the invariant zero is present in the matrix A_{11} , it is not possible to apply the output feedback results to the triple (A_{11}, A_{12}, C_1) . The matrices A_{11} and A_{12} are partitioned according to equations (4.56) and (4.55)

respectively. A new subsystem $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ is generated by eliminating the invariant zero:

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 25 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A_{122} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix C_1 is reduced to $\tilde{C}_1 = [0 \ 1]$. The matrix A_{122} is rank deficient but it is in the form of equation (4.59). Hence the transformation T_m in equation (4.59) is not essential. The matrix $\tilde{B}_1$ is obtained as $\tilde{B}_1 = [-1 \ 0]^T$. It is observed that the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ cannot be stabilised using output feedback since it does not satisfy the ‘Kimura-Davison’ condition. It is thus not possible to find the stabilisable gain matrix for the reduced order sliding mode dynamics by an arbitrary pole placement using static output feedback. A compensator of minimum first order is necessary to satisfy the ‘Kimura-Davison’ condition. Using the compensator dynamics defined in equation (5.2), the augmented reduced order sliding dynamic triple is formed as in equation (5.13), giving extra freedom to use the output feedback results. These are as follows

$$\hat{A}_1 = \begin{bmatrix} 0 & 25 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \hat{C}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix pairs $(\hat{A}_1, \hat{B}_1)$ and $(\hat{A}_1, \hat{C}_1)$ satisfy Lemma 5.1 and also the ‘Kimura-Davison’ condition, hence the triple is output feedback stabilisable. Any output feedback results may be utilised. The robust output feedback pole assignment technique [17] is applied to the augmented reduced order sliding dynamic as described in Section §4.6. The stabilising gain matrix K_q will determine the robust hyperplane matrix parameters K_1, K_{c1} and the compensator matrix parameters $\tilde{D}_1, \tilde{H}_1$. Thereafter the controller matrix N is then calculated to satisfy the reachability condition. The optimal matrices P and Q of Theorem 1 are chosen as

$$P = \begin{bmatrix} 0.4029 & -1.1280 & 1.1254 \\ 0.0035 & 0.2623 & 0.0531 \\ -0.1399 & -0.1471 & 0.4516 \end{bmatrix}, \quad Q = \begin{bmatrix} 3.6127 & 3.5396 & -10.0100 \\ 3.5396 & 28.5134 & -8.4241 \\ -10.0100 & -8.4241 & 27.8122 \end{bmatrix}$$

The closed loop poles of the augmented reduced order system are at $\{-0.4443 \pm 1.8304i, -2.0000\}$ and the gain matrix K_q is calculated as

$$K_q = \begin{bmatrix} -30.3251 & -4.2247 \\ -1.9615 & -2.8887 \end{bmatrix}$$

Hence the matrices $K_1 = -30.3251$, $K_{c1} = -4.2247$ and $\tilde{D}_1 = -1.9615$, $\tilde{H}_1 = -2.8887$ are isolated. Since K_2 and K_{c2} have no effect on the sliding dynamics these are chosen arbitrarily as $K_2 = 0$, $K_{c2} = 0$. The switching surface parameters

$$K = \begin{bmatrix} -30.3251 \\ 0 \end{bmatrix} \quad \text{and} \quad K_c = \begin{bmatrix} -4.2247 \\ 0 \end{bmatrix}$$

are computed. Since the computed $\tilde{H}_1$ is a stable matrix, $D_2 = [1 \ 0]$ will define a stable compensator state matrix $H = -7.1134$ and input partition matrix $D_1 = -32.2866$. The compensator input matrix D is calculated from equation (5.8) as

$$D = [-32.2866 \quad 1 \quad 0]$$

The matrix F_2 has no affect on the sliding dynamics. This can be obtained in the process of controller design which gives some extra freedom to calculate the matrix N . This is given as

$$F_2 = \begin{bmatrix} -0.1206 & -0.3926 \\ -0.0282 & -0.7046 \end{bmatrix}$$

and the switching surface matrices in equation (4.31) and (5.15) are given as

$$F = \begin{bmatrix} 3.6566 & -0.1206 & -0.3926 \\ 0.8537 & -0.0282 & -0.7046 \end{bmatrix} \quad \text{and} \quad F_c = \begin{bmatrix} 0.5094 \\ 0.1183 \end{bmatrix}$$

In order to quantify the robustness of the plant closed-loop dynamics, the non-normal parameter measurements may be taken to evaluate the normal measure of the matrix A_c as described in Section §4.6.2. For this example, the measure parameters are compared in Table 5.1. If a matrix is normal, then the measure parameters are equal to the optimal values; thus the greater the value of the above parameters the worse the normal measure of the system. The measurement of these parameters for both the open-loop system matrices are relatively large. The reduced order closed-loop sub-system $\tilde{A}_{11}^d$ is optimised in the design. The table shows that although the measure parameters $\Delta_2(A_c)$ and $\Delta_F(A_c)$ are not generally reduced, the parameters $K(W_{A_c})$ and $\nu(A_c)$ are reduced

No.	Measure of Robustness	Optimal Value	Open-loop Matrix A	Open-loop Matrix $\tilde{A}$	Closed-loop Matrix $\tilde{A}_{11}^d$	Closed-loop Matrix A_{11}^d
1	$\Delta_2(A_c)$	0	0.9979	0.9997	0.9688	0.9999
2	$\Delta_F(A_c)$	0	0.9070	0.9541	1.1336	1.1775
3	$K(W_{A_c})$	1	1.4747×10^5	1.4755×10^5	5.3328	48.1333
4	$MS(A_c)$	0	1	0.9923	0.9560	0.2103
5	$\delta(A_c)$	0	0.6689	0.8357	0.8755	0.8593
6	$\nu(A_c)$	1	1.6669×10^5	1.8268×10^5	6.5251	55.9815

Table 5.1: Comparison of Robustness Measures of Example 1

significantly, when compared with both the open-loop system matrices. The dynamical behaviour of the plant depends on the behaviour of the reduced order closed loop matrix A_{11}^d . It is observed that the non-normal measurements of $\tilde{A}_{11}^d$ and A_{11}^d are not exactly matched due to the presence of the invariant zero in the closed loop system A_{11}^s . The controller design matrix N is evaluated to meet the reaching condition in equation (4.81) as

$$N = \begin{bmatrix} -4.2969 & -39.4633 & 1.3066 & 11.3990 \\ 0 & 1.0000 & -0.0001 & -0.0003 \\ -0.0332 & 0.9093 & -0.1339 & -28.1755 \\ -26.1279 & -186.3348 & 6.1028 & 54.4605 \\ 0.9387 & 5.7776 & 0.1657 & -1.9221 \\ -0.0004 & -0.0001 & -0.0001 & -0.0044 \end{bmatrix}$$

with initial value of $\alpha = 0.0$. The simulation results are based on a perturbed plant model.

The controller design is based on the nominal plant presented above and implemented on a significantly perturbed system, which induces unmatched and matched uncertainty into the matrices A and B in order to illustrate the robustness. The perturbed plant

matrices are given by,

$$A_p = \begin{bmatrix} -0.0880 & 0 & 0 & 1.1000 & 1.1000 \\ 0 & 0 & 27.5000 & -1.1000 & 0 \\ 0 & 1.1000 & 0 & 0 & 0 \\ 0 & 32.5096 & 6.3821 & -0.2562 & -0.0009 \\ 0 & -32.4945 & 1.1632 & -0.0508 & -0.0043 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.0199 & 0 \\ 0 & 0.0199 \end{bmatrix}$$

and the output matrix C is unperturbed. The amount of structural uncertainty can similarly be calculated and it will be tolerated by the robust consideration as described in Section §4.6. At this point it is important to note that as in Chapter §4 the control law is not augmented with unmatched uncertainty bound parameters. However the method can tolerate unmatched uncertainty. This clearly reflects the effect of a robust design approach which will be helpful to control the uncertain plant. In order to control the matched uncertainty, the discontinuity vector parameters K_g and K_α are chosen as 0.001 and 3.0 respectively. A value of $\alpha = 15.0$ is sufficient to bring the outputs onto the switching surface. Due to the discontinuous control action in equation (5.21), the responses show some chattering. To eliminate this effect a small parameter $\delta = 10^{-5}$ is added to the denominator in equation (5.21) to smooth the discontinuity.

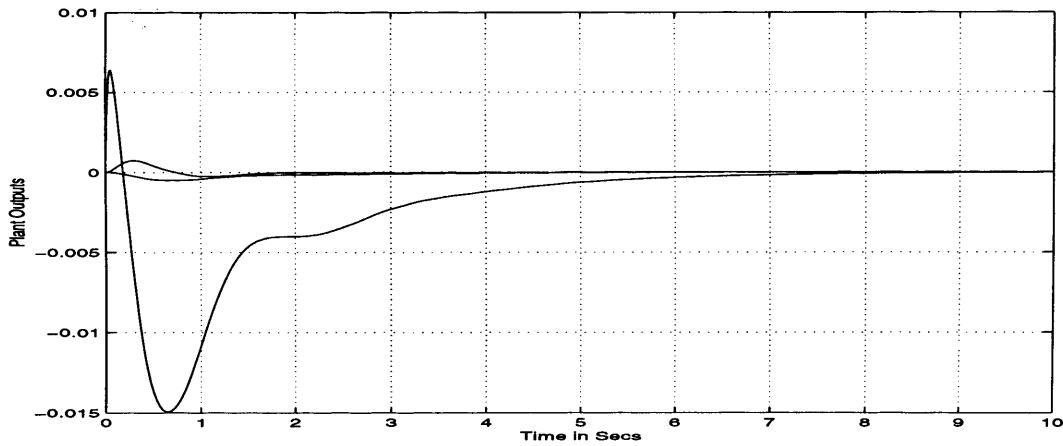


Figure 5.1: Time Response of Output Vector in Example 1

Figure 5.1 represents the perturbed plant output responses. The control action is given in Figure 5.2; this shows reasonable control effort is applied. The switching surface response of the perturbed plant is shown in Figure 5.3. A sliding mode is attained within 4.0 seconds approximately.

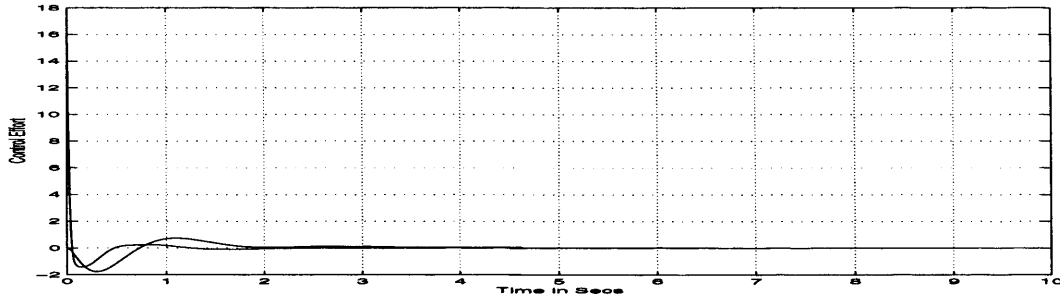


Figure 5.2: Actuator Demand in Example 1

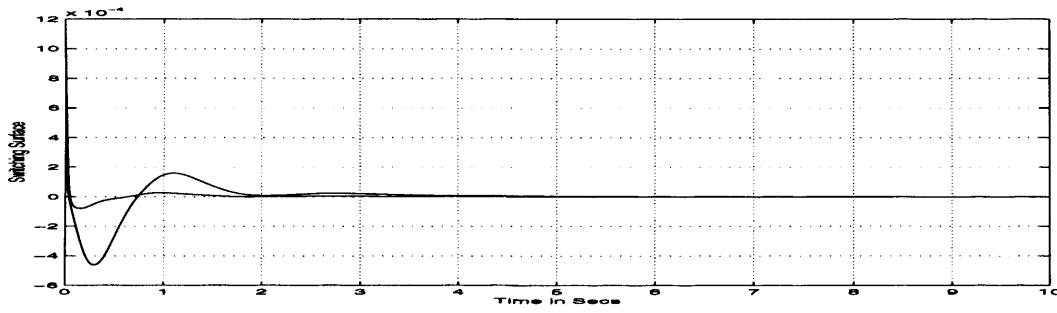


Figure 5.3: Sliding Surface Attainment in Example 1

5.5.2 Example 2

Consider the system below which satisfies the static output feedback sliding mode criteria. A controller can be designed without any additional dynamics. The system triple (A, B, C) is given in the canonical form of equations (4.6) - (4.9) as

$$A = \begin{bmatrix} -1.0000 & 1.4142 & 0 & 1.4142 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 4.0383 & -5.0000 & 0 & 0.8080 \\ -0.7071 & 1.0000 & 0 & 1.0000 & 0 \\ -0.7071 & 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

From the canonical form, the reduced order system matrices are isolated as

$$A_{11} = \begin{bmatrix} -1.0000 & 1.4142 & 0 \\ 0 & 0 & 0 \\ 0 & 4.0383 & -5.0000 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1.4142 & 0 \\ 0 & 1.0000 \\ 0 & 0.8080 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is showing that the matrix A_{11} has an invariant zero at -1 , hence it is clear that an arbitrary pole placement is not possible at this stage. Eliminating the appropriate row and column a new reduced order sub-system

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0 \\ 4.0383 & -5.0000 \end{bmatrix}, \quad A_{122} = \begin{bmatrix} 0 & 1.0000 \\ 0 & 0.8080 \end{bmatrix}$$

and $\tilde{C}_1 = I_2$ is formed. However, the matrix A_{122} shows rank deficiency of unity. Using the transformation $T_{m'}$ of equation (4.60), giving

$$T_{m'} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and the matrix $\tilde{B}_1 = [-1 \quad -0.8080]^T$. The system triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is completely controllable and observable and the ‘Kimura-Davison’ condition is satisfied. Hence an arbitrarily pole placement is possible. Using the robust switching surface design method presented in Section §4.6, the gain matrix $K_1 = [570.1885 \quad -711.9109]$ is found which places the poles at $\{-4.0, -6.0\}$. Using the transformation $T_{m'}$, the gain matrix

$$K = \begin{bmatrix} 0 & 0 \\ -570.1885 & 711.9109 \end{bmatrix}$$

is obtained, where $K_2 = [0 \quad 0]$ is considered. Consider the robustness property of the reduced order closed-loop system A_{11}^s and the sub-system $\tilde{A}_{11}^s$. Table 5.2 shows the measurements of robustness as in Subsection §4.6.2. These show that the closed-loop sub-system $\tilde{A}_{11}^s$ has not attained its optimal values of the measurements. The measurements also show some variations between the reduced order systems A_{11}^s and the sub-system $\tilde{A}_{11}^s$ where the robust design is performed. This variation is due to the addition of the invariant zero into the new sub-system and cannot be avoided as in the other cases. This shows that the attainment of the robust measurement are very poor. A dynamic compensator is added to give some extra degrees of freedom in design which

No.	Measure of Robustness	Optimal Value	Open-loop Matrix A	Closed-loop Matrix $\tilde{A}_{11}^s$	Closed-loop Matrix A_{11}^s
1	$\Delta_2(A_c)$	0	0.8248	1.0	1.0
2	$\Delta_F(A_c)$	0	0.9023	1.1892	1.1892
3	$K(W_{A_c})$	1	2.0379×10^{12}	1.1766×10^3	1.2748×10^3
4	$MS(A_c)$	0	0.6786	1.0	1.0
5	$\delta(A_c)$	0	0.9983	0.9999	0.9986
6	$\nu(A_c)$	1	2.2783×10^{12}	1.1766×10^3	1.5130×10^3

Table 5.2: Robustness Measures of Example 2 in Static VSCOF

should increase the robustness. Consider the new reduced order subsystem $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ where a single order compensator is introduced to form the augmented reduced order sliding mode triple as in equation (5.13). These are given as follows

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 4.0383 & -5.0000 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} -1.0000 & 0 \\ -0.8080 & 0 \\ 0 & -1.0000 \end{bmatrix} \text{ and } \hat{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ satisfies the assumptions for output feedback sliding mode control. The triple also satisfies Lemma 5.1 and also the ‘Kimura-Davison’ condition, hence output feedback results can be utilised. Applying the robust output feedback pole assignment technique the stabilising gain matrix K_q is determined as

$$K_q = \begin{bmatrix} -3.7606 & 1.0000 & 0.0343 \\ 0.0343 & 0.0343 & -6.0000 \end{bmatrix}$$

which places the augmented reduced order closed-loop poles at $\{-2.9526, -4.9993, -6.000\}$ which gives the robust hyperplane matrix parameters K_1 , K_{c1} and the compensator matrix parameters $\tilde{D}_1$, $\tilde{H}_1$ as follows $K_1 = [-3.7606 \ 1.0000]$, $K_{c1} = 0.0343$ and $\tilde{D}_1 = [0.0343 \ 0.0343]$, $\tilde{H}_1 = -6.0000$. For comparison purposes one of the eigenvalues is kept in its original position as in the case of the static output feedback sliding mode control. There was rank deficiency in the matrix A_{122} . Hence, using the inverse transformation of equation (4.59), the transformed stabilising gain matrices are given

as

$$K = \begin{bmatrix} 0 & 0 \\ 3.7606 & -1.0000 \end{bmatrix}, \quad K_c = \begin{bmatrix} 0 \\ -0.0343 \end{bmatrix}$$

where the matrices $K_2 = [0 \ 0]$ and $K_{c2} = 0$ are considered. The matrix $D_2 = [0 \ -145.9854]$ is computed using a state feedback control law giving a stable compensator state matrix $H = -1.0$ and input partition matrix $D_1 = [-548.9528 \ 146.0197]$. The compensator input matrix D is then calculated from equation (5.8) as

$$D = [-146.0197 \ 548.9528 \ 145.9854 \ 0]$$

The matrix F_2 which does not affect the sliding dynamics is taken as the identity matrix

No.	Measure of Robustness	Optimal Value	Open-loop Matrix $\tilde{A}$	Closed-loop Matrix $\tilde{A}_{11}^d$	Closed-loop Matrix A_{11}^d
1	$\Delta_2(A_c)$	0	1.0	0.0195	0.3613
2	$\Delta_F(A_c)$	0	1.1892	0.0166	0.2994
3	$K(W_{A_c})$	1	4.5603×10^{45}	1.0053	1.7865
4	$MS(A_c)$	0	1.0	0.0011	0.2348
5	$\delta(A_c)$	0	0.9988	0.2895	0.3387
6	$\nu(A_c)$	1	5.5853×10^{45}	3.0	4.3515

Table 5.3: Robustness Measures of Example 2 in Dynamic VSCOF

and the switching surface matrices in equation (4.31) and (5.15) are given as

$$F = \begin{bmatrix} 0 & 0 & 0 & -1.0000 \\ 1.0000 & -3.7606 & -1.0000 & 0 \end{bmatrix} \quad \text{and} \quad F_c = \begin{bmatrix} 0 \\ -0.0343 \end{bmatrix}$$

In order to compare the robustness of the closed-loop dynamics between the two design methods, the non-normal parameter measurements are tabulated in Table 5.3. Comparing the non-normal measures between Tables 5.2 and 5.3, shows that the parameters have changed. For comparison one of the eigenvalues was kept in the same position where the remaining eigenvalues of the reduced order closed loop dynamics are placed at lower convergent rate, so that the control action is kept to a minimum. The results may be improved with different sets of eigenvalues. This indicates that for the non-scalar case if the robust performance is not satisfactory then an additional compensator

may be added to improve the performance in the static output feedback sliding mode case. The controller design matrix N is given to meet the reaching condition in equation (4.81) as

$$N = \begin{bmatrix} 1.9658 & -0.0652 & 0.4295 & -1.1284 & 1.1393 \\ -0.2234 & 1.0212 & -0.0045 & 0.0246 & 0.0513 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0.2602 & -20.0373 & 71.1697 & 18.7713 & 161.8689 \\ 1.0323 & -8.9429 & 42.8343 & 12.2561 & -286.4579 \\ 0.5306 & 0.7905 & 0.2067 & -0.6086 & 658.2398 \end{bmatrix}$$

with initial value of $\alpha = 0.0$. The simulation results are presented based on the perturbed plant model

$$A_p = \begin{bmatrix} -1.0800 & 1.4142 & 0 & 2.4142 & 1.0000 \\ 0 & 0 & 5.0000 & -1.0000 & 1.0000 \\ 0 & 5.0383 & -5.0000 & 0 & 0.8080 \\ -0.7071 & 26.0074 & 4.9093 & 0.8029 & -0.0007 \\ -0.7071 & -24.9958 & 0.8948 & 0.9609 & -0.0033 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1.25 \\ -1.00 & 0 \end{bmatrix}$$

with initial conditions $[1.000, 0.025, 0.500, 0.100, 0.000]$. The uncertainty boundeds

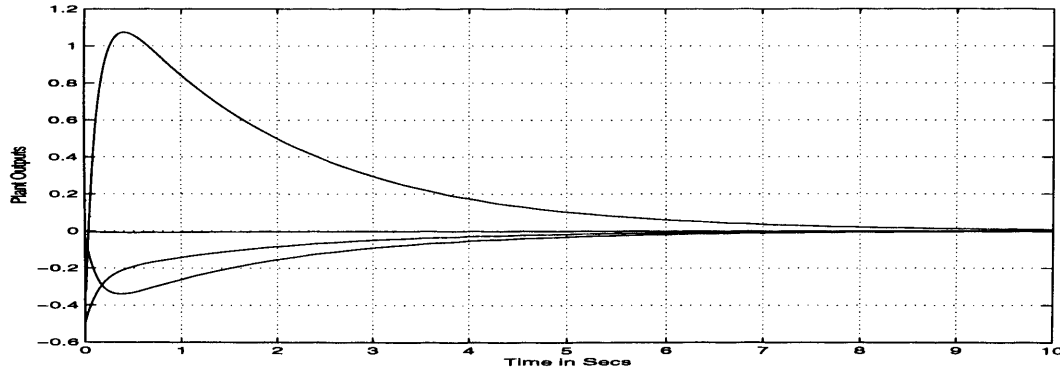


Figure 5.4: Output trajectory of perturbed plant in Example 2

$K_g = 0.05$ and $K_\alpha = 2.0$ are chosen to control the uncertainty and $\alpha = 0.0$, since the reaching time is sufficiently fast. A parameter $\delta = 0.10$ is used in the denominator of the discontinuous control part to avoid chattering. More over it is seen that the parameters K_f and K_d are not used in this control law. However the unmatched uncertainty in the perturbed plant is controlled. This also signifies the use of the robust design.

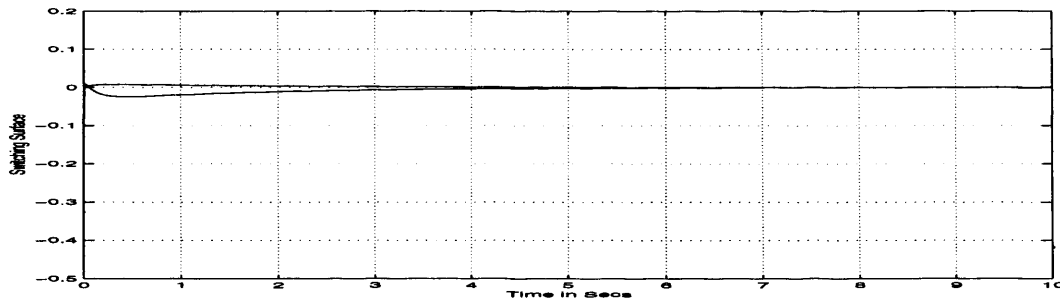


Figure 5.5: Switching surface of perturbed plant in Example 2

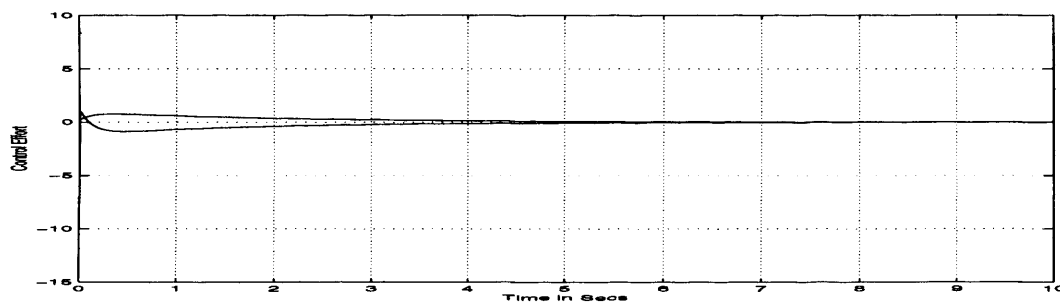


Figure 5.6: Control effort of perturbed plant in Example 2

5.6 Summary

A framework for variable structure dynamic output feedback controller design is described for the case of linear multivariable time invariant uncertain systems which may possess stable invariant zeros and cannot be solved by the static output feedback method described in Chapter §4. It is also useful when more design freedom is required to meet performance specifications. Here, a method is described which tolerates the presence of stable invariant zeros and the dimensional constraint. It is shown that a compensator may be added to the output feedback sliding mode control framework and the resulting reduced order dynamics is further formulated as a static output feedback problem. A robust approach to output feedback stabilisation is used to determine both the switching surface and compensator in dynamic output feedback sliding mode controller design. It is mentioned that the switching surface design for the square plant does not improve using this type of compensator dynamics.

The controller design is similar to that described in static output feedback case. The controller gain and reachability condition are satisfied for the augmented system. The

numerical examples demonstrate the effectiveness of the proposed method. The simulation results demonstrate the applicability of the method; the proposed controllers guarantee the attainment of a sliding mode despite the presence of uncertainty. The latter example shows that as one would expect, the greater the design freedom in calculating the output feedback gain, the better the attainable normal measure of the closed loop system. The presence of unstable invariant zeros cannot be dealt with using this type of compensator. The next chapter will describe the case of unstable invariant zeros using another type of dynamic output feedback sliding mode control.

Chapter 6

Dynamic Output Feedback VSC with Unstable Transmission Zeros

6.1 Introduction

A static output feedback sliding mode control strategy was developed in Chapter §4. The dynamic output feedback sliding mode control strategy which circumvents problems caused in developing static output feedback sliding mode controllers for systems which do not satisfy the ‘Kimura-Davison’ conditions and/or contain stable invariant zeros has been explicitly investigated in Chapter §5. It is shown that both the switching surface design problem for the static case and the switching surface and compensator design for the dynamic case may be formulated as a static output feedback problem for particular system triples [7]. However, the case of unstable invariant zeros has not been discussed. It has been seen that the unstable transmission zeros in the system cause difficulties in developing an output feedback sliding mode controller. Essentially the reduced order dynamics when sliding are seen to have the invariant zeros amongst the poles of the closed-loop system. Unstable transmission zeros thus lead to an unstable sliding mode dynamic using these techniques. In this chapter, a technique for systems which possess unstable transmission zeros is described. The presence of a class of uncertainty is considered for design and analysis. In this context, the work of Diong and Medanic [28] is useful. They have developed a simplex control strategy for non-minimum phase linear systems. Further, they have investigated the case of uncertain systems where the switching surface is designed using an H_∞ technique and implemented with a simplex

controller [27]. This work is limited to square systems and the invariance properties do not strictly hold when simplex control is used.

In this work an augmented system will be formed from the plant dynamics and the compensator dynamics. A nonlinear estimator/observer type dynamic compensator is designed first and then a state feedback controller is formulated which produces more design freedom. The system requirement is that the linear system matrices (A, B, C) must be completely controllable and observable. It is mentioned that during sliding the system is sensitive to any unmatched uncertainty. However, effects can be minimised by ensuring that the reduced order closed-loop sliding dynamics are maximally robust to such uncertainty. A robust technique based on state feedback design may be utilised to design the reduced order sliding dynamics. A controller strategy guaranteeing quadratic stability onto the switching surface is also described. The whole design process may be divided into three main steps as described in later sections.

The outline of the chapter is as follows: The system definition and necessary assumptions are stated in Section §6.2. Section §6.3 describes the dynamic compensator parameterisation. The controller design and associated reachability criteria are discussed in Section §6.4. The closed-loop analysis and switching surface design are presented in Section §6.5. Section §6.6 gives a brief discussion regarding the use of this technique and the effect of the uncertainty class on the compensator and plant dynamics. Numerical examples illustrate the technique in Section §6.7.

6.2 System Description

Consider a linear time invariant state space model with some uncertainties in the system

$$\dot{x}(t) = Ax(t) + Bu(t) + h(t, u, x) \quad (6.1a)$$

$$y(t) = Cx(t) \quad (6.1b)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ with $m \leq p < n$. The nominal linear system triple (A, B, C) is assumed known and the input and output matrices B and C are both of full rank so that the regular form in equation (3.7) of the triple (A, B, C) exists. In

addition the following assumptions are considered.

Assumptions:

- A3) the system triple (A, B, C) is completely controllable and observable;
- A4) The unknown function $h(t, u, x)$ in equation (6.1a) represents nonlinearities plus model uncertainties which may be written as

$$h(t, u, x) = K\xi(t, u, x) \quad (6.2)$$

where $K \in \mathbb{R}^{n \times p}$ is a matrix such that the triple (A, K, C) is minimum phase and the matrix CK has full rank. The function $\xi(t, u, x)$ is assumed bounded and has the structure below

$$\tilde{\xi}(t, u, x) = \begin{bmatrix} \tilde{\xi}_1(t, x) \\ \tilde{\xi}_2(t, u, x) \end{bmatrix} = T^{-1}\xi(t, u, x) \quad (6.3)$$

where the functions $\tilde{\xi}_1(t, x)$ and $\tilde{\xi}_2(t, u, x)$ are the unmatched and matched parts respectively and the matrix $T \in \mathbb{R}^{p \times p}$ is an orthogonal matrix. The matrices K and T are partitioned as follows

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{matrix} \uparrow_{n-m} \\ \uparrow_m \end{matrix} \quad T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{matrix} \xleftrightarrow{p'} & \xleftrightarrow{p-p'} \end{matrix}$$

where the index p' is the rank of K_1 and should be as small as possible so that a minimum amount of unmatched uncertainty enter into the reduced order sliding dynamics. The matrix KT has the following structure

$$KT = \begin{bmatrix} K_1 T_1 & K_1 T_2 \\ K_2 T_1 & K_2 T_2 \end{bmatrix} \begin{matrix} \xleftrightarrow{p'} & \xleftrightarrow{p-p'} \\ \xleftrightarrow{p'} & \xleftrightarrow{p-p'} \end{matrix} = \begin{bmatrix} \mathcal{K}_1 & 0 \\ \mathcal{K}' & \mathcal{K}_2 \end{bmatrix} \begin{matrix} \uparrow_{n-m} \\ \uparrow_m \end{matrix} \quad (6.4)$$

The functions $\tilde{\xi}_1(t, x) \in \mathbb{R}^{p'}$ and $\tilde{\xi}_2(t, u, x) \in \mathbb{R}^{p-p'}$ are defined as follows

$$\tilde{\xi}_1(t, x) = F_1(t, x)y(t) + F_2(t, x) \quad (6.5a)$$

$$\tilde{\xi}_2(t, u, x) = G_1(t, u, x)u(t) + G_2(t, x) \quad (6.5b)$$

where the functions are bounded as

$$\begin{aligned} \|F_1(t, x)\| &< K_f; & \|F_2(t, x)\| &< K_d \\ \|G_1(t, u, x)\| &< K_g; & \|G_2(t, x)\| &< K_\alpha \end{aligned}$$

This implies that the function

$$\|\xi(t, u, x)\| \leq \rho_o(t, u, y) = K_f\|y\| + K_g\|u\| + K_a \quad (6.6)$$

where $K_a = K_d + K_\alpha$. Hence by definition $\rho_o(t, u, y)$ is a positive scalar function. The uncertainty $h(t, u, x)$ in equation (6.1a) is thus bounded and comprised of unmatched and matched components with respect to the plant. Further discussion about the appropriateness of the class of uncertainty is available in a later section.

Using the above uncertainty structure, the sections below develop a method to solve the difficulties associated in the design of sliding mode controllers for non-minimum phase systems.

6.3 Parameterisation of Dynamic Compensator

With the above assumptions it is possible to consider a nonsingular transformation [33] $\tilde{T}$ such that the system equations (6.1a) and (6.1b) are written as

$$\dot{x}_1(t) = \mathcal{A}_{11}x_1(t) + \mathcal{A}_{12}y(t) + \mathcal{B}_1u(t) \quad (6.7a)$$

$$\dot{y}(t) = \mathcal{A}_{21}x_1(t) + \mathcal{A}_{22}y(t) + \mathcal{B}_2u(t) + \tilde{K}_2\xi(t, u, x_1, y) \quad (6.7b)$$

where $x_1 \in \mathbb{R}^{(n-p)}$, $y \in \mathbb{R}^p$ and the matrix $\mathcal{A}_{11}$ is stable.

Proposition 6.1 *With the help of successive linear nonsingular transformation of the coordinates (A, B, C, K) , the observer canonical matrices represented by the coordinates $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \tilde{K})$ can be formed.*

Proof: Consider the transformation as defined in equation (4.10) in Chapter 4, rewritten as

$$T_c = \begin{bmatrix} N_c \\ C \end{bmatrix} \quad (6.8)$$

where $N_c \in \mathbb{R}^{(n-p) \times n}$ is any matrix whose rows span the null space of C . Performing the transformation, the new coordinates are partitioned as follows

$$\begin{aligned} T_c A T_c^{-1} &= \begin{bmatrix} \overset{n-p}{\leftrightarrow} & \overset{p}{\leftrightarrow} \\ \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{matrix} \uparrow_{n-p} \\ \uparrow_p \end{matrix} & T_c B &= \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \begin{matrix} \uparrow_{n-p} \\ \uparrow_p \end{matrix} \\ C T_c^{-1} &= \begin{bmatrix} 0 & I_p \end{bmatrix} & T_c K &= \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \begin{matrix} \uparrow_{n-p} \\ \uparrow_p \end{matrix} \end{aligned}$$

By assumption the matrix CK has full rank. This implies that $\text{rank}(\tilde{K}_2) = p$, hence $\tilde{K}_2$ is invertible. Setting a further transformation

$$T_b = \begin{bmatrix} I_{(n-p)} & -\tilde{K}_1 \tilde{K}_2^{-1} \\ 0 & I_p \end{bmatrix}$$

the observer canonical form is available as follows

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{21} & \tilde{A}_{12} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{22} + \mathcal{A}_{11} \tilde{K}_1 \tilde{K}_2^{-1} \\ \tilde{A}_{21} & \tilde{A}_{22} + \tilde{A}_{21} \tilde{K}_1 \tilde{K}_2^{-1} \end{bmatrix} \\ \mathcal{B} &= \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{B}_2 \\ \tilde{B}_2 \end{bmatrix} \end{aligned}$$

The matrices

$$\mathcal{C} = \begin{bmatrix} 0 & I_p \end{bmatrix} \quad \text{and} \quad \tilde{K} = \begin{bmatrix} 0 \\ \tilde{K}_2 \end{bmatrix}$$

and the total transformation may be represented by $\tilde{T} = T_b T_c$. The partitioned matrices with the above structure are used for the observer based compensator design. ■

Define a dynamic compensator

$$\dot{\tilde{x}}_1(t) = \mathcal{A}_{11} \tilde{x}_1(t) + \mathcal{A}_{12} \tilde{y}(t) - \mathcal{A}_{12} e_y(t) + \mathcal{B}_1 u(t) \quad (6.9a)$$

$$\dot{\tilde{y}}(t) = \mathcal{A}_{21} \tilde{x}_1(t) + \mathcal{A}_{22} \tilde{y}(t) - (\mathcal{A}_{22} - \mathcal{A}_{22}^s) e_y(t) + \mathcal{B}_2 u(t) - \tilde{K}_2 \nu(e_y) \quad (6.9b)$$

where the matrix $\mathcal{A}_{22}^s \in \mathbb{R}^{p \times p}$ is any stable matrix and the discontinuous parameter $\nu(e_y)$ is defined as

$$\nu(e_y) = \begin{cases} -\rho(t, u, y) \frac{\tilde{K}_2^T \tilde{P}_2 e_y(t)}{\|\tilde{K}_2^T \tilde{P}_2 e_y\|} & \text{if } \|\tilde{K}_2^T \tilde{P}_2 e_y\| \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.10)$$

where $\rho(t, u, y)$ is a positive scalar function satisfying the condition

$$\rho(t, u, y) \geq \rho_o(t, u, y) + \|\tilde{K}_2\|^{-1}\gamma_o$$

where $\tilde{P}_2 \in \mathbb{R}^{p \times p}$ is a symmetric positive definite solution of the Lyapunov equation

$$\tilde{P}_2 \mathcal{A}_{22}^s + (\mathcal{A}_{22}^s)^T \tilde{P}_2 = -\tilde{Q}_2 \quad (6.11)$$

where $\tilde{Q}_2$ is a symmetric positive definite matrix. The positive scalar constant γ_o is defined later in the section. If the state estimation errors are defined as $e_1 = x_1 - \tilde{x}_1$ and $e_y = y - \tilde{y}$ then the error dynamics can be written as follows

$$\dot{e}_1(t) = \mathcal{A}_{11}e_1(t) \quad (6.12a)$$

$$\dot{e}_y(t) = \mathcal{A}_{21}e_1(t) + \mathcal{A}_{22}^s e_y(t) + \tilde{K}_2 \xi(t, u, x_1, y) + \tilde{K}_2 \nu(e_y) \quad (6.12b)$$

The equation (6.12a) shows that the class of uncertainty $\xi(t, u, x)$ does not affect the reduced order error dynamics. It is proved in [33] that if an appropriate $\nu(e_y)$ is designed, the error outputs in equation (6.12b) are stabilised in the presence of bounded uncertainty $\xi(t, u, x_1, y)$ which also ensures a stable sliding motion on the surface

$$\mathcal{S}_e = \{e_y \in \mathbb{R}^p : e_y = Ce = 0\} \quad (6.13)$$

for a positive constant value $\gamma_o \geq \|\mathcal{A}_{21}e_1\|$. The proofs of stability and the reachability condition for the above error dynamics have been explicitly described in Edwards and Spurgeon [33] and further discussion is omitted here. With the above formulation, the nonlinear dynamic compensator can be conveniently written as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - LCe(t) - K\nu(e_y) \quad (6.14a)$$

$$\tilde{y}(t) = C\tilde{x}(t) \quad (6.14b)$$

where the gain matrix L is given as

$$L = \tilde{T}^{-1} \begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{22} - \mathcal{A}_{22}^s \end{bmatrix} \quad (6.15)$$

The discontinuous parameter $\nu(e_y)$ and the matrix K are as previously defined. The final objective of this work is to regulate the plant outputs rather than observe the states. It is thus necessary to construct the sliding mode control law which will regulate the plant.

6.4 Controller Formulation

If an appropriate discontinuous vector $\nu(e_y)$ is designed then the error in equation (6.12a) is asymptotically stable and the output error in equation (6.12b) goes to zero.

If the inverse of the matrix $\tilde{T}$ is partitioned as

$$\tilde{T}^{-1} = \begin{array}{cc} \begin{array}{c} n-p \\ \leftrightarrow \end{array} & \begin{array}{c} p \\ \leftrightarrow \end{array} \\ \left[\begin{array}{cc} \tilde{T}_1 & \tilde{T}_2 \end{array} \right] \end{array} \quad (6.16)$$

then in the original coordinate the error states

$$e(t) = \tilde{T}_1 e_1(t) \quad \text{for all } t \rightarrow [t_s, \infty] \quad (6.17)$$

and the plant states

$$\begin{aligned} x(t) &= \tilde{x}(t) + e(t) \\ &= \tilde{x}(t) + \tilde{T}_1 e_1(t) \quad \text{for all } t \rightarrow [t_s, \infty] \end{aligned} \quad (6.18)$$

where t_s is the time taken for the error system to reach a sliding mode.

The aim is to develop a control law which ensures stability and produces desired performance of the plant in equation (6.1a) while sliding on the switching surface

$$\mathcal{S} = \{\tilde{x} \in \mathbb{R}^n : s(\tilde{x}) = S\tilde{x}(t) = 0\} \quad (6.19)$$

where the matrix S is the switching surface matrix. This will be designed to ensure the stability and performance of the closed-loop system as discussed in the next section.

Rewrite equation (6.14a) as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - LCe(t) + Bu(t) - K\nu(e_y) \quad (6.20)$$

Using the sliding condition $S\dot{\tilde{x}}(t) = 0$ an equivalent control law can be formed as

$$u_{eq}(t) = -(SB)^{-1} \{SA\tilde{x}(t) - SLe_y(t) - SK\nu(e_y)\} \quad (6.21)$$

which is sufficient to attain and maintain the sliding motion on $\mathcal{S}$. This gives a controller structure similar to that employed by DeCarlo *et al.*[25]. However, some times it is necessary to manipulate the reaching time onto the switching surface. For this purpose, consider the control law as defined below

$$u(t) = -(SB)^{-1} \{G_l \tilde{x}(t) - SLe_y(t) - SK\nu(e_y)\} \quad (6.22)$$

where the linear gain matrix $G_l = [SA + \frac{\alpha}{2}S]$ and the positive scalar α is a design parameter. The effect of α has been discussed in Chapters §4 & §5. The Lemma below will prove the compensator dynamics attain a sliding mode.

Lemma 6.1 *The control law defined in equation (6.22) quadratically stabilises the system in equation (6.20) and induces a stable sliding motion on S for any positive scalar value of α .*

Proof: Consider the Lyapunov function

$$V(s) = \frac{1}{2}s^T(\tilde{x})s(\tilde{x}) \quad (6.23)$$

The reachability condition can be derived from the derivative of the Lyapunov function of equation (6.23) along the state trajectory and using the equation (6.20), gives.

$$\dot{V}(s) = \tilde{x}^T S^T S \{A\tilde{x}(t) - LCe(t) + Bu(t) - K\nu(e_y)\}$$

Inserting the control law from equation (6.22) into the above equation, gives

$$\dot{V}(s) = -\frac{\alpha}{2}\tilde{x}^T S^T S \tilde{x} \leq -\frac{\alpha}{2}\|S\tilde{x}\|^2 \quad (6.24)$$

which is always a negative scalar quantity for all positive scalar values of α . Equation (6.24) satisfies the reaching condition, i.e. $\dot{V}(s) \leq 0$; if and only if the parameter α is chosen to be a positive scalar quantity. ■

Using the equivalent control law from equation (6.21), equivalent compensator dynamics can be written as

$$\dot{\tilde{x}}(t) = [I_n - B(SB)^{-1}S] A\tilde{x}(t) + [I_n - B(SB)^{-1}S] K\nu(e_y) \quad (6.25)$$

Note that the dynamics in equation (6.25) will be completely invariant to the matched uncertainty if the matrix $K = BR$ where the matrix $R \in \mathbb{R}^{m \times p}$. In this case the invariant zeros are required to be stable. The section below describes the switching surface design which maintains the stability and performance of the plant and analyses the closed-loop dynamics of the augmented system.

6.5 The Closed-loop Analysis and Switching Surface Design

In this section the effort of using an estimated state control law will be explored. In particular, the stability of the combined closed-loop system is observed. Assume the control action in equation (6.22) is available to both the compensator and plant and also that the compensator dynamics attain and maintain a sliding motion. Partitioning the matrix S and the compensator states, the equation (6.19) can be written as

$$\begin{array}{c} \xleftrightarrow{n-m} \quad \xleftrightarrow{m} \\ \left[\begin{array}{cc} S_1 & S_2 \end{array} \right] \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = 0 \end{array} \quad (6.26)$$

Then the dynamics $\tilde{x}_2(t)$ can be written as

$$\begin{aligned} \tilde{x}_2(t) &= -S_2^{-1}S_1\tilde{x}_1(t) \\ &= -K_s\tilde{x}_1(t) \end{aligned} \quad (6.27)$$

where $\tilde{x}_1 \in \mathbb{R}^{n-m}$, $\tilde{x}_2 \in \mathbb{R}^m$ and the gain matrix $K_s = S_2^{-1}S_1 \in \mathbb{R}^{m \times (n-m)}$ is a design parameter to be discussed later in this section. The switching surface matrix S is then represented using the gain matrix K_s as

$$S = S_2 [K_s \quad I_m] \quad (6.28)$$

where the matrix S_2 does not affect the closed-loop stability, but merely scales the switching surface $\mathcal{S}$. The plant states in equation (6.18) can be represented using the equation (6.27) as

$$\begin{aligned} x(t) &= \tilde{x}(t) + \tilde{T}_1 e_1(t) \\ &= T_s \tilde{x}_1(t) + \tilde{T}_1 e_1(t) \end{aligned} \quad (6.29)$$

where $T_s = [I_{n-m} \quad -K_s^T]^T$. This shows that the behaviour of the plant dynamics depends on the reduced order compensator and error dynamics. Hence, it is useful to observe the stability of the augmented reduced order closed-loop dynamics instead of the plant dynamics. First rewrite the compensator dynamics (6.20) using an equivalent $\nu(e_y)$ from equation (6.12b) as the derivative of the output error $\dot{e}_y(t) = 0$ while in the sliding mode, and substitute the uncertainty $\xi(t, u, x_1, y)$, giving

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + Bu(t) + K\tilde{K}_2^{-1}\mathcal{A}_{21}e_1(t) + K\xi(t, u, x_1, y) \\ &= A\tilde{x}(t) + Bu(t) + K\tilde{K}_2^{-1}\mathcal{A}_{21}e_1(t) + KT\tilde{\xi}(t, u, x_1, y) \end{aligned} \quad (6.30)$$

Using the partitioned matrices A and B in equation (3.7), the matrix KT and the uncertainty structure in equations (6.4) and (6.3) respectively, the reduced order compensator sliding dynamics can be written as

$$\dot{\tilde{x}}_1(t) = A_{11}\tilde{x}_1(t) + A_{12}\tilde{x}_2(t) + K_1\tilde{K}_2^{-1}\mathcal{A}_{21}e_1(t) + \mathcal{K}_1\tilde{\xi}_1(t, x_1, y) \quad (6.31)$$

Substituting for $\tilde{x}_2(t)$ from equation (6.27), the sliding dynamics in equation (6.31) can be written as

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= (A_{11} - A_{12}K_s)\tilde{x}_1(t) + K_1\tilde{K}_2^{-1}\mathcal{A}_{21}e_1(t) + \mathcal{K}_1\tilde{\xi}_1(t, x_1, y) \\ &= A_{11}^s\tilde{x}_1(t) + A_e e_1(t) + \mathcal{K}_1\tilde{\xi}_1(t, x_1, y) \end{aligned} \quad (6.32)$$

where the matrix $A_e = K_1\tilde{K}_2^{-1}\mathcal{A}_{21}$. The closed-loop matrix $A_{11}^s = (A_{11} - A_{12}K_s)$ is stabilised using the gain matrix K_s which can be designed based on a state feedback approach. To carry out such techniques the matrix pair (A_{11}, A_{12}) must be completely controllable which holds since the matrix pair (A, B) is completely controllable. It is seen that the uncertainty $\tilde{\xi}_1(t, x_1, y)$ affects the reduced order sliding dynamics. Thus a robust environment is necessary to minimise the effect of this uncertainty on the reduced order sliding dynamics. Hence a robust technique is used during the design of the closed loop matrix A_{11}^s . One such robust method using state feedback control technique is presented in Appendix C.

The augmented reduced order closed-loop dynamics is then written using equations (6.12b) and (6.32) as follows

$$\dot{z}_1 = A_c z_1 + \Gamma \tilde{\xi}_1(t, x_1, y) \quad (6.33)$$

where the states $z_1 = [e_1^T \quad \tilde{x}_1^T]^T$ and the matrices

$$A_c = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ A_e & A_{11}^s \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ \mathcal{K}_1 \end{bmatrix} \quad (6.34)$$

The closed-loop matrix A_c is made stable by independently stabilising the closed-loop matrices $\mathcal{A}_{11}$ and A_{11}^s in equations (6.12a) and (6.32) respectively. Substituting the plant states $x(t)$ in equation (6.29), the uncertainty in equation (6.5a) may be written as

$$\tilde{\xi}_1(t, x_1, y) = F_e(t, e_1, \tilde{x}_1)e_1(t) + F_{\tilde{x}}(t, e_1, \tilde{x}_1)\tilde{x}_1(t) + \tilde{F}_2(t, e_1, \tilde{x}_1) \quad (6.35)$$

where

$$\begin{aligned} F_e(t, e_1, \tilde{x}_1) &= F_1(t, \tilde{T}_1 e_1 + T_s \tilde{x}_1) C \tilde{T}_1 \\ F_{\tilde{x}}(t, e_1, \tilde{x}_1) &= F_1(t, \tilde{T}_1 e_1 + T_s \tilde{x}_1) C T_s \\ \tilde{F}_2(t, e_1, \tilde{x}_1) &= F_2(t, \tilde{T}_1 e_1 + T_s \tilde{x}_1) \end{aligned} \quad (6.36)$$

Substituting the uncertainty structure in equation (6.35) into the equation (6.33), gives

$$\begin{aligned} \dot{z}_1 &= A_c z_1 + \Gamma_c z_1 + \Gamma \tilde{F}_2(t, e_1, \tilde{x}_1) \\ &= \hat{A}_c z_1 + \Gamma \tilde{F}_2(t, e_1, \tilde{x}_1) \end{aligned} \quad (6.37)$$

where

$$\begin{aligned} \Gamma_c &= \Gamma [F_e(t, e_1, \tilde{x}_1) \quad F_{\tilde{x}}(t, e_1, \tilde{x}_1)] \\ \hat{A}_c &= A_c + \Gamma_c \end{aligned}$$

The Lemma below determines the boundedness of these augmented reduced order dynamics.

Lemma 6.2 *If the augmented reduced order closed-loop matrix A_c is stable then the dynamics in equation (6.33) is globally uniformly ultimately bounded for all $e_1(t)$ and $\tilde{x}_1(t)$ with respect to the ellipsoid*

$$\mathcal{E}(r) = \left\{ z_1 \in \mathbb{R}^{2n-p-m} : \frac{1}{2} z_1^T P_1 z_1 \leq r \right\} \quad (6.38)$$

where

$$r = \varepsilon + 2 \frac{K_r^2}{v^2} \|P_1\|^2 \quad (6.39)$$

with $\varepsilon > 0$ defined to be small constant and

$$K_r = \sup_{\tilde{F}_2} \|P_1^{\frac{1}{2}} \Gamma \tilde{F}_2\| \quad (6.40)$$

for a Lyapunov equation

$$P_1 \hat{A}_c + (\hat{A}_c)^T P_1 \leq -v I_{n-p} \quad (6.41)$$

where $P_1 \in \mathbb{R}^{(n-p) \times (n-p)}$ is a symmetric positive definite matrix and the parameter $v > 0$.

Proof: Consider the Lyapunov function

$$V(z_1) = \frac{1}{2} z_1^T P_1 z_1 \quad (6.42)$$

which is determined from the unique solution of the Lyapunov equation

$$P_1 A_c + A_c^T P_1 + I_{2n-p-m} = 0 \quad (6.43)$$

The derivatives of equation (6.42) along the state trajectory $z_1(t)$ can be written using the uncertainty defined in equation (6.35) as

$$\begin{aligned} \dot{V}(z_1) &= \frac{1}{2} z_1^T (P_1 \hat{A}_c + (\hat{A}_c)^T P_1) z_1 + z_1^T P_1 \Gamma \tilde{F}_2(\cdot) \\ &\leq -\frac{1}{2} v \|z_1\|^2 + z_1^T P_1 \Gamma \tilde{F}_2(\cdot) \end{aligned} \quad (6.44)$$

It follows that

$$\dot{V}(z_1) \leq -v V_1(z_1) \|P_1\|^{-1} + [2V_1(z_1)]^{\frac{1}{2}} \|P_1^{\frac{1}{2}} \Gamma \tilde{F}_2\| \quad (6.45)$$

Therefore if there exists

$$V_1(z_1) > r - \varepsilon$$

where r is defined in equation (6.39), then the derivative of the Lyapunov function $V_1(z_1)$ is a negative scalar, i.e. $\dot{V}_1(z_1) < 0$. Then the motion is globally uniformly ultimately bounded¹ [88, 98] for all t . ■

In this way, a globally uniformly ultimately bounded solution of the non-minimum phase uncertain system may be available using output based sliding mode control. Some remarks and discussion regarding the application of this method, the class of uncertainty and its effect on the dynamics are given below. The subsequent section demonstrates the applicability of the proposed theory using numerical examples. The application of this theory is further discussed in Chapter §8 using a bench-mark aircraft problem.

6.6 Discussion

It is seen that an additional assumption relating to ‘the observability of the matrix pair (A, C) ’ is imposed while the assumption that the ‘rank of CB is equal to the number of

¹The definition of globally uniformly ultimately bounded motion is given in Appendix B

inputs' is relaxed. However the case where the rank of CB equals to zero may not be solved using this method. It is further assumed that the triple (A, K, C) is minimum phase and the rank of CK is equal to the number of outputs. Essentially this does not differ from the original assumptions on (A, B, C) except here the design triple has been modified. Hence the uncertainty is also matched with respect to the linear design triple (A, K, C) . Alternatively it can be said that the uncertainty is acting through the matrix K instead of the matrix B . This gives the appropriateness of the uncertainty structure used in this chapter and the matched uncertainty considered elsewhere in this thesis. If the uncertainty is considered as $B\xi(t, u, x)$ then the uncertainty $\xi(t, u, x)$ will appear in the reduced order error dynamics as well as the augmented reduced order dynamics. Hence a strong boundary is necessary for matched uncertainty where the plant can easily tolerate this uncertainty. This gives more conservative analysis. However if the uncertainty is considered as $K\xi(t, u, x)$ then the analysis shows that the strong boundary is necessary only for the unmatched uncertainty and is less restrictive to the matched uncertainty for a particular class of uncertainty structure. Moreover, from the structure of the matrix KT , it is seen that some of the unmatched uncertainty may appear in the input channel where the effect can be easily tolerated. This leads to well known results in output feedback sliding mode control [33]. This gives wider classes of uncertainty and is different in analysis but is not different in philosophy.

The selection of the matrix K is an important issue which must be discussed as this impacts on the uncertainty channels. In many practical examples, such as electrical and mechanical systems where the uncertainty structure is known, an uncertainty distribution matrix K can be obtained and if that matrix satisfies the necessary assumptions then it can be used for the compensator design. Alternatively, if there is no information about the uncertainty distribution or the requirements are not satisfied, it can be considered to be any matrix such that the observer canonical matrix $\mathcal{A}_{11}$ is stable which will be demonstrated in the examples below. In equation (6.25) of Section §6.4, it is seen that the system is completely invariant to the matched uncertainty if the matrix $K = BR$ and this means the system triple (A, B, C) minimum phase. In practice for a non-minimum phase system it is difficult to achieve a $K = BR$, hence this gives a contribution of the control action and the matched uncertainty into the reduced order

sliding dynamics which spoils the novel inherent properties of the sliding mode technique. However this effect may be minimised in the case where K is a design parameter. This will lead to a multi-objective numerical problem [10, 11, 12, 13, 83, 84] such that a matrix K close to the matrix B is sought, i.e. $\|B_1\|$ is minimised which will give minimum contribution of the unmatched uncertainty into the reduced order dynamics. In Section §6.2 it is mentioned that the size of p' should be as small as possible. This also reflects that the matrix K may be chosen as close as possible to the matrix B while simultaneously rendering the triple (A, K, C) minimum phase. One such method will be summarised in the Appendix C. The next section demonstrates the technique with examples.

6.7 Numerical Examples

The examples below will demonstrate the theory of this chapter. For the ease of understanding first consider an example of a square system containing two unstable transmission zeros. The second is a non-square system.

6.7.1 Example 1

Consider the example of Diong and Medanic [28] which contains two unstable transmission zeros at the origin and the system has equal numbers of inputs and outputs. The plant is completely controllable and observable and has poles at $\{\pm 1, -2, -3\}$ and zeros at $\{\pm i\}$, where i represents the square root of -1 . Hence it is both open loop unstable and a non-minimum phase system. Static output feedback sliding mode control is not possible but the dynamic output feedback sliding mode design method of this chapter may be applied. The system also satisfies the assumption of $\text{rank}(CB)$ equals to the numbers of inputs although it is not necessary. The system triple (A, B, C) is transformed into the regular form of equation (3.7) and the partitioned matrices are obtained as

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -5 & -5 \\ 1 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

First consider the design of the dynamic compensator parameters. In this case the matrix K is obtained so that the closed loop matrix $\mathcal{A}_{11} = (\tilde{A}_{11} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{21})$ is stable as in Proposition 6.1 which gives

$$K = \begin{bmatrix} -5.6867 & 0.1873 \\ 1.6894 & -0.4318 \\ 0 & -1.0000 \\ -2.6894 & 0.4318 \end{bmatrix}$$

Here p' is equal to 2, which means more uncertainty will appear on the reduced order dynamics. However, this was found to be the minimum requirement to overcome the non-minimum phase problem and to get a robust controller with rapid decay of the error dynamics. The transformation $\tilde{T}$ is obtained as

$$\tilde{T} = \begin{bmatrix} 0.7071 & -6.7105 & 0.5643 & -5.7105 \\ 0.7071 & -1.3317 & -0.2994 & -2.3317 \\ 0 & -1.0000 & 0 & -1.0000 \\ 0 & 0 & -1.0000 & 0 \end{bmatrix}$$

which gives the observer canonical form and the partitioned matrices as

$$\begin{aligned} \mathcal{A}_{11} &= \begin{bmatrix} -5.5000 & 0 \\ 0.0 & -4.0000 \end{bmatrix}, \quad \mathcal{A}_{12} = \begin{bmatrix} 33.5221 & 5.4284 \\ 7.1228 & 2.0323 \end{bmatrix} \\ \mathcal{A}_{21} &= \begin{bmatrix} -0.7071 & -0.7071 \\ 1.9645 & -9.0355 \end{bmatrix}, \quad \mathcal{A}_{22} = \begin{bmatrix} 5.6867 & 0.8127 \\ 4.8497 & -1.1867 \end{bmatrix} \end{aligned}$$

where the matrix $\mathcal{A}_{11}$ is verified as stable and the poles are at $\{-5.500, -4.00\}$ and the matrix $\tilde{K}_2$ is the identity. The matrix $\mathcal{A}_{11}$ is a normal matrix, and hence achieves high insensitivity properties. The stable matrix $\mathcal{A}_{22}^s$ in equation (6.9b) is taken to be

$$\mathcal{A}_{22}^s = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

which gives the Lyapunov solution of equation (6.11) as

$$\tilde{P}_2 = \begin{bmatrix} 0.50 & 0 \\ 0 & 0.25 \end{bmatrix}$$

for a symmetric positive definite matrix $\tilde{Q}_2 = I_2$. Finally the gain matrix L is calculated using equation (6.15) as

$$L = \begin{bmatrix} -8.3761 & 0.8065 \\ -3.9972 & -0.6763 \\ -4.8497 & -0.8133 \\ -2.6894 & -0.1364 \end{bmatrix}$$

The second stage of the design is to find an appropriate switching surface matrix S as in equation (6.28), based on estimated states which stabilises the matrix pair (A_{11}, A_{12}) . From the reduced order system in equation (6.32) the gain matrix K_s is obtained as

$$K_s = \begin{bmatrix} 0 & 0 \\ 7 & 12 \end{bmatrix}$$

which gives the switching surface matrix of equation (6.28) as

$$S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 7 & 12 & 0 & 1 \end{bmatrix}$$

where the matrix S_2 is considered as the identity. Finally the control law is constructed

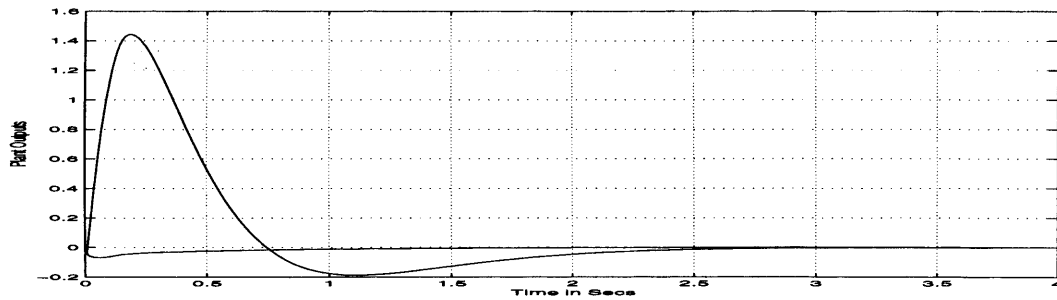


Figure 6.1: Time Response of Output Vector in Example 1

as in equation (6.22). The simulation results are obtained using an arbitrary perturbed plant representation where the uncertainties are matched to the plant input channels. These are given as

$$A_p = \begin{bmatrix} 0 & 0 & 0 & 1.0000 \\ 1.0000 & 0 & 0 & 0 \\ 5.0716 & 7.3420 & -5.0716 & -5.0000 \\ 0 & 0 & 1.0143 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.9 & 1.25 \\ 0 & 1 \end{bmatrix}$$

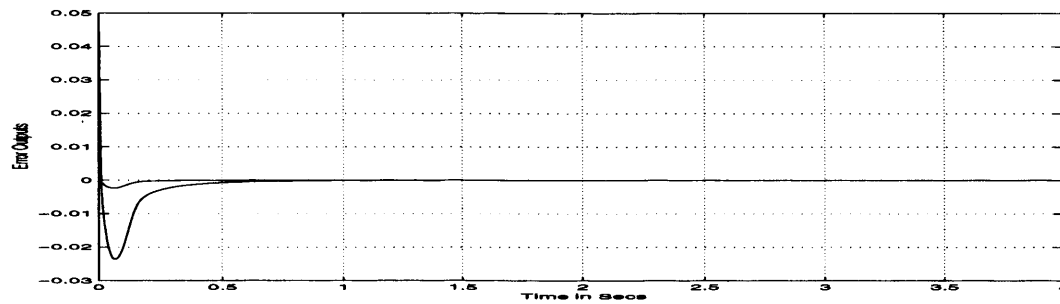


Figure 6.2: Time Response of Error Output Vector in Example 1

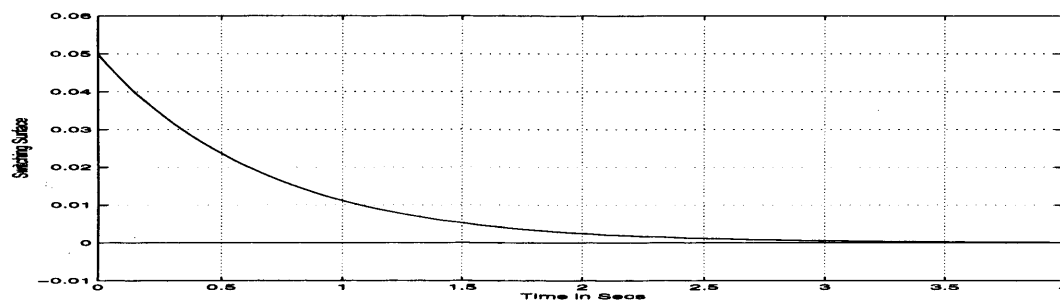


Figure 6.3: Sliding Surface Attainment of the Compensator Dynamic in Example 1

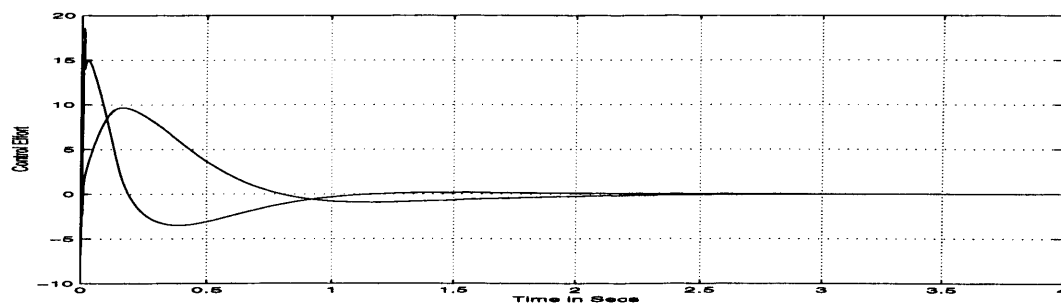


Figure 6.4: Actuator Demand in Example 1

The controller parameter $\alpha = 2.50$ and the positive scalar constants $K_f = 0.0$, $K_g = 0.50$ and $K_\gamma = K_a + \gamma_o = 2.5$ are chosen to control the uncertainty. The simulation results show reasonable response of the perturbed plant. The parameter $\delta = 0.001$ is added to the denominator of the discontinuous component of the control law during implementation to avoid chattering. The plant outputs in Figure 6.1 have gone to zero in 3.0 seconds and the error in Figure 6.2 has decayed rapidly and attained zero at 0.75 seconds approximately. The sliding mode is attained within 3.5 seconds as shown in Figure 6.3. The control action in Figure 6.4 shows that a reasonable control effort is applied.

6.7.2 Example 2

Consider a system configuration which is non-square and contains an unstable invariant zero. The reduced order dynamics cannot be stabilised by static output feedback due to the presence of the unstable invariant zero. The plant triple is presented in the regular form of equation (3.7) for ease of exposition

$$A = \begin{bmatrix} -0.3000 & 0 & 0.2000 & 4.0000 & 3.0000 \\ -0.1000 & 1.2000 & 1.0000 & 5.0000 & 2.0000 \\ 1.0000 & 0 & -0.2000 & -3.0000 & -1.0000 \\ 0.0005 & 5.0000 & -0.0329 & -0.0036 & -0.0161 \\ 0.0021 & 2.5000 & -0.0302 & 0.0064 & 0.0285 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Since the unstable invariant zero 1.2 will appear in the matrix A_{11} if the system is transformed into the canonical form of equations (4.6)-(4.9), it is not possible to apply static output feedback results. The dynamic output feedback sliding mode method may be used in this case, since the matrix pair (A, C) is observable. For the purpose of parameterisation of the compensator dynamics, define the uncertainty distribution

matrix

$$K = \begin{bmatrix} -1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.34 \\ 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & -1.00 \\ 0 & 0 & -1.00 & 0 \end{bmatrix}$$

which stabilises the closed loop matrix $\mathcal{A}_{11} = (\tilde{A}_{11} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{21})$ as in Proposition 6.1 where the matrix $\tilde{K}_2$ is chosen as the identity matrix. The matrix K is close to the matrix B in the sense that $\|\mathcal{B}_1\|$ is equal to 1.34 and thus small. The optimisation method described in Appendix C is used to obtain the matrix K . This shows that the size of $p' = 3$. Naturally higher amount of uncertainty will appear in the reduced order sliding dynamics. However it is considered to be tolerable due to minimisation of the norm of $\mathcal{B}_1$. Hence the inherent robustness properties of the sliding mode technique are maximised. The transformation matrix $\tilde{T}$ is constructed as

$$\tilde{T} = \begin{bmatrix} 0 & -1.00 & 0 & 1.34 & 0 \\ -1.00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.00 \\ 0 & 0 & 0 & -1.00 & 0 \end{bmatrix}$$

The observer canonical matrices are partitioned as defined in Proposition 6.1, giving the matrices

$$\begin{aligned} \mathcal{A}_{11} &= \begin{bmatrix} -5.5 \end{bmatrix}, \quad \mathcal{A}_{12} = \begin{bmatrix} -0.1007 & -1.0441 & 2.0216 & -2.3652 \end{bmatrix} \\ \mathcal{A}_{21} &= \begin{bmatrix} 0 \\ 0 \\ 2.5 \\ 5.0 \end{bmatrix}, \quad \mathcal{A}_{22} = \begin{bmatrix} -0.3000 & -0.2000 & 3.0000 & 4.0000 \\ -1.0000 & -0.2000 & 1.0000 & 3.0000 \\ 0.0021 & 0.0302 & 0.0285 & 3.3564 \\ 0.0005 & 0.0329 & -0.0161 & 6.6964 \end{bmatrix} \end{aligned}$$

where the reduced order system matrix $\mathcal{A}_{11}$ is stable. The reduced order error dynamics becomes a scalar problem. The other important stable matrix of the compensator dynamics in equation (6.9b) is chosen as $\mathcal{A}_{22}^s = -5 \times I_4$ which produces the Lyapunov

solution of the equation (6.11) as $\tilde{P}_2 = 0.10 \times I_4$ for the symmetric positive definite matrix $\tilde{Q}_2 = I_4$. The compensator gain matrix L in equation (6.15) is given as

$$L = \begin{bmatrix} -4.7000 & 0.2000 & -3.0000 & -4.0000 \\ 0.1000 & 1.0000 & -2.0000 & -13.3080 \\ -1.0000 & 4.8000 & 1.0000 & 3.0000 \\ -0.0005 & -0.0329 & 0.0161 & -11.6964 \\ -0.0021 & -0.0302 & -5.0285 & -3.3564 \end{bmatrix}$$

The above design ensures the stability of the error dynamics but it does not guarantee the stability and performance of the plant. For the case of a regulator problem, consider a switching surface $\mathcal{S}$ as in equation (6.19) such that $S\tilde{x} = 0$ for some matrix S and a control law $u(t)$ in equation (6.22) exists. A reduced order sliding dynamics is formed as in equation (6.32). The reduced order system matrices are identified as

$$A_{11} = \begin{bmatrix} -0.30 & 0 & 0.20 \\ -0.10 & 1.20 & 1.00 \\ 1.00 & 0 & -0.20 \end{bmatrix} \quad \text{and} \quad A_{12} = \begin{bmatrix} 4 & 3 \\ 5 & 2 \\ -3 & -1 \end{bmatrix}$$

The reduced order closed-loop gain matrix K_s is then calculated using the controllable pair (A_{11}, A_{12}) . This is given as

$$K_s = \begin{bmatrix} -1.3994 & -0.3559 & -2.7414 \\ 3.5881 & 4.6999 & 8.8111 \end{bmatrix}$$

which places the poles of the closed-loop matrix A_{11}^s at $[-4.5 \quad -4.0 \quad -3.0]$. The matrix S_2 does not affect the properties of the switching surface $\mathcal{S}$ and it is taken as the identity matrix which produces the matrix in equation (6.28) as

$$S = \begin{bmatrix} -1.3994 & -0.3559 & -2.7414 & 1.0000 & 0 \\ 3.5881 & 4.6999 & 8.8111 & 0 & 1.0000 \end{bmatrix}$$

The control law defined in equation (6.22) is used. The simulation results are obtained on a linearly perturbed plant model. The controller design is based on the nominal plant presented above and implemented on a significantly perturbed system, which induces matched uncertainty into the matrices A and B in order to illustrate the technique.

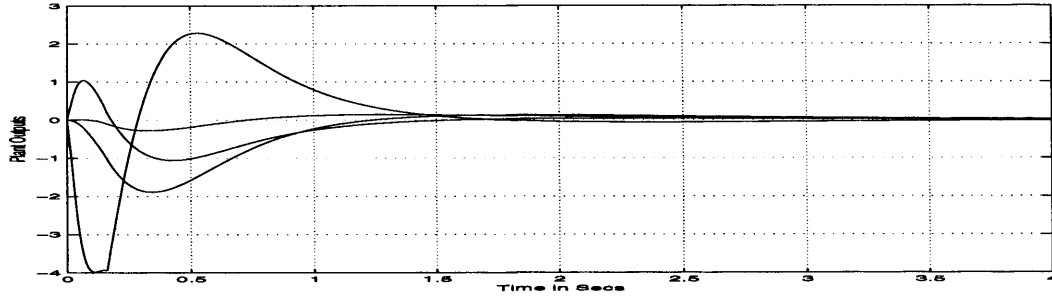


Figure 6.5: Time Response of Output Vector in Example 2

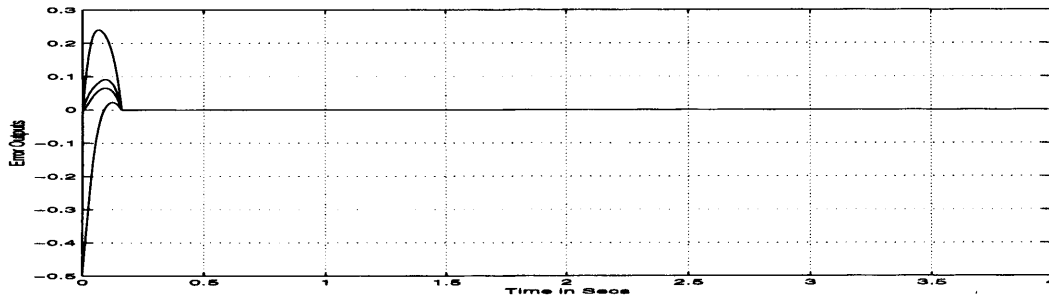


Figure 6.6: Time Response of Error Output Vector in Example 2

The perturbed plant matrices are given by,

$$A_p = \begin{bmatrix} 1.2000 & -0.1000 & -1.0000 & 2.0000 & 5.0000 \\ 0 & -0.3000 & -0.2000 & 3.0000 & 4.0000 \\ 0 & -1.0000 & -0.2000 & 1.0000 & 3.0000 \\ 3.5810 & 0.0030 & 0.0433 & 0.0408 & 0.0092 \\ 7.1620 & 0.0007 & 0.0471 & -0.0231 & -0.0052 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.025 & 0 \\ 0 & 1.025 \end{bmatrix}$$

and the output matrix C is unperturbed. In order to control the matched uncertainty, the norm bound parameters in Section §6.2 are chosen as $K_f = 0.0$, $K_g = 0.50$ and $K_\gamma = K_a + \gamma_o = 10.0$. A value of $\alpha = 2.50$ is sufficient to bring the outputs onto the switching surface in reasonable time. Due to the discontinuous control action in equation (6.10), the responses show some chattering. To eliminate this effect a small parameter $\delta = 0.0001$ is added to the denominator in equation (6.10) to smooth the discontinuity. The simulation results are presented. Figures 6.5 & 6.6 represent the perturbed plant and error output responses respectively. It demonstrates the efficiency of the compensator dynamics. The switching surface response of the perturbed plant is shown in Figure 6.7. A sliding mode is attained within 3.75 seconds approximately.

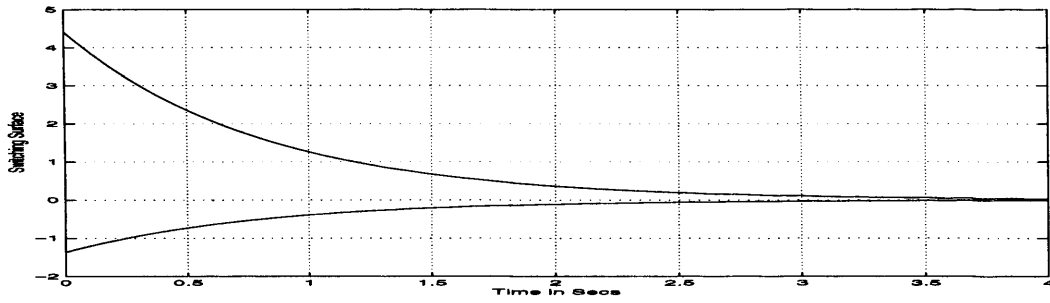


Figure 6.7: Sliding Surface Attainment in Example 2

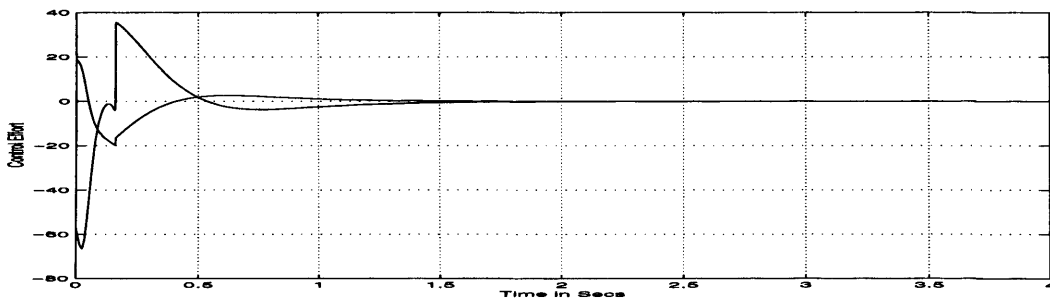


Figure 6.8: Actuator Demand in Example 2

The control action is given in Figure 6.8; this shows comparatively higher control effort is applied, however this is only a numerical example to demonstrate the technique.

6.8 Summary

This chapter presents a framework for sliding mode output feedback control design for the case of multivariable linear time invariant uncertain systems which may possess unstable invariant zeros. The switching surface design is equivalent to an output feedback design problem for both the static and dynamic output feedback sliding mode control as described in previous chapters. The latter is used for systems where the sliding mode design triple lacks freedom in design due to not fulfilling the output feedback design criteria. However it does not tolerate the unstable invariant zeros. It follows that the sliding dynamics become unstable if the system possesses any unstable invariant zero. Here, a method is described which tolerates the presence of unstable invariant zeros. It is shown that a compensator may be designed in the VSCOF framework where with appropriate choice of a gain matrix, the modified system triple is minimum phase. Further

it is seen that while the modified triple has stable transmission zeros, the sliding dynamics may lose its inherent invariance properties. However, it is completely invariant to a particular class of uncertainty and in addition an appropriate robust design may be performed so that sensitivity decreases. A robust approach may be used for the design of the reduced order closed-loop matrices for both the plant and compensator dynamics. The method presented here may also be utilised for systems for which the $\text{rank}(CB)$ is greater than zero. This relaxes the previous assumption that $\text{rank}(CB)$ must be equal to the number of inputs. However the method increases the total dynamics. The closed-loop configuration has shown boundedness instead of absolute stability. The controller design is straightforward. The controller gain can be varied to influence the reaching time to the switching surface. The numerical examples have shown the effectiveness of the technique. The proposed controllers guarantee the attainment of a sliding mode despite the presence of uncertainty. It can be concluded that although there are some inherent difficulties in the design of sliding mode controllers for non-minimum phase systems, some bounded results between error and plant may be achieved using output feedback sliding mode design.

Chapter 7

Output Feedback Sliding Mode Control of Rotorcraft

7.1 Introduction

This chapter considers the use of output feedback sliding mode controller design for rotorcraft dynamics. The dynamics represent a highly nonlinear multivariable system which has two invariant zeros and model uncertainty. It has open loop unstable poles, exhibits high levels of cross coupling among the states and variations in handling characteristics with flight condition. The use of advanced automatic controllers for stabilisation and reduction of cross coupling is a challenging task. This chapter is presented to demonstrate the ability of the output feedback sliding mode controllers developed in Chapter §4 to control the fully nonlinear helicopter model. It has been shown that the switching surface design problem is formulated as a static output feedback dynamic problem for a particular subsystem design triple and the VSC design methodology itself can tolerate the matched uncertainty. In addition the robust design procedure presented in Section §4.6 is also used to solve this static output feedback problem to minimise the effects of the unmatched uncertainty which will affect the reduced order sliding motion. A model following control configuration is used to perfect the tracking of outputs. The ideal model is designed based on an H^∞ one degree of freedom technique such that internal cross coupling of the states are minimised. A controller is synthesised to attain and maintain the sliding mode in the presence of unmatched uncertainty. Most of the previous work for designing a rotorcraft controller using the sliding mode method employs

full state feedback. If the state vector cannot be measured directly, an estimator is used [47, 48]. A state feedback VSC is more complex to implement and increases dynamics. The alternative, an output feedback based VSC, is presented here and implemented on a fully nonlinear rotorcraft simulation. The rotorcraft is the DRA, Bedford's Rationalised Helicopter Model (RHM), representing a 'Lynx-like' high performance military helicopter [59]. The ability of an output feedback sliding mode controller to guarantee quadratic stability and force the output error onto the switching surface is demonstrated using this fully nonlinear RHM model.

The outline of this chapter is as follows: The mathematical model of the helicopter and the source of uncertainty are presented in Section §7.2. Section 7.3 describes the VSC model following output tracking, the design objectives and the ideal model design. The switching surface, robustness measurements and controller design are described in Sections §7.4 & §7.5. The simulation results of the fully nonlinear helicopter model are presented in Section §7.6. The summary of the chapter is given in Section §7.7.

7.2 Mathematical Model of Helicopter and Uncertainties

A rigid body nonlinear rotorcraft may be mathematically modelled as a linear time invariant state space model with some uncertainties in the system of equations (4.1a)-(4.1b) as defined in Section §4.2. The helicopter model is linearised at different operating points in the flight envelope to produce a number of (A, B, C) triples. The linearised data obtained at hover is considered as the nominal model for the purpose of design and it satisfies the assumptions required to develop a static output feedback sliding mode controller.

The most common source of uncertainties in this system are due to linearisation at various operating points, change of weather, rotor dynamics, actuator dynamics, error in modelling and residual terms of higher order. It is assumed that the unknown function $h(t, u, x) : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ which represents the system nonlinearities plus all model uncertainties in the system satisfies the norm bound condition in Section §4.2. In Chapter §4 it is demonstrated that in the sliding mode approach, the reduced order sliding dynamics mostly affected by the unmatched uncertainty and it is possible to

minimise the effect using robust design of the closed-loop system. It is shown that it may be sufficient to use the matched uncertainty and a robust sliding mode design approach. The matched uncertainty in Section §4.2 can then conveniently be written as

$$g(t, u, x) = B\xi(t, u, x)$$

where the unknown function $\xi(t, u, x) : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\|\xi(t, u, x)\| < K_g \|u\| + K_\alpha$$

for some known positive constants $0 \leq K_g < \frac{1}{k}$ and K_α . The constant parameters K_g, K_α and k are defined in Chapter §4. This gives the nonlinear equations (4.1a)-(4.1b) in the form as

$$\dot{x}(t) = Ax(t) + Bu(t) + B\xi(t, u, x) \quad (7.1a)$$

$$y(t) = Cx(t) \quad (7.1b)$$

The states of the linearised helicopter model are given in Table 7.1 and the output

No.	State	Description
1	θ	Pitch Attitude
2	ϕ	Roll Attitude
3	p	Roll Rate
4	q	Pitch Rate
5	r	Yaw Rate
6	u	Forward Velocity
7	v	Lateral Velocity
8	w	Vertical Velocity

Table 7.1: State Vector

vector representing information available to the control law is given in Table 7.2. For this application, any control scheme must decouple the inherent internal cross coupling of the states; that is each input should be tracked by a single output without affecting the other outputs. This helicopter problem has 4 inputs and 6 measured outputs. It is

No.	Con. O/P	Description	Pilot I/P	Units
1	$\dot{H}(y1)$	Heave Vel.	Coll.	Ft/s
2	$\Theta(y2)$	Pitch Atti	Long.	Rad.
3	$\Phi(y3)$	Roll Atti	Latt.	Rad
4	$\dot{\Psi}(y4)$	Head Rate	Pedal	Rad/s
5	$p(y5)$	Roll Rate	—	Rad/s
6	$q(y6)$	Pitch Rate	—	Rad/s

Table 7.2: Output Vector

therefore necessary to define a ‘Controlled Variable’ subset of four outputs which are to be independently controlled by the inputs. These are denoted in Table 7.2 in terms of ‘Pilot I/P’.

7.3 Model Following Control in VSC Output Feedback

Model following control is a technique whereby the plant is required to follow the dynamic behaviour of a specified model. Linear model-following control (LMFC) is an efficient control method that avoids the difficulties of specifying a performance index [124] and also many other problems like de-coupling of states, which are usually encountered in the application of optimal control to multivariable control systems. The model that specifies the design objectives is part of the plant dynamics. The LMFC systems are not adequate when there are large parameter variations or disturbances. This has led to the development of so-called adaptive model following control systems (AMFC) [73, 124]. Most of the work available in this field using sliding mode ideas assumes that all plant states are available; for example, the work of Ambrosino *et al.*[3], Spurgeon and Patton [99] and Zinober *et al.*[120]. Work using only input - output information has appeared within the adaptive variable structure control literature. The method was utilised by Hsu *et al.*[60, 61] and also the theory was used in nonlinear and time varying control systems by Slotine and Sastry [95] and Balestrino *et al.*[8]. In this section, a method of model following is considered which uses only output information. The in-

tention is to formulate a control law using the model states and the plant outputs which forces the plant outputs to follow the exact path of the model outputs. To effect the de-coupling and for performance specification, an ideal model response characteristic is defined. The output feedback variable structure scheme is then employed to ensure the output vector of the plant faithfully follows that of the ideal model response. The error between the ideal model and plant will be controlled using sliding mode control.

7.3.1 Output Tracking and Design Objectives

Model following is a method of control in which the plant should behave like an ideal model which is also called the reference model. The control system design problem is thus that of determining a feedback scheme whereby the output vector of the plant faithfully follows that of the ideal model. Mathematically the scheme can be represented as described below. Consider the ideal model response

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \quad (7.2a)$$

$$y_m(t) = C x_m(t) \quad (7.2b)$$

where $x_m \in \mathbb{R}^n$ is the state vector and $r \in \mathbb{R}^m$ and $y_m \in \mathbb{R}^p$ are the reference signal and output vector respectively of the ideal model. An appropriate gain matrix L_x can be designed based on full state feedback control techniques to meet desired performance requirements, ensure the required levels of de-coupling are attained and that the model matrix $A_m = A + B L_x$ is stable. The input matrix B_m is designed so that each reference input affects the corresponding output correctly. One way to choose the gain matrix L_r is such that the transfer function from the reference input to the model output

$$G(s) = C_m(sI - A_m)^{-1} B L_r$$

has unity D.C gain, where C_m is the output control matrix of dimension $m \times n$ to be controlled independently by the inputs. Therefore, at steady state, the output from an ideal model tracks the reference signal perfectly. This gives

$$L_r = [C_m(-A_m)^{-1} B]^{-1}$$

An alternative method of designing the gain matrix L_r is using the theory of O' Brien and Broussard [81]. Then the model input matrix $B_m = B L_r$ will ensure steady-state

output tracking. This structure of L_x and B_m will ensure the conditions for perfect model following of Chan [16] and Erzberger[44] are satisfied, i.e. $\text{rank}[B : BL_x] = \text{rank}[B] = \text{rank}[B : B_m]$.

The error dynamics can then be written as

$$\dot{e}(t) = Ae(t) + Bu_e(t) - B\xi(t, u, x) \quad (7.3a)$$

$$e_y(t) = Ce(t) \quad (7.3b)$$

where the error states $e(t) = x_m(t) - x(t)$ and $u_e(t)$ is the proposed control law which depends only on the output error $e_y(t)$ and the ideal model response and is given by

$$u_e(t) = L_x x_m(t) + L_r r(t) - u(t) \quad (7.4)$$

The plant control action $u(t)$ can be derived from equation (7.4). The uncertain function $\xi(t, u, x)$ still satisfies the matching condition. For the purposes of ideal model design, an eigenvalue-eigenvector assignment or an H_∞ technique may be appropriate since these produce good de-coupling properties. One such design method and associated implementation issues are considered in the next section.

To develop the variable structure control scheme, first consider a switching surface based on error output measurements which will generate a robust stable sliding motion on the surface

$$\mathcal{S}_e = \{e_y \in \mathbb{R}^p : s(e_y) = Fe_y(t) = 0\} \quad (7.5)$$

for a selected matrix $F \in \mathbb{R}^{m \times p}$ and also stabilise the uncertain error system defined in equation (7.3a). Once the switching surface is designed, the second stage is then to design a controller which can induce a sliding motion on the error switching surface $\mathcal{S}_e$. The proposed variable structure control component in equation (7.4) is represented by

$$u_e(t) = Ge_y(t) - \nu(e_y) \quad (7.6)$$

where G is a fixed gain matrix and the discontinuity vector $\nu(e_y)$ is dependent on error outputs $e_y(t)$ only. The key problem is to design an ideal model, the switching surface matrix F and the gain matrix G , so that the closed-loop system dynamics are both stable and robust to the uncertainty.

7.3.2 Ideal Model Design based on an H_∞ 1-DOF Design

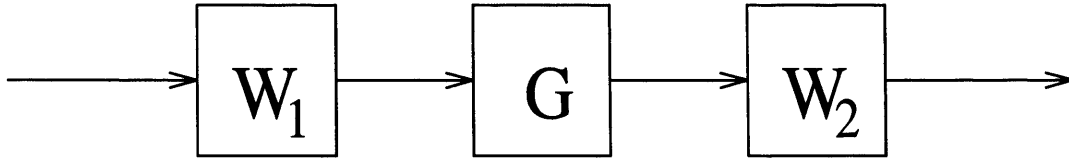
The ideal model in equation (7.2a) may be designed using various methods. A robust approach is preferable for a good performance of the closed-loop sliding mode controller so that the ideal model response is minimally affected by noise. Foster [47] has presented different methodologies for designing the ideal model response including eigenstructure assignment and H_∞ design methods. The eigenstructure methodology has been widely applied to the helicopter system [36, 49, 50, 51] and many other aerospace controller design problems [45, 62]. The theory of eigenstructure assignment for model following control is well established in the literature and further discussion is omitted here. Further details of the method and its use with a sliding mode state feedback controller is available in Foster [47]. The H_∞ design technique is well established in theory but its application with model following sliding mode control based upon output feedback is not addressed in the literature. In this section the ideal model design based on an H_∞ one degree of freedom technique and then implemented with a sliding mode output feedback controller will be discussed.

The loop shaping design procedure (LSDP) described in MacFarlane and Glover [79] is used to obtain performance/robustness trade-offs, with a robust stabilisation technique as a means of guaranteeing the closed-loop system stability. For application of the LSDP the plant is weighted which produces a shaped system which has higher order than the plant. Hence for design of the sliding mode controller it is necessary to use the shaped plant. From a mathematical point of view, to demonstrate the ideal model matching technique, it is useful that the controller produced by the LSDP can be separated into a state feedback and a state observer (Kalman filter structure [65]). The LSDP may be summarised in the following stages.

1. The shaped plant model, G_s : The singular values of the nominal plant are shaped using filters W_1 and W_2 to give a desired open-loop frequency response. The nominal plant and the shaping weights are combined in Figure 7.1 to form the shaped plant

$$G_s = \begin{bmatrix} A_s & B_s \\ C_s & 0 \end{bmatrix} = W_2 G W_1$$

where G is the nominal plant.

Figure 7.1: The shaped plant model G_s

2. Design the robust H_∞ stabilising controller, K_∞ : The robust H_∞ loop shaping techniques using normalised coprime factors [79] are now considered to synthesise the robust H_∞ stabilising controller. It is well known that the shaped plant system G_s has a normalised left and right coprime factorisation

$$G_s = \tilde{M}^{-1}\tilde{N} = NM^{-1} \quad (7.7)$$

that satisfy

$$\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I, \quad N^*N + M^*M = I \quad (7.8)$$

The right and left coprime factor realisations are

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A_s + B_s F & B_s S^{-\frac{1}{2}} \\ \hline F & S^{-\frac{1}{2}} \\ C_s + D_s F & D_s S^{-\frac{1}{2}} \end{array} \right] \quad (7.9a)$$

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A_s + H C_s & H & B_s + H D_s \\ \hline R^{-\frac{1}{2}} C_s & R^{-\frac{1}{2}} & R^{-\frac{1}{2}} D_s \end{array} \right] \quad (7.9b)$$

where

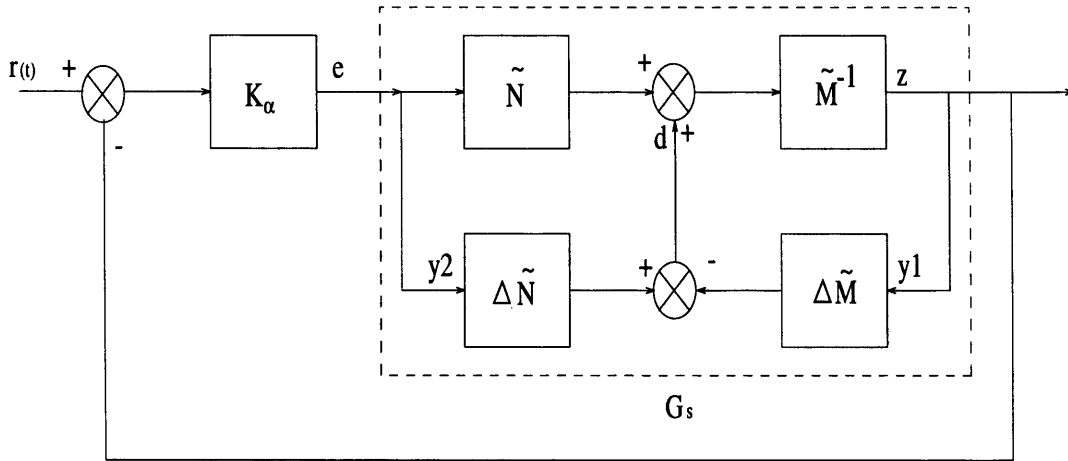
$$\begin{aligned} F &= -S^{-1}(D_s^* C_s + B_s^* X), \quad H = -(B_s D_s^* + Z C_s^*) R^{-1} \\ S &= I + D_s^* D_s, \quad R = I + D_s D_s^* \end{aligned}$$

where $[A_s, B_s, C_s, D_s]$ is the state space realisation of the shaped plant G_s and the star $*$ represents the transpose of the matrix. The matrix X is the solution to the control Riccati equation and Z is the solution to the filter Riccati equation, given by

$$\Phi^* X + X \Phi - X B_s S^{-1} B_s^* X + C_s^* R C_s = 0 \quad (7.10a)$$

$$\Phi Z + Z \Phi^* - Z C_s^* R^{-1} C_s Z + B_s S B_s^* = 0 \quad (7.10b)$$

where $\Phi = A_s - B_s S^{-1} D_s^* C_s = A_s - B_s D_s^* R^{-1} C_s$.

Figure 7.2: Perturbed left coprime factored of plant G_s in closed-loop feedback system

Let the plant G_s be perturbed due to the uncertainty as shown in the feedback closed-loop system of Figure 7.2 and represented by

$$G_s = (\tilde{M} + \Delta_{\tilde{M}})^{-1}(\tilde{N} + \Delta_{\tilde{N}}) \quad (7.11)$$

where $\Delta_{\tilde{M}}$ and $\Delta_{\tilde{N}}$ are left coprime factor perturbations or uncertainty. Assume that the H_∞ norm of the perturbations is bounded by

$$\left\| \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \right\|_\infty < \epsilon, \quad \epsilon > 0 \quad (7.12)$$

Consider the feedback control system in Figure 7.2. It can be shown that

$$\begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} (I + G_s K_\infty)^{-1} \tilde{M}^{-1} \\ K_\infty (I + G_s K_\infty)^{-1} \tilde{M}^{-1} \end{bmatrix} [d] \quad (7.13)$$

and that

$$d = - \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} \quad (7.14)$$

As the norms of the perturbations are bounded by ϵ as given in equation (7.12) then a stabilising controller must be found that satisfies

$$\|T_{yd}\|_\infty = \left\| \begin{bmatrix} (I + G_s K_\infty)^{-1} \tilde{M}^{-1} \\ K_\infty (I + G_s K_\infty)^{-1} \tilde{M}^{-1} \end{bmatrix} \right\|_\infty \quad (7.15a)$$

$$= \left\| \begin{bmatrix} S_o \tilde{M}^{-1} \\ K_\infty S_o \tilde{M}^{-1} \end{bmatrix} \right\|_\infty < \epsilon^{-1} \quad (7.15b)$$

where $S_o = (I + G_s K_\infty)^{-1}$. It can be seen that this is a special case of the mixed sensitivity minimisation problem. The standard problem of the mixed sensitivity system could be constructed and solved iteratively, but in the special case that $\gamma = \epsilon^{-1}$ then the optimum central controller can be calculated exactly and without iteration as

$$\gamma_o = \sqrt{1 + \lambda_{\max}(ZX)} \quad (7.16)$$

The central controller can be obtained from the set of all stabilising controllers

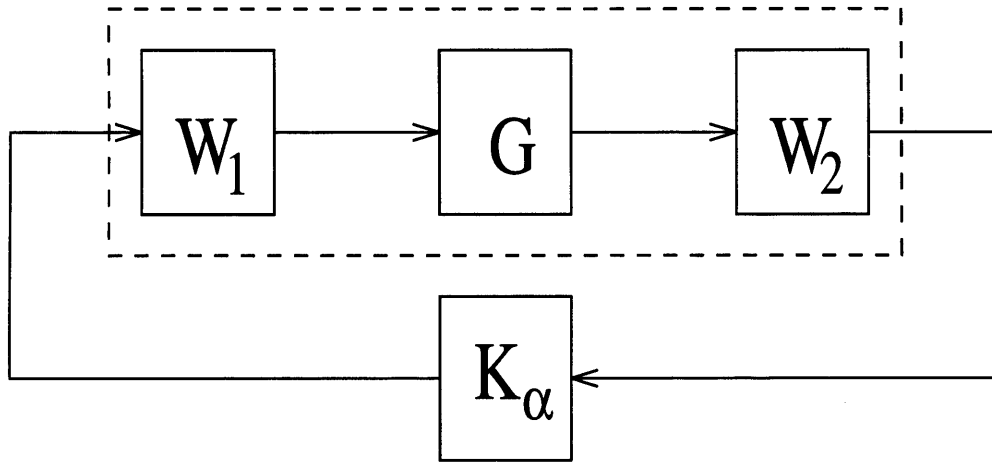


Figure 7.3: H_∞ robust stabilisation of plant G_s ,

from [79] as

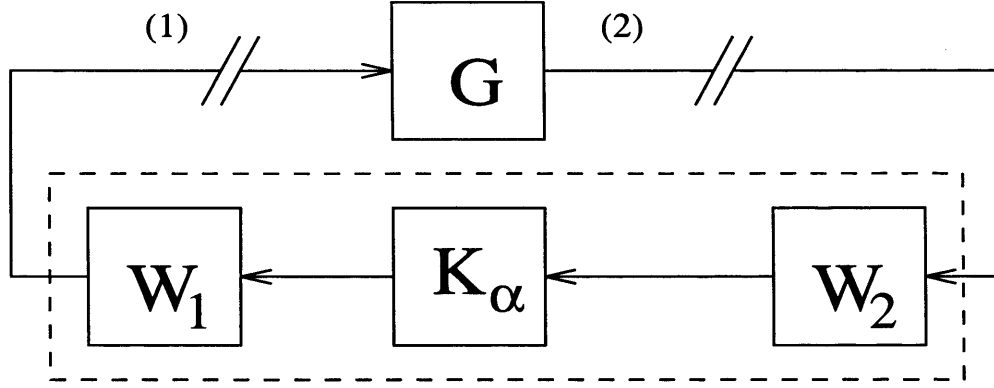
$$K_\infty = \left[\begin{array}{c|c} \frac{A + BF + \gamma^2 W^{*-1} \gamma^2 Z C^* (C + DF)}{B^* X} & \gamma^2 W^{*-1} Z C^* \\ \hline & -D^* \end{array} \right] \quad (7.17)$$

where $W = I + XZ - \gamma^2 I$, which can stabilise the plant G as shown in Figure 7.3.

3. Develop the feedback controller, K : Finally the feedback controller K is obtained using the robust H_∞ stabilising controller, K_∞ , in combination with the shaping weights W_1 and W_2 as shown in Figure 7.4, giving

$$K = W_1 K_\infty W_2$$

4. Form the final closed-loop system : The final closed-loop system is as shown in Figure 7.5 which also includes a constant gain matrix pre-filter, K_o , which provides

Figure 7.4: The feedback controller K

the necessary steady state gain on the reference to give zero error when compared with the K_∞ outputs.

In which case, as shown in [91], the H_∞ 1-DOF loop shaping controller can be realised as an observer for the shaped plant plus a state feedback control law. The equations are

$$\begin{aligned}\dot{\hat{x}}_s &= A_s \hat{x}_s + H_s (C_s \hat{x}_s - y_s) + B_s u_s \\ u_s &= K_s \hat{x}_s\end{aligned}$$

where $\hat{x}_s$ is the observer state, u_s and y_s are respectively the input and output of the shaped plant, and the matrices

$$K_s = -B_s^T [I - \gamma^{-2} I - \gamma^{-2} X Z]^{-1} X \quad (7.18a)$$

$$H_s = -Z C_s^T \quad (7.18b)$$

stabilise the closed-loop matrices $[A_s + B_s K_s]$ and $[A_s + H_s C_s]$ respectively.

For the purpose of design of the ideal model for the RHM, the linear nominal model at hover knots is used with the pre-filter weights W_1 and W_2 ;

$$W_1 = \text{diagonal} \left[\frac{s+5}{s}, \frac{s+4.95}{s}, \frac{s+4.88}{s}, \frac{s+4.98}{s} \right]$$

$$W_2 = \text{diagonal} [1, 1, 1, 1, 0.2, 0.2]$$

The components of the weight W_1 are chosen differently to avoid repetitions of the eigenvalues in the shaped plant since it is well known that such systems produce ill-conditioning. An alignment gain is used to shift the frequency response

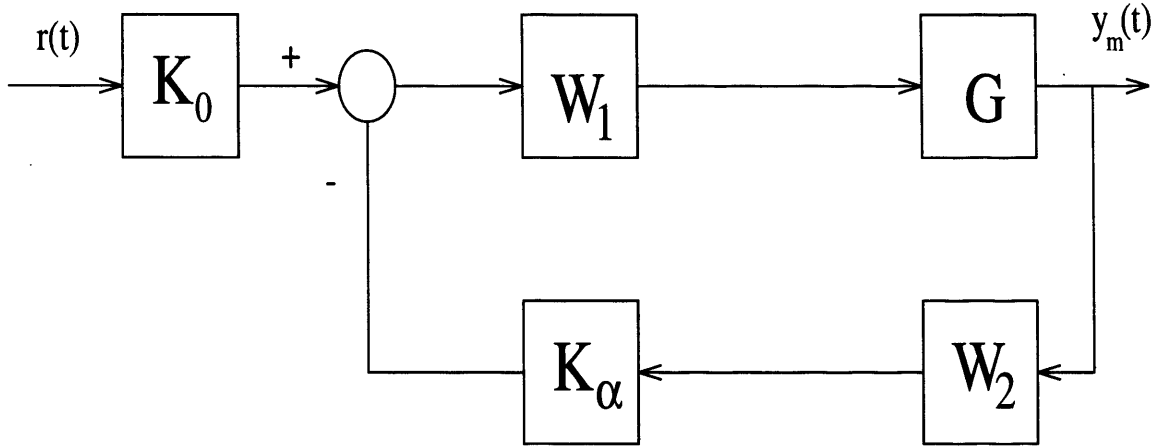


Figure 7.5: Closed-loop system for ideal model with LSDP controller

vertically, thus giving an obtainable 'cut off' frequency of 5.0 rad/sec. The extra pre-multiplying gain is used to provide more de-coupling and decrease the steady state error in certain channels. Therefore 0.1 and 0.5 are used in the main rotor collective channel and tail rotor collective channel respectively. The controller is then formulated using robust stabilisation of the normalised left coprime factor of the shaped plant and split into the feedback controller K_s . An optimal value of $\gamma_o = 2.5837$ and the gain matrices K_s and H_s are obtained as follows

$$K_s = \begin{bmatrix} 52.6872 & -98.3176 & 30.1309 & -2.4832 & -30.0078 & -98.1902 \\ -0.3944 & 5.5035 & -0.8229 & -0.1045 & 0.8202 & 5.4996 \\ -0.1040 & 1.5452 & -0.8081 & -0.0621 & 0.8022 & 1.5461 \\ 0.1420 & 0.0370 & 0.1616 & -0.5978 & -0.1674 & 0.0490 \\ -2.0952 & 75.3235 & -10.3949 & -0.4066 & 10.3778 & 75.2462 \\ 1.0389 & -13.6655 & 4.6319 & -0.8289 & -4.6169 & -13.6388 \\ 0.0469 & -0.9984 & 0.4745 & -0.3591 & -0.4759 & -0.9906 \\ -0.3265 & 15.9034 & -2.1995 & -0.1894 & 2.1946 & 15.8892 \\ -0.0988 & 1.5361 & -0.7478 & 1.9085 & 0.7662 & 1.4961 \\ 0.1440 & 0.3246 & -0.0034 & -0.0075 & 0.0036 & 0.3243 \\ 0.1542 & 0.2649 & -0.0009 & -0.0021 & 0.0011 & 0.2646 \\ -2.3275 & -5.1888 & 0.1295 & 0.1133 & -0.1321 & -5.1840 \end{bmatrix}^T$$

$$H_s = \begin{bmatrix} -0.0793 & -0.0758 & 0.0130 & -0.0304 & 0.0193 & -0.1327 \\ -1.1335 & -14.1267 & 1.3628 & -1.3090 & 2.4231 & -22.6909 \\ 0.1965 & 0.6274 & 6.0121 & 0.5374 & 6.0958 & 0.1327 \\ -0.0635 & 1.8625 & -2.5915 & 8.7098 & -3.4444 & 3.4563 \\ -0.3606 & -4.2081 & 0.1367 & -0.0764 & 0.2537 & -2.3327 \\ -0.0698 & 0.1367 & -3.5714 & 0.2025 & -1.5139 & 0.0610 \\ -0.1643 & 1.2687 & -7.5695 & -2.3838 & -10.5405 & 1.3467 \\ -1.1254 & -11.6634 & 0.3051 & -0.6778 & 1.3467 & -15.4656 \\ -0.1267 & -0.7002 & 0.2188 & -4.4846 & -0.4047 & -0.9627 \\ 0.5761 & 45.3116 & -0.6507 & 0.6215 & -1.8422 & 17.5082 \\ -4.1243 & 1.5085 & -34.9540 & 3.7767 & -3.4829 & 1.5205 \\ 1.5525 & 3.1471 & -1.8375 & 0.3058 & -0.2632 & 1.3522 \end{bmatrix}$$

It is clear that the H_∞ technique increases the number of states due to the augmentation of the plant. Thus using the shaped plant G_s , the error dynamics in equation (7.3a) have higher dimension than the original plant. Hence, for the model matching condition to be satisfied, it is necessary to consider the error dynamics for the shaped plant. This will be discussed in the next section. Thus the H_∞ method produces the matrices $L_x = K_s$ and

$$L_r = \begin{bmatrix} -0.4405 & 0.0535 & 0.0179 & 0.0085 & -0.0108 & 0.0063 \\ -0.0369 & -0.3779 & 0.0323 & -0.0274 & 0.0106 & -0.1155 \\ -0.0098 & -0.0252 & -0.3664 & 0.0032 & -0.0854 & 0.0003 \\ -0.0013 & 0.0123 & -0.0651 & -0.4789 & 0.0022 & 0.0049 \\ 0.0097 & 0.0248 & 0.3626 & -0.0090 & 0.0848 & -0.0003 \\ -0.0368 & -0.3776 & 0.0343 & -0.0175 & 0.0108 & -0.1154 \end{bmatrix}$$

Since the design aim is to control the first four measured outputs via the pilot inputs; the final two reference signals are considered to be zero.

The next section will describe the sliding mode controller design of u_e based on output feedback for the shaped plant parameters (A_s, B_s, C_s) .

7.4 Design of Switching Surface

It is necessary to assume that the nonlinear plant equations (7.1a) -(7.1b) when multiplied with the known weights W_1 and W_2 can be written using shaped plant parameters (A_s, B_s, C_s) . The shaped plant does not satisfy the assumption of $\text{rank}(C_s B_s)$ equal to the number of inputs. There is a rank deficiency of 2. Since the pilot has four inputs and is required to track only four outputs, a transformation of the input matrix B_s can easily solve the problem as follows:

$$B_s T_s = [B_{st} \quad 0] \quad (7.19)$$

where the matrix B_{st} has full rank and also satisfies the assumption that the rank of $(C_s B_{st})$ is equal to m . Hence the input $u_e(t)$ is transformed as

$$u_{st}(t) = T_s^{-1} u_e(t) \quad (7.20)$$

where $u_{st}(t)$ can be defined as

$$u_{st}(t) = \begin{bmatrix} u_{usc} \\ 0 \end{bmatrix} \quad (7.21)$$

The parameter $u_{usc}(t)$ is designed using the output feedback sliding mode design method described in Chapter §4 for the triple (A_s, B_{st}, C_s) . Then the error dynamics in equation (7.3a) can be represented as

$$\dot{e}(t) = A_s e(t) + B_{st} u_{usc}(t) + B_{st} \xi(t, u, x) \quad (7.22a)$$

$$e_y(t) = C_s e(t) \quad (7.22b)$$

where the uncertainty $\xi(t, u, x)$ is further matched and bounded. The numerical values of the parameters T_s and B_{st} are obtained as follows:

$$T_s = \begin{bmatrix} -0.4153 & 0.8585 & -0.0565 & 0.2955 & -0.0000 & -0.0000 \\ -0.6371 & -0.3061 & -0.0322 & -0.0122 & -0.0009 & -0.7065 \\ 0.0447 & -0.0208 & -0.7082 & -0.0122 & 0.7041 & 0.0003 \\ -0.1259 & 0.2673 & -0.0079 & -0.9552 & -0.0086 & 0.0146 \\ -0.0464 & 0.0236 & 0.7023 & 0.0005 & 0.7100 & -0.0013 \\ -0.6337 & -0.3112 & -0.0304 & 0.0075 & 0.0002 & 0.7075 \end{bmatrix}$$

$$B_{st} = \begin{bmatrix} 0.1764 & 0 & 0 & 0 \\ 24.2438 & 11.7219 & 0 & 0 \\ 0.5625 & -0.2964 & -8.5813 & 0 \\ -5.4418 & 0.3401 & 3.1603 & -8.3751 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5780 & 1.7801 & 23.5633 & 0.1498 \\ 11.5198 & 5.5648 & -0.1226 & 0.0000 \\ 1.2621 & 0.2522 & 3.6074 & 1.7315 \\ -13.1594 & -6.3778 & 0.1406 & -0.0000 \\ -1.1908 & 0.4331 & 5.4150 & -1.9668 \\ -0.8595 & -0.0045 & 0.0000 & -0.0000 \end{bmatrix}$$

The VSC canonical form of the linear shaped model (A_s, B_{st}, C_s) are obtained with the orthogonal matrix

$$T_o = \begin{bmatrix} -0.0000 & 0.0000 & 0.7869 & -0.0710 & 0.6130 & -0.0043 \\ -0.8769 & -0.4807 & -0.0000 & 0 & 0 & 0 \\ 0.4807 & -0.8769 & 0.0000 & 0 & 0 & 0 \\ 0.0000 & -0.0000 & -0.5029 & -0.5854 & 0.5759 & -0.2697 \\ -0.0000 & 0.0000 & 0.3367 & -0.7815 & -0.5230 & -0.0483 \\ -0.0000 & -0.0000 & 0.1206 & 0.2037 & -0.1380 & -0.9617 \end{bmatrix} \quad (7.23)$$

The system has more outputs than inputs and satisfies the conditions for VSC output feedback design. The reduced order matrices are easily be partitioned. This shows six invariant zeros are present in the state matrix A_{11} . Two of these are from original plant and the remaining four appear because of the addition of states due to the weight W_1 . Since the invariant zeros are present in the matrix A_{11} , the reduced order triple (A_{11}, A_{12}, C_1) cannot be stabilised using an arbitrary pole placement technique, where the matrix C_1 is defined as $C_1 = [0_{2 \times 6} \quad I_2]$. Eliminating appropriate rows and columns corresponding to the invariant zeros, the new reduced order sub-system $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ is constructed as follows

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{122} = \begin{bmatrix} 0.2889 & -2.7622 & -0.6619 & 4.1115 \\ -1.7273 & 2.9817 & 2.5803 & 2.5476 \end{bmatrix}$$

and $\tilde{C}_1 = I_2$. Further this shows rank deficiency of 2 in the matrix A_{122} . Using the transformation in equation (4.59) in Section §4.4 produces the matrices

$$T_{m'} = \begin{bmatrix} -0.0577 & 0.3454 & 0.0633 & -0.93455 \\ 0.5518 & -0.5968 & -0.5059 & -0.28895 \\ 0.1322 & -0.5160 & 0.8343 & -0.14235 \\ -0.8214 & -0.5083 & -0.2100 & -0.1513 \end{bmatrix}$$

The matrix $\tilde{B}_1$ is given by

$$\tilde{B}_1 = \begin{bmatrix} -5.0055 & 0 \\ -0.0063 & -5.0023 \end{bmatrix}$$

It is now clear that the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ satisfies the pole placement criteria including the ‘Kimura-Davison’ condition, so an arbitrary pole placement is possible.

The normal matrix design approach to robust pole placement presented in Section §4.6 is used. The optimal matrices P & Q of Theorem 4.1 are computed as

$$P = \begin{bmatrix} 1.9476 & -0.0046 \\ 2.7393 & 1.0852 \end{bmatrix}, \quad Q = \begin{bmatrix} 11.1869 & 9.1581 \\ 9.1581 & 7.5298 \end{bmatrix}$$

giving closed-loop poles at $\{-3.0, -3.5\}$ with gain matrix

$$K_1 = \begin{bmatrix} -0.6044 & 0.0225 \\ 0.0218 & -0.6935 \end{bmatrix}$$

Subsequently the matrix K_1 is transformed into its original coordinates using the transformation matrix $T_{m'}$, giving

$$K = \begin{bmatrix} 0.0424 & -0.2408 \\ -0.3465 & 0.4263 \\ -0.0912 & 0.3608 \\ 0.4854 & 0.3340 \end{bmatrix}$$

where the matrix $K_2 = 0_{2 \times 2}$ is considered since it does not affect the reduced order closed-loop system. The attainment of normal matrix properties of the reduced order closed-loop sub-matrix

$$\tilde{A}_{11}^s = \begin{bmatrix} -3.0253 & 0.1129 \\ 0.1051 & -3.4689 \end{bmatrix}$$

and the reduced order closed-loop matrix

$$A_{11}^s = \begin{bmatrix} -0.0019 & -0.0021 & -0.0216 & 0.0650 & -0.0382 & -0.0018 & 28.5890 & 15.1935 \\ -0.0014 & -0.0148 & -0.2276 & 0.0764 & -0.0357 & 0.0014 & -15.2422 & 28.9643 \\ -0.0241 & -0.1681 & -4.9721 & -0.0008 & 0.0013 & 0.0002 & -9.4462 & 8.7516 \\ 0.0295 & 0.1105 & -0.0031 & -4.9480 & 0.0006 & 0.0003 & -13.7987 & -12.6113 \\ 0.0000 & -0.0000 & -0.0000 & 0.0017 & -4.8802 & 0.0040 & 0.9884 & -9.3917 \\ 0.0000 & -0.0000 & 0.0002 & 0.0003 & 0.0040 & -4.9999 & 0.1063 & -0.1693 \\ -0.0000 & 0.0000 & -0.0000 & 0 & -0.0000 & 0.0000 & -3.0253 & 0.1129 \\ 0.0000 & 0.0000 & 0.0000 & 0 & -0.0000 & -0.0000 & 0.1051 & -3.4689 \end{bmatrix}$$

which reflects the robust properties of the reduced order system and the robust switching surface $\mathcal{S}_e$ is verified. The measure parameters are compared in Table 7.3. This

No.	Measure of Robustness	Optimal Value	Open-loop Matrix A_s	Closed-loop Matrix $\tilde{A}_{11}^s$	Closed-loop Matrix A_{11}^s
1	$\Delta_2(A_c)$	0	0.9996	0.0177	0.9963
2	$\Delta_F(A_c)$	0	0.8689	0.0160	0.9813
3	$K(W_{A_c})$	1	291.3011	1.0158	32.4170
4	$MS(A_c)$	0	0.9967	0.0022	0.9940
5	$\delta(A_c)$	0	1.0	0.0624	0.9955
6	$\nu(A_c)$	1	727.9331	2.0002	84.0554

Table 7.3: : Comparison of Robustness Measurement

shows that due to the addition of the invariant zeros and transformation into reduced order closed-loop system A_{11}^s , there is some variation in the optimised closed-loop system $\tilde{A}_{11}^s$ parameters measure and the reduced order closed-loop system A_{11}^s parameters measure. At this point it is necessary to say that if the components of the weight W_1 are indifferent then the measure parameters of A_{11}^s increase due to repetition of eigenvalues and eigenvectors. Hence the weights are chosen at slightly different locations. Using equation (4.31) in Section §4.4 and the value of T_o from equation (7.23), the switching surface matrix F can be calculated for an arbitrary choice of F_2 since this does not affect the switching surface and merely acts as a scaling. However in this case the matrix F_2

is obtained as

$$F_2 = \begin{bmatrix} 0.0091 & 0.0126 & -0.0268 & -0.0152 \\ -0.0111 & 0.0089 & 0.0016 & -0.0607 \\ -0.0024 & 0.0017 & -0.0040 & -0.0365 \\ -0.0234 & -0.0054 & 0.0032 & 0.0437 \end{bmatrix}$$

which produces the switching surface matrix

$$F = \begin{bmatrix} -0.0101 & 0.0134 & 0.0059 & -0.0233 & 0.0080 & 0.0220 \\ -0.0081 & 0.0354 & -0.0043 & 0.0177 & -0.0086 & 0.0586 \\ -0.0043 & 0.0217 & 0.0021 & 0.0078 & 0.0017 & 0.0357 \\ -0.0163 & -0.0283 & -0.0063 & 0.0050 & -0.0074 & -0.0464 \end{bmatrix}$$

The above switching surface matrix F will show robustness to the bounded uncertainty defined in equation (7.1a). The next part of this design approach is choice of the appropriate controller parameter N .

7.5 Design of the Controller

Without loss of generality it can be assumed that the plant triple (A_s, B_{st}, C_s) and the switching surface matrix F are known. Define the control law as

$$u_{usc}(t) = -(F_2 B_2)^{-1} [G_l e_y(t) + \nu(e_y)] \quad (7.24)$$

The gain matrix G_l is defined as

$$G_l = \left[F C_s A_s N + \frac{\alpha}{2} F \right]$$

where $N \in \mathbb{R}^{n \times p}$ is a design matrix and α is some positive constant to be chosen accordingly. Details of these are described in Chapter §4. Consider a Lyapunov candidate based on error switching function $s(e_y) = F e_y(t)$ as

$$V(s) = \frac{1}{2} s^T(e_y) s(e_y) \quad (7.25)$$

The reachability condition is satisfied if $\dot{V}(s) < 0$, i.e. if $s^T(e_y) \dot{s}(e_y) < 0$ for all $e_y(t)$. It is proved in Chapter §4 of Lemma 4.2 that the reachability condition can be attained by appropriate choice of N for a certain value of α . The component $\nu(e_y)$ is defined by

$$\nu(e_y) = \begin{cases} \rho(t, u_{usc}, e_y) \frac{s(e_y)}{\|s(e_y)\|} & \text{if } s(e_y) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.26)$$

Here $\rho(t, u_{vsc}, e_y)$ is a positive scalar quantity defined by the uncertainty bound parameters such that

$$\rho(t, u_{vsc}, e_y) = \frac{kK_g \|G_l e_y\| + K_\alpha \|FC_s\|}{(1 - kK_g)} \quad (7.27)$$

where the denominator is considered to be always positive. It follows that K_g is limited by $0 \leq K_g < \frac{1}{k}$ and k is defined as $k = \| (FC_s) \| \cdot \| (F_2 B_2)^{-1} \|$. Clearly lower values of k increase the limiting parameter K_g , which gives a higher bound on the *matched uncertainty* and its choice depends on the matrix F . The control law in equation (7.24) induces a sliding mode on the relevant error switching surface $\mathcal{S}_e$ if and only if the matrix $\mathcal{L}(N)$ satisfies the condition

$$\mathcal{L}(N) = (FC_s)^T \left[FC_s A_s (I - NC_s) - \frac{\alpha}{2} FC_s \right] \leq 0$$

by design of the matrix N for a certain value of α , i.e. the matrix $\mathcal{L}(N)$ is required to be negative semi definite. An optimisation approach of Heck *et al.*[57] is used to produce the matrix

$$N = \begin{bmatrix} 3.4620 & 0.0469 & 0.2627 & 0.4098 & 0.0281 & 0.0017 \\ -0.0147 & 1.1801 & -0.0291 & -0.0044 & -0.0006 & 0.0284 \\ -0.2204 & -0.1100 & 0.7025 & -0.4726 & 0.0415 & 0.0045 \\ -0.0385 & -0.1563 & 0.2127 & 0.7312 & 0.0482 & -0.0096 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 1.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ -0.1646 & -0.1132 & -0.0704 & -0.4371 & 0.0516 & 0.0037 \\ -0.0167 & 0.0772 & -0.0952 & -0.0547 & 0.0019 & 0.0144 \\ -0.0259 & -0.1015 & 0.1600 & -0.1933 & 0.0352 & -0.0051 \\ 0.0002 & 0.0023 & 0.0018 & 0.0036 & 0.0510 & 0.0002 \\ -0.0012 & 0.0025 & -0.0050 & -0.0002 & -0.0003 & 0.0004 \\ 0.0290 & -0.0018 & 0.0065 & 0.0007 & 0.0011 & -0.0001 \end{bmatrix}$$

which makes $\mathcal{L}(N)$ negative semi-definite with the maximum eigenvalue of $\lambda_{\mathcal{L}(N)} = 3.051 \times 10^{-19}$ which is considered equal to zero for an initial value of $\alpha = 5.0$. The section below presents the full nonlinear simulation results obtained using the nonlinear RHM model.

7.6 Nonlinear Helicopter Model Simulation Results

The VSC controller $u_{vsc}(t)$ produces only four input signals corresponding to the control outputs. For implementation with the overall control structure, a transformation as in equation (7.20) is necessary to obtain the VSC controller $u_{vsc}(t)$ which produces the control action $u_e(t)$ and also the RHM is required to be weighted. The simulation results presented are based on the ideal model responses and the VSC controller designed at hover is implemented on the fully nonlinear RHM model with initial conditions at hover, 60 knots, 100 knots and 120 knots forward speed flight condition which gives a significantly perturbed system, to illustrate the robustness. In order to control the matched and unmatched uncertainties, the norm bound parameters in Section §7.2 are chosen as follows: $K_g = 0.002$, $K_\alpha = 20.0$. The unmatched uncertainty will be partially tolerated by the robust design consideration as discussed in Section §4.6. A value of $\alpha = 20.0$ is sufficient to bring the outputs onto the switching surface. Due to the discontinuous control action in equation (7.24), the responses show some chattering. To eliminate this effect a parameter $\delta = 0.10$ is added to the denominator in equation (7.26) to smooth the discontinuity into a approximately continuous.

7.6.1 Hover Flight Condition

The following set of time responses are obtained at the hover initial condition of the fully nonlinear model of the RHM. The dashed line represents the ideal model response and the solid line represents the plant responses. Figure (7.6) represents the step demand responses of heave velocity and the other three controlled outputs, which shows minimal coupling among outputs. This shows satisfactory tracking of the ideal model responses by the plant outputs. The actuator demands for each axis are shown in Figure (7.7). These demonstrate the low control effort. The switching surfaces for each axis response are shown in Figure (7.8). It is seen that the system has attained close to an ideal sliding mode in a reasonably short period. The pitch axis step demand is given in Figure (7.9) which also shows good tracking of the ideal model response. The heave axis response shows very small error which does not affect the handling qualities. The low controller effort is represented in Figure (7.10). Figure (7.11) represents the roll

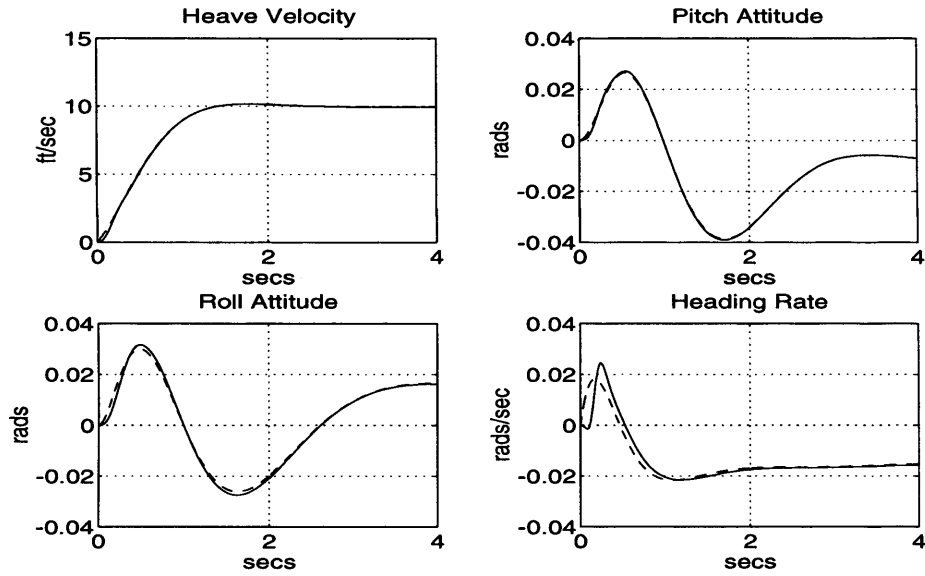


Figure 7.6: Heave Axis Step Demand

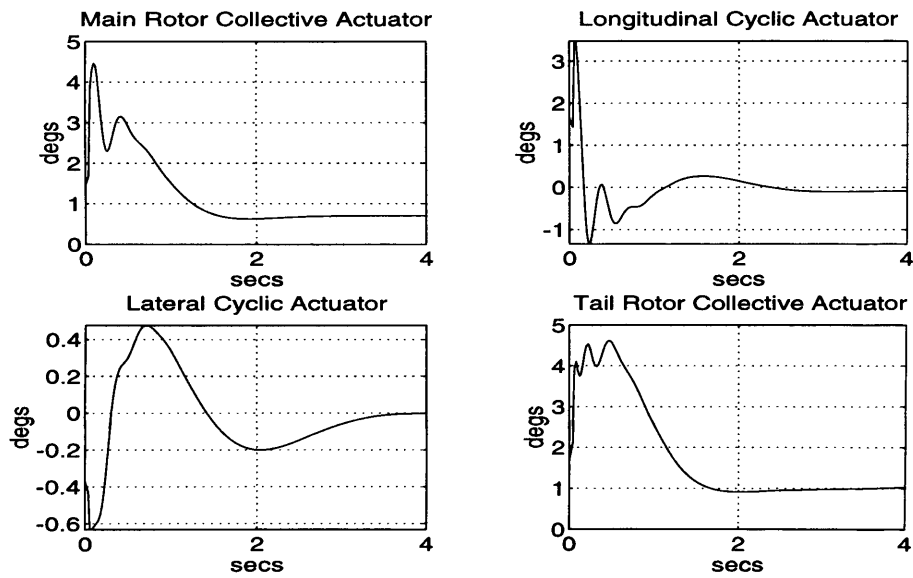


Figure 7.7: Heave Axis Actuator Demands

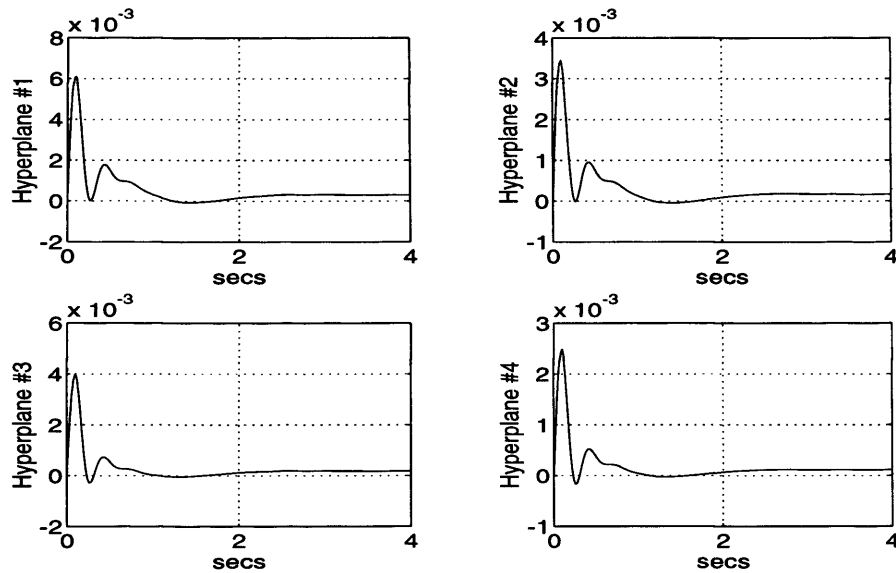


Figure 7.8: Sliding Surface for Heave Axis Step Demand

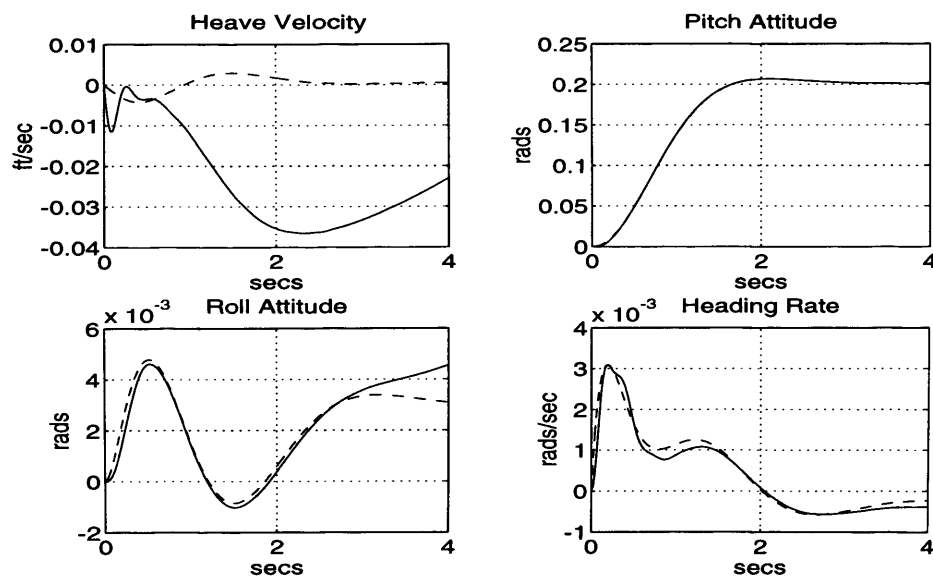


Figure 7.9: Pitch Axis Step Demand

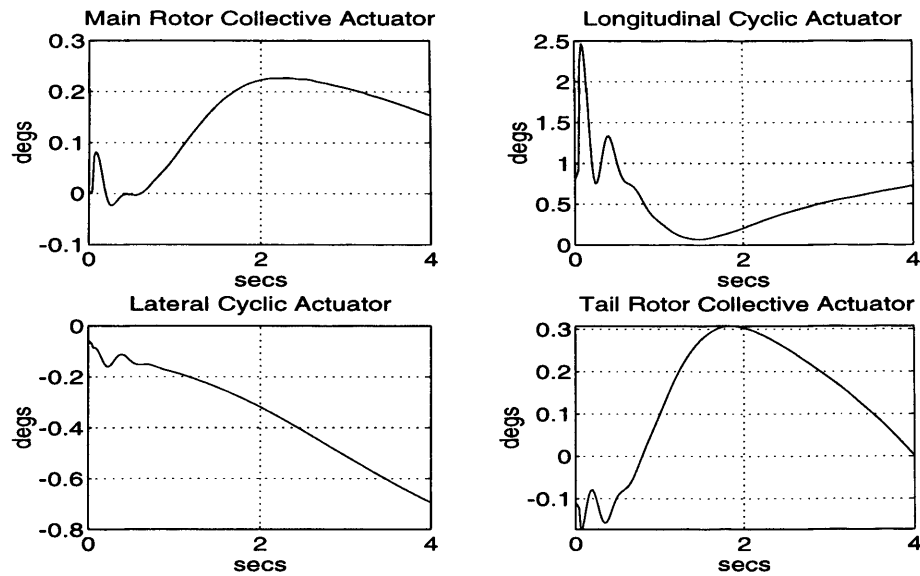


Figure 7.10: Pitch Axis Actuator Demands

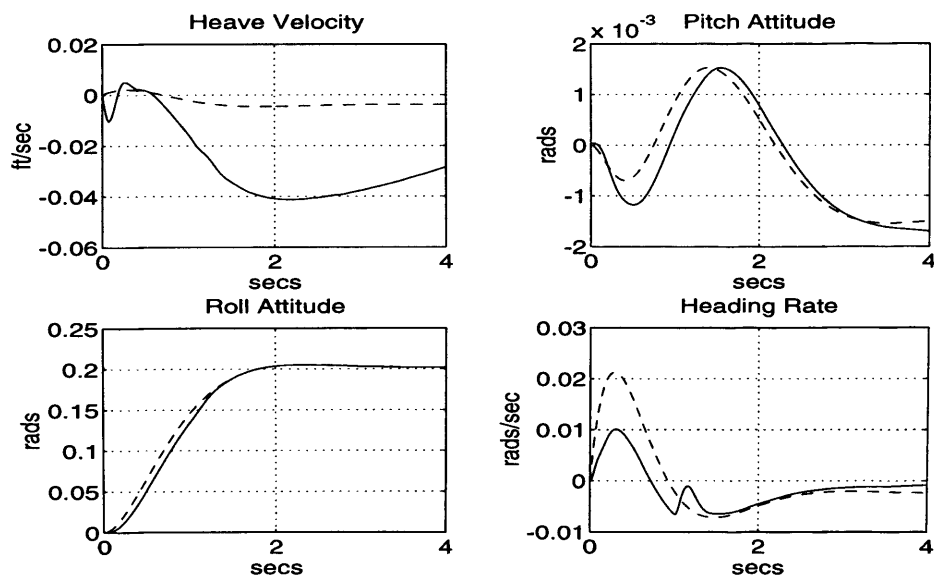


Figure 7.11: Roll Axis Step Demand

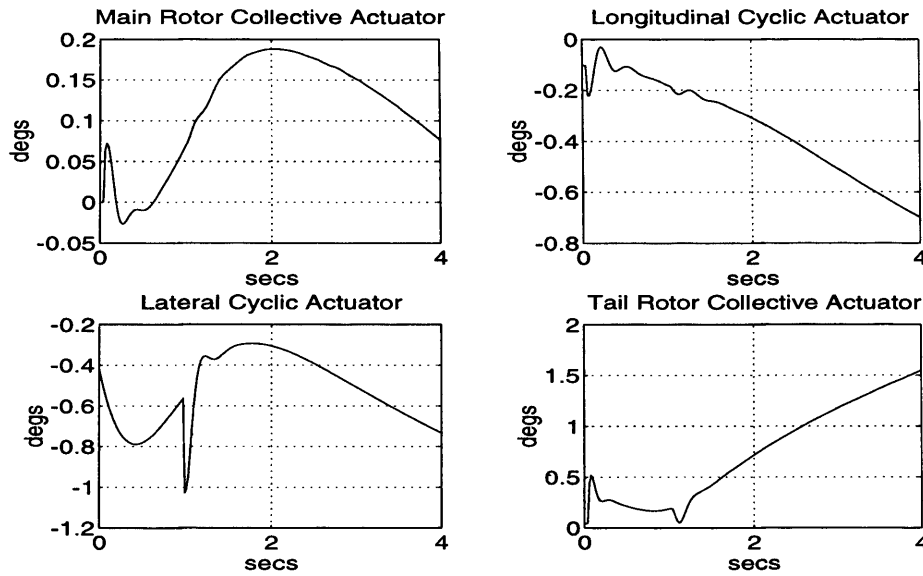


Figure 7.12: Roll Axis Actuator Demands

axis step demand and shows good tracking. The controller effort is small enough as shown in Figure (7.12). Figure (7.13) gives the yaw axis step responses showing perfect tracking in all channels except the heave velocity which shows overshoot in the first two seconds. The switching surface attainment of the outputs for both the pitch axis

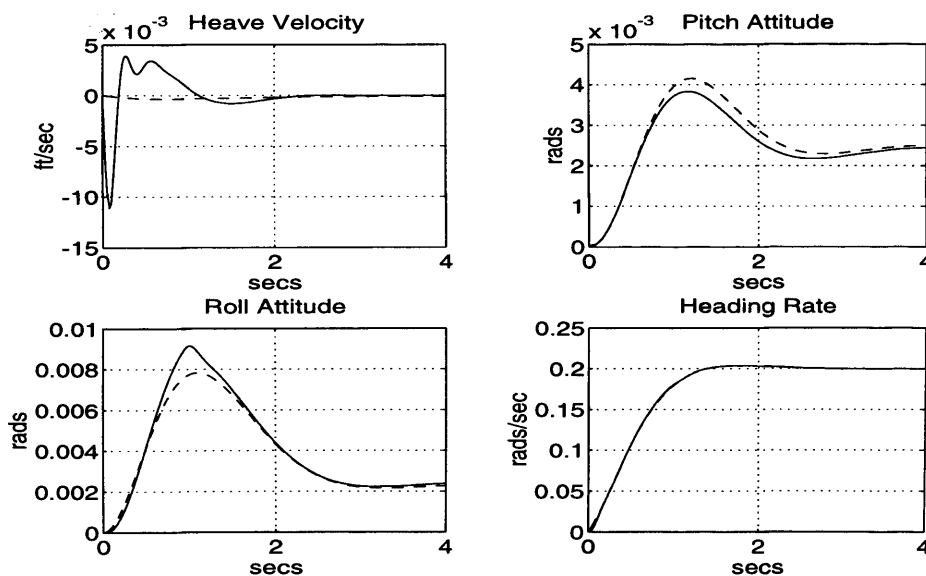


Figure 7.13: Yaw Axis Step Demand

and roll axis are also reasonable. The sliding mode is attained in a short period with very small magnitude. Figure (7.14) and Figure (7.15) represent the control effort and

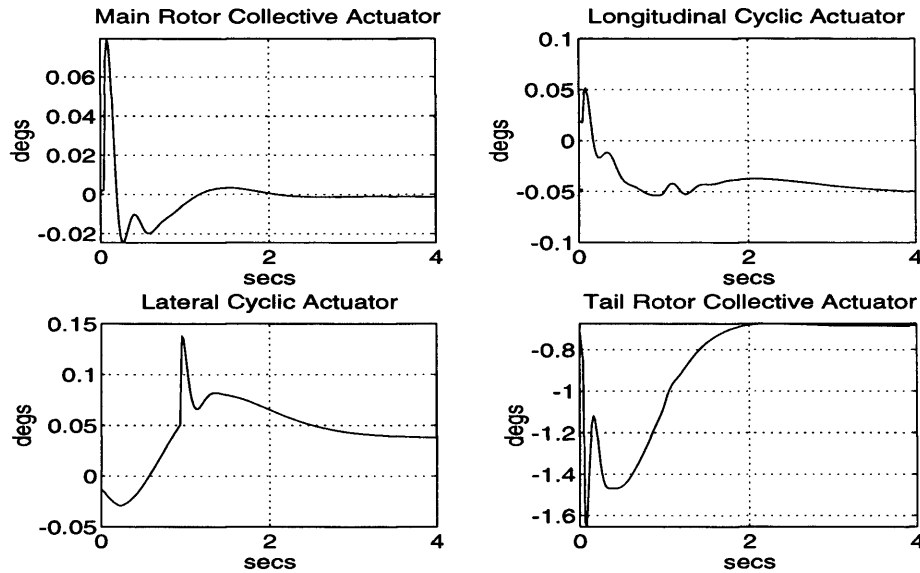


Figure 7.14: Yaw Axis Actuator Demands

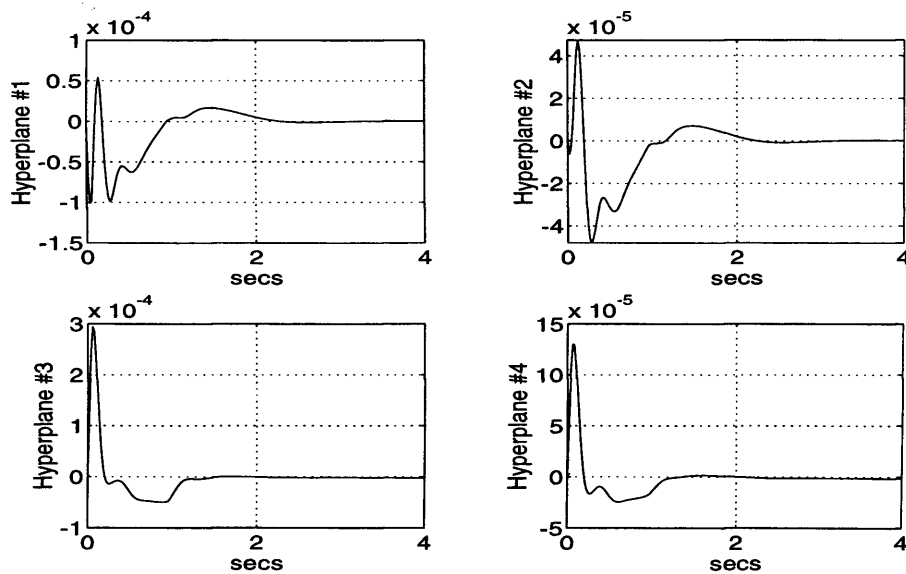


Figure 7.15: Sliding Surface for Yaw Axis Step Demand

switching surface attainment respectively which are reasonable.

7.6.2 60 knots Forward Flight Condition

This subsection shows the simulation results at 60 knots forward flight condition. Figure (7.16) shows good tracking of the outputs and the controller effort is within the limits as in Figure (7.17). The pitch axis step demand at 60 knots is given in Figure (7.18)

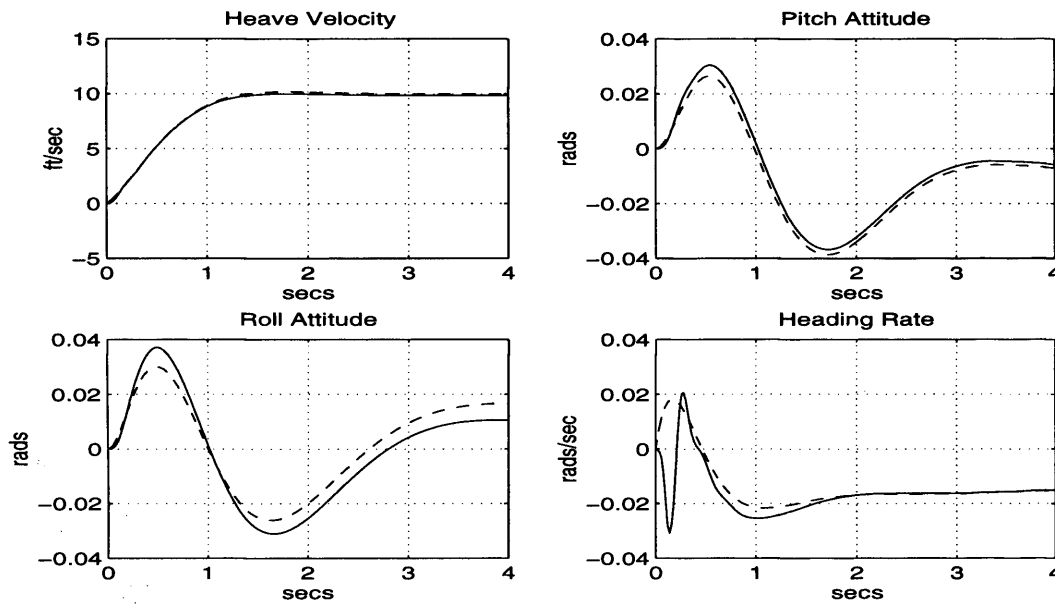


Figure 7.16: Heave Axis Step Demand

and tracks the ideal model response but the other three channels show some tracking error. In the short term analysis these are within the range of tolerance of the handling qualities. The controller effort and sliding mode in Figures (7.19) and (7.20) are both within limits. The next three Figures are the roll axis step demands, controller effort and switching surface attainments which are self explanatory. Figure (7.24) shows the yaw axis step demand which does not track satisfactory but has reached a stable motion with some steady state error. This may be tolerable by tuning when implemented or using an integral control action. Other channels show small error but are within the handling qualities requirements. The control effort and switching surface behaviour in yaw axis is satisfactory.

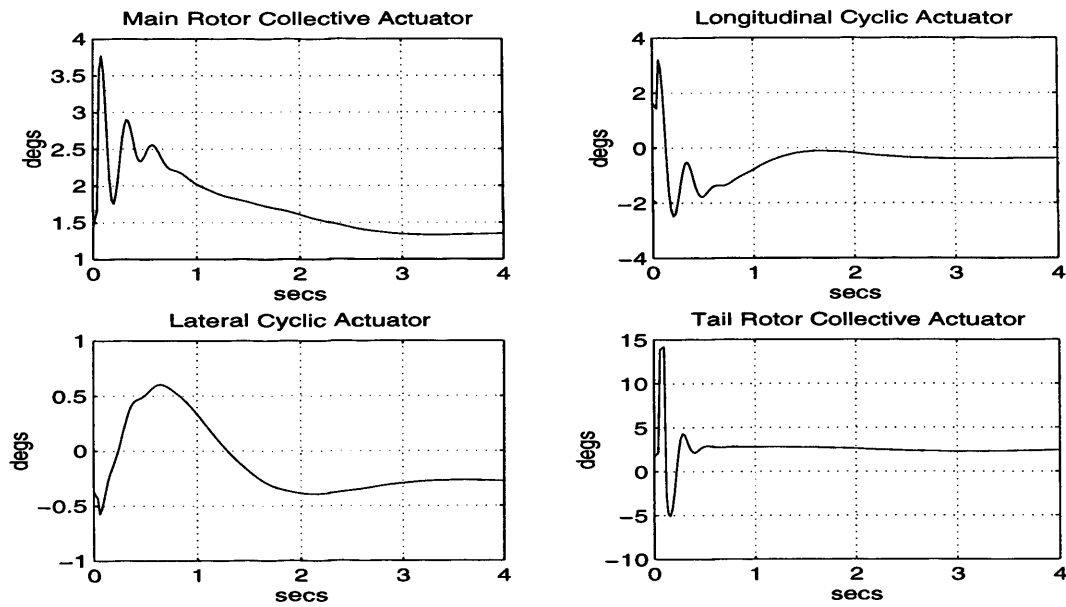


Figure 7.17: Heave Axis Actuator Demands

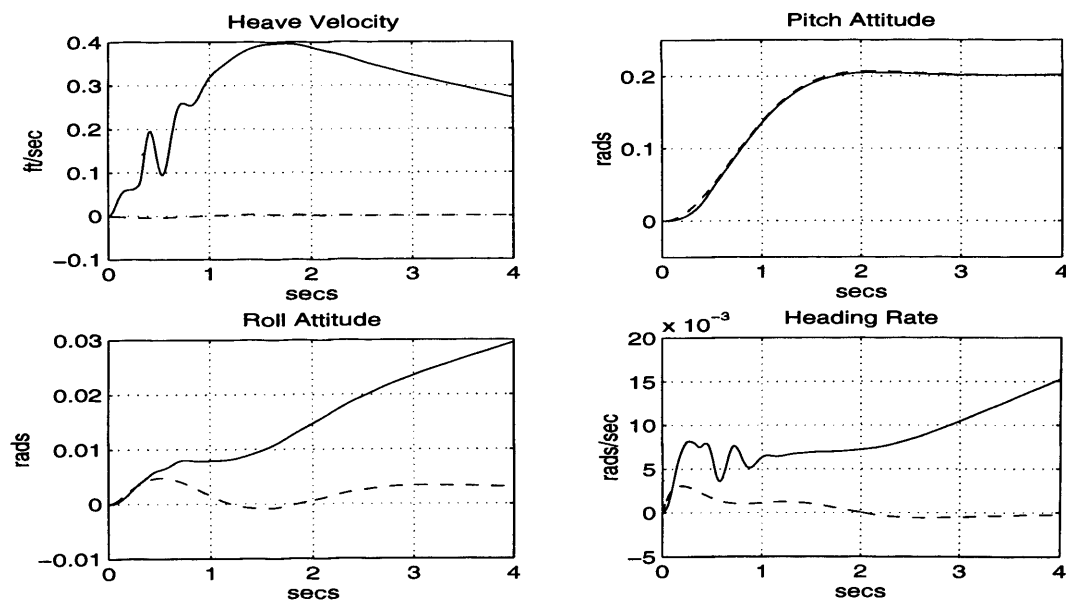


Figure 7.18: Pitch Axis Step Demand

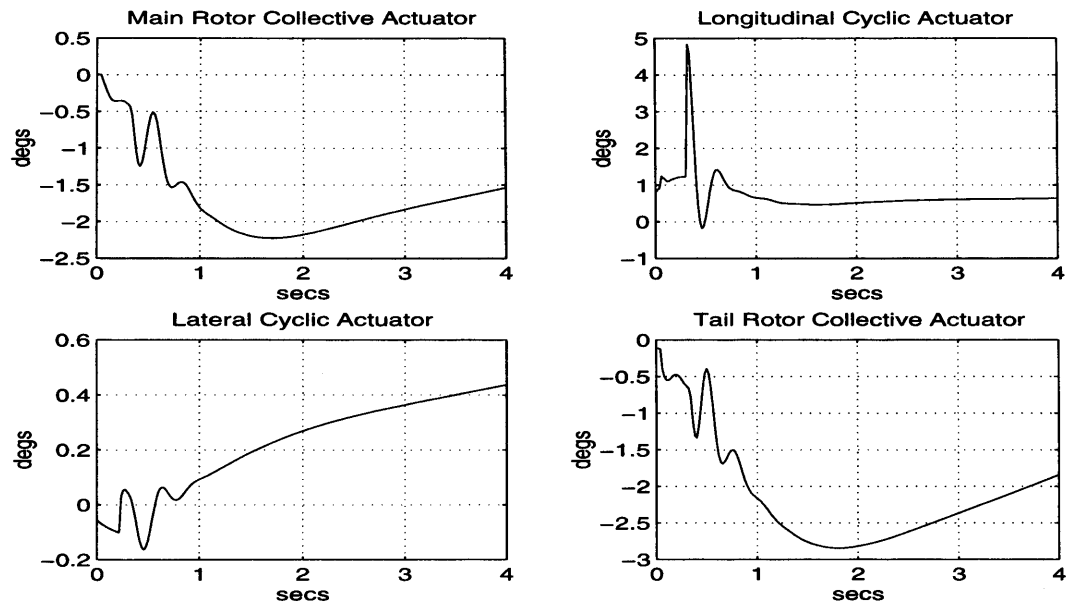


Figure 7.19: Pitch Axis Actuator Demands

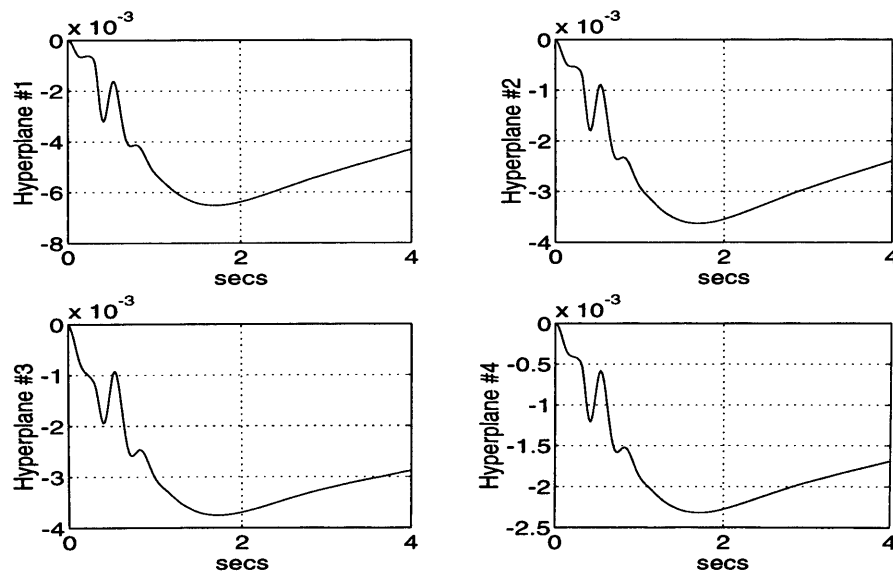


Figure 7.20: Sliding Surface for Pitch Axis Step Demand

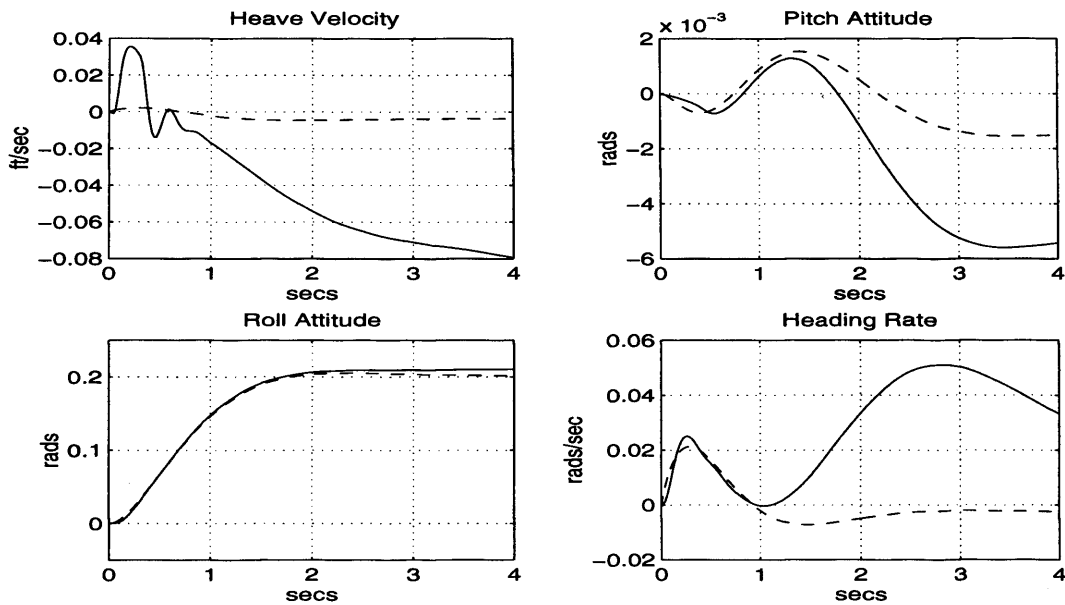


Figure 7.21: Roll Axis Step Demand

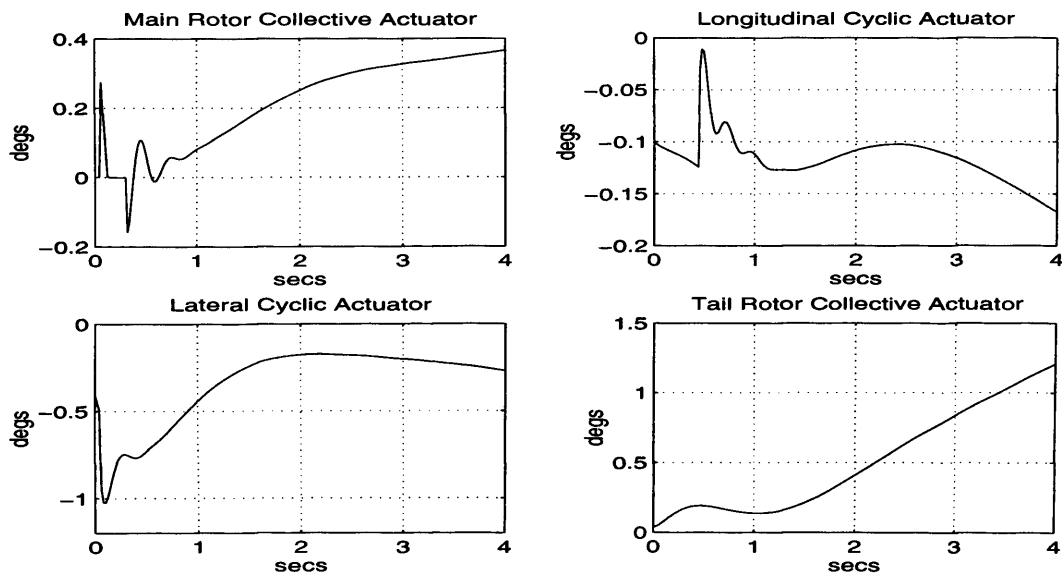


Figure 7.22: Roll Axis Actuator Demands

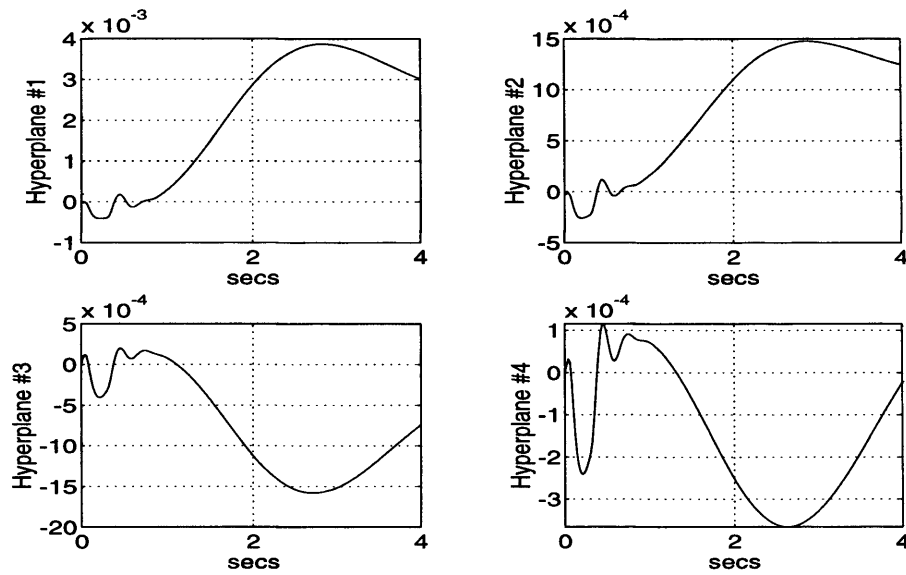


Figure 7.23: Sliding Surface for Roll Axis Step Demand

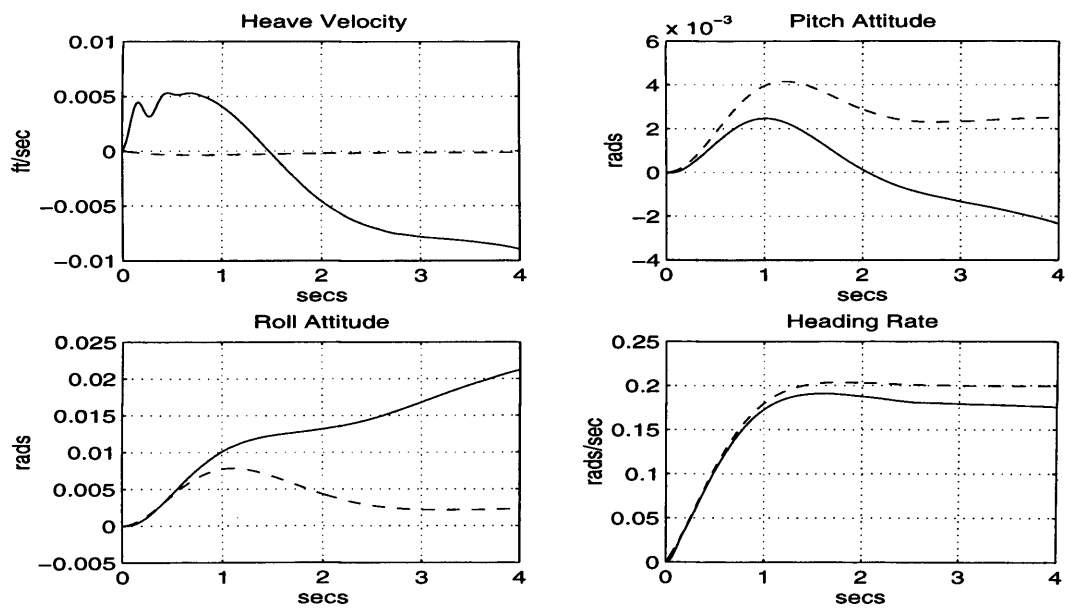


Figure 7.24: Yaw Axis Step Demand

7.6.3 100 knots Forward Flight Condition

The next few pages show the simulation results for 100 knots and 120 knots initial conditions. The results are self explanatory. These show good tracking in the step

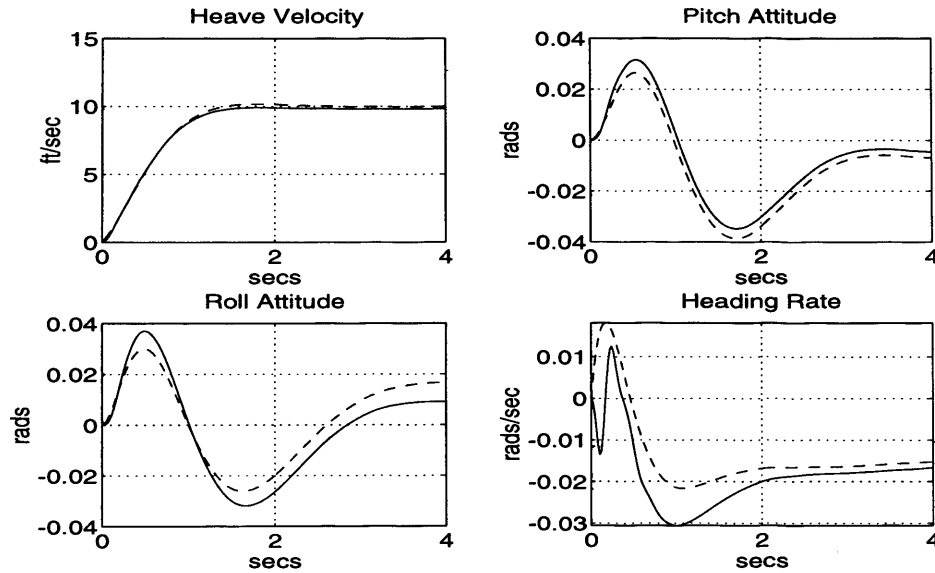


Figure 7.25: Heave Axis Step Demand

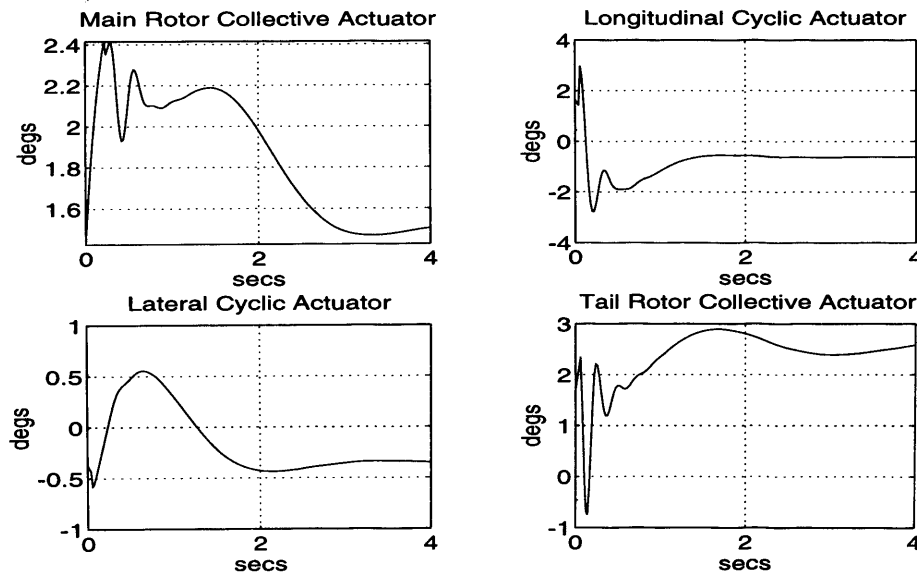


Figure 7.26: Heave Axis Actuator Demands

input channels and the remaining channels do not exceed the handling quality limits.

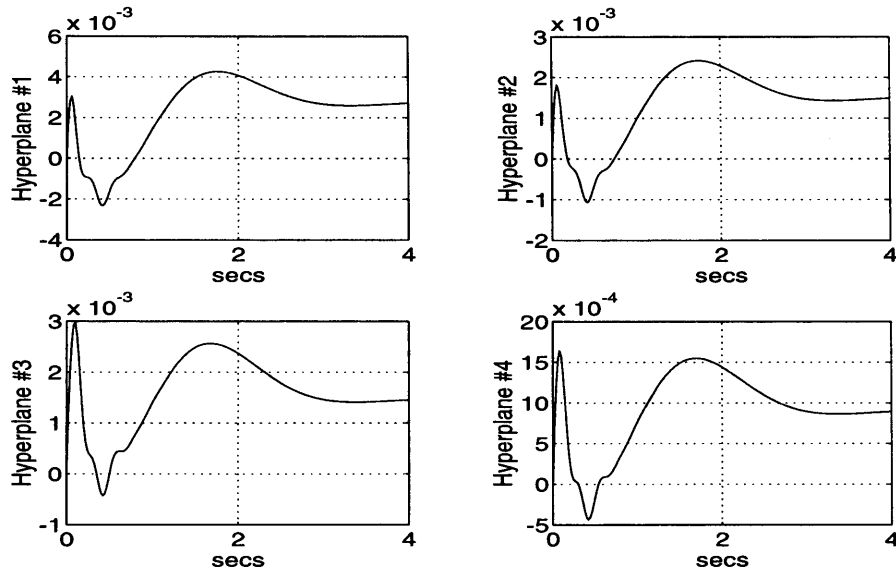


Figure 7.27: Sliding Surface for Heave Axis Step Demand

The control actions do not hit the actuator limits in all cases. The switching surface is also shown to be attained in reasonable short period.

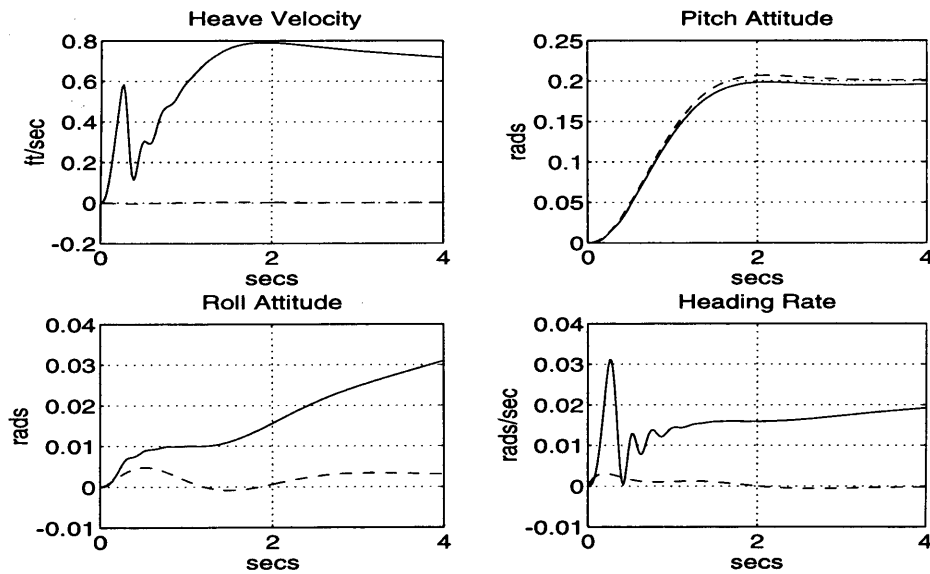


Figure 7.28: Pitch Axis Step Demand

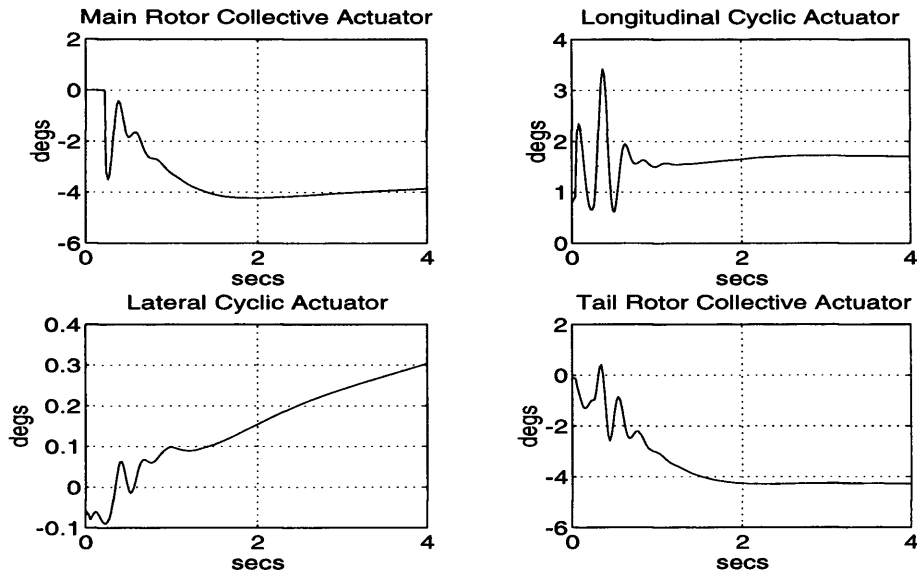


Figure 7.29: Pitch Axis Actuator Demands

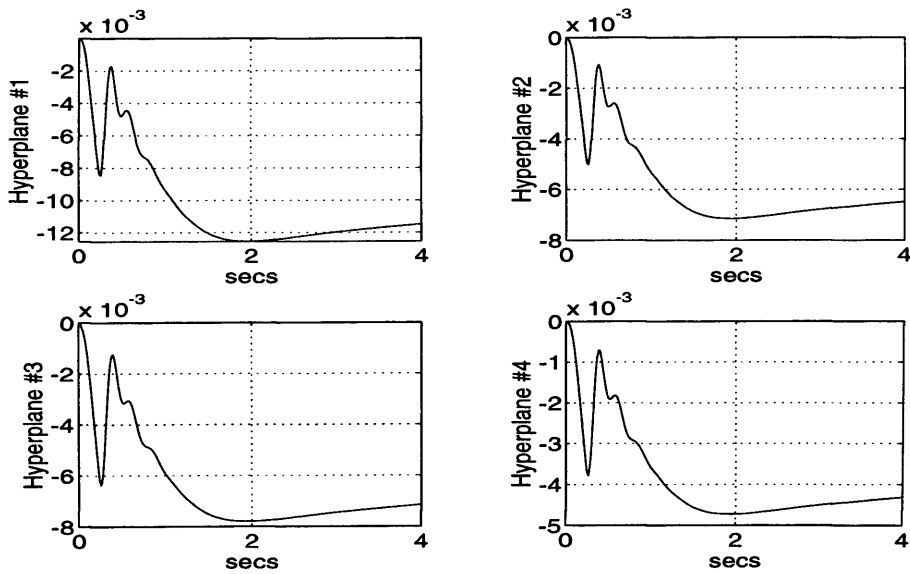


Figure 7.30: Sliding Surface for Pitch Axis Step Demand

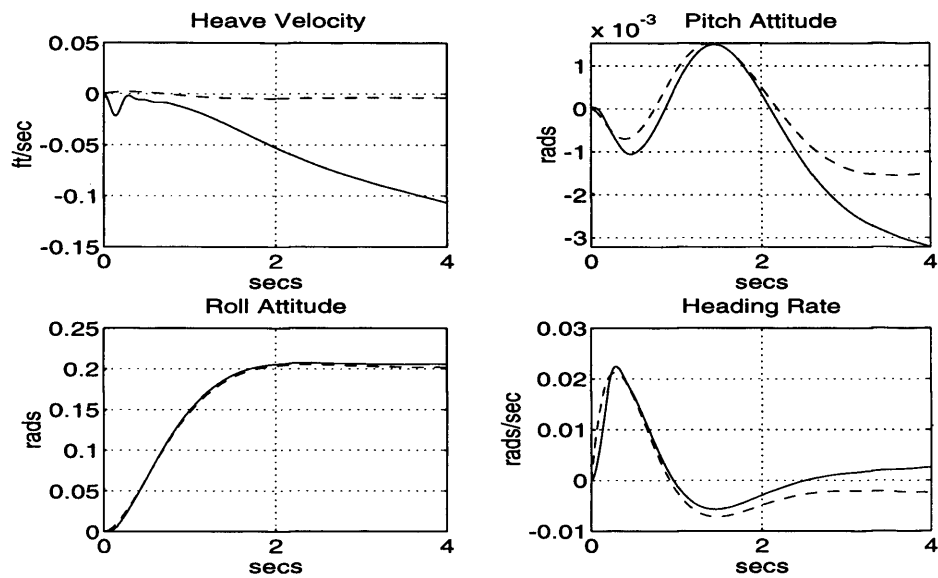


Figure 7.31: Roll Axis Step Demand

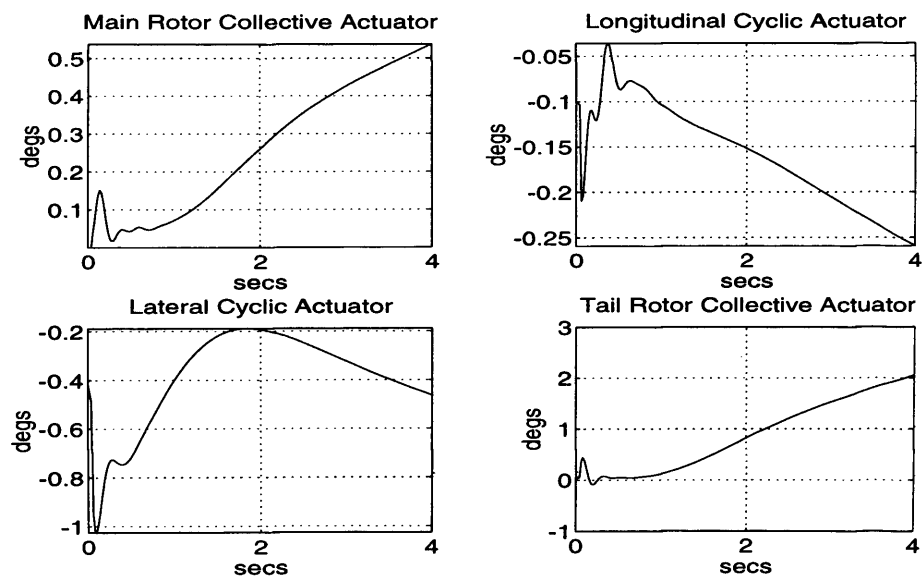


Figure 7.32: Roll Axis Actuator Demands

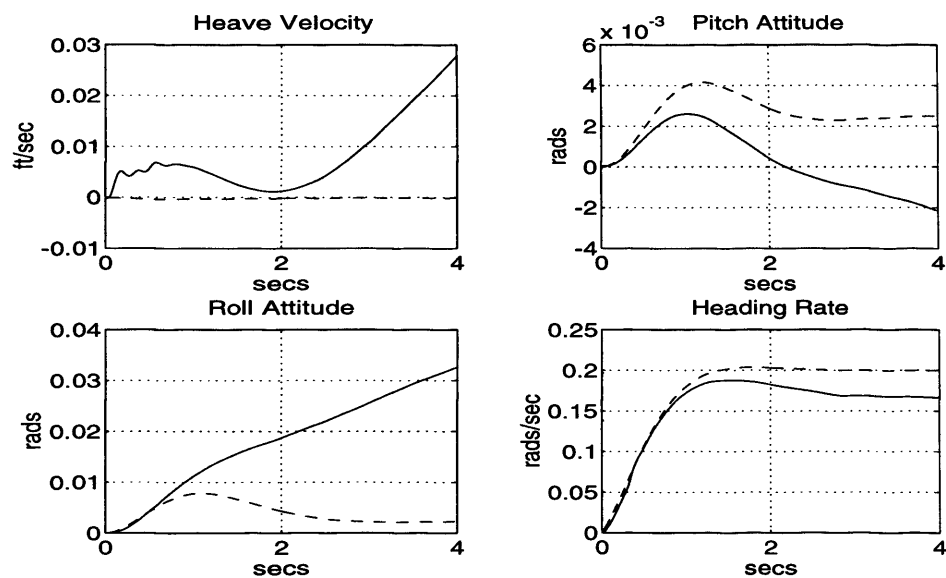


Figure 7.33: Yaw Axis Step Demand

7.6.4 120 knots Forward Flight Condition

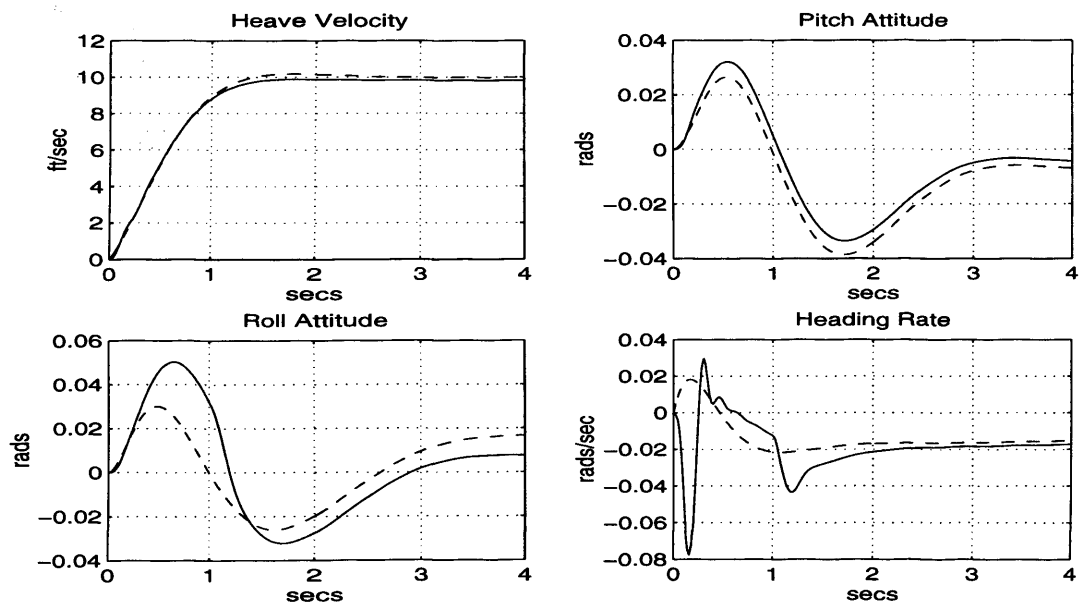


Figure 7.34: Heave Axis Step Demand

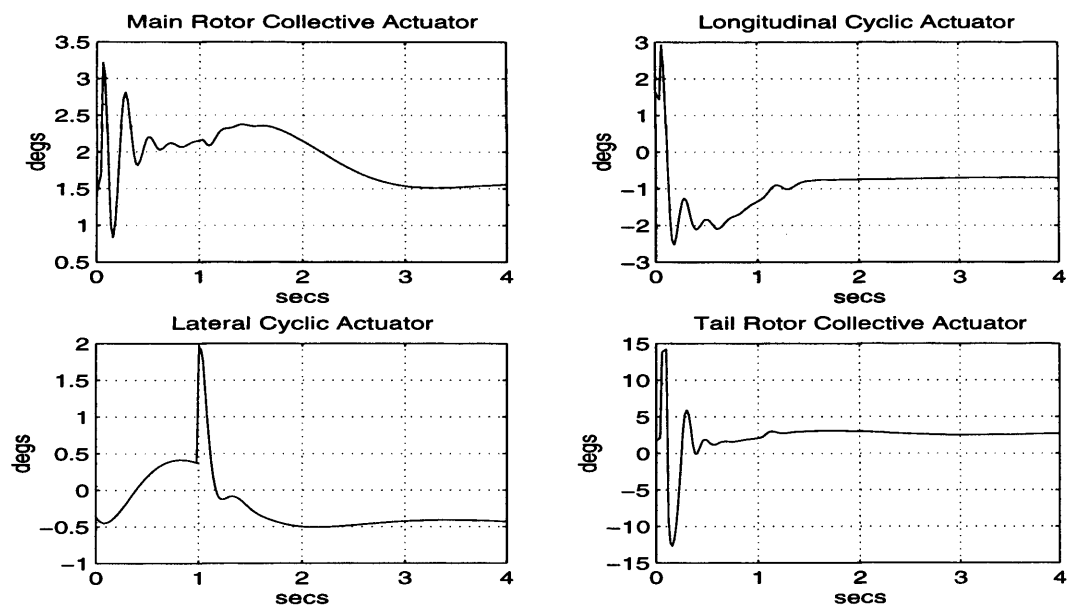


Figure 7.35: Heave Axis Actuator Demands

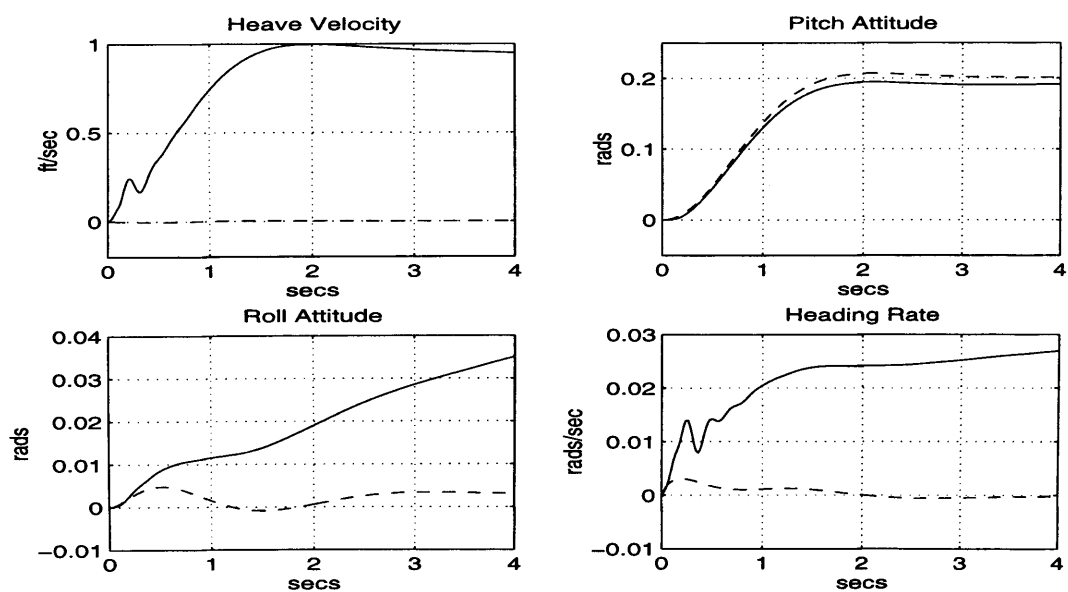


Figure 7.36: Pitch Axis Step Demand

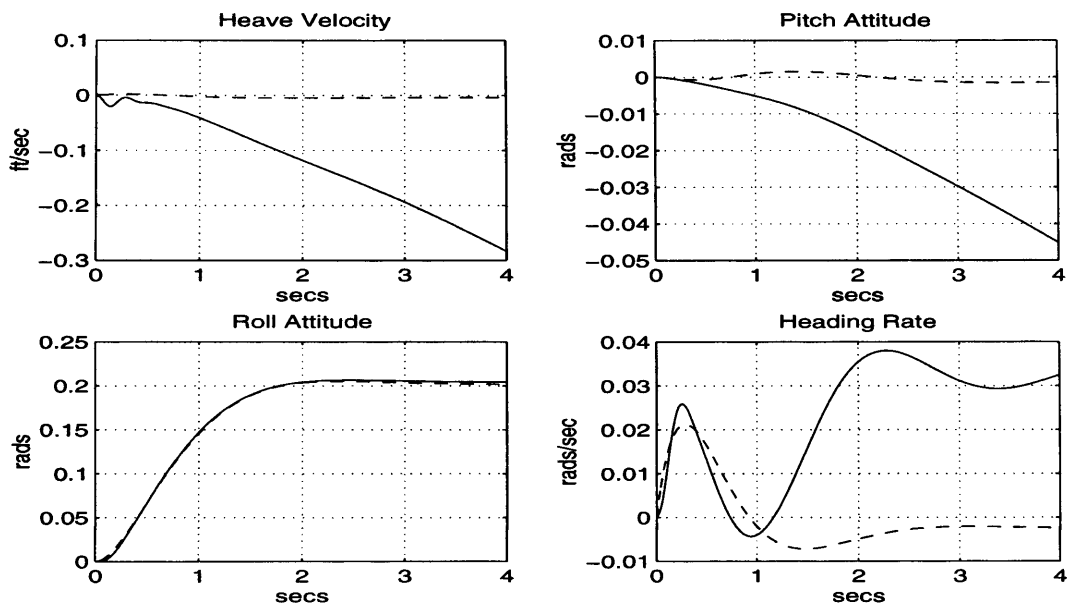


Figure 7.37: Roll Axis Step Demand

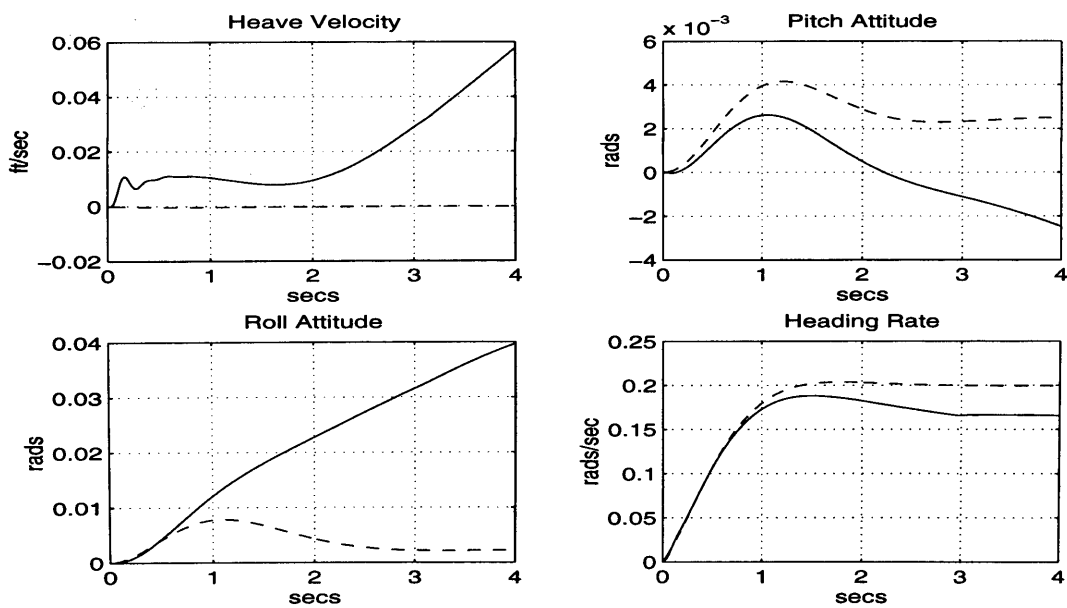


Figure 7.38: Yaw Axis Step Demand

7.7 Summary

This work considers the application of a robust output feedback sliding mode controller design strategy to a helicopter control problem. Most of the mathematical and theoretical developments are presented in Chapter §4. Other mathematical models for tracking, ideal model response design and implementation are presented in this chapter. The ideal model is designed based on an H_∞ 1-DOF technique. A method of implementation of an H_∞ 1-DOF system with the VSC technique as a model matching problem has been discussed. The ideal model following technique enables de-coupling of the internal states. The switching surface is designed based on a robust output feedback pole placement technique. The robustness properties of the switching surface are examined. The controller structure in Section §4.7 is implemented. The controller gain is varied to influence the reaching time to the switching surface. The technique is enable to control the nonlinear multivariable rotorcraft dynamics. The helicopter example shows the effectiveness of the technique. The simulation results demonstrate the applicability of the method; the proposed controllers guarantee the attainment of a sliding mode despite the presence of matched and unmatched uncertainties.

Chapter 8

Dynamic Output Feedback Sliding Mode Control of Aircraft

8.1 Introduction

This chapter is presented to demonstrate an application of the dynamic output feedback sliding mode controller design procedure described in Chapter §6. A system with unstable transmission zeros is considered here. A bench-mark problem of the Group for Aeronautical Research and Technology in Europe (GARTEUR) is considered. It is a six degree of freedom nonlinear aircraft model called the High Incidence Research Model (HIRM). The original system as presented in [1] has 16 states, 11 inputs, including turbulence inputs, and 20 measured outputs although some of these are available only for simulation. The aircraft is basically stable both longitudinally and laterally. There are however combinations of angle of attack and control surface deflection which may cause the aircraft to be unstable longitudinally and/or laterally. The control variables are separated into longitudinal and lateral controls for the purpose of design. The system has unstable transmission zeros in both the longitudinal and the lateral dynamics. The static output feedback sliding mode control described in Chapter §4 is not applicable due to the presence of unstable transmission zeros but the dynamic output feedback sliding mode control method as presented in Chapter §6 is applicable. Robustness considerations are applied during the design. The simulation results are obtained with linearly perturbed systems relating to different operating conditions. The simulation results show the ability of the dynamic output feedback sliding mode controller to con-

trol linear perturbed systems with unstable transmission zeros. The aircraft controller design is a tracking control problem. However the results here are presented in regulator form. Hence, it is not strictly possible to compare the results obtained with the aircraft handling quality requirements. In particular the rate limit of the control action is not considered during design. However the aim of this chapter is clearly demonstrated.

The chapter is presented as follows: The aircraft model is described in Section 8.2. Sections §8.3 & §8.4 present the longitudinal channel controller and simulation results respectively and Sections §8.5 & §8.6 present the lateral channel controller and simulation results respectively. Finally, a brief summary of the chapter and the utility of the method is given in Section §8.7.

8.2 Description of Aircraft Model

The mathematical model of the aircraft dynamics may be represented as a linear time invariant state space model with uncertainties as defined in equations (6.1a)-(6.1b). The HIRM dynamics show redundancy in the control surfaces which creates difficulties in design. For simplicity it is considered that the minimum number of control surfaces are available. These are identified as the symmetrical taileron deflection δ_{TD} , the differential taileron deflection δ_{TS} , the rudder deflection δ_R and the engine throttle δ_{TH} where both the left and right engines have equal command. The nominal linear system triple (A, B, C) is the linearised data at level flight with mach number $\mathcal{M} = 0.3$ at height $h = 5000$ ft., and it is denoted as `init3005`. Similar notation will be used through out this chapter for defining the respective operating point. The longitudinal requirements are given in terms of response to commands in flight path angle and airspeed which are directly available to the controller. The measurement signals for this channel are the pitch rate q and the total velocity V . The longitudinal controller uses the symmetrical command δ_{TS} and the engine commands δ_{TH} assuming that both the engines are identical. The lateral requirements relate to command of sideslip angle β and roll angle ϕ , which are also directly available to the controller. The outputs of the controller are differential taileron command δ_{TD} and rudder command δ_R respectively. The below sections will demonstrate the controller design and simulation results for both the

channels.

8.3 Longitudinal Channel Control

The states, inputs and outputs associated with the longitudinal channels are selected from the complete model. The states associated with the longitudinal dynamics are the longitudinal velocity u , the normal velocity w , the pitch rate q , the pitch angle θ and the engine states $E_{f_{1,2}}$. The latter are merged to achieve equal control action. The two inputs ($\delta_{TS}, \delta_{TH} = \delta_{TH_1} + \delta_{TH_2}$) and two outputs (q, v) are isolated respectively. This gives a six state square plant. The system shows that the matrix CB has rank equal to zero which gives difficulties in the use of output feedback sliding mode techniques. However a balanced model reduction produces the five state square system triple (A, B, C) , essentially removing the second state of the engine, which satisfies the static output sliding mode design criteria; i.e. the input and output matrices B and C both have full rank. It also satisfies the assumptions

A5) the system triple (A, B, C) is completely controllable and observable;

A6) the rank of matrix CB is equal to the number of inputs.

However the static output feedback sliding mode design is not possible since the triple shows mixed (unstable and stable) transmission zeros at $\{21.3254, -0.5128, 0.0018\}$. Hence, the dynamic output feedback sliding mode design is used where the latter assumption is not essential. The controller is simulated at different operating conditions relating to six state linearly perturbed systems. The system regular form is first obtained and the system matrices are partitioned as in equation (3.7) for the design of the compensator parameters and switching surface. The important partitioned matrices are given as

$$\begin{aligned} A_{11} &= \begin{bmatrix} -0.3112 & 0.3388 & 0.2149 \\ -0.1997 & -0.3183 & 0.8459 \\ 0.1295 & -0.8799 & -0.4635 \end{bmatrix} & A_{12} &= \begin{bmatrix} 0.8231 & 0.1082 \\ -0.0865 & -0.4823 \\ 0.2125 & 0.0635 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 1.0387 & -0.1895 & -0.3946 \\ -0.1953 & 0.3384 & 0.3796 \end{bmatrix} & A_{22} &= \begin{bmatrix} -3.3719 & 0.0844 \\ -0.2476 & -0.1344 \end{bmatrix} \end{aligned}$$

$$B_2 = \begin{bmatrix} -0.0000 & 1.9185 \\ 12.4902 & -0.5375 \end{bmatrix} \quad C_1 = \begin{bmatrix} -0.0972 & 8.9589 \\ -0.1487 & 2.8235 \\ 0.4544 & 8.4386 \end{bmatrix}^T \quad C_2 = \begin{bmatrix} -0.0879 & -0.2609 \\ -0.4511 & -0.1172 \end{bmatrix}$$

First consider the compensator parameterisation. In this respect it is necessary to say that the uncertainty distribution is unknown, hence the matrix K has to be obtained such that it stabilises the closed loop matrix $\mathcal{A}_{11} = (\tilde{A}_{11} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{21})$ as in Proposition 6.1. Using an estimator design technique for the observable pair (A_{11}, A_{21}) , the matrix

$$K = \begin{bmatrix} 9.9671 & -1.2126 \\ -13.5034 & 1.3306 \\ 0.1252 & 0.0328 \\ -5.3592 & 0.0672 \\ 0.1864 & -0.0203 \end{bmatrix}$$

is obtained where the matrix $\tilde{K}_2$ is considered as the identity matrix. The size of the index p' is equal to 2. This contributes higher uncertainty to the reduced order sliding dynamics. However, this is accepted since the error dynamics is required to decay rapidly. The transformation $\tilde{T}$ which transforms the system to observer canonical form is then set as

$$\tilde{T} = \begin{bmatrix} 0.1724 & 0.1746 & 0.0671 & -0.0838 & 0.9787 \\ 6.8150 & 6.2953 & 2.3123 & -3.1760 & -1.2239 \\ -12.6854 & -11.8899 & 1.6484 & 6.4940 & 2.5751 \\ -0.0074 & -0.0117 & 0.5369 & -0.1576 & 0.0250 \\ -8.4247 & -7.1573 & 5.2625 & 2.5513 & 1.8045 \end{bmatrix}$$

which produces the observer canonical partitioned matrices in equations (6.7a)-(6.7b)

as

$$\begin{aligned} \mathcal{A}_{11} &= \begin{bmatrix} -3.4740 & 0.0034 & -0.0571 \\ 5.1649 & -2.6995 & -0.0665 \\ -13.2756 & -0.7737 & -2.8266 \end{bmatrix}, & \mathcal{A}_{12} &= \begin{bmatrix} 0.1909 & 0.0155 \\ 22.8482 & -2.0720 \\ -10.5314 & 3.5119 \end{bmatrix} \\ \mathcal{A}_{21} &= \begin{bmatrix} -0.2121 & -0.5200 & -0.2610 \\ -9.8126 & -4.5579 & -3.8164 \end{bmatrix}, & \mathcal{A}_{22} &= \begin{bmatrix} 2.4629 & -0.0335 \\ 12.1035 & 1.9379 \end{bmatrix} \end{aligned}$$

which verify that the eigenvalues of the matrix $\mathcal{A}_{11}$ are stable and the poles are at $\{-4.0, -3.0, -2.0\}$. The stable matrix $\mathcal{A}_{22}^s$ in equation (6.9b) is considered as

$$\mathcal{A}_{22}^s = \begin{bmatrix} -1.0 & 0 \\ 0 & -2.0 \end{bmatrix}$$

which produces the Lyapunov solution of equation (6.11) as

$$P_2 = \begin{bmatrix} 0.50 & 0 \\ 0 & 0.25 \end{bmatrix}$$

for the symmetric positive definite matrix $Q_2 = I_2$. Then the gain matrix L in equation (6.15) is given as

$$L = \begin{bmatrix} 11.3627 & -2.5627 \\ -12.3898 & 2.5256 \\ 5.3603 & -0.0009 \\ -3.3710 & 0.1475 \\ -0.2519 & 0.0293 \end{bmatrix}$$

The state feedback gain matrix K_s is obtained for the controllable matrix pair (A_{11}, A_{12}) as

$$K_s = \begin{bmatrix} 3.0764 & 1.9234 & 0.0329 \\ -1.1534 & -5.4680 & 0.3484 \end{bmatrix}$$

which gives poles at $\{-1.0, -2.0, -3.0\}$ for the reduced order compensator dynamics in equation (6.32). The switching surface is then calculated using equation (6.28) as

$$S = \begin{bmatrix} 3.0764 & 1.9234 & 0.0329 & 1.0000 & 0 \\ -1.1534 & -5.4680 & 0.3484 & 0 & 1.0000 \end{bmatrix}$$

where the matrix S_2 is considered as the identity matrix. Its effect has been discussed in previous chapters and examples. It merely scales the switching surface and has no effect on the sliding mode dynamics.

8.4 Longitudinal Channel Simulation Results

The above compensator and controller parameters are designed with the nominal model init3005 and then simulated at various flight conditions. The uncertainty bound parameters in Section §6.2 are obtained by trial and error as follows $K_f = 0.02$, $K_g = 0.50$

and $K_\gamma = K_a + \gamma_o = 3.50$ which are sufficient to control the range of uncertainties. The positive scalar constant $\alpha = 2.5$ is chosen to bring the state trajectories on the switching surface $\mathcal{S}$ with in reasonable time. The system shows chattering as is natural with a discontinuous control action as explained. As a remedy the small parameter $\delta = 0.01$ is used in the denominator of the discontinuous control vector $\nu(e_y)$. The simulation results at different flight conditions are presented below.

8.4.1 Level Flight with $\mathcal{M} = 0.30$ at $h = 5000$ ft.

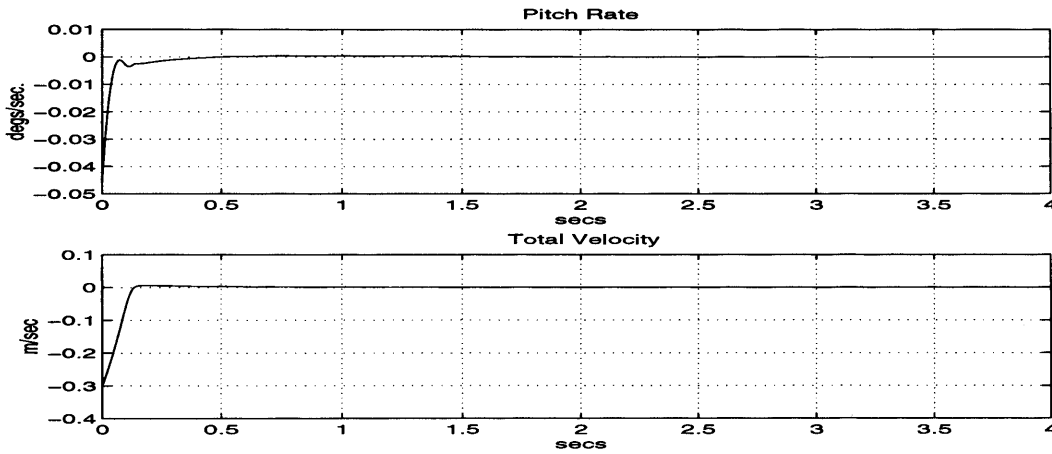


Figure 8.1: Time Response of Error Outputs

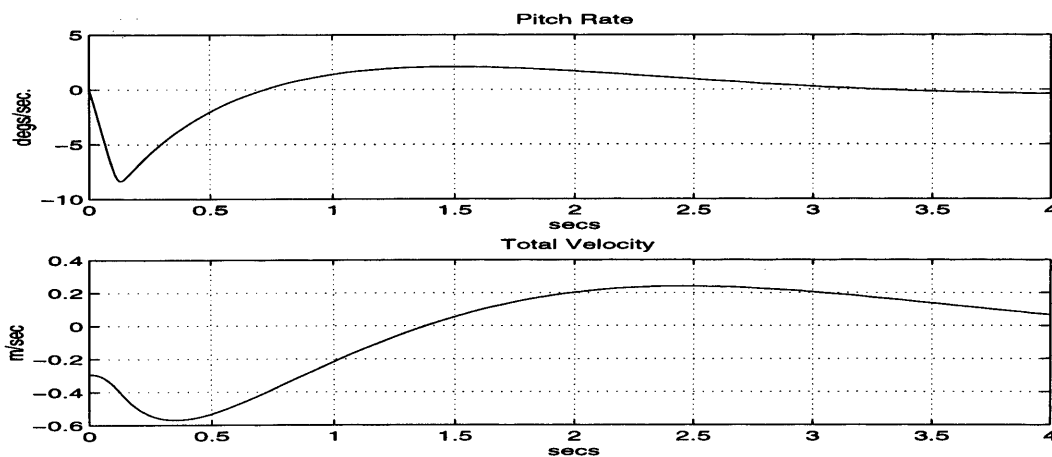


Figure 8.2: Time Response of Plant Outputs

Figure 8.1 shows that the error outputs are small and have gone to zero within 0.75 seconds and 0.25 seconds respectively. This shows the efficiency of the compensator

design. The output of the plant in Figure 8.2 has decreased close to the zero value within reasonable time. The total velocity is small in magnitude. It is difficult to

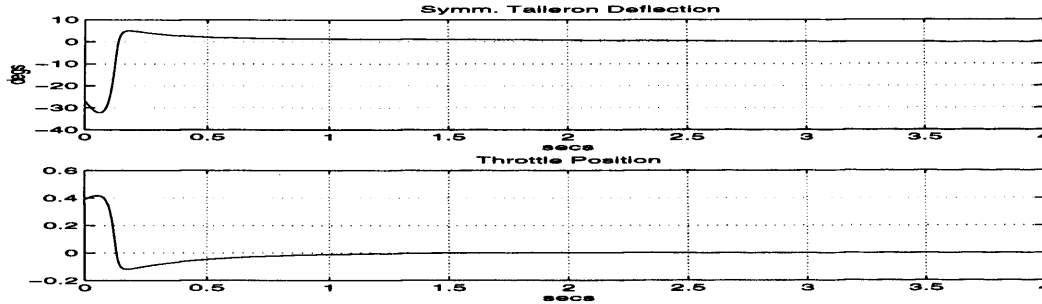


Figure 8.3: Actuator Demands

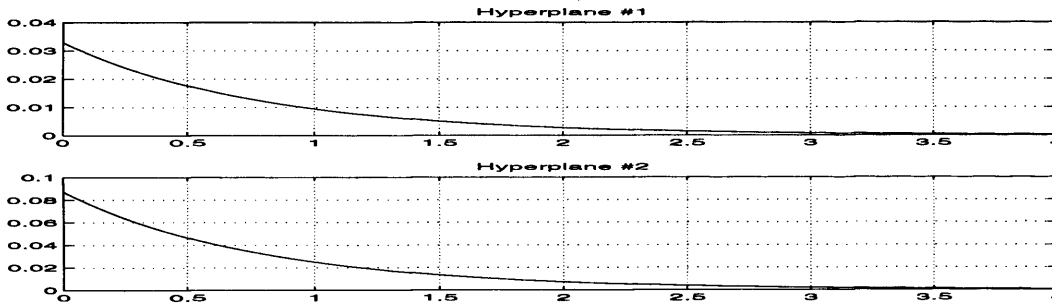


Figure 8.4: Switching Surface Behaviour

understand its appropriateness due to the reduction of the model and elimination of a control surface. However, it clearly demonstrates the ability of the control technique. The control effort is shown in Figure 8.3. This indicates that the control action has reached the maximum limit $[10, -40]$ for symmetrical taileron deflection where the throttle position is reasonable although a slight negative value is achieved which is undesirable since the throttle position is bounded between zero and two. This may be due to numerical error. Figure 8.4 gives the switching surface behaviour of the compensator states. It shows very small values and rapid decay to zero.

8.4.2 Level Flight with $\mathcal{M} = 0.45$ at $h = 10000$ ft.

These simulation results are obtained with linear plant parameters at operating condition init4510. The error outputs in Figure 8.5 are small and reach zero within 0.50 seconds and 0.25 seconds respectively. Figure 8.6 shows the plant output response and

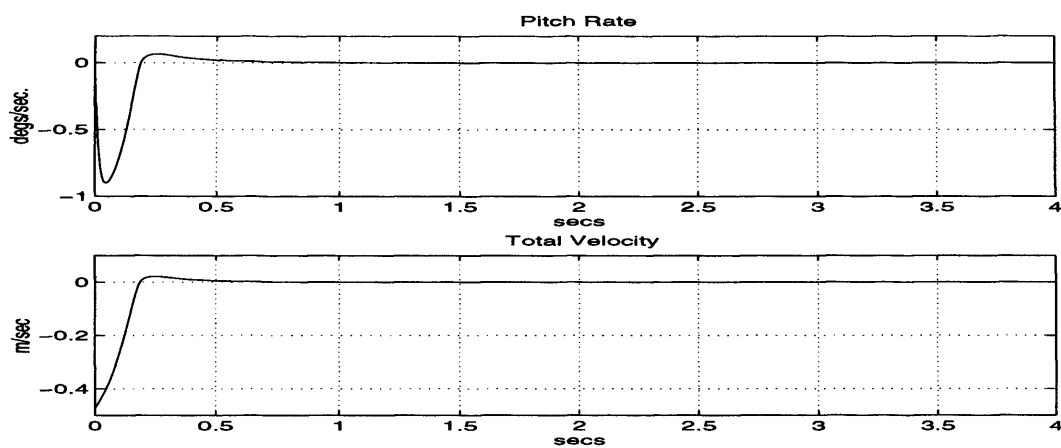


Figure 8.5: Time Response of Error Outputs

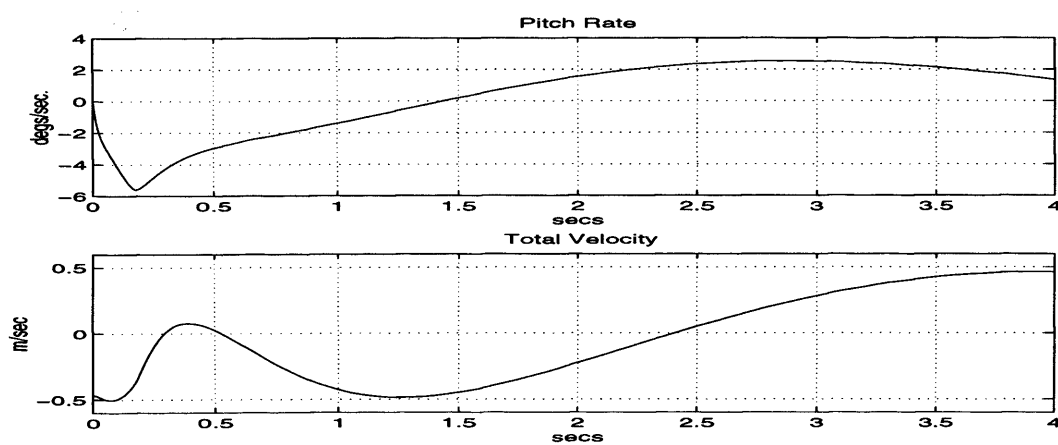


Figure 8.6: Time Response of Plant Outputs

the control action is shown in Figure 8.7. Further the switching surface is plotted in

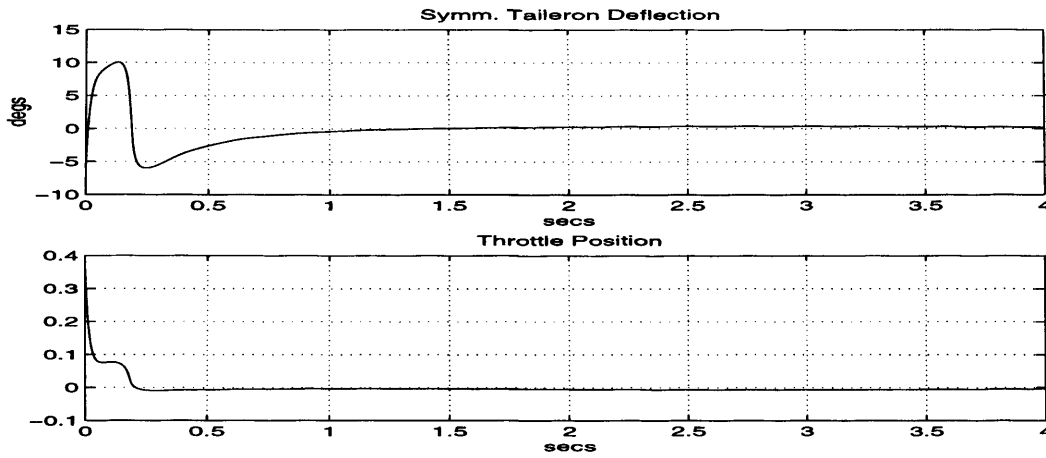


Figure 8.7: Actuator Demands

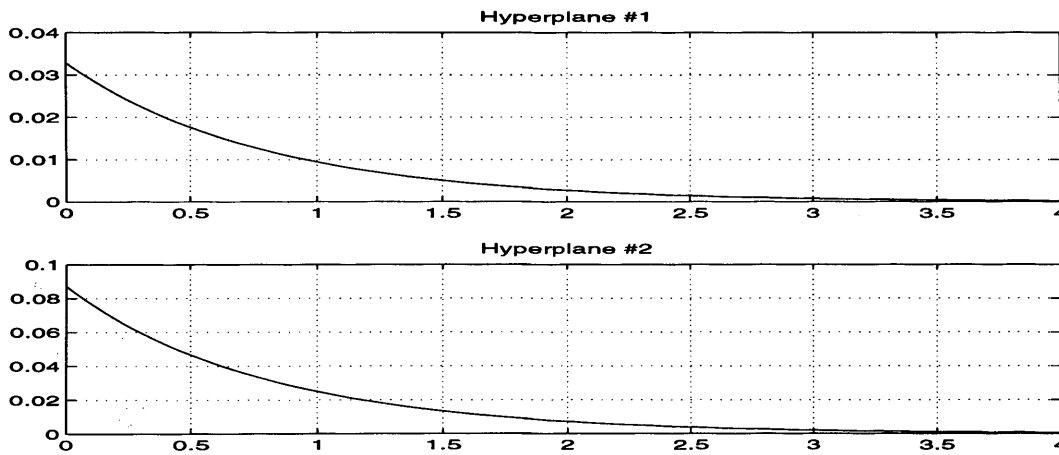


Figure 8.8: Switching Surface Behaviour

Figure 8.8. These responses indicate the ability of the method to control uncertain systems containing unstable invariant zeros.

8.4.3 Level Flight with $M = 0.50$ at $h = 15000$ ft.

These results are obtained at operating condition init5015. These Figures show that there is no significant changes in the behaviour with the results obtained in init4510. This indicates that the controller may be robust enough to control the linear perturbations at different operating conditions. However the controller demands has slightly changed in behaviour when compared with Figures 8.3 & 8.7.

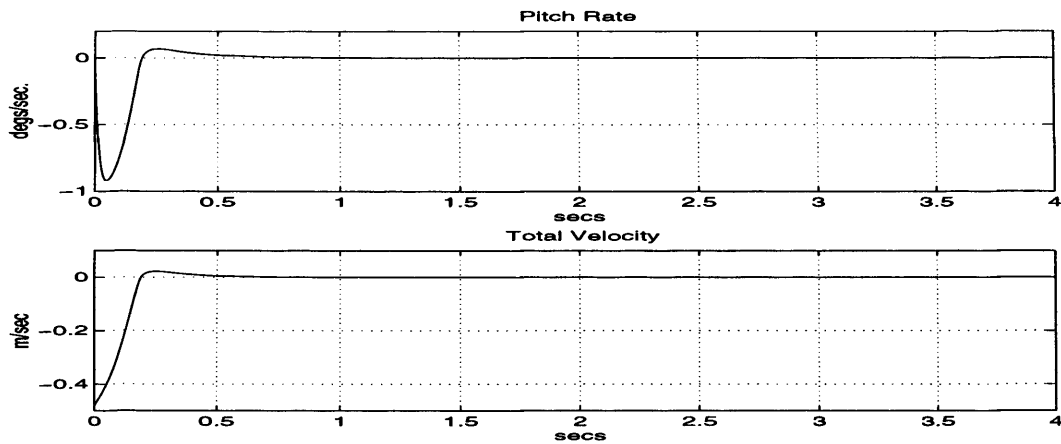


Figure 8.9: Time Response of Error Outputs

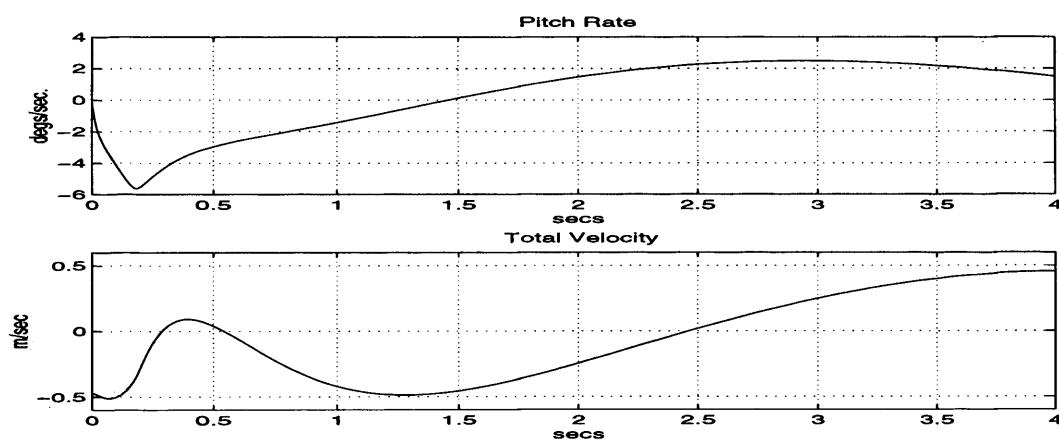


Figure 8.10: Time Response of Plant Outputs

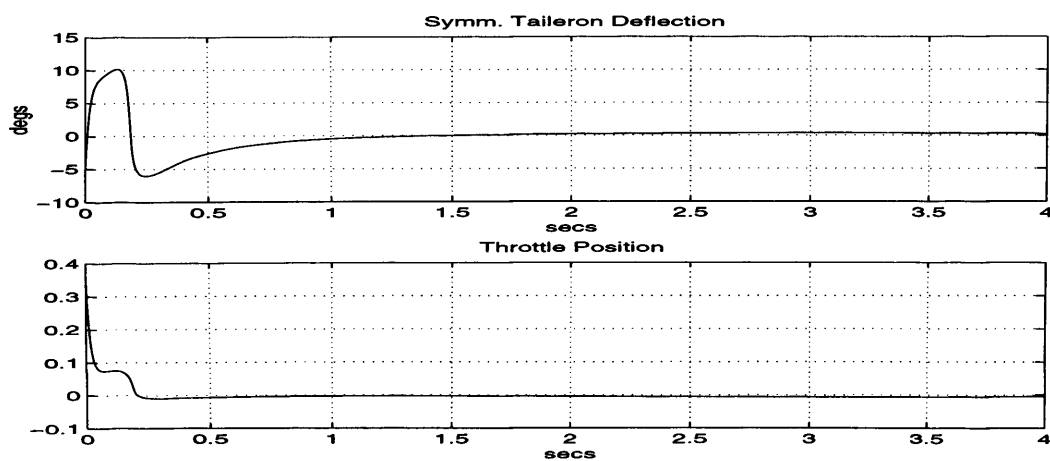


Figure 8.11: Actuator Demands

8.5 Lateral Channel Control

The lateral controller is designed analogously to the longitudinal controller. As for the longitudinal channel, the lateral channel model is also extracted from the complete model. There are four states: the lateral velocity v , the roll rate p , the yaw rate r , and the bank angle ϕ . The two inputs are (δ_{TD}, δ_R) and the two outputs are (p, β) . The lateral system triple (A, B, C) satisfies the design assumptions and contains one unstable and one stable transmission zeros at $\{0.0208, -42.1313\}$ respectively. Hence static output feedback sliding mode design is not possible due to the presence of the unstable transmission zero. The dynamic output feedback sliding mode design is applicable. Obtaining the regular form as in equation (3.7), the partitioned matrices are given as follows

$$\begin{aligned} A_{11} &= \begin{bmatrix} -0.0000 & 0.2037 \\ 3.7909 & -35.4625 \end{bmatrix} & A_{12} &= \begin{bmatrix} -0.2339 & 0.9741 \\ -17.1892 & 3.3176 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} -8.3558 & 77.5905 \\ -2.7992 & 26.6074 \end{bmatrix} & A_{22} &= \begin{bmatrix} 37.4828 & -7.1488 \\ 13.4748 & -3.8492 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 0.0000 & -5.7662 \\ -6.8951 & 1.7364 \end{bmatrix} & C_1 &= \begin{bmatrix} 0 & 0.0088 \\ 0 & 0.0039 \end{bmatrix} & C_2 &= \begin{bmatrix} -0.3140 & 0.9494 \\ -0.0087 & -0.0029 \end{bmatrix} \end{aligned}$$

The matrix K is further obtained to stabilise the closed loop matrix $\mathcal{A}_{11} = (\tilde{A}_{11} - \tilde{K}_1 \tilde{K}_2^{-1} \tilde{A}_{21})$ as in Proposition 6.1, gives

$$K = \begin{bmatrix} 35.5884 & 25.6458 \\ 0.4895 & 38.2335 \\ -0.1164 & -87.9597 \\ 1.0103 & -29.4509 \end{bmatrix}$$

where matrix $\tilde{K}_2$ equals to the identity matrix. In this case the index p' is equal to 2 and the closed loop matrix $\mathcal{A}_{11}$ is a normal matrix which is designed based on multiobjective optimisation technique presented in Appendix C. The transformation matrix

$$\tilde{T} = \begin{bmatrix} -0.9512 & 0.6778 & -10.6797 & 31.9482 \\ 0.3086 & 0.7473 & 4.0202 & -10.7679 \\ 0 & 0.0088 & -0.3140 & 0.9494 \\ 0 & 0.0039 & -0.0087 & -0.0029 \end{bmatrix}$$

is chosen which gives the observer canonical form where the closed-loop matrix $\mathcal{A}_{11}$ is given as

$$\begin{aligned}\mathcal{A}_{11} &= \begin{bmatrix} -2.5000 & 0.0 \\ 0.0 & -2.0000 \end{bmatrix}, & \mathcal{A}_{12} &= \begin{bmatrix} 35.2977 & -362.7966 \\ -5.9610 & 133.5716 \end{bmatrix} \\ \mathcal{A}_{21} &= \begin{bmatrix} 0.2166 & 0.6677 \\ -0.3907 & -0.8946 \end{bmatrix}, & \mathcal{A}_{22} &= \begin{bmatrix} -1.2110 & -13.6616 \\ 3.1047 & 3.8821 \end{bmatrix}\end{aligned}$$

The poles are at $\{-2.5, -2.0\}$. The closed-loop matrix $\mathcal{A}_{11}$ is a symmetric matrix, and hence achieves high robustness due to its normal matrix properties. The stable matrix in equation (6.9b) is

$$\mathcal{A}_{22}^s = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

which gives the Lyapunov solution of equation (6.11) as

$$P_2 = \begin{bmatrix} 0.50 & 0 \\ 0 & 0.25 \end{bmatrix}$$

for the symmetric positive definite matrix $Q_2 = I_2$. The compensator gain matrix in equation (6.15) is obtained as

$$L = \begin{bmatrix} 36.6994 & 50.9661 \\ 123.3956 & 232.0877 \\ -271.0906 & -510.0889 \\ -91.0409 & -185.2755 \end{bmatrix}$$

The state feedback gain matrix K_s is obtained which stabilises the controllable matrix pair (A_{11}, A_{12}) as

$$K_s = \begin{bmatrix} 0.0804 & 2.0836 \\ 1.5592 & 0.7094 \end{bmatrix}$$

which gives the poles at $\{-2.0, -1.5\}$. The switching surface matrix as in equation (6.28) is then given as

$$S = \begin{bmatrix} 0.0804 & 2.0836 & 1.0000 & 0 \\ 1.5592 & 0.7094 & 0 & 1.0000 \end{bmatrix}$$

where the matrix S_2 is further considered as the identity matrix since it does not affect the sliding dynamics.

8.6 Lateral Channel Simulation Results

The simulation results at various operating points are obtained using the above compensator and controller parameters designed with the nominal lateral model of init3005. The uncertainty bound parameters are chosen as $K_f = 0.02$, $K_g = 0.050$ and $K_\gamma = K_a + \gamma_o = 3.0$. The positive scalar constant $\alpha = 3.50$ is chosen which is sufficient for attainment of the switching surface $\mathcal{S}$ within reasonable time. A small parameter $\delta = 0.01$ is again used in the denominator of the discontinuous control vector $\nu(e_y)$ of equation (6.10) to avoid chattering. The simulation results below are presented at different flight conditions.

8.6.1 Level Flight with $\mathcal{M} = 0.30$ at $h = 5000$ ft.

The simulation results are presented with the nominal linear plant model. The efficiency of the compensator design is demonstrated in Figure 8.12. The error outputs have attained the zero value within less than 0.12 seconds. The plant outputs in Figure 8.13

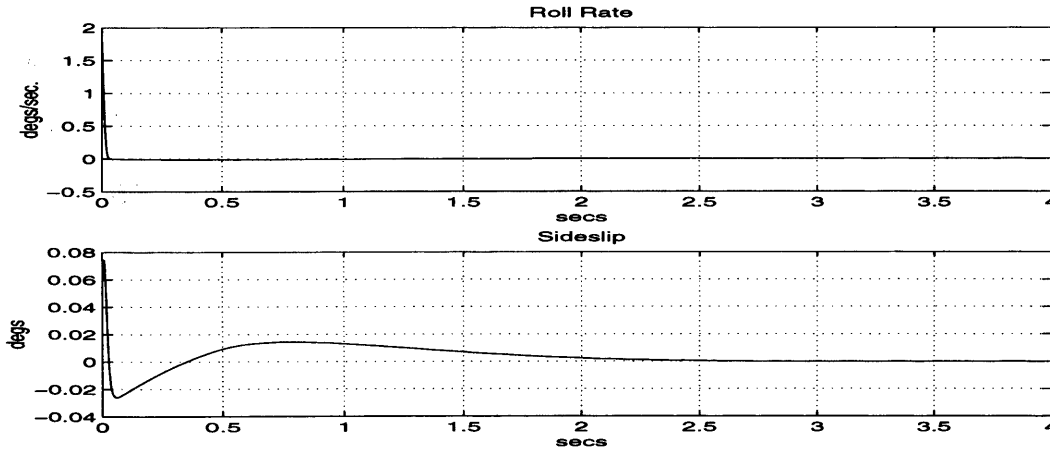


Figure 8.12: Time Response of Error Outputs

has achieved the zero value in 2.5 seconds and 2.0 seconds respectively. Sideslip response is little higher when compared with other operating conditions. However it is within reasonable limits. The control action is shown in Figure 8.14 and is reasonable for the differential taileron deflection but the rudder deflection has reached the maximum limit of $[30, -30]$. The control action is too steep, since the rate limit is not considered during design to demonstrate that the outputs reach to zero in reasonable time. The

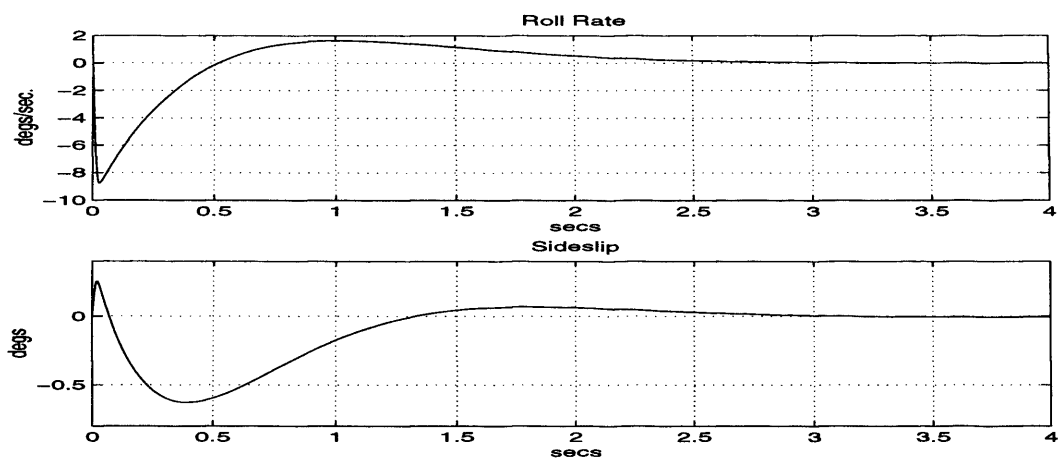


Figure 8.13: Time Response of Plant Outputs

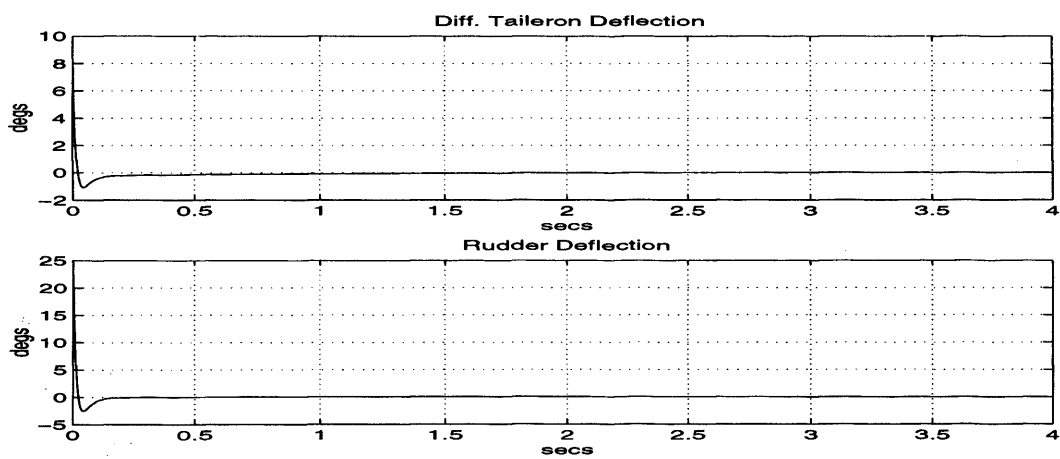


Figure 8.14: Actuator Demands

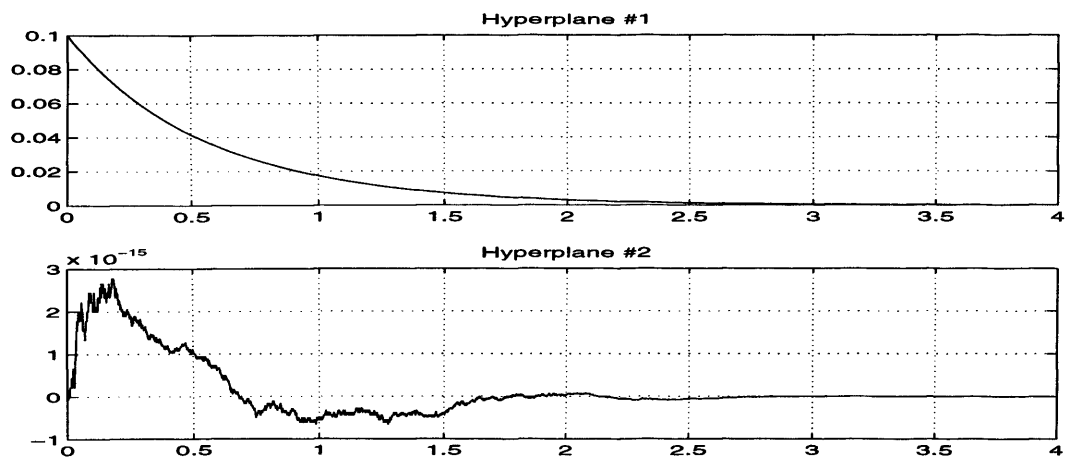


Figure 8.15: Switching Surface Behaviour

switching surface in Figure 8.15 shows small magnitude and the system has attained a sliding mode at 3.0 seconds.

8.6.2 Level Flight with $M = 0.45$ at $h = 10000$ ft.

These results are obtained at the operating condition of init4510 for the lateral control channel. The Figures 8.16, 8.17 & 8.18 show the error output, plant output and control effort respectively. The error shows small value and has attained zero within less than

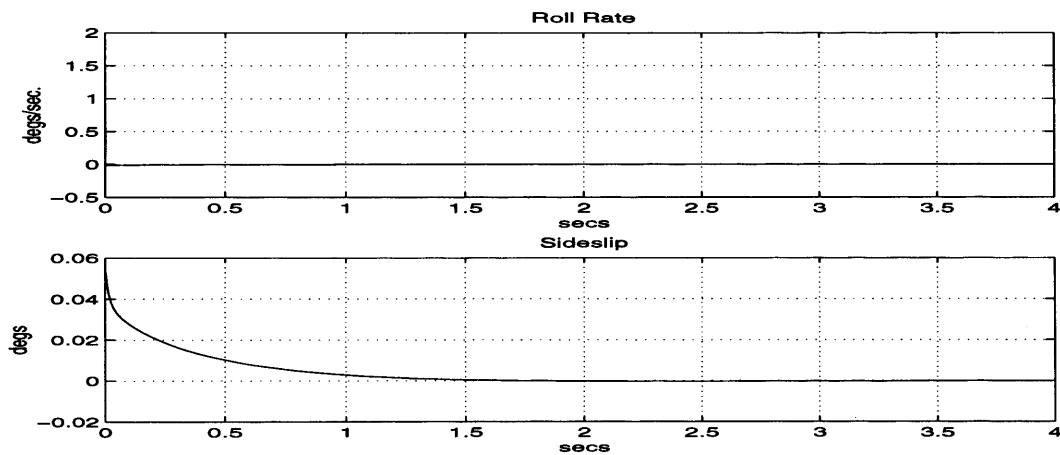


Figure 8.16: Time Response of Error Outputs

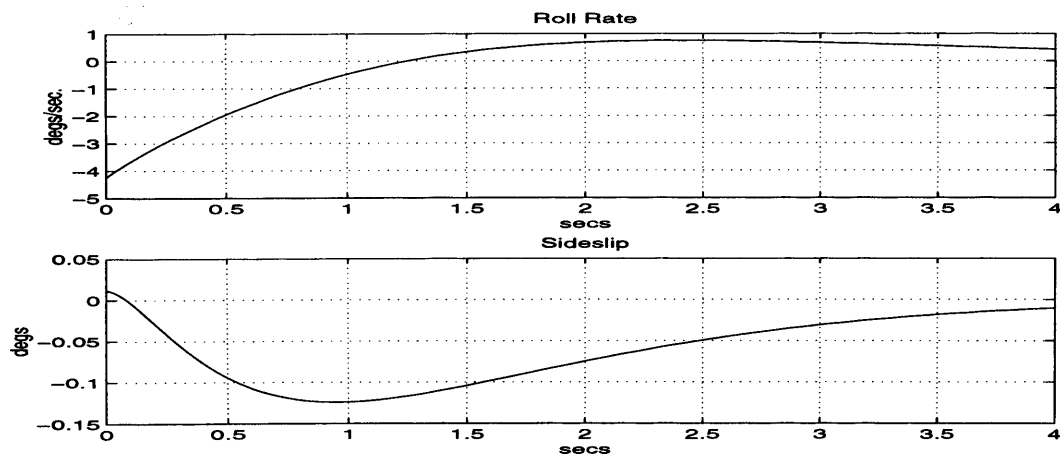


Figure 8.17: Time Response of Plant Outputs

0.12 seconds. The roll rate has reached a constant value and the sideslip is small as necessary. The controller demand for rudder deflection has reached the maximum limit

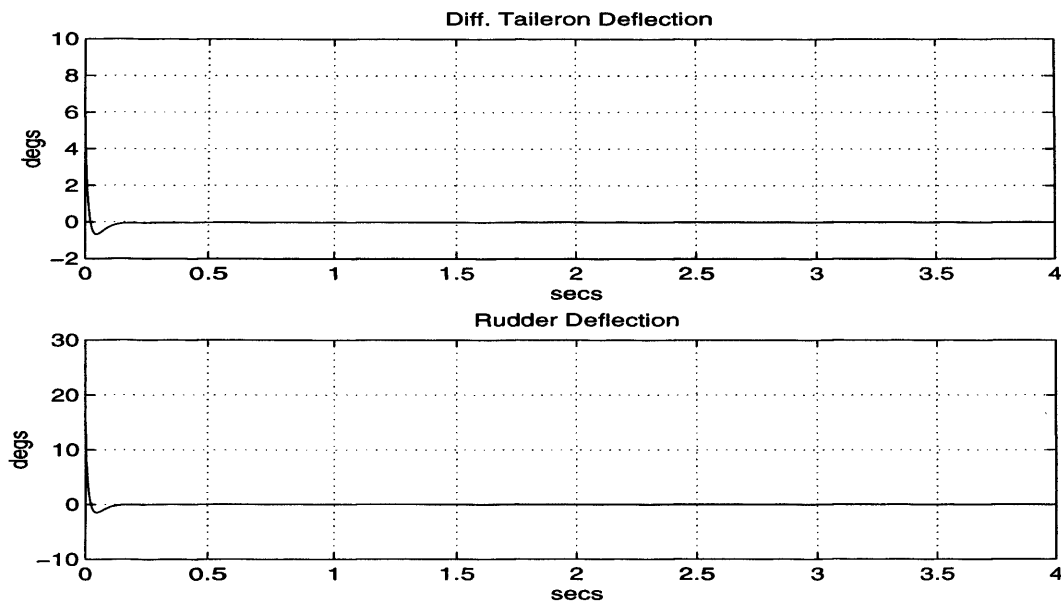


Figure 8.18: Actuator Demands

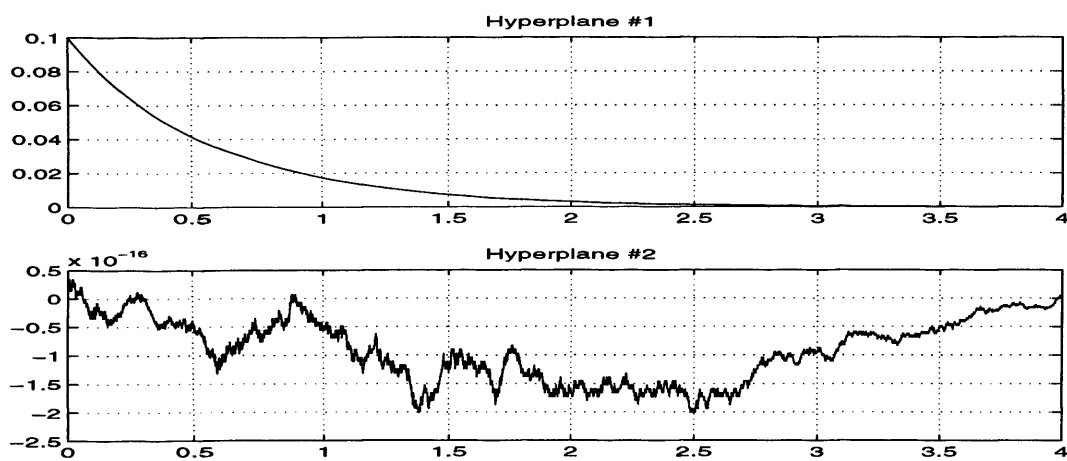


Figure 8.19: Switching Surface Behaviour

and the differential taileron deflection shows reasonable control action is applied. The switching surface attains its sliding mode within 3.0 seconds.

8.6.3 Level Flight with $\mathcal{M} = 0.5$ at $h = 15000$ ft.

The simulation results presented in Figures 8.20, 8.21 & 8.22 are represented as error outputs, plant outputs and control action respectively for the linear plant model at operating condition init5015. Further the controller demand for rudder deflection has

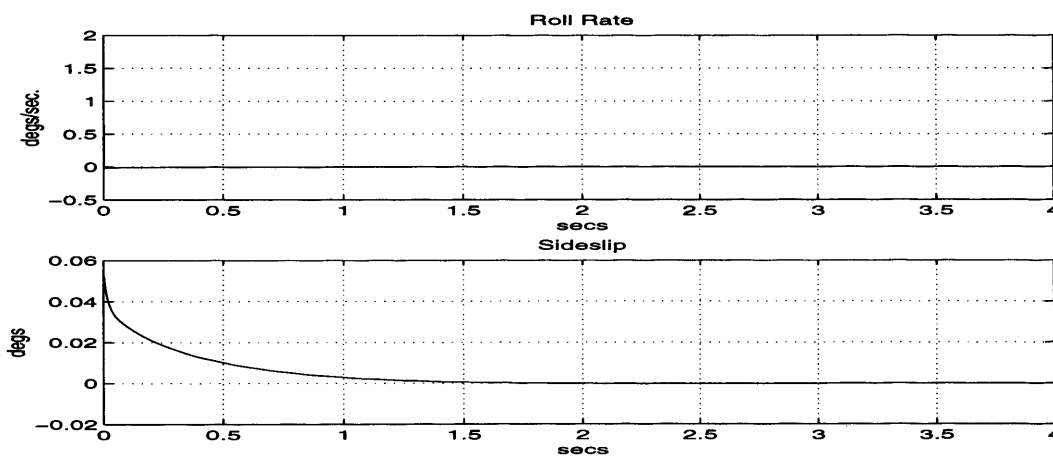


Figure 8.20: Time Response of Error Outputs

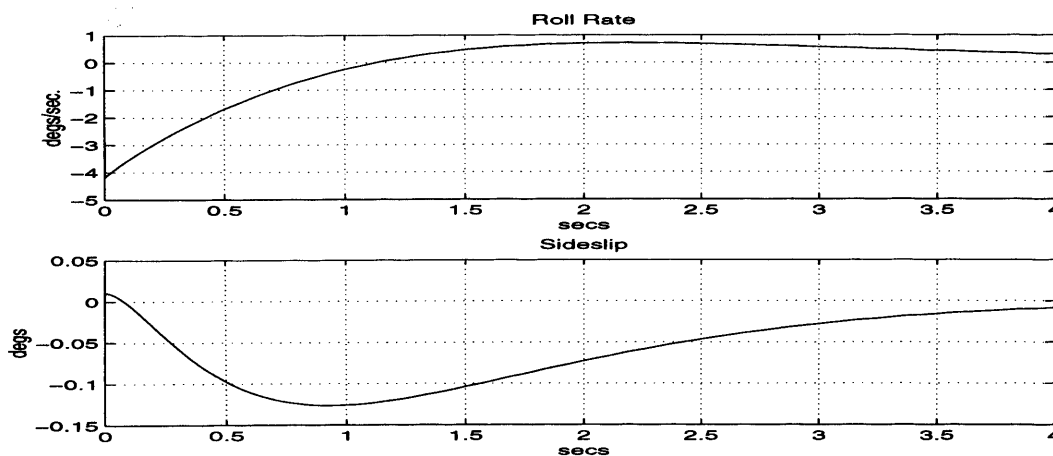


Figure 8.21: Time Response of Plant Outputs

attained maximum value. Figure 8.23 shows the switching surface. The results show the efficiency of the method.

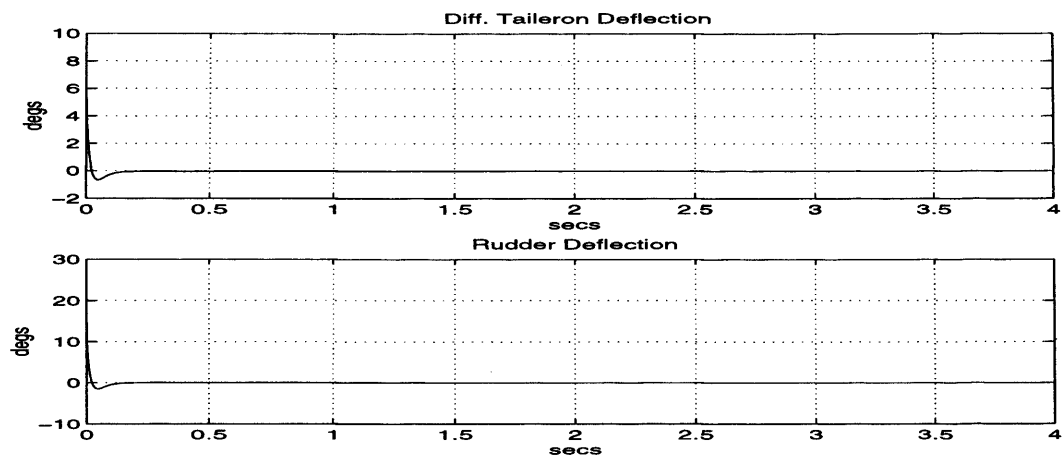


Figure 8.22: Actuator Demands

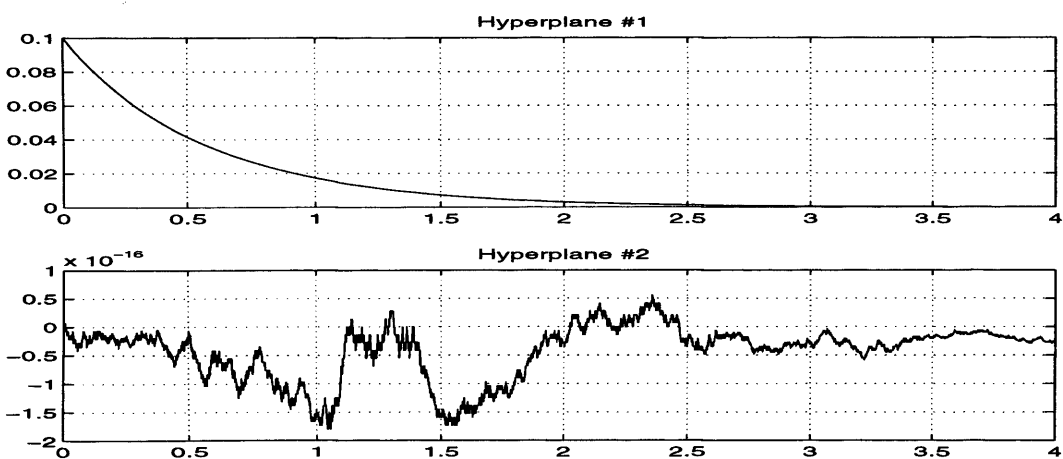


Figure 8.23: Switching Surface Behaviour

8.7 Summary

This chapter is presented to illustrate the application of the theory in Chapter §6 with the industrial example of an aircraft model which is nonlinear in nature and has variations in handling characteristics. For ease of design, the complete aircraft dynamics is divided into two sub-dynamics although this is not essential. Both the system dynamics have unstable transmission zero. The theoretical developments in Chapter §6 are then utilised to design the controllers. During design it is seen that the indexing parameter p' has not been reduced. However the system dynamics has maintained stability. This reflects the use of the robust design method. The simulation results are obtained at various operating points. The simulation results show the regulation of the outputs of the plant although exact specifications are not met. The response shows the efficiency of the technique. The results strongly demonstrate that the method is applicable to systems with unstable invariant zeros. These results justify the theoretical developments.

Chapter 9

Conclusions and Future Work

9.1 Concluding Remarks

This thesis has considered the problem of designing robust sliding mode controllers using output information. The theoretical developments are presented in the early chapters and the applications are presented in later chapters. Two different approaches have been proposed: firstly, the development of static output feedback sliding mode controllers, where no additional dynamics are used except the plant dynamics, has been considered; and secondly, the development of dynamic output feedback sliding mode controllers for systems which do not fall into the first category have been developed. The resulting closed loop system has additional dynamics other than the plant dynamics which gives extra freedom in design and performance. Two different industrial examples involving helicopter and aircraft control have been used to show the effectiveness of the theory. The major contributions of this work are outlined below:

- The first method uses only the plant dynamics, i.e. only the plant outputs are fed thorough to the controller. The analysis of the reduced order closed-loop sliding dynamics is an output feedback problem and the unmatched uncertainty affects the stability and performance. Results for quadratic stability have been achieved. It has been shown that the unmatched uncertainty effects can be reduced by using a robust output feedback design method. It is known that for a normal matrix the eigenvalues are insensitive to matrix perturbations, and hence the insensitivity of the reduced order closed loop sliding dynamics may be maximised by using a

normal matrix design approach. Such a normal matrix design method has been proposed to prescribe robustness and examples show the effectiveness.

- The static output feedback sliding mode techniques require a particular design triple to satisfy the ‘Kimura-Davison’ condition and any transmission zeros to be stable. The stable invariant zeros problem has been solved. Essentially the transmission zeros are left in their respective positions. The problems with the ‘Kimura-Davison’ condition and unstable transmission zeros have been solved using additional dynamics which leads to the development of dynamic output feedback sliding mode design methods.
- The second method uses a dynamic compensator driven by the plant outputs. A minimum order dynamic compensator is used to solve the ‘Kimura-Davison’ problem. This method may also be used to generate extra freedom in controller design even if the system does not need a compensator. The reduced order closed-loop sliding dynamics is formulated as a static output feedback problem. The unmatched uncertainty affects the reduced order sliding dynamics as in the case of static output feedback sliding mode control. Hence the compensator parameters and sliding parameters are obtained together using a robust output feedback design method. The examples demonstrate the applicability of the theory. Both the unmatched and matched perturbations are considered in the examples. However the controller has been formulated based only on matched uncertainty and the robust approach has been used in the closed-loop design. This reflects the novelty of the robust technique and the use of a dynamic compensator to handle the unmatched uncertainty and ‘Kimura-Davison’ problems.
- The other type of compensator uses the full order dynamics of the plant. This method basically uses a state feedback sliding mode controller and extends the idea to non-minimum phase uncertain systems. Basically an observer type non-linear dynamic compensator is designed and the augmented closed loop dynamics are considered. This work shows that if a gain matrix is obtained such that the linear compensator design triple is minimum phase then a quadratically stable sliding motion can be achieved for the plant dynamics. More over, it is seen

that a particular class of uncertainty may be decomposed into matched and unmatched uncertainties, and only the unmatched uncertainty affects the reduced order augmented sliding dynamics. Further a robust design approach is necessary to maximise the insensitivity to such unmatched uncertainty. A normal matrix design based on a state feedback control is developed. The examples further show the effectiveness of the technique. The closed-loop analysis shows that the outputs of the plant exactly equal the outputs of the compensator. Hence, for a regulation problem, if the compensator outputs are stable then the plant outputs are also stable. An estimated state control law is used and the stability of the combined closed loop system is observed. It is demonstrated that the controller and compensator can be designed independently. In other words, the well known separation principle for linear systems holds for this class of uncertain systems controller and compensator design. The examples reflect the use of this robust technique in minimising the uncertainties.

- The novelty of the robust static output feedback sliding mode design approach has been reflected in the controllers designed for a helicopter system. The dynamics are nonlinear in nature and have variations in operating conditions. The helicopter dynamics satisfy the conditions for static output feedback sliding mode controller design and have stable invariant zeros. A nominal linear model has been obtained at the hover operating condition and a robust controller design has been performed as stated in Chapter §4. In addition, a model following technique is utilised so that the plant outputs faithfully track the ideal model outputs. Fully nonlinear simulation results are obtained at various operating conditions which show good output tracking in the model following sliding mode control. This industrial example reflects the applicability of the robust static output feedback sliding mode technique.
- Another industrial example is the aircraft dynamics which demonstrates the use of the robust dynamic output feedback sliding mode technique. The dynamics are again nonlinear and vary with operating conditions. The aircraft dynamics have been separated into two sub-problems, i.e. the longitudinal channel and the lateral channel. A linear nominal model is obtained and both longitudinal and

lateral dynamics satisfy the output feedback sliding mode conditions but have unstable invariant zeros. Hence the modified dynamic output feedback sliding mode control is applicable. The robust technique is used to design the controllers. The problem is formulated to regulate the plant outputs as presented in Chapter §6 although the aircraft control is a tracking problem. Hence the aircraft handling quality specifications are not followed. However the results demonstrate the use of dynamic output feedback sliding mode techniques to handle non-minimum phase uncertain systems. The simulation results are obtained using linearly perturbed systems at various operating points. The results show the use of a dynamic compensator for a non-minimum phase system.

Both the results above demonstrate the applicability and the effectiveness of robust output feedback sliding mode control in real nonlinear multivariable industrial problems. These two industrial multivariable control problems should convince the reader of the merits of introducing ‘robust sliding mode control using output information’ into a real system where the robustness and practicality of the procedures are substantiated by the examples.

9.2 Recommendations for Future Work

- Throughout the thesis, especially from a theoretical standpoint, in Chapters [§4, §5, §6] where ‘discontinuous’ control components are considered in all proofs and induce ideal sliding motions. However, for simulation (and/or practical implementation), the discontinuous component has been replaced by an approximate ‘continuous’ control component. This avoids the chattering caused by the discontinuous control but it also reduces the robust properties. A ‘trade-off’ is necessary in such cases [21]. An alternative approach may be studied by reformulating these controllers, the effect of this modification could be rigourously explored using a ‘practical stability’ argument as used in Ryan and Corless [88] and Spurgeon and Davies [98] and formal performance bounds could be achieved.
- One major step towards verifying such output feedback sliding mode controllers is to implement them on a real plant. This would provide a series of exciting and

challenging practical issues. From an implementation viewpoint, it is relevant to examine the effect of output noise, vibration, anti-windup, complex nonlinearities etc. on the sliding mode performance which has not been addressed in this thesis. Some relevant work has appeared in the literature [29, 93, 117].

- In Chapters §4 and §5 it is seen that the square plant has no design freedom in output feedback sliding mode design unless it is used with a full order compensator/estimator as in Chapter §6. This increases the total dynamics. Hence it is useful to study a minimum order dynamic compensator which can produce some design freedom and/or circumvent the problems associated in the controller design of the square plant and/or the non-minimum phase systems.
- Output tracking for non-minimum phase systems is not addressed in this thesis. An appropriate tracking controller formulation for this system may be studied. The complete nonlinear behaviour of a non-minimum phase industrial problem with dynamic output feedback sliding mode controller has not been done. This represents an area of future research.
- In Chapters §4 and §5, it is assumed that the system satisfies the controllability assumption and the rank of (CB) is equal to the number of inputs. In Chapter §6, an additional assumption of observability is added. Many practical systems do not satisfy the assumption 'rank(CB) equal to the numbers of inputs'. This may give some clue to future work using dynamic output feedback sliding mode control.
- A more divergent and speculative area of future work is to investigate the discrete time versions of the robust output feedback sliding mode control considered in this thesis. From a practical viewpoint, it is important that the sampling period required for sliding mode approach is very small due to the nonlinear nature of the control structure. A discrete time framework would consequently be beneficial for this type controller.

Appendix A

Mathematical Preliminaries

A.1 Matrix Properties

Consider the matrix partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where the sub-blocks $P_{11} \in \mathbb{R}^{l \times l}$ and $P_{22} \in \mathbb{R}^{r \times r}$. The following properties of the matrix P hold.

1. If the matrix $P_{21} = 0$ then

- $\det(P) = \det(P_{11}) \det(P_{22})$
- $\sigma(P) = \sigma(P_{11}) \cup \sigma(P_{22})$

2. If the matrix $P = P^T$ and $P_{12} = P_{21}^T$, i.e. the matrix is symmetric and positive definite then

- $P > 0 \Rightarrow P_{11} > 0$ and $P_{22} > 0$
- $P_{11} > P_{12}P_{22}^{-1}P_{21}$ and $P_{22} > P_{21}P_{11}^{-1}P_{12}$

where $P > 0$ implies the eigenvalues of the matrix P are positive.

A.2 Controllability and Observability

Consider the linear time invariant system in equations (2.16) and (2.18) then the below definition and properties are exist.

Definition A.1 *The system is said to be completely controllable if any given initial condition $x(t_0)$ there exists an input function such that in finite time t the system $x(t) = 0$.*

If the above definition holds then the following conditions are equivalent :

- the matrix $\text{rank}[B \ AB \ A^2B \ \dots \ A^{n-1}B] = n$
- the matrix $[sI_n - A \ B]$ has full rank for all $s \in \mathbb{C}$
- the spectrum of $(A + BF)$ can be assigned arbitrarily by choice of gain matrix $F \in \mathbb{R}^{m \times n}$

Definition A.2 *The system is said to be completely observable if the output function $y(t)$ of equation (2.18) uniquely determines the initial condition $x(t_o)$ over a small time interval $[t_o, \ t]$.*

The matrix pair (A, C) is completely observable if and only if the matrix pair (A^T, C^T) is completely controllable then the above properties are true for the matrix pair (A^T, C^T) .

Appendix B

Stability and Boundedness

B.1 Lyapunov Stability

Consider the nonlinear system defined as

$$\dot{x}(t) = F(t, x) \quad (\text{B.1})$$

where $x \in \mathbb{R}^n$ and $F : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $F(t, 0) = 0$, i.e. the system has equilibrium point at origin. If a generalised energy function can be found which is non zero except at an equilibrium point and whose total time derivative decreases along the system trajectories then the equilibrium point is stable [94]. Define a scalar function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the quadratic form

$$V(x) = x^T P x \quad (\text{B.2})$$

where $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite matrix and the function is non zero except at origin.

Definition B.1 *The equilibrium point of the nonlinear system in equation (B.1) is said to be quadratically stable if there exists asymmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$ such that the total time derivative satisfies*

$$\dot{V}(x) = 2x^T P F(t, x) \leq -x^T Q x \quad (\text{B.3})$$

This implies that $\dot{V} \leq \gamma V$ where $\gamma = \lambda_{\min}(Q)/\lambda_{\max}(P)$ and hence the state is asymptotically convergence to the origin with rate of at least γ . If the equation (B.1) represents

a linear function as $F(t, x) = Ax(t)$ where the matrix A must have stable eigenvalues if and only if it satisfies the Lyapunov equation

$$PA + A^T P = -Q \quad (\text{B.4})$$

Therefore the linear system is quadratically stable. However in practice for an uncertain system, it may not be possible to guarantee asymptotic stability but bounded stability may be defined.

B.2 Ultimate Boundedness

Consider $\mathcal{E} \in \mathbb{R}^n$ as a bounded set such that if the trajectories $x(t)$ enter $\mathcal{E}$ in finite time and remain there for all subsequent time, i.e. the system energy always decreases outside the set $\mathcal{E}$ then the system is unable to escape the boundary of the set $\mathcal{E}$. This may be defined as

Definition B.2 *An uncertain system with state $x(t)$ is said to be globally uniformly ultimately bounded with respect to the bounded set $\mathcal{E} \in \mathbb{R}^n$ if:*

- *for each uncertainty realisation $F(t, x)$ and for each $(t_o, x(t_o)) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists at least one solution $x(\cdot) : [t_o, t_1) \rightarrow \mathbb{R}^n, t_1 > t_o$;*
- *given any real number $\delta > 0$, there exists a real number $d(\delta) > 0$ such that, for any solution $x(\cdot) : [t_o, t_1) \rightarrow \mathbb{R}^n$ with $\|x(t_o)\| \leq \delta, \|x(t)\| \leq d(\delta)$ for all $t \in [t_o, t_1)$; all solutions can thus be continued over $[t_o, \infty]$;*
- *for every $x(t_o) \in \mathbb{R}^n$ there exists a non-negative constant $T(x(t_o), \mathcal{E}) \in \mathbb{R}_+$ such that, for every solution $x(\cdot) : [t_o, \infty] \rightarrow \mathbb{R}^n$ with $t_o \in \mathbb{R}_+$ arbitrary, $x(t) \in \mathcal{E}$ for all $t \geq t_o + T(x(t_o), \mathcal{E})$.*

The set $\mathcal{E}$ is usually small boundary around origin and the concept is often called *practical stability*.

Appendix C

Numerical Algorithms

C.1 Method for Nonlinear Least-Square Solution

Solution of nonlinear least square problem using general Gauss-Newton iterative method is summarised using the cost function defined as

$$J(x) = \frac{1}{2}(FX)^T(FX) \quad (C.1)$$

Using the iterative formula of Ortega and Rheinboldt [82], the above equation (C.1) can be written as

$$x^{i+1} = x^i - \left[F'(x^i)^T F'(x^i) \right]^{-1} F'(x^i)^T F x^i \quad (C.2)$$

where, $i = 0, 1, 2, \dots$, the point x^* , the minimum of the function x can be derived, assuming that the inverse of the above expression is exists, where $F'(x^i)$ is the Frechet derivative with respect to x^i . The corrected form of the general Gauss-Newton iterative formula can be transformed to improve the convergence rate with the help of two weighted parameter ω_i and λ_i , such that

$$x^{i+1} = x^i - \omega_i \left[F'(x^i)^T F'(x^i) + \lambda_i I \right]^{-1} F'(x^i)^T F x^i \quad (C.3)$$

In this equation $F'(x^i)^T F'(x^i)$ is a symmetric positive semi-definite matrix. If $\lambda_i > 0$, then the inverse in above equation is always exists and the parameter ω_i can be chosen so that the $J(x^{i+1}) \leq J(x^i)$, hence the convergence rate can be controlled.

C.2 Method for Symmetric Matrix Design

Consider the matrix defined in equation (4.88) and setting the initial value of $\alpha = 0.0$, giving

$$\mathcal{L}(N) = (FC)^T FCA(I - NC) \quad (\text{C.4})$$

which is required to be negative semi-definite in order to satisfy the reachability condition. The symmetric matrix may be written as

$$\mathcal{L}(M) = M_o + \sum_{i=1}^{np} n_i M_i \quad (\text{C.5})$$

where the matrix

$$M_o = \left[\frac{(FC)^T FCA + (FCA)^T FC}{2} \right] \quad (\text{C.6})$$

is the constant part of the symmetric matrix $\mathcal{L}(M)$ and the let M_i is defined as

$$M_i = \left[\frac{(FC)^T FCAN_i C + C^T N_i^T (FCA)^T FC}{2} \right] \quad (\text{C.7})$$

The value of n_i is defined as the i^{th} element of the vector formed by stacking the columns of the N matrix : $[n_{11}, n_{21}, \dots, n_{n1}; n_{12}, n_{22}, \dots, n_{n2}; \dots; n_{1p}, n_{2p}, \dots, n_{np}]^T$. Since the matrix N is a gain matrix and straightforward minimisation of the eigenvalue might yield too large gain. To avoid this problem define the weighted average of the norm of matrix N and then minimise the maximum eigenvalue of the matrix $\mathcal{L}(M)$. This is defined as the quantity to be minimised :

$$J(N) = \lambda_{max}\{\mathcal{L}(M)\} + \rho \|N\|^r \quad (\text{C.8})$$

where $\rho \geq 0$ is a constant and r is the type of norm to be minimised [57].

C.3 Method for Robust Multi-objective Matrix Design

Consider the state feedback closed-loop matrix

$$A_c = A_q - B_q K_q \quad (\text{C.9})$$

where the matrices $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$ and the matrix gain $K_q \in \mathbb{R}^{m \times n}$ is to be designed assuming that the matrix pair (A_q, B_q) is completely controllable. The normal

matrix design approach described in Section §4.6 is applicable in this case. Theorem 4.1 is to be valid for the closed-loop matrix A_c . Using Theorem 4.1 and equation (C.9), the equation (4.64) can be rewritten as

$$\left[P^T B_q K_q + B_q K_q P \right] = \left[Q + P^T A_q + A_q P \right] = \tilde{Q} \quad (\text{C.10})$$

Using the Kronecker product, the above equation becomes

$$(P^T B_q \otimes I_n + B_q \otimes P^T) rs K_q = rs \tilde{Q} \quad (\text{C.11})$$

For simplicity, this is further written as

$$\begin{aligned} Y rs K_q &= rs \tilde{Q} \\ rs K_q &= Y^\dagger rs \tilde{Q} \end{aligned} \quad (\text{C.12})$$

where $Y = (P^T B_q \otimes I + B_q \otimes P^T) \in \mathbb{R}^{nn \times mn}$, $rs K_q \in \mathbb{R}^{mn \times 1}$ and $rs \tilde{Q} \in \mathbb{R}^{nn \times 1}$ represent the column vectors spanned by the rows of the matrices K_q and $\tilde{Q}$ respectively and $Y^\dagger = (Y^T Y)^{-1} Y^T$ represents the pseudo inverse of Y . The solution of equation (C.12) can be written as nonlinear least-square solution as defined in equation (4.73), rewriting

$$J_1 = \|Y(P) rs K_q - rs \tilde{Q}(P, Q)\|^2 \quad (\text{C.13})$$

for all matrices $P > 0$ and $Q = Q^T > 0$. Introducing the second constrain as

$$B_1 = B_1 - K_q B_2 \quad (\text{C.14})$$

must be minimum where B_1 and B_2 are two appropriate matrices. This can be written using the Kronecker product of equation (C.14) and substituting the equation (C.12), the solution can be written as follows

$$J_2 = \|rs B_1 - I \otimes B_2^T Y^\dagger rs \tilde{Q}\|^2 \quad (\text{C.15})$$

The overall cost function is then given as

$$J = \alpha J_1 + \beta J_2 \quad (\text{C.16})$$

where α and β are positive weighting factors. If the equation (C.16) is minimised using matrices $P > 0$ and $Q = Q^T > 0$, then the corresponding solution of K_q produces a robust stable closed-loop matrix of equation (C.9), hence minimise the effect of uncertainty and also minimising $\|B_1\|$, which gives minimum contributions of the matched uncertainty into the reduced order channel. This way the closed-loop system may be design for an optimal performance.

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