Finite generation of Ext and (D, A)-stacked algebras

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Abstract. We introduce the class of (D, A)-stacked algebras, which generalise the classes of Koszul algebras, d-Koszul algebras and (D, A)-stacked monomial algebras. We show that the Ext algebra of a (D, A)-stacked algebra is finitely generated in degrees $0, 1, 2$ and 3. After investigating some general properties of $E(\Lambda)$ for this class of algebras, we look at a regrading of $E(\Lambda)$ and give examples for which the regraded Ext algebra is a Koszul algebra. Following this we give a general construction of a (D, A)-stacked algebra $\tilde{\Lambda}$ from a d-Koszul algebra Λ, setting $D = dA$, with $A \geq 1$. From this construction we relate the homological properties of $\tilde{\Lambda}$ and Λ, including the projective resolutions and the structure of the Ext algebra.

This thesis is dedicated to my greatest achievement in this world, my children, without whom I would not have found the motivation to succeed.

To Lorren, Kirsty, Leyton, Riley and Reece, everything worth having is worth fighting for.

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Contents

1. Introduction

With complicated, non-commutative algebras came the need to try and simplify them. Representation theory is a vast branch of Mathematics, with the fundamental idea that we can represent something complex by something that is easier to understand but still retaining the properties that we wish to understand. There are various ways of doing this but the concept we use in this thesis is to consider the modules and cohomology of an algebra given by quiver and relations.

In particular, we are interested in the Ext algebra, $E(\Lambda)$, of a finite-dimensional algebra Λ, where $\Lambda = K\mathcal{Q}/I$ for K a field, $\mathcal{Q}$ a finite quiver and I an admissible ideal. This was introduced into the mainstream by Gabriel in 1972, (see [1]) with his widely known theorem and classification of finite quivers. In this thesis we look at when the Ext algebra is finitely generated as an algebra. One class of algebras where it is known that the Ext algebra is finitely generated is the class of Koszul algebras, which were introduced by Priddy in 1970, [25]. He introduced the notion of a Koszul algebra whilst studying Steenrod algebras and defined this class of algebras, which were a subset of the quadratic algebras, and for which the calculation of $E(\Lambda)$ was simple to determine via the Koszul resolution. Since then, Koszul algebras and their generalisations have occurred in many different places in algebra and they have been the focus of numerous papers, such as the two papers by Green and Martínez-Villa, [15, 16].

In this thesis we introduce and study a class of finite-dimensional algebras, which are a generalisation of Koszul algebras, by extending the (D, A)-stacked monomial algebras of Green and Snashall, introduced in [18]. There are many generalisations of Koszul algebras in the literature, including D-Koszul algebras which were introduced by Berger [5] and the δ-Koszul algebras of Green and Marcos [12]. The class of (D, A)-stacked monomial algebras of [18] are a natural extension of Koszul monomial algebras and D-Koszul monomial algebras; we now extend the theory to non-monomial (D, A)-stacked algebras.

We begin the thesis with some background information on finite-dimensional algebras given by quiver and relations, and an introduction to Koszul algebras and their generalisations. In Chapter 2 we remind the reader of the projective resolution of a module over such an algebra. In Chapter 3, we describe the construction of the Ext groups $\mathrm{Ext}^n_\Lambda(M, N)$ for modules M, N over a finite-dimensional algebra Λ, and the minimal projective resolution of $\Lambda/\mathfrak{r}$ of Green, Solberg and Zacharia, given in [20], where $\mathfrak{r}$ is the Jacobson radical of Λ. This is followed by an explicit example of an algebra $\Lambda = K\mathcal{Q}/I$, where we calculate the minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module and compute its Ext groups. This algebra will be shown to be a (D, A)-stacked algebra further on, in Chapter 5.

Chapter 4 looks at Koszul algebras and some of their generalisations. A natural question to ask is when is the Ext algebra of a finite-dimensional algebra itself finitely generated as an algebra? It is well known that the Ext algebra of a Koszul algebra is finitely generated in degrees 0 and 1. The D-Koszul algebras of [5] were shown to have their Ext algebra generated in degrees $0, 1$ and 2 in [13] by Green, Marcos, Martínez-Villa and Zhang. In [18], Green and Snashall introduced (D, A)-stacked monomial algebras and proved in [19] that the Ext algebra of these algebras is generated in degrees $0, 1, 2$ and 3. After reviewing these classes of algebras, we then give explicit examples, firstly of a 3-Koszul algebra and then a Koszul algebra.

In Chapter 5, we define the new class of (D, A)-stacked algebras. We show that these algebras include the previously mentioned Koszul, D-Koszul and (D, A)-stacked monomial algebras. The main result of this chapter is Theorem 5.7 where we show that the Ext algebra of a (D, A)-stacked algebra is finitely generated as an algebra.

Theorem 5.7 Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra with $D \geq 2$ and $A \geq 1$. Then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3.

In Chapter 6 we explicitly look at some general properties of the Ext algebra for our class of (D, A)-stacked algebras and provide a characterisation of (D, A)-stacked algebras in Theorem 6.8.

Theorem 6.8 Let $\Lambda = K\mathcal{Q}/I$ where I is generated by homogeneous elements of length $D \geq 2$. Then $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is length graded. Suppose, in the minimal projective resolution, (P^n, d^n), of Λ_0 that P^3 is generated in a single degree, $D + A$, for $A \geq 1$. Then Λ is a (D, A)-stacked algebra if and only if $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3 and the following conditions hold:

i) $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0$, if $D \neq 2$;

ii) $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, for all n odd, $n \geq 1$, if $D > 2, D \neq A + 1$;

iii) $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, for all n even, $n \geq 2$, if $D > 2, A > 1$; and

iv) $\mathrm{Ext}^{2m+1}_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^{2n+1}_\Lambda(\Lambda_0, \Lambda_0) = 0$, for all $m, n \geq 1$, if $D \neq 2A, D > 2$.

Chapter 7 is motivated by the work of Green, Marcos, Martínez-Villa and Zhang in [13]. In this paper the authors take the Ext algebra of a D-Koszul algebra and show that after regrading it is a Koszul algebra. This prompts the question 'Can the Ext algebra of a (D, A)-stacked algebra be regraded as a Koszul algebra?' We give a regrading of the Ext algebra, and then devote the rest of the chapter to discussing when this regraded algebra is Koszul. In particular, we give a $(6, 2)$-stacked non-monomial algebra, and through the working of this example and the use of Gröbner bases, we prove that our regrading of its Ext algebra is Koszul. This is followed by an example of a $(6, 2)$-stacked monomial algebra, where we also prove that its regraded Ext algebra is a Koszul algebra. We finish this chapter with a $(4, 2)$-stacked algebra whose regraded Ext algebra is not Koszul, along with a general subclass of (D, A)-stacked algebras, namely the $(2m, m)$-stacked algebras, with $m \geq 2$, where the Ext algebra cannot be regraded as a Koszul algebra.

In the rest of the thesis we take a different approach to (D, A)-stacked algebras. The aim of Chapter 8 is to give a precise construction of a (D, A)-stacked algebra, $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$, from a given d-Koszul algebra, $\Lambda = K\mathcal{Q}/I$, where $D = dA$. For any chosen $A \geq 1$, this construction gives a unique algebra. We begin with the construction of the new quiver $\tilde{\mathcal{Q}}$ and the new ideal $\tilde{I}$ thus defining a new algebra $\tilde{\Lambda}$ from a d-Koszul algebra Λ. We then define a map $\theta : \Lambda \to \tilde{\Lambda}$, which with a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, allows us to explicitly describe a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module. Our main result is:

Theorem 8.15 Let Λ be a d-Koszul algebra. Let $A \geq 1$ and set $D = dA$. With the given construction, the algebra $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ is a (D, A)-stacked algebra.

Chapter 9 begins with the question 'What is the relationship between $E(\Lambda)$ and $E(\tilde{\Lambda})$?'. Throughout this chapter we investigate this question, and construct a K-algebra homomorphism $\Psi : \mathrm{Ext}_{\Lambda}^{\geqslant 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \mathrm{Ext}_{\tilde{\Lambda}}^{\geqslant 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ in Theorem 9.15.

As a next stage, it is natural to ask what the relationship is between the Hochschild cohomology rings of Λ and $\tilde{\Lambda}$. Although we are not yet able to fully answer this question, in the final chapter of this thesis, Chapter 10, we construct a minimal projective bimodule resolution of $\tilde{\Lambda}$ from a given minimal projective bimodule resolution of Λ. We use the map $\theta : \Lambda \to \tilde{\Lambda}$ given in Chapter 8, to define a new map $\phi : \Lambda^e \to \tilde{\Lambda}^e$. Our construction of a minimal projective bimodule resolution of $\tilde{\Lambda}$ is given in Theorem 10.16, which is the main result of this chapter. This naturally leads to the question of how $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$ are related. This question would be a good topic for further research.

2. Preliminaries

This chapter recalls some of the basic definitions needed to look at algebras given by quiver and relations, and their modules. The definitions and concepts here can be found in many good books on representation theory, including [1], [2] and [6].

We begin with a description of an algebra by quiver and relations.

Definition 2.1. A quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$, consists of two sets, $\mathcal{Q}_0$ which is a set of vertices and $\mathcal{Q}_1$ which is a set of arrows, together with two maps $s, t : \mathcal{Q}_1 \longrightarrow \mathcal{Q}_0$, which associate to each arrow $\alpha \in \mathcal{Q}_1$ its source $s(\alpha)$ and its target $t(\alpha)$.

Definition 2.2.
(1) A quiver $\mathcal{Q}$ is said to be finite if $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are both finite sets.
(2) The underlying graph of the quiver $\mathcal{Q}$ is denoted $\overline{\mathcal{Q}}$ and means $\mathcal{Q}$ with no orientation on the edges (arrows).
(3) The quiver $\mathcal{Q}$ is said to be connected if $\overline{\mathcal{Q}}$ is connected.

Definition 2.3.
(1) Let $\mathcal{Q}$ be a quiver and let $a, b \in \mathcal{Q}_0$. A path from a to b is a sequence of arrows α_i such that the first arrow has source a and the last arrow has target b, and if the α_i are ordered so that the path is $\alpha_1\alpha_2\cdots\alpha_n$ then $s(\alpha_1) = a$ and $t(\alpha_n) = b$, with $s(\alpha_{i+1}) = t(\alpha_i)$, for each $1 \leq i \leq n-1$. The length of the path is n and each arrow is of length 1.
(2) The set of paths of length n is denoted by $\mathcal{Q}_n$; this ties in well since the set of paths of length 1 is $\mathcal{Q}_1$.
(3) Associated with each vertex a is a trivial path e_a; this is a path of length 0.

Note that the above definition means that we write our paths from left to right.

Definition 2.4. Let $\mathcal{Q}$ be a finite quiver and let K be a field. The path algebra $K\mathcal{Q}$ of $\mathcal{Q}$ is the K-algebra whose underlying vector space has the set of all paths in $\mathcal{Q}$ as its basis. Multiplication is given by concatenation of paths. Thus the product of two paths β_1 and β_2 is defined as $\beta_1\beta_2$ if $t(\beta_1) = s(\beta_2)$, and $\beta_1\beta_2 = 0$ if $t(\beta_1) \neq s(\beta_2)$.

We assume throughout this thesis that $\mathcal{Q}$ is a finite quiver and K is a field.

It follows that $K\mathcal{Q}$ has a direct sum decomposition,

$$K\mathcal{Q} = K\mathcal{Q}_0 \oplus K\mathcal{Q}_1 \oplus \cdots \oplus K\mathcal{Q}_n \oplus \cdots$$

where $K\mathcal{Q}_i$ is the subspace generated by $\mathcal{Q}_i$, the set of paths of length i.

Given any two of these subspaces $K\mathcal{Q}_n$ and $K\mathcal{Q}_m$ then $K\mathcal{Q}_n \cdot K\mathcal{Q}_m \subseteq K\mathcal{Q}_{n+m}$, so $K\mathcal{Q}$ is a graded K-algebra.

Definition 2.5. Let $\mathcal{Q}$ be a quiver. The two sided ideal of the path algebra $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$ is called the arrow ideal of $K\mathcal{Q}$, denoted $R_{\mathcal{Q}}$.

Definition 2.6. Let $\mathcal{Q}$ be a finite quiver, $R_{\mathcal{Q}}$ the arrow ideal of $K\mathcal{Q}$. A two sided ideal I of $K\mathcal{Q}$ is admissible if there exists $m \geq 2$ such that $R_{\mathcal{Q}}^m \subseteq I \subseteq R_{\mathcal{Q}}^2$.

One of the main reasons for using quivers to describe our algebras, is that, essentially, every finite-dimensional algebra can be described in this way. This is due to the following result by Gabriel, for which see [7].

Theorem 2.7. *Any basic finite-dimensional algebra over an algebraically closed field K is isomorphic to $K\mathcal{Q}/I$ for some unique quiver $\mathcal{Q}$ and admissible ideal I.*

We say that an algebra of the form $K\mathcal{Q}/I$ for some admissible ideal I of $K\mathcal{Q}$ is given by quiver and relations. If I is generated by a set of paths in $K\mathcal{Q}$ then we say $K\mathcal{Q}/I$ is a monomial algebra. Throughout this thesis all our algebras $K\mathcal{Q}/I$ are finite-dimensional.

We now look at modules, as one of the main tools we will be using is the projective resolution of a simple module.

Definition 2.8. Let R be a ring. An R-module M is simple if the only submodules of M are 0 and M.

Definition 2.9. Let R be a ring. An R-module M is projective if whenever $f : X \longrightarrow Y$ is an epimorphism and $g : M \longrightarrow Y$ is a homomorphism then there exists $h : M \longrightarrow X$ such that $fh = g$, that is, the diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, "h"'] \arrow[d, "g"] & \\ X \arrow[r, "f"'] & Y \arrow[r] & 0 \end{tikzcd}$$

Proposition 2.10. *Let R be a ring and let e be an idempotent in R, that is, $e^2 = e$. Then eR is projective as a right R-module. In particular, R is a projective R-module.*

Definition 2.11. Let R be a ring. A complex is a sequence of R-modules M_n, together with homomorphisms d^n

$$\cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \xrightarrow{d^{n+2}} \cdots$$

such that $d^{n+1} \circ d^n = 0$ for all n.

Definition 2.12. An exact sequence of R-modules is a complex

$$\cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \xrightarrow{d^{n+2}} \cdots$$

such that $\operatorname{Im} d^n = \operatorname{Ker} d^{n+1}$ for all n.

Definition 2.13. Let R be a ring. A projective resolution of an R-module M is an exact sequence

$$\cdots \longrightarrow P^m \xrightarrow{d^m} P^{m-1} \xrightarrow{d^{m-1}} \cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

such that all the P^m are projective R-modules.

In the next chapter we look at minimal projective resolutions for modules over a finite-dimensional algebra $\Lambda = K\mathcal{Q}/I$, following the construction of Green, Solberg and Zacharia [20].

3. Minimal Projective Resolutions and the Ext Algebra

In this chapter we look at minimal projective resolutions and the Ext algebra. We start with a definition of the Ext groups, taken from [1], [24] and [26].

Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra over a field K with Jacobson radical $\mathfrak{r}$.

Definition 3.1. Let M, N be right Λ-modules, and let

$$\cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

be a projective resolution of M as a right Λ-module. Apply $\operatorname{Hom}_\Lambda(-, N)$ to give the complex

$$\cdots \longrightarrow \operatorname{Hom}_\Lambda(P^{n-1}, N) \xrightarrow{d^{n-1*}} \operatorname{Hom}_\Lambda(P^n, N) \xrightarrow{d^{n*}} \operatorname{Hom}_\Lambda(P^{n+1}, N) \longrightarrow \cdots$$

where $d^{n*} : \operatorname{Hom}_\Lambda(P^n, N) \to \operatorname{Hom}_\Lambda(P^{n+1}, N)$ is the map induced from $d^{n+1} : P^{n+1} \to P^n$. The module $\operatorname{Ext}^n_\Lambda(M, N)$ is defined by $\operatorname{Ext}^n_\Lambda(M, N) = \operatorname{Ker} d^{n*}/\operatorname{Im} d^{n-1*}$ for all $n \geq 0$.

Theorem 3.2. *Let M, N be right Λ-modules. The module $\operatorname{Ext}^n_\Lambda(M, N)$ is independent of the choice of projective resolution of M, for all $n \geq 0$.*

The above theorem means that we can use any projective resolution of M as a right Λ-module to determine the Ext groups $\operatorname{Ext}^n_\Lambda(M, N)$. We will consider minimal projective resolutions. A projective resolution

$$\cdots \longrightarrow P^m \xrightarrow{d^m} P^{m-1} \xrightarrow{d^{m-1}} \cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

of a Λ-module M is minimal if $\operatorname{Im} d^m \subseteq \operatorname{rad}(P^{m-1}) = P^{m-1}\mathfrak{r}$ for all $m \geq 1$.

We are interested in the Ext groups $\operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ for $n \geq 0$, so we use a minimal projective resolution of the right Λ-module $\Lambda/\mathfrak{r}$. We use the resolution of Green, Solberg and Zacharia, given in [20]. We give a brief summary of the construction in this paper. We begin with hereditary algebras, for which see [6], and the definition of a uniform element.

Definition 3.3. Let R be a ring. Then R is hereditary if every submodule of a projective module is also projective.

Theorem 3.4. *Let $K\mathcal{Q}$ be a path algebra. Then $K\mathcal{Q}$ is hereditary.*

Corollary 3.5. *Let $K\mathcal{Q}$ be a path algebra. For every $x \in K\mathcal{Q}$, the $K\mathcal{Q}$-module $xK\mathcal{Q}$ is projective.*

Definition 3.6. Let x be an element in $K\mathcal{Q}$. Then x is uniform if there exist vertices $e_i, e_j \in \mathcal{Q}_0$ such that $x = e_i x = x e_j$.

We now look at [20] and a minimal projective resolution (P^n, d^n) for $\Lambda/\mathfrak{r}$.

Let $\Lambda = K\mathcal{Q}/I$ and let $\Lambda/\mathfrak{r} = \bigoplus_i S_i$, a finite sum of simple modules. Choose a set of elements $\{g^0\}$ of $K\mathcal{Q}$ such that $\bigoplus_i g_i^0 K\mathcal{Q}/\bigoplus_i g_i^0 I$ is a projective Λ-module and there is a surjective map $d : \bigoplus_i g_i^0 K\mathcal{Q}/\bigoplus_i g_i^0 I \mapsto \Lambda/\mathfrak{r}$. Without loss of generality, we can choose $g_i^0 = e_i$ for each vertex e_i of $K\mathcal{Q}$. Let $d^0 : \bigoplus_i g_i^0 K\mathcal{Q} \to \Lambda/\mathfrak{r}$ be the canonical surjection of $K\mathcal{Q}$-modules; then we have

$$0 \to \operatorname{Ker} d^0 \to \bigoplus_{i \in \mathcal{Q}_0} g_i^0 K\mathcal{Q} \to \Lambda/\mathfrak{r} \to 0.$$

We can now choose a set of elements $\{g_i^{1*}\}$ of $\bigoplus_{i\in\mathcal{Q}_0} g_i^0 K\mathcal{Q}$ such that $\bigoplus_i g_i^{1*} K\mathcal{Q} = \operatorname{Ker} d^0$. From this set we eliminate the elements that are contained in the set $\bigoplus_i g_i^0 I$, and let $\{g_i^1\}$ be the subset of $\{g_i^{1*}\}$ containing the elements that are not in $\bigoplus_i g_i^0 I$. Clearly we have $\bigoplus_i g_i^1 K\mathcal{Q} \subseteq \bigoplus_i g_i^0 K\mathcal{Q}$.

The remaining terms are defined inductively. We assume that we have the sets $\{g_i^k\}$, for all $k = 0, \dots, n$. To find $\{g_i^{n+1}\}$ we proceed as follows. Consider all the elements in the intersection $(\bigoplus_i g_i^n K\mathcal{Q}) \bigcap (\bigoplus_j g_j^{n-1} I)$. We stop if this intersection is zero and set it equal to $\bigoplus_l g_l^{n+1*} K\mathcal{Q}$ otherwise. From this set $\{g_l^{n+1*}\}$ we eliminate the elements that are contained in the set $\bigoplus_i g_i^n I$ and take $\{g_i^{n+1}\}$ to be the subset of $\{g_l^{n+1*}\}$ containing the elements that are not in $\bigoplus_i g_i^n I$. If each g_l^{n+1*} is in $\bigoplus_i g_i^n I$ we again stop.

We assume, for each n, that the elements g_i^n are uniform.

Setting $T^n = \bigoplus_i g_i^n K\mathcal{Q}$, the construction of [20] gives a filtration

$$\cdots \subseteq T^n \subseteq T^{n-1} \cdots \subseteq T^2 \subseteq T^1 \subseteq T^0.$$

For each $n \geq 0$, let $P^n = \bigoplus_i g_i^n K\mathcal{Q} / \bigoplus_i g_i^n I$. Then $P^n \cong \bigoplus_i t(g_i^n)\Lambda$ and is a projective right Λ-module. Let $d^n : P^n \to P^{n-1}$ be the Λ-homomorphism induced from the inclusion $\bigoplus_i g_i^n K\mathcal{Q} \subseteq \bigoplus_j g_j^{n-1} K\mathcal{Q}$.

Theorem 3.7. [20, Theorem 1.2] *Let (P^n, d^n) be the resolution*

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

with P^n and d^n as given above. Then (P^n, d^n) is a projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module.

For our setting, [20] shows that this construction gives a minimal projective resolution of $\Lambda/\mathfrak{r}$.

Theorem 3.8. [20, Theorem 2.4] *Let (P^n, d^n) be a projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, as given above. Then the sets $\{g_i^n\}$ can be chosen so that for each n, no proper K-linear combination of a subset of $\{g_i^n\}$ lies in $\bigoplus_j g_j^{n-1} I + \bigoplus_i g_i^{n*} J$, where J denotes the arrow ideal of $K\mathcal{Q}$. Moreover, in this case the resolution (P^n, d^n) is minimal.*

To summarize the resolution of [20] to give a minimal projective resolution of $\Lambda/\mathfrak{r}$

- Let g^0 be the set of vertices in $\mathcal{Q}$.
- Let g^1 be the set of arrows in $\mathcal{Q}$.
- Let g^2 be a minimal set of uniform relations in the generating set of I.

- For $n \geq 3$ each $x \in g^n$ is a uniform element with

$$x = \sum_{y \in g^{n-1}} y r_y = \sum_{z \in g^{n-2}} z s_z$$

  where r_y and s_z are unique elements in $K\mathcal{Q}$.
- These sets g^n are such that there is a minimal projective resolution of $\Lambda/\mathfrak{r}$ with the following properties:
  (1) for each $n \geq 0, P^n = \bigoplus_{x \in g^n} t(x)\Lambda$
  (2) for $x \in g^n$ there are unique elements $r_j \in K\mathcal{Q}$ with

$$x = \sum_{j=1}^{m} g_j^{n-1} r_j$$

  where $m = |g^{n-1}|$.
  (3) For each $n \geq 1$, the map $P^n \longrightarrow P^{n-1}$ is given by

$$t(x)\lambda \mapsto \sum_{j=1}^{m} t(g_j^{n-1}) r_j t(x)\lambda$$

  where $t(g_j^{n-1}) r_j t(x)\lambda$ is in the component of P^{n-1} corresponding to $t(g_j^{n-1})$.

In examples it may be simpler to start by computing a projective resolution for each simple Λ-module. The following theorem shows that given a minimal projective resolution for each simple Λ-module, we can write down a minimal projective resolution of $\Lambda/\mathfrak{r}$, since $\Lambda/\mathfrak{r} = \bigoplus_i S_i$, where $\{S_i\}$ is a complete set of pairwise non-isomorphic simple modules for Λ.

Proposition 3.9. *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence and $P_1, \bar{P}_1$ are projective resolutions of A and C respectively, then there exists a projective resolution of B such that the following diagram is commutative:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2 \oplus \bar{P}_2 \arrow[r] \arrow[d] & P_1 \oplus \bar{P}_1 \arrow[r] \arrow[d] & P_0 \oplus \bar{P}_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \bar{P}_2 \arrow[r] \arrow[d] & \bar{P}_1 \arrow[r] \arrow[d] & \bar{P}_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

with exact rows and columns.

We now give an example where we compute the minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module and give the sets g^n of the construction of [20].

Example 3.10. Let $\mathcal{Q}$ be the quiver given by

$$
\begin{tikzcd}
 & 2 \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_7"'] & & 3 \arrow[dd, "\alpha_3"] \\
 & 7 \arrow[ur, "\alpha_8"'] & \\
6 \arrow[uu, "\alpha_6"] & & 4 \arrow[dl, "\alpha_4"] \\
 & 5 \arrow[ul, "\alpha_5"] &
\end{tikzcd}
$$

and let $\Lambda = K\mathcal{Q}/I$, where $I = \langle(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\rangle$.

The indecomposable projective modules are $e_i\Lambda$ and the simple modules are $S_i = e_i\Lambda/e_i\mathfrak{r}$, for $i = 1, \ldots, 7$. We begin by finding projective resolutions of the simple modules.

For S_1, the resolution begins

$$e_1\Lambda \xrightarrow{d^0} e_1\Lambda/e_1\mathfrak{r} \longrightarrow 0$$

where $P^0 = e_1\Lambda$ and d^0 is the canonical surjection. So $\operatorname{Ker} d^0 = e_1\mathfrak{r} = \alpha_1\Lambda + \alpha_7\Lambda$. Let $P^1 = e_2\Lambda \oplus e_7\Lambda$ and let $d^1 : P^1 \longrightarrow P^0$ be given by $d^1(e_2\lambda_1, e_7\lambda_2) = \alpha_1\lambda_1 + \alpha_7\lambda_2$, where $\lambda_1, \lambda_2 \in \Lambda$. Then we have $\operatorname{Ker} d^1 = \{(e_2\lambda_1, e_7\lambda_2) \in P^1 | \alpha_1\lambda_1 + \alpha_7\lambda_2 = 0\}$.

It is straightforward to show that $\operatorname{Ker} d^1 = (\alpha_2\alpha_3\alpha_4, -\alpha_8\alpha_3\alpha_4)\Lambda$. Let $P^2 = e_5\Lambda$, and define $d^2 : P^2 \longrightarrow P^1$ by $d^2(e_5\lambda) = (\alpha_2\alpha_3\alpha_4, -\alpha_8\alpha_3\alpha_4)\lambda$. Here $\operatorname{Ker} d^2 = \alpha_5\alpha_6\Lambda$. Let $P^3 = e_1\Lambda$ and define $d^3 : P^3 \longrightarrow P^2$ by $d^3(e_1\lambda) = \alpha_5\alpha_6\lambda$. Then $\operatorname{Ker} d^3 = (\alpha_1\alpha_2 - \alpha_7\alpha_8)\Lambda$. Continuing in this way, we let $P^4 = e_3\Lambda$ and define $d^4 : P^4 \longrightarrow P^3$ by $d^4(e_3\lambda) = (\alpha_1\alpha_2 - \alpha_7\alpha_8)\lambda$, giving $\operatorname{Ker} d^4 = \alpha_3\alpha_4\Lambda$. Let $P^5 = e_5\Lambda$ and define $d^5 : P^5 \longrightarrow P^4$ by $d^5(e_5\lambda) = \alpha_3\alpha_4\lambda$, giving $\operatorname{Ker} d^5 = \alpha_5\alpha_6\Lambda = \operatorname{Ker} d^2$.

We can see inductively that the projective resolution (P^n, d^n) for S_1 then repeats with

- $P^n = e_1\Lambda, d^n(e_1\lambda) = \alpha_5\alpha_6\lambda, \operatorname{Ker} d^n = (\alpha_1\alpha_2 - \alpha_7\alpha_8)\Lambda$, if $n = 3m, m \in \mathbb{N}$.
- $P^n = e_3\Lambda, d^n(e_3\lambda) = (\alpha_1\alpha_2 - \alpha_7\alpha_8)\lambda, \operatorname{Ker} d^n = \alpha_3\alpha_4\Lambda$, if $n = 3m+1, m \in \mathbb{N}$.
- $P^n = e_5\Lambda, d^n(e_5\lambda) = \alpha_3\alpha_4\lambda, \operatorname{Ker} d^n = \alpha_5\alpha_6\Lambda$, if $n = 3m+2, m \in \mathbb{N}$.

Note that we use the convention that $0 \notin \mathbb{N}$.

This sequence is a minimal projective resolution (P^n, d^n) for S_1 as a Λ-module.

For S_2, let $P^0 = e_2\Lambda$ and d^0 be the canonical surjection, $P^0 \to S_2$. Then $\operatorname{Ker} d^0 = e_2\mathfrak{r} = \alpha_2\Lambda$. Let $P^1 = e_3\Lambda$ and define $d^1 : P^1 \longrightarrow P^0$ by $d^1(e_3\lambda) = \alpha_2\lambda$. Now, $\operatorname{Ker} d^1 = \{0\}$ and so we have $P^2 = 0$ and we are finished, that is,

$$0 \longrightarrow e_3\Lambda \xrightarrow{d^1} e_2\Lambda \xrightarrow{d^0} e_2\Lambda/e_2\mathfrak{r} \longrightarrow 0$$

is a projective resolution of S_2 as a Λ-module.

In this way we can see that the simple modules S_2, S_4, S_6 and S_7 have finite projective resolution, whilst S_1, S_3 and S_5 have infinite projective resolution. We can now write down a projective resolution $(\bar{P}^n, \bar{d}^n)$ of $\Lambda/\mathfrak{r}$:

$$\cdots \longrightarrow \bar{P}^n \xrightarrow{\bar{d}^n} \bar{P}^{n-1} \xrightarrow{\bar{d}^{n-1}} \cdots \longrightarrow \bar{P}^2 \xrightarrow{\bar{d}^2} \bar{P}^1 \xrightarrow{\bar{d}^1} \bar{P}^0 \xrightarrow{\bar{d}^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where $\bar{P}^0 = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \oplus e_4\Lambda \oplus e_5\Lambda \oplus e_6\Lambda \oplus e_7\Lambda$, $\bar{P}^1 = e_2\Lambda \oplus e_7\Lambda \oplus e_3\Lambda \oplus e_4\Lambda \oplus e_5\Lambda \oplus e_6\Lambda \oplus e_1\Lambda \oplus e_3\Lambda$ and $\bar{P}^2 = e_5\Lambda \oplus e_1\Lambda \oplus e_3\Lambda$. The remaining terms of the resolution are given by:

• If $n = 3m, m \in \mathbb{N}, \bar{P}^n = e_1\Lambda \oplus e_3\Lambda \oplus e_5\Lambda$,

$\bar{d}^n(\lambda_1, \lambda_2, \lambda_3) = (\alpha_5\alpha_6\lambda_1, (\alpha_1\alpha_2 - \alpha_7\alpha_8)\lambda_2, \alpha_3\alpha_4\lambda_3)$,

$\operatorname{Ker} \bar{d}^n = (\alpha_1\alpha_2 - \alpha_7\alpha_8)\Lambda \oplus \alpha_3\alpha_4\Lambda \oplus \alpha_5\alpha_6\Lambda$.

• If $n = 3m+1, m \in \mathbb{N}, \bar{P}^n = e_3\Lambda \oplus e_5\Lambda \oplus e_1\Lambda$,

$\bar{d}^n(\lambda_1, \lambda_2, \lambda_3) = ((\alpha_1\alpha_2 - \alpha_7\alpha_8)\lambda_1, \alpha_3\alpha_4\lambda_2, \alpha_5\alpha_6\lambda_3)$,

$\operatorname{Ker} \bar{d}^n = \alpha_3\alpha_4\Lambda \oplus \alpha_5\alpha_6\Lambda \oplus (\alpha_1\alpha_2 - \alpha_7\alpha_8)\Lambda$.

• If $n = 3m+2, m \in \mathbb{N}, \bar{P}^n = e_5\Lambda \oplus e_1\Lambda \oplus e_3\Lambda$,

$\bar{d}^n(\lambda_1, \lambda_2, \lambda_3) = (\alpha_3\alpha_4\lambda_1, \alpha_5\alpha_6\lambda_2, (\alpha_1\alpha_2 - \alpha_7\alpha_8)\lambda_3)$,

$\operatorname{Ker} \bar{d}^n = \alpha_5\alpha_6\Lambda \oplus (\alpha_1\alpha_2 - \alpha_7\alpha_8)\Lambda \oplus \alpha_3\alpha_4\Lambda$.

We remark that $\bar{P}^n \cong \bar{P}^2$ for $n \geq 2$. Since $\operatorname{Im} \bar{d}^n \subseteq \bar{P}^{n-1}\mathfrak{r}$ for $n \geq 1$, the sequence $(\bar{P}^n, \bar{d}^n)$ is a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module.

For the above algebra, the sets g^n are given as follows:

- $g^0 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
- $g^1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$
- $g^2 = \{(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\}$
- $g^3 = \{(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_3\alpha_4\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8),$
  $\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4\}$
- If $n = 3m, m \in \mathbb{N}, g^n = \{((\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4\alpha_5\alpha_6)^m,$
  $(\alpha_3\alpha_4\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8))^m, (\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4)^m\}$.
- If $n = 3m+1, m \in \mathbb{N}, g^n = \{((\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4\alpha_5\alpha_6)^m(\alpha_1\alpha_2 - \alpha_7\alpha_8),$
  $(\alpha_3\alpha_4\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8))^m\alpha_3\alpha_4, (\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4)^m\alpha_5\alpha_6\}$.

- If $n = 3m+2, m \in \mathbb{N}, g^n = \{((\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4\alpha_5\alpha_6)^m(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4, (\alpha_3\alpha_4\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8))^m\alpha_3\alpha_4\alpha_5\alpha_6, (\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4)^m\alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\}$.

Given the minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module and the sets g^n for all $n \geq 0$, we now have all the information we need to look at the Ext algebra of Λ.

Definition 3.11. The Ext algebra of Λ is

$$E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \bigoplus_{n \geq 0} \operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}).$$

In order to see that $E(\Lambda)$ is an algebra, we need to describe the product structure of $E(\Lambda)$. The product is given by the Yoneda product.

We start with a minimal projective resolution (P^n, d^n) of $\Lambda/\mathfrak{r}$ as a right Λ-module, so $\operatorname{Im} d^n \subseteq P^{n-1}\mathfrak{r}$. As above, let $d^{n*} : \operatorname{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r}) \to \operatorname{Hom}_\Lambda(P^{n+1}, \Lambda/\mathfrak{r})$ be the map induced from $d^{n+1} : P^{n+1} \to P^n$. Then $\operatorname{Im} d^{n*} = \{0\}$ for all $n \geq 0$, so that $\operatorname{Ker} d^{n*} \cong \operatorname{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$ and $\operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \cong \operatorname{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$ for all n.

Given $g \in \operatorname{Hom}_\Lambda(P^m, \Lambda/\mathfrak{r})$ and $f \in \operatorname{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$ which represent elements in $\operatorname{Ext}^m_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ and $\operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ respectively, the product $g \cdot f$ is the element in $\operatorname{Ext}^{m+n}_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ given by the map $g \circ \mathcal{L}^m f : P^{m+n} \to \Lambda/\mathfrak{r}$ where $\mathcal{L}^m f$ is the m-th lifting of f. Thus

$$\begin{tikzcd}
\cdots \arrow[r] & P^{m+n} \arrow[r, "d^{m+n}"] \arrow[d, "\mathcal{L}^m f"'] & P^{m+n-1} \arrow[r] \arrow[d, "\mathcal{L}^{m-1} f"'] & \cdots \arrow[r] & P^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "\mathcal{L}^1 f"'] & P^n \arrow[d, "\mathcal{L}^0 f"'] \arrow[dr, "f"] & & \\
\cdots \arrow[r] & P^m \arrow[r, "d^m"'] \arrow[dr, "g"'] & P^{m-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[r, "d^1"'] & P^0 \arrow[r, "d^0"'] & \Lambda/\mathfrak{r} \arrow[r] & 0 \\
& & \Lambda/\mathfrak{r} & & & & &
\end{tikzcd}$$

The liftings are not unique; nevertheless the product $g \cdot f$ in $\mathrm{Ext}_{\Lambda}^{n+m}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is independent of the choice of liftings. In this way $E(\Lambda)$ has an algebra structure. We have seen that $\mathrm{Ext}_{\Lambda}^{m}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \times \mathrm{Ext}_{\Lambda}^{n}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \subseteq \mathrm{Ext}_{\Lambda}^{m+n}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, so $E(\Lambda)$ is a graded algebra, with the homomlogical degree. When we refer to $E(\Lambda)$ as a graded algebra, we mean graded by the homological degree, unless otherwise specified.

The main aim of this thesis is to consider how $E(\Lambda)$ is generated and whether or not it is finitely generated. Do we need the basis elements of $\mathrm{Ext}_{\Lambda}^{n}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ for all n or can we find a finite n such that $E(\Lambda)$ is generated in degrees $0, 1, 2, \ldots, n$? In the next chapter we look at Koszul algebras and some generalisations where it is known that $E(\Lambda)$ is finitely generated.

4. Koszul algebras and some generalisations

In this chapter we define Koszul algebras and some of their generalisations. All the algebras have their Ext algebra finitely generated. The algebras in this chapter motivate the (D, A)-stacked algebras which we introduce in Chapter 5 and which will form the main objects of study of this thesis.

Koszul algebras play an important role in algebra and in topology, see [3], [15], [16] and [25]. They are graded algebras. We start with some definitions.

Definition 4.1.

(1) Let Λ be a K-algebra, for some field K. Then Λ is a graded algebra if $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$, with $\Lambda_m \cdot \Lambda_n \subseteq \Lambda_{m+n}$ for all $m, n \geq 0$.

(2) Let Λ be a graded algebra, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, and let M be a right Λ-module. Then M is a graded Λ-module if

$$M = \oplus_i \, M_i \text{ and } M_i \Lambda_j \subseteq M_{i+j}, \text{ for all } i, j \text{ with } j \geq 0.$$

(3) Let Λ be a graded algebra, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, and let M be a graded Λ-module, $M = M_i \oplus M_{i+1} \oplus M_{i+2} \oplus \cdots$. Then M is generated in degree i if for each $j \geq 0$, $M_{i+j} = M_i \Lambda_j$.

Definition 4.2. [15] Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra. Then Λ is a Koszul algebra if Λ is a graded algebra $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ and if $\Lambda/\mathfrak{r} \cong \Lambda_0$ considered as a graded Λ-module in degree 0 has a graded projective resolution

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \Lambda_0 \longrightarrow 0$$

such that P^i is generated in degree i. In this case, we say that Λ_0 has a linear resolution.

A finite-dimensional algebra $K\mathcal{Q}/I$ is graded whenever I is a homogeneous ideal of $K\mathcal{Q}$. We note that [15, Corollary 7.3] shows that if a finite-dimensional graded algebra $K\mathcal{Q}/I$ is Koszul then I is a quadratic ideal. In this case I is generated by linear combinations of paths of length 2.

Theorem 4.3. [15] *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra and let I be generated by length homogeneous elements. Then Λ is a Koszul algebra if and only if $E(\Lambda)$ is generated in degrees 0 and 1, that is, by $\mathrm{Ext}^0_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ and $\mathrm{Ext}^1_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$.*

The class of D-Koszul algebras was introduced by Berger in [5] as a generalisation of Koszul algebras. In a D-Koszul algebra $K\mathcal{Q}/I$, the ideal I is homogeneous (so $K\mathcal{Q}/I$ is graded), and is generated by linear combinations of paths of length $D \geq 2$.

Definition 4.4. [5] Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra. Then Λ is a D-Koszul algebra if, for each $n \geq 0$, the nth projective P^n in a minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in exactly one degree, $\delta(n)$, where

$$\delta(n) = \begin{cases} \frac{n}{2}D & \text{if } n \text{ is even} \\ \\ (\frac{n-1}{2})D + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Green, Marcos, Martínez-Villa and Zhang showed that the Ext algebra of a D-Koszul algebra is finitely generated in the following theorem.

Theorem 4.5. [13, Theorem 4.1] *Let $\Lambda = K\mathcal{Q}/I$ where I is generated by homogeneous elements of length D for some $D \geq 2$. Then Λ is D-Koszul if and only if the Ext algebra $E(\Lambda)$ can be generated in degrees $0, 1$ and 2.*

We now give two examples, the first is a D-Koszul monomial algebra, and the second is a Koszul algebra.

Example 4.6. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd} & 2 \arrow[dr, "\alpha_2"] & \\ 1 \arrow[ur, "\alpha_1"] & & 3 \arrow[ll, "\alpha_3"] \end{tikzcd}$$

and let $I = \langle \alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_1, \alpha_3\alpha_1\alpha_2 \rangle$. Let $\Lambda = K\mathcal{Q}/I$.

The minimal projective resolution of $\Lambda/\mathfrak{r}$ is given by

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{d^{n-1}} \cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where we define the terms P^n and the maps d^n as follows;

- Let $P^0 = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$. We define the map $d^0 : P^0 \to \Lambda/\mathfrak{r}$ by $d^0(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3 + \mathfrak{r}$ so we have $\operatorname{Ker} d^0 = \alpha_1\Lambda \oplus \alpha_2\Lambda \oplus \alpha_3\Lambda$.
- Let $P^1 = e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$. We define the map $d^1 : P^1 \to P^0$ by $d^1(e_2\lambda_1, e_3\lambda_2, e_1\lambda_3) = (\alpha_1\lambda_1, \alpha_2\lambda_2, \alpha_3\lambda_3)$ so then $\operatorname{Ker} d^1 = \alpha_2\alpha_3\Lambda \oplus \alpha_3\alpha_1\Lambda \oplus \alpha_1\alpha_2\Lambda$.
- For all $n \geq 2, n$ even, we have $P^n = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$. We define the map $d^n : P^n \to P^{n-1}$ by $d^n(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = (\alpha_2\alpha_3\lambda_1, \alpha_3\alpha_1\lambda_2, \alpha_1\alpha_2\lambda_3)$ so $\operatorname{Ker} d^n = \alpha_1\Lambda \oplus \alpha_2\Lambda \oplus \alpha_3\Lambda$.
- For all $n \geq 3, n$ odd, we have $P^n = e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$. We define the map $d^n : P^n \to P^{n-1}$ by $d^n(e_2\lambda_1, e_3\lambda_2, e_1\lambda_3) = (\alpha_1\lambda_1, \alpha_2\lambda_2, \alpha_3\lambda_3)$ so $\operatorname{Ker} d^n = \alpha_2\alpha_3\Lambda \oplus \alpha_3\alpha_1\Lambda \oplus \alpha_1\alpha_2\Lambda$.

To be able to use this resolution to find the Ext algebra, we need the sets g^n. They are:

- $g^0 = \{e_1, e_2, e_3\}$.
- $g^1 = \{\alpha_1, \alpha_2, \alpha_3\}$.
- $g^2 = \{\alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_1, \alpha_3\alpha_1\alpha_2\}$.
- Then, for all $n \geq 3, n$ odd, the set $g^n = \{(\alpha_1\alpha_2\alpha_3)^{(n-1)/2}\alpha_1, (\alpha_2\alpha_3\alpha_1)^{(n-1)/2}\alpha_2, (\alpha_3\alpha_1\alpha_2)^{(n-1)/2}\alpha_3\}$.
- For $n \geq 4, n$ even, the set $g^n = \{(\alpha_1\alpha_2\alpha_3)^{n/2}, (\alpha_2\alpha_3\alpha_1)^{n/2}, (\alpha_3\alpha_1\alpha_2)^{n/2}\}$.

We can now see that the sets g^n have length $\delta(n)$ for $D = 3$. Hence each projective term P^n is generated in degree $\delta(n)$, so Λ is a 3-Koszul algebra.

Example 4.7. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
1 \arrow[rr, bend left=10, "\alpha_1"] \arrow[rr, bend right=10, "\alpha_4"'] & & 2 \arrow[dl, bend left, "\alpha_2"] \\
& 3 \arrow[ul, bend left, "\alpha_3"] &
\end{tikzcd}$$

and let $I = \langle (\alpha_1 - \alpha_4)\alpha_2, \alpha_2\alpha_3, \alpha_3(\alpha_1 - \alpha_4)\rangle$. Let $\Lambda = K\mathcal{Q}/I$.

We note that Λ is a monomial algebra, with a change of generators, $\gamma_1 = \alpha_1 - \alpha_4, \gamma_2 = \alpha_1$. However, we will come back to this example in Chapter 8 and so choose to give I with non-monomial generators.

The sets g^n are given as follows;

- The set g^0 is given by $\{e_1, e_2, e_3\}$.
- The set g^1 is given by $\{\alpha_1, \alpha_4, \alpha_2, \alpha_3\}$.
- The set g^2 is given by $\{(\alpha_1 - \alpha_4)\alpha_2, \alpha_2\alpha_3, \alpha_3(\alpha_1 - \alpha_4)\}$.
- The set g^3 is given by $\{(\alpha_1 - \alpha_4)\alpha_2\alpha_3, \alpha_2\alpha_3(\alpha_1 - \alpha_4), \alpha_3(\alpha_1 - \alpha_4)\alpha_3\}$.

  For $n \geq 3$, we have;
- If $n = 3m, m \in \mathbb{N}, g^n = \{((\alpha_1 - \alpha_4)\alpha_2\alpha_3)^m, (\alpha_2\alpha_3(\alpha_1 - \alpha_4))^m, (\alpha_3(\alpha_1 - \alpha_4)\alpha_2)^m\}$.
- If $n = 3m+1, m \in \mathbb{N}, g^n = \{((\alpha_1 - \alpha_4)\alpha_2\alpha_3)^m(\alpha_1 - \alpha_4), (\alpha_2\alpha_3(\alpha_1 - \alpha_4))^m\alpha_2,$
  $(\alpha_3(\alpha_1 - \alpha_4)\alpha_2)^m\alpha_3\}$.
- If $n = 3m + 2, m \in \mathbb{N}, g^n = \{((\alpha_1 - \alpha_4)\alpha_2\alpha_3)^m(\alpha_1 - \alpha_4)\alpha_2,$
  $(\alpha_2\alpha_3(\alpha_1 - \alpha_4))^m\alpha_2\alpha_3, (\alpha_3(\alpha_1 - \alpha_4)\alpha_2)^m\alpha_3(\alpha_1 - \alpha_4)\}$.

We label the elements of the set g^n by $g_1^n, g_2^n, \ldots$ in the order they are given here.

A minimal projective resolution of $\Lambda/\mathfrak{r}$ is;

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{d^{n-1}} \cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where

- $P^0 = \bigoplus e_i\Lambda = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$

  $d^0(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = (e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3 + \mathfrak{r})$

- $P^1 = \bigoplus t(\alpha_i)\Lambda = e_2\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$

  $d^1(e_2\lambda_1, e_2\lambda_2, e_3\lambda_3, e_1\lambda_4) = (\alpha_1\lambda_1 + \alpha_4\lambda_2, \alpha_2\lambda_3, \alpha_3\lambda_4)$
- $P^2 = \bigoplus t(g_i^2)\Lambda = e_3\Lambda \oplus e_1\Lambda \oplus e_2\Lambda$

  $d^2(e_3\lambda_1, e_1\lambda_2, e_2\lambda_3) = (\alpha_2\lambda_1, -\alpha_2\lambda_1, \alpha_3\lambda_2, (\alpha_1 - \alpha_4)\lambda_3)$.
- For $n = 3m, m \in \mathbb{N}, P^n = \bigoplus t(g_i^n)\Lambda = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$,

  $d^n(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = (\alpha_3\lambda_1, (\alpha_1 - \alpha_4)\lambda_2, \alpha_2\lambda_3)$.
- For $n = 3m+1, m \in \mathbb{N}, P^n = \bigoplus t(g_i^n)\Lambda = e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$,

  $d^n(e_2\lambda_1, e_3\lambda_2, e_1\lambda_3) = ((\alpha_1 - \alpha_4)\lambda_1, \alpha_2\lambda_2, \alpha_3\lambda_3)$.
- For $n = 3m+2, m \in \mathbb{N}, P^n = \bigoplus t(g_i^n)\Lambda = e_3\Lambda \oplus e_1\Lambda \oplus e_2\Lambda$,

  $d^n(e_3\lambda_1, e_1\lambda_2, e_2\lambda_3) = (\alpha_2\lambda_1, \alpha_3\lambda_2, (\alpha_1 - \alpha_4)\lambda_3)$.

Since each g^n has length n, we have shown that $\Lambda/\mathfrak{r}$ has a linear resolution. Hence Λ is a Koszul algebra, and thus the Ext algebra $E(\Lambda)$ is generated in degrees 0 and 1. We now look at a basis of the Ext algebra. For each $n \geq 0$ and each $g_i^n \in g^n$, we let $f_i^n \in \mathrm{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$ be the map given by $t(g_j^n) \mapsto \begin{cases} t(g_i^n) + \mathfrak{r} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$
The elements f_i^n for all n and all i form a K-basis for $E(\Lambda)$.

Let us now consider $\mathrm{Ext}^2_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. Then $f_1^2 = f_2^1 \cdot f_1^1 = f_2^1 \circ \mathcal{L}^1 f_1^1$, where the lifting $\mathcal{L}^1 f_1^1$ can be chosen as $\mathcal{L}^1 f_1^1 : P^2 \longrightarrow P^1,\ t(g_1^2)\lambda = e_3\lambda \mapsto e_3\lambda = t(g_2^1)\lambda, \text{else } \mapsto 0$. Similarly $f_2^2 = f_3^1 \cdot f_2^1 = f_3^1 \circ \mathcal{L}^1 f_2^1$, where the lifting $\mathcal{L}^1 f_2^1$ can be chosen as $\mathcal{L}^1 f_3^1 : P^2 \longrightarrow P^1,\ t(g_2^2)\lambda = e_1\lambda \mapsto e_1\lambda = t(g_3^1)\lambda, \text{else } \mapsto 0$. And $f_3^2 = f_1^1 \cdot f_3^1 = f_1^1 \circ \mathcal{L}^1 f_3^1$, where the lifting $\mathcal{L}^1 f_3^1$ can be chosen as $\mathcal{L}^1 f_3^1 : P^2 \longrightarrow P^1,\ t(g_3^2)\lambda = e_2\lambda \mapsto e_2\lambda = t(g_1^1)\lambda, \text{else } \mapsto 0$.

For this algebra, and all $n \geq 2$, we can write each element f_i^n as a product of the form $f_j^1 \cdot f_k^{n-1}$ for suitable j, k. Thus $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \mathrm{Ext}^1_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \times \mathrm{Ext}^{n-1}_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ in this case. This provides a second proof that $E(\Lambda)$ is generated in degrees 0 and 1.

Another class of algebras which generalise Koszul monomial algebras were introduced by Green and Snashall in [18], and see [19].

Definition 4.8. [18, Definition 3.1] Let $\Lambda = KQ/I$ be a finite-dimensional monomial algebra. Then Λ is said to be a (D,A)-stacked monomial algebra if there is some $D \geq 2$ and $A \geq 1$ such that, for all $n \geq 2$ and $g_i^n \in g^n$, where the g^n are the sets of a minimal projective resolution of $\Lambda/\mathfrak{r}$ from [20], then

$$l(g_i^n) = \begin{cases} \frac{n}{2}D & \text{if } n \text{ is even} \\ \\ (\frac{n-1}{2})D + A & \text{if } n \text{ is odd.} \end{cases}$$

In particular I is generated by paths of length D.

Remark. Let Λ be a (D,A)-stacked monomial algebra.

(1) For $D = 2, A = 1$,the $(2,1)$-stacked monomial algebras are precisely the quadratic monomial algebras, or equivalently, the Koszul monomial algebras. In this case $E(\Lambda)$ is generated in degrees 0 and 1.

(2) For $A = 1$, the $(D,1)$-stacked monomial algebras for $D \geq 2$ are the D-Koszul monomial algebras. In this case, $E(\Lambda)$ is generated in degrees $0, 1$ and 2. We note that this new class of algebras include both non-monomial and monomial algebras.

Proposition 4.9. [18, Proposition 3.3] *Let Λ be a (D,A)-stacked monomial algebra. Then*

(1) if $\operatorname{gldim}\Lambda \geq 3$ *then $D > A$;*

(2) if $\operatorname{gldim}\Lambda \geq 4$ *then $D = dA$ for some $d \geq 2$.*

The Ext algebra of a (D,A)-stacked monomial algebra is finitely generated as the following result shows.

Theorem 4.10. [19, Theorem 3.6] *Let Λ be a (D,A)-stacked monomial algebra. Then $E(\Lambda)$ is generated in degrees* $0, 1, 2$ *and* 3.

Moreover, for monomial algebras of infinite global dimension, the (D,A)-stacked monomial algebras are precisely the monomial algebras for which every projective module in the minimal projective resolution of $\Lambda/\mathfrak{r}$ over Λ is generated in a single degree and for which the Ext algebra of Λ is finitely generated. For a (D, A)-stacked monomial algebra of infinite global dimension, the nth projective P^n in the minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in degree

$$\begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n}{2}D & \text{if } n \geq 2, n \text{ even} \\ (\frac{n-1}{2})D + A & \text{if } n \geq 3, n \text{ odd.} \end{cases}$$

This class of monomial (D, A)-stacked algebras leads to our (D, A)-stacked algebras which we now introduce.

5. (D, A)-Stacked Algebras

In this chapter we introduce a new class of algebras, which we call (D, A)-stacked algebras. This class is motivated by the (D, A)-stacked monomial algebras of [18] and includes all the Koszul algebras, D-Koszul algebras and (D, A)-stacked monomial algebras of Chapter 4. We start with the definition of a (D, A)-stacked algebra.

Definition 5.1. Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional algebra. We define Λ to be a (D, A)-stacked algebra if there is some $D \geq 2$ and some $A \geq 1$ such that for all n the projective module P^n in a minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in degree $\delta(n)$, where

$$\delta(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n}{2}D & \text{if } n = 2r, r \in \mathbb{N} \\ \left(\frac{n-1}{2}\right) D + A & \text{if } n = 2r+1, r \in \mathbb{N}. \end{cases}$$

Remark. Let Λ be a (D, A)-stacked algebra. If $A = 1$, then Λ is D-Koszul and we have seen in Theorem 4.5 that $E(\Lambda)$ is generated in degrees $0, 1$ and 2. Moreover, if $A = 1$ and $D = 2$ then Λ is Koszul and $E(\Lambda)$ is generated in degrees 0 and 1.

Clearly the (D, A)-stacked algebras contain the (D, A)-stacked monomial algebras of [18]. Thus (D, A)-stacked algebras are a natural generalisation of Koszul algebras, containing all the Koszul algebras, D-Koszul algebras and (D, A)-stacked monomial algebras. We note that this new class of algebras contains both monomial and non-monomial algebras.

The main result in this chapter is Theorem 5.7 where we show that $E(\Lambda)$ is finitely generated and moreover is generated in degrees $0, 1, 2$ and 3. In order to prove this, we need the following results from [12] and [13].

Proposition 5.2. [13, Proposition 3.6] *Let Λ be a graded algebra and let*

$$\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$$

be a minimal graded projective resolution of Λ_0 as a right Λ-module. Suppose that P^i is finitely generated with generators in degree d_i, for $i = \alpha, \beta, \alpha + \beta$. Assume that

$$d_{\alpha+\beta} = d_\alpha + d_\beta.$$

Then the Yoneda map

$$\mathrm{Ext}_\Lambda^\alpha(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^\beta(\Lambda_0, \Lambda_0) \to \mathrm{Ext}_\Lambda^{\alpha+\beta}(\Lambda_0, \Lambda_0)$$

is surjective. Thus

$$\begin{aligned}\mathrm{Ext}_\Lambda^{\alpha+\beta}(\Lambda_0, \Lambda_0) &= \mathrm{Ext}_\Lambda^\alpha(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^\beta(\Lambda_0, \Lambda_0) \\ &= \mathrm{Ext}_\Lambda^\beta(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^\alpha(\Lambda_0, \Lambda_0).\end{aligned}$$

If Λ is a (D, A)-stacked algebra then the projective module P^2 in a minimal projective resolution of Λ_0 is generated in degree D. Hence the ideal I of $K\mathcal{Q}$ is generated by homogeneous elements of length D. Thus there is a length grading on Λ so that Λ is a graded algebra, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_m$, with m finite since Λ is finite-dimensional. In particular, each element $\lambda_i \in \Lambda_i$ is homogeneous of length i.

Definition 5.3. [12] Let $\cdots \to P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ be a minimal graded projective resolution of Λ_0 over Λ. We say that Λ is δ-resolution determined if there is a function $\delta : \mathbb{N} \to \mathbb{N}$ such that for all $n \geq 0, P^n$ is generated in degree $\delta(n)$.

It is clear that a (D, A)-stacked algebra is δ-resolution determined, with δ as in Definition 5.1.

Definition 5.4. [12] A δ-resolution determined algebra Λ is δ-Koszul if $E(\Lambda)$ is finitely generated as an algebra.

Theorem 5.5. [12, Theorem 3.6] *Let $\Lambda = K\mathcal{Q}/I$ be a graded algebra where I is an ideal generated by length homogeneous elements in $K\mathcal{Q}$ and the grading is induced from the length grading in $K\mathcal{Q}$. Assume that Λ is δ-resolution determined. Then*

Λ is a δ-Koszul algebra if and only if there is some positive integer t, such that, if $k > t$, then there exists i, with $0 < i < k$, such that $\delta(i) + \delta(k - i) = \delta(k)$.

We are now ready to consider the Ext algebra of a (D, A)-stacked algebra.

Theorem 5.6. *Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra with $D > 2$ and $A > 1$. Then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3.*

Proof. Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra, with $D > 2$ and $A > 1$. Then I is generated by homogeneous elements of length D, so Λ is length graded and is δ-resolution determined.

We now show that there exists some appropriate positive integer t, namely $t = 3$, so that the hypotheses of Theorem 5.5 are satisfied and hence Λ is δ-Koszul. We have,

$$\delta(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n}{2}D & \text{if } n = 2r, r \in \mathbb{N} \\ \left(\frac{n-1}{2}\right) D + A & \text{if } n = 2r+1, r \in \mathbb{N}. \end{cases}$$

Let $t = 3$ and $i = 2$. Then, for all $n > 3$, we have:

- if n odd, $\delta(n-2) + \delta(2) = \left(\left(\frac{(n-2)-1}{2}\right)D + A\right) + \left(\frac{2}{2}\right)D$

$$= \left(\frac{n}{2}D - D - \frac{1}{2}D + A\right) + D$$

$$= \frac{n}{2}D - \frac{1}{2}D + A$$

$$= \frac{(n-1)}{2}D + A$$

$$= \delta(n),$$

and

$$\bullet \text{ if } n \text{ even, } \delta(n-2)+\delta(2) = \left(\frac{n-2}{2}\right)D + \frac{2}{2}D$$
$$= \frac{n}{2}D - D + D$$
$$= \frac{n}{2}D$$
$$= \delta(n).$$

So Λ is a δ-Koszul algebra (with $t = 3, i = 2$). Hence $E(\Lambda)$ is finitely generated.

We note that we need $t = 3$, since if $n = 3$ then necessarily $t = 2$ and $i = 1$. However, $\delta(3) = D + A$ and $\delta(1) + \delta(2) = D + 1$ but $A > 1$, so $\delta(1) + \delta(2) \neq \delta(3)$.

It remains to show that $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3. For all $n > 3$, we have $\delta(2) + \delta(n-2) = \delta(n)$ and so from Proposition 5.2 we have

$$\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^{n-2}_\Lambda(\Lambda_0, \Lambda_0).$$

Thus, $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3. $\square$

Using the fact that the Ext algebra of a Koszul algebra is generated in degrees 0 and 1, and the Ext algebra of a D-Koszul algebra is generated in degrees $0, 1$ and 2, we have the following theorem.

Theorem 5.7. *Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra with $D \geq 2$ and $A \geq 1$. Then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3.*

Note that the comment following Definition 5.3 together with Theorem 5.7 shows that every (D, A)-stacked algebra is a δ-Koszul algebra.

In the next chapter we give a full characterisation of (D, A)-stacked algebras.

6. Properties of the Ext Algebra

In this chapter we look at some of the general properties of the Ext algebra, $E(\Lambda)$, where Λ is a (D, A)-stacked algebra, as given in Definition 5.1. We start by recalling some well known definitions, see for example, [13], [14], [26].

Definition 6.1. (1) Let M, N be graded modules. A homomorphism of degree n is a homomorphism $f : M \to N$ such that $f(M_i) \subseteq N_{i+n}$, for all i.

(2) Let Λ be a graded algebra, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, and let M be a graded Λ-module, $M = \oplus_i M_i$. We define the nth shift of M, denoted $M[n]$, to be the graded Λ-module $X = \oplus_i X_i$, where $X_i = M_{i-n}$.
If M is generated in degree n, so $M = M_n \oplus M_{n+1} \oplus \cdots$, then $M[-n] = X_0 \oplus X_1 \oplus \cdots$, where $X_0 = M_n, X_1 = M_{n+1}, \ldots$, and hence $M[-n]$ is generated in degree 0.

Notation Let $Gr(\Lambda)$ be the category of graded right Λ-modules together with the set of degree 0 homomorphisms and let F denote the forgetful functor, $F : Gr(\Lambda) \to \mathrm{Mod}_\Lambda$.

Definition 6.2. [13] Let Λ be a graded algebra, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$. Let $M = \oplus_i M_i$ and $N = \oplus_i N_i$ be graded Λ-modules. If M is finitely generated then the abelian group $\mathrm{Hom}_\Lambda(F(M), F(N))$ can be graded as follows;

$$\mathrm{Hom}_\Lambda(F(M), F(N))_i = \mathrm{Hom}_{Gr(\Lambda)}(M, N[i]).$$

This is called the hom-grading. More generally, suppose we have a graded projective resolution (Q^n, d^n) of M where each Q^n is finitely generated. We define

$$\mathrm{Ext}^n_\Lambda(F(M), F(N))_i = \mathrm{Ext}^n_{Gr(\Lambda)}(M, N[i])$$

which is the homology of the complex obtained by applying $\mathrm{Hom}_{Gr(\Lambda)}(-, N[i])$ to (Q^n, d^n). This is called the shift-grading.

Proposition 6.3. [13, Theorem 2.1] *Let Λ be a graded algebra. Let (Q^n, d^n) be a minimal graded projective resolution of a graded Λ-module M. Assume that each Q^n is finitely generated. Suppose that N is a graded Λ-module such that* $\operatorname{rad} N = (0)$, *where* $\operatorname{rad} N$ *denotes the radical of N. Then*

$$\operatorname{Ext}^n_\Lambda(F(M), F(N))_i \cong \operatorname{Hom}_{Gr(\Lambda)}(\Omega^n(M), N[i]) \cong \operatorname{Hom}_{Gr(\Lambda)}(\Omega^n(M)[-i], N),$$

where Ω^n denotes the nth syzygy of M with respect to the resolution (Q^n, d^n).

We will now look at when products in the Ext algebra $E(\Lambda)$ are zero, for a (D, A)-stacked algebra Λ. Since $\operatorname{rad} \Lambda_0 = (0)$, we will use the above proposition and Proposition 5.2 to show that if $\delta(n) + \delta(m) \neq \delta(n+m)$ then we have $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0) = 0$. Throughout this chapter, we let (P^n, d^n) denote a minimal projective resolution of Λ_0 for our (D, A)-stacked algebra Λ.

Proposition 6.4. *Let Λ be a (D, A)-stacked algebra with $D \neq 2$. Then*

$$\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0.$$

Proof. The projective module P^1 is generated in degree 1 and $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \cong \operatorname{Hom}(P^1, \Lambda_0)$, so every element of $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as a short exact sequence of graded modules of the form $0 \to \Lambda_0[-1] \to E \to \Lambda_0 \to 0$. Let

$$0 \to \Lambda_0[-1] \to E \to \Lambda_0 \to 0 \tag{1}$$

and

$$0 \to \Lambda_0[-1] \to \hat{E} \to \Lambda_0 \to 0 \tag{2}$$

be two short exact sequences in $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$. We can shift the sequence (2) by -1 to get

$$0 \to \Lambda_0[-2] \to \hat{E}[-1] \to \Lambda_0[-1] \to 0. \tag{3}$$

We then splice the sequences (1) and (3) together to obtain

$$0 \to \Lambda_0[-2] \to \hat{E}[-1] \to E \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)_2$. However, we know that P^2 is generated in degree $D \neq 2$, so $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)_2 = 0$. Therefore, $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0$, when $D \neq 2$. $\square$

Proposition 6.5. *Let Λ be a (D, A)-stacked algebra with $D > 2$.*

i) If $D \neq A+1$ then $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, for all n odd, $n \geq 1$.

ii) If $A > 1$ then $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, for all n even, $n \geq 2$.

Proof. The case $n = 1$ follows from Proposition 6.4. Thus we may assume $n \geq 2$. The projective module P^n is generated in degree $\delta(n)$. So, since $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \cong \mathrm{Hom}(P^n, \Lambda_0)$, each extension can be viewed as an exact sequence of graded modules of the form

$$0 \to \Lambda_0[-\delta(n)] \to E_n \to \cdots \to E_1 \to \Lambda_0 \to 0.$$

Using the shift-grading, we can shift this sequence by -1 to obtain

$$0 \to \Lambda_0[-\delta(n) - 1] \to E_n[-1] \to \cdots \to E_1[-1] \to \Lambda_0[-1] \to 0.$$

We can then splice this with an extension from $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$,

$$0 \to \Lambda_0[-1] \to E' \to \Lambda_0 \to 0$$

to obtain

$$0 \to \Lambda_0[-\delta(n) - 1] \to E_n[-1] \to \cdots \to E_1[-1] \to E' \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$ lies in $\mathrm{Ext}^{n+1}_\Lambda(\Lambda_0, \Lambda_0)_{\delta(n)+1}$. However, we know the projective module P^{n+1} is generated in degree $\delta(n+1)$, so

$\operatorname{Ext}_\Lambda^{n+1}(\Lambda_0, \Lambda_0) = \operatorname{Ext}_\Lambda^{n+1}(\Lambda_0, \Lambda_0)_{\delta(n+1)}$. If $n = 2r + 1$ is odd, then $\delta(n+1) = \delta(2r+2) = (r+1)D$ and $\delta(n)+1 = \delta(2r+1)+1 = rD+A+1$, and since $D \neq A+1$ we have $\operatorname{Ext}_\Lambda^{n+1}(\Lambda_0, \Lambda_0)_{\delta(n)+1} = 0$, for n odd.

On the other hand, if $n = 2r$ is even, the projective module P^{n+1} is generated in degree $\delta(n+1) = \delta(2r+1) = rD+A$ and $\delta(n)+1 = \delta(2r)+1 = rD+1$, and since $A > 1$ we have $\operatorname{Ext}_\Lambda^{n+1}(\Lambda_0, \Lambda_0)_{\delta(n)+1} = 0$ for n even. Hence $\operatorname{Ext}_\Lambda^{n}(\Lambda_0, \Lambda_0) \times \operatorname{Ext}_\Lambda^{1}(\Lambda_0, \Lambda_0) = 0$ for all n even, $n \geq 2$.

The case for $\operatorname{Ext}_\Lambda^{1}(\Lambda_0, \Lambda_0) \times \operatorname{Ext}_\Lambda^{n}(\Lambda_0, \Lambda_0) = 0$ is similar. This completes the proof. $\square$

Proposition 6.6. *Let Λ be a (D, A)-stacked algebra with $D > 2, D \neq 2A$. Then* $\operatorname{Ext}_\Lambda^{2m+1}(\Lambda_0, \Lambda_0) \times \operatorname{Ext}_\Lambda^{2n+1}(\Lambda_0, \Lambda_0) = 0$, *for all $m, n \geq 1$.*

Proof. Let $m \geq 1, n \geq 1$. The projective modules P^{2m+1} and P^{2n+1} are generated in degrees $\delta(2m+1)$ and $\delta(2n+1)$, respectively. So each extension in $\operatorname{Ext}_\Lambda^{2m+1}(\Lambda_0, \Lambda_0)$ can be given as an exact sequence of graded modules of the form

$$0 \to \Lambda_0[-(\delta(2m+1))] \to E_{2m+1} \to \cdots \to E_1 \to \Lambda_0 \to 0 \tag{4}$$

and each extension in $\operatorname{Ext}_\Lambda^{2n+1}(\Lambda_0, \Lambda_0)$ can be given by an exact sequence of graded modules of the form

$$0 \to \Lambda_0[-(\delta(2n+1))] \to E'_{2n+1} \to \cdots \to E'_1 \to \Lambda_0 \to 0. \tag{5}$$

Shifting the sequence (5) by $-(\delta(2m+1))$ we get

$$\begin{aligned} 0 \to \Lambda_0[-\delta(2n+1) - \delta(2m+1)] &\to E'_{2n+1}[-\delta(2m+1)] \\ &\to \cdots \to E'_1[-\delta(2m+1)] \to \Lambda_0[-\delta(2m+1)] \to 0. \end{aligned} \tag{6}$$

Then splicing together (4) and (6), we get

$$0 \to \Lambda_0[-\delta(2m+1)-\delta(2n+1)] \to E'_{2n+1}[-\delta(2m+1)]$$

$$\to \cdots \to E'_1[-\delta(2m+1)] \to E_{2m+1} \to \cdots \to E_1 \to \Lambda_0 \to 0$$

which is an extension in $\mathrm{Ext}_\Lambda^{2m+2n+2}(\Lambda_0, \Lambda_0)_{\delta(2m+1)+\delta(2n+1)}$. However, $P^{2(m+n+1)}$ is generated in degree $\delta(2(m+n+1)) = (m+n+1)D$ and $\delta(2m+1)+\delta(2n+1) = mD + A + nD + A = (m+n)D + 2A$. Since $D \neq 2A$, we have $\mathrm{Ext}_\Lambda^{2(m+n+1)}(\Lambda_0, \Lambda_0)_{\delta(2m+1)+\delta(2n+1)} = 0$, so $\mathrm{Ext}_\Lambda^{2m+1}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{2n+1}(\Lambda_0, \Lambda_0) = 0$.

Hence, for all $m, n \geq 1$, we have $\mathrm{Ext}_\Lambda^{2m+1}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{2n+1}(\Lambda_0, \Lambda_0) = 0$ and this completes the proof. □

We summarise Propositions 6.4, 6.5 and 6.6 in the following result.

Theorem 6.7. *Let Λ be a (D, A)-stacked algebra. Then*

i) $\mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) = 0,$ *if $D \neq 2$,*

ii) $\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0),$ *for all n odd, $n \geq 1$, if $D > 2, D \neq A+1$,*

iii) $\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0),$ *for all n even, $n \geq 2$, if $D > 2, A > 1$, and*

iv) $\mathrm{Ext}_\Lambda^{2m+1}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{2n+1}(\Lambda_0, \Lambda_0) = 0,$ *for all $n, m \geq 1$, if $D > 2$, $D \neq 2A$.*

We now use Theorems 6.7 and 5.7 to give the following characterisation of (D, A)-stacked algebras.

Theorem 6.8. *Let $\Lambda = KQ/I$ where I is generated by homogeneous elements of length $D \geq 2$. Then $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is length graded. Suppose, in the minimal projective resolution, (P^n, d^n), of Λ_0 that P^3 is generated in a single degree, $D + A$, for $A \geq 1$. Then Λ is a (D, A)-stacked algebra if and only if $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3 and the following conditions hold:*

i) $\mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) = 0,$ *if $D \neq 2$;*

ii) $\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0),$ *for all n odd, $n \geq 1$, if $D > 2, D \neq A+1$;*

iii) $\mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)=0=\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)$, *for all* n *even,* $n\geq 2$, *if* $D>2, A>1$; *and*

iv) $\mathrm{Ext}^{2m+1}_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0)=0$, *for all* $m,n\geq 1$, *if* $D\neq 2A, D>2$.

Proof. Suppose $\Lambda = K\mathcal{Q}/I$ is a (D,A)-stacked algebra. Then from Theorem 5.7 we know that $E(\Lambda)$ is generated in degrees $0,1,2$ and 3. From Theorem 6.7 we know that conditions $(i),(ii),(iii)$ and (iv) hold.

To show the other direction, we will look at this in 3 cases.

Case 1: $D=2, A=1$.

Assume $\Lambda = K\mathcal{Q}/I$ where I is generated by homogeneous elements of length 2 and $E(\Lambda)$ is generated in degrees $0,1,2$ and 3. We know that in a minimal projective resolution (P^n,d^n) of Λ_0 we have P^0 is generated in degree 0, P^1 is generated in degree 1 and P^2 is generated in degree 2. By hypothesis P^3 is generated in degree $D+A=3$. By Proposition 5.2 with $\alpha=1,\beta=1$ we have $\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)=\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$. Putting $\alpha=1,\beta=2$ we have $\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)=\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$. Therefore $E(\Lambda)$ is generated in degrees 0 and 1, so Λ is Koszul and therefore a $(2,1)$-stacked algebra.

Case 2: $D>2, A=1$. Assume $\Lambda = K\mathcal{Q}/I$ where I is generated by homogeneous elements of length D and, in the minimal projective resolution of Λ_0, P^3 is generated in degree $D+1$. Assume $E(\Lambda)$ is generated in degrees $0,1,2$ and 3. We know that in a minimal projective resolution of Λ_0 we have P^0 is generated in degree 0, P^1 is generated in degree 1 and P^2 is generated in degree D. By Proposition 5.2 with $\alpha=1,\beta=2$ we have $\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)=\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$. Therefore $E(\Lambda)$ is generated in degrees $0,1$ and 2 and by [13, Theorem 4.1] Λ is D-Koszul and hence, Λ is a $(D,1)$-stacked algebra.

Case 3: $D>2, A>1$.

Suppose that P^3 is generated in degree $D+A$ and $E(\Lambda)$ is generated in degrees $0,1,2$ and 3, with conditions $(i),(ii),(iii)$ and (iv) holding. We know that

$\mathrm{Ext}^0_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^0_\Lambda(\Lambda_0, \Lambda_0)_0$ and $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)_1$ because the projective modules P^0 and P^1 in the minimal resolution of Λ_0 are generated in degrees 0 and 1 respectively. By hypothesis, we have $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)_D$ and $\mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)_{D+A}$.

We now need to look at $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)$. Since $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3 we have

$$\begin{aligned}\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) + \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)\\ + \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0).\end{aligned}$$

Now, either $D = A + 1$ or $D \neq A + 1$. Assume first that $D \neq A + 1$. Then from condition (ii), $\mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)$. So $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$.

An element of $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence

$$0 \to \Lambda_0[-D] \to E_2 \to E_1 \to \Lambda_0 \to 0 \tag{7}$$

and, using the shift grading, we can shift the sequence (7) by $-D$ to obtain

$$0 \to \Lambda_0[-2D] \to E_2[-D] \to E_1[-D] \to \Lambda_0[-D] \to 0. \tag{8}$$

Let

$$0 \to \Lambda_0[-D] \to E_2' \to E_1' \to \Lambda_0 \to 0 \tag{9}$$

be another element of $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ then we can splice sequences (8) and (9) together to obtain

$$0 \to \Lambda_0[-2D] \to E_2[-D] \to E_1[-D] \to E_2' \to E_1' \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ lies in $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{2D}$, and P^4 is generated in degree $2D$.

Now assume that $D = A+1$. We can similarly show that the image of $\mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$ and the image of $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)$ lies in $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{D+A+1}$ but $D = A+1$ so $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{D+A+1} = \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{2D}$. The same argument above also shows that the image of $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ lies in $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{2D}$. Hence $\mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0)_{2D}$ and P^4 is generated in degree $2D$.

Let $n = 5$. Then

$$\begin{aligned}\mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0) &= \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) + \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \\ &\quad + \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) + \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0).\end{aligned}$$

By condition (iii), since $D > 2$ and $A > 1$, $\mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) = 0 = \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$. So $\mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) + \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$. As before, an element in $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as a exact sequence of the following form

$$0 \to \Lambda_0[-D] \to E_2 \to E_1 \to \Lambda_0 \to 0 \tag{10}$$

and an element in $\mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)$ can be given by

$$\Lambda_0[-D-A] \to E'_3 \to E'_2 \to E'_1 \to \Lambda_0 \to 0 \tag{11}$$

The sequence (11) can be shifted by $-D$ to obtain

$$\Lambda_0[-2D-A] \to E'_3[-D] \to E'_2[-D] \to E'_1[-D] \to \Lambda_0[-D] \to 0 \tag{12}$$

and we can splice the sequences (10) and (12) to obtain

$$0 \to \Lambda_0[-2D-A] \to E'_3[-D] \to E'_2[-D] \to E'_1[-D] \to \Lambda_0[-D] \to E_2 \to E_1 \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0)_{2D+A}$. Similarly, the image of $\mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0)_{2D+A}$. Hence $\mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0) = \mathrm{Ext}^5_\Lambda(\Lambda_0, \Lambda_0)_{2D+A}$ and P^5 is generated in degree $2D+A$.

Let $n = 6$. Then

$$\begin{aligned}\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) &= \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)+\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)\\ &+\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)+\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\\ &+\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0).\end{aligned}$$

We begin by looking at $\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)$. Using exact sequences, we have

$$0\to\Lambda_0[-D]\to E_2\to E_1\to\Lambda_0\to 0 \tag{13}$$

is an element of $\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$ and

$$0\to\Lambda_0[-2D]\to E'_4\to E'_3\to E'_2\to E'_1\to\Lambda_0\to 0 \tag{14}$$

is an element of $\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)$. We can shift the sequence (14) by $-D$ to obtain

$$0\to\Lambda_0[-3D]\to E'_4[-D]\to E'_3[-D]\to E'_2[-D]\to E'_1[-D]\to\Lambda_0[-D]\to 0 \tag{15}$$

and splicing together the sequences (13) and (15) we obtain

$$0\to\Lambda_0[-3D]\to E'_4[-D]\to E'_3[-D]\to E'_2[-D]\to E'_1[-D]\to E_2\to E_1\to\Lambda_0\to 0.$$

Thus the image of $\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$. Similarly, the image of $\mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)\times\mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$.

Again, we can view the elements of $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$ as exact sequences. Let

$$0\to\Lambda_0[-D-A]\to E_3\to E_2\to E_1\to\Lambda_0\to 0 \tag{16}$$

and

$$0\to\Lambda_0[-D-A]\to E'_3\to E'_2\to E'_1\to\Lambda_0\to 0 \tag{17}$$

be two elements of $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$. We can shift the sequence (17) by $-D-A$ to obtain

$$0\to\Lambda_0[-2D-2A]\to E'_3[-D-A]\to E'_2[-D-A]\to E'_1[-D-A]\to\Lambda_0[-D-A]\to 0 \tag{18}$$

and splicing together the sequences (16) and (18) we obtain

$$0 \to \Lambda_0[-2D-2A] \to E_3'[-D-A] \to E_2'[-D-A] \to E_1'[-D-A]$$
$$\to E_3 \to E_2 \to E_1 \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{2D+2A}$.

In the same way we can show that the image of $\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$ and the image of $\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)$ are contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{2D+A+1}$

Now, we have 3 cases to consider:

(1) Let $D = 2A$. If $D = A+1$, then $A = 1$, which is a contradiction, so we must have $D \neq A+1$. Then by condition (ii), we have $\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)$. So $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0) + \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) + \mathrm{Ext}^4_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$. Now, the image of $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{2D+2A}$ but $D = 2A$, so $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$. It follows from above that $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$.

(2) Let $D \neq 2A$ and $D = A+1$. Since $D \neq 2A$ then by condition (iv), we have $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) = 0$. Now, the images of $\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$ and $\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)$ are contained in $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{2D+A+1}$ and $2D + A + 1 = 3D$, since $D = A+1$. Hence $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$.

(3) Let $D \neq 2A$ and $D \neq A+1$ then by condition (ii), $\mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) = 0 = \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)$ and by condition (iv), $\mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) = 0$. So $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$.

Therefore $\mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)_{3D}$ and P^6 is generated in degree $3D$.

We now use induction on n to show that $\mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)_{\delta(n)}$ for $n \geq 7$. We can assume that for $2 \leq m \leq 6$ we have $\mathrm{Ext}^m_\Lambda(\Lambda_0,\Lambda_0) = \mathrm{Ext}^m_\Lambda(\Lambda_0,\Lambda_0)_{\delta(m)}$ and P^m is generated in degree $\delta(m)$.

We consider two cases. First suppose that n is odd, so let $n = 2r + 1$, with $r \geq 3$. Assume that for $m < n$ we have $\operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0) = \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)_{\delta(m)}$ and P^m is generated in degree $\delta(m)$. Since $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3, we have

$$\begin{aligned}\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) \quad &= \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) + \operatorname{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^{n-2}_\Lambda(\Lambda_0, \Lambda_0)\\ &+ \cdots + \operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0) + \cdots +\\ &\operatorname{Ext}^{n-2}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) + \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0).\end{aligned}$$

By condition (iii) since $D > 2, A > 1$, $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) = 0 = \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$, so we need to look at $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ for $m \geq 2$ and $n - m \geq 2$. We begin by supposing m is odd, so $n - m$ is even. Then P^m is generated in degree $((m-1)/2)D + A$ and P^{n-m} is generated in degree $((n-m)/2)D$. An element of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((n-m)/2)D] \to E_{n-m} \to \cdots \to E_1 \to \Lambda_0 \to 0 \tag{19}$$

and an element of $\operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((m-1)/2)D - A] \to E'_m \to \cdots \to E'_1 \to \Lambda_0 \to 0. \tag{20}$$

We can shift the sequence (19) by $-((m-1)/2)D - A$ to obtain

$$\begin{aligned}0 \to \Lambda_0[-((n-1)/2)D - A] \to E_{n-m}[-((m-1)/2)D - A]\\ \to \cdots \to \Lambda_0[-((m-1)/2)D - A] \to 0.\end{aligned} \tag{21}$$

Splicing the sequences (20) and (21) we get

$$0 \to \Lambda_0[-((n-1)/2)D - A] \to E_{n-m}[-((m-1)/2)D - A] \to \cdots \to E'_1 \to \Lambda_0 \to 0.$$

Thus the image of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{((n-1)/2)D+A}$.

Similarly, if m is even then $n-m$ is odd and again we get that $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0)$ is contained in $\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)_{((n-1)/2)D+A}$. Thus for all $m \geq 2, n-m \geq 2$ we have $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0)$ is contained in $\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)_{((n-1)/2)D+A}$. Hence for n odd, P^n is generated in degree $((n-1)/2)D + A$.

Now we need to consider the even case. Let $n = 2r$ with $r \geq 4$. Again we assume that for $m < n$ we have $\mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0) = \mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0)_{\delta(m)}$ and P^m is generated in degree $\delta(m)$. We have

$$\begin{aligned}\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) &= \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{n-1}(\Lambda_0, \Lambda_0) + \mathrm{Ext}_\Lambda^2(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{n-2}(\Lambda_0, \Lambda_0) \\ &\quad + \cdots + \mathrm{Ext}_\Lambda^{n/2}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^{n/2}(\Lambda_0, \Lambda_0) + \cdots + \\ &\quad \mathrm{Ext}_\Lambda^{n-2}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^2(\Lambda_0, \Lambda_0) + \mathrm{Ext}_\Lambda^{n-1}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^1(\Lambda_0, \Lambda_0.)\end{aligned}$$

We begin with the case $m \geq 2$. Suppose m is even then $n-m$ is even. So P^m is generated in degree $((m)/2)D$ and P^{n-m} is generated in degree $((n-m)/2)D$. An element of $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((n-m)/2)D] \to E_{n-m} \to \cdots \to E_1 \to L_0 \to 0 \tag{22}$$

and an element of $\mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((m)/2)D] \to E'_m \to \cdots \to E'_1 \to L_0 \to 0. \tag{23}$$

We can shift the sequence (22) by $-((m)/2)D$ and we obtain

$$\Lambda_0[-(n/2)D] \to E_{n-m}[-(m/2)D] \to \cdots \to \Lambda_0[-(m/2)D] \to 0. \tag{24}$$

We can then splice the sequences (23) and (24) together to get

$$0 \to \Lambda_0[-(n/2)D] \to E_{n-m}[-(m/2)D] \to \cdots \to E'_m \to \cdots \to E'_1 \to \Lambda_0 \to 0.$$

Thus the image of $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0, \Lambda_0) \times \mathrm{Ext}_\Lambda^m(\Lambda_0, \Lambda_0)$ is contained in
$\mathrm{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)_{(n/2)D}$.

Now suppose m is odd, $m \geq 3$, so $n-m$ is also odd. Then P^m is generated in degree $((m-1)/2)D + A$ and P^{n-m} is generated in degree $((n-m-1)/2)D + A$. An element of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((n-m-1)/2)D - A] \to E_{n-m} \to \cdots \to E_1 \to \Lambda_0 \to 0 \tag{25}$$

and an element of $\operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ can be viewed as an exact sequence of the form

$$0 \to \Lambda_0[-((m-1)/2)D - A] \to E'_m \to \cdots \to E'_1 \to \Lambda_0 \to 0. \tag{26}$$

We can shift the sequence (25) by $-((m-1)/2)D - A$ and we obtain

$$\begin{aligned} 0 \to \Lambda_0[-((n-2)/2)D - 2A] &\to E_{n-m}[-((m-1)/2)D - A] \\ &\to \cdots \to \Lambda_0[-((m-1)/2)D - A] \to 0. \end{aligned} \tag{27}$$

We then splice together the sequences (25) and (27) to get the following sequence

$$0 \to \Lambda_0[-((n-2)/2)D - 2A] \to E_{n-m}[-((m-1)/2)D - A] \to \cdots \to E'_1 \to \Lambda_0 \to 0.$$

Thus the image of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{((n-2)/2)D+2A}$.

In a similar way we can show that the images of $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0)$ and $\operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$ are contained in $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{(\frac{n-2}{2})D+A+1}$.

Now, we have 3 cases to consider:

(1) Let $D = 2A$ so $D \neq A+1$. Then by condition (ii) $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) = 0 = \operatorname{Ext}^{n-1}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0)$. The image of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{(\frac{n}{2})D}$ for m even and $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{(\frac{n-2}{2})D+2A}$ for m odd, $m \geq 3$. However, $D = 2A$, so for all $m \geq 2$, the image of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ is contained in $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{(\frac{n}{2})D}$. So $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0) = \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)_{(\frac{n}{2})D}$.

(2) Let $D \neq 2A$, $D = A+1$. Then the image of $\operatorname{Ext}^{n-m}_\Lambda(\Lambda_0, \Lambda_0) \times \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0) = 0$ for m odd, $m \geq 3$, by condition (iv). The images of $\operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \times$

$\mathrm{Ext}_\Lambda^{n-1}(\Lambda_0,\Lambda_0)$ and $\mathrm{Ext}_\Lambda^1(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^{n-1}(\Lambda_0,\Lambda_0)$ are contained in $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n-2}{2})D+A+1}=\mathrm{Ext}_\Lambda^1(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$, since $D=A+1$. The image of $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^m(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$, for m even. So $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)=\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$.

(3) Let $D\neq 2A, D\neq A+1$. By conditions (ii) and (iv), $\mathrm{Ext}_\Lambda^1(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^{n-1}(\Lambda_0,\Lambda_0)=0=\mathrm{Ext}_\Lambda^{n-1}(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^1(\Lambda_0,\Lambda_0)$ and $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^m(\Lambda_0,\Lambda_0)=0$, for m odd. The image of $\mathrm{Ext}_\Lambda^{n-m}(\Lambda_0,\Lambda_0)\times\mathrm{Ext}_\Lambda^m(\Lambda_0,\Lambda_0)$ is contained in $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$, for m even. So $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)=\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$.

Hence for n even, $\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)=\mathrm{Ext}_\Lambda^n(\Lambda_0,\Lambda_0)_{(\frac{n}{2})D}$, and therefore P^n is generated in degree $\frac{n}{2}D$.

Thus, for all $n\geq 0$, P^n is generated in degree $\delta(n)$, where $\delta(n)$ is given by

$$\delta(n)=\begin{cases} 0 & \text{if} \quad n=0\\ 1 & \text{if} \quad n=1\\ \frac{n}{2}D & \text{if} \quad n=2r, r\in\mathbb{N}\\ \left(\frac{n-1}{2}\right)D+A & \text{if} \quad n=2r+1, r\in\mathbb{N}.\end{cases}$$

Therefore, Λ is a (D,A)-stacked algebra. $\square$

7. Regrading of the Ext Algebra

In this section we consider the following question. Given a (D, A)-stacked algebra, Λ, can the Ext algebra, $E(\Lambda)$, be regraded as a Koszul algebra? This is inspired by the work of [13], where the authors take the Ext algebra of a D-Koszul algebra Λ, and show that there is a regrading, given by

$$\begin{aligned}
\hat{E}(\Lambda)_0 &= \operatorname{Ext}^0_\Lambda(\Lambda_0, \Lambda_0) \\
\hat{E}(\Lambda)_1 &= \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \oplus \operatorname{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \\
\hat{E}(\Lambda)_n &= \operatorname{Ext}^{2n-1}_\Lambda(\Lambda_0, \Lambda_0) \oplus \operatorname{Ext}^{2n}_\Lambda(\Lambda_0, \Lambda_0), \text{for } n \geq 2.
\end{aligned}$$

With this hat-degree grading, $\hat{E}(\Lambda) = \oplus_{n \geq 0} \hat{E}(\Lambda)_n$ is a Koszul algebra, [13, section 7]. We note that for $D = 2, A = 1$, then Λ is a Koszul algebra and it is well known that $E(\Lambda)$ is a Koszul algebra without any regrading.

We now define a grading on our (D, A)-stacked algebras with certain conditions on D and A, which we also call the hat-degree grading.

Definition 7.1. Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra, with $D > 2, A > 1$, $D \neq 2A$ and $D \neq A + 1$.

$$\begin{aligned}
\text{We define } \hat{E}(\Lambda)_0 &= \operatorname{Ext}^0_\Lambda(\Lambda_0, \Lambda_0) \\
\hat{E}(\Lambda)_1 &= \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \oplus \operatorname{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \oplus \operatorname{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \\
\hat{E}(\Lambda)_n &= \operatorname{Ext}^{2n}_\Lambda(\Lambda_0, \Lambda_0) \oplus \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0, \Lambda_0), \text{for } n \geq 2. \\
\text{Let } \hat{E}(\Lambda) &= \oplus_{n \geq 0} \hat{E}(\Lambda)_n.
\end{aligned}$$

We now proceed to show that the hat-degree gives a well defined grading, with certain conditions on D and A. This will be followed by explicit examples.

Theorem 7.2. *Let Λ be a (D, A)-stacked algebra with $D > 2, D \neq 2A, D \neq A + 1, A > 1$, with $\hat{E}(\Lambda) = \oplus_{n \geq 0} \hat{E}(\Lambda)_n$. Then the Ext algebra $\hat{E}(\Lambda)$ is graded in this hat-degree.*

Proof. We need to show $\hat{E}(\Lambda)_m \times \hat{E}(\Lambda)_n = \hat{E}(\Lambda)_{m+n}$, for all $m, n \geq 0$. This is clearly true for either $m = 0$ or $n = 0$. We start with the case $m = n = 1$. Then

$$\begin{aligned}
\hat{E}(\Lambda)_1 \times \hat{E}(\Lambda)_1 &= \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\qquad \times \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&= \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \\
&+ \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \\
&+ \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) + \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\oplus \Big(\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) + \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\oplus \Big(\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \Big).
\end{aligned}$$

From Theorem 6.7, we find $\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) = 0$, $\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) = 0 = \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$, $\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) = 0 = \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$ and $\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) = 0$. This leaves us with only $\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$, $\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$ and $\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$.

Now, by Proposition 5.5, $\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) = \operatorname{Ext}^4_\Lambda(\Lambda_0,\Lambda_0)$ and $\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) = \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) = \operatorname{Ext}^5_\Lambda(\Lambda_0,\Lambda_0)$. Thus $\hat{E}(\Lambda)_1 \times \hat{E}(\Lambda)_1 = \hat{E}(\Lambda)_2$, as required.

Let $m = 1, n \geq 2$. Then

$$\begin{aligned}
\hat{E}(\Lambda)_1 \times \hat{E}(\Lambda)_n &= \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\qquad \times \Big(\operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&= \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \\
&+ \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \\
&+ \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \Big)
\end{aligned}$$

Using Proposition 5.2 and Theorem 6.7, we have

$$\hat{E}(\Lambda)_1 \times \hat{E}(\Lambda)_n = \operatorname{Ext}^{2n+2}_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^{2n+3}_\Lambda(\Lambda_0,\Lambda_0) = \hat{E}(\Lambda)_{n+1}, \text{ as required.}$$

Similarly, $\hat{E}_n(\Lambda) \times \hat{E}_1(\Lambda) = \hat{E}_{n+1}(\Lambda)$, for $n \geq 2$.

Finally, let $m \geq 2, n \geq 2$. Then

$$\begin{aligned}
\hat{E}(\Lambda)_m \times \hat{E}(\Lambda)_n &= \Big(\operatorname{Ext}^{2m}_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^{2m+1}_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\qquad \times \Big(\operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&= \Big(\operatorname{Ext}^{2m}_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \Big) \oplus \Big(\operatorname{Ext}^{2m}_\Lambda(\Lambda_0,\Lambda_0) \times \\
&\quad \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) + \operatorname{Ext}^{2m+1}_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n}_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&\quad \oplus \Big(\operatorname{Ext}^{2m+1}_\Lambda(\Lambda_0,\Lambda_0) \times \operatorname{Ext}^{2n+1}_\Lambda(\Lambda_0,\Lambda_0) \Big) \\
&= \Big(\operatorname{Ext}^{2m+2n}_\Lambda(\Lambda_0,\Lambda_0) \oplus \operatorname{Ext}^{2m+2n+1}_\Lambda(\Lambda_0,\Lambda_0) \Big), \\
&\qquad \text{from Proposition 5.2 and Theorem 6.7,} \\
&= \hat{E}(\Lambda)_{m+n}, \text{ as required.}
\end{aligned}$$

This completes the proof. $\square$

Now that we have shown Definition 7.1 gives a grading on our algebra $E(\Lambda)$, we will look at some specific examples of (D, A)-stacked algebras and their Ext algebras. In particular we ask the question, with the regrading of Definition 7.1 do we obtain a Koszul algebra? We begin by considering a $(6, 2)$-stacked algebra in Section 7.1, in which we find that we need to use Gröbner Bases; these are introduced in Section 7.2. We return to this example in Section 7.3 and show that, after regrading, the Ext algebra is indeed a Koszul algebra. In Section 7.4 we consider an example of a $(6, 2)$-stacked monomial algebra which has the same underlying quiver as the previous example but with monomial relations. After these two examples we then briefly look at an example of a $(4, 2)$-stacked algebra in which Definition 7.1 does not define a grading. We conclude this chapter with Theorem 7.25, (in Section 7.5), in which we show that there are some (D, A)-stacked algebras for which there is no regrading such that $\hat{E}(\Lambda)$ is Koszul. Finally, we give an example of a (D, A)-stacked algebra which justifies the hypothesis of Theorem 7.25.

7.1. **Example 1: A $(6, 2)$-stacked algebra.**

Example 7.3. Let $\mathcal{Q}$ be the quiver given by

$$
\begin{tikzcd}
 & & 3 \arrow[r, "\alpha_3"] & 4 \arrow[r, "\alpha_4"] & 5 \arrow[dr, "\alpha_5"] & & & & 14 \arrow[dd, "\alpha_{15}"] \\
 & 2 \arrow[ur, "\alpha_2"] & & & & 6 \arrow[dr, "\alpha_6"] & & 13 \arrow[ur, "\alpha_{14}"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_7"'] & & & & & & 12 \arrow[ur, "\alpha_{13}"] & & 15 \arrow[dd, "\alpha_{16}"] \\
 & 7 \arrow[dr, "\alpha_8"'] & & & & 11 \arrow[ur, "\alpha_{12}"'] & & 17 \arrow[ul, "\alpha_{18}"] & \\
 & & 8 \arrow[r, "\alpha_9"'] & 9 \arrow[r, "\alpha_{10}"'] & 10 \arrow[ur, "\alpha_{11}"'] & & & & 16 \arrow[ul, "\alpha_{17}"]
\end{tikzcd}
$$

and let $\Lambda = K\mathcal{Q}/I$ where $I = \langle \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 - \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}, \alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}, \alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}, \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}, \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\rangle$.

We start by finding a minimal projective resolution of $\Lambda/\mathfrak{r}$ in order to show that Λ is a $(6, 2)$-stacked algebra.

A minimal projective resolution for $\Lambda/\mathfrak{r}$ as a right Λ-module is given by

$$\cdots \to P^n \to \cdots \to P^2 \to P^1 \to P^0 \to \Lambda/\mathfrak{r} \to 0$$

where

- $P^0 = \bigoplus_{i=1}^{17} e_i\Lambda$ and d^0 is the canonical surjection , $d^0 : e_i\lambda \mapsto e_i\lambda + \mathfrak{r}$.
- $P^1 = \bigoplus_{i=1}^{18} t(\alpha_i)\Lambda$ and $d^1 : t(\alpha_i)\lambda \mapsto \alpha_i\lambda$.
- $P^2 = e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda$ and

$d^2(e_{12}\lambda_1, e_{14}\lambda_2, e_{16}\lambda_3, e_{14}\lambda_4, e_{16}\lambda_5, e_{12}\lambda_6, e_{14}\lambda_7, e_{16}\lambda_8)$

$= (\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\lambda_1, 0, \alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}\lambda_2, 0, \alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_3, 0, -\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\lambda_1, 0, \alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\lambda_4, 0, \alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_5, 0, \alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_6, 0, \alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_7, 0, \alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_8)$.

- $P^3 = e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda$ and

$d^3(e_{14}\lambda_1, e_{16}\lambda_2, e_{12}\lambda_3, e_{16}\lambda_4, e_{12}\lambda_5, e_{14}\lambda_6, e_{16}\lambda_7, e_{12}\lambda_8)$

$= (\alpha_{13}\alpha_{14}\lambda_1, \alpha_{15}\alpha_{16}\lambda_2, \alpha_{17}\alpha_{18}\lambda_3, \alpha_{15}\alpha_{16}\lambda_4, \alpha_{17}\alpha_{18}\lambda_5, \alpha_{13}\alpha_{14}\lambda_6, \alpha_{15}\alpha_{16}\lambda_7, \alpha_{17}\alpha_{18}\lambda_8)$.

- For $n \geq 4$, if n is even, $P^n = e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda$

and $d^n(e_{12}\lambda_1, e_{14}\lambda_2, e_{16}\lambda_3, e_{14}\lambda_4, e_{16}\lambda_5, e_{12}\lambda_6, e_{14}\lambda_7, e_{16}\lambda_8)$

$= (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_1, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_2, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_3, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_4,$
$\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_5, \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_6, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_7, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_8)$.

- For $n \geq 5$, if n is odd, $P^n = e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda$

and $d^n(e_{14}\lambda_1, e_{16}\lambda_2, e_{12}\lambda_3, e_{16}\lambda_4, e_{12}\lambda_5, e_{14}\lambda_6, e_{16}\lambda_7, e_{12}\lambda_8)$

$= (\alpha_{13}\alpha_{14}\lambda_1, \alpha_{15}\alpha_{16}\lambda_2, \alpha_{17}\alpha_{18}\lambda_3, \alpha_{15}\alpha_{16}\lambda_4, \alpha_{17}\alpha_{18}\lambda_5, \alpha_{13}\alpha_{14}\lambda_6, \alpha_{15}\alpha_{16}\lambda_7, \alpha_{17}\alpha_{18}\lambda_8)$.

The sets g^n are given as follows;

- $g^0 = \{e_1, e_2, \ldots, e_{17}\}$.

- $g^1 = \{g^1_1 = \alpha_1, g^1_2 = \alpha_2, \ldots, g^1_{18} = \alpha_{18}\}$.
- $g^2 = \{g^2_1 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 - \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}, g^2_2 = \alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14},$

  $g^2_3 = \alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, g^2_4 = \alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}, g^2_5 = \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16},$

  $g^2_6 = \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}, g^2_7 = \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}, g^2_8 = \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\}$
- $g^3 = \{g^3_1 = (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 - \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12})\alpha_{13}\alpha_{14},$

  $g^3_2 = \alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, g^3_3 = \alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18},$

  $g^3_4 = \alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, g^3_5 = \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18},$

  $g^3_6 = \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}, g^3_7 = \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16},$

  $g^3_8 = \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\}$
- For $n \geq 4$, if $n = 2r, n \geq 2$,

  $g^n_1 = \{g^n_1 = (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 - \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1},$

  $g^n_2 = (\alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1},$

  $g^n_3 = (\alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1},$

  $g^n_4 = (\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1},$

  $g^n_5 = (\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1},$

  $g^n_6 = (\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^r,\ g^n_7 = (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^r,$

  $g^n_8 = (\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^r\}$.
- For $n \geq 5$, if $n = 2r+1, r \geq 2$,

  $g^n_1 = \{g^n_1 = (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 - \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1}\alpha_{13}\alpha_{14},$

  $g^n_2 = (\alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1}\alpha_{15}\alpha_{16},$

  $g^n_3 = (\alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1}\alpha_{17}\alpha_{18},$

  $g^n_4 = (\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1}\alpha_{15}\alpha_{16},$

  $g^n_5 = (\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1}\alpha_{17}\alpha_{18},$

  $g^n_6 = (\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^r\alpha_{13}\alpha_{14}, g^n_7 = (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^r\alpha_{15}\alpha_{16},$

  $g^n_8 = (\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^r\alpha_{17}\alpha_{18}\}$.

Looking at the length of g^n, it is clear from Definition 5.1 that Λ is a $(6,2)$-stacked algebra. Then we can use Theorem 5.7 to say that $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3. We now describe $E(\Lambda)$ by quiver and relations.

Since (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$ then $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \cong \mathrm{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$, as discussed in Chapter 3. We can take a basis f^n of $\mathrm{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$ for each $n \geq 0$ as follows.

Let f^n_i be the Λ-module homomorphism $P^n \to \Lambda/\mathfrak{r}$ given by

$$t(g^n_j)\lambda_j \mapsto \begin{cases} t(g^n_i)\lambda_i + \mathfrak{r} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We set $f^n = \{f^n_i\}$ so that $|f^n| = |g^n|$.

We have the following products in the Ext algebra. If $n = 2r$ and $r \geq 2$ then, $f^n_1 = f^2_6 \cdot f^{n-2}_1$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_1 = f^2_6 \circ \mathcal{L}^2 f^{n-2}_1$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_1$ can be chosen as $\mathcal{L}^2 f^{n-2}_1 : P^n \longrightarrow P^2,\ t(g^n_1)\lambda = e_{12}\lambda \mapsto e_{12}\lambda = t(g^2_6)\lambda, \text{else } \mapsto 0$.

$f^n_2 = f^2_7 \cdot f^{n-2}_2$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_2 = f^2_7 \circ \mathcal{L}^2 f^{n-2}_2$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_2$ can be chosen as $\mathcal{L}^2 f^{n-2}_2 : P^n \longrightarrow P^2,\ t(g^n_2)\lambda = e_{14}\lambda \mapsto e_{14}\lambda = t(g^2_7)\lambda, \text{else } \mapsto 0$.

$f^n_3 = f^2_8 \cdot f^{n-2}_3$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_3 = f^2_8 \circ \mathcal{L}^2 f^{n-2}_3$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_3$ can be chosen as $\mathcal{L}^2 f^{n-2}_3 : P^n \longrightarrow P^2,\ t(g^n_3)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g^2_8)\lambda, \text{else } \mapsto 0$.

$f^n_4 = f^2_7 \cdot f^{n-2}_4$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_4 = f^2_7 \circ \mathcal{L}^2 f^{n-2}_4$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_4$ can be chosen as $\mathcal{L}^2 f^{n-2}_4 : P^n \longrightarrow P^2,\ t(g^n_4)\lambda = e_{14}\lambda \mapsto e_{14}\lambda = t(g^2_7)\lambda, \text{else } \mapsto 0$.

$f^n_5 = f^2_8 \cdot f^{n-2}_5$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_5 = f^2_8 \circ \mathcal{L}^2 f^{n-2}_5$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_5$ can be chosen as $\mathcal{L}^2 f^{n-2}_5 : P^n \longrightarrow P^2,\ t(g^n_5)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g^2_8)\lambda, \text{else } \mapsto 0$.

$f^n_6 = f^2_6 \cdot f^{n-2}_6$ in $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f^n_6 = f^2_6 \circ \mathcal{L}^2 f^{n-2}_6$ as maps, where the lifting $\mathcal{L}^2 f^{n-2}_6$ can be chosen as $\mathcal{L}^2 f^{n-2}_6 : P^n \longrightarrow P^2,\ t(g^n_6)\lambda = e_{12}\lambda \mapsto e_{12}\lambda = t(g^2_6)\lambda, \text{else } \mapsto 0$.

$f_7^n = f_7^2 \cdot f_7^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_7^n = f_7^2 \circ \mathcal{L}^2 f_7^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_7^{n-2}$ can be chosen as $\mathcal{L}^2 f_7^{n-2} : P^n \longrightarrow P^2$, $t(g_7^n)\lambda = e_{14}\lambda \mapsto e_{14}\lambda = t(g_7^2)\lambda, \text{else} \mapsto 0$.

$f_8^n = f_8^2 \cdot f_8^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_8^n = f_8^2 \circ \mathcal{L}^2 f_8^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_8^{n-2}$ can be chosen as $\mathcal{L}^2 f_8^{n-2} : P^n \longrightarrow P^2$, $t(g_8^n)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g_8^2)\lambda, \text{else} \mapsto 0$.

If $n = 2r + 1$ and $r \geq 2$ then, $f_1^n = f_7^2 \cdot f_1^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_1^n = f_7^2 \circ \mathcal{L}^2 f_1^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_1^{n-2}$ can be chosen as $\mathcal{L}^2 f_1^{n-2} : P^n \longrightarrow P^2$, $t(g_1^n)\lambda = e_{14}\lambda \mapsto e_{14}\lambda = t(g_7^2)\lambda, \text{else} \mapsto 0$.

$f_2^n = f_8^2 \cdot f_2^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_2^n = f_8^2 \circ \mathcal{L}^2 f_2^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_2^{n-2}$ can be chosen as $\mathcal{L}^2 f_2^{n-2} : P^n \longrightarrow P^2$, $t(g_2^n)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g_8^2)\lambda, \text{else} \mapsto 0$.

$f_3^n = f_6^2 \cdot f_3^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_3^n = f_6^2 \circ \mathcal{L}^2 f_3^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_3^{n-2}$ can be chosen as $\mathcal{L}^2 f_3^{n-2} : P^n \longrightarrow P^2$, $t(g_3^n)\lambda = e_{12}\lambda \mapsto e_{12}\lambda = t(g_6^2)\lambda, \text{else} \mapsto 0$.

$f_4^n = f_8^2 \cdot f_4^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_4^n = f_8^2 \circ \mathcal{L}^2 f_4^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_4^{n-2}$ can be chosen as $\mathcal{L}^2 f_4^{n-2} : P^n \longrightarrow P^2$, $t(g_4^n)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g_8^2)\lambda, \text{else} \mapsto 0$.

$f_5^n = f_6^2 \cdot f_5^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_5^n = f_6^2 \circ \mathcal{L}^2 f_5^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_5^{n-2}$ can be chosen as $\mathcal{L}^2 f_5^{n-2} : P^n \longrightarrow P^2$, $t(g_5^n)\lambda = e_{12}\lambda \mapsto e_{12}\lambda = t(g_6^2)\lambda, \text{else} \mapsto 0$.

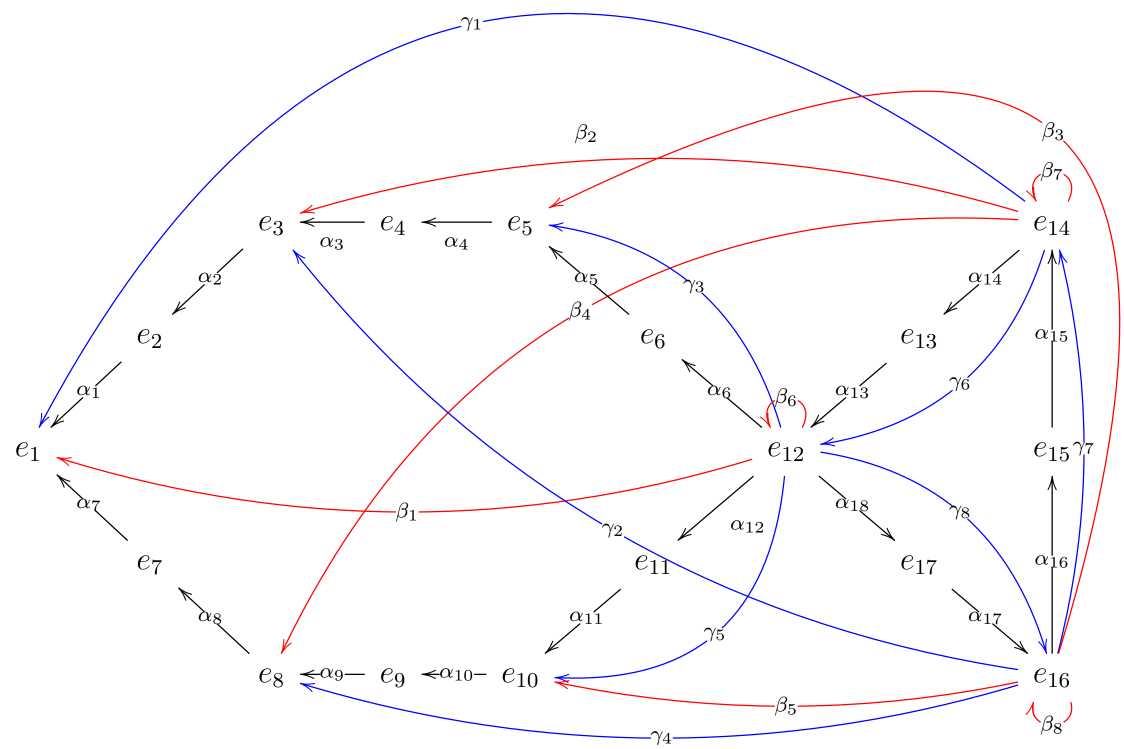
$f_6^n = f_7^2 \cdot f_6^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_6^n = f_7^2 \circ \mathcal{L}^2 f_6^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_6^{n-2}$ can be chosen as $\mathcal{L}^2 f_6^{n-2} : P^n \longrightarrow P^2$, $t(g_6^n)\lambda = e_{14}\lambda \mapsto e_{14}\lambda = t(g_7^2)\lambda, \text{else} \mapsto 0$.

$f_7^n = f_8^2 \cdot f_7^{n-2}$ in $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_7^n = f_8^2 \circ \mathcal{L}^2 f_7^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_7^{n-2}$ can be chosen as $\mathcal{L}^2 f_7^{n-2} : P^n \longrightarrow P^2$, $t(g_7^n)\lambda = e_{16}\lambda \mapsto e_{16}\lambda = t(g_8^2)\lambda, \text{else} \mapsto 0$.

$f_8^n = f_6^2 \cdot f_8^{n-2}$ in $\mathrm{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, since $f_8^n = f_6^2 \circ \mathcal{L}^2 f_8^{n-2}$ as maps, where the lifting $\mathcal{L}^2 f_8^{n-2}$ can be chosen as $\mathcal{L}^2 f_8^{n-2} : P^n \longrightarrow P^2$, $t(g_8^n)\lambda = e_{12}\lambda \mapsto e_{12}\lambda = t(g_6^2)\lambda, \text{else } \mapsto 0$.

Now that we have this information, the Ext algebra, $E(\Lambda)$, can be represented by quiver and relations.

Let Γ be the quiver given by



and let $A = K\Gamma/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by

- $\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_8\alpha_7, \alpha_9\alpha_8, \alpha_{10}\alpha_9, \alpha_{11}\alpha_{10}, \alpha_{12}\alpha_{11}, \alpha_{13}\alpha_6,$ $\alpha_{13}\alpha_{12}, \alpha_{13}\alpha_{18}, \alpha_{14}\alpha_{13}, \alpha_{15}\alpha_{14}, \alpha_{16}\alpha_{15}, \alpha_{17}\alpha_{16}, \alpha_{18}\alpha_{17},$
- $\alpha_{13}\beta_1, \alpha_{15}\beta_2, \beta_2\alpha_2, \alpha_{17}\beta_3, \beta_3\alpha_4, \alpha_{15}\beta_4, \beta_4\alpha_8, \alpha_{17}\beta_5, \beta_5\alpha_{10}, \alpha_{13}\beta_6, \beta_6\alpha_6,$ $\beta_6\alpha_{12}, \beta_6\alpha_{18}, \alpha_{15}\beta_7, \beta_7\alpha_{14}, \alpha_{17}\beta_8, \beta_8\alpha_{16},$
- $\alpha_{15}\gamma_1, \alpha_{17}\gamma_2, \gamma_2\alpha_2, \alpha_{13}\gamma_3, \gamma_3\alpha_4, \alpha_{17}\gamma_4, \gamma_4\alpha_8, \alpha_{13}\gamma_5, \gamma_5\alpha_{10}, \alpha_{15}\gamma_6, \gamma_6\alpha_6, \gamma_6\alpha_{12},$ $\gamma_6\alpha_{18}, \alpha_{17}\gamma_7, \gamma_7\alpha_{14}, \alpha_{13}\gamma_8, \gamma_8\alpha_{16},$
- $\beta_7\gamma_1 - \gamma_6\beta_1,\ \beta_8\gamma_2 - \gamma_7\beta_2,\ \beta_6\gamma_3 - \gamma_8\beta_3,\ \beta_8\gamma_4 - \gamma_7\beta_4,\ \beta_6\gamma_5 - \gamma_8\beta_5,\ \beta_7\gamma_6 - \gamma_6\beta_6,$ $\beta_8\gamma_7 - \gamma_7\beta_7,\ \beta_6\gamma_8 - \gamma_8\beta_8,$
- $\gamma_7\gamma_1, \gamma_8\gamma_2, \gamma_6\gamma_3, \gamma_8\gamma_4, \gamma_6\gamma_5, \gamma_7\gamma_6, \gamma_8\gamma_7, \gamma_6\gamma_8.$

This algebra A is the algebra $E(\Lambda)$ given by quiver and relations, where we write α_i for f_i^1, β_i for f_i^2 and γ_i for f_i^3. This gives the following result.

Proposition 7.4. *Let Λ be the algebra given in Example 7.3, and let $A = K\Gamma/\mathcal{I}$ as defined above. Then $A \cong \hat{E}(\Lambda)$ where α_i, β_i and γ_i are all in degree 1, corresponding to the elements of $\hat{E}(\Lambda)_1$.*

With this hat-degree grading, we wish to show A is a Koszul algebra. From Definition 4.2, we know that if A is Koszul then there is a linear minimal projective resolution of $A/\mathfrak{r}$ as a right A-module. We start by trying to construct a minimal projective resolution of $A/\mathfrak{r}$ as a right A-module:

$$\cdots \longrightarrow P^n \xrightarrow{d^n} \cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A/\mathfrak{r} \longrightarrow 0.$$

The projective modules are denoted by P^n in the usual way, however these are not to be confused with the projective modules in the minimal projective resolution of $\Lambda/\mathfrak{r}$.

We know $P^0 = e_1A \oplus e_2A \oplus \cdots \oplus e_{17}A$, d^0 is the canonical surjection given by $d^0 : e_i\lambda \mapsto e_i\lambda + \mathfrak{r}$. Then $\operatorname{Ker} d^0 \cong \mathfrak{r}$, so $P^1 = t(\alpha_1)A \oplus \cdots \oplus t(\alpha_{18})A \oplus t(\beta_1)A \oplus \cdots \oplus t(\beta_8)A \oplus t(\gamma_1)A \oplus \cdots \oplus t(\gamma_8)A$ and $d^1(t(p_i)\lambda) = p_i\lambda$, where p_i is an arrow in $\Gamma_1, \lambda \in A$.

Let g^2 be the set of minimal generators of I, labelled g_i^2 for $i = 1, \ldots, 68$. For each i we have $g_i^2 = \sum_j g_j^1 q_{i,j}$, as in [20]. Then $P^2 = \bigoplus_{i=1}^{68} t(g_i^2)A$, $d^2(t(g_i^2)\lambda) = \sum_j q_{i,j}\lambda$, where $g_i^2 = \sum_j g_j^1 q_{i,j}$, for $q_{i,j} \in \mathcal{Q}_1$.

Let g^3 be the set of g_i^3, as given in [20]. So $g_i^3 = \sum_j g_j^2 q_{i,j}$, where $q_{i,j} \in K\mathcal{Q}$. In particular, for A, the set g^3 is given by

$\{\alpha_3\alpha_2\alpha_1, \alpha_4\alpha_3\alpha_2, \alpha_5\alpha_4\alpha_3, \alpha_6\alpha_5\alpha_4, \alpha_9\alpha_8\alpha_7, \alpha_{10}\alpha_9\alpha_8, \alpha_{11}\alpha_{10}\alpha_9, \alpha_{12}\alpha_{11}\alpha_{10},$
$\alpha_{13}\alpha_6\alpha_5, \alpha_{13}\alpha_{12}\alpha_{11}, \alpha_{13}\alpha_{18}\alpha_{17}, \alpha_{14}\alpha_{13}\alpha_6, \alpha_{14}\alpha_{13}\alpha_{12}, \alpha_{14}\alpha_{13}\alpha_{18}, \alpha_{14}\alpha_{13}\beta_1,$
$\alpha_{14}\alpha_{13}\beta_6, \alpha_{14}\alpha_{13}\gamma_3, \alpha_{14}\alpha_{13}\gamma_5, \alpha_{14}\alpha_{13}\gamma_8, \alpha_{15}\alpha_{14}\alpha_{13}, \alpha_{16}\alpha_{15}\alpha_{14}, \alpha_{17}\alpha_{16}\alpha_{15},$
$\alpha_{18}\alpha_{17}\alpha_{16}, \alpha_{18}\alpha_{17}\beta_3, \alpha_{18}\alpha_{17}\beta_5, \alpha_{18}\alpha_{17}\beta_8, \alpha_{18}\alpha_{17}\gamma_2, \alpha_{18}\alpha_{17}\gamma_4, \alpha_{18}\alpha_{17}\gamma_7,$
$\alpha_{15}\beta_2\alpha_2, \beta_2\alpha_2\alpha_1, \alpha_{17}\beta_3\alpha_4, \beta_3\alpha_4\alpha_3, \alpha_{15}\beta_4\alpha_8, \beta_4\alpha_8\alpha_7, \alpha_{17}\beta_5\alpha_{10}, \beta_5\alpha_{10}\alpha_9,$
$\alpha_{13}\beta_6\alpha_6, \alpha_{13}\beta_6\alpha_{12}, \alpha_{13}\beta_6\alpha_{18}, \beta_6\alpha_6\alpha_5, \beta_6\alpha_{12}\alpha_{11}, \beta_6\alpha_{18}\alpha_{17}, \alpha_{15}\beta_7\alpha_{14}, \beta_7\alpha_{14}\alpha_{13},$
$\alpha_{17}\beta_8\alpha_{16}, \beta_8\alpha_{16}\alpha_{15}, \alpha_{17}\gamma_2\alpha_2, \gamma_2\alpha_2\alpha_1, \alpha_{13}\gamma_3\alpha_4, \gamma_3\alpha_4\alpha_3, \alpha_{17}\gamma_4\alpha_8, \gamma_4\alpha_8\alpha_7,$
$\alpha_{13}\gamma_5\alpha_{10}, \gamma_5\alpha_{10}\alpha_9, \alpha_{15}\gamma_6\alpha_6, \alpha_{15}\gamma_6\alpha_{12}, \alpha_{15}\gamma_6\alpha_{18}, \alpha_{15}\gamma_6\gamma_3, \alpha_{15}\gamma_6\gamma_5, \alpha_{15}\gamma_6\gamma_8,$
$\gamma_6\alpha_6\alpha_5, \gamma_6\alpha_{12}\alpha_{11}, \gamma_6\alpha_{18}\alpha_{17}, \alpha_{17}\gamma_7\alpha_{14}, \alpha_{17}\gamma_7\gamma_1, \alpha_{17}\gamma_7\gamma_6, \gamma_7\alpha_{14}\alpha_{13}, \alpha_{13}\gamma_8\alpha_{16},$
$\alpha_{13}\gamma_8\gamma_2, \alpha_{13}\gamma_8\gamma_4, \alpha_{13}\gamma_8\gamma_7, \gamma_8\alpha_{16}\alpha_{15}, \gamma_8\gamma_2\alpha_2, \gamma_6\gamma_3\alpha_4, \gamma_8\gamma_4\alpha_8, \gamma_6\gamma_5\alpha_{10},$
$\gamma_7\gamma_6\alpha_6, \gamma_7\gamma_6\alpha_{12}, \gamma_7\gamma_6\alpha_{18}, \gamma_7\gamma_6\gamma_3, \gamma_7\gamma_6\gamma_5, \gamma_7\gamma_6\gamma_8, \gamma_8\gamma_7\alpha_{14}, \gamma_8\gamma_7\gamma_1, \gamma_8\gamma_7\gamma_6,$
$\gamma_6\gamma_8\alpha_{16}, \gamma_6\gamma_8\gamma_2, \gamma_6\gamma_8\gamma_4, \gamma_6\gamma_8\gamma_7, (\beta_8\gamma_2 - \gamma_7\beta_2)\alpha_2, (\beta_6\gamma_3 - \gamma_8\beta_3)\alpha_4, (\beta_8\gamma_4 - \gamma_7\beta_4)\alpha_8,$
$(\beta_6\gamma_5 - \gamma_8\beta_5)\alpha_{10}, (\beta_7\gamma_6 - \gamma_6\beta_6)\alpha_6, (\beta_7\gamma_6 - \gamma_6\beta_6)\alpha_{12}, (\beta_7\gamma_6 - \gamma_6\beta_6)\alpha_{18},$
$(\beta_8\gamma_7 - \gamma_7\beta_7)\alpha_{14}, (\beta_6\gamma_8 - \gamma_8\beta_8)\alpha_{16})\}.$

From this set we can write down the projective module P^3 and the map $d^3 : P^3 \to P^2$. We label the elements of g^3 by $g^3_1, g^3_2, \ldots, g^3_{99}$ in the order they are listed above. Then $P^3 = \bigoplus_{i=1}^{99} t(g^3_i)A$ and the map $d^3 : P^3 \to P^2$ is the A-module homomorphism where $d^3(t(g^3_i))$ has entry $t(g^2_j)q_{i,j}$ in the component of P^2 corresponding to $t(g^2_j)$, where $g^3_i = \sum_j g^2_j q_{i,j}$, with $q_{i,j} \in K\Gamma$.

We now find the set g^4. Each g^4_i is found following the same method, from [20], in particular $g^4_i = \sum_j g^3_j p_{i,j}$, for some $p_{i,j} \in K\Gamma$. The following table gives the elements g^4_i that begin at e_5, e_6, e_{10} and e_{11} respectively; they end at the vertex with which the column is indexed. In all cases, the element g^4_i is given by the element g^3_j, as indexed by the rows, composed with the single elements given within the rows.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}
$\alpha_4\alpha_3\alpha_2$	α_1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_5\alpha_4\alpha_3$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

$\alpha_{10}\alpha_9\alpha_8$	α_7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{11}\alpha_{10}\alpha_9$	-	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-	-	-

The following table shows the elements of g^4 that begin at e_{12}, end at e_k, for $k = \{1, 2, \ldots, 17\}$, as the columns are indexed.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}
$\alpha_6\alpha_5\alpha_4$	-	-	α_3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{12}\alpha_{11}\alpha_{10}$	-	-	-	-	-	-	-	α_9	-	-	-	-	-	-	-	-	-
$\alpha_{18}\alpha_{17}\alpha_{16}$	-	-	-	-	-	-	-	-	-	-	-	-	α_{15}	-	-	-	-
$\alpha_{18}\alpha_{17}\beta_3$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{18}\alpha_{17}\beta_5$	-	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-	-
$\alpha_{18}\alpha_{17}\beta_8$	-	-	-	-	-	-	-	-	-	-	-	-	α_{16}	-	-	-	-
$\alpha_{18}\alpha_{17}\gamma_2$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{18}\alpha_{17}\gamma_4$	-	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-	-	-
$\alpha_{18}\alpha_{17}\gamma_7$	γ_1	-	-	-	-	-	-	-	-	-	-	γ_6	α_{14}	-	-	-	-
$\beta_6\alpha_6\alpha_5$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-	-	-
$\beta_6\alpha_{12}\alpha_{11}$	-	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-	-
$\beta_6\alpha_{18}\alpha_{17}$	-	-	γ_2	α_4	β_3	-	-	-	-	β_5	-	-	-	γ_7	α_{16}	β_8	-
$\gamma_3\alpha_4\alpha_3$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_5\alpha_{10}\alpha_9$	-	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-	-	-
$\gamma_8\alpha_{16}\alpha_{15}$	γ_1	-	β_2	-	-	-	-	β_4	-	-	-	γ_6	α_{14}	β_7	-	-	-
$\gamma_8\gamma_2\alpha_2$	α_1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_8\gamma_4\alpha_8$	α_7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_8\gamma_7\alpha_{14}$	-	-	-	-	-	-	-	-	-	-	-	α_{13}	-	-	-	-	-
$\gamma_8\gamma_7\gamma_6$	-	-	-	-	γ_3	α_6	-	-	-	γ_5	α_{12}	-	-	-	-	γ_8	α_{18}
$(\beta_6\gamma_3 - \gamma_8\beta_3)\alpha_4$	-	-	α_3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$(\beta_6\gamma_5 - \gamma_8\beta_5)\alpha_{10}$	-	-	-	-	-	-	-	α_9	-	-	-	-	-	-	-	-	-
$(\beta_6\gamma_8 - \gamma_8\beta_8)\alpha_{16}$	-	-	-	-	-	-	-	-	-	-	-	-	-	α_{15}	-	-	-

The next table consists of the elements g_i^4 with $s(g_i^4) = e_{13}$. Where there is no column headed e_j, this means there is no element of g^4 starting at e_{13} and ending at e_j.

	e_1	e_2	e_3	e_4	e_5	e_7	e_8	e_9	e_{10}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}
$\alpha_{13}\alpha_6\alpha_5$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-
$\alpha_{13}\alpha_{12}\alpha_{11}$	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-
$\alpha_{13}\alpha_{18}\alpha_{17}$	-	γ_2	-	-	β_3	-	-	γ_4	β_5	-	-	γ_7	α_{16}	β_8
$\alpha_{13}\beta_6\alpha_6$	-	-	-	-	α_5	-	-	-	-	-	-	-	-	-
$\alpha_{13}\beta_6\alpha_{12}$	-	-	-	-	-	-	-	-	α_{11}	-	-	-	-	-
$\alpha_{13}\beta_6\alpha_{18}$	-	-	-	-	-	-	-	-	-	-	-	-	-	α_{17}
$\alpha_{13}\gamma_3\alpha_4$	-	-	α_3	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{13}\gamma_5\alpha_{10}$	-	-	-	-	-	-	α_9	-	-	-	-	-	-	-
$\alpha_{13}\gamma_8\alpha_{16}$	-	-	-	-	-	-	-	-	-	-	-	α_{15}	-	-
$\alpha_{13}\gamma_8\gamma_2$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{13}\gamma_8\gamma_4$	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-
$\alpha_{13}\gamma_8\gamma_7$	γ_1	-	-	-	-	-	-	-	-	γ_6	α_{14}	-	-	-

The next table consists of the elements g_i^4 with $s(g_i^4) = e_{14}$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}
$\alpha_{14}\alpha_{13}\alpha_6$	-	-	-	-	α_5	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{14}\alpha_{13}\alpha_{12}$	-	-	-	-	-	-	-	-	-	α_{11}		-	-	-	-	-	-
$\alpha_{14}\alpha_{13}\alpha_{18}$	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		α_{18}
$\alpha_{14}\alpha_{13}\beta_6$	-	-	-	-	-	α_6	-	-	-	-	α_{12}	-	-	-	-	-	α_{18}
$\alpha_{14}\alpha_{13}\gamma_3$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{14}\alpha_{13}\gamma_5$	-	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-	-
$\alpha_{14}\alpha_{13}\gamma_8$	-	-	γ_2	-	-	-	-	γ_4	-	-	-	-	-	γ_7	α_{16}	-	-

$\beta_7\alpha_{14}\alpha_{13}$	β_1	-	-	-	γ_3	α_6	-	-	-	γ_5	α_{12}	β_6	-	-	-	γ_8	α_{18}
$(\beta_7\gamma_6-\gamma_6\beta_6)\alpha_6$	-	-	-	-	α_5	-	-	-	-	-	-	-	-	-	-	-	-
$(\beta_7\gamma_6-\gamma_6\beta_6)\alpha_{12}$	-	-	-	-	-	-	-	-	-	α_{11}	-	-	-	-	-	-	-
$(\beta_7\gamma_6-\gamma_6\beta_6)\alpha_{18}$	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	α_{17}	-
$\gamma_6\alpha_6\alpha_5$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_6\alpha_{12}\alpha_{11}$	-	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-	-
$\gamma_6\alpha_{18}\alpha_{17}$	-	-	γ_2	-	β_3	-	-	γ_4	-	β_5	-	-	-	γ_7	α_{16}	β_8	-
$\gamma_6\gamma_3\alpha_4$	-	-	α_3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_6\gamma_5\alpha_{10}$	-	-	-	-	-	-	-	α_9	-	-	-	-	-	-	-	-	-
$\gamma_6\gamma_8\alpha_{16}$	-	-	-	-	-	-	-	-	-	-	-	-	-	α_{15}	-	-	-
$\gamma_6\gamma_8\gamma_2$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_6\gamma_8\gamma_4$	-	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-	-	-
$\gamma_6\gamma_8\gamma_7$	γ_1	-	-	-	-	-	-	-	-	-	-	γ_6	α_{14}	-	-	-	-

The next table consists of the elements g_i^4 with $s(g_i^4) = e_{15}$.

	e_1	e_3	e_4	e_5	e_6	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{14}	e_{15}	e_{16}	e_{17}
$\alpha_{15}\alpha_{14}\alpha_{13}$	β_1	-	-	γ_3	α_6	-	-	γ_5	α_{12}	β_6	-	-	γ_8	α_{18}
$\alpha_{15}\beta_2\alpha_2$	α_1	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{15}\beta_4\alpha_8$	α_7	-	-	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{15}\beta_7\alpha_{14}$	-	-	-	-	-	-	-	-	-	α_{13}	-	-	-	-
$\alpha_{15}\gamma_6\alpha_6$	-	-	-	α_5	-	-	-	-	-	-	-	-	-	-
$\alpha_{15}\gamma_6\alpha_{12}$	-	-	-	-	-	-	-	α_{11}	-	-	-	-	-	-
$\alpha_{15}\gamma_6\alpha_{18}$	-	-	-	-	-	-	-	-	-	-	-	-	α_{17}	-
$\alpha_{15}\gamma_6\gamma_3$	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{15}\gamma_6\gamma_5$	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-
$\alpha_{15}\gamma_6\gamma_8$	-	γ_2	-	-	-	γ_4	-	-	-	-	γ_7	α_{16}	-	-

The next table consists of the elements g_i^4 with $s(g_i^4) = e_{16}$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}
$\alpha_{16}\alpha_{15}\alpha_{14}$	-	-	-	-	-	-	-	-	-	-	-	α_{13}	-	-	-	-	-
$\beta_3\alpha_3\alpha_4$	-	α_2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\beta_5\alpha_{10}\alpha_9$	-	-	-	-	-	-	α_8	-	-	-	-	-	-	-	-	-	-
$\beta_8\alpha_{16}\alpha_{15}$	γ_1	-	β_2	-	-	-	-	β_4	-	-	-	γ_6	α_{14}	β_7	-	-	-
$(\beta_8\gamma_2 - \gamma_7\beta_2)\alpha_2$	α_1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$(\beta_8\gamma_4 - \gamma_7\beta_4)\alpha_8$	α_7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$(\beta_8\gamma_7 - \gamma_7\beta_7)\alpha_{14}$	-	-	-	-	-	-	-	-	-	-	-	α_{13}	-	-	-	-	-
$\gamma_7\alpha_{14}\alpha_{13}$	β_1	-	-	-	γ_3	α_6	-	-	-	γ_5	α_{12}	β_6	-	-	-	γ_8	α_{18}
$\gamma_7\gamma_6\alpha_6$	-	-	-	-	α_5	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_7\gamma_6\alpha_{12}$	-	-	-	-	-	-	-	-	-	α_{11}	-	-	-	-	-	-	-
$\gamma_7\gamma_6\alpha_{18}$	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	α_{17}	-
$\gamma_7\gamma_6\gamma_3$	-	-	-	α_4	-	-	-	-	-	-	-	-	-	-	-	-	-
$\gamma_7\gamma_6\gamma_5$	-	-	-	-	-	-	-	-	α_{10}	-	-	-	-	-	-	-	-
$\gamma_7\gamma_6\gamma_8$	-	-	γ_2	-	-	-	-	γ_4	-	-	-	-	-	γ_7	α_{16}	-	-

The next table consists of the elements g_i^4 with $s(g_i^4) = e_{17}$.

	e_1	e_3	e_5	e_6	e_8	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{16}	e_{17}
$\alpha_{17}\alpha_{16}\alpha_{15}$	γ_1	β_2	-	-	β_4	-	-	γ_6	α_{14}	β_7	-	-
$\alpha_{17}\beta_3\alpha_4$	-	α_3	-	-	-	-	-	-	-	-	-	-
$\alpha_{17}\beta_5\alpha_{10}$	-	-	-	-	α_9	-	-	-	-	-	-	-
$\alpha_{17}\beta_8\alpha_{16}$	-	-	-	-	-	-	-	-	-	α_{15}	-	-

$\alpha_{17}\gamma_2\alpha_2$	α_1	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{17}\gamma_4\alpha_8$	α_7	-	-	-	-	-	-	-	-	-	-	-
$\alpha_{17}\gamma_7\alpha_{14}$	-	-	-	-	-	-	-	α_{13}	-	-	-	-
$\alpha_{17}\gamma_7\gamma_6$	-	-	γ_3	α_6	-	γ_5	α_{12}	-	-	-	γ_8	α_{18}

We have now given the sets g^n for $n = 0, 1, 2, 3$ and 4. From this it is clear that the minimal projective resolution so far looks to be linear. However, since the sets are growing very large, it is not possible to determine the whole resolution of $A/\mathfrak{r}$, and therefore we cannot show it is a linear resolution. This means that we need to find another way to show that our new algebra is Koszul. Let us look instead at Gröbner bases.

7.2. **Gröbner Bases.** We give an introduction to Gröbner bases following [8] and [9].

In this section we assume that Γ is a finite quiver, and let $\mathcal{B}$ be the basis of the path algebra $K\Gamma$ which consists of all paths in $K\Gamma$. We remark that $\mathcal{B}$ is a multiplicative basis of $K\Gamma$, that is, if $p, q \in \mathcal{B}$ then $p \cdot q \in \mathcal{B}$ or $p \cdot q = 0$.

Definition 7.5. [9] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis of all paths. We say $>$ is a well-order on $\mathcal{B}$ if $>$ is a total order on $\mathcal{B}$ and every non-empty subset of $\mathcal{B}$ has a minimal element.

Definition 7.6. [9] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis of all paths. An admissible ordering on $\mathcal{B}$ is a well-order $>$ on $\mathcal{B}$ that satisfies the following properties:

(1) if $p, q, r \in \mathcal{B}$ and $p > q$ then $pr > qr$ if both are not zero and $rp > rq$ if both are not zero;
(2) if $p, q, r \in \mathcal{B}$ and $p = qr$ then $p \geq q$ and $p \geq r$.

Definition 7.7. [9] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis consisting of all paths. The left length lexicographic order is an admissible order defined as follows. Arbitrarily order the vertices and arrows such that every vertex is less than every arrow. For paths of length greater than 1, if $p = \alpha_1\alpha_2 \cdots \alpha_n$ and $q = \beta_1\beta_2 \cdots \beta_m$ where the α_i and β_i are arrows and $p, q \in \mathcal{B}$, then $p > q$ if $n > m$ or, if $n = m$, then there is some $1 \leq i \leq n$ with $\alpha_j = \beta_j$ for $j < i$ and $\alpha_i > \beta_i$.

Given an admissible order $>$, we are now concerned with finding a Gröbner basis of I, where I is an ideal in $K\Gamma$.

Definition 7.8. [11] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis of all paths, with admissible order $>$.

(1) Let x be an element of $K\Gamma$, so x is a linear combination of paths p_i. Then $\mathrm{Tip}(x)$ is the largest p_i, in the ordering $>$, occurring in x.
(2) If I is an ideal in $K\Gamma$ then $\mathrm{Tip}(I)$ is the set of paths that occur as Tips of non-zero elements of I.
(3) We let $\mathrm{Nontip}(I)$ be the set of finite paths in $K\Gamma$ that are not in $\mathrm{Tip}(I)$.
(4) Let $\mathrm{CTip}(x)$ denote the coefficient of $\mathrm{Tip}(x)$.
(5) An element $x \in I$ is sharp if $x = p + \sum_i \alpha_i q_i$ where $\mathrm{Tip}(x) = p, \alpha_i \in K$ and $q_i \in \mathrm{Nontip}(I)$ for all i.

Definition 7.9. [9] Let $K\Gamma$ be a path algebra with basis $\mathcal{B}$ consisting of all paths. Every element γ of $K\Gamma$ can be written as a linear combination of elements in $\mathcal{B}$. Those elements of $\mathcal{B}$ which occur in γ with non-zero coefficients are called the support of γ, denoted $\mathrm{Supp}(\gamma)$.

Definition 7.10. [8] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis consisting of all paths. Let $a, b \in \mathcal{B}$. We say a divides b if there exist $u, v \in \mathcal{B}$ such that $b = uav$.

Definition 7.11. [8] Let $K\Gamma$ be a path algebra and let I be an ideal of $K\Gamma$. A Gröbner basis for I is a non-empty subset $G \subseteq I$ such that the tip of each nonzero element of I is divisible by the tip of some element in G.

Definition 7.12. [8] Let $K\Gamma$ be a path algebra and let $\mathcal{B}$ be the basis consisting of all paths. Let a be a non-zero element of $K\Gamma$. A simple (algebra) reduction for

a is determined by a 4-tuple (λ, u, f, v) where $\lambda \in K^*, f \in K\Gamma \backslash \{0\}$ and $u, v \in \mathcal{B}$, satisfying

(1) $u \operatorname{Tip}(f) v \in \operatorname{Supp}(a)$ and

(2) $u \operatorname{Tip}(f) v \notin \operatorname{Supp}(a - \lambda u f v)$.

We say that a reduces over f to $a - \lambda u f v$. We say that a reduces to a' over a set $X = \{f_1, \dots, f_n\}$ and write $a \Rightarrow_X a'$ if there is a finite sequence so that a reduces to a_1 over f_1, a_1 reduces to a_2 over f_2, and so on, with a_{n-1} reducing to a' over f_n.

Definition 7.13. [9] Let $K\Gamma$ be a path algebra, let $\mathcal{B}$ be the basis consisting of all paths of $K\Gamma$ and let $>$ be an admissible order on $\mathcal{B}$. Let $\xi_1, \xi_2 \in K\Gamma$ and suppose there are elements $p, q \in \mathcal{B}$ such that

(1) $\operatorname{Tip}(\xi_1) p = q \operatorname{Tip}(\xi_2)$,

(2) $\operatorname{Tip}(\xi_1)$ does not divide q and $\operatorname{Tip}(\xi_2)$ does not divide p.

Then the overlap difference of ξ_1 and ξ_2 by p, q is

$$o(\xi_1, \xi_2, p, q) = (1/\operatorname{CTip}(\xi_1))\ \xi_1 p - (1/\operatorname{CTip}(\xi_2))\ q \xi_2.$$

The next theorem uses the concept of uniform elements for which we refer the reader back to Definition 3.6.

Theorem 7.14. [8, Theorem 13] *Let $K\Gamma$ be a path algebra and let $\mathcal{H} = \{h_j : j \in \mathcal{J}\}$ be a subset of non-zero uniform elements in $K\Gamma$, which generates the ideal I. Assume that the following conditions hold;*

i) $\operatorname{CTip}(h_j)$ *is* 1, *for each $j \in \mathcal{J}$,*

ii) h_i does not reduce over h_j for $i \neq j$, and

iii) every overlap difference for two (not necessarily distinct) members of $\mathcal{H}$ always reduces to zero over $\mathcal{H}$.

Then $\mathcal{H}$ is a reduced Gröbner basis of I.

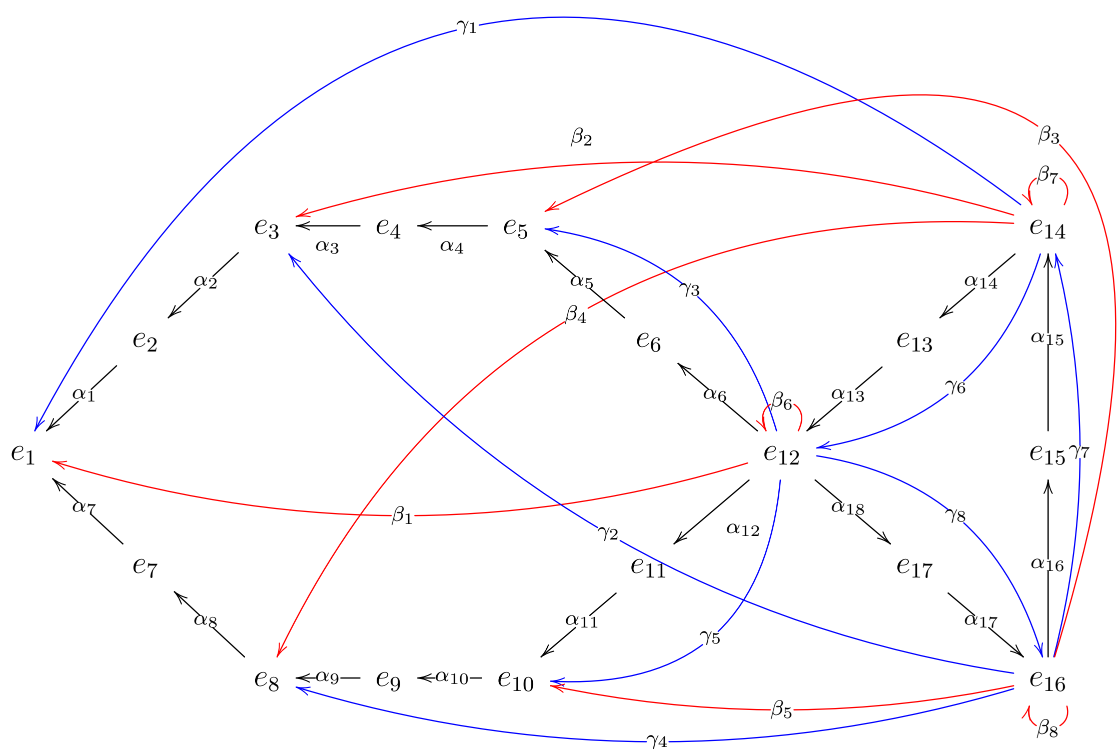
We note that in [8] a reduced Gröbner basis of I is called MINSHARP(I).

Theorem 7.15. [11, Theorem 3] *Let $K\Gamma$ be a path algebra and I a quadratic ideal in the path algebra $K\Gamma$. Fix an admissible ordering and let $A = K\Gamma/I$. Then:*

(1) The reduced Gröbner basis of I consists of homogeneous elements.

(2) If the reduced Gröbner basis of I consists of quadratic elements then A is a Koszul algebra.

7.3. **A Return to Example 1.** Now we are in a position to use Gröbner bases to continue looking at the example introduced in Section 7.1. We want to show that the set $\mathcal{H}$ given below is indeed a reduced Gröbner basis for $\mathcal{I}$, where $\mathcal{I}$ is the ideal for the Ext algebra given as $A = K\Gamma/\mathcal{I}$. Recall that Γ is the quiver given by



and let K be a field. Let $\mathcal{B}$ be the basis of $K\Gamma$ which consists of all paths.

Let the vertices be ordered $e_1 > e_2 > \cdots > e_{17}$ and let the arrows be ordered $\alpha_1 > \alpha_2 > \cdots > \alpha_{18} > \beta_1 > \beta_2 > \cdots > \beta_8 > \gamma_1 > \gamma_2 > \cdots > \gamma_8$. Let the admissible order on $\mathcal{B}$ be the left length lexicographic order as given in Definition 7.7.

Let $\mathcal{H} \subseteq \mathcal{I}$ be the minimal generating set for $\mathcal{I}$ as given in Section 7.1, that is, $\mathcal{H}$ is the set consisting of

- $\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_8\alpha_7, \alpha_9\alpha_8, \alpha_{10}\alpha_9, \alpha_{11}\alpha_{10}, \alpha_{12}\alpha_{11}, \alpha_{13}\alpha_6, \alpha_{13}\alpha_{12}, \alpha_{13}\alpha_{18}, \alpha_{13}\beta_1, \alpha_{13}\beta_6, \alpha_{13}\gamma_3, \alpha_{13}\gamma_5, \alpha_{13}\gamma_8, \alpha_{14}\alpha_{13}, \alpha_{15}\alpha_{14}, \alpha_{15}\beta_2, \alpha_{15}\beta_4, \alpha_{15}\beta_7, \alpha_{15}\gamma_1, \alpha_{15}\gamma_6, \alpha_{16}\alpha_{15}, \alpha_{17}\alpha_{16}, \alpha_{17}\beta_3, \alpha_{17}\beta_5, \alpha_{17}\beta_8, \alpha_{17}\gamma_2, \alpha_{17}\gamma_4, \alpha_{17}\gamma_7, \alpha_{18}\alpha_{17}$,
- $\beta_2\alpha_2, \beta_3\alpha_4, \beta_4\alpha_8, \beta_5\alpha_{10}, \beta_6\alpha_6, \beta_6\alpha_{12}, \beta_6\alpha_{18}, \beta_6\gamma_3-\gamma_8\beta_3, \beta_6\gamma_5-\gamma_8\beta_5, \beta_6\gamma_8-\gamma_8\beta_8, \beta_7\alpha_{14}, \beta_7\gamma_1 - \gamma_6\beta_1, \beta_7\gamma_6 - \gamma_6\beta_6, \beta_8\alpha_{16}, \beta_8\gamma_2 - \gamma_7\beta_2, \beta_8\gamma_4 - \gamma_7\beta_4, \beta_8\gamma_7 - \gamma_7\beta_7$,
- $\gamma_2\alpha_2, \gamma_3\alpha_4, \gamma_4\alpha_8, \gamma_5\alpha_{10}, \gamma_6\alpha_6, \gamma_6\alpha_{12}, \gamma_6\alpha_{18}, \gamma_6\gamma_3, \gamma_6\gamma_5, \gamma_6\gamma_8, \gamma_7\alpha_{14}, \gamma_7\gamma_1, \gamma_7\gamma_6, \gamma_8\alpha_{16}, \gamma_8\gamma_2, \gamma_8\gamma_4, \gamma_8\gamma_7$

and let us label the elements in the order presented here as h_i for $i \in \{1, \dots, 68\}$.

Proposition 7.16. *Let $K\Gamma$ be the path algebra, $\mathcal{I}$ be the ideal of $K\Gamma$ and $\mathcal{H}$ be the subset of $\mathcal{I}$ as given above. Then $\mathcal{H}$ is a reduced Gröbner basis of $\mathcal{I}$.*

Proof. We will use Theorem 7.14 to show that $\mathcal{H}$ is a reduced Gröbner basis of $\mathcal{I}$. It is clear that $\mathcal{H}$ is a generating set for $\mathcal{I}$. We examine each of the conditions in turn.

Condition *i)* $\mathrm{CTip}(h_i)$ is 1 for all $i \in \{1, \dots, 68\}$. This is clear from looking at the set $\mathcal{H}$.

Condition *ii)* h_i does not reduce over h_j for $i \neq j$.
We now show that this condition is satisfied for all $i \neq j$. Assume h_i reduces over h_j. Then there are elements $u, v \in \mathcal{B}$ such that $u\,\mathrm{Tip}(h_j)v \in \mathrm{Supp}(h_i)$. Looking at the element $\mathrm{Tip}(h_j)$ and the elements that are in $\mathrm{Supp}(h_i)$, it is clear that they are all of length 2 for all i, j, hence $u, v \in \Gamma_0$. In particular, $u = s(\mathrm{Tip}(h_j))$ and $v = t(\mathrm{Tip}(h_j))$. Therefore if h_i reduces over h_j then $\mathrm{Tip}(h_j) \in \mathrm{Supp}(h_i)$. By inspection of $\mathcal{H}$ we can see this is not the case. So h_i does not reduce over h_j for $i \neq j$.

Condition *iii)* every overlap difference for two (not necessarily distinct) members of $\mathcal{H}$ always reduces to zero over $\mathcal{H}$.
We now show that the elements of $\mathcal{H}$ satisfy this condition. Let us consider the set

$\mathcal{H}$ as two distinct subsets, let $\mathcal{H}_1$ be the set of monomial elements of $\mathcal{H}$ and let $\mathcal{H}_2$ be the set of non-monomial elements of $\mathcal{H}$. Clearly, $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$.

Now, consider two arbitrary elements of $\mathcal{H}_1$. It follows that in this set, $h_i = \mathrm{Tip}(h_i)$. Assume we have elements $p, q \in \mathcal{B}$ such that $\mathrm{Tip}(h_i)p = q\,\mathrm{Tip}(h_j)$, with $\mathrm{Tip}(h_i)$ does not divide q and $\mathrm{Tip}(h_j)$ does not divide p. Then the overlap difference is defined as $o(h_i, h_j, p, q) = (\frac{1}{\mathrm{CTip}(h_i)})h_i p - (\frac{1}{\mathrm{CTip}(h_j)})q h_j$. Now, by condition i), we know $\mathrm{CTip}(h_i) = 1$ and $\mathrm{CTip}(h_j) = 1$. So $o(h_i, h_j, p, q) = h_i p - q h_j$ and since $\mathrm{Tip}(h_i)p = h_i p$, $q\,\mathrm{Tip}(h_j) = q h_j$, we have $o(h_i, h_j, p, q) = 0$.

Now we are left with two possibilities, $h_i, h_j \in \mathcal{H}_2$ and $h_i \in \mathcal{H}_1, h_j \in \mathcal{H}_2$. Before looking at these explicitly we can look at a more general property of the overlap difference.

An overlap difference of two elements $h_i, h_j \in \mathcal{H}$ requires $\mathrm{Tip}(h_i)p = q\,\mathrm{Tip}(h_j)$. Note that $\mathrm{Tip}(h)$ has length 2 for all $h \in \mathcal{H}$. Now, $l(p) = l(q) < 2$, since we also require $\mathrm{Tip}(h_i)$ does not divide q and $\mathrm{Tip}(h_j)$ does not divide p. If p and q have length 0, then $\mathrm{Tip}(h_i) = \mathrm{Tip}(h_j)$; by looking at the elements of $\mathcal{H}$ we can see this is true if and only if $i = j$. In this case, the overlap difference will be 0. So if $\mathrm{Tip}(h_i)p = q\,\mathrm{Tip}(h_j)$ then we may assume $l(p) = 1 = l(q)$ and p, q are arrows.

Let us consider two elements $h_i, h_j \in \mathcal{H}_2$. The second arrow of $\mathrm{Tip}(h_i)p$ is γ_k and the second arrow of $q\,\mathrm{Tip}(h_j)$ is β_l, for some k, l. This cannot happen, so there are no overlap differences except when $i = j$, which will be 0.

Finally, we look at the overlap difference of one monomial element and one non-monomial element. Let us first consider $o(h, h', p, q)$ where $h \in \mathcal{H}_1$ and $h' \in \mathcal{H}_2$. We will consider $o(h', h, p, q)$ for $h \in \mathcal{H}_1$ and $h' \in \mathcal{H}_2$ afterwards. There will be some non-zero overlap differences here and we can work through the possibilities to show that every overlap difference reduces to zero over $\mathcal{H}$.

The elements of $\mathcal{H}_2$ are: $h_{42} = \beta_6\gamma_3 - \gamma_8\beta_3, h_{43} = \beta_6\gamma_5 - \gamma_8\beta_5, h_{44} = \beta_6\gamma_8 - \gamma_8\beta_8, h_{46} = \beta_7\gamma_1 - \gamma_6\beta_1, h_{47} = \beta_7\gamma_6 - \gamma_6\beta_6, h_{49} = \beta_8\gamma_2 - \gamma_7\beta_2, h_{50} = \beta_8\gamma_4 - \gamma_7\beta_4, h_{51} = \beta_8\gamma_7 - \gamma_7\beta_7$.

We start with $h' = h_{42} = \beta_6\gamma_3 - \gamma_8\beta_3$. From above, we know that the second arrow of the term $q\,\mathrm{Tip}(h_{42})$ is β_6. So we need to find all overlap differences with h', and it is easy to see that there is only one possible overlap. Let $h = h_{15} = \alpha_{13}\beta_6$. Then $\mathrm{Tip}(h_{42}) = \beta_6\gamma_3$ and $\mathrm{Tip}(h_{15}) = \alpha_{13}\beta_6$. Let $p = \gamma_3$ and $q = \alpha_{13}$. The overlap difference is given by

$$\begin{aligned} o(h, h', p, q) &= o(h_{15}, h_{42}, \gamma_3, \alpha_{13}) \\ &= \alpha_{13}\beta_6\gamma_3 - \alpha_{13}\beta_6\gamma_3 + \alpha_{13}\gamma_8\beta_3 \\ &= \alpha_{13}\gamma_8\beta_3. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = 1, u = e_{13}, f = h_{18} = \alpha_{13}\gamma_8$ and $v = \beta_3$. Then $o(h_{15}, h_{42}, \gamma_3, \alpha_{13}) - \lambda ufv = 0$, so $o(h_{15}, h_{42}, \gamma_3, \alpha_{13}) \Rightarrow_{\mathcal{H}} 0$. Hence this overlap difference reduces to zero over $\mathcal{H}$.

Let $h' = h_{43} = \beta_6\gamma_5 - \gamma_8\beta_5$ and $h = h_{15} = \alpha_{13}\beta_6$. Then $\mathrm{Tip}(h_{43}) = \beta_6\gamma_5$ and $\mathrm{Tip}(h_{15}) = \alpha_{13}\beta_6$. Let $p = \gamma_5$ and $q = \alpha_{13}$. The overlap difference is given by

$$\begin{aligned} o(h_{15}, h_{43}, \gamma_5, \alpha_{13}) &= \alpha_{13}\beta_6\gamma_5 - \alpha_{13}\beta_6\gamma_5 + \alpha_{13}\gamma_8\beta_5 \\ &= \alpha_{13}\gamma_8\beta_5. \end{aligned}$$

Let $\lambda = 1, u = e_{13}, f = h_{18} = \alpha_{13}\gamma_8$ and $v = \beta_5$. Then $o(h_{15}, h_{43}, \gamma_5, \alpha_{13}) - \lambda ufv = 0$. Hence this overlap difference reduces to zero over $\mathcal{H}$.

Similarly $o(h_{15}, h_{44}, \gamma_8, \alpha_{13}), o(h_{23}, h_{46}, \gamma_1, \alpha_{15}), o(h_{23}, h_{47}, \gamma_6, \alpha_{15}), o(h_{20}, h_{49}, \gamma_2, \alpha_{17}), o(h_{20}, h_{50}, \gamma_4, \alpha_{17}), o(h_{20}, h_{51}, \gamma_7, \alpha_{17})$ all have an element of $\mathcal{H}_1$ as a subpath and so these overlap differences all reduce to zero over $\mathcal{H}$.

We now consider the overlap difference $o(h', h, p, q)$ where $h' \in \mathcal{H}_2, h \in \mathcal{H}_1$.

The overlap differences $o(h_{42}, h_{53}, \alpha_4, \beta_6), o(h_{43}, h_{55}, \alpha_{10}, \beta_6), o(h_{44}, h_{65}, \alpha_{16}, \beta_6), o(h_{47}, h_{56}, \alpha_6, \beta_7), o(h_{47}, h_{57}, \alpha_{12}, \beta_7), o(h_{47}, h_{58}, \alpha_{18}, \beta_7), o(h_{49}, h_{52}, \alpha_2, \beta_8), o(h_{50}, h_{54}, \alpha_8, \beta_8), o(h_{51}, h_{62}, \alpha_{14}, \beta_8)$ also have an element of $\mathcal{H}_1$ as a subpath and so reduce to zero over $\mathcal{H}$.

Finally, $o(h_{44}, h_{66}, \gamma_2, \beta_6), o(h_{44}, h_{67}, \gamma_4, \beta_6), o(h_{44}, h_{68}, \gamma_7, \beta_6), o(h_{47}, h_{59}, \gamma_3, \beta_7), o(h_{47}, h_{60}, \gamma_5, \beta_7), o(h_{47}, h_{61}, \gamma_8, \beta_7), o(h_{51}, h_{64}, \gamma_6, \beta_8)$ need more than a simple reduction. We now give explicit details for these.

Let us take $h_{66} = \gamma_8\gamma_2$, and let $p = \gamma_2, q = \beta_6$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{44}, h_{66}, \gamma_2, \beta_6) = \beta_6\gamma_8\gamma_2 - \gamma_8\beta_8\gamma_2 - \beta_6\gamma_8\gamma_2 = -\gamma_8\beta_8\gamma_2$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_8, f_1 = h_{49} = \beta_8\gamma_2 - \gamma_7\beta_2$ and $v_1 = e_3$. Then $o(h_{44}, h_{66}, \gamma_2, \beta_6) - \lambda_1 u_1 f_1 v_1 = -\gamma_8\gamma_7\beta_2$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{68} = \gamma_8\gamma_7$ and $v_2 = \beta_2$. Then we have $o(h_{44}, h_{66}, \gamma_2, \beta_6) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{67} = \gamma_8\gamma_4$, and let $p = \gamma_4, q = \beta_6$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{44}, h_{67}, \gamma_4, \beta_6) = \beta_6\gamma_8\gamma_4 - \gamma_8\beta_8\gamma_4 - \beta_6\gamma_8\gamma_4 = -\gamma_8\beta_8\gamma_4$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_8, f_1 = h_{50} = \beta_8\gamma_4 - \gamma_7\beta_4$ and $v_1 = e_8$. Then $o(h_{44}, h_{67}, \gamma_4, \beta_6) - \lambda_1 u_1 f_1 v_1 = -\gamma_8\gamma_7\beta_4$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{68} = \gamma_8\gamma_7$ and $v_2 = \beta_4$. Then we have $o(h_{44}, h_{67}, \gamma_4, \beta_6) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{68} = \gamma_8\gamma_7$, and let $p = \gamma_7, q = \beta_6$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{44}, h_{68}, \gamma_7, \beta_6) = \beta_6\gamma_8\gamma_7 - \gamma_8\beta_8\gamma_7 - \beta_6\gamma_8\gamma_7 = -\gamma_8\beta_8\gamma_7$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_8, f_1 = h_{51} = \beta_8\gamma_7 - \gamma_7\beta_7$ and $v_1 = e_{14}$. Then $o(h_{44}, h_{68}, \gamma_7, \beta_6) - \lambda_1 u_1 f_1 v_1 = -\gamma_8\gamma_7\beta_7$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{68} = \gamma_8\gamma_7$ and $v_2 = \beta_7$. Then we have $o(h_{44}, h_{67}, \gamma_4, \beta_6) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{59} = \gamma_6\gamma_3$, and let $p = \gamma_3, q = \beta_7$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{47}, h_{59}, \gamma_3, \beta_7) = \beta_7\gamma_6\gamma_3 - \gamma_6\beta_6\gamma_3 - \beta_7\gamma_6\gamma_3 = -\gamma_6\beta_6\gamma_3$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_6, f_1 = h_{42} = \beta_6\gamma_3 - \gamma_8\beta_3$ and $v_1 = e_5$. Then $o(h_{47}, h_{59}, \gamma_3, \beta_7) - \lambda_1 u_1 f_1 v_1 = -\gamma_6\gamma_8\beta_3$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{14}, f_2 = h_{61} = \gamma_6\gamma_8$ and $v_2 = \beta_3$. Then we have $o(h_{47}, h_{59}, \gamma_3, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{60} = \gamma_6\gamma_5$, and let $p = \gamma_5, q = \beta_7$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{47}, h_{60}, \gamma_5, \beta_7) = \beta_7\gamma_6\gamma_5 - \gamma_6\beta_6\gamma_5 - \beta_7\gamma_6\gamma_5 = -\gamma_6\beta_6\gamma_5$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_6, f_1 = h_{43} = \beta_6\gamma_5 - \gamma_8\beta_5$ and $v_1 = e_{10}$. Then $o(h_{47}, h_{60}, \gamma_5, \beta_7) - \lambda_1 u_1 f_1 v_1 = -\gamma_6\gamma_8\beta_5$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{14}, f_2 = h_{61} = \gamma_6\gamma_8$ and $v_2 = \beta_5$. Then we have $o(h_{47}, h_{60}, \gamma_5, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{61} = \gamma_6\gamma_8$, and let $p = \gamma_8, q = \beta_7$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{47}, h_{61}, \gamma_8, \beta_7) = \beta_7\gamma_6\gamma_8 - \gamma_6\beta_6\gamma_8 - \beta_7\gamma_6\gamma_8 = -\gamma_6\beta_6\gamma_8$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_6, f_1 = h_{44} = \beta_6\gamma_8 - \gamma_8\beta_8$ and $v_1 = e_{16}$. Then $o(h_{47}, h_{61}, \gamma_8, \beta_7) - \lambda_1 u_1 f_1 v_1 = -\gamma_6\gamma_8\beta_8$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{14}, f_2 = h_{61} = \gamma_6\gamma_8$ and $v_2 = \beta_8$. Then we have $o(h_{47}, h_{61}, \gamma_8, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Let us take $h_{64} = \gamma_7\gamma_6$, and let $p = \gamma_6, q = \beta_8$. The overlap difference is $o(h_i, h_j, p, q) = o(h_{51}, h_{64}, \gamma_6, \beta_8) = \beta_8\gamma_7\gamma_6 - \gamma_7\beta_7\gamma_6 - \beta_8\gamma_7\gamma_6 = -\gamma_7\beta_7\gamma_6$. This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_7, f_1 = h_{47} = \beta_7\gamma_6 - \gamma_6\beta_6$ and $v_1 = e_{12}$. Then $o(h_{51}, h_{64}, \gamma_6, \beta_8) - \lambda_1 u_1 f_1 v_1 = -\gamma_7\gamma_6\beta_6$. This can be further reduced, let $\lambda_2 = -1, u_2 = e_{16}, f_2 = h_{64} = \gamma_7\gamma_6$ and $v_2 = \beta_6$. Then we have $o(h_{51}, h_{64}, \gamma_6, \beta_8) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Therefore this overlap difference can be reduced to zero.

Thus we have now shown that, for all elements h_i, h_j in the set $\mathcal{H}$, the overlap difference of two elements reduces to zero. Therefore the set $\mathcal{H}$ satisfies the hypothesis of Theorem 7.14, and $\mathcal{H}$ is a reduced Gröbner basis of the ideal $\mathcal{I}$. $\square$

Remark. We have shown above that the overlap difference of two monomial relations is always zero.

Now we have a reduced Gröbner basis of $\mathcal{I}$, we can use Theorem 7.15 to give the following result.

Theorem 7.17. *Let $A = K\Gamma/\mathcal{I}$ be the algebra given in Section* 7.1*. Then A is a Koszul algebra.*

Proof. It is clear by looking at $\mathcal{H}$ that it consists of quadratic elements, and the proof immediately follows from Theorem 7.15, part 2). $\square$

We recall from Proposition 7.4 that $A \cong \hat{E}(\Lambda)$ where Λ is the $(6,2)$-stacked algebra of Example 7.3. This gives the following result.

Theorem 7.18. *Let Λ be the $(6,2)$-stacked algebra of Example 7.3. Let $A = \hat{E}(\Lambda)$ be the Ext algebra of Λ with the hat-degree grading. Then A is a Koszul algebra.*

7.4. **Example 2: A $(6,2)$-stacked monomial algebra.** We now give an example of an algebra with the same underlying quiver as Example 7.3. The ideal I is now generated by monomial relations, this produces a monomial algebra. For monomial algebras, the projective resolution of $\Lambda/\mathfrak{r}$ and the Ext algebra were studied by Green and Zacharia in [21]. Here we give a single example.

Example 7.19. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
 & & 3 \arrow[r, "\alpha_3"] & 4 \arrow[r, "\alpha_4"] & 5 \arrow[dr, "\alpha_5"] & & & & 14 \arrow[dd, "\alpha_{15}"] \\
 & 2 \arrow[ur, "\alpha_2"] & & & & 6 \arrow[dr, "\alpha_6"] & & 13 \arrow[ur, "\alpha_{14}"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_7"'] & & & & & & 12 \arrow[ur, "\alpha_{13}"] & & 15 \arrow[dd, "\alpha_{16}"] \\
 & 7 \arrow[dr, "\alpha_8"'] & & & & 11 \arrow[ur, "\alpha_{12}"'] & & 17 \arrow[ul, "\alpha_{18}"'] & \\
 & & 8 \arrow[r, "\alpha_9"'] & 9 \arrow[r, "\alpha_{10}"'] & 10 \arrow[ur, "\alpha_{11}"'] & & & & 16 \arrow[ul, "\alpha_{17}"']
\end{tikzcd}$$

and let $\Lambda = K\mathcal{Q}/I$ where I is the admissible ideal given by

$I = \langle \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}, \alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12},$

$\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}, \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}, \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14},$
$\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\rangle$.

A minimal projective resolution for $\Lambda/\mathfrak{r}$ as a right Λ-module is given by

$$\cdots \to P^n \to \cdots \to P^2 \to P^1 \to P^0 \to \Lambda/\mathfrak{r} \to 0$$

where

$\bullet P^0 = \bigoplus_{i=1}^{17} e_i\Lambda$ and d^0 is the canonical surjection , $d^0 : e_i\lambda \mapsto e_i\lambda + e_i\mathfrak{r}$.

$\bullet P^1 = \bigoplus_{i=1}^{18} t(\alpha_i)\Lambda$ and $d^1 : t(\alpha_i)\lambda \mapsto \alpha_i\lambda$.

$\bullet P^2 = \bigoplus_{i=1}^{9} t(g_i^2)\Lambda = e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda$ and

$d^2(e_{12}\lambda_1, e_{14}\lambda_2, e_{16}\lambda_3, e_{12}\lambda_4, e_{14}\lambda_5, e_{16}\lambda_6, e_{12}\lambda_7, e_{14}\lambda_8, e_{16}\lambda_9)$
$= (\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\lambda_1, , \alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}\lambda_2, \alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_3, \alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\lambda_4$
$\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\lambda_5, \alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_6, \alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_7, \alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_8,$
$\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_9)$.

For $n \geq 3$,

• If n is odd, $P^n = e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda$ and $d^n(e_{14}\lambda_1, e_{16}\lambda_2, e_{12}\lambda_3, e_{14}\lambda_4, e_{16}\lambda_5, e_{12}\lambda_6, e_{14}\lambda_7, e_{16}\lambda_8, e_{12}\lambda_9) = (\alpha_{13}\alpha_{14}\lambda_1, \alpha_{15}\alpha_{16}\lambda_2,$
$\alpha_{17}\alpha_{18}\lambda_3, \alpha_{13}\alpha_{14}\lambda_4, \alpha_{15}\alpha_{16}\lambda_5, \alpha_{17}\alpha_{18}\lambda_6, \alpha_{13}\alpha_{14}\lambda_7, \alpha_{15}\alpha_{16}\lambda_8, \alpha_{17}\alpha_{18}\lambda_9)$.

• If n is even, $P^n = e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda \oplus e_{12}\Lambda \oplus e_{14}\Lambda \oplus e_{16}\Lambda$ and $d^n(e_{12}\lambda_1, e_{14}\lambda_2, e_{16}\lambda_3, e_{12}\lambda_4, e_{14}\lambda_5, e_{16}\lambda_6, e_{12}\lambda_7, e_{14}\lambda_8, e_{16}\lambda_9)$

$$= (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_1, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_2, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_3, \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_4, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_5,$$
$$\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_6, \alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\lambda_7, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\lambda_8, \alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\lambda_9).$$

The sets g^n are given as follows;

- $g^0 = \{e_1, e_2, \ldots, e_{17}\}$.
- $g^1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{18}\}$.

- $g^2 = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14}, \alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12},$
  $\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}, \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}, \alpha_{13}\alpha_{14}\alpha_{15}\alpha 16\alpha_{17}\alpha_{18},$
  $\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}, \alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\}$
- For $n \geq 3$ if $n = 2r+1, r \in \mathbb{N}$, the set $\{g^n\}$ is given by;
  $\{(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6)(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1}\alpha_{13}\alpha_{14},$
  $(\alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1}\alpha_{15}\alpha_{16},$
  $(\alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1}\alpha_{17}\alpha_{18},$
  $(\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1}\alpha_{13}\alpha_{14},$
  $(\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1}\alpha_{15}\alpha_{16},$
  $(\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1}\alpha_{17}\alpha_{18},$
  $(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r}\alpha_{13}\alpha_{14}, (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r}\alpha_{15}\alpha_{16},$
  $(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r}\alpha_{17}\alpha_{18}\}.$
- For $n \geq 4$ if $n = 2r, r \in \mathbb{N}$, the set $\{g^n\}$ is given by;

$\{(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6)(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1}, (\alpha_3\alpha_4\alpha_5\alpha_6\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1},$
$(\alpha_5\alpha_6\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1}, (\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r-1}$
$(\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14})(\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r-1}, (\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})(\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r-1},$
$(\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18})^{r}, (\alpha_{15}\alpha_{16}\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14})^{r}, (\alpha_{17}\alpha_{18}\alpha_{13}\alpha_{14}\alpha_{15}\alpha_{16})^{r}\}.$

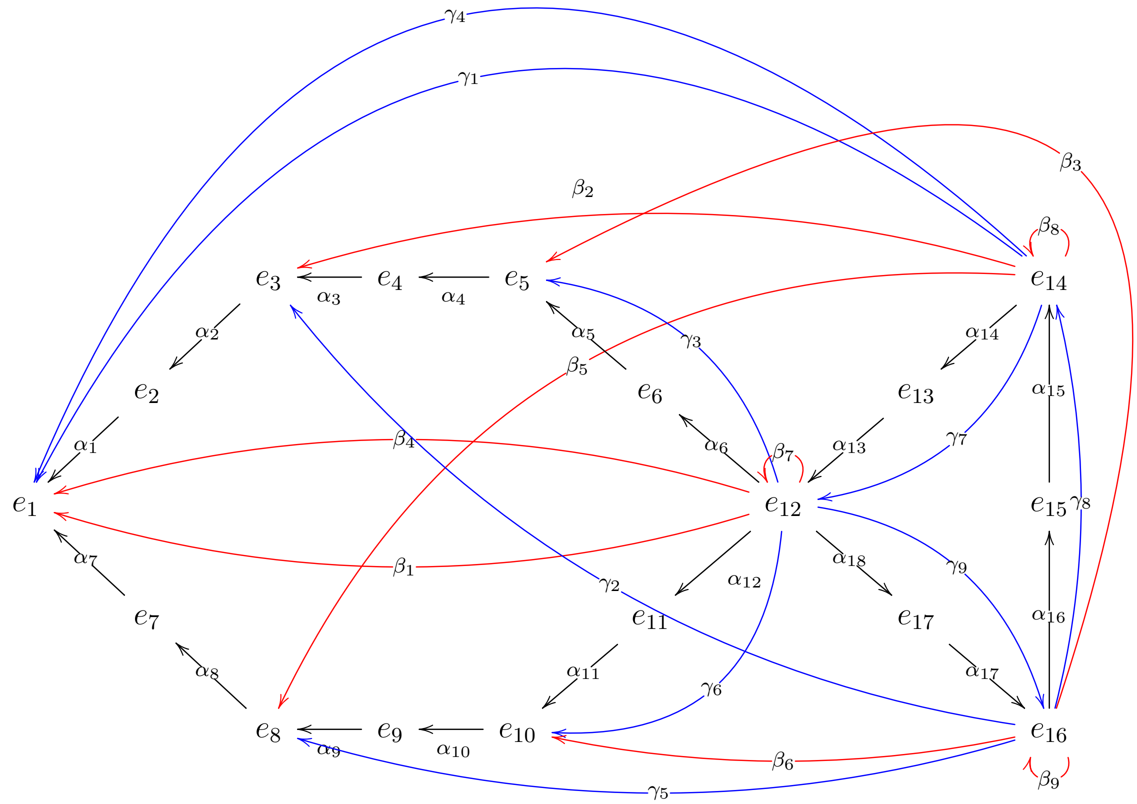
Looking at the length of the elements g^n it is clear that this is a $(6,2)$-stacked monomial algebra. From [18], we know that since Λ is a $(6,2)$-stacked algebra then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3. We have

The sets f^n are given by

- $f^0 = \{f_i^0, \text{ for } i = 1, \ldots, 17\}, f_i^0 : e_i\lambda \mapsto e_i\lambda + \mathfrak{r}, \text{ else } \mapsto 0.$
- $f^1 = \{f_i^1, \text{ for } i = 1, \ldots, 18\}, f_i^1 : t(\alpha_i)\lambda \mapsto t(\alpha_i)\lambda + \mathfrak{r}, \text{ else } \mapsto 0.$
- $f^2 = \{f_i^2, \text{ for } i = 1, \ldots, 9\}, f_i^2 : t(g_i^2)\lambda \mapsto t(g_i^2)\lambda + \mathfrak{r}, \text{ else } \mapsto 0.$
- $f^3 = \{f_i^3, \text{ for } i = 1, \ldots, 9\}, f_i^3 : t(g_i^3)\lambda \mapsto t(g_i^3)\lambda + \mathfrak{r}, \text{ else } \mapsto 0.$

These are the generators of $E(\Lambda)$. For $n \geq 4$ the products are given in [18].

The Ext algebra, $E(\Lambda)$, can be represented by quiver and relations, as described in [21]. Let Γ be the quiver given by



and let $A = K\Gamma/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by;

- $\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_8\alpha_7, \alpha_9\alpha_8, \alpha_{10}\alpha_9, \alpha_{11}\alpha_{10}, \alpha_{12}\alpha_{11}, \alpha_{13}\alpha_6,$ $\alpha_{13}\alpha_{12}, \alpha_{13}\alpha_{18}, \alpha_{14}\alpha_{13}, \alpha_{15}\alpha_{14}, \alpha_{16}\alpha_{15}, \alpha_{17}\alpha_{16}, \alpha_{18}\alpha_{17},$
- $\alpha_{13}\beta_1, \alpha_{15}\beta_2, \beta_2\alpha_2, \alpha_{17}\beta_3, \beta_3\alpha_4, \alpha_{13}\beta_4, \alpha_{15}\beta_5, \beta_5\alpha_8, \alpha_{17}\beta_6, \beta_6\alpha_{10}, \alpha_{13}\beta_7,$ $\beta_7\alpha_6, \beta_7\alpha_{12}, \beta_7\alpha_{18}, \alpha_{15}\beta_8, \beta_8\alpha_{14}, \alpha_{17}\beta_9, \beta_9\alpha_{16},$
- $\alpha_{15}\gamma_1, \alpha_{17}\gamma_2, \gamma_2\alpha_2, \alpha_{13}\gamma_3, \gamma_3\alpha_4, \alpha_{15}\gamma_4, \alpha_{17}\gamma_5, \gamma_5\alpha_8, \alpha_{13}\gamma_6, \gamma_6\alpha_{10}, \alpha_{15}\gamma_7,$ $\gamma_7\alpha_6, \gamma_7\alpha_{12}, \gamma_7\alpha_{18}, \alpha_{17}\gamma_8, \gamma_8\alpha_{14}, \alpha_{13}\gamma_9, \gamma_9\alpha_{16},$
- $\gamma_8\gamma_1, \gamma_9\gamma_2, \gamma_7\gamma_3, \gamma_8\gamma_4, \gamma_9\gamma_5, \gamma_7\gamma_6, \gamma_8\gamma_7, \gamma_9\gamma_8, \gamma_7\gamma_9,$
- $\beta_8\gamma_1 - \gamma_7\beta_1, \beta_9\gamma_2 - \gamma_8\beta_2, \beta_7\gamma_3 - \gamma_9\beta_3, \beta_8\gamma_4 - \gamma_7\beta_4, \beta_9\gamma_5 - \gamma_8\beta_5,$ $\beta_7\gamma_6 - \gamma_9\beta_6, \beta_8\gamma_7 - \gamma_7\beta_7, \beta_9\gamma_8 - \gamma_8\beta_8, \beta_7\gamma_9 - \gamma_9\beta_9.$

This algebra A is the algebra $E(\Lambda)$ given by quiver and relations, where we write α_i for f_i^1, β_i for f_i^2 and γ_i for f_i^3. This gives the following result.

Proposition 7.20. *Let Λ be the algebra given in Example 7.19 and let $A = K\Gamma/\mathcal{I}$ as defined above. Then $A \cong \hat{E}(\Lambda)$ where α_i, β_i and γ_i are all in degree 1, corresponding to the elements of $\hat{E}(\Lambda)_1$.*

With this hat-degree grading, A is a Koszul algebra and we can show this by using Gröbner bases, as in the previous example.

Let $\mathcal{B}$ be the basis of $K\Gamma$ which consists of finite paths. Let the vertices be ordered $e_1 > e_2 > \cdots > e_{17}$ and let the arrows be ordered $\alpha_1 > \alpha_2 > \cdots > \alpha_{18} > \beta_1 > \beta_2 > \cdots > \beta_9 > \gamma_1 > \gamma_2 > \cdots > \gamma_9$. Let the admissible order on $\mathcal{B}$ be the length lexicographic order as given in Definition 7.7.

Let $\mathcal{H}$ be the minimal generating set for $\mathcal{I}$ as given above, that is, $\mathcal{H}$ is the set consisting of

- $\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_5\alpha_4, \alpha_6\alpha_5, \alpha_8\alpha_7, \alpha_9\alpha_8, \alpha_{10}\alpha_9, \alpha_{11}\alpha_{10}, \alpha_{12}\alpha_{11}, \alpha_{13}\alpha_6, \alpha_{13}\alpha_{12}, \alpha_{13}\alpha_{18}, \alpha_{13}\beta_1, \alpha_{13}\beta_4, \alpha_{13}\beta_7, \alpha_{13}\gamma_3, \alpha_{13}\gamma_6, \alpha_{13}\gamma_9, \alpha_{14}\alpha_{13}, \alpha_{15}\alpha_{14}, \alpha_{15}\beta_2, \alpha_{15}\beta_5, \alpha_{15}\beta_8, \alpha_{15}\gamma_1, \alpha_{15}\gamma_4, \alpha_{15}\gamma_7, \alpha_{16}\alpha_{15}, \alpha_{17}\alpha_{16}, \alpha_{17}\beta_3, \alpha_{17}\beta_6, \alpha_{17}\beta_9, \alpha_{17}\gamma_2, \alpha_{17}\gamma_5, \alpha_{17}\gamma_8, \alpha_{18}\alpha_{17}$,
- $\beta_2\alpha_2, \beta_3\alpha_4, \beta_5\alpha_8, \beta_6\alpha_{10}, \beta_7\alpha_6, \beta_7\alpha_{12}, \beta_7\alpha_{18}, \beta_7\gamma_3 - \gamma_9\beta_3, \beta_7\gamma_6 - \gamma_9\beta_6, \beta_7\gamma_9 - \gamma_9\beta_9, \beta_8\alpha_{14}, \beta_8\gamma_1 - \gamma_7\beta_1, \beta_8\gamma_4 - \gamma_7\beta_4, \beta_8\gamma_7 - \gamma_7\beta_7, \beta_9\alpha_{16}, \beta_9\gamma_2 - \gamma_8\beta_2, \beta_9\gamma_5 - \gamma_8\beta_5, \beta_9\gamma_8 - \gamma_8\beta_8$,
- $\gamma_2\alpha_2, \gamma_3\alpha_4, \gamma_5\alpha_8, \gamma_6\alpha_{10}, \gamma_7\alpha_6, \gamma_7\alpha_{12}, \gamma_7\alpha_{18}, \gamma_7\gamma_3, \gamma_7\gamma_6, \gamma_7\gamma_9, \gamma_8\alpha_{14}, \gamma_8\gamma_1, \gamma_8\gamma_4, \gamma_8\gamma_7, \gamma_9\alpha_{16}, \gamma_9\gamma_2, \gamma_9\gamma_5, \gamma_9\gamma_8$.

and let us index these elements h_i, for $i \in \{1, \ldots, 72\}$.

Proposition 7.21. *The set $\mathcal{H}$ as given above is a reduced Gröbner basis of $\mathcal{I}$.*

Proof. We will show that the set $\mathcal{H}$ is a reduced Gröbner basis using Theorem 7.14. It is clear that $\mathcal{H}$ is a subset of A consisting of non-zero elements, and $\langle\mathcal{H}\rangle = \langle\mathcal{I}\rangle$.

Condition $i)$ $\mathrm{CTip}(h_i)$ is 1 for all $i \in \{1, \ldots, 72\}$. This is clear from looking at the set $\mathcal{H}$.

Condition *ii*) h_i does not reduce over h_j for $i \neq j$.

We now show that this condition is satisfied for all $i \neq j$. Assume h_i reduces over h_j then there are elements $u, v \in \mathcal{B}$ such that $u\,\mathrm{Tip}(h_j)v \in \mathrm{Supp}(h_i)$. Looking at the element $\mathrm{Tip}(h_j)$ and the elements that are in $\mathrm{Supp}(h_i)$, it is clear that they are all of length 2 for all i, j, hence $u, v \in \Gamma_0$. In particular, $u = s(\mathrm{Tip}(h_j))$ and $v = t(\mathrm{Tip}(h_j))$. Therefore if h_i reduces over h_j then $\mathrm{Tip}(h_j) \in \mathrm{Supp}(h_i)$. By inspection of $\mathcal{H}$ we can see this is not the case. So h_i does not reduce over h_j for $i \neq j$.

Condition *iii*) every overlap difference for two (not necessarily distinct) members of $\mathcal{H}$ always reduces to zero over $\mathcal{H}$.

We now show that the elements of $\mathcal{H}$ satisfy this condition. Let us consider the set $\mathcal{H}$ as two distinct subsets, let $\mathcal{H}_1$ be the set of monomial elements of $\mathcal{H}$ and let $\mathcal{H}_2$ be the set of non-monomial elements of $\mathcal{H}$. Clearly, $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$.

Now, consider two arbitrary elements of $\mathcal{H}_1$, it follows that in this set, $h_i = \mathrm{Tip}(h_i)$. Assume we have elements $p, q \in \mathcal{B}$ such that $\mathrm{Tip}(h_i)p = q\,\mathrm{Tip}(h_j)$, with; $\mathrm{Tip}(h_i)$ does not divide q and $\mathrm{Tip}(h_j)$ does not divide p. Then the overlap difference is defined as $o(h_i, h_j, p, q) = (\frac{1}{\mathrm{CTip}(h_i)})h_i p - (\frac{1}{\mathrm{CTip}(h_j)})q h_j$. Now, by condition *i*), we know $\mathrm{CTip}(h_i) = 1$ and $\mathrm{CTip}(h_j) = 1$. So $o(h_i, h_j, p, q) = h_i p - q h_j$ and since $\mathrm{Tip}(h_i)p = h_i p$, $q\,\mathrm{Tip}(h_j) = q h_j$, we have $o(h_i, h_j, p, q) = 0$.

Now we are left with the possibility of $h_i, h_j \in \mathcal{H}_2$ and $h_i \in \mathcal{H}_1, h_j \in \mathcal{H}_2$. We will use the property $l(p) = 1 = l(q)$, that is, p, q are arrows.

Let us consider two elements $h_i, h_j \in \mathcal{H}_2$. The second arrow of $\mathrm{Tip}(h_i)p$ is γ_k and the second arrow of $q\,\mathrm{Tip}(h_j)$ is β_l. This cannot happen, so there are no overlap differences except when $i = j$, which will be 0.

Finally, we look at the overlap difference of one monomial element and one non-monomial element. Let us first consider $h_i \in \mathcal{H}_1$ and $h_j \in \mathcal{H}_2$, we will later consider $h_i \in \mathcal{H}_2, h_j \in \mathcal{H}_1$. There will be some non-zero overlap differences here and we can work through the possibilities to show that every overlap difference reduces to zero over $\mathcal{H}$.

Now, let $h_j = h_{43} = \beta_7\gamma_3 - \gamma_9\beta_3$. We know that the second arrow of the term $q\,\mathrm{Tip}(h_{43})$ is β_7. So we need to find all h_i with this property. For these next 9 relations, it is easy to see that there is only one possible overlap. Let $h_i = h_{16} = \alpha_{13}\beta_7$. Then $\mathrm{Tip}(h_{43}) = \beta_7\gamma_3$ and $\mathrm{Tip}(h_{16}) = \alpha_{13}\beta_7$. Let $p = \gamma_3$ and $q = \alpha_{13}$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{16}, h_{43}, \gamma_3, \alpha_{13}) \\ &= \alpha_{13}\beta_7\gamma_3 - \alpha_{13}\beta_7\gamma_3 + \alpha_{13}\gamma_9\beta_3 \\ &= \alpha_{13}\gamma_9\beta_3. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = 1, u = e_{13}, f = h_{19} = \alpha_{13}\gamma_9$ and $v = \beta_3$. Then $o(h_{16}, h_{43}, \gamma_3, \alpha_{13}) - \lambda u f v = 0$. Hence this overlap can be reduced to zero.

Let $h_j = h_{44} = \beta_7\gamma_6 - \gamma_9\beta_6$. Let $h_i = h_{16} = \alpha_{13}\beta_7$, $p = \gamma_6$ and $q = \alpha_{13}$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{16}, h_{44}, \gamma_6, \alpha_{13}) \\ &= \alpha_{13}\beta_7\gamma_6 - \alpha_{13}\beta_7\gamma_6 + \alpha_{13}\gamma_9\beta_6 \\ &= \alpha_{13}\gamma_9\beta_6. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = 1, u = e_{13}, f = h_{19} = \alpha_{13}\gamma_9$ and $v = \beta_6$. Then $o(h_{16}, h_{44}, \gamma_6, \alpha_{13}) - \lambda u f v = 0$. Hence this overlap can be reduced to zero.

Similarly $o(h_{16}, h_{45}, \gamma_9, \alpha_{13}), o(h_{24}, h_{47}, \gamma_1, \alpha_{15}), o(h_{24}, h_{48}, \gamma_4, \alpha_{15}), o(h_{24}, h_{49}, \gamma_7, \alpha_{15})$, $o(h_{22}, h_{51}, \gamma_2, \alpha_{17}), o(h_{22}, h_{52}, \gamma_5, \alpha_{17}), o(h_{22}, h_{53}, \gamma_8, \alpha_{17})$ all have a single element of $\mathcal{H}_1$ as a subpath and so these overlap differences all reduce to zero over $\mathcal{H}$.

We now consider the overlap difference for $h_i \in \mathcal{H}_2, h_j \in \mathcal{H}_1$.

Let $h_i = h_{43} = \beta_7\gamma_3 - \gamma_9\beta_3$. The possibilities for h_j are those which have γ_3 as the first arrow, in this case, $h_{56} = \gamma_3\alpha_4$ is the only possible overlap. Let $p = \alpha_4$ and $q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{43}, h_{56}, \alpha_4, \beta_7) \\ &= \beta_7\gamma_3\alpha_4 - \gamma_9\beta_3\alpha_4 - \beta_7\gamma_3\alpha_4 \\ &= -\gamma_9\beta_3\alpha_4. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = -1, u = \gamma_9, f = h_{38} = \beta_3\alpha_4$ and $v = e_4$. Then $o(h_{43}, h_{56}, \alpha_4, \beta_7) - \lambda ufv = 0$. Hence this overlap can be reduced to zero.

Let $h_i = h_{44} = \beta_7\gamma_6 - \gamma_9\beta_6$. The possibilities for h_j are those which have γ_6 as the first arrow, in this case, $h_{58} = \gamma_6\alpha_{10}$ is the only possible overlap. Let $p = \alpha_{10}$ and $q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{44}, h_{58}, \alpha_{10}, \beta_7) \\ &= \beta_7\gamma_6\alpha_{10} - \gamma_9\beta_6\alpha_{10} - \beta_7\gamma_6\alpha_{10} \\ &= -\gamma_9\beta_6\alpha_{10}. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = -1, u = \gamma_9, f = h_{40} = \beta_6\alpha_{10}$ and $v = e_9$. Then $o(h_{44}, h_{58}, \alpha_{10}, \beta_7) - \lambda ufv = 0$. Hence this overlap can be reduced to zero.

Let $h_i = h_{45} = \beta_7\gamma_9 - \gamma_9\beta_9$. The possibilities for h_j are those which have γ_9 as the first arrow, in this case there are four possibilities; $h_{69} = \gamma_9\alpha_{16}, h_{70} = \gamma_9\gamma_2, h_{71} = \gamma_9\gamma_5, h_{72} = \gamma_9\gamma_8$. These need to be looked at separately.

i) Let us take $h_j = h_{69} = \gamma_9\alpha_{16}$ and let $p = \alpha_{16}, q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{45}, h_{69}, \alpha_{16}, \beta_7) \\ &= \beta_7\gamma_9\alpha_{16} - \gamma_9\beta_9\alpha_{16} - \beta_7\gamma_9\alpha_{16} \\ &= -\gamma_9\beta_9\alpha_{16}. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda = -1, u = \gamma_9, f = h_{50} = \beta_9\alpha_{16}$ and $v = e_{15}$. Then $o(h_{45}, h_{69}, \alpha_{16}, \beta_7) - \lambda ufv = 0$. Hence this overlap can be reduced to zero.

ii) Let us take $h_j = h_{70} = \gamma_9\gamma_2$ and let $p = \gamma_2, q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{45}, h_{70}, \gamma_2, \beta_7) \\ &= \beta_7\gamma_9\gamma_2 - \gamma_9\beta_9\gamma_2 - \beta_7\gamma_9\gamma_2 \\ &= -\gamma_9\beta_9\gamma_2. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_9, f_1 = h_{51} = \beta_9\gamma_2 - \gamma_8\beta_2$ and $v_1 = e_3$. Then

$$\begin{aligned} o(h_{45}, h_{70}, \gamma_2, \beta_7) - \lambda_1 u_1 f_1 v_1 &= -\gamma_9\beta_9\gamma_2 + \gamma_9\beta_9\gamma_2 - \gamma_9\gamma_8\beta_2 \\ &= -\gamma_9\gamma_8\beta_2. \end{aligned}$$

This can be further reduced. Let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{72} = \gamma_9\gamma_8$ and $v_2 = \beta_2$. Then $o(h_{45}, h_{70}, \gamma_2, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Hence this overlap can be reduced to zero.

iii) Let us take $h_j = h_{71} = \gamma_9\gamma_5$ and let $p = \gamma_5, q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{45}, h_{71}, \gamma_5, \beta_7) \\ &= \beta_7\gamma_9\gamma_5 - \gamma_9\beta_9\gamma_5 - \beta_7\gamma_9\gamma_5 \\ &= -\gamma_9\beta_9\gamma_5. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_9, f_1 = h_{52} = \beta_9\gamma_5 - \gamma_8\beta_5$ and $v_1 = e_8$. Then

$$\begin{aligned} o(h_{45}, h_{71}, \gamma_5, \beta_7) - \lambda_1 u_1 f_1 v_1 &= -\gamma_9\beta_9\gamma_5 + \gamma_9\beta_9\gamma_5 - \gamma_9\gamma_8\beta_5 \\ &= -\gamma_9\gamma_8\beta_5. \end{aligned}$$

This can be further reduced. Let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{72} = \gamma_9\gamma_8$ and $v_2 = \beta_5$. Then $o(h_{45}, h_{71}, \gamma_5, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Hence this overlap can be reduced to zero.

iv) Let us take $h_j = h_{72} = \gamma_9\gamma_8$ and let $p = \gamma_8, q = \beta_7$. The overlap difference is given by

$$\begin{aligned} o(h_i, h_j, p, q) &= o(h_{45}, h_{72}, \gamma_8, \beta_7) \\ &= \beta_7\gamma_9\gamma_8 - \gamma_9\beta_9\gamma_8 - \beta_7\gamma_9\gamma_8 \\ &= -\gamma_9\beta_9\gamma_8. \end{aligned}$$

This non-zero overlap difference can be reduced. Let $\lambda_1 = -1, u_1 = \gamma_9, f_1 = h_{53} = \beta_9\gamma_8 - \gamma_8\beta_8$ and $v_1 = e_{14}$. Then

$$\begin{aligned} o(h_{45}, h_{72}, \gamma_8, \beta_7) - \lambda_1 u_1 f_1 v_1 &= -\gamma_9\beta_9\gamma_8 + \gamma_9\beta_9\gamma_8 - \gamma_9\gamma_8\beta_8 \\ &= -\gamma_9\gamma_8\beta_8. \end{aligned}$$

This can be further reduced. Let $\lambda_2 = -1, u_2 = e_{12}, f_2 = h_{72} = \gamma_9\gamma_8$ and $v_2 = \beta_8$. Then $o(h_{45}, h_{72}, \gamma_8, \beta_7) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$. Hence this overlap can be reduced to zero.

For $h_i = h_{47}$ or $h_i = h_{48}$ there are no possible h_j, therefore no possible overlaps.

For $h_i = h_{50} = \beta_8\gamma_7 - \gamma_7\beta_7$. The possibilities for h_j are those which have γ_7 as the first arrow, in this case there are six possibilities; $h_{59} = \gamma_7\alpha_6, h_{60} = \gamma_7\alpha_{12}, h_{61} = \gamma_7\alpha_{18}, h_{62} = \gamma_7\gamma_3, h_{63} = \gamma_7\gamma_6, h_{64} = \gamma_7\gamma_9$. In the same way as above for h_{45} we look at the overlaps separately and we see that we get;

- $o(h_{50}, h_{59}, \alpha_6, \beta_8) = -\gamma_7\beta_7\alpha_6$, which reduces using $\lambda = -1, u = \gamma_7$, $f = h_{41} = \beta_7\alpha_6, v = e_6$ then $o(h_{50}, h_{59}, \alpha_6, \beta_8) - \lambda ufv = 0$
- $o(h_{50}, h_{60}, \alpha_{12}, \beta_8) = -\gamma_7\beta_7\alpha_{12}$, which reduces using $\lambda = -1, u = \gamma_7$, $f = h_{42} = \beta_7\alpha_{12}, v = e_{11}$ then $o(h_{50}, h_{60}, \alpha_{12}, \beta_8) - \lambda ufv = 0$
- $o(h_{50}, h_{61}, \alpha_{18}, \beta_8) = -\gamma_7\beta_7\alpha_{18}$, which reduces using $\lambda = -1, u = \gamma_7$, $f = h_{43} = \beta_7\alpha_{18}, v = e_{17}$ then $o(h_{50}, h_{61}, \alpha_{18}, \beta_8) - \lambda ufv = 0$
- $o(h_{50}, h_{62}, \gamma_3, \beta_8) = -\gamma_7\beta_7\gamma_3$, which reduces using $\lambda_1 = -1, u_1 = \gamma_7$, $f_1 = h_{44} = \beta_7\gamma_3 - \gamma_9\beta_3, v_1 = e_5$ then $o(h_{50}, h_{62}, \alpha_6, \beta_8) - \lambda_1 u_1 f_1 v_1 = -\gamma_7\gamma_9\beta_3$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{14} f_2 = h_{64} = \gamma_7\gamma_9, v_2 = \beta_3$ then $o(h_{50}, h_{62}, \gamma_3, \beta_8) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$
- $o(h_{50}, h_{63}, \gamma_6, \beta_8) = -\gamma_7\beta_7\gamma_6$, which reduces using $\lambda_1 = -1, u_1 = \gamma_7$, $f_1 = h_{45} = \beta_7\gamma_6 - \gamma_9\beta_6, v_1 = e_{10}$ then $o(h_{50}, h_{63}, \gamma_6, \beta_8) - \lambda_1 u_1 f_1 v_1 = -\gamma_7\gamma_9\beta_6$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{14} f_2 = h_{64} = \gamma_7\gamma_9, v_2 = \beta_6$ then $o(h_{50}, h_{63}, \gamma_6, \beta_8) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$
- $o(h_{50}, h_{64}, \gamma_9, \beta_8) = -\gamma_7\beta_7\gamma_9$, which reduces using $\lambda_1 = -1, u_1 = \gamma_7$, $f_1 = h_{46} = \beta_7\gamma_9 - \gamma_9\beta_9, v_1 = e_{16}$ then $o(h_{50}, h_{64}, \gamma_9, \beta_8) - \lambda_1 u_1 f_1 v_1 = -\gamma_7\gamma_9\beta_9$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{14} f_2 = h_{64} = \gamma_7\gamma_9, v_2 = \beta_9$ then $o(h_{50}, h_{64}, \gamma_9, \beta_8) - \lambda_1 u_1 f_1 v_1) - \lambda_2 u_2 f_2 v_2 = 0$

For $h_i = h_{52} = \beta_9\gamma_2 - \gamma_8\beta_2$. The only possible $h_j = h_{55} = \gamma_2\alpha_2$ and $o(h_{52}, h_{55}, \alpha_2, \beta_9) = -\gamma_8\beta_2\alpha_2$, which reduces using $\lambda = -1, u = \gamma_8, f = h_{37} = \beta_2\alpha_2, v = e_2$ then $o(h_{52}, h_{55}, \alpha_2, \beta_9) - \lambda ufv = 0$.

For $h_i = h_{53} = \beta_9\gamma_5 - \gamma_8\beta_5$. The only possible $h_j = h_{57} = \gamma_5\alpha_8$ and $o(h_{53}, h_{57}, \alpha_8, \beta_9)$ $= -\gamma_8\beta_5\alpha_8$, which reduces using $\lambda = -1, u = \gamma_8, f = h_{39} = \beta_5\alpha_8, v = e_7$ then $o(h_{53}, h_{57}, \alpha_8, \beta_9) - \lambda ufv = 0$.

For $h_i = h_{54} = \beta_9\gamma_8 - \gamma_8\beta_8$. The possibilities for h_j are those which have γ_8 as the first arrow, in this case there are four possibilities; $h_{65} = \gamma_8\alpha_{14}, h_{66} = \gamma_8\gamma_1, h_{67} = \gamma_8\gamma_4, h_{68} = \gamma_8\gamma_7$. In the same way as above we look at the overlaps separately and see that we get;

- $o(h_{54}, h_{65}, \alpha_{14}, \beta_9) = -\gamma_8\beta_8\alpha_{14}$, which reduces using $\lambda = -1, u = \gamma_8$, $f = h_{47} = \beta_7\alpha_6, v = e_{13}$ then $o(h_{54}, h_{65}, \alpha_{14}, \beta_9) - \lambda ufv = 0$
- $o(h_{54}, h_{66}, \gamma_1, \beta_9) = -\gamma_8\beta_8\gamma_1$, which reduces using $\lambda_1 = -1, u_1 = \gamma_8$, $f_1 = h_{48} = \beta_8\gamma_1 - \gamma_7\beta_1, v_1 = e_1$ then $o(h_{54}, h_{66}, \gamma_1, \beta_9) - \lambda_1u_1f_1v_1 = -\gamma_8\gamma_7\beta_1$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{16}f_2 = h_{68} = \gamma_8\gamma_7, v_2 = \beta_1$ then $o(h_{54}, h_{66}, \gamma_1, \beta_9) - \lambda_1u_1f_1v_1) - \lambda_2u_2f_2v_2 = 0$
- $o(h_{54}, h_{67}, \gamma_4, \beta_9) = -\gamma_8\beta_8\gamma_4$, which reduces using $\lambda_1 = -1, u_1 = \gamma_8$, $f_1 = h_{49} = \beta_8\gamma_4 - \gamma_7\beta_4, v_1 = e_1$ then $o(h_{54}, h_{67}, \gamma_4, \beta_9) - \lambda_1u_1f_1v_1 = -\gamma_8\gamma_7\beta_4$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{16}f_2 = h_{68} = \gamma_8\gamma_7, v_2 = \beta_4$ then $o(h_{54}, h_{67}, \gamma_4, \beta_9) - \lambda_1u_1f_1v_1) - \lambda_2u_2f_2v_2 = 0$
- $o(h_{54}, h_{68}, \gamma_7, \beta_9) = -\gamma_8\beta_8\gamma_7$, which reduces using $\lambda_1 = -1, u_1 = \gamma_8$, $f_1 = h_{50} = \beta_8\gamma_7 - \gamma_7\beta_7, v_1 = e_{12}$ then $o(h_{54}, h_{68}, \gamma_7, \beta_9) - \lambda_1u_1f_1v_1 = -\gamma_8\gamma_7\beta_7$, this can be further reduced using $\lambda_2 = -1, u_2 = e_{16}f_2 = h_{68} = \gamma_8\gamma_7, v_2 = \beta_7$ then $o(h_{54}, h_{68}, \gamma_7, \beta_9) - \lambda_1u_1f_1v_1) - \lambda_2u_2f_2v_2 = 0$.

Thus we have now shown that for all elements h_i, h_j in the set $\mathcal{H}$, the overlap difference of two elements reduces to zero. Therefore the set $\mathcal{H}$ satisfies the hypothesis of Theorem 7.14, and $\mathcal{H}$ is a reduced Gröbner basis of the ideal $\mathcal{I}$. $\square$

Now we have a reduced Gröbner basis of $\mathcal{I}$, we can use Theorem 7.15 to give the following result.

Theorem 7.22. *Let Λ be the $(6,2)$-stacked monomial algebra of Example 7.19. Let $A = \hat{E}(\Lambda)$ be the Ext algebra of Λ with the hat-degree grading. Then A is a Koszul algebra.*

Proof. It is clear by looking at $\mathcal{H}$ that it consists of quadratic elements, and the proof immediately follows from Theorem 7.15, part 2). $\square$

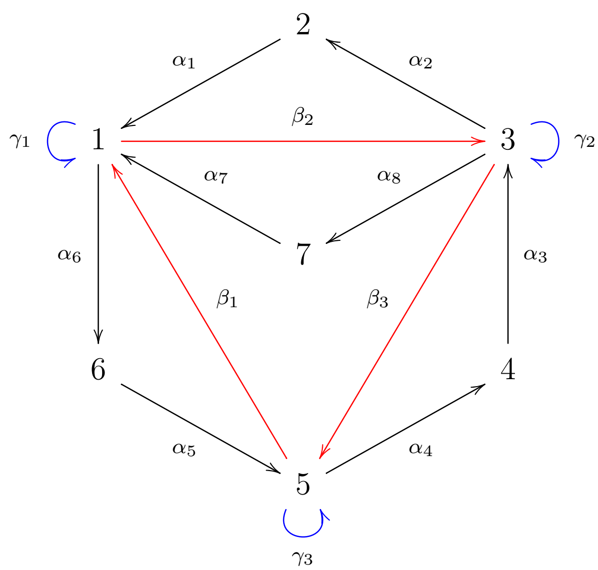
7.5. **Further Examples and a Generalisation.** We have seen that we can regrade $E(\Lambda)$ in the hat-degree, to give a Koszul algebra, for two examples. This raises the question, can this be done for all (D,A)-stacked algebras? The answer is no. If Λ is a (D,A)-stacked algebra with $D = 2A$, $A \neq 1$ and gldim ≥ 6, then we cannot regrade $E(\Lambda)$ in this way. We illustrate this with an example, with a generalised theorem following in Theorem 7.25.

Example 7.23. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
 & 2 \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_7"] & & 3 \arrow[dd, "\alpha_3"] \\
 & 7 \arrow[ur, "\alpha_8"] & \\
6 \arrow[uu, "\alpha_6"] & & 4 \arrow[dl, "\alpha_4"] \\
 & 5 \arrow[ul, "\alpha_5"] &
\end{tikzcd}$$

and let $\Lambda = K\mathcal{Q}/I$, where $I = \langle(\alpha_1\alpha_2 - \alpha_7\alpha_8)\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \alpha_5\alpha_6(\alpha_1\alpha_2 - \alpha_7\alpha_8)\rangle$. Using Definition 8.1, we can construct this algebra from Example 4.7, which is a Koszul algebra (so $d = 2$). We set $A = 2, D = dA$ to obtain this $(4,2)$-stacked algebra Λ. This is also the algebra of Example 3.10, where we constructed a minimal projective resolution of $\Lambda/\mathfrak{r}$.

We can represent the Ext algebra, $E(\Lambda)$, by quiver and relations. Let Γ be the quiver given by



and let $A = K\Gamma/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by;

- $\alpha_i\alpha_j$ for all $i, j \in \{1, \ldots, 8\}$
- $\alpha_i\beta_j,\ \beta_j\alpha_i,$ for all $i \in \{1, \ldots, 8\}$ and for all $j \in \{1, 2, 3\}$
- $\alpha_i\gamma_j,\ \gamma_j\alpha_i$ for all $i \in \{1, \ldots, 8\}$ and for all $j \in \{1, 2, 3\}$
- $\beta_3\gamma_3 - \gamma_2\beta_3,\ \beta_2\gamma_2 - \gamma_1\beta_2,\ \beta_1\gamma_1 - \gamma_3\beta_1$
- $\beta_1\beta_2\beta_3 - (\gamma_3)^2,\ \beta_2\beta_3\beta_1 - (\gamma_1)^2, \beta_3\beta_1\beta_2 - (\gamma_2)^2.$

Then $A \cong E(\Lambda)$, where we write α_i for f_i^1, β_i for f_i^2 and γ_i for f_i^3.

Proposition 7.24. *Let Λ be the algebra of Example 7.23 and let $A = K\Gamma/\mathcal{I}$ as given above. Then there is no regrading so that A is Koszul.*

Proof. Assume that we can give this algebra a grading so that it is Koszul. By definition we need the generators to be in degree 0 and 1. So we have to have

$$\hat{E}_0(\Lambda) = \mathrm{Ext}^0_\Lambda(\Lambda_0, \Lambda_0),$$

$$\hat{E}_1(\Lambda) = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \oplus \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \oplus \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0)$$

with

$$\hat{E}_2 \supseteq \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda)$$

and

$$\hat{E}_3 \supseteq \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda).$$

Now consider the element $\beta_1\beta_2\beta_3$ in A. We know that $\beta_i \in \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \subset \hat{E}_1(\Lambda)$ so $\beta_1\beta_2\beta_3 \in \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \subseteq \hat{E}_3(\Lambda)$. However $\beta_1\beta_2\beta_3 = (\gamma_3)^2$ and $(\gamma_3)^2 \in \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \subset \hat{E}_1(\Lambda)$ so $(\gamma_3)^2 \in \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \subseteq \hat{E}_2(\Lambda)$. Therefore $\beta_1\beta_2\beta_3 \in \hat{E}_2(\Lambda) \cap \hat{E}_3(\Lambda)$, which is a contradiction. $\square$

Example 7.23 is not an isolated case. There is a subset of (D, A)-stacked algebras for which we cannot regrade the Ext algebra to be Koszul. This is made clear in the following theorem.

Theorem 7.25. *Let Λ be a (D, A)-stacked algebra, with $D = 2A, A > 1$ and $\operatorname{gldim} \Lambda \geq 6$. Then there is no regrading such that the Ext algebra is Koszul.*

Proof. Let Λ be a (D, A)-stacked algebra, with $D = 2A, A > 1$ and $\operatorname{gldim} \Lambda \geq 6$. For $E(\Lambda)$ to be Koszul we need a hat-degree grading $\hat{E}(\Lambda)$ such that $\hat{E}(\Lambda)$ is generated in degrees 0 and 1. From Theorem 5.6 we know that $E(\Lambda)$ is generated in degrees $0, 1, 2, 3$ and cannot be generated in degrees $0, 1$ and 2. So we can assume that we require $\hat{E}(\Lambda)$ to have the following structure:

$$\begin{aligned} \hat{E}_0(\Lambda) &= \mathrm{Ext}^0_\Lambda(\Lambda_0, \Lambda_0) \\ \hat{E}_1(\Lambda) &= \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \oplus \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \oplus \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0). \end{aligned}$$

Assume that there is a grading, so $\hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \subseteq \hat{E}_2(\Lambda)$ and $\hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \times \hat{E}_1(\Lambda) \subseteq \hat{E}_3(\Lambda)$.

Now, let us consider $\mathrm{Ext}^6_\Lambda(\Lambda_0, \Lambda_0)$. This is non-empty, since $\operatorname{gldim} \Lambda \geq 6$. From Proposition 5.2 and using the fact that $D = 2A$, we know

$$\begin{aligned} \mathrm{Ext}^6_\Lambda(\Lambda_0, \Lambda_0) &= \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^3_\Lambda(\Lambda_0, \Lambda_0), \\ \mathrm{Ext}^6_\Lambda(\Lambda_0, \Lambda_0) &= \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \text{ and} \\ \mathrm{Ext}^4_\Lambda(\Lambda_0, \Lambda_0) &= \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \times \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0). \end{aligned}$$

$$\begin{aligned}\text{So, } \operatorname{Ext}^6_\Lambda(\Lambda_0,\Lambda_0) &= \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)\\ &= \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0).\end{aligned}$$

We can assume that there exists a non-zero element $z \in \operatorname{Ext}^6_\Lambda(\Lambda_0,\Lambda_0)$ such that $z = x_1x_2x_3 = \sum y_iy_i'$, for $x_j \in \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)$ and $y_i, y_i' \in \operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$. But $y_i, y_i' \in \hat{E}_1(\Lambda)$ so $\sum y_iy_i' \in \hat{E}_2(\Lambda)$ and $x_1, x_2, x_3 \in \hat{E}_1(\Lambda)$ so $x_1x_2x_3 \in \hat{E}_3(\Lambda)$. This contradicts the definition of grading. $\square$

The necessity of the hypothesis gldim ≥ 6 is illustrated in the following example.

Example 7.26. Let $\mathcal{Q}$ be the quiver given by

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5 \xrightarrow{\alpha_5} 6 \xrightarrow{\alpha_6} 7$$

and let $I = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6\rangle$. Let $\Lambda = K\mathcal{Q}/I$. Then Λ is a monomial algebra, so using [18] we have that Λ is a $(4,2)$-stacked algebra with a minimal projective resolution given by

$$0 \longrightarrow P^3 \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \Lambda_0 \longrightarrow 0.$$

This algebra has global dimension 3. It is clear that $\operatorname{Ext}^n_\Lambda(\Lambda_0,\Lambda_0) \cong \operatorname{Hom}(P^n,\Lambda_0) = 0$ for $n \geq 4$. Let $\hat{E}_0(\Lambda) = \operatorname{Ext}^0_\Lambda(\Lambda_0,\Lambda_0)$ and let $\hat{E}_1(\Lambda) = \operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\oplus\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\oplus\operatorname{Ext}^3_\Lambda(\Lambda_0,\Lambda_0)$. Since Λ is a $(4,2)$-stacked algebra we know from Propositions 6.4 and 6.5 that $\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) = 0$ and $\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0) = 0 = \operatorname{Ext}^2_\Lambda(\Lambda_0,\Lambda_0)\times\operatorname{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$. Thus the Ext algebra $\hat{E}(\Lambda) = \hat{E}_0(\Lambda)\oplus\hat{E}_1(\Lambda)$ is a graded algebra with this hat-degree grading.

We have $g^1_i = \alpha_i$ for $i = 1,\ldots,6$, $g^2_1 = \alpha_1\alpha_2\alpha_3\alpha_4, g^2_2 = \alpha_3\alpha_4\alpha_5\alpha_6$ and $g^3_1 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$. Let f^n_i be the Λ-module homomorphism $P^n \to \Lambda/\mathfrak{r}$ given by

$$t(g^n_j) \mapsto \begin{cases} t(g^n_i) & \text{if } i = j\\ 0 & \text{otherwise}\end{cases}$$

so that $\{f_i^n\}$ is a K-basis of $E(\Lambda)$. Then $f_i^n f_j^m = 0$ for all $m \geq 1, n \geq 1$, and all i and j. Thus $\hat{E}(\Lambda)$ is a quadratic monomial algebra and hence is a Koszul algebra with this regrading.

We have now shown 3 examples of our regrading that result in a Koszul algebra, and a collection of (D, A)-stacked algebras whose Ext algebra is not Koszul under regrading. A future research project is to investigate the regrading further. In particular the use of overlaps to construct the Ext algebra of a monomial algebra as given by Green and Zacharia in [21] may provide a way of showing that our regrading on the Ext algebra of a (D, A)-stacked monomial algebra with $D \neq 2A$ is such that this regraded Ext algebra is Koszul.

In the next chapter we take a different approach in studying (D, A)-stacked algebras, and give a precise method to construct a (D, A)-stacked algebra from a d-Koszul algebra.

8. Constructing (D, A)-stacked Algebras

Our aim in this section is to construct a (D, A)-stacked algebra, $\tilde{\Lambda}$, from a given d-Koszul algebra, Λ, where $D = dA$, $A \geq 1$ and $d \geq 2$. Given this relationship between Λ and $\tilde{\Lambda}$, then, if we know the structure of the smaller algebra Λ, we hope to obtain homological information about the infinite family of algebras $\tilde{\Lambda}$. We will consider this question later in Chapter 9.

Fix $d \geq 2$. We assume throughout this section that $\Lambda = K\mathcal{Q}/I$ is a d-Koszul algebra. Thus I is an ideal of $K\mathcal{Q}$ generated by homogeneous elements of length d. Let $A \geq 1$ and set $D = dA$. To construct our (D, A)-stacked algebra, we begin by using the quiver $\mathcal{Q}$ and ideal I of $K\mathcal{Q}$ to define a new quiver $\tilde{\mathcal{Q}}$ and ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$. We then set $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$. In order to show that $\tilde{\Lambda}$ is a (D, A)-stacked algebra, we construct a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module. We will then be able to see that each of the projective modules in this resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ is generated in the correct degree. Our main result is Theorem 8.15 in which we show that $\tilde{\Lambda}$ is a (D, A)-stacked algebra. We set $\mathfrak{r}$ to be the Jacobson radical of Λ and $\tilde{\mathfrak{r}}$ to be the Jacobson radical of $\tilde{\Lambda}$.

In Chapter 3, we described the sets g^n from [20] which determine a minimal projective resolution of $\Lambda/\mathfrak{r}$. We recall that

- g^0 is the set of vertices of $\mathcal{Q}$,
- g^1 is the set of arrows of $\mathcal{Q}$, and
- g^2 is a minimal generating set of I consisting of uniform elements.

We start with our construction of the quiver $\tilde{\mathcal{Q}}$ and ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$.

Definition 8.1. Let $\mathcal{Q}$ be a finite quiver and let I be an ideal of $K\mathcal{Q}$ which is generated by homogeneous elements of length d where $d \geq 2$. Let $A \geq 1$ and set $D = dA$.

- We construct the new quiver $\tilde{\mathcal{Q}}$ as follows. For each arrow α in $\mathcal{Q}$ we have A arrows $\alpha_1, \alpha_2, \ldots, \alpha_A$ in $\tilde{\mathcal{Q}}$ and additional vertices $v_1, v_2, \ldots, v_{A-1}$ in $\tilde{\mathcal{Q}}$, in such a way that :

$$\begin{aligned} s(\alpha_1) &= s(\alpha) \\ t(\alpha_1) &= s(\alpha_2) = v_1 \\ t(\alpha_2) &= s(\alpha_3) = v_2 \\ \vdots \quad & \quad \vdots \\ t(\alpha_{A-1}) &= s(\alpha_A) = v_{A-1} \\ t(\alpha_A) &= t(\alpha) \end{aligned}$$

and the only arrows incident with the vertex v_j are α_j and α_{j+1}. In this way the arrow α in $\mathcal{Q}$ corresponds to a path $\alpha_1 \cdots \alpha_A$ of length A in $\tilde{\mathcal{Q}}$ and the set of vertices of $\mathcal{Q}$ is contained in the set of vertices of $\tilde{\mathcal{Q}}$.

The following diagram illustrates this process of defining $\tilde{\mathcal{Q}}$ from $\mathcal{Q}$.

in $\mathcal{Q}$ | in $\tilde{\mathcal{Q}}$

$$e_1 \xrightarrow{\alpha} e_2 \qquad\qquad e_1 \xrightarrow{\alpha_1} v_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{A-1}} v_{A-1} \xrightarrow{\alpha_A} e_2$$

- We construct the ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$ as follows. Let $g^2 = \{g_1^2, g_2^2, \ldots, g_m^2\}$ be the minimal generating set of uniform elements of I. Write $g_i^2 = \sum c_j \alpha_{j_1}\alpha_{j_2} \cdots \alpha_{j_d}$, for $i = 1, \ldots, m$ where $c_i \in K$ and the α_{j_k} are arrows in $\mathcal{Q}$. The arrow α_{j_k} corresponds to the path $\alpha_{j_k,1}\alpha_{j_k,2} \cdots \alpha_{j_k,A}$ in $\tilde{\mathcal{Q}}$.
Define $\tilde{g}_i^2 = \sum c_j(\alpha_{j_1,1}\alpha_{j_1,2} \cdots \alpha_{j_1,A})(\alpha_{j_2,1}\alpha_{j_2,2} \cdots \alpha_{j_2,A}) \cdots (\alpha_{j_d,1}\alpha_{j_d,2} \cdots \alpha_{j_d,A})$, and let $\tilde{g}^2 = \{\tilde{g}_1^2, \tilde{g}_2^2, \ldots, \tilde{g}_m^2\}$. Define $\tilde{I}$ to be the ideal of $K\tilde{\mathcal{Q}}$ generated by the set $\tilde{g}^2$.
- Let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$.

We now have our new algebra $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$. If m_0 is the number of vertices of $\mathcal{Q}$, and if m_1 the number of arrows of $\mathcal{Q}$ then the quiver $\tilde{\mathcal{Q}}$ has $m_0 + m_1(A-1)$ vertices and $m_1 A$ arrows. It is clear from the construction that $\tilde{\Lambda}$ is again a finite-dimensional algebra.

Proposition 8.2. *With the notation above, each of the elements $\tilde{g}_i^2$ in the minimal generating set of $\tilde{I}$ has length D.*

Proof. Since every arrow in $\mathcal{Q}$ has been replaced by A arrows in $\tilde{\mathcal{Q}}$, it follows that $\tilde{g}_i^2$ is homogeneous with $l(\tilde{g}_i^2) = d \cdot A = D$. $\square$

We now illustrate this construction with two examples. Our first example uses a monomial d-Koszul algebra.

Example 8.3. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
& e_2 \arrow[dr, "\alpha_2"] & \\
e_1 \arrow[ur, "\alpha_1"] & & e_3 \arrow[ll, "\alpha_3"]
\end{tikzcd}$$

and let $I = \langle \alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_1, \alpha_3\alpha_1\alpha_2 \rangle$. Let $\Lambda = K\mathcal{Q}/I$. We have previously seen in Example 4.6 that this is a 3-Koszul monomial algebra.

Let $A \geq 1$ and let $D = 3A$. Let $\tilde{\mathcal{Q}}$ be the quiver given by

$$\begin{tikzcd}
& & & e_2 \arrow[dr, "\alpha_{2,1}"] & & & \\
& & \cdot^{\cdot^{\cdot}} \arrow[ur, "\alpha_{1,A}"] & & v_{2,1} \arrow[dr, "\alpha_{2,2}"] & & \\
& v_{1,1} \arrow[ur, "\alpha_{1,2}"] & & & & \ddots \arrow[dr, "\alpha_{2,A}"] & \\
e_1 \arrow[ur, "\alpha_{1,1}"] & & \cdots \arrow[ll, "\alpha_{3,A}"] & & v_{3,1} \arrow[ll, "\alpha_{3,2}"] & & e_3 \arrow[ll, "\alpha_{3,1}"]
\end{tikzcd}$$

and let $\tilde{I} = \langle \alpha_{1,1} \cdots \alpha_{1,A}\alpha_{2,1} \cdots \alpha_{2,A}\alpha_{3,1} \cdots \alpha_{3,A}, \;\; \alpha_{2,1} \cdots \alpha_{2,A}\alpha_{3,1} \cdots \alpha_{3,A}\alpha_{1,1} \cdots \alpha_{1,A}$, $\alpha_{3,1} \cdots \alpha_{3,A}\alpha_{1,1} \cdots \alpha_{1,A}\alpha_{2,1} \cdots \alpha_{2,A} \rangle$. Let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$. Then $\tilde{\Lambda}$ is the related algebra.

It is clear that $\tilde{\Lambda}$ is a monomial algebra. The fact that $\tilde{\Lambda}$ is a (D, A)-stacked algebra follows from Chapter 4 where we discussed the (D, A)-stacked monomial algebras of [19].

Our second example begins with the Koszul algebra of Example 4.7.

Example 8.4. Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
1 \arrow[rr, bend left=10, "\alpha_1"] \arrow[rr, bend right=10, "\alpha_4"'] & & 2 \arrow[dl, "\alpha_2"] \\
& 3 \arrow[ul, "\alpha_3"] &
\end{tikzcd}$$

and let $I = \langle (\alpha_1 - \alpha_4)\alpha_2, \alpha_2\alpha_3, \alpha_3(\alpha_1 - \alpha_4)\rangle$. Let $\Lambda = K\mathcal{Q}/I$. We have already seen in Chapter 4 that this is a Koszul algebra and we remark again, that this is a monomial algebra with the specified change of generators. The ideal is given by non-monomial generators.

Let $\tilde{\mathcal{Q}}$ be the quiver given by

$$\begin{tikzcd}
& v_{1,1} \arrow[r] & \cdots \arrow[r] & v_{1,A-1} \arrow[dr, "\alpha_{1,A}"] & \\
1 \arrow[ur, "\alpha_{1,1}"] \arrow[dr, "\alpha_{4,1}"] & & & & 2 \arrow[d, "\alpha_{2,1}"] \\
v_{3,A-1} \arrow[u, "\alpha_{3,A}"] & v_{4,1} \arrow[r] & \cdots \arrow[r] & v_{4,A-1} \arrow[ur, "\alpha_{4,A}"] & v_{2,1} \arrow[dl, "\alpha_{2,2}"] \\
& \ddots \arrow[ul, "\alpha_{3,A-1}"] & & \iddots \arrow[dl, "\alpha_{2,A}"] & \\
& & 3 \arrow[ul, "\alpha_{3,1}"] & &
\end{tikzcd}$$

and let $\tilde{I} = \langle (\alpha_{1,1} \cdots \alpha_{1,A} - \alpha_{4,1} \cdots \alpha_{4,A})\alpha_{2,1} \cdots \alpha_{2,A}, \alpha_{2,1} \cdots \alpha_{2,A}\alpha_{3,1} \cdots \alpha_{3,A},$ $\alpha_{3,1} \cdots \alpha_{3,A}(\alpha_{1,1} \cdots \alpha_{1,A} - \alpha_{4,1} \cdots \alpha_{4,A})\rangle$. Let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$. It is clear that $\tilde{\Lambda}$ is not a monomial algebra. This construction produces a non-monomial $(2A, A)$-stacked algebra, $\tilde{\Lambda}$. The proof of this will follow as we prove the general case.

Let $\Lambda = K\mathcal{Q}/I$ be a d-Koszul algebra and let $\tilde{\Lambda}$ be the algebra given by the construction above, for some $A \geq 1$ and $D = dA$. In order to prove $\tilde{\Lambda}$ is a (D, A)-stacked algebra, we begin by defining a map $\theta : \Lambda \to \tilde{\Lambda}$. Then given a minimal projective resolution, (P^n, d^n), of $\Lambda/\mathfrak{r}$ as a right Λ-module, we use the maps d^n and θ to define projective $\tilde{\Lambda}$-modules $\tilde{P}^n$ and maps $\tilde{d}^n : \tilde{P}^n \to \tilde{P}^{n-1}$. We then show that $(\tilde{P}^n, \tilde{d}^n)$ is indeed a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module. This will enable us to show $\tilde{\Lambda}$ is a (D, A)-stacked algebra.

Definition 8.5. We keep the above notation. Let $\theta^* : K\mathcal{Q} \to K\tilde{\mathcal{Q}}$ be the map defined as follows;

$$\begin{aligned}
\theta^*(e) &= e \text{ for } e \in \mathcal{Q}_0,\\
\theta^*(\alpha) &= \alpha_1\alpha_2\cdots\alpha_A \text{ for each arrow in } \mathcal{Q},\\
\theta^*(\gamma_1\gamma_2\cdots\gamma_r) &= \theta^*(\gamma_1)\theta^*(\gamma_2)\cdots\theta^*(\gamma_r), \text{ where } \gamma_i \text{ are arrows in } K\mathcal{Q},\\
\theta^*(c_1\gamma_1 + c_2\gamma_2) &= c_1(\theta^*(\gamma_1)) + c_2(\theta^*(\gamma_2)), \text{ for } c_1, c_2 \in K, \gamma_1, \gamma_2 \in K\mathcal{Q}.
\end{aligned}$$

The map θ^* is a K-algebra homomorphism and is $1-1$, by construction. It follows that $\tilde{g}_i^2 = \theta^*(g_i^2)$ so the ideal $\tilde{I}$ is generated by the elements $\theta^*(g_i^2)$, where $\{g_i^2\}$ is a minimal generating set of uniform elements of I.

Definition 8.6. We keep the above notation. Define $\theta : K\mathcal{Q}/I \to K\tilde{\mathcal{Q}}/\tilde{I}$ by $\theta(x + I) = \theta^*(x) + \tilde{I}$ for all $x \in K\mathcal{Q}$.

From the definition of $\tilde{I}$ it is straightforward to verify that θ is well-defined, a K-algebra homomorphism and $1-1$.

Let (P^n, d^n) be a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module as constructed by Green, Solberg and Zacharia in [20]. We recall from Chapter 3 that

- $P^n = \bigoplus_i t(g_i^n)\Lambda$,
- $d^0 : P^0 \to \Lambda/\mathfrak{r}$ is the canonical surjection given by $d^0(e\lambda) = e\lambda + \mathfrak{r}$ for $e \in \mathcal{Q}_0$.
- $d^1 : P^1 \to P^0$ is defined by $d^1(t(\alpha)\lambda) = \alpha\lambda$, where the entry $\alpha\lambda$ is in the summand of P^0 corresponding to $s(\alpha)$, for each $\alpha \in \mathcal{Q}_1$.
- Let $g_i^2 \in g^2$. Then $g_i^2 = \sum_j \alpha_j\beta_j$, where α_j is an arrow in $\mathcal{Q}$ and $\beta_j \in K\mathcal{Q}$. The

map $d^2 : P^2 \to P^1$ is such that $d^2(t(g_i^2)\lambda)$ has entry $\beta_j\lambda$ in the summand of P^1 corresponding to $t(\alpha_j)$.

• Let $n \geq 3$ with $g_i^n \in g^n$. Then $g_i^n = \sum_j g_j^{n-1}q_j$, with $q_j \in K\mathcal{Q}$. The map $d^n : P^n \to P^{n-1}$ is such that $d^n(t(g_i^n)\lambda)$ has entry $t(g_j^{n-1})q_j\lambda$ in the summand of P^{n-1} corresponding to $t(g_j^{n-1})$. Moreover, if n is odd, $n = 2r+1, r \geq 1$, we have $l(q_j) = 1$, since $l(g_i^n) = rd+1$ and $l(g_j^{n-1}) = rd$. If n is even, $n = 2r, r \geq 2$, then $l(q_j) = d-1$, since $l(g_i^n) = rd$ and $l(g_j^{n-1}) = (r-1)d+1$.

We now use P^n, d^n and θ^* to find a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$.

Definition 8.7. Define the sets $\tilde{g}^n$, for $n \geq 0$, as follows. Let

- $\tilde{g}^0$ be the set of vertices of $\tilde{\mathcal{Q}}$,
- $\tilde{g}^1$ be the set of arrows of $\tilde{\mathcal{Q}}$,
- $\tilde{g}^2$ be the generating set of $\tilde{I}$ as given in Definition 8.1.
- For $n \geq 3$, we define $\tilde{g}_i^n = \theta^*(g_i^n)$ for each $g_i^n \in g^n$, and set $\tilde{g}^n = \{\tilde{g}_i^n\}$.

Observe that $\tilde{g}_i^2 = \theta^*(g_i^2)$ for each $g_i^2 \in g^2$. Moreover, for $n \geq 3$, it follows from above that we have

$$\begin{aligned}\tilde{g}_i^n &= \theta^*(g_i^n)\\ &= \theta^*(\textstyle\sum_j g_j^{n-1}q_j)\\ &= \textstyle\sum_j \theta^*(g_j^{n-1}q_j)\\ &= \textstyle\sum_j \tilde{g}_j^{n-1}\theta^*(q_j),\end{aligned}$$

where $g_i^n = \sum_j g_j^{n-1}q_j$ for some $q_j \in K\mathcal{Q}$.

Proposition 8.8. *Let Λ be a d-Koszul algebra, with $d \geq 2$, and $\tilde{\Lambda}$ be the related algebra as defined above, with $D = dA$ and $A \geq 1$. Let $n \geq 2$, let $g_i^n \in K\mathcal{Q}$ and $\tilde{g}_i^n \in K\tilde{\mathcal{Q}}$. Then $s(g_i^n) = s(\tilde{g}_i^n)$ and $t(g_i^n) = t(\tilde{g}_i^n)$ under the identification of $\mathcal{Q}_0$ as a subset of $\tilde{\mathcal{Q}}_0$.*

Proof. For $n \geq 2$,

$$
\begin{aligned}
\tilde{g}_i^n &= \theta^*(g_i^n) \\
&= \theta^*(s(g_i^n) g_i^n t(g_i^n)) \\
&= \theta^*(s(g_i^n))\theta^*(g_i^n)\theta^*(t(g_i^n)) \\
&= s(g_i^n)\tilde{g}_i^n t(g_i^n) \text{ by definition of } \theta^*.
\end{aligned}
$$

Hence $s(g_i^n) = s(\tilde{g}_i^n)$ and $t(g_i^n) = t(\tilde{g}_i^n)$. □

Since each of the sets g^n consists of uniform elements, it follows that each of the sets $\tilde{g}^n$ also consists of uniform elements. So we may define $\tilde{P}^n = \bigoplus_i t(\tilde{g}_i^n)\tilde{\Lambda}$, for all $n \geq 0$.

We now need to define maps $\tilde{d}^n : \tilde{P}^n \to \tilde{P}^{n-1}$. Then we will show in Proposition 8.14 that $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module.

Given the maps θ and d^n, we can construct the following diagram:

$$
\begin{array}{ccccccccccccccccc}
\cdots & \longrightarrow & P^n & \xrightarrow{d^n} & P^{n-1} & \xrightarrow{d^{n-1}} & \cdots & \longrightarrow & P^2 & \xrightarrow{d^2} & P^1 & \xrightarrow{d^1} & P^0 & \xrightarrow{d^0} & \Lambda/\mathfrak{r} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n-1}} & & & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \bar{\theta}} & & \\
\cdots & \longrightarrow & \tilde{P}^n & \longrightarrow & \tilde{P}^{n-1} & \longrightarrow & \cdots & \longrightarrow & \tilde{P}^2 & \longrightarrow & \tilde{P}^1 & \longrightarrow & \tilde{P}^0 & \longrightarrow & \tilde{\Lambda}/\tilde{\mathfrak{r}} & \longrightarrow & 0
\end{array}
$$

where the maps $\theta_n : P^n \to \tilde{P}^n$ and $\bar{\theta} : \Lambda/\mathfrak{r} \to \tilde{\Lambda}/\tilde{\mathfrak{r}}$ are induced from the map θ as given in Definition 8.6, that is, θ_n and $\bar{\theta}$ are K-module homomorphisms given by:

$$
\begin{aligned}
\theta_0 &: P^0 \to \tilde{P}^0,\ \theta_0(e\lambda) = e\theta(\lambda) \\
\theta_1 &: P^1 \to \tilde{P}^1,\ \theta_1(t(\alpha)\lambda) = t(\alpha_A)\theta(\lambda) \\
\theta_n &: P^n \to \tilde{P}^n,\ \theta_n(t(g_i^n)\lambda) = t(\tilde{g}_i^n)\theta(\lambda),\ \text{for all } n \geq 2 \\
\bar{\theta} &: \Lambda/\mathfrak{r} \to \tilde{\Lambda}/\tilde{\mathfrak{r}},\ \bar{\theta}(\lambda + \mathfrak{r}) = \theta(\lambda) + \tilde{\mathfrak{r}}
\end{aligned}
$$

for all $e \in \mathcal{Q}_0$, all arrows $\alpha \in \mathcal{Q}_1$, with $\theta^*(\alpha) = \alpha_1 \cdots \alpha_A$, and all $\lambda \in \Lambda$.

It is clear that since θ is well-defined and $1-1$, then θ_n, for all $n \geq 0$, and $\bar{\theta}$ are also well-defined and $1-1$.

The first stage is to show that $\bar{\theta}$ and θ_n are Λ-module homomorphisms, for all $n \geq 0$. We start by showing that we can define $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ and $\tilde{P}^n$ as right Λ-modules.

Definition 8.9. We define $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ to be a non-unital right Λ-module via the map θ by $(\tilde{\lambda}+\tilde{\mathfrak{r}})\cdot\mu = (\tilde{\lambda}+\tilde{\mathfrak{r}})\theta(\mu)$, for all $\tilde{\lambda}+\tilde{\mathfrak{r}} \in \tilde{\Lambda}/\tilde{\mathfrak{r}}$ and all $\mu \in \Lambda$.

We define $\tilde{P}^n$ to be a non-unital right Λ-module via the map θ, where for all $e\tilde{\lambda} \in \tilde{P}^n$ and all $\mu \in \Lambda$ we have $e\tilde{\lambda}\cdot\mu = e\tilde{\lambda}\,\theta(\mu)$.

Proposition 8.10. *The maps $\bar{\theta}$ and θ_n for $n \geq 0$, are right Λ-module homomorphisms.*

Proof. We start with $\bar{\theta}$ and need to show that $\bar{\theta}((\lambda+\mathfrak{r})\mu) = \bar{\theta}(\lambda+\mathfrak{r})\cdot\mu$ for all $\lambda,\mu \in \Lambda$. We have $\bar{\theta}((\lambda+\mathfrak{r})\mu) = \bar{\theta}(\lambda\mu+\mathfrak{r}) = \theta(\lambda\mu)+\tilde{\mathfrak{r}}$. On the other side we have $\bar{\theta}(\lambda+\mathfrak{r})\cdot\mu = (\theta(\lambda)+\tilde{\mathfrak{r}})\cdot\mu = (\theta(\lambda)+\tilde{\mathfrak{r}})\theta(\mu) = \theta(\lambda)\theta(\mu)+\tilde{\mathfrak{r}} = \theta(\lambda\mu)+\tilde{\mathfrak{r}}$. Hence $\bar{\theta}$ is a right Λ-module homomorphism.

Let $n \geq 0$. We now consider θ_n and need to show that $\theta_n(e\lambda\mu) = \theta_n(e\lambda)\cdot\mu$, where $e \in \mathcal{Q}_0$ and $\lambda,\mu \in \Lambda$. From the definition, we have $\theta_n(e\lambda\mu) = e\theta(\lambda\mu) = e\theta(\lambda)\theta(\mu)$. Now $\theta_n(e\lambda)\cdot\mu = \theta_n(e\lambda)\theta(\mu) = e\theta(\lambda)\theta(\mu)$. Hence $\theta_n(e\lambda\mu) = \theta_n(e\lambda)\mu$ and θ_n is a right Λ-module homomorphism as required. $\square$

Now we define $\tilde{\Lambda}$-module homomorphisms $\tilde{d}^n : \tilde{P}^n \to \tilde{P}^{n-1}$, that are analogous to the maps $d^n : P^n \to P^{n-1}$.

Definition 8.11. Keeping the same notation, let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$, let $\tilde{P}^n = \bigoplus_i t(\tilde{g}_i^n)\tilde{\Lambda}$ and let $\tilde{\lambda} \in \tilde{\Lambda}$.

- For $n = 0$, we define $\tilde{d}^0 : \tilde{P}^0 \to \tilde{\Lambda}/\tilde{\mathfrak{r}}$ to be the canonical surjection given by $\tilde{d}^0(\tilde{e}\tilde{\lambda}) = \tilde{e}\tilde{\lambda}+\tilde{\mathfrak{r}}$, for all $\tilde{e} \in \tilde{\mathcal{Q}}_0$.
- Let $n = 1$ and let $\tilde{\alpha}$ be an arrow in $\tilde{\mathcal{Q}}_1$. We define $\tilde{d}^1 : \tilde{P}^1 \to \tilde{P}^0$ to be the $\tilde{\Lambda}$-module homomorphism given by $\tilde{d}^1(t(\tilde{\alpha})\tilde{\lambda}) = \tilde{\alpha}\tilde{\lambda}$, where the entry $\tilde{\alpha}\tilde{\lambda}$ is in the summand of $\tilde{P}^0$ corresponding to $s(\tilde{\alpha})$.
- Let $n = 2$ and let $\tilde{g}_i^2 \in \tilde{g}^2$. We can write $\tilde{g}_i^2 = \sum_j \tilde{\alpha}_j\tilde{\eta}_j$, where $\tilde{\eta}_j \in K\tilde{\mathcal{Q}}$ and $\tilde{\alpha}_j$ is an arrow in $\tilde{\mathcal{Q}}$. We define the map $\tilde{d}^2 : \tilde{P}^2 \to \tilde{P}^1$ to be the $\tilde{\Lambda}$-module homomorphism such that $\tilde{d}^2(t(\tilde{g}_i^2)\tilde{\lambda})$ has entry $\tilde{\eta}_j\tilde{\lambda}$ in the summand of $\tilde{P}^1$ corresponding to $t(\tilde{\alpha}_j)$.

- Let $n \geq 3$ and let $\tilde{g}_i^n \in \tilde{g}^n$. We may write $\tilde{g}_i^n = \sum_j \tilde{g}_j^{n-1}\theta^*(q_j)$. Then we define $\tilde{d}^n : \tilde{P}^n \to \tilde{P}^{n-1}$ to be the $\tilde{\Lambda}$-module homomorphism such that $\tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda})$ has entry $t(\tilde{g}_j^{n-1})\theta(q_j)\tilde{\lambda}$ in the summand of $\tilde{P}^{n-1}$ corresponding to $t(\tilde{g}_j^{n-1})$. Thus $\tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda}) = \theta_{n-1}(d^n(t(g_i^n)))\tilde{\lambda}$.

Remark. The $\tilde{\Lambda}$-module homomorphisms $\tilde{d}^n : \tilde{P}^n \to \tilde{P}^{n-1}$ are also Λ-module homomorphisms when we consider $\tilde{P}^n$ and $\tilde{P}^{n-1}$ as right Λ-modules. For, suppose $\tilde{\lambda} \in \tilde{\Lambda}$ and $\mu \in \Lambda$. Then

$$\begin{aligned}\tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda}\cdot\mu) &= \tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda}\theta(\mu)) \\ &= \tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda})\theta(\mu) \\ &= \tilde{d}^n(t(\tilde{g}_i^n)\tilde{\lambda})\cdot\mu.\end{aligned}$$

We now look at whether we have a commutative diagram using the maps we have defined, that is, do we have $\theta_{n-1} \circ d^n = \tilde{d}^n \circ \theta_n$.

Proposition 8.12. *Let $A > 1$. In the following diagram of right Λ-modules*

$$\begin{tikzcd}
\cdots \arrow[r] & P^n \arrow[r, "d^n"] \arrow[d, "\theta_n"] \arrow[dr, phantom, "n"] & P^{n-1} \arrow[r] \arrow[d, "\theta_{n-1}"] & \cdots \arrow[r] & P^2 \arrow[r, "d^2"] \arrow[d, "\theta_2"] \arrow[dr, phantom, "2"] & P^1 \arrow[r, "d^1"] \arrow[d, "\theta_1"] \arrow[dr, phantom, "1"] & P^0 \arrow[r, "d^0"] \arrow[d, "\theta_0"] \arrow[dr, phantom, "0"] & \Lambda/\mathfrak{r} \arrow[r] \arrow[d, "\bar{\theta}"] & 0 \\
\cdots \arrow[r] & \tilde{P}^n \arrow[r, "\tilde{d}^n"'] & \tilde{P}^{n-1} \arrow[r] & \cdots \arrow[r] & \tilde{P}^2 \arrow[r, "\tilde{d}^2"'] & \tilde{P}^1 \arrow[r, "\tilde{d}^1"'] & \tilde{P}^0 \arrow[r, "\tilde{d}^0"'] & \tilde{\Lambda}/\tilde{\mathfrak{r}} \arrow[r] & 0
\end{tikzcd}$$

the squares, n, commute for $n = 0$ and $n \geq 3$. The squares 1 and 2 do not commute.

Proof. Let $n = 0$. Then the square labelled 0 is given as follows;

$$\begin{tikzcd}
P^0 \arrow[r, "d^0"] \arrow[d, "\theta_0"'] & \Lambda/\mathfrak{r} \arrow[d, "\bar{\theta}"] \\
\tilde{P}^0 \arrow[r, "\tilde{d}^0"'] & \tilde{\Lambda}/\tilde{\mathfrak{r}}
\end{tikzcd}$$

Let y be an element of P^0 with entry $e\lambda$ in the summand corresponding to e and 0 otherwise. Then $d^0(y) = e\lambda + \mathfrak{r}$. So $\bar{\theta} \circ d^0(y) = \theta(e\lambda) + \tilde{\mathfrak{r}}$.

On the other hand, $\theta_0(y)$ has entry $\theta(e\lambda)$ in the summand of $\tilde{P}^0$ corresponding to e and 0 otherwise. So $\tilde{d}^0 \circ \theta_0(y) = \theta(e\lambda) + \tilde{\mathfrak{r}}$.

It is now clear that this square commutes.

Let $n = 1$. Then the square 1 is given as follows;

$$\begin{tikzcd} P^1 \arrow[r, "d^1"] \arrow[d, "\theta_1"'] & P^0 \arrow[d, "\theta_0"] \\ \tilde{P}^1 \arrow[r, "\tilde{d}^1"'] & \tilde{P}^0 \end{tikzcd}$$

Let α be an arrow in Q and let $\theta(\alpha) = \alpha_1\alpha_2\cdots\alpha_A$. Let x be an element of P^1 with entry $t(\alpha)\lambda$ in the summand corresponding to $t(\alpha)$ and 0 otherwise. Now $d^1(x)$ has entry $\alpha\lambda$ in the summand of P^0 corresponding to $s(\alpha)$ and 0 otherwise.

So $\theta_0 \circ d^1(x)$ has entry $\begin{cases} \theta(\alpha\lambda) & \text{in the } s(\alpha)\text{-summand of } \tilde{P}^0 \\ 0 & \text{otherwise.} \end{cases}$

On the other hand, $\theta_1(x)$ has entry $\theta(\lambda)$ in the summand of $\tilde{P}^1$ corresponding to $t(\alpha_A)$ and 0 otherwise.

So $\tilde{d}^1 \circ\ \theta_1(x)$ has entry $\begin{cases} \alpha_A\theta(\lambda) & \text{in the } s(\alpha_A)\text{-summand of } \tilde{P}^0 \\ 0 & \text{otherwise.} \end{cases}$

So this square does not commute for $A > 1$.

Let $n = 2$. The square 2 is given as follows;

$$\begin{tikzcd} P^2 \arrow[r, "d^2"] \arrow[d, "\theta_2"'] & P^1 \arrow[d, "\theta_1"] \\ \tilde{P}^2 \arrow[r, "\tilde{d}^2"'] & \tilde{P}^1 \end{tikzcd}$$

Let $g_i^2 \in g^2$ with $g_i^2 = \sum_j \alpha_j\beta_j$, where α_j is an arrow in $\mathcal{Q}_1$ and $\beta_j \in K\mathcal{Q}$. Let $\tilde{g}_i^2 = \theta^*(g_i^2) = \theta^*(\sum_j \alpha_j\beta_j) = \sum_j \theta^*(\alpha_j)\theta^*(\beta_j) = \sum_j \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,A}\theta^*(\beta_j)$. Thus $\tilde{g}_i^2 = \sum_j \alpha_{j,1}\tilde{\eta}_j$, where $\tilde{\eta}_j = \alpha_{j,2}\cdots\alpha_{j,A}\ \theta^*(\beta_j)$.

Let y be an element in P^2 with entry $t(g_i^2)\lambda$ in the summand corresponding to $t(g_i^2)$ and 0 otherwise. Now $d^2(y)$ has entry $t(\alpha_j)\beta_j\lambda$ in the summand of P^1 corresponding to $t(\alpha_j)$.

So $\theta_1 \circ\ d^2(y)$ has entry $\begin{cases} t(\alpha_{j,A})\theta(\beta_j)\theta(\lambda) & \text{in the } t(\alpha_{j,A})\text{-summand of } \tilde{P}^1 \\ \\ 0 & \text{otherwise.} \end{cases}$

On the other hand, $\theta_2(y)$ has entry $t(\tilde{g}_i^2)\theta(\lambda)$ in the summand of $\tilde{P}^2$ corresponding to $t(\tilde{g}_i^2)$ and 0 otherwise.

So $\tilde{d}^2 \circ \theta_2(y)$ has entry $\begin{cases} t(\alpha_{j,1})\tilde{\eta}_j\theta(\lambda) & \text{in the } t(\alpha_{j,1})\text{-summand of } \tilde{P}^1 \\ \\ 0 & \text{otherwise.} \end{cases}$

Since $t(\alpha_{j,1}) \neq t(\alpha_{j,A})$, this square does not not commute for $A > 1$.

Let $n = 3$. Then the square 3 is given as follows;

$$\begin{tikzcd} P^3 \arrow[r, "d^3"] \arrow[d, "\theta_3"'] & P^2 \arrow[d, "\theta_2"] \\ \tilde{P}^3 \arrow[r, "\tilde{d}^3"'] & \tilde{P}^2 \end{tikzcd}$$

Let $g_i^3 \in g^3$ and write $g_i^3 = \sum_{j=1}^{m_2} g_j^2 q_j$ where m_2 is the number of elements in the set g^2. Let $\tilde{g}_i^3 = \theta^*(g_i^3) = \theta^*(\sum_{j=1}^{m_2} g_j^2 q_j) = \sum_{j=1}^{m_2} \theta^*(g_j^2)\theta^*(q_j) = \sum_{j=1}^{m_2} \tilde{g}_j^2 \theta^*(q_j)$. Let x be an element in P^3 with entry $t(g_i^3)\lambda$ in the summand of P^3 corresponding to $t(g_i^3)$ and 0 otherwise. Now, $d^3(x)$ has entry $t(g_j^2)q_j\lambda$ in the summand of P^2 corresponding to $t(g_j^2)$, for $j = 1, 2, \ldots, m_2$. So $\theta_2 \circ d^3(x)$ has entry $t(\tilde{g}_j^2)\theta(q_j)\theta(\lambda),$ in the $t(\tilde{g}_j^2)$-summand of $\tilde{P}^2$, for $j = 1, 2, \ldots, m_2$.

On the other hand, $\theta_3(x)$ has entry $t(\tilde{g}_i^3)\theta(\lambda)$ in the $t(\tilde{g}_i^3)$-summand of $\tilde{P}^3$ and 0 otherwise. So, $\tilde{d}^3 \circ \theta_3(x)$ has entry $t(\tilde{g}_j^2)\theta(q_j)\theta(\lambda)$ in the $t(\tilde{g}_j^2)$-summand of $\tilde{P}^2$, for $j = 1, 2, \ldots, m_2$. Hence the square 3 is commutative.

Let $n \geq 4$. Then the square n is given as follows;

$$\begin{tikzcd} P^n \arrow[r, "d^n"] \arrow[d, "\theta_n"'] & P^{n-1} \arrow[d, "\theta_{n-1}"] \\ \tilde{P}^n \arrow[r, "\tilde{d}^n"'] & \tilde{P}^{n-1} \end{tikzcd}$$

Let $g_i^n \in g^n$, with $g_i^n = \sum_{j=1}^{m_{n-1}} g_j^{n-1} q_j$ where m_{n-1} is the number of elements in the set g^{n-1}. Let $\tilde{g}_i^n = \theta^*(g_i^n) = \sum_{j=1}^{m_{n-1}} \theta^*(g_j^{n-1})\theta^*(q_j) = \sum_{j=1}^{m_{n-1}} \tilde{g}_j^{n-1}\theta^*(q_j)$.

Let y be an element in P^n with entry $t(g_i^n)\lambda$ in the summand of P^n corresponding to $t(g_i^n)$ and 0 otherwise. Now $d^n(y)$ has entry $t(g_j^{n-1})q_j\lambda$ in the summand of P^{n-1} corresponding to $t(g_j^{n-1})$, for $j = 1, 2, \ldots, m_{n-1}$. So, $\theta_{n-1} \circ d^n(y)$ has entry $t(\tilde{g}_j^{n-1})\theta(q_j)\theta(\lambda)$ in the $t(\tilde{g}_j^{n-1})$-summand of $\tilde{P}^{n-1}$, for $j = 1, 2, \ldots, m_{n-1}$.

On the other hand, $\theta_n(y)$ has entry $t(\tilde{g}_i^n)\theta(\lambda)$ in the summand of $\tilde{P}^n$ corresponding to $t(\tilde{g}_i^n)$ and 0 otherwise. So, $\tilde{d}^n \circ \theta_n(y)$ has entry $t(\tilde{g}_j^{n-1})\theta(q_j)\theta(\lambda)$ in the $t(\tilde{g}_j^{n-1})$-summand of $\tilde{P}^{n-1}$, for $j = 1, 2, \ldots, m_{n-1}$.

Hence, the square n is commutative for $n \geq 4$.

In summary, the diagram

$$\begin{tikzcd} P^n \arrow[r, "d^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\text{n}"] & P^{n-1} \arrow[d, "\theta_{n-1}"] \\ \tilde{P}^n \arrow[r, "\tilde{d}^n"'] & \tilde{P}^{n-1} \end{tikzcd}$$

is commutative for $n \geq 3$ and $n = 0$, but not for $n = 1$ or 2. □

Proposition 8.13. *Let $\tilde{P}^n$ and $\tilde{d}^n$ be as given above. Then $(\tilde{P}^n, \tilde{d}^n)$ is a complex.*

Proof. To show $(\tilde{P}^n, \tilde{d}^n)$ is a complex we need to show, for all $n \geq 0$, that we have $\tilde{d}^n \circ \tilde{d}^{n+1} = 0$.

Let $n = 0$. Let $\tilde{\alpha}$ be an arrow in $\tilde{\mathcal{Q}}_1$. We can write $\tilde{\alpha} = e\tilde{\alpha}$, where e is a vertex in $\tilde{\mathcal{Q}}_0$. Let x be an element of $\tilde{P}^1$ with entry $t(\tilde{\alpha})\tilde{\lambda}$ in the summand of $\tilde{P}^1$

corresponding to $t(\tilde{\alpha})$ and 0 otherwise. Then $\tilde{d}^1(x)$ has entry $\tilde{\alpha}\tilde{\lambda}$ in the summand of $\tilde{P}^0$ corresponding to $s(\tilde{\alpha})$ and 0 otherwise. Now $\tilde{d}^0$ is the canonical surjection, so $\tilde{d}^0 \circ \tilde{d}^1(x) = \tilde{\alpha}\tilde{\lambda} + \tilde{\mathfrak{r}} = 0$. Hence $\tilde{d}^0 \circ \tilde{d}^1 = 0$.

Let $n = 1$. Let $\tilde{g}^2_i \in \tilde{g}^2$ and write $\tilde{g}^2_i = \sum_j \tilde{\alpha}_j \tilde{\eta}_j$ with $\tilde{\alpha}_j$ an arrow in $\tilde{\mathcal{Q}}_1$ and $\tilde{\eta}_j \in K\tilde{\mathcal{Q}}$. Let y be an element of $\tilde{P}^2$ with entry $t(\tilde{g}^2_i)\tilde{\lambda}$ in the summand of $\tilde{P}^2$ corresponding to $t(\tilde{g}^2_i)$ and 0 otherwise. Then $\tilde{d}^2(y)$ has entry $t(\tilde{\alpha}_j)\tilde{\eta}_j\tilde{\lambda}$ in the summand of $\tilde{P}^1$ corresponding to $t(\tilde{\alpha}_j)$. Since $\tilde{g}^2_i$ is uniform, we have that all these arrows $\tilde{\alpha}_j$ start at the same vertex, namely $s(\tilde{g}^2_i)$. So $\tilde{d}^1 \circ \tilde{d}^2(y)$ has entry $\sum_j \tilde{\alpha}_j\tilde{\eta}_j\tilde{\lambda}$ in the summand of $\tilde{P}^0$ corresponding to $s(\tilde{g}^2_i)$. But $\sum_j \tilde{\alpha}_j\tilde{\eta}_j\tilde{\lambda} = \tilde{g}^2_i\tilde{\lambda} = 0$ in $\tilde{\Lambda}$. Hence $\tilde{d}^1 \circ \tilde{d}^2(y) = 0$. It follows that $\tilde{d}^1 \circ \tilde{d}^2 = 0$.

Let $n = 2$. Let $\tilde{g}^3_i \in \tilde{g}^3$ and $g^3_i \in g^3$ where $g^3_i = \sum_{j=1}^{m_2} g^2_j q_{j,i}$ with $q_{j,i} \in K\mathcal{Q}$ and where m_2 is the number of elements in g^2. Then $\tilde{g}^3_i = \theta^*(g^3_i) = \theta^*(\sum_{j=1}^{m_2} g^2_j q_{j,i}) = \sum_{j=1}^{m_2} \tilde{g}^2_j \theta(q_{j,i})$. For each $g^2_j \in g^2$, write $g^2_j = \sum_{k=1}^{r_1} \alpha_k \beta_{k,j}$ with α_k an arrow in $\mathcal{Q}_1$, $\beta_{k,j} \in K\mathcal{Q}$ and r_1 is the number of arrows in $\mathcal{Q}_1$. Then $\tilde{g}^2_j = \theta^*(g^2_j) = \theta^*(\sum_{k=1}^{r_1} \alpha_k\beta_{k,j}) = \sum_{k=1}^{r_1} \alpha_{k,1}\alpha_{k,2}\cdots\alpha_{k,A}\theta^*(\beta_{k,j})$ with $\alpha_{k,l}$ an arrow in $\tilde{\mathcal{Q}}_1$. Thus we can write $\tilde{g}^2_j = \sum_{k=1}^{r_1} \alpha_{k,1}\tilde{\eta}_{k,j}$ where $\tilde{\eta}_{k,j} = \alpha_{k,2}\cdots\alpha_{k,A}\theta^*(\beta_{k,j})$.

Let x be an element of $\tilde{P}^3$ with entry $t(\tilde{g}^3_i)\tilde{\lambda}$ in the summand of $\tilde{P}^3$ corresponding to $t(\tilde{g}^3_i)$ and 0 otherwise. Then $\tilde{d}^3(x) = (t(\tilde{g}^2_1)\theta(q_{1,i}), t(\tilde{g}^2_2)\theta(q_{2,i}), \ldots, t(\tilde{g}^2_{m_2})\theta(q_{m_2,i}))\tilde{\lambda}$ in $\tilde{P}^2$. Then $\tilde{d}^2 \circ \tilde{d}^3(x)$ has entry $\sum_{j=1}^{m_2} t(\alpha_{k,1})\tilde{\eta}_{k,j}\theta(q_{j,i})\tilde{\lambda}$ in the summand of $\tilde{P}^1$ corresponding to $t(\alpha_{k,1})$, for $k = 1, \ldots, r_1$, and 0 otherwise.

Now, let y be the element of P^3 with $t(g^3_i)$ in the summand of P^3 corresponding to $t(g^3_i)$ and 0 otherwise. Then $d^2 \circ d^3(y)$ has entry $\sum_{j=1}^{m_2} t(\alpha_k)\beta_{k,j}q_{j,i}$ in the summand of P^1 corresponding to $t(\alpha_k)$ for $k = 1, \ldots, r_1$, and 0 otherwise. But (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$, so $d^2 \circ d^3 = 0$. In particular, $d^2 \circ d^3(y) = 0$. Hence

$$\begin{aligned} \textstyle\sum_{j=1}^{m_2} t(\alpha_{k,1})\tilde{\eta}_{k,j}\theta(q_{j,i})\tilde{\lambda} &= \textstyle\sum_{j=1}^{m_2} t(\alpha_{k,1})\alpha_{k,2}\cdots\alpha_{k,A}t(\tilde{\alpha}_{k,A})\theta^*(\beta_{k,j})\theta(q_{j,i})\tilde{\lambda} \\ &= t(\alpha_{k,1})\alpha_{k,2}\cdots\alpha_{k,A}t(\alpha_{k,A})\theta^*(\textstyle\sum_{j=1}^{m_2} \beta_{k,j}q_{j,i})\tilde{\lambda} \\ &= 0 \end{aligned}$$

for $k = 1, \ldots, r_1$. So $\tilde{d}^2 \circ \tilde{d}^3(x) = 0$ and it follows that $\tilde{d}^2 \circ \tilde{d}^3 = 0$.

Let $n \geq 3$. Let y be an element of $\tilde{P}^{n+1}$ with entry $t(\tilde{g}_i^{n+1})\tilde{\lambda}$ in the summand of $\tilde{P}^{n+1}$ corresponding to $t(\tilde{g}_i^{n+1})$ and 0 otherwise. Then $\tilde{d}^{n+1}(y) = \tilde{d}^{n+1}(t(\tilde{g}_i^{n+1}))\tilde{\lambda} = \theta_n(d^{n+1}(t(g_i^{n+1})))\tilde{\lambda}$. So $\tilde{d}^n \circ \tilde{d}^{n+1}(y) = \tilde{d}^n \theta_n(d^{n+1}(t(g_i^{n+1})))\tilde{\lambda} = \theta_{n-1} d^n(d^{n+1}(t(g_i^{n+1})))\tilde{\lambda}$ using Proposition 8.12. But (P^n, d^n) is a resolution of $\Lambda/\mathfrak{r}$, so $d^n \circ d^{n+1} = 0$. Hence $\tilde{d}^n \circ \tilde{d}^{n+1}(y) = 0$. It follows that $\tilde{d}^n \circ \tilde{d}^{n+1} = 0$.

Hence, for all $n \geq 0, \tilde{d}^n \circ \tilde{d}^{n+1} = 0$ and $(\tilde{P}^n, \tilde{d}^n)$ is a complex. $\square$

We are now able to show that $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module.

Theorem 8.14. *Let $\tilde{P}^n$ and $\tilde{d}^n$ be as given above. Then $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module.*

Proof. From Proposition 8.13 we have that $(\tilde{P}^n, \tilde{d}^n)$ is a complex, so, for $(\tilde{P}^n, \tilde{d}^n)$ to be a projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ it remains to show $\operatorname{Ker} \tilde{d}^n \subseteq \operatorname{Im} \tilde{d}^{n+1}$.

In the cases $n = 0$ and $n = 1$, we use [20] to see that the complex is exact at $\tilde{P}^0$ and $\tilde{P}^1$, since $\tilde{g}^0$ is the set of vertices of $\tilde{\mathcal{Q}}$, $\tilde{g}^1$ is the set of arrows of $\tilde{\mathcal{Q}}$ and $\tilde{g}^2$ is a minimal set of uniform elements which generate the ideal $\tilde{I}$.

Let $n = 2$. Let $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$, so $\tilde{x}$ is an element of $\tilde{P}^2$ with entry $t(\tilde{g}_i^2)\tilde{\lambda}_i$ in the summand of $\tilde{P}^2$ corresponding to $t(\tilde{g}_i^2)$, that is, $\tilde{x} = (t(\tilde{g}_1^2)\tilde{\lambda}_1, t(\tilde{g}_2^2)\tilde{\lambda}_2, \ldots, t(\tilde{g}_{m_2}^2)\tilde{\lambda}_{m_2})$, where m_2 is the number of elements in the set $\tilde{g}^2$. We recall that $g_i^2 = \sum_{j=1}^{r_1} \alpha_j \beta_{j,i}$ where r_1 is the number of arrows in $\mathcal{Q}_1$, and $\tilde{g}_i^2 = \theta^*(g_i^2) = \theta^*(\sum_{j=1}^{r_1} \alpha_j \beta_{j,i}) = \sum_{j=1}^{r_1} \alpha_{j,1} \cdots \alpha_{j,A} \theta^*(\beta_{j,i})$, so $\tilde{g}_i^2 = \sum_{j=1}^{r_1} \alpha_{j,1} \tilde{\eta}_{j,i}$ where $\tilde{\eta}_{j,i} = \alpha_{j,2} \cdots \alpha_{j,A} \theta(\beta_{j,i})$.

Assume first that for each i we have $\tilde{\lambda}_i = \theta(\lambda_i)$ for some $\lambda_i \in \Lambda$. Then $\tilde{x} = \theta_2(x)$, where $x = (t(g_1^2)\lambda_1, t(g_2^2)\lambda_2, \ldots, t(g_{m_2}^2)\lambda_{m_2}) \in P^2$. Since $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$, we have $0 = \tilde{d}^2(\tilde{x})$ and $\tilde{d}^2(\tilde{x})$ has entry $\sum_{i=1}^{m_2} \tilde{\eta}_{j,i}\tilde{\lambda}_i$ in the summand of $\tilde{P}^1$ corresponding to $t(\alpha_{j,1})$. Now $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$ so, for each $j = 1, \ldots, r_1$, the entry $\sum_{i=1}^{m_2} \tilde{\eta}_{j,i}\tilde{\lambda}_i = \sum_{i=1}^{m_2} \alpha_{j,2} \cdots \alpha_{j,A} \theta(\beta_{j,i})\theta(\lambda_i) = \alpha_{j,2} \cdots \alpha_{j,A} \sum_{i=1}^{m_2} \theta(\beta_{j,i}\lambda_i) = 0$. The path $\alpha_{j,2} \cdots \alpha_{j,A} \in \tilde{\Lambda}$ has no proper subpath $\tilde{q}$ such that $\tilde{q} = \theta(q)$ for $q \in \Lambda$, and $t(\alpha_{j,A}) = t(\alpha_j)$. Since the ideal $\tilde{I}$ is generated by uniform elements $\tilde{g}_i^2$, which all begin at a vertex in $\mathcal{Q}_0$, we must have $\sum_{i=1}^{m_2} \theta(t(\alpha_j)\beta_{j,i}\lambda_i) = 0$. As θ is a monomorphism, we have $\sum_{i=1}^{m_2} \beta_{j,i}\lambda_i = 0$, so $x \in \operatorname{Ker} d^2$. As (P^n, d^n) is a resolution, then $\operatorname{Ker} d^2 = \operatorname{Im} d^3$, so $x = d^3(y)$ for some $y \in P^3$. We now have $\tilde{x} = \theta_2(x) = \theta_2(d^3(y)) = \tilde{d}^3(\theta_3(y))$, by Proposition 8.12. Therefore $\tilde{x} \in \operatorname{Im} \tilde{d}^3$ as required.

Now suppose $\tilde{x} = (t(\tilde{g}_1^2)\tilde{\lambda}_1, t(\tilde{g}_2^2)\tilde{\lambda}_2, \ldots, t(\tilde{g}_{m_2}^2)\tilde{\lambda}_{m_2})$ is an arbitrary element in $\operatorname{Ker} \tilde{d}^2$. We can write $\tilde{x} = \tilde{x}e + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \tilde{x}w$, where $e = \sum_{v \in \mathcal{Q}_0} v$ and $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Since $\operatorname{Ker} \tilde{d}^2$ is a right $\tilde{\Lambda}$-module, then $\tilde{x}e \in \operatorname{Ker} \tilde{d}^2$. From the construction of $K\tilde{\mathcal{Q}}$, recall that if an element $\tilde{p} \in \tilde{\Lambda}$ has $s(\tilde{p}) \in \mathcal{Q}_0$ and $t(\tilde{p}) \in \mathcal{Q}_0$ then $\tilde{p} = \theta(p)$ for some $p \in K\mathcal{Q}$. Thus we may write $t(\tilde{g}_i^2)\tilde{\lambda}_i e = t(\tilde{g}_i^2)\theta(\lambda_i)$ for some $\lambda_i \in \Lambda$ and for all $i = 1, \ldots, m_2$. So

$$\tilde{x}e = (t(\tilde{g}_1^2)\theta(\lambda_1), t(\tilde{g}_2^2)\theta(\lambda_2), \ldots, t(\tilde{g}_{m_2}^2)\theta(\lambda_{m_2})) = \theta_2(z)$$

where $z = (t(g_1^2)\lambda_1, t(g_2^2)\lambda_2, \ldots, t(g_{m_2}^2)\lambda_{m_2}) \in P^2$. The above argument now gives that $\tilde{x}e \in \operatorname{Im} \tilde{d}^3$.

Now consider $\tilde{x}w$, where $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Then

$$\tilde{x}w = (t(\tilde{g}_1^2)\tilde{\lambda}_1 w, t(\tilde{g}_2^2)\tilde{\lambda}_2 w, \ldots, t(\tilde{g}_{m_2}^2)\tilde{\lambda}_{m_2} w).$$

By construction of the quiver $\tilde{\mathcal{Q}}$, for each $i = 1, \ldots, m_2$, the element $t(\tilde{g}_i^2)\tilde{\lambda}_i w = t(\tilde{g}_i^2)\theta(\mu_i)\tilde{p}_w$ where $\mu_i \in \Lambda$, and $\tilde{p}_w$ is the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at a vertex in $\mathcal{Q}_0$ and ends at w. Note that $\tilde{p}_w$ contains no proper subpath $\tilde{q}$ such that $\tilde{q} = \theta(q)$ for some $q \in \Lambda$, and that $l(\tilde{p}_w) < A$. Hence

$$\tilde{x}w = (t(\tilde{g}_1^2)\theta(\mu_1)\tilde{p}_w, \ldots, t(\tilde{g}_{m_2}^2)\theta(\mu_{m_2})\tilde{p}_w) = (t(\tilde{g}_1^2)\theta(\mu_1), \ldots, t(\tilde{g}_{m_2}^2)\theta(\mu_{m_2}))\tilde{p}_w.$$

The element $\tilde{d}^2(\tilde{x}w)$ has entry $t(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\sum_{i=1}^{m_2}\theta(\beta_{j,i})\theta(\mu_i)\tilde{p}_w$ in the summand of $\tilde{P}^1$ corresponding to $t(\alpha_{j,1})$, and 0 otherwise. But $\tilde{x}w \in \operatorname{Ker} \tilde{d}^2$. Hence $\tilde{d}^2(\tilde{x}w) = 0$ gives that $\alpha_{j,2}\cdots\alpha_{j,A}\left(\sum_{i=1}^{m_2}\theta(\beta_{j,i})\theta(\mu_i)\right)\tilde{p}_w = 0$ for all $j = 1, \ldots, r_1$. Since the ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$ is generated by uniform elements $\theta(g_1^2), \ldots, \theta(g_{m_2}^2)$, which all start and end at a vertex in $\mathcal{Q}_0$, we must have that $\sum_{i=1}^{m_2} t(\alpha_{j,A})\theta(\beta_{j,i})\theta(\mu_i) = 0$, for all $j = 1, \ldots, r_1$. Now θ is a monomorphism, so it follows that $\sum_{i=1}^{m_2} \beta_{j,i}\mu_i = 0$ for $j = 1, \ldots, r_1$. Let $\xi_w = (t(g_1^2)\mu_1, t(g_2^2)\mu_2, \ldots, t(g_{m_2}^2)\mu_{m_2}) \in P^2$. Then $\tilde{x}w = \theta_2(\xi_w)\tilde{p}_w$, and $d^2(\xi_w) = 0$. Hence $\xi_w \in \operatorname{Ker} d^2$. Since (P^n, d^n) is a resolution, $\operatorname{Ker} d^2 = \operatorname{Im} d^3$, so $\xi_w \in \operatorname{Im} d^3$ and thus $\xi_w = d^3(y_w)$ for some $y_w \in P^3$. Then $\theta_2(\xi_w) = \theta_2(d^3(y_w)) = \tilde{d}^3\theta_3(y_w)$ by Proposition 8.12. So $\theta_2(\xi_w) \in \operatorname{Im} \tilde{d}^3$. But $\tilde{x}w = \theta_2(\xi_w)\tilde{p}_w$, so $\tilde{x}w \in \operatorname{Im} \tilde{d}^3$ for all $w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0$.

We have shown that $\tilde{x}e \in \operatorname{Im} \tilde{d}^3$ and $\tilde{x}w \in \operatorname{Im} \tilde{d}^3$, for all $w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0$. Hence $\tilde{x} \in \operatorname{Im} \tilde{d}^3$ and $\operatorname{Ker} \tilde{d}^2 \subseteq \operatorname{Im} \tilde{d}^3$ as required.

Let $n \geq 3$. Let $\tilde{x} \in \operatorname{Ker} \tilde{d}^n$, so $\tilde{x}$ is an element of $\tilde{P}^n$ with entry $t(\tilde{g}_i^n)\tilde{\lambda}_i$ in the summand of $\tilde{P}^n$ corresponding to $t(\tilde{g}_i^n)$, that is, $\tilde{x} = (t(\tilde{g}_1^n)\tilde{\lambda}_1, \ldots, t(\tilde{g}_{m_n}^n)\tilde{\lambda}_{m_n})$, where m_n is the number of elements in the set $\tilde{g}^n$.

Assume first that for each i we have $\tilde{\lambda}_i = \theta(\lambda_i)$ for some $\lambda_i \in \Lambda$. Then $\tilde{x} = \theta_n(x)$ where $x = (t(g_1^n)\lambda_1, \ldots, t(g_{m_n}^n)\lambda_{m_n}) \in P^n$. Since $\tilde{x} \in \operatorname{Ker} \tilde{d}^n$, we have $0 = \tilde{d}^n(\tilde{x}) = \tilde{d}^n(\theta_n(x)) = \theta_{n-1}(d^n(x))$, by Proposition 8.12. Now, θ_{n-1} is $1-1$ so $d^n(x) = 0$. Hence $x \in \operatorname{Ker} d^n$. As (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, then $x \in \operatorname{Im} d^{n+1}$, so $x = d^{n+1}(y)$ for some $y \in P^{n+1}$. Then $\tilde{x} = \theta_n(x) = \theta_n(d^{n+1}(y)) = \tilde{d}^{n+1}(\theta_{n+1}(y))$ by Proposition 8.12. Therefore $\tilde{x} \in \operatorname{Im} \tilde{d}^{n+1}$ as required.

Now suppose that $\tilde{x} = (t(\tilde{g}_1^n)\tilde{\lambda}_1, \ldots, t(\tilde{g}_{m_n}^n)\tilde{\lambda}_{m_n}) \in \operatorname{Ker} \tilde{d}^n$. Again, we can write $\tilde{x} = \tilde{x}e + \sum_{w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0} \tilde{x}w$, where $e = \sum_{v \in \mathcal{Q}_0} v$ is an element in $\tilde{\Lambda}$. Since $\operatorname{Ker} \tilde{d}^n$ is a right $\tilde{\Lambda}$-module, $\tilde{x}e \in \operatorname{Ker} \tilde{d}^n$. From the construction of $K\tilde{\mathcal{Q}}$, if $\tilde{p}$ is an element of $K\tilde{\mathcal{Q}}$ with $s(\tilde{p}) \in \mathcal{Q}_0$ and $t(\tilde{p}) \in \mathcal{Q}_0$ then $\tilde{p} = \theta(p)$ for some $p \in K\mathcal{Q}$. Thus we may write $t(\tilde{g}_i^n)\tilde{\lambda}_i e = t(\tilde{g}_i^n)\theta(\lambda_i)$ for some $\lambda_i \in \Lambda$ and for all $i = 1, \ldots, m_n$. So $\tilde{x}e = (t(\tilde{g}_1^n)\theta(\lambda_1), \ldots, t(\tilde{g}_{m_n}^n)\theta(\lambda_{m_n})) = \theta_n(z)$ where $z = (t(g_1^n)\lambda_1, \ldots, t(g_{m_n}^n)\lambda_{m_n}) \in P^n$. Our argument above now gives that $\tilde{x}e \in \operatorname{Im} \tilde{d}^{n+1}$.

We now show that $\tilde{x}w \in \operatorname{Im} \tilde{d}^{n+1}$ for all $w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0$. Since $\tilde{x} \in \operatorname{Ker} \tilde{d}^n$, we have $\tilde{x}w \in \operatorname{Ker} \tilde{d}^n$ for all $w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0$ and $\tilde{x}w = (t(\tilde{g}_1^n)\tilde{\lambda}_1 w, \ldots, t(\tilde{g}_{m_n}^n)\tilde{\lambda}_{m_n} w)$. By construction of the quiver $\tilde{\mathcal{Q}}$, for each $i = 1, \ldots, m_n$, the element $t(\tilde{g}_i^n)\tilde{\lambda}_i w =$

$t(\tilde{g}_i^n)\theta(\mu_i)\tilde{p}_w$ where $\mu_i \in \Lambda$, and $\tilde{p}_w$ is the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at a vertex in $\mathcal{Q}_0$ and ends at w, and $\tilde{p}_w$ contains no proper subpath $\tilde{q}$ such that $\tilde{q} = \theta(q)$ for some $q \in \Lambda$. Hence $\tilde{x}w = (t(\tilde{g}_1^n)\theta(\mu_1)\tilde{p}_w, \ldots, t(\tilde{g}_{m_n}^n)\theta(\mu_{m_n})\tilde{p}_w) = (t(\tilde{g}_1^n)\theta(\mu_1), \ldots, t(\tilde{g}_{m_n}^n)\theta(\mu_{m_n}))\tilde{p}_w$.

We may write $g_i^n = \sum_{j=1}^{m_{n-1}} g_j^{n-1} q_{j,i}$ so that $\tilde{g}_i^n = \sum_{j=1}^{m_{n-1}} \tilde{g}_j^{n-1}\theta^*(q_{j,i})$. Then the element $\tilde{d}^n(\tilde{x}w)$ has entry $\sum_{i=1}^{m_n} t(\tilde{g}_j^{n-1})\theta(q_{j,i})\theta(\mu_i)\tilde{p}_w$ in the component of $\tilde{P}^{n-1}$ corresponding to $t(\tilde{g}_j^{n-1})$. Hence $(\sum_{i=1}^{m_n} t(\tilde{g}_j^{n-1})\theta(q_{j,i})\theta(\mu_i))\tilde{p}_w = 0$ for all $j = 1, \ldots, m_{n-1}$. Since the ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$ is generated by uniform elements $\theta(g_1^2), \ldots, \theta(g_{m_2}^2)$, which all start and end at a vertex in $\mathcal{Q}_0$, we must have that $\sum_{i=1}^{m_n} t(\tilde{g}_j^{n-1})\theta(q_{j,i})\theta(\mu_i) = 0$, for all $j = 1, \ldots, m_{n-1}$. Hence the element $(t(\tilde{g}_1^n)\theta(\mu_1), \ldots, t(\tilde{g}_{m_n}^n)\theta(\mu_{m_n}))$ has image 0 under the map $\tilde{d}^n$. Let $\tilde{\xi}_w = (t(\tilde{g}_1^n)\theta(\mu_1), \ldots, t(\tilde{g}_{m_n}^n)\theta(\mu_{m_n}))$. Then $\tilde{d}^n(\tilde{\xi}_w) = 0$ and it follows that we have $\tilde{\xi}_w = \theta_n(\xi_w)$ where $\xi_w = (t(g_1^n)\mu_1, \ldots, t(g_{m_n}^n)\mu_{m_n}) \in P^n$. Again our previous argument gives that $\tilde{\xi}_w \in \operatorname{Im}\tilde{d}^{n+1}$. But $\tilde{x}w = \tilde{\xi}_w\tilde{p}_w$ so $\tilde{x}w \in \operatorname{Im}\tilde{d}^{n+1}$.

We have shown that if $\tilde{x} \in \operatorname{Ker}\tilde{d}^n$ then $\tilde{x}e$ and $\tilde{x}w$ are in $\operatorname{Im}\tilde{d}^{n+1}$ for all $w \in \tilde{\mathcal{Q}}_0 \backslash \mathcal{Q}_0$. Hence $\tilde{x} \in \operatorname{Im}\tilde{d}^{n+1}$ and $\operatorname{Ker}\tilde{d}^n \subseteq \operatorname{Im}\tilde{d}^{n+1}$ for all $n \geq 3$, as required.

Therefore, for all $n \geq 0$, we have $\operatorname{Ker}\tilde{d}^n = \operatorname{Im}\tilde{d}^{n+1}$. Hence $(\tilde{P}^n, \tilde{d}^n)$ is a projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module.

From the definition of the maps $\tilde{d}^n$ it is clear that $\operatorname{Im}\tilde{d}^n \subseteq \tilde{P}^{n-1}\tilde{\mathfrak{r}}$ for all $n \geq 0$, Hence the resolution $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module. $\square$

We finish this section with our main result, which shows that the algebra $\tilde{\Lambda}$ we have constructed is indeed a (D, A)-stacked algebra.

Theorem 8.15. *Let Λ be a d-Koszul algebra. Let $A \geq 1$ and set $D = dA$. With the above construction, the algebra $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ is a (D, A)-stacked algebra.*

Proof. Looking at the length of each $\tilde{g}_i^n$ it is clear that we have

$$l(\tilde{g}_i^n) = \delta(n) = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ \frac{n}{2}D & \text{if} \quad n = 2r, r \in \mathbb{N} \\ \frac{n-1}{2}D + A & \text{if} \quad n = 2r+1, r \in \mathbb{N}. \end{cases}$$

Hence $\tilde{\Lambda}$ is a (D, A)-stacked algebra. □

In the next chapter we look at the relationship between $E(\Lambda)$ and $E(\tilde{\Lambda})$.

9. Properties of the Ext Algebra under this Construction

We have shown, in Chapter 8, that we can construct a (D, A)-stacked algebra $\tilde{\Lambda}$ from a given d-Koszul algebra Λ, with $D = dA$. We also constructed a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ from a minimal projective resolution of $\Lambda/\mathfrak{r}$. In Chapter 6, we looked at some general properties of the Ext algebra of a (D, A)-stacked algebra. This begs the question, 'What is the relationship between $E(\Lambda)$ and $E(\tilde{\Lambda})$?'

Let $d \geq 2, A \geq 1$ and set $D = dA$. We assume throughout this section that Λ is a d-Koszul algebra, with $d \geq 2$ and that $\tilde{\Lambda}$ is the related (D, A)-stacked algebra using the construction of Chapter 8.

We have seen in Chapter 3 that $\mathrm{Ext}^m_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \cong \mathrm{Hom}_\Lambda(P^m, \Lambda_0)$, so we can take a basis f^m of $\mathrm{Hom}_\Lambda(P^m, \Lambda_0)$ for each $m \geq 0$.

Definition 9.1. Let f^m_i be the Λ-module homomorphism, $P^m \to \Lambda_0$, given by

$$t(g^m_j) \mapsto \begin{cases} t(g^m_i) + \mathfrak{r} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

We set $f^m = \{f^m_i\}$ so that $|f^m| = |g^m|$.

Let $\tilde{f}^m_i$ be the $\tilde{\Lambda}$-module homomorphism, $\tilde{P}^m \to \tilde{\Lambda}_0$, given by

$$t(\tilde{g}^m_j) \mapsto \begin{cases} t(\tilde{g}^m_i) + \tilde{\mathfrak{r}} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

We set $\tilde{f}^m = \{\tilde{f}^m_i\}$ so that $|\tilde{f}^m| = |\tilde{g}^m|$.

Recall from Chapter 8, that $|g^m| = |\tilde{g}^m|$ for $m \geq 2$. So $|f^m| = |g^m| = |\tilde{g}^m| = |\tilde{f}^m|$ for $m \geq 2$.

Remark. With the maps $\theta_m, \bar{\theta}$ from Chapter 8 and with $m \geq 2$ we have

$$\tilde{f}^m_i(t(\tilde{g}^m_j)) = \begin{cases} \bar{\theta}(t(g^m_i) + \mathfrak{r}) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\tilde{f}_i^m\theta_m(t(g_j^m)) = \bar{\theta}f_i^m(t(g_j^m))$, so the diagram below commutes as right Λ-modules, for $m \geq 2$.

$$\begin{tikzcd} P^m \arrow[r, "f_i^m"] \arrow[d, "\theta_m"'] & \Lambda_0 \arrow[d, "\bar{\theta}"] \\ \tilde{P}^m \arrow[r, "\tilde{f}_i^m"'] & \tilde{\Lambda}_0 \end{tikzcd}$$

Definition 9.2. Let Λ be a d-Koszul algebra, with $d \geq 2$ and let $\tilde{\Lambda}$ be the related (D, A)-stacked algebra, with $A \geq 1$, $D = dA$. Let $\mathrm{Ext}_\Lambda^{\geqslant 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ $= \bigoplus_{m\geqslant 2} \mathrm{Ext}_\Lambda^m(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ and let $\mathrm{Ext}_{\tilde{\Lambda}}^{\geqslant 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}}) = \bigoplus_{m\geqslant 2} \mathrm{Ext}_{\tilde{\Lambda}}^m(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$.

We define a K-module homomorphism $\Psi : \mathrm{Ext}_\Lambda^{\geqslant 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \mathrm{Ext}_{\tilde{\Lambda}}^{\geqslant 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ as follows;

$$\Psi(f_i^m) = \tilde{f}_i^m \text{ for } m \geq 2.$$

The aim of this chapter is to show that Ψ is a K-algebra homomorphism, and we do this in Theorem 9.15.

We remind the reader that the product structure in $E(\tilde{\Lambda})$ is given by the Yoneda product, and was discussed in Chapter 3. We will show that if we have all the liftings required for elements of $E(\Lambda)$, then we can use these to give the liftings for elements of $E(\tilde{\Lambda})$.

We start by looking at liftings of f_i^m in $E(\Lambda)$.

Definition 9.3. Let Λ be a d-Koszul algebra, with $d \geq 2$. Let $f_i^m \in \mathrm{Ext}_\Lambda^m(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, with $m \geq 2$. Define $\mathcal{L}^0 f_i^m$ to be the right Λ-module homomorphism as follows;

$$\mathcal{L}^0 f_i^m : P^m \to P^0,\ t(g_j^m) \mapsto \begin{cases} t(g_i^m) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 9.4. *The lifting $\mathcal{L}^0 f_i^m$, as defined above, is a lifting of f_i^m, for $m \geq 2$.*

Proof. To show $\mathcal{L}^0 f_i^m$ is a lifting of f_i^m, we need to show that $d^0 \circ \mathcal{L}^0 f_i^m(t(g_j^m)) = f_i^m(t(g_j^m))$, for all j, that is, the following diagram commutes:

$$\begin{tikzcd} P^m \arrow[d, "\mathcal{L}^0 f_i^m"'] \arrow[dr, "f_i^m"] & \\ P^0 \arrow[r, "d^0"'] & \Lambda_0 \end{tikzcd}$$

The map $d^0 : P^0 \to \Lambda/\mathfrak{r}$, is the canonical surjection. Thus,

$$d^0 \circ \mathcal{L}^0 f_i^m(t(g_j^m)\lambda) = \begin{cases} d^0(t(g_i^m)\lambda) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} t(g_i^m)\lambda + \mathfrak{r} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

So we have $d^0 \circ \mathcal{L}^0 f_i^m(t(g_j^m)\lambda) = f_i^m(t(g_j^m)\lambda)$. Hence $d^0 \circ \mathcal{L}^0 f_i^m = f_i^m$ and $\mathcal{L}^0 f_i^m$ is a lifting of f_i^m. $\square$

We can now look at the first lifting $\mathcal{L}^1 f_i^m$ of f_i^m.

Definition 9.5. Fix i and consider the element $f_i^m \in \mathrm{Ext}_\Lambda^m(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ where $m \geq 2$. To define a lifting we need to consider the elements of g^{m+1} in which the element g_i^m occurs. For each $g_j^{m+1} \in g^{m+1}$, write $g_j^{m+1} = \sum_{k=1}^{r_m} g_k^m q_{j,k} = g_i^m q_{j,i} + \sum_{k \neq i} g_k^m q_{j,k}$. Now $q_{j,i}$ is in the arrow ideal of $K\mathcal{Q}$, so write $q_{j,i} = \sum_\alpha \alpha\gamma_{j,i,\alpha}$ where each α is an arrow in $\mathcal{Q}$.

Define the map $\mathcal{L}^1 f_i^m : P^{m+1} \to P^1$ to be the right Λ-module homomorphism given by $t(g_j^{m+1}) \mapsto \sum_\alpha t(\alpha)\gamma_{j,i,\alpha} t(g_j^{m+1})$ where each $t(\alpha)\gamma_{j,i,\alpha}t(g_j^{m+1})$ is in the $t(\alpha)$ component of P^1.

Proposition 9.6. *With the definition above, $\mathcal{L}^1 f_i^m$ is a lifting of f_i^m, for $m \geq 2$.*

Proof. To show $\mathcal{L}^1 f_i^m$ is a lifting of f_i^m we need to show that the diagram below commutes as right Λ-module modules.

$$\begin{tikzcd} P^{m+1} \arrow[r, "d^{m+1}"] \arrow[d, "\mathcal{L}^1 f_i^m"'] & P^m \arrow[d, "\mathcal{L}^0 f_i^m"] \\ P^1 \arrow[r, "d^1"'] & P^0 \end{tikzcd}$$

Recall that $d^{m+1} : t(g_j^{m+1})\lambda \mapsto \sum_k t(g_k^m)q_{j,k}t(g_j^{m+1})\lambda$. Then the entry $\mathcal{L}^0 f_i^m \circ d^{m+1}(t(g_j^{m+1})\lambda)$ in the $t(g_k^m)$ component of P^0 is $\begin{cases} t(g_k^m)q_{j,k}\lambda & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$

We know $d^1 : P^1 \to P^0,\ t(\alpha)\lambda \mapsto \alpha\lambda$, where $\alpha\lambda$ is in the $s(\alpha)$ component of P^0. So $d^1 \circ \mathcal{L}^1 f_i^m(t(g_j^{m+1})\lambda) = \sum_\alpha \alpha\gamma_{j,i,\alpha}\lambda$ with $\alpha\gamma_{j,i,\alpha}$ in the $s(\alpha)$ component of P^0.

Since $\sum_\alpha \alpha\gamma_{j,i,\alpha} = q_{j,i}$ from Definition 9.5 and $q_{j,i}$ is a uniform element, then each α in the element $\sum_\alpha \alpha\gamma_{j,i,\alpha}$ will start at the same vertex, namely $s(q_{j,i}) = t(g_i^m)$. So we have $d^1 \circ \mathcal{L}^1 f_i^m(t(g_j^{m+1})\lambda)$ has entry $t(g_i^m)q_{j,i}\lambda$ in the $t(g_i^m)$ component of P^0 and 0 otherwise.

Therefore we have $\mathcal{L}^0 f_i^m \circ\ d^{m+1}(t(g_j^{m+1})) = d^1 \circ \mathcal{L}^1 f_i^m(t(g_j^{m+1}))$ so that $\mathcal{L}^0 f_i^m \circ d^{m+1} = d^1 \circ \mathcal{L}^1 f_i^m$. Hence $\mathcal{L}^1 f_i^m$ is a lifting for f_i^m. $\square$

Definition 9.7. Let $m \geq 2, n \geq 2$, and let $f_i^m \in \operatorname{Ext}_\Lambda^m(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. Since we know that liftings of f_i^m exist, we can fix liftings $\mathcal{L}^n f_i^m$. We write $\mathcal{L}^n f_i^m : P^{m+n} \to P^n$ as the right Λ-module homomorphism such that $\mathcal{L}^n f_i^m(t(g_j^{m+n}))$ has entry $t(g_k^n)\sigma_{i,j,k}^n t(g_j^{m+n})$ in the $t(g_k^n)$ component of P^n, for some $\sigma_{i,j,k}^n \in \Lambda$ and for all j, k. Since this is a lifting, we have $d^n \circ \mathcal{L}^n f_i^m = \mathcal{L}^{n-1} f_i^m \circ d^{m+n}$.

Although liftings are not unique, we have now fixed a lifting $\mathcal{L}^n f_i^m$ of f_i^m for all $n \geq 0$.

We will now look at the liftings of the $\tilde{f}_i^m$.

Definition 9.8. Let Λ be a d-Koszul algebra, with $d \geq 2$ and let $\tilde{\Lambda}$ be the related (D, A)-stacked algebra, with $D = dA$ and $A > 1$. Let $\tilde{f}_i^m \in \operatorname{Ext}_{\tilde{\Lambda}}^m(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$, with $m \geq 2$. Let $\mathcal{L}^0 \tilde{f}_i^m$ be the $\tilde{\Lambda}$-module homomorphism defined as follows:

$$\mathcal{L}^0 \tilde{f}_i^m : \tilde{P}^m \to \tilde{P}^0,\ t(\tilde{g}_j^m) \mapsto \theta_0 \mathcal{L}^0 f_i^m(t(g_j^m))$$

for all j, since $t(\tilde{g}_j^m) = \theta_0(t(g_j^m))$ for $m \geq 2$, and where $\mathcal{L}^0 f_i^m$ is the lifting of Definition 9.3.

Proposition 9.9. *Let $\mathcal{L}^0\tilde{f}_i^m : \tilde{P}^m \to \tilde{P}^0$ be the map as given above. Then $\mathcal{L}^0\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$ for all i and all $m \geq 2$.*

Proof. We need to show that $\tilde{d}^0 \circ \mathcal{L}^0\tilde{f}_i^m(t(\tilde{g}_j^m)) = \tilde{f}_i^m(t(\tilde{g}_j^m))$ for all j. We have

$$\begin{aligned}
\tilde{d}^0 \circ \mathcal{L}^0\tilde{f}_i^m(t(\tilde{g}_j^m)) &= \tilde{d}^0 \circ \theta_0(\mathcal{L}^0 f_i^m(t(g_j^m))) \\
&= \bar{\theta} \circ d^0(\mathcal{L}^0 f_i^m(t(g_j^m))), \text{ by Proposition 8.12,} \\
&= \bar{\theta}(d^0 \circ \mathcal{L}^0 f_i^m(t(g_j^m))) \\
&= \bar{\theta}(f_i^m(t(g_j^m))), \text{ by Proposition 9.4} \\
&= \tilde{f}_i^m(t(\tilde{g}_j^m)).
\end{aligned}$$

Hence $\mathcal{L}^0\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$. □

Now that we have $\mathcal{L}^0\tilde{f}_i^m$, we can look at the next lifting, $\mathcal{L}^1\tilde{f}_i^m$. We use the notation of Definition 9.5.

Definition 9.10. Fix i and consider the element $\tilde{f}_i^m \in \operatorname{Ext}_{\tilde{\Lambda}}^m(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ for $m \geq 2$. Let $\tilde{g}_j^{m+1} \in \tilde{g}^{m+1}$; since $m \geq 2$, $\tilde{g}_j^{m+1} = \theta^*(g_j^{m+1})$. Write $g_j^{m+1} = \sum_{k=1}^{r_m} g_k^m q_{j,k}$. Then $\tilde{g}_j^{m+1} = \theta^*(\sum_{k=1}^{r_m} g_k^m q_{j,k}) = \sum_{k=1}^{r_m} \tilde{g}_k^m \theta^*(q_{j,k})$. Let $\theta^*(q_{j,k}) = \tilde{q}_{j,k}$, so that $\tilde{g}_j^{m+1} = \sum_{k=1}^{r_m} \tilde{g}_k^m \tilde{q}_{j,k} = \tilde{g}_i^m \tilde{q}_{j,i} + \sum_{k \neq i} \tilde{g}_k^m \tilde{q}_{j,k}$.

Using the notation of Definition 9.5, we have

$$\begin{aligned}
\tilde{q}_{j,i} &= \theta(q_{j,i}) \\
&= \theta(\textstyle\sum_\alpha \alpha\gamma_{j,i,\alpha}), \text{ where each } \alpha \text{ is an arrow in } \mathcal{Q}, \\
&= \textstyle\sum_\alpha \theta(s(\alpha))\theta(\alpha)\theta(\gamma_{j,i,\alpha}) \\
&= \textstyle\sum_\alpha \theta(s(\alpha))\tilde{\alpha}_1\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta(\gamma_{j,i,\alpha}),
\end{aligned}$$

where for each $\alpha \in \mathcal{Q}_1$, the image $\theta(\alpha)$ is the path, $\tilde{\alpha}_1\tilde{\alpha}_2 \cdots \tilde{\alpha}_A$, of A arrows in $\tilde{\mathcal{Q}}$.

Define $\mathcal{L}^1\tilde{f}_i^m$ to be the Λ-module homomorphism $\mathcal{L}^1\tilde{f}_i^m : \tilde{P}^{m+1} \to \tilde{P}^1$ where $\mathcal{L}^1\tilde{f}_i^m(t(\tilde{g}_j^{m+1}))$ has entry $t(\tilde{\alpha}_1)\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta(\gamma_{j,i,\alpha})$ in the $t(\tilde{\alpha}_1)$ component of $\tilde{P}^1$, and 0 otherwise.

Proposition 9.11. *With the above notation $\mathcal{L}^1\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$, for all i and all $m \geq 2$.*

Proof. To show $\mathcal{L}^1\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$ we need to show that the diagram below commutes as right $\tilde{\Lambda}$-modules.

$$\begin{tikzcd} \tilde{P}^{m+1} \arrow[r, "\tilde{d}^{m+1}"] \arrow[d, "\mathcal{L}^1\tilde{f}_i^m"'] & \tilde{P}^m \arrow[d, "\mathcal{L}^0\tilde{f}_i^m"] \\ \tilde{P}^1 \arrow[r, "\tilde{d}^1"'] & \tilde{P}^0 \end{tikzcd}$$

We first consider $\mathcal{L}^0\tilde{f}_i^m \circ \tilde{d}^{m+1}(t(\tilde{g}_j^{m+1}))$. Keeping the above notation and from Definition 8.11, $\tilde{d}^{m+1}(t(\tilde{g}_j^{m+1}))$ has entry $t(\tilde{g}_k^m)\theta(q_{j,k})$ in the summand of $\tilde{P}^m$ corresponding to $t(\tilde{g}_k^m)$. Using Definition 9.8, we can see that the entry of $\mathcal{L}^0\tilde{f}_i^m \circ \tilde{d}^{m+1}(t(\tilde{g}_j^{m+1}))$ in the summand of $\tilde{P}^0$ corresponding to $t(\tilde{g}_i^m)$ is $t(\tilde{g}_i^m)\tilde{q}_{j,i}$ and is 0 otherwise.

We now consider $\tilde{d}^1 \circ \mathcal{L}^1\tilde{f}_i^m(t(\tilde{g}_j^{m+1}))$. We know $\tilde{d}^1 : \tilde{P}^1 \to \tilde{P}^0,\ t(\tilde{\alpha}) \mapsto \tilde{\alpha}$, for all $\tilde{\alpha} \in \tilde{\mathcal{Q}}_1$, where $\tilde{\alpha}$ is in the $s(\tilde{\alpha})$ component of $\tilde{P}^0$. Then $\tilde{d}^1 \circ \mathcal{L}^1\tilde{f}_i^m(t(\tilde{g}_j^{m+1}))$ has entry $\tilde{\alpha}_1\tilde{\alpha}_2\cdots\tilde{\alpha}_A\theta(\gamma_{j,i,\alpha}) = \theta(\alpha\gamma_{j,i,\alpha})$ in the $s(\tilde{\alpha}_1)$ component of $\tilde{P}^0$ and 0 otherwise. Now, $\tilde{g}_i^m$ is uniform, so $t(\tilde{g}_i^m) = s(\tilde{q}_{j,i,})$. But $\tilde{q}_{j,i} = \sum_\alpha \theta(\alpha\gamma_{j,i,\alpha})$; so each $\theta(\alpha\gamma_{j,i,\alpha})$ lies in the same component of $\tilde{P}^0$, namely in the $t(\tilde{g}_i^m)$ component of $\tilde{P}^0$. Hence $\tilde{d}^1 \circ \mathcal{L}^1\tilde{f}_i^m(t(\tilde{g}_j^{m+1}))$ has entry $\sum_\alpha \theta(\alpha\gamma_{j,i,\alpha}) = \tilde{q}_{j,i}$ in the $t(\tilde{g}_i^m)$ component of $\tilde{P}^0$ and 0 otherwise. Thus $\mathcal{L}^0\tilde{f}_i^m \circ \tilde{d}^{m+1}(t(\tilde{g}_j^{m+1})) = \tilde{d}^1 \circ \mathcal{L}^1\tilde{f}_i^m(t(\tilde{g}_j^{m+1}))$.
Hence $\mathcal{L}^0\tilde{f}_i^m \circ \tilde{d}^{m+1} = \tilde{d}^1 \circ \mathcal{L}^1\tilde{f}_i^m$ and $\mathcal{L}^1\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$. $\square$

Definition 9.12. Let Λ be a d-Koszul algebra, with $d \geq 2$ and let $\tilde{\Lambda}$ be the related (D,A)-stacked algebra, with $D = dA$ and $A > 1$. Let $m,n \geq 2$, let $\tilde{f}_i^m \in \operatorname{Ext}_{\tilde{\Lambda}}^m(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ and let $\mathcal{L}^n f_i^m$ be the lifting of Definition 9.7. We define $\mathcal{L}^n\tilde{f}_i^m$ to be the right $\tilde{\Lambda}$-module homomorphism such that:

$$\mathcal{L}^n\tilde{f}_i^m : \tilde{P}^{m+n} \to \tilde{P}^n,\ t(\tilde{g}_j^{m+n}) \mapsto \theta_n\mathcal{L}^n f_i^m(t(g_j^{m+n}))$$

for all j.

In order to prove that $\mathcal{L}^n\tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$, we consider the cases $n = 2$ and $n \geq 3$ separately.

Proposition 9.13. *Let $m \geq 2$ and let $n = 2$. Let $\mathcal{L}^2 \tilde{f}_i^m : \tilde{P}^{m+2} \to \tilde{P}^2$ be the map as given above. Then $\mathcal{L}^2 \tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$.*

Proof. If $\mathcal{L}^2 \tilde{f}_i^m$ is a lifting then the following diagram commutes:

$$\begin{tikzcd} \tilde{P}^{m+2} \arrow[r, "\tilde{d}^{m+2}"] \arrow[d, "\mathcal{L}^2 \tilde{f}_i^m"'] & \tilde{P}^{m+1} \arrow[d, "\mathcal{L}^1 \tilde{f}_i^m"] \\ \tilde{P}^2 \arrow[r, "\tilde{d}^2"'] & \tilde{P}^1 \end{tikzcd}$$

So we need to show $\mathcal{L}^1 \tilde{f}_i^m \circ \tilde{d}^{m+2}(t(\tilde{g}_j^{m+2})) = \tilde{d}^2 \circ \mathcal{L}^2 \tilde{f}_i^m(t(\tilde{g}_j^{m+2}))$ for all j. Let us consider $\mathcal{L}^1 \tilde{f}_i^m \circ \tilde{d}^{m+2}(t(\tilde{g}_j^{m+2}))$. Since $m \geq 2$, $\tilde{g}_j^{m+2} = \theta^*(g_j^{m+2})$, and we may write $g_j^{m+2} = \sum_{k=1}^{r_{m+1}} g_k^{m+1} p_{j,k}$. Then $\tilde{g}_j^{m+2} = \theta^*(\sum_{k=1}^{r_{m+1}} g_k^{m+1} p_{j,k}) = \sum_{k=1}^{r_{m+1}} \tilde{g}_k^{m+1} \theta^*(p_{j,k})$. Let $\theta^*(p_{j,k}) = \tilde{p}_{j,k}$, so $\tilde{g}_j^{m+2} = \sum_{k=1}^{r_{m+1}} \tilde{g}_k^{m+1} \tilde{p}_{j,k}$. We know that $\tilde{d}^{m+2}(t(\tilde{g}_j^{m+2}))$ has entry $t(\tilde{g}_k^{m+1})\tilde{p}_{j,k}$ in the summand of $\tilde{P}^{m+1}$ corresponding to $t(\tilde{g}_k^{m+1})$. So $\mathcal{L}^1 \tilde{f}_i^m \circ \tilde{d}^{m+2}(t(\tilde{g}_j^{m+2})) = \mathcal{L}^1 \tilde{f}_i^m(t(\tilde{g}_1^{m+1})\tilde{p}_{j,1}, t(\tilde{g}_2^{m+1})\tilde{p}_{j,2}, \ldots, t(\tilde{g}_{r_{m+1}}^{m+1})\tilde{p}_{j,r_{m+1}})$.
Keeping the notation of Definitions 9.5 and 9.10, $g_k^{m+1} = \sum_{l=1}^{r_m} g_l^m q_{k,l}, q_{k,i} = \sum_\alpha \alpha\gamma_{k,i,\alpha}$ and $\theta(\alpha) = \tilde{\alpha}_1\tilde{\alpha}_2 \cdots \tilde{\alpha}_A$.

So, $\mathcal{L}^1 \tilde{f}_i^m(t(\tilde{g}_k^{m+1}))$ has entry $t(\tilde{\alpha}_1)\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta(\gamma_{k,i,\alpha})$ in the $t(\tilde{\alpha}_1)$ component of $\tilde{P}^1$ and 0 otherwise. So, for each α the entry of the $t(\tilde{\alpha}_1)$ component of $\tilde{P}^1$ of $\mathcal{L}^1 \tilde{f}_i^m \circ \tilde{d}^{m+2}(t(\tilde{g}_j^{m+2}))$ is

$$\sum_{k=1}^{r_{m+1}} t(\tilde{\alpha}_1)\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta(\gamma_{k,i,\alpha})\tilde{p}_{j,k} = \sum_{k=1}^{r_{m+1}} t(\tilde{\alpha}_1)\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta(\gamma_{k,i,\alpha} p_{j,k}).$$

We now consider $\tilde{d}^2 \circ \mathcal{L}^2 \tilde{f}_i^m(t(\tilde{g}_j^{m+2}))$. With the notation of Definition 9.7,

$$\begin{aligned} \mathcal{L}^2 \tilde{f}_i^m(t(\tilde{g}_j^{m+2})) &= \theta_2(\mathcal{L}^2 f_i^m(t(g_j^{m+2}))) \\ &= \theta_2(t(g_1^2)\sigma_{i,j,1}^2, t(g_2^2)\sigma_{i,j,2}^2, \ldots, t(g_{r_2}^2)\sigma_{i,j,r_2}^2) \\ &= t(\tilde{g}_1^2)\theta(\sigma_{i,j,1}^2), t(\tilde{g}_2^2)\theta(\sigma_{i,j,2}^2), \ldots, t(\tilde{g}_{r_2}^2)\theta(\sigma_{i,j,r_2}^2)) \end{aligned}.$$

For $l = 1, \ldots, m_2$, write $g_l^2 = \sum_{\alpha \in \mathcal{Q}_1} \alpha\beta_{l,\alpha}$. Then

$$\tilde{g}_l^2 = \theta^*(g_l^2) = \sum_{\alpha \in \mathcal{Q}_1} \tilde{\alpha}_1\tilde{\alpha}_2 \cdots \tilde{\alpha}_A \theta^*(\beta_{l,\alpha})$$

where $\theta(\alpha)$ is the path $\tilde{\alpha}_1\tilde{\alpha}_2\cdots\tilde{\alpha}_A$ of arrows in $\tilde{\mathcal{Q}}$. So $\tilde{d}^2 \circ \mathcal{L}^2 \tilde{f}_i^m(t(\tilde{g}_j^{m+2}))$ has entry

$$\sum_{l=1}^{r_2} t(\tilde{\alpha}_1)\tilde{\alpha}_2\cdots\tilde{\alpha}_A\theta(\beta_{l,\alpha})\theta(\sigma^2_{i,j,l})$$

in the $t(\tilde{\alpha}_1)$ component of $\tilde{P}^1$ for all α.

Now, $\mathcal{L}^2 f_i^m$ is a lifting of f_i^m, so $d^2 \circ \mathcal{L}^2 f_i^m = \mathcal{L}^1 f_i^m \circ d^{m+1}$. We have that $d^2 \circ \mathcal{L}^2 f_i^m(t(g_j^{m+2}))$ has entry

$$\sum_{l=1}^{r_2} t(\alpha)\beta_{l,\alpha}\sigma^2_{i,j,l}$$

in the $t(\alpha)$ component of P^1, for all $\alpha \in \mathcal{Q}_1$. And $\mathcal{L}^1 f_i^m \circ d^{m+1}(t(g_j^{m+2}))$ has entry

$$\sum_{k=1}^{r_{m+1}} t(\alpha)\gamma_{k,i,\alpha}p_{j,k}$$

in the $t(\alpha)$ component of P^1, for all $\alpha \in \mathcal{Q}_1$. Thus

$$\sum_{l=1}^{r_2} t(\alpha)\beta_{l,\alpha}\sigma^2_{i,j,l} = \sum_{k=1}^{r_{m+1}} t(\alpha)\gamma_{k,i,\alpha}p_{j,k}$$

for all $\alpha \in \mathcal{Q}_1$. Now, $t(\alpha) = t(\tilde{\alpha}_A)$ for all $\alpha \in \mathcal{Q}_1$. Thus $\sum_{k=1}^{r_{m+1}} t(\tilde{\alpha}_1)\tilde{\alpha}_2\cdots\tilde{\alpha}_A\theta(\gamma_{k,i,\alpha}p_{j,k})$ $= \sum_{l=1}^{r_2} t(\tilde{\alpha}_1)\tilde{\alpha}_2\cdots\tilde{\alpha}_A\theta(\beta_{l,\alpha})\theta(\sigma_{i,j,l})$ for all arrows $\tilde{\alpha}$. Hence we have

$$\tilde{d}^2 \circ \mathcal{L}^2 \tilde{f}_i^m(t(\tilde{g}_j^{m+2})) = \mathcal{L}^1 \tilde{f}_i^m \circ \tilde{d}^{m+2}(t(\tilde{g}_j^{m+2}))$$

for all j, and $\mathcal{L}^2 \tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$ as required. $\square$

Proposition 9.14. *Let $m \geq 2$ and let $n \geq 3$. Let $\mathcal{L}^n \tilde{f}_i^m : \tilde{P}^{m+n} \to \tilde{P}^n$ be the map as given in Definition 9.12. Then $\mathcal{L}^n \tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$.*

Proof. Let $n \geq 3$ and assume that $\mathcal{L}^0 \tilde{f}_i^m, \mathcal{L}^1 \tilde{f}_i^m, \ldots, \mathcal{L}^{n-1} \tilde{f}_i^m$ are liftings of $\tilde{f}_i^m$. In order to show $\mathcal{L}^n \tilde{f}_i^m$ is a lifting, we need $\tilde{d}^n \circ \mathcal{L}^n \tilde{f}_i^m(t(\tilde{g}_j^{m+n})) = \mathcal{L}^{n-1} \tilde{f}_i^m \circ \tilde{d}^{m+n}(t(\tilde{g}_j^{m+n}))$ for all j. On the left hand side we have

$$
\begin{aligned}
\tilde{d}^n \circ \mathcal{L}^n \tilde{f}_i^m(t(\tilde{g}_j^{m+n})) &= \tilde{d}^n(\theta_n \mathcal{L}^n f_i^m(t(g_j^{m+n}))) \\
&= \theta_{n-1} \circ d^n(\mathcal{L}^n f_i^m(t(g_j^{m+n}))), \text{ by Proposition 8.12,} \\
&= \theta_{n-1}(\mathcal{L}^{n-1} f_i^m \circ d^{m+n}(t(g_j^{m+n}))), \text{ since } \mathcal{L}^n f_i^m \text{ is a lifting,} \\
&= \theta_{n-1} \circ \mathcal{L}^{n-1} f_i^m(d^{m+n}(t(g_j^{m+n}))).
\end{aligned}
$$

On the right hand side, we have

$$
\mathcal{L}^{n-1} \tilde{f}_i^m \circ \tilde{d}^{m+n}(t(\tilde{g}_j^{m+n})) = \mathcal{L}^{n-1} \tilde{f}_i^m(\theta_{m+n-1}(d^{m+n}(t(g_j^{m+n})))) \text{ from Definition 8.11.}
$$

Now, write $g_j^{m+n} = \sum_{k=1}^{s} g_k^{m+n-1} q_k$, with $q_k \in K\mathcal{Q}$, where $s = |g^{m+n-1}|$. Then

$$
\begin{aligned}
&\mathcal{L}^{n-1} \tilde{f}_i^m \circ \tilde{d}^{m+n}(t(\tilde{g}_j^{m+n})) \\
&= \mathcal{L}^{n-1} \tilde{f}_i^m(\theta_{m+n-1}(t(g_1^{m+n-1})q_1, t(g_2^{m+n-1})q_2, \ldots, t(g_s^{m+n-1})q_s), \\
&= \mathcal{L}^{n-1} \tilde{f}_i^m(t(\tilde{g}_1^{m+n-1})\theta(q_1), t(\tilde{g}_2^{m+n-1})\theta(q_2), \ldots, t(\tilde{g}_s^{m+n-1})\theta(q_s)) \\
&= \textstyle\sum_{k=1}^{s} \mathcal{L}^{n-1} \tilde{f}_i^m(t(\tilde{g}_k^{m+n-1}))\theta(q_k), \text{ since } \mathcal{L}^{n-1} \tilde{f}_i^m \text{ is a} \\
&\qquad\qquad\qquad\qquad\qquad \tilde{\Lambda}\text{-module homomorphism,} \\
&= \textstyle\sum_{k=1}^{s} \theta_{n-1} \mathcal{L}^{n-1} f_i^m(t(g_k^{m+n-1}))\theta(q_k) \\
&= \textstyle\sum_{k=1}^{s} \theta_{n-1}(\mathcal{L}^{n-1} f_i^m(t(g_k^{m+n-1})q_k)) \\
&= \theta_{n-1} \mathcal{L}^{n-1} f_i^m(\textstyle\sum_{k=1}^{s} t(g_k^{m+n-1})q_k) \\
&= \theta_{n-1} \mathcal{L}^{n-1} f_i^m(d^{m+n}(t(g_j^{m+n}))).
\end{aligned}
$$

Hence, we have the equality and $\mathcal{L}^n \tilde{f}_i^m$ is a lifting of $\tilde{f}_i^m$. □

We now have liftings $\mathcal{L}^n \tilde{f}_i^m$ for all $n \geq 0, m \geq 2$ and for all i. Our final result in this chapter uses these liftings in order to show that the map $\Psi : \mathrm{Ext}_{\Lambda}^{\geqslant 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \mathrm{Ext}_{\tilde{\Lambda}}^{\geqslant 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ as given in Definition 9.2, is a K-algebra homomorphism.

Theorem 9.15. *Let* $\Psi : \mathrm{Ext}_{\Lambda}^{\geqslant 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \mathrm{Ext}_{\tilde{\Lambda}}^{\geqslant 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ *be the map given in Definition 9.2. Then* Ψ *is a* K*-algebra homomorphism.*

Proof. We need to show that $\Psi(f_i^m \circ f_k^n) = \Psi(f_i^m) \circ \Psi(f_k^n)$, for all $m \geq 2, n \geq 2, i$ and k. We have $\Psi(f_i^m \circ f_k^n) = \Psi(f_i^m(\mathcal{L}^m f_k^n))$. Now $\mathcal{L}^m f_k^n(t(g_j^{m+n}))$ has entry $t(g_l^m)\sigma_{k,j,l}^m$ in the summand corresponding to $t(g_l^m)$. So $f_i^m(\mathcal{L}^m f_k^n(t(g_j^{m+n})) = t(g_i^m)\sigma_{k,j,i}^m + \mathfrak{r}$. Now, $t(g_i^m)\sigma_{k,j,i}^m + \mathfrak{r} = t(g_i^m)\sigma_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r}$ since f_i^m and $\mathcal{L}^m f_k^n$ are Λ-module homomorphisms. So $t(g_i^m)\sigma_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r}$ is non-zero precisely when $t(g_i^m) = t(g_j^{m+n})$ and $t(g_j^{m+n})\sigma_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r} = c_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r}$ for some $c_{k,j,i}^m \in K \setminus \{0\}$. So

$f_i^m \circ f_k^n = \sum c_{k,j,i}^m f_j^{m+n}$, where the sum is over j such that $t(g_i^m) = t(g_j^{m+n})$ and $t(g_j^{m+n})\sigma_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r} = c_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r}$ for some $c_{k,j,i}^m \in K \setminus \{0\}$. Hence $\Psi(f_i^m \circ f_k^n) = \Psi(\sum c_{k,j,i}^m f_j^{m+n}) = \sum c_{k,j,i}^m \tilde{f}_j^{m+n}$, where the sum is over those j described above.

On the right hand side we have $\Psi(f_i^m) \circ \Psi(f_k^n) = \tilde{f}_i^m \circ \tilde{f}_k^n = \tilde{f}_i^m(\mathcal{L}^m \tilde{f}_k^n)$. Now

$$\begin{aligned}
\tilde{f}_i^m(\mathcal{L}^m \tilde{f}_k^n(t(\tilde{g}_j^{m+n})) &= \tilde{f}_i^m(\theta_m \mathcal{L}^m f_k^n(t(g_j^{m+n}))) \\
&= \tilde{f}_i^m(\theta_m(\textstyle\sum_{k=1}^s t(g_l^m)\sigma_{k,j,l}^m) \\
&= \tilde{f}_i^m(\textstyle\sum_{k=1}^s t(\tilde{g}_l^m)\theta(\sigma_{k,j,l}^m)) \\
&= t(\tilde{g}_i^m)\theta(\sigma_{k,j,i}^m) + \mathfrak{r}.
\end{aligned}$$

Now, $t(\tilde{g}_i^m)\theta(\sigma_{k,j,i}^m) + \tilde{\mathfrak{r}} = t(\tilde{g}_i^m)\theta(\sigma_{k,j,i}^m)t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}}$ since $\tilde{f}_i^m$ and $\mathcal{L}^m \tilde{f}_k^n$ are $\tilde{\Lambda}$-module homomorphisms. Then $t(\tilde{g}_i^m)\theta(\sigma_{k,j,i}^m)t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}}$ is non-zero when $t(\tilde{g}_i^m) = t(\tilde{g}_j^{m+n})$ and $t(\tilde{g}_i^m)\theta(\sigma_{k,j,i}^m)t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}} = d_{k,j,i}^m t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}}$, for some $d_{k,j,i}^m \in K \setminus \{0\}$. So $\tilde{f}_i^m(\mathcal{L}^m \tilde{f}_k^n) = \sum d_{k,j,i}^m \tilde{f}_j^{m+n}$ where the sum is over j such that $t(\tilde{g}_i^m) = t(\tilde{g}_j^{m+n})$ and $t(\tilde{g}_j^{m+n})\theta(\sigma_{k,j,i}^m)t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}} = d_{k,j,i}^m t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}}$ for some $d_{k,j,i}^m \in K \setminus \{0\}$. However, $t(\tilde{g}_i^m) = t(\tilde{g}_j^{m+n})$ and $t(\tilde{g}_j^{m+n})\theta(\sigma_{k,j,i}^m)t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}} = d_{k,j,i}^m t(\tilde{g}_j^{m+n}) + \tilde{\mathfrak{r}}$ precisely when $t(g_i^m) = t(g_j^{m+n})$ and $t(g_j^{m+n})\sigma_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r} = c_{k,j,i}^m t(g_j^{m+n}) + \mathfrak{r}$, since $m \geq 2$ and θ is a K-algebra monomorphism, so that $d_{k,j,i}^m = c_{k,j,i}^m$. Hence, $\Psi(f_i^m \circ f_k^n) = \sum c_{k,j,i}^m \tilde{f}_j^{m+n} = \Psi(f_i^m) \circ \Psi(f_k^n)$ and Ψ is a K-algebra homomorphism. $\square$

The above result means that given the product structure of $E(\Lambda)$ for a d-Koszul algebra Λ, we also know the product structure of $E(\tilde{\Lambda})$ for the related (D, A)-stacked algebra $\tilde{\Lambda}$. From Chapter 5 we know that $\operatorname{Ext}^2_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}}) \times \operatorname{Ext}^{n-2}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}}) \to \operatorname{Ext}^n_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ is surjective. Hence for all $n \geq 4$, if $f_i^n = \sum_{j,k} c_{j,k} f_j^2 \circ f_k^{n-2}$ with $c_{j,k} \in K$ then $\tilde{f}_i^n = \Psi(f_i^n) = \Psi(\sum_{j,k} c_{j,k} f_j^2 \circ f_k^{n-2}) = \Psi(\sum_{j,k} c_{j,k} f_j^2) \circ \Psi(f_k^{n-2}) = \sum_{j,k} c_{j,k} \tilde{f}_j^2 \circ \tilde{f}_k^{n-2}$.

This chapter now ends our study of the Ext algebra of (D, A)-stacked algebras and in the next chapter we show how we can construct a bimodule resolution of $\tilde{\Lambda}$ over $\tilde{\Lambda}^e$ from a given bimodule resolution of Λ over Λ^e.

10. A Bimodule Resolution

We have spent a considerable amount of time in this thesis concerned with the Ext algebra of a finite-dimensional algebra Λ. The Ext algebra is obtained by taking the cohomology of the complex gained by applying the functor $\mathrm{Hom}_\Lambda(-, \Lambda/\mathfrak{r})$ to the deleted projected resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module. The Ext algebra is then given a product structure via the Yoneda product.

If we follow the same process, but beginning with a minimal projective resolution of Λ as a right Λ^e-module and applying the functor $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$, we obtain the Hochschild cohomology groups, $\mathrm{HH}^n(\Lambda)$. The Hochschild cohomology ring, $\mathrm{HH}^*(\Lambda)$, also has a product structure given by the Yoneda product. After the work of Chapter 9 the immediate question arises 'What is the relationship between $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$?' where Λ is a d-Koszul algebra and $\tilde{\Lambda}$ is the related (D, A)-stacked algebra.

We start this chapter by reviewing the construction of the beginning of a bimodule as given by Green and Snashall in [17]. Using the ideas of Chapter 8, in which we constructed a minimal projective resolution $(\tilde{P}^n, \tilde{d}^n)$ of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module from a given resolution (P^n, d^n) of $\Lambda/\mathfrak{r}$ as a right Λ-module, we then use a similar method to construct a minimal projective resolution for $\tilde{\Lambda}$ as a right $\tilde{\Lambda}^e$-module from a given minimal projective resolution for Λ as a right Λ^e-module. Throughout this chapter we write $\otimes$ instead of $\otimes_K$.

We begin with some background information and some definitions; these are taken from [26].

Definition 10.1. Let Λ be a K-algebra. Then Λ^{op} is a K-algebra with the same underlying vector space structure as Λ. The multiplication in Λ^{op} is defined as $\lambda * \mu = \mu\lambda$, for all $\lambda, \mu \in \Lambda$.

The enveloping algebra of Λ is defined to be

$$\Lambda^e = \Lambda^{op} \otimes \Lambda$$

with multiplication in Λ^e given by $(\lambda_1 \otimes \mu_1)(\lambda_2 \otimes \mu_2) = \lambda_2\lambda_1 \otimes \mu_1\mu_2$, for all $\lambda_1, \lambda_2 \in \Lambda^{op}$ and for all $\mu_1, \mu_2 \in \Lambda$.

Sometimes it is easier to work with Λ-Λ-bimodules rather than right Λ^e-modules. With this in mind it is advantageous to realise that given a Λ-Λ-bimodule M, this is equivalent to M being a right Λ^e-module, and then all the properties of modules can be applied to bimodules.

Proposition 10.2. *Let Λ be a K-algebra. If M is a Λ-Λ-bimodule, then M is a right Λ^e-module with scalar multiplication $m(\lambda \otimes \mu) = (\lambda m)\mu = \lambda(m\mu)$, for all $\lambda \in \Lambda^{op}, \mu \in \Lambda$ and $m \in M$.*

The projectives in a minimal projective resolution of Λ as a Λ-Λ-bimodule were given by Happel, [22].

Proposition 10.3. [22] *Let Λ be a finite-dimensional algebra and let*

$$\cdots \longrightarrow Q^n \longrightarrow Q^{n-1} \longrightarrow \cdots \longrightarrow Q^2 \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow \Lambda \longrightarrow 0$$

be a minimal projective resolution of Λ as a Λ-Λ-bimodule. Then

$$Q^n = \bigoplus_{i,j} P(i,j)^{dim Ext^n_\Lambda(S_i,S_j)}$$

where $P(i,j)$ is the projective Λ-Λ-bimodule $\Lambda(e_i \otimes e_j)\Lambda$, and S_i, S_j are the simple modules corresponding to $e_i\Lambda$ and $e_j\Lambda$ respectively.

If $\Lambda/\mathfrak{r}$ has a minimal projective resolution (P^n, d^n) as a right Λ-module, following [20], with $P^n = \bigoplus_i t(g_i^n)\Lambda$ then the nth projective Q^n in a minimal projective bimodule resolution of Λ is given by $Q^n = \bigoplus_i \Lambda s(g_i^n) \otimes t(g_i^n)\Lambda$.

Having the projective modules Q^n for all n, it remains to determine the maps $\delta^n : Q^n \to Q^{n-1}$. The maps δ^n, for $n = 0, 1, 2$ and 3 for an arbitrary finite-dimensional algebra Λ are given in [17]. We give a brief introduction to this paper.

Definition 10.4. [17] Write $Q^0 = \bigoplus_i \Lambda e_i \otimes e_i\Lambda$. The map $\delta^0 : Q^0 \to \Lambda$ is the multiplication map, that is, $\delta^0(\lambda e_i \otimes e_i\mu) = \lambda e_i \mu$, for all i and all $\lambda, \mu \in \Lambda$.

We have $Q^1 = \bigoplus_\alpha \Lambda s(\alpha) \otimes t(\alpha)\Lambda$. The map $\delta^1 : Q^1 \longrightarrow Q^0$ is defined by $\delta^1(s(\alpha) \otimes t(\alpha)) = s(\alpha) \otimes \alpha - \alpha \otimes t(\alpha)$ for all arrows α in the quiver $\mathcal{Q}$, where $s(\alpha) \otimes \alpha$ lies in the $s(\alpha) \otimes s(\alpha)$ component of Q^0 and $\alpha \otimes t(\alpha)$ lies in the $t(\alpha) \otimes t(\alpha)$ component of Q^0.

The map δ^1, and indeed δ^n for all $n \geq 1$, is easier to use in matrix form. We represent the map δ^1 by the matrix A_1, which has rows indexed by g^0 and columns indexed by g^1. So A_1 is a $|g^0| \times |g^1|$ matrix.

For $e \in g^0$ and $\alpha \in g^1$, the (e, α) entry of A_1 is

$$\begin{cases} s(\alpha) \otimes \alpha \text{ if } s(\alpha) = e \text{ and } t(\alpha) \neq e \\ -\alpha \otimes t(\alpha) \text{ if } t(\alpha) = e \text{ and } s(\alpha) \neq e \\ s(\alpha) \otimes \alpha - \alpha \otimes t(\alpha) \text{ if } s(\alpha) = e \text{ and } t(\alpha) = e \\ 0 \ \text{ otherwise.} \end{cases}$$

.

Definition 10.5. [17] Write $Q^2 = \bigoplus_{x \in g^2} \Lambda s(x) \otimes t(x)\Lambda$. Let the matrix A_2 represent the map $\delta^2 : Q^2 \longrightarrow Q^1$. Then A_2 is a $|g^1| \times |g^2|$ matrix with rows indexed by g^1 and columns indexed by g^2. Let $\alpha \in g^1$ and let $x \in g^2$ be an arbitrary element of g^2 given by $x = \sum_{j=1}^{r} c_j \alpha_{1,j}\alpha_{2,j} \cdots \alpha_{s_j,j}$. The (α, x) entry of A_2 is given by

$$\sum_{j=1}^{r} c_j \sum_{k=1}^{s_j} \varepsilon_{k,j}\alpha_{1,j} \cdots \alpha_{k-1,j} \otimes \alpha_{k+1,j} \cdots \alpha_{s_j,j}$$

where

$$\varepsilon_{kj} = \begin{cases} 1 \text{ if } \alpha_{kj} & = \alpha \\ 0 \ \text{ otherwise.} \end{cases}$$

Definition 10.6. [17] Write $Q^3 = \bigoplus_{y \in g^3} \Lambda s(y) \otimes t(y)\Lambda$. Let the matrix A_3 represent the map $\delta^3 : Q^3 \longrightarrow Q^2$. Then A_3 is a $|g^2| \times |g^3|$ matrix with the rows indexed by g^2 and the columns indexed by g^3. Let y be an arbitrary element of g^3 given by

$y = \sum_i g_i^2 p_i = \sum_i q_i g_i^2 r_i$. The (g_i^2, y) entry is given by

$$s(g_i^2) \otimes p_i - q_i \otimes r_i.$$

We now have the information required to give the start of a minimal bimodule resolution.

Theorem 10.7. [17, Theorem 2.9] *With the above definitions, the following sequence forms part of a minimal projective bimodule resolution of* Λ*:*

$$Q^3 \longrightarrow Q^2 \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow \Lambda \longrightarrow 0$$

with maps $A_i : Q^i \to Q^{i-1}$ *for* $i = 1, 2, 3$.

For a more detailed account and the proof of minimality and exactness, see [17].

In the same way, we can construct the beginning of a minimal projective bimodule resolution $(\tilde{Q}^n, \tilde{\delta}^n)$ of $\tilde{\Lambda}$. Thus, we have

$$\tilde{Q}^3 \xrightarrow{\tilde{\delta}^3} \tilde{Q}^2 \xrightarrow{\tilde{\delta}^2} \tilde{Q}^1 \xrightarrow{\tilde{\delta}^1} \tilde{Q}^0 \xrightarrow{\tilde{\delta}^0} \tilde{\Lambda} \longrightarrow 0$$

where $\tilde{Q}^0 = \bigoplus_{\tilde{e} \in \tilde{g}^0} \tilde{\Lambda}\tilde{e} \otimes \tilde{e}\tilde{\Lambda}$, $\tilde{Q}^1 = \bigoplus_{\tilde{\alpha} \in \tilde{g}^1} \tilde{\Lambda} s(\tilde{\alpha}) \otimes t(\tilde{\alpha})\tilde{\Lambda}$, $\tilde{Q}^2 = \bigoplus_{\tilde{x} \in \tilde{g}^2} \tilde{\Lambda} s(\tilde{x}) \otimes t(\tilde{x})\tilde{\Lambda}$ and $\tilde{Q}^3 = \bigoplus_{\tilde{y} \in \tilde{g}^3} \tilde{\Lambda} s(\tilde{y}) \otimes t(\tilde{y})\tilde{\Lambda}$.

It now remains to use a minimal projective bimodule resolution of Λ to determine the higher maps $\tilde{\delta}^n : \tilde{Q}^n \to \tilde{Q}^{n-1}$, for $n \geq 4$. We remark that there is no explicit formula for a map $\delta^n : Q^n \to Q^{n-1}$, for $n \geq 4$, in terms of the g^n that works for all finite-dimensional algebras. However, there has been extensive work on certain classes of algebras for which the minimal bimodule resolution has been explicitly constructed. For example, the minimal projective bimodule resolution of a Koszul algebra was given by Green, Hartman, Marcos and Solberg in [10] and the minimal projective bimodule resolution of a monomial algebra was given by Bardzell in [4]. In [28] Snashall and Taillefer described a minimal bimodule resolution for a class of special biserial algebras. Other examples include a minimal bimodule resolution for Hecke algebras of type A, constructed by Schroll and Snashall in [27].

We now wish to describe a method for constructing a minimal bimodule resolution of $\tilde{\Lambda}$ as a $\tilde{\Lambda}$-$\tilde{\Lambda}$ bimodule from that of Λ as a Λ-Λ-bimodule. We already have the projectives for all $n \geq 0$, and we have the maps $\tilde{Q}^n \to \tilde{Q}^{n-1}$ for $n = 0, 1, 2$ and 3. In order to define maps $\tilde{\delta}^n : \tilde{Q}^n \to \tilde{Q}^{n-1}$, for $n \geq 4$, we first use the map θ from Definition 8.6 to define a map $\phi : \Lambda^e \to \tilde{\Lambda}^e$.

Definition 10.8. Let Λ be a d-Koszul algebra and let $\tilde{\Lambda}$ be the related (D, A)-stacked algebra with $D = dA$. Let $\eta \in \Lambda^e$, so $\eta = \sum_i \eta_i \otimes \eta_i'$ for some $\eta_i \in \Lambda^{op}$ and $\eta_i' \in \Lambda$. The map θ clearly induces a K-algebra homomorphism $\Lambda^{op} \to \tilde{\Lambda}^{op}$ which we also call θ. We define a K-module homomorphism $\phi : \Lambda^e \to \tilde{\Lambda}^e$ by

$$\phi(\eta) = \sum_i \theta(\eta_i) \otimes \theta(\eta_i').$$

Proposition 10.9. *Let $\phi : \Lambda^e \to \tilde{\Lambda}^e$ be as defined above. Then ϕ is a monomorphism.*

Proof. Fix a K-basis $\mathcal{B}$ of Λ consisting of paths. Then Λ^e has a K-basis $\mathfrak{B} = \{b \otimes b' | \ b, b' \in \mathcal{B}\}$. However, we know that for each $b \in \mathcal{B}$, $\theta(b)$ is a non-zero path in $\tilde{\Lambda}$, and thus $\tilde{\Lambda}^e$ has a K-basis $\tilde{\mathfrak{B}}$ which contains the set $\{\theta(b) \otimes \theta(b') | \ b, b' \in \mathcal{B}\}$.

Let $\eta \in \Lambda^e$ and write $\eta = \sum_i c_i b_i \otimes b_i'$, where $c_i \in K$ and $b_i, b_i' \in \mathcal{B}$. Suppose that $\phi(\eta) = 0$. Then $\phi(\sum_i c_i b_i \otimes b_i') = \sum_i c_i \theta(b_i) \otimes \theta(b_i') = 0$. Since for each i, $\theta(b_i) \otimes \theta(b_i')$ is an element of a K-basis for $\tilde{\Lambda}^e$ then the elements $\theta(b_i) \otimes \theta(b_i')$ are linearly independent and thus $c_i = 0$ for all i. Hence $\eta = 0$ and ϕ is a monomorphism. $\square$

For $n \geq 2$, the map ϕ from Definition 10.8 induces K-module homomorphisms $\phi_n : Q^n \to \tilde{Q}^n$ by

$$\phi_n(\eta_1 s(g_i^n) \otimes t(g_i^n)\eta_2) = \theta(\eta_1) s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n)\theta(\eta_2), \text{ for } \eta_1, \eta_2 \in \Lambda.$$

It is clear that ϕ_n is also a monomorphism.

We now wish to show that ϕ_n is a Λ-Λ-bimodule homomorphism, but we first need to give $\tilde{Q}^n$ the structure of a Λ-Λ-bimodule.

Definition 10.10. Let $n \geq 2$ and let $\tilde{Q}^n = \bigoplus_i \tilde{\Lambda} s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n)\tilde{\Lambda}$. Then we may write $\tilde{Q}^n = \bigoplus_i (s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n))\tilde{\Lambda}^e$. We define $\tilde{Q}^n$ to be a right Λ^e-module via the map ϕ in the following way

$$(s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n))(\tilde{\lambda}_1 \otimes \tilde{\lambda}_2) \cdot \eta = (s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n))(\tilde{\lambda}_1 \otimes \tilde{\lambda}_2)\phi(\eta)$$

for all $\eta \in \Lambda^e$. Thus, as a Λ-Λ-bimodule

$$\eta_1 \cdot (\tilde{\lambda}_1 s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n)\tilde{\lambda}_2) \cdot \eta_2 = \theta(\eta_1)\tilde{\lambda}_1 s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n)\tilde{\lambda}_2\theta(\eta_2)$$

for all $\eta_1, \eta_2 \in \Lambda$.

Proposition 10.11. *Let $n \geq 2$ and let $\phi_n : Q^n \to \tilde{Q}^n$ be as defined above. Then ϕ_n is a Λ-Λ-bimodule homomorphism.*

The proof is straightforward and can easily be verified.

Using these maps we now define the maps $\tilde{\delta}^n : \tilde{Q}^n \to \tilde{Q}^{n-1}$ for $n \geq 3$.

Definition 10.12. Let (Q^n, δ^n) be a minimal projective bimodule resolution for Λ with the part up to Q^3 as given by [17]. Let $n \geq 3$. Define $\tilde{\delta}^n : \tilde{Q}^n \to \tilde{Q}^{n-1}$ to be the $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule homomorphism given by

$$\tilde{\delta}^n(s(\tilde{g}_i^n) \otimes t(\tilde{g}_i^n)) = \phi_{n-1}(\delta^n(s(g_i^n) \otimes t(g_i^n))).$$

Proposition 10.13. *The definition of $\tilde{\delta}^3$ from Definition 10.12 coincides with that of Definition 10.6.*

Proof. From Definition 10.12 we have $\tilde{\delta}^3(s(\tilde{g}_j^3) \otimes t(\tilde{g}_j^3)) = \phi_2(\delta^3(s(g_j^3) \otimes t(g_j^3)))$. Let $g_j^3 = \sum_i g_i^2 p_{i,j} = \sum_i q_{i,j} g_i^2 r_{i,j}$. Now $\tilde{g}_j^3 = \theta^*(g_j^3)$, so $\tilde{g}_j^3 = \theta^*(\sum_i g_i^2 p_{i,j}) = \theta^*(\sum_i q_{i,j} g_i^2 r_{i,j}) = \sum_i \tilde{g}_i^2 \theta^*(p_{i,j}) = \sum_i \theta^*(q_{i,j})\tilde{g}_i^2\theta^*(r_{i,j})$. Now $\delta^3(s(g_j^3) \otimes t(g_j^3))$ is given by the matrix A_3 where each (g_i^2, g_j^3)-entry is given by $s(g_i^2) \otimes p_{i,j} - q_{i,j} \otimes r_{i,j}$.

Then we have $\phi_2(s(g_i^2) \otimes p_{i,j} - q_{i,j} \otimes r_{i,j}) = s(\tilde{g}_i^2) \otimes \theta(p_{i,j}) - \theta(q_{i,j}) \otimes \theta(r_{i,j})$. So the matrix representation of $\tilde{\delta}^3 : \tilde{Q}^3 \to \tilde{Q}^2$ from Definition 10.12 has $(\tilde{g}_i^2, \tilde{g}_j^3)$-entry $s(\tilde{g}_i^2) \otimes \theta(p_{i,j}) - \theta(q_{i,j}) \otimes \theta(r_{i,j})$.

From Definition 10.6, the matrix $\tilde{A}_3$ has $(\tilde{g}_i^2, \tilde{g}_j^3)$-entry $s(\tilde{g}_i^2) \otimes \theta(p_{i,j}) - \theta(q_{i,j}) \otimes \theta(r_{i,j})$. Hence the two definitions coincide. $\square$

We have now defined $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule homomorphisms $\tilde{\delta}^n : \tilde{Q}^n \to \tilde{Q}^{n-1}$, for all $n \geq 0$, giving us a sequence

$$\cdots \longrightarrow \tilde{Q}^n \xrightarrow{\tilde{\delta}^n} \tilde{Q}^{n-1} \xrightarrow{\tilde{\delta}^{n-1}} \cdots \longrightarrow \tilde{Q}^2 \xrightarrow{\tilde{\delta}^2} \tilde{Q}^1 \xrightarrow{\tilde{\delta}^1} \tilde{Q}^0 \xrightarrow{\tilde{\delta}^0} \tilde{\Lambda} \longrightarrow 0.$$

We need to show that this is indeed a minimal projective bimodule resolution of $\tilde{\Lambda}$. We start by considering the commutativity of the following diagram of Λ-Λ-bimodules.

$$\begin{tikzcd}
\cdots \arrow[r] & Q^n \arrow[r, "\delta^n"] \arrow[d, "\phi_n"'] & Q^{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r] & Q^3 \arrow[r, "\delta^3"] \arrow[d, "\phi_3"'] & Q^2 \arrow[r, "\delta^2"] \arrow[d, "\phi_2"'] & \cdots \\
\cdots \arrow[r] & \tilde{Q}^n \arrow[r, "\tilde{\delta}^n"'] & \tilde{Q}^{n-1} \arrow[r, "\tilde{\delta}^{n-1}"'] & \cdots \arrow[r] & \tilde{Q}^3 \arrow[r, "\tilde{\delta}^3"'] & \tilde{Q}^2 \arrow[r, "\tilde{\delta}^2"'] & \cdots
\end{tikzcd}$$

Proposition 10.14. *For all $n \geq 2$, the following square is commutative as Λ-Λ-bimodules.*

$$\begin{tikzcd}
Q^{n+1} \arrow[r, "\delta^{n+1}"] \arrow[d, "\phi_{n+1}"'] & Q^n \arrow[d, "\phi_n"] \\
\tilde{Q}^{n+1} \arrow[r, "\tilde{\delta}^{n+1}"'] & \tilde{Q}^n
\end{tikzcd}$$

Proof. Let $n \geq 2$ and let x be the element of Q^{n+1} with entry $s(g_i^{n+1}) \otimes t(g_i^{n+1})$ in the $\Lambda s(g_i^{n+1}) \otimes t(g_i^{n+1})\Lambda$ component and 0 otherwise. Then

$$\begin{aligned}
\phi_n \circ \delta^{n+1}(x) &= \phi_n \circ \delta^{n+1}(s(g_i^{n+1}) \otimes t(g_i^{n+1})) \\
&= \tilde{\delta}^{n+1}(s(\tilde{g}_i^{n+1}) \otimes t(\tilde{g}_i^{n+1})), \text{ by Definition 10.12.} \\
&= \tilde{\delta}^{n+1}(\theta(s(g_i^{n+1})) \otimes \theta(t(g_i^{n+1}))) \\
&= \tilde{\delta}^{n+1} \circ \phi_{n+1}(s(g_i^{n+1}) \otimes t(g_i^{n+1}))) \\
&= \tilde{\delta}^{n+1} \circ \phi_{n+1}(x).
\end{aligned}$$

Hence for all $n \geq 2$, $\tilde{\delta}^{n+1} \circ \phi_{n+1} = \phi_n \circ \delta^{n+1}$ and the square commutes. □

Proposition 10.15. *Let $(\tilde{Q}^n, \tilde{\delta}^n)$ be as given above. Then $(\tilde{Q}^n, \tilde{\delta}^n)$ is a complex.*

Proof. For $n = 0, 1, 2$ it follows from [17, Theorem 2.9] that $\tilde{\delta}^n \circ \tilde{\delta}^{n+1} = 0$.

For $n \geq 3$, we have

$$\begin{aligned}
\tilde{\delta}^n \circ \tilde{\delta}^{n+1}(s(\tilde{g}_i^{n+1}) \otimes t(\tilde{g}_i^{n+1})) &= \tilde{\delta}^n(\phi_n(\delta^{n+1}(s(g_i^{n+1}) \otimes t(\tilde{g}_i^{n+1})))) \\
&= \phi_{n-1}(\delta^n(\delta^{n+1}(s(g_i^{n+1}) \otimes t(g_i^{n+1})))), \\
&\qquad \text{by Proposition 10.14,} \\
&= 0, \text{ since } \delta^n \circ \delta^{n+1} = 0.
\end{aligned}$$

Hence $(\tilde{Q}^n, \tilde{\delta}^n)$ is a complex. □

Theorem 10.16. *Let $(\tilde{Q}^n, \tilde{\delta}^n)$ be as given above. Then $(\tilde{Q}^n, \tilde{\delta}^n)$ is a minimal projective bimodule resolution of $\tilde{\Lambda}$ as a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule.*

Proof. Let $(\tilde{Q}^n, \tilde{\delta}^n)$ be the complex of Proposition 10.15, so we have $\operatorname{Im} \tilde{\delta}^{n+1} \subseteq \operatorname{Ker} \tilde{\delta}^n$ for all $n \geq 0$. From [17] we know that the complex is exact at $n = 0, 1$ and 2. It remains to show that $\operatorname{Ker} \tilde{\delta}^n \subseteq \operatorname{Im} \tilde{\delta}^{n+1}$ for $n \geq 3$.

Let $n \geq 3$. Let $\tilde{x} \in \operatorname{Ker} \tilde{\delta}^n$ and write

$$\tilde{x} = (\tilde{\lambda}_1 s(\tilde{g}_1^n) \otimes t(\tilde{g}_1^n)\tilde{\mu}_1, \ldots, \tilde{\lambda}_{m_n} s(\tilde{g}_{m_n}^n) \otimes t(\tilde{g}_{m_n}^n)\tilde{\mu}_{m_n})$$

with $\tilde{\lambda}_i, \tilde{\mu}_i \in \tilde{\Lambda}$.

First let us assume that there are λ_i and μ_i in Λ such that $\theta(\lambda_i) = \tilde{\lambda}_i$ and $\theta(\mu_i) = \tilde{\mu}_i$ for each i. Then we have $\tilde{x} = (\theta(\lambda_1 s(g_1^n)) \otimes \theta(t(g_1^n)\mu_1), \ldots, \theta(\lambda_{m_n} s(g_{m_n}^n)) \otimes \theta(t(g_{m_n}^n)\mu_{m_n}))$, so $\tilde{x} = \phi_n(x)$ where $x = (\lambda_1 s(g_1^n) \otimes t(g_1^n)\mu_1, \ldots, \lambda_{m_n} s(g_{m_n}^n) \otimes t(g_{m_n}^n)\mu_{m_n})$ $\in Q^n$. Now $\tilde{x} \in \operatorname{Ker} \tilde{\delta}^n$, so $\tilde{\delta}^n(\tilde{x}) = 0 = \tilde{\delta}^n(\phi_n(x))$ and from Proposition 10.14,

$\tilde{\delta}^n(\phi_n(x)) = \phi_{n-1}(\delta^n(x))$. We know that ϕ_n is $1-1$, so $\delta^n(x) = 0$. Hence $x \in \operatorname{Ker}\delta^n = \operatorname{Im}\delta^{n+1}$, and we have $x = \delta^{n+1}(y)$ for some $y \in Q^{n+1}$. Therefore $\tilde{x} = \phi_n(x) = \phi_n(\delta^{n+1}(y)) = \tilde{\delta}^{n+1}(\phi_{n+1}(y))$ and $\tilde{x} \in \operatorname{Im}\tilde{\delta}^{n+1}$.

Now let $\tilde{x}$ be an arbitrary element of $\operatorname{Ker}\tilde{\delta}^n$, and write

$$\tilde{x} = (\tilde{\lambda}_1 s(\tilde{g}_1^n) \otimes t(\tilde{g}_1^n)\tilde{\mu}_1, \ldots, \tilde{\lambda}_{m_n} s(\tilde{g}_{m_n}^n) \otimes t(\tilde{g}_{m_n}^n)\tilde{\mu}_{m_n}).$$

Then $\tilde{x} = e\tilde{x}e + \sum_{w,w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0}(e\tilde{x}w + w\tilde{x}e + w\tilde{x}w')$ where $e = \sum_{v \in \mathcal{Q}_0} v$ is an element in $\tilde{\Lambda}$. Now $\operatorname{Ker}\tilde{\delta}^n$ is a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule. So $e\tilde{x}e, e\tilde{x}w, w\tilde{x}e, w\tilde{x}w' \in \operatorname{Ker}\tilde{\delta}^n$ for all $w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. From the construction of $K\tilde{\mathcal{Q}}$, if $\tilde{p}$ is an element of $K\tilde{\mathcal{Q}}$ with $s(\tilde{p}) \in \mathcal{Q}_0$ and $t(\tilde{p}) \in \mathcal{Q}_0$ then $\tilde{p} = \theta(p)$ for some $p \in K\mathcal{Q}$. Thus we can write $e\tilde{\lambda}_i s(g_i^n) \otimes t(g_i^n)\tilde{\mu}_i e = e\theta(\lambda_i)s(g_i^n) \otimes t(g_i^n)\theta(\mu_i)e$ for some $\lambda_i, \mu_i \in \Lambda$ and for all $i = 1, \ldots, m_n$. So $e\tilde{x}e = \phi_n(z)$ where $z = (\lambda_1 s(g_1^n) \otimes t(g_1^n)\mu_1, \ldots, \lambda_{m_n} s(g_{m_n}^n) \otimes t(g_{m_n}^n)\mu_{m_n}) \in Q^n$. The argument above now gives that $e\tilde{x}e \in \operatorname{Im}\tilde{\delta}^{n+1}$.

We now show that $e\tilde{x}w \in \operatorname{Im}\tilde{\delta}^{n+1}$ where $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. By construction of the quiver $\tilde{\mathcal{Q}}$, for each $i = 1, \ldots, m_n$, we can write $t(g_i^n)\tilde{\mu}_i w = t(g_i^n)\theta(\eta_i)\tilde{p}_w$ where $\eta_i \in \Lambda$ and $\tilde{p}_w$ is the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at a vertex in $\mathcal{Q}_0$ and ends at w, and $\tilde{p}_w$ contains no proper subpath $\tilde{q}$ such that $\tilde{q} = \theta(q)$ for some $q \in \Lambda$. Hence $e\tilde{x}w = (e\theta(\lambda_1)s(g_1^n) \otimes t(g_1^n)\theta(\eta_1)\tilde{p}_w, \ldots, e\theta(\lambda_{m_n})s(g_{m_n}^n) \otimes t(g_{m_n}^n)\theta(\eta_{m_n})\tilde{p}_w)$ for $\lambda_i \in \Lambda$. Let $z = (\lambda_1 s(g_1^n) \otimes t(g_1^n)\eta_1, \ldots, \lambda_{m_n} s(g_{m_n}^n) \otimes t(g_{m_n}^n)\eta_{m_n}) \in Q^n$. Then $e\tilde{x}w = \phi_n(z)\tilde{p}_w$. We now show that $z \in \operatorname{Ker}\delta^n$. The entry of $\delta^n(z)$ in the component of Q^{n-1} corresponding to $s(g_i^{n-1}) \otimes t(g_i^{n-1})$ may be written $\sum_j c_{i,j} b_{i,j} \otimes b'_{i,j}$ for some $c_{i,j} \in K$ and $b_{i,j}, b'_{i,j} \in \mathcal{B}$, where $\mathcal{B}$ is a K-basis of Λ consisting of paths. Then $\phi_{n-1} \circ \delta^n(z)$ has entry $\sum_j c_{i,j}\theta(b_{i,j}) \otimes \theta(b'_{i,j})$ in the component of $\tilde{Q}^{n-1}$ corresponding to $s(\tilde{g}_i^{n-1}) \otimes t(\tilde{g}_i^{n-1})$. However, $\tilde{\delta}^n(e\tilde{x}w) = 0$ and $\tilde{\delta}^n(e\tilde{x}w) = \tilde{\delta}^n(\phi_n(z))\tilde{p}_w = (\phi_{n-1} \circ \delta^n(z))\tilde{p}_w$ from Proposition 10.14. Thus $(\phi_{n-1} \circ \delta^n(z))\tilde{p}_w = 0$ and so $\sum_j c_{i,j}\theta(b_{i,j}) \otimes \theta(b'_{i,j})\tilde{p}_w = 0$ for all $i = 1, \ldots, m_n$. Now $t(\theta(b'_{i,j})) = s(\tilde{p}_w)$, so by the construction of $\tilde{\Lambda}$, $\theta(b'_{i,j})\tilde{p}_w$ is a non-zero path in $\tilde{\Lambda}$, and thus $\{\theta(b_{i,j}) \otimes \theta(b'_{i,j})\tilde{p}_w\}$ is a linearly independent set in $\tilde{\Lambda}^e$. Therefore, $c_{i,j} = 0$ for all j and all $i = 1, \ldots, m_{n-1}$ so we must have that $\delta^n(z) = 0$. Since (Q^n, δ^n) is a projective resolution for Λ as a Λ-Λ-bimodule, $\operatorname{Im}\delta^{n+1} = \operatorname{Ker}\delta^n$ so $z \in \operatorname{Im}\delta^{n+1}$. Thus $z = \delta^{n+1}(y)$ for some $y \in Q^{n+1}$. Therefore $\phi_n(z) = \phi_n \circ \delta^{n+1}(y) = \tilde{\delta}^{n+1} \circ \phi_{n+1}(y)$ from Proposition 10.14. So $e\tilde{x}w = \phi_n(z)\tilde{p}_w = (\tilde{\delta}^{n+1} \circ \phi_{n+1}(y))\tilde{p}_w = \tilde{\delta}^{n+1}(\phi_{n+1}(y)\tilde{p}_w)$ so $e\tilde{x}w \in \operatorname{Im}\tilde{\delta}^{n+1}$.

A similar argument shows that $w\tilde{x}e$ and $w\tilde{x}w'$ are in $\operatorname{Im}\tilde{\delta}^{n+1}$ for all $w, w' \in \tilde{\mathcal{Q}}_0\backslash\mathcal{Q}_0$. Hence $\tilde{x} \in \operatorname{Im}\tilde{\delta}^{n+1}$ and $\operatorname{Ker}\tilde{\delta}^n \subseteq \operatorname{Im}\tilde{\delta}^{n+1}$, for $n \geq 3$, as required.

Therefore, for all $n \geq 0$, we have $\operatorname{Ker}\tilde{\delta}^n = \operatorname{Im}\tilde{\delta}^{n+1}$. Hence $(\tilde{Q}^n, \tilde{\delta}^n)$ is a projective bimodule resolution of $\tilde{\Lambda}$ as a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule. From Happel [22] (see Proposition 10.3) we know that the $\tilde{Q}^n$ are the projectives of a minimal projective bimodule resolution, and hence $(\tilde{Q}^n, \tilde{\delta}^n)$ is a minimal projective bimodule resolution of $\tilde{\Lambda}$. $\square$

In this thesis we have introduced a new class of algebras called (D, A)-stacked algebras, which are motivated by and generalise the Koszul algebras, D-Koszul algebras and (D, A)-stacked monomial algebras. We have shown that the Ext algebra is always finitely generated as an algebra and given a characterisation of these algebras. We have also given an explicit construction for a family of (D, A)-stacked algebras $\tilde{\Lambda}$ from a d-Koszul algebra Λ, where $D = dA$, for $A \geq 1$. Included in this is an explicit construction of a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module from a given minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, and a minimal projective bimodule resolution of $\tilde{\Lambda}$ as a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule from a given minimal projective bimodule resolution of Λ as a Λ-Λ-bimodule.

Future directions for research would be to investigate whether every (D, A)-stacked algebra arises from a d-Koszul algebra with $D = dA$ via our construction, and to investigate the relationship between $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$.

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