Blocks of fat category $\mathcal{O}$

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Abstract

We generalize the category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand to the so called fat category $\mathcal{O}$, $\mathcal{O}^{(n)}$, and derive some of its properties.

From a Lie theoretic point of view, $\mathcal{O}^{(n)}$ contains a significant amount of indecomposable representations that do not belong to $\mathcal{O}$ (although it fails to add new simple ones) such as the fat Verma modules. These modules have simple top and socle and may be viewed as standard objects once a block decomposition of $\mathcal{O}^{(n)}$ is obtained and each block is seen to be equivalent to a category of finite dimensional modules over a finite dimensional standardly stratified algebra.

We describe the Ringel dual of these algebras (concluding that principal blocks are self dual) and we obtain the character formulae for their tilting modules. Furthermore, a double centralizer property is proved, relating each block with the corresponding fat algebra of coinvariants. As a byproduct we obtain a classification of all blocks of $\mathcal{O}^{(n)}$ in terms of their representation type.

In the process of determining the quiver and relations which characterize the basic algebras associated to each block of $\mathcal{O}^{(n)}$, we prove (for root systems of small rank) a formula establishing the dimension of the Ext^1 spaces between simple modules. By borrowing from Soergel some results describing the behaviour of the combinatorial functor $\mathbb{V}$, we are able to compute examples.

Chapter 1

Introduction

1.1 Motivation

For any finite dimensional complex semisimple Lie algebra $\mathcal{G}$, the Verma modules $\Delta(\lambda)$ constitute a well known class of well behaved representations of $\mathcal{G}$. They are indecomposable, cyclic and $\Delta(\lambda)/Rad(\Delta(\lambda))$ is simple thus providing a vast family of simple representations of $\mathcal{G}$ - a major quest in Lie theory. Chapter 7 of Dixmier's book *Universal enveloping algebras* provides a comprehensive treatment of these Verma modules including a detailed proof of the BGG theorem (see [16], Proposition 7.6.23) due to Bernstein, Gelfand and Gelfand.

In our work we define the so called "fat" Verma modules, $\Delta^{(n)}(\lambda)$ (for $n = 1$ we recover the Verma module $\Delta(\lambda)$) and try to work out some of their main properties. The initial idea was to obtain modules which are filtered by (isomorphic copies of) Verma modules. Needless to say, some (probably most) of the properties of Verma modules and results concerning them still hold for their fat version. For example, fat Verma modules have simple socle. Nevertheless there are exceptions, the most remarkable one being the above mentioned BGG theorem.

It was clear from the beginning that a categorical framework to work in was needed. Is is well known that the celebrated category $\mathcal{O}$ (introduced by Bernstein, Gelfand and Gelfand around 1976 in [9]) encapsulates a significant amount of information about representations of $\mathcal{G}$. In particular, its blocks are equivalent to categories of finite dimensional modules over finite dimensional quasi-hereditary algebras where Verma modules play the role of standard objects. This led us to the definition of the "fat" category $\mathcal{O}^{(n)}$ (where, as before, we recover the category $\mathcal{O}$ by setting $n = 1$). Now, blocks of $\mathcal{O}^{(n)}$ will be equivalent to categories of finite dimensional modules over finite dimensional properly stratified algebras. This implies the existence of another class of distinguished objects (adding to simples, projectives, injectives, standard, costandard, proper standard and proper costandard modules): the tilting and cotilting modules.

1.2 Further motivation

In [33], dealing with problems arising from the representation theory of complex semisimple Lie groups, Soergel introduced the categories $\mathcal{O}^I$ (the "thick" category $\mathcal{O}$) where I stands for an ideal of finite codimension of $U(\mathcal{H})$ and $\mathcal{H}$

is a fixed Cartan subalgebra of $\mathcal{G}$. Soergel's idea (among others) was to generalize category $\mathcal{O}$ to categories closed under taking subquotients and tensor products with finite dimensional $\mathcal{G}$-modules and where all the nice properties of his "combinatorial functor" $\mathbb{V}$ would still hold.

"Our" category $\mathcal{O}^{(n)}$ is no other than Soergel's $\mathcal{O}^I$ for a particular choice of ideal I. It is easy to see that $\bigcup \mathcal{O}^I$ (where I runs through the set of ideals of finite codimension of $U(\mathcal{H})$) equals $\bigcup \mathcal{O}^{(n)}$ (where n runs through positive integers) hence $\mathcal{O}^{(n)}$ may play an interesting role (not exploited in our work) in the representation theory of complex semisimple Lie groups.

1.3 Description of results

In Chapter 2 we introduce the main objects of study of our work: the fat Verma modules $\Delta^{(n)}(\lambda)$ lying in the fat category $\mathcal{O}^{(n)}$. Fat Verma modules $\Delta^{(n)}(\lambda)$ are easily seen to be cyclic, indecomposable and filtered by n^m Verma modules $\Delta(\lambda)$ (where m is the rank of the Lie algebra $\mathcal{G}$). Furthermore, we are able to prove in Proposition 2.2.8 that the fat Verma modules have simple socle. We proceed at cruise speed to prove that $\mathcal{O}^{(n)}$ is closed under taking subquotients and tensor products with finite dimensional vector spaces and we describe the simple modules in $\mathcal{O}^{(n)}$, which turn out to be the same as in $\mathcal{O}$. By proving that $\mathcal{O}^{(n)}$ has enough projectives and each object has finite length we open way to the subsequent Chapter 3: the study of blocks of fat category $\mathcal{O}$, $\mathcal{O}_\lambda^{(n)}$, as categories of (finitely generated) modules over finite dimensional algebras $A_{\lambda,n}$. There, in Section 3.2, an example is worth taking a look at: the description of the principal block of $\mathcal{O}^{(n)}$ for $\mathcal{G} = sl_2(\mathbb{C})$. After proving that the algebras $A_{\lambda,n}$ are standardly stratified (they are even properly stratified) we borrow from [1] some relevant features of this class of algebras regarding the existence and behaviour of standard, proper standard and tilting modules. Observing that, for $n > 1$, the algebras $A_{\lambda,n}$ have infinite global dimension, motivates our next step: with the help of a theorem of Platzeck and Reiten (Theorem 3.4.6) we conclude in Corollary 3.4.7 that the full subcategory of $A_{\lambda,n} - mod$ consisting of modules filtered by standard objects is properly contained in the full subcategory whose objects have finite projective dimension.

The largest part of Chapter 3 is taken to investigate the nature of the endomorphism rings of tilting modules. To do so we use Arkhipov's functor $\mathcal{A}$ to conclude in Corollary 3.8.5 that the algebras $A_{0,n}$ (i.e. the principal blocks of $\mathcal{O}^{(n)}$) coincide with their Ringel duals. As a byproduct we obtain in Corollary 3.8.7 the character formulae for fat tilting modules.

The next step reveals a property which is, in general, not shared by Soergel's thick category $\mathcal{O}^{I}$: the existence of an indecomposable projective-injective object, i.e. an object which is simultaneously projective and injective. This is proved in Proposition 3.9.1 and enables us to verify in Proposition 3.9.4 the validity of a double centralizer property relating the principal block of $\mathcal{O}^{(n)}$ with the fat algebra of coinvariants. Chapter 3 ends with a success and a failure. We succeed in classifying the algebras $A_{\lambda,n}$ with respect to their representation type (see Theorem 3.10.6) but fail to relate them in any way with their Ext-algebra (recall that regular blocks of category $\mathcal{O}$ coincide with their homological duals (see [7])).

As the title indicates, Chapter 4 is oriented towards the determination of the basic algebra $A_{\lambda,n}$ by means of quiver and relations. To compute the quivers we make use of Theorem 4.3.1, where we compute (for root systems of rank 1 and 2) the dimension of the Ext^1 spaces between isomorphic simple modules in $\mathcal{O}^{(n)}$, and Theorem 4.4.1, complementing the information on the extensions of non isomorphic simple modules. By borrowing from [33] results describing the behaviour of Soergel's combinatorial functor $\mathbb{V}$ we reprove the above mentioned double centralizer property and we are able to compute not only the quiver but also the relations for all the principal fat blocks of type A_1 and all the singular blocks of type A_2. Here, with a little bit of computer programming expertise we might have gone a little bit further...

Chapter 2

The category $\mathcal{O}^{(n)}$

2.1 Main definitions

Let $\mathcal{G}$ be a complex semisimple Lie algebra of finite dimension. Fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, a root system Φ for the pair $(\mathcal{G}, \mathcal{H})$ and the corresponding Weyl group W. Then, as usual, we have a triangular decomposition of the form

$$\mathcal{G} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+,$$

with respect to Φ, where $\mathcal{B} = \mathcal{H} \oplus \mathcal{N}_+$ is a Borel subalgebra. The corresponding universal enveloping algebras are denoted by $U(\mathcal{G})$, $U(\mathcal{B})$ and so on.

Let $\alpha_1, ..., \alpha_m$ be the simple roots of Φ. It is well known that $U(\mathcal{H})$ is isomorphic to the polynomial ring $\mathbb{C}[H_1, ..., H_m]$, where m is the rank of $\mathcal{G}$ and $H_1, ..., H_m$ is a basis of $\mathcal{H}$ where H_i is the coroot associated to α_i.

Throughout this work, all weights $\lambda \in \mathcal{H}^*$ are taken to be *integral.* For such a weight λ, we consider the ideal

$$I_{n,\lambda} := ((H_1 - \lambda(H_1))^n, ..., (H_m - \lambda(H_m))^n)$$

and the algebra $C_{n,\lambda} = \mathbb{C}[H_1, ...H_m]/I_{n,\lambda}$, which may be viewed as a finite dimensional quotient of $U(\mathcal{H})$.

We will proceed to define the more relevant categories in this work:

Definition 2.1.1 *The objects of category $\mathcal{O}^{(n)}$ (the so called* fat category $\mathcal{O}$) *are left $\mathcal{G}$-modules M with the following properties:*

- *M is finitely generated (as a module over $U(\mathcal{G})$);*
- *M is a direct sum of $C_{n,\lambda}$-modules (as a module over $U(\mathcal{H})$);*
- *M is locally $\mathcal{N}_+$-finite (i.e. for each $m \in M$ the vector space $U(\mathcal{N}_+)m$ is finite dimensional over $\mathbb{C}$).*

The morphisms in $\mathcal{O}^{(n)}$ are arbitrary $\mathcal{G}$-module homomorphisms.

Remark

Observe that under our previous assumptions on weights, all our modules will be *integrally supported.* This means that the decomposition of a module M in $\mathcal{O}^{(n)}$ as a direct sum of $C_{n,\lambda}-$modules is such that λ is integral.

□

There is an alternative definition of $\mathcal{O}^{(n)}$, due to Soergel (see [33]), which may be seen as a particular situation inside the more general framework of "deformation theory": Let I be an ideal of $U(\mathcal{H}) \cong \mathbb{C}[H_1, ..., H_m]$ of finite codimension. On any $\mathcal{G}$-module M which is locally finite over $\mathcal{H}$, the nilpotent part of the $\mathcal{H}$-action gives rise to a new action of $\mathcal{H}$ on M. Let $\mathcal{O}^I$ denote the category whose objects are the locally $U(\mathcal{B})$-finite and finitely generated $\mathcal{G}$-modules such that this "new" action of $\mathcal{H}$ factors through I.

We claim that "our" category $\mathcal{O}^{(n)}$ is no other than $\mathcal{O}^I$ for $I := I_{n,0} = (H_1^n, ..., H_m^n)$. Let us prove this fact starting with some basic knowledge about the representation theory of $\mathbb{C}[H_1, ..., H_m]$. The simple $U(\mathcal{H})$-modules are one-dimensional with (central) character λ for some $\lambda \in \mathcal{H}^*$ (i.e. all elements $h \in \mathcal{H}$ act by scalar multiplication with $\lambda(h)$). Moreover, if M is an indecomposable finite dimensional $U(\mathcal{H})$-module there is an element $\lambda \in \mathcal{H}^*$ (the generalized character of M) such that for each element $h \in \mathcal{H}$ one may find a basis of M in such a way that h acts by multiplication with a direct sum of Jordan blocks having diagonal elements equal to $\lambda(h)$.

Hence this "new" action, $*$, of $\mathcal{H}$ in any finite dimensional indecomposable module M may be described by

$$h * x = (h - \lambda(h))x$$

for all $x \in M$, where λ is the generalized character of M.

But, if $M = \bigoplus_{i \in J} M_i$ is a decomposition of a module M in $\mathcal{O}^{I_{n,0}}$ (in Soergel's sense) as a direct sum of (necessarily finite dimensional) indecomposable $U(\mathcal{H})$-modules, we have that

$$I_{n,0} * M = 0 \iff I_{n,0} * M_i = 0 \text{ for all } i \in J \iff I_{n,\lambda_i} M_i = 0 \text{ for all } i \in J,$$

where λ_i is the character of M_i. Thus we conclude that M is an object of $\mathcal{O}^{(n)}$ in the sense of Definition 2.1.1. Going backwards in the above argument we convince ourselves that both definitions are actually equivalent.

Special objects in $\mathcal{O}^{(n)}$ are the *fat Verma modules* defined as

$$\Delta^{(n)}(\lambda) := U(\mathcal{G}) \otimes_{U(\mathcal{B})} C_{n,\lambda}$$

where the element $b = h + n$ of $U(\mathcal{B})$ acts on the n^m-th dimensional complex vector space $C_{n,\lambda}$ by left multiplication by h.

An important remark is that, for $n = 1$, $\mathcal{O}^{(n)}$ is no other than the classical category $\mathcal{O}$ as defined by Bernstein, Gelfand and Gelfand in [9]. In this setting, we will use the adjective *fat* to characterize objects (or even properties) of $\mathcal{O}^{(n)}$ *but only for* $n > 1$. Objects (or properties) of $\mathcal{O}$ will often be referred to as *classical.*

2.2 Basic properties of $\mathcal{O}^{(n)}$

In the sequel we will try to establish some analogies with the category $\mathcal{O}$. Recalling that we have denoted the rank of $\mathcal{G}$ by m, we have

Proposition 2.2.1 *Every fat Verma module, $\Delta^{(n)}(\lambda)$, is cyclic and filtered by n^m "classical" Verma modules.*

Proof

Let us start by defining some distinguished elements $v_{\lambda,i_1,...,i_m}$ in $\Delta^{(n)}(\lambda)$ in the following way:

$$v_{\lambda,i_1,...,i_m} := 1 \otimes (H_1 - \lambda(H_1))^{i_1} ... (H_m - \lambda(H_m))^{i_m},$$

for all $i_s \in \{0, ..., n-1\}$ and $s \in \{1, ..., m\}$. It is clear that $v_{\lambda,0,...,0}$ generates $\Delta^{(n)}(\lambda)$. The space $U(\mathcal{G})v_{\lambda,n-1,...,n-1}$ is clearly $\mathcal{H}$-invariant and free over $U(\mathcal{N}_-)$ and $U(\mathcal{N}_+)$ annihilates $v_{\lambda,n-1,...,n-1}$, hence $v_{\lambda,n-1,...,n-1}$ generates a classical Verma module. Factoring it out, the result follows inductively.

□

Remark

1) For simplicity, we will abbreviate the above mentioned generator of $\Delta^{(n)}(\lambda)$, $v_{\lambda,0,...,0}$, by v_λ and call it the *canonical generator of* $\Delta^{(n)}(\lambda)$.

2) Define the λ, n*-weight space* (or n*-fat* λ*-weight space*) of a $\mathcal{G}$-module M as

$$M_\lambda^{(n)} := \{x \in M : (H_i - \lambda(H_i))^n x = 0, \text{for all } i \in \{1, ..., m\}\}.$$

The vector space $M_\lambda^{(n)}$ is evidently a $C_{n,\lambda}$-module, hence the second condition in the definition of $\mathcal{O}^{(n)}$ can be replaced by the more suggestive

• M is the direct sum of its n-fat weight spaces (or fat weight spaces, for short).

3) Observe that to each non-zero element in $M_\lambda^{(n)}$ one can attach a radical (and a socle) layer where it naturally lives. One further observes that

the indecomposable summands of the fat weight spaces of fat Verma modules $\Delta^{(n)}(\lambda)$ have $m(n-1)+1$ layers, the first and the last ones being one dimensional.

4) It is easy to see that, for all $s \geq 1$, we have $dim(\Delta^{(n)}(\lambda))_{\lambda}^{(s)} = s^m$.

5) Consider the truncated polynomial ring $\mathbb{C}[H_1, ...H_m]/(H_1^n, ..., H_m^n)$ which is our guiding light for the fat weight spaces of fat Verma modules. As a module over $\mathbb{C}[H_1, ...H_m]$, it is generated by (the equivalence class of) 1, which will be called homogeneous of degree 0, while $H_1, ..., H_m$ will be called homogeneous of degree 1, and so on. The point is that this terminology may (and will) be naturally transported to the fat weight spaces of our fat Verma modules.

$\square$

There is an alternative construction of "fat" Verma modules by "generators and relations". To achieve this start by defining $J_{\lambda,n}$ as the left ideal of $U(\mathcal{G})$ generated by $\mathcal{N}^+$ together with the elements of the form $(H_i - \lambda(H_i))^n$ where $i \in \{1, ..., m\}$. Then we have

Proposition 2.2.2 *1) Denoting the left coset of* 1 *by* $\bar{1}$, *the map*

$$U(\mathcal{G})/J_{\lambda,n} \longrightarrow \Delta^{(n)}(\lambda)$$

$$\bar{1} \mapsto v_{\lambda}$$

is an isomorphism of $U(\mathcal{G})$-modules.

2) For every sequence $i_1, ..., i_m$ where $i_s \in \{0, ..., n-1\}$ for all $s \in \{1, ..., m\}$, the map

$$\Phi_{i_1,...,i_m} : \Delta^{(n)}(\lambda) \longrightarrow \Delta^{(n)}(\lambda)$$

$$v_\lambda \mapsto v_{\lambda,i_1,...,i_m}$$

is a well defined homomorphism of $\mathcal{G}$-modules.

Proof

Starting by 1), it is clear that $J_{\lambda,n}$ annihilates v_λ thus proving that the map is well defined and, therefore, surjective.

It is also clear that $U(\mathcal{N}_-)$ acts injectively on both modules (i.e. left multiplication by any fixed element of $U(\mathcal{N}_-)$ is an injective map). Hence, by applying the Poincare-Birkhoff-Witt theorem (which, from now on, will be referred to as "PBW theorem") it is enough to show that for all $u_\mathcal{H} \in U(\mathcal{H})$ we have

$$u_\mathcal{H} v_\lambda = 0 \Longrightarrow \bar{u_\mathcal{H}} = 0$$

which follows immediately from the fact that $Ann_{U(\mathcal{H})}(v_\lambda) = I_{\lambda,n}$.

Assertion 2) follows immediately from 1) since $v_{\lambda,i_1,...,i_m}$ is easily seen to be annihilated by $J_{\lambda,n}$. □

Proposition 2.2.3 *Every fat Verma module is indecomposable.*

Proof

To prove the indecomposability of $\Delta^{(n)}(\lambda)$ observe that by restricting to the $U(\mathcal{H})$-module $C_{n,\lambda}$, one has the following isomorphism of vector spaces:

$$End_{U(\mathcal{G})}(U(\mathcal{G})\otimes_{U(\mathcal{B})}C_{n,\lambda})$$

$$\cong Hom_{U(\mathcal{H})}(C_{n,\lambda}, \Delta^{(n)}(\lambda)).$$

But morphisms in $\mathcal{O}^{(n)}$ preserve fat weight spaces, thus

$$Hom_{U(\mathcal{H})}(C_{n,\lambda}, \Delta^{(n)}(\lambda)) \cong End_{U(\mathcal{H})}(C_{n,\lambda}) \cong C_{n,\lambda},$$

which is a local ring.

By part 2) of the previous proposition we may conclude that the morphisms $\Phi_{i_1,\dots,i_m}$ with $i_s \in \{0, \dots, n-1\}$ and $s \in \{1, \dots, m\}$ form a basis of $End_{U(\mathcal{G})}(\Delta^{(n)}(\lambda))$. This allows us to conclude that $End_{U(\mathcal{G})}(\Delta^{(n)}(\lambda))$ and $C_{n,\lambda}$ are actually isomorphic as rings.

□

Example 2.2.4 *Let* $Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *be the standard generators of the Lie algebra* $\mathcal{G} = sl_2(\mathbb{C})$.

Below we will try to depict some Verma modules, namely $\Delta(0)$, $\Delta^{(2)}(0)$, $\Delta(-1)$ *and* $\Delta^{(2)}(-1)$. *The reason for choosing these highest weights will (hopefully) become more clear in subsequent sections.*

In our pictures, (fat) weight spaces will be labelled by the corresponding weight while the action of the generators X, Y and H will be represented (respectively) by arrows $\rightarrow$, $\leftarrow$ and $\circlearrowleft$ (plus its matrix realization). We denote by I_2 the identity matrix of order 2.

Recalling that X acts like 0 on (fat) highest weight spaces, we represent

$$
\begin{tikzcd}
\Delta(0) : \cdots \arrow[r, shift left, "\cdots"] & -4 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "-2"] \arrow[loop below, "-4"] & -2 \arrow[l, shift left, "1"] \arrow[r, shift left, "0"] \arrow[loop below, "-2"] & 0 \arrow[l, shift left, "1"] \arrow[loop below, "0"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta(-1) : \cdots \arrow[r, shift left, "\cdots"] & -5 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "-4"] \arrow[loop below, "-5"] & -3 \arrow[l, shift left, "1"] \arrow[r, shift left, "-1"] \arrow[loop below, "-3"] & -1 \arrow[l, shift left, "1"] \arrow[loop below, "-1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\Delta^{(2)}(0) : \cdots \arrow[r, shift left, "\cdots"] & -4 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}"] \arrow[loop below, "\begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}"] & -2 \arrow[l, shift left, "I_2"] \arrow[r, shift left, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[loop below, "\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}"] & 0 \arrow[l, shift left, "I_2"] \arrow[loop below, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\Delta^{(2)}(-1) : \cdots \arrow[r, shift left, "\cdots"] & -5 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\begin{pmatrix} -4 & 2 \\ 0 & -4 \end{pmatrix}"] \arrow[loop below, "\begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix}"] & -3 \arrow[l, shift left, "I_2"] \arrow[r, shift left, "\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}"] \arrow[loop below, "\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}"] & -1 \arrow[l, shift left, "I_2"] \arrow[loop below, "\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}"]
\end{tikzcd}
$$

The following lemma will be crucial for several results throughout this chapter:

Lemma 2.2.5 *Let M be a $\mathcal{G}$-module which is the direct sum of its fat k-weight spaces and let N be a submodule of M. Then N is also the direct sum of its fat k-weight spaces.*

Remark Note that all $\mathcal{G}$-modules contain the (direct) sum of its fat weight spaces as a submodule. □

In the sequel, we will use the "hat" symbol in a product to express an absence. For instance, the sequence $a\hat{b}c\hat{d}e$ should be read as ace.

Proof

As indicated, assume that

$$M = \bigoplus_{\lambda \in \mathcal{H}^*} M_\lambda^{(k)}$$

and suppose that x is an element of N of the form

$$x = m_{\lambda_1} + m_{\lambda_2} + ... + m_{\lambda_n}$$

where $m_{\lambda_j} \in M_{\lambda_j}^{(k)}$ for $j = 1, 2, ..., n$. This means that

$$(H_i - \lambda_j(H_i))^k m_{\lambda_j} = 0 \ (1)$$

for $i = 1, ..., m$ and $j = 1, 2..., n$.

To ease notation, set $a_{p,i} = -\lambda_p(H_i)$ for $i = 1, ..., m$ and $p = 1, ..., n$. By (1) we have that

$$((H_{i_1} + a_{1,i_1})^k ... (\widehat{H_{i_r} + a_{r,i_r}})^k ... (H_{i_n} + a_{n,i_n})^k) m_{\lambda_r} \in N$$

for all $i_s \in \{1, .., m\}$ and $s \in \{1, ..., n\}$.

Now, observe that for all $i_2 = 1, .., m$,

$$0 = (H_{i_2} + a_{1,i_2})^k m_{\lambda_1} = (H_{i_2} + a_{2,i_2} + a_{1,i_2} - a_{2,i_2})^k m_{\lambda_1} =$$

$$= ((H_{i_2} + a_{2,i_2})^k + (H_{i_2} + a_{2,i_2})^{k-1}(a_{1,i_2} - a_{2,i_2}) + ... + (a_{1,i_2} - a_{2,i_2})^k) m_{\lambda_1} \quad (2)$$

Multiplying (2) by $(\widehat{H_{i_1} + a_{1,i_1}})^k (H_{i_2} + a_{2,i_2})^{k-1} ... (H_{i_n} + a_{n,i_n})^k$ and using (1) one concludes that

$$(a_{1,i_2} - a_{2,i_2})^k (\widehat{H_{i_1} + a_{1,i_1}})^k (H_{i_2} + a_{2,i_2})^{k-1} ... (H_{i_n} + a_{n,i_n})^k m_{\lambda_1} \in N \quad (3)$$

for all $i_s \in \{1, .., m\}$ and $s \in \{2, ..., n\}$.

Exactly the same way we may conclude that, for all $r, t = 1, ..., n$

$$(a_{r,i_s} - a_{s,i_s})^k (H_{i_1} + a_{1,i_1})^k (\widehat{H_{i_r} + a_{r,i_r}})^k ... (H_{i_t} + a_{s,i_t})^{k-1} ... (H_{i_n} + a_{n,i_n})^k m_{\lambda_r} \in N \quad (4)$$

for all $i_s \in \{1, .., m\}$ and $s \in \{1, ..., n\}$.

Comparing (4) (or (3) which is a particular case of (4)) with (1) we observe that we managed to decrease by 1 the degree of our polynomial. We repeat this process until we reach a constant polynomial. At that point we will have, for all $r = 1, ..., n$,

$$((a_{r,i_1} - a_{1,i_1})^k)...(\widehat{a_{r,i_r} - a_{r,i_r}})^k...(a_{r,i_n} - a_{n,i_n})^k)m_{\lambda_r} \in N$$

for all $i_s \in \{1,..,m\}$ and $s \in \{1,...,n\}$.

Finally, choosing i_s such that $a_{r,i_s} \neq a_{s,i_s}$ (recall that $\lambda_r \neq \lambda_s$) we conclude that $m_{\lambda_r} \in N$ for all $r = 1,..,n$.

□

Proposition 2.2.6 *1) $\mathcal{O}^{(n)}$ is closed under taking subquotients and tensor products with finite dimensional vector spaces.*

2) In $\mathcal{O}^{(n)}$, fat weight spaces and homomorphism spaces are finite dimensional.

Proof

If N is a submodule of a module M in $\mathcal{O}^{(n)}$ then, by Proposition 2.2.5, it is also the direct sum of its weight spaces. It is obvious that N is $U(\mathcal{B})$-finite and to claim it is finitely generated one only needs to invoke the well known fact that $U(\mathcal{G})$ is noetherian (since submodules of finitely generated modules over noetherian rings are also finitely generated).

Now let $f : M \longrightarrow N$ be an epimorphism of $\mathcal{G}$-modules. It is a straightforward exercise to verify that N is an object of $\mathcal{O}^{(n)}$ if M is.

Finally, what can be said about $M \otimes E$ for a $\mathcal{G}$-module M in $\mathcal{O}^{(n)}$ and a finite dimensional $\mathcal{G}$-module E? Start by recalling the classical fact that all finite dimensional $\mathcal{G}$-modules belong to the category $\mathcal{O}$. Now, if $(e_1, ..., e_s)$ is a $\mathbb{C}$-basis of E where each e_i has weight λ_i and $\{m_1, ..., m_r\}$ is a generating set for M, then the set $\{m_j \otimes e_i : 1 \leq j \leq r, 1 \leq i \leq s\}$ clearly generates $M \otimes E$. Again, $U(\mathcal{B})$-finiteness is obvious and, if m_λ is a vector of M having fat weight λ, then the identity

$$(H - (\lambda + \lambda_i)(H))^n (m_\lambda \otimes e_i) = ((H - (\lambda)(H))^n m_\lambda) \otimes e_i$$

shows us that $M \otimes E$ is a direct sum of its fat weight spaces. This proves 1).

To prove 2), again suppose that M is an object of $\mathcal{O}^{(n)}$ admitting $\{m_1, ..., m_r\}$ as a generating set with each m_j an element with fat weight λ_j.

Since $M = U(\mathcal{G})m_1 + ... + U(\mathcal{G})m_r$, we can restrict ourselves to proving that the cyclic modules $U(\mathcal{G})m_i$ have finite dimensional fat weight spaces.

Using PBW Theorem, we observe that, as vector spaces,

$$U(\mathcal{G})m_i = U(\mathcal{N}_-) \otimes (U(\mathcal{B})m_i).$$

But $U(\mathcal{B})m_i$ has finite dimension p_i (say), hence all fat weight spaces $(U(\mathcal{G})m_i)_\lambda^{(n)}$ (with $\lambda \leq \lambda_i$) will have dimension smaller than $p_i k(\lambda - \lambda_i)$, where k is the Kostant partition function.

Now it is easy to prove that the dimension of $Hom_{\mathcal{O}^{(n)}}(M, N)$ is finite if N is another element of $\mathcal{O}^{(n)}$: if f is such a homomorphism, then it is determined by the image of the generators of M. But $f(M_\lambda^{(n)}) \subseteq N_\lambda^{(n)}$, hence

$$Hom_{\mathcal{O}^{(n)}}(M, N) \subseteq Hom_{\mathbb{C}}(M_{\lambda_1}^{(n)} \oplus ... \oplus M_{\lambda_r}^{(n)}, N_{\lambda_1}^{(n)} \oplus ... \oplus N_{\lambda_r}^{(n)}),$$

which, as we have just seen, is finite dimensional.

□

Proposition 2.2.7 *The module top*$\Delta^{(n)}(\lambda)$ *is simple and all simples in* $\mathcal{O}^{(n)}$ *occur in this way.*

Proof

Let v_λ be the canonical generator of $\Delta^{(n)}(\lambda)$ and N one of its proper submodules. Clearly $\mathbb{C}v_\lambda \cap N = \{0\}$ and, by Lemma 2.2.5, one concludes that v_λ does not belong to the sum of all proper submodules of $\Delta^{(n)}(\lambda)$. This forces the existence of a unique maximal submodule (which has to be $Rad(\Delta^{(n)}(\lambda))$). It is obvious that $top\Delta^{(n)}(\lambda)$ is an object of $\mathcal{O}^{(n)}$. In fact, if L is a simple object in $\mathcal{O}^{(n)}$, it is generated by a highest weight vector (which is the same as saying that such a vector has to exist...), hence it is a quotient of some Verma module in the category $\mathcal{O}$.

□

Remark The previous proposition, combined with the fact that fat Verma modules are filtered by Verma modules of category $\mathcal{O}$, implies that simple objects of $\mathcal{O}^{(n)}$ are the same for all n (in particular, they coincide with the simple objects of $\mathcal{O}$).

Hence, from now on, we will denote the *top* of $\Delta^{(n)}(\lambda)$ by $L(\lambda)$.

□

The next proposition expresses further similarities with $\mathcal{O}$:

Proposition 2.2.8 *Let λ be a weight. Then fat Verma modules $\Delta^{(n)}(\lambda)$ have simple socle $L(\alpha)$ where α is the unique antidominant weight in the orbit of λ under the dot action of the Weyl group.*

Proof

For category $\mathcal{O}$ (i.e. $n = 1$) this result is well known: it follows from the fact that Verma modules are isomorphic to $U(\mathcal{N}_-)$ (as $U(\mathcal{N}_-)$-modules) and any two proper (left) ideals of $U(\mathcal{N}_-)$ intersect non trivially.

Now the fat Verma modules are filtered by Verma modules, hence their socle is a direct sum of isomorphic simple modules. Consequently, each simple component of the socle of $\Delta^{(n)}(\lambda)$ is generated by a vector of fat weight α. But the structure of $(\Delta^{(n)}(\lambda))_\alpha^{(n)}$ as $U(\mathcal{H})$-module is well understood: it consists of a certain number of copies of $C_{n,\alpha}$ (given by the Kostant partition function). From this we conclude that the socle of $\Delta^{(n)}(\lambda)$ must be generated by vectors of weight α (in other words: vectors belonging to the socle of $(\Delta^{(n)}(\lambda))_\alpha^{(n)}$, considered as a module over $U(\mathcal{H})$). But

$$\Delta(\lambda) \cong \sum_{\mu \in \mathcal{H}^*} (\Delta^{(n)}(\lambda))_\mu,$$

since the RHS is clearly a submodule of $\Delta^{(n)}(\lambda)$ generated by $v_{\lambda,n-1,\dots,n-1}$. Thus $soc(\Delta^{(n)}(\lambda)) \subseteq soc(\Delta(\lambda))$ which is simple.

Finally, $L(\alpha) \cong \Delta(\alpha)$, since $U(\mathcal{N}_-)$ acts injectively on fat Verma modules (and, therefore, on its submodules), hence, by a classical result (see [16], Proposition 7.6.24), we know that α is antidominant.

□

2.3 Block decomposition

Let us recall some facts from classical Lie theory. Apart from the natural action of the Weyl group W on $\mathcal{H}^*$ we have the so called *dot action* of W on $\mathcal{H}^*$ which, for $w \in W$ and $\lambda \in \mathcal{H}^*$, is defined as

$$w.\lambda := w(\lambda + \rho) - \rho,$$

where ρ is the half sum of the positive roots of Φ.

Now, let $\phi : U(\mathcal{G}) \longrightarrow \mathcal{U}(\mathcal{H})$ be the Harish-Chandra homomorphism (i.e. the projection onto $U(\mathcal{H})$ along the decomposition

$$U(\mathcal{G}) = (\mathcal{N}_-\mathcal{U}(\mathcal{G}) \oplus \mathcal{U}(\mathcal{G})\mathcal{N}_+) \oplus \mathcal{U}(\mathcal{H}))$$

and view an element $\lambda \in \mathcal{H}^*$ as an algebra homomorphism from $U(\mathcal{H})$ to $\mathbb{C}$. Construct the central character χ_λ as follows:

$$\chi_\lambda := (\lambda \circ \phi)_{|Z(\mathcal{G})} : Z(\mathcal{G}) \longrightarrow \mathbb{C}$$

It is well known (see [23], Chapter 3) that every central character of $U(\mathcal{G})$ (i.e. a homomorphism from the center of $U(\mathcal{G})$ to $\mathbb{C}$) is of the form χ_λ for some $\lambda \in \mathcal{H}^*$ and, furthermore, $\chi_\lambda = \chi_\mu$ if and only if $\lambda = w.\mu$ for some element w in the Weyl group W.

Now let M be a $\mathcal{G}$-module and χ be a central character of $U(\mathcal{G})$. Denote by M_χ the submodule of M defined by

$$M_\chi := \{m \in M : \forall z \in Z(\mathcal{G}) \exists k \in \mathbb{N} : (z - \chi(z))^k m = 0\}$$

and let $\mathcal{O}_\chi^{(n)}$ be the full subcategory of $\mathcal{O}^{(n)}$ whose objects are the $\mathcal{G}$-modules M which coincide with M_χ.

By what has been said above, one may relabel $\mathcal{O}_{\chi_\lambda}^{(n)}$ by $\mathcal{O}_\lambda^{(n)}$, bearing in mind that $\mathcal{O}_\lambda^{(n)} = \mathcal{O}_\mu^{(n)}$ if and only if λ and μ lie in the same orbit under the dot action of the Weyl group.

Proposition 2.3.1 *Consider the fat category $\mathcal{O}^{(n)}$. Then*

1) Exactly as for category $\mathcal{O}$, we have a block decomposition of the form

$$\mathcal{O}^{(n)} = \bigoplus_\lambda \mathcal{O}_\lambda^{(n)},$$

where λ runs through a set of representatives of orbits in $\mathcal{H}^*$ for the dot action of the Weyl group. Consequently, all composition factors of modules in $\mathcal{O}_\lambda^{(n)}$ are of the form $L(\alpha)$, where α lies in the orbit of λ under the dot action of the Weyl group.

2) All objects of $\mathcal{O}^{(n)}$ have finite length.

Remark

Calling $\mathcal{O}_\lambda^{(n)}$ a *block* does not constitute an abuse of language since the categories $\mathcal{O}_\lambda^{(n)}$ cannot decompose further: in fact, any two simples in $\mathcal{O}_\lambda^{(n)}$ are homomorphic images of the corresponding Verma modules which have the same socle.

□

Proof

Since M is the direct sum of its fat weight spaces, it suffices to show that for all $\lambda \in \mathcal{H}^*$ there exist central characters $\chi_1, ..., \chi_k$ such that

$$M_\lambda^{(n)} \subseteq M_{\chi_1} + ... + M_{\chi_k},$$

whose sum is easily seen to be direct.

It is clear that fat weight spaces are $Z(\mathcal{G})$-modules, thus multiplication by central elements yields endomorphisms of fat weight spaces.

Now suppose that such an element $z \in Z(\mathcal{G})$ acts on $M_\lambda^{(n)}$ by a sum of Jordan blocks of the form

$$J_{a_z} := \begin{pmatrix} a_z & 1 & & \\ 0 & \ddots & \ddots & \\ & & \ddots & 1 \\ & & 0 & a_z \end{pmatrix}.$$

Recalling the fact that any family of commuting endomorphisms of a finite dimensional vector space admits a common eigenvector, one is able to define a map

$$\chi : Z \longrightarrow \mathbb{C}$$

$$z \mapsto \chi(z)$$

where $\chi(z)$ is the eigenvalue corresponding to the above cited common eigenvector of the endomorphims induced by the elements $z \in Z$. It is straightforward to check that χ is actually a central character of Z: in effect, for elements $z_1, z_2 \in Z$ and denoting by v the common central eigenvector, we have

$$\chi(z_1 z_2)v = (z_1 z_2)v = z_1(z_2 v) = z_1(\chi(z_2)v) =$$

$$= \chi(z_2)(z_1 v) = \chi(z_2)\chi(z_1)v = \chi(z_1)\chi(z_2)v.$$

Moreover, since we are talking about eigenvalues, we have that

$$im\chi \subseteq \{a_z : z \in Z\},$$

hence for each weight λ there exist a character χ such that

$$M_\lambda^{(n)} \cap M_\chi \neq 0$$

which implies that

$$dim(M/M_\chi)_\lambda^{(n)} \leq dim M_\lambda^{(n)}.$$

By induction one concludes that there exist central characters $\chi_1, ..., \chi_k$ such that

$$M_\lambda^{(n)} \subseteq M_{\chi_1} + ... + M_{\chi_k}.$$

Finally, we observe that $M_\chi \cap M_{\chi'} = 0$ if $\chi \neq \chi'$, from which it follows that

$$M = \bigoplus_{\chi} M_\chi,$$

where χ runs through the set of all central characters. Note that M is finitely generated, hence only a finite number of central characters yield non zero summands in the above decomposition.

To prove 2), observe that for an object M in $\mathcal{O}^{(n)}$, only finitely many non isomorphic composition factors can occur. Thus, if the length of M is not finite, then it admits an infinite number of isomorphic composition factors yielding the existence of infinite dimensional weight spaces. □

Remark

1) Let W_λ denote, as usual, the stabilizer of $\lambda \in \mathcal{H}^*$ for the natural action of the Weyl group W. A block $\mathcal{O}_\lambda^{(n)}$ is called *regular* if λ is regular (i.e. $|W_\lambda| = 1$); otherwise it is called *singular*. Usually the most "interesting" block is the *principal block*, $\mathcal{O}_0^{(n)}$, the one containing the trivial representation.

2) The block decomposition proved above allows us to define a "projection onto the block" functor, Pr_λ, from $\mathcal{O}^{(n)}$ to $\mathcal{O}_\lambda^{(n)}$, by which an element M in $\mathcal{O}^{(n)}$ is sent to its maximal summand in $\mathcal{O}_\lambda^{(n)}$. Furthermore, for dominant weights λ and μ, one may construct the so called "translation functor", T_λ^μ, from $\mathcal{O}_\lambda^{(n)}$ to $\mathcal{O}_\mu^{(n)}$, as follows:

$$T_\lambda^\mu : \mathcal{O}_\lambda^{(n)} \longrightarrow \mathcal{O}_\mu^{(n)}$$

$$M \mapsto Pr_\mu(E(\mu - \lambda) \otimes M),$$

where $E(\mu - \lambda)$ stands for the unique finite dimensional simple module admitting $\mu - \lambda$ as highest weight. Since tensoring with finite dimensional $\mathcal{G}$-modules and projecting onto blocks are both exact functors, we conclude that translation functors are exact as well.

□

2.4 Projective objects in $\mathcal{O}^{(n)}$

The aim of this section is to present some properties of the projective objects of $\mathcal{O}^{(n)}$ which generalize the corresponding ones in $\mathcal{O}$. Firstly, let us prove a kind of "recognition" result which tell us how to acknowledge being in the presence of a fat Verma module:

Lemma 2.4.1 *("recognition lemma")*

Let M be a cyclic (left) $U(\mathcal{G})$-module generated by a vector w_λ of highest n-fat weight λ. Then

a) There is an epimorphism $f : \Delta^{(n)}(\lambda) \longrightarrow M$ defined by sending v_λ, the canonical generator of $\Delta^{(n)}(\lambda)$, to w_λ.

b) The above map is an isomorphism if and only if two things happen:

i) $U(\mathcal{N}_-)$ *acts injectively on* M *and*

ii) $U(\mathcal{H})w_\lambda \cong C_{n,\lambda}$.

Proof

The map f is obviously well defined since our assumptions are just another way of saying that

$$Ann_{U(\mathcal{G})}v_\lambda \subseteq Ann_{U(\mathcal{G})}w_\lambda$$

and consequently it is surjective since generator is mapped to generator. This proves a). To prove b), assume that $U(\mathcal{N}_-)$ acts injectively on M and that the $U(\mathcal{H})$-module generated by w_λ is isomorphic to $C_{n,\lambda}$. Then we construct a map

$$g : M \longrightarrow \Delta^{(n)}(\lambda)$$

$$xw_\lambda \mapsto xv_\lambda,$$

for all $x \in U(\mathcal{G})$, which, so we claim, is well defined: In fact, if x is an element of $U(\mathcal{G})$ of the form

$$x = x_{n_-}x_h,$$

where $x_{n_-} \in U(\mathcal{N}_-)$ and $x_h \in U(\mathcal{H})$, then, since $U(\mathcal{N}_-)$ acts injectively on M, we have

$$xw_\lambda = 0 \iff x_h w_\lambda = 0$$

which, when combined with the fact that the $U(\mathcal{H})$-module generated by w_λ is isomorphic to $C_{n,\lambda}$, forces

$$Ann_{U(\mathcal{G})} w_\lambda \subseteq Ann_{U(\mathcal{G})} v_\lambda.$$

□

Proposition 2.4.2 *Let E be a finite dimensional $\mathcal{G}$-module and $(e_1, ..., e_k)$ a basis of E such that, for all i, we have the vector e_i of weight λ_i. Assuming the indexing satisfies*

$$\lambda_i \leq \lambda_j \Longrightarrow j \leq i$$

and fixing $\lambda \in \mathcal{H}^*$, we have:

The $\mathcal{G}$-module $\Delta^{(n)}(\lambda) \otimes E$ has a filtration

$$\Delta^{(n)}(\lambda) \otimes E = M_k \supseteq M_{k-1} \supseteq ... \supseteq M_0 = 0$$

such that, for all $i \in \{1, 2..., k\}$, M_i/M_{i-1} is isomorphic to $\Delta^{(n)}(\lambda + \lambda_i)$.

Proof

Let v_λ be the canonical generator of $\Delta^{(n)}(\lambda)$ and set $w_i := v_\lambda \otimes e_i$. Further, construct, for $i = 1, ..., k$,

$$M_i = U(\mathcal{G})w_1 + ... + U(\mathcal{G})w_i.$$

One is easily convinced that $\Delta^{(n)}(\lambda) \otimes E = M_k$ given that $v_\lambda \otimes E \subseteq M_k$ and v_λ generates $\Delta^{(n)}(\lambda)$.

Additionally, $w_i + M_{i-1}$ is a vector of fat n-weight $\lambda + \lambda_i$ generating M_i/M_{i-1}.

Moreover, if α is a positive root and $x_\alpha \in \mathcal{G}_\alpha$, then the routine calculation

$$x_\alpha w_i = v_\lambda \otimes x_\alpha e_i \in M_{i-1}$$

shows that $\mathcal{N}^+$ kills $w_i + M_{i-1}$ and, consequently,

$$M_i = U(\mathcal{B}_-)w_1 + \ldots + U(\mathcal{B}_-)w_i$$

Where, as usual, $\mathcal{B}_-$ stands for the Lie algebra $\mathcal{N}_- \oplus \mathcal{H}$. To invoke successfully the recognition lemma we only need to show that $U(\mathcal{N}_-)$ acts injectively on M_i/M_{i-1} (which follows immediately from the fact that $U(\mathcal{N}_-)$ acts injectively on v_λ) and that the $U(\mathcal{H})$-module generated by $w_i + M_{i-1}$ is isomorphic to $C_{n,\lambda+\lambda_i}$. For this we use the formula

$$(H_j - (\lambda + \lambda_i)(H_j))(v_\lambda \otimes e_i) = (H_j - \lambda(H_j)v_\lambda) \otimes e_i),$$

which implies

$$Ann_{U(\mathcal{H})}((v_\lambda \otimes e_i) + M_{i-1}) = Ann_{U(\mathcal{H})}(v_{\lambda+\lambda_i})$$

allowing us to construct a map

$$\psi : U(\mathcal{H})(w_i + M_{i-1}) \longrightarrow C_{n,\lambda+\lambda_i}$$

$$v_\lambda \otimes e_i \mapsto v_{\lambda+\lambda_i},$$

which is well defined and injective. It is also surjective since it sends generator to generator. Clearly ψ preserves the $U(\mathcal{H})$-module structure hence it is an isomorphism.

□

Remark The above result tells us precisely how many fat Verma subquotients of a given weight α appear in such a decomposition: As many as the dimension of the subspace of E consisting of vectors of weight $\alpha - \lambda$. □

Proposition 2.4.3 *Let λ be a dominant weight. Then $\Delta^{(n)}(\lambda)$ is projective in $\mathcal{O}^{(n)}$ and in $\mathcal{O}_\lambda^{(n)}$.*

Proof

Let us show, equivalently, that the functor $Hom_{\mathcal{O}^{(n)}}(\Delta^{(n)}, -)$ is exact.

Firstly, we observe that proving for $\mathcal{O}_\lambda^{(n)}$ implies proving for $\mathcal{O}^{(n)}$.

Now let $M \in \mathcal{O}^{(n)}$, let $M_\lambda^{(n)}$ denote the subspace of M consisting of vectors with fat weight λ and let v_λ be the canonical generator of $\Delta^{(n)}(\lambda)$.

Since taking fat weight spaces is clearly an exact functor from $\mathcal{O}_\lambda^{(n)}$ to $\mathbb{C} - mod$, it is enough to show that the map

$$\Phi : Hom_{\mathcal{O}^{(n)}}(\Delta^{(n)}(\lambda), M) \longrightarrow M_\lambda^{(n)}$$

$$f \longrightarrow f(v_\lambda)$$

is a bijection. But v_λ is a generator of fat weight λ of $\Delta^{(n)}(\lambda)$, hence Φ is injective and well defined.

In order to check surjectivity, let us consider the following sets:

$$A = \{f(v_\lambda) : f \in Hom_{\mathcal{O}^{(n)}}(\Delta^{(n)}(\lambda), M\}$$

and

$$B = \{v \in M_\lambda^{(n)} : \mathcal{N}^+ v = 0\}.$$

We will be done if we manage to show that the following statements are true:

$$\bullet A = Im\Phi;$$

$$\bullet A = B;$$

$$\bullet B = M_\lambda^{(n)}.$$

The first one is obvious and it is also obvious that $A \subseteq B$ (since v_λ is killed by $\mathcal{N}^+$). To show that $B \subseteq A$, let w_λ be an element of $M_\lambda^{(n)}$. Then we have

$$Ann_{U(\mathcal{G})}(v_\lambda) = \langle \mathcal{N}^+, ((H_1 - \lambda(H_1))^n, ..., (H_m - \lambda(H_m))^n) \rangle \subseteq Ann_{U(\mathcal{G})}(w_\lambda).$$

Therefore, there exists a morphism $f \in Hom_{\mathcal{O}^{(n)}}(\Delta^{(n)}(\lambda), M)$ defined by

$$f(v_\lambda) = w_\lambda.$$

Finally, we prove that $\mathcal{N}^+ M^{(n)}{}_\lambda = 0$. In fact, given that M is a module in the block $\mathcal{O}_\lambda^{(n)}$, all its composition factors are of the form $L(\alpha)$ with $\alpha \leq \lambda$ because λ is dominant, thus proving the result.

□

Remark There exists a canonical isomorphism of vector spaces

$$Hom_{\mathcal{G}}(M \otimes E, N) \cong Hom_{\mathcal{G}}(M, N \otimes E^*).$$

Since tensoring with finite-dimensional vector spaces is an exact functor in $\mathcal{G} - mod$ and $\mathcal{O}^{(n)}$ is closed under tensor products, we observe that $M \otimes E$ is projective if M is so. □

Proposition 2.4.4 *Let $\lambda \in \mathcal{H}^*$ be a weight. Then*

1) There exists a unique (up to isomorphism) projective module $P^{(n)}(\lambda) \in \mathcal{O}_\lambda^{(n)}$ such that

$$P^{(n)}(\lambda)/Rad(P^{(n)}(\lambda)) \cong L(\lambda).$$

In particular, $P^{(n)}(\lambda)$ is indecomposable.

2) The set $\{P^{(n)}(\lambda) : \lambda \in \mathcal{H}^*\}$ is a complete set of representatives of isomorphism classes of projective indecomposable modules in $\mathcal{O}^{(n)}$.

Proof

Let $\lambda \in \mathcal{H}^*$ and choose a dominant weight $\alpha \in \mathcal{H}^*$ in such a way that $\alpha - \lambda$ is a dominant integral weight (one can always do this since the set of dominant weights is a cone in an euclidean space). Define furthermore

$E := L(\alpha - \lambda)$ so that α is the highest fat weight of $L(\lambda) \otimes E$. Hence $\Delta^{(n)}(\alpha)$ maps onto a submodule of $L(\lambda) \otimes E$ and for this reason

$$Hom_{\mathcal{G}}(\Delta^{(n)}(\alpha), L(\lambda) \otimes E) = Hom_{\mathcal{G}}(\Delta^{(n)}(\alpha) \otimes E^*, L(\lambda)) \neq 0.$$

By the previous remark, we can now say that every simple module is a homomorphic image of a projective module. Since we are working inside a category where all objects have finite length, we derive that all simples (and consequently all modules) admit a projective cover thus proving 1) and 2).

□

Definition 2.4.5 *A $U(\mathcal{G})$-module M is said to have an* n-fat Verma flag *if there exists a chain of submodules*

$$0 \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_k = M$$

such that, for all $i \in \{1, ..., k-1\}$, we have $M_{i+1}/M_i \cong \Delta^{(n)}(\lambda_{i+1})$ for some weight $\lambda_{i+1} \in \mathcal{H}^*$.

Further, define $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ as the full subcategory of $\mathcal{O}^{(n)}$ consisting of objects that have an n-fat Verma flag.

Proposition 2.4.6 *The category $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ is closed under taking direct summands.*

Proof

Let M be an object in $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ and let λ be a maximal weight among those which appear in the Verma flag of M. Since M is the direct sum of its fat weight spaces, we may pick an element v living in the lowest layer of $M_\lambda^{(n)}$. Our first aim is to show that the $U(\mathcal{G})$-module generated by v is isomorphic to $\Delta^{(n)}(\lambda)$.

By the definition of Verma flag there exists a number $j \in \{1, ..., k\}$ such that $v \in M_j$ but $v \notin M_{j-1}$ (assume $M_{-1} = 0$). Then $v + M_{j-1}$ is an element of $(M_j/M_{j-1})_\lambda^{(n)}$. But $M_j/M_{j-1} \cong \Delta^{(n)}(\lambda_j)$ hence we may conclude that $\lambda = \lambda_j$ and that $v + M_{j-1}$ is actually a generator of M_j/M_{j-1}. From here we deduce that the $U(\mathcal{H})$-module generated by v is isomorphic to $C_{n,\lambda}$. This last assertion, combined with the obvious observation that $\mathcal{N}^+ v = 0$, and combined with the fact that $U(\mathcal{N}_-)$ acts injectively on v (since it does so on $v + M_{j-1}$), allows us to conclude that $U(\mathcal{G})v \cong \Delta^{(n)}(\lambda)$.

Now we proceed to prove that all summands of M are still objects of $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ by induction on the length of M . To that effect, suppose that $M = M_1 \oplus M_2$. The element v picked above decomposes as $v = v_1 + v_2$, where $v_i \in M_i$ for $i \in \{1, 2\}$, and at least one of the v_i's is non zero and lives in the lowest layer of $(M_i)_\lambda^{(n)}$. Assume it is v_1. As we have deduced above, $U(\mathcal{G})v_1 \cong \Delta^{(n)}(\lambda)$, hence

$$M/U(\mathcal{G})v_1 = M_1/U(\mathcal{G})v_1 \oplus M_2$$

is a direct decomposition of an element of $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ of length smaller than the one of M. By induction, we derive that both $M_1/U(\mathcal{G})v_1$ (and also M_1) and M_2 belong to $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$.

□

Remark The above result tells us that $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ is *resolving* as defined in [4]

□

The next two corollaries give us a good description on how the projective modules of $\mathcal{O}^{(n)}$ look like:

Corollary 2.4.7 *Every projective (indecomposable) object in $\mathcal{O}^{(n)}$ is a summand of $\Delta^{(n)}(\alpha)\otimes E$, where E is a finite dimensional $\mathcal{G}$-module and α is a dominant weight.*

Proof

This is an immediate consequence *of the proof* of Proposition 2.4.4: there we showed that for all weights λ there exists a finite dimensional $\mathcal{G}$-module E (depending on λ) such that $\Delta^{(n)}(\alpha) \otimes E$ maps onto $L(\lambda)$, thus implying that $P^{(n)}(\lambda)$ is a direct summand of $\Delta^{(n)}(\alpha) \otimes E$. □

Corollary 2.4.8 *Every projective module in $\mathcal{O}^{(n)}$ is filtered by fat Verma modules. This filtration can be arranged in such a way that, from top to* *bottom, the indices of the fat Verma modules appear in a non-decreasing order.*

Proof

Consequence of Proposition 2.4.2 and the proof of Proposition 2.4.6. □

Example 2.4.9 *Let $\mathcal{O}_0^{(2)}$ be the principal block of $\mathcal{O}^{(2)}$ for $sl_2(\mathbb{C})$.*

Using previous results and looking back at Example 2.2.4, we are able to describe some relevant objects:

The simples in this block are the one dimensional module $L(0)$ and $L(-2)$ which is isomorphic to $\Delta(-2)$. Further, we have the fat Verma $\Delta^{(2)}(0)$, which coincides with $P^{(2)}(0)$. If we take $E(1)$ to be the two dimensional simple $sl_2(\mathbb{C})$-module of highest weight one, we have

$$P^{(2)}(-2) \cong \Delta^{(2)}(-1) \otimes E(1).$$

The next result reveals one of the nicest features of $\mathcal{O}^{(n)}$:

Theorem 2.4.10 *Every fat projective module admits a filtration the subquotients of which are n^m isomorphic classical projective modules (i.e. modules that are projective when considered as objects of $\mathcal{O}$).*

Proof

Let λ be a dominant weight. Proposition 2.2.1 guarantees the existence of a short exact sequence of the form

$$0 \longrightarrow \Delta(\lambda) \longrightarrow \Delta^{(n)}(\lambda) \longrightarrow N \longrightarrow 0,$$

where N is filtered by $n^m - 1$ classical Verma modules. Since tensoring with finite dimensional $\mathcal{G}$-modules is an exact functor within the category of all $U(\mathcal{G})$-modules, we get our result for projectives of the form $\Delta^{(n)}(\lambda)\otimes E$. We will be done if we show that the projective indecomposable $P(\alpha)$ appears as a summand of $\Delta(\lambda) \otimes E$ whenever the fat projective $P^{(n)}(\alpha)$ appears as a summand of $\Delta^{(n)}(\lambda)\otimes E$, with equal multiplicities. In effect, since $\Delta^{(n)}(\lambda)\otimes E$ maps surjectively onto $\Delta(\lambda) \otimes E$ and projective indecomposables $P^{(n)}(\alpha)$ are generated by vectors of fat weight α, we conclude that $P(\alpha)$ has got to occur in the decomposition of $\Delta(\lambda) \otimes E$ into projective indecomposables. But combining Proposition 2.4.2 with Proposition 2.4.8 forces the multiplicities to be the same.

The result is, therefore, true for *all* projectives in $\mathcal{O}^{(n)}$. □

Remark 1) Denote by $(P^{(n)}(\lambda) : \Delta^{(n)}(\alpha))$ the number of occurrences of $\Delta^{(n)}(\alpha)$ in a Verma filtration of $P^{(n)}(\lambda)$ and denote by $[\Delta^{(n)}(\alpha) : L(\lambda)]$ the number of composition factors of $\Delta^{(n)}(\alpha)$ isomorphic to $L(\lambda)$.

2) Recall the *BGG reciprocity* that holds for $\mathcal{O}$ (see [23]):

$$(P(\lambda) : \Delta(\alpha)) = [\Delta(\alpha) : L(\lambda)].$$

□

Proposition 2.4.11 *(BGG reciprocity) With the above notations we have*

$$n^m(P^{(n)}(\lambda) : \Delta^{(n)}(\alpha)) = [\Delta^{(n)}(\alpha) : L(\lambda)].$$

Proof Follows directly from Theorem 2.4.10 and the previous remark. □

Obviously, the BGG reciprocity does not tell us *which* simples appear as composition factors of Verma modules. To achieve that one needs to quote a famous theorem of Bernstein, Gelfand and Gelfand (usually known as the "BGG theorem", see [16], Theorem 7.6.23), which tells us the following:

Theorem 2.4.12 *(BGG)*

Let λ and μ be weights. Then the following assertions are equivalent:

i) There is an inclusion $\Delta(\lambda) \subseteq \Delta(\mu)$;

ii) the simple module $L(\lambda)$ is a composition factor of $\Delta(\mu)$;

iii) there are positive roots $\gamma_1, ..., \gamma_k$ such that there is a chain of inequalities

$$\mu \geq s_{\gamma_1}.\mu \geq ... \geq s_{\gamma_n}...s_{\gamma_k}.\mu = \lambda$$

(where s_{γ_i} denotes the reflection associated with γ_i).

Remark 1) Omitting *i)* one may rewrite the above theorem in the context of "fat" Verma modules.

2) Unfortunately, BGG theorem tells us nothing about multiplicities...

□

To finish this section, we will try to convince the reader about the existence of a remarkably "well behaved" projective object - the "big fat projective" - inside the principal block. We are talking about the projective cover in $\mathcal{O}^{(n)}$ of the simple module $L(-2\rho)$. One reason (perhaps the most important one) for its prominent role is the following

Proposition 2.4.13 *For all $w \in W$, we have*

$$[P^{(n)}(-2\rho) : \Delta^{(n)}(w \cdot 0)] = 1.$$

Proof

Since all fat Verma modules have simple socle, we know, by BGG reciprocity, that the Verma modules in the above formula *do* occur in a Verma filtration of the "big" fat projective. We need to show that such filtrations are multiplicity free.

For that, consider the module

$$\Delta^{(n)}(-\rho) \otimes L(\rho),$$

where $L(\rho)$ is the finite dimensional simple $\mathcal{G}$-module having ρ as an extremal (highest) weight. It is projective (since $\Delta^{(n)}(-\rho)$ is projective) and, by Proposition 2.4.2, it is filtered by fat Verma modules, the top one being $\Delta^{(n)}(-2\rho)$. From this, we conclude that $P^{(n)}(-2\rho)$ is a summand of $\Delta^{(n)}(-\rho) \otimes L(\rho)$.

But, again by Proposition 2.4.2, the highest fat weights of the Verma modules appearing in the above mentioned Verma flag are of the form $-\rho+\alpha$, where α is a weight of $L(\rho)$.

Now, by definition of the dot orbit, we have

$$-\rho + \alpha \in W \cdot 0 \iff \alpha \text{ is an extremal weight for } L(\rho),$$

in which case $L(\rho)_\alpha$ is one dimensional and therefore $\Delta^{(n)}(\alpha)$ appears with multiplicity one in the Verma filtration of $P^{(n)}(-2\rho)$.

□

Combining the BGG reciprocity with the previous result, we have

Corollary 2.4.14 *For all $\alpha \in W \cdot 0$, we have*

$$[\Delta^{(n)}(\alpha) : L(-2\rho)] = n^m.$$

Remark

The two previous results are still true if we replace -2ρ and 0 by, respectively, any (integral) antidominant weight α and any weight in the orbit of α under the dot action of the Weyl group.

□

Now that a substantial amount of information about projective objects is available, what can we say about injective objects?

Given a $\mathcal{G}$-module M, one may equip its $\mathbb{C}$-dual, M^*, with a structure of $\mathcal{G}$-module by defining an action in the following way:

$$(x_\alpha f)(m) = f(x_{-\alpha} m),$$

$$(hf)(m) = f(hm)$$

where x_α is a basis element of $\mathcal{G}$ indexed by a simple root α, h is an element of $\mathcal{H}$, f belongs to M^* and m is an element of M.

In other words: $\mathcal{G}$ acts on M^* via the Chevalley anti-isomorphism.

Recall that all objects M in $\mathcal{O}^{(n)}$ are the direct sum of their fat weight spaces, i.e.

$$M = \bigoplus_{\lambda \in \mathcal{H}^*} M_\lambda^{(n)}$$

and, accordingly, define the vector space $i(M)$ as

$$i(M) := \bigoplus_{\lambda \in \mathcal{H}^*} (M^*)_\lambda^{(n)}.$$

We aim to prove that

Proposition 2.4.15 *With the above notation, $i(M)$ is a $\mathcal{G}$-module (that is, a submodule of M^*).*

Proof

Let x_α be an element of $\mathcal{G}$ indexed by a simple root α, f an element of

$$(M^*)^{(n)}_\lambda$$

and m an element of M. Let us show that $x_\alpha f$ belongs to $(M^*)^{(n)}_{\lambda+\alpha}$. We have, for a basis element H_i of $\mathcal{H}$,

$$((H_i - (\lambda+\alpha)(H_i))^n(x_\alpha f))(m) = (x_\alpha f)((H_i - (\lambda+\alpha)(H_i))^n m)$$

$$= f(x_{-\alpha}((H_i - (\lambda+\alpha)(H_i))^n)m)$$

$$= f(((H_i\lambda(H_i))^n x_{-\alpha})m)$$

$$= ((H_i - \lambda(H_i))^n f)(x_{-\alpha} m)$$

$$= 0,$$

thus proving the result.

□

The previous proposition ensures that the map i induces an endofunctor for $\mathcal{O}^{(n)}$. Moreover, since i preserves the dimension of fat weight spaces, it follows that simple objects are also preserved and, consequently, one concludes that i preserves the blocks of $\mathcal{O}^{(n)}$.

One concludes that i is a duality as

Lemma 2.4.16 *For all weights $\lambda \in \mathcal{H}^*$,*

$$(M^*)^{(n)}_\lambda = (M^{(n)}_\lambda)^*.$$

Remark If N is a subspace of M, we will view N^* as the subspace of M^* consisting of functionals in M vanishing outside of N.

□

Proof

Let us start with the easy implication. Let H_i be a basis vector of $\mathcal{H}$. If f is a functional on M vanishing outside $M^{(n)}_\lambda$, then, since fat weight spaces are invariant under $U(\mathcal{H})$, $(H_i - \lambda(H_i))^n f$ also vanishes outside $M^{(n)}_\lambda$. But all the elements of $M^{(n)}_\lambda$ are killed by $(H_i - \lambda(H_i))^n$, hence it vanishes everywhere.

For the other direction note that, for all $i = 1, ..., m$, for all $m_\beta \in M^{(n)}_\beta$ with $\beta \neq \lambda$ and for all $f \in (M^*)^{(n)}_\lambda$, we have

$$0 = ((H_i - \lambda(H_i))^n f)(m_\beta) = f((H_i - \lambda(H_i))^n m_\beta) = ((\beta - \lambda)(H_i))^n f(m_\beta).$$

Since $\beta \neq \lambda$, there exists $j \in \{1, ..., m\}$ such that $\beta(H_j) \neq \lambda(H_j)$. One may multiply the above by $1/((\beta - \lambda)(H_i))^n$ to reach the conclusion that f vanishes outside $M^{(n)}_\lambda$. □

Summarizing all the information we have just obtained, one may state the following theorem:

Theorem 2.4.17 *There exists a duality*

$$i : \mathcal{O}_\lambda^{(n)} \longrightarrow \mathcal{O}_\lambda^{(n)}$$

which fixes the simple modules.

From now on we will denote the images of $\Delta^{(n)}(\lambda)$ under the above defined functor i by $\nabla^{(n)}(\lambda)$ and often refer to them as "dual fat Verma modules". We will also denote by $I^{(n)}(\lambda)$ the image of the projective indecomposable $P^{(n)}(\lambda)$ under i. Obviously, $I^{(n)}(\lambda)$ will be the injective envelope of $L(\lambda)$. Note that all theorems involving projective objects that have been proved so far admit a natural dual statement regarding the injective counterparts. We will refrain from stating them except for two results which will be used later:

Proposition 2.4.18 *If λ is a dominant weight, then $\nabla^{(n)}(\lambda)$ is an injective indecomposable object in $\mathcal{O}^{(n)}$.*

Proof

Dual of 2.4.3.

□

Proposition 2.4.19 *Every injective module in $\mathcal{O}^{(n)}$ is filtered by fat dual Verma modules. This filtration can be arranged in such a way that, from top to bottom, the indices of the fat Verma modules appear in a non-increasing order.*

Proof

Dual of 2.4.10.

□

Chapter 3

Blocks of $\mathcal{O}^{(n)}$ as finite dimensional algebras

3.1 An equivalence of categories

In the previous chapter we have seen that all blocks of $\mathcal{O}^{(n)}$ contain a finite number of simple objects, each object has finite length and there exist enough projectives (and enough injectives). This is strong evidence that, as for $\mathcal{O}$, one has the following equivalence of categories:

Theorem 3.1.1 *Let λ be a weight and let n be a natural number. Then*

$$\mathcal{O}_{\lambda}^{(n)} \sim mod - A_{\lambda,n}$$

for some finite dimensional algebra $A_{\lambda,n}$.

Proof

Recall from Proposition 2.4.4 that, for every weight α, the simple module $L(\alpha) \in \mathcal{O}^{(n)}$ admits a projective cover $P^{(n)}(\alpha)$. Thus, in similar fashion as for category $\mathcal{O}$, we construct a projective generator (i.e. a projective object surjecting onto each simple object)

$$P_\lambda^{(n)} := \bigoplus_x P^{(n)}(x.\lambda),$$

where x runs through a complete set of representatives of W/W_λ, and define $A_{\lambda,n}$ to be the ring of endomorphisms of $P_\lambda^{(n)}$.

By a standard general reasoning (see, for instance, [6]) the fact that $\mathcal{O}^{(n)}$ is an abelian category (by being a full subcategory of $U(\mathcal{G}) - mod$) where all its objects have finite length implies that the functor

$$\mathcal{O}_\lambda^{(n)} \longrightarrow mod - End_{\mathcal{O}_\lambda^{(n)}}(P_\lambda^{(n)})$$

$$M \mapsto Hom_{\mathcal{O}_\lambda^{(n)}}(P_\lambda^{(n)}, M)$$

is an exact equivalence of categories.

□

Remark The duality constructed in Theorem 2.4.17 tells us that, on the level of finite dimensional algebras, $A_{\lambda,n}$ and $A_{\lambda,n}^{op}$ are Morita equivalent.

□

3.2 Example: The principal block for $sl_2(\mathbb{C})$

1) Let us look at the principal block of $\mathcal{O}^{(n)}$ for $\mathcal{G} = sl_2(\mathbb{C})$ and $n = 2$. In the previous chapter we saw that $P^{(2)}(0) = \Delta^{(2)}(0)$ and $P^{(2)}(-2) = \Delta^{(2)}(-1) \otimes E(1)$. Then (using minor brute force linear algebra) we have that $A_{0,2}$ is given by the following factor algebra $\mathbb{C}Q/J_{0,2}$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \end{tikzcd}$$

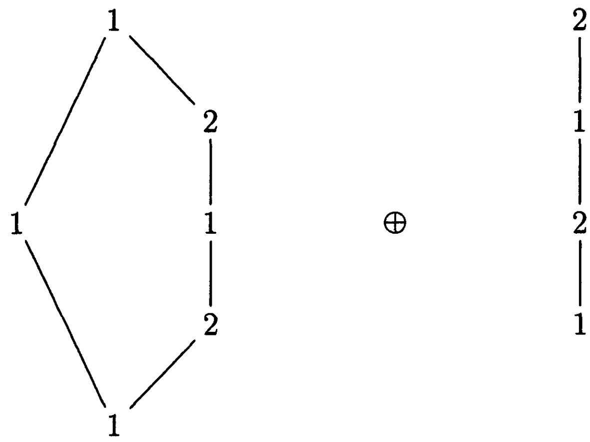
and $J_{0,2}$ the ideal of relations $J_{0,2} = \langle xz, zy, (z+yx)^2, (xy)^2 \rangle$.

2) For general n, we may still describe the principal block of $\mathcal{O}^{(n)}$ for $\mathcal{G} = sl_2(\mathbb{C})$ by means of quiver and relations. In fact (this time using major brute force linear algebra), we have that $A_{0,n}$ is given by the following factor algebra $\mathbb{C}Q/J_{0,n}$ where Q is the quiver

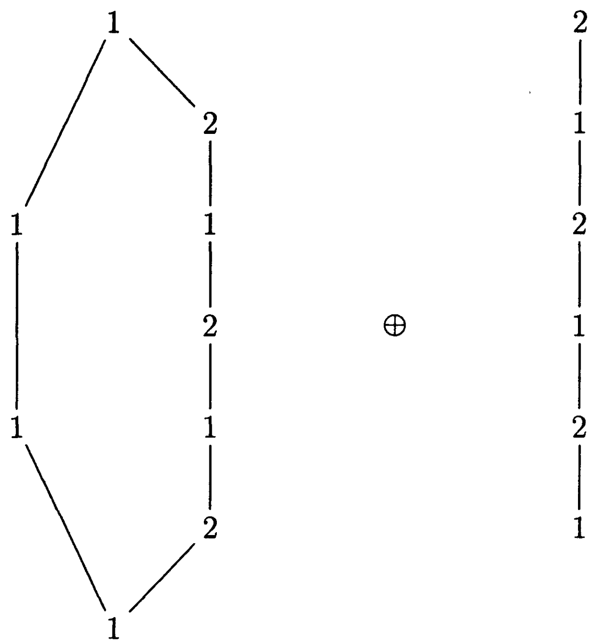
$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \end{tikzcd}$$

and $J_{0,n}$ the ideal of relations $J_{0,n} = \langle xz, zy, (z+yx)^n, (xy)^n \rangle$.

3) Making use of 1) and 2), one obtains the decomposition of $A_{0,2}$ into projective indecomposables:



For $A_{0,3}$ the picture is the following:



Remark In Chapter 4 we will be able to describe some blocks of $\mathcal{O}^{(n)}$ (including the previous ones) using sophisticated methods due to Soergel.

$\square$

3.3 Extensions of fat Verma modules

The next proposition clarifies how two fat Verma modules may (or may not) extend:

Proposition 3.3.1 *Consider the weights α, β and λ:*

i) If $\alpha \leq \beta$ then $Ext^1{}_{\mathcal{O}}(\Delta(\beta), \Delta(\alpha)) = 0$;

ii) If $n \geq 2$ then $Ext^1{}_{\mathcal{O}^{(n)}}(\Delta(\lambda), \Delta(\lambda)) \neq 0$;

iii) If $\alpha \leq \beta$ then $Ext^1{}_{\mathcal{O}^{(n)}}(\Delta^{(n)}(\beta), \Delta^{(n)}(\alpha)) = 0$.

Proof

Observe that *i)* is a well-known result for category $\mathcal{O}$ and that it is a particular case of *iii)*. So, let us start with *iii)*.

It suffices to prove the result for blocks of $\mathcal{O}^{(n)}$ since there are no non trivial extensions between objects from different blocks. So let

$$0 \longrightarrow \Delta^{(n)}(\alpha) \longrightarrow M \longrightarrow \Delta^{(n)}(\beta) \longrightarrow 0$$

be a short exact sequence in $\mathcal{O}_\lambda^{(n)}$ with $\alpha \leq \beta$, let v_β be the canonical generator of $\Delta^{(n)}(\beta)$ and pick one of its preimages $m_\beta \in M$ lying in $M_\beta^{(n)}$. As we have already seen before in Proposition 2.2.2, the annihilator of v_β in

$U(\mathcal{G})$ is generated by the $(H_i - \lambda(H_i))^n$'s and by $\mathcal{N}_+$. The exactness of the above sequence implies that $Ann_{U(\mathcal{G})}(v_\beta)m_\beta$ is contained in an isomorphic copy of $\Delta^{(n)}(\alpha)$ but, clearly, all the $(H_i - \lambda(H_i))^n$ kill m_β whereas $\mathcal{N}_+ m_\beta$ is either zero or contains vectors of (fat) weight higher than β. But the latter is impossible since $\alpha \leq \beta$.

Hence $Ann_{U(\mathcal{G})}(v_\beta) \subseteq Ann_{U(\mathcal{G})}(m_\beta)$ and the sequence splits.

For *ii*), recall that, by Proposition 2.2.1, $\Delta^{(2)}(\lambda)$ is filtered by some copies of $\Delta(\lambda)$. This proves the result since $\Delta^{(2)}(\lambda) \in \mathcal{O}^{(n)}$ for all $n \geq 2$.

□

Corollary 3.3.2 *For all weights λ the following multiplicity rule holds:*

$$(P^{(n)}(\lambda) : \Delta^{(n)}(\lambda)) = 1.$$

Proof

By Corollary 2.4.8 we know that $\Delta^{(n)}(\lambda)$ appears on the top of a fat Verma filtration of $P^{(n)}(\lambda)$ and the indices of the Verma filtration are in non decreasing order. But fat Verma modules do not self extend (as we have just seen) and $P^{(n)}(\lambda)$ is indecomposable hence has simple top. From this we conclude that our multiplicity law holds. □

3.4 Standardly stratified algebras

While the blocks of category $\mathcal{O}$ are known to be (equivalent to) module categories of *quasi-hereditary algebras* (or *highest weight categories* in the sense of Cline-Parshall-Scott, see [14]), the appropriate class of finite dimensional algebras, sufficient to describe blocks of $\mathcal{O}^{(n)}$ will be shown to be the class of *properly stratified algebras*, introduced in [15]. This is a subclass of the class of *standardly stratified algebras* wherein most of the properties we are interested in may be derived. In what follows we will be borrowing from [1] which in its turn may be viewed as a natural continuation of Ringel's fundamental paper ([29]) on quasi-hereditary algebras.

We recall the main definitions and properties.

Denote by $(A, \leq)$ the basic algebra A together with a fixed (total) ordering on a complete set of primitive orthogonal idempotents $\{e_1, ..., e_m\}$. For $1 \leq i \leq m$, let $L(i)$ denote the simple top of the projective indecomposable $P(i) = Ae_i$.

The *standard module* $\Delta(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $L(j)$ with $j > i$. Furthermore, denote by $\Delta^{(k)}(i)$ the maximal factor module of $\Delta(i)$ such that the multiplicity condition

$$[\Delta^{(k)}(i) : L(i)] = k$$

holds.

Note firstly that $\Delta^{(k)}(i)$ may or may not exist but $\Delta^{(1)}(i)$ always exists and is called the *proper standard module* associated with the simple $L(i)$. Secondly, observe that $\Delta(i) = \Delta^{(r)}(i)$ for some $r \leq dimA$.

Dually, we define the *costandard modules* $\nabla^{(r)}(i)$ and *proper costandard modules* $\nabla^{(1)}(i)$ for all natural numbers r and $1 \leq i \leq m$.

Some basic properties of (proper) standard and (proper) costandard modules are as follows:

Lemma 3.4.1 *With the above notation, we have*

$$i)\ Hom_A(\Delta(i), \Delta(j)) = 0 \textit{ for } i > j;$$

$$ii)\ Ext^1_A(\Delta(i), \Delta(j)) = 0 \textit{ for } i \geq j;$$

$$iii)\ Hom_A(\nabla(i), \nabla(j)) = 0 \textit{ for } i < j;$$

$$iv)\ Ext^1_A(\nabla^{(1)}(i), \nabla^{(1)}(j)) = 0 \textit{ for } i < j;$$

$$v)\ Hom_A(\Delta(i), \nabla^{(1)}(j)) = 0 \textit{ for } i \neq j;$$

$$vi)\ Ext^1_A(\Delta(i), \nabla^{(1)}(j)) = 0 \textit{ for every } i, j.$$

Proof

The statement can be found in [1] (Lemma 1.2), and the proofs are, essentially, contained in [29]. For completeness, let us check some of the above properties:

To prove $i)$ it is enough to observe that a non zero homomorphism between $\Delta(i)$ and $\Delta(j)$ would force the existence of a composition factor $L(i)$ of $\Delta(j)$ which is impossible since $i > j$. For $ii)$, consider the short exact sequece $0 \longrightarrow \Delta(j) \longrightarrow N \longrightarrow \Delta(i) \longrightarrow 0$, where N is an A-module. Taking the projective cover $P(i)$ of $\Delta(i)$ yields the following commutative diagram:

$$\begin{tikzcd} & & & P(i) \arrow[dl, "\beta"'] \arrow[d, "\pi"] & \\ 0 \arrow[r] & \Delta(j) \arrow[r, "f"] & N \arrow[r, "g"] & \Delta(i) \arrow[r] & 0 \end{tikzcd}$$

where $g\beta = \pi$ and thus $g(im\beta) = \Delta(i)$. Now, by the maximality of the standard object $\Delta(i)$, we have got only two possibilities left: Either $im(\beta)$ is isomorphic to $\Delta(i)$ (forcing a splitting of the above short exact sequence) or there exists a composition factor $L(k)$ of $im(\beta) \cap Ker(g)$ with $k \geq i$. But the latter is impossible since $Ker(g) = im(f) \cong \Delta(j)$ and $\Delta(j)$ has no such composition factors.

The other statements are proved in somewhat similar ways. For instance, in $vi)$, if we consider a non split short exact sequence

$$0 \longrightarrow \nabla^{(1)}(j) \longrightarrow N \longrightarrow \Delta(i) \longrightarrow 0$$

then, by maximality of $\nabla^{(1)}(j)$, we have $j \leq i$ and, by maximality of $\Delta(i)$, we have $j > i$. Hence no such exact sequence can occur.

□

Given a category $\mathcal{C}$ and a subcategory $\mathcal{B}$, define $\mathcal{B}^{\perp}$ (respectively, ${}^{\perp}\mathcal{B}$) in the following way:

$$\mathcal{B}^{\perp} = \{C \in \mathcal{C} : Ext^1{}_{\mathcal{C}}(M, C) = 0 \text{ for all } M \in \mathcal{B}\}$$

(respectively,

$${}^{\perp}\mathcal{B} = \{C \in \mathcal{C} : Ext^1{}_{\mathcal{C}}(C, M) = 0 \text{ for all } M \in \mathcal{B}\}).$$

Let $\mathcal{F}_A(\Delta)$ (respectively, $\mathcal{F}_A(\Delta^{(1)})$) denote the full subcategory of $A{-}mod$ whose objects are modules admitting a filtration the subquotients of which are standard (respectively proper standard) modules. Further, we will consider subcategories $\mathcal{F}_A(\Delta)_j$ (respectively, $\mathcal{F}_A(\Delta^{(1)})_j$) defined in a similar way as above but where the choice of standard (respectively, proper standard) modules is confined to set $\{\Delta(1), ..., \Delta(j)\}$ (respectively, $\{\Delta^{(1)}(1), ..., \Delta^{(1)}(j)\}$).

Our main definition is

Definition 3.4.2 *The pair $(A, \leq)$ is said to be* standardly stratified *if ${}_AA \in \mathcal{F}(\Delta)$.*

Going back to the Lie theoretic framework and recalling that $\mathcal{O}^{(n)}{}_{\lambda}$ is equivalent to $mod - A_{\lambda,n}$ for some finite dimensional algebra $A_{\lambda,n}$, we have the following expected result:

Proposition 3.4.3 *Let $\preceq$ be a total order extending the usual partial order $\leq$ on the set of weights. Then the algebra $(A_{\lambda,n}, \preceq)$ is standardly stratified where for $w \in W/W_\lambda$, $\Delta^{(n)}(w.\lambda)$ (respectively, $\Delta(w.\lambda)$) is a complete list of standard (respectively, proper standard) modules.*

Proof

By Corollary 2.4.8 we know that, for each μ lying in the dot orbit of λ, there exists a short exact sequence

$$0 \to N \to P^{(n)}(\mu) \to \Delta^{(n)}(\mu) \to 0$$

where the composition factors of the *top* of N are of the form $L(\beta)$ with $\mu < \beta$ (and hence $\mu \prec \beta$). Now, since all fat Verma modules are filtered by "classical" Verma modules and, by the BGG theorem, the composition factors of Verma modules are known, we conclude that $\Delta^{(n)}(\mu)$ is the standard object of weight μ. $\square$

Remark It is well known (see [1], Theorem 2.4.) that if $(A, \leq)$ is standardly stratified then

$$(A, \leq) \text{ is quasi-hereditary} \iff gl.dim.A < \infty.$$

But, if δ is an anti-dominant weight then $\Delta(\delta)$ is simple, hence, since $\Delta^{(n)}(\delta)$ is filtered by $\Delta(\delta)$'s, we have that

$$Ext^1_{\mathcal{O}^{(n)}}(L(\delta), L(\delta)) \neq 0,$$

for $n > 1$ and consequently we conclude that $A_{\lambda,n}$ has infinite global dimension if $n > 1$.

$\square$

Now that we know that blocks of $\mathcal{O}^{(n)}$ are equivalent to standardly stratified algebras, we may (and will) move freely between both settings. In particular, we may (and will) use for $\mathcal{O}^{(n)}$ all the results obtained in [1], in the context of finite dimensional standardly stratified algebras. We will proceed to quote some of those results:

Proposition 3.4.4 *In the above setting, if A is standardly stratified then*

1) $^{\perp}\mathcal{F}_A(\nabla^{(1)}) = \mathcal{F}(\Delta)$;

2) $\mathcal{F}_A(\nabla^{(1)}) = \mathcal{F}_A(\Delta)^{\perp}$;

3) The statements ii), iv) and vi) in Lemma 3.4.1 will also hold if we replace Ext^1 by Ext^k for all $k > 1$.

Proof

See [1] for details. $\square$

In the sequel we investigate the relationship between two important subcategories of blocks of $\mathcal{O}^{(n)}$, namely $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ and $P^{<\infty}_{\mathcal{O}^{(n)}_\lambda}$. The latter is defined to be the full subcategory of $\mathcal{O}^{(n)}_\lambda$ whose objects are the modules having finite projective dimension. In this context the following well known result is useful:

Proposition 3.4.5 *Let* $(A, \leq)$ *be a standardly stratified algebra. Then we have*

$$\mathcal{F}_A(\Delta) \subseteq P_A^{<\infty}.$$

Proof

Let $P(1), ..., P(m)$ be the corresponding ordered set of indecomposable projective modules and assume m is maximal. It is obvious that $\Delta(m) = P(m)$ and thus it has finite projective dimension. For $i < n$ consider the exact sequence $0 \longrightarrow K(i) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0$ where $K(i)$ is filtered by $\Delta(j)$'s with $j > i$. Assume by induction that $proj.dim_A \Delta(j) < \infty$ for $j > i$. Then $K(i)$ will also have finite projective dimension and the same will happen to $\Delta(i)$. $\square$

The previous result and the next theorem have both been borrowed from [24]. In particular, the next result gives a sufficient condition for the inclusion in Proposition 3.4.5 to be proper:

Theorem 3.4.6 *Let* $(A, \leq)$ *be a standardly stratified algebra such that* 1 *is minimal.*

If $\mathcal{F}_A(\Delta) = P_A^{<\infty}$ *then* $\Delta(1) \not\subseteq rad(X)$ *for all* X *in* $P_A^{<\infty}$.

Proof See [24], Theorem 2.4. $\square$

An immediate consequence of this is the following corollary.

Corollary 3.4.7 *Let $\mathcal{O}_\lambda^{(n)}$ be a block of $\mathcal{O}^{(n)}$. Then $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)}) \subsetneqq P^{<\infty}_{\mathcal{O}_\lambda^{(n)}}$.*

Proof Let us prove it for the principal block, the arguments being the same for any other block. By Corollary 2.4.8 the self duality of the projective indecomposable $P^{(n)}(-2\rho)$ forces it to be the injective envelope of $\Delta^{(n)}(-2\rho)$. Since projectives have simple top, we conclude that $\Delta^{(n)}(-2\rho) \subseteq rad(P^{(n)}(-2\rho))$. $\square$

3.5 Tilting modules for standardly stratified algebras

Keeping the notation from the previous section, let us recall some classic terminology from *tilting theory*:

Definition 3.5.1 *An A-module M is said to be a* generalised tilting module *if*

i) $proj.dimM < \infty$;

ii) $Ext^i_A(M, M) = 0$ for all $i > 0$;

iii) there exists an exact sequence $0 \to A \to M^0 \to \ldots \to M^p \to 0$ where $M^k \in add(M)$ for all k.

Remark

The notion of *generalized cotilting module* is defined dually. For more details on tilting theory see [4] or [22].

□

Definition 3.5.2 *An indecomposable A-module $T(i)$ satisfying the following two properties*

i) $T(i)$ is filtered by proper costandard modules;

ii) $T(i)$ is filtered by standard modules where the bottom subquotient is isomorphic to $\Delta(i)$ is called a tilting module with parameter i.

Dually, we define the cotilting module with parameter i, $K(i)$.

Remark

In fact, if such modules do exist they need not generalized tilting modules in the sense of Definition 3.5.1!

□

Theorem 3.5.3 *There exists a unique tilting (respectively, cotilting) module $T(i)$ (respectively, $K(i)$) for all $i \in \{1, ..., n\}$.*

A sketch of proof (which can be found in [1]) goes as follows:

Proof

Choose a total order that extends $\leq$ and pick j, the successor of i. Imitating [29], we build the *universal extension from below* of $\Delta(j)$ by copies of $\Delta(i)$ and proceed inductively up the chain. The module we eventually get is easily seen to be a tilting module with parameter i.

□

Remark

The module $T := \bigoplus_{i\in\{1,\dots,n\}} T(i)$ is called the *characteristic tilting module* of the algebra A. By definition we know that $T \in \mathcal{F}_A(\Delta) \cap \mathcal{F}_A(\nabla^{(1)})$. In fact, we may quote from [1] a much stronger statement:

$$add({}_AT) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla^{(1)})$$

which, by the way, has been implicitly used in the uniqueness assertion of Theorem 3.5.3.

□

3.6 Properly stratified algebras

Although some of the more relevant properties of blocks of $\mathcal{O}^{(n)}$ appearing in our work may be viewed in the context of *standardly stratified algebras*, the appropriate class of finite dimensional algebras sufficient to describe the blocks of $\mathcal{O}^{(n)}$ is the class of *properly stratified algebras.* As mentioned before, this is a subclass of the above mentioned class of *standardly stratified algebras* and was introduced by Dlab in [15]. For completeness, we quote its main results keeping the notation from the previous sections.

Theorem 3.6.1 *([15], Theorem 5)*

The following statements are equivalent for a finite dimensional basic algebra A and a complete set $(e_1, e_2, ..., e_m)$ of primitive ortogonal idempotents:

i) $_AA \in \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta^{(1)})$;

ii) $_AA \in \mathcal{F}(\Delta)$ *and* $\Delta(i)$ *is filtered by copies of* $\Delta^{(1)}(i)$ *for all* $i \in \{1, ..., m\}$;

iii) $Ext_A^2(\Delta(i), \nabla^{(1)}(j)) = 0$ *for all* $i, j \in \{1, ..., m\}$;

iv) $\mathcal{F}(\Delta) =^{\perp} \mathcal{F}(\nabla^{(1)})$ *and* $\mathcal{F}(\Delta^{(1)}) =^{\perp} \mathcal{F}(\nabla)$;

v) $dimA = \sum_{i=1}^{m} dim\Delta(i)dim\, \nabla^{(1)}\, (i) = \sum_{i=1}^{m} dim\Delta^{(1)}(i)dim\, \nabla\, (i)$.

Proof

See [15].

□

Definition 3.6.2 *A finite dimensional algebra A is said to be* properly stratified *if its basic algebra satisfies any of the conditions stated in the previous theorem.*

Remark

The definition of properly stratified algebras in [15] is actually given by recurrence in terms of *properly stratifying sequences of idempotents* in a similar way as for the quasi-hereditary situation.

□

The expected result now is

Proposition 3.6.3 *For every weight λ and for every natural number n, the algebra $A_{\lambda,n}$ is properly stratified.*

Proof

Looking at statement *i*) in Theorem 3.6.1, we need only invoke Proposition 3.4.3 and Proposition 2.2.1.

□

3.7 Arkhipov's functor and the semiregular bimodule

Recall the definition of $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ as the full subcategory of category $\mathcal{O}^{(n)}$ whose objects are modules having a fat Verma flag. The aim of this section is to establish an equivalence between $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$ and $\mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})^{op}$. For $n = 1$ (i.e. for category $\mathcal{O}$) this was done by Soergel in [32]. There, the equivalence consists on tensoring over $U(\mathcal{G})$ with a certain bimodule - the *semi-regular bimodule* - which was firstly introduced by Arkhipov in [2]. Its construction will be reviewed below and, for $n \geq 1$, our equivalence will follow similar patterns to the one established by Soergel in [32].

Let us start by making some general considerations about graded modules over graded Lie algebras.

It is well known that the Serre relations (and, of course, the PBW theorem) equip $U(\mathcal{G})$ (considered as a $\mathcal{G}$-module under the adjoint action) with a natural grading: this is done by giving grade -1 and 1 to basis vectors corresponding to simple roots lying respectively in $\mathcal{N}_-$ and $\mathcal{N}_+$, and by giving grade 0 to basis vectors lying in $\mathcal{H}$. It is also clear how $U(\mathcal{N}_-)$ inherits that same grading.

In general, if M and N are $\mathbb{Z}$-graded modules, denote by $grHom(M, N)$ the set of all $\mathbb{Z}$-graded homomorphisms from M to N (i.e. the direct sum of its homogeneous components of degree $j \in \mathbb{Z}$ which consist of homomorphisms f satisfying $f(M_i) \subseteq N_{i+j}$ for all $i \in \mathbb{Z}$). In particular, by considering the trivial module $\mathbb{C}$ as being concentrated in grade 0, we denote by $M^{\circledast}$ the $\mathbb{Z}$-graded dual of M, in other words,

$$(M^{\circledast})_i = (M_{-i})^*,$$

where the left action of $\mathcal{G}$ is given by the formula

$$(xf)(m) = -f(xm),$$

for all $x \in \mathcal{G}$ and for all $m \in M$.

It is a routine task to verify that $U(\mathcal{N}_-)^{\circledast}$ is a $U(\mathcal{N}_-) - U(\mathcal{N}_-)-$ graded bimodule via

$$(fn)(m) = f(nm)$$

$$(nf)(m) = f(mn),$$

for all $m, n \in U(\mathcal{N}_-)$.

In fact, if $x_{-\alpha}$ is an element of $U(\mathcal{N}_-)$ with grade -1 (associated to a simple negative root $-\alpha$) and f_i is an homogeneous element of $U(\mathcal{N}_-)^{\circledast}$ with grade i then for all homogeneous elements $u_j \in (U(\mathcal{N}_-))_j$ with $j \neq 1 - i$ we have

$$(x_{-\alpha}f_i)(u_j) = f_i(u_j x_{-\alpha}) = 0$$

because $u_j x_{-\alpha}$ lies outside $U(\mathcal{N}_-)_{-i}$.

Consider (again) the left action given by the formula

$$(xf)(m) = -f(xm)$$

for all $x \in \mathcal{N}^-$ and for all $m \in U(\mathcal{N}_-)$. We will show that the transpose of the principal automorphism Γ of $U(\mathcal{N}_-)^{\circledast}$ identifies in a natural way the two left actions mentioned above. Here the principal automorphism multiplies by -1 the sign of elements of grade 1 while the transpose of the element $X_{-\alpha_1}...X_{-\alpha_s}$, where α_i is a simple root for all $i \in \{1, ..., s\}$, is defined to be $X_{-\alpha_s}...X_{-\alpha_1}$. We will denote the transpose of the principal automorphism Γ by Γ^t.

Hence, if $X_{-\alpha}$ is a basis element of $U(\mathcal{N}_-)$ of degree -1, f is an element of $U(\mathcal{N}_-)^{\circledast}$ and $X_{-\alpha_1}...X_{-\alpha_p}$ is an element of grade $-p$ then

$$(\Gamma^t(X_{-\alpha}f))(X_{-\alpha_1}...X_{-\alpha_p}) = (-1)^p(X_{-\alpha}f)(X_{-\alpha_p}...X_{-\alpha_1}) =$$

$$= (-1)^p f(X_{-\alpha_p}...X_{-\alpha_1}X_{-\alpha}) = (-1)^{p+1} f(-X_{-\alpha_p}...X_{-\alpha_1}X_{-\alpha}) =$$

$$= (\Gamma^t(f))(-X_{-\alpha}X_{-\alpha_1}...X_{-\alpha_p}) = (X_{-\alpha}(\Gamma^t(f)))(X_{-\alpha_1}...X_{-\alpha_p}).$$

Now set $S := U(\mathcal{N}_-)^{\circledast} \otimes U(\mathcal{B})$ and consider right and left actions of $U(\mathcal{G})$ on S given respectively by the transport of structure via the following isomorphisms of vector spaces:

$$U(\mathcal{N}_-)^{\circledast} \otimes U(\mathcal{B}) \cong U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} U(\mathcal{G})$$

and

$$U(\mathcal{N}_-)^{\circledast} \otimes U(\mathcal{B}) (\cong grHom_{\mathbb{C}}(U(\mathcal{N}_-), U(\mathcal{B}))) \cong grHom_{U(\mathcal{B})}(U(\mathcal{G}), \mathbb{C}_{2\rho} \otimes U(\mathcal{B})),$$

where the last tensor product is the tensor of the 1-dimensional left $\mathcal{B}$-module $\mathbb{C}_{2\rho}$ (the element $b = h + ...$ acting like scalar multiplication by $2\rho(h)$) by the left regular $\mathcal{B}$-module $U(\mathcal{B})$.

The left action of $U(\mathcal{G})$ on the above bimodule can be made more explicit via the formula:

$$U(\mathcal{N}_-)^{\circledast} \otimes U(\mathcal{B}) \cong grHom_{U(\mathcal{B})}(U(\mathcal{G}), \mathbb{C}_{2\rho} \otimes U(\mathcal{B}))$$

$$f \otimes b_1 \longrightarrow (b \otimes n \longrightarrow f(n) b (1 \otimes b_1)),$$

for all $b, b_1 \in U(\mathcal{B}), n \in U(\mathcal{N}_-), f \in U(\mathcal{N}_-)^{\circledast}$.

Define the map i in the following natural way:

$$i : U(\mathcal{N}_-)^{\circledast} \longrightarrow S$$

$$i(f) = f \otimes 1.$$

Then, in the above setting, one has:

Theorem 3.7.1 *(Soergel)*

1) S is an $U(\mathcal{G})$-$U(\mathcal{G})$-bimodule;

2) The map $i : U(\mathcal{N}_-)^{\circledast} \longrightarrow S$ is an inclusion of $U(\mathcal{N}_-)$-$U(\mathcal{N}_-)$-bimodules satisfying

$$hi(f) - i(f)h = 2\rho(h)i(f) - f(ad_h) \otimes 1.$$

Proof

This is done in [32] and it is much harder to prove than one might expect! Let us check property 2) which will be used a number of times in the sequel:

We start by showing that

$$h(f \otimes 1) = -(f(ad_h) \otimes 1) + (f \otimes (-2\rho(h) + h)),$$

for all $f \in U(\mathcal{N}_-)^{\circledast}, h \in U(\mathcal{H})$.

This is because

$$h(f \otimes 1)(bn) = (f \otimes 1)(bnh)$$

$$= (f \otimes 1)(-bhn + bnh + bhn) = (f \otimes 1)(-b(hn - nh) + bhn)$$

$$= (f \otimes 1)(-b(ad_h(n)) + (f \otimes 1)(bhn)$$

$$= f(-ad_h(n)b(1 \otimes 1) + f(n)bh(1 \otimes 1))$$

$$= (f(-ad_h) \otimes 1)(bn) + f(n)b(2\rho(h) \otimes 1 + 1 \otimes h)$$

$$= (f(-ad_h) \otimes 1 + f \otimes (h + 2\rho(h)))(bn).$$

□

Now consider the following full subcategories $\mathcal{M}, \mathcal{M}_{\mathcal{H}}, \mathcal{K}, \mathcal{K}_{\mathcal{H}}$ of $\mathcal{G} - mod$ whose objects are defined by

$\mathcal{M} = \{M \in \mathcal{G}{-}mod : M$ is a free $\mathbb{Z}-$graded $U(\mathcal{N}_-)-$module of finite rank$\}$,

$\mathcal{K} = \{M \in \mathcal{G}{-}mod : M$ is a cofree $\mathbb{Z}-$graded $U(\mathcal{N}_-)-$module of finite rank$\}$

and $\mathcal{M}_{\mathcal{H}}$ (respectively, $\mathcal{K}_{\mathcal{H}}$) denotes the full subcategory of $\mathcal{M}$ (respectively, $\mathcal{K}$) whose objects are $\mathcal{H}$-diagonalizable (i.e. objects which are direct sums of their weight spaces).

A big result that we will use in this section is the following:

Theorem 3.7.2 *(Soergel) The functor $\mathcal{A} : M \longmapsto S \otimes_{U(\mathcal{G})} M$ defines an exact equivalence of categories $\mathcal{M} \sim \mathcal{K}$.*

The above defined functor was introduced in [2] and since then called "Arkhipov's functor". In [32], Soergel used it to prove the above theorem. A sketch of the proof goes as follows:

Proof

By the construction of $\mathcal{A}$ it is easy to see that it is exact and maps $\mathcal{M}$ into $\mathcal{K}$.

Define the candidate inverse functor

$$\mathcal{B} : \mathcal{K} \longrightarrow \mathcal{M}$$

as $\mathcal{B} := Hom_{U(\mathcal{G})}(S, -) \cong Hom_{U(\mathcal{N}_-)}(U(\mathcal{N}_-)^{\circledast}, -)$.

Note that under $\mathcal{B}$, $U(\mathcal{N}_-)^{\circledast}$ is sent to $End_{U(\mathcal{N}_-)}(U(\mathcal{N}_-)^{\circledast}) \cong U(\mathcal{N}_-)$. In general, for $M \in \mathcal{M}$, one has

$$\mathcal{B}(\mathcal{A}(M)) \cong Hom_{U(\mathcal{G})}(S, S \otimes_U M)$$

$$\cong Hom_{U(\mathcal{N}_-)}(U(\mathcal{N}_-)^{\circledast}, U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} M).$$

But M is $U(\mathcal{N}_-)$-free of finite rank, hence the canonical map

$$M \longrightarrow \mathcal{B}(\mathcal{A}(M)),$$

which sends an element $m \in M$ to a map

$$U(\mathcal{N}_-)^{\circledast} \longrightarrow U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} M$$

$$f \mapsto f \otimes m,$$

is an isomorphism.

The other direction follows similar patterns. Indeed, for $K \in \mathcal{K}$,

$$\mathcal{A}(\mathcal{B}(K)) = S \otimes_{U(\mathcal{N}_-)} Hom_{U(\mathcal{N}_-)}(S, K)$$

$$= U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} Hom_{U(\mathcal{N}_-)}(U(\mathcal{N}_-)^{\circledast}, K) \cong K$$

by adjointness.

□

The next result is a more detailed version of the previous theorem.

Proposition 3.7.3 *The functor $M \longmapsto (S \otimes_{U(\mathcal{G})} M)^{\circledast}$ defines an equivalence of categories $\mathcal{M} \sim \mathcal{M}^{op}$ with the following properties:*

1) It is covariant and exact;

2) If E is a finite dimensional $\mathcal{H}$-module then $U(\mathcal{G}) \otimes_{U(\mathcal{B})} E$ is mapped to $U(\mathcal{G}) \otimes_{U(\mathcal{B})} (\mathbb{C}_{-2\rho} \otimes E^*)$;

3) Restriction yields an equivalence $\mathcal{M}_{\mathcal{H}} \sim \mathcal{M}_{\mathcal{H}}{}^{op}$.

Proof For the proofs see [32]. Later we will prove its "fat" version.

□

If, furthermore, we define $\mathcal{F}(\Delta)$ as the full subcategory of $\mathcal{O}$ whose objects have a Verma flag, then the previous proposition can (or better, could) be continued in the following way:

Proposition 3.7.4 *(Soergel)*

With the same notation as in the previous result we have:

4) $\mathcal{F}(\Delta) = \mathcal{O} \cap \mathcal{M}_{\mathcal{H}}$;

5) Restriction induces an exact equivalence $\mathcal{F}(\Delta) \sim \mathcal{F}(\Delta)^{op}$.

Proof

Let us try to look through Soergel's ideas . We will prove 4) and 5) follows easily. The inclusion $\subseteq$ is the easy one:

Let us use induction on the length of the Verma flag (which is a well known invariant) the initial cases being trivial.

If $M \in \mathcal{F}(\Delta)$ and admits a submodule $N \cong \Delta$ for some Verma module Δ then, by induction, M/Δ is a free (hence projective) $U(\mathcal{N}_-)$-module K of finite rank thus forcing $M = \Delta \oplus K$.

The other direction is harder:

Suppose $M = M_1 \oplus ... \oplus M_k$ where all $M_i's$ are isomorphic to $U(\mathcal{N}_-)$ as $U(\mathcal{N}_-)$-modules. Furthermore, let m_i be a free generator of M_i. But M_i is semisimple as $U(\mathcal{H})$-module (i.e. M_i is the direct sum of its weight spaces) so we can write

$$m_i = a_{\lambda-i,1} + ... + a_{\lambda_{i,s_i}},$$

where $a_{\lambda_{i,j}} \in (M_i)_{\lambda_j}$ for all $j \in \{1, ..., j_i\}$. Now pick $\lambda_{k,s}$ maximal among all the weights appearing in the decompositions of all the $m_i's$ and consider the map

$$\Delta(\lambda_{k,s}) \longrightarrow < a_{\lambda_{k,s}} >$$

$$v_{\lambda_{k,s}} \longrightarrow a_{\lambda_{k,s}}.$$

We observe that the above map is surjective by the universality of $\Delta(\lambda_{k,s})$ and the maximality of $\lambda_{k,s}$ and is injective by the freeness of M.

By factoring out $\Delta(\lambda_{k,s})$ from M, we obtain a module which is in $\mathcal{O}$ (since $\mathcal{O}$ is closed under quotients) and is (as $U(\mathcal{N}_-)$-module) free of finite rank. The result now follows inductively.

□

What can be said in the more general context of $\mathcal{O}^{(n)}$?

As we have seen before, the fat Verma modules $\Delta^{(n)}(\lambda)$ in the blocks of $\mathcal{O}^{(n)}$ are the standard objects for the algebra corresponding to such block (which we have shown to be standardly stratified) hence it makes sense to define, for $i \leq n$,

$$\mathcal{F}(\Delta^{(i)}) = \{M \in \mathcal{O}^{(n)} : M \text{ has a fat Verma flag in } \mathcal{O}^{(i)}\},$$

where a fat Verma flag is, by definition, a filtration whose subquotients are fat Verma modules.

On the other hand, we denote by $\mathcal{M}_{\mathcal{H},(n)}$ the obvious "fat" generalization of $\mathcal{M}_{\mathcal{H}}$ consisting of objects of $\mathcal{M}$ which are the direct sum of its fat weight spaces.

Remark Observe that, as usual, $\mathcal{F}(\Delta^{(1)}) = \mathcal{F}(\Delta)$. Furthermore, $\mathcal{F}(\Delta^{(n)})$ stands for the subcategory of $\mathcal{O}^{(n)}$ whose objects are filtered by the standard objects (objects having "good filtrations" in the sense of [29]).

Observe as well that $\mathcal{M}_{\mathcal{H},(1)} = \mathcal{M}_{\mathcal{H}}$. □

With all these considerations we are now in a position to prove

Proposition 3.7.5 *The restriction of Arkhipov's functor induces an equivalence $\mathcal{M}_{\mathcal{H},(n)} \sim \mathcal{M}_{\mathcal{H},(n)}{}^{op}$.*

Proof

Clearly, it suffices to verify that Arkhipov's functor sends objects of $\mathcal{M}_{\mathcal{H},(n)}$ to objects of $\mathcal{M}_{\mathcal{H},(n)}$. Accordingly, let M be an object of $\mathcal{M}_{\mathcal{H},(n)}$ and let us see what happens to

$$(S \otimes_{U(\mathcal{G})} M)^{\circledast}$$

$$= (U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} U(\mathcal{G}) \otimes_{U(\mathcal{G})} M)^{\circledast}$$

$$= (U(\mathcal{N}_-)^{\circledast} \otimes_{U(\mathcal{N}_-)} M)^{\circledast}.$$

For that, let $v_{1,1}, ..., v_{1,i_1}, v_{2,1}, ..., v_{2,i_2}, ..., v_{k,1}, ..., v_{k,i_k}$ be a basis of $M_\lambda^{(n)}$ where the left part of the index is labelling the layer (thus $k \leq n$) and the right part of the index counts the number of independent elements in each layer. Suppose, furthermore, that f_i is a basis vector of $U(\mathcal{N}_-)^{\circledast}$ dual to a basis vector of $U(\mathcal{N}_-)$ of grade -1, say $x_{-\alpha_i}$, where α_i is a positive root basis element. Then, by part 2 of Theorem 3.7.1, we have

$$H_j(f_i \otimes v_{r,p}) = -(f_i(ad_{H_j}) \otimes v_{r,p}) + f_i \otimes (2\rho(H_j) + H_j)v_{r,p}) =$$

$$= (\alpha_i + \lambda + 2\rho)(H_j) f_i \otimes v_{r,p} + f_i \otimes v_{\text{layer } r+1}.$$

Hence $B_i := \{f_i \otimes v_{r,p} : 1 \leq r \leq n, 1 \leq p \leq i_r\}$ is a basis of $(S \otimes_{U(\mathcal{G})} M)^n_{-\alpha_i+\lambda+2\rho}$ and clearly $B_i^{\circledast}$ (consisting of duals of elements of B_i) is then a basis of

$$((S \otimes_{U(\mathcal{G})} M)^{\circledast})^{(n)}_{-\alpha_i+\lambda+2\rho}.$$

For an element $x = x_{-\alpha_1}{}^{n_1} \ldots x_{-\alpha_k}{}^{n_k}$ of $U(\mathcal{N}_-)$ of grade smaller than -1 and a corresponding dual vector f_x, we use the identity

$$f_x(ad_h) = -(n_1\alpha_1(h) + \ldots + n_k\alpha_k(h)) f_x,$$

for all $h \in \mathcal{H}$, to prove the result in exactly the same way. In fact, if $\gamma = n_1\alpha_1 + \ldots + n_k\alpha_k$, we have

$$H_j(f_x \otimes v_{r,p}) = -(f_x(ad_{H_j}) \otimes v_{r,p}) + f_x \otimes (2\rho(H_j) + H_j)v_{r,p})$$

$$= (\gamma + \lambda + 2\rho)(H_j) f_x \otimes v_{r,p} + f_x \otimes v_{\text{layer } r+1}.$$

Hence we may conclude that $(S \otimes_{U(\mathcal{G})} M)^{\circledast} \in \mathcal{M}_{\mathcal{H},(n)}$.

□

As in the classical case, we will be able to show that

Proposition 3.7.6 *With the above notation, we have:*

1) $\mathcal{F}(\Delta^{(n)}) = \mathcal{O}^{(n)} \cap \mathcal{M}_{\mathcal{H},(n)}$;

2) Arkhipov's functor sends $\Delta^{(n)}(\lambda)$ *to* $\Delta^{(n)}(-2\rho - \lambda)$;

3) Restriction induces an exact equivalence $\mathcal{F}(\Delta^{(n)}) \sim \mathcal{F}(\Delta^{(n)})^{op}$.

Before launching into a long and tedious proof, we will need the following insultingly easy result of linear algebra:

Lemma 3.7.7 *Let* $C_{n,\lambda}$ *(respectively,* $C_{\lambda,n}$*) denote the* n^m*-th dimensional complex vector space with an additional structure of* $U(\mathcal{H})$*-module given by*

$$H_{i_k} v_{i_1 \ldots i_m} = v_{i_1 \ldots i_k - 1 \ldots i_m} + \lambda(H_{i_k}) v_{i_1 \ldots i_m}$$

(respectively,

$$H_{i_k} v_{i_1 \ldots i_m} = -v_{i_1 \ldots i_k + 1 \ldots i_m} + \lambda(H_{i_k}) v_{i_1 \ldots i_m})$$

where $k \in \{1, \ldots, m\}$, $i_k \in \{1, \ldots, n\}$ *and* $(v_{a_1 \ldots a_m})_{a_1, \ldots, a_m \in \{1, \ldots, n\}}$ *is a basis of the underlying vector space (assuming* $v_{negatives} = 0$*).*

Then $C_{n,\lambda} \cong C_{\lambda,n}$ *as* $U(\mathcal{H})$*-modules.*

Proof

The reason behind this is that the matrices $\begin{pmatrix} a & 1 & & \\ 0 & \ddots & \ddots & \\ & & \ddots & 1 \\ & & 0 & a \end{pmatrix}$

and $\begin{pmatrix} a & 0 & & \\ -1 & \ddots & \ddots & \\ & & \ddots & 0 \\ & & -1 & a \end{pmatrix}$ are similar.

□

We are now in a position to prove Proposition 3.7.6:

Proof

To prove 1 all we have to do is to mimic the proof of Proposition 3.7.4 , while 3) follows from 1) and 2).

Let us prove 2) showing that

$$\Delta^{(n)}(-\lambda - 2\rho) \cong (S \otimes_{U(\mathcal{G})} \Delta^{(n)}(\lambda))^{\circledast}.$$

Consider the map

$$\Psi : \Delta^{(n)}(-\lambda - 2\rho) \longrightarrow ((U(\mathcal{N}_-)^{\circledast} \otimes U(\mathcal{B})) \otimes_{U(\mathcal{B})} C_{\lambda,n})^{\circledast}$$

$$n \otimes v_{i_1,\dots,i_m} \longrightarrow \Psi_{n,v_{i_1,\dots,i_m}} : f \otimes 1 \otimes w_{j_1,\dots,w_{j_m}} \to \delta_{i_1\dots i_m,j_1\dots j_m} f(n)$$

where $(v_{a_1\dots a_m})_{a_1,\dots,a_m\in\{1,\dots,n\}}$ is a basis for $C_{n,-\lambda-2\rho}$, $(w_{b_1\dots b_m})_{b_1,\dots,b_m\in\{1,\dots,n\}}$ is a basis for $C_{\lambda,n}$, $n \in U(\mathcal{N}_-)$ and $\delta_{i,j}$ is the Kronecker symbol.

It is clear that Ψ is a vector space isomorphism since it maps basis of the LHS onto basis of the RHS and it is also quite easy to convince oneself that Ψ preserves the $U(\mathcal{N}_-)$ -action. The trickiest thing to check is actually the action of the Cartan subalgebra. For $H_k \in \mathcal{H}$, one has

$$\Psi(H_k(n \otimes v_{i_1,\ldots,i_m}))(f \otimes 1 \otimes v_{j_1,\ldots,j_m})$$

$$= \Psi((nH_k + ad_{H_k}(n)) \otimes v_{i_1,\ldots,i_m})(f \otimes 1 \otimes v_{j_1,\ldots,j_m})$$

$$= \Psi(n \otimes (v_{i_1,\ldots,i_k-1,\ldots,i_m} - (2\rho + \lambda)(H_k)v_{i_1,\ldots,i_m}) +$$

$$+ad_{H_k}(n) \otimes v_{i_1,\ldots,i_m})(f \otimes 1 \otimes v_{j_1,\ldots,j_m})$$

$$= \delta_{i_1\ldots i_k-1\ldots i_m, j_1\ldots j_m} f(n) - \delta_{i_1\ldots i_m, j_1\ldots j_m}(2\rho + \lambda)(H_k)f(n) +$$

$$+\delta_{i_1\ldots i_m, j_1\ldots j_m} f(ad_{H_k}(n)),$$

whereas

$$(H_k \Psi_{n, v_{i_1,\ldots,i_m}})(f \otimes 1 \otimes v_{j_1,\ldots,j_m})$$

$$= -\Psi_{n, v_{i_1,\ldots,i_m}}(H_k(f \otimes 1 \otimes v_{j_1,\ldots,j_m}))$$

$$= -\Psi_{n, v_{i_1,\ldots,i_m}}((-f(ad_{H_k}) \otimes 1 +$$

$$+f \otimes (2\rho(H_k) + H_k)) \otimes v_{j_1,\ldots,j_m})$$

$$= -\Psi_{n, v_{i_1,\ldots,i_m}}(-f(ad_{H_k}) \otimes 1 \otimes v_{j_1,\ldots,j_m} +$$

$$+f \otimes 2\rho(H_k) \otimes v_{j_1,\ldots,j_m} +$$

$$+f \otimes 1 \otimes \lambda(H_k)v_{j_1,\ldots,,m} - v_{j_1,\ldots,j_k+1,\ldots,j_m})$$

$$= \delta_{i_1\ldots i_m, j_1\ldots j_m} f(ad_{H_k}(n)) - \delta_{i_1\ldots i_m, j_1\ldots j_m}(2\rho + \lambda)(H_k)f(n) +$$

$$+\delta_{i_1\ldots i_m, j_1\ldots j_k+1\ldots j_m} f(n).$$

□

3.8 Ringel self duality and characters of tilting modules

Definition 3.8.1 *Let λ be a weight. An indecomposable object $T^{(n)}(\lambda)$ in $\mathcal{O}^{(n)}$ is called the* tilting module with parameter λ *if it satisfies the two following properties:*

i) $Ext^1{}_{\mathcal{O}^{(n)}}(\Delta^{(n)}(\alpha), T^{(n)}(\lambda)) = 0$ for all weights α;

ii) $T^{(n)}(\lambda)$ admits a fat Verma flag with bottom subquotient isomorphic to $\Delta^{(n)}(\lambda)$.

Dually, we define the cotilting module with parameter λ, $K^{(n)}(\lambda)$.

Remark 1) Note that from now on (until the end of this section) each statement will be equipped with its dual. We will often refrain from displaying it.

2) Clearly, $T^{(n)}(\lambda) \in \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)}) \cap \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})^{\perp}$.

3) By Proposition 3.4.4, the previous definition is just a redecoration of Definition 3.5.2!

□

As an application of Theorem 3.5.3 we get

Theorem 3.8.2 *For every weight λ there exists a unique tilting module with parameter λ.*

Let A be a (finite dimensional) standardly stratified algebra and let Λ be a set indexing the simple A-modules. Then recall that $T := \oplus_{i\in\Lambda} T(i)$ is the characteristic tilting module for A.

Definition 3.8.3 *Keeping the above notation, the algebra $End_A(T)$ is called the* Ringel dual *of A.*

In the case of a fat block of $\mathcal{O}^{(n)}$, $\mathcal{O}_\lambda^{(n)}$, we have that the characteristic tilting module is

$$T_\lambda^{(n)} := \oplus_x T^{(n)}(x.\lambda),$$

where x runs through a complete set of representatives of W/W_λ.

The main result of this section exhibits the Ringel dual of the blocks of $\mathcal{O}^{(n)}$:

Theorem 3.8.4 *(Ringel duality for $\mathcal{O}^{(n)}$)*

Let $\mathcal{O}_\lambda^{(n)}$ be a block of $\mathcal{O}^{(n)}$ and $A_{\lambda,n}$ its basic algebra. Then,

$$A_{\lambda,n} \cong End_{\mathcal{O}^{(n)}}(T_{-\lambda-2\rho}^{(n)}).$$

Proof

Consider the equivalence

$$F : \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)}) \longrightarrow \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})^{op},$$

established in Proposition 3.7.3, and a projective module $P^{(n)}(\alpha)$ in $\mathcal{O}_\lambda^{(n)}$. By the properties of projective modules, we know that

$$P^{(n)}(\alpha) \in \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)}).$$

Moreover, by the definition of F and Proposition 2.4.8, we have

$\bullet F(P^{(n)}(\alpha)) \in \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$;

$\bullet 0 = Ext^1{}_{\mathcal{O}^{(n)}}(P^{(n)}(\alpha), M) = Ext^1{}_{\mathcal{O}^{(n)}}(F(M), F(P^{(n)}(\alpha)))$, for all $M \in \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})$. Consequently,

$$F(P^{(n)}(\alpha)) \in \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)}) \cap \mathcal{F}_{\mathcal{O}^{(n)}}(\Delta^{(n)})^{\perp}.$$

But F is contravariantly exact and preserves indecomposability, hence the Verma filtration of $F(P^{(n)}(\alpha))$ has $\Delta^{(n)}(-2\rho-\alpha)$ at the bottom, from which we conclude that $F(P^{(n)}(\alpha)) \cong T^{(n)}(-2\rho-\alpha)$ by the definion of fat tilting modules.

Now it is easy to see that the bijection

$$\lambda \mapsto -\lambda - 2\rho,$$

from $\mathcal{H}^*$ to itself, permutes the dot orbits. In fact, for $w \in W$ and $\lambda, \mu \in \mathcal{H}^*$, we have

$$-\lambda - 2\rho = w.(-\mu - 2\rho) \iff -\lambda - \rho = w(-\mu - \rho) \iff$$

$$\iff \lambda + \rho = w(\mu + \rho) \iff \lambda = w.(\mu).$$

Finally, up to Morita equivalence, we have

$$A_{\lambda,n} \cong End_{\mathcal{O}^{(n)}}(\oplus_x P^{(n)}(x.\lambda))$$

$$\cong End_{\mathcal{O}^{(n)}}(\oplus_x T^{(n)}(-x.\lambda - 2\rho)) \cong End_{\mathcal{O}^{(n)}}(T^{(n)}_{-\lambda-2\rho}),$$

where x runs through a complete set of representatives of W/W_λ, thus concluding the proof.

□

Corollary 3.8.5 *The principal blocks of $\mathcal{O}^{(n)}$ coincide with their Ringel dual.*

Proof

Obvious consequence of the previous theorem and the fact that 0 and -2ρ lie in the same dot orbit. □

Corollary 3.8.6 *If $\mathcal{G} = sl_2(\mathbb{C})$ then* all *blocks of $\mathcal{O}^{(n)}$ coincide with their Ringel dual.*

Proof

For the root system A_1, the Weyl group is the symmetric group of order 2 and ρ equals 1. If $w \in W$ is not the identity, we have, for all weights $\lambda \in \mathbb{Z}$,

$$w.\lambda = w(\lambda + 1) - 1 = -\lambda - 2,$$

hence λ and $-\lambda - 2\rho$ always lie in the same dot orbit.

□

Now that we are in the mood of collecting our gains we can combine Proposition 3.7.3 with BGG reciprocity and the exactness of F in order to state

Corollary 3.8.7 *The character formulae for tilting modules look like*

$$n^m(P^{(n)}(\lambda) : \Delta^{(n)}(\alpha)) = [\Delta^{(n)}(\alpha) : L(\lambda)] = n^m[T^{(n)}(-2\rho - \lambda) : L(-2\rho - \alpha)].$$

Finally, we can combine the exactness of F with Theorem 2.4.10 to obtain a surprising structural result:

Corollary 3.8.8 *Let $T_\lambda^{(n)}$ be a tilting module (with parameter λ) of a given block of $\mathcal{O}^{(n)}$.*

Then there exists a filtration

$$0 \subseteq N_1 \subseteq \ldots \subseteq N_{n^m-1} \subseteq N_{n^m} = T_\lambda^{(n)},$$

where the subquotients N_i/N_{i-1} are tilting modules (with parameter λ) of $\mathcal{O}$.

3.9 Existence of projective-injective objects and double centralizer properties

We will start by deriving the existence of a unique indecomposable projective-injective object P in each block of $\mathcal{O}^{(n)}$. Afterwards, we proceed to show that blocks of $\mathcal{O}^{(n)}$ have a "double centralizer property" with respect to P.

Proposition 3.9.1 *For every weight λ there exists a unique projective-injective indecomposable object in each block of $\mathcal{O}^{(n)}{}_{\lambda}$.*

Proof

It is enough to show the existence of such object at $\mathcal{O}^{(n)}_{-\rho}$, where ρ denotes the half-sum of the positive roots, and then use translation functors. Now $\mathcal{O}^{(n)}{}_{-\rho}$ contains only one simple object and only one projective indecomposable which happens to be injective but (except for category $\mathcal{O}$, i.e. for $n = 1$) not simple.

In fact,

$$A_{-\rho,n} \cong End_{\mathcal{O}^{(n)}}(\Delta^{(n)}(-\rho)) \cong C_{-\rho,n}.$$

But $C_{-\rho,n}$ is selfinjective, so the above isomorphism tells us that the regular representation is the unique projective-injective representation of $A_{-\rho,n}$.

As for uniqueness, observe that, by Proposition 2.2.8, fat Verma modules have simple socle. On the other hand, by Proposition 2.4.8 fat projectives are filtered by fat Verma modules hence the socle of each projective module $P^{(n)}(\lambda)$ is a direct sum of a certain number (depending on λ) of copies of the same simple module. The result now follows from the fact that all injective modules are the injective envelope of their socle.

□

Now let A be an algebra, M a (left) A-module and $B := End_A(M)$. Then M is a (left) B-module and we set $C := End_B(M)$ (so the elements of C commute with all elements of B).

A ring homomorphism can be established

$$\phi : A \longrightarrow C$$

$$a \mapsto \phi_a,$$

where ϕ_a consists of left multiplication by an element $a \in A$.

Then A is said to have a double centralizer property (with respect to M) if ϕ is an epimorphism (in which case an isomorphism is induced by factoring out the annihilator of M from A).

Our final aim in this section is to generalize to $\mathcal{O}^{(n)}$ Soergel's double centralizer theorem (see [31]), which relates the principal block of the category $\mathcal{O}$ with the endomorphism ring of its unique indecomposable projective-injective object.

Later in Chapter 4 we will achieve this by closely following Soergel's methods but at the present moment we will mimic the ideas of [27] with tools borrowed from ring theory ($QF-3$ rings, dominant dimension, etc). In contrast with the traditional proofs (e.g. by the fundamental theorems of invariant theory), these techniques are virtually computation free.

Definition 3.9.2 *1) The* dominant dimension *of a finite dimensional algebra A (domdim$A$ for short) is the supremum of all $n \in \mathbb{N}$ such that there exists an exact sequence of the form*

$$0 \longrightarrow A \longrightarrow K_1 \longrightarrow ... \longrightarrow K_n,$$

where the K_i's are projective-injective objects in $A - mod$.

2) An algebra is said to be $QF - 3$ if there exists a faithful projective-injective A-module.

The proof of the next crucial theorem can be found in [27] .

Theorem 3.9.3 *Let A be a $QF - 3$ algebra and pick a minimal faithful projective-injective A-module (necessarily of the form Ae for some idempotent e). The following statements are equivalent:*

i) $domdimA \geq 2$;

ii) $A \cong End(Ae_{eAe})$.

Remark It is well known that $eAe^{op} \cong End({}_AAe)$, hence *ii)* does assert the validity of a double centralizer property for A.

□

As an application of the previous theorem, one is able to establish another analogy with $\mathcal{O}$:

Proposition 3.9.4 *(Double centralizer property)*

Let A be a block of $\mathcal{O}^{(n)}$ and Ae its unique indecomposable projective-injective object. Then

a) A is $QF - 3$ and $domdimA \geq 2$;

b) $A \cong End(Ae_{eAe})$.

Proof

Let $A = A_{\lambda,n}$ be the algebra corresponding to the block of $\mathcal{O}_\lambda^{(n)}$ and let λ_{min} and λ_{max} be, respectively, the dominant and the antidominant weights in the orbit of λ under the dot action of the Weyl group. As we have seen before, the socle of A is a direct sum of copies of $L_{\lambda_{min}}$ (where $L_{\lambda_{min}}$ is the simple A-module corresponding to $L(\lambda_{min})$ under the equivalence described in 3.1.1) and consequently the injective envelope of A is a direct sum of copies of $Ae_{\lambda_{min}}$ (where, again, this is the projective A-module corresponding to $P^{(n)}_{\lambda_{min}}$).

Thus $Ae_{\lambda_{min}}$ is faithful and A is $QF-3$.

Now, by Proposition 2.4.10, there exists a short exact sequence

$$0 \longrightarrow \Delta^{(n)}(\lambda_{max}) \longrightarrow P^{(n)}(\lambda_{min}) \longrightarrow M \longrightarrow 0,$$

where M is filtered by fat Verma modules (each occurring with multiplicity one) forcing its socle to be a direct sum of copies of $L(\lambda_{min})$ thus providing an exact sequence of the form

$$0 \longrightarrow \Delta^{(n)}(\lambda_{max}) \longrightarrow P^{(n)}(\lambda_{min}) \longrightarrow (P^{(n)}(\lambda_{min}))^k,$$

for some natural number k (e.g. take $k = |W/W_\lambda| - 1$) where $\Delta^{(n)}(\lambda_{max})$ is projective and $P^{(n)}(\lambda_{min})$ is projective-injective.

Adequately tensoring with finite dimensional vector spaces produces exact sequences of the form

$$0 \longrightarrow P \longrightarrow P_1 \longrightarrow P_2,$$

where, as before, P is projective and the P_i's are projective-injectives in $\mathcal{O}_\lambda^{(n)}$.

But, by 2.4.7, all projective objects in $\mathcal{O}^{(n)}$ may be obtained in a similar way to P, hence A has dominant dimension higher or equal to two.

By Theorem 3.9.3, $b)$ follows from $a)$.

□

3.10 Auslander's framework and representation type

We start this section by presenting two classical results due to Auslander (see [3], section 5). In the sequel, we assume that A is a finite dimensional algebra and $\{L(\lambda) : \lambda \in \Lambda\}$ is a complete collection of non isomorphic simple A-modules. Let $\Lambda_1 \subseteq \Lambda$ and define both $P(\Lambda_1) := \bigoplus_{\lambda \in \Lambda_1} P(\lambda)$ and $A_{\Lambda_1} = End_A(P(\Lambda_1))^{op}$.

Proposition 3.10.1 *(Auslander, [3])*

With the above notation, the following statements are equivalent for an A-module M:

1) M is Λ_1-projectively presented (i.e. there exists an exact sequence of the form

$$P_2 \to P_1 \to M \to 0,$$

where the P_i's are quotients of direct sums of $P(\lambda)$'s with $\lambda \in \Lambda_1$);

2) There exists an A-module N such that $M = P(\Lambda_1) \otimes_{A_{\Lambda_1}} N$

(in fact, up to Morita equivalence, $A_{\Lambda_1} = eAe$ for some idempotent e).

Proposition 3.10.2 *(Auslander, [3])*

The full subcategory $B(\Lambda_1)$ of Λ_1-projectively presented modules is equivalent to the category of A_{Λ_1}-mod (via induction and restriction).

An immediate consequence of the previous propositions is the following

Corollary 3.10.3 *Let $A_{\lambda,n}$ be a block of $\mathcal{O}^{(n)}$ and $A_{\lambda,n}e_\lambda$ the 'big projective'. Then $e_\lambda A_{\lambda,n}e_\lambda$-mod embeds in $A_{\lambda,n}$-mod .*

Furthermore, the above embedding preserve indecomposability.

Remark With similar arguments to those in [11] it could be shown that under the same assumptions of Corollary 3.10.3 one has that $e_\lambda A_{\lambda,n}e_\lambda$ -mod embeds in $\mathcal{F}(\Delta^{(n)})$.

□

Example 3.10.4 *Let A be the principal block of $\mathcal{O}^{(2)}(sl_2(\mathbb{C}))$. Then, according to the example described in Section 3.2 , the local algebra e_1Ae_1-mod is isomorphic to $\mathbb{C}[X,Y]/(X^2,Y^2)$. Thus e_1Ae_1-mod is a well known tame algebra.*

We will now proceed to classify the blocks of $\mathcal{O}^{(n)}$ according to their representation type. For $n = 1$ this task is much harder and has been done in [19] and [11].

Recall, after Drozd (see [17]), that the representation type of every finite dimensional algebra (over an algebraically closed field) falls into one, and only one, of the following cases: finite, tame or wild. Further, if A and B are two finite dimensional algebras of infinite representation type (i.e. with an infinite number of indecomposable modules) and there exists an embedding from $A - mod$ to $B - mod$ preserving indecomposability then

- If B is tame then A is tame.
- If A is wild then B is wild.

Having this in mind and looking back at Corollary 3.10.3, we conclude that the classification of the algebras $e_\lambda A_{\lambda,n} e_\lambda$ according to their representation type is of fundamental importance to achieve our main goal. For $n = 1$, the most difficult situation, this has been achieved in [21].

By deep results of Soergel (see [31]) the algebra $e_\lambda A_{\lambda,1} e_\lambda$ is a well known geometric object: the algebra of coinvariants (associated with the Cartan subalgebra $\mathcal{H}$). Until the end of this section, we will assume the validity of an important structural fact that will appear in more detail in the next chapter: then, we will be able to quote from Soergel that

$$e_\lambda A_{\lambda,n} e_\lambda \cong U(\mathcal{H})^{W_\lambda} \otimes_{U(\mathcal{H})^W} U(\mathcal{H})/I_{\lambda,n}.$$

where $U(\mathcal{H})^W$ is the subalgebra of $U(\mathcal{H})$ generated by all elements which are fixed under the (natural) action of W.

We denote the algebra on the RHS of the above isomorphism by $C_\lambda^{(n)}$ and call it the "fat algebra of coinvariants" of the block $\mathcal{O}_\lambda^{(n)}$. For $n = 1$ we will denote the algebra of coinvariants by C_λ.

Observe that $C_\lambda^{(n)}$ is a finite dimensional, local, symmetric and commutative algebra (since C_λ is and the ring structure of $C_\lambda^{(n)}$ follows easily from the ones of C_λ and $U(\mathcal{H})/I_{\lambda,n}$). But the tame, local, symmetric, commutative algebras are classified in [18], Theorem III.1 (following ideas from [30]). They have the form $K[X,Y]/I$, where I is an ideal of the following type:

1) $I = (X^m + Y^n, XY)$ where $m \geq n \geq 2$ and $m + n > 4$;

2) $I = (X^2, Y^2)$;

3) $I = (X^2, Y^2 - XY)$, where K is a field of characteristic two.

In particular, any such algebra with minimal number of generators greater than two is wild.

The theory described above seems to have been tailor made for our situation. In fact,the previous observations immediately imply our next result which describes the representation type of the fat algebra of coinvariants.

Theorem 3.10.5 *Let $n > 1$ and let $C_\lambda^{(n)}$ be the fat algebra of coinvariants. Then*

i) $C_\lambda^{(n)}$ is of finite type if and only if $rank(\mathcal{G}) = 1$ and $W = W_\lambda$.

ii) $C_\lambda^{(n)}$ is tame if and only if $rank(\mathcal{G}) = 1$ and $W_\lambda = 1$ or $rank(\mathcal{G}) = 2$, $n = 2$ and $W = W_\lambda$.

Proof

For all situations apart from the above cited ones, it is easy to spot three or more generators for $C_\lambda^{(n)}$. In fact, $C_\lambda^{(n)}$ is graded (e.g. by giving grade 1 to basis vectors of $\mathcal{H}$), hence all independent vectors of grade 1 must be contained in some minimal generating set.

Now in *i)* we are talking about $\mathbb{C}[X]/(X^n)$ which is clearly of finite type. In *ii)*, if $rank(\mathcal{G} = 1)$ and $W_\lambda = 1$, we have the algebra $\mathbb{C}[X,Y]/(X^n, X^2 - Y^2)$ which for $n = 2$ falls into case 2 (see above) and for $n > 2$ falls into case 1. The last assertion is less obvious. To see it we have to check the following isomorphism of algebras:

$$\mathbb{C}[x,y]/(x^n, x^2 - y^2) \to \mathbb{C}[X,Y]/(X^n + Y^n, XY)$$

$$x \mapsto X + Y$$

$$y \mapsto X - Y.$$

In the last situation, if $rank(\mathcal{G}) = 2$, $n = 2$ and $W = W_\lambda$, we are in the presence of $\mathbb{C}[X,Y]/(X^2, Y^2)$, so, again, we fall into case 2.

□

Now the classification for the blocks of $\mathcal{O}^{(n)}$ is just a matter of "copy and paste":

Theorem 3.10.6 *Let $n > 1$ and let $\mathcal{O}_\lambda^{(n)}$ be a block of $\mathcal{O}^{(n)}$. Then*

i) $\mathcal{O}_\lambda^{(n)}$ is of finite type if and only if $rank(\mathcal{G}) = 1$ and $W = W_\lambda$;

ii) $\mathcal{O}_\lambda^{(n)}$ is tame if and only if $rank(\mathcal{G}) = 1$ and $W_\lambda = 1$ or $rank(\mathcal{G}) = 2$, $n = 2$ and $W = W_\lambda$.

Proof

There is only one situation to analyse (since, in the two others, the blocks coincide with their fat algebras of coinvariants): the regular blocks of $sl_2(\mathbb{C})$. But these algebras were described in Section 3.2. They constitute an example of *special biserial algebras* which are well known to be tame (see [18] again).

□

3.11 Koszul duality: the first drawback

We start this section with the definition of *Koszul ring*:

Definition 3.11.1 *A Koszul ring is a positively graded ring $A = \oplus_{j \geq 0} A_j$ such that*

i) A_0 is semisimple;

ii) As a graded left A-module, $A_0 (\cong \frac{A}{\oplus_{j>0} A_j})$ *admits a graded projective resolution*

$$\dots \to P_2 \to P_1 \to P_0 \to A_0 \to 0$$

such that every projective module P_i *is generated by its degree* i *component.*

For an algebra A, denote by $E(A)$ the vector space

$$\oplus_{n \geq 0} Ext^n_A(A/Rad(A), A/Rad(A))$$

made into a ring via the Yoneda multiplication (see [8] or [13] for more details). We will refer to $E(A)$ as the *Ext-algebra* of A.

If $A_{0,1}$ denotes the basic algebra corresponding to the principal block of category $\mathcal{O}$ then Beilinson, Ginzburg and Soergel proved in [7] the following theorem:

Theorem 3.11.2 *1) There exists an isomorphism of* $\mathbb{C}$*-algebras*

$$A_{0,1} \cong E(A_{0,1});$$

2) $E(A_{0,1})$ *is a Koszul ring.*

Proof This is Theorem 1.1.1. of [7]. □

Since the algebras $A_{0,n}$ have infinite global dimension for $n > 1$, there is no hope of finding an isomorphism between $A_{0,n}$ and its Ext-algebra. In fact, the following example illustrates how "far from isomorphic" can $A_{0,n}$ and $E(A_{0,n})$ be:

3.11.1 Example: The Ext-algebra of $A_{0,2}$

Going back to Section 3.2, recall that the basic algebra corresponding to the 2−fat principal block of $sl_2(\mathbb{C})$, $A_{0,2}$, is given by the following factor algebra $\mathbb{C}Q/J_{0,2}$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \end{tikzcd}$$

and $J_{0,2}$ is the ideal of relations $J_{0,2} = \langle xz, zy, (z+yx)^2, (xy)^2 \rangle$.

The projective resolutions of the simple modules $L(1)$ and $L(2)$ are

$$\ldots \to P(1) \oplus P(2) \to P(1) \oplus P(2) \to P(1) \to L(1) \to 0$$

and

$$\ldots \to P(1) \oplus P(2) \to P(1) \oplus P(2) \to P(1) \to P(2) \to L(2) \to 0$$

where $P(i)$ is the projective cover of $L(i)$ and all the terms in the above resolutions which are not on display are of the form $P(1) \oplus P(2)$.

After some not very difficult calculations we reach the conclusion that $E(A_{0,2})$ is given by following factor algebra $\mathbb{C}Q/K$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \arrow[loop right, "k"] \end{tikzcd}$$

and K the ideal of relations $K = \langle xy, yx, yk - z^2y, kx - xz^2 \rangle$.

The algebra $\mathbb{C}Q/K$ is not even finite dimensional!

It is obviously a graded algebra (for being an Ext-algebra) where its generators x, y and z have grade 1, while k has grade 2.

The graded layers of $A_{0,2}$ may be depicted in the following table:

	$1 \to 1$	$1 \to 2$	$2 \to 1$	$2 \to 2$
grade 1	z	x	y	
grade 2	z^2	xz	zy	k
grade 3	z^3	xz^2	z^2y	xzy
grade 4	z^4	xz^3	z^3y	k^2
grade 5	...	...	...	xz^3y
...	...	...	...	...

Observe that $E(A_{0,2})$ is not generated by the set elements of grade 0 or 1, hence $A_{0,2}$ is not Koszul.

3.12 Example

To illustrate our previous results let us look at what happens in the principal block for $n = 1$ and $n = 2$.

In the following table, observe how fat Verma (respectively fat projective, fat tilting) modules are filtered by Verma (respectively projective, tilting) modules from category $\mathcal{O}$. Observe, as well, that projective and tilting modules are filtered by Verma modules with multiplicities agreeing with the character formulae described in Corollary 3.8.7.

	$A_{0,1}$	$A_{0,2}$
$\Delta(-2)$	1	1 1
$\Delta(0)$	2 1	2 1 2 1
$P(-2)$	1 2 1	1 2 1 1 2 1
$P(0)$	2 1	2 1 2 1
$T(-2)$	1	1 1
$T(0)$	1 2 1	1 2 1 1 2 1

Chapter 4

Towards examples

4.1 Generators of Verma submodules

We begin this chapter describing some (quite elementary) features of Verma modules in category $\mathcal{O}$.

The aim of this section is to investigate the existence of "nice" generators of submodules of Verma modules which are themselves isomorphic to Verma modules. By a "nice" generator we mean a generator of the form

$$X^{k_1}_{-\alpha_{i_1}} \dots X^{k_s}_{-\alpha_{i_s}} v_\lambda,$$

where the α_j's are simple roots and v_λ is the canonical generator of $\Delta(\lambda)$.

Our main tool is the following very well known result

Proposition 4.1.1 *Let λ be a weight such that $s_\alpha.\lambda \leq \lambda$ for some simple root α. Then*

$$X^{i_{\lambda,\alpha}}_{-\alpha} v_\lambda,$$

where $i_{\lambda,\alpha} = \langle \lambda + \rho, s_\alpha(\lambda + \rho) \rangle$, generates a submodule isomorphic to $\Delta(s_\alpha.\lambda)$.

Proof

This is Proposition 7.1.15 of [16] .

□

In further computations with root systems of rank 2, the following lemma is useful:

Lemma 4.1.2 *Let α_1 and α_2 be simple roots. Then*

$$s_{\alpha_2}.s_{\alpha_1}.\lambda \leq s_{\alpha_1}.\lambda \iff (\lambda+\rho)(H_2) - (\lambda+\rho)(H_1)\langle s_{\alpha_1}, s_{\alpha_2}\rangle \geq 0.$$

Proof

This is straightforward since

$$\begin{aligned} &s_{\alpha_2}.s_{\alpha_1}.\lambda - s_{\alpha_1}.\lambda \\ &= \lambda - \langle\lambda+\rho,\alpha_1\rangle\alpha_1 - \langle\lambda - \langle\lambda+\rho,\alpha_1\rangle\alpha_1 + \rho, \alpha_2\rangle\alpha_2 - (\lambda - \langle\lambda+\rho,\alpha_1\rangle\alpha_1) \\ &= \lambda - \langle\lambda+\rho,\alpha_1\rangle(\alpha_1 - \langle\alpha_1,\alpha_2\rangle\alpha_2) - \langle\lambda+\rho,\alpha_2\rangle\alpha_2 - (\lambda - \langle\lambda+\rho,\alpha_1\rangle\alpha_1) \\ &= (\langle\lambda+\rho,\alpha_1\rangle\langle\alpha_1,\alpha_2\rangle - \langle\lambda+\rho,\alpha_2\rangle)\alpha_2. \end{aligned}$$

The next lemma allows us to compute our desired generators of Verma submodules for all Verma modules of rank 2.

Lemma 4.1.3 *Let γ be a positive root of the form*

$$\gamma = n_1\alpha_1 + n_2\alpha_2,$$

where n_1 and n_2 are non zero positive integers. Suppose that

$$(\lambda+\rho)(H_1) \geq 0,\ (\lambda+\rho)(H_2) \leq 0 \text{ and } (\lambda+\rho)(H_\gamma) \geq 0.$$

Then,

$$s_{\alpha_2}.s_{\alpha_1}.\lambda \leq s_{\alpha_1}.\lambda.$$

Proof

Two cases may occur:

Case 1: $\gamma = n_1\alpha_1 + n_2\alpha_2$ with $0 < n_1 \leq n_2$.

In this case, since $(\lambda+\rho)(H_\gamma) \geq 0$ we have that $(\lambda+\rho)(H_1 + H_2) \geq 0$ and consequently $(\lambda+\rho)(H_2) - (\lambda+\rho)(H_1)\langle s_{\alpha_1}, s_{\alpha_2}\rangle \geq 0$. Hence, by Lemma 4.1.2, we conclude that $s_{\alpha_2}.s_{\alpha_1}.\lambda \leq s_{\alpha_1}.\lambda$.

Case 2: $\gamma = 2\alpha_1 + \alpha_2$, $\gamma = 3\alpha_1 + \alpha_2$ or $\gamma = 3\alpha_1 + 2\alpha_2$.

The first situation may happen in types B_2 and G_2 while the second and third situations happen only in type G_2 (in all cases α_1 is the short simple root). In all three situations we have to verify directly that every weight λ lying in the region defined by the hypothesis of the lemma satisfies $s_{\alpha_2}.s_{\alpha_1}.\lambda \leq s_{\alpha_1}.\lambda$.

□

4.2 General formulae for fat Verma modules

We want to understand how reflections act inside the fat Verma modules. More accurately, we want to derive a formula describing $X^i_{\alpha_j}(X^i_{-\alpha_j}v_\lambda)$ where i is a positive integer, v_λ is the canonical generator of $\Delta^{(n)}(\lambda)$ and α_j is a simple root.

To get there let us start by fixing weights α and λ and define recursively the weights $m_i(\alpha, \lambda)$ in the following way:

$$m_1(\alpha, \lambda) = \lambda,$$

$$m_2(\alpha, \lambda) = 2\lambda - \alpha,$$

$$m_3(\alpha, \lambda) = 3\lambda - 3\alpha,$$

$$m_4(\alpha, \lambda) = 4\lambda - 6\alpha,$$

$$\vdots$$

$$m_i(\alpha, \lambda) = m_{i-1}(\alpha, \lambda) + \lambda - (i-1)\alpha.$$

Observe that $m_i(\alpha, \lambda)$ are still integral weights. Observe, as well, that the m_i's could, equivalently, be described in the following way, for all positive integers i:

$$m_i(\alpha, \lambda) = i\lambda - \frac{i(i-1)}{2}\alpha.$$

Then, if v_λ denotes the canonical generator of $\Delta(\lambda)$ and α_j is a positive root, we have a first formula for the action of the X_{α_j}'s on the generators of Verma modules in category $\mathcal{O}$:

Proposition 4.2.1 *With the above notation*

$$X^i_{\alpha_j}(X^i_{-\alpha_j}v_\lambda) = m_1(\alpha_j, \lambda)(H_{\alpha_j})m_2(\alpha_j, \lambda)(H_{\alpha_j})...m_i(\alpha_j, \lambda)(H_{\alpha_j})v_\lambda,$$

or, equivalently,

$$X^i_{\alpha_j}X^i_{-\alpha_j}v_\lambda = m_i(\alpha_j, \lambda)(H_{\alpha_j})X^{i-1}_{\alpha_j}X^{i-1}_{-\alpha_j}v_\lambda.$$

Proof

Let us prove the formula by induction on i. The case $i = 1$ tells us what we already know: $X_{\alpha_j}X_{-\alpha_j}v_\lambda = H_{\alpha_j}v_\lambda = \lambda(H_{\alpha_j})v_\lambda$. Now, assuming the validity of the formula for $i-1$, one has

$$X^i_{\alpha_j}X^i_{-\alpha_j}v_\lambda = X^{i-1}_{\alpha_j}X_{-\alpha_j}X_{\alpha_j}X^{i-1}_{-\alpha_j}v_\lambda + (\lambda - (i-1)\alpha_j)(H_{\alpha_j})X^{i-1}_{\alpha_j}X^{i-1}_{-\alpha_j}v_\lambda$$

$$= X_{\alpha_j}^{i-1} X_{-\alpha_j}^{2} X_{\alpha_j} X_{-\alpha_j}^{i-2} v_\lambda + (\lambda - (i-1)\alpha_j + \lambda - (i-2)\alpha_j)(H_{\alpha_j}) X_{\alpha_j}^{i-1} X_{-\alpha_j}^{i-1} v_\lambda$$

$$= \ldots$$

$$= (\lambda - (i-1)\alpha_j + \lambda - (i-2)\alpha_j + \ldots + (\lambda - \alpha_j) + \lambda)(H_{\alpha_j}) X_{\alpha_j}^{i-1} X_{-\alpha_j}^{i-1} v_\lambda$$

$$= m_1(\alpha_j, \lambda)(H_{\alpha_j}) m_2(\alpha_j, \lambda)(H_{\alpha_j}) \ldots m_i(\alpha_j, \lambda)(H_{\alpha_j}) v_\lambda.$$

□

The next lemma hides the "easy part" of the BGG theorem (see Theorem 2.4.12).

Lemma 4.2.2 *Let α_j be a positive simple root and $i := i_{\lambda,\alpha_j} = \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle$. Then $m_i(\alpha_j, \lambda)(H_{\alpha_j}) = 0$.*

Proof

In fact,

$$m_i(\alpha_j, \lambda)(H_{\alpha_j}) = \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle \lambda(H_{\alpha_j}) - (\langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle(\langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle - 1))$$

$$= \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle(\lambda(H_{\alpha_j}) + 1 - \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle)$$

$$= \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle(\lambda(H_{\alpha_j}) - (\lambda+\rho)(H_{\alpha_j} + \rho(H_{\alpha_j})) = 0.$$

□

With this in mind, fix weights α and λ together with a positive integer i and, for $j \in \{1, ..., m-1\}$, define weights $A_j(\alpha, \lambda, i)$ in the following way:

$$A_0(\alpha, \lambda, i) = m_1(\alpha, \lambda)m_2(\alpha, \lambda)...m_i(\alpha, \lambda)$$

$$A_1(\alpha, \lambda, i) = \sum_{1 \leq s_1 \leq i} s_1 m_1(\alpha, \lambda)m_2(\alpha, \lambda)...\widehat{m_{s_1}}(\alpha, \lambda)...m_i(\alpha, \lambda)$$

$$A_2(\alpha, \lambda, i) = \sum_{1 \leq s_1 \leq s_2 \leq i} s_1 s_2 m_1(\alpha, \lambda)m_2(\alpha, \lambda)...\widehat{m_{s_1}}(\alpha, \lambda)...\widehat{m_{s_2}}(\alpha, \lambda)...m_i(\alpha, \lambda)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$A_j(\alpha, \lambda, i) = \sum_{1 \leq s_1 \leq ... \leq s_j \leq i} s_1...s_j m_1(\alpha, \lambda)m_2(\alpha, \lambda)...\widehat{m_{s_1}}(\alpha, \lambda)...\widehat{m_{s_j}}(\alpha, \lambda)...m_i(\alpha, \lambda).$$

With the above notation, we can prove the following auxiliary result which will be very helpful later on:

Lemma 4.2.3 *For all simple roots α_j and all weights λ and all positive integers i we have*

1) $A_1(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0$;

2) If $(\lambda + \rho)(H_{\alpha_j}) \leq 0$ *then* $A_0(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0$.

Proof

Let us start by proving 1): denote the integer $\lambda(H_{\alpha_j})$ by a. Then

$$A_1(\alpha_j, \lambda, i) = \sum_{s=1}^{i} s(a(2a-2)(3a-6)...\widehat{s(a-(s-1))}...(i(a-(i-1))))$$

$$= \sum_{s=1}^{i} s(2.3...\hat{s}...i)a(a-1)...(a-\widehat{(s-1)})...(a-(i-1))$$

$$= \sum_{s=0}^{i-1} i!a(a-1)...\widehat{(a-s)}...(a-(i-1)).$$

Therefore, $A_1(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0$ if $a > i-1$ (in which case it is always positive) or $a < 0$ (here the sign depends on the parity of i). If $a \in \{0, ..., i-1\}$, it annihalates all factors of the above expression except one. Thus the result follows.

To prove 2) it suffices to observe that

$$A_0(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0 \iff \lambda(H_{\alpha_j}) \notin \{0, 1, ..., i_{\lambda,\alpha_j} - 1\}.$$

This follows from simple calculations since, by definition, $A_0(\alpha_j, \lambda, i)(H_{\alpha_j})$ is non zero if and only if $m_s(\alpha_j, \lambda)(H_{\alpha_j})$ is non zero for all $s \in \{1, ..., i\}$. But $m_s(\alpha_j, \lambda)(H_{\alpha_j}) = s\lambda(H_{\alpha_j}) - s(s-1) = s(\lambda(H_{\alpha_j}) - s + 1)$ hence the result follows.

□

Now recall that the homogeneous elements $v_{\lambda,k_1,...,k_m} \in \Delta^{(n)}(\lambda)$ were defined in Chapter 2 as

$$v_{\lambda,k_1,...,k_m} = (H_{\alpha_1} - \lambda(H_{\alpha_1}))^{k_1}...(H_{\alpha_m} - \lambda(H_{\alpha_m}))^{k_m} v_\lambda,$$

where $\alpha_1, ..., \alpha_m$ are the simple roots.

The version of Proposition 4.2.1 for fat Verma modules goes as follows:

Proposition 4.2.4 *Let α_j be a simple root. With the above notation, we have, for every positive integer i,*

$$X^i_{\alpha_j} X^i_{-\alpha_j} v_{\lambda,k_1,\ldots,k_m} = \sum_{p=0}^{i} A_p(\alpha_j, \lambda, i)(H_{\alpha_j}) v_{\lambda,k_1,\ldots(k_j+p),\ldots,k_m}$$

or, equivalently,

$$X^i_{\alpha_j} X^i_{-\alpha_j} v_{\lambda,k_1,\ldots,k_m} = m_i(\alpha_j, \lambda)(H_{\alpha_j}) X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_{\lambda,k_1,\ldots,k_m} + i X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_{\lambda,k_1,\ldots(k_j+p),\ldots,k_m}.$$

Proof

In the same lines of the proof of Proposition 4.2.1, we proceed by induction. For $i = 1$, we have

$$X_{\alpha_j} X_{-\alpha_j} v_{\lambda,k_1,..,k_m} = H_{\alpha_j} v_{\lambda,k_1,..,k_m}$$

$$= \lambda(H_{\alpha_j}) v_{\lambda,k_1,..,k_m} + v_{\lambda,k_1,\ldots,(k_j+1),\ldots,k_m}.$$

Now, assuming the validity of the statement for $i-1$, we have

$$X^i_{\alpha_j}(X^i_{-\alpha_j} v_\lambda) = X^{i-1}_{\alpha_j} X_{-\alpha_j} X_{\alpha_j}(X^{i-1}_{-\alpha_j} v_\lambda) +$$

$$+(\lambda - (i+1)\alpha_j)(H_{\alpha_j}) X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_\lambda + X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_{\lambda,k_1,\ldots,(k_j+1),\ldots,k_m}$$

$$= \ldots$$

$$= m_i(\alpha_j, \lambda)(H_{\alpha_j}) X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_{\lambda,k_1,\ldots,k_m} + i X^{i-1}_{\alpha_j} X^{i-1}_{-\alpha_j} v_{\lambda,k_1,\ldots,(k_j+1),\ldots,k_m}.$$

□

Later on we will make use of the following corollary: Using Lemma 4.2.2, a particular case of the above proposition is given by the following corollary:

Corollary 4.2.5 *1) If α_j is a positive root and $i := i_{\lambda,\alpha_j} = \langle \lambda+\rho, s_{\alpha_j}(\lambda+\rho)\rangle$ then, keeping the above notation, we have*

$$X^i_{\alpha_j} X^i_{-\alpha_j} v_\lambda = \sum_{p=1}^{i} A_p(\alpha_j, \lambda, i)(H_{\alpha_j})(H_{\alpha_j} - \lambda(H_{\alpha_j}))^p v_\lambda,$$

where $A_1(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0$;

2) If α_s and α_j are simple roots, $i := i_{\lambda,\alpha_j} = \langle \lambda + \rho, s_{\alpha_j}(\lambda + \rho)\rangle$ and r is a positive integer then, keeping the above notation, we have

$$X^r_{\alpha_s} X^i_{\alpha_j} X^r_{-\alpha_s} X^i_{-\alpha_j} v_\lambda$$

$$= \sum_{q=0}^{r} A_q(\alpha_s, \lambda, r)(H_{\alpha_s}) \sum_{p=1}^{i} A_p(\alpha_j, \lambda, i)(H_{\alpha_j})(H_{\alpha_s} - \lambda(H_{\alpha_s}))^q (H_{\alpha_j} - \lambda(H_{\alpha_j}))^p v_\lambda,$$

where $A_0(\alpha_s, \lambda, r)(H_{\alpha_s}) A_1(\alpha_j, \lambda, i)(H_{\alpha_j}) \neq 0$.

Proof

Assertion 1) is the combined application of Proposition 4.2.4 and Lemma 4.2.2 while assertion 2) follows directly from Proposition 4.2.4 and Lemma 4.2.3.

□

4.3 Self extensions of simples in $\mathcal{O}^{(n)}$

We are now in a position to prove one of the main result of this chapter: the determination of the dimension of the Ext^1 spaces between *all* simple modules in each block of $\mathcal{O}^{(n)}$ for semisimple Lie algebras of rank one and two (and a sharp bound in the general case). Note that the choice of cohomology is being made by identifying an object in $Ext^1_{\mathcal{O}^{(n)}}(L(\alpha), L(\beta))$ with a(n equivalence class of a) short exact sequence

$$0 \longrightarrow L(\beta) \longrightarrow M \longrightarrow L(\alpha) \longrightarrow 0$$

with $M \in \mathcal{O}^{(n)}$.

To relieve a bit the burden of notation, we will split the result into two theorems.

Firstly let us introduce some terminology. If $a \leq b$ for some partial order $\leq$, let us write $a \prec b$ whenever a is an immediate predecessor of b. If $w \in W$, define

$$H(w) = \{H_\gamma : s_\gamma w \prec w\}.$$

Denote the cardinality of $H(w)$ by $h(w)$ and observe that, by the BGG theorem, the elements of $H(w)$ parametrize part of the composition factors of the top of the radical of the Verma module $\Delta(w.\lambda)$, where λ is a dominant weight . Observe, as well, that $h(w) \leq m$. In fact, $H(w)$ consists of linearly independent vectors of $\mathcal{H}^*$ and therefore we may construct a basis

$$\mathcal{B}(w) := H(w) \dot{\cup} \{H_{j_1}, ..., H_{j_{k(w)}}\},$$

where $\{j_1, ..., j_{k(w)}\} \subseteq \{1, ..., m\}$. Observe that $k(w) = m - h(w)$.

Now, if γ, γ_1 and γ_2 are positive roots such that $\gamma = \gamma_1 + \gamma_2$, we have that $H_\gamma \in H(w)$ forces $H_{\gamma_1} \in H(w)$ or $H_{\gamma_2} \in H(w)$. Hence, we may conclude that if $H_\gamma \in H(w)$ and $\gamma = \sum_{s=1}^{k} n_{i_s}\alpha_{i_s}$, where the n_i's are positive integers and the α_i' are simple roots, then $H_{\alpha_{i_s}}$ is an integral linear combination of the elements of $H(w)$. This allows us to rewrite

$$\mathcal{B}(w) := \{H_{i_1}, ..., H_{i_k}\}\dot{\cup}\{H_{j_1}, ..., H_{j_{k(w)}}\}.$$

With these observations in mind, we are now in the position to prove the main result of this chapter:

Theorem 4.3.1 *With the above notation, we have, for $n > 1$, $w \in W$ and λ dominant,*

1) $dimExt^1_{\mathcal{O}_\lambda^{(n)}}(L(w.\lambda), L(w.\lambda)) \geq k(w)$;

2) If $rank(\mathcal{G}) \leq 2$ then $dimExt^1_{\mathcal{O}_\lambda^{(n)}}(L(w.\lambda), L(w.\lambda)) = k(w)$.

Remark

It is well known that for every weight λ we have $dimExt^1_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$. This follows immediately from the fact that the simple modules in $\mathcal{O}$ have one dimensional highest weight spaces and $\mathcal{H}$ acts diagonally on them.

□

Proof

To make notation slightly easier, set $\beta := w.\lambda$. We are trying to count the number of composition factors in the top of the radical of $P^n(\beta)$ that are isomorphic to $L(\beta)$ but since we are dealing with a properly stratified algebra this number equals the number of occurences of $L(\beta)$ in the top of the radical of $\Delta^n(\beta)$. But simple modules are quotients of fat Verma modules which are themselves filtered by (usual) Verma modules . Hence, if such a self extension does exist it is isomorphic to a quotient of $\Delta^{(n)}(\beta)$ and its generalized β-highest weight space admits the images of v_β and $(H-\beta(H))v_\beta$ as a basis, for some $H \in \mathcal{B}(w)$. Now, by the "recognition" Lemma 2.4.1, each highest weight vector of the form $(H - \beta(H))v_\beta$, with $H \in \mathcal{B}(w)$, generates a submodule of $\Delta^{(n)}(\beta)$ with simple top $L(\beta)$ (it is easy to check that $\langle (H - \beta(H))v_\beta \rangle \cong \Delta^{(n-1)}(\beta)$, for all $H \in \mathcal{B}(w)$). In fact, denoting by N the submodule of $\Delta^{(n)}(\beta)$ generated by all vectors of the form

$$(H - \beta(H))v_\beta,$$

where $H \in B(w)$, there is a short exact sequence

$$0 \longrightarrow N \longrightarrow \Delta^{(n)}(\beta) \longrightarrow \Delta(\beta) \longrightarrow 0,$$

where, by the PBW, the elements $(H - \beta(H))v_\beta$, with $H \in B(w)$, lie in the top of N. Hence, viewing $\Delta(\beta)$ as the top Verma subquotient of $\Delta^{(n)}(\beta)$, we have

$$(H-\beta(H))v_\beta \in top(Rad(\Delta^{(n)}(\beta)) \iff (H-\beta(H))v_\beta \notin \sum_{i=1}^{p} U(\mathcal{G})w_i,$$

where $\{w_1, ..., w_p\}$ are preimages of a basis of $top(Rad(\Delta(\beta)))$.

To prove 1), start by observing that, again by the BGG theorem, we may pick sums of elements of the form

$$X_{-\beta_1}^{k_1}...X_{-\beta_l}^{k_l}v_\beta,$$

where $\Gamma := -k_1\beta_1 - ... - k_l\beta_l$ is of the form $\beta \geq s_{\gamma_1}.\beta \geq ... \geq s_{\gamma_n}...s_{\gamma_k}.\beta = \Gamma$ for some positive roots $\gamma_1, ..., \gamma_n$, as preimages of a basis of $Rad(\Delta(\beta))$. In particular, all simple roots appearing in the decomposition of $-\Gamma$ belong to $\{\alpha_{i_1}, ..., \alpha_{i_k}\}$.

Now if $u \in U(\mathcal{G})$ is such that $uX_{-\beta_1}^{k_1}...X_{-\beta_l}^{k_l}v_\beta$ belongs to the fat β-weight space of $\Delta^n(\beta)$ then, by the PBW Theorem, we have

$$uX_{-\beta_1}^{k_1}...X_{-\beta_l}^{k_l}v_\beta \in \mathbb{C}[H_{\alpha_{i_1}}, ..., H_{\alpha_{ik}}]v_\beta.$$

This shows that $(H-\beta(H))v_\beta \in top(Rad(\Delta^{(n)}(\beta))$ for all $H \in \{H_{j_1}, ..., H_{j_{k(w)}}\}$ thus proving 1) since the elements $H_{j_1}, ..., H_{j_{k(w)}}$ are linearly independent.

To prove 2), suppose γ is a positive root satisfying $H_\gamma \in H(w)$. Suppose, further, that γ is of the form

$$\gamma = n_1\alpha_1 + n_2\alpha_2,$$

where the α_i's are simple roots and the n_i's are non negative integers. Then two situations may occur: If one of the n_i's is zero, say n_1, then $X_{-\alpha_2}^{i_{\beta,\alpha_2}} v_\beta$ generates a submodule of $\Delta^{(n)}(\beta)$ with simple top $L(s_{\alpha_2}.\beta)$. By Corollary 4.2.5 1), we conclude that

$$(H_2 - \beta(H_2))v_\beta \notin top(Rad(\Delta^{(n)}(\beta)).$$

If both n_1 and n_2 are non zero then we may assume that $H_1 \in H(w)$ and $H_2 \notin H(w)$. In other words, we may assume that $(\beta + \rho)(H_1) > 0$ and $(\beta + \rho)(H_2) \leq 0$. Then we argue as above to conclude that

$$(H_1 - \beta(H_1))v_\beta \notin top(Rad(\Delta^{(n)}(\beta))$$

and, since Lemma 4.1.3 tells us that

$$s_{\alpha_2}.s_{\alpha_1}.\beta \leq s_{\alpha_1}.\beta \leq \beta,$$

we use Corollary 4.2.5 2) to conclude that

$$(H_2 - \beta(H_2))v_\beta \notin top(Rad(\Delta^{(n)}(\beta))),$$

thus finishing the proof.

□

4.4 Extensions of non isomorphic simples

Now suppose that α and β are two different weights yielding non isomorphic simple modules $L(\alpha)$ and $L(\beta)$ in $\mathcal{O}$. Then we have

Theorem 4.4.1 *There is an isomorphism of vector spaces*

$$Ext^1_{\mathcal{O}^{(n)}}(L(\alpha), L(\beta)) \cong Ext^1_{\mathcal{O}}(L(\alpha), L(\beta)).$$

Proof

Suppose there is a short exact sequence

$$0 \longrightarrow L(\beta) \longrightarrow M \longrightarrow L(\alpha) \longrightarrow 0$$

with $M \in \mathcal{O}^{(n)}$.

The result is obvious if M is a trivial extension and is also obvious if $\alpha \not\leq \beta$ or $\beta \not\leq \alpha$ in which case $L(\alpha)$ and $L(\beta)$ have different central characters and thus cannot extend non trivially. Now if M is a non trivial extension and $\beta < \alpha$ then M is cyclic generated by any non zero element v such that $v \notin L(\beta)$. In particular, M is generated by any non zero element belonging to the one dimensional generalized highest α-weight space. Hence, $\mathcal{H}$ acts diagonally on the generators of M and consequently on the whole M. Finally, by using the duality i of $\mathcal{O}^{(n)}$ (see Chapter 2, Theorem 2.4.17) we observe that

$$Ext^1_{\mathcal{O}^{(n)}}(L(\alpha), L(\beta)) \cong Ext^1_{\mathcal{O}^{(n)}}(L(\beta), L(\alpha)),$$

thus proving the result.

□

Remark Since

$$dimExt^1_{\mathcal{O}}(L(\alpha), L(\beta))$$

is given by the Kazhdan-Lusztig conjectures (a theorem for category $\mathcal{O}$, see [12]) , the above result enables us to construct the quiver for all blocks of $\mathcal{O}^{(n)}$ for Lie algebras of rank one or two. □

4.5 Quivers of $\mathcal{O}^{(n)}$

Using the results of the previous section, one can display the quivers of *all* blocks of $\mathcal{O}^{(n)}$ for semisimple Lie algebras of rank 1 or 2. Here are some examples:

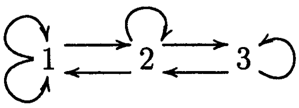
Type A_1, singular block

$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

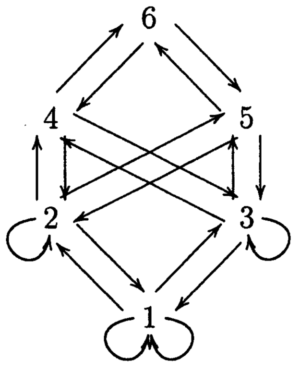
Type A_1, principal block

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r, shift left] & 2 \arrow[l, shift left] \end{tikzcd}$$

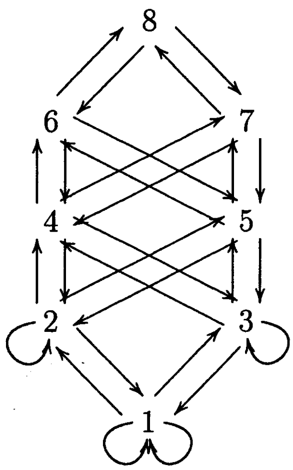
Type A_2, singular block



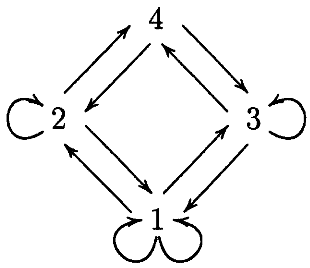
Type A_2, principal block



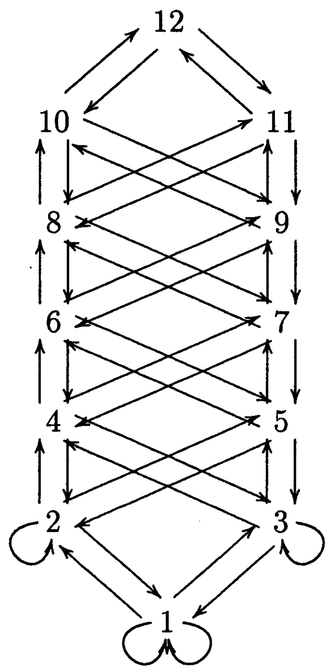
Type B_2, principal block



Type B_2, singular block



Type G_2, principal block



Remark The quiver corresponding to the singular blocks of type G_2 coincides with the quiver corresponding to the principal block of type A_2. Similarly, the quiver corresponding to the singular blocks of type B_2 coincides with the quiver corresponding to the principal block of type $A_1 \times A_1$. $\square$

4.6 Quiver AND relations for $\mathcal{O}^{(n)}$

4.6.1 Painting a stolen car

Our next aim is to describe combinatorially the fat blocks of $\mathcal{O}^{(n)}$. For that we will need three theorems (and a corollary) due to Soergel. Actually, *all* the results in this section were borrowed from [33] and [31].

Firstly, let us see how the endomorphism ring of the self dual indecomposable projectives of $\mathcal{O}^{(n)}$ looks like.

Let λ be an integral antidominant weight and let w_0^λ denote the longest coset element in W/W_λ. Then

Theorem 4.6.1 *(Soergel, [33], Theorem 9)*

With previous notation,

$$End_{\mathcal{O}^{(n)}} P^{(n)}(\lambda) = U(\mathcal{H})^{W_\lambda} \otimes_{U(\mathcal{H})^W} U(\mathcal{H})/I_{n,\lambda}.$$

If $n = 1$, the above ring is the algebra of coinvariants associated with the vector space $\mathcal{H}$. For arbitrary n we will simply call it the "fat algebra of coinvariants" and denote it by $C_\lambda^{(n)}$ (or simply $C^{(n)}$ if it is clear we are dealing with regular weights).

It is clear that Theorem 4.6.1 allows us to construct the functor

$$\mathbb{V} := \mathbb{V}_\lambda := Hom_{\mathcal{O}^{(n)}}(P^{(n)}(\lambda), -) : \mathcal{O}_\lambda^{(n)} \longrightarrow U(\mathcal{H})^{W_\lambda} - mod - U(\mathcal{H})/I_{n,\lambda},$$

with the remarkable property that

Theorem 4.6.2 *(Soergel, [33], Theorem 11)*

The functor $\mathbb{V}$ *is fully faithful on projectives.*

In other words, there is an isomorphism of vector spaces

$$Hom_{\mathcal{G}}(P, Q) \cong Hom_{C_\lambda^{(n)}}(\mathbb{V}(P), \mathbb{V}(Q))$$

for all projective modules P and Q in $\mathcal{O}_\lambda^{(n)}$.

Remark On the level of finite dimensional algebras, the functor $\mathbb{V}$ could be (re)defined as

$$\mathbb{V} : A_{\lambda,n} \longrightarrow C_\lambda^{(n)} - mod$$

$$M \mapsto e_\lambda M$$

where $A_{\lambda,n}e_\lambda$ is the 'big projective'.

Observe that by setting $P = Q = A_{\lambda,n}$, the definition of $\mathbb{V}$ together with the above theorem imply the validity of the double centralizer property which has been shown in Proposition 3.9.4.

□

If one wants to compute examples it is therefore of great importance to obtain a description of the right hand side of the above isomorphism.

Let us see how that can be achieved. To save energies start by assuming that λ is regular and for each simple reflection s consider the translation functor through the s-wall, θ_s. Now for $x \in W$, $x = s_r...s_2s_1$ being a reduced expression and λ dominant, we have

$$(*)\ \theta_{s_1}...\theta_{s_r}(\Delta^{(n)}(\lambda)) \doteq P^{(n)}(x.\lambda) \oplus (\oplus \text{projectives of the form } P^{(n)}(y.\lambda)),$$

with $y < x$ (for the Bruhat order on W).

Finally, the above identity stresses the need for a somewhat simpler description of the composition functor $\mathbb{V} \circ \theta_s$ which is given by the next result:

Theorem 4.6.3 *(Soergel, [33], Lemma 10)*

There exists a natural isomorphism of functors from $\mathcal{O}_\lambda^{(n)}$ to $C^{(n)} - mod$,

$$\mathbb{V} \circ \theta_s \cong C^{(n)} \otimes_{(C^{(n)})^s} \mathbb{V},$$

where $(C^{(n)})^s$ denotes the invariants of $C^{(n)}$ under the action of the reflection s.

The dream corollary (modùlo the sentence "a summand of ") is

Corollary 4.6.4 *(Soergel)*

If $x = s_r...s_2s_1$ is a reduced expression then the module $\mathbb{V}(P^{(n)}(\lambda))$ is a summand of

$$C^{(n)} \otimes_{(C^{(n)})^{s_1}} C^{(n)} \otimes_{(C^{(n)})^{s_2}} ... \otimes_{(C^{(n)})^{s_r}} U(\mathcal{H})/I_{n,\lambda}.$$

Remark Before giving examples, let us stress that this method works (up to minimal modifications on the multiplicities in formula (*)) in the singular situation as well. See [31] for more details.

□

4.6.2 Example: Regular blocks of type A_1

Basically, we will try to describe all $\mathbb{V}(P)'s$ for all projective indecomposable modules P and all $C^{(n)}$-homomorphisms between them. In the $sl_2(\mathbb{C})$ situation, if we identify the coroot $\alpha^\vee$ with the indeterminate X, we have, by Theorem 4.6.1, that

$$C^{(n)} = \mathbb{C}[X] \otimes_{\mathbb{C}[X^2]} \mathbb{C}[X]/(X^n),$$

which is isomorphic to the polynomial algebra $\mathbb{C}[X,Y]/(X^2 - Y^2, Y^n)$ and its radical (and socle) layers can be depicted as follows:

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
X \arrow[d] \arrow[drr] & & Y \arrow[d] \arrow[dll] \\
XY \arrow[d] \arrow[drr] & & Y^2 \arrow[d] \arrow[dll] \\
\vdots & & \vdots \\
XY^{n-2} \arrow[dr] & & Y^{n-1} \arrow[dl] \\
 & XY^{n-1} &
\end{tikzcd}$$

where the arrows represent the action (by multiplication) of the generators X and Y.

Following Soergel's theory, we can now identify

$$\mathbb{V}(\Delta^{(n)}(0)) \cong < X + Y >$$

as $C^{(n)}$-modules. As $\mathbb{C}$-basis one may pick $(X+Y, XY+Y^2, ..., XY^{n-2}+Y^{n-1}, XY^{n-1})$.

If n=2 then $C^{(2)}$ is represented by

$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& XY &
\end{tikzcd}$$

while $M := \mathbb{V}(\Delta^{(2)}(0)) \cong < X+Y, XY >$. It is easy to find a $\mathbb{C}$-basis of homomorphisms between these two modules which is listed below:

	$C^{(2)}$	M
$C^{(2)}$	id_{C^2} $\mathrm{a}_{1,1} : 1 \mapsto X$ $\mathrm{b}_{1,1} : 1 \mapsto Y$ $\mathrm{c}_{1,1} : 1 \mapsto XY$	$\mathrm{a}_{1,2} : 1 \mapsto X+Y$ $\mathrm{b}_{1,2} : 1 \mapsto XY$
M	$\mathrm{a}_{2,1} : X+Y \mapsto X+Y$ $\mathrm{b}_{2,1} : X+Y \mapsto XY$	id_M $\mathrm{a}_{2,2} : X+Y \mapsto XY$

If we denote $x := a_{1,2}$, $y := a_{2,1}$ and $z := a_{2,1}a_{1,2} - 2a_{1,1}$, the $\mathbb{C}$-algebra $A_{0,2}$ is Morita equivalent to the factor algebra $\mathbb{C}Q/J_{0,2}$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \end{tikzcd}$$

and $J_{0,2}$ is the ideal of relations $J_{0,2} = \langle zy, xz, (yx - z)^2, (xy)^2 \rangle$.

Observe that $A_{0,2}$ is graded by giving grade 1 to x and y, and grade 2 to z. The grading filtration of $A_{0,2}$ looks like

$$\begin{array}{cccc}
 & 1 & & \\
 & & & 2 \\
 & 2 & & \\
 & & & 1 \\
1 & & 1 \oplus & \\
 & & & 2 \\
 & 2 & & \\
 & & & 1 \\
 & 1 & &
\end{array}$$

while the radical filtration of $A_{0,2}$ may be depicted in the following way:

$$\begin{array}{cccc}
 & 1 & & \\
 & & & 2 \\
1 & & 2 & \\
 & & & 1 \\
 & 1 & \oplus & \\
 & & & 2 \\
 & 2 & & \\
 & & & 1 \\
 & 1 & &
\end{array}$$

In the same spirit, some slightly harder computations would show us that for $n > 1$ the algebra $A_{0,n}$ is Morita equivalent to the factor algebra $\mathbb{C}Q/J_{0,n}$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "y"] \end{tikzcd}$$

and $J_{0,n}$ is the ideal of relations $J_{0,n} = \langle zy, xz, (yx - z)^n, (xy)^n \rangle$.

4.6.3 Example: Singular blocks of type A_2

Start by identifying X and Y, respectively, with the coroots $\alpha^\vee$ and $\beta^\vee$ in such a way that $s_\alpha(Y) = X + Y = s_\beta(X)$. Furthermore, consider a singular weight λ (different from $-\rho$) sitting on the s_α-wall.

Borrowing from [34], we know that for $sl_3(\mathbb{C})$ the fat algebra of coinvariants is given by

$$C^{(n)} = \mathbb{C}[X, Y] \otimes_{\mathbb{C}[X^2+XY+Y^2, 2X^3+3X^2Y-3XY^2-2Y^3]} \mathbb{C}[X, Y]/(X^n, Y^n).$$

Now, according to [34], the elements

$$1 \otimes 1, X \otimes 1, Y \otimes 1, X^2 \otimes 1, Y^2 \otimes 1, X^3 \otimes 1$$

form a basis of $C^{(1)}$ and, denoting $X + 2Y$ by a, the elements

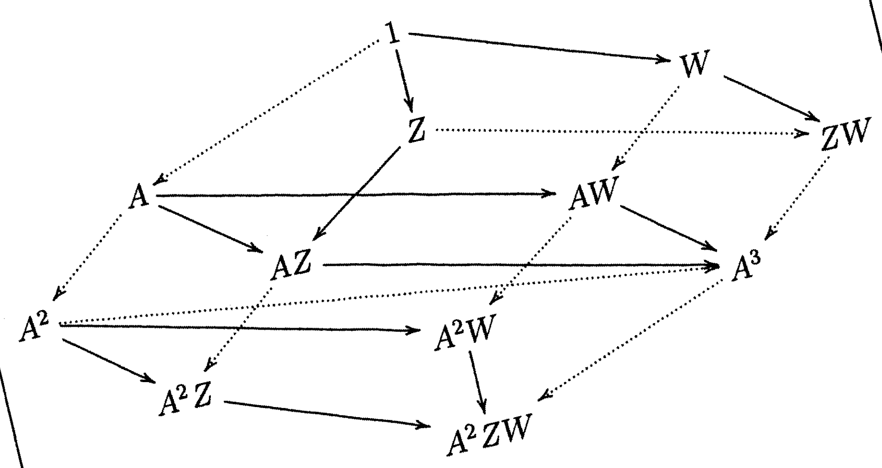
$$1, a \otimes 1, a^2 \otimes 1$$

form a basis of $C^{(1)}_\lambda$. From here we can easily build a basis for $C^{(n)}$ and $C^{(n)}_\lambda$ and we observe that the elements $A := a \otimes 1$, $Z := 1$ and $W := 1 \otimes Y$ generate $C^{(n)}_\lambda$ as an algebra.

Again following Corollary 4.6.4, we are able to describe the indecomposable $C^{(n)}_\lambda$-modules $\mathbb{V}(P^{(n)}(\lambda))$.

Their structure is described in the tables below. We will abbreviate the expression $a \otimes_{(C^{(n)})^{s_\alpha}} b \otimes_{(C^{(n)})^{s_\beta}} \otimes c$ simply by (a, b, c) and on the bottom of the table one finds the (non obvious) action of the generator A. The action of A may also be seen in the pictures as represented by a dotted arrow.

$C_\lambda^{(2)}$



$A^3 = 3AZW$

$\mathbb{V}(\Delta^{(2)}(s_\beta.\lambda))$

$$\begin{tikzcd}
 & (1,1,1) \arrow[dl] \arrow[d] \arrow[dr, dotted] & \\
(1,1,Z) \arrow[d] \arrow[dr, dotted] & (1,1,W) \arrow[dl] \arrow[dr, dotted] & (1,X,1) \arrow[dll] \arrow[dl] \arrow[d] \\
(1,1,ZW) \arrow[dr, dotted] & (1,X,Z) \arrow[d] & (1,X,W) \arrow[dl] \\
 & (1,X,ZW) &
\end{tikzcd}$$

A.(1,1,1)=4(1,1,Z)+2(1,1,W)-3(1,X,1)

$A^2.(1,1,1) = 7(1,1,ZW) - 6(1,X,Z) - 3(1,X,W)$

A.(1,X,1)=3(1,1,ZW)-2(1,X,Z)-(1,X,W)

$\mathbb{V}(\Delta^{(2)}(\lambda))$

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
Z \arrow[dr] & & W \arrow[dl] \\
 & ZW &
\end{tikzcd}$$

A.1=2W+Z

A.Z=2ZW

A.W=ZW

After some long and tedious calculations, we are in a position to exhibit a set of independent generators of the homomorphism spaces between the $\mathbb{V}(P)$'s described above. They are listed in the table below. Let us stress that our calculations were made much easier by the knowledge of the cardinality of the above mentioned generating set, which was given by Theorem 4.3.1.

	$C_\lambda^{(2)}$	$\mathbb{V}(\Delta^{(n)}(s_\beta.\lambda))$	$\mathbb{V}(\Delta^{(n)}(\lambda))$
$C_\lambda^{(2)}$	$\mathrm{id}_{C_\lambda^{(2)}}$ $\mathrm{c}_{1,1} : 1 \mapsto Z$ $\mathrm{d}_{1,1} : 1 \mapsto W$	$\mathrm{x}_{1,2} : 1 \mapsto (1,1,1)$	
$\mathbb{V}(\Delta^{(n)}(s_\beta.\lambda))$	$y_{2,1} : (1,1,1) \mapsto$ $a + 2Z + W$	$\mathrm{id}_{\mathbb{V}(\Delta^{(n)}(s_\beta.\lambda))}$ $\mathrm{b}_{2,2} : (1,1,1) \mapsto (1,1,W)$	$\mathrm{x}_{2,3} : (1,1,1) \mapsto 1$
$\mathbb{V}(\Delta^{(n)}(\lambda))$		$y_{3,2} : 1 \mapsto (1,1,Z)+$ $+(1,1,W) - (1,X,1)$	$\mathrm{id}_{\mathbb{V}(\Delta^{(n)}(\lambda))}$ $\mathrm{a}_{3,3} : 1 \mapsto Z$

After determining the relations between these generators, one may produce the basic algebra $A_{(\lambda,2)}$ corresponding to the singular block of $\mathcal{O}^{(2)}$ of $sl_3(\mathbb{C})$ indexed by the antidominant weight λ:

Proposition 4.6.5 *The algebra $A_{\lambda,2}$ is Morita equivalent to the factor algebra $\mathbb{C}Q/J_{(\lambda,2)}$ where Q is the quiver*

$$\begin{tikzcd} 1 \arrow[loop above, "c"] \arrow[loop below, "d"] \arrow[r, shift left, "X"] & 2 \arrow[l, shift left, "Y"] \arrow[loop above, "b"] \arrow[r, shift left, "x"] & 3 \arrow[l, shift left, "y"] \arrow[loop right, "a"] \end{tikzcd}$$

and $J_{(\lambda,2)}$ *is the ideal generated by the relations*

$$\langle c^2, d^2, b^2, a^2,$$

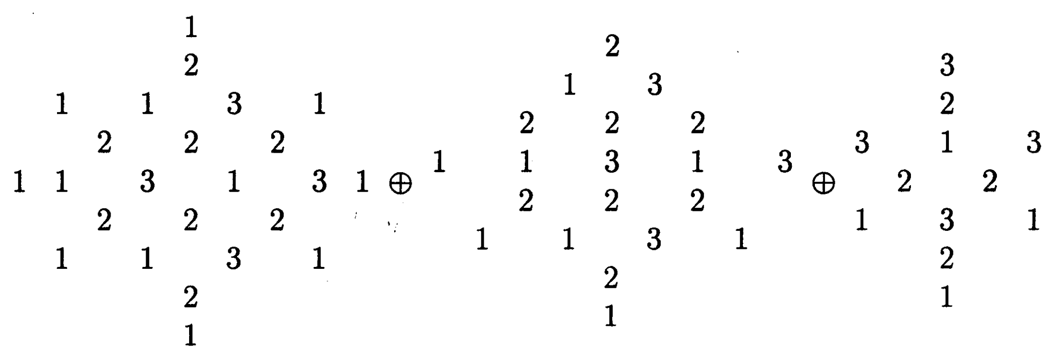
$$cd - dc, bXXd, Yb - dY,$$

$$YXc - cYX, XcY - yex, xye - exy, xby, xXc - exX, Yye - cYy,$$

$$XYX - (yxX + Xc), YXY - (Yyx + cY), xyx - xb, yxy - by,$$

$$xXY - (xb + ex), XYy - (ye + by)\rangle$$

This algebra is graded by giving grade 1 to $\{x, y, X, Y\}$ and grade 2 to $\{a, b, c, d\}$. The grading filtration of $A_{\lambda,2}$ looks like



With similar calculations one could still show that

Proposition 4.6.6 *The algebra* $A_{\lambda,3}$ *is Morita equivalent to the factor algebra* $\mathbb{C}Q/J_{(\lambda,3)}$ *where* Q *is the quiver*

$$
\begin{tikzcd}
1 \arrow[loop above, "c"] \arrow[loop below, "d"] \arrow[r, shift left, "X"] & 2 \arrow[l, shift left, "Y"] \arrow[loop above, "b"] \arrow[r, shift left, "x"] & 3 \arrow[l, shift left, "y"] \arrow[loop right, "a"]
\end{tikzcd}
$$

and

$J_{(\lambda,3)}$ *is the ideal generated by the relations*

$$\langle c^3, d^3, b^3, a^3,$$

$$cd - dc, bXXd, Yb - dY,$$

$$YXc - cYX, Xc^2Y - ye^2x, xye - exy, xb^2y, xXc - exX, Yye - cYy,$$

$$XYX - (yxX + Xc), YXY - (Yyx + cY), xyx - xb, yxy - by,$$

$$xXY - (xb + ex), XYy - (ye + by)\rangle$$

Again one could draw the grading (and radical, and socle) layers of this algebra but, at this point, let us just leave it as the first exercise.

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