ON INDECOMPOSABLE MODULES OVER CLUSTER-TILTED ALGEBRAS OF TYPE A

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

by

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September 2007

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Abstract

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Gabriel's Theorem describes the dimension vectors of the finitely generated indecomposable modules over the path algebra of a simply-laced Dynkin quiver. It shows that they can be obtained from the expressions for the positive roots of the corresponding root system in terms of the simple roots. Here, we present a method for finding the dimension vectors of the finitely generated indecomposable modules over a cluster-tilted algebra of Dynkin type A.

It is known that the quiver of a cluster-tilted algebra of Dynkin type A is given by an exchange matrix of the corresponding cluster algebra. We define a companion basis for such a quiver to be a $\mathbb{Z}$-basis of roots of the integral root lattice of the corresponding root system whose associated matrix of inner products is a positive quasi-Cartan companion of the corresponding exchange matrix.

Our main result establishes that the dimension vectors of the finitely generated indecomposable modules over a cluster-tilted algebra of Dynkin type A arise from expressions for the positive roots of the corresponding root system in terms of a companion basis (for the quiver of that algebra). This can be regarded as a generalisation of part of Gabriel's Theorem in the Dynkin type A case. The proof uses the fact that the quivers of the cluster-tilted algebras of Dynkin type A have a particularly nice description in terms of triangulations of regular polygons.

In addition, we give an explicit combinatorial procedure for constructing a companion basis for the quiver of any cluster-tilted algebra of Dynkin type A.

Acknowledgements

I would like to start by thanking my supervisor Robert Marsh for all of his guidance, support, encouragement and enthusiasm throughout the past few years. Our meetings and discussions were always a source of both great enjoyment and valuable insight. Thank you also to my examiners Peter Jørgensen and Nicole Snashall for their helpful comments and interesting suggestions.

I am extremely grateful to the Engineering and Physical Sciences Research Council for funding my research.

The working environment in Leicester was amazing. For this, I primarily wish to thank all of the wonderful people with whom I had the pleasure of sharing the room G1. A big thank you also to all of the other members of the Departments of Mathematics and Computer Science who readily offered help and advice when required.

Finally, thank you very much to my family and friends for all of their support and encouragement, and moreover, for some fun and very welcome distractions.

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] in an attempt to better understand the dual canonical basis of the quantised enveloping algebra of a finite dimensional semisimple Lie algebra. In [MRZ], a link between cluster algebras and representations of quivers was established. This subsequently led to the introduction of cluster categories in [BMRRT], which were intended to give a categorical model of cluster algebras. (Note that independently of [BMRRT], a geometric definition of cluster categories of Dynkin type A was given in [CCS1].) A key development in the study of cluster categories was the creation of a generalised version of APR-tilting theory (refer to [APR]), called cluster-tilting theory. In this theory, a key role is played by the cluster-tilted algebras, as introduced in [BMR1]. These cluster-tilted algebras are the main object of study of this thesis.

An important goal of representation theory is to understand the finitely generated modules over an algebra. For this, it is in fact enough to understand the finitely generated indecomposable modules. One possible way of starting to understand the finitely generated indecomposable modules is to describe their dimension vectors.

Consider the path algebra of a simply-laced Dynkin quiver. In this case, Gabriel's Theorem [Gab] describes the dimension vectors of the finitely generated indecomposable modules. Gabriel's Theorem states that there is a one-to-one correspondence between the finitely generated indecomposable modules and the positive roots of the corresponding root system of simply-laced Dynkin type. Moreover, it says that the dimension vector of an indecomposable module is the vector whose components are the coefficients appearing in the expression for the corresponding positive root as an integral linear combination of simple roots.

The aim of this thesis is to work towards giving a description of the dimension vectors of the finitely generated indecomposable modules over a cluster-tilted algebra of simply-laced Dynkin type. The cluster-tilted algebras of simply-laced Dynkin type are in fact closely related to the path algebras of simply-laced Dynkin quivers, as they arise from the cluster categories associated to these algebras. Given this link, it might be expected that the dimension vectors of the finitely generated indecomposable modules over a cluster-tilted algebra of simply-laced Dynkin type can also be described in terms of the positive roots of the corresponding root system. The main result of this thesis establishes that this is indeed the case for cluster-tilted algebras of Dynkin type A, providing a generalisation of part of Gabriel's Theorem in this case.

In [BMR2], an important link between cluster-tilted algebras and cluster algebras was established. Each cluster algebra is associated with an equivalence class of skew-symmetrizable integer matrices. These matrices are called the exchange matrices of that cluster algebra. In particular, the exchange matrices of a cluster algebra of simply-laced Dynkin type are skew-symmetric, and can therefore be represented as quivers. It was shown in [BMR2] (and independently in [CCS2]) that the quivers of the cluster-tilted algebras of a given simply-laced Dynkin type are precisely the quivers of the exchange matrices of the cluster algebra of that type.

In addressing the problem of recognising the cluster algebras of finite type, the paper [BGZ] considered a class of matrices closely related to the Cartan matrices, called the positive quasi-Cartan matrices. Given a cluster-tilted algebra of simply-laced Dynkin type, it follows from the main result of [BGZ] that the exchange matrix associated to its quiver must have a positive quasi-Cartan companion. (That is, there exists some positive quasi-Cartan matrix for which the absolute values of the off-diagonal entries match the absolute values of the off-diagonal entries of the exchange matrix.) Moreover, it follows from the classification of the positive quasi-Cartan matrices, that this positive quasi-Cartan companion arises as the matrix of inner products associated to some $\mathbb{Z}$-basis of roots of the integral root lattice of the corresponding root system of simply-laced Dynkin type.

From above, given a cluster-tilted algebra of Dynkin type A, there is an exchange matrix of the corresponding cluster algebra associated to its quiver. Also, there is a $\mathbb{Z}$-basis of roots of the integral root lattice of the corresponding root system whose associated matrix of inner products is a positive quasi-Cartan companion of this exchange matrix. The main result of this thesis shows that the dimension vectors of the finitely generated indecomposable modules over the given cluster-tilted algebra can be obtained from the coefficients of the expressions for the positive roots in terms of any such $\mathbb{Z}$-basis.

There now follows an outline of this thesis.

In Chapter 1, the required background material on cluster algebras is presented. This includes the definition of a cluster algebra, an outline of the classification of the cluster algebras of finite type (as given in [FZ2]), and the main results of [BGZ] on recognising cluster algebras of finite type. The latter of these uses the concept of positive quasi-Cartan matrices. Also, a simple but motivational corollary of a result classifying the positive quasi-Cartan matrices is established.

Following [BMRRT], a brief introduction to cluster categories of simply-laced Dynkin type is given in Chapter 2. In particular, this includes a description of some aspects of the relationship between such a cluster category and the corresponding cluster algebra. Also, cluster-tilted algebras are defined, and it is noted that the quiver of any cluster-tilted algebra of simply-laced Dynkin type is given by an exchange matrix of the corresponding cluster algebra.

Chapter 3 focusses on the study of those $\mathbb{Z}$-bases of the integral root lattice of a root system of simply-laced Dynkin type, consisting of roots, whose associated matrix of inner products is a positive quasi-Cartan companion of an exchange matrix of the corresponding cluster algebra. These sets are termed companion bases and are associated with the quivers of the cluster-tilted algebras of simply-laced Dynkin type. Given a cluster-tilted algebra of simply-laced Dynkin type, a method for finding a companion basis for its quiver is outlined. Also, a complete description of all of the companion bases for that quiver in terms of an arbitrary initial such companion basis is given.

The main result of this thesis, which gives a description of the dimension vectors of the finitely generated indecomposable modules over a cluster-tilted algebra of Dynkin type A, is established in Chapter 4. The proof relies on the structure of the quivers of the cluster-tilted algebras of Dynkin type A. These are examined by making use of a known description of these quivers in terms of triangulations of regular polygons. It is conjectured that the main result generalises to the Dynkin type D and E cases, and some possible strategies for proving this conjecture are briefly discussed.

In Chapter 5, a method is presented for explicitly constructing a companion basis for the quiver of any given cluster-tilted algebra of Dynkin type A. The key to this is the introduction of a procedure for labelling the vertices of such quivers. By studying these explicitly constructed companion bases, a more explicit alternative proof of the main result is then presented.

Finally, in Chapter 6, detailed consideration is given to one of the results of Chapter 3. This leads to an interesting consequence of the main result being established.

Chapter 1

Cluster Algebras

Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] in 2001. A cluster algebra is a subring of the field of rational functions in n indeterminates, generated by cluster variables. These cluster variables are obtained via a process of mutation, starting from some "initial seed".

Cluster algebras having only finitely many cluster variables are called cluster algebras of finite type. One of the early key results in the development of the theory of cluster algebras was the classification of the cluster algebras of finite type, given in [FZ2].

We start this chapter by giving the definition of a cluster algebra and the classification of the cluster algebras of finite type. After briefly introducing root systems, we then describe some aspects of the relationship between a cluster algebra of finite type and its corresponding root system. We also present the main results of Barot, Geiss and Zelevinsky [BGZ] on recognising cluster algebras of finite type. In these results, an important role is played by the so-called positive quasi-Cartan matrices. In the concluding part of this chapter, we establish a simple consequence of the classification of the positive quasi-Cartan matrices which is important in motivating our subsequent work.

1.1 What is a Cluster Algebra?

In this section, following [FZ2], we give the definition of a cluster algebra and outline the classification of the cluster algebras of finite type.

Firstly, we need a preliminary definition.

DEFINITION 1.1.1 *A square integer matrix B is said to be skew-symmetrizable (resp. symmetrizable) if there is some diagonal matrix D with positive integer diagonal entries such that DB is skew-symmetric (resp. symmetric).*

We can now give the definition of a cluster algebra.

Let $n \in \mathbb{N}$ and let $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n)$ be the field of rational functions in n indeterminates. Let $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq \mathbb{F}$ be a transcendence basis for $\mathbb{F}$ over $\mathbb{Q}$, and let $B = (b_{xy})_{x,y \in \mathbf{x}}$ be an $n \times n$ skew-symmetrizable integer matrix with rows and columns indexed by the entries of $\mathbf{x}$.

We call the pair $(\mathbf{x}, B)$ a seed in $\mathbb{F}$, and we obtain more seeds from this "initial seed" via a mutation process.

Let $z \in \mathbf{x}$. We obtain a new transcendence basis $\mathbf{x}' = (\mathbf{x} \setminus \{z\}) \cup \{z'\}$ for $\mathbb{F}$ over $\mathbb{Q}$, where z' is obtained using the exchange relation

$$zz' = \prod_{x \in \mathbf{x},\, b_{xz} > 0} x^{b_{xz}} + \prod_{x \in \mathbf{x},\, b_{xz} < 0} x^{-b_{xz}}.$$

Similarly, we obtain a skew-symmetrizable matrix (refer to [FZ1, Proposition 4.5]) B' from B with entries given by

$$b'_{xy} = \begin{cases} -b_{xy} & \text{if } x = z \text{ or } y = z, \\ b_{xy} + \frac{|b_{xz}|b_{zy} + b_{xz}|b_{zy}|}{2} & \text{otherwise.} \end{cases}$$

The row and column labelled z in B are relabelled z' in B'. The pair $(\mathbf{x}', B')$ then form a seed which we call the mutation of $(\mathbf{x}, B)$ in the direction z. It can easily be checked that by mutation of $(\mathbf{x}', B')$ in the direction z', we recover the seed $(\mathbf{x}, B)$.

By iterated mutations of the initial seed $(\mathbf{x}, B)$ in all directions, we obtain a set of seeds $\mathcal{S}$. The transcendence bases appearing in these seeds are called clusters, and their elements are called cluster variables. The matrices appearing in these seeds are called exchange matrices. The set of all cluster variables is the union of all of the transcendence bases appearing in seeds in $\mathcal{S}$, and is denoted by χ.

We then define the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ to be the $\mathbb{Z}$-subalgebra (subring) of $\mathbb{F}$ generated by χ.

Note: In the general definition of cluster algebras given in [FZ1] and [FZ2], certain coefficients appear in the exchange relations. In the definition we have given above, all of these coefficients have been set equal to one. This is the case of greatest interest to us, since this is the case that has been studied most extensively in connection with representation theory.

Two cluster algebras $\mathcal{A}'$ and $\mathcal{A}''$, contained in fields of rational functions $\mathbb{F}'$ and $\mathbb{F}''$ respectively, are said to be isomorphic (as cluster algebras) if there is a $\mathbb{Z}$-algebra isomorphism $\iota\colon \mathbb{F}' \to \mathbb{F}''$ taking some seed $(\mathbf{y}, C)$ of $\mathcal{A}'$ to a seed $(\iota(\mathbf{y}), C)$ of $\mathcal{A}''$. (We note that the terminology "strongly isomorphic" is used in [FZ2].)

Up to isomorphism of cluster algebras, $\mathcal{A}(\mathbf{x}, B)$ does not depend on the choice of transcendence basis $\mathbf{x}$ for $\mathbb{F}$, and so we can denote this cluster algebra by $\mathcal{A}(B)$. In fact, we can go further than this. It is clear that the mutation of skew-symmetrizable matrices outlined above gives rise to an equivalence relation on the set of $n \times n$ skew-symmetrizable integer matrices. Up to isomorphism of cluster algebras, the cluster algebra $\mathcal{A}(B)$ only depends on the mutation equivalence class of B.

We have the following definition from [FZ2].

DEFINITION 1.1.2 *A cluster algebra is said to be of finite type if it has only finitely many cluster variables.*

The cluster algebras of finite type were classified in [FZ2]. Before stating the classification result, we first need to introduce some terminology.

DEFINITION 1.1.3 *An $n \times n$ integer matrix $A = (a_{ij})$ is called a generalised Cartan matrix if*

(i) $a_{ii} = 2$ for all $1 \leq i \leq n$,

(ii) $a_{ij} \leq 0$ for all $i \neq j$,

(iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ for all $i \neq j$.

If in addition to these properties, all principal minors of A are positive, then we call A a Cartan matrix of finite type.

The Cartan-Killing classification of the Cartan matrices of finite type is a well known result, see [Kac, Chapter 4] for example. Indeed, the Cartan matrices of finite type can be encoded by Dynkin diagrams. If $A = (a_{ij})$ is a Cartan matrix of finite type, then the associated Dynkin diagram has vertices indexed by the rows and columns of A. Distinct vertices i and j are joined by $a_{ij}a_{ji}$ edges, and these edges are equipped with an arrow pointing from i towards j if $|a_{ij}| < |a_{ji}|$. The Dynkin diagrams of the Cartan matrices of finite type are listed in [Kac, Section 4.8].

DEFINITION 1.1.4 *Let $B = (b_{ij})$ be an $n \times n$ integer matrix. Then the Cartan counterpart of B is defined to be the matrix $A(B) = (a_{ij})$ given by $a_{ii} = 2$ for all $1 \leq i \leq n$, and $a_{ij} = -|b_{ij}|$ otherwise.*

Note: It is clear that the Cartan counterpart of a skew-symmetrizable matrix must be a symmetrizable matrix.

We can now give the classification result of the cluster algebras of finite type. The result was originally given in [FZ2], but we give the version as stated in [BGZ].

THEOREM 1.1.5 *Let $\mathcal{F}$ be a mutation equivalence class of skew-symmetrizable matrices. Then the following are equivalent:*
(i) The cluster algebra associated to $\mathcal{F}$ is of finite type.
(ii) There is some matrix $B \in \mathcal{F}$ such that the Cartan counterpart of B is a Cartan matrix of finite type.
(iii) For every matrix $B = (b_{ij}) \in \mathcal{F}$, $|b_{ij}b_{ji}| \leq 3$ for all $i \neq j$.

Furthermore, the Cartan-Killing type of the Cartan matrix in (ii) is uniquely determined by the equivalence class $\mathcal{F}$ (and is referred to as being the type of the cluster algebra associated to $\mathcal{F}$).

Since the Cartan matrices of finite type can be represented by Dynkin diagrams, we can consider this result as giving as giving a classification of the cluster algebras of finite type by Dynkin diagrams.

If $B = (b_{ij})$ is an $n \times n$ skew-symmetric integer matrix, then we can associate to B a quiver $\Gamma(B)$ with vertices corresponding to the rows and columns of B, and b_{ij} arrows from the vertex i to the vertex j whenever $b_{ij} > 0$.

It is easy to show that mutation in any direction of a skew-symmetric matrix results in a skew-symmetric matrix, and hence the mutation equivalence class of B must consist entirely of skew-symmetric matrices. Therefore, we can associate a quiver to each matrix belonging to the mutation equivalence class of B.

In view of Theorem 1.1.5, we see that the cluster algebras of Dynkin types A, D and E are those associated with equivalence classes of skew-symmetric matrices which contain a matrix whose associated quiver is a Dynkin quiver of the same type.

REMARK 1.1.6 *Suppose $C = (c_{ij})$ is a matrix appearing in a seed of a cluster algebra of (simply-laced) Dynkin type A, D or E. Then, we have that C is a skew-symmetric integer matrix, and also, from Theorem 1.1.5, we see that C must satisfy $|c_{ij}c_{ji}| \leq 3$ for all $i \neq j$. It follows that the entries of C must all belong to the set $\{0, \pm 1\}$.*

1.2 Root Systems

We saw in the previous section that the cluster algebras of finite type are associated with the Cartan matrices of finite type. Also associated with the Cartan matrices of finite type are root systems. This suggests that there may be some kind of relationship between the cluster algebras of finite type and the corresponding root systems. We will start to consider this relationship in the next section, after first giving a brief introduction to root systems in this section. (For standard terminology and results regarding root systems, refer to [Hum1], for example.)

Also in this section, we introduce the compatibility degree (of Fomin and Zelevinsky, see [FZ3]) which assigns a non-negative integer to each given pair of "almost positive" roots. This is a useful tool for describing the clusters of a cluster algebra of finite type.

Let V be a Euclidean space. That is, let V be a finite dimensional real vector space together with a positive definite symmetric bilinear form $(\ ,\ )$.

Given any non-zero vector $\alpha \in V$, we can define an orthogonal linear transformation $s_\alpha : V \to V$ (called a reflection of V) by setting

$$s_\alpha \beta = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

for all $\beta \in V$. It is clear that s_α takes α to $-\alpha$, and fixes pointwise the orthogonal complement to α. Notice also that $s_\alpha = s_{c\alpha}$ for all non-zero $c \in \mathbb{R}$.

We now introduce the definition of a root system in V.

DEFINITION 1.2.1 *We call a subset $\Phi \subseteq V$ a root system in V if it satisfies:*

(i) Φ is finite, spans V and does not contain 0.

(ii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

(iii) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

(iv) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Let Φ be a root system in V.

The Weyl group W_Φ of Φ is defined to be the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Phi$. (Recall that $GL(V)$ consists of all invertible linear transformations of V.)

We call $\dim V$ the rank of the root system Φ.

DEFINITION 1.2.2 *A subset $\Pi \subseteq \Phi$ is called a simple system of Φ if:*

(i) Π is a vector space basis of V over $\mathbb{R}$.

(ii) Each $\alpha \in \Phi$ can be written (uniquely) as an integral linear combination of Π with either all coefficients being non-negative, or all coefficients being non-positive.

If $\Pi \subseteq \Phi$ is a simple system of Φ, we call the elements of Π simple roots.

Note from [Hum1, p.48-49] that every root system has a simple system.

Let Π be a simple system of Φ and suppose $\beta \in \Phi$. So, we can write $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ with $c_\alpha \in \mathbb{Z}$ for all $\alpha \in \Pi$. If $c_\alpha \geq 0$ for all $\alpha \in \Pi$, then we call β a positive root, and if $c_\alpha \leq 0$ for all $\alpha \in \Pi$, then we call β a negative root. We denote the set of positive roots in Φ (relative to Π) by Φ^+, and we denote the set of negative roots in Φ by Φ^-. It is clear that $\Phi^- = -\Phi^+$.

Also, we define the set of almost positive roots (relative to Π) to be $\Phi_{\geq -1} = \Phi^+ \cup (-\Pi)$.

For each $\alpha \in \Phi$, set $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Then, $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ is also a root system in V (called the dual root system), and $\Pi^\vee = \{\alpha^\vee : \alpha \in \Pi\}$ is a simple system of $\Phi^\vee$.

We now define what it means for two root systems to be isomorphic.

DEFINITION 1.2.3 *Let Φ and Φ' be root systems in the Euclidean spaces V and V' respectively. We say that Φ is isomorphic to Φ' if there is a vector space isomorphism $\phi: V \to V'$ sending Φ onto Φ' such that $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2(\phi(\beta),\phi(\alpha))}{(\phi(\alpha),\phi(\alpha))}$ for all $\alpha, \beta \in \Phi$.*

Now, every root system has an associated Cartan matrix which determines that root system up to isomorphism, and the classification of the root systems turns out to be identical to the classification of the Cartan matrices of finite type (see [Hum1, Chapter 11]). In particular, every root system has a well defined Dynkin type.

We will later be mostly focussing on root systems of simply-laced Dynkin type. We always choose these in such a way that the squared length of each root is 2. In particular, this implies that each root is its own dual. Also, we then have the following well known and useful result.

LEMMA 1.2.4 *Suppose that the root system $\Phi \subseteq V$ is of simply-laced Dynkin type. If $\alpha, \beta \in \Phi$ are non-proportional roots, then $(\alpha, \beta) \in \{0, \pm 1\}$.*

We conclude this section by introducing the compatibility degree of Fomin and Zelevinsky (see [FZ3]).

Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix of finite type. Recall that the Coxeter graph of A has vertices corresponding to the rows and columns of A, and an edge joining two distinct vertices i and j whenever $a_{ij} \neq 0$ (or equivalently, whenever $a_{ji} \neq 0$). (Note that we omit the edge labels as we have no need for them here.) Denote the set of vertices of the Coxeter graph of A by I.

Suppose further that A is indecomposable and let Φ be the (irreducible) root system associated to A. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ and let $\Phi_{\geq -1}$ be the corresponding set of almost positive roots. The compatibility degree, which we define below, assigns a non-negative integer to each pair of almost positive roots.

Since A is an indecomposable Cartan matrix of finite type, the Coxeter graph of A is a tree (this follows from the classification of the Cartan matrices of finite type by Dynkin diagrams [Kac, Chapter 4]). Therefore, we can write I as a disjoint union

$I = I_+ \sqcup I_-$ where $I_+, I_- \subseteq I$, and the full subgraphs of the Coxeter graph of A on each of I_+ and I_- are totally disconnected.

For each $1 \leq i \leq n$, let $s_i = s_{\alpha_i}$ and let τ_+ and τ_- be the permutations of $\Phi_{\geq -1}$ defined by

$$\tau_+(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, i \in I_-, \\ \left(\prod_{i \in I_+} s_i\right)(\alpha) & \text{otherwise,} \end{cases}$$

and

$$\tau_-(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, i \in I_+, \\ \left(\prod_{i \in I_-} s_i\right)(\alpha) & \text{otherwise.} \end{cases}$$

For each $1 \leq i \leq n$ and each almost positive root α, using $[\alpha\colon \alpha_i]$ to denote the coefficient of α_i in the expansion of α in terms of the simple roots, we define

$$(-\alpha_i || \alpha) = \max\{[\alpha\colon \alpha_i], 0\}.$$

The definition of $(\ ||\ )$ is then extended to all pairs of almost positive roots by specifying that it is τ_+- and τ_--invariant. We note that due to [FZ2, Theorem 3.1], we can obtain a negative simple root from any given almost positive root by iteratively applying τ_+ and τ_-.

We say that two almost positive roots α and β are compatible if $(\alpha||\beta) = 0$.

Note: From [FZ3, Proposition 3.3] we have that if the Dynkin diagram corresponding to the Cartan matrix A is simply-laced, then the compatibility degree function is symmetric.

1.3 Correspondence Between Cluster Variables and Almost Positive Roots

The main purpose of this section is to start to consider the relationship between the cluster algebras of finite type and their associated root systems. In particular, we will state an important result from [FZ2] which says that there is a one-to-one correspondence between the set of cluster variables of a cluster algebra of finite type, and the set of almost positive roots of the associated root system.

Let B be an $n \times n$ skew-symmetrizable matrix, and suppose that $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ is a cluster algebra of finite type. Also, suppose that $\mathcal{F}$ is the mutation equivalence class of B.

Then, by Theorem 1.1.5, there is some skew-symmetrizable matrix $B' \in \mathcal{F}$ such that the Cartan counterpart $A = A(B')$ is a Cartan matrix of finite type.

Now, from [FZ2, Theorem 1.6] we have that there must be some skew-symmetrizable matrix $B_0 = (b_{ij}) \in \mathcal{F}$ such that $A(B_0) = A$ and $b_{ij}b_{ik} \geq 0$ for all $1 \leq i, j, k \leq n$.

Since B_0 is mutation equivalent to B, there must be some seed which contains B_0. Let $(\mathbf{x}_0, B_0)$ be such a seed where $\mathbf{x}_0 = \{x_1, \ldots, x_n\}$ is a transcendence basis for $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n)$ over $\mathbb{Q}$. We then have that $\mathcal{A} = \mathcal{A}(\mathbf{x}_0, B_0) = \mathcal{A}(\mathbf{x}, B)$.

Let Φ be the root system associated to A and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ. We then have the following result from [FZ2, Theorem 1.9].

THEOREM 1.3.1 *There is a unique bijection $\alpha \mapsto x[\alpha]$ between the almost positive roots in Φ and the cluster variables in $\mathcal{A}$, such that for any $\alpha \in \Phi_{\geq -1}$, the cluster variable $x[\alpha]$ is expressed in terms of $\mathbf{x}_0 = \{x_1, \ldots, x_n\}$ as*

$$x[\alpha] = \frac{P_\alpha(\mathbf{x}_0)}{\prod_{i=1}^{n} x_i^{a_i}}$$

where P_α is an integer polynomial with non-zero constant term, and $\alpha = \sum_{i=1}^{n} a_i \alpha_i$. Under this bijection, $x[-\alpha_i] = x_i$ for each i, $1 \le i \le n$.

We end this section by noting from [FZ2] that maximal pairwise compatible subsets of $\Phi_{\geq -1}$ correspond to clusters in $\mathcal{A}$.

1.4 Recognising Cluster Algebras of Finite Type

Given a skew-symmetrizable matrix B, the classification theorem for the cluster algebras of finite type provides two conditions for checking whether or not the cluster algebra $\mathcal{A} = \mathcal{A}(B)$ is of finite type. The paper [BGZ] highlights that both of these conditions can be difficult to check in general. The focus of that paper is to solve this problem by giving a method for determining whether or not the cluster algebra $\mathcal{A}(B)$ is of finite type, based solely on consideration of the matrix B itself.

In this section, we will state the main result of [BGZ], and also two further results from [BGZ] which highlight the usefulness of the main result.

Note: An alternative method for recognising cluster algebras of finite type was given in [Sev].

In order to state the main result of [BGZ], we need to introduce some terminology.

DEFINITION 1.4.1 *A symmetrizable matrix $A = (a_{ij})$ is said to be quasi-Cartan if $a_{ii} = 2$ for all i.*

DEFINITION 1.4.2 *A quasi-Cartan matrix A is said to be positive if the symmetrized matrix DA is positive definite.*

So, we see that a quasi-Cartan matrix A is positive if and only if the principal minors of A are all positive.

The following definition provides a quasi-Cartan analogue of Cartan counterparts.

DEFINITION 1.4.3 *Let B be a skew-symmetrizable matrix. A quasi-Cartan companion of B is a quasi-Cartan matrix A such that $|a_{ij}| = |b_{ij}|$ for all $i \neq j$.*

It is clear that opposite entries have opposite signs in skew-symmetrizable matrices. (Note also that opposite entries have the same sign in symmetrizable matrices.) Therefore, given any skew-symmetrizable matrix $B = (b_{ij})$, we can associate a quiver $\widetilde{\Gamma}(B)$ to B as follows. The vertices of $\widetilde{\Gamma}(B)$ correspond to the rows and columns of B, and there is an arrow from the vertex i to the vertex j whenever $b_{ij} > 0$.

DEFINITION 1.4.4 *We define a chordless cycle in $\widetilde{\Gamma}(B)$ to be a (not necessarily oriented) cycle in $\widetilde{\Gamma}(B)$ such that the full subquiver on its vertices is also a cycle in $\widetilde{\Gamma}(B)$.*

We can now state the main result from [BGZ] on recognising cluster algebras of finite type ([BGZ, Theorem 1.2]).

THEOREM 1.4.5 *Let B be a skew-symmetrizable matrix. Then, the cluster algebra associated to (the mutation equivalence class of) B is of finite type if and only if*

(i) every chordless cycle in $\widetilde{\Gamma}(B)$ is cyclically oriented, and

(ii) B has a positive quasi-Cartan companion.

It is clear that condition (i) is easy to check for a given skew-symmetrizable matrix B, however, condition (ii) could be harder to check as B could have many quasi-Cartan companions. In fact, if B has N non-zero above diagonal entries, then there are 2^N different quasi-Cartan companions. The results of the following two propositions ([BGZ, Proposition 1.4] and [BGZ, Proposition 1.5] respectively) demonstrate the power of the above theorem. Indeed, it turns out that the positivity of only one (carefully chosen) quasi-Cartan companion of B has to be checked.

PROPOSITION 1.4.6 *Let B be skew-symmetrizable, and let $A = (a_{ij})$ be a quasi-Cartan companion of B. For A to be positive, it must satisfy:*

$$\textit{For all chordless cycles } Z \textit{ in } \widetilde{\Gamma}(B), \prod_{\{i,j\}\in Z} (-a_{ij}) < 0. \qquad (\blacktriangle)$$

Now, when choosing a quasi-Cartan companion A for B, for each $b_{ij} \neq 0$ with $i \neq j$, we must either choose $a_{ij}, a_{ji} > 0$ or $a_{ij}, a_{ji} < 0$. This can be considered as making a sign choice for each arrow in $\widetilde{\Gamma}(B)$. By Proposition 1.4.6, in order to have any chance of getting a positive quasi-Cartan companion, these signs must be chosen such that each chordless cycle has an odd number of arrows assigned positive sign.

The following proposition tells us that if all chordless cycles in $\widetilde{\Gamma}(B)$ are cyclically oriented, then such a choice of quasi-Cartan companion exists. Furthermore, only the positivity of this companion needs to be checked to determine whether or not $\mathcal{A}(B)$ is of finite type. (This is because performing simultaneous sign changes in the rows and columns of a matrix does not affect the positivity of that matrix.)

PROPOSITION 1.4.7 *Let B be a skew-symmetrizable matrix. If every chordless cycle in $\widetilde{\Gamma}(B)$ is cyclically oriented, then B has a quasi-Cartan companion satisfying $(\blacktriangle)$, unique up to simultaneous sign changes in rows and columns.*

1.5 A Motivational Result

In this section we prove a corollary of a result appearing in [BGZ]. The result of this corollary will turn out to be a key motivating factor for our later research.

We start by introducing a standard notion of equivalence for quasi-Cartan matrices.

DEFINITION 1.5.1 *Let A and A' be two quasi-Cartan matrices. If there is some diagonal matrix D with positive integer diagonal entries such that $C = DA$ and $C' = DA'$ are symmetric, and there is some integer matrix E with determinant ± 1 such that $C' = E^T CE$, then we say that A and A' are equivalent.*

It is clear that if A is a positive quasi-Cartan matrix and A' is a quasi-Cartan matrix equivalent to A, then A' is also a positive quasi-Cartan matrix. A result classifying the equivalence classes of positive quasi-Cartan matrices by Cartan-Killing types is given in [BGZ, Proposition 2.9]. We now state this result.

Let $A = (a_{ij})$ be an $n \times n$ quasi-Cartan matrix. For each i, $1 \leq i \leq n$, define an automorphism s_i of the lattice $\mathbb{Z}^n$ by setting $s_i(e_j) = e_j - a_{ij}e_i$ where $\{e_1, \dots, e_n\}$ is the standard basis in $\mathbb{Z}^n$. Let $W(A) \subseteq GL_n(\mathbb{Z})$ be the group generated by $s_1, \dots, s_n$.

PROPOSITION 1.5.2 *The following conditions on a quasi-Cartan matrix A are equivalent:*

(i) A is positive.

(ii) The group $W(A)$ is finite.

(iii) There is a root system Φ and a linearly independent subset $\{\beta_1, \dots, \beta_n\} \subseteq \Phi$ such that $a_{ij} = (\beta_i^\vee, \beta_j)$ for all $1 \leq i, j \leq n$.

(iv) A is equivalent to a Cartan matrix A^0 of finite type.

Under these conditons, if $\Phi_0 \subseteq \Phi$ is the smallest root subsystem of Φ that contains the set $\{\beta_1, \ldots, \beta_n\}$ in (iii), then the Cartan-Killing type of Φ_0 is the same as the Cartan-Killing type of the matrix A^0 in (iv), and it characterises A up to equivalence. Furthermore, $W(A)$ is naturally identified with the Weyl group of Φ_0.

We now give further consideration to the positive quasi-Cartan companions of those skew-symmetrizable matrices which give rise to cluster algebras of finite type.

Let B be a skew-symmetrizable matrix and suppose that the cluster algebra $\mathcal{A}(B)$ is of finite type. Then, by Theorem 1.4.5, all chordless cycles in $\widetilde{\Gamma}(B)$ are cyclically oriented and B has a positive quasi-Cartan companion A. By Proposition 1.4.7, we see that all positive quasi-Cartan companions for B can be obtained from A by performing simultaneous sign changes in rows and columns. So, it is straightforward to see that all positive quasi-Cartan companions for B are equivalent.

If B' is mutation equivalent to B, then $\mathcal{A}(B')$ is also of finite type. (In fact, $\mathcal{A}(B)$ and $\mathcal{A}(B')$ are isomorphic cluster algebras.) So, all chordless cycles in $\widetilde{\Gamma}(B')$ must be cyclically oriented and B' must have a positive quasi-Cartan companion. Then, applying [BGZ, Corollary 3.3], we see that there is a positive quasi-Cartan companion of B' which is equivalent to A, and hence all positive quasi-Cartan companions of B' are equivalent to A.

In particular, by Theorem 1.1.5, there must be some skew-symmetrizable matrix B_0, mutation equivalent to B, with Cartan counterpart $A_0 = A(B_0)$ a Cartan matrix of finite type. It is clear that A_0 is a positive quasi-Cartan companion of B_0. Therefore, we see that A_0 is equivalent to A.

Now, let Φ be a root system of the same Cartan-Killing type as A_0 (which is the Cartan-Killing type of $\mathcal{A}(B)$) in some Euclidean space V with positive definite symmetric bilinear form $(\, , \,)$.

We have the following corollary of Proposition 1.5.2.

COROLLARY 1.5.3 *Let $A = (a_{ij})$ be a positive quasi-Cartan companion of B. Then, there is a subset $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ which is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$ such that $a_{ij} = (\beta_i^\vee, \beta_j)$ for all $1 \leq i, j \leq n$.*

Proof By Proposition 1.5.2, we have that there is a root system Φ' (in some Euclidean space V' with positive definite symmetric bilinear form $(\ ,\ )'$) and a linearly independent subset $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi'$ such that $a_{ij} = (\beta_i^\vee, \beta_j)'$ for all $1 \leq i, j \leq n$. Furthermore, if $\Phi_0' \subseteq \Phi'$ is the smallest root subsystem of Φ' that contains $\{\beta_1, \ldots, \beta_n\}$, then the Cartan-Killing type of Φ_0' is the same as the Cartan-Killing type of A_0, and hence Φ_0' and Φ are isomorphic root systems.

It remains to be seen that $\{\beta_1, \ldots, \beta_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi_0'$. The argument is standard, but we include it for the convenience of the reader.

Let $W_{\Phi'}$ be the Weyl group of Φ', and suppose that W is the subgroup of $W_{\Phi'}$ generated by $s_{\beta_1}, \ldots, s_{\beta_n}$. Let $\tilde{\Phi} = W\{\beta_1, \ldots, \beta_n\} \subseteq \Phi'$.

We will show that $\tilde{\Phi} = \Phi_0'$ and that $\{\beta_1, \ldots, \beta_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\tilde{\Phi}$. In order to show the former, it is sufficient to check that $\tilde{\Phi}$ is a root system in $\mathrm{span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_n\}$. (Because, if $\tilde{\Phi} \subseteq \Phi'$ is a root system, then it's clearly the smallest root subsystem of Φ' containing $\{\beta_1, \ldots, \beta_n\}$.)

Firstly, we check that $\tilde{\Phi}$ is a root system in $\mathrm{span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_n\}$.

(i) Since $\tilde{\Phi} \subseteq \Phi'$, we see that $\tilde{\Phi}$ is finite and does not contain 0. Also, it is clear that $\tilde{\Phi}$ spans $\mathrm{span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_n\}$.

(ii) Let $\alpha, \beta \in \tilde{\Phi}$. We must show that $s_\alpha(\beta) \in \tilde{\Phi}$.

Write $\alpha = w\beta_i$ and $\beta = v\beta_j$ for some $w, v \in W$ and $1 \leq i, j \leq n$. Then, using [Hum1, Lemma 9.2], we have $s_\alpha(\beta) = s_{w\beta_i}(v\beta_j) = ws_{\beta_i}w^{-1}(v\beta_j) \in \tilde{\Phi}$. (Note that this implies $s_\alpha\tilde{\Phi} = \tilde{\Phi}$ since s_α acts injectively on $\tilde{\Phi}$.)

(iii) Suppose $\alpha \in \tilde{\Phi}$. Then, $\alpha = w\beta_i$ for some $w \in W$ and $1 \leq i \leq n$. So, $-\alpha = ws_{\beta_i}(\beta_i) \in \tilde{\Phi}$. Combining this with the fact that $\tilde{\Phi} \subseteq \Phi'$, we see that $\mathbb{R}\alpha \cap \tilde{\Phi} = \{\alpha, -\alpha\}$ for all $\alpha \in \tilde{\Phi}$.

(iv) Let $\alpha, \beta \in \tilde{\Phi}$. Then, $\frac{2(\beta,\alpha)'}{(\alpha,\alpha)'} \in \mathbb{Z}$ since $\alpha, \beta \in \Phi'$.

Therefore, $\tilde{\Phi}$ is a root system in $\mathrm{span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_n\}$.

We will now check that $\{\beta_1, \ldots, \beta_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\tilde{\Phi}$.

We already have that $\{\beta_1, \ldots, \beta_n\}$ is a linearly independent set. So, to show that $\{\beta_1, \ldots, \beta_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\tilde{\Phi}$, we must show that every element of $\tilde{\Phi}$ is an integral linear combination of $\{\beta_1, \ldots, \beta_n\}$. (It then follows immediately that every element of $\mathbb{Z}\tilde{\Phi}$ is an integral linear combination of $\{\beta_1, \ldots, \beta_n\}$.)

Note that for any $1 \leq i, j \leq n$, we have $s_{\beta_i}(\beta_j) = \beta_j - \frac{2(\beta_j,\beta_i)'}{(\beta_i,\beta_i)'}\beta_i \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$, since $\frac{2(\beta_j,\beta_i)'}{(\beta_i,\beta_i)'} \in \mathbb{Z}$.

Now, elements of $\tilde{\Phi}$ are of the form $s_{\beta_{i_k}}s_{\beta_{i_{k-1}}} \cdots s_{\beta_{i_1}}(\beta_j)$ with $k \in \mathbb{N}$ and $1 \leq j, i_1, \ldots, i_k \leq n$. We will show that $s_{\beta_{i_k}} \cdots s_{\beta_{i_1}}(\beta_j) \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$ by induction on k.

In the initial case when $k = 1$, we have from above that $s_{\beta_1}(\beta_j) \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$.

Suppose that $s_{\beta_{i_{k-1}}} \cdots s_{\beta_{i_1}}(\beta_j) \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$. Then we can write

$$s_{\beta_{i_{k-1}}} \cdots s_{\beta_{i_1}}(\beta_j) = a_1\beta_1 + \ldots + a_n\beta_n$$

with $a_i \in \mathbb{Z}$ for all $1 \leq i \leq n$.

But then,

$$\begin{aligned} s_{\beta_{i_k}}(s_{\beta_{i_{k-1}}} \cdots s_{\beta_{i_1}}(\beta_j)) &= s_{\beta_{i_k}}(a_1\beta_1 + \ldots + a_n\beta_n) \\ &= a_1 s_{\beta_{i_k}}(\beta_1) + \ldots + a_n s_{\beta_{i_k}}(\beta_n). \end{aligned}$$

Therefore, since $s_{\beta_{i_k}}(\beta_1), \ldots, s_{\beta_{i_k}}(\beta_n) \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$, we see that $s_{\beta_{i_k}} \cdots s_{\beta_{i_1}}(\beta_j) \in \mathbb{Z}\{\beta_1, \ldots, \beta_n\}$.

Therefore, we have that $\tilde{\Phi}$ is a root system in $\mathrm{span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_n\}$ and that $\{\beta_1, \ldots, \beta_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\tilde{\Phi}$.

Thus $\tilde{\Phi} = \Phi_0'$, and the result follows since Φ_0' is isomorphic to Φ. ■

Chapter 2

Cluster Categories and Cluster-tilted Algebras

In this chapter, we present the background material that we require on cluster categories and cluster-tilted algebras. We start by giving the definition of a cluster category of simply-laced Dynkin type, and explaining that the (isomorphism classes of the) indecomposable objects in such a category are in one-to-one correspondence with the cluster variables of the corresponding cluster algebra. We also give the definition of cluster-tilting objects in cluster categories, and the definition of cluster-tilted algebras, which are associated to these. We explain that the one-to-one correspondence between the indecomposable objects in a cluster category of simply-laced Dynkin type and the cluster variables of the corresponding cluster algebra induces a one-to-one correspondence between basic cluster-tilting objects and clusters (and hence seeds). Finally, we discuss how for any given basic cluster-tilting object in a cluster category of simply-laced Dynkin type, the quiver of the associated cluster-tilted algebra is the quiver of the matrix appearing in the seed corresponding to that basic cluster-tilting object.

2.1 Cluster Categories of Simply-laced Dynkin Type

In this section, following [BMRRT], we give a brief introduction to cluster categories of simply-laced Dynkin type. We start by defining cluster categories of simply-laced Dynkin type and stating what the indecomposable objects are in these categories. Also, we explain that there is a one-to-one correspondence between the set of indecomposable objects in a cluster category of simply-laced Dynkin type and the set of almost positive roots of the associated root system. It then follows that there is a one-to-one correspondence between the set of indecomposable objects in such a cluster category and the set of cluster variables of the corresponding cluster algebra. Further links between cluster categories and the corresponding cluster algebras will be considered in subsequent sections of this chapter.

Note: Refer to [ARS] for general results in representation theory, and [Hap] for results on derived categories of finite dimensional algebras.

Let k be an algebraically closed field and let Q be a simply-laced quiver of Dynkin type with underlying graph Δ. Let $\mathcal{D} = \mathcal{D}^b(kQ\text{−mod})$ be the bounded derived category of the category of finitely generated left kQ-modules with shift functor $[1]$. Also, let τ be the AR-translation in $\mathcal{D}$, and define $F = \tau^{-1}[1]$. (Note that F is an autoequivalence of $\mathcal{D}$, since both $[1]$ and τ are autoequivalences of $\mathcal{D}$.)

The cluster category $\mathcal{C} = \mathcal{C}(kQ)$ is then defined to be the factor category

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ\text{−mod})}{F}$$

whose objects are the objects of $\mathcal{D}$, and where morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, F^i Y)$$

for objects X, Y in $\mathcal{C}$.

Note: It is easy to show that for any object X of $\mathcal{D}$, we have X is isomorphic to FX in $\mathcal{C}$.

From [BMRRT, Proposition 1.2], we have that the category $\mathcal{C}$ is a Krull-Schmidt category. Also, it is shown in [Kel] that $\mathcal{C}$ is a triangulated category, with the shift in $\mathcal{C}$ induced by the shift in $\mathcal{D}$. We will use [1] also to denote the shift functor in $\mathcal{C}$. Note that [1] is an autoequivalence of $\mathcal{C}$.

For each finitely generated indecomposable left kQ-module M, recall that the stalk complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

with M appearing in degree zero is an indecomposable object in $\mathcal{D}$. By identifying M with this stalk complex, we can consider M as an indecomposable object in $\mathcal{D}$. Let $\mathcal{Z}$ be the set consisting of these indecomposable objects in $\mathcal{D}$ and the indecomposable objects in $\mathcal{D}$ of the form $P[1]$ for P a finitely generated indecomposable projective left kQ-module.

It is noted in [BMRRT, Section 1] that $\mathcal{Z}$ contains exactly one representative from each F-orbit on the set of isomorphism classes of indecomposable objects in $\mathcal{D}$. Furthermore, considering the elements of $\mathcal{Z}$ as objects in $\mathcal{C}$, we have from [BMRRT, Proposition 1.6] that the elements of $\mathcal{Z}$ are, up to isomorphism, the indecomposable objects in $\mathcal{C}$.

Let Φ be the root system of Dynkin type Δ. Suppose that $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ is a simple system of Φ, and that Φ^+, $\Phi_{\geq -1}$ are respectively the corresponding sets of positive and almost positive roots.

It is clear that Gabriel's Theorem (see [Gab] and [BGP]) induces a one-to-one correspondence between $\mathcal{Z}$ and the set of almost positive roots $\Phi_{\geq -1}$: Let $X \in \mathcal{Z}$. If X is (corresponds to) an indecomposable kQ-module, then define $\gamma_Q(X)$ to be the positive root corresponding to X as given by Gabriel's Theorem. If X is of the form $P_i[1]$ where P_i is the indecomposable projective kQ-module corresponding to the vertex i of Q, then define $\gamma_Q(X)$ to be $-\alpha_i$.

It immediately follows that the map $\gamma_Q \colon \mathcal{Z} \to \Phi_{\geq -1}$ induces a one-to-one correspondence between the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ and $\Phi_{\geq -1}$.

Let $\mathcal{A}$ be the cluster algebra of Dynkin type Δ.

Recall that Theorem 1.3.1 gave a one-to-one correspondence between the set of cluster variables of $\mathcal{A}$ and $\Phi_{\geq -1}$. Combining this with the above correspondence provides us with a one-to-one correspondence between the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ and the set of cluster variables in $\mathcal{A}$.

2.2 Cluster-tilting Objects and Cluster-tilted Algebras

In this section, again following [BMRRT], we define cluster-tilting sets and cluster-tilting objects in cluster categories. Also, we introduce cluster-tilted algebras (as defined in [BMR1]).

Let k be an algebraically closed field, let Q be a simply-laced quiver of Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ\text{−mod})}{F}$$

be the corresponding cluster category.

Firstly, we need a preliminary definition.

DEFINITION 2.2.1 *For $\mathcal{E}$ equal to either $\mathcal{D}$ or $\mathcal{C}$, and for objects U, V in $\mathcal{E}$, we define* $\mathrm{Ext}^1_{\mathcal{E}}(U, V)$ *to be* $\mathrm{Hom}_{\mathcal{E}}(U, V[1])$.

It is clear that if X and Y are objects in $\mathcal{C}$, then

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^1_{\mathcal{D}}(X, F^i Y).$$

Also, we note from [BMRRT, Proposition 1.7] that $\mathrm{Ext}^1_{\mathcal{C}}$ is symmetric in $\mathcal{C}$. That is, for all $X, Y \in \mathcal{C}$ we have $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \cong D\mathrm{Ext}^1_{\mathcal{C}}(Y, X)$, where D is the duality $D = \mathrm{Hom}_k(\ , k)$.

We can now define cluster-tilting sets and cluster-tilting objects in $\mathcal{C}$.

DEFINITION 2.2.2 *A set $\mathcal{T}$ of non-isomorphic indecomposable objects in $\mathcal{C}$ is called a cluster-tilting set if* $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0$ *for all $X, Y \in \mathcal{T}$, and it is a maximal such set.*

An object T in $\mathcal{C}$ is called a cluster-tilting object if $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$ *and T has a maximal number of non-isomorphic indecomposable direct summands.*

A cluster-tilting object is said to be basic if all of its direct summands are non-isomorphic. From the above definition, it is therefore clear that an object in $\mathcal{C}$ is a basic cluster-tilting object if and only if it is the direct sum of all objects in some cluster-tilting set. Note (from [BMRRT, Theorem 3.3]) that all cluster-tilting sets in $\mathcal{C}$ are finite. In fact, the number of objects in any cluster-tilting set is equal to the number of simple kQ-modules (i.e. the number of vertices of Q).

DEFINITION 2.2.3 *Let T be a cluster-tilting object in $\mathcal{C}$. We call the algebra* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ *a cluster-tilted algebra (of Dynkin type Δ).*

2.3 Basic Cluster-tilting Objects Correspond to Clusters

The main focus of this section is a result from [BMRRT] which demonstrates another important link between cluster categories of simply-laced Dynkin type and cluster algebras.

Recall from Section 2.1, that the indecomposable objects in a cluster category of simply-laced Dynkin type are in one-to-one correspondence with the cluster variables of the associated cluster algebra. The result from [BMRRT] we state here shows that under this correspondence, the basic cluster-tilting objects in such a cluster category correspond to the clusters of the associated cluster algebra.

With the use of a result from [FZ2], we also make some further remarks on the link between cluster categories of simply-laced Dynkin type and cluster algebras.

Let k be an algebraically closed field, let Q be an alternating quiver of Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ\text{−mod})}{F}$$

be the corresponding cluster category. Let $\mathcal{A}$ be the cluster algebra of Dynkin type Δ. Suppose Φ is the root system of Dynkin type Δ, with simple system $\Pi = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ and corresponding set of almost positive roots $\Phi_{\geq -1}$.

Let I be the set of vertices of Δ. Since Δ is a tree, we can write I as a disjoint union $I = I_+ \sqcup I_-$, where each of the subsets $I_+, I_- \subseteq I$ is totally disconnected. We then have the compatibility degree $(\ ||\ )$ on pairs of almost positive roots, as defined in Section 1.2.

We saw in Section 2.1 that there is a one-to-one correspondence between the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ and the set of almost positive roots $\Phi_{\geq -1}$. If α is an almost positive root, denote the (isomorphism class of the) indecomposable object in $\mathcal{C}$ corresponding to α by M_α.

We have the following important result from [BMRRT, Corollary 4.3].

THEOREM 2.3.1 *Let* $\alpha, \beta \in \Phi_{\geq -1}$. *Then,* $(\alpha||\beta) = \dim \operatorname{Ext}^1_{\mathcal{C}}(M_\alpha, M_\beta)$.

Suppose T is a basic cluster-tilting object in $\mathcal{C}$. Then, each of the direct summands of T corresponds to an almost positive root, which in turn corresponds to a cluster variable in $\mathcal{A}$. Theorem 2.3.1 shows that the set of almost positive roots corresponding to the set of direct summands of T is a maximal compatible set. That T corresponds to a cluster of the cluster algebra $\mathcal{A}$ then follows from the fact that maximal compatible subsets of almost positive roots correspond to clusters (refer to [FZ2]).

Theorem 2.3.1 therefore establishes that there is a one-to-one correspondence between the set of basic cluster-tilting objects in $\mathcal{C}$ and the set of clusters of the cluster algebra $\mathcal{A}$ (see [BMRRT, Theorem 4.5]).

From [FZ2, Theorem 1.12], we have the following.

THEOREM 2.3.2 *Every seed* $(\mathbf{x}, B)$ *in* $\mathcal{A}$ *is uniquely determined by its cluster* $\mathbf{x}$. *For any cluster* $\mathbf{x}$ *and any* $x \in \mathbf{x}$, *there is a unique cluster* $\mathbf{x}'$ *with* $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} \setminus \{x\}$.

Note: Theorem 2.3.2 is known to hold for all cluster algebras of finite type.

As an immediate consequence of this result, we see that there is a one-to-one correspondence between the set of basic cluster-tilting objects in $\mathcal{C}$ and the set of seeds of $\mathcal{A}$.

DEFINITION 2.3.3 *If $\overline{T} \oplus X$ is a basic cluster-tilting object in $\mathcal{C}$ and X is an indecomposable object in $\mathcal{C}$, then we call $\overline{T}$ an almost complete basic cluster-tilting object in $\mathcal{C}$, and we call X a complement of $\overline{T}$.*

As a consequence of Theorem 2.3.2, we see that if $\overline{T}$ is an almost complete basic cluster-tilting object in $\mathcal{C}$, then there are (up to isomorphism) exactly two ways to complete $\overline{T}$ to a basic cluster-tilting object in $\mathcal{C}$. That is, up to isomorphism, there are exactly two complements M, M^* of $\overline{T}$ such that $M \ncong M^*$. (For a representation theoretic proof of this fact, refer to [BMRRT, Theorem 5.1].)

2.4 The Quivers of Cluster-tilted Algebras

We have now seen that for any basic cluster-tilting object T in a cluster category, there is a corresponding seed in the associated cluster algebra. We conclude this chapter by considering the main result of [BMR2] which shows that the quiver of the cluster-tilted algebra associated to T is the same as the quiver associated to the seed corresponding to T. (Note that this result was also independently obtained in [CCS2, Theorem 3.1].)

Let k be an algebraically closed field, let Q (with n vertices) be an alternating quiver of Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ\text{-mod})}{F}$$

be the corresponding cluster category.

Let T be a basic cluster-tilting object in $\mathcal{C}$, let Λ be the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, and suppose that Q_Λ is the quiver of Λ. From [BMR2, Proposition 3.2], we have that Q_Λ has no loops and no oriented cycles of length two. Therefore, we can associate an $n \times n$ skew-symmetric integer matrix $X_\Lambda = (x_{ij})$ to Q_Λ, with rows and columns indexed by the vertices of Q_Λ. If there is at least one arrow from i to j in Q_Λ, we set x_{ij} to be the number of arrows from i to j in Q_Λ. If there are no arrows between i and j, we set $x_{ij} = 0$. Otherwise, we set $x_{ij} = -x_{ji}$.

Clearly there is a one-to-one correspondence between the set of quivers with n vertices, no loops and no oriented cycles of length two, and the set of $n \times n$ skew-symmetric integer matrices with rows and columns indexed by the vertices of these quivers.

From [BMR2, Theorem 1.3] we have the following important result.

THEOREM 2.4.1 *Let $\overline{T}$ be an almost complete basic cluster-tilting object in $\mathcal{C}$ with complements M and M^*. Let Λ, Λ' be the cluster-tilted algebras $\Lambda = \mathrm{End}_{\mathcal{C}}(\overline{T} \oplus M)^{\mathrm{op}}, \Lambda' = \mathrm{End}_{\mathcal{C}}(\overline{T} \oplus M^*)^{\mathrm{op}}$ and suppose that their quivers are Q_Λ and $Q_{\Lambda'}$ respectively. Fix an ordering on the vertices of Q_Λ and suppose that the vertex of Q_Λ corresponding to M is l. Then, Q_Λ and $Q_{\Lambda'}$, or equivalently the matrices $X_\Lambda = (x_{ij})$ and $X_{\Lambda'} = (x'_{ij})$, are related by the formulas*

$$x'_{ij} = \begin{cases} -x_{ij} & \text{if } i = l \text{ or } j = l, \\ x_{ij} + \frac{|x_{il}|x_{lj} + x_{il}|x_{lj}|}{2} & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be the cluster algebra of Dynkin type Δ with initial seed $(\{x_1, \ldots, x_n\}, B_0)$ such that the quiver associated to B_0 is Q.

We then have the following corollary of Theorem 2.4.1.

COROLLARY 2.4.2 *Let T be a basic cluster-tilting object in $\mathcal{C}$ and suppose that the seed of the cluster algebra $\mathcal{A}$ corresponding to T is $(\mathbf{y}, B)$. If X_Λ is the matrix associated to the quiver Q_Λ of the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, then $X_\Lambda = B$ (identifying indecomposable objects in $\mathcal{C}$ with the corresponding cluster variables in $\mathcal{A}$). Hence, Q_Λ is the same as the quiver associated to the skew-symmetric matrix B.*

Proof From [BMR2, Section 6].

In view of Theorem 2.4.1, we just need to show that there is some basic cluster-tilting object T_0 in $\mathcal{C}$ such that the quiver of the cluster-tilted algebra $\Lambda_0 = \mathrm{End}_{\mathcal{C}}(T_0)^{\mathrm{op}}$ is the same as the quiver of the seed corresponding to T_0. (The result then follows by induction using Theorem 2.4.1.)

For each $i \in \{1, \ldots, n\}$, let P_i be the indecomposable projective kQ-module corresponding to the vertex i of Q.

Consider the cluster-tilted algebra $\Lambda_0 = \mathrm{End}_{\mathcal{C}}(T_0)^{\mathrm{op}}$ where T_0 is the cluster-tilting object $T_0 = P_1[1] \oplus \ldots \oplus P_n[1]$. Since $[1]$ is an autoequivalence of $\mathcal{C}$ and since $kQ = P_1 \oplus \ldots \oplus P_n$, we have that $\Lambda_0 \cong \mathrm{End}_{\mathcal{C}}(kQ)^{\mathrm{op}}$. Therefore, $\Lambda_0 \cong \mathrm{End}_{kQ}(kQ)^{\mathrm{op}} \cong kQ$, and hence the quiver of Λ_0 is Q.

Under the correspondence between indecomposable objects in $\mathcal{C}$ and cluster variables in $\mathcal{A}$, recall that for each i, the cluster variable corresponding to $P_i[1]$ is x_i. Therefore, the seed corresponding to T_0 is $(\{x_1, \ldots, x_n\}, B_0)$. This completes the proof since the quiver of B_0 is also Q. ∎

Let T be a basic cluster-tilting object in $\mathcal{C}$ and suppose that the seed of the cluster algebra $\mathcal{A}$ corresponding to T is $(\mathbf{y}, B)$. From Corollary 2.4.2, we have that the quiver associated to B is the same as the quiver of the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. But also, from Corollary 1.5.3, we have that if $A = (a_{ij})$ is a positive quasi-Cartan companion of B and Φ is the root system associated to $\mathcal{A}$, then there is a subset $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ which is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$ such that $a_{ij} = (\beta_i^\vee, \beta_j)$ for all $1 \leq i, j \leq n$.

This establishes a link between any given cluster-tilted algebra and the $\mathbb{Z}$-bases (consisting of roots) of the root lattice of the associated root system that give rise to positive quasi-Cartan companions of the exchange matrix from the corresponding seed of the associated cluster algebra.

Our research in subsequent chapters will focus on further examination of this link.

Chapter 3

Companion Bases

The main focus of this chapter is the study of a particular collection of $\mathbb{Z}$-bases of the integral root lattice of a root system of simply-laced Dynkin type. To be precise, we study those $\mathbb{Z}$-bases of roots whose associated matrix of inner products is a positive quasi-Cartan companion of an exchange matrix of the corresponding cluster algebra. The definition of these $\mathbb{Z}$-bases, which we call companion bases, is motivated by Corollary 1.5.3.

In view of Corollary 2.4.2, companion bases are associated with the quivers of the cluster-tilted algebras of simply-laced Dynkin type. Moreover, the elements of a companion basis for the quiver of a given cluster-tilted algebra of simply-laced Dynkin type are naturally indexed by the vertices of that quiver.

We start this chapter by giving the definition of a companion basis for the quiver of a given cluster-tilted algebra of simply-laced Dynkin type. Our initial aim is then to find a method which allows us, given a cluster-tilted algebra of simply-laced Dynkin type, to construct a companion basis for its quiver. The latter part of this chapter is devoted to examining the relationship between different companion bases for the same quiver, with the main result here giving a complete (theoretical) description of all of the companion bases for any given quiver.

Throughout this chapter, we keep the following set-up.

Let k be an algebraically closed field, let Q (with n vertices) be an alternating quiver of simply-laced Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ - \mathrm{mod})}{F}$$

be the corresponding cluster category.

Let Λ be the cluster-tilted algebra (of simply-laced Dynkin type) given by $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, where T is a basic cluster-tilting object in $\mathcal{C}$. Let $\mathcal{A}$ be the cluster algebra of Dynkin type Δ, and suppose that $(\mathbf{x}, B)$ is the seed in $\mathcal{A}$ corresponding to the basic cluster-tilting object T. We then have by Corollary 2.4.2 that $\Gamma = \Gamma(B)$ is the quiver of Λ. Write $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices of Γ, and Γ_1 is the set of arrows of Γ.

Let $\Phi \subseteq V$ be the root system of Dynkin type Δ where V is a Euclidean space with positive definite symmetric bilinear form $(\ ,\ )$, and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ.

3.1 Companion Bases and Companion Basis Mutation

We start by giving the definition of a companion basis for Γ. Then, after introducing a standard quiver mutation procedure, we obtain the main result of this section. This result establishes a companion basis mutation procedure that, given a companion basis for Γ, produces a companion basis for any mutation of Γ.

In the following section, we explain how iterated companion basis mutation gives a method for finding a companion basis for the quiver of any given cluster-tilted algebra of simply-laced Dynkin type.

The companion basis mutation procedure we introduce here also has some other interesting consequences, as we will see when we give it further consideration in Chapter 6.

DEFINITION 3.1.1 *We call a subset* $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ *a companion basis for* $\Gamma = \Gamma(B)$ *if it satisfies the following properties:*
(i) $\{\gamma_x : x \in \Gamma_0\}$ *is a* $\mathbb{Z}$*-basis for* $\mathbb{Z}\Phi$.
(ii) The matrix $A = (a_{xy})$ *given by* $a_{xy} = (\gamma_x, \gamma_y)$ *for all* $x, y \in \Gamma_0$ *is a positive quasi-Cartan companion of* B.

In this case, we will also often refer to $\{\gamma_x : x \in \Gamma_0\}$ *as a companion basis for* Γ *giving rise to the positive quasi-Cartan companion* A *of* B.

We note that the elements of any candidate companion basis for Γ are indexed by the vertices of Γ. But also, the vertices of Γ correspond to the rows and columns of the matrix B, and hence to the elements (cluster variables) of the cluster $\mathbf{x}$. So, we can consider the elements of any candidate companion basis for Γ as being indexed by the cluster variables of $\mathbf{x}$. We note that it therefore makes sense to ask whether or not the matrix of inner products defined by any given candidate companion basis for Γ is a positive quasi-Cartan companion of B.

We can slightly simplify Definition 3.1.1 as a consequence of the following simple result.

LEMMA 3.1.2 *If* $\{z_1, \ldots, z_n\} \subseteq V$ *is a basis for* V*, then the matrix* $C = (c_{ij})$ *given by* $c_{ij} = (z_i, z_j)$ *for all* $1 \leq i, j \leq n$ *is positive definite.*

Proof The result follows since the symmetric bilinear form $(\ , \ )$ is positive definite. ∎

We have the following corollary.

COROLLARY 3.1.3 *If $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Phi$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, then the matrix $A = (a_{ij})$ given by $a_{ij} = (\gamma_i, \gamma_j)$ for all $1 \leq i, j \leq n$ is a positive quasi-Cartan matrix.*

Proof It is clear that $\{\gamma_1, \ldots, \gamma_n\}$ must be a basis for V since $V = \mathrm{span}_{\mathbb{R}}\{\gamma_1, \ldots, \gamma_n\}$ and $\dim_{\mathbb{R}} V = n$. (Note that $\Phi \subseteq \mathrm{span}_{\mathbb{R}}\{\gamma_1, \ldots, \gamma_n\}$ and so $V = \mathrm{span}_{\mathbb{R}}\Phi \subseteq \mathrm{span}_{\mathbb{R}}\{\gamma_1, \ldots, \gamma_n\} \subseteq V$.) We can therefore apply Lemma 3.1.2 to deduce that A must be positive definite. Also, we see that A must be symmetric with $a_{ii} = 2$ for all $1 \leq i \leq n$. ∎

We can thus modify the definition of a companion basis for Γ by replacing condition (ii) above with:

(ii)$'$ The matrix $A = (a_{xy})$ given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$ is a companion of B. That is, $|a_{xy}| = |b_{xy}|$ for all $x \neq y$, $x, y \in \Gamma_0$.

Corollary 1.5.3 gives the existence of a companion basis for Γ. Moreover, it tells us that for any positive quasi-Cartan companion A' of B, there is a companion basis for Γ giving rise to A'. We will now start working towards obtaining a method for constructing a companion basis for the quiver of any given cluster-tilted algebra of simply-laced Dynkin type. In preparation for introducing our companion basis mutation procedure, we must first outline the process of quiver mutation.

Let k be a vertex of Γ. The vertex k corresponds to a row and column of B, and hence to a cluster variable $x_k \in \mathbf{x}$. Mutating the seed $(\mathbf{x}, B)$ in the direction x_k, we obtain a new seed $(\mathbf{x}', B')$. We then call $\Gamma' = \Gamma(B')$ the quiver obtained from Γ by mutating at the vertex k.

Suppose T' is the basic cluster-tilting object in $\mathcal{C}$ corresponding to the seed $(\mathbf{x}', B')$. Then, we have that Γ' is the quiver of the cluster-tilted algebra $\Lambda' = \mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$.

Note: It's a trivial observation that if we mutate Γ' at the vertex k, we recover Γ.

We now give an explicit description (refer to [FZ2, Section 8]) of the quiver Γ' obtained from Γ by mutating at the vertex k, relying solely on the quiver Γ itself.

Recall (from Remark 1.1.6) that all matrices appearing in seeds of cluster algebras of Dynkin types A, D and E are skew-symmetric integer matrices whose entries belong to the set $\{0, \pm 1\}$. (This means that between any pair of vertices in the quiver of a cluster-tilted algebra of simply-laced Dynkin type there is at most one arrow, and there are no loops on any vertex.) Making use of this information, the following can be obtained via the matrix mutation formula which appears in the definition of a cluster algebra:

Suppose Γ' is the quiver obtained from Γ by mutating at the vertex k. We may identify the vertices of Γ' with those of Γ. Then, the arrows appearing in Γ' are the same as those appearing in Γ, except in the following cases:

(i) All arrows incident with k in Γ are replaced in Γ' by the corresponding reversed arrows.

(ii) Suppose x and y are vertices such that there is an arrow from x to k in Γ and an arrow from k to y in Γ. Then,

(a) there is an arrow from x to y in Γ' if x and y are not joined by an arrow in Γ,

(b) x and y are not joined by an arrow in Γ' if there is an arrow from y to x in Γ.

The following result introduces the concept of companion basis mutation which gives us a way of obtaining companion bases for Γ' (and hence companion bases for any quiver that can be obtained from Γ by performing a sequence of quiver mutations) from companion bases for Γ. Apart from helping us with our initial aim, this concept will also play an important role in the proof of our main result of Chapter 6.

THEOREM 3.1.4 *Let $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ. Then,*

(i) the set $\{\gamma'_x : x \in \Gamma'_0\} \subseteq \Phi$ given by

$$\gamma'_x = \begin{cases} s_{\gamma_k}(\gamma_x) & \text{if there is an arrow from } x \text{ to } k \text{ in } \Gamma, \\ \gamma_x & \text{otherwise} \end{cases}$$

is a companion basis for Γ';

(ii) the set $\{\gamma''_x : x \in \Gamma'_0\} \subseteq \Phi$ given by

$$\gamma''_x = \begin{cases} s_{\gamma_k}(\gamma_x) & \text{if there is an arrow from } k \text{ to } x \text{ in } \Gamma, \\ \gamma_x & \text{otherwise} \end{cases}$$

is a companion basis for Γ'.

We refer to $\{\gamma'_x : x \in \Gamma'_0\}$ as the companion basis for Γ' obtained from $\{\gamma_x : x \in \Gamma_0\}$ by mutating inwardly at k, and we refer to $\{\gamma''_x : x \in \Gamma'_0\}$ as the companion basis for Γ' obtained from $\{\gamma_x : x \in \Gamma_0\}$ by mutating outwardly at k.

Proof We only need prove (i); the proof of (ii) is similar.

We have that $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, and the matrix $A = (a_{xy})$ given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$ is a positive quasi-Cartan companion of B. We must show that $\{\gamma'_x : x \in \Gamma'_0\} \subseteq \Phi$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, and the matrix $A' = (a'_{xy})$ given by $a'_{xy} = (\gamma'_x, \gamma'_y)$ for all $x, y \in \Gamma'_0$ is a companion of B'.

We start by checking that $\{\gamma'_x : x \in \Gamma'_0\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$.

Let $z \in \mathbb{Z}\Phi$. Since $\{\gamma_x : x \in \Gamma_0\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, we can write $z = \sum_{x \in \Gamma_0} a_x \gamma_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$.

Let x be a vertex of Γ. If there is an arrow from x to k in Γ, we have $\gamma'_x = s_{\gamma_k}(\gamma_x) = \gamma_x - (\gamma_x, \gamma_k)\gamma_k$. Otherwise, we have $\gamma'_x = \gamma_x$.

Therefore, we have

$$\begin{aligned}
z = \sum_{x \in \Gamma_0} a_x \gamma_x &= \sum_{x \not\to k} a_x \gamma_x + \sum_{x \to k} a_x \gamma_x \\
&= \sum_{x \not\to k} a_x \gamma'_x + \sum_{x \to k} a_x (\gamma'_x + (\gamma_x, \gamma_k)\gamma'_k) \\
&= \sum_{x \not\to k} a_x \gamma'_x + \sum_{x \to k} a_x \gamma'_x + \sum_{x \to k} a_x (\gamma_x, \gamma_k)\gamma'_k \\
&= \sum_{x \neq k} a_x \gamma'_x + \left(a_k + \sum_{x \to k} a_x (\gamma_x, \gamma_k) \right) \gamma'_k,
\end{aligned}$$

thus showing that we can write z as an integral linear combination of the roots γ'_x for $x \in \Gamma'_0$.

To establish the $\mathbb{Z}$-linear independence of the set $\{\gamma'_x : x \in \Gamma'_0\}$, we note that if $\sum_{x \in \Gamma'_0} c_x \gamma'_x = 0$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma'_0$, then $\sum_{x \neq k} c_x \gamma_x + (c_k - \sum_{x \to k} c_x (\gamma_x, \gamma_k)) \gamma_k = 0$. Therefore, we must have $c_x = 0$ for all $x \in \Gamma'_0$, due to the $\mathbb{Z}$-linear independence of the set $\{\gamma_x : x \in \Gamma_0\}$.

In order to check that A' is a companion of B', it is sufficient to check that $(\gamma_i', \gamma_j') = \pm 1$ whenever the vertices i and j are joined by an arrow in Γ', and $(\gamma_i', \gamma_j') = 0$ otherwise. (We note that all entries of B' belong to the set $\{0, \pm 1\}$, and moreover, that i and j are joined by an arrow in Γ' if and only if $b_{ij}' = \pm 1$.)

Let x be a vertex in Γ, $x \neq k$.

If x and k are not joined by an arrow in Γ, then they are not joined by an arrow in Γ'. Also, we have $\gamma_k' = \gamma_k$ and $\gamma_x' = \gamma_x$, and so $(\gamma_k', \gamma_x') = (\gamma_k, \gamma_x) = 0$.

If there is an arrow from k to x in Γ, then there is an arrow from x to k in Γ'. Also, we have $\gamma_k' = \gamma_k$ and $\gamma_x' = \gamma_x$, and so $(\gamma_k', \gamma_x') = (\gamma_k, \gamma_x) = \pm 1$.

If there is an arrow from x to k in Γ, then there is an arrow from k to x in Γ'. Also, we have $\gamma_k' = \gamma_k$ and $\gamma_x' = s_{\gamma_k}(\gamma_x)$, and so, $(\gamma_k', \gamma_x') = (\gamma_k, s_{\gamma_k}(\gamma_x)) = (s_{\gamma_k}(\gamma_k), \gamma_x) = -(\gamma_k, \gamma_x) \in \{\pm 1\}$.

Let y be another vertex in Γ, $y \neq k, x$.

If there is no arrow from x to k in Γ, and no arrow from y to k in Γ, we see that $\gamma_x' = \gamma_x$ and $\gamma_y' = \gamma_y$, and so $(\gamma_x', \gamma_y') = (\gamma_x, \gamma_y)$. Note also that the full subquiver of Γ on the vertices x and y must be the same as the full subquiver of Γ' on the vertices x and y.

Therefore, from now on, we will assume (without loss of generality) that x is fixed such that there is an arrow from x to k in Γ.

We want to compute (γ_x', γ_y') and show that it is equal to ± 1 if and only if x and y are joined by an arrow in Γ', and equal to 0 if and only if x and y are not joined by an arrow in Γ'.

Case 1: We will start by considering the case where x and y are joined by an arrow in Γ. By assumption, we have $(\gamma_x, \gamma_y) = \pm 1$.

There are two possibilities.

Case 1.1: Suppose k and y are not joined by an arrow in Γ. Then, the arrow joining x and y in Γ also appears in Γ'.

$$\begin{array}{cc} \Gamma & \Gamma' \\ \begin{tikzcd} x \arrow[rr] \arrow[dr, dash] & & k \\ & y & \end{tikzcd} & \begin{tikzcd} x \arrow[dr, dash] & & k \arrow[ll] \\ & y & \end{tikzcd} \end{array}$$

We have $\gamma'_k = \gamma_k$, $\gamma'_x = s_{\gamma_k}(\gamma_x) = \gamma_x - (\gamma_x, \gamma_k)\gamma_k$, and $\gamma'_y = \gamma_y$. So,

$$\begin{aligned} (\gamma'_x, \gamma'_y) = (s_{\gamma_k}(\gamma_x), \gamma_y) &= (\gamma_x, \gamma_y) - (\gamma_x, \gamma_k)(\gamma_k, \gamma_y) \\ &= (\gamma_x, \gamma_y) \end{aligned}$$

since $(\gamma_k, \gamma_y) = 0$.

Hence, $(\gamma'_x, \gamma'_y) = \pm 1$.

Case 1.2: Suppose k and y are joined by an arrow in Γ. Then, there is a triangle in Γ on the vertices k, x, y. This triangle must be cyclically oriented (since it is a chordless cycle in Γ). Therefore, since the arrow joining k and x has head k, the arrow joining k and y must have head y, and the arrow joining x and y must have head x. It follows that x and y are not joined by an arrow in Γ'.

$$\begin{array}{cc} \Gamma & \Gamma' \\ \begin{tikzcd} x \arrow[rr] & & k \arrow[dl] \\ & y \arrow[ul] & \end{tikzcd} & \begin{tikzcd} x & & k \arrow[ll] \\ & y \arrow[ur] & \end{tikzcd} \end{array}$$

We have $\gamma_k' = \gamma_k$, $\gamma_x' = s_{\gamma_k}(\gamma_x)$ and $\gamma_y' = \gamma_y$. So, $(\gamma_x', \gamma_y') = (\gamma_x, \gamma_y) - (\gamma_x, \gamma_k)(\gamma_k, \gamma_y)$.

Now, since A is a positive quasi-Cartan companion of B, it follows from Proposition 1.4.6 that an odd number of (γ_x, γ_y), (γ_x, γ_k) and (γ_k, γ_y) are positive. We thus have the following possibilities:

(γ_x, γ_y)	(γ_x, γ_k)	(γ_k, γ_y)	(γ_x', γ_y')
1	-1	-1	0
-1	1	-1	0
-1	-1	1	0
1	1	1	0

Therefore, we see that we must have $(\gamma_x', \gamma_y') = 0$.

Case 2: We will now consider the case where x and y are not joined by an arrow in Γ. By assumption, we have $(\gamma_x, \gamma_y) = 0$.

There are three possibilities.

Case 2.1: Suppose k and y are not joined by an arrow in Γ. Then, x and y are not joined by an arrow in Γ'.

$$\Gamma \qquad\qquad\qquad\qquad \Gamma'$$

$$\begin{tikzcd} x \arrow[r] & k \\ y & \end{tikzcd} \qquad\qquad \begin{tikzcd} x & k \arrow[l] \\ y & \end{tikzcd}$$

We have $\gamma_k' = \gamma_k$, $\gamma_x' = s_{\gamma_k}(\gamma_x)$ and $\gamma_y' = \gamma_y$. So,

$$\begin{aligned} (\gamma_x', \gamma_y') &= (\gamma_x, \gamma_y) - (\gamma_x, \gamma_k)(\gamma_k, \gamma_y) \\ &= 0 \end{aligned}$$

since $(\gamma_x, \gamma_y) = 0$ and $(\gamma_k, \gamma_y) = 0$.

Case 2.2: Suppose that there is an arrow from y to k in Γ. Again, we see that x and y are not joined by an arrow in Γ'.

$$\Gamma \qquad\qquad\qquad\qquad \Gamma'$$

$$\begin{tikzcd} x \arrow[rr] & & k \\ & y \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd} x & & k \arrow[ll] \arrow[dl] \\ & y & \end{tikzcd}$$

We have $\gamma_k' = \gamma_k$, $\gamma_x' = s_{\gamma_k}(\gamma_x)$ and $\gamma_y' = s_{\gamma_k}(\gamma_y)$. So, $(\gamma_x', \gamma_y') = (s_{\gamma_k}(\gamma_x), s_{\gamma_k}(\gamma_y)) = (\gamma_x, \gamma_y) = 0$.

Case 2.3: Finally, we suppose that there is an arrow from k to y in Γ. In this case, we see that k, x, y are the vertices of a cyclically oriented triangle in Γ'. In particular, we have that x and y are joined by an arrow (from x to y) in Γ'.

$$\Gamma \qquad\qquad\qquad\qquad \Gamma'$$

$$\begin{tikzcd} x \arrow[rr] & & k \arrow[dl] \\ & y & \end{tikzcd} \qquad\qquad \begin{tikzcd} x \arrow[dr] & & k \arrow[ll] \\ & y \arrow[ur] & \end{tikzcd}$$

We have $\gamma_k' = \gamma_k$, $\gamma_x' = s_{\gamma_k}(\gamma_x)$ and $\gamma_y' = \gamma_y$. So,

$$\begin{aligned} (\gamma_x', \gamma_y') &= (\gamma_x, \gamma_y) - (\gamma_x, \gamma_k)(\gamma_k, \gamma_y) \\ &= -(\gamma_x, \gamma_k)(\gamma_k, \gamma_y) \in \{\pm 1\} \end{aligned}$$

since $(\gamma_x, \gamma_y) = 0$ and $(\gamma_x, \gamma_k), (\gamma_k, \gamma_y) \in \{\pm 1\}$.

This completes the proof that $\{\gamma_x' : x \in \Gamma_0'\}$ is a companion basis for Γ'. ■

3.2 A Companion Basis Construction Procedure

We can now complete our initial aim by showing how we can construct a companion basis for Γ. The key step here is to exhibit a companion basis for a quiver (of a cluster-tilted algebra of simply-laced Dynkin type) from which we can obtain Γ by applying a sequence of quiver mutations. We may then apply the companion basis mutation procedure introduced in Theorem 3.1.4 to yield the desired result.

Due to the classification of the cluster algebras of finite type (Theorem 1.1.5), we have that there is some seed $(\mathbf{x}_0, B_0)$ in $\mathcal{A}$ such that the Cartan counterpart $A(B_0)$ is a Cartan matrix of type Δ. In particular, the quiver $\Gamma(B_0)$ must be an orientation of Δ.

By Theorems 2.3.1 and 2.3.2, the seed $(\mathbf{x}_0, B_0)$ corresponds to a basic cluster-tilting object T_0 in $\mathcal{C}$. Let $\Lambda_0 = \mathrm{End}_{\mathcal{C}}(T_0)^{\mathrm{op}}$ be the associated cluster-tilted algebra. It follows from Corollary 2.4.2 that $\Gamma^0 = \Gamma(B_0)$ is the quiver of Λ_0.

Recall that $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a simple system of Φ. We therefore have by definition that $\{\alpha_1, \dots, \alpha_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$. So, it follows from Corollary 3.1.3 that the matrix $\widetilde{A} = (\widetilde{a}_{ij})$ given by $\widetilde{a}_{ij} = (\alpha_i, \alpha_j)$ for all $1 \leq i, j \leq n$ is a positive quasi-Cartan matrix. Moreover, we see that $\widetilde{A}$ must be a companion of B_0 (up to simultaneous reordering of the rows and columns of $\widetilde{A}$, that is, up to reordering of $\alpha_1, \dots, \alpha_n$) and therefore, we have that Π is a companion basis for Γ^0.

Note: To see that $\widetilde{A}$ is a companion of B_0 (up to simultaneous reordering of rows and columns) we firstly note that all entries of B_0 must belong to the set $\{0, \pm 1\}$. Secondly, we note that all of the off-diagonal entries of $\widetilde{A}$ must also belong to the set $\{0, \pm 1\}$, since inner products of pairs of non-proportional roots always belong to this set in root systems of Dynkin types A, D and E (refer to Lemma 1.2.4).

We now have, at least in theory, a method for finding a companion basis for Γ. Since B and B_0 are both matrices appearing in seeds in $\mathcal{A}$, then they are mutation equivalent. In particular, we must be able to obtain Γ from Γ^0 by applying a sequence of quiver mutations. So, by applying the corresponding sequence of companion basis mutations to the companion basis Π for Γ^0 (with iterated applications of Theorem 3.1.4(i)), it follows that we obtain a companion basis for Γ.

Of course, in practice, this method of finding a companion basis for Γ could be difficult to apply, as it requires us to find a sequence of quiver mutations that takes us from Γ^0 to Γ. However, at the very least, we have given another proof of the existence of a companion basis for the quiver of any given cluster-tilted algebra of simply-laced Dynkin type.

3.3 Sign Changes in Companion Bases

Having now seen a method enabling us to construct a companion basis for Γ, our main focus throughout the remainder of this chapter will be on attempting to give a complete description of all of the companion bases for Γ. We will do this by examining the relationship between arbitrary companion bases for Γ.

We start, in this section, by showing that the companion bases for Γ giving rise to a chosen positive quasi-Cartan companion of B differ from those giving rise to a distinct chosen positive quasi-Cartan companion of B only by the signs of their elements. In the next section, we will then consequently only need to consider the relationship between the companion bases for Γ which give rise to some fixed positive quasi-Cartan companion of B.

Let $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ. It is natural to ask if we can use this companion basis to help us to find further companion bases for Γ. Here, we show that from this companion basis we can exhibit, for any positive quasi-Cartan companion of B, a companion basis for Γ giving rise to that positive quasi-Cartan companion of B.

Let $A = (a_{xy})$ be the matrix given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$. We have that A is a positive quasi-Cartan companion of B. Let $\overline{A}$ be another positive quasi-Cartan companion of B.

Since $\mathcal{A} = \mathcal{A}(B)$ is a cluster algebra of finite type, we see that A is the unique positive quasi-Cartan companion of B up to simultaneous sign changes in rows and columns (due to the results of Section 1.4). That is, by applying simultaneous sign changes in the rows and columns of A, we can obtain all positive quasi-Cartan companions of B. Note also, that all matrices obtained in this way are positive quasi-Cartan companions of B.

In particular, $\overline{A}$ can be obtained from A by applying simultaneous sign changes in some collection I of rows and columns of A.

Suppose that $\widetilde{A} = (\tilde{a}_{xy})$ is the matrix obtained by simultaneously changing signs in row and column k of the matrix A for some k. Also, suppose that $\{\tilde{\gamma}_x : x \in \Gamma_0\} \subseteq \Phi$ is obtained by setting $\tilde{\gamma}_x = \gamma_x$ for $x \neq k$, and $\tilde{\gamma}_k = -\gamma_k$. We then have the following simple result.

LEMMA 3.3.1 $\{\tilde{\gamma}_x : x \in \Gamma_0\}$ *is a companion basis for* Γ *giving rise to the positive quasi-Cartan companion* $\widetilde{A}$ *of* B.

Proof Since $\{\tilde{\gamma}_x : x \in \Gamma_0\}$ is clearly a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, we need only check that $(\tilde{\gamma}_x, \tilde{\gamma}_y) = \tilde{a}_{xy}$ for all $x, y \in \Gamma_0$.

If $x, y \neq k$, then $(\tilde{\gamma}_x, \tilde{\gamma}_y) = (\gamma_x, \gamma_y) = a_{xy} = \tilde{a}_{xy}$.

For $y \neq k$, we have $(\tilde{\gamma}_k, \tilde{\gamma}_y) = (-\gamma_k, \gamma_y) = -a_{ky} = \tilde{a}_{ky}$, and similarly, $(\tilde{\gamma}_y, \tilde{\gamma}_k) = \tilde{a}_{yk}$.

Also, $(\tilde{\gamma}_k, \tilde{\gamma}_k) = \tilde{a}_{kk} = 2$. ■

As an immediate consequence of Lemma 3.3.1, we have the following.

COROLLARY 3.3.2 *The subset* $\{\bar{\gamma}_x : x \in \Gamma_0\} \subseteq \Phi$ *given by*

$$\bar{\gamma}_x = \begin{cases} \gamma_x & \text{if } x \notin I, \\ -\gamma_x & \text{if } x \in I \end{cases}$$

is a companion basis for Γ *giving rise to* $\overline{A}$.

This result tells us that in order to give a complete description of all of the companion bases for Γ, we now only need describe those that give rise to some fixed positive quasi-Cartan companion of B. (This is because an arbitrary companion basis for Γ differs from one giving rise to the fixed positive quasi-Cartan companion of B, only by the signs of its elements.)

3.4 A Description of all Companion Bases for Γ

We now focus our attention on the companion bases for Γ that give rise to some fixed positive quasi-Cartan companion A of B. We start by fixing an arbitrary companion basis for Γ giving rise to A, and then proceed to show that we can obtain all other such companion bases for Γ from this initial companion basis. This enables us to give a complete description of all of the companion bases for Γ, in terms of the given initial companion basis.

Let $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ and suppose that $A = (a_{xy})$ is the positive quasi-Cartan companion of B given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$.

Firstly, we show that we can use the companion basis $\{\gamma_x : x \in \Gamma_0\}$ to find more companion bases for Γ that give rise to A.

Let W_Φ be the Weyl group of Φ. We then have the following simple result.

LEMMA 3.4.1 *For any $w \in W_\Phi$, both $\{w\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ and $\{-w\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ are companion bases for Γ giving rise to the positive quasi-Cartan companion A of B.*

Proof Let $w \in W_\Phi$. Then, w is an orthogonal linear transformation of V which permutes the set of roots Φ. From this, we can deduce that $\{w\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ must be a companion basis for Γ giving rise to A. It then follows immediately that $\{-w\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is also a companion basis for Γ giving rise to A. ■

The result of Lemma 3.4.1 leads us to ask whether or not we can describe all of the companion bases for Γ that give rise to A in terms of the "initial" companion basis $\{\gamma_x : x \in \Gamma_0\}$. The following well known result will provide the key in helping us to answer this question.

PROPOSITION 3.4.2 *Suppose $\varphi : V \to V$ is an orthogonal linear transformation which permutes the set of roots Φ. Then, there is some $w \in W_\Phi$ such that $\varphi\Pi = w\Pi$.*

Proof Since φ permutes the set of roots Φ, it is easy to see that $\varphi\Pi$ is a vector space basis for V. If $\alpha \in \Phi$, we can write $\alpha = \sum_{i=1}^{n} c_i\alpha_i$ with either $c_i \geq 0$ for all $1 \leq i \leq n$, or $c_i \leq 0$ for all $1 \leq i \leq n$. But then, since $\varphi\alpha = \sum_{i=1}^{n} c_i\varphi(\alpha_i)$, and again using the fact that φ permutes the set of roots Φ, we see that each $\alpha \in \Phi$ is a linear combination of $\varphi\Pi$ with all coefficients being non-negative, or all coefficients being non-positive. Therefore, $\varphi\Pi$ is a simple system.

Now, from [Hum2, Theorem 1.4] we have that any two simple systems of Φ are conjugate under W_Φ. Therefore, since $\varphi\Pi$ is a simple system, we see that there is some $w \in W_\Phi$ such that $\varphi\Pi = w\Pi$. ■

Let $\{\delta_x : x \in \Gamma_0\} \subseteq \Phi$ be another companion basis for Γ giving rise to A. We aim to describe this companion basis in terms of the companion basis $\{\gamma_x : x \in \Gamma_0\}$.

Define a map $T: V \to V$ by specifying $T(\gamma_x) = \delta_x$ for all $x \in \Gamma_0$, and extending linearly. By definition, we have that T is an invertible linear transformation. So, in order to be able to apply Proposition 3.4.2 to T, we must show that T is an orthogonal transformation which permutes the set of roots Φ.

Checking the orthogonality of T is a simple task.

LEMMA 3.4.3 *The map $T: V \to V$ is an orthogonal transformation.*

Proof The result follows as an easy consequence of the fact that $(\gamma_x, \gamma_y) = (\delta_x, \delta_y)$ for all $x, y \in \Gamma_0$. ■

The following result provides the main step towards establishing that T permutes the set of roots Φ. The proof we present is an analogue of the first two parts of the proof of [Hum2, Theorem 1.5]. Before stating this result, we first give a preliminary definition.

DEFINITION 3.4.4 *Let $\alpha \in \Phi$. Since $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is a companion basis, we can write α uniquely in the form $\alpha = \sum_{x \in \Gamma_0} c_x \gamma_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. We then define $\sum_{x \in \Gamma_0} |c_x|$ to be the height of α with respect to the companion basis $\{\gamma_x : x \in \Gamma_0\}$.*

PROPOSITION 3.4.5 *Let W_Φ be the Weyl group of Φ and suppose that W' is the subgroup of W_Φ generated by the reflections s_{γ_x} for $x \in \Gamma_0$. Then, for any $\alpha \in \Phi$ there exists $w \in W'$ and $x \in \Gamma_0$ such that $\alpha = w\gamma_x$.*

Proof Let $\alpha \in \Phi$ and consider the non-empty subset $W'\alpha \subseteq \Phi$.

Let δ be an element of $W'\alpha$ of minimal height with respect to the companion basis $\{\gamma_x : x \in \Gamma_0\}$. We claim that $\delta = \pm\gamma_y$ for some $y \in \Gamma_0$.

Since $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is a companion basis, we can write $\delta = \sum_{x \in \Gamma_0} c_x \gamma_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. We have $0 < (\delta, \delta) = \sum_{x \in \Gamma_0} c_x(\delta, \gamma_x)$, and therefore, there must be some y such that $c_y(\delta, \gamma_y) > 0$.

If $\delta = \pm\gamma_y$, then we are done. Suppose that this is not the case and consider the root

$$\begin{aligned} s_{\gamma_y}(\delta) &= \delta - (\delta, \gamma_y)\gamma_y \\ &= \sum_{x \in \Gamma_0} c_x \gamma_x - (\delta, \gamma_y)\gamma_y \\ &= \sum_{x \neq y} c_x \gamma_x + (c_y - (\delta, \gamma_y))\gamma_y \in W'\alpha. \end{aligned}$$

Now, we have $c_y(\delta, \gamma_y) > 0$. So, there are two possibilities:

(i) $c_y > 0$ and $(\delta, \gamma_y) > 0$,

(ii) $c_y < 0$ and $(\delta, \gamma_y) < 0$.

In case (i), using Lemma 1.2.4, we see that $c_y - (\delta, \gamma_y) = c_y - 1 < c_y$, and moreover, that $|c_y - (\delta, \gamma_y)| < |c_y|$.

In case (ii), using Lemma 1.2.4, we see that $|c_y - (\delta, \gamma_y)| = |c_y + 1| < |c_y|$.

Therefore, in either case we see that the height of $s_{\gamma_y}(\delta)$ with respect to the companion basis $\{\gamma_x : x \in \Gamma_0\}$ is less than the height of δ with respect to the companion basis $\{\gamma_x : x \in \Gamma_0\}$. This is a contradiction.

So, we must have that $\delta = \pm\gamma_y$.

In particular, there is some $w \in W'$ such that either $w\alpha = \gamma_y$ or $w\alpha = -\gamma_y$. In the former case, we write $\alpha = w'\gamma_y$ by taking $w' = w^{-1} \in W'$, and in the latter case, we write $\alpha = w'\gamma_y$ by taking $w' = (s_{\gamma_y}w)^{-1} = w^{-1}s_{\gamma_y} \in W'$. ■

As an immediate consequence of the above, we have the following corollary.

COROLLARY 3.4.6 *If $\alpha \in \Phi$, then we can write α in the form $\alpha = s_{\gamma_{x_1}} \cdots s_{\gamma_{x_t}}(\gamma_y)$ with $x_1, \ldots, x_t, y \in \Gamma_0$.*

Note: Let W' be as given in Proposition 3.4.5. The third part of the proof of [Hum2, Theorem 1.5] establishes that $W' = W_\Phi$.

Recall that we had defined an orthogonal linear transformation $T: V \to V$ by extending linearly the map sending γ_x to δ_x for all $x \in \Gamma_0$. We were hoping to show that this map permutes the set of roots Φ. We will be able to do this by combining the result of the following lemma with that of Corollary 3.4.6.

LEMMA 3.4.7 *If γ is a root, then $Ts_\gamma = s_{T(\gamma)}T$.*

Proof The result follows from the fact that T is an orthogonal linear transformation. ■

PROPOSITION 3.4.8 *T permutes the set of roots Φ.*

Proof Let α be a root. From Corollary 3.4.6, we see that we can write α in the form $\alpha = s_{\gamma_{x_1}} \cdots s_{\gamma_{x_t}}(\gamma_y)$ with $x_1, \ldots, x_t, y \in \Gamma_0$. But then, (repeatedly) applying Lemma 3.4.7, we see that

$$T\alpha = s_{T(\gamma_{x_1})} \cdots s_{T(\gamma_{x_t})}(T(\gamma_y)) = s_{\delta_{x_1}} \cdots s_{\delta_{x_t}}(\delta_y).$$

In particular, $T\alpha$ is a root since $\delta_{x_1}, \ldots, \delta_{x_t}, \delta_y$ are all roots.

The result then follows since T is invertible. ∎

By applying Proposition 3.4.2 to T, we are now able to prove the following result which gives useful insight into the relationship between the companion bases $\{\gamma_x : x \in \Gamma_0\}$ and $\{\delta_x : x \in \Gamma_0\}$.

LEMMA 3.4.9 *There is some $w \in W_\Phi$ and some orthogonal linear transformation $\sigma: V \to V$ which permutes Π such that $\delta_x = w\sigma\gamma_x$ for all $x \in \Gamma_0$.*

Proof By Proposition 3.4.2, we have that there is some $w \in W_\Phi$ such that $T\Pi = w\Pi$.

Define $\sigma = w^{-1}T: V \to V$. It is immediate that σ is an orthogonal linear transformation. Moreover, since $T\Pi = w\Pi$, we see that $w^{-1}T\Pi = \Pi$, and so $w^{-1}T$ permutes the set of simple roots Π. The proof is completed by noting that $w\sigma(\gamma_x) = T(\gamma_x) = \delta_x$ for all $x \in \Gamma_0$. ∎

Note: Given Proposition 3.4.8, the fact that Lemma 3.4.9 holds is essentially contained in [Sam, p.87].

We will now see that the converse of Lemma 3.4.9 also holds.

LEMMA 3.4.10 *Let $w \in W_\Phi$ and let ρ be an orthogonal linear transformation of V that permutes Π. Then, the subset $\{w\rho\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is a companion basis for Γ giving rise to A.*

Proof We start by checking that $\{w\rho\gamma_x : x \in \Gamma_0\} \subseteq \Phi$.

Given any root α, we have from [Hum2, Theorem 1.5 & Corollary 1.5] that α can be written in the form $\alpha = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}(\alpha_j)$ for some $\alpha_{i_1}, \ldots, \alpha_{i_t}, \alpha_j \in \Pi$, with $t \in \mathbb{N}$ and $1 \leq i_1, \ldots, i_t, j \leq n$.

Since ρ is an orthogonal linear transformation, it is easily seen that

$$\rho\alpha = s_{\rho(\alpha_{i_1})} \cdots s_{\rho(\alpha_{i_t})}(\rho(\alpha_j)),$$

and therefore, since ρ permutes Π, we deduce that $\rho\alpha \in \Phi$. Consequently, we see that $\{w\rho\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ and that ρ permutes the set of roots Φ.

Since w and ρ are both orthogonal linear transformations, it is clear that $(w\rho\gamma_x, w\rho\gamma_y) = (\gamma_x, \gamma_y) = a_{xy}$ for all $x, y \in \Gamma_0$. Therefore, it just remains to be checked that $\{w\rho\gamma_x : x \in \Gamma_0\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$.

Let $z \in \mathbb{Z}\Phi$. We want to write z as an integral linear combination of the roots $w\rho\gamma_x$ for $x \in \Gamma_0$.

Since $w^{-1} \in W_\Phi$ is an orthogonal linear transformation of V that permutes Φ, we see that $w^{-1}z \in \mathbb{Z}\Phi$. Furthermore, it is clear that ρ is invertible (since ρ permutes Π), and that the inverse map $\rho^{-1}: V \to V$ is also an orthogonal linear transformation of V that permutes Π, and moreover, permutes Φ. (Note that if $u, v \in V$, then $(\rho^{-1}u, \rho^{-1}v) = (\rho\rho^{-1}u, \rho\rho^{-1}v) = (u, v)$, using the orthogonality of ρ.) We thus have that $\rho^{-1}w^{-1}z \in \mathbb{Z}\Phi$.

Therefore, since $\{\gamma_x : x \in \Gamma_0\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$, we can write $\rho^{-1}w^{-1}z = \sum_{x\in\Gamma_0} a_x\gamma_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. In particular, we have

$$\begin{aligned} z &= w\rho(\rho^{-1}w^{-1}z) \\ &= w\rho\left(\sum_{x\in\Gamma_0} a_x\gamma_x\right) \\ &= \sum_{x\in\Gamma_0} a_x w\rho\gamma_x \end{aligned}$$

as required.

In order to check the $\mathbb{Z}$-linear independence of $\{w\rho\gamma_x : x \in \Gamma_0\}$, we suppose that $\sum_{x\in\Gamma_0} a_x w\rho\gamma_x = 0$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. The result then follows from the $\mathbb{Z}$-linear independence of $\{\gamma_x : x \in \Gamma_0\}$ since $\rho^{-1}w^{-1}\sum_{x\in\Gamma_0} a_x w\rho\gamma_x = \sum_{x\in\Gamma_0} a_x\gamma_x = 0$. ■

We have now established the following result.

THEOREM 3.4.11 *The companion bases for Γ that give rise to A are precisely the sets of the form $\{w\sigma\gamma_x : x \in \Gamma_0\} \subseteq \Phi$, where $w \in W_\Phi$ and σ is an orthogonal linear transformation of V that permutes Π.*

In view of Section 3.3, we also have the following corollary.

COROLLARY 3.4.12 *The companion bases for Γ are precisely the sets of the form $\{\varepsilon_x w\sigma\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ where $w \in W_\Phi$, σ is an orthogonal linear transformation of V that permutes Π, and $\varepsilon_x \in \{\pm 1\}$ for all $x \in \Gamma_0$.*

We note that a complete description of the orthogonal linear transformations of V that permute Π is well known (and easily obtained) in each of the simply-laced Dynkin cases. (Refer to [Bou, Chapter VI Section 4], for example.) In each case, these may be interpreted as the graph automorphisms of the corresponding Dynkin diagram.

We have now given a complete description of all of the companion bases for Γ in terms of the (arbitrary) initial companion basis $\{\gamma_x : x \in \Gamma_0\}$.

Note: In principle, this establishes a method of constructing all of the companion bases for Γ. Recall that in Section 3.2 we outlined a procedure for constructing an initial companion basis for Γ. Therefore, using Corollary 3.4.12 (together with a description of the orthogonal linear transformations of V that permute Π), we may then construct all of the other companion bases for Γ from this initial companion basis.

3.5 A Refined Description of Companion Bases in Type A

For this section, we restrict our attention to only the Dynkin type A_n case. We show that in this case, we can refine the result of Theorem 3.4.11.

Let $\{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ. Also, suppose that $A = (a_{xy})$ is the positive quasi-Cartan companion of B given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$.

From Theorem 3.4.11, we have that the companion bases for Γ giving rise to A are precisely the sets of the form $\{w\sigma\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ where $w \in W_\Phi$ and σ is an orthogonal linear transformation of V permuting Π. (Note that since Φ is the root system of Dynkin type A_n, then $W_\Phi = S_{n+1}$, the symmetric group on $n+1$ letters.) By considering the possible choices for σ, we can refine this description.

Suppose the simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of Φ is chosen such that $(\alpha_i, \alpha_{i+1}) = -1$ for all $1 \leq i \leq n-1$, and $(\alpha_i, \alpha_j) = 0$ for all $1 \leq i < j \leq n$ otherwise.

For $n \geq 2$, it is then easily established that there are just two orthogonal linear transformations of V that permute Π. These are the identity map on V and the map $\sigma: V \to V$ given by setting $\sigma(\alpha_i) = \alpha_{n+1-i}$ for all $1 \leq i \leq n$ and extending linearly. (Note that in the Dynkin type A_1 case, the identity map on V is the only orthogonal linear transformation of V that permutes Π.)

Using this information, we obtain the following result.

THEOREM 3.5.1 *The companion bases for Γ giving rise to A are precisely the sets of the form $\pm\{w\gamma_x : x \in \Gamma_0\}$ with $w \in W_\Phi$.*

Proof We start by noting that all sets of the form $\pm\{w\gamma_x : x \in \Gamma_0\}$ are companion bases for Γ giving rise to A by Lemma 3.4.1. We therefore just need to prove that all of the companion bases for Γ giving rise to A are of the stated form.

In the $n = 1$ case, this is clear.

Suppose $n \geq 2$ and let $\{\delta_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ giving rise to A. We have that $\delta_x = w\sigma\gamma_x$ for all $x \in \Gamma_0$, for some $w \in W_\Phi$ and some orthogonal linear transformation $\sigma: V \to V$ permuting Π.

We saw above that there are only two possible choices for σ. One possibility is that σ is the identity map on V, in which case it follows immediately that $\delta_x = w\gamma_x$ for all $x \in \Gamma_0$. The other case to consider is where σ is the map given by setting $\sigma(\alpha_i) = \alpha_{n+1-i}$ for all $1 \leq i \leq n$, and extending linearly. In this case, we will show that there is some $v \in W_\Phi$ such that $\delta_x = -v\gamma_x$ for all $x \in \Gamma_0$.

Now, if w_0 is the longest element in W_Φ, then it is well known (and not hard to show) that $w_0(\alpha_i) = -\alpha_{n+1-i}$ for all $1 \le i \le n$. Therefore, $w_0\sigma(\alpha_i) = -\alpha_i$ for all $1 \le i \le n$. So, $w_0\sigma$ is minus the identity map on V. But, there is a unique element $v \in W_\Phi$ such that $w = vw_0$. For this choice of v we see that $\delta_x = w\sigma\gamma_x = vw_0\sigma\gamma_x = -v\gamma_x$ for all $x \in \Gamma_0$.

This completes the proof. ∎

Since there are $(n+1)!$ elements in W_Φ, we see that there can be at most $2(n+1)!$ companion bases for Γ giving rise to A. For $n \ge 2$, we will now check that there are exactly $2(n+1)!$ such sets.

PROPOSITION 3.5.2 *For $n \ge 2$, there are precisely $2(n+1)!$ companion bases for Γ giving rise to A.*

Proof Firstly, suppose $w, v \in W_\Phi$ and $w\gamma_x = v\gamma_x$ for all $x \in \Gamma_0$. Then, $v^{-1}w\gamma_x = \gamma_x$ for all $x \in \Gamma_0$. If $\alpha \in \Phi$, then we can write α in the form $\alpha = \sum_{x\in\Gamma_0} c_x\gamma_x$ where $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Therefore, we have $v^{-1}w\alpha = v^{-1}w\sum_{x\in\Gamma_0} c_x\gamma_x = \sum_{x\in\Gamma_0} c_x(v^{-1}w\gamma_x) = \sum_{x\in\Gamma_0} c_x\gamma_x = \alpha$. So, $v^{-1}w \in W_\Phi$ sends no positive roots to negative roots, and hence we must have $v^{-1}w = 1$. Thus, $v = w$. (This also shows that if $w, v \in W_\Phi$ and $-w\gamma_x = -v\gamma_x$ for all $x \in \Gamma_0$, then $w = v$.)

Finally, suppose $w, v \in W_\Phi$ and $w\gamma_x = -v\gamma_x$ for all $x \in \Gamma_0$. Then, $v^{-1}w\gamma_x = -\gamma_x$ for all $x \in \Gamma_0$. Therefore, for every $\alpha \in \Phi$ we see that $v^{-1}w\alpha = -\alpha$. So, $v^{-1}w \in W_\Phi$ sends every positive root to a negative root, and hence $v^{-1}w = w_0$, the longest element in W_Φ. But $w_0(\alpha_i) = -\alpha_{n+1-i}$ for all $1 \le i \le n$, whereas $v^{-1}w(\alpha_i) = -\alpha_i$ for all $1 \le i \le n$. This is a contradicition. Therefore, we see that there are no elements $w, v \in W_\Phi$ such that $w\gamma_x = -v\gamma_x$ for all $x \in \Gamma_0$. ∎

Note: In the $n = 1$ case, it is clear that there are only two companion bases for Γ (the quiver consisting of only one vertex and no arrows) giving rise to A (the 1×1 matrix whose sole entry is 2).

Chapter 4

Dimension Vectors via Companion Bases

In this chapter, we start to consider the significance of companion bases. In particular, we establish the main result of this thesis. This result can be regarded as a generalisation, in the Dynkin type A case, of part of Gabriel's Theorem. Suppose we are given a cluster-tilted algebra of Dynkin type A, and suppose further that we are given a companion basis for the quiver of this cluster-tilted algebra. By expressing the positive roots of the corresponding root system in terms of this companion basis, and taking the absolute values of the coefficients appearing in these expressions, we associate a vector to each positive root. Our main result establishes that the vectors obtained in this way are the dimension vectors of the finitely generated indecomposable modules over the given cluster-tilted algebra.

We start this chapter by seeing how Gabriel's Theorem motivates the introduction of these vectors associated to the positive roots. The first step in the proof of the main result is then to describe these vectors. We do this by studying the structure of the quiver of the given cluster-tilted algebra, and showing that the positive roots can be associated with certain "unoriented paths" in this quiver. Our attention then turns to the finitely generated indecomposable modules over the given cluster-tilted algebra. We show that these are associated with the same unoriented paths, which then enables us to deduce that their dimension vectors are the same as the vectors associated to the positive roots.

We conclude this chapter by conjecturing that analogues of the main result hold in the Dynkin type D and E cases, and discussing some possible strategies for proving this conjecture.

4.1 Motivation and Main Result

Here, we look at the motivation for, and give a statement of, our main result.

Let k be an algebraically closed field, let Q (with n vertices) be an alternating quiver of simply-laced Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ - \text{mod})}{F}$$

be the corresponding cluster category. Suppose $\mathcal{A}$ is the cluster algebra of Dynkin type Δ. Also, suppose that $\Phi \subseteq V$ is the root system of Dynkin type Δ, where V is a Euclidean space with positive definite symmetric bilinear form $(\, , \,)$. Finally, let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ.

Due to the classification of the cluster algebras of finite type (Theorem 1.1.5), we have that there is some seed $(\mathbf{x}_0, B_0)$ in $\mathcal{A}$ such that the Cartan counterpart $A(B_0)$ is a Cartan matrix of type Δ.

By Theorems 2.3.1 and 2.3.2, the seed $(\mathbf{x}_0, B_0)$ corresponds to a basic cluster-tilting object T_0 in $\mathcal{C}$. Let $\Lambda_0 = \text{End}_{\mathcal{C}}(T_0)^{\text{op}}$ be the associated cluster-tilted algebra. It follows from Corollary 2.4.2 that $\Gamma^0 = \Gamma(B_0)$ is the quiver of Λ_0.

Since the Cartan counterpart $A(B_0)$ is a Cartan matrix of type Δ, we have that the quiver Γ^0 must be an orientation of Δ. In particular, the graph underlying Γ^0 is a tree. It therefore follows from [BMR3, Theorem 4.2] that $\Lambda_0 \cong k\Gamma^0$, and hence Λ_0 is (isomorphic to) a path algebra of finite representation type.

Now, applying Gabriel's Theorem to Λ_0 will help us to deduce a little more information about the companion basis Π for Γ^0. (We saw in Section 3.2 that Π is a companion basis for Γ^0.)

Recall that Gabriel's Theorem [Gab] may be stated as follows.

THEOREM 4.1.1 *Let $\widetilde{Q}$ be a connected quiver and let K be a field. Then, $K\widetilde{Q}$ has finite representation type if and only if the underlying graph of $\widetilde{Q}$ is a Dynkin diagram of one of the following types: A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7 or E_8.*

In this case, if $\tilde{\Phi}$ is the root system of the corresponding Dynkin diagram with simple system $\widetilde{\Pi} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ and corresponding positive system $\tilde{\Phi}^+$, then there is a bijective correspondence between the set of isoclasses of the finitely generated indecomposable $K\widetilde{Q}$-modules and the set of positive roots $\tilde{\Phi}^+$.

Under this correspondence, if M is a finitely generated indecomposable $K\widetilde{Q}$-module, and the dimension vector of M is $(d_1, \ldots, d_n)$, then the positive root corresponding to M is $\alpha = d_1\tilde{\alpha}_1 + \ldots + d_n\tilde{\alpha}_n$.

Since Π is a simple system of Φ, we can write each positive root $\alpha \in \Phi^+$ uniquely as an integral linear combination of $\alpha_1, \ldots, \alpha_n$. Moreover, all of the coefficients in these expressions must be non-negative. In particular, we can associate a vector to each $\alpha \in \Phi^+$, where the components of this vector are the coefficients appearing in the expression for α in terms of $\alpha_1, \ldots, \alpha_n$. Gabriel's Theorem then tells us that the vectors obtained in this way are the dimension vectors of the finitely generated indecomposable Λ_0-modules.

Let Λ be the cluster-tilted algebra of simply-laced Dynkin type given by $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ where T is a basic cluster-tilting object in $\mathcal{C}$. Suppose that $(\mathbf{x}, B)$ is the seed corresponding to T in the cluster algebra $\mathcal{A}$, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. Write $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices of Γ, and Γ_1 is the set of arrows of Γ. Let $\Psi = \{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ. Then, Ψ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$. So, we can write each root in Φ uniquely as an integral linear combination of the elements of the companion basis Ψ. This enables us to assign a vector to each root, as follows.

DEFINITION 4.1.2 *Let $\alpha \in \Phi$ and suppose that $\alpha = \sum_{x \in \Gamma_0} c_x \gamma_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. We define d_α^Ψ to be the vector $d_\alpha^\Psi = (|c_x|)_{x \in \Gamma_0}$.*

Note that for every $\alpha \in \Phi$, the vector d_α^Ψ associated to α is the same as the vector $d_{-\alpha}^\Psi$ associated to $-\alpha$. For this reason, we will usually restrict our attention to the vectors d_α^Ψ for $\alpha \in \Phi^+$.

Gabriel's Theorem tells us that the vectors d_α^Π for $\alpha \in \Phi^+$ are the dimension vectors of the finitely generated indecomposable Λ_0-modules. Motivated by this fact, we introduce the following definition.

DEFINITION 4.1.3 *We call Ψ a strong companion basis for Γ if the vectors d_α^Ψ for $\alpha \in \Phi^+$ are the dimension vectors of the finitely generated indecomposable Λ-modules.*

Having already seen how to find all of the companion bases for Γ, it is natural to ask whether or not we can decide which of these companion bases are strong.

We will start by considering the Dynkin type A_n case, with $n \in \mathbb{N}$ fixed but arbitrary. We then have that Q is an alternating quiver whose underlying graph is the Dynkin diagram of type A_n, that the cluster algebra $\mathcal{A}$ is of Dynkin type A_n, and that the root system Φ is of Dynkin type A_n. The main result we will establish is the following important result.

THEOREM 4.1.4 *Let $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ be a cluster-tilted algebra of Dynkin type A_n, where T is a basic cluster-tilting object in $\mathcal{C}$. Suppose that $(\mathbf{x}, B)$ is the seed corresponding to T in the cluster algebra $\mathcal{A}$, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. Then, all companion bases for Γ are strong. That is, if $\Psi \subseteq \Phi$ is a companion basis for Γ, then the vectors d_α^Ψ for $\alpha \in \Phi^+$ are precisely the dimension vectors of the finitely generated indecomposable Λ-modules.*

Giving a proof of this result will be our main aim of the following four sections, throughout which we will keep the current set-up.

We will later conjecture that the equivalent result holds in each of the Dynkin type D_n ($n \geq 4$), E_6, E_7 and E_8 cases, and discuss what progress we can make towards proving this conjecture.

4.2 Quivers of Cluster-tilted Algebras of Dynkin Type A

The aim of this section and the next is to develop our understanding of the structure of the quivers of the cluster-tilted algebras of Dynkin type A_n. We do this by making use of a known alternative description of these quivers.

We start this section by considering the triangulations of a regular $(n+3)$-gon. Using a result from [FZ3], we will see that these triangulations correspond to the seeds of the cluster algebra of Dynkin type A_n, and hence to the basic cluster-tilting objects in $\mathcal{C}$. Also, we have a natural way of associating quivers to these triangulations (from [CCS1]).

We will see that for any given triangulation of a regular $(n+3)$-gon, the quiver associated to this triangulation is the quiver of the associated cluster-tilted algebra, therefore establishing that the quivers associated to the triangulations of a regular $(n+3)$-gon are precisely the quivers of the cluster-tilted algebras of Dynkin type A_n.

In the next section, we then start to examine some of the basic properties of the quivers associated to the triangulations of a regular $(n+3)$-gon.

Let $\mathbb{P}_{n+3}$ be a regular $(n+3)$-gon. We have that $|\Phi_{\geq -1}| = \frac{1}{2}n(n+1)+n = \frac{1}{2}n(n+3)$ which is equal to the number of diagonals of $\mathbb{P}_{n+3}$.

Following [FZ3], let the vertices of $\mathbb{P}_{n+3}$ be $P_1, P_2, \ldots, P_{n+3}$, labelled in the anti-clockwise direction, and identify the almost positive roots with the diagonals of $\mathbb{P}_{n+3}$ as follows. (Recall that $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a simple system of Φ.)

For $1 \leq i \leq \frac{n+1}{2}$, identify $-\alpha_{2i-1} \in \Phi_{\geq -1}$ with the diagonal joining P_i and P_{n+3-i}, and for $1 \leq i \leq \frac{n}{2}$, identify $-\alpha_{2i} \in \Phi_{\geq -1}$ with the diagonal joining P_{i+1} and P_{n+3-i}. Finally, for each $1 \leq i \leq j \leq n$, identify the positive root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ with the unique diagonal that crosses precisely the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$. (Note that two diagonals are said to cross if they are distinct and have a common interior point.)

We then have the following result from [FZ3, Proposition 3.14].

PROPOSITION 4.2.1 *Let $\alpha, \beta \in \Phi_{\geq -1}$. Then,*

$$(\alpha||\beta) = \begin{cases} 1 & \textit{if the diagonals } \alpha \textit{ and } \beta \textit{ cross,} \\ 0 & \textit{otherwise.} \end{cases}$$

So, compatible sets are collections of mutually non-crossing diagonals. Therefore, there is a one-to-one correspondence between the set of clusters (or equivalently, seeds) of $\mathcal{A}$ and the set of triangulations of $\mathbb{P}_{n+3}$ by non-crossing diagonals.

Let $\mathbb{T}$ be a triangulation of $\mathbb{P}_{n+3}$. Then, we can associate a connected quiver $Q_{\mathbb{T}}$ to $\mathbb{T}$ as in [CCS1, Section 3.2]. Take the vertices of $Q_{\mathbb{T}}$ to be the midpoints of the diagonals in $\mathbb{T}$. Let i and j be vertices in $Q_{\mathbb{T}}$ lying on diagonals d_i and d_j respectively. Then, there is an arrow from i to j in $Q_{\mathbb{T}}$ if d_i and d_j bound a common triangle (from the triangulation), and the angle of minimal rotation about the common point of d_i and d_j taking the line through d_i to the line through d_j is in the anticlockwise direction.

Suppose that $(\mathbf{x}, B)$ is the seed of $\mathcal{A}$ corresponding to the triangulation $\mathbb{T}$ of $\mathbb{P}_{n+3}$ as given by Proposition 4.2.1 above. It follows immediately from [FZ2, Proposition 12.5] that $Q_{\mathbb{T}}$ is the quiver associated to B (taking the vertices of $Q_{\mathbb{T}}$ to be indexed by the corresponding cluster variables in $\mathcal{A}$).

Now, we have that the seed $(\mathbf{x}, B)$ corresponds to some basic cluster-tilting object T in $\mathcal{C}$. In view of Corollary 2.4.2, we then have that $Q_{\mathbb{T}}$ is the quiver of the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

Therefore, the quivers associated to the triangulations of $\mathbb{P}_{n+3}$ are precisely the quivers of the cluster-tilted algebras associated to the basic cluster-tilting objects in $\mathcal{C}$.

4.3 The Structure of Quivers Associated to Triangulations

In this section, we start to examine the structure of the quivers associated to the triangulations of a regular $(n+3)$-gon. In particular, we introduce the concept of shortest unoriented paths in these quivers. Shortest unoriented paths will play a key role in the proof of Theorem 4.1.4.

Keeping the triangulation $\mathbb{T}$ of $\mathbb{P}_{n+3}$ from the previous section, recall that $Q_{\mathbb{T}}$ denotes the quiver associated to $\mathbb{T}$.

Since $Q_{\mathbb{T}}$ is the quiver of a cluster-tilted algebra, we will now consider its structure more closely. Firstly however, we note that all of the triangles in the triangulation $\mathbb{T}$ are of the following three types:

(I) Triangles that consist of one diagonal and two boundary edges of $\mathbb{P}_{n+3}$.

(II) Triangles that consist of two diagonals and one boundary edge of $\mathbb{P}_{n+3}$.

(III) Triangles that consist of three diagonals of $\mathbb{P}_{n+3}$.

Note: Since $n \geq 1$, at least one side of any given triangle in $\mathbb{T}$ must be a diagonal of $\mathbb{P}_{n+3}$.

By the definition of $Q_{\mathbb{T}}$, we have that a triangle in $\mathbb{T}$ of type (I) gives rise to a vertex in $Q_{\mathbb{T}}$, a triangle in $\mathbb{T}$ of type (II) gives rise to an arrow between two vertices in $Q_{\mathbb{T}}$, and a triangle in $\mathbb{T}$ of type (III) gives rise to an oriented 3-cycle in $Q_{\mathbb{T}}$.

Let x be a vertex in $Q_{\mathbb{T}}$, and let d_x be the corresponding diagonal of $\mathbb{P}_{n+3}$ in $\mathbb{T}$. Then, d_x must bound precisely two triangles in $\mathbb{T}$. If d_x bounds two triangles of type (I), then we must have $n = 1$, and x is the only vertex of $Q_{\mathbb{T}}$. If d_x bounds a triangle of type (I) and a triangle of type (II), then x has valency one. If d_x bounds a triangle of type (I) and a triangle of type (III), then x lies on a 3-cycle in $Q_{\mathbb{T}}$ and has valency two. If d_x bounds two triangles of type (II), then x has valency two. If d_x bounds a triangle of type (II) and a triangle of type (III), then x is a vertex at which an arrow meets a 3-cycle, and x has valency three. Finally, if d_x bounds two triangles of type (III), then x is a vertex at which two 3-cycles meet, and x has valency four. This covers all possible cases for the vertex x.

Suppose d is a diagonal in $\mathbb{T}$, joining the vertices P_a and P_b of $\mathbb{P}_{n+3}$. The diagonal d divides the polygon $\mathbb{P}_{n+3}$ into two parts, $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$. We can consider $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$ as two polygons which have been "glued together" along their shared boundary edge d. So, if $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$ have l and m sides respectively, then $l + m = (n + 3) + 2 = n + 5$.

Note: For a triangulation X of a polygon $\mathbb{P}$, we will write $\text{diag}(X)$ to denote the subset of the set of diagonals of $\mathbb{P}$ that appear (i.e. form the boundaries of the triangles) in X.

The triangulation $\mathbb{T}$ of $\mathbb{P}_{n+3}$ induces triangulations $\mathbb{T}^{d^+}$ and $\mathbb{T}^{d^-}$ of $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$ respectively. Each element of $\text{diag}(\mathbb{T}) \setminus \{d\}$ appears in precisely one of $\text{diag}(\mathbb{T}^{d^+})$ or $\text{diag}(\mathbb{T}^{d^-})$. We have that $d \notin \text{diag}(\mathbb{T}^{d^+}), \text{diag}(\mathbb{T}^{d^-})$, since d is a boundary edge of both $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$.

It is clear that no diagonal in $\mathbb{T}^{d^+}$ can bound the same triangle (in $\mathbb{T}$) as any diagonal in $\mathbb{T}^{d^-}$. As an immediate consequence of this, we see that there can be no cycles in the underlying (unoriented) graph of $Q_{\mathbb{T}}$ containing vertices corresponding to the elements of $\text{diag}(\mathbb{T}^{d^+})$ and vertices corresponding to the elements of $\text{diag}(\mathbb{T}^{d^-})$. This proves the following.

LEMMA 4.3.1 *In the underlying graph of $Q_{\mathbb{T}}$, the only cycles are 3-cycles arising from triangles in $\mathbb{T}$ of type (III).*

As a result of Lemma 4.3.1, we see that two 3-cycles in $Q_{\mathbb{T}}$ cannot meet in an arrow.

Note: The above description of the possibilities at a vertex in $Q_{\mathbb{T}}$ and Lemma 4.3.1 were shown independently in [BV].

Recall that $Q_{\mathbb{T}}$ is the quiver of the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ where T is the basic cluster-tilting object in $\mathcal{C}$ corresponding to (the seed corresponding to) the triangulation $\mathbb{T}$.

Let $I_{\mathbb{T}}$ be the ideal of the path algebra $kQ_{\mathbb{T}}$ generated by the set of all paths in $Q_{\mathbb{T}}$ consisting of two consecutive arrows in any given 3-cycle. It is clear that $I_{\mathbb{T}}$ is an admissible ideal of $kQ_{\mathbb{T}}$, and we have the following.

PROPOSITION 4.3.2 *The cluster-tilted algebra Λ is isomorphic to $\frac{kQ_{\mathbb{T}}}{I_{\mathbb{T}}}$.*

Proof The result follows immediately from [BMR3, Theorem 4.2] due to the result of Lemma 4.3.1. ∎

Closely related to the paths in a quiver are the unoriented paths in that quiver. These are defined as follows.

DEFINITION 4.3.3 *Let Q' be a quiver and suppose that i and j are vertices of Q'. We define an unoriented path in Q' from i to j to be a sequence of consecutive arrows of Q' leading from i to j, where the orientations of the arrows are ignored. Trivial paths on the vertices of Q' are also considered to be unoriented paths in Q'.*

We define the length of an unoriented path in Q' from i to j to be the number of arrows appearing in that path (including multiplicities). An unoriented path in Q'

from i to j of minimal length will be called a shortest unoriented path in Q' from i to j.

We have the following.

LEMMA 4.3.4 *Let i and j be vertices of $Q_{\mathbb{T}}$. Then, there is a unique shortest unoriented path in $Q_{\mathbb{T}}$ from i to j.*

Proof Firstly, note that the existence of an unoriented path in $Q_{\mathbb{T}}$ from i to j is a trivial consequence of the fact that $Q_{\mathbb{T}}$ is connected.

Suppose by way of contradiction that p and q are distinct shortest unoriented paths in $Q_{\mathbb{T}}$ from i to j. Moving along p starting from i and heading towards j, suppose that the vertices of p are $i = p_1, p_2, \ldots, p_r = j$. Likewise, suppose that the vertices of q are $i = q_1, q_2, \ldots, q_r = j$.

Let $k \in \mathbb{N}$ be the smallest index such that $p_k \neq q_k$. We know that such a k exists since p and q are distinct. Now, we have that $p_r = q_r (= j)$. So, let $s, t \in \mathbb{N} \setminus \{1, \ldots, k-1\}$ be the smallest indices such that $p_s = q_t$. Then, there is an unoriented cycle in $Q_{\mathbb{T}}$ on the vertices $p_{k-1}, p_k, p_{k+1}, \ldots, p_{s-1}, p_s = q_t, q_{t-1}, \ldots, q_{k+1}, q_k, p_{k-1}$. But, we have already seen that the only (unoriented) cycles in $Q_{\mathbb{T}}$ are 3-cycles. So, this cycle must be a 3-cycle.

This contradicts p and q both being shortest unoriented paths in $Q_{\mathbb{T}}$ from i to j.

Therefore, there is a unique shortest unoriented path in $Q_{\mathbb{T}}$ from i to j. ■

Let i and j be vertices of $Q_{\mathbb{T}}$. From Lemma 4.3.4 we have that there is a unique shortest unoriented path in $Q_{\mathbb{T}}$ from i to j. But also, we have that there is a unique shortest unoriented path in $Q_{\mathbb{T}}$ from j to i. Moreover, it is clear that this must be the reverse of the shortest unoriented path from i to j.

Throughout the remainder of this chapter, we will identify the shortest unoriented paths in the quiver of a cluster-tilted algebra of Dynkin type A with their reverse paths. So, the shortest unoriented paths in such a quiver can be thought of as corresponding to (unordered) pairs of vertices.

4.4 Description of Vectors Associated to the Positive Roots

We briefly recall our current set-up. We have that $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is a cluster-tilted algebra of Dynkin type A_n, with T a basic cluster-tilting object in $\mathcal{C}$. The seed of $\mathcal{A}$ corresponding to T is $(\mathbf{x}, B)$, meaning that $\Gamma = \Gamma(B)$ is the quiver of Λ. But also, the seed $(\mathbf{x}, B)$ corresponds to the triangulation $\mathbb{T}$ of $\mathbb{P}_{n+3}$, and we have that Γ is the same as the quiver $Q_{\mathbb{T}}$ associated to this triangulation.

Let $\Psi = \{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ, where Γ_0 denotes the set of vertices of Γ. We have already seen that we can associate a vector d_α^Ψ to each root $\alpha \in \Phi$. Recall that, for a given root α, this is the vector whose components are the absolute values of the coefficients of the γ_x's in the (unique) expression for α in terms of the roots γ_x for $x \in \Gamma_0$.

Our aim is to show that the vectors d_α^Ψ for $\alpha \in \Phi^+$ are the dimension vectors of the finitely generated indecomposable Λ-modules.

Our strategy for achieving this aim will be as follows. Firstly, we will establish a bijection between the set of shortest unoriented paths in Γ and the set of positive roots Φ^+. Using this bijection, we will be able to give a complete description of the vectors d_α^Ψ for $\alpha \in \Phi^+$. Secondly, we will note that (the isomorphism classes of) the finitely generated indecomposable Λ-modules are also in bijection with the shortest unoriented paths in Γ. This will then establish a bijection between the set of positive roots Φ^+ and the set of (isomorphism classes of the) finitely generated indecomposable Λ-modules. The final step will then be to show that for any given positive root α, the vector d_α^Ψ is the dimension vector of the finitely generated indecomposable Λ-module corresponding to α.

We are going to need some way of associating shortest unoriented paths in Γ to (positive) roots. The following definition provides this.

DEFINITION 4.4.1 *Let $\alpha \in \Phi$. We can write α uniquely in the form $\alpha = \sum_{x\in\Gamma_0} c_x\gamma_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Let $I = \{x \in \Gamma_0 : c_x \neq 0\}$. If the elements of I are precisely the vertices of a shortest unoriented path p in Γ, then we say that α has support p.*

Straight away, we see that no root can have two distinct shortest unoriented paths as support.

LEMMA 4.4.2 *Let α be a root. Then, α has at most one shortest unoriented path as support.*

Proof This is immediate due to the fact that Ψ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$. ■

Given any shortest unoriented path in Γ, we will now show that we can exhibit a positive root which has that shortest unoriented path as support.

We can deal with the trivial shortest unoriented paths immediately. Suppose that p is the trivial shortest unoriented path on the vertex $z \in \Gamma_0$. Then, γ_z is a root with support p. Therefore, either γ_z or $-\gamma_z$ is a positive root with support p.

We must now consider the non-trivial shortest unoriented paths in Γ.

Let p be a non-trivial shortest unoriented path in Γ, and suppose that the vertices of p, taken in consecutive order from one end of p to the other, are $x_0, x_1, \ldots, x_t$ ($t \geq 1$). Supposing that $0 \leq j < i \leq t$, we see that x_i is joined by an arrow in Γ to x_j if and only if $j = i - 1$. (This follows immediately from the fact that p is a shortest unoriented path in Γ.)

Now, $\Psi = \{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ is a companion basis for Γ. Therefore, supposing that $0 \leq j < i \leq t$, we then have that $\left(\gamma_{x_i}, \gamma_{x_j}\right) = \pm 1$ if and only if $j = i - 1$, and $\left(\gamma_{x_i}, \gamma_{x_j}\right) = 0$ otherwise.

The key result enabling us to find a positive root with support p is the following.

PROPOSITION 4.4.3 *If* $1 \leq r \leq t$, *then*

$$s_{\gamma_{x_r}} \cdots s_{\gamma_{x_1}}(\gamma_{x_0}) = \gamma_{x_0} + \sum_{k=1}^{r} (-1)^k \left(\prod_{l=1}^{k} \left(\gamma_{x_l}, \gamma_{x_{l-1}}\right) \right) \gamma_{x_k}.$$

Proof We proceed by induction on r.

In the initial case when $r = 1$, we have $s_{\gamma_{x_1}}(\gamma_{x_0}) = \gamma_{x_0} - \left(\gamma_{x_1}, \gamma_{x_0}\right) \gamma_{x_1}$, so the result holds in this case.

For our induction hypothesis, we suppose that $r \geq 2$ and

$$s_{\gamma_{x_{r-1}}} \cdots s_{\gamma_{x_1}}(\gamma_{x_0}) = \gamma_{x_0} + \sum_{k=1}^{r-1} (-1)^k \left(\prod_{l=1}^{k} \left(\gamma_{x_l}, \gamma_{x_{l-1}}\right) \right) \gamma_{x_k}.$$

Now, $s_{\gamma_{x_r}}(s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})) = s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0}) - \left(\gamma_{x_r}, s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})\right)\gamma_{x_r}$.

We have,

$$\begin{aligned}
\left(\gamma_{x_r}, s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})\right) &= \left(\gamma_{x_r}, \gamma_{x_0} + \sum_{k=1}^{r-1}(-1)^k\left(\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\gamma_{x_k}\right) \\
&= \left(\gamma_{x_r},\gamma_{x_0}\right) + \sum_{k=1}^{r-1}(-1)^k\left(\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\left(\gamma_{x_r},\gamma_{x_k}\right).
\end{aligned}$$

Therefore, since $(\gamma_{x_r},\gamma_{x_0}) = 0$ and since $(\gamma_{x_r},\gamma_{x_k}) \neq 0$ only for $k = r-1$, we see that

$$\begin{aligned}
\left(\gamma_{x_r}, s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})\right) &= (-1)^{r-1}\left(\prod_{l=1}^{r-1}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\left(\gamma_{x_r},\gamma_{x_{r-1}}\right) \\
&= (-1)^{r-1}\prod_{l=1}^{r}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right).
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
s_{\gamma_{x_r}}s_{\gamma_{x_{r-1}}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0}) &= \gamma_{x_0} + \sum_{k=1}^{r-1}(-1)^k\left(\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\gamma_{x_k} \\
&- (-1)^{r-1}\left(\prod_{l=1}^{r}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\gamma_{x_r} \\
&= \gamma_{x_0} + \sum_{k=1}^{r}(-1)^k\left(\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\gamma_{x_k}
\end{aligned}$$

as required. ∎

By taking $r = t$ in Proposition 4.4.3, we have that

$$s_{\gamma_{x_t}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0}) = \gamma_{x_0} + \sum_{k=1}^{t}(-1)^k\left(\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right)\right)\gamma_{x_k}.$$

We claim that the root $s_{\gamma_{x_t}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})$ has support p. Indeed, we have already noted that $\left(\gamma_{x_l},\gamma_{x_{l-1}}\right) = \pm 1$ for all $1 \leq l \leq t$. So, for each k, $1 \leq k \leq t$, we have that $(-1)^k\prod_{l=1}^{k}\left(\gamma_{x_l},\gamma_{x_{l-1}}\right) = \pm 1$. Therefore, $s_{\gamma_{x_t}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0})$ is of the form

$$s_{\gamma_{x_t}}\cdots s_{\gamma_{x_1}}(\gamma_{x_0}) = \gamma_{x_0} \pm \gamma_{x_1} \pm \cdots \pm \gamma_{x_t},$$

showing that $s_{\gamma_{x_t}} \cdots s_{\gamma_{x_1}}(\gamma_{x_0})$ has support p.

We have thus exhibited a positive root with support p, for if $s_{\gamma_{x_t}} \cdots s_{\gamma_{x_1}}(\gamma_{x_0})$ is not a positive root, then $-s_{\gamma_{x_t}} \cdots s_{\gamma_{x_1}}(\gamma_{x_0})$ is a positive root with support p.

We have now shown that for any given shortest unoriented path p in Γ, there is some positive root α which has support p. Moreover, we have shown that we can choose α such that all of the coefficients in the unique expression for α in terms of Ψ belong to the set $\{0, \pm 1\}$. (Note that whether p is a trivial or a non-trivial shortest unoriented path in Γ, the positive root with support p obtained as above satisfies this property.) In particular, the vector d_α^Ψ has a one in each position corresponding to a vertex of Γ lying on p, and zeros everywhere else.

We will now see that the positive root with support p that we have exhibited is in fact the unique positive root with support p. In view of the fact that any positive root has at most one shortest unoriented path in Γ as support (refer to Lemma 4.4.2), we are able to show this by establishing that the number of distinct shortest unoriented paths in Γ is equal to the number of positive roots in Φ.

To this end, observe firstly that the number of (distinct) non-trivial shortest unoriented paths in Γ is given by $\binom{n}{2}$, since there is a unique shortest unoriented path in Γ associated to each pair of distinct vertices of Γ. Also, there are n trivial shortest unoriented paths in Γ, one associated to each vertex. Therefore, the total number of shortest unoriented paths in Γ is

$$\binom{n}{2} + n = \frac{n!}{2!(n-2)!} + n = \frac{1}{2}n(n-1) + n = \frac{1}{2}n(n+1),$$

which is equal to the number of positive roots in Φ.

We have now established the following result.

PROPOSITION 4.4.4 *There is a bijective correspondence between the set of shortest unoriented paths in Γ and the set of positive roots Φ^+, with each shortest unoriented path in Γ corresponding to the unique positive root which has that shortest unoriented path as support.*

As a consequence of this result, we see that we have now given a complete description of the vectors d_α^Ψ for $\alpha \in \Phi^+$.

Let α be a positive root. Then, there is some shortest unoriented path p in Γ such that α has support p. The vector d_α^Ψ has a one in each position corresponding to a vertex of Γ lying on p, and zeros everywhere else. In this way, the vectors associated to the positive roots correspond precisely to the shortest unoriented paths in Γ.

4.5 Completion of the Proof of Theorem 4.1.4

Recall that we are trying to show that the vectors d_α^Ψ for all positive roots α are the same as the dimension vectors of the finitely generated indecomposable Λ-modules.

In the previous section, we were able to completely describe the vectors d_α^Ψ for $\alpha \in \Phi^+$ by associating a positive root to each shortest unoriented path in Γ. Here, we will see how we can associate a finitely generated indecomposable Λ-module to each shortest unoriented path in Γ, and we will complete the proof of the desired result.

Firstly, we recall from Proposition 4.3.2 that we have an isomorphism of k-algebras $\Lambda \cong \frac{k\Gamma}{I}$, where I is the admissible ideal of $k\Gamma$ generated by the set of all paths in Γ consisting of two consecutive arrows in any given triangle (3-cycle). It is then a standard result that the category Λ-mod of finitely generated Λ-modules is equivalent to the category $\mathrm{rep}_k(\Gamma, I)$ of finite dimensional representations of Γ over k which satisfy the relations in I.

In order to associate finitely generated indecomposable Λ-modules to the shortest unoriented paths in Γ, we start by considering a method of associating a representation R_p of Γ over k to each shortest unoriented path p in Γ.

Let p be a shortest unoriented path in Γ. To each vertex of Γ lying on p we associate the vector space k, and to each vertex of Γ not lying on p we associate the zero space. Also, to each arrow of Γ lying on p we associate the identity map on k, and to each arrow of Γ not lying on p we associate the zero map. In this way, we obtain a representation R_p of Γ over k.

It is easy to see that R_p is an indecomposable representation of Γ over k. Also, because p is a shortest unoriented path in Γ, we have that p never passes through two consecutive arrows in any given triangle in Γ. In particular, R_p never has two consecutive identity maps in any given triangle, and therefore, R_p satisfies the relations in I.

So, the above gives us a way of assigning a different indecomposable representation in $\mathrm{rep}_k(\Gamma, I)$, and hence a different finitely generated indecomposable Λ-module, to each different shortest unoriented path in Γ.

We now complete the proof of Theorem 4.1.4 with the following proposition.

PROPOSITION 4.5.1 *The vectors d_α^Ψ for $\alpha \in \Phi^+$ are the dimension vectors of the finitely generated indecomposable Λ-modules. That is, Ψ is a strong companion basis for Γ.*

Proof Firstly, we note that since Λ is cluster-tilted from an algebra with simply-laced quiver of Dynkin type, then as a consequence of [BMR1, Theorem A], we have that the number of (isomorphism classes of) finitely generated indecomposable Λ-modules is equal to $|\Phi^+|$. Therefore, the described method of assigning finitely generated indecomposable Λ-modules to shortest unoriented paths in Γ provides us with a bijective correspondence between the set of all (isomorphism classes of) finitely generated indecomposable Λ-modules and the set of all shortest unoriented paths in Γ.

We now complete the proof by showing that for each positive root α, the dimension vector of the finitely generated indecomposable Λ-module corresponding to the shortest unoriented path in Γ corresponding to α is equal to d_α^Ψ.

Let α be a positive root, and let p be the shortest unoriented path in Γ corresponding to α. (So, α has support p.) Then by construction, we have that the dimension vector of the finitely generated indecomposable Λ-module corresponding to p has a one in every position corresponding to a vertex of Γ lying on p, and a zero in every position corresponding to a vertex of Γ not lying on p. So, we see that the dimension vector of the finitely generated indecomposable Λ-module corresponding to p is equal to d_α^Ψ. ■

EXAMPLE 4.5.2 Let Γ be the following quiver.

$$\begin{tikzcd} & & 3 \arrow[dr, "c"] & \\ 1 \arrow[r, "a"'] & 2 \arrow[ur, "b"] & & 4 \arrow[ll, "d"] \end{tikzcd}$$

Using Section 4.2, it is easily checked that Γ arises from a triangulation of a regular 7-gon, and therefore Γ is the quiver of a cluster-tilted algebra Λ of Dynkin type A_4.

In fact, $\Lambda = \frac{k\Gamma}{I}$ where I is the admissible ideal of $k\Gamma$ given by $I = \langle cb, dc, bd \rangle$ (refer to Proposition 4.3.2).

Let Φ be the root system of Dynkin type A_4, and suppose that $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subseteq \Phi$ is a simple system of Φ.

Firstly, we will show that the set $\Psi = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \subseteq \Phi$ given by $\gamma_1 = -\alpha_1$, $\gamma_2 = -\alpha_2 - \alpha_3$, $\gamma_3 = \alpha_3$, $\gamma_4 = \alpha_4$ is a companion basis for Γ.

That Ψ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$ is clear.

Let $B = (b_{ij})$ be the skew-symmetric integer matrix associated to the quiver Γ. (Recall that the rows and columns of B are indexed by the vertices of Γ.) Then,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

We must check that the matrix $A = (a_{ij})$ given by $a_{ij} = (\gamma_i, \gamma_j)$ for $1 \leq i, j \leq 4$ is a positive quasi-Cartan companion of B.

We have

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix}.$$

Since $|a_{ij}| = |b_{ij}|$ for all $i \neq j$, it follows from Corollary 3.1.3 that A is a positive quasi-Cartan companion of B.

We will now check that the vectors associated to the positive roots with respect to Ψ are the dimension vectors of the finitely generated indecomposable Λ-modules.

Firstly, we find the vectors associated to the positive roots with respect to Ψ:

$$
\begin{array}{ll}
\alpha_1 = -\gamma_1 & d^{\Psi}_{\alpha_1} = (1,0,0,0) \\
\alpha_2 = -\gamma_2 - \gamma_3 & d^{\Psi}_{\alpha_2} = (0,1,1,0) \\
\alpha_3 = \gamma_3 & d^{\Psi}_{\alpha_3} = (0,0,1,0) \\
\alpha_4 = \gamma_4 & d^{\Psi}_{\alpha_4} = (0,0,0,1) \\
\alpha_1 + \alpha_2 = -\gamma_1 - \gamma_2 - \gamma_3 & d^{\Psi}_{\alpha_1+\alpha_2} = (1,1,1,0) \\
\alpha_2 + \alpha_3 = -\gamma_2 & d^{\Psi}_{\alpha_2+\alpha_3} = (0,1,0,0) \\
\alpha_3 + \alpha_4 = \gamma_3 + \gamma_4 & d^{\Psi}_{\alpha_3+\alpha_4} = (0,0,1,1) \\
\alpha_1 + \alpha_2 + \alpha_3 = -\gamma_1 - \gamma_2 & d^{\Psi}_{\alpha_1+\alpha_2+\alpha_3} = (1,1,0,0) \\
\alpha_2 + \alpha_3 + \alpha_4 = -\gamma_2 + \gamma_4 & d^{\Psi}_{\alpha_2+\alpha_3+\alpha_4} = (0,1,0,1) \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\gamma_1 - \gamma_2 + \gamma_4 & d^{\Psi}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} = (1,1,0,1)
\end{array}
$$

The indecomposable representations of Γ that satisfy the relations in I are as follows:

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
k \arrow[r] & 0 \arrow[ur] & & 0 \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
0 \arrow[r] & k \arrow[ur] & & 0 \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & k \arrow[dr] & \\
0 \arrow[r] & 0 \arrow[ur] & & 0 \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
0 \arrow[r] & 0 \arrow[ur] & & k \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
k \arrow[r, "1"'] & k \arrow[ur] & & 0 \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & k \arrow[dr] & \\
0 \arrow[r] & k \arrow[ur, "1"] & & 0 \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & k \arrow[dr, "1"] & \\
0 \arrow[r] & 0 \arrow[ur] & & k \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
0 \arrow[r] & k \arrow[ur] & & k \arrow[ll, "1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & k \arrow[dr] & \\
k \arrow[r, "1"'] & k \arrow[ur, "1"] & & 0 \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[dr] & \\
k \arrow[r, "1"'] & k \arrow[ur] & & k \arrow[ll, "1"]
\end{tikzcd}
$$

Therefore, we see that the vectors d^{Ψ}_{α} for $\alpha \in \Phi^+$ are the dimension vectors of the finitely generated indecomposable Λ-modules.

4.6 A Conjectured Generalisation

With the proof of Theorem 4.1.4 now completed, we conjecture here that the result of this theorem holds for all cluster-tilted algebras of simply-laced Dynkin type. The main result of this section shows that the problem of proving this conjecture can be reduced to that of proving an equivalent but simpler one. We conclude the section by commenting on some possible strategies for solving this reduced problem.

We need to return to a more general set-up.

Let k be an algebraically closed field, let Q (with n vertices) be an alternating quiver of simply-laced Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ - \mathrm{mod})}{F}$$

be the corresponding cluster category. Suppose that $\mathcal{A}$ is the cluster algebra of Dynkin type Δ. Also, suppose that $\Phi \subseteq V$ is the root system of Dynkin type Δ, where V is a Euclidean space with positive definite symmetric bilinear form $(\ , \ )$. Finally, let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ.

Motivated by the result of Theorem 4.1.4, we make the following conjecture.

CONJECTURE 4.6.1 *Let $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ be a cluster-tilted algebra of simply-laced Dynkin type, where T is a basic cluster-tilting object in $\mathcal{C}$. Suppose $(\mathbf{x}, B)$ is the seed corresponding to T in the cluster algebra $\mathcal{A}$, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. Then, all companion bases for Γ are strong. That is, if $\Psi \subseteq \Phi$ is a companion basis for Γ, then the vectors d_α^Ψ for $\alpha \in \Phi^+$ are precisely the dimension vectors of the finitely generated indecomposable Λ-modules.*

The main focus of this section will be to see what progress we can make towards proving Conjecture 4.6.1.

Let Λ, T, $(\mathbf{x}, B)$ and Γ be as in the statement of Conjecture 4.6.1. Given a companion basis for Γ, we have seen that we can associate a vector to each positive root. We will start by showing that the collection of vectors associated to the set of positive roots is the same, regardless of the chosen companion basis for Γ. This then shows that if one companion basis for Γ is strong, then all of the companion bases for Γ are strong, thereby reducing the problem of proving Conjecture 4.6.1 to the problem of showing that we can find a strong companion basis for Γ. We will then proceed by considering some possible strategies to try to solve this reduced problem.

PROPOSITION 4.6.2 *Let* $\Psi = \{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ *and* $\Theta = \{\delta_x : x \in \Gamma_0\} \subseteq \Phi$ *be two companion bases for* Γ. *Then,* $\{d_\alpha^\Psi : \alpha \in \Phi^+\} = \{d_\alpha^\Theta : \alpha \in \Phi^+\}$. *In particular, the set of vectors associated to the set of positive roots, with respect to a given companion basis for* Γ, *does not depend on the chosen companion basis.*

Proof Suppose that Ψ gives rise to the positive quasi-Cartan companion A of B, and suppose that Θ gives rise to the positive quasi-Cartan companion A' of B. (So, $A = (a_{xy})$ and $A' = (a'_{xy})$ are respectively given by $a_{xy} = (\gamma_x, \gamma_y)$ and $a'_{xy} = (\delta_x, \delta_y)$ for all $x, y \in \Gamma_0$.)

By applying sign changes to the elements of Θ, we may obtain a companion basis for Γ giving rise to A (refer to Section 3.3).

Let $\widetilde{\Theta} = \{\varepsilon_x \delta_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ giving rise to A, where $\varepsilon_x \in \{\pm 1\}$ for all $x \in \Gamma_0$.

We start by showing that $\{d_\alpha^\Theta : \alpha \in \Phi^+\} = \{d_\alpha^{\widetilde{\Theta}} : \alpha \in \Phi^+\}$.

Let $\alpha \in \Phi^+$. Then, we can write α uniquely in the form $\alpha = \sum_{x\in\Gamma_0} c_x\delta_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Since $\varepsilon_x \in \{\pm 1\}$ for all $x \in \Gamma_0$, we have that $\alpha = \sum_{x\in\Gamma_0} \varepsilon_x^2 c_x\delta_x$, and so $\alpha = \sum_{x\in\Gamma_0} \varepsilon_x c_x(\varepsilon_x\delta_x)$.

Therefore,

$$\begin{aligned} d_\alpha^{\tilde{\Theta}} &= (|\varepsilon_x c_x|)_{x\in\Gamma_0} \\ &= (|\varepsilon_x|\,|c_x|)_{x\in\Gamma_0} \\ &= (|c_x|)_{x\in\Gamma_0} \\ &= d_\alpha^{\Theta}. \end{aligned}$$

This shows that $\{d_\alpha^{\Theta} : \alpha \in \Phi^+\} = \{d_\alpha^{\tilde{\Theta}} : \alpha \in \Phi^+\}$.

We may therefore complete the proof by checking that $\{d_\alpha^{\Psi} : \alpha \in \Phi^+\} = \{d_\alpha^{\tilde{\Theta}} : \alpha \in \Phi^+\}$.

Let T be the (invertible) orthogonal linear transformation of V defined by specifying $T(\gamma_x) = \varepsilon_x\delta_x$ for all $x \in \Gamma_0$, and extending linearly. We have already seen that T must permute the set of roots Φ (refer to Section 3.4 and in particular, Proposition 3.4.8).

Let $\alpha \in \Phi$. Then, we can write α uniquely in the form $\alpha = \sum_{x\in\Gamma_0} c_x\gamma_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. We have $T\alpha = \sum_{x\in\Gamma_0} c_x T(\gamma_x) = \sum_{x\in\Gamma_0} c_x(\varepsilon_x\delta_x)$, and thus $d_\alpha^{\Psi} = d_{T\alpha}^{\tilde{\Theta}}$. Because T permutes the set of roots Φ, it therefore follows that

$$\{d_\alpha^{\Psi} : \alpha \in \Phi\} = \{d_\alpha^{\tilde{\Theta}} : \alpha \in \Phi\}. \tag{4.1}$$

Also, we have $-\alpha = \sum_{x\in\Gamma_0}(-c_x)\gamma_x$ and so $d_\alpha^{\Psi} = d_{-\alpha}^{\Psi}$. Since $\Phi^- = -(\Phi^+)$, this shows that $\{d_\alpha^{\Psi} : \alpha \in \Phi^+\} = \{d_\alpha^{\Psi} : \alpha \in \Phi^-\}$. Therefore, $\{d_\alpha^{\Psi} : \alpha \in \Phi\} = \{d_\alpha^{\Psi} : \alpha \in \Phi^+\}$. Similarly, we may see that $\{d_\alpha^{\tilde{\Theta}} : \alpha \in \Phi\} = \{d_\alpha^{\tilde{\Theta}} : \alpha \in \Phi^+\}$.

It then follows from (4.1) that $\{d_\alpha^\Psi : \alpha \in \Phi^+\} = \{d_\alpha^{\widetilde{\Theta}} : \alpha \in \Phi^+\}$.

This completes the proof. ■

It follows immediately that in order to prove Conjecture 4.6.1, it is enough to prove the following.

CONJECTURE 4.6.3 *Let $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ be a cluster-tilted algebra of simply-laced Dynkin type, where T is a basic cluster-tilting object in $\mathcal{C}$. Suppose $(\mathbf{x}, B)$ is the seed corresponding to T in the cluster algebra $\mathcal{A}$, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. Then, there exists a strong companion basis for Γ. That is, there is some companion basis $\Psi \subseteq \Phi$ such that the vectors d_α^Ψ for $\alpha \in \Phi^+$ are precisely the dimension vectors of the finitely generated indecomposable Λ-modules.*

We conclude this section by mentioning two possible strategies for proving Conjecture 4.6.3.

One possibility would be to try to make use of the companion basis mutation procedure introduced in Chapter 3 (Theorem 3.1.4 in particular), to construct a strong companion basis for Γ.

Recall that there must be some seed $(\mathbf{x}_0, B_0)$ of $\mathcal{A}$ such that the Cartan counterpart $A(B_0)$ is a Cartan matrix of type Δ. Let T_0 be the corresponding basic cluster-tilting object in $\mathcal{C}$. Then, the cluster-tilted algebra $\Lambda_0 = \mathrm{End}_{\mathcal{C}}(T_0)^{\mathrm{op}}$ has quiver $\Gamma^0 = \Gamma(B_0)$ which is an orientation of Δ. Moreover, recall that the simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a strong companion basis for Γ^0 (for some indexing of the simple roots by the vertices of Γ^0).

By applying companion basis mutations to Π, we are able to obtain a companion basis for Γ. Therefore, if we can show that companion basis mutation preserves strongness, then it will follow that we can find a strong companion basis for Γ.

So, we may prove Conjecture 4.6.3, and hence also Conjecture 4.6.1, by proving that companion basis mutation preserves strongness.

As an alternative approach to proving Conjecture 4.6.3, we could try to give a procedure for explicitly constucting a strong companion basis for Γ. In Chapter 5, we will see that we can do this in the case where Λ is a cluster-tilted algebra of Dynkin type A, by making use of the description of the quivers of the cluster-tilted algebras of Dynkin type A given in Sections 4.2 and 4.3. As a consequence, this will in fact provide us with an alternative proof of Theorem 4.1.4.

Chapter 5

A More Explicit Approach

In this chapter, we again focus our attention on the Dynkin type A case. However, here we place particular emphasis on more explicit considerations than those of the previous chapter. In particular, we obtain an alternative (more explicit) proof of our main result, Theorem 4.1.4.

In Section 3.2, we presented a method for finding a companion basis for the quiver of any given cluster-tilted algebra of simply-laced Dynkin type. A weakness of this method is the following. In order to find a companion basis for the quiver of a given cluster-tilted algebra, we have to find a sequence of quiver mutations taking us from that quiver to another quiver for which we already have a companion basis (for example, a Dynkin quiver). Here, we show how to explicitly construct a strong companion basis for the quiver of any given cluster-tilted algebra of Dynkin type A. Of key importance is a procedure we introduce for labelling the vertices of any such quiver. Detailed consideration of these labelled quivers will reveal a number of useful properties, enabling us to give a simple method for constructing an explicit companion basis for each such quiver. The advantage of the method that we present here is that the desired companion bases may simply be read off from the labelled quivers.

We prove directly that these constructed explicit companion bases are strong. That this provides us with an alternative proof of Theorem 4.1.4 then follows from Proposition 4.6.2.

Throughout this chapter, we keep the following set-up.

Let k be an algebraically closed field and let Q be an alternating quiver whose underlying graph is a Dynkin diagram of type A_n, $n \in \mathbb{N}$. Let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ - \mathrm{mod})}{F}$$

be the corresponding cluster category. Suppose $\mathcal{A}$ is the cluster algebra of Dynkin type A_n. Also, suppose that $\Phi \subseteq V$ is the root system of Dynkin type A_n, where V is a Euclidean space with positive definite symmetric bilinear form $(\ ,\ )$. Finally, let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a simple system of Φ.

5.1 Preliminaries for the Labelling Procedure

Recall that the quivers of the cluster-tilted algebras of Dynkin type A_n are precisely the quivers associated to the triangulations of a regular $(n+3)$-gon. We have already seen that these quivers are connected and made up of linear sections (i.e. subquivers whose underlying graphs are Dynkin diagrams of type A) and cyclically oriented triangles. Vertices have valency four if and only if they lie at a point where two triangles meet, valency three if and only if they lie at a point where a linear section meets a triangle, valency one if and only if they lie at the end of a linear section (not meeting a triangle), and valency two otherwise. Also, the only cycles in these quivers are cyclically oriented triangles.

In this section, we continue the study of these quivers. In particular, we cover the preliminary results and introduce the terminology required in preparation for introducing our procedure for labelling the vertices of such quivers in the next section.

Let $\mathbb{P}_{n+3}$ be a regular $(n+3)$-gon. Let the vertices of $\mathbb{P}_{n+3}$ be $P_1, \ldots, P_{n+3}$, labelled in the anticlockwise direction, and suppose that the almost positive roots are identified with the diagonals of $\mathbb{P}_{n+3}$, as in Section 4.2.

Let $\mathbb{T}$ be a triangulation of $\mathbb{P}_{n+3}$ and suppose that $d \in \operatorname{diag}(\mathbb{T})$. Suppose also that d joins the vertices P_a and P_b of $\mathbb{P}_{n+3}$. We saw in Section 4.3 that d divides $\mathbb{P}_{n+3}$ into two polygons $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$ with respective triangulations $\mathbb{T}^{d^+}$ and $\mathbb{T}^{d^-}$ induced from the triangulation $\mathbb{T}$ of $\mathbb{P}_{n+3}$. We noted also that $d \notin \operatorname{diag}(\mathbb{T}^{d^+}), \operatorname{diag}(\mathbb{T}^{d^-})$, and that no diagonal in $\mathbb{T}^{d^+}$ bounds the same triangle (in $\mathbb{T}$) as any diagonal in $\mathbb{T}^{d^-}$. Therefore, if $Q_{\mathbb{T}^{d^+}}$ and $Q_{\mathbb{T}^{d^-}}$ are the quivers corresponding respectively to the triangulations $\mathbb{T}^{d^+}$ and $\mathbb{T}^{d^-}$ of $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$, then we see that $Q_{\mathbb{T}^{d^+}}$ and $Q_{\mathbb{T}^{d^-}}$ together form the full subquiver of $Q_{\mathbb{T}}$ on all vertices of $Q_{\mathbb{T}}$ except the vertex corresponding to $d \in \operatorname{diag}(\mathbb{T})$.

For $\varepsilon \in \{+,-\}$, let $\widetilde{\mathbb{P}}^{d^\varepsilon}_{n+3}$ be the convex polygon obtained from $\mathbb{P}^{d^\varepsilon}_{n+3}$ by adding an extra vertex x^{d^ε} and two extra boundary edges, one from P_a to x^{d^ε}, and the other from x^{d^ε} to P_b, so that d becomes an interior diagonal of $\widetilde{\mathbb{P}}^{d^\varepsilon}_{n+3}$. (Note that if $\mathbb{P}^{d^+}_{n+3}$ and $\mathbb{P}^{d^-}_{n+3}$ have l and m sides respectively, then $\widetilde{\mathbb{P}}^{d^+}_{n+3}$ and $\widetilde{\mathbb{P}}^{d^-}_{n+3}$ have $l+1$ and $m+1$ sides respectively.) By construction, we see that there are triangulations $\widetilde{\mathbb{T}}^{d^+}$ and $\widetilde{\mathbb{T}}^{d^-}$ of $\widetilde{\mathbb{P}}^{d^+}_{n+3}$ and $\widetilde{\mathbb{P}}^{d^-}_{n+3}$ respectively, defined by $\operatorname{diag}(\widetilde{\mathbb{T}}^{d^+}) = \operatorname{diag}(\mathbb{T}^{d^+}) \cup \{d\}$ and $\operatorname{diag}(\widetilde{\mathbb{T}}^{d^-}) = \operatorname{diag}(\mathbb{T}^{d^-}) \cup \{d\}$.

Let $Q_{\widetilde{\mathbb{T}}^{d+}}$ and $Q_{\widetilde{\mathbb{T}}^{d-}}$ be the quivers corresponding to these triangulations of $\widetilde{\mathbb{P}}^{d+}_{n+3}$ and $\widetilde{\mathbb{P}}^{d-}_{n+3}$. Both of these quivers have a vertex corresponding to $d \in \mathrm{diag}(\mathbb{T})$. It is then clear that the connected quiver obtained from $Q_{\widetilde{\mathbb{T}}^{d+}}$ and $Q_{\widetilde{\mathbb{T}}^{d-}}$ by identifying these two vertices is $Q_{\mathbb{T}}$.

We now introduce a definition, and make use of the above observations in the proof of the lemma which follows it.

DEFINITION 5.1.1 *Suppose that x, y and z are the three vertices of a triangle (3-cycle) in $Q_{\mathbb{T}}$. We define the section of $Q_{\mathbb{T}}$ above x to be the full subquiver of $Q_{\mathbb{T}}$ on all vertices that can be reached on unoriented paths starting at x which do not pass through y or z.*

Note: With x, y and z as in Definition 5.1.1 above, it is a simple observation that the section of $Q_{\mathbb{T}}$ above x, the section of $Q_{\mathbb{T}}$ above y, and the section of $Q_{\mathbb{T}}$ above z are pairwise disjoint.

LEMMA 5.1.2 *Let x, y and z be the three vertices of a triangle in $Q_{\mathbb{T}}$. Then, the section of $Q_{\mathbb{T}}$ above x arises as the quiver associated to a triangulation of a regular m-gon $\mathbb{P}_m$ for some $m \geq 4$.*

Proof Suppose that $d_x \in \mathrm{diag}(\mathbb{T})$ is the diagonal of $\mathbb{P}_{n+3}$ corresponding to the vertex x. Associated to the diagonal d_x, there are polygons $\widetilde{\mathbb{P}}^{d_x^+}_{n+3}$ and $\widetilde{\mathbb{P}}^{d_x^-}_{n+3}$ with triangulations $\widetilde{\mathbb{T}}^{d_x^+}$ and $\widetilde{\mathbb{T}}^{d_x^-}$ respectively. Furthermore, it is clear from above that the section of $Q_{\mathbb{T}}$ above x is given by either $Q_{\widetilde{\mathbb{T}}^{d_x^+}}$ or $Q_{\widetilde{\mathbb{T}}^{d_x^-}}$. The result follows. ■

In the following two definitions, we introduce some useful terminology distinguishing certain vertices in $Q_{\mathbb{T}}$.

DEFINITION 5.1.3 *We call any vertex of $Q_{\mathbb{T}}$ belonging to a cyclically oriented triangle a triangle vertex.*

DEFINITION 5.1.4 *A vertex of $Q_{\mathbb{T}}$ is said to be an end vertex if it has valency zero, valency one, or is a triangle vertex of valency two.*

So, the end vertices of $Q_{\mathbb{T}}$ are precisely those vertices corresponding to diagonals in $\mathbb{T}$ which bound a triangle in $\mathbb{T}$ of type (I). (Refer to Section 4.3 for a description of the types of triangles that may appear in any given triangulation of $\mathbb{P}_{n+3}$.)

Now, it is well known (and easy to show using a counting argument) that any triangulation of a regular polygon with at least four sides must contain at least two triangles of type (I). Also, no diagonal in a triangulation of a regular polygon with at least five sides can bound two triangles of type (I). Therefore, any quiver associated to a triangulation of a regular polygon with at least five sides must have at least two end vertices. Note that any triangulation of a regular 4-gon must consist of a single diagonal bounding two triangles of type (I), and so the associated quiver must consist of a solitary end vertex (of valency zero).

We conclude this section by introducing some terminology concerning shortest unoriented paths in $Q_{\mathbb{T}}$.

Let i and j be vertices in $Q_{\mathbb{T}}$ and consider the (unique) shortest unoriented path p in $Q_{\mathbb{T}}$ from i to j. It is clear that p does not pass through two consecutive arrows of any given triangle in $Q_{\mathbb{T}}$, and that none of the arrows appearing in p appear more than once. (Likewise, none of the vertices appearing in p appear more than once.)

Starting from i and moving along p towards j, we may pass through a number of triangle vertices. We make the following definitions.

DEFINITION 5.1.5 *For any triangle with two vertices appearing in* p*, we call the first vertex of that triangle appearing in* p *(when moving from* i *towards* j*) a left triangle vertex relative to* p*. We call the second vertex of that trianglc appearing in* p *a right triangle vertex relative to* p*, and we call the vertex of that triangle not appearing in* p *a top triangle vertex relative to* p*.*

DEFINITION 5.1.6 *Suppose* x*,* y *and* z *are the three vertices of some triangle in* $Q_{\mathbb{T}}$*, and suppose further that* x*,* y *and* z *are respectively left, right and top triangle vertices relative to* p*. Then we call* y *the right triangle vertex corresponding to* x *relative to* p*, and we call* z *the top triangle vertex corresponding to* x *relative to* p*.*

It is worth noting that since two triangles can meet in a vertex in $Q_{\mathbb{T}}$, then it is possible for a vertex to be a right triangle vertex relative to p with respect to one triangle, and a left triangle vertex relative to p with respect to another triangle.

5.2 The Labelling Procedure

Throughout the remainder of this chapter, we take Λ to be the cluster-tilted algebra of Dynkin type A_n given by $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, where T is a basic cluster-tilting object in $\mathcal{C}$. We suppose also that $(\mathbf{x}, B)$ is the seed of $\mathcal{A}$ corresponding to T, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. (We note that Γ is the quiver $Q_{\mathbb{T}}$ associated to some triangulation $\mathbb{T}$ of a regular $(n+3)$-gon $\mathbb{P}_{n+3}$.)

Our aim is to construct explicitly a strong companion basis for Γ. We will proceed as follows. Firstly, we will outline a procedure for labelling the vertices of the quiver Γ. After examining some properties of the labelled quiver Γ, we will then introduce a candidate companion basis for Γ. The proof that this candidate companion basis is indeed a companion basis for Γ will rely on these properties. Again making use of the properties of the labelled quiver Γ, we will be able to complete our aim by proving (independently of the proof of Theorem 4.1.4) that the companion basis for Γ that we have constructed is strong.

We will now outline a procedure for labelling the vertices of Γ using the labels $1, \dots, n$.

In the previous section, we saw that for $n \geq 2$, Γ must have at least two distinct end vertices.

Consider Γ together with a choice of an ordered pair of end vertices, distinct if possible. We will now outline a procedure for labelling the vertices of Γ, given this initial choice of end vertices, by induction on the number n of vertices of Γ.

If Γ has a single vertex (i.e. if $n = 1$), then we label that vertex 1, and the labelling is complete.

Fix $k \geq 2$ and suppose that in the cases where $n < k$, we have labelled the vertices of all possible choices for Γ, for any given initial choice of an ordered pair of end vertices in Γ (distinct if possible).

Suppose now that $n = k$ and that we have chosen an ordered pair of distinct end vertices in Γ. Label the first vertex of this ordered pair 1, and consider the shortest unoriented path p in Γ from 1 to the other chosen end vertex. Starting from 1, move along p, labelling subsequent vertices consecutively $2, 3, 4, \dots,$ up to and including the first left triangle vertex i relative to p. (Note that 1 could be a left triangle vertex relative to p. Note also that there may be no left triangle vertices relative to p, in which case, the labelling procedure ends here.)

Denote the section of Γ above the top triangle vertex corresponding to i relative to p by Γ'. As a consequence of Lemma 5.1.2, it follows that Γ' is the quiver of a cluster-tilted algebra of Dynkin type A. Suppose that there are a vertices in Γ'. We have that the top triangle vertex corresponding to i relative to p is an end vertex in Γ'. We then obtain an ordered pair of end vertices in Γ' by choosing another end vertex in Γ', distinct if possible (i.e. if $a > 1$), and setting this chosen end vertex to be the first vertex in the ordering.

By induction, we have a labelling of the vertices of Γ' using the labels 1 up to a, given this choice of an ordered pair of end vertices in Γ'. Add i to each of the vertex labels in this labelling for Γ', and then assign the labels thus obtained to the corresponding vertices in Γ.

Label the right triangle vertex corresponding to i relative to p with $i + a + 1$. Then, from $i + a + 1$, continue along p labelling subsequent vertices consecutively $i+a+2, i+a+3, \dots$, and proceed as above for each subsequent left triangle vertex relative to p. The second of our initially chosen end vertices of Γ will be the last vertex to be labelled, and will be labelled n.

Suppose the vertex labelled j is a left triangle vertex relative to some path p considered in the labelling procedure, and that there are b vertices in the section of Γ above the top triangle vertex corresponding to j relative to p. Then, due to the inductive nature of the labelling procedure, we see that the top triangle vertex corresponding to j relative to p will be labelled $j + b$. Also, by construction, the right triangle vertex corresponding to j relative to p will be labelled $j + b + 1$.

Note: To label (the vertices of) Γ according to the above procedure, we initially choose two end vertices (distinct if $n \geq 2$) in Γ. But we also make a further choice of an end vertex for each left triangle vertex relative to some path considered in the labelling procedure. Because of these choices, there are potentially many different labellings of Γ that can be obtained using the outlined procedure. This is not a problem however, as all of the labellings of Γ that may be obtained using the outlined procedure share the important properties that we will need to enable us to explicitly construct a strong companion basis for Γ. The focus of the next section will be on studying these properties.

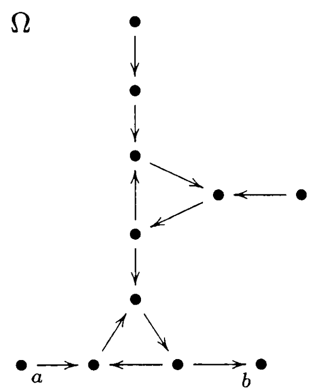
The labelling of Γ that we obtain using the above procedure may seem a little unnatural. However, it is a labelling that can be obtained by constructing Γ from the Dynkin quiver of type A_n (oriented and labelled as shown)

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n-1 \longleftarrow n$$

by applying quiver mutations.

We conclude this section by giving a detailed example showing the labelling procedure in action.

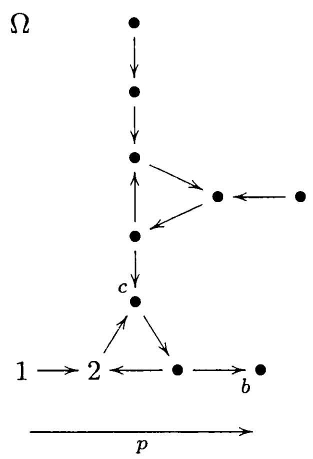
EXAMPLE 5.2.1 Let Ω be the following quiver.



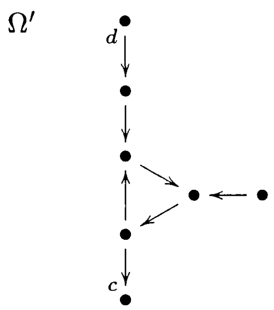
It is easily checked that Ω arises from a triangulation of a regular 14-gon, and is therefore the quiver of a cluster-tilted algebra of Dynkin type A_{11}. (Note that we don't need to worry about relations, as they play no role in the labelling procedure.)

We will use the prescribed labelling procedure to obtain a labelling of (the vertices of) Ω.

The first step is to choose an ordered pair (a, b) of end vertices in Ω (as shown above). We label the first vertex in this pair 1, and then consider the shortest unoriented path p in Ω from 1 to the other chosen end vertex. Starting from 1 and moving along p, labelling subsequent vertices in increments of one, we have that 2 is the first left triangle vertex relative to p.

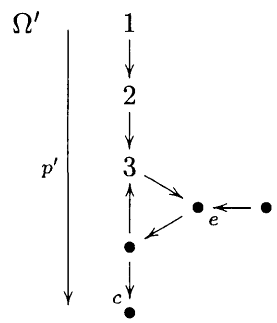


We must now consider the section Ω' of Ω above the top triangle vertex c corresponding to 2 relative to p.

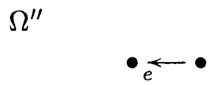


(Note that Ω' arises from a triangulation of a regular 10-gon.)

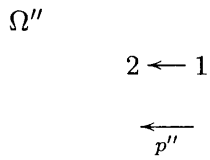
Since c is an end vertex in Ω', by choosing another end vertex d in Ω', we obtain an ordered pair (d, c) of end vertices. We now start to label the vertices of Ω', proceeding in the same manner as above. We label the first vertex of our ordered pair 1, and consider the shortest unoriented path p' in Ω' from 1 to the other chosen end vertex. Starting from 1 and moving along p', labelling subsequent vertices in increments of one, we have that 3 is the first left triangle vertex relative to p'.



We must now consider the section Ω'' of Ω' above the top triangle vertex e corresponding to 3 relative to p'.



By following the labelling procedure, we obtain the following labelling of the vertices of Ω''.



Having now completed the labelling of Ω'', we add 3 to each of the labels of the vertices of Ω'', and assign the labels thus obtained to the corresponding vertices in

Ω'.

$$\begin{tikzcd}
\Omega' & & & \\
{} \arrow[ddddd, "p'"] & 1 \arrow[d] & & \\
 & 2 \arrow[d] & & \\
 & 3 \arrow[dr] & & \\
 & & 5 \arrow[dl] & 4 \arrow[l] \\
 & \bullet \arrow[uu] \arrow[d, "c"'] & & \\
{} & \bullet & &
\end{tikzcd}$$

We label the right triangle vertex corresponding to 3 relative to p' with the label 6. Then, starting from 6, we proceed along p' labelling subsequent vertices in increments of one. This gives us the following labelling of the vertices of Ω'.

$$\begin{tikzcd}
\Omega' & 1 \arrow[d] & & \\
 & 2 \arrow[d] & & \\
 & 3 \arrow[dr] & & \\
 & & 5 \arrow[dl] & 4 \arrow[l] \\
 & 6 \arrow[uu] \arrow[d] & & \\
 & 7 & &
\end{tikzcd}$$

We now add 2 to each of the labels of the vertices of Ω', and assign the labels thus obtained to the corresponding vertices in Ω.

$$\begin{tikzcd}
\Omega & & 3 \arrow[d] & & \\
 & & 4 \arrow[d] & & \\
 & & 5 \arrow[dr] & & \\
 & & & 7 \arrow[dl] & 6 \arrow[l] \\
 & & 8 \arrow[uu] \arrow[d] & & \\
 & & 9 \arrow[dr] & & \\
1 \arrow[r] & 2 \arrow[ur] & & \bullet \arrow[ll] \arrow[r, "b"'] & \bullet \\
{} \arrow[rrrr, "p"'] & & & & {}
\end{tikzcd}$$

Finally, by labelling the right triangle vertex corresponding to 2 relative to p with the label 10, and then proceeding (from 10) along p, we complete the labelling of Ω.

$$
\begin{tikzcd}
 & \Omega & 3 \arrow[d] & & \\
 & & 4 \arrow[d] & & \\
 & & 5 \arrow[dr] & & \\
 & & & 7 \arrow[dl] & 6 \arrow[l] \\
 & & 8 \arrow[uu] \arrow[d] & & \\
 & & 9 \arrow[dr] & & \\
1 \arrow[r] & 2 \arrow[ur] & & 10 \arrow[ll] \arrow[r] & 11
\end{tikzcd}
$$

5.3 Properties of the Labelled Quiver

For the purposes of this section, we suppose that the vertices of the quiver Γ have been labelled according to the labelling procedure outlined above. In fact, throughout the remainder of this chapter, whenever we refer to the quiver of any cluster-tilted algebra of Dynkin type A, we will automatically suppose that its vertices have been labelled according to the outlined labelling procedure. Also, by a minor abuse of notation, we will usually refer to the vertices of that quiver by their labels, whilst still regarding the labels as numerical values.

The main focus of this section is on examining some of the properties of the labelled quiver Γ. In particular, we prove two results regarding the vertex labels of the shortest unoriented paths in Γ (from vertices labelled i to vertices labelled j, with $i \leq j$), which in fact classify these shortest unoriented paths into two distinct types. These results will subsequently be very useful. They will help us firstly to find an explicit companion basis for Γ, and then moreover, to give a direct proof that this companion basis for Γ must be strong.

During the procedure for labelling the vertices of Γ, a number of shortest unoriented paths in Γ are considered. In fact, these shortest unoriented paths determine the given labelling of the vertices of Γ. We therefore call them the labelling paths for the given labelling of (the vertices of) Γ.

We will now describe these labelling paths for the given labelling of Γ.

Recall that the first step in the labelling procedure is to consider a shortest unoriented path joining two initially chosen end vertices of Γ. In the labelled quiver Γ, this is the shortest unoriented path from 1 to n. We call this labelling path the 0-labelling path for Γ.

Due to the inductive nature of the labelling procedure, it is a simple matter to describe the other labelling paths for the given labelling of Γ. We will call these other labelling paths m-labelling paths, where for a given labelling path, $m \in \mathbb{N}$ depends on how "close" that labelling path is to the 0-labelling path for Γ. For $m \in \mathbb{N}$, we define m-labelling paths by using $(m-1)$-labelling paths. (We are able to do this, given that we have already introduced the 0-labelling path for Γ.)

Let $m \in \mathbb{N}$ and suppose that i is a left triangle vertex relative to an $(m-1)$-labelling path p for Γ. Suppose also that there are a vertices in the section of Γ above the top triangle vertex corresponding to i relative to p. Due to the inductive nature of the labelling procedure, we then have that the shortest unoriented path in Γ from $i+1$ to $i+a$ is a labelling path for Γ. We call this labelling path an m-labelling path for Γ. Note that the vertex labelled $i+a$ is the top triangle vertex corresponding to i relative to p, and the vertex labelled $i+1$ is another end vertex in the section of Γ above this top triangle vertex.

It is clear that every vertex of Γ lies on exactly one labelling path. Furthermore, for any given triangle in Γ, it is easily seen that there is a unique labelling path which passes through exactly two vertices of that triangle.

Fix an arbitrary triangle in Γ and let p be the labelling path which passes through two vertices of that triangle. Relative to p, we may consider the vertices of the given triangle as a left, a right and a top triangle vertex.

Suppose that the left triangle vertex relative to p is labelled j and that there are b vertices in the section of Γ above the top triangle vertex corresponding to j relative to p. The top and right triangle vertices corresponding to j relative to p must then be labelled $j+b$ and $j+b+1$ respectively. In particular, the label of the left triangle vertex relative to p is lower than the label of the top triangle vertex relative to p, which in turn is lower than the label of the right triangle vertex relative to p.

Since this holds for every triangle in Γ, we make the following definition, in which reference to specific labelling paths for Γ is dropped.

DEFINITION 5.3.1 *If the vertices of a triangle in the labelled quiver Γ have labels i, j, k with $i < j < k$, then we call i a left triangle vertex in Γ, j a top triangle vertex in Γ, and k a right triangle vertex in Γ.*

Furthermore, we call j the top triangle vertex in Γ corresponding to i, and we call k the right triangle vertex in Γ corresponding to i.

In the situation where two triangles meet in a vertex, we see that that vertex can be both a right triangle vertex in Γ and either a left or a top triangle vertex in Γ.

The following consequence of Definition 5.3.1 is immediate.

COROLLARY 5.3.2 *A given vertex of* Γ *is a left (resp. top, right) triangle vertex in* Γ *if and only if it is a left (resp. top, right) triangle vertex relative to some labelling path for the given labelling of* Γ.

We make the following definitions.

DEFINITION 5.3.3 *We call the shortest unoriented path in* Γ *from* 1 *to* n *the trunk of* Γ.

DEFINITION 5.3.4 *Suppose that* i *is a left triangle vertex in* Γ *and suppose that there are* a *vertices in the section of* Γ *above the top triangle vertex in* Γ *corresponding to* i. *Then, that top triangle vertex is labelled* $i + a$, *and we call the shortest unoriented path in* Γ *from* $i + 1$ *to* $i + a$ *a branch of* Γ. *(In the case where* $a = 1$, *we call the branch a trivial branch of* Γ.*)*

Also, we call the triangle on vertices $i, i + a, i + a + 1$ *the base triangle of the branch.*

Note: We have that the trunk of Γ is the 0-labelling path for Γ. Also, the branches of Γ are the m-labelling paths for Γ, for $m \geq 1$.

We now take a closer look at some of the properties of the labelled quiver Γ, starting with the following remark.

REMARK 5.3.5 *We note that no vertex of* Γ *can be both a top triangle vertex in* Γ *and a left triangle vertex in* Γ. *Indeed, given any top triangle vertex in* Γ, *we see (by construction) that there is a branch of* Γ *starting at some end vertex of* Γ *and ending at that top triangle vertex in* Γ. *By definition, the given top triangle vertex in* Γ *cannot be a left triangle vertex relative to this branch. However, the given top triangle vertex in* Γ *lies on no other labelling path for* Γ.

Consider the shortest unoriented path in Γ from 1 to n (i.e. the trunk of Γ). By construction, as we move along this path from 1 towards n, the labels of subsequent vertices strictly increase. In fact, if k lies on this path and is not a left triangle vertex in Γ, then the vertex following k is $k+1$. Whereas, if k lies on this path and is a left triangle vertex in Γ, then the vertex following k is $k+a+1$ where a is the number of vertices in the section of Γ above the top triangle vertex corresponding to k.

Due to the inductive nature of the labelling procedure, we have that each of the branches of Γ satisfies properties analogous to these.

Making use of these properties, we obtain the following useful result.

PROPOSITION 5.3.6 *Let i be a vertex in Γ and consider the shortest unoriented path p in Γ from i to n. As we move along this path from i towards n, the labels of subsequent vertices strictly increase. If j lies on p and is not a left triangle vertex in Γ, then the vertex following j is $j+1$. Also, if j lies on p and is a left triangle vertex in Γ, then the vertex following j is $j+a+1$, where a is the number of vertices in the section of Γ above the top triangle vertex corresponding to j.*

Proof We proceed by (reverse) induction on the vertex labels.

As an initial case, we see that the result is clear for the shortest unoriented path in Γ from n to n.

For our induction hypothesis, we assume that the result holds for all vertices l with $n \geq l > i$.

From the comments made prior to stating this proposition, we see that if i lies on the shortest unoriented path in Γ from 1 to n, then the result holds.

So, suppose that i lies on a branch. Suppose that the left triangle vertex of the base triangle of this branch is m, and suppose that there are c vertices in the section of Γ above the top triangle vertex in Γ corresponding to m. Then, i lies on the shortest unoriented path in Γ from $m+1$ to $m+c$, and the vertices of the base triangle of this branch are m, $m+c$ and $m+c+1$.

Again due to the comments made prior to stating this proposition, we see that as we travel along the shortest unoriented path in Γ from i to $m+c$, the labels of subsequent vertices strictly increase. Also, if j' lies on this path and is not a left triangle vertex in Γ, then the vertex following j' is $j'+1$. And if j' lies on this path and is a left triangle vertex in Γ, then the vertex following j' is $j'+a'+1$, where a' is the number of vertices in the section of Γ above the top triangle vertex corresponding to j'.

Also, by the induction hypothesis, we have that the shortest unoriented path in Γ from $m+c+1$ to n satisfies the required result.

It therefore follows that the unoriented path p' in Γ constructed by combining the shortest unoriented path in Γ from i to $m+c$, the shortest unoriented path (of length one) in Γ between $m+c$ and $m+c+1$, and the shortest unoriented path in Γ between $m+c+1$ and n, has the desired properties (since the vertex labelled $m+c$ cannot be a left triangle vertex in Γ). So, all that remains to be checked is that p' is the shortest unoriented path in Γ from i to n (i.e. that $p'=p$).

It is clear that p' is an unoriented path in Γ from i to n. Suppose x $(m<x<m+c)$ is the vertex preceeding $m+c$ in p', and suppose y $(y>m+c+1)$ is the vertex following $m+c+1$ in p'. It is enough to check that x, $m+c$ and $m+c+1$ are not the three vertices of a triangle in Γ, and that $m+c$, $m+c+1$ and y are not the three vertices of a triangle in Γ (whenever such vertices x and y exist).

In both cases this is clear, since m, $m+c$ and $m+c+1$ are the three vertices of the base triangle for the branch containing i, and no two triangles in Γ can share an arrow.

Therefore, p' is the shortest unoriented path in Γ from i to n, and so the result holds. ■

It is clear that we have the following corollary of Proposition 5.3.6.

COROLLARY 5.3.7 *Let i be a vertex in Γ and consider the shortest unoriented path p in Γ from i to n. If j lies on p and is a left triangle vertex in Γ, then the vertex following j (on p) is the corresponding right triangle vertex in Γ.*

We now turn our attention to shortest unoriented paths in Γ from vertices i to j with $i \leq j$, and with the aid of Proposition 5.3.6, we obtain another useful result.

PROPOSITION 5.3.8 *Let i and j be vertices in Γ, $i \leq j$, and suppose p is the shortest unoriented path in Γ from i to j. Then, there can be at most one instance of a left triangle vertex in Γ lying on p being followed by the corresponding top triangle vertex in Γ.*

Proof By Corollary 5.3.7, no left triangle vertex in Γ lying on the shortest unoriented path in Γ from i to n can be followed (on that path) by the corresponding top triangle vertex in Γ. So, if j lies on the shortest unoriented path in Γ from i to n, then no left triangle vertex in Γ lying on p is followed (on p) by the corresponding top triangle vertex in Γ.

Suppose j does not lie on the shortest unoriented path in Γ from i to n. Since $j \geq i$, there must be some left triangle vertex k in Γ, lying on the shortest unoriented path in Γ from i to n, such that j lies in the section of Γ above the top triangle vertex in Γ corresponding to k.

Note that k lies on the shortest unoriented path in Γ from i to n, so no left triangle vertex in Γ lying on the shortest unoriented path in Γ from i to k can be followed by the corresponding top triangle vertex in Γ.

Suppose there are a vertices in the section of Γ above the top triangle vertex corresponding to k. Then the top triangle vertex in Γ corresponding to k is labelled $k + a$, and the right triangle vertex in Γ corresponding to k is labelled $k + a + 1$.

We get an unoriented path in Γ from j to n by combining the shortest unoriented path in Γ from j to $k + a$, the shortest unoriented path (of length one) in Γ from $k + a$ to $k + a + 1$, and the shortest unoriented path in Γ from $k + a + 1$ to n. It is easily checked that this path is in fact the shortest unoriented path in Γ from j to n. Therefore, by the result of Proposition 5.3.6, we have that as we travel along the shortest unoriented path in Γ from j to $k + a$, the labels of subsequent vertices strictly increase.

So, as we travel along the shortest unoriented path in Γ from $k + a$ to j, the labels of subsequent vertices will strictly decrease. As an immediate consequence of this we see that no left triangle vertex in Γ lying on the shortest unoriented path in Γ from $k + a$ to j can be followed by the corresponding top triangle vertex in Γ.

Now, the shortest unoriented path in Γ from i to j is the path obtained by combining the shortest unoriented path in Γ from i to k, the shortest unoriented path (of length one) in Γ from k to $k + a$, and the shortest unoriented path in Γ from $k + a$ to j. So, if j doesn't lie on the shortest unoriented path in Γ from i to n, then exactly one left triangle vertex in Γ lying on p is followed by the corresponding top triangle vertex in Γ.

Therefore, we see that there can be at most one instance of a left triangle vertex in Γ lying on p being followed by the corresponding top triangle vertex in Γ. ∎

REMARK 5.3.9 *Suppose i and j are vertices in Γ with $i \leq j$. From the proofs of Propositions 5.3.6 and 5.3.8, we see that the label of every vertex lying on the shortest unoriented path p in Γ from i to j must be greater than or equal to i. Moreover, if some left triangle vertex k in Γ lying on p is followed by the corresponding top triangle vertex, labelled $k + a$, where a is the number of vertices in the section of Γ above that top triangle vertex, then the label of every vertex lying on p must also be less than or equal to $k + a$. On top of this, the vertex $k + a$ must lie on the shortest unoriented path in Γ from j to n, and so no left triangle vertex in Γ lying on the shortest unoriented path in Γ from j to $k + a$ can be followed by the corresponding top triangle vertex.*

5.4 An Explicit Companion Basis for Γ

Recall that we want to construct explicitly a strong companion basis for (the labelled quiver) Γ. We will firstly specify a positive quasi-Cartan companion of B, and then introduce a candidate companion basis for Γ, defined using the labelling of the vertices of Γ. In order to prove that this candidate companion basis is indeed a companion basis for Γ, we start by showing that it gives rise to the specified positive quasi-Cartan companion of B, before showing that it is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$. The proof that the constructed companion basis is strong will be given in the next section.

Constructing a positive quasi-Cartan companion of B is a simple task.

DEFINITION 5.4.1 *Let $A_\Gamma = (a_{ij})$ be the matrix whose entries are given as follows: Suppose i and j are vertices of Γ, $i \neq j$.*

(i) If i and j are joined by an arrow in Γ, i is a left triangle vertex in Γ and j is the corresponding right triangle vertex in Γ, we set $a_{ij} = a_{ji} = 1$.
(ii) Otherwise, if i and j are joined by an arrow in Γ, we set $a_{ij} = a_{ji} = -1$.
(iii) If i and j are not joined by an arrow in Γ, we set $a_{ij} = a_{ji} = 0$.
(iv) For all vertices k in Γ, we set $a_{kk} = 2$.

LEMMA 5.4.2 *The matrix A_Γ introduced in Definition 5.4.1 is a positive quasi-Cartan companion of B.*

Proof By construction, we have that A_Γ is a quasi-Cartan companion of B. Hence, it just remains to establish the positivity of A_Γ.

Treating the matrix A_Γ as an assignment of signs to the arrows of Γ (in the natural way), we see that each triangle in Γ has exactly one (an odd number) arrow assigned positive sign. Therefore, since B is a matrix appearing in a seed of the cluster algebra $\mathcal{A}$ of (finite) Dynkin type A_n, we see by Theorem 1.4.5 and Propositions 1.4.6 and 1.4.7 that the matrix A_Γ must be positive. ■

We now introduce a candidate companion basis $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ for Γ. (Recall that $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a simple system of Φ.)

DEFINITION 5.4.3 *For each vertex i in the quiver Γ, let m_i be the number of left triangle vertices appearing in the shortest unoriented path in Γ from i to n. If i is not a left triangle vertex in Γ, then set $\beta_i = (-1)^{m_i}\alpha_i$. If i is a left triangle vertex in Γ, then set $\beta_i = (-1)^{m_i}(\alpha_i + \ldots + \alpha_{i+a})$, where a is the number of vertices in the section of Γ above the top triangle vertex corresponding to i.*

Notice that for each i, β_i is a root, since it is plus or minus a sum of consecutive simple roots.

The following simple lemma will help us to prove that our candidate companion basis for Γ gives rise to the matrix A_Γ. In other words, that $(\beta_i, \beta_j) = a_{ij}$ for all $1 \leq i, j \leq n$.

LEMMA 5.4.4 *Let $i \neq n$ be a vertex in Γ.*

(i) If i is not a left triangle vertex in Γ, then β_i is positive (resp. negative) if and only if β_{i+1} is positive (resp. negative).

(ii) If i is a left triangle vertex in Γ, and there are a vertices in the section of Γ above the corresponding top triangle vertex, then β_i is positive (resp. negative) if and only if β_{i+a} and β_{i+a+1} are negative (resp. positive).

Proof (i) Suppose i is not a left triangle vertex in Γ. By Proposition 5.3.6 we see that i is followed (immediately) by $i+1$ on the shortest unoriented path in Γ from i to n. As an immediate consequence of this we see that β_i is positive (resp. negative) if and only if β_{i+1} is positive (resp. negative).

(ii) On the other hand, suppose that i is a left triangle vertex in Γ, and that there are a vertices in the section of Γ above the top triangle vertex corresponding to i. Then, the top and right triangle vertices in Γ corresponding to i will be labelled $i+a$

and $i+a+1$ respectively. Again by Proposition 5.3.6, we see that i is followed by $i+a+1$ on the shortest unoriented path in Γ from i to n, and that $i+a$ is followed by $i+a+1$ on the shortest unoriented path in Γ from $i+a$ to n (noting that $i+a$ cannot be a left triangle vertex in Γ). Therefore, we see that β_i is positive (resp. negative) if and only if β_{i+a} and β_{i+a+1} are negative (resp. positive). ■

PROPOSITION 5.4.5 *With $A_\Gamma = (a_{ij})$ and $\{\beta_1, \dots, \beta_n\}$ as given in Definitions 5.4.1 and 5.4.3 respectively, we have $a_{ij} = (\beta_i, \beta_j)$ for all $1 \leq i, j \leq n$.*

Proof Suppose i is a left triangle vertex in Γ. Suppose also that there are $a > 1$ vertices in the section of Γ above the top triangle vertex in Γ corresponding to i. Then, the said top triangle vertex will be labelled $i+a$ and we must have $\beta_i = \pm(\alpha_i + \ldots + \alpha_{i+a})$.

If $j \neq i+a$ is a vertex in the section of Γ above $i+a$ (i.e. if $i+1 \leq j < i+a$), then by construction we see that $\beta_j = \pm(\alpha_j + \ldots + \alpha_r)$, where $r < i+a$. Therefore, since $(\alpha_i + \ldots + \alpha_{i+a}, \alpha_j + \ldots + \alpha_r) = 0$, we see that $(\beta_i, \beta_j) = 0 = a_{ij}$ in this case.

By Lemma 5.4.4, we see that if β_i is a positive (resp. negative) root, then β_{i+a} and β_{i+a+1} are negative (resp. positive) roots.

The top triangle vertex $i+a$ cannot be a left triangle vertex in Γ (by Remark 5.3.5), and so $\beta_{i+a} = \pm\alpha_{i+a}$. Therefore, since $(\alpha_i + \ldots + \alpha_{i+a}, -\alpha_{i+a}) = -1$, we see that $(\beta_i, \beta_{i+a}) = -1 = a_{i,i+a}$.

Also, $\beta_{i+a+1} = \pm(\alpha_{i+a+1} + \ldots + \alpha_{i+a+1+c})$, for some $c \geq 0$, with $c = 0$ if $i+a+1$ is not a left triangle vertex in Γ. Therefore, since $(\alpha_i + \ldots + \alpha_{i+a}, -\alpha_{i+a+1} - \ldots - \alpha_{i+a+1+c}) = 1$, we see that $(\beta_i, \beta_{i+a+1}) = 1 = a_{i,i+a+1}$.

For $j > i + a + 1$, it is clear that we have $(\beta_i, \beta_j) = 0 = a_{ij}$.

In the $a = 1$ case, we have that i is a left triangle vertex in Γ, and that the corresponding top and right triangle vertices are labelled $i+1$ and $i+2$ respectively. Furthermore, we have $\beta_i = \pm(\alpha_i + \alpha_{i+1})$, $\beta_{i+1} = \mp\alpha_{i+1}$ and $\beta_{i+2} = \mp(\alpha_{i+2} + \ldots + \alpha_{i+2+c})$, for some $c \geq 0$, with $c = 0$ if $i + 2$ is not a left triangle vertex in Γ. Therefore, $(\beta_i, \beta_{i+1}) = -1 = a_{i,i+1}$ and $(\beta_i, \beta_{i+2}) = 1 = a_{i,i+2}$. Also, for $j > i + 2$ it is clear that $(\beta_i, \beta_j) = 0 = a_{ij}$.

So, if i is a left triangle vertex in Γ, we have shown that $(\beta_i, \beta_j) = a_{ij}$ for all $j > i$.

Now suppose i is not a left triangle vertex in Γ. So, $\beta_i = \pm\alpha_i$. Then, $\beta_{i+1} = \pm(\alpha_{i+1} + \ldots + \alpha_{i+1+c})$, for some $c \geq 0$, with $c = 0$ if $i + 1$ is not a left triangle vertex in Γ. Also, β_{i+1} must have the same sign as β_i by Lemma 5.4.4. Therefore, $(\beta_i, \beta_{i+1}) = -1 = a_{i,i+1}$. For $j \geq i + 2$, it is clear that $(\beta_i, \beta_j) = 0 = a_{ij}$.

Therefore, for every vertex i, we have shown that $(\beta_i, \beta_j) = a_{ij}$ for all $j > i$. Furthermore, since $a_{ij} = a_{ji}$ for all i and j, and $(\beta_i, \beta_j) = (\beta_j, \beta_i)$ for all i and j, we have $(\beta_i, \beta_j) = a_{ij}$ for all $i \neq j$.

Finally, since $(\beta_i, \beta_i) = 2 = a_{ii}$ for all $1 \leq i \leq n$, we see that $a_{ij} = (\beta_i, \beta_j)$ for all $1 \leq i, j \leq n$. ∎

With the following result, we establish that the candidate companion basis $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ is indeed a companion basis for Γ.

PROPOSITION 5.4.6 *The set $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ as obtained from Definition 5.4.3 is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi$.*

Proof It is clear from the construction that $\{\beta_1, \ldots, \beta_n\}$ is a linearly independent set. (Note that for each i, β_i is of the form $\beta_i = \pm(\alpha_i + \ldots + \alpha_{i+c})$, with $c \geq 0$. Therefore, we see that the $n \times n$ matrix whose (i,j)-entry is defined to be the coefficient of α_j in β_i is upper triangular.)

In order to complete the proof, it is enough to show that each simple root $\alpha_i \in \Pi$ can be written as an integral linear combination of $\{\beta_1, \ldots, \beta_n\}$. This being the case, it then follows immediately that the $\mathbb{Z}$-span of $\{\beta_1, \ldots, \beta_n\}$ must contain $\mathbb{Z}\Phi$. Furthermore, since $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$, the $\mathbb{Z}$-span of $\{\beta_1, \ldots, \beta_n\}$ must in fact be equal to $\mathbb{Z}\Phi$.

If i is not a left triangle vertex in Γ, then $\beta_i = \pm\alpha_i$ and so $\alpha_i = \pm\beta_i$.

Suppose now that i is a left triangle vertex in Γ. If there are $a \geq 1$ vertices in the section of Γ above the top triangle vertex corresponding to i, then $\beta_i = \pm(\alpha_i + \ldots + \alpha_{i+a})$, and so $\alpha_i = \pm\beta_i - \alpha_{i+1} - \ldots - \alpha_{i+a}$.

Now, consider the shortest unoriented path in Γ from $i+1$ to $i+a$ (the top triangle vertex in Γ corresponding to i). Starting from $i+1$ and moving towards $i+a$, suppose that the vertices appearing in this path are respectively $i_1, i_2, \ldots, i_t$ (so $i_1 = i+1$ and $i_t = i+a$).

Consider i_k, $1 \leq k < t$. If i_k is not a left triangle vertex in Γ, then $\beta_{i_k} = \pm\alpha_{i_k}$, and $i_{k+1} = i_k + 1$. If i_k is a left triangle vertex in Γ, then by construction, $\beta_{i_k} = \pm(\alpha_{i_k} + \alpha_{i_k+1} + \ldots + \alpha_{i_{k+1}-1})$. Also, $\beta_{i_t} = \pm\alpha_{i+a}$.

Therefore, we see that $\beta_{i_1} + \ldots + \beta_{i_t} = \sum_{j=i+1}^{i+a} c_j\alpha_j$ with $c_j = \pm 1$ for all j. Moreover, we can write $\alpha_{i+1} + \ldots + \alpha_{i+a} = \sum_{j=1}^{t} b_j\beta_{i_j}$ with $b_j = \pm 1$ for all j.

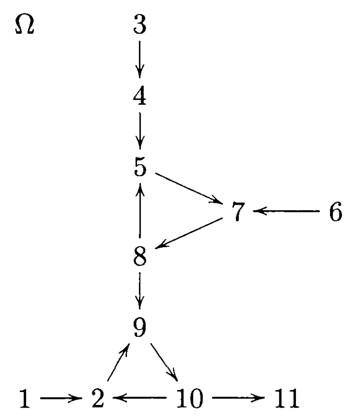
Therefore, $\alpha_i = \pm\beta_i - \sum_{j=1}^{t} b_j \beta_{i_j}$ where $i_1, \ldots, i_t$ are the vertices of the shortest unoriented path in Γ from $i+1$ to $i+a$.

This completes the proof. ■

We have now proved that the following result holds.

COROLLARY 5.4.7 *The set* $\{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ *as obtained from Definition 5.4.3 is a companion basis for* Γ.

EXAMPLE 5.4.8 Here, we construct a companion basis for the labelled quiver Ω considered in Example 5.2.1.



As noted previously, the quiver Ω arises from a triangulation of a regular 14-gon and is therefore the quiver of a cluster-tilted algebra of Dynkin type A_{11}. So, by applying Corollary 5.4.7, we are able to obtain a companion basis for Ω.

Take Φ to be the root system of Dynkin type A_{11}, with simple system $\Pi = \{\alpha_1, \ldots, \alpha_{11}\}$.

We start by noting that 2 and 5 are the only left triangle vertices in Ω.

Consider the vertex labelled 1. We have that 1 is not a left triangle vertex in Ω and that 2 is the only left triangle vertex in Ω lying on the shortest unoriented path from 1 to 11. Therefore, we set $\beta_1 = -\alpha_1$.

Now consider the vertex labelled 2. This vertex is a left triangle vertex in Ω, and is the only left triangle vertex in Ω lying on the shortest unoriented path from 2 to 11. We have that the top triangle vertex in Ω corresponding to 2 is 9. Therefore, as there are seven vertices in the section of Ω above 9, we set $\beta_2 = -(\alpha_2 + \cdots + \alpha_9)$.

Continuing in this way, we also obtain $\beta_3 = -\alpha_3$, $\beta_4 = -\alpha_4$, $\beta_5 = -(\alpha_5 + \alpha_6 + \alpha_7)$, $\beta_6 = \alpha_6$, $\beta_7 = \alpha_7$, $\beta_8 = \alpha_8$, $\beta_9 = \alpha_9$, $\beta_{10} = \alpha_{10}$ and $\beta_{11} = \alpha_{11}$.

We have that $\{\beta_1, \ldots, \beta_{11}\} \subseteq \Phi$ is a companion basis for Ω.

(We note that a routine check establishes that the constructed companion basis for Ω does indeed give rise to the matrix A_Ω that we obtain from Definition 5.4.1.)

5.5 The Constructed Companion Basis for Γ is Strong: A Direct Proof

Theorem 4.1.4 tells us that all companion bases for Γ must be strong. In particular, the (explicit) companion basis $\Upsilon = \{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ for Γ that we constructed in the previous section must be strong.

Recall that in Section 4.4 we established, for each companion basis for Γ, a bijective correspondence between the set of shortest unoriented paths in Γ (identifying each shortest unoriented path with its reverse) and the set of positive roots, depending on that companion basis. This was in fact one of the key steps in the proof of Theorem 4.1.4.

In this section, we construct this bijective correspondence explicitly for the companion basis Υ. Furthermore, we do this independently of the general construction given in Section 4.4. This then enables us to give a direct proof of the fact that Υ is a strong companion basis for Γ. We conclude this section by noting that as a consequence of Proposition 4.6.2, we therefore obtain an alternative proof of our main result, Theorem 4.1.4.

Rather than identifying shortest unoriented paths in Γ with their corresponding reverse paths, as in Section 4.4, we will here just consider shortest unoriented paths in Γ from vertices i to j with $i \leq j$. For this reason, we make the following definition.

DEFINITION 5.5.1 *Let i and j be vertices in Γ with $i \leq j$. Then, we call the shortest unoriented path in Γ from i to j a positive shortest unoriented path in Γ (or an su$^+$-path in Γ for short). We denote the set of all su$^+$-paths in Γ by* $\mathrm{su}^+(\Gamma)$.

In order to construct the desired bijective correspondence between the set of su$^+$-paths in Γ and the set of positive roots, it is enough to show that for each su$^+$-path p in Γ, we can find explicitly a positive root with support p. This is because the number of (distinct) su$^+$-paths in Γ is equal to the number of positive roots, and no positive root can have two different su$^+$-paths as support (refer to Section 4.4).

Now, Proposition 5.3.8 tells us that for any su$^+$-path p in Γ, there can be at most one instance of a left triangle vertex in Γ lying on p being followed by the corresponding top triangle vertex in Γ. So, there are essentially two types of su$^+$-paths in Γ. Those for which every left triangle vertex in Γ is followed by the corresponding right triangle vertex in Γ, and those where exactly one left triangle vertex in Γ is followed by the corresponding top triangle vertex in Γ.

In the following proposition, we examine the relationship between the positive roots and the first of these types of su$^+$-paths in Γ.

PROPOSITION 5.5.2 *Let i and j be vertices in Γ with $i \leq j$. Let p be the shortest unoriented path in Γ from i to j. Suppose that whenever p passes through a left triangle vertex in Γ, it does not then proceed through the corresponding top triangle vertex in Γ. Then:*

(i) If j is not a left triangle vertex in Γ, then the positive root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ has support p.

(ii) If j is a left triangle vertex in Γ and there are a vertices in the section of Γ above the top triangle vertex in Γ corresponding to j, then the positive root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j + \ldots + \alpha_{j+a}$ has support p.

Proof We start by noting that the vertex j must lie on the shortest unoriented path in Γ from i to n. (Because otherwise, the proof of Proposition 5.3.8 shows that some left triangle vertex in Γ lying on p must be followed (on p) by the corresponding top triangle vertex in Γ.)

Starting from i and moving towards j, suppose that the vertices in the shortest unoriented path from i to j are respectively $i_1, i_2, \ldots, i_t$. (So, $i_1 = i$ and $i_t = j$.)

Fix k, $1 \leq k < t$. If i_k is not a left triangle vertex in Γ, then $\beta_{i_k} = \pm\alpha_{i_k}$ and $i_{k+1} = i_k + 1$.

If i_k is a left triangle vertex in Γ, then $\beta_{i_k} = \pm(\alpha_{i_k} + \ldots + \alpha_{i_k+m})$, where m is the number of vertices in the section of Γ above the top triangle vertex corresponding to i_k. By assumption, p must proceed through the right triangle vertex in Γ corresponding to i_k. But this vertex must be labelled $i_k + m + 1$, and so we have $i_{k+1} = i_k + m + 1$.

In case (i), we have that j is not a left triangle vertex in Γ, and hence $\beta_{i_t} = \pm\alpha_{i_t}$. It is then clear that we can write $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ in the form $c_1\beta_{i_1} + \ldots + c_t\beta_{i_t}$ with $c_i = \pm 1$ for all i. Therefore, we see that $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ has support p.

In case (ii), we have that j is a left triangle vertex in Γ, and $\beta_{i_t} = \pm(\alpha_{i_t} + \ldots + \alpha_{i_t+a})$. So, in a similar manner, we see that the positive root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j + \ldots + \alpha_{j+a}$ has support p. ■

Now we complete the picture by examining the relationship between the positive roots and the su$^+$-paths in Γ that weren't covered by the above proposition.

PROPOSITION 5.5.3 *Let i and j be vertices in Γ with $i < j$, and let p be the shortest unoriented path in Γ from i to j. Suppose that at some point, p travels through a left triangle vertex in Γ and then proceeds through the corresponding top triangle vertex in Γ. Then, the positive root $\alpha_i + \ldots + \alpha_{j-1}$ has support p.*

Proof Starting from i and moving towards j, suppose that the vertices in the shortest unoriented path in Γ from i to j are respectively $i_1, i_2, \ldots, i_t$. (So, $i_1 = i$ and $i_t = j$.)

We have from Proposition 5.3.8 that only on one occasion can p travel through a left triangle vertex in Γ and then proceed through the corresponding top triangle vertex in Γ.

Suppose x is a left triangle vertex in Γ lying on p, suppose that there are a vertices in the section of Γ above the top triangle vertex corresponding to x, and suppose $i_k = x$ and $i_{k+1} = x + a$ for some k, where $1 \leq k < t$. (Note that x must lie on the shortest unoriented path in Γ from i to n.)

Whenever the shortest unoriented path in Γ from i to x passes through a left triangle vertex in Γ, then it does not proceed through the corresponding top triangle vertex in Γ. Therefore, from the proof of Proposition 5.5.2, we see that we can write the positive root $\alpha_i + \ldots + \alpha_x + \ldots + \alpha_{x+a}$ in the form $\alpha_i + \ldots + \alpha_x + \ldots + \alpha_{x+a} = \sum_{l=1}^{k} c_l \beta_{i_l}$ with $c_l = \pm 1$ for all l.

Whenever the shortest unoriented path in Γ from j to $x + a$ travels through a left triangle vertex in Γ, then it must proceed through the corresponding right triangle vertex in Γ (see the last part of Remark 5.3.9). Therefore, again using the proof of Proposition 5.5.2, we see that we can write the positive root $\alpha_j + \ldots + \alpha_{x+a}$ in the form $\alpha_j + \ldots + \alpha_{x+a} = \sum_{l=k+1}^{t} c_l \beta_{i_l}$ with $c_l = \pm 1$ for all l.

Therefore, $\alpha_i + \ldots + \alpha_{j-1} = \sum_{l=1}^{k} c_l \beta_{i_l} - \sum_{l=k+1}^{t} c_l \beta_{i_l}$ with $c_l = \pm 1$ for all l, and so we see that the positive root $\alpha_i + \ldots + \alpha_{j-1}$ has support p. ∎

In view of the paragraph following Definition 5.5.1, we have now constructed an explicit bijective correspondence (for the companion basis Υ for Γ) between the set of su$^+$-paths in Γ and the set of positive roots. Under this correspondence, a given su$^+$-path p in Γ corresponds to the unique positive root that has support p.

We now give an example.

EXAMPLE 5.5.4 Again, we consider the labelled quiver Ω introduced in Example 5.2.1.

$$
\begin{tikzcd}
\Omega & & 3 \arrow[d] & & & \\
& & 4 \arrow[d] & & & \\
& & 5 \arrow[drr] & & & \\
& & & & 7 \arrow[dll] & 6 \arrow[l] \\
& & 8 \arrow[uu] \arrow[d] & & & \\
& & 9 \arrow[dr] & & & \\
1 \arrow[r] & 2 \arrow[ur] & & 10 \arrow[ll] \arrow[r] & 11 &
\end{tikzcd}
$$

Suppose Φ is the root system of Dynkin type A_{11} and that $\Pi = \{\alpha_1, \ldots, \alpha_{11}\} \subseteq \Phi$ is a simple system of Φ.

In Example 5.4.8, we constructed a companion basis $\{\beta_1, \ldots, \beta_{11}\} \subseteq \Phi$ for Ω, given by $\beta_1 = -\alpha_1$, $\beta_2 = -(\alpha_2 + \cdots + \alpha_9)$, $\beta_3 = -\alpha_3$, $\beta_4 = -\alpha_4$, $\beta_5 = -(\alpha_5 + \alpha_6 + \alpha_7)$ and $\beta_i = \alpha_i$ for all $6 \leq i \leq 11$.

Here, we consider two su$^+$-paths in Ω. For each of these, we find the positive root which has that su$^+$-path as support.

Let p_1 be the shortest unoriented path in Ω from 4 to 10. We see that every left triangle vertex in Ω lying on p_1 is followed by the corresponding right triangle vertex in Ω (since 5 is followed by 8). Therefore, since 10 is not a left triangle vertex in Ω, we have from part (i) of Proposition 5.5.2 that the positive root $\alpha_4 + \cdots + \alpha_{10}$ has support p_1. Indeed, by expressing $\alpha_4 + \cdots + \alpha_{10}$ in terms of the given companion basis for Ω, we obtain $\alpha_4 + \cdots + \alpha_{10} = -\beta_4 - \beta_5 + \beta_8 + \beta_9 + \beta_{10}$.

Let p_2 be the shortest unoriented path in Ω from 1 to 5. We see that the left triangle vertex 2 lies on p_2 and is followed by 9, the corresponding top triangle vertex in Ω. Hence, by Proposition 5.5.3, we have that the positive root $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ has support p_2. We conclude this example by noting that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\beta_1 - \beta_2 + \beta_5 - \beta_8 - \beta_9$, thus confirming this.

Using the established bijective correspondence (for the companion basis Υ for Γ) between the set of su$^+$-paths in Γ and the set of positive roots, we may now describe the vectors d_α^Υ for $\alpha \in \Phi^+$. This will then enable us to complete a direct proof of the fact that Υ is a strong companion basis for Γ.

Let α be a positive root and suppose that p is the su$^+$-path in Γ which corresponds to α. Then, α has support p. The following lemma tells us that the vector d_α^Υ has a one in each position corresponding to a vertex of Γ lying on p, and zeros everywhere else.

LEMMA 5.5.5 *Let α be a positive root and suppose that $\alpha = \sum_{i=1}^{n} c_i\beta_i$. Then, $c_i \in \{0, \pm 1\}$ for all $1 \leq i \leq n$.*

Proof The result follows immediately from the proofs of Propositions 5.5.2 and 5.5.3. ■

COROLLARY 5.5.6 *The companion basis $\Upsilon = \{\beta_1, \ldots, \beta_n\} \subseteq \Phi$ for Γ is strong.*

Proof Having now completed the description of the vectors d_α^Υ for $\alpha \in \Phi^+$, the fact that Υ is a strong companion basis for Γ follows by proceeding exactly as in Section 4.5. ■

We have now seen that we can construct explicitly a strong companion basis for the quiver of any given cluster-tilted algebra of Dynkin type A. Therefore, by Proposition 4.6.2, we have provided an alternative proof of our main result, Theorem 4.1.4.

Chapter 6

Companion Basis Mutation and Dimension Vectors

In the final chapter of this thesis, we return to considering the procedure of companion basis mutation (introduced in Theorem 3.1.4). We start by giving an outline of the contents of this chapter.

Suppose that Λ is a cluster-tilted algebra of simply-laced Dynkin type, and that Γ is the quiver of Λ. By Proposition 4.6.2, the set of vectors associated to the positive roots of the corresponding root system is the same with respect to any companion basis for Γ. We may therefore consider this set of vectors as the set of vectors associated to the positive roots with respect to Γ.

Suppose that by applying a single quiver mutation to Γ, we obtain the quiver Γ', and let Λ' be the cluster-tilted algebra associated to Γ'.

Via inward companion basis mutation, we will see that each companion basis for Γ induces a map from the set of vectors associated to the positive roots with respect to Γ, to the set of vectors associated to the positive roots with respect to Γ'. Moreover, we will show that this map does not depend on the choice of companion basis for Γ.

In the Dynkin type A case, we will give an explicit description of this map. Finally, in this case, as a consequence of our main result (Theorem 4.1.4), we will then deduce that given the dimension vectors of the finitely generated indecomposable Λ-modules, we can immediately write down the dimension vectors of the finitely generated indecomposable Λ'-modules.

Our set-up for this chapter is as follows.

Let k be an algebraically closed field, let Q (with n vertices) be an alternating quiver of simply-laced Dynkin type, with underlying graph Δ, and let

$$\mathcal{C} = \frac{\mathcal{D}^b(kQ - \text{mod})}{F}$$

be the corresponding cluster category. Suppose that $\mathcal{A}$ is the cluster algebra of Dynkin type Δ. Also, suppose that $\Phi \subseteq V$ is the root system of Dynkin type Δ, where V is a Euclidean space with positive definite symmetric bilinear form $(\ ,\ )$, and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system of Φ.

Let $\Lambda = \text{End}_{\mathcal{C}}(T)^{\text{op}}$ be a cluster-tilted algebra, where T is some basic cluster-tilting object in $\mathcal{C}$. Suppose that $(\mathbf{x}, B)$ is the seed of $\mathcal{A}$ which corresponds to T, so that $\Gamma = \Gamma(B)$ is the quiver of Λ. Write $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices of Γ, and Γ_1 is the set of arrows of Γ.

Finally, let $\Psi = \{\gamma_x : x \in \Gamma_0\} \subseteq \Phi$ be a companion basis for Γ, and suppose that $A = (a_{xy})$ is the positive quasi-Cartan companion of B given by $a_{xy} = (\gamma_x, \gamma_y)$ for all $x, y \in \Gamma_0$.

Note: Although we only consider inward companion basis mutation in this chapter, we note that analogues of the results established here may be obtained similarly with regard to outward companion basis mutation.

6.1 Maps Induced by Mutation

In this section, we study a collection of maps associated to the companion bases for Γ, induced by companion basis mutation. In order to enable us to define these maps, we start by showing that the set $\{d_\alpha^\Psi : \alpha \in \Phi^+\}$ has $|\Phi^+|$ distinct elements. That is, each vector d_α^Ψ uniquely determines the positive root α.

PROPOSITION 6.1.1 $\left|\{d_\alpha^\Psi : \alpha \in \Phi^+\}\right| = |\Phi^+|$. *That is, the vector d_α^Ψ is different for each positive root α.*

Proof Let $\alpha, \beta \in \Phi^+$ and suppose $d_\alpha^\Psi = d_\beta^\Psi$. We will show that $\alpha = \beta$.

Suppose $\alpha = \sum_{x \in \Gamma_0} a_x \gamma_x$ and $\beta = \sum_{x \in \Gamma_0} b_x \gamma_x$ with $a_x, b_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Since $d_\alpha^\Psi = d_\beta^\Psi$, we have $|a_x| = |b_x|$ for all $x \in \Gamma_0$.

Let $I = \{x : b_x = a_x\}$ and $J = \{x : b_x \neq a_x\}$ (i.e. $J = \{x : a_x \neq 0 \text{ and } b_x = -a_x\}$). Define $\gamma = \sum_{x \in I} a_x \gamma_x \in \mathbb{Z}\Phi$ and $\delta = \sum_{x \in J} a_x \gamma_x \in \mathbb{Z}\Phi$.

Then, $\alpha = \gamma + \delta$ and $\beta = \gamma - \delta$.

Since $\alpha, \beta \in \Phi$, we have

$$2 = (\alpha, \alpha) = (\gamma + \delta, \gamma + \delta) = (\gamma, \gamma) + 2(\gamma, \delta) + (\delta, \delta)$$

and

$$2 = (\beta, \beta) = (\gamma - \delta, \gamma - \delta) = (\gamma, \gamma) - 2(\gamma, \delta) + (\delta, \delta).$$

It therefore follows that we must have $(\gamma, \gamma) + (\delta, \delta) = 2$.

Now, for any $0 \neq z \in \mathbb{Z}\Phi$, it is clear that we must have $(z, z) \in \mathbb{N}$. (Note that $(\ ,\ )$ is positive definite, and (z, z) is clearly an integer.) But also, we cannot have $(z, z) = 1$ (refer to [CS]). Therefore, there are two cases to consider.

Case 1: Suppose $(\gamma, \gamma) = 2$ and $(\delta, \delta) = 0$. In this case, we must have that $\delta = 0$ and hence $\alpha = \beta$.

Case 2: Suppose $(\gamma, \gamma) = 0$ and $(\delta, \delta) = 2$. In this case, we must have that $\gamma = 0$ and hence $\alpha = -\beta$. But this contradicts the fact that α and β are both positive roots.

Therefore, case 2 cannot arise, and so we must have $\alpha = \beta$ as required. ■

By Proposition 4.6.2, the set of vectors associated to the positive roots is the same with respect to any companion basis for Γ. For notational convenience, we will call this set $D(\Gamma)$. In particular, we have $D(\Gamma) = \{d^{\Psi}_{\alpha} : \alpha \in \Phi^{+}\}$.

Let k be a vertex in Γ and suppose that Γ' is the quiver obtained from Γ by applying quiver mutation at the vertex k. The vertex k corresponds to a row and column of B and hence to a cluster variable $x_k \in \mathbf{x}$. We have that $\Gamma' = \Gamma(B')$, where $(\mathbf{x}', B')$ is the seed of $\mathcal{A}$ obtained from $(\mathbf{x}, B)$ by mutating in the direction x_k. The seed $(\mathbf{x}', B')$ corresponds to some basic cluster-tilting object T' in $\mathcal{C}$, and so Γ' is the quiver of the cluster-tilted algebra $\Lambda' = \mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$.

Denote the set of vertices of Γ' by Γ'_0, and let $\Psi' = \{\gamma'_x : x \in \Gamma'_0\} \subseteq \Phi$ be the companion basis for Γ' obtained from Ψ by mutating inwardly at k (refer to Theorem 3.1.4).

In view of Proposition 6.1.1, associated to the companion basis Ψ for Γ, we have the following bijective map

$$\begin{aligned} \phi^{\Psi}_{\mathrm{in}} : D(\Gamma) &\longrightarrow D(\Gamma') \\ d^{\Psi}_{\alpha} &\longmapsto d^{\Psi'}_{\alpha}. \end{aligned}$$

Likewise, we have such a bijective map associated to each companion basis for Γ. We will show that these bijective maps are all the same.

That they all have the same domain, and all have the same codomain, is immediate.

Let $\Theta = \{\nu_x : x \in \Gamma_0\} \subseteq \Phi$ be another companion basis for Γ, and suppose that $\Theta' = \{\nu'_x : x \in \Gamma'_0\} \subseteq \Phi$ is the companion basis for Γ' obtained from Θ by mutating inwardly at k.

We will show that the maps ϕ^{Ψ}_{in} and $\phi^{\Theta}_{\text{in}}$ are equal.

Recall that by applying sign changes to the elements of Θ, we can obtain a companion basis $\Upsilon = \{\delta_x : x \in \Gamma_0\} \subseteq \Phi$ for Γ giving rise to A (refer to Corollary 3.3.2). (Note that $\delta_x = \pm\nu_x$ for all $x \in \Gamma_0$.) Let $\Upsilon' = \{\delta'_x : x \in \Gamma'_0\} \subseteq \Phi$ be the companion basis for Γ' obtained from Υ by mutating inwardly at k.

It follows as a consequence of the following result that $\phi^{\Upsilon}_{\text{in}} = \phi^{\Theta}_{\text{in}}$.

LEMMA 6.1.2 *Fix $z \in \Gamma_0$ and let $\Omega = \{\xi_x : x \in \Gamma_0\} \subseteq \Phi$ be the companion basis for Γ given by*

$$\xi_x = \begin{cases} -\nu_x & \text{if } x = z, \\ \nu_x & \text{otherwise.} \end{cases}$$

Then: (i) The companion basis $\Omega' = \{\xi'_x : x \in \Gamma'_0\} \subseteq \Phi$ for Γ' obtained from Ω by mutating inwardly at k is given by

$$\xi'_x = \begin{cases} -\nu'_x & \text{if } x = z, \\ \nu'_x & \text{otherwise.} \end{cases}$$

(ii) $\phi^{\Omega}_{\text{in}} = \phi^{\Theta}_{\text{in}}$.

Proof (i) There are three cases to consider.

Case 1: Suppose $z \neq k$ and there is no arrow in Γ from z to k. In this case, we have $\xi'_z = \xi_z = -\nu_z = -\nu'_z$, and $\xi'_x = \nu'_x$ for all $x \neq z$.

Case 2: Suppose there is an arrow in Γ from z to k. (Note that we must then have $z \neq k$.) In this case, we have $\xi'_z = s_{\xi_k}(\xi_z) = s_{\nu_k}(-\nu_z) = -s_{\nu_k}(\nu_z) = -\nu'_z$, and $\xi'_x = \nu'_x$ for all $x \neq z$.

Case 3: Suppose $z = k$. In this case, we have $\xi'_z = -\nu'_z$ since $\xi'_k = \xi_k = -\nu_k = -\nu'_k$. Also, if $x \neq z$ and there is no arrow in Γ from x to k, then $\xi'_x = \xi_x = \nu_x = \nu'_x$, and if there is an arrow in Γ from x to k, then $\xi'_x = s_{\xi_k}(\xi_x) = s_{-\nu_k}(\nu_x) = s_{\nu_k}(\nu_x) = \nu'_x$.

Thus, in each case we see that $\xi'_x = \nu'_x$ for all $x \neq z$, and $\xi'_z = -\nu'_z$.

(ii) Let $\alpha \in \Phi^+$ and write $\alpha = \sum_{x\in\Gamma_0} a_x\nu_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Then, we have $\alpha = \sum_{x\in\Gamma_0} b_x\xi_x$ where $b_x = a_x$ for all $x \neq z$, and $b_z = -a_z$. Furthermore, if $\alpha = \sum_{x\in\Gamma'_0} c_x\nu'_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma'_0$, then $\alpha = \sum_{x\in\Gamma'_0} d_x\xi'_x$ where $d_x = c_x$ for all $x \neq z$, and $d_z = -c_z$. In particular, $d^{\Theta}_{\alpha} = d^{\Omega}_{\alpha}$ and $d^{\Theta'}_{\alpha} = d^{\Omega'}_{\alpha}$ for all $\alpha \in \Phi^+$.

It follows immediately that the maps $\phi^{\Theta}_{\text{in}}$ and $\phi^{\Omega}_{\text{in}}$ are equal. ■

The following is an obvious consequence of Lemma 6.1.2.

COROLLARY 6.1.3 $\phi^{\Theta}_{\text{in}} = \phi^{\Upsilon}_{\text{in}}$.

With the following result, we now prove that all of the bijective maps associated to the companion bases for Γ are the same.

PROPOSITION 6.1.4 $\phi^{\Psi}_{\text{in}} = \phi^{\Theta}_{\text{in}}$.

Proof From Corollary 6.1.3 we have that $\phi_{\text{in}}^{\Theta} = \phi_{\text{in}}^{\Upsilon}$. So, we will prove the desired result by showing that $\phi_{\text{in}}^{\Psi} = \phi_{\text{in}}^{\Upsilon}$.

Since $\Psi = \{\gamma_x : x \in \Gamma_0\}$ and $\Upsilon = \{\delta_x : x \in \Gamma_0\}$ are both companion bases for Γ giving rise to A, we have from Theorem 3.4.11 that there is some orthogonal linear transformation $\sigma : V \to V$ (which permutes the set of simple roots Π) and some $w \in W_\Phi$ (the Weyl group of Φ) such that $\delta_x = w\sigma\gamma_x$ for all $x \in \Gamma_0$.

Let $\alpha \in \Phi^+$ and write $\alpha = \sum_{x \in \Gamma_0} a_x \gamma_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Since w and σ both permute the set of roots Φ (refer to the proof of Lemma 3.4.10), we see that $w\sigma\alpha \in \Phi$. In particular, either $w\sigma\alpha \in \Phi^+$ or $-w\sigma\alpha \in \Phi^+$. Suppose without loss of generality that $w\sigma\alpha \in \Phi^+$.

We have $w\sigma\alpha = \sum_{x \in \Gamma_0} a_x w\sigma\gamma_x = \sum_{x \in \Gamma_0} a_x \delta_x$, and therefore $d_\alpha^\Psi = d_{w\sigma\alpha}^\Upsilon$ $(= d_{-w\sigma\alpha}^\Upsilon)$. By Proposition 6.1.1, $w\sigma\alpha$ must be the unique element of Φ^+ having this property. So, it only remains to check that $d_\alpha^{\Psi'} = d_{w\sigma\alpha}^{\Upsilon'}$.

We start by showing that $w\sigma\gamma'_x = \delta'_x$ for all $x \in \Gamma'_0$.

There are two cases to consider.

Case 1: Suppose that there is no arrow in Γ from x to k. Then, $w\sigma\gamma'_x = w\sigma\gamma_x = \delta_x = \delta'_x$.

Case 2: Suppose that there is an arrow in Γ from x to k. Then,

$$\begin{aligned} w\sigma\gamma'_x &= w\sigma(s_{\gamma_k}(\gamma_x)) = w\sigma(\gamma_x - (\gamma_x, \gamma_k)\,\gamma_k) \\ &= w\sigma\gamma_x - (\gamma_x, \gamma_k)\, w\sigma\,\gamma_k \\ &= \delta_x - (\gamma_x, \gamma_k)\,\delta_k \end{aligned}$$

$$
\begin{aligned}
&= \delta_x - (\delta_x, \delta_k)\, \delta_k \\
&= s_{\delta_k}(\delta_x) \\
&= \delta'_x.
\end{aligned}
$$

Finally, we see that if $\alpha = \sum_{x\in\Gamma'_0} c_x\gamma'_x$ with $c_x \in \mathbb{Z}$ for all $x \in \Gamma'_0$, then $w\sigma\alpha = \sum_{x\in\Gamma'_0} c_x w\sigma\gamma'_x = \sum_{x\in\Gamma'_0} c_x\delta'_x$. Therefore, $d^{\Psi'}_{\alpha} = d^{\Upsilon'}_{w\sigma\alpha}$ $(= d^{\Upsilon'}_{-w\sigma\alpha})$ and hence $\phi^{\Psi}_{\text{in}} = \phi^{\Upsilon}_{\text{in}}$.

This completes the proof. ■

We have now seen that the map $\phi^{\Psi}_{\text{in}}: D(\Gamma) \to D(\Gamma')$ does not depend on the companion basis Ψ for Γ. That is, if we replace Ψ with any other companion basis for Γ, then we still get the same map. We will therefore call this map $\phi^{\Gamma}_{\text{in}}$.

6.2 Towards an Explicit Description of $\phi^{\Gamma}_{\text{in}}$

We now aim to describe the map $\phi^{\Gamma}_{\text{in}}: D(\Gamma) \to D(\Gamma')$ explicitly. That is, to find a rule that enables us to directly compute the image of any given vector in $D(\Gamma)$ under $\phi^{\Gamma}_{\text{in}}$. We start by showing that whenever $\phi^{\Gamma}_{\text{in}}$ (equivalently ϕ^{Ψ}_{in}) is applied to a vector in $D(\Gamma)$, then the resultant vector differs from the initial vector in at most one component.

Let $\alpha \in \Phi^+$ and write $\alpha = \sum_{x\in\Gamma_0} a_x\gamma_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. Using this expression for α in terms of Ψ, we can obtain an expression for α in terms of Ψ'.

Suppose x is a vertex of Γ. If there is no arrow in Γ from x to k, then $\gamma'_x = \gamma_x$. On the other hand, if there is an arrow in Γ from x to k, then $\gamma'_x = s_{\gamma_k}(\gamma_x) = \gamma_x - (\gamma_x, \gamma_k)\, \gamma_k$, and hence $\gamma_x = \gamma'_x + (\gamma_x, \gamma_k)\,\gamma'_k$ (since $\gamma'_k = \gamma_k$). Therefore, we have

$$\begin{aligned}
\alpha &= \sum_{x \not\to k} a_x \gamma_x + \sum_{x \to k} a_x \gamma_x \\
&= \sum_{x \not\to k} a_x \gamma'_x + \sum_{x \to k} a_x (\gamma'_x + (\gamma_x, \gamma_k)\,\gamma'_k) \\
&= \sum_{x \neq k} a_x \gamma'_x + a_k \gamma'_k + \sum_{x \to k} a_x\,(\gamma_x, \gamma_k)\,\gamma'_k \\
&= \sum_{x \neq k} a_x \gamma'_x + \left(a_k + \sum_{x \to k} a_x\,(\gamma_x, \gamma_k)\right) \gamma'_k.
\end{aligned}$$

Notice in particular that d_α^Ψ and $d_\alpha^{\Psi'}$ differ in at most one component, the k-component. Therefore, in order to describe the map ϕ_{in}^Γ, we should aim to express the k-component of $d_\alpha^{\Psi'}$ solely in terms of the components of d_α^Ψ. That is, we want to express $|a_k + \sum_{x \to k} a_x\,(\gamma_x, \gamma_k)|$ solely in terms of the integers $|a_x|$ for $x \in \Gamma_0$.

The following lemma will subsequently enable us to do this in the Dynkin type A_n case.

LEMMA 6.2.1 $|-|a_k| + \sum_{x \to k} |a_x|| - |a_k + \sum_{x \to k} a_x(\gamma_x, \gamma_k)|$ *is an even integer.*

Proof We will show that

$$-|a_k| + \sum_{x \to k} |a_x| - \left(a_k + \sum_{x \to k} a_x(\gamma_x, \gamma_k)\right) = -|a_k| - a_k + \sum_{x \to k} (|a_x| - a_x(\gamma_x, \gamma_k))$$

is even. The result then follows.

Firstly, since $a_k \in \mathbb{Z}$, we have that $-|a_k| - a_k$ is even.

Let x be a vertex in Γ, and suppose that there is an arrow in Γ from x to k. Since Ψ is a companion basis for Γ, we must have $(\gamma_x, \gamma_k) = \pm 1$. Thus, $|a_x| - a_x(\gamma_x, \gamma_k) = |a_x| \mp a_x$. Now, since $a_x \in \mathbb{Z}$, $|a_x| - a_x$ and $|a_x| + a_x$ are both even, and hence $|a_x| - a_x(\gamma_x, \gamma_k)$ is even.

Therefore, $-|a_k| - a_k + \sum_{x \to k} (|a_x| - a_x(\gamma_x, \gamma_k))$ is even. ■

6.3 A Description of $\phi_{\text{in}}^{\Gamma}$ in Dynkin Type A_n

In this section we restrict our attention to only the Dynkin type A_n case. (That is, we suppose that Δ is a Dynkin diagram of type A_n.) In this case, we are able to give an explicit description of the map $\phi_{\text{in}}^{\Gamma}: D(\Gamma) \to D(\Gamma')$, by using Lemma 6.2.1. We finish by highlighting a consequence of this description due to Theorem 4.1.4.

PROPOSITION 6.3.1 *Let $\alpha \in \Phi^+$ and suppose that $\alpha = \sum_{x \in \Gamma_0} a_x \gamma_x$ with $a_x \in \mathbb{Z}$ for all $x \in \Gamma_0$. We then have*

$$\left| a_k + \sum_{x \to k} a_x (\gamma_x, \gamma_k) \right| = \left| -|a_k| + \sum_{x \to k} |a_x| \right| .$$

Proof We have that all of the components of both d_α^{Ψ} and $d_\alpha^{\Psi'}$ must belong to the set $\{0, 1\}$ (refer to Section 4.4). Therefore, $|a_x| \in \{0, 1\}$ for all vertices x in Γ, and $|a_k + \sum_{x \to k} a_x (\gamma_x, \gamma_k)| \in \{0, 1\}$.

Since $|a_x| \in \{0, 1\}$ for all vertices x in Γ, and due to the bijective correspondence between the set of positive roots and the set of shortest unoriented paths in Γ established in Section 4.4, a simple case-by-case analysis establishes that $-|a_k| + \sum_{x \to k} |a_x| \in \{0, \pm 1\}$.

We have now shown that $|a_k + \sum_{x \to k} a_x (\gamma_x, \gamma_k)|, |-|a_k| + \sum_{x \to k} |a_x|| \in \{0, 1\}$. It therefore follows from Lemma 6.2.1 that

$$\left| a_k + \sum_{x \to k} a_x (\gamma_x, \gamma_k) \right| = \left| -|a_k| + \sum_{x \to k} |a_x| \right|$$

as required. ■

We have proved the following.

COROLLARY 6.3.2 *Let $\alpha \in \Phi^+$ and suppose that $d_\alpha^\Psi = (d_x)_{x \in \Gamma_0}$. Then, $d_\alpha^{\Psi'}$ is given by*

$$\left(d_\alpha^{\Psi'}\right)_z = \begin{cases} d_z & \text{if } z \neq k, \\ \left|-d_k + \sum_{x \to k} d_x\right| & \text{if } z = k. \end{cases}$$

By Theorem 4.1.4, we have that the dimension vectors of the finitely generated indecomposable Λ-modules are precisely the elements of the set $D(\Gamma)$. Likewise, the dimension vectors of the finitely generated indecomposable Λ'-modules are precisely the elements of the set $D(\Gamma')$. We have therefore established the following.

COROLLARY 6.3.3 *Given the dimension vectors of the finitely generated indecomposable Λ-modules, we can simply write down the dimension vectors of the finitely generated indecomposable Λ'-modules.*

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