A classification of toral and planar attractors and substitution tiling spaces

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Sheila Margaret McCann

Department of Mathematics

University of Leicester

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Abstract We focus on dynamical systems which are one-dimensional expanding attractors with a local product structure of an arc times a Cantor set. We define a class of Denjoy continua and show that each one of the class is homeomorphic to an orientable DA attractor with four complementary domains which in turn is homeomorphic to a tiling space consisting of aperiodic substitution tilings. The planar attractors are non-orientable as is the Plykin attractor in the 2-sphere which we describe.

We classify these attractors and tiling spaces up to homeomorphism and the symmetries of the underlying spaces up to isomorphism. The criterion for homeomorphism is the irrational slope of the expanding eigenvector of the defining matrix from whence the attractor was formed whilst the criterion for isomorphism is the matrix itself. We find that the permutation groups arising from the 4 'special points' which serve as the repelling set of an attractor are isomorphic to subgroups of S_4. Restricted to these 4 special points, we show that the isotopy class group of the self-homeomorphisms of an attractor, and likewise those of a tiling space, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.

To the memory of my dear parents Anna and Jack.

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List of abbreviations and symbols

card	cardinality
DA	derived from Anosov
det	determinant
IFS	iterated function system
Int	interior
ker	kernel
mod	modulo
o-p	orientation-preserving
o-r	orientation-reversing
par	parity
PF	Perron-Frobenius
μ	the golden ratio
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	the set of positive integers including zero
$\equiv$	congruent; equivalent
$:=$	definition
$\cong$	homeomorphic; isomorphic
$\simeq$	isotopic

Introduction

The ultimate results of this thesis are due in no small part to *serendipity*, a word attributed to Walpole[1] to describe folklore heroes who made "discoveries . . . of things they were not in quest of" (see for example [13]); in a mathematical context could be added the criterion for an unexpected result to be a "prepared mind" [49]. Suffice to say that our intended path took a diversion.

The main theme of this thesis is the construction and classification of one-dimensional hyperbolic expanding attractors with four complementary domains and their homeomorphic substitution tiling spaces. Before giving a synopsis of each chapter we describe here the content of our research which we consider to be original. To that end we prepare the ground in chapter 5 by constructing an attractor Λ (Def. 5.7) with one complementary domain. To do so we use the hyperbolic toral automorphism $\mathcal{C}$ (1.10), known as Arnold's[2] Cat map [6], as an example of an Anosov[3] diffeomorphism (Def. 5.3) on which we apply surgery first introduced in 1967 by Smale[4] in his paper [51]. This yields a 'derived from Anosov' (DA) diffeomorphism f (5.2) and its attractor Λ. By considering the torus $\mathbb{T}^2$ as a 2-fold branched covering of the 2-sphere S^2 together with a DA map f_Π (5.4), the Plykin[5] attractor $\Lambda_\Pi \subseteq S^2$ has four complementary domains. Plykin first conceived his attractor directly on S^2 in his 1974 paper

[1]Horace Walpole (1717 - 1797) English art historian, man of letters, antiquarian and Whig politician.

[2]Vladimir Igorevich Arnold (1937 - 2010) Russian mathematician.

[3]Dmitri Victorovich Anosov (1936 -) Russian mathematician.

[4]Stephen Smale (1930 -) American mathematician.

[5]Romen Vasilievich Plykin (1935 - 2011) Russian mathematician.

[42]. In chapter 6 we use Yi's algorithm [57] to lift the Plykin attractor's planar projection Λ_P to $\tilde{\Lambda}_P \subseteq \mathbb{T}^2$. The dynamics of a self-homeomorphism of $\tilde{\Lambda}_P$ are modelled by the induced orientation-preserving self-map $\tilde{g}_*$ (6.11) of the branched 1-manifold $\tilde{\mathcal{M}}_*$ which gives the presentation of a Williams' solenoid [54], $\tilde{\Sigma}_* = \varprojlim(\tilde{\mathcal{M}}_*, \tilde{g}_*)$ with shift map $\tilde{\sigma}_*$. In order to reduce $\tilde{\mathcal{M}}_*$ to an elementary branched 1-manifold consisting of a wedge of circles with a single branch point, we devise an original method which bypasses the need for using Williams' Lemma 5.3 and his §5.5 *Four models for reduction* both given in [54]. We find that by simply iterating the map (6.11) and locating the returns to one of the four branch points of $\tilde{\mathcal{M}}_*$ chosen to be the origin, this gives the information to form what Barge and Diamond call a rose [7] such that the rose K_* is equipped with a self-map r_* (6.12) (Def. 6.14). The maps $\tilde{g}_*$ and r_* commute (Prop. 6.25). An elementary presentation of a Williams' solenoid is then $\Omega = \varprojlim(K_*, r_*)$ with shift map ω, defined on page 115. From recognising the pattern of letters appearing as words in the rose map we form a proper substitution map and hence a tiling space $\mathcal{T}_{\omega^2}$ homeomorphic to $\tilde{\Lambda}_P$ (Theorem 6.32).

In chapter 7 we create a novel method to construct a toral attractor $\mathfrak{A}_\alpha$ (Def. 7.8) with four complementary domains. To develop and illustrate the method we use six examples of Anosov diffeomorphisms, each of the form F (Def. 7.1), chosen merely by reason of the defining matrix M belonging to one of the six equivalence classes whose union forms a group $\mathbb{M} = \dot{\bigcup}_{i=0}^{5} \bar{M}_i$ (Def 7.11), isomorphic to the quotient group $GL(2, \mathbb{Z}/2\mathbb{Z})$ (Theorem 7.13). In summary the ingredients of our method involve a sequence of operations which uses a secondary Markov[6] partition $\ddot{\mathcal{P}}$ which is a finer version of a standard partition $\mathcal{P}$; a 2θ-space $\ddot{\mathcal{M}}$ still with two branch points but having double the number of edges than that of a θ-space which we used to represent an attractor with one

[6]Andrey A. Markov (1856 - 1922) Russian mathematician.

complementary domain. The self-map $\ddot{\vartheta} : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$ is derived from the linear transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$, induced by the DA map $\mathfrak{f}$ (7.7) which yields $\mathfrak{A}_\alpha$ (Lemma 7.20). Iterating the map $\ddot{\vartheta}$ locates the five distinct return words to the chosen origin then using these return words we can generate the rose map $r : \ddot{K} \to \ddot{K}$ which is the elementary presentation of a solenoid $\ddot{\Omega}$ (Def. 7.24). Then once again the patterns of letters which occur in the words of a rose map provide the mappings for a proper substitution $\ddot{\omega}$ (Def. 7.28) which then leads to a tiling space $\mathcal{T}_{\alpha(\ddot{\omega})}$ (Def. 7.29) where $\mathfrak{A}_\alpha \cong \mathcal{T}_{\alpha(\ddot{\omega})}$ (Lemma 7.31).

The classification of spaces appears in chapter 8. Up to homeomorphism the main result is Theorem 8.4. This states that attractors $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are homeomorphic if and only if the quadratic irrational slopes α and β of the expanding eigenvectors of the matrices associated to the attractors are equivalent. The equivalence of the slopes is determined by the continued fraction expansions of α and β as explained on page 146. Involved in the proof of Theorem 8.4 is the Denjoy[7] continuum $\mathbb{D}_\alpha^B$, defined on page 142, such that B serves as the repelling set of points which shapes an attractor $\mathfrak{A}_\alpha$. Furthermore, by reason of an attractor's matrix M permuting the points of set B, M remains the key criterion for the classification of spaces up to isomorphism. In particular, the permutation group of the points of set B is isomorphic to either the symmetric group A_4 or D_4 (Lemma 8.11), the symmetry group $S(\mathfrak{A}_\alpha)$ is isomorphic to either the rotation subgroup $S_r(\mathscr{T})$ of a regular tetrahedron or to that of a square D_4 (Theorem 8.15), while the class subgroup of a self-homeomorphism $\mathfrak{K}_0(\mathfrak{A}_\alpha)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ (Corollary 8.7). Similar results hold for a non-orientable attractor $P\mathfrak{A}_\alpha$ and a tiling space $\mathcal{T}_{\alpha(\ddot{\omega})}$.

Chapter 1. Broadly speaking a **dynamical system** has an initial state which evolves over discrete or continuous time according to some rule. Such a system can range from celestial mechanics to population growth, the

[7] Arnaud Denjoy (1884 - 1974) French mathematician.

latter often modelled by a logistic map of the quadratic family $f_k(x) = kx(1-x)$. We introduce the notion of **symbolic dynamics** first in the context of shift spaces (Def. 1.1, 1.2) then with respect to a **β-expansion** (Def. 1.3) whose symbolic alphabet $\{0,1\}$ contributes to the plotting of the **Rauzy**[8] **fractal** $\mathcal{T}_\beta$ (Def. 1.13). The term fractal was first introduced by Mandelbrot[9] in his 1975 Essay [33]. In his subsequent publication [34] he writes "A fractal is by definition a set for which the Hausdorff[10] Besicovitch[11] dimension strictly exceeds the topological dimension". (We raise the question of dimension in our conclusions of chapter 9.) As yet the term fractal evades a consistent definition, being dependent upon an individual author. However, relevant to this thesis in which Cantor[12] sets are prominent, Mandelbrot cites the original **Cantor set** as a fractal since its non-integer dimension is $\log 2/\log 3 \approx 0.6309 > 0$ while its topological dimension is 0. The Rauzy fractal can also be described as the attractor of an **iterated function system** (Prop. 1.15). With regards to the term **attractor**, a simple description which suits our introduction is given by Devaney in [20] where "an attractor is an invariant set to which all nearby orbits converge". The dynamical systems which we present display such behaviour so in the words of Mandelbrot we could describe our work as that of "fractal attractors". The notion of an attractor is partnered with that of a **repeller** which as the word implies admits opposing behaviour to that of an attractor.

A **symbolic sequence** is equipped with a natural shift map and we show that a **Sturmian**[13] **shift space** (Σ_α, S) is topologically conjugate to the closure of a **rotation sequence** $(\overline{\mathcal{R}}_\alpha, \rho_\alpha)$ (Theorem 1.28). The Fibonacci

[8] Gérard Rauzy (1938 - 2009) French mathematician.
[9] Benoit B. Mandelbrot (1924 - 2010) Polish mathematician.
[10] Felix Hausdorff (1868 - 1942) German mathematician.
[11] Abram Samoilovitch Besicovitch (1891 - 1970) Ukrainian mathematician.
[12] Georg Cantor (1845 - 1918) German mathematician.
[13] Jacques C. F. Sturm (1803 - 1855) French mathematician.

map φ (1.7) and the Cat map γ (1.9) are examples of **Sturmian substitutions** which have **incidence matrices** (Def. 1.34) $M_\varphi = \left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right]$ respectively. Informally an incidence matrix shows the frequency that a letter occurs in a word under the first iteration of its substitution map. The matrices M_φ and A also serve to define the **hyperbolic toral automorphisms** F_φ (1.8) and $\mathcal{C}$ (1.10) respectively. These maps play a significant role throughout this thesis.

Mandelbrot gives an example tiling in [34] using Gosper islands derived from a hexagon to cover the plane with self-similar tiles. He comments that most tiles cannot be subdivided into equal tiles similar to the whole but that some fractal tiles allow subdivision into different number of parts. The Rauzy fractal is an example of the latter type since the central tile subdivides into three self-similar tiles with a scale factor of $|\alpha| \approx |-0.4+0.6i|$. In contrast, our work continues by concentrating on one-dimensional tilings (Def. 1.40) and in particular **one-dimensional substitution tilings** which are **aperiodic** (Def. 2.12), loosely meaning that the tiling has no translational symmetry.

Chapter 2. The **circle dynamics** described here involve a **Denjoy map** D_α (2.2) which is a non-transititve C^1 orientation-preserving diffeomorphism of the circle with an irrational rotation number lying in the unit interval. This map is the consequence of a construction often referred to in texts as the 'Denjoy example' [14] [32]. The construction yields a set C_α and by the topological conjugacy of $(C_\alpha, D_{\alpha\restriction_{C_\alpha}})$ to (Σ_α, S) (Theorem 2.11) we prove that C_α and the Sturmian subshift Σ_α are minimal and indeed Cantor sets (Remark 2.24). The Denjoy minimal set C_α plays a key role in the next and future chapters.

Chapter 3. Informally a **suspension** (Def. 3.2) construction turns a map into a flow. By this process we embed the Denjoy map and hence its minimal map in a torus which crucially leads to a **Denjoy continuum** $\mathbb{D}_\alpha$ (Def. 3.9). We prove that the suspension of a **symbolic full shift space** $\mathcal{A}^{\mathbb{Z}}$ and a **full tiling space** $\mathcal{T}_{\mathscr{P}}$ support flows which are topologically conjugate (Prop. 3.15). We supply an illustrated Example 3.16 of these flows.

Chapter 4. The Plykin attractor formulated in detail in §5.2 requires some knowledge of a **branched covering** (Def. 4.1) so we describe such a structure in this chapter. In particular we consider a genus 2 surface $\mathbb{T}^2\sharp\mathbb{T}^2$ as a 2-fold branched covering of a torus $\mathbb{T}^2$. The space $\mathbb{T}^2\sharp\mathbb{T}^2$ is given both a combinatorial representation in the plane and a Euclidean[14] representation in $\mathbb{R}^3$. Then we compare a **covering flow** (Prop. 4.11) on $\mathbb{T}^2\sharp\mathbb{T}^2$ projected to a linear flow on $\mathbb{T}^2$, with supporting diagrams.

Chapter 5. Smale states in Theorem (3.3) of [51] that an Anosov diffeomorphism (Def. 5.3) of a compact manifold M is **structurally stable** (Def. 5.1). His horseshoe map with infinitely many periodic orbits is also structurally stable. The Smale solenoid is another example of a structurally stable attractor but is defined in a solid torus. The term **solenoid** is applied to a large class of spaces described as an **inverse limit** (Def. 6.3) with which we will have many dealings. In the 1950s, Williams gave the first example of an expansive homeomorphism on a compact connected metric space, that is the shift on the dyadic solenoid Σ_2. These ideas have strong links to the development of our work which begins in this chapter with the construction of a **DA diffeomorphism** (Def. 5.6) and its **DA attractor** (Def. 5.7). The DA attractor Λ is homeomorphic to $\mathbb{D}_\alpha$ (Remark 5.9). Then by realising the sphere as a quotient of the torus we describe the **Plykin attractor** Λ_Π and its projection Λ_P in the plane.

[14]Euclid of Alexandria (c. 325 BC - 265 BC) Greek mathematician.

Chapter 6. Theorem 6.20 : The attractor $\Lambda \subseteq \mathbb{T}^2$ is homeomorphic to the tiling space $\mathcal{T}_\gamma \subseteq \mathbb{T}^2$.

Theorem 6.32 : The lifted Plykin attractor $\tilde{\Lambda}_P \subseteq \mathbb{T}^2$ is homeomorphic to the tiling space $\mathcal{T}_{\omega^2} \subseteq \mathbb{T}^2$.

These two theorems arise in part from the coordination of several constructs from dynamical systems begun in the 1998 paper [3] of Anderson and Putnam. They showed that a substitution tiling homeomorphism is topologically conjugate to the shift on an inverse limit, deemed equivalent to one of Williams' generalised solenoids of [52], and that for a one-dimensional space the underlying structure is that of a **branched 1-manifold** (Def. 6.9). In order to differentiate between two different styles of representation of a branched 1-manifold, we call our first example a **complex** (Def. 6.2) K_γ after Barge and Diamond in their paper [8]. Then by comparison we call the branched 1-manifold $\mathcal{M}_0$, derived from a **Markov partition** (Def. 6.8), a **θ-space** as given in [48]. By using a Williams' construction which he explains in [54] we show that both our spaces reduce to an equivalent **elementary branched 1-manifold** (Def. 6.11), called a **rose** in the vocabulary of Barge and Diamond [7]. From the rose we extract equivalent **proper substitutions** (Def. 6.15) leading to Theorem 6.20, stated above.

The branched 1-manifold $\mathcal{M}_*$ in §6.2 represents the Plykin attractor which is non-orientable so cannot be homeomorphic to a substitution tiling space. But by using Yi's algorithm described in [57] we construct an **orientable double cover** representation $\tilde{\mathcal{M}}_*$. Then by using the self-map on $\tilde{\mathcal{M}}_*$ and **combinatorics** we are able to identify five distinct **first returns** to a branch point in $\tilde{\mathcal{M}}_*$ nominated as the origin. The solenoid $\tilde{\Sigma}_*$ associated to $\tilde{\mathcal{M}}_*$, the suspension $\mathcal{W}_c$ of the first return map and the solenoid Ω associated to the rose K_* are mutually homeomorphic spaces

(Remark 6.29). The consequence of this together with the construction of a proper substitution yield Theorem 6.32, stated above.

We feel it pertinent to paraphrase here Robinson's comment in [48] that in 1974 Plykin proved that if a hyperbolic attractor Λ in the plane with an attracting region N is not just a periodic orbit, then N must have at least three holes removed; but on the surface of a sphere four holes need to be removed. The latter describes our situation.

Chapter 7. On page 124 we define set B which consists of the **4 special points** in the torus which serve as a repelling set for the attractors with four fundamental domains. We prepare the classification criteria which evolve from the defining matrix M of a hyperbolic toral automorphism F (Def. 7.1). In particular $M \in GL(2, \mathbb{Z})$, the **general linear group**, where M has non-negative entries and so by the Perron[15]-Frobenius[16] Theorem (see for example Theorem 1.9.11 in [32]), M has a largest positive simple PF eigenvalue λ with an eigenvector $\mathbf{v}^u$ of strictly positive components. In our context $\mathbf{v}^u$ is expanding and its slope α is a **quadratic irrational**. We consider two subgroups of $GL(2, \mathbb{Z})$, namely $\mathscr{G} \cong \mathbb{Z} \oplus \mathbb{Z}_2$ (Theorem 7.6) and the quotient group $\mathbb{M} \cong GL(2, \mathbb{Z}/2\mathbb{Z})$ which partitions $GL(2, \mathbb{Z})$ into six matrix **types** (Def. 7.11) according to the **parity** of the matrix entries (Theorem 7.13). Then through commentary, supporting diagrams and tables of data we develop our construction method for building a toral attractor (Lemma 7.20) and its homeomorphic tiling space (Lemma 7.31). The **algorithm** on page 144 summarises this original method.

Chapter 8. This chapter describes our main results for a **classification of attractors and tiling spaces**. The **continued fraction expansion** of the slope α distinguishes homeomorphic attractors (Theorem 8.4).

[15]Oskar Perron (1880 - 1975) German mathematician.
[16]Ferdinand Georg Frobenius (1849 - 1917) German mathematician.

The **isotopy class** (Def. 8.6) of an attractor's self-homeomorphism corresponds to an element of $\mathbb{Z} \oplus \mathbb{Z}_2$ (Corollary 8.7). Determined by matrix type, the **permutation groups** arising from the repelling set are found to be isomorphic to subgroups of S_4 (Lemma 8.11) whilst the **'symmetries'** of the asymptotic path-components of the attractors produce groups which are isomorphic to subgroups of a regular tetrahedron or of a square (Theorems 8.15 and 8.18). Companion classifications are given for a substitution tiling space (Theorems 8.19 and 8.24). The chapter closes by classifying specific spaces which have already been discussed in the document.

Chapter 9. In this concluding chapter we interpret significant features of our research and suggest three further lines of enquiry which would be interesting to pursue. The construction of the attractors can be summarised by the algorithm to which we refer and which is listed precisely on page 144. We note that no arithmetical computation is required other than finding the images of the Markov partition under the linear transformation M. The remaining operations are combinatorial in nature. With regard to classification, the matrix M persists in importance in that it determines the group structure of our spaces up to isomorphism and the slope of the matrix eigenvector classifies the spaces up to homeomorphism.

The research in this thesis has concentrated on attractors emanating from set B consisting of four special points in the torus. Thus we feel it would be of interest to question what happens if we choose a different set of four points in the torus. Also, since our planar attractors can be lifted to the torus it would be of interest to see if we have accounted for all such attractors and if not, what lifts are possible. These questions are listed as 1. and 2. respectively. Question 3. raises the issue of Hausdorff dimension and whether this property could be exploited to classify attractors.

Chapter 1

Symbolic dynamics

This chapter introduces the notion of symbolic dynamics and coding. These serve to model an abstract dynamical system such as sequences in a shift space or a more concrete system such as tilings in a one-dimensional substitution tiling space whereas the digits of a radix β-expansion contribute to the plotting of the Rauzy fractal in the plane. The letters in a symbolic alphabet may or may not have numerical significance.

Throughout the document, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The definitions in §1.1 are adapted from [11], [14], [23] and [35].

1.1 Shift spaces

Definition 1.1. *Over a finite* alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$ *let the* full two-sided shift space *be* $\mathcal{A}^{\mathbb{Z}} = \{(u_i)_{i\in\mathbb{Z}} \,|\, u_i \in \mathcal{A}, \forall\, i \in \mathbb{Z}\}$, *endowed with the shift map* S (1.1).

Definition 1.2. *Over a finite* alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$ *let the* full one-sided shift space *be* $\mathcal{A}^{\mathbb{N}_0} = \{(u_i)_{i\in\mathbb{N}_0} \,|\, u_i \in \mathcal{A}, \forall\, i \in \mathbb{N}_0\}$, *endowed with the shift map* S (1.1).

For a one-sided or two-sided sequence $u = (u_i)$, let a *shift map* be defined by

$$S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}},\ S((u_i)) = (u_{i+1}),\ \forall\, i \in \mathbb{Z}. \tag{1.1}$$

The space $(\mathcal{A}^{\mathbb{Z}}, S)$ is a compact invertible dynamical system whereas the space $(\mathcal{A}^{\mathbb{N}_0}, S)$ is non-invertible since for a one-sided sequence, the leftmost symbol disappears under the shift and every point has n pre-images [14].

Let a *metric* on a shift space be defined by

$$\begin{aligned} &d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}, \\ &d(u,v) := \begin{cases} 2^{-k} & \text{if } u \neq v,\ k = \inf\{i \in \mathbb{N}_0, u_i \neq v_i \text{ or } u_{-i} \neq v_{-i}\}, \\ 0 & \text{if } u = v. \end{cases} \end{aligned} \tag{1.2}$$

The metric induces the product topology on $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{A}^{\mathbb{N}_0}$. In the product topology, periodic points are dense, and there are dense orbits. A closed shift-invariant subset of a full shift is called a *subshift*. The closure of a subshift $\overline{X} := \overline{\{S^n((u_i)_{i\in\mathbb{Z}}) \,|\, n \in \mathbb{Z}\}} \subseteq \mathcal{A}^{\mathbb{Z}}$ is *periodic* if $\exists\, u = (u_i) \in X$ and an integer k such that $X = \{u, S(u), \ldots, S^k(u) = u\}$. Otherwise it is *aperiodic* (almost-periodic). Let a *word* be a finite sequence of *letters* from an alphabet $\mathcal{A}$. Let $\mathcal{F}$ be a collection of words over $\mathcal{A}$ which we call *forbidden words.* A *shift of finite type* is a shift space X that can be described by some finite set $\mathcal{F}$ of forbidden word(s). That is, no word belonging to $\mathcal{F}$ occurs in a sequence $u \in X$.

The content and definitions of §1.1.1 and §1.1.2 have been sourced predominantly from [11].

1.1.1 β-expansion

Let the *floor* and *ceiling* functions be floor$(x) = \lfloor x \rfloor$, the largest integer not greater than x and ceiling$(x) = \lceil x \rceil$, the smallest integer not less than x.

Definition 1.3. *[11] Let $\beta > 1$ be a real number. A* β-expansion *of a real number $x \in [0,1]$ is the sequence $(x_i)_{i\geq 1}$ with values in $\mathcal{A}_\beta = \{0, 1, \ldots, \lceil\beta\rceil - 1 = m\}$ produced by the* β-transformation

$$T_\beta : x \mapsto \beta x \pmod 1,\ u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor,\ \forall\, i \geq 1,\ \textit{so } x = \sum_{i\geq 1} u_i \beta^{-i}.$$

We denote the β*-expansion of* 1 by $d_\beta(1) = (t_i)_{i\geq 1}$.

Definition 1.4. *A number β such that $d_\beta(1)$ is ultimately periodic is called a* Parry[1] number*. If $d_\beta(1)$ is finite then β is called a* simple Parry number*.*

For a simple Parry number we omit the ending zeros when writing $d_\beta(1)$.

Definition 1.5. *[11] Suppose β is a Parry number. Let $d^*_\beta(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*_\beta(1) = (t_1 \ldots t_{n-1}(t_n - 1))^\infty$ if $d_\beta(1) = t_1 \ldots t_{n-1} t_n$ is finite $(t_n \neq 0)$.*

Definition 1.6. *Let $\mathcal{A}_\beta^{\mathbb{N}} = \{(u_i)_{i\in\mathbb{N}} \,|\, u_i \in \mathcal{A}_\beta, \forall i \in \mathbb{N}\}$ be endowed with a one-sided* β-shift *$S : \mathcal{A}_\beta^{\mathbb{N}} \to \mathcal{A}_\beta^{\mathbb{N}}$, $S((u_i)) = (u_{i+1})$, $\forall i \in \mathbb{N}$.*

It suits to let $\bar{d}$ be a metric on $\mathcal{A}_\beta^{\mathbb{N}}$ similarly defined as in (1.2), but with $i \geq 1$. Now consider two sequences v and w with $v \neq w$. We say that v is *lexicographically* less than w whenever for $k \geq 1$, the first pair of non-matching digits has index k such that $v_k <_{\text{lex}} w_k$.

Definition 1.7. *Let $U_\beta = \{(u_i)_{i\in\mathbb{N}} \,|\, u_i \in \mathcal{A}_\beta \,|\, \forall k \geq 1,\ (u_i)_{i\geq k} <_{lex} d^*_\beta(1)\}$ be the shift invariant set of β-expansions.*

Definition 1.8. *Let $\overline{U}_\beta = \{(u_i) \in \mathcal{A}_\beta^{\mathbb{N}} \,|\, \forall k \geq 1, (u_i)_{i\geq k} \leq_{lex} d^*_\beta(1)\}$ denote the closure of U_β.*

[1] William (Bill) Parry, FRS (1934 - 2006) British mathematician.

Proposition 1.9. $(y_i)_{i\geq 1} \in \overline{U}_\beta \Leftrightarrow \forall j \in \mathbb{N}, (y_i)_{i\geq j} \leq_{lex} d^*_\beta(1)$.

Proof. Let $(y_i) \in \overline{U}_\beta$ then $\exists \forall n \in \mathbb{N}$ a point $(x_i^n) \in U_\beta$ such that $\bar{d}((y_i),(x_i^n)) < \frac{1}{2^n}$. If for $k \in \mathbb{N}, (y_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1)$ is false then $(y_i)_{i\geq k} >_{\text{lex}} d^*_\beta(1)$. Let us write $d^*_\beta(1) = (t_i)_{i\geq 1}$. For $j \geq k$, let n be the first index j such that $y_{k+n} \neq t_{1+n}$ then $y_{k+n} > t_{1+n}$, and $y_{k+j} = t_{1+j}$ for $0 \leq j < n$. But $\bar{d}((y_i),(x_i^n)) < \frac{1}{2^n} \Rightarrow y_i = x_i$ for $i = 1,\ldots,n$ giving $x_i^n = t_{1+n}$ for $1 \leq i < k+n$. This leads to the contradiction $(x_i^n)_{i\geq k} >_{\text{lex}} d^*_\beta(1)$. Thus $(y_i)_{i\geq k} \not>_{\text{lex}} d^*_\beta(1), (y_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1)$.

Conversely, suppose $(y_i)_{i\geq j} \leq_{\text{lex}} d^*_\beta(1)$. If $\forall j \in \mathbb{N}^+, (y_i)_{i\geq j} <_{\text{lex}} d^*_\beta(1)$, then $(y_i)_{i\geq j} \in U_\beta \subset \overline{U}_\beta$ by definition. Otherwise, $y = y_1y_2\ldots y_{j-1}t_1t_2\ldots$ and for $k < j, (y_i)_{i\geq k} <_{\text{lex}} d^*_\beta(1)$. For $m \in \mathbb{N}$, let $(x_i^m) = y_1y_2\ldots y_{j-1}t_1t_2\ldots t_m\overline{0}\ldots$ then for $n \geq m, S^n(x_i^m) = \overline{0}\ldots$. But $t_i \geq 1$ for at least one $i = q > m$ since (t_i) does not have a tail of zeros. Therefore $S^n(x_i^m)_{i\geq q} <_{\text{lex}} (t_i) = d^*_\beta(1) \Rightarrow S^n(x_i^m) \in U_\beta$. Since U_β is shift-invariant $\Rightarrow (x_i^m) \in U_\beta, \forall m \in \mathbb{N}$. $\square$

Definition 1.10. *[11] An algebraic integer $\beta > 1$ is a Pisot[2]-Vijayaraghavan[3], or* Pisot number, *if all its algebraic conjugates α satisfy $|\alpha| < 1$.*

In homage to the *Fibonacci*[4] polynomial $X^2 - X - 1$, the so called *Tribonacci* polynomial $X^3 - X^2 - X - 1$ is the characteristic polynomial of its real root $\beta \approx 1.8 > 1$ whose complex conjugate roots α and $\overline{\alpha}$ both have modulus less than 1. This classifies β as a Pisot number, which is also a Parry number. Furthermore, $1 = 1/\beta + 1/\beta^2 + 1/\beta^3$ so $d_\beta(1) = 111$ and $d^*_\beta(1) = (110)^\infty$. So let $\overline{U}_\beta$ be a subshift over the alphabet $\mathcal{A}_\beta = \{0,1\}$ then let $L := \{(u_i)_{i\in\mathbb{N}} \,|\, u_i \in \{0,1\} \,|\, \forall k \geq 1,\ (u_i)_{i\geq k} <_{\text{lex}} d^*_\beta(1)\}$ and $T := \{(u_i)_{i\in\mathbb{N}} \,|\, \forall i \in \mathbb{N},\ u_i \in \{0,1\},\ u_iu_{i+1}u_{i+2} = 0\}$.

[2]Charles Pisot (1910 - 1984) French mathematician.
[3]Tirukkannapuram Vijayaraghavan (1902 - 1955) Indian mathematician.
[4]Leonardo Pisano (1170 - 1250) Italian mathematician.

Proposition 1.11. *The sets T and L are equivalent descriptions for $\overline{U}_\beta$.*

Proof. Suppose $(u_i) \in L$. Assume $(u_i) \notin T$ so $\exists\, k_0 \in \mathbb{N}$ such that $u_{k_0}u_{k_0+1}u_{k_0+2} = 1$. By hypothesis, $(u_i) \in L$ so $(u_i)_{i\geq k_0} \leq_{\text{lex}} (t_i) = (110\overline{110})$. So $(u_{k_0}u_{k_0+1}u_{k_0+2}u_{k_0+3}\ldots) = (111u_{k_0+3}\ldots) \leq_{\text{lex}} (110\overline{110})$, which is false. Therefore $(u_i) \in T \Rightarrow L \subset T$. Conversely, let $(u_i) \in T$. Suppose $\exists\, k \in \mathbb{N}$ such that $(u_i)_{i\geq k} >^{\text{lex}} (110\overline{110})$. Let $l = \min\{j \geq k | u_j \neq t_{l-k+2}\}$. By hypothesis, $(u_i)_{i\geq k} >^{\text{lex}} (110\overline{110}) \Rightarrow u_l > t_{l-k+1}$. Thus $u_l = 1$ and $t_{l-k+1} = 0 \Rightarrow l - k + 1 \geq 3 \Rightarrow l \geq k+2$. So $u_{l-1}u_{l-2} = t_{l-k}t_{l-k-1} = 11$ giving $u_l = u_{l-1} = u_{l-2} = 1$. This contradicts $(u_i) \in T$. Thus $(u_i)_{i\geq k} \leq_{\text{lex}} (t_i) = (110\overline{110})$ and $T \subset L$. Therefore $L = T$ and the two descriptions for $\overline{U}_\beta$ are equivalent. □

Remark 1.12. *$\overline{U}_\beta$ is a subshift of finite type with forbidden word $\mathcal{F} = \{111\}$ when β is the Pisot root of the Tribonacci polynomial.*

1.1.2 The Rauzy fractal

The Rauzy fractal is introduced in his paper [47].

Definition 1.13. *The Tribonacci polynomial has a real root $\beta > 1$. Let α be one of the two complex conjugate roots, $|\alpha| < 1$. Then the* Rauzy fractal *(central tile) is a compact subset of $\mathbb{C}$ defined by*

$$\mathcal{T}_\beta = \Big\{ \sum_{i\geq 0} u_i\alpha^i \mid \forall i \in \mathbb{N}_0,\, u_i \in \{0,1\} \mid u_iu_{i+1}u_{i+2} = 0 \Big\}.$$

Define a set $A := \{(u_i) \in \{0,1\}^{\mathbb{N}_0} \,|\, \forall i \in \mathbb{N}_0,\, u_iu_{i+1}u_{i+2} = 0\}$ which is compact since $A \subset \mathcal{A}^{\mathbb{N}_0}$. Let the *$\beta$-shift map* be $S : \{0,1\}^{\mathbb{N}_0} \to \{0,1\}^{\mathbb{N}_0}$, $(u_i) \mapsto (u_{i+1})$, $\forall i \geq 0$ then define $B := \{S^n((u_i)_{i\in\mathbb{N}_0}) \,|\, n \in \mathbb{N}_0\} \subseteq A$ which is dense in A. Let a continuous *plot* function be defined for fixed $\alpha \in \mathbb{C}$, $|\alpha| < 1$, by

$$f : A \to \mathcal{T}_\beta,\ f((u_i)) = \sum_{i\geq 0} u_i\alpha^i. \tag{1.3}$$

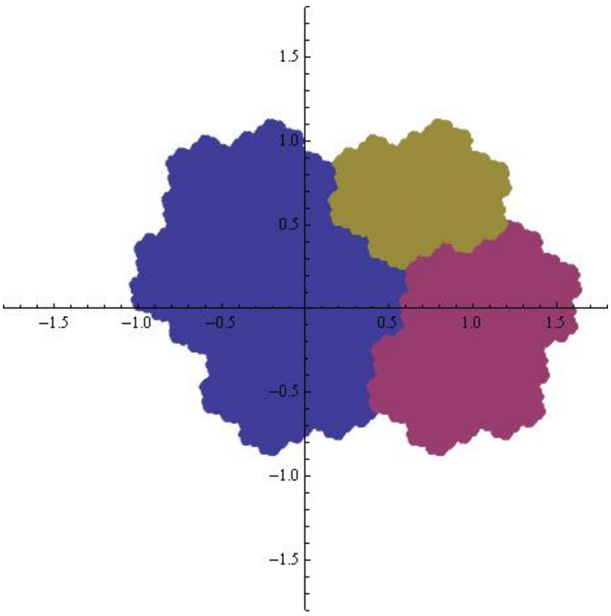


Figure 1.1: The Rauzy fractal, $\mathcal{T}_\beta$.

For $n = 0, \ldots, k-1$ and $\alpha \approx -0.4 + 0.6i$, $f\big(S^n((u_i)_{i \in \mathbb{N}_0})\big)$ generates the Rauzy fractal of Figure 1.1. This shows the division of the central tile into its three coloured *basic tiles* according to the first three digits of the sequence (u_i). That is, $f(u_0u_1u_2)$ generates $f(011) = 0 + \alpha + \alpha^2$ which leads to $\mathcal{T}_\beta(1)$ (blue); $f(101) = 1 + 0 + \alpha^2$ which leads to $\mathcal{T}_\beta(2)$ (red); and $f(110) = 1 + \alpha + 0$ which leads to $\mathcal{T}_\beta(3)$ (green). Applying f with respect to the conjugate root $\overline{\alpha}$ reflects the plotted image in the horizontal axis.

Note. We used [55] to plot $\mathcal{T}_\beta$. To avoid a *null* output from our *Mathematica* program for terms with large i in f, we modified the plot function to read $f^*((u_j)) = \sum_{j=0}^{19} u_j\, \alpha^j$, where $j = i \,(\mathrm{mod}\ 20)$, $i \geq 0$. This led to a map $f^*(S^m((u_j)))$, for $m = 0, \ldots, k-1$ where $k = \frac{|u_i|}{20}$. This was a sufficient number of terms to produce the fractal image of Figure 1.1.

The central tile as attractor

As seen in Figure 1.1, $\mathcal{T}_\beta = \mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2) \cup \mathcal{T}_\beta(3)$, where the basic tiles are determined by the leading digits in the β-expansion. If we now consider the central tile as an attractor with contracting scale factor $|\alpha| < 1$ (Def. 1.14) then we have the following computations by using the images of the function f found above:

$\alpha\big((0+\alpha+\alpha^2)+(1+0+\alpha^2)+(1+\alpha+0)\big)$ gives the terms $\{0, \alpha, \alpha^2, \alpha^3\}$ where the first three terms correspond to $\mathcal{T}_\beta(1)$;

$1 + \alpha(0 + \alpha + \alpha^2)$ gives the terms $\{1, 0, \alpha^2, \alpha^3\}$ where the first three terms correspond to $\mathcal{T}_\beta(2)$;

$1 + \alpha(1 + 0 + \alpha^2)$ gives the terms $\{1, \alpha, 0, \alpha^3\}$ where the first three terms correspond to $\mathcal{T}_\beta(3)$.

It follows from these results that

$$\begin{cases} \mathcal{T}_\beta(1) &= \alpha(\mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2) \cup \mathcal{T}_\beta(3)), \\ \mathcal{T}_\beta(2) &= \alpha(\mathcal{T}_\beta(1)) + 1, \\ \mathcal{T}_\beta(3) &= \alpha(\mathcal{T}_\beta(2)) + 1. \end{cases}$$

This suggests that we can define a family of contracting similarities $\{\Lambda_1, \Lambda_2\}$ called an *iterated function system* (IFS) as follows:

Definition 1.14. *For* $z \in \mathcal{T}_\beta = \bigcup_{n=1}^{3} \mathcal{T}_\beta(n)$ *and* $|\alpha| < 1$ *let*

$$\Lambda_1, \Lambda_2 : \mathbb{R}^2 \to \mathbb{R}^2,\ \Lambda_1(z) = \alpha z,\ \Lambda_2(z) = \alpha z + 1.$$

Then $\Lambda_1(z) = z' \in \mathcal{T}_\beta(1)$ *and* $\Lambda_2(z) = \begin{cases} z' \in \mathcal{T}_\beta(2) & \text{if } z \in \mathcal{T}_\beta(1), \\ z' \in \mathcal{T}_\beta(3) & \text{if } z \in \mathcal{T}_\beta(2). \end{cases}$

Proposition 1.15. *The set $\mathcal{T}_\beta$ is the unique attractor of the IFS* $\{\Lambda_1, \Lambda_2\}$.

Proof. By Definition 1.14, $\mathcal{T}_\beta = \bigcup_{i=1}^{2} \Lambda_i(\mathcal{T}_\beta)$ so $\mathcal{T}_\beta$ is invariant under $\{\Lambda_1, \Lambda_2\}$. Now $\mathcal{T}_\beta$ is a non-empty compact subset of $\mathbb{R}^2$ and each similarity Λ_1, Λ_2 has a contraction ratio of $|\alpha| < 1$ on $\mathbb{R}^2$. Thus, by the uniqueness of the invariant set, $\mathcal{T}_\beta$ is the unique attractor for the IFS (see for example Theorem 9.1 in [21]). $\square$

The definitions in §1.1.3 and §1.1.4 follow those of chapter 6 in [23].

1.1.3 The Sturmian family

Definition 1.16. *Let u be a sequence over a finite alphabet then the* complexity function *of u is $p_u(n)$ which, to each $n \in \mathbb{N}$, associates the number of distinct words of length n that occur in u.*

Definition 1.17. *[23] A sequence is called* Sturmian *if $p_u(n) = n + 1$.*

Definition 1.18. *Let the set of* one-sided Sturmian *sequences be*

$$\Sigma^+ = \{(u_i)_{i\in\mathbb{N}_0} \mid u_i \in \{0,1\}, \forall\, i \in \mathbb{N}_0\}$$

and the set of bi-infinite Sturmian *sequences be*

$$\Sigma = \{(u_i)_{i\in\mathbb{Z}} \mid u_i \in \{0,1\}, \forall\, i \in \mathbb{Z}\}.$$

Definition 1.19. *A Sturmian sequence is of* type 0 *if* 1 *is isolated so that* 11 *does not occur in any word. A* type 1 *sequence is one in which* 0 *is isolated.*

Remark 1.20. *Denote the* length *of a finite word u by $|u|$ and the* number of times *letter $a \in \mathcal{A}$ appears in u as $|u|_a$. Define the* frequency *of the letter* 1 *occurring in the positive semi-orbit of $u \in \Sigma_\alpha$ as the limit of $\frac{|u_0 u_1 \ldots u_{n-1}|_1}{n}$*

as n tends to infinity. Since Sturmian sequences are balanced*, that is* $||v|_1 - |w|_1| \leq 1$ *for subwords* v, w *of equal length in* u*, then* $\lim_{n\to\infty} \frac{|u_0u_1\ldots u_{n-1}|_1}{n} = \lim_{n\to\infty} \frac{|u_{-n+1}\ldots u_{-1}u_0|_1}{n}$*, this limit being well defined and irrational (see for example Prop. 6.1.10 in [23]). Call this* limit α*.*

Definition 1.21. *Denote the* orbit closure *of* $(u_i) \in \Sigma$ *under the shift map as* $\Sigma_\alpha = \overline{\{S^n((u_i)_{i\in\mathbb{Z}}) \mid n \in \mathbb{Z}\}}$.

Proposition 1.22. *The closure* Σ_α *is invariant under the shift map* S*.*

Proof. Let $(u_i)_{i\in\mathbb{Z}}, u_iu_{i+1} \neq 11, \forall i \in \mathbb{Z}$, be a type 0 sequence in Σ. Let $(w_i)_{i\in\mathbb{Z}} = S((u_i)_{i\in\mathbb{Z}})$ then $\forall i \in \mathbb{Z}$, $(w_i) = (u_{i+1}) \Rightarrow w_iw_{i+1} \neq 11$ so (w_i) is of type 0 and $(w_i) \in \Sigma$ since $(u_{i+1}) \in \Sigma$. Similarly, if $u_iu_{i+1} \neq 00, \forall i \in \mathbb{Z}$, then (u_i) is a type 1 sequence in Σ and then so too is $(w_i) \in \Sigma$. It is known that the closure of a shift-invariant set is also shift-invariant. Thus $S(\Sigma_\alpha) \subseteq \Sigma_\alpha$. □

The invertible Sturmian dynamical system $(\Sigma_\alpha, S) \subseteq (\mathcal{A}^{\mathbb{Z}}, S)$.

1.1.4 Rotating sequences

For $\alpha \in [0, 1] - \mathbb{Q}$ let an anticlockwise rotation of the circle be defined by

$$R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ R_\alpha([x]) = [x + \alpha], \tag{1.4}$$

where $[\cdot]$ denotes modulo 1. For all $[x] \in \mathbb{R}/\mathbb{Z}$ and irrational α, the orbit $\{R_\alpha^n([x]) \mid n \in \mathbb{Z}\}$ is dense in $\mathbb{R}/\mathbb{Z}$. With slight abuse of terminology, let a *partition* of $\mathbb{R}/\mathbb{Z}$ be given by $\mathcal{P} := \{[0, \alpha], [\alpha, 1]\} = \{J_0, J_1\}$, with respect to limit α (Remark 1.20).

Definition 1.23. *The* coding κ*, relative to* $\mathcal{P}$*, is defined over the union of two sets:*

1. For $[x] \in \mathbb{R}/\mathbb{Z} - \{R^n_\alpha([0]) \mid n \in \mathbb{Z}\}$, $\kappa([x]) = (x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, *where*

$$\forall n \in \mathbb{Z},\ x_n = \begin{cases} 0 & \text{if } R^n_\alpha([x]) \in J_0, \\ 1 & \text{if } R^n_\alpha([x]) \in J_1. \end{cases}$$

2. For $[x] \in \{R^n_\alpha([0]) \mid n \in \mathbb{Z}\}$,

(i) if $R^n_\alpha([0]) \in (J_0 \cup J_1) \setminus (J_0 \cap J_1)$ *then code as in 1, otherwise*

(ii) when $R^n_\alpha([0]) \in J_0 \cap J_1$ *assign two codes to each point of intersection as follows:*

for some $n \in \mathbb{Z}$, *at* $[0]$ *set* $x^a_n = 1$ *and* $x^b_n = 0$ *and then at* $[\alpha]$ *set* $x^a_{n+1} = 0$ *and* $x^b_{n+1} = 1$. *Denote the resulting coded sequences as* $\kappa([x]^a) = (x^a_n)_{n\in\mathbb{Z}}$ *and* $\kappa([x]^b) = (x^b_n)_{n\in\mathbb{Z}}$ *where* $(x^a_n)_{n\in\mathbb{Z}}, (x^b_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$.

For all $[x]$ in the orbit $\{R^n([0]) \mid n \in \mathbb{Z}\}$, if $R^n([x]) \in (J_0 \cup J_1) \setminus (J_0 \cap J_1)$ then the code assigned to x_n is unique. However each $[x]$ carries two positions of ambiguity in the coding of its itinerary, namely when its orbit lands consecutively on the common endpoints $J_0 \cap J_1 = [0]$ and $J_0 \cap J_1 = [\alpha]$ in either order. At this time, to each position, we assign two codes such that the sequence complies with existing constraints. This event happens when $\exists\, n \geq 0$ in the positive semi-orbit with $R^n_\alpha([x]) = [0] \Rightarrow R^{n+1}_\alpha([x]) = [\alpha]$ and when in the negative semi-orbit $\exists\, n < 0$ with $R^n_\alpha([x]) = [\alpha] \Rightarrow R^{n-1}_\alpha([x]) = [0]$. Significantly, the coding of an orbit $\{R^n_\alpha([x])\}$ generates a rotation sequence, defined below.

Example 1.24. Coding the orbit $\{R^n_\alpha([0])\} = \{\ldots, [x]_{-1}, [0], [\alpha], [x]_2, \ldots\}$ by κ gives $(x^a_n) = \ldots 0 \cdot 100 \ldots$ and $(x^b_n) = \ldots 0 \cdot 010 \ldots$, as seen in Figure 1.2.

Denote the integer part of x by its floor $\lfloor x \rfloor := \sup\{n \in \mathbb{Z} \mid n \leq x\}$.

Definition 1.25. *A* rotation sequence *is a sequence* u *with* $\alpha \in [0,1] - \mathbb{Q}$ *and* $\iota \in \mathbb{R}$ *such that* $u_n = \lfloor (n+1)\alpha + \iota \rfloor - \lfloor n\alpha + \iota \rfloor$, $\forall n \in \mathbb{Z}$. *We call* α *the* angle *of the rotation sequence and* ι *the* initial point.

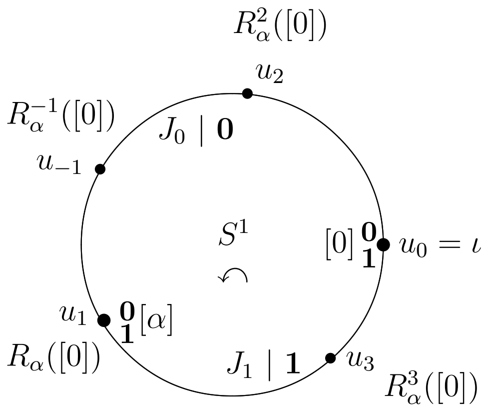


Figure 1.2: Coded rotations.

Proposition 1.26. *A rotation sequence $(u_n)_{n\in\mathbb{Z}}$ is such that $u_n \in \{0,1\}, \forall n \in \mathbb{Z}$.*

Proof. Consider

$$\begin{aligned} u_n &= \lfloor (n+1)\alpha + \iota \rfloor - \lfloor n\alpha + \iota \rfloor \\ &= \sup\{p \in \mathbb{Z} \mid p \leq (n+1)\alpha + \iota\} - \sup\{q \in \mathbb{Z} \mid q \leq n\alpha + \iota\} \\ &\Rightarrow |p - q| \leq (n+1)\alpha + \iota - (n\alpha + \iota) \leq \alpha \\ &\Rightarrow |p - q| \in \{0, 1\}. \end{aligned}$$

□

Given $\alpha \in [0,1] - \mathbb{Q}$, let the set of α-rotation sequences be defined by $\mathcal{R}_\alpha := \{(u_n)_{n\in\mathbb{Z}} \mid u_n = \lfloor (n+1)\alpha + \iota \rfloor - \lfloor n\alpha + \iota \rfloor,\ \forall n \in \mathbb{Z},\ \iota \in \mathbb{R}\}$. Then define a map

$$\rho_\alpha : \mathcal{R}_\alpha \to \mathcal{R}_\alpha,\ \rho_\alpha((u_n)) = (u_{n+1}), \forall n \in \mathbb{Z}. \tag{1.5}$$

Definition 1.27. *Let $\overline{\mathcal{R}}_\alpha = \overline{\{\rho_\alpha^k((u_n)_{n\in\mathbb{Z}}) \mid k \in \mathbb{Z}\}}$ denote the* orbit closure *of $(u_n)_{n\in\mathbb{Z}} \in \mathcal{R}_\alpha$.*

Then $(\overline{\mathcal{R}}_\alpha, \rho_\alpha) \subseteq \mathcal{A}^{\mathbb{Z}}$ over alphabet $\mathcal{A} = \{0, 1\}$. The construction of coding κ shares the properties of a Sturmian sequence. Since parameter α describes the frequency of a letter in a sequence $v \in \Sigma_\alpha$, the partition $\mathcal{P}$ will determine the type of Sturmian sequence generated by the orbit of a point $[x] \in \mathbb{R}/\mathbb{Z}$ under a rotation R_α. It follows that when length $l(J_1) < l(J_0)$ or $l(J_1) > l(J_0)$ the Sturmian will be a type 0 or type 1 sequence respectively. The connection between a rotation sequence and a Sturmian shift sequence is made explicit in the following theorem.

Theorem 1.28. *For $u \in \overline{\mathcal{R}}_\alpha$, with fixed $\alpha \in [0, 1] - \mathbb{Q}$, there exists $v \in \Sigma_\alpha$ such that $u = v$.*

We give a proof that a rotation sequence is a Sturmian sequence. For a proof of the converse, see for example Theorem 6.4.22 in [23].

Proof. Consider a sequence $(x_n)_{n \in \mathbb{N}_0}$ obtained from the coding κ in following the positive semi-orbit of $R_\alpha^n([0]), n \geq 0$. Partitioning of the circle into two intervals J_0 and J_1 imposes a cardinality of two on an alphabet $\mathcal{A}$, let it be $\{0, 1\}$. The letter $x_n \in \mathcal{A}$ identifies which interval contains $R_\alpha^n([0])$; or knowing which interval $R_\alpha^n([0])$ belongs to determines the letter x_n. However, when $R_\alpha^n([0]) \in J_0 \cap J_1$, x_n is an arbitrary letter belonging to $\mathcal{A}$. Now the first n letters of the sequence gives the itinerary of $[0]$ for its first n iterates under the rotation. Similarly, for any subword w of length m in (x_n), its letters will give the position of each iterate m of w with respect to partition $\mathcal{P}$. Observe that the first iteration $R_\alpha([0])$ divides the circle into two subintervals and yields two choices for a subword of length one $x_0 = 0$ or $x_0 = 1$. The second iteration $R_\alpha^2([0])$ divides the circle into three subintervals and yields three choices which are, depending on the value of α, a subword of length two $x_0x_1 = 00$ or 01 or 10 alternatively the three choices are $x_0x_1 = 11$ or 10 or 01. That is, after n iterations $R_\alpha^n([0])$ divides the circle into $n+1$ subintervals yielding $n+1$ choices for a subword of length n. This means that there can be no more than $n+1$ different words of length n in the coding of a semi-orbit under R_α^n. Iterating along the negative semi-orbit will likewise produce a subword of length $|n|$ for $n<0$ in $R_\alpha^n([0])$. Thus over the whole orbit of $R_\alpha^n([0])$, κ will code a maximum of $n+1$ unique words of length n. That is, for a bi-infinite rotation sequence $x=(x_i)_{i\in\mathbb{Z}} \in \overline{\mathcal{R}}_\alpha$ it has complexity $p_x(n)=n+1, n\in\mathbb{N}$, over an alphabet of two symbols. Hence by definition x is Sturmian. $\square$

Definition 1.29. *[14] Let $f: X\to X$ and $g: Y\to Y$ be topological dynamical systems. A* topological semi-conjugacy *from g to f is a surjective continuous map $h: Y\to X$ such that $f\circ h = h\circ g$. If h is a homeomorphism, it is called a* topological conjugacy *and f and g are said to be* topologically conjugate*.*

Remark 1.30. *A topological conjugacy follows from Theorem 1.28. That is $(\overline{\mathcal{R}}_\alpha, \rho_\alpha) \cong (\Sigma_\alpha, S)$.*

The content and definitions in §1.2 and §1.3 are adapted from [7] and the books [23], [50].

1.2 Substitutions

Definition 1.31. *With a finite alphabet $\mathcal{A}=\{a_1,\dots,a_n\}$ and $\mathcal{A}^*=\{$non-empty set of finite words over $\mathcal{A}\}$ define a* substitution map *$\sigma:\mathcal{A}\to\mathcal{A}^*$ where to each $a\in\mathcal{A}$, $\sigma(a)\in\mathcal{A}^*$.*

The map extends naturally to $\sigma:\mathcal{A}^*\to\mathcal{A}^*$ and $\sigma(u\,v)=\sigma(u)\sigma(v)$ for $u,v\in\mathcal{A}^*$ which extends to bi-infinite words $\mathcal{A}^{\mathbb{Z}}$ by concatenation. Namely, for $\{w_i\}_{i\in\mathbb{Z}} \subseteq \mathcal{A}^*$, $\sigma(\cdots w_{-2}w_{-1}.\,w_0w_1\cdots) = \cdots\sigma(w_{-2})\sigma(w_{-1}).\,\sigma(w_0)\sigma(w_1)\cdots$. Endowed with the metric (1.2), $(\mathcal{A}^*\cup\mathcal{A}^{\mathbb{Z}}, d)$ is a metric space.

Definition 1.32. *A substitution σ is* irreducible *if for any $a,b\in\mathcal{A}$ there is $n(a,b)\in\mathbb{N}$ such that $\sigma^{n(a,b)}(a)$ contains b.*

Definition 1.33. *A substitution σ is* primitive *if $\exists\, n \in \mathbb{N}$ such that for every $a, b \in \mathcal{A}$ the letter a occurs in $\sigma^n(b)$.*

Denote the *length* of a finite word u by $|u|$ and the number of times letter $a \in \mathcal{A}$ appears in u as $|u|_a$.

Definition 1.34. *An $n \times n$* incidence matrix *M_σ associated with substitution σ with* $\mathrm{card}(\mathcal{A}) = n$ *is given by*

$$a, b \in \mathcal{A},\ [M_\sigma]_{a,b} = |\sigma(b)|_a.$$

Definition 1.35. *An $n \times n$ matrix M is* irreducible *if for each $i, j\ \exists\, k = k(i,j) > 0$ such that $[M^k]_{i,j} > 0$, where $[M^k]_{i,j}$ denotes the (i,j)th entry of the kth power of M.*

Definition 1.36. *An $n \times n$ matrix M is* aperiodic *if $\exists\, k > 0$ such that $\forall\, i, j,\ [M^k]_{i,j} > 0$.*

Matrices which are both irreducible and aperiodic are *primitive*. A substitution is primitive if its incidence matrix is primitive. Primitivity ensures that there exists a *periodic point* $(a_i)_{i \geq 0} \in \mathcal{A}^{\mathbb{N}_0}$ such that $\sigma^n((a_i)) = (a_i)_{i \geq 0}$ is fixed for some iterate $n \in \mathbb{N}$ and similarly in $\mathcal{A}^{\mathbb{Z}}$ there exists a *bi-infinite periodic point* $(a_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that $\sigma^m((a_i)) = (a_i)_{i \in \mathbb{Z}}$ is fixed for some iterate $m \in \mathbb{N}$. Further, if $\sigma(a) = a \ldots$ and $|\sigma^n(a)| \to \infty$ as $n \to \infty$ then $\sigma^\infty(a)$ is a *fixed point* of σ.

Let the *Tribonacci substitution* be defined over $\mathcal{A} = \{0, 1, 2\}$ by

$$\tau : \mathcal{A} \to \mathcal{A}^*,\ 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0. \tag{1.6}$$

Under its forward semi-orbit $\tau^\infty(0)$ is the unique fixed point of the primitive substitution τ. Since τ is defined over a three letter alphabet it cannot be Sturmian.

Definition 1.37. *A substitution σ is* Sturmian *if the image by σ of any Sturmian sequence is a Sturmian sequence.*

Let the *Fibonacci substitution* be defined over $\mathcal{A} = \{0, 1\}$ by

$$\varphi : \mathcal{A} \to \mathcal{A}^*,\ 0 \mapsto 01,\ 1 \mapsto 0. \tag{1.7}$$

Proposition 1.38. *The Fibonacci substitution φ is Sturmian.*

Proof. The substitution is defined over a two letter alphabet. The words $\varphi(01) = 010$ and $\varphi(10) = 001$ are of equal length and differ only in their second and third indices, with 10 replaced by 01. Conversely, let $v = v_0 \dots v_j \in \mathcal{A}^*$, $j \in \mathbb{N}$, and suppose that words $u, w \in \mathcal{A}^*$ are such that $|u| = |w|$ where for $i \in \mathbb{N}$, $i > j$, $u = vu_iu_{i+1} = v10$ and $w = vw_iw_{i+1} = v01$. Then the pre-image $v_j = u_{i-1} = 0 = w_{i-1}$ where $010 = \varphi(01)$ and $001 = \varphi(10)$. Thus φ is Sturmian (Prop. 6.7.8 in [23]). □

Let u be a one-sided Fibonnaci Sturmian sequence of type 0. Since $\varphi(0) = 0\,1$ and the length $|\varphi^n(0)| \to \infty$ as $n \to \infty$ then $u = \varphi^\infty(0)$ is the unique fixed point of φ under its forward semi-orbit. Its two pre-images $01.u$ and $10.u$ are fixed points of period 2 of the substitution.

Denote the incidence matrix of the Fibonacci substitution φ by $M_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The determinant $\det(M_\varphi) = -1$ so φ is unimodular and orientation-reversing. The Perron-Frobenius eigenvalue of M_φ is the *golden ratio* $\mu = \frac{1}{2}(1 + \sqrt{5})$ with left eigenvector $\mathbf{v}^u = [\mu, 1]$ while $\bar{\mu} = \frac{1}{2}(1 - \sqrt{5}) = -\frac{1}{\mu}$ with eigenvector $\mathbf{v}^s = [-1, \mu]$. Since $\mu > 1$ is an algebraic integer with $|\bar{\mu}| < 1$, φ is a *Pisot type* substitution. Furthermore, since neither eigenvalue has modulus 1, the linear transformation $M_\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ induces a hyperbolic toral automorphism defined in (1.8).

Define the torus $\mathbb{T}^2 := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and let $\bar{[\cdot]}$ denote an equivalence class in $\mathbb{T}^2$.

$$F_\varphi : \mathbb{T}^2 \to \mathbb{T}^2,\ F_\varphi\left(\overline{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \overline{M_\varphi \begin{bmatrix} x \\ y \end{bmatrix}}. \tag{1.8}$$

Let the *Cat substitution* be defined over $\mathcal{A} = \{0, 1\}$ by

$$\gamma : \mathcal{A} \to \mathcal{A}^*,\ 0 \mapsto 010,\ 1 \mapsto 01. \tag{1.9}$$

The substitution γ is also of Pisot type and is Sturmian since $\gamma(0) = \varphi^2(0) = 010$ and $\gamma(1) = \varphi^2(1) = 01$. Let the incidence matrix be $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = M_\varphi^2$, $\det(A) = 1$, PF eigenvalue $\mu^2 = \frac{1}{2}(3 + \sqrt{5}) > 1$, $\mathbf{v}^u = [\mu, 1]$ and $0 < \mu^{-2} = \frac{1}{2}(3 - \sqrt{5}) < 1$, $\mathbf{v}^s = [-1, \mu]$. Then A induces the *Cat map* defined by

$$\mathcal{C} : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathcal{C}\left(\overline{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \overline{A \begin{bmatrix} x \\ y \end{bmatrix}}. \tag{1.10}$$

1.3 One-dimensional tilings

Definition 1.39. *A* tile $T \subset \mathbb{R}$ *is a closed interval.*

Let a finite collection of *prototiles* be $\mathscr{P} = \{P_1, \ldots, P_n\}, n \in \mathbb{N}$.

Definition 1.40. *A* tiling T *is made up of prototiles from* $\mathscr{P}$ *where* $T = \{T_i\}_{i\in\mathbb{Z}}$ *with* $\cup_{i\in\mathbb{Z}} T_i = \mathbb{R}$ *and where each tile is a translate of some prototile* $P \in \mathscr{P}$ *satisfying the following conditions: when* $i \neq j$ *either* $T_i \cap T_j = \emptyset$ *or otherwise it is a point. Taking a natural ordering, we place* T_i *to the left of* T_j *when* $i < j$ *and let* $0 \in T_0 \setminus T_1$*.*

Definition 1.41. *Let* $\mathcal{T}_{\mathscr{P}}$ *be the* full tiling space *of all possible tilings of* $\mathbb{R}$ *by prototiles from the set* $\mathscr{P}$*.*

Define the *tiling metric* by

$$\underline{d} : \mathcal{T}_{\mathscr{P}} \times \mathcal{T}_{\mathscr{P}} \to \mathbb{R}, \tag{1.11}$$
$$\underline{d}(T, T') := \inf\Big(\{1\} \cup \{\epsilon > 0 \mid T + v \text{ and } T' + v' \text{ agree on } B_{1/\epsilon}(0) \text{ for some } |v|, |v'| < \epsilon/2\}\Big),$$

where $B_r(0)$ is the open ball of radius r around the origin. We say that two tilings are 'ϵ-close' if, after a small translation less than or equal to ϵ, the tilings agree on a ball of radius $r = 1/\epsilon$ around the origin. The tiling topology induced by $\underline{d}$ on $\mathcal{T}_{\mathscr{P}}$ is compact and metric. By the continuous $\mathbb{R}$-action on a tiling T define a flow

$$\phi : \mathbb{R} \times \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}},\ \phi(t, T) = T - t. \tag{1.12}$$

Definition 1.42. *The* orbit *of a tiling T is the set $\mathcal{O}(T) = \{T - t \mid t \in \mathbb{R}\}$ of translates of T.*

Definition 1.43. *A* tiling space $\mathcal{T}$ *of $T \in \mathcal{T}_{\mathscr{P}}$ is the orbit closure $\overline{\mathcal{O}(T)}$.*

1.3.1 Substitution tilings

A substitution rule σ induces an inflation map on a tiling, characterised by expansion and translation. Our description follows that of [7]. Take a primitive substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$, where $\operatorname{card}(\mathcal{A}) = j$, matrix M_σ has a PF eigenvalue λ_σ with left eigenvector $\mathbf{v}_\sigma = [v_1, \ldots, v_j]$. Let a tiling T be made from a set of prototiles $\mathscr{P}$ and let the length $[0, v_j]$ of each $P_j \in \mathscr{P}$ be equal to the entry v_j in $\mathbf{v}_\sigma$, that is $|P_j| = v_j$. For $a_j \in \mathcal{A}$, if $\sigma(a_j) = a_{j_1} a_{j_2} \ldots a_{j_n} \in \mathcal{A}^*$ then $\lambda_\sigma v_j = \sum_{i=1}^n v_{j_i}$. So the length $|\lambda_\sigma P_j| = \sum_{i=1}^n |P_{j_i}|$ and $\lambda_\sigma P_j$ is tiled by $\{T_i\}_{i=1}^n$ where $T_i = P_{j_i} + \sum_{k=1}^{i-1} v_{j_k}$. This process of inflating, substituting and suitably translating each T_i extends to the map (1.13) which takes a tiling $T = \{T_i\}_{i \in \mathbb{Z}}$ of $\mathbb{R}$ by prototiles to a new tiling $T' = F_\sigma(T)$ of $\mathbb{R}$ by prototiles.

Definition 1.44. *Define $\mathcal{T}_\sigma$ to be the* substitution tiling space *arising from a substitution σ.*

Then for $w = w_1 \dots w_n \in \mathcal{A}^*$ and $t \in \mathbb{R}$ define the *inflation and substitution homeomorphism* [7] by

$$F_\sigma : \mathcal{T}_\sigma \to \mathcal{T}_\sigma,\ \mathcal{P}_w + t = \{P_{w_1} + t, P_{w_2} + t + |P_{w_1}|, \dots, P_{w_n} + t + \Sigma_{i<n}|P_{w_i}|\}. \quad (1.13)$$

Then $F_\sigma(P_i + t) = \mathcal{P}_{\sigma(i)} + \lambda_\sigma t$ and $F_\sigma(\{P_{k_i} + t_i\}_{i\in\mathbb{Z}}) = \bigcup_{i\in\mathbb{Z}}(\mathcal{P}_{\sigma(k_i)} + \lambda_\sigma t_i)$ where $P_k \in \mathscr{P}$ and i is the position of P_k in the tiling T. Let the inflation and substitution homeomorphism of the Cat substitution (1.9) be defined by

$$F_\gamma : \mathcal{T}_\gamma \to \mathcal{T}_\gamma, \quad (1.14)$$

$$\begin{aligned}
F_\gamma(\mathbf{0} + t) &= \mathcal{P}_{\gamma(\mathbf{0})} + \mu^2 t \\
&= \mathcal{P}_{\mathbf{010}} + \mu^2 t \\
&= \{P_{\mathbf{0}} + \mu^2 t, P_{\mathbf{1}} + \mu^2 t + |P_{\mathbf{0}}|, P_{\mathbf{0}} + \mu^2 t + |P_{\mathbf{0}}| + |P_{\mathbf{1}}|\} \\
&= \{\mathbf{0} + \mu^2 t, \mathbf{1} + \mu^2 t + \mu, \mathbf{0} + \mu^2 t + \mu + 1\}, \\
F_\gamma(\mathbf{1} + t) &= \mathcal{P}_{\gamma(\mathbf{1})} + \mu^2 t \\
&= \mathcal{P}_{\mathbf{01}} + \mu^2 t \\
&= \{P_{\mathbf{0}} + \mu^2 t, P_{\mathbf{1}} + \mu^2 t + |P_{\mathbf{0}}|\} \\
&= \{\mathbf{0} + \mu^2 t, \mathbf{1} + \mu^2 t + \mu\}.
\end{aligned}$$

We return to the Cat substitution tiling in §6.1.1.

Chapter 2

Circle dynamics

In this chapter we consider the Denjoy construction which is unique in that it realises a non-transitive C^1 orientation-preserving homeomorphism of the circle without periodic points. The construction realises a cantor minimal set C_α which we show is homeomorphic to a Sturmian shift space Σ_α and whose maps are topologically conjugate (Theorem 2.11).

2.1 Disconnecting the circle

Ternary to binary

Consider the *middle third Cantor set* $C := \left\{\sum_{i=1}^{\infty} \frac{x_i}{3^i} \mid \forall i \in \mathbb{N}, x_i \in \{0,2\}\right\}$. Let

$$f : C \to [0,1],\ f(x) = \sum_{i=1}^{\infty} \frac{x_i/2}{2^i} \in [0,1]. \tag{2.1}$$

The function f transforms a ternary sequence into a binary sequence. For example, $f(\cdots 0\overline{2}\cdots) = \cdots 0\overline{1}\cdots$ and $f(\cdots 2\overline{0}\cdots) = \cdots 1\overline{0}\cdots$ and significantly $\cdots 0\overline{1}\cdots = \cdots 1\overline{0}\cdots$. For $0 < x < 1$, f is a two-to-one mapping but is one-to-one on the extreme endpoints 0 and 1. In Figure 2.1 we see how f 'closes up' a deleted interval on level E_1.

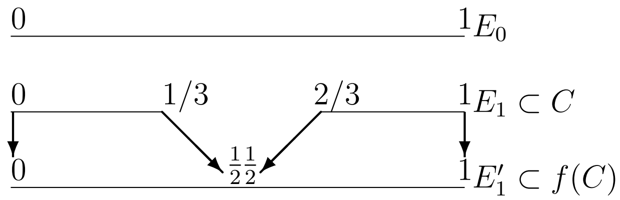


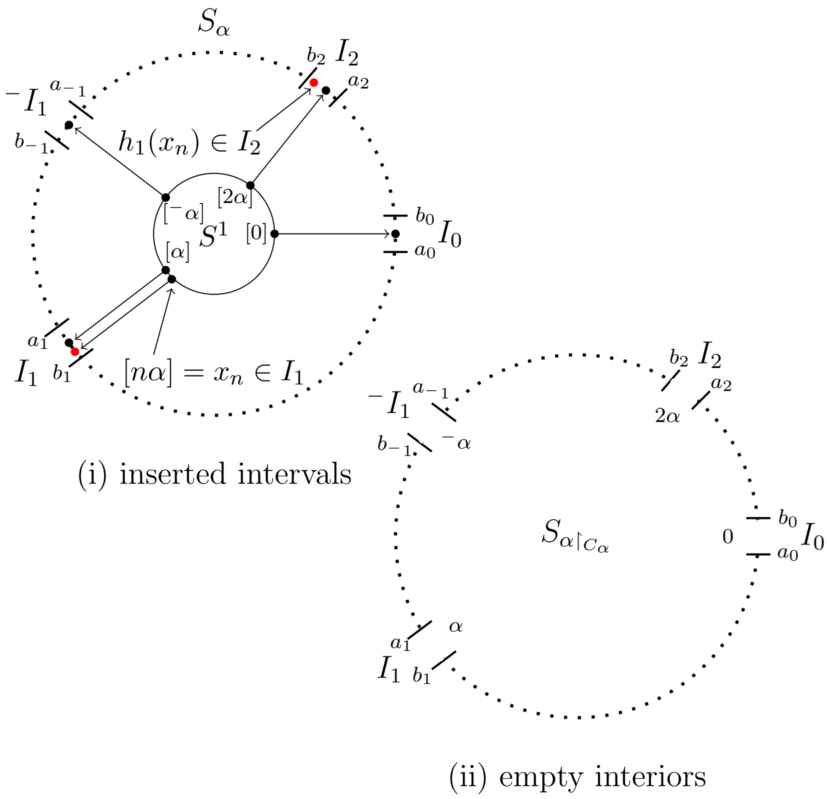
Figure 2.1: $f(E_1)$.

2.1.1 The Denjoy map

Construction of the Denjoy homeomorphism (2.2)**.** We follow in part the descriptions given on pp 4-5 of §15 [37] and the proof of Theorem 7.2.3 in [14]. Let $S^1 := \mathbb{R}/\mathbb{Z}$ and consider again the rotation (1.4) with a fixed α. Set $[x_0] = [0]$ then insert at each point $[n\alpha]$ of $R_\alpha^n([0])$ an interval $I_n = [a_n, b_n]$, of length $l_n = b_n - a_n$, $n \in \mathbb{Z}$. These lengths $l_n > 0$ must satisfy $\sum_{-\infty}^{\infty} l_n \leq 1$ so to simplify the construction, let us choose lengths $l_n > 0$, $\sum_{n\in\mathbb{Z}} l_n = 1$ such that l_n is decreasing as $n \to \pm\infty$. Since the inserted intervals take the same order as that of the orbit $\{R_\alpha^n([0]) \,|\, n \in \mathbb{Z}\}$ we have no choice about the order of the orbit of any point but we do have a choice about the spacing between points in the orbit. Additional constraints ensure that the intervals I_n are pairwise disjoint and the details of these constraints can be read on page 162 of [14]. Now define the endpoints of the required intervals $I_n = [a_n, b_n]$ by

$$a_n = \sum_{\{k\in\mathbb{Z} | R_\alpha^k([0]) \in [0, R_\alpha^n([0])]\}} l_k, \quad b_n = a_n + l_n.$$

Since $\sum_{n\in\mathbb{Z}} l_n = 1$, the union of these intervals covers a set of measure 1 in $[0,1]$, and is therefore dense [14]. Denote this *modified circle* by S_α. Figure 2.2 is not to scale but (i) illustrates the idea of the construction while (ii) conveys the formation of the set C_α which is defined later on page 43 (Def. 2.7). (The intervals I_n are analogous to the interstices of the middle third Cantor set C.)



(i) inserted intervals

(ii) empty interiors

Figure 2.2: Modified circles.

Definition 2.1. *A homeomorphism $f : X \to X$ is said to be* topologically transitive *if there exists a point $x \in X$ such that its orbit $\{f^n(x) \,|\, n \in \mathbb{Z}\}$ is dense in X.*

The above construction equips S_α with a non-transitive orientation-preserving homeomorphism of the circle without periodic points which we define as the *Denjoy map*,

$$D_\alpha : S_\alpha \to S_\alpha,\ D_\alpha(x) = \begin{cases} R_\alpha(x) & \text{if } x \notin \cup_{n\in\mathbb{Z}} I_n, \\ h_n(x) & \text{if } x \in I_n, \end{cases} \tag{2.2}$$

where for all $n \in \mathbb{Z},\ h_n : I_n \to I_{n+1}$ is an o-p homeomorphism.

Remark 2.2. *If we want D_α to be a C^1 diffeomorphism the lengths of the intervals must be chosen so that* $\lim_{|n|\to\infty} l(I_{n+1})/l(I_n) = 1$. *In addition each* *homeomorphism h_n needs to be a diffeomorphism with derivative equal to* $+1$ *at the endpoints of an interval, with derivative converging uniformly to* $+1$ *as* $|n| \to \infty$ *(see [37] for details).*

Quotient maps

Consider the intervals $\{I_n\}_{n\in\mathbb{Z}} \subset S_\alpha$. Let an equivalence relation on S_α be given by $x \sim y$ if $x = y$ or $x \sim y \Leftrightarrow \exists\, n \in \mathbb{Z}$ such that $x, y \in I_n$. Let $\tilde{x}$ denote the $\sim$-class of x then define a quotient map

$$p : S_\alpha \to \tilde{S}_\alpha,\ p(x) = \tilde{x}, \tag{2.3}$$

where $\tilde{S}_\alpha := \{I_n \mid n \in \mathbb{Z}\} \cup \{\{x\} \mid x \in S_\alpha - \cup_{n\in\mathbb{Z}} I_n\}$ is the set of equivalence classes which partition S_α. That is, $\tilde{S}_\alpha$ is one-point sets $\{\tilde{x}\}$ representing $\{I_n\}$ together with points of S^1. Let q be defined piecewise by

$$q : S_\alpha \to \mathbb{R}/\mathbb{Z},\ q(x) = \begin{cases} [x] & \text{if } x \notin \cup_{n\in\mathbb{Z}} I_n, \\ R_\alpha^n([0]) & \text{if } x \in I_n,\ \text{for some } n \in \mathbb{Z}. \end{cases} \tag{2.4}$$

Proposition 2.3. *The map q is continuous.*

Proof. The collection of classes of open intervals $(a, b) \in \mathbb{R}/\mathbb{Z}$ is a basis for a topology on $\mathbb{R}/\mathbb{Z}$. There are two cases to consider: (i) $q(x) = [x] \in (a, b)$ such that $q^{-1}((a, b)) \ni x,\ x \notin \cup_{n\in\mathbb{Z}} I_n$. Now the complement of $\{R_\alpha^n([0])\}$ is dense in $\mathbb{R}/\mathbb{Z}$ so $\exists\, y \in q^{-1}((a, b)) \cap I_n$ with $y \neq x$ and $y \neq R_\alpha^n([0])$ for any $n \in \mathbb{Z}$ then $q^{-1}((a, b))$ is open in S_α; (ii) $q(x) = R_\alpha^n([0]) \in (a, b) \Rightarrow q^{-1}((a, b)) \ni x,\ x \in I_{n_0}$ for $n_0 \in \mathbb{Z}$. Then by the density of $\{R_\alpha^n([0])\}$, $\exists\, y \in I_{n_0}$ such that $x \neq y$ and $q^{-1}((a, b)) \subset I_{n_0}$ is open in I_{n_0}. Thus by (i) and (ii), the pre-image of every open basis element of $\mathbb{R}/\mathbb{Z}$ is open in S_α. Hence q is continuous. $\square$

Define a map by

$$r : \tilde{S}_\alpha \to \mathbb{R}/\mathbb{Z},\ r(\tilde{x}) = \begin{cases} [x] & \text{if } \tilde{x} \notin \cup_{n\in\mathbb{Z}} I_n, \\ R_\alpha^n([0]) & \text{if } \tilde{x} = I_n,\ \text{for some } n \in \mathbb{Z}. \end{cases} \tag{2.5}$$

Proposition 2.4. *The quotient space $\tilde{S}_\alpha$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$ via the homeomorphism $r : \tilde{S}_\alpha \to \mathbb{R}/\mathbb{Z}$.*

Proof. The compact metric space S_α is mapped continuously onto the compact metric space $\mathbb{R}/\mathbb{Z}$ by q (Prop. 2.3). By the construction of (2.3), $\tilde{S}_\alpha$ is a quotient space of S_α. Let $\tilde{S}_\alpha = \{q^{-1}([y]) \mid [y] \in \mathbb{R}/\mathbb{Z}\}$ then $\tilde{S}_\alpha$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$ (see for example Theorem 3.21 in [39]). Thus r is a homeomorphism. $\square$

Remark 2.5.

$$\begin{tikzcd} S_\alpha \arrow[d, "p"'] \arrow[dr, "q"] & \\ \tilde{S}_\alpha \arrow[r, "r"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Since r is a homeomorphism it confirms that q is a quotient map and that the maps commute $r \circ p = q$ (see for example Corollary 22.3 (a) in [38]).

Factor maps

Proposition 2.6. *The two dynamical systems $D_\alpha : S_\alpha \to S_\alpha$ and $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ are semi-conjugate (Def. 1.29) and $\forall x \in S_\alpha$, $q \circ D_\alpha(x) = R_\alpha \circ q(x)$.*

$$\begin{tikzcd} S_\alpha \arrow[r, "D_\alpha"] \arrow[d, "q"'] & S_\alpha \arrow[d, "q"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "R_\alpha"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Proof. The map q is a continuous surjection. Now consider the composite $R_\alpha \circ q$ and take a point $x \notin \cup_{n\in\mathbb{Z}} I_n$ then $R_\alpha \circ q(x) = R_\alpha([x]) = [x + \alpha]$

whilst $q \circ D_\alpha(x) = q(R_\alpha(x)) = [x + \alpha]$. Next suppose point $x \in I_{n_0}$ for some $n_0 \in \mathbb{Z}$ then $R_\alpha \circ q(x) = R_\alpha(R_\alpha^{n_0}([0])) = R_\alpha^{n_0+1}([0])$. In the other direction, $q \circ D_\alpha(x) = q \circ h_{n_0}(x) = q(R_\alpha^{n_0+1}(0))$ where $R_\alpha^{n_0+1}(0) \in I_{n_0+1}$. Then $q(R_\alpha^{n_0+1}(0)) = R_\alpha^{n_0+1}([0])$. Thus the semi-conjugacy follows. $\square$

Definition 2.7. *Let* $C_\alpha = S_\alpha - \bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n)$ *be the* Denjoy minimal set *and let the collection of* endpoints *of the intervals* $\{I_n\}$ *be* $E = \{\{a_n, b_n\} \mid n \in \mathbb{Z}\}$.

In §2.1.2 we show that C_α is a Cantor set but presently note that C_α contains infinitely many points from the original circle S^1 plus a collection E of two element sets, where an endpoint of each set has a countably infinite orbit. Furthermore, the closed set $C_\alpha \subset S_\alpha$ so that C_α is compact and the map q is continuous which implies that $q(C_\alpha)$ is compact and hence closed in $\mathbb{R}/\mathbb{Z}$. Now let $D_{\alpha\restriction_{C_\alpha}}$ be the restriction of D_α to C_α and define a homeomorphism,

$$D_{\alpha\restriction_{C_\alpha}} : C_\alpha \to C_\alpha,\ D_{\alpha\restriction_{C_\alpha}}(x) = \begin{cases} R_\alpha(x) & \text{if } x \notin \{a_n, b_n\}, \\ \hbar_n(x) & \text{otherwise,} \end{cases} \tag{2.6}$$

where for all $n \in \mathbb{Z}$, $\hbar_n : a_n \to a_{n+1}$ and $b_n \to b_{n+1}$.

Proposition 2.8. *The set* C_α *is invariant under* $D_{\alpha\restriction_{C_\alpha}}$ *and thus* $D_{\alpha\restriction_{C_\alpha}}$ *is well-defined.*

Proof. Let $x \in C_\alpha$ and let $y = D_{\alpha\restriction_{C_\alpha}}(x)$. If $D^{-1}_{\alpha\restriction_{C_\alpha}}(y) \notin \{a_n, b_n\}$ then $D_{\alpha\restriction_{C_\alpha}}(x) = R_\alpha(x) = y \notin \{a_n, b_n\}$. However, if $D^{-1}_{\alpha\restriction_{C_\alpha}}(y) \in \{a_n, b_n\} \Rightarrow x = a_n$ or $x = b_n$ for some $n \in \mathbb{Z}$. Then $\hbar_n(a_n) = a_{n+1} \Rightarrow y = a_{n+1}$ or $\hbar_n(b_n) = b_{n+1} \Rightarrow y = b_{n+1}$ where $\{a_{n+1}, b_{n+1}\} \subset E$. Thus $D_{\alpha\restriction_{C_\alpha}}(C_\alpha) \subseteq C_\alpha$. $\square$

Recall the map (2.4) and let its restricted map be defined by

$$q_{\restriction_{C_\alpha}} : S_{\alpha\restriction_{C_\alpha}} \to \mathbb{R}/\mathbb{Z},\ q_{\restriction_{C_\alpha}}(x) = \begin{cases} [x] & \text{if } x \notin \{a_n, b_n\}, \\ R_\alpha^n([0]) & \text{otherwise.} \end{cases} \tag{2.7}$$

Proposition 2.9. *The map $q_{\restriction C_\alpha}$ semi-conjugates $D_{\alpha\restriction C_\alpha}$ to R_α and $\forall x \in C_\alpha$, $q_{\restriction C_\alpha} \circ D_{\alpha\restriction C_\alpha}(x) = R_\alpha \circ q_{\restriction C_\alpha}(x)$.*

$$\begin{tikzcd}
S_{\alpha\restriction C_\alpha} \arrow[r, "D_{\alpha\restriction C_\alpha}"] \arrow[d, "q_{\restriction C_\alpha}"'] & S_{\alpha\restriction C_\alpha} \arrow[d, "q_{\restriction C_\alpha}"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "R_\alpha"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

Proof. The map $D_{\alpha\restriction C_\alpha}$ and the set $S_{\alpha\restriction C_\alpha}$ are respectively mere restrictions of a predefined map and a set. As such, and by similar reasoning to the proof of Proposition 2.6, $\forall x \in C_\alpha \subset S_\alpha$, $q_{\restriction C_\alpha} \circ D_{\alpha\restriction C_\alpha}(x) = R_\alpha \circ q_{\restriction C_\alpha}(x)$. We now show that $q_{\restriction C_\alpha}$ maps C_α onto $\mathbb{R}/\mathbb{Z}$. If $y \neq R_\alpha^n([0])$ for any $n \in \mathbb{Z}$ then $q_{\restriction C_\alpha}^{-1}(y) = x \in S_\alpha - \bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n) - E \subset C_\alpha$. But if $y = R_\alpha^n([0])$ for some $n \in \mathbb{Z}$ then $q_{\restriction C_\alpha}^{-1}(y) = x = a_n$ or $x = b_n$ where $\{a_n, b_n\} \subset E$. Thus $\forall\, y \in \mathbb{R}/\mathbb{Z}$, $\exists\, x \in C_\alpha$ such that $q_{\restriction C_\alpha}(x) = y$. Hence $q_{\restriction C_\alpha}$ with domain C_α remains surjective. $\square$

Towards conjugacy

Loosely speaking the map $D_{\alpha\restriction C_\alpha}$ behaves like a rotation on the restricted circle space $S_{\alpha\restriction C_\alpha}$. We have seen that for points not in an interval, $D_{\alpha\restriction C_\alpha}$ rotates them as 'normal'. More particularly, for a pair of interval endpoints, $D_{\alpha\restriction C_\alpha}$ carries their individual orbits around the circle in the manner of a rotation but, being a pair, when they land on an ambiguous point of the circle partition, they pick up the duplicate coding assigned by κ then continue on their separate itineraries. Consequently, we may treat the orbit of an endpoint in C_α as mimicking the shift orbit of a point in a Sturmian sequence: both systems iterate the two specially coded points $[0]$ and $[\alpha]$. These ideas are made precise in Lemma 2.10 and Theorem 2.11.

Recall the initial point $\iota \in \mathbb{R}$ of a rotation sequence $(u_n)_{n\in\mathbb{Z}} \in \overline{\mathcal{R}}_\alpha$ with map (1.5). By the map (2.7) either $q_{\restriction C_\alpha}^{-1}([\iota]) = x \notin \{a_n, b_n\}$ or $q_{\restriction C_\alpha}^{-1}([\iota]) = x \in \{a_n, b_n\}$.

Let a homeomorphism be defined by

$$\bar{q} : C_\alpha \to \overline{\mathcal{R}}_\alpha,\ \bar{q}(x) = \begin{cases} \iota & \text{if } x \notin \{a_n, b_n\}, \\ \text{otherwise} & \\ \iota^{a_n} & \text{if } x = a_n, \\ \iota^{b_n} & \text{if } x = b_n, \end{cases} \tag{2.8}$$

where we let $\iota, \iota^{a_n}, \iota^{b_n}$ be the initial points of $(u_n)_{n\in\mathbb{Z}}, (u_n^{a_n})_{n\in\mathbb{Z}}, (u_n^{b_n})_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ respectively.

Lemma 2.10. *The dynamical systems $(C_\alpha, D_{\alpha\restriction C_\alpha})$ and $(\overline{\mathcal{R}}_\alpha, \rho_\alpha)$ are topologically conjugate such that the maps commute $\bar{q} \circ D_{\alpha\restriction C_\alpha} = \rho_\alpha \circ \bar{q}$.*

$$\begin{tikzcd} C_\alpha \arrow[r, "D_{\alpha\restriction C_\alpha}"] \arrow[d, "\bar{q}"'] & C_\alpha \arrow[d, "\bar{q}"] \\ \overline{\mathcal{R}}_\alpha \arrow[r, "\rho_\alpha"] & \overline{\mathcal{R}}_\alpha \end{tikzcd}$$

Proof. Firstly $\bar{q}$ is a homeomorphism because

(i) surjection: this follows from $q_{\restriction C_\alpha}(C_\alpha) \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ since $\bar{q}(C_\alpha) \to \overline{\mathcal{R}}_\alpha \subseteq \mathbb{R}/\mathbb{Z}$.

(ii) injection : for $x, y \notin E$, if $\bar{q}(x) = \bar{q}(y)$ then $\iota = \iota' \Rightarrow x = y$; for $x, y \in E$, if $\bar{q}(x) = \bar{q}(y)$ then $\bar{q}(a_n) = \bar{q}(a_n')$ and $\bar{q}(b_n) = \bar{q}(b_n')$ since x and y are the chosen representatives of their respective $\sim$-class of $a_n \sim b_n$ and $a_n' \sim b_n'$. But by definition, $\bar{q}(a_n) = \iota^{a_n}$ and $\bar{q}(a_n') = \iota^{a_n'} \Rightarrow \iota^{a_n} = \iota^{a_n'}$. Similarly, $\iota^{b_n} = \iota^{b_n'}$. Thus $x = y$.

(iii) continuity: $\bar{q}$ only reassigns each image point of $q_{\restriction C_\alpha}$ to have a particular significance in its codomain, in other words $\bar{q}$ identifies a unique point in $\mathbb{R}/\mathbb{Z}$ to be regarded as the initial point of a rotation sequence in $\overline{\mathcal{R}}_\alpha$. As such $\bar{q}$ is still a quotient map and so is continuous. Both C_α and $\overline{\mathcal{R}}_\alpha$ are compact in a metric space and thus $(\bar{q})^{-1}$ is also continuous.

Next consider commutativity: Let $(u_n)_{n\in\mathbb{Z}} \in \overline{\mathcal{R}}_\alpha$. For $x \notin \{a_n, b_n\}$, $\rho_\alpha \circ \bar{q}(x) = \rho_\alpha(\iota) = \rho_\alpha(u_0) = u_1$ while $\bar{q} \circ D_{\alpha\restriction C_\alpha}(x) = \bar{q}(R_\alpha(x)) = \bar{q}(x+\alpha) = \iota + \alpha$. But $\lfloor (n+1)\alpha + (\iota+\alpha)\rfloor - \lfloor n\alpha + (\iota+\alpha)\rfloor = \lfloor (n+2)\alpha + \iota\rfloor - \lfloor (n+1)\alpha + \iota\rfloor = u_{n+1}$. So if $u_n = u_0 \Rightarrow \bar{q} \circ D_{\alpha\restriction C_\alpha}(x) = u_1$. For $x \in \{a_n, b_n\}$, $\rho_\alpha \circ \bar{q}(x) = \rho_\alpha(\bar{q}(a_n)) = \rho_\alpha(\iota^{a_n}) = \rho_\alpha(u_0^{a_n}) = u_1^{a_n}$, similarly $\rho_\alpha(\bar{q}(b_n)) = u_1^{b_n}$. In the other direction we have $\bar{q} \circ D_{\alpha\restriction C_\alpha}(x) = \bar{q}(\hbar_n(x))$ where $\bar{q}(\hbar_n(a_n)) = \bar{q}(a_{n+1}) = \iota^{a_{n+1}} = u_1^{a_n}$ and similarly $\bar{q}(\hbar_n(b_n)) = u_1^{b_n}$. Hence $\bar{q}$ conjugates the maps as shown. □

The conjugacies $(\overline{\mathcal{R}}_\alpha, \rho_\alpha) \cong (\Sigma_\alpha, S)$ (Remark 1.30) and $(C_\alpha, D_{\alpha\restriction C_\alpha}) \cong (\overline{\mathcal{R}}_\alpha, \rho_\alpha)$ (Lemma 2.10) yield the next theorem.

Theorem 2.11. *The dynamical systems $(C_\alpha, D_{\alpha\restriction C_\alpha})$ and (Σ_α, S) are topologically conjugate.*

2.1.2 Minimal Cantor sets

We set out to prove that the Sturmian subshift Σ_α is minimal. With respect to shift spaces, [35] writes that there are many equivalent characterisations of minimality. For instance, a minimal shift contains no proper subshift; every orbit is dense in a minimal shift; minimal shifts display almost-periodic behaviour. The route that we take is to prove that C_α is minimal and then by conjugacy that Σ_α is minimal. This implies that any point in Σ_α will show almost-periodic behaviour. We close the section with the conclusion that Σ_α and C_α are both Cantor sets.

Definition 2.12. *[26] A point $x \in (X, d)$ is* almost-periodic *under a continuous map f provided that to each $\epsilon > 0$ there corresponds an $N \in \mathbb{N}$ with the property that in every set of consecutive Ns there appears $n \in \mathbb{Z}$ such that $d(x, f^n(x)) < \epsilon$.*

An equivalent but alternative definition of almost-periodic appears on page 49.

Definition 2.13. *A homeomorphism $f : X \to X$ is* minimal *if the orbit of every point $x \in X$ is dense in X or equivalently, if f has no proper closed invariant sets. A closed invariant set is* minimal *if it contains no proper closed invariant subsets or equivalently, if it is the orbit closure of any of its points.*

Lemma 2.14. *Let X be a compact metric space and let $f : X \to X$ be a continuous map. If $x \in X$ is almost-periodic under f then the orbit closure of x is minimal. Conversely, if X is minimal under f then each $x \in X$ is almost-periodic.*

Proof. See for example Lemmas 3 and 4 in [26]. □

Corollary 2.15. *A point $[x] \in \mathbb{R}/\mathbb{Z}$ is almost-periodic under an irrational rotation of the circle, R_α (1.4).*

Proof. Each $[x] \in \mathbb{R}/\mathbb{Z}$ has a dense orbit under R_α so by definition R_α is minimal. Thus a point $[x] \in \mathbb{R}/\mathbb{Z}$ is almost-periodic under R_α (Lemma 2.14). □

In light of the following analysis, it is useful to recall $C_\alpha = S_\alpha - \bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n)$; $E = \{\{a_n, b_n\} \mid n \in \mathbb{Z}\}$ the collection of endpoints of the intervals $\{I_n\} \subseteq S_\alpha$; the maps q (2.4), $D_{\alpha\restriction_{C_\alpha}}$ (2.6) and $q_{\restriction_{C_\alpha}}$ (2.7).

Lemma 2.16. *For any interval I in S_α with $I \cap C_\alpha \neq \emptyset$, there exists an open interval $J \subseteq I$ in S_α such that $q^{-1}(q(J)) = J$. (Fig. 2.3)*

Proof. With $a_I < b_I$ define $I := (a_I, b_I)$ to be an interval in S_α such that $I \cap C_\alpha \neq \emptyset$. Let $x \in C_\alpha$ and choose this point x to lie in I but where $x \notin \{a_n, b_n\}$. This is possible since E is a countably infinite collection of sets which leaves an uncountably infinite choice of points in C_α.

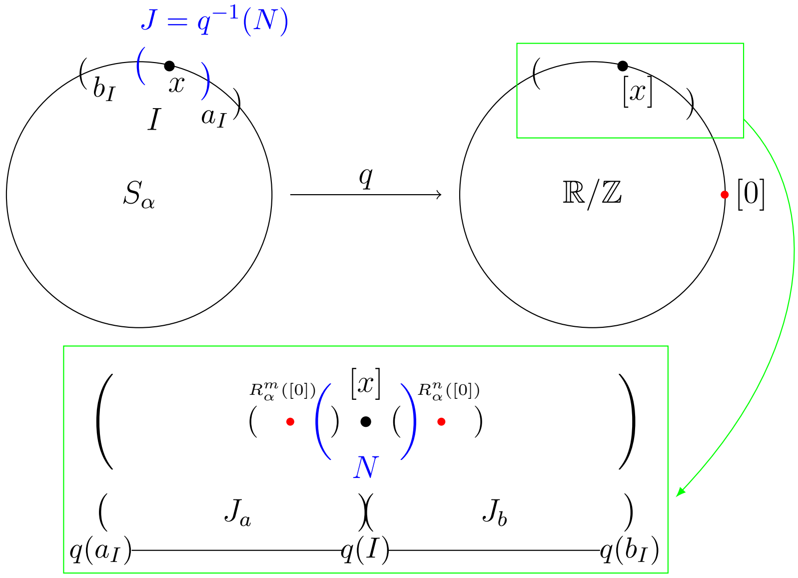


Figure 2.3: A representation of Lemma 2.16.

$$\begin{tikzcd} C_\alpha \arrow[r, hook, "i"] \arrow[dr, dashed, "q_{\restriction C_\alpha}"'] & S_\alpha \arrow[d, "q"] \\ & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Let $i : C_\alpha \hookrightarrow S_\alpha$ be the inclusion map so $q_{\restriction C_\alpha}(x) = q \circ i(x) = q(x)$. Then mapping into $\mathbb{R}/\mathbb{Z}$ gives $q(a_I) \neq q(x)$ and $q(b_I) \neq q(x)$ which means that $q(a_I) \neq q(b_I)$ because $I \cap C_\alpha \neq \emptyset$. That is, I is not a proper subset of any I_n, $n \in \mathbb{Z}$, in S_α. Moreover, q preserves order so $q(a_I) < q(x) = [x] < q(b_I)$ which splits into two intervals $([q(a_I), x])$ and $([x, q(b_I)])$ where $([q(a_I), q(b_I)]) = [q(I)]$. Now let these two non-empty open intervals in $\mathbb{R}/\mathbb{Z}$ be $J_a := ([q(a_I), x])$ and $J_b := ([x, q(b_I)])$. By the minimality of R_α, $\exists\, m \in \mathbb{Z}$ such that $R_\alpha^m([0]) \in J_a$ and $\exists\, n \in \mathbb{Z}$ such that $R_\alpha^n([0]) \in J_b$. Thus there exists an open interval containing $[x]$, call it N, such that N lies between the two aforesaid iterates $[m\alpha]$ and $[n\alpha]$ and where this $N \subseteq [q(I)]$. Finally define $J := q^{-1}(N)$ then $J \subseteq I$ and $q^{-1}(q(J)) = J$. $\square$

Proposition 2.17. *The set C_α is minimal.*

Proof. Let I be an open interval such that $I \cap C_\alpha \neq \emptyset$. We show that $\forall\, x \in C_\alpha$, the orbit $\{D^n_{\alpha\restriction_{C_\alpha}}(x) \,|\, n \in \mathbb{Z}\}$ is dense in C_α. That is, for any $x \in C_\alpha$, $\exists\, n \in \mathbb{Z}$ such that $D^n_{\alpha\restriction_{C_\alpha}}(x) \in I$. Next there exists a J belonging to a basis for C_α by Lemma 2.16. Now for $x \in C_\alpha$, either $x \notin \{a_n, b_n\} \Rightarrow q_{\restriction_{C_\alpha}}(x) = [x] \in \mathbb{R}/\mathbb{Z}$ or else $x \in \{a_n, b_n\} \Rightarrow q_{\restriction_{C_\alpha}}(x) = R^n_\alpha([0])$ for some $n \in \mathbb{Z}$. In either case, by the minimality of R_α, $\exists\, n_0 \in \mathbb{Z}$ such that $R^{n_0}_\alpha(q_{\restriction_{C_\alpha}}(x)) \in [q(J)]$ which is open in $\mathbb{R}/\mathbb{Z}$. Now the semi-conjugacy of Proposition 2.9 gives $q_{\restriction_{C_\alpha}}(D^{n_0}_{\alpha\restriction_{C_\alpha}}(x)) = R^{n_0}_\alpha(q_{\restriction_{C_\alpha}}(x)) \Rightarrow q_{\restriction_{C_\alpha}}(D^{n_0}_{\alpha\restriction_{C_\alpha}}(x)) \in q_{\restriction_{C_\alpha}}(J) \Rightarrow D^{n_0}_{\alpha\restriction_{C_\alpha}}(x) \in q^{-1}_{\restriction_{C_\alpha}}(q_{\restriction_{C_\alpha}}(J)) = J \subseteq I$. Hence C_α is minimal. $\square$

Remark 2.18. *The conjugacy $(C_\alpha, D_{\alpha\restriction_{C_\alpha}}) \cong (\Sigma_\alpha, S)$ then implies that Σ_α is minimal.*

Remark 2.19. *Since Σ_α is minimal, a point $u \in (\Sigma_\alpha, S)$ is almost-periodic (Lemma 2.14).*

Definition 2.12 may be expressed otherwise, by saying that a point x belonging to a subshift X is *almost-periodic* if and only if every allowed finite subword of x appears in x with bounded gaps (see for example [35]). We use this approach to prove the next proposition.

A set A is *perfect* if A is closed and every point of A is a limit point of A.

Proposition 2.20. *The Sturmian subshift Σ_α is perfect.*

Proof. Let $u = (u_i)_{i\in\mathbb{Z}} \in \Sigma_\alpha$ then u is almost-periodic but not periodic. Given $\epsilon > 0$, let $1/2^{k+1} < \epsilon$ for $k \in \mathbb{N}_0$. By almost-periodicity let $w = u_{-k} \dots u_k$ appear again in u at position $u_{-k+l} \dots u_{k+l}$ for some $l \neq 0$. Then $d(S^l(u), u) \leq 1/2^{k+1} < \epsilon$. Now by shift invariance $S^l(u) \in \Sigma_\alpha$ with $S^l(u) \neq u$. So u is a limit point of Σ_α and thus Σ_α is perfect. $\square$

Definition 2.21. *A* totally disconnected space *is a space X where for every two points $x_1, x_2 \in X$ there exist disjoint open sets $O_1, O_2 \subset X$, containing x_1, x_2 respectively, whose union is X.*

Proposition 2.22. *The space $\mathcal{A}^{\mathbb{Z}}$ is totally disconnected.*

Proof. Let $\mathcal{A}^{\mathbb{Z}}$ have a finite alphabet $\mathcal{A}$ of N distinct letters. Let D be a non-empty subset of $\mathcal{A}^{\mathbb{Z}}$ with points $(u_i)_{i\in\mathbb{Z}} \neq (v_i)_{i\in\mathbb{Z}} \in D$ and where $u_k \neq v_k$ for some $k \in \mathbb{Z}$. So D contains more than one point. Let $U_a = \{(x_i) \,|\, x_k = a, a \in \mathcal{A}\}$ then $\bigcup_{n=1}^{N} U_{a_n} = \mathcal{A}^{\mathbb{Z}}$ and $U_{a_1} \cap U_{a_2} \neq \emptyset \Leftrightarrow a_1 = a_2$. Now $(u_i) \in U_{a_1} \Rightarrow u_k = a_1$ and $(v_i) \in U_{a_2} \Rightarrow v_k = a_2$, $a_1 \neq a_2$. Further, $U_{a_1} \cap D \neq \emptyset$ and $U_{a_2} \cap D \neq \emptyset$ so U_{a_1} and U_{a_2} are pairwise disjoint in D. Thus D is disconnected. Since D is arbitrary and the only connected subsets of $\mathcal{A}^{\mathbb{Z}}$ are points, $\mathcal{A}^{\mathbb{Z}}$ is totally disconnected. $\square$

Remark 2.23. *Since $\Sigma_\alpha \subseteq \mathcal{A}^{\mathbb{Z}} = \{0,1\}^{\mathbb{Z}}$, Σ_α is totally disconnected.*

Remark 2.24. *Every perfect, totally disconnected, compact metric space is a* Cantor set *(see for example Theorem A.1.38 in [28]). Thus Σ_α is a Cantor set and by conjugacy so too is C_α.*

Chapter 3

Flows and suspensions

Following the construction of an orientable attractor which is homeomorphic to a solenoid in §6.2, the proof of Proposition 6.24 appeals to the fact that a solenoid and a suspension are homeomorphic spaces. This is one good reason for explaining in this chapter the suspension construction and how it induces a flow and a first return map. Additionally, in §3.3 by suspending the Denjoy minimal map we realise a Denjoy continuum and in §3.4 we find a conjugacy between the suspension flow of a full shift space and a tiling space (Prop. 3.15).

3.1 A toral linear flow

Let a metric on $\mathbb{R}^2$ be given by $d'((x,y),(x_0,y_0)) := \max\{|x-x_0|, |y-y_0|\}$. For $(x_0,y_0) \in \mathbb{R}^2$ and $\epsilon > 0$ denote the set of open ϵ-balls centred at (x_0,y_0) as $B_\epsilon((x_0,y_0)) = \big\{(x,y) \in \mathbb{R}^2 \,|\, d'\big((x,y),(x_0,y_0)\big) < \epsilon\big\}$. Now consider a covering map $p_2 : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $p_2\big(B_\epsilon((x_0,y_0))\big) = B_\epsilon(\overline{(x_0,y_0)})$ where $B_\epsilon(\overline{(x_0,y_0)}) = \big\{\overline{(x,y)} \in \mathbb{T}^2 \,|\, d^*\big(\overline{(x,y)}, \overline{(x_0,y_0)}\big) < \epsilon\big\}$. Then let a metric on $\mathbb{T}^2$ be $\overline{d^*\big((x,y),(x_0,y_0)\big)} := \min\{d((u,v),(u_0,v_0)) \,|\, (u,v) \in \overline{(x,y)}, (u_0,v_0) \in \overline{(x_0,y_0)}\}$.

Fix $\alpha \in [0,1] - \mathbb{Q}$ and define a *linear flow* by

$$\phi_\alpha : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2,\ \phi_\alpha\left(t, \overline{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \overline{t\begin{bmatrix} \alpha \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}} = \overline{\begin{bmatrix} x+\alpha t \\ y+t \end{bmatrix}}. \tag{3.1}$$

Proposition 3.1. *The flow ϕ_α^t is (i) continuous and (ii) well-defined.*

Proof. (i) Given $\epsilon > 0$ choose $\epsilon < 1/2$ and take $0 < \delta < \epsilon$. Let $(u,v) \in \overline{(x,y)}$ and $(u_0, v_0) \in \overline{(x_0, y_0)}$ then $B_\epsilon(\phi_\alpha^t(u_0,v_0)) = B_\epsilon((u_0+\alpha t, v_0+t))$. Now $(\phi_\alpha^t)^{-1}\big(B_\epsilon((u_0+\alpha t, v_0+t))\big) = B_\epsilon((u_0+\alpha t, v_0+t)) - (\alpha t, t) = B_\epsilon((u_0,v_0))$. But this ϵ-ball contains some $B_\delta((u_0,v_0))$ such that $\phi_\alpha^t\big(B_\delta((u_0,v_0))\big) = B_\delta((u_0,v_0)) + (\alpha t, t) = B_\delta((u_0+\alpha t, v_0+t)) \subset B_\epsilon\big(\phi_\alpha^t((u_0,y_0))\big)$. Thus ϕ_α^t is continuous.

(ii) If $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$ determine the same point in $\mathbb{T}^2$ then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} x_0+m \\ y_0+n \end{bmatrix}$ for some $m,n \in \mathbb{Z}$. Now $\phi_\alpha^t\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \overline{\begin{bmatrix} x+\alpha t \\ y+t \end{bmatrix}}$ and $\phi_\alpha^t\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) = \phi_\alpha^t\left(\overline{\begin{bmatrix} x_0+m \\ y_0+n \end{bmatrix}}\right) = \overline{\begin{bmatrix} x_0+m+\alpha t \\ y_0+n+t \end{bmatrix}} = \overline{\begin{bmatrix} x+\alpha t \\ y+t \end{bmatrix}}$. That is, ϕ_α^t is independent of the chosen point and thus is well-defined. □

3.2 The suspension construction

Informally, suspension is a construction which turns a map into a flow and when the term suspension is used it implies that a flow exists.

Definition 3.2. *Let X be a simple closed curve homeomorphic to S^1 endowed with a homeomorphism $h : X \to X$. On $X \times [0,1]$ define the relation $\sim$ where $(x,1)$ is identified with $(h(x),0)$. Let the* quotient space *be $X_c(h) = X \times [0,1]/_\sim = X \times \mathbb{R}/_\approx$ where the equivalence relation $(x,s) \approx (y,t) \Leftrightarrow \exists\, n \in \mathbb{Z}$ such that $y = h^n(x)$ and $s - t = n$.*

Throughout this chapter the notation $(\widetilde{\cdots,\cdots})$ indicates the equivalence class of the quotient space. Let a *suspension flow* of $h : X \to X$ be defined by

$$\bar{\phi} : \mathbb{R} \times (X \times \mathbb{R}/_{\approx}) \to X \times \mathbb{R}/_{\approx},\ \bar{\phi}(t, \widetilde{(x, s)}) = (\widetilde{x, s+t}). \tag{3.2}$$

Theorem 3.3. *Let $h : S^1 \to S^1$ be an orientation-preserving homeomorphism then the suspension $X_c(h)$ is homeomorphic to the torus $\mathbb{T}^2$.*

We verify this theorem via the next lemma and proposition.

Let a point in $S^1 \times [0, 1]$ have unique parametric coordinates $([x] + t(h([x]) - [x]), t)$, $0 \le t \le 1$, where $[x] = x$ modulo 1. Now define $\tilde{H}$ by

$$\tilde{H} : S^1 \times [0, 1] \to S^1 \times [0, 1],\ \tilde{H}\big(([x] + t(h([x]) - [x]), t)\big) = ([x], t). \tag{3.3}$$

Lemma 3.4. *The map $\tilde{H}$ is a homeomorphism.*

Proof. Let l and l' be two line segments in $S^1 \times [0, 1]$ which join points $([x], 0)$ to $(h([x]), 1)$ and $([x], 0)$ to $([x], 1)$ respectively. Then for $0 \le t \le 1$, a point on l satisfies $([x] + t(h([x]) - [x]), t)$ while a point on l' satisfies $([x], t)$. Now $\tilde{H}$, which sends l onto l', is

(i) injective: for $0 \le s, t \le 1$, let two points on l be $p = ([u] + t(h([u]) - [u]), t)$ and $q = ([v] + s(h([v]) - [v]), s)$ such that $\tilde{H}(p) = \tilde{H}(q) \Rightarrow ([u], t) = ([v], s) \Rightarrow p = q$;

(ii) surjective: let $q = ([v], s)$ be an arbitrary point on l' in the codomain $S^1 \times [0, 1]$. Let $\tilde{H}(([u] + t(h([u]) - [u]), t)) = ([u], t) = ([v], s)$ then there exists a point p such that $\tilde{H}(p) = q$;

(iii) $\tilde{H}$ is continuous and since $S^1 \times [0, 1]$ is compact in a metric space, the inverse $\tilde{H}^{-1}$ is continuous. Thus by (i), (ii) and (iii) $\tilde{H}$ is a homeomorphism. $\square$

Now consider two quotient maps where $\forall\, [x] \in S^1$, $q_{\tilde{c}} : S^1 \times [0,1] \to \mathbb{T}^2 = S^1 \times [0,1]/_\sim$ identifies $([x],0) \sim ([x],1)$ and $q_c : S^1 \times [0,1] \to X_c(h) = S^1 \times [0,1]/_\sim$ identifies $([x],0) \sim (h([x]),1)$.

Proposition 3.5. *(i) $\tilde{H}$ respects the quotient maps q_c and $q_{\tilde{c}}$ and (ii) the uniquely determined map H is a homeomorphism satisfying $q_{\tilde{c}} \circ \tilde{H} = H \circ q_c$.*

$$\begin{tikzcd}
S^1 \times [0,1] \arrow[r, "\tilde{H}"] \arrow[d, "q_c"'] \arrow[dr, dashed, "f"] & S^1 \times [0,1] \arrow[d, "q_{\tilde{c}}"] \\
X_c(h) \arrow[r, "H"] & \mathbb{T}^2
\end{tikzcd}$$

Proof. (i) Let $f : S^1 \times [0,1] \to \mathbb{T}^2$ be the composite function $q_{\tilde{c}} \circ \tilde{H}$. Since $q_{\tilde{c}}$ is a closed continuous surjective map of the homeomorphism $\tilde{H}$, f is closed continuous and surjective and hence is a quotient map by definition.

(ii) Let $X_c(h) = \big\{f^{-1}\big(\{([x],t)\}\big) \mid ([x],t) \in \mathbb{T}^2\big\}$ be the collection of subsets of $S^1 \times [0,1]$ where $X_c(h)$ takes the quotient topology. Then f induces a homeomorphism H (see for example Corollary 22.3 in [38]). Now for parameter $0 \le t \le 1$ let a point be $([x] + t(h([x]) - [x]), t) \in X_c(h)$ then define H by

$$H : X_c(h) \to \mathbb{T}^2,\ H\big(([x] + t(\widetilde{h([x])} - [x]), t)\big) = \widetilde{([x], t)}. \tag{3.4}$$

To show the commutativity $q_{\tilde{c}} \circ \tilde{H} = H \circ q_c$ let $p = ([x] + t(h([x]) - [x]), t)$ and consider $q_{\tilde{c}} \circ \tilde{H}(p) = q_{\tilde{c}}\big(([x],t)\big)$ which identifies $([x],0) \sim ([x],1)$. Whereas for $H \circ q_c(p)$ first apply q_c which identifies $([x],0) \sim (h([x]),1)$ then map $H\big(\widetilde{([x],0)}\big) = \widetilde{([x],0)}$ and $H\big(\widetilde{(h([x]),1)}\big) = \widetilde{([x],1)} \Rightarrow ([x],0) \sim ([x],1)$. $\square$

Thus Lemma 3.4 and Proposition 3.5 yield Theorem 3.3.

Now let a suspension flow on the torus be defined by

$$\phi : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2,\ \phi^t\Big(H\big(\widetilde{([x], s)}\big)\Big) = H\big(\widetilde{([x], s+t)}\big). \tag{3.5}$$

Proposition 3.6. *The dynamical systems $(X_c(h), \bar{\phi}^t)$ and $(\mathbb{T}^2, \phi^t)$ are topologically conjugate such that the maps commute $H \circ \bar{\phi}^t = \phi^t \circ H$, $\forall\, t \in \mathbb{R}$.*

$$\begin{tikzcd} X_c(h) \arrow[r, "\bar{\phi}^t"] \arrow[d, "H"'] & X_c(h) \arrow[d, "H"] \\ \mathbb{T}^2 \arrow[r, "\phi^t"] & \mathbb{T}^2 \end{tikzcd}$$

Proof. The homeomorphism H conjugates the flow on $X_c(h)$ as follows:
$H \circ \bar{\phi}^t\widetilde{([x], s)} = H\big(\widetilde{([x], s+t)}\big)$ whilst $\phi^t \circ H\big(\widetilde{([x], s)}\big) = H\big(\widetilde{([x], s+t)}\big)$. □

The following corollary follows from Theorem 3.3.

Corollary 3.7. *The toral linear flow ϕ^t_α is conjugate to the suspension flow of the circle rotation R_α on the torus $\mathbb{T}^2$.*

Let the unit square in Figure 3.1 represent the quotient space $S^1_c(R_\alpha)$ which is homeomorphic to $\mathbb{T}^2$. Informally the sketch imitates the flow ϕ^t_α carrying a point $([x], 0)$ up to $([x], 1)$ which is then identified with $(R_\alpha([x]), 0)$. Thereafter the flow continues in a similar fashion.

Let $A := X \times \{0\} \subset X \times \mathbb{R}/_\approx$ be a *cross-section* for the flow $\psi : \mathbb{R} \times X \times \mathbb{R}/_\approx \to X \times \mathbb{R}/_\approx$ such that $\psi^t(\widetilde{x, s}) = a \in A$, $\forall (\widetilde{x, s}) \in X \times \mathbb{R}/_\approx$ and $t \in \mathbb{R}^+$. Now let $t_0 > 0$ be the minimum time for a to return to A under the flow then define the *first return map* by

$$\rho : A \to A,\ \rho(a) = \psi^{t_0}(a). \tag{3.6}$$

The return map R_α to the cross-section $S^1 \times \{0\}$ of the flow ϕ^t_α has a constant return time of 1.

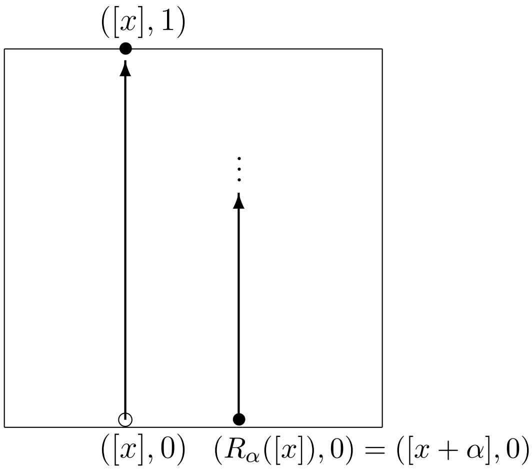


Figure 3.1: The suspension flow ϕ_α^t.

3.3 Embedding the Denjoy map

Definition 3.8. *An* embedding *of the topological space X in the topological space Y is a continuous injective function $f : X \to Y$ which is a homeomorphism onto its image and where $f(X)$ takes the subspace topology from Y.*

Recall the Denjoy map (2.2) then let the quotient space $S_{\alpha_c}(D_\alpha)$ be similarly derived as in Definition 3.2. Now (2.2) is an o-p homeomorphism and S_α is homeomorphic to S^1 (see for example page 689 in [36]). Thus we have an embedding

$$\hat{H} : S_{\alpha_c}(D_\alpha) \to \mathbb{T}^2,\ \hat{H}\big(\widetilde{(D_\alpha^t(u), t)}\big) = \widetilde{(u, t)},\ 0 \le t \le 1, \tag{3.7}$$

from which we may define the Denjoy suspension flow

$$\hat{\phi}_\alpha : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2,\ \hat{\phi}_\alpha^t\Big(\hat{H}\big(\widetilde{(u, s)}\big)\Big) = \hat{H}\big(\widetilde{(u, s+t)}\big), \tag{3.8}$$

Definition 3.9. *The suspension of the Denjoy minimal map $D_{\alpha \restriction C_\alpha}$ (2.6) is known as a* Denjoy continuum *which we denote by $\mathbb{D}_\alpha$.*

3.4 Shift and tiling spaces

Definition 3.10. *Consider a full shift space $\mathcal{A}^{\mathbb{Z}}$ with finite $\mathcal{A} = \{a_1, \ldots, a_n\}$. Let a* symbolic cylinder *be $[a] = \{(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \mid u_0 = a \in \mathcal{A}\}$. Then $\mathcal{A}^{\mathbb{Z}} = [a_1]\dot{\cup}[a_2]\dot{\cup}\ldots\dot{\cup}[a_n]$ is a partition of $\mathcal{A}^{\mathbb{Z}}$ into a disjoint union of n clopen sets.*

Definition 3.11. *Consider a full tiling space $\mathcal{T}_{\mathscr{P}}$ made from a finite set of prototiles $\mathscr{P} = \{P_1, \ldots, P_n\}$. Let a* tiling cylinder *be $[P] = \{(T_i)_{i\in\mathbb{Z}} \in \mathcal{T}_{\mathscr{P}} \mid T_0 = P \in \mathscr{P},$ such that the origin lies at the left hand end of $T_0\}$.*

Definition 3.12. *Let* $\mathrm{card}(\mathcal{A}) = \mathrm{card}(\mathscr{P}) = N$ *and let each $a_j \in \mathcal{A}$ be matched to $P_j \in \mathscr{P}$, $1 \leq j \leq N$. Then define a map*

$$\tau : \mathcal{A}^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}},\ \tau((u_i)) = (T_i),\ \forall\, i \in \mathbb{Z}, \tag{3.9}$$

such that the origin 0 lies at the left hand end of tile T_0 and where T_i is a translate of the prototile matched with u_i. It follows that $[a_j]$ is identified with $[P_j]$. Now assign each pair of cylinders to have equal length $l[a_j] = l[P_j] = l_j \in (0, \infty)$.

In order to cater for the suspension of a map involving cylinders with different lengths, we define a *ceiling function* c which takes the form

$$c : \mathcal{A}^{\mathbb{Z}} \to (0, \infty),\ c((u_i)_{i\in\mathbb{Z}}) = l_j,\ \text{if and only if } (u_i)_{i\in\mathbb{Z}} \in [a_j], \tag{3.10}$$

where $l_j \in (0, \infty)$ is the fixed length of the cylinder $[a_j]$, $1 \leq j \leq n$.

Definition 3.13 (Symbolic quotient space)**.** *Consider the shift map $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. Let $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, $\mathcal{A} = \{a_1, \ldots, a_n\}$, and let $((u_i), s) \in [a_j] \times [0, l_j] \Rightarrow u_0 = a_j \in \mathcal{A}$ for some $1 \leq j \leq n$. Then*

- *In the positive semi-orbit, $t \geq 0$, let $\sim$ denote the equivalence relation which identifies the point $\big((u_i), c((u_i))\big)$ with $\big(S((u_i)), 0\big) \Leftrightarrow s + t = l[u_0]$, $s \in [0, l_j]$. That is, $((u_i), l_j) \sim ((u_{i+1}), 0) \in ([u_1], 0)$;*

- *in the negative semi-orbit, $t \leq 0$, let $\sim$ denote the equivalence relation which identifies the point $\big((u_i), c((u_i))\big)$ with $\big(S^{-1}((u_i)), 0\big) \Leftrightarrow s + |t| = l[u_0]$, $s \in [0, l_j]$. That is, $((u_i), l_j) \sim ((u_{i-1}), 0) \in ([u_{-1}], 0)$.*

Then define $\mathcal{A}_c^{\mathbb{Z}} := \big([a_1] \times [0, l_1] \dot\cup [a_2] \times [0, l_2] \dot\cup \ldots \dot\cup [a_n] \times [0, l_n]\big)/_\sim$.

Definition 3.14 (Symbolic flow)**.** *Let $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. We treat separately the positive and negative trajectories of a suspension flow Φ^t. So consider*

- *the positive trajectory, $t \geq 0$. Set $0 \leq s < l[u_0] \Rightarrow ((u_i), s) \in [u_0] \times [0, l[u_0]]$ then choose a unique integer $n \in \mathbb{N}_0$ such that $0 < s + t - \sum_{i=0}^{n} l[u_i] \leq l[u_{n+1}]$. Then let the positive semi-flow be*

$$\Phi : \mathbb{R}^+ \cup \{0\} \ \times \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{A}_c^{\mathbb{Z}},\ \Phi^t\big(\widetilde{(u_i), s}\big) = (\widetilde{(u_{i+(n+1)}), s^+}),$$

  where $(u_{i+(n+1)}) \in [u_{n+1}]$ and $0 \leq s^+ = s + t - \sum_{i=0}^{n} l[u_i]$;

- *the negative trajectory, $t \leq 0$. Set $0 \leq s < l[u_0] \Rightarrow ((u_i), s) \in [u_0] \times [0, l[u_0]]$ then choose a unique integer $n \in \mathbb{N}_0$ such that $0 < |s + t + \sum_{i=-n}^{-1} l[u_i]| \leq l[u_{-(n+1)}]$. Then let the negative semi-flow be*

$$\Phi : \mathbb{R}^- \cup \{0\} \times \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{A}_c^{\mathbb{Z}},\ \Phi^t\big(\widetilde{(u_i), s)}\big) = (\widetilde{(u_{i-(n+1)}), s^-}),$$

  where $(u_{i-(n+1)}) \in [u_{-(n+1)}]$ and $0 \leq s^- = l[u_{-(n+1)}] - |t + s + \sum_{i=-n}^{-1} l[u_i]|$.

Let $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}} \in \mathcal{T}_{\mathscr{P}}$ satisfy Definition 3.12 and set $l_j = c((u_i))$ with $0 \leq s < l[u_0]$. Then define an o-p homeomorphism by

$$\tau_c : \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}},\ \tau_c\big(\widetilde{(u_i)_{i\in\mathbb{Z}}}, s\big) = \phi^s\big(\tau\big((u_i)_{i\in\mathbb{Z}}\big)\big). \tag{3.11}$$

Proposition 3.15. *Induced by τ_c, $\Phi : \mathbb{R} \times \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{A}_c^{\mathbb{Z}}$ is topologically conjugate to $\phi : \mathbb{R} \times \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}}$ such that the maps commute $\tau_c \circ \Phi^t = \phi^t \circ \tau_c$, $\forall\, t \in \mathbb{R}$.*

$$\begin{tikzcd}
\mathcal{A}_c^{\mathbb{Z}} \arrow[r, "\Phi^t"] \arrow[d, "\tau_c"'] & \mathcal{A}_c^{\mathbb{Z}} \arrow[d, "\tau_c"] \\
\mathcal{T}_{\mathscr{P}} \arrow[r, "\phi^t"] & \mathcal{T}_{\mathscr{P}}
\end{tikzcd}$$

Proof. By construction the map (3.9) bestows a one-to-one correspondence between a symbolic sequence $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and a tiling $(T_i)_{i\in\mathbb{Z}} \in \mathcal{T}_{\mathscr{P}}$ so (3.9) is a bijection. For all $t \in \mathbb{R}$ the flow (1.12) is a homeomorphism in the tiling space $\mathcal{T}_{\mathscr{P}}$. Thus the composite map τ_c is a homeomorphism. With respect to commutativity, consider firstly the positive trajectory of a point $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}_c^{\mathbb{Z}}$. That is for $t \geq 0$ and $\forall\, i \in \mathbb{Z}$, $\tau_c \circ \Phi\big(\widetilde{(u_i), s}\big) = \tau_c\big(\widetilde{(u_{i+(n+1)}), s^+}\big) = \phi^{s+}(\tau((u_{i+(n+1)}))) = \phi^{s+}\big((T_{i+(n+1)})\big) = (T_{i+(n+1)}) - s^+$, whereas $\phi^t \circ \tau_c\big(\widetilde{(u_i), s}\big) = \phi^t \circ \phi^s\big(\tau((u_i))\big) = \phi^{t+s}\big((T_i)\big) = (T_i) - (t+s) = (T_{i+(n+1)}) - s^+$ since $0 \leq s^+ = s + t - \sum_{i=0}^{n} l[u_i]$. Now for the negative trajectory where $t \leq 0$ and $\forall\, i \in \mathbb{Z}$, $\tau_c \circ \Phi\big(\widetilde{(u_i), s}\big) = \tau_c\big(\widetilde{(u_{i-(n+1)}), s^-}\big) = \phi^{s-}(\tau((u_{i-(n+1)}))) = \phi^{s-}\big((T_{i+(n+1)})\big) = (T_{i+(n+1)}) - s^-$, whereas $\phi^t \circ \tau_c\big(\widetilde{(u_i), s}\big) = \phi^t \circ \phi^s\big(\tau((u_i))\big) = \phi^{t+s}\big((T_i)\big) = (T_i) - |t+s| = (T_{i-(n+1)}) - s^-$ since $0 \leq s^- = l[u_{-(n+1)}] - |t + s + \sum_{i=-n}^{-1} l[u_i]|$. Thus τ_c commutes the flows which are hence topologically conjugate. $\square$

Example 3.16. Consider a full shift $\{a, b\}^{\mathbb{Z}}$ with a particular point $(u_i) = \ldots u_{-1} \,.\, u_0 u_1 u_2 u_3 \ldots = \ldots b \,.\, abba \ldots$. Consider a full tiling space $\mathcal{T}_{\mathscr{P}}$ whose tiling $(T_i)_{i\in\mathbb{Z}}$ consists of prototiles from $\mathscr{P} = \{\mathbf{a} = [0,1], \mathbf{b} = [0,1/2]\}$ such that $(T_i) = \ldots T_{-1} \,.\, T_0 T_1 T_2 T_3 \ldots = \mathbf{b} \,.\, \mathbf{abba} \ldots$, with the origin positioned at the left hand end of T_0. Use (3.9) to put the symbolic cylinders $[a] = \{(u_i) \in \{a,b\}^{\mathbb{Z}} \,|\, u_0 = a\}$ and $[b] = \{(u_i) \in \{a,b\}^{\mathbb{Z}} \,|\, u_0 = b\}$ in correspondence with the tiling cylinders $[\mathbf{a}]$ and $[\mathbf{b}]$. Then for both cylinder sets, the lengths are $l_1 = 1$ and $l_2 = 1/2$. Now let $\big(\widetilde{(u_i), 0}\big) \in \mathcal{A}_c^{\mathbb{Z}}$ and $t = \frac{9}{4}$, say, then consider $\Phi\big(\widetilde{(u_i), 0}\big)$. With $s = 0$ and $u_0 = a \Rightarrow (u_i) \in [a]$ then $0 \leq s^+ = \frac{9}{4} - \sum_{i=0}^{n} l[u_i] \Rightarrow$

$\sum_{i=0}^{n} l[u_i] = l[u_0] + l[u_1] + l[u_2] = 2 \Rightarrow n = 2$ and $s^+ = \frac{1}{4}$. Thus $\Phi^{\frac{9}{4}}\big(\widetilde{(u_i),0}\big) = \big(\widetilde{(u_{i+3})}, \frac{1}{4}\big) \in [a]\times[0,1]$. Next by (3.11), $\forall\, i \in \mathbb{Z}$, $\tau_c\big(\widetilde{(u_i),0}\big) = \phi^0\big(\tau((u_i))\big) = (T_i)$ then $\phi^{\frac{9}{4}}\big((T_i)\big) = (T_i) - \frac{9}{4} = (T_{i+3}) - \frac{1}{4}$ where the image tiling $(T_{i+3}) - \frac{1}{4} \in [\mathbf{a}]$.

We can imagine the symbolic flow as being represented by Figure 3.2 (i) while the corresponding tiling flow produces the pattern in 3.2 (ii), the cylinders being colour coded as shown.

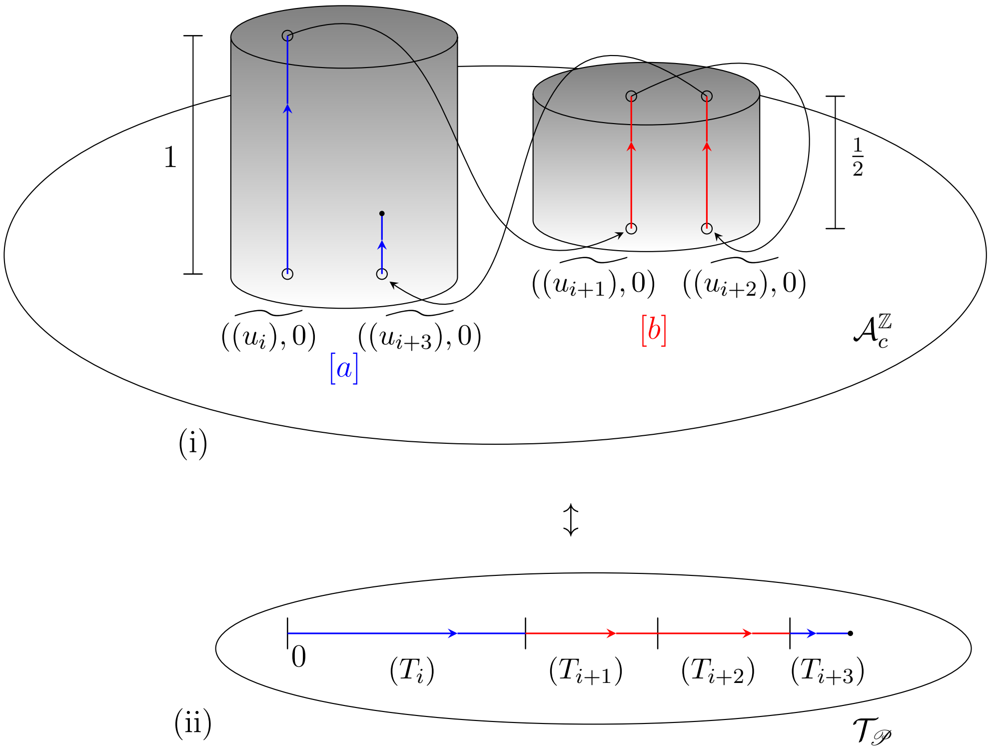


Figure 3.2: Flows over time $t = \frac{9}{4}$ (cf. Ex.3.16).

Definition 3.17. *Two maps are* flow equivalent *if there is a homeomorphism between their suspensions taking trajectories of one to trajectories of the other and preserving orientation.*

In our context of one-dimensional tilings we can say that a *simple* tiling of $\mathbb{R}$ is a tiling for which there are only a finite number of tile types (prototiles) up to translation and each tile is an interval.

Definition 3.18. *[50] A homeomorphism between (simple) tiling spaces is a continuous map $f : \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}'}$ that is 1-1 and onto.*

Since $\mathcal{T}_{\mathscr{P}}$ is compact, f^{-1} is automatically continuous, so Definition 3.18 agrees with the usual topological definition of homeomorphism. Let $\text{card}(\mathscr{P}) = \text{card}(\mathscr{P}') = N$ and let the tilings $T \in \mathcal{T}_{\mathscr{P}}$ and $T' \in \mathcal{T}_{\mathscr{P}'}$. Now $\tau : \mathcal{A}^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}} \Rightarrow (T_i) = \tau((u_i))$, $\forall i \in \mathbb{Z}$, so let $\tau' : \mathcal{A}^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}'} \Rightarrow (T_i') = \tau'((u_i'))$, $\forall i \in \mathbb{Z}$. Then define

$$\bar{\tau} : \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}'},\ \bar{\tau}((T_i)) = (T_i'),\ \forall i \in \mathbb{Z}, \tag{3.12}$$

where $\bar{\tau} = \tau' \circ \bar{\tau} \circ \tau^{-1}$.

Proposition 3.19. *The tiling spaces $\mathcal{T}_{\mathscr{P}}$ and $\mathcal{T}_{\mathscr{P}'}$ are topologically conjugate such that $\phi'^t = \bar{\tau} \circ \phi^t \circ \bar{\tau}^{-1}$, $\forall\, t \in \mathbb{R}$.*

$$\begin{tikzcd} \mathcal{T}_{\mathscr{P}} \arrow[r, "\phi^t"] \arrow[d, "\bar{\tau}"'] & \mathcal{T}_{\mathscr{P}} \arrow[d, "\bar{\tau}"] \\ \mathcal{T}_{\mathscr{P}'} \arrow[r, "\phi'^t"] & \mathcal{T}_{\mathscr{P}'} \end{tikzcd}$$

Proof. Let cylinder sets $[P_j] \subseteq \mathcal{T}_{\mathscr{P}}$ and $[P_j'] \subseteq \mathcal{T}_{\mathscr{P}'}$ have lengths l_j and l_j' respectively. We consider two cases.

(i) Suppose that $l_j = l_j'$ for all $1 \leq j \leq N$. Let $((T_i), s) \in \mathcal{T}_{\mathscr{P}} \times [0, l[T_0]]$ then for $t \geq 0$, $\bar{\tau} \circ \phi^t\big((T_i), s\big) = \bar{\tau}((T_{i+(n+1)}), s^+) = (T'_{i+(n+1)}, s^+)$ while $\phi'^t \circ \bar{\tau}((T_i), s) = \phi'^t\big((T_i'), s\big) = ((T'_{i+(n+1)}, s^+)$. For $t \leq 0$, $\bar{\tau}$ commutes the flows accordingly.

(ii) Suppose that $l_j \neq l'_j$ for any $1 \leq j \leq N$. Then let us scale all cylinders in each of the two tiling spaces to have unit lengths. Let a *scaling* function be defined by $\rho : [P] \times [0, \infty] \to [P] \times [0, 1]$, $((T_i), s) \mapsto ((T_i), s.(l[T_i])^{-1})$, $0 \leq s \leq l[T_i]$. Now let $((T_i), s) \in \mathcal{T}_{\mathscr{P}} \times [0, l[T_0]]$ and by examining its semi-orbit under the flow ϕ^t scaled by ρ we reach the following definition. For $t \geq 0$, set $0 \leq s < l[T_0] \Rightarrow ((T_i), s) \in [T_0] \times [0, l[T_0]]$. Next choose a unique integer $n \in \mathbb{N}_0$ such that $0 < s.(l[T_0])^{-1} + t - (n+1) \leq 1$. Then $\phi^t \circ \rho((T_i), s) = ((T_{i+(n+1)}), s^+)$ where $0 \leq s^+ = s.(l[T_0])^{-1} + t - (n+1)$. For $t \leq 0$ the negative semi-orbit may be derived in a similar way. Identically, let $((T'_i), s) \in \mathcal{T}_{\mathscr{P}'} \times [0, l[T'_0]]$ then its semi-orbit for $t \geq 0$ satisfies $\phi'^t \circ \rho((T'_i), s) = ((T'_{i+(n+1)}), s^+)$ with the same conditions on its parameters as those given for $(T_i) \in \mathcal{T}_{\mathscr{P}}$, since ρ scales each cylinder set in both spaces to be of unit length and (3.12) is a bijection between $\mathcal{T}_{\mathscr{P}}$ and $\mathcal{T}_{\mathscr{P}'}$. Moreover the map $\bar{\tau}$ will commute the 'scaled' flows similarly as described in (i).

Since $\bar{\tau}$ conjugates the homeomorphisms τ' and τ, $\bar{\tau}$ is a homeomorphism. Therefore by (i) and (ii), the tiling spaces $\mathcal{T}_{\mathscr{P}}$ and $\mathcal{T}_{\mathscr{P}'}$ are flow equivalent and $\mathcal{T}_{\mathscr{P}}$ and $\mathcal{T}_{\mathscr{P}'}$ are homeomorphic tiling spaces whose flows conjugate over $\bar{\tau}$. $\square$

Definition 3.20. *[50] A* factor map *between tiling spaces is a map that commutes with the action of the translation group. A* topological conjugacy *between tiling spaces is a homeomorphism that is also a factor map.*

Remark 3.21. *There is not necessarily a conjugacy between two tiling spaces $\mathcal{T}_{\mathscr{P}}$ and $\mathcal{T}_{\mathscr{P}'}$ since it is not always possible to rescale time such that their flows are conjugate. For example, consider two tilings $T \in \mathcal{T}_{\mathscr{P}}$ and $T' \in \mathcal{T}_{\mathscr{P}'}$ with prototile sets $\mathscr{P} = \{P_1 = [0,1], P_2 = [0,1]\}$ and $\mathscr{P}' = \{P'_1 = [0,1], P'_2 = [0,1/2]\}$ respectively. However, if the tiling spaces are derived from a Pisot substitution such as the Fibonacci substitution* (1.7) *then it is possible to realise a conjugacy between their flows after rescaling time [17].*

Remark 3.22. *We know that the Cat substitution γ is Pisot (Def. 1.10) since its PF eigenvalue $\mu^2 > 1$ is an algebraic integer with a conjugate eigenvalue $|\mu^{-2}| < 1$ and its minimal subshift is Sturmian $\Sigma_{\alpha=1/\mu}$ (see page 35). Its tiling space is $\mathcal{T}_\gamma$. Thus (i) $\mathcal{T}_\gamma$ embeds in an orientable surface and (ii) $\mathcal{T}_\gamma$ embeds in a torus (Theorem 5 in [29]).*

Chapter 4

Branched covering spaces

In preparation for constructing a Plykin attractor via the torus, treated as a 2-fold branched covering of the sphere, and the emergence in the following chapters of branched 1-manifolds as representations of attractors, we define and discuss branched coverings herewith. A branched covering is distinguished from an unbranched covering by the following definition.

Definition 4.1. *[4] Let $\tilde{X}$ and X be complex manifolds. Consider a k-fold, $k \in \mathbb{N}$, covering map $p : \tilde{X} \to X$, where $x \in X$ has a neighbourhood $\mathcal{U} \ni x$ such that the complete inverse image of disjoint neighbourhoods is $p^{-1}(\mathcal{U}) = \bigcup_{i=1}^{k} V_i$. Then p is called a* branched covering *if, for any point $x \in X$, the restriction $p_{|V_i}$ is topologically conjugate to some mapping $\delta_k : z \mapsto z^k$, $z \in \mathbb{C}$. That is, there exist homeomorphisms $h : \mathcal{U} \to \mathbb{C}$ and $h_i : V_i \mapsto \mathbb{C}$ such that $\delta_k \circ h_{i|V_i} = h \circ p_{|V_i}$. The* deficiency *$k - 1$ in the number of covering points of a branch point $x_0 \in X$ is called the* order *of the branch point.*

Our particular interest lies in 2-fold coverings of the standard torus. For this purpose we consider representations of a genus 2 surface, the double torus, in two ways. Firstly by a polygon in $\mathbb{R}^2$ with combinatorial edge equations and secondly by a geometric representation in Euclidean 3-space. The next section consists of necessary definitions.

4.1 Combinatorial definitions of a surface

We base these definitions on [12].

Definition 4.2. *Define the set* $\Delta = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ *where the disc* Δ *is a polygon with* $n \in \mathbb{N}$ *sides and* n *evenly spaced vertices labelled* $P_1 = (1, 0), P_2, \ldots, P_n$*, which divide the circumference of* Δ *into directed edges labelled* $a_1, a_2, \ldots, a_n$ *(Fig. 4.1). An edge labelled* a_i^{-1} *takes the reverse direction to that of edge* a_i*.*

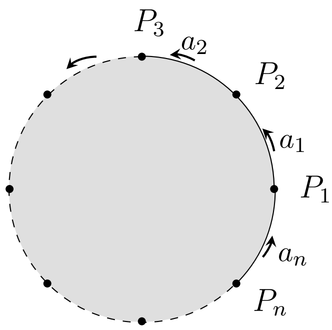


Figure 4.1: Disc Δ.

Definition 4.3. *Let* $f : \Delta \to \Pi$ *be an orientation-preserving homeomorphism which defines an* n*-sided* topological polygon (Π, f) *in a Euclidean space which, for* $i = 1, 2, \ldots, n$*, consists of* vertices *labelled* $Q_i = f(P_i)$ *and curved or straight* edges $b_i = f(a_i)$*. An* edge equation *for* (Π, f) *takes the form* $b_1 b_2 \ldots b_n = 1$*.*

In order to allow for edge identifications we relax the condition that f is a homeomorphism and modify the definition to be

Definition 4.4. *A continuous mapping* $\bar{f}$ *of the disc* Δ *onto a set* Π *defines a* singular topological polygon $(\Pi, \bar{f})$ *if* $\bar{f}$ *satisfies the following conditions:*

1. *Every point in* Π *is* $\bar{f}(P)$ *for some point* $P \in \Delta$*.*

2. *If* P *is an interior point of* Δ *and* Q *is any other point in* Δ*, then* $\bar{f}(Q) \neq \bar{f}(P)$*.*

3. *If* a_j*,* $1 \leq j \leq n$*, is an edge of* Δ *then either (i) for every non-vertex point* P *of* a_j*, there is no other point* Q *in* Δ *such that* $\bar{f}(Q) = \bar{f}(P)$*; or (ii) for every point* P *of* a_j*, other than* P_j *or* P_{j+1}*, there is a unique point* $P'(\neq P)$ *in* Δ *such that* $\bar{f}(P') = \bar{f}(P)$*. Furthermore, as* P *moves from* P_j *to* P_{j+1}*,* P' *moves along an edge* a_k*,* $k \neq j$*, either from* P_k *to* P_{k+1} *or from* P_{k+1} *to* P_k*. (Note that* $P_{n+1} = P_1$*).*

From now on *polygon* will mean singular topological polygon.

Definition 4.5. *A set of points in a Euclidean space is a* surface *if the set can be subdivided into a finite number of polygons such that*

1. *Polygons intersect only in edges and vertices.*
2. *Polygons have common vertices only to the extent required by the common edges.*
3. *No curve is used more than twice as a polygonal edge.*
4. *The polygons cannot be divided into two sets of polygons with no edge in common.*

Definition 4.5 requires that a surface is *closed* which means that it is a compact connected Hausdorff space in which each point has a neighbourhood homeomorphic to the plane.

Definition 4.6. *A* combinatorial representation of a surface *is a system of edge equations such that (a) no letter appears more than twice; (b) if the system is divided into two sets of equations, there is at least one letter that appears in an equation of each set.*

Note: $aba^{-1}b^{-1} = 1$ and $efe^{-1}f^{-1} = 1 \not\Rightarrow aba^{-1}b^{-1} = efe^{-1}f^{-1}$ necessarily.

4.2 2-fold branched coverings of $\mathbb{T}^2$

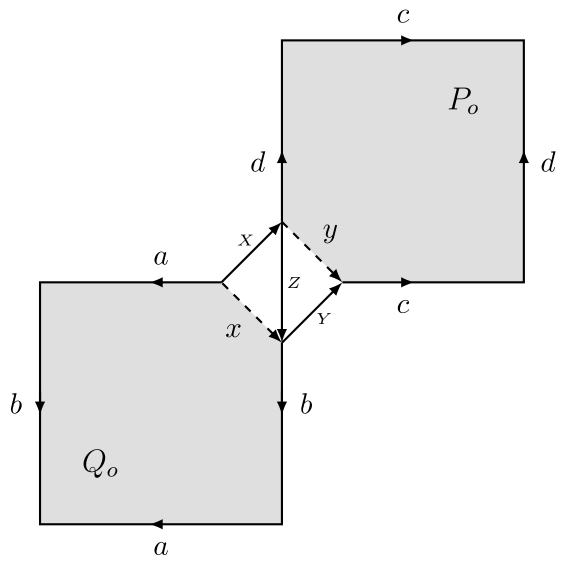
A planar representation of $\mathbb{T}^2\sharp\mathbb{T}^2$

Define $S_2' := \mathbb{T}^2\sharp\mathbb{T}^2 \subset \mathbb{R}^2$ to be a *double torus*. We construct S_2' by the following surgery. Let polygonal discs P and Q represent two 2-tori with edge equations $aba^{-1}b^{-1} = 1$ and $cdc^{-1}d^{-1} = 1$ respectively. Now remove two discs with boundaries $x = XZ$ from Q and $y = ZY$ from P, which results in respective edge equations for $Q_o : aba^{-1}b^{-1}x^{-1} = 1$ and for $P_o : cdc^{-1}d^{-1}y = 1$ (Fig. 4.2(i)). Identifying the two disc boundaries x and y forms a genus 2 surface. Moreover, when $x = y$ then the latter edge equation becomes $cdc^{-1}d^{-1}x = 1 \Rightarrow cdc^{-1}d^{-1} = x^{-1} \Rightarrow aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$ which is a single edge equation (Fig. 4.2(ii)).

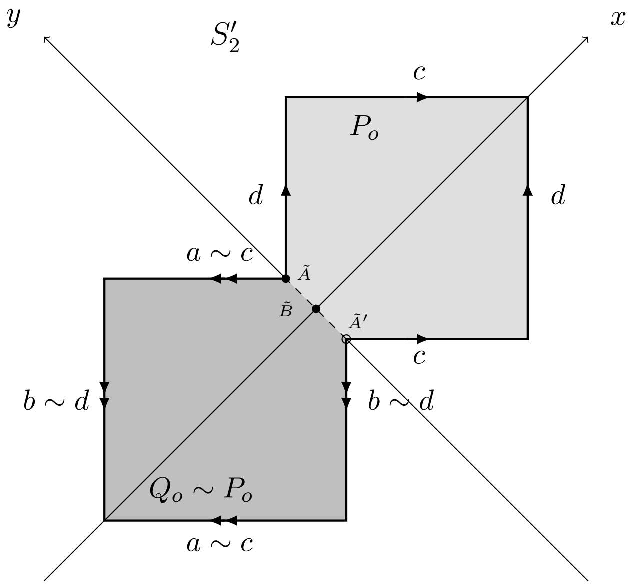
Consider S_2' in $\mathbb{R}^2$. Let $\tilde{B} = (0,0)$ be the centre of symmetry of S_2' with $\tilde{A} = (0,y)$ and $\tilde{A}' = (0,-y)$ where the line segment $\widetilde{AA'}$ represents the common loop x, with basepoint $\tilde{A} = \tilde{A}'$, joining the two tori with discs removed. Define a linear transformation by $R' : \mathbb{R}^2 \to \mathbb{R}^2,\ R'(x,y) = (-x,-y)$. Then $\tilde{B}$ is a fixed point of R' as is $\tilde{A} = \tilde{A}' = R'(\tilde{A})$. The rotation R' maps the boundary $cdc^{-1}d^{-1}x$ onto $aba^{-1}b^{-1}x^{-1}$, which we write as $acbda^{-1}c^{-1}b^{-1}d^{-1}x^{-1}x = acbda^{-1}c^{-1}b^{-1}d^{-1}$. Then the map π' identifies $a \sim c$ and $b \sim d$ giving 2 copies of the edge equation $aba^{-1}b^{-1} = 1$. That is, two copies of $\mathbb{T}^2$.

Definition 4.7. *The space S_2' is a* 2-fold branched covering *of the torus $\mathbb{T}^2$, with branch points A and B, induced by a quotient map $\pi' : S_2' \to S_2'/_\sim$, where the equivalence relation $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists\, n \in \mathbb{Z}$ such that $\mathbf{y} = (R')^n(\mathbf{x})$.*

Then $S_2'/_\sim \cong \mathbb{T}^2$ as a surface in $\mathbb{R}^2$.



(i) Remove a disc x from torus Q to give Q_o and a disc y from torus P to give P_o



(ii) Edges identified after a rotation through angle π.

Figure 4.2: $S_2' = \mathbb{T}^2 \sharp \mathbb{T}^2$ covers $\mathbb{T}^2$.

Proposition 4.8. $\mathbb{T}^2\sharp\mathbb{T}^2$ *is not homeomorphic to* $\mathbb{T}^2$.

Proof. We show that the fundamental group $\pi_1(\mathbb{T}^2\sharp\mathbb{T}^2)$ is distinct from $\pi_1(\mathbb{T}^2)$. Let a, b, c, d be four loops with a common basepoint x and boundary loop labelled γ. The fundamental group of γ is a free group on four generators, that is, $\pi_1(\gamma, x) = \langle a, b, c, d\rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Let the region $\mathcal{R}$ have a boundary labelled $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ representing $\mathbb{T}^2\sharp\mathbb{T}^2$. Let discs $D \subset U \subset \mathcal{R}$ with $U \setminus D \ni x$ and let $V = \mathcal{R} \setminus D$ such that $\gamma \subset U \cap V$. Now remove the interior of D and deform retract V onto the boundary of $\mathcal{R}$. Let the inclusion map $i : U \cap V \hookrightarrow \mathcal{R} \Rightarrow i(\gamma) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$ then by the Seifert[1]-van Kampen[2] theorem,

$$\pi_1(\mathbb{T}^2\sharp\mathbb{T}^2, x) \cong \frac{\pi_1(U, x) * \pi_1(V, x)}{N[i_*\pi_1(U \cap V, x)]} \cong \frac{e * \pi_1(V, x)}{N[i_*(\gamma)]} \cong \frac{\langle a, b, c, d\rangle}{N[i_*(\gamma)]},$$

where N is the least normal subgroup containing the element $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Thus the fundamental group of the double torus is

$$\pi_1(\mathbb{T}^2\sharp\mathbb{T}^2, x) = \langle a, b, c, d \,|\, aba^{-1}b^{-1}cdc^{-1}d^{-1} = e\rangle.$$

Now $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2 \ncong \pi_1(\mathbb{T}^2\sharp\mathbb{T}^2)$ and since homeomorphic spaces have isomorphic groups, $\mathbb{T}^2\sharp\mathbb{T}^2$ is not homeomorphic to $\mathbb{T}^2$. □

A representation of $\mathbb{T}^2\sharp\mathbb{T}^2$ in $\mathbb{R}^3$

Define $S_2 := \mathbb{T}^2\sharp\mathbb{T}^2 \subset \mathbb{R}^3$ (Fig. 4.3). By surgery analogous to the polygonal surgery described above remove a (shaded) disc from each 2-torus then identify the boundaries of each disc. Now in $\mathbb{R}^3$ position $\tilde{A} = (0, 0, z)$ and $\tilde{B} = (0, 0, 0)$ then rotate S_2 about the z-axis through an angle π by a linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3,\ R(x, y, z) = (-x, -y, z)$. Then $\tilde{A}$ and $\tilde{B}$ are fixed points of R.

[1]Herbert Karl Johannes Seifert (1907 - 1996) German mathematician.
[2]Egbert Rudolf van Kampen (1908 - 1942) Belgian mathematician.

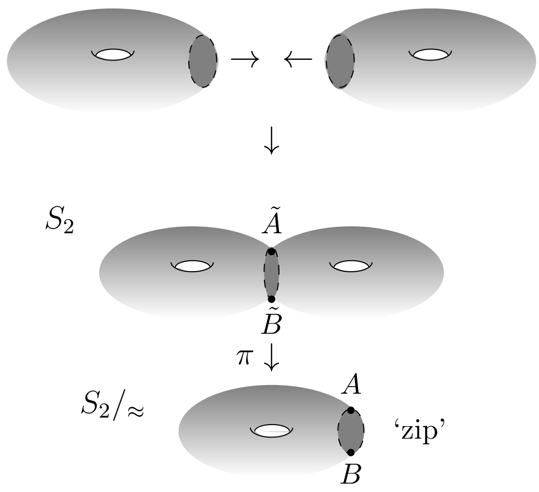


Figure 4.3: $S_2 = \mathbb{T}^2 \sharp \mathbb{T}^2$ covers $\mathbb{T}^2$.

Definition 4.9. *The space* $S_2 = \mathbb{T}^2 \sharp \mathbb{T}^2$ *is a* 2-fold branched covering *of* $\mathbb{T}^2$, *with branch points* A *and* B, *induced by a quotient map* $\pi : S_2 \to S_2/_{\approx}$, *where the equivalence relation* $\mathbf{x} \approx \mathbf{y} \Leftrightarrow \exists\, n \in \mathbb{Z}$ *such that* $\mathbf{y} = R^n(\mathbf{x})$.

Then $S_2/_{\approx} \cong \mathbb{T}^2$ as a surface in $\mathbb{R}^3$.

For a k-fold covering $\tilde{X}$ the *Euler*[3] *characteristic* $e(\tilde{X}) := k \cdot e(X) - \delta$ where the *total deficiency* δ is the sum of the orders of the branch points in $\tilde{X}$. In both cases $\pi^{-1}(\mathbf{x})$ and $(\pi')^{-1}(\mathbf{x})$ each have two pre-images in S_2 and S_2' respectively with the exceptions of a single pre-image $\pi^{-1}(A) = \tilde{A}$ and $\pi^{-1}(B) = \tilde{B}$ in S_2 and $(\pi')^{-1}(A) = \tilde{A}(= \tilde{A}')$ and $(\pi')^{-1}(B) = \tilde{B}$ in S_2'. So A and B are branch points of order 1. Thus $e(\mathbb{T}^2 \sharp \mathbb{T}^2) = k \cdot e(\mathbb{T}^2) - \delta = -2$ is satisfied.

Remark 4.10. *There exists a natural homeomorphism* $h : S_2 \to S_2'$ *which conjugates the rotations* $h \circ R = R' \circ h$.

We give a *heuristic argument* in justification of Remark 4.10. Consider S_2 embedded symmetrically in $\mathbb{R}^3$. Let a non-branch point of S_2 be $\tilde{p}^+ = (x, y, z)$ with its counterpart $\tilde{p}_- = (x, y, -z)$. Then let $(x, y, 0)$ be their common representative point, considered as a projection onto the horizontal xy plane through

[3]Leonhard Euler (1707 - 1783) Swiss mathematician and physicist.

the origin and denoted by $\tilde{p} = (x, y)$. So $R(\tilde{p}^+)$ and $R(\tilde{p}_-)$ is represented by $R(x, y, 0) = (-x, -y, 0)$, that is $\tilde{p} = (x, y) \mapsto (-x, -y)$. Let $\tilde{p}' = (x, y) \in S_2'$ then $R'(x, y) = (-x, -y)$. The fixed points $\tilde{A}, \tilde{B} \in S_2$ and $\tilde{A}', \tilde{B}' \in S_2'$ concur in both spaces. Now by the construction of $\mathbb{T}^2 \sharp \mathbb{T}^2$ it is possible to assign a one-one correspondence between $\tilde{p} \sim \tilde{p}'$, $\tilde{A} \sim \tilde{A}'$ and $\tilde{B} \sim \tilde{B}'$, from which the fixed angle rotations R and R' transform their respective points in an equivalent manner. So we may deduce that the rotations on each surface are topologically conjugate.

4.3 A covering flow

Proposition 4.11. *Let ϕ^t, $t \in \mathbb{R}$, be a smooth flow on a closed orientable surface X and let $\mathbf{V} \subseteq \mathbb{R}^2$ be its associated vector field. Then on a 2-fold branched covering of X, the flow $\tilde{\phi}^t$, $t \in \mathbb{R}$, with vector field $\mathbf{V}^* \subseteq \mathbb{R}^2$ is a* covering flow *for ϕ^t.*

Proof. Consider the branched covering $p : \tilde{X} \to X$. Let the set of branch points $\{x_i \in X \mid i \in \mathbb{N}\}$ be fixed points of ϕ^t then denote the restricted flow on the set $X \setminus \{x_i\}$ by $\mathring{\phi}^t$ with its associated vector field $\mathring{\mathbf{V}}$. Since $p_{|\tilde{X} \setminus p^{-1}(\{x_i\})} : \tilde{X} \setminus p^{-1}(\{x_i\}) \to X \setminus \{x_i\}$ is an unbranched covering, there is a vector field $\mathring{\mathbf{V}}^*$ on $\tilde{X} \setminus p^{-1}(\{x_i\})$ which covers $\mathring{\mathbf{V}}$ (see for example [4]). At the points $p^{-1}(\{x_i\})$, map the vector field to the zero vector, that is $\mathring{\mathbf{V}}^* = \mathbf{0}$. Then since the branch points $\{x_i\}$ are designated as fixed points of ϕ^t, the vector field $\mathbf{V}^*$ on the cover $\tilde{X}$ is continuous and covers $\mathbf{V}$, by the definition of a branched covering. Thus the flow $\tilde{\phi}^t$ determined by $\mathbf{V}^*$ is a covering flow for ϕ^t. $\square$

We work with the planar definition of the surfaces, $S_2' = \mathbb{T}^2 \sharp \mathbb{T}^2 \subset \mathbb{R}^2$ and $S_2'/_\sim \cong \mathbb{T}^2$. Fix $\alpha \in [0,1] - \mathbb{Q}$ and define a smooth flow by

$$\Phi_\alpha : \mathbb{R} \times S_2'/_\sim \to S_2'/_\sim, \text{ with associated vector field } \mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}. \tag{4.1}$$

Now for $t \in \mathbb{R}$ let the flow $\mathring{\Phi}_\alpha^t$ on $S_2'/_\sim \setminus \{A, B\}$ have the associated vector field $\mathring{\mathbf{V}}$ then at $(\pi')^{-1}(\{A, B\}) \in S_2'$ set $\mathring{\mathbf{V}}^* = \mathbf{0}$. Then by Proposition 4.11 define a covering flow for (4.1) by

$$\tilde{\Phi}_\alpha : \mathbb{R} \times S_2' \to S_2', \text{ with vector field } \mathbf{V}^*. \tag{4.2}$$

Now $\tilde{\Phi}_\alpha^t$ is defined by the solution set of a pair of simultaneous first-order autonomous equations $\dot{x} = x$ and $\dot{y} = -y$ while Φ_α^t satisfies the solution set of the paired equations $\dot{x} = x^2 + y^2$ and $\dot{y} = 0$. Consequently, $\tilde{\Phi}_\alpha^t$ is a family of hyperbolic integral curves, $xy =$ constant in $\mathbb{R}^2$, along which the points move with time t. The flow has a single fixed point at $\tilde{B} = (0,0)$ to which the points on the x-axis (y-axis) approach asymptotically for $t < 0$ ($t > 0$). Now define a complex valued function by

$$\delta_2 : \mathbb{C} \to \mathbb{C}, \; z \mapsto z^2. \tag{4.3}$$

This function projects $\tilde{\Phi}_\alpha^t$ onto Φ_α^t in real space by $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$, which is the solution set of the system $\dot{x} = x^2 + y^2$, $\dot{y} = 0$. This set forms a family of horizontal lines $y =$ constant such that $\forall\,(x, y) \in \mathbb{R}^2 - (0,0)$, $x \to \infty$ as $t \to \infty$. For points on the x-axis, when $x < 0$, $(x, 0) \to (0,0)$ as $t \to \infty$ whilst for $x > 0$, $x \to \infty$ as $t \to \infty$. The sole fixed point $\tilde{B} = (0,0)$ of the hyperbolic solution set is an unstable saddle point whilst the family of solution curves $y =$ constant has a stable fixed point at $B = (0,0)$. Having two branch points in the covering space, the covering flow $\tilde{\Phi}^t_\alpha$ has exactly two saddle points on $\mathbb{T}^2\sharp\mathbb{T}^2$, one at $(\pi')^{-1}(A)$ and the other at $(\pi')^{-1}(B)$. We lift the linear flow Φ^t_α of $S'_2/_\sim$ to the hyperbolic flow $\tilde{\Phi}^t_\alpha$ of the covering space S'_2 then examine the nature of the flow on the quotient space $S'_2/_\sim$.

Figure 4.4. Sketch (i) shows directed linear flow line segments of $\mathcal{O}_A(\mathbf{x})$ (green) and $\mathcal{O}_B(\mathbf{x})$ (blue). For the sake of clarity in (ii) only blue lifted flow line segments of $(\pi')^{-1}(\mathcal{O}_B(\mathbf{x}))$ are shown where the $+,-$ indices act only to distinguish the two 'branches' of the hyperbolic system. Then in (iii) the blue segments of the lifted and rotated flow lines which either leave or limit to $\tilde{B}$ are shown. Sketch (iv) with green and blue two-headed arrows indicate superposition of the two branches of the flow sourced and limiting to A and B respectively, identified under the quotient map yielding a linear flow on the 2-fold torus.

In general consider a flow ϕ^t on X. For $x \in X$, an orbit $\mathcal{O}(x) = \{\phi^t(x) \,|\, t \in \mathbb{R}\}$ partitions X into equivalence classes which we shall call *congruent orbits*. Thus, congruent orbits written $\mathcal{O}(x) \equiv \mathcal{O}(y) \Leftrightarrow x \equiv y \Rightarrow y = \phi^t(x)$ for some $t \in \mathbb{R}$. Furthermore, each point $x \in X$ lies in exactly one congruent orbit and no two congruent orbits intersect. In the present context of our branched covering, let forward orbits be $\mathcal{O}_A(\mathbf{x}) := \{\Phi^t_\alpha(\mathbf{x}) \,|\, t > 0,\, \mathbf{x} \neq A\}$ and $\mathcal{O}_B(\mathbf{x}) := \{\Phi^t_\alpha(\mathbf{x}) \,|\, t > 0,\, \mathbf{x} \neq B\}$. Then $\mathcal{O}_A(\mathbf{x}) \cap \mathcal{O}_B(\mathbf{x}) = \emptyset$ and likewise the complete inverses $(\pi')^{-1}(\mathcal{O}_A(\mathbf{x})) \cap (\pi')^{-1}(\mathcal{O}_B(\mathbf{x})) = \emptyset$. In general, each *trajectory* l belonging to a congruent orbit, and hence a flow, has the same tangent vector $\mathbf{v}_p$ at a point $p \in \mathcal{O}(x)$. Call the associated vector field $\mathbf{V}$ the *generator* of the flow. Then given a smooth vector field $\mathbf{V}$ on $\mathbb{R}^2$ there exists a congruence of orbits in $\mathbb{R}^2$ such that $\mathbf{V}$ is the generator for the corresponding flow. The existence of one of these constructions: a vector field, flow, congruence, assures the existence of the other two (see for example [19]).

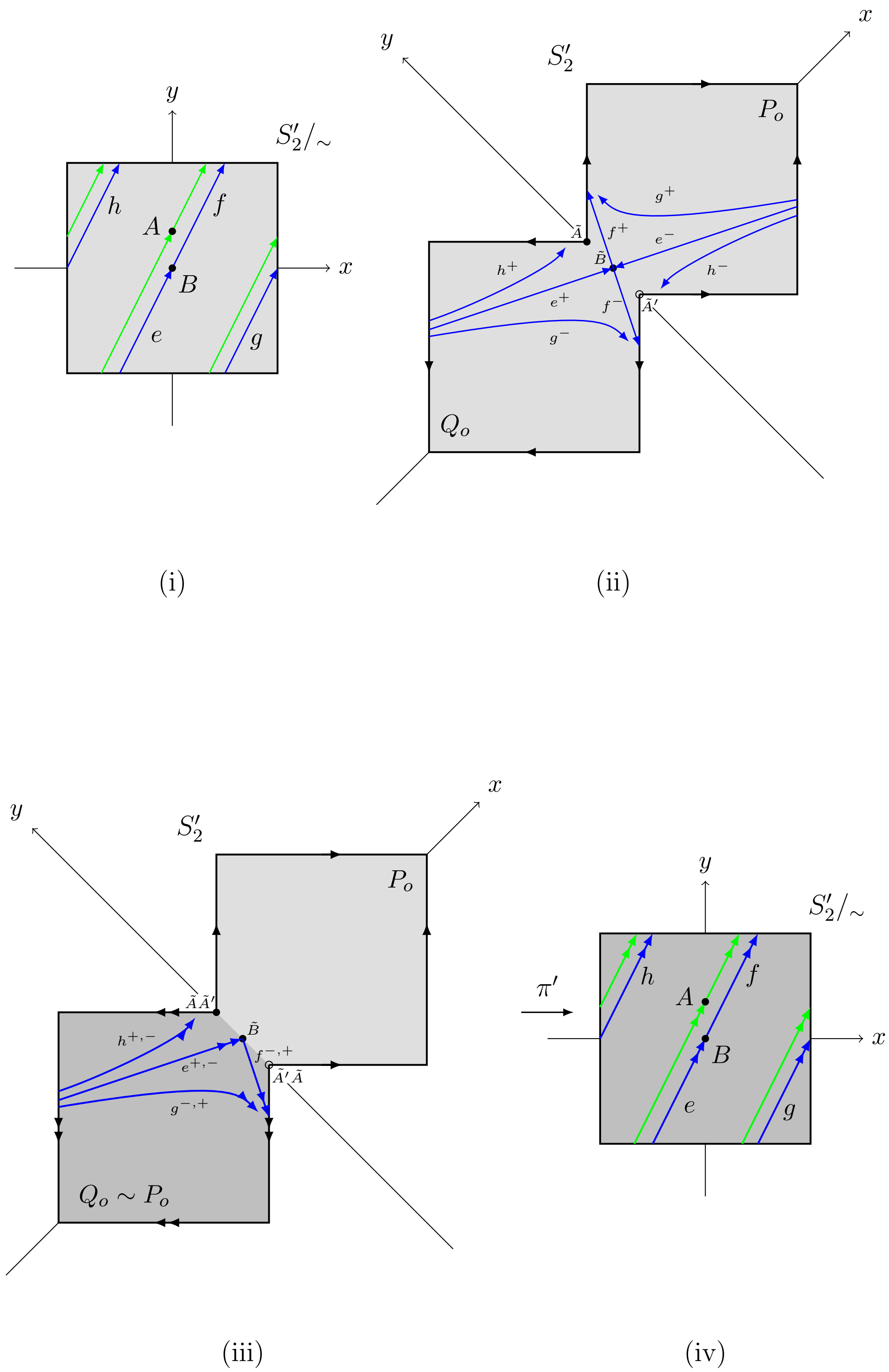


Figure 4.4: Lifted and projected flowlines.

Definition 4.12. *A flow ϕ^t on a closed orientable surface X is* transitive *if and only if ϕ^t has a trajectory that is dense in X.*

Any trajectory l of Φ^t_α different from that of a fixed point is dense in $S'_2/_\sim$ and so a trajectory l_A of a point from $\mathcal{O}_A(\mathbf{x})$ or l_B of a point from $\mathcal{O}_B(\mathbf{x})$ is dense in $S'_2/_\sim$. Consequently, as for example stated in [4], the complete inverse $(\pi')^{-1}(l_A)$, consisting of two trajectories $\tilde{l}_{\tilde{A}_1}$ and $\tilde{l}_{\tilde{A}_2}$ of $\tilde{\Phi}^t_\alpha$ is dense in S'_2; as is $(\pi')^{-1}(l_B)$ with trajectories $\tilde{l}_{\tilde{B}_1}$ and $\tilde{l}_{\tilde{B}_2}$. However, consider the sufficiency of a single trajectory cited in the following proposition.

Proposition 4.13. *A single trajectory $\tilde{l}$ of $\tilde{\Phi}^t_\alpha$ is dense in the covering space S'_2 and thus the flow $\tilde{\Phi}^t_\alpha$ is transitive.*

Proof. Let $l \subset S'_2/_\sim$ and $\tilde{l} \subset S'_2$ be respective trajectories of the flow Φ^t_α and its covering flow $\tilde{\Phi}^t_\alpha$; l is not a trajectory of a fixed point A or B. The closure $\bar{\tilde{l}}$ is compact, connected and invariant and so $\pi'(\bar{\tilde{l}})$ is compact, connected and invariant. It follows that $\pi'(\bar{\tilde{l}}) = S'_2/_\sim$ then $l = \pi'(\tilde{l}) \Rightarrow \bar{l} = S'_2/_\sim$. Let an open invariant set be $U := S'_2 - \bar{\tilde{l}} \Rightarrow \pi'(U)$ is open and invariant which then implies that $\pi'(U) = S'_2/_\sim$. The boundary of U is $\partial U = \bar{U} \cap \bar{\tilde{l}} \subseteq \bar{\tilde{l}}$. Let x be a limit point of $\bar{l}$. Call $(\pi')^{-1}_{i=1,2}(x) = p_{i=1,2}$ and set $p_1 \in \partial U$. Now for $\epsilon > 0$, $B_\epsilon(p_1) \cap U - \{p_1\} \neq \emptyset \Rightarrow \exists\, y \in U, y \neq p_1,\ y \notin \tilde{l}$. Since p_1 is also a limit point of $\tilde{l}$ this contradicts $\tilde{l}$ being closed. Thus $U :\neq S'_2 - \bar{\tilde{l}} \Rightarrow \bar{\tilde{l}} = S'_2$. In the context of the flow, if $\lim_{t\to\infty} \tilde{\Phi}^t_\alpha(z) = p_1 \in \bar{U} \Rightarrow \exists\, t_0 < t$ such that $\tilde{\Phi}^{t_0}_\alpha(z) \in U$. So part of the trajectory of z lies in U which our argument shows is impossible. In conclusion, $\bar{\tilde{l}} = S'_2 \Rightarrow \bar{\tilde{l}}$ is dense in S'_2. Hence the flow $\tilde{\Phi}^t_\alpha$ is transitive. $\square$

Proposition 4.14. *The function δ_2 (4.3) induces a bijective continuous map $h' : S_2'/_\sim \to \mathbb{T}^2$ which is a homeomorphism. The maps satisfy $h' \circ \pi' = \delta_2$.*

$$\begin{tikzcd} S_2' \arrow[d, "\pi'"'] \arrow[dr, "\delta_2"] & \\ S_2'/_\sim \arrow[r, "h'"] & \mathbb{T}^2 \end{tikzcd}$$

Proof. By definition, π' is a quotient map. Let $\tilde{\mathcal{U}} \subset S_2'$ be an open neighbourhood of the branch point $\tilde{B} = (0,0)$. Let other points $\tilde{p}, \tilde{q} \in \tilde{\mathcal{U}}$ with $\tilde{p} \in P_0$, $\tilde{q} \in Q_0$ and $|\tilde{p} - \tilde{B}| = |\tilde{q} - \tilde{B}|$ in the Euclidean metric. Now $\pi'(\tilde{\mathcal{U}}) = \mathcal{U} \subset S_2'/_\sim$ and $\pi'(\tilde{B}) = B$. Let $s \in \mathcal{U}$ denote the identification point $\pi'(\tilde{p}) \sim \pi'(\tilde{q})$ then $|s - B| = |\tilde{p} - \tilde{B}|$ by the isometry R'.

Consider S_2' as a surface with polar coordinates $(r, \theta) \in \mathbb{R}^2$. The function $\delta_2 : S_2' \to \mathbb{T}^2$ is a surjective map which is continuous everywhere since $\delta_2(z_0) = z_0^2$ and $\lim_{z \mapsto z_0} z^2 = z_0^2$. Let $\mathcal{U}$ be open in $\mathbb{T}^2$ if and only if $\delta_2^{-1}(\mathcal{U}) = \tilde{\mathcal{U}}$ is open in S_2', then δ_2 is a quotient map. We show that δ_2 is constant on each set $(\pi')^{-1}(\{s\})$, $s \in S_2'/_\sim$. Firstly, take the branch point, $\delta_2((\pi')^{-1}(B)) = (0,0) = B$. Next, if one of the two pre-images of $s \in \mathcal{U}$ is $\tilde{p} = (r, \theta) \Rightarrow \tilde{q} = (r, \theta + \pi)$ which means that $\delta_2(\tilde{p}) = (r^2, 2\theta) \equiv (r^2, 2(\theta + \pi)) = \delta_2(\tilde{q})$, modulo 2π. Then the constancy of δ_2 on each pre-image of π' follows. Hence h' is a homeomorphism (see for example Corollary 22.3 in [38]). $\square$

Remark 4.15. *Proposition 4.14 confirms $\pi' : S_2' \to S_2'/_\sim$ as a 2-fold branched covering complying with Definition 4.1. In other words, a double torus $\mathbb{T}^2 \sharp \mathbb{T}^2$ is a 2-fold branched covering of $\mathbb{T}^2$.*

Chapter 5

Hyperbolic attractors

This chapter begins a key topic in the thesis. In §5.1.2 we define and construct a toral hyperbolic attractor Λ with one complementary domain. In §5.2 we extend the process to define and construct the Plykin attractor Λ_Π with four complementary domains. The minimal Denjoy flow $\mathbb{D}_\alpha$ (Def. 3.9) is homeomorphic to the attractor Λ (Remark 5.9) which in turn is homeomorphic to the tiling space $\mathcal{T}_\gamma$ (Theorem 6.20) where $\mathcal{T}_\gamma$ is the tiling space derived from the Cat substitution γ (1.9).

A *diffeomorphism* is a one-to-one differentiable mapping with a diferentiable inverse. Let M be a closed smooth manifold then in the C^r topology, $\mathrm{Diff}^r(M) := \{f : M \to M \mid f \text{ is a } C^r \text{ diffeomorphism}\}$.

Definition 5.1. *[20] The map $f \in \mathrm{Diff}^r(M)$ is* structurally stable *if there is a neighbourhood U of f, $U \subset \mathrm{Diff}^r(M)$, such that $\forall\, g \in U$, g is topologically conjugate to f.*

Definition 5.2. *[28] A* stable manifold *is defined* $W^s(x) = \{z \in M | d(f^n z, f^n x)\} \to 0$ *as* $n \to \infty$. *An* unstable manifold *is defined* $W^u(x) = \{z \in M \mid d(f^n z, f^n x)\} \to 0$ *as* $n \to -\infty$.

Definition 5.3. *[40] A diffeomorphism f on a manifold M is* Anosov *if*

1. *The tangent bundle $TM = E^s \oplus E^u$. (The tangent subspaces E^s and E^u span the whole tangent space and they are respectively tangents to the stable(unstable) manifolds.)*

2. *$E^{s(u)}$ are invariant: $\forall\, p \in M$, $\forall\, \mathbf{v} \in E_p^{s(u)}$, $Df_p(\mathbf{v}) \in E_{f(p)}^{s(u)}$.*

3. *$\exists\, 0 < \lambda < 1$ such that (for some Riemannian[1] metric on M), $\forall\, p \in M$, $\|Df_p(\mathbf{v})\| \leq \lambda\|\mathbf{v}\|$, $\forall\, \mathbf{v} \in E^s$, and $\|Df_p^{-1}(\mathbf{v})\| \leq \lambda\|\mathbf{v}\|$, $\forall\, \mathbf{v} \in E^u$.*

Definition 5.4. *[20] A fixed point p of f is* hyperbolic *if $Df_p : T_pM \to T_pM$ has no eigenvalues of modulus 1, where T_pM denotes the set of tangent vectors to M at p. If p is periodic of period n, then p is hyperbolic if Df_p^n has no eigenvalues of modulus 1.*

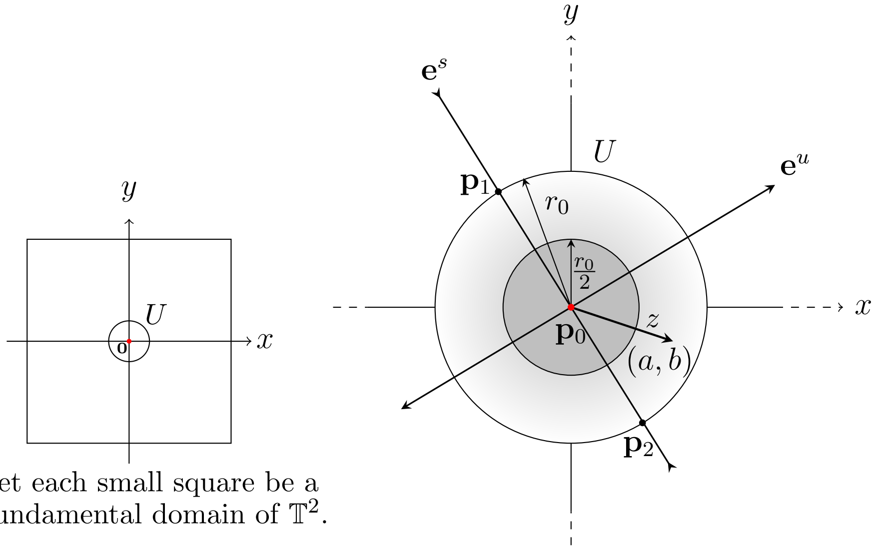
Definition 5.5. *[28] A point $p \in M$ is* non-wandering *if for every neighbourhood $U \ni p$ and for every $n_0 \in \mathbb{N}$, $\exists\, n \in \mathbb{Z}$ such that $|n| > n_0$ and $f^n(U) \cap U \neq \emptyset$.*

5.1 The DA construction

Recall the hyperbolic toral Cat map $\mathcal{C}$ (1.10) with the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which we can now identify as an Anosov diffeomorphism and any Anosov diffeomorphism is structurally stable (see for example Theorem (3.3) in [51]). Thus $\mathcal{C}$ is structurally stable. By perturbing such a diffeomorphism one may construct a *derived from Anosov (DA) diffeomorphism.* This technique was first demonstrated by Smale who applied a type of surgery to a 2-torus which can be read on pages 788 -789 of his paper [51]. For our purposes we take guidance from [48] and [32] and apply a similar technique to the map $\mathcal{C}$ in $\mathbb{T}^2$ resulting in the DA diffeomorphism (5.2).

[1] Georg Friedrich Bernhard Riemann (1826 - 1866) German mathematician.

5.1.1 A DA diffeomorphism



(i) Let each small square be a fundamental domain of $\mathbb{T}^2$.

(ii) Enlarged view of the neighbourhood U.

Figure 5.1: Constructing a DA diffeomorphism.

Figure 5.1 illustrates the following analysis. Let $\mathbf{p}_0$ be a fixed point of $\mathcal{C}$ corresponding to $\mathbf{0}$ in $\mathbb{R}^2$. Let U be a relatively small neighbourhood of $\mathbf{p}_0$ in which (a, b) are the coordinates with respect to the diagonalisation $A = \frac{1}{\mu^2+1}\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}\begin{bmatrix} \mu^2 & 0 \\ 0 & \mu^{-2} \end{bmatrix}\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$. Let $\mathbf{e}^u$ and $\mathbf{e}^s$ be the normalized orthogonal eigenvectors in the unstable and stable direction corresponding to the eigenvalues μ^2 and μ^{-2} respectively. Thus any point $(a\mathbf{e}^u, 0)$ lies in the expanding unstable manifold of $[\mathbf{0}]$ while the point $(0, b\mathbf{e}^s)$ lies in the contracting stable manifold of $[\mathbf{0}]$ and then $\mathcal{C}(a, b) = (\mu^2 a, \mu^{-2} b)$ on U. Let $z = (a, b)$ and $|z|$ its length. For $r_0 > 0$ let the ball $B_{r_0}(\mathbf{p}_0) \subset U$. Let a C^∞ *bump function* be defined by

$$\delta : \mathbb{R}^2 \to [0, 1],\ \delta(z) = \begin{cases} 0 & \text{if } |z| \geq r_0 \\ 1 & \text{if } |z| \leq \frac{r_0}{2}, \end{cases} \tag{5.1}$$

with $\delta'(z) < 0$ if $\frac{r_0}{2} < \delta(z) < r_0$. Consider the paired differential equations $\dot{a} = 0$ and $\dot{b} = b\,\delta(z)$. Let ϕ^t be the flow of these differential equations then

$\phi^t(a, b) = (a, \phi_b^t(a, b))$. So at the point $\mathbf{p}_0$, a is constant and $\dot{b} = b \Rightarrow b = \mathbf{p}_0 e^t$. The derivative of the flow at $\mathbf{p}_0$ with respect to the basis $\{\mathbf{e}^u, \mathbf{e}^s\}$ is $D\phi^t_{\mathbf{p}_0} = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}$.

Definition 5.6. *[48] For a fixed $\tau > 0$, define the map*

$$f : \mathbb{T}^2 \to \mathbb{T}^2, \;\; f = \phi^\tau \circ \mathcal{C}, \tag{5.2}$$

such that $e^\tau \mu^{-2} > 1$, where μ^{-2} is the stable eigenvalue of the matrix A which determines $\mathcal{C}$. The map f is a DA diffeomorphism.

Note that the derivative of f at $\mathbf{p}_0$ is $Df_{\mathbf{p}_0} = D\phi^\tau_{\mathbf{p}_0} D\mathcal{C}_{\mathbf{p}_0} = \begin{bmatrix} 1 & 0 \\ 0 & e^\tau \end{bmatrix} \begin{bmatrix} \mu^2 & 0 \\ 0 & \mu^{-2} \end{bmatrix} = \begin{bmatrix} \mu^2 & 0 \\ 0 & e^\tau \mu^{-2} \end{bmatrix}$ making $\mathbf{p}_0$ a source. Outside of U we have $f_{\restriction_{\mathbb{T}^2 \setminus U}} = \mathcal{C}_{\restriction_{\mathbb{T}^2 \setminus U}}$.

5.1.2 A DA attractor

Definition 5.7. *For a compact set M, $N \subset M$ is a* basin of attraction *for $f : M \to M$ if $f(N) \subset \mathrm{Int}(N)$; and $\Lambda \subset M$ is an* attractor *if $\Lambda = \bigcap_{n \in \mathbb{N}} f^n(N)$ for some basin N.*

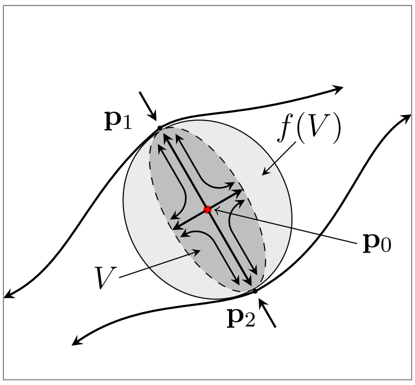


Figure 5.2: Developing a DA attractor.

Figure 5.2 illustrates the following analysis. The DA map (5.2) has three fixed points. As previously demonstrated $f(\mathbf{p}_0) = \mathbf{p}_0$ is a source. Outside of U, f preserves the stable manifold $W^s(0)$ while Df preserves the stable tangent subspace E^s for $\mathcal{C}$. Thus there must be two fixed points $\mathbf{p}_1, \mathbf{p}_2 \in W^s(0)$, lying one on either side of $\mathbf{p}_0$. Then $\mathbf{p}_1$ and $\mathbf{p}_2$ are saddle points which lie at the intersection of stable and unstable manifolds.

Now let $U \supset V$ be a small open neighbourhood of $\mathbf{p}_0$ with $\mathbf{p}_1, \mathbf{p}_2 \notin V$ such that $f(V) \supset V$. Then $V \subset W^u = \bigcup_{n\in\mathbb{N}} f^n(V)$ which makes V a basin of repulsion. Let its complement $N := \mathbb{T}^2 \setminus V$ be a basin of attraction for f with expanding attractor $\Lambda := \bigcap_{n\in\mathbb{N}} f^n(N)$ of topological dimension one. Moreover, the restricted map $f_{\restriction_\Lambda}$ is transitive whose periodic points are dense in Λ (see for example Theorem 8.1 in [48]) and since $W^u(\mathbf{p}_0)$ is an open dense set in $\mathbb{T}^2$, Λ has an empty interior (Def. 5.8).

Definition 5.8. *[38] Let A be a subset of a space X. A has* empty interior *if every point of A is a limit point of the complement of A, that is, if the complement of A is dense in X.*

Remark 5.9. *[53] The attractor Λ and the Denjoy continuum $\mathbb{D}_\alpha$ are homeomorphic.*

Informally we can compare Λ and $\mathbb{D}_\alpha$ in the following manner. The unstable manifold $W^u(\mathbf{p}_0)$ of f is a 'split open' version of the unstable manifold $W^u(0)$ of $\mathcal{C}$ such that the pair of forward(backward) components of $W^u(\mathbf{p}_0)$ approach each other asymptotically. This 'splitting open' is analogous to 'inserting intervals' into S^1 during the Denjoy construction. That is, the set $W^u(\mathbf{p}_0)$ corresponds to $\bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n)$ while their complements form Λ and $\mathbb{D}_\alpha$ respectively. In §6.1.2 we illustrate the construction of Λ through a series of diagrams. But for now our task is to construct the Plykin attractor, see §5.2.

5.2 The Plykin attractor

We describe below a construction of the Plykin attractor Λ_Π, so named after R. V. Plykin who first introduced his attractor directly on the 2-sphere [42]. Our approach is to realise the torus as a 2-fold branched covering of the sphere with four branch points. At each branch point we apply the DA construction of §5.1.

So to begin the process let an *involution* be defined by

$$i : \mathbb{T}^2 \to \mathbb{T}^2,\ i\left(\overline{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \overline{\begin{bmatrix} -x \\ -y \end{bmatrix}},\ (\text{mod } 1). \tag{5.3}$$

Label the four points $\left\{\mathbf{b}_0 = \overline{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \mathbf{b}_1 = \overline{\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}}, \mathbf{b}_2 = \overline{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}, \mathbf{b}_3 = \overline{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}\right\} \subset \mathbb{T}^2$.

Proposition 5.10. *The involution satisfies $i(i(\bar{\mathbf{x}})) = \bar{\mathbf{x}}$, $\forall\, \bar{\mathbf{x}} \in \mathbb{T}^2$. Each $\mathbf{b}_j \in \mathbb{T}^2$, $j = 0, \ldots, 3$, is fixed by i.*

Proof. Let $\bar{\mathbf{x}} = \overline{\begin{bmatrix} x \\ y \end{bmatrix}} \in \mathbb{T}^2$ then $i(\bar{\mathbf{x}}) = \overline{\begin{bmatrix} -x \\ -y \end{bmatrix}}$, $i(i(\bar{\mathbf{x}})) = i\left(\overline{\begin{bmatrix} -x \\ -y \end{bmatrix}}\right) = \overline{\begin{bmatrix} x \\ y \end{bmatrix}} = \bar{\mathbf{x}}$. Let $\epsilon, \delta \in \{0, 1/2\}$ and let $\begin{bmatrix} \epsilon \\ \delta \end{bmatrix} \in \mathbb{R}^2$ then $\begin{bmatrix} \epsilon \\ \delta \end{bmatrix} = \begin{bmatrix} -\epsilon \\ -\delta \end{bmatrix}$ (mod 1). So for $\overline{\begin{bmatrix} \epsilon \\ \delta \end{bmatrix}} \in \mathbb{T}^2$, $i\left(\overline{\begin{bmatrix} \epsilon \\ \delta \end{bmatrix}}\right) = \overline{\begin{bmatrix} -\epsilon \\ -\delta \end{bmatrix}} = \overline{\begin{bmatrix} \epsilon \\ \delta \end{bmatrix}}$. Thus $i(\mathbf{b}_j) = \mathbf{b}_j$, $j = 0, \ldots, 3$. $\square$

Let $\overline{\begin{bmatrix} x \\ y \end{bmatrix}} \sim \overline{\begin{bmatrix} 1-x \\ 1-y \end{bmatrix}}$ be the equivalence relation $\sim$ which is induced by i on $\mathbb{T}^2$.

Definition 5.11. *The torus $\mathbb{T}^2$ is a* 2-fold branched covering *of the 2-sphere S^2, with* branch points $\mathbf{b}_j$, $j = 0, \ldots, 3$, *induced by a* quotient map $\Pi : \mathbb{T}^2 \to \mathbb{T}^2/i$, *where i is the involution defined above.*

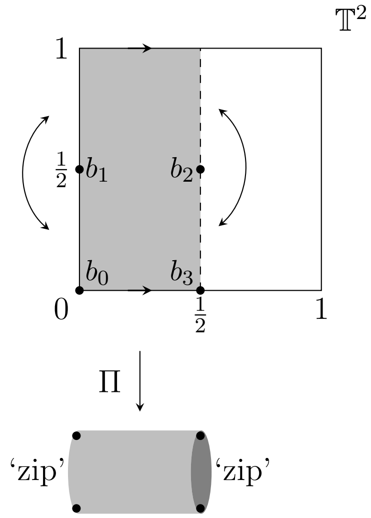


Figure 5.3: A sketch of $\mathbb{T}^2/i$.

Then $\mathbb{T}^2/i \cong S^2$. Observe that the construction satisfies the Euler characteristic $e(\mathbb{T}^2) = k \cdot e(S^2) - \delta = 2 \cdot 2 - 4 = 0$ and is illustrated in Figure 5.3. A comparable illustration of the construction is given in §3.3 of [4]. Now consider the DA map (5.2) which is i-invariant since $\mathcal{C}$ is orientation-preserving and on $\mathbb{T}^2$, i behaves like the rotation R' on S_2'. Further, $A^3\mathbf{b}_j = \mathbf{b}_j \Rightarrow \mathcal{C}^3(\mathbf{b}_j) = \mathbf{b}_j$, $\forall\, 0 \leq j \leq 3$. Thus $\mathcal{C}^3$ preserves the four fixed points of i. Now repeat the construction of the DA diffeomorphism described in §5.1.1 but substitute $\mathcal{C}^3$ for $\mathcal{C}$ which endows (5.4) with four fixed repelling points. So let the DA diffeomorphism induced by the covering map Π be defined by

$$f_\Pi : S^2 \to S^2, \; f_\Pi = \phi^\tau \circ \mathcal{C}^3, \tag{5.4}$$

such that for a fixed $\tau > 0$, $e^\tau \mu^{-6} > 1$. To form an attractor, repeat the process of §5.1.2 at each of the four points by setting $\mathbf{p}_{0_j}$ to correspond with $\mathbf{b}_j$, $0 \leq j \leq 3$. Put simultaneously $\mathbf{p}_{0_j}$, $0 \leq j \leq 3$, in its small open neighbourhood $V_j \subset (\mathbb{T}^2 \setminus \Lambda)/i$, $V_j \cap V_k = \emptyset$, $j \neq k$. Then for $0 \leq j \leq 3$, $V_j \subset W^u(\mathbf{p}_{0_j})$ and $W^u(\mathbf{p}_{0_j}) = \bigcup_{n\in\mathbb{N}} f_\Pi^n(V_j)$ so that $\dot{\bigcup}_{j=0}^3 V_j \subset \bigcup_{n\in\mathbb{N}} f_\Pi^n(\dot{\bigcup}_{j=0}^3 V_j)$ is a set of four basins of repulsion for f_Π. Let its complement be $N_\Pi := (\mathbb{T}^2 \setminus \dot{\bigcup}_{j=0}^3 V_j)/i$

which defines an attracting region for f_Π. Then $\Lambda_\Pi := \bigcap_{n\in\mathbb{N}} f_\Pi^n(N_\Pi)$ defines the *Plykin attractor*.

Figure 5.4 is a template of a regular tetrahedron which serves as a Euclidean representation for the Plykin attractor Λ_Π. The faces, embossed with the (computer generated) orbits induced by the four repelling fixed points, can be cut and pasted to form a tangible model of Λ_Π. Explanatory details and a picture of one face are given in [18]. On page 165 of this document we conclude that the symmetry of a regular tetrahedron reflects the structural symmetry of the Plykin attractor, made precise in Theorem 8.18.

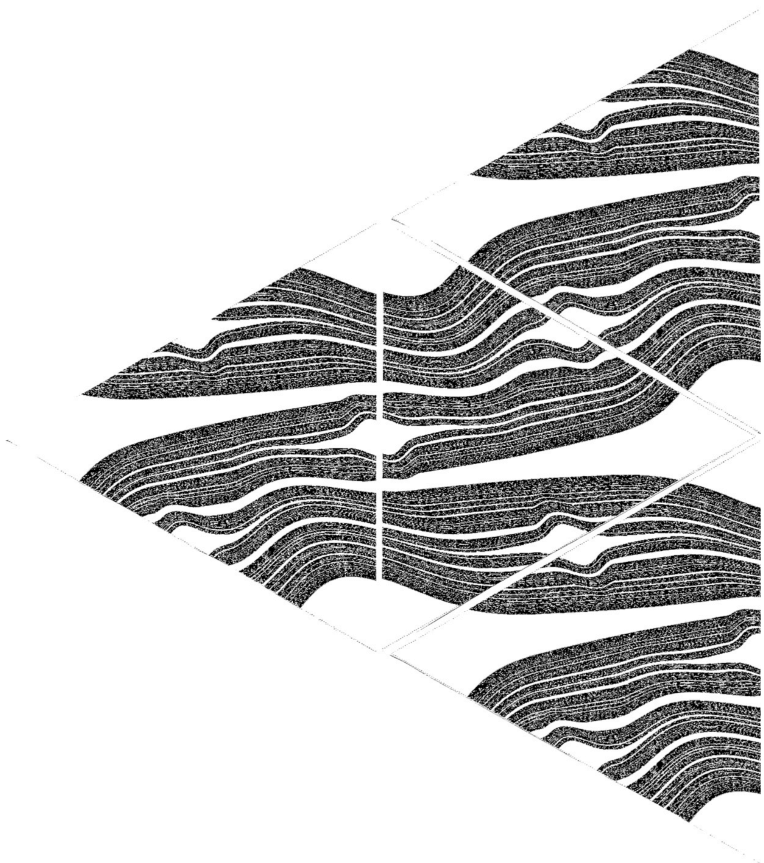


Figure 5.4: A Plykin template.

Chapter 6

Substitution tiling spaces on a torus

We loosely say that a torus has n *hole(s)* when a hyperbolic attractor on the torus has a basin of attraction resulting from a DA diffeomorphism with n repelling fixed point(s). So this extensive chapter divides naturally into two parts, §6.1 and §6.2, which consider tori with one and four holes respectively.

In §6.1.1 we use known methods to analyse the construction of the attractor Λ with one complementary domain. We introduce branched 1-manifolds which in §6.1.3 we reduce to elementary branched 1-manifolds which represent solenoids as inverse limit spaces, for example $\Sigma_2 = \varprojlim(\mathcal{M}_2, g_2)$. This results in the substitution tiling space $\mathcal{T}_\gamma$ which is homeomorphic to Λ (Theorem 6.20).

On page 113 of §6.2.1, after lifting the planar Plykin attractor Λ_P to the torus, we create a combinatorial method to derive an elementary branched 1-manifold K_*, called a rose, and the solenoid $\Omega = \varprojlim(K_*, r_*)$. Ultimately we show that the lifted Plykin attractor $\tilde{\Lambda}_\Pi$ is homeomorphic to the tiling space $\mathcal{T}_{\omega^2}$ (Theorem 6.32).

6.1 A torus with one hole

6.1.1 The Cat complex

We describe the construction of a complex K by a method outlined in [8]. Consider a substitution tiling induced by $\sigma : \mathcal{A} \to \mathcal{A}^*$ (see §1.3.1). The PF eigenvalue λ_σ and left eigenvector $\mathbf{v}_\sigma = [v_1, \ldots, v_j]$ determines $0 < \epsilon = \min\{\frac{v_a}{2\lambda_\sigma}\}_{a\in\mathcal{A}}$ then a complex $K = K_\sigma$ has three main ingredients :

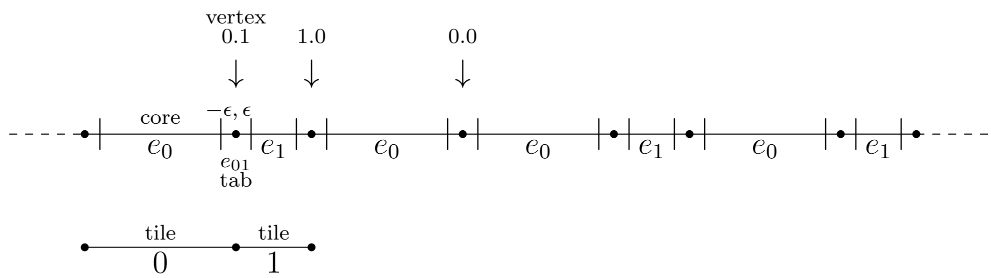
- a collection of 1-cells representing the letters of $\mathcal{A}$. We call $a \in \mathcal{A}$ a *tile* which has length $l(a) = v_a = [0, v_i]$ where v_i, $1 \le i \le j$, is a component in the eigenvector $\mathbf{v}_\sigma$. Let a cell $e_a := [\epsilon, v_a - \epsilon] \times \{a\}$ be called the *core* of a tile where $l(e_a) = l(a) - 2\epsilon$.

- a collection of 1-cells representing the allowed transitions between letters. Let the *language* $\mathcal{L} = \mathcal{L}_\sigma$ be the set of finite allowed words in σ then for words of length two $ab \in \mathcal{L}$ let a cell $e_{ab} := [-\epsilon, \epsilon] \times \{ab\}$ be called a *tab* with length $l(e_{ab}) = 2\epsilon$. Call the midpoint of e_{ab} the *vertex* $a.b$.

- identifications. For all $a, b \in \mathcal{A}$, $(v_a - \epsilon, a) \sim (-\epsilon, ab)$ and $(\epsilon, b) \sim (\epsilon, ab)$.

The diagram in Example 6.1 below and Figure 6.1 may clarify the composition of these three ingredients.

Example 6.1. Recall the Cat substitution (1.9): $\gamma(0) = 010$ and $\gamma(1) = 01$, $\mathcal{A} = \{0, 1\}$, $j = 2$. The word 11 is not in $\mathcal{L}_\gamma$. A typical subword is $\ldots 0100101 \ldots$ which gives the annotated diagram below (not to scale):



Definition 6.2. *The* complex $K = \left[\left(\bigcup_{a \in \mathcal{A}} e_a\right) \cup \left(\bigcup_{ab \in \mathcal{L}} e_{ab}\right) \right] / \sim$ *and its* subcomplex $K_s = \bigcup_{ab \in \mathcal{L}} e_{ab} / \sim$.

The calculations associated to the complex K_γ, derived from the substitution γ, with subcomplex K_s (dashed), (Fig. 6.1) are the following. The eigenvalue μ^2 has left eigenvector $\mathbf{v}^u = [\mu, 1]$ so let $l(0) = v_0 = \mu$ and $l(1) = v_1 = 1$ then $\epsilon = \min\left\{\frac{\mu}{2\mu^2}, \frac{1}{2\mu^2}\right\} = \frac{1}{2\mu^2}$. The cores: $e_0 = [\frac{1}{2\mu^2}, \mu - \frac{1}{2\mu^2}] \times \{0\}$ and $e_1 = [\frac{1}{2\mu^2}, 1 - \frac{1}{2\mu^2}] \times \{1\}$. The tabs: $e_{00} = [-\frac{1}{2\mu^2}, \frac{1}{2\mu^2}] \times \{00\}$, $e_{01} = [-\frac{1}{2\mu^2}, \frac{1}{2\mu^2}] \times \{01\}$ and $e_{10} = [-\frac{1}{2\mu^2}, \frac{1}{2\mu^2}] \times \{10\}$. The identifications: firstly, the right end point of a core is identified with the left end point of a tab: $(\mu - \frac{1}{2\mu^2}, 0) \sim (-\frac{1}{2\mu^2}, 00)$, $(\mu - \frac{1}{2\mu^2}, 0) \sim (-\frac{1}{2\mu^2}, 01)$, $(1 - \frac{1}{2\mu^2}, 1) \sim (-\frac{1}{2\mu^2}, 10)$; then the left end point of a core is identified with the right end point of a tab: $(\frac{1}{2\mu^2}, 0) \sim (\frac{1}{2\mu^2}, 00)$, $(\frac{1}{2\mu^2}, 0) \sim (\frac{1}{2\mu^2}, 10)$, $(\frac{1}{2\mu^2}, 1) \sim (\frac{1}{2\mu^2}, 01)$.

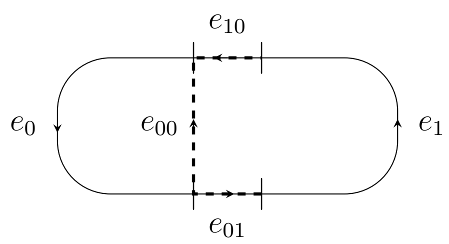


Figure 6.1: The Cat complex K_γ.

Definition 6.3. *Let* $g : K \to K$, $K^{\mathbb{N}} = \Pi_{i=1}^{\infty} K$, *then the* inverse limit space *is given by*

$$K_\infty = \varprojlim\{(K, g)\} = \{(x_i) \in K^{\mathbb{N}} \,|\, \forall i \in \mathbb{N}, g(x_{i+1}) = x_i\}.$$

Definition 6.4. *Let the* shift *be* $\sigma : K_\infty \to K_\infty$, $(x_i) \mapsto (y_i)$; $y_i = x_{i+1}$, *that is* $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. *Then the inverse* $\sigma^{-1} : K_\infty \to K_\infty$, $(x_i) \mapsto (y_i)$; $y_i = x_{i-1}$ *and for* $i \geq 2$, $y_1 = g(y_2) = g(x_1)$. *That is* $(x_1, x_2, \ldots) \mapsto (g(x_1), x_1, x_2, \ldots)$.

The shift σ is a homeomorphism because K_∞ is compact and since σ is continuous on its coordinates, σ^{-1} is continuous. Denote a *solenoid* as the inverse limit $\Sigma = K \overset{g}{\leftarrow} K \overset{g}{\leftarrow} K \overset{g}{\leftarrow} \dots$ and the shift map $\sigma : \Sigma \to \Sigma$ then with the terminology of [54] call $g : K \to K$ a *presentation* of (Σ, σ). We call g a *bonding map* of the inverse limit.

From tiling space to complex

Consider a substitution tiling derived from the Cat substitution γ, $T = \{T_i\}_{i\in\mathbb{Z}} \in \mathcal{T}_\gamma$, $t \in \mathbb{R}$ and $a, b, c \in \mathcal{A}$, such that $0 \in T_0 = [0, v_a] - t$, $T_{-1} = [0, v_b] - t - v_b$, and $T_1 = [0, v_c] - t + v_a$. Then let a continuous surjection p be defined by

$$p : \mathcal{T}_\gamma \to K_\gamma,\ p(T) = \begin{cases} [(t, a)] & \text{if } \epsilon \leq t \leq v_a - \epsilon, \\ [(t, ba)] & \text{if } 0 \leq t \leq \epsilon, \\ [(t - v_a, ac)] & \text{if } v_a - \epsilon \leq t \leq v_a. \end{cases} \tag{6.1}$$

Now set T to be the *fixed tiling* where the origin $0 \in T_0 \setminus T_{-1}$ is at the vertex $0.0 \Rightarrow t = 0$. Then $T_0 = [0, v_0] = [0, \mu]$, $T_{-1} = [0, v_0] - v_0 = [0, \mu] - \mu$ and $T_1 = [0, v_1] + v_0 = [0, 1] + \mu$. We now relabel the cells of K_γ with the set $\mathcal{E} = \{a, \dots, h\}$, defined below, in order to distinguish the dashed negative and positive tabs and the two core tiles: b is blue and e is red (Fig. 6.2).

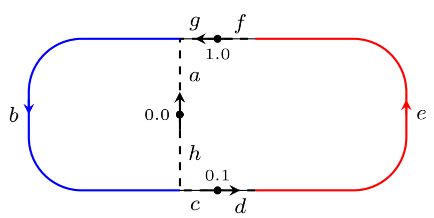


Figure 6.2: Labelled cells of K_γ.

In accordance with the definition of $p(T)$ the cells are defined by

$$
\begin{array}{l|l}
a := \{[(t,00)]\,|\,0 \le t \le \epsilon\} & e := \{[(t,1)]\,|\,\epsilon \le t \le v_1 - \epsilon\} \\
b := \{[(t,0)]\,|\,\epsilon \le t \le v_0 - \epsilon\} & f := \{[(t-v_1,10)]\,|\,v_1 - \epsilon \le t \le v_1\} \\
c := \{[(t-v_0,01)]\,|\,v_0 - \epsilon \le t \le v_0\} & g := \{[(t,10)]\,|\,0 \le t \le \epsilon\} \\
d := \{[(t,01)]\,|\,0 \le t \le \epsilon\} & h := \{[(t-v_0,00)]\,|\,v_0 - \epsilon \le t \le v_0\}
\end{array}
$$

The self-map c (6.2) is a piecewise linear map determined by the parameter $0 \le t \le v_a$, $a \in \mathcal{A}$, and the scale factor $\mu^2 > 0$ so that $\epsilon = \frac{1}{2\mu^2}$ (see Ex. 6.1).

Cell	h	a	b	c	d	e	f	g	b
Lower cell limit, t	$-\epsilon$	0	ϵ	$\mu-\epsilon$	μ	$\mu+\epsilon$	$\mu+1-\epsilon$	$\mu+1$	$\mu+1+\epsilon$
$c(t)=\mu^2 t$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\mu^3-\frac{1}{2}$	μ^3	$\mu^3+\frac{1}{2}$	$\mu^3+\mu^2-\frac{1}{2}$	$\mu^3+\mu^2$	$\mu^3+\mu^2+\frac{1}{2}$

Table 6.1: Values for the plot $c(t)$.

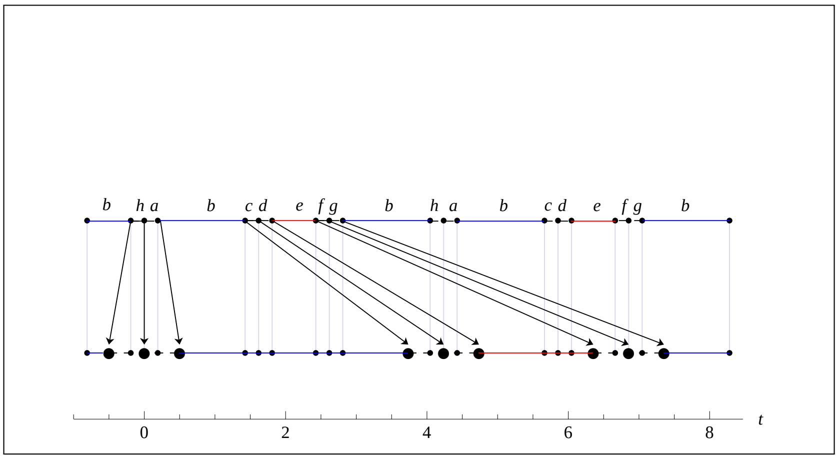


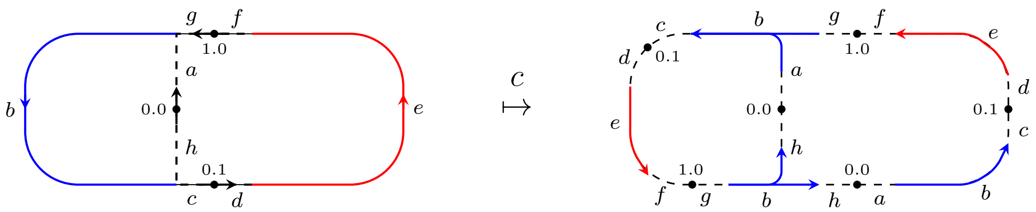
Figure 6.3: A plot of $c(t) = \mu^2 t$.

The mappings for (6.2) are derived from the values shown in Table 6.1. But in defining the map c we remark that for $i, i' \in \mathcal{E}$, $c(i)$ is not necessarily onto the first or last cell of its image *block* $i \dots i'$ which is clearly seen in Figure 6.3

which is a plot $c(t) = \mu^2 t$ using [55]. So we use the notation $i\wr$ and $\wr i$ to indicate partial transition through a cell. A 'complete' cell $i = i\wr\wr i$, $i \in \mathcal{E}$.

$$c : K_\gamma \to K_\gamma, \begin{cases} a \mapsto ab\wr & e \mapsto \wr bcde\wr \\ b \mapsto \wr bcdefgb\wr & f \mapsto \wr ef \\ c \mapsto \wr bh & g \mapsto gb\wr \\ d \mapsto ab\wr & h \mapsto \wr bh \end{cases} \tag{6.2}$$

We repeat Figure 6.2 in Figure 6.4 for ease of comparison. The image cells of (i) are shown in (ii) (not to scale) where for example the original blue core cell b of (i) is replaced by a representation of its image $c(b) = \wr bcdefgb\wr$ which is blue, dashed, red, dashed, blue. The two fixed points of map c are $0.0 = h \cap a$ and $1.0 = f \cap g$ while $c(0.1) = c(c \cap d) = (0.0)$.



(i) (ii)

Figure 6.4: The mapping of cells by $c : K_\gamma \to K_\gamma$.

Consider the Cat tiling homeomorphism F_γ (1.13) where $p \circ F_\gamma = c \circ p$ as shown in the commuting diagram below. Let $T \in \mathcal{T}_\gamma$ then significantly the tiling orbit $t.T = 0.T$ so $F_\gamma(0.T) = 0.T = T$ which means that T is fixed by F_γ.

$$\begin{tikzcd} \mathcal{T}_\gamma \arrow[d, "p"'] & \mathcal{T}_\gamma \arrow[l, "F_\gamma"'] \arrow[d, "p"] \\ K_\gamma & K_\gamma \arrow[l, "c"'] \end{tikzcd}$$

Let the fixed point at the tiling origin be denoted by $x_0 = 0.0$ then it follows that $F_\gamma(x_0) = x_0 \Rightarrow c(p(x_0)) = p(x_0)$. The map $c : K_\gamma \to K_\gamma$ is a presentation of the inverse limit space $\Gamma_0 = K_\gamma \xleftarrow{c} K_\gamma \xleftarrow{c} K_\gamma \xleftarrow{c} \ldots$ and the shift map $\gamma : \Gamma_0 \to \Gamma_0$ where $\gamma^{-1}(x_1, x_2, \ldots) = (c(x_1), x_1, x_2, \ldots)$.

Remark 6.5. $\Gamma_0 = \varprojlim(K_\gamma, c) \cong \mathcal{T}_\gamma$ *(see for example Lemma 2 in [8]).*

6.1.2 The Cat branched 1-manifold

Definition 6.6. *A homeomorphism $h : (X, d) \to (X, d)$ is* expansive *if $\exists\, \delta > 0$ such that $\forall\, x, y \in X$ $(x \neq y)$ $\exists\, n \in \mathbb{Z}$ such that $d(h^n(x), h^n(y)) > \delta$.*

Remark 6.7. *The restriction of a diffeomorphism to a hyperbolic set is expansive (see for example Corollary 10.1.10 in [28]).*

A Markov partition

Definition 6.8. *[35] Let (M, ϕ) be an invertible dynamical system. A topological partition $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$ of M gives a* symbolic representation *of (M, ϕ) if for every x in the shift space $X_{\mathcal{P},\phi}$ the intersection $\bigcap_{n=0}^{\infty} \bar{D}_n(x)$ consists of exactly one point, where $\bar{D}_n(x)$ are the closures of the sets $D_n(x) = \bigcap_{k=-n}^{n} \phi^{-k}(P_{x_k}) \subseteq M$. We call $\mathcal{P}$ a* Markov partition *for (M, ϕ) if $\mathcal{P}$ gives a symbolic representation of (M, ϕ) and furthermore $X_{\mathcal{P},\phi}$ is a shift of finite type.*

Let $\mathcal{P}$ be a Markov partition (Fig. 6.5) which gives a symbolic representation of the invertible dynamical system (1.10). Steps in the construction of $\mathcal{P}$ are shown in Figure 6.6 and for the commentary we follow [35]. Firstly consider the natural quotient map $q : \mathbb{R}^2 \to \mathbb{T}^2$, $q((s,t)) = (s,t) + \mathbb{Z}^2$. Let $R_i = \{R_1 = A, R_2, = B, R_3 = C\}$ be a set of three open rectangles in the plane such that the closures $\bar{R}_i$ of $q(R_i)$, $i = 1, 2, 3$, cover the torus. The boundaries of the R_i, $i = 1, 2, 3$, consist of segments parallel to the contracting and expanding eigenvectors of the matrix A, $\mathbf{v}^s = [-1, \mu]$ and $\mathbf{v}^u = [\mu, 1]$ respectively. That is, the sides of the rectangles run along leaves of the stable and unstable foliations of the torus. As R_i covers $\mathbb{T}^2$ it forms a topological partition of the torus so that we may denote $\mathbb{T}^2$ as the union of the closed rectangles $\bigcup_{i=1}^{3} \bar{R}_i$ with the edges identified using q. The colour coding of Figure 6.6 shows coloured line segments to represent the stable leaves of the foliation and orthogonal black line segments to represent the unstable leaves.

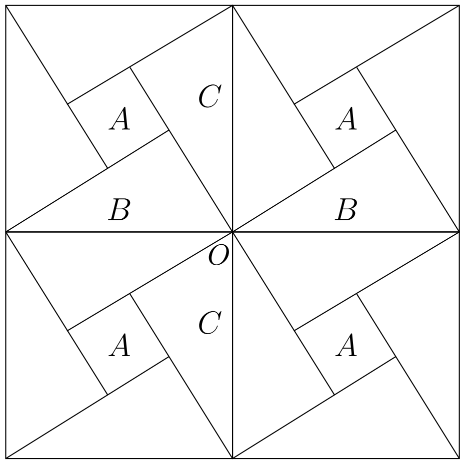


Figure 6.5: A Markov partition $\mathcal{P}$ over 4 fundamental domains of $\mathbb{T}^2$.

The *intersection property* of $\mathcal{P}$ requires that $\mathcal{C}(R_i) \cap R_j,\ i, j = 1, 2, 3,$ is a single connected strip parallel to $\mathbf{v}^u$ which cuts completely through R_j in the unstable direction and does not contain an expanding boundary segment of R_j. We derive the images of $\mathcal{C}(R_i),\ i = 1, 2, 3,$ by using its matrix A as a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ and computing $A(R_i)$ for each $i = 1, 2, 3$. These planar images with good intersections are shown in Figure 6.7. Note that since the eigenvalue $\mu^{-2} < 1$, a segment S contained in the stable foliation satisfies $A(S) \subset S$ while a segment U contained in the unstable foliation satisfies $A(U) \supset U$ due to the eigenvalue $\mu^2 > 1$.

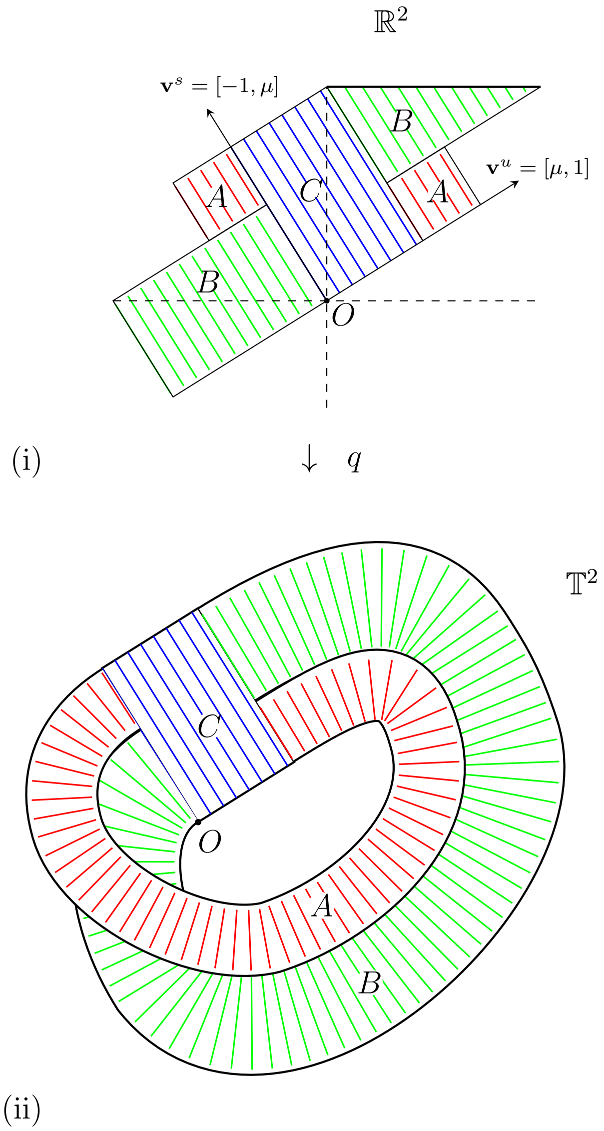


Figure 6.6: Topological discs in the plane and on the torus.

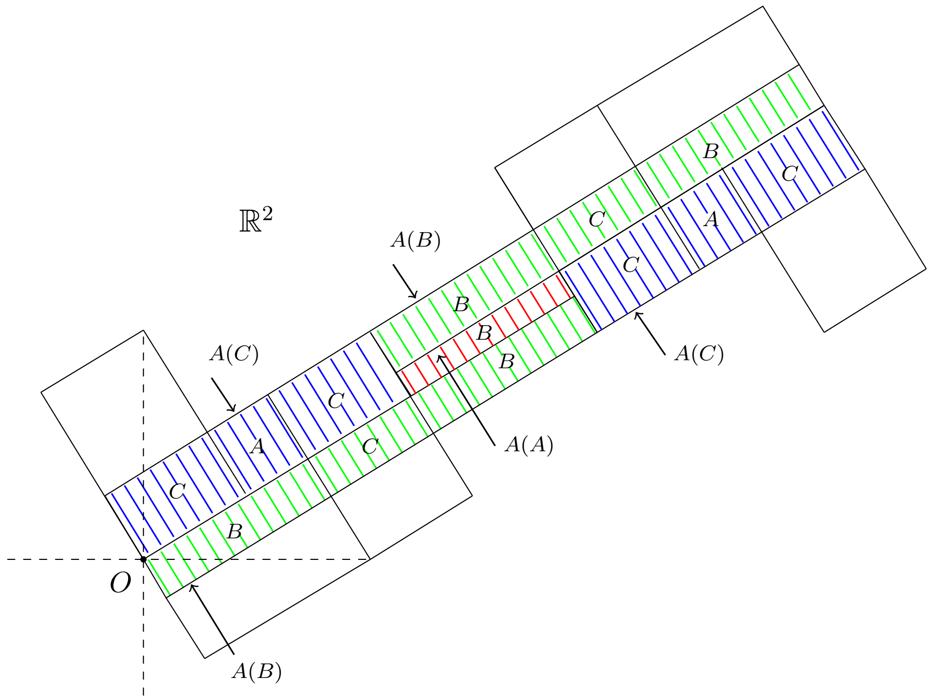


Figure 6.7: The image of $\mathcal{P}$ in $\mathbb{R}^2$.

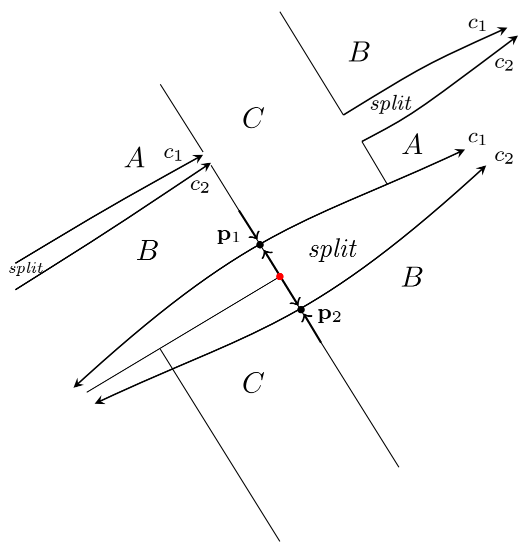


Figure 6.8: Splitting open the unstable manifold.

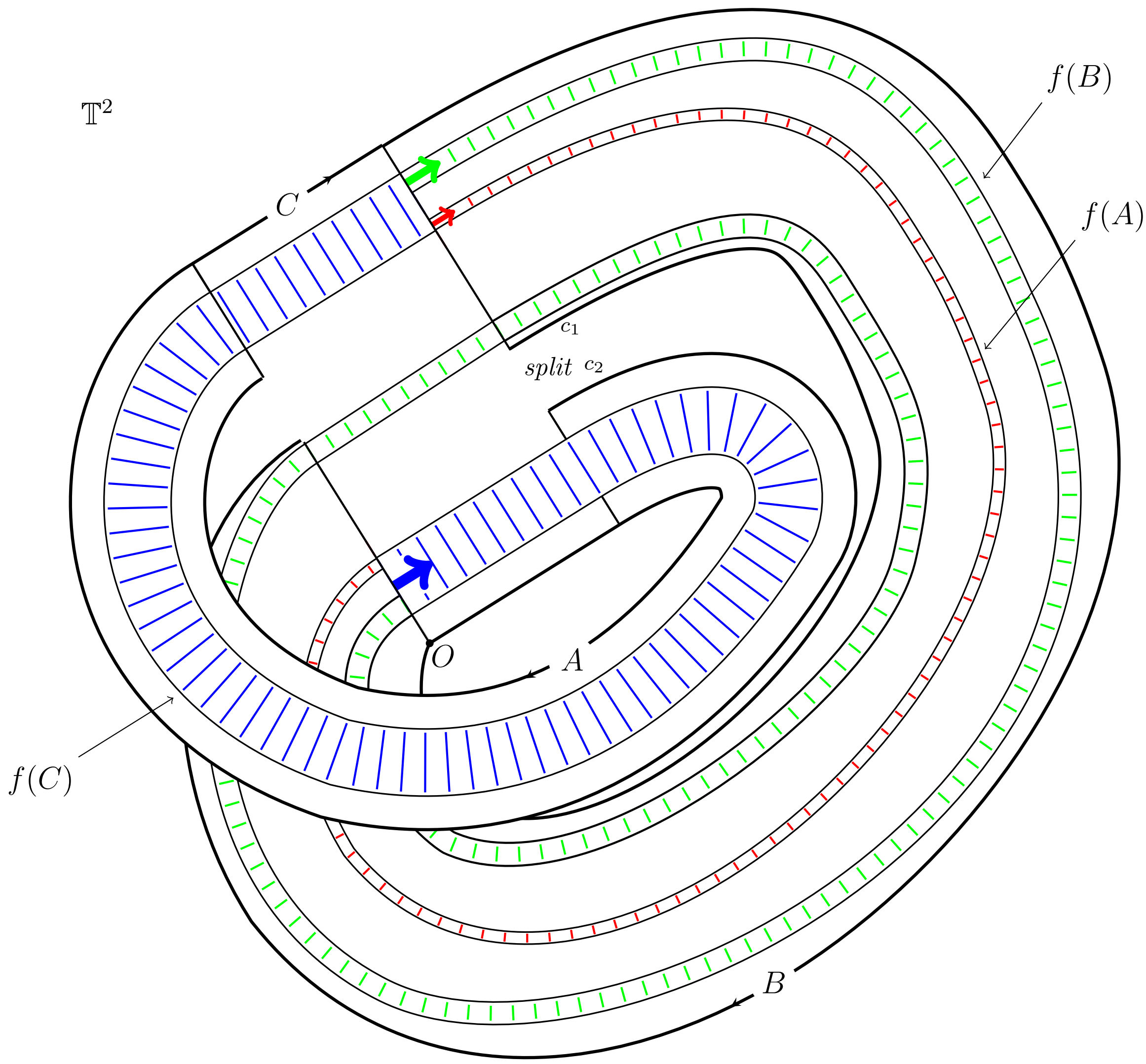


Figure 6.9: The attractor $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(N)$.

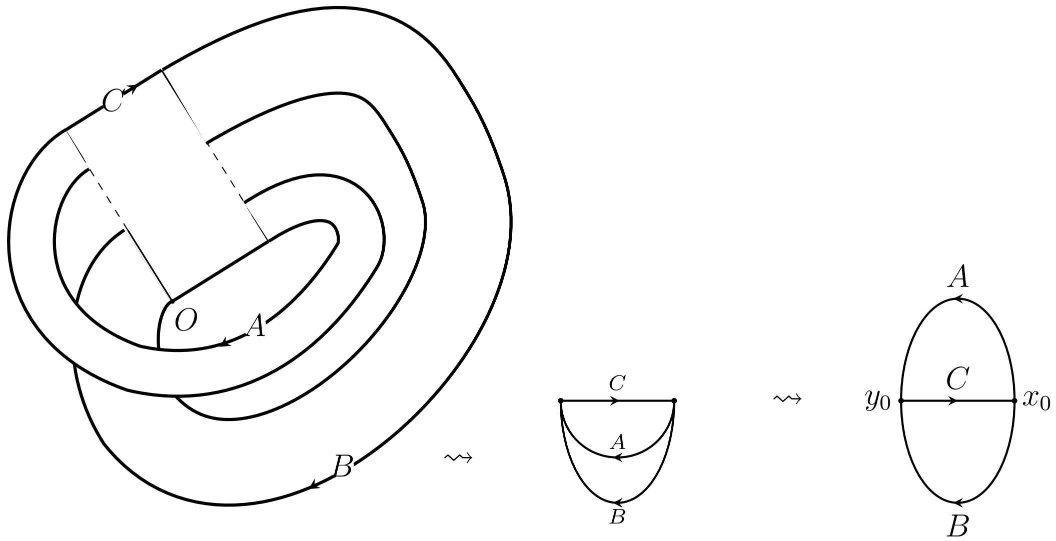
A geometric and topological model of the attractor Λ

In Figure 5.2 we saw how the repelling action of the source $\mathbf{p}_0$, induced by the DA diffeomorphism f (5.2), 'split open' the unstable manifold of the origin. Now observe Figure 6.8 which takes an extract from the partition $\mathcal{P}$ and replicates the repelling effect of the map f to form a pair of (forward) asymptotic path-components labelled c_1 and c_2 of the attractor Λ.

The next diagram of the series Figure 6.9 is modelled on Figure 6.6. This Figure 6.9 illustrates the hyperbolic structure of the map f which simultaneously stretches the discs A, B and C across the leaves of the stable foliation while contracting these leaves in the stable direction. Also seen is $f(N) \subset N$ illustrating the attracting set Λ.

Definition 6.9. *Let $\mathcal{M}$ be a* branched 1-manifold *consisting of a finite set $\mathcal{E} = \{e_1, \ldots, e_k\}$ of 1-cells and a finite set of vertices $\mathcal{V}$ and where each $e \in \mathcal{E}$ joins $v_i, v_j \in \mathcal{V}$ such that $i \neq j$ for at least one pair of vertices.*

Any two points on the same leaf of the foliation of $\mathbb{T}^2$ behave identically under iteration by f which also preserves the fixed points $\mathbf{p}_1$ and $\mathbf{p}_2$. So in order to describe the dynamics of f globally, consider the behaviour of the leaves under f. To that end let the equivalence relation $\sim$ identify q with p if q lies in the *component* $\mathrm{comp}_p(W^s(p, f) \setminus V)$ then $q \sim p$ collapses the stable manifold of p to a point, leaving the expanding direction intact. Let $q : N \to \mathcal{M}_0$ denote the quotient map and denote the quotient space by the branched 1-manifold $\mathcal{M}_0 := \{\mathrm{comp}_p(W^s(p, f) \setminus V)/_\sim \,|\, p \in N\}$ (Fig. 6.10 (iii)). Note that $\mathcal{M}_0$ contains the unstable manifolds of $\mathbf{p}_{i=1,2}$ in the complement of V.



(i) (ii) (iii) $\mathcal{M}_0$

Figure 6.10: The Cat branched 1-manifold $\mathcal{M}_0$.

In Figure 6.10, (i) is a skeleton of the expanding leaves of the foliation which reduces to diagram (ii) which in turn leads to a θ-*space* in (iii). Let this θ-space be $\mathcal{M}_0$ whose induced symbolic map (6.3) will model the dynamics of f. Label the set of 1-cells $\mathcal{E} = \{A, B, C\} \subset \mathcal{M}_0$ and let the two branch points be $x_0, y_0 \in \mathcal{M}_0$ which correspond to $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{T}^2$. Then define

$$g_0 : \mathcal{M}_0 \to \mathcal{M}_0, \begin{cases} A \mapsto B, \\ B \mapsto BCB, \\ C \mapsto CAC. \end{cases} \tag{6.3}$$

Set the length of the three 1-cells $e_k \in \mathcal{E}$ to be $|A| = 1$ and $|C| = \mu$, which are the components of the eigenvector $\mathbf{v}^u$, and set $|B| = \mu^2$. Then these assigned values are consistent with the map g_0, with a stretch factor of μ^2, whilst satisfying the Fibonacci relation $\mu^2 - \mu - 1 = 0$. That is $g_0(|A|) = \mu^2 = |B|$; $g_0(|B|) = \mu^4 = (1+\mu)^2 = \mu^2 + \mu + \mu^2 = |B| + |C| + |B|$ and $g_0(|C|) = \mu^3 = \mu(1+\mu) = \mu + 1 + \mu = |C| + |A| + |C|$. The two points x_0

and y_0 are fixed by g_0. Note that the language of two-letter words in g_0 is $\{AC, BC, CA, CB\}$. Set the origin of $\mathcal{M}_0$ to be x_0 at the vertex $C.B$ such that $x_0 \in B \setminus C$. Let $\bar{p} : \Lambda \to \mathcal{M}_0$ be the restriction of q to Λ. Then by the parameterisation detailed above, the calculated values in Table 6.2 generate the plot $g_0(t) = \mu^2 t$ (Fig. 6.11).

Cell	B	C	B	C	A	C
Lower cell limit, t	0	μ^2 $= 1 + \mu$	$\mu^2 + \mu$ $= 1 + 2\mu$	$2\mu^2 + \mu$ $= 2 + 3\mu$	$2\mu^2 + 2\mu$ $= 2 + 4\mu$	$2\mu^2 + 2\mu + 1$ $= 3 + 4\mu$
$g_0(t) = \mu^2 t$	0	μ^4 $= 2 + 3\mu$	$\mu^4 + \mu^3$ $= 3 + 5\mu$	$2\mu^4 + \mu^3$ $= 5 + 8\mu$	$2\mu^4 + 2\mu^3$ $= 6 + 10\mu$	$2\mu^4 + 2\mu^3 + \mu^2$ $= 7 + 11\mu$

Table 6.2: Values for the plot $g_0(t)$.

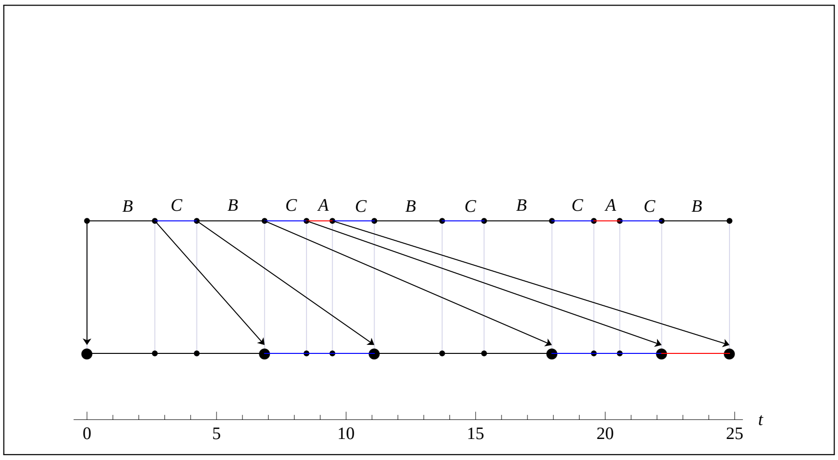


Figure 6.11: A plot of $g_0(t) = \mu^2(t)$.

The maps satisfy $\bar{p} \circ f = g_0 \circ \bar{p}$ shown in the commuting diagram below.

$$\begin{tikzcd}
\Lambda \arrow[d, "\bar{p}"'] & \Lambda \arrow[l, "f"'] \arrow[d, "\bar{p}"] \\
\mathcal{M}_0 & \mathcal{M}_0 \arrow[l, "g_0"']
\end{tikzcd}$$

The map $g_0 : \mathcal{M}_0 \to \mathcal{M}_0$ is a presentation of the inverse limit space $\Sigma_0 = \mathcal{M}_0 \xleftarrow{g_0} \mathcal{M}_0 \xleftarrow{g_0} \mathcal{M}_0 \xleftarrow{g_0} \ldots$ and the shift map $\sigma_0 : \Sigma_0 \to \Sigma_0$ where $\sigma_0^{-1}(x_1, x_2, \ldots) = (g_0(x_1), x_1, x_2, \ldots)$.

Remark 6.10. $\Sigma_0 = \varprojlim(\mathcal{M}_0, g_0) \cong \Lambda$.

6.1.3 Elementary branched 1-manifolds

Definition 6.11. *[54] An* elementary branched 1-manifold $\mathcal{M}$ *is one which is topologically a wedge of circles with a single vertex b and where the 1-cells form 1-cycles which may or may not be orientable.*

Definition 6.12. *[32] An* immersion *of a manifold M into a manifold N is a differentiable map $f : M \to V$ onto a subset of N whose differential is injective everywhere.*

Let K be a compact branched 1-manifold and $g : K \to K$ an immersion then Axioms $1, 2, 3°$ (referred to below) and listed by Williams in [54], serve as hypotheses for his Theorem 5.2 which states the existence of an elementary branched 1-manifold. In this section we construct two elementary branched 1-manifolds: K_3 derived from the complex K_γ and $\mathcal{M}_2$ derived from the branched 1-manifold $\mathcal{M}_0$. We recall that K_γ is derived from the Cat substitution map γ while $\mathcal{M}_2$ is derived from the Cat toral map $\mathcal{C}$ where both maps are associated to the matrix $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Our construction follows the strategy given in the proof of Lemma 5.3 in [54].

From K_γ to K_3.

To validate the construction, we show that the presentation $c : K_\gamma \to K_\gamma$ of (Γ_0, γ) satisfies Axioms $1, 2$ and $3°$ of [54]:

1. The map c is an expansion with stretch factor μ^2.

2. Let $\Omega(c)$ denote the non-wandering set of c and let point $p \in U \subset K_\gamma$. The map c is induced by the primitive substitution map γ. By the periodic property of primitivity, $\forall n_0 \in \mathbb{N}$, $\exists\, n \in \mathbb{Z}$, $|n| > n_0$ such that $c^n(p) \in U \Rightarrow c^n(U) \cap U \neq \emptyset$. Thus $\Omega(c) = K_\gamma$.

3° (equivalent to the *flattening axiom* 2 in [57]). Denote the branch points of K_γ by $b_1 = a \cap b \cap g$ and $b_2 = h \cap b \cap c$ and let b_1, b_2 lie in open neighbourhoods U_1, U_2 respectively. Observe that $\forall\, n \in \mathbb{Z}$, $n \neq 0$ and $i = 1, 2$, $c^n(b_i) \in c^n(U_i)$ which is homeomorphic to an open interval $(-\delta, \delta)$.

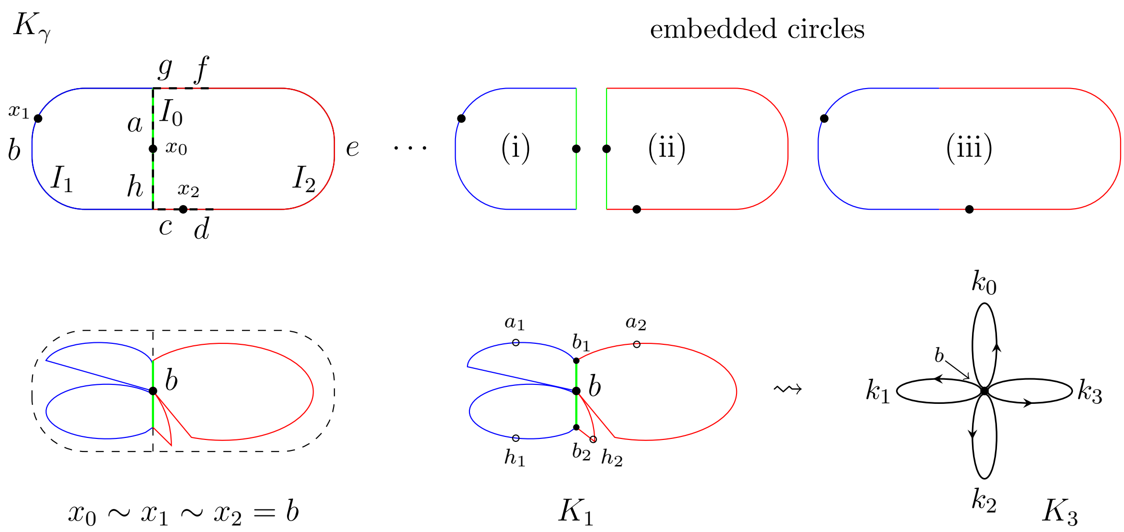


Figure 6.12: Stages of the construction from K_γ to K_3.

The construction of K_3 is illustrated with a sequence of topological diagrams in Figure 6.12. Let a set of three open 1-cells in K_γ be $I_0 \subset h \cup a$, $I_1 \subset b$ and $I_2 \subset c \cup d \cup e \cup f \cup g$. Then there exist three embedded circles labelled (i), (ii) and (iii) in K_γ, each of which contains some I_i, $i = 0, 1, 2$, namely (i) $h \cup a \cup b \supset I_0 \cup I_1$, (ii) $a \cup h \cup c \cup d \cup e \cup f \cup g \supset I_0 \cup I_2$ and (iii) $b \cup c \cup d \cup e \cup f \cup g \supset I_1 \cup I_2$. It follows that for $n = 2$, $c^n(I_i) \supset I_0$, $i = 0, 1, 2$. Choose the fixed point $x_0 = 0.0 \in I_0$ with period $s = 1$ and let $m = ns = 2$ then $c^2(x_0) = x_0$ and for each $i = 0, 1, 2$, $x_0 \in c^2(I_i)$ so that $c^{-2}(x_0) \cap I_i \neq \emptyset$. That is, *each embedded circle in K_γ contains a point of $c^{-2}(x_0)$*.

Next let two further points be $x_1 = \frac{1}{\mu} \in b$ and $x_2 = \mu = c.d.$ Then $c^2(x_1) = c(x_2) = x_0 = 0.0$, that is $[(\frac{1}{\mu}, 0)] \mapsto [(\mu, 01)] \mapsto [(\mu^3, 00)]$ while $c^2(x_2) = c(x_0) = x_0$, that is $[(\mu, 01)] \mapsto [(\mu^3, 00)] \mapsto [(\mu^5, 00)]$. Further, each circle (i), (ii) and (iii) intersects the set $\{x_0, x_1, x_2\}$. Now identify the points x_0, x_1, x_2 to a single point b which forms a new complex K_1 with b a point of high ramification. Denote the induced map on K_1 by c_1 which satisfies Axioms $1, 2, 3$ for all points in K_1 with the exception of the point b. At this point, c_1 satisfies Axioms $1, 2$ but only the weaker Axiom $3°$ since it takes more than one iteration of c_1 to become 'locally flat' at b. Now we have reached the result that *each embedded circle of* K_1 *contains* b.

To complete the construction we need to remove the original two branch points of K_γ which remain and are labelled b_1 and b_2 (see diagram K_1). To do so we use twice the 'move' number (2) in §5.5 of [54]. This opens up the *stem* $a = [b, a_1] \cap [b, a_2]$ which removes the branch point b_1 to form a new complex K_2 then repeating the move on the stem $h = [b, h_1] \cap [b, h_2]$ removes b_2. This is the sought after elementary branched 1-manifold K_3 which is a wedge of four circles labelled $k_0, \ldots, k_3$ with a common vertex b. In forming the sequence of complexes, $c : K_\gamma \to K_\gamma$ passes to a shift equivalent $c_i : K_i \to K_i$ thereafter to a shift equivalent $c_{i+1} : K_{i+1} \to K_{i+1}$, $i = 1, 2$.

Definition 6.13. *[54] If* $g : \mathcal{M} \to \mathcal{M}$ *satisfies Axioms* $1, 2, 3°$, g *is called an* elementary presentation *of the solenoid* $\mathcal{M} \overset{g}{\leftarrow} \mathcal{M} \overset{g}{\leftarrow} \mathcal{M} \overset{g}{\leftarrow} \ldots$ *and the shift map* σ*.*

The map $c_3 : K_3 \to K_3$ is an elementary presentation of the connected solenoid with shift map $(\Gamma_3, \tilde{\gamma}^2)$ (Theorem 5.2 in [54]).

From $\mathcal{M}_0$ to $\mathcal{M}_2$.

Firstly we verify that the presentation $g_0 : \mathcal{M}_0 \to \mathcal{M}_0$ of (Σ_0, σ_0) satisfies Axioms $1, 2$ and $3°$ of [54].

1. The map g_0 is expansive with stretch factor μ^2. (The quotient space $\mathcal{M}_0$ retains only the expanding direction of the unstable manifolds.)

2. Let $\Omega(g_0)$ denote the non-wandering set of g_0. The map g_0 models the dynamics of the DA map f which by definition admits a non-wandering domain. Thus $\Omega(g_0) = \mathcal{M}_0$.

$3°$ (equivalent to the *flattening axiom* 2 in [57]). Consider the branch points $x_0, y_0 = A \cap C \cap B \in \mathcal{M}_0$ and let x_0, y_0 lie in open neighbourhoods U_x, U_y respectively. Observe that $\forall\, n \in \mathbb{Z}$, $n \neq 0$, $g_0^n(A \cap C \cap B) \neq A \cap C \cap B$ thus $\forall\, n \in \mathbb{Z}$, $n \neq 0$, $g_0^n(x_0) \in g_0^n(U_x)$, $g_0^n(y_0) \in g_0^n(U_y)$ such that $g_0^n(U_x)$, $g_0^n(U_y)$ are homeomorphic to an open interval $(-\delta, \delta)$.

Let three open 1-cells in the θ-space $\mathcal{M}_0$ be $I_0 \subset C \cup B$, $I_1 \subset A$ and $I_2 \subset B$. Then form three embedded circles in $\mathcal{M}_0$, (i) $B \cup C \supset I_0$ (ii) $A \cup C \supset I_1$ and (iii) $A \cup B \supset I_1 \cup I_2$. Observe that for $n = 2$, $g_0^2(BC) \supset I_0$, $g_0^2(A) \supset I_0$ and $g_0^2(B) \supset I_0$ so that $g_0^2(I_i) \supset I_0$, $i = 0, 1, 2$. Now put the fixed point $x_0 = [(0, B)]$ with period $s = 1$ in I_0, then $m = ns = 2$. It follows that $g_0^2(x_0) = x_0$ and that for $i = 0, 1, 2$, $x_0 \in g_0^2(I_i) \Rightarrow g_0^2(x_0) \cap I_i \neq \emptyset$. Thus *each embedded circle in $\mathcal{M}_0$ contains a point of $g_0^{-2}(x_0)$.*

Next, let $x_1 = \frac{1}{\mu} \in A$ and $x_2 = \mu \in B$ then $g_0^2(x_1) = g_0(x_2) = C.B$ since $[(\frac{1}{\mu}, A)] \mapsto [(\mu, B)] \mapsto [(\mu^3, CB)]$ and $g_0^2(x_2) = g_0(x_0) = (x_0)$. So $g_0^2(x_{i=1,2}) = x_0$ and each embedded circle intersects $\{x_0, x_1, x_2\}$. Identify the points x_0, x_1, x_2 to a single point b then $\mathcal{M}_1$ is a new branched 1-manifold with two remaining branch points $x_0 \sim b$ and y_0. The shift g_1 satisfies Axioms $1, 2, 3$ while $g_1(b)$ satisfies Axioms $1, 2, 3°$. Then *each embedded circle of $\mathcal{M}_1$ contains b.*

Finally, label two points either side of y_0 as y_1 and y_2. Then use the 'move' number (2) in [54] to open up the stem $y = [b, y_1] \cap [b, y_2]$ which removes the branch point y_0. The branch point b remains the single vertex of the wedge of circles labelled $m_0, \ldots, m_3$ which forms the elementary branched 1-manifold $\mathcal{M}_2$ (Fig. 6.13). The map $g_{i+1} : \mathcal{M}_{i+1} \to \mathcal{M}_{i+1}$ is shift equivalent to $g_i : \mathcal{M}_i \to \mathcal{M}_i$, $0 \le i \le 2$.

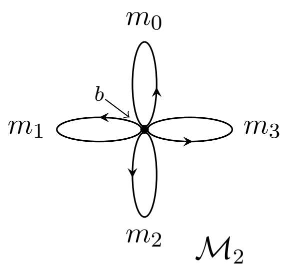


Figure 6.13: The elementary branched 1-manifold $\mathcal{M}_2$.

The map $g_2 : \mathcal{M}_2 \to \mathcal{M}_2$ is an elementary presentation of the connected solenoid with shift map (Σ_2, σ_0^2) (Theorem 5.2 in [54]).

6.1.4 Rose maps

Recall from §1.2 a primitive substitution σ with card$(\mathcal{A}) = j$, M_σ with PF eigenvalue λ_σ and left eigenvector $[v_1, \ldots, v_j]$ then from §1.3 its tiling space $\mathcal{T}_\sigma$ with associated prototiles P_j from set $\mathscr{P}$ where length $|P_j| = v_j$.

Definition 6.14. *[7] Let K be an elementary branched* 1*-manifold , also called a* rose*, which consists of a wedge of j oriented circles $k_0, \ldots, k_{j-1}$ called* petals*, each with a circumference $k_{j-1} = v_j$. Then let $r_\sigma : K \to K$ be a* rose map *which is a linear expansion with stretch factor λ_σ and which adheres to the ordered pattern of the substitution σ.*

Let the map $c_3 : K_3 \to K_3$ now be denoted by $r_{\gamma^2} : K_3 \to K_3$ in order to distinguish it as belonging to the family of rose maps. The petals $k_0, \ldots, k_3$ are those of the rose K_3 and $a, \ldots, h \in \mathcal{E} \subset K_\gamma$. Then $k_0 \subset a \cup b$, $k_1 \subset b \cup c \cup h$, $k_2 \subset d \cup e$ and $k_3 \subset e \cup f \cup g \cup b$. The petals have circumferences $|k_0| = |k_2| = \frac{1}{\mu}$ and $|k_1| = |k_3| = 1$. It follows that $\sum_{j=0}^{3} |k_j| = 2\mu = |K_3|$. Table 6.3 shows the values used to derive the map (6.4) and by inspection of the table, we see that for the branch point $b = 0.0 \in K_3$, $r_{\gamma^2}(k_j)$, $j = 0, \ldots, 3$ always returns to b.

Petal	k_0	k_1	k_2	k_3
$t \in [k_j, k_{j+1}]$	$\left[0, \frac{1}{\mu}\right]$	$\left[\frac{1}{\mu}, 1 + \frac{1}{\mu}\right]$	$\left[1 + \frac{1}{\mu}, 1 + \frac{2}{\mu}\right]$	$\left[1 + \frac{2}{\mu}, 2 + \frac{2}{\mu}\right]$
$r_{\gamma^2}(t) = \mu^4 t$ origin $b = 0.0$	$[0, \mu^3]$ $= [0, 2\mu + 1]$ =[0.0,0.0]	$[\mu^3, \mu^4 + \mu^3]$ $= [2\mu + 1, 5\mu + 3]$ =[0.0,0.0]	$[\mu^4 + \mu^3, \mu^4 + 2\mu^3]$ $= [5\mu + 3, 7\mu + 4]$ =[0.0,0.0]	$[\mu^4 + 2\mu^3, 2\mu^4 + 2\mu^3]$ $= [7\mu + 4, 10\mu + 6]$ =[0.0,0.0]

Table 6.3: Values for the rose map r_{γ^2}.

$$r_{\gamma^2} : K_3 \to K_3, \begin{cases} k_0 \mapsto k_0 k_1 k_2 k_3 k_1, \\ k_1 \mapsto k_0 k_1 k_2 k_3 k_1 k_2 k_3 k_1 \\ k_2 \mapsto k_0 k_1 k_2 k_3 k_1, \\ k_3 \mapsto k_0 k_1 k_2 k_3 k_1 k_2 k_3 k_1. \end{cases} \tag{6.4}$$

That is, $r_{\gamma^2}(k_0) = r_{\gamma^2}(k_2)$ and $r_{\gamma^2}(k_1) = r_{\gamma^2}(k_3)$.

Definition 6.15. *[8] For a substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$ with precisely one periodic, hence fixed bi-infinite word, σ is called* proper *if and only if there are $b, e \in \mathcal{A}$ such that for all sufficiently large n and all $i \in \mathcal{A}$, $\sigma^n(i) = b \ldots e$.*

Let a proper substitution $\tilde{\gamma}^2$ be defined over $\mathcal{A} = \{0, 1, 2, 3\}$ which is the set of subscripts of $k_i \in K_3$. Then $\forall\, i \in \mathcal{A}$ let the word $\tilde{\gamma}^2(i)$ have the same pattern as $r_{\gamma^2}(k_i)$ in K_3.

$$\tilde{\gamma}^2 : \mathcal{A} \to \mathcal{A}^*, \begin{cases} 0 \mapsto 01231 \\ 1 \mapsto 01231231 \\ 2 \mapsto 01231 \\ 3 \mapsto 01231231. \end{cases} \tag{6.5}$$

Then $\tilde{\gamma}^2(0) = \tilde{\gamma}^2(2)$ and $\tilde{\gamma}^2(1) = \tilde{\gamma}^2(3)$. The incidence matrix $M_{\tilde{\gamma}^2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ has PF eigenvalue μ^4 with left eigenvector $\left[\frac{1}{\mu}, 1, \frac{1}{\mu}, 1\right]$.

Definition 6.16. *[3] Consider two tilings $T, T' \in (\mathcal{T}_\sigma, F_\sigma)$.* Forcing the border *means that there is a fixed positive integer N such that for any tile $\bar{T}$ contained in T and T', $F_\sigma^N(T)$ and $F_\sigma^N(T')$ coincide on all $\bar{T}$ and on all tiles that meet $F_\sigma^N(\bar{T})$.*

Remark 6.17. *Since $\tilde{\gamma}^2$ is proper the substitution forces the border [8]. Thus the tiling space $\mathcal{T}_{\tilde{\gamma}^2}$ is homeomorphic to $\varprojlim(K_3, r_{\gamma^2})$ (Theorem 4.3 in [3]). We know from Remark 6.5 that $\varprojlim(K_\gamma, c)$ is homeomorphic to $\mathcal{T}_\gamma$. Although γ is not proper, $\varprojlim(K_\gamma, c)$ is homeomorphic to a quotient of $\mathcal{T}_\gamma$, namely $\varprojlim(K_3, r_{\gamma^2})$. Thus $\mathcal{T}_{\tilde{\gamma}^2} \cong \mathcal{T}_\gamma$ (Theorem 3.10 in [7]).*

Consider again the elementary presentation $g_2 : \mathcal{M}_2 \to \mathcal{M}_2$ which we now define by a rose map $r_{\sigma_0^2}$ in (6.6) derived from the following criteria. Let the petals of $\mathcal{M}_2$ have circumferences $|m_0| = |m_2| = \frac{1}{\mu}$ and $|m_1| = |m_3| = 1$. The expanding map (6.3) gives $g_0^2(B) = BCBCACBCB$ so position the origin at the fixed point $b \sim x_0 = C.B$, that is at the branch point $b \in \mathcal{M}_2$. Now apply the stretch factor μ^4 to the circumference of each petal leading to the entries in Table 6.4.

Petal	m_0	m_1	m_2	m_3
$t \in [m_j, m_{j+1}]$	$\left[0, \frac{1}{\mu}\right]$	$\left[\frac{1}{\mu}, 1 + \frac{1}{\mu}\right]$	$\left[1 + \frac{1}{\mu}, 1 + \frac{2}{\mu}\right]$	$\left[1 + \frac{2}{\mu}, 2 + \frac{2}{\mu}\right]$
$r_{\sigma_0^2}(t) = \mu^4 t$ origin $b = C.B$	$[0, \mu^3]$ $= [0, 2\mu + 1]$ $= [C.B, C.B]$	$[\mu^3, \mu^4 + \mu^3]$ $= [2\mu + 1, 5\mu + 3]$ $= [C.B, C.B]$	$[\mu^4 + \mu^3, \mu^4 + 2\mu^3]$ $= [5\mu + 3, 7\mu + 4]$ $= [C.B, C.B]$	$[\mu^4 + 2\mu^3, 2\mu^4 + 2\mu^3]$ $= [7\mu + 4, 10\mu + 6]$ $= [C.B, C.B]$

Table 6.4: Values for the rose map $r_{\sigma_0^2}$.

Observe that the returns to the origin $b = C.B$ are identically spaced to the returns to the origin 0.0 under the map r_{γ^2} shown in Table 6.3.

$$r_{\sigma_0^2} : \mathcal{M}_2 \to \mathcal{M}_2, \begin{cases} m_0 \mapsto m_0 m_1 m_2 m_3 m_1, \\ m_1 \mapsto m_0 m_1 m_2 m_3 m_1 m_2 m_3 m_1, \\ m_2 \mapsto m_0 m_1 m_2 m_3 m_1, \\ m_3 \mapsto m_0 m_1 m_2 m_3 m_1 m_2 m_3 m_1. \end{cases} \tag{6.6}$$

That is, $r_{\sigma_0^2}(m_0) = r_{\sigma_0^2}(m_2)$ and $r_{\sigma_0^2}(m_1) = r_{\sigma_0^2}(m_3)$.

Similar to forming the proper substitution (6.5), let a proper substitution be defined over $\mathcal{A} = \{0, 1, 2, 3\}$ by

$$\sigma_0^2 : \mathcal{A} \to \mathcal{A}^*, \begin{cases} 0 \mapsto 01231 \\ 1 \mapsto 01231231 \\ 2 \mapsto 01231 \\ 3 \mapsto 01231231. \end{cases} \tag{6.7}$$

Then $\sigma_0^2(0) = \sigma_0^2(2)$ and $\sigma_0^2(1) = \sigma_0^2(3)$.

Since the substitutions σ_0^2 and $\tilde{\gamma}^2$ have identical definitions, $M_{\sigma_0^2} = M_{\tilde{\gamma}^2}$.

6.1.5 An attracting tiling space

Define a homeomorphism

$$\rho : K_3 \to \mathcal{M}_2,\ \rho(k_j) = m_j,\ 0 \le j \le 3. \tag{6.8}$$

Proposition 6.18. *The rose maps $r_{\gamma^2} : K_3 \to K_3$ and $r_{\sigma_0^2} : \mathcal{M}_2 \to \mathcal{M}_2$ are topologically conjugate such that the maps commute $\rho \circ r_{\gamma^2} = r_{\sigma_0^2} \circ \rho$.*

$$\begin{tikzcd} K_3 \arrow[r, "r_{\gamma^2}"] \arrow[d, "\rho"'] & K_3 \arrow[d, "\rho"] \\ \mathcal{M}_2 \arrow[r, "r_{\sigma_0^2}"] & \mathcal{M}_2 \end{tikzcd}$$

Proof. Let $k_j \in K_3$, $0 \le j \le 3$, be an oriented petal parametrised by its circumference with the origin at the single vertex $b \in K_3$. A petal $m_j \in \mathcal{M}_2$, $0 \le j \le 3$, has identical properties. By construction, $\forall\, j$, $|k_j| = |m_j|$. So the continuous bijection $\rho : K_3 \to \mathcal{M}_2$ in a compact metric space has a continuous inverse and is therefore a homeomorphism. Thus K_3 and $\mathcal{M}_2$ are isomorphic roses with corresponding proper substitutions $\tilde{\gamma}^2$ and σ_0^2 over $\mathcal{A}$. With respect to commutativity, consider $k_j \in K_3$, $j = 0, 2$, where $\rho \circ r_{\gamma^2}(k_j) = \rho(k_0k_1k_2k_3k_1) = m_0m_1m_2m_3m_1$ while $r_{\sigma_0^2} \circ \rho(k_j) = r_{\sigma_0^2}(m_j) = m_0m_1m_2m_3m_1$. Then for $j = 1, 3$, $\rho \circ r_{\gamma^2}(k_j) = \rho(k_0k_1k_2k_3k_1k_2k_3k_1) = m_0m_1m_2m_3m_1m_2m_3m_1$ while $r_{\sigma_0^2} \circ \rho(k_j) = r_{\sigma_0^2}(m_j) = m_0m_1m_2m_3m_1m_2m_3m_1$. Thus ρ conjugates the rose maps as given. $\square$

Proposition 6.19. *The inverse limits $\varprojlim(K_\gamma, c)$ and $\varprojlim(\mathcal{M}_0, g_0)$ are homeomorphic spaces.*

Proof. Consider two sequences defined by

$$K_s := K_\gamma \xleftarrow{c} K_\gamma \xleftarrow{c} K_\gamma \xleftarrow{c} \dots \text{ and } K_s' := K_\gamma \xleftarrow{c^2} K_\gamma \xleftarrow{c^2} K_\gamma \xleftarrow{c^2} \dots$$

and recall the shift $\gamma^{-1}((x_i)) = (c(x_i), x_i)$, $i \geq 1$, associated to the bonding map c. Let h_K be a homeomorphism defined by $h_K : \varprojlim(K_s, c) \to \varprojlim(K'_s, c^2)$, $h_K((x_1, x_2, x_3, \ldots)) = (x_1, x_3, x_5, \ldots)$ then $h_K^{-1}((y_1, y_2, y_3, \ldots)) = (y_1, c(y_2), y_2, c(y_3), y_3, \ldots)$. So $\varprojlim(K'_s, c^2)$ is homeomorphic to $\varprojlim(K_\gamma, c)$. Similarly, define the sequences

$$\mathcal{M}_s := \mathcal{M}_0 \overset{g_0}{\leftarrow} \mathcal{M}_0 \overset{g_0}{\leftarrow} \mathcal{M}_0 \overset{g_0}{\leftarrow} \ldots \text{ and } \mathcal{M}'_s := \mathcal{M}_0 \overset{g_0^2}{\leftarrow} \mathcal{M}_0 \overset{g_0^2}{\leftarrow} \mathcal{M}_0 \overset{g_0^2}{\leftarrow} \ldots$$

where the shift $\sigma_0^{-1}((x_i)) = (g_0(x_i), x_i)$, $i \geq 1$, is relative to the bonding map g_0. Then let $h_{\mathcal{M}} : \varprojlim(\mathcal{M}_s, g_0) \to \varprojlim(\mathcal{M}'_s, g_0^2)$ realise a homeomorphism between $\varprojlim(\mathcal{M}_s, g_0)$ and $\varprojlim(\mathcal{M}'_s, g_0^2)$. Now from the conjugacy of Proposition 6.18, $\varprojlim(K'_s, c^2)$ is homeomorphic to $\varprojlim(\mathcal{M}'_s, g_0^2)$ and so in conclusion $\varprojlim(K_\gamma, c)$ is homeomorphic to $\varprojlim(\mathcal{M}_0, g_0)$. $\square$

Theorem 6.20. *The attractor $\Lambda \subseteq \mathbb{T}^2$ is homeomorphic to the tiling space $\mathcal{T}_\gamma \subseteq \mathbb{T}^2$.*

Proof. Let the symbol $\cong$ signify homeomorphic spaces. By Remarks 6.10 and 6.5 respectively, $\varprojlim(\mathcal{M}_0, g_0) \cong \Lambda$ and $\varprojlim(K_\gamma, c) \cong \mathcal{T}_\gamma$. Then by Proposition 6.19, $\varprojlim(\mathcal{M}_0, g_0) \cong \varprojlim(K_\gamma, c) \Rightarrow \Lambda \cong \mathcal{T}_\gamma$. $\square$

Remark 6.21. *It was concluded in Remark 6.17 that $\varprojlim(K_3, r_{\gamma^2}) \cong \mathcal{T}_{\tilde{\gamma}^2} \cong \mathcal{T}_\gamma$. So we may say that topological conjugacy is achieved between the DA map f^2 yielding the attractor Λ and the inflation and substitution homeomorphism of the tiling space $\mathcal{T}_{\tilde{\gamma}^2}$ when the bonding maps on their inverse limit spaces are defined as g_0^2 and c^2 respectively.*

6.2 A torus with four holes

The Plykin attractor on a plane

Recall the Plykin attractor Λ_Π on S^2. A stereographic projection of the sphere onto the plane sends the repelling fixed point $(0,0) \in \mathbb{T}^2$ to infinity such that the map obtained from f_Π is a diffeomorphism of the plane for which infinity is repelling. Denote this diffeomorphism by P whose period 3 repelling orbit forms a basin of repulsion. Let $R \subset \mathbb{R}^2$ be a foliated region with 3 open holes and where the leaves are segments of the stable manifolds. Let $P : R \to R$ be a contracting map on the leaves then $P(R) \subset R$ is a basin of attraction for the Plykin attractor $\Lambda_P := \bigcap_{n\in\mathbb{N}} P^n(R)$ in the plane (see for example [20] and [48] for detailed diagrams). Now form equivalence classes determined by $\sim$ which identifies $q \sim p$ if $q \in \mathrm{comp}(W^s(p) \cap R)$. This collapses each leaf to a point and the resulting quotient space $\mathcal{M}_* = R/\sim$ is a branched 1-manifold for P (Fig. 6.14). Denote the 1-cells by $\mathcal{E} = \{a, b, c, d\}$ and label the two branch points b_1 and b_2.

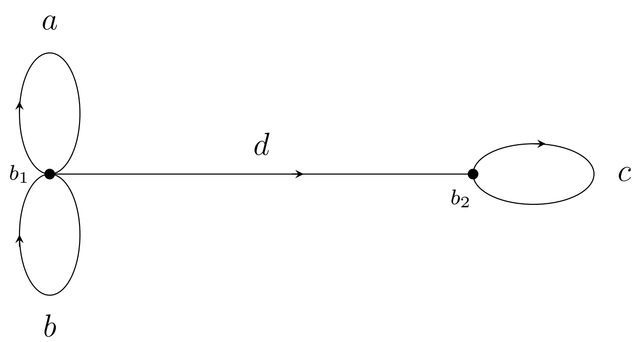


Figure 6.14: The branched 1-manifold $\mathcal{M}_*$.

Similar to that given in [20] define an expanding map g_* but let $e_i \in \mathcal{E}$ follow the direction of the arrow while e_i^{-1} takes the reverse direction to that of e_i,

$$g_* : \mathcal{M}_* \to \mathcal{M}_*, \begin{cases} a \mapsto b, \\ b \mapsto bdcd^{-1}b^{-1}, \\ c \mapsto a, \\ d \mapsto dc^{-1}d^{-1}. \end{cases} \tag{6.9}$$

Let $\Sigma_* := \mathcal{M}_* \overset{g_*}{\leftarrow} \mathcal{M}_* \overset{g_*}{\leftarrow} \mathcal{M}_* \overset{g_*}{\leftarrow} \ldots$ with shift map $\sigma_* : \Sigma_* \to \Sigma_*$ then $\Lambda_P \cong \varprojlim(\mathcal{M}_*, g_*)$. Since the map g_* is not orientation-preserving it cannot represent a substitution map.

6.2.1 Lifting the Plykin attractor

By using the algorithm and covering map presented in [57], we construct an orientable double cover $\tilde{\mathcal{M}}_*$ of $\mathcal{M}_*$ represented by the branched 1-manifold of Figure 6.15. For this purpose let the edges $E_{i,1}$, $E_{i,2}$ be the liftings of e_i in $\tilde{\mathcal{M}}_*$ such that $E_{i,1}$ corresponds to $e_i \in \mathcal{E}$ and that $E_{i,2}$ corresponds to e_i^{-1}. Then let the map p be defined by a $2:1$ local homeomorphism, $t \in [0, 1]$,

$$p : \tilde{\mathcal{M}}_* \to \mathcal{M}_*, \begin{cases} E_{i,1}(t) \mapsto e_i(t), \\ E_{i,2}(t) \mapsto e_i(1-t). \end{cases} \tag{6.10}$$

Let E denote an oriented *edge*, or *arc*, of $\tilde{\mathcal{M}}_*$ which comprises the set $\tilde{\mathcal{E}} = \{A_i, B_i, C_i, D_i \,|\, i = 1, 2\}$. The four branch points are labelled p_i, q_i, $i = 1, 2$.

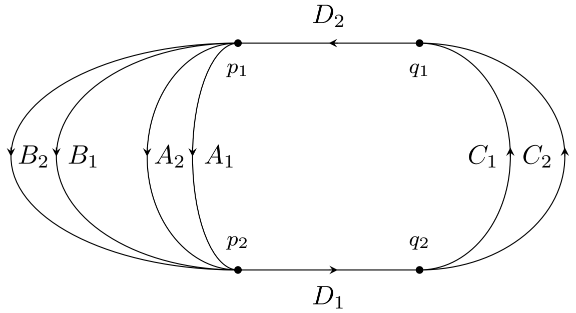


Figure 6.15: The branched 1-manifold $\tilde{\mathcal{M}}_*$.

Let the lift of (6.9) be defined by

$$\tilde{g}_* : \tilde{\mathcal{M}}_* \to \tilde{\mathcal{M}}_*, \begin{cases} A_1 \mapsto B_1, & A_2 \mapsto B_2, \\ B_1 \mapsto B_1D_1C_1D_2B_2, & B_2 \mapsto B_1D_1C_2D_2B_2, \\ C_1 \mapsto A_1, & C_2 \mapsto A_2, \\ D_1 \mapsto D_1C_2D_2, & D_2 \mapsto D_1C_1D_2. \end{cases} \tag{6.11}$$

Let the matrices M_1 and M_2 be such that for $i, j \in \mathcal{E}$, $[M_1]_{i,j} = |g_*(j)|_i$ when $g_*(j)$ has positive index and $[M_2]_{i,j} = |g_*(j)|_i$ when $g_*(j)$ has negative index.

$$M_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then the matrices $M = M_1 + M_2$ and $\tilde{M} = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ (Prop. 3.9 in [57]).

That is,

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix}, \quad \tilde{M} = \begin{array}{c} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The *incidence matrix* $\tilde{M}$ of the branched 1-manifold $\tilde{\mathcal{M}}_*$ is such that an entry i, j gives the number of times the ith edge of $\tilde{\mathcal{M}}_*$ is covered by the jth edge of $\tilde{\mathcal{M}}_*$ under the map $\tilde{g}_*$ (as in Def. 1.34). The matrix $\tilde{M}$ has rank 6, zero determinant and its PF eigenvalue μ^2 admits a left eigenvector $[\mu, \mu^3, 1/\mu, 1, \mu, \mu^3, 1/\mu, 1]$. The map $\tilde{g}_*$ is a well-defined o-p continuous map such that $p \circ \tilde{g}_* = g_* \circ p$ (Prop. 3.3 in [57]). Furthermore Proposition 3.6 (2) in [57] allows the following definition.

Definition 6.22. *The map $\tilde{g}_* : \tilde{\mathcal{M}}_* \to \tilde{\mathcal{M}}_*$ is an* orientable *presentation of the solenoid $\tilde{\Sigma}_* = \tilde{\mathcal{M}}_* \xleftarrow{\tilde{g}_*} \tilde{\mathcal{M}}_* \xleftarrow{\tilde{g}_*} \tilde{\mathcal{M}}_* \xleftarrow{\tilde{g}_*} \dots$ and the shift map $\tilde{\sigma}_* : \tilde{\Sigma}_* \to \tilde{\Sigma}_*$ where $\tilde{\sigma}_*^{-1}(x_1, x_2, \dots) = (\tilde{g}_*(x_1), x_1, x_2, \dots)$.*

Remark 6.23. *The attractor Λ_P in the plane does not support a flow. But significantly, since $\tilde{\Sigma}_*$ is orientable, it does support a flow ϕ_*^t without rest points (Theorems 1.4 in [24], 11 in [1]).*

Constructing a rose by combinatorics

We describe a *word* in a branched 1-manifold X as a sequence $e_0 \ldots e_n$ of oriented edges of X such that the final vertex of e_i equals the initial vertex of e_{i+1}, for $i = 0, \ldots, n-1$. We may call a closed word which starts and finishes at the same vertex a *loop*, or in a rose a *petal*, or just a word if the meaning is clear. In $\tilde{\mathcal{M}}_*$ let a loop $w := D_1 \ldots B_2$ be a word which starts with the edge D_1 and finishes with the first occurrence of B_2 inclusive. Now position the origin at the branch point $p_2 = B_2 \cap D_1 \in \tilde{\mathcal{M}}_*$ then iterate $\tilde{g}_*^5(D_1) = D_1C_2D_2 \ldots D_1C_1D_2$. Locate the successive first returns to p_2 which occur on the fifth iterate. This produces a set $\mathcal{W}$ of 5 distinct words in $\tilde{\mathcal{M}}_*$ listed below.

$$\mathcal{W} := \left\{ \begin{array}{ll} w_0 = & D_1C_2D_2A_2D_1C_1D_2B_2, \\ w_1 = & D_1C_2D_2A_1D_1C_1D_2B_1D_1C_2D_2B_2, \\ w_2 = & D_1C_2D_2A_2D_1C_1D_2B_1D_1C_2D_2A_1D_1C_1D_2B_1D_1C_1D_2B_2, \\ w_3 = & D_1C_2D_2A_2D_1C_1D_2B_1D_1C_2D_2B_2, \\ w_4 = & D_1C_2D_2A_1D_1C_1D_2B_1D_1C_1D_2B_2 \end{array} \right\}.$$

Proposition 6.24. *The solenoid $\tilde{\Sigma}_*$ is homeomorphic to the suspension $\mathcal{W}_c$ (defined in the proof).*

Proof. Consider the first projection $p_{*1} : \tilde{\Sigma}_* \to \tilde{\mathcal{M}}_*$. Then for the branch point $p_2 \in \tilde{\mathcal{M}}_*$, $\mathscr{C} := p_{*1}^{-1}(p_2)$ is a Cantor set cross-section of the flow ϕ_*^t on $\tilde{\Sigma}_*$. This set $\mathscr{C}$ admits the partition into clopen sets $p_{*1}^{-1}(B_2 \cap D_1) \dot\cup p_{*1}^{-1}(B_1 \cap D_1) \dot\cup p_{*1}^{-1}(A_1 \cap D_1) \dot\cup p_{*1}^{-1}(A_2 \cap D_1)$. For arcs α_1, α_2 of $\tilde{\mathcal{M}}_*$ that intersect in p_2, let

$$\tilde{\alpha}_1 \cap \tilde{\alpha}_2 := \{x \in \mathscr{C} \mid \text{for sufficiently small } \epsilon > 0 \text{ and all } 0 < t < \epsilon, \\ p_{*1}(\phi_*^{-t}(x)) \in \alpha_1 \text{ and } p_{*1}(\phi_*^t(x)) \in \alpha_2\}.$$

Note that $\tilde{\alpha}_1 \cap \tilde{\alpha}_2$ need not be identical to $\tilde{\alpha}_2 \cap \tilde{\alpha}_1$. Then $\tilde{B}_2 \cap \tilde{D}_1$ is a clopen subset of $\mathscr{C}$ and $\tilde{\Sigma}_*$ is homeomorphic to the *suspension* $\mathcal{W}_c$ of the first return map ρ_* of ϕ_*^t to $\tilde{B}_2 \cap \tilde{D}_1$ (Theorems 3.3 in [24], 17 in [1]). Further, $\tilde{B}_2 \cap \tilde{D}_1$ can be partitioned by 5 clopen subsets. For $0 \leq j \leq 4$,

$$[w_j] := \{x \in \tilde{B}_2 \cap \tilde{D}_1 \,|\, t \geq 0,\ p_{*1}(\phi_*^t(x)) \text{ follows the sequence of arcs}$$
$$\text{described by } w_j \text{ in } \mathcal{W}\}.$$

Call $[w_j]$ a 'cylinder' and assign it to have length l_j. $\square$

Having found the set $\mathcal{W}$ of 5 loops, we may represent the elementary branched 1-manifold K_* as a rose of 5 petals labelled $k_0, \ldots, k_4$ with a single branch point b (Fig. 6.16).

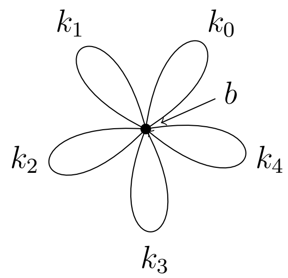


Figure 6.16: The elementary branched 1-manifold K_*.

Let K_* be endowed with an expanding map r_* defined by

$$r_* : K_* \to K_*, \quad \begin{cases} k_0 \mapsto k_0 k_1, \\ k_1 \mapsto k_2 k_3, \\ k_2 \mapsto k_0 k_4 k_2 k_1, \\ k_3 \mapsto k_0 k_4 k_3, \\ k_4 \mapsto k_2 k_1. \end{cases} \tag{6.12}$$

The incidence matrix $M_{r_*} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$ has determinant 1, rank 5, and PF eigenvalue μ^2 with left eigenvector $[1/\mu, 1, \mu, 1, 1]$. By the construction of K_* we recognise that there exists a map h defined by

$$h : K_* \to \tilde{\mathcal{M}}_*, \; h(k_j) = w_j, \; 0 \leq j \leq 4. \tag{6.13}$$

Proposition 6.25. *The maps commute $h \circ r_* = \tilde{g}_* \circ h$; the map h is surjective.*

Proof. Consider the petal $k_0 \in K_*$ so that

$$\begin{aligned} h \circ r_*(k_0) &= h(k_0 k_1) = h(k_0)h(k_1) \\ &= w_0 w_1 \\ &= D_1C_2D_2A_2D_1C_1D_2B_2D_1C_2D_2A_1D_1C_1D_2B_1D_1C_2D_2B_2 \\ \text{whilst} \quad \tilde{g}_* \circ h(k_0) &= \tilde{g}_*(w_0) \\ &= \tilde{g}_*(D_1C_2D_2A_2D_1C_1D_2B_2) \\ &= D_1C_2D_2A_2D_1C_1D_2B_2D_1C_2D_2A_1D_1C_1D_2B_1D_1C_2D_2B_2. \end{aligned}$$

It is easily shown that $\tilde{g}_*(w_1) = w_2w_3$, $\tilde{g}_*(w_2) = w_0w_4w_2w_1$, $\tilde{g}_*(w_3) = w_0w_4w_3$ and $\tilde{g}_*(w_4) = w_2w_1$. So that $h \circ r_*(k_1) = h(k_2k_3) = w_2w_3 = \tilde{g}_*(w_1) = \tilde{g}_* \circ h(k_1)$ and the maps commute similarly for each k_2, k_3, $k_4 \in K_*$. Thus for each $k \in K_*$, $h \circ r_* = \tilde{g}_* \circ h$. Now by inspection, for each edge $E \in \tilde{\mathcal{M}}_*$, $\exists\, k \in K_*$ such that $E \subset h(k)$. So h is surjective. $\square$

Define the solenoid $\Omega := \varprojlim(K_*, r_*)$ with shift map $\omega : \Omega \to \Omega$, $\omega((x_i)) = (x_{i+1})$, $\forall\, i \in \mathbb{N}$.

Proposition 6.26. *The map h induces a continuous surjection $\hat{h} : \underleftarrow{\lim}(K_*, r_*) \to \underleftarrow{\lim}(\tilde{\mathcal{M}}_*, \tilde{g}_*)$.*

$$\begin{tikzcd}
K_* \arrow[d, "h"'] & K_* \arrow[l, "r_*"'] \arrow[d, "h"'] & K_* \arrow[l, "r_*"'] \arrow[d, "h"'] & \cdots \arrow[l, "r_*"'] \\
\tilde{\mathcal{M}}_* & \tilde{\mathcal{M}}_* \arrow[l, "\tilde{g}_*"'] & \tilde{\mathcal{M}}_* \arrow[l, "\tilde{g}_*"'] & \cdots \arrow[l, "\tilde{g}_*"']
\end{tikzcd}$$

Proof. We know that $\forall\, i \in \mathbb{N}$, $K_{*i} = K_*$ is a compact metric space and that $\forall\, i \in \mathbb{N}$, $\tilde{\mathcal{M}}_{*i} = \tilde{\mathcal{M}}_*$ is a compact metric space. Then $r_{*i} : K_{*(i+1)} \to K_{*i}$ is r_* and $\tilde{g}_{*i} : \tilde{\mathcal{M}}_{*(i+1)} \to \tilde{\mathcal{M}}_{*i}$ is $\tilde{g}_*$. The maps commute as in the ladder diagram above and the projection h into each factor $h_i : K_{*_i} \to \tilde{\mathcal{M}}_{*i}$ is continuous $\forall\, i \in \mathbb{N}$. So with $\underleftarrow{\lim}(K_*, r_*)$ considered as a subspace of $\Pi_{i\in\mathbb{N}}\underleftarrow{\lim}(K_*, r_*)$ in the product topology, the map $\hat{h} : \underleftarrow{\lim}(K_*, r_*) \to \underleftarrow{\lim}(\tilde{\mathcal{M}}_*, \tilde{g}_*)$, $\hat{h}((x_i)) = (h_i(x_i))$, is well-defined and continuous. Let $(y_i) \in \tilde{\Sigma}_*$. For each $n \in \mathbb{N}$, let $\Omega(n) := \{(x_i) \in \Omega \,|\, h(x_n) = y_n\}$. By the surjection h, $\forall n \in \mathbb{N}$, $\Omega(n) \neq \emptyset$. Then by the commutativity of $h \circ r_* = \tilde{g}_* \circ h$, $\forall\, n \in \mathbb{N}$, $\Omega(n) \supseteq \Omega(n+1)$ which is true by induction: let $(x_i) = x_1x_2x_3\ldots \in \Omega(2)$ then $\forall\, i \in \mathbb{N}$, $r_*(x_{i+1}) = x_i$. That is, under the shift $\omega : \Omega \to \Omega$, $\omega^{-1}(x_1x_2x_3\ldots) = (r_*(x_1)x_1x_2x_3\ldots) \in \Omega(1)$. So the inclusion is true for $n = 1$. Suppose it is true for $n = j$, then $\Omega(j) \supseteq \Omega(j+1)$. So let $(x_{i+j}) = x_{1+j}x_{2+j}x_{3+j}\ldots \in \Omega(j+2)$, $\forall\, i, j \in \mathbb{N}$, then $w^{-1}(x_{1+j}x_{2+j}x_{3+j}\ldots) = (r_*(x_{1+j})x_{1+j}x_{2+j}x_{3+j}\ldots) \in \Omega(j+1) \Rightarrow \Omega(j+1) \supseteq \Omega(j+2)$. Thus inductively $\Omega(n) \supseteq \Omega(n+1)$ is true $\forall\, n \in \mathbb{N}$. Since Ω is compact, $\bigcap_{n\in\mathbb{N}} \Omega(n) \neq \emptyset$. So any $(x_i) \in \bigcap_{n\in\mathbb{N}} \Omega(n)$ satisfies $\hat{h}((x_i)) = (y_i)$. Thus $\hat{h}$ is surjective. $\square$

Three homeomorphic spaces

Consider the setting shown in Figure 6.17 where $F = \{p_*^{-1}(\tilde{B}_2 \cap \tilde{D}_1)\}$ represents all the fibres of the origin $p_2 = \tilde{B}_2 \cap \tilde{D}_1$. Measured from the point b we choose to make the circumference of a petal equal to the length of the corresponding

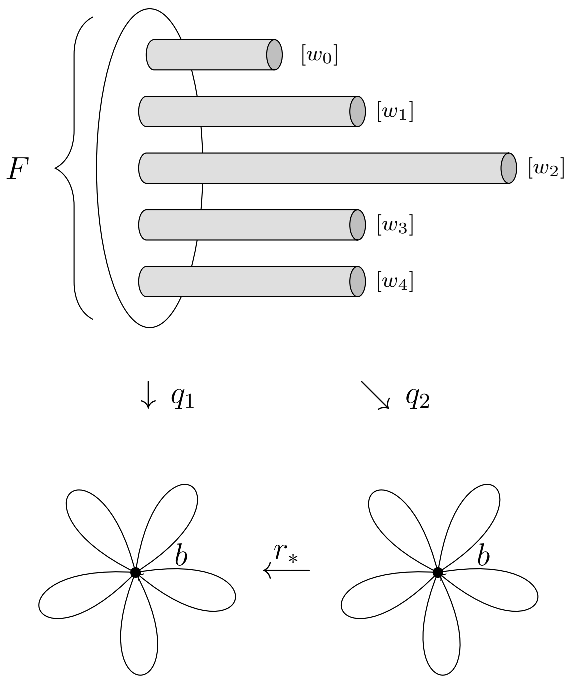


Figure 6.17: A schematic view of suspension cylinders projecting onto roses.

'cylinder'. That is for $0 \leq j \leq 4$, $k_j \subset K_*$, let $l(k_j) = l_j = l[w_j]$, $[w_j] \subset \tilde{\mathcal{M}}_*$. For some $0 \leq j \leq 4$ let $(x, s) \in [w_j] \times [0, l_j] \subset \mathcal{W}_c$ then define an o-p homeomorphism $\epsilon : \mathcal{W}_c \to \tilde{\Sigma}_*$, $(x, s) \mapsto x_i$, $i \in \mathbb{N}$, followed by a linear projection defined by

$$q : \tilde{\Sigma}_* \to K_*,\ q(x_i) = (s, k_j) = y_i. \tag{6.14}$$

Next let $\hat{q}$ be defined by

$$\hat{q} : \tilde{\Sigma}_* \to \Omega,\ \hat{q}\big((x_i)\big) = \big(y_i, r_*^{-1}(y_i), \dots\big) = \big(q(x_i), q(\tilde{\sigma}_*(x_i)), \dots\big). \tag{6.15}$$

Proposition 6.27. *The map $\hat{q}$ is a continuous and well-defined projection which commutes the shift maps $\hat{q} \circ \tilde{\sigma}_* = \omega \circ \hat{q}$.*

$$\begin{tikzcd}
\tilde{\Sigma}_* \arrow[r, "\tilde{\sigma}_*"] \arrow[d, "\hat{q}"'] & \tilde{\Sigma}_* \arrow[d, "\hat{q}"] \\
\Omega \arrow[r, "\omega"] & \Omega
\end{tikzcd}$$

Proof. For each $i \in \mathbb{N}$, $q_i(x_l) = y_i$ is a continuous onto projection from $\tilde{\Sigma}_*$ to K_{*i}. The shift $\tilde{\sigma}_*$ is a homeomorphism. Thus for each $i \in \mathbb{N}$, $q_i \circ \tilde{\sigma}_*(x_l) = q_i(x_{l+1}) = y_{i+1}$ is continuous and onto with $y_{i+1} \in K_{*i+1}$. Now let $(y_i) = \big(q(x_i), q(\tilde{\sigma}_*(x_i)), \ldots\big) = (y_1, y_2, \ldots)$ and assume that $(y_i) \in \Omega$, then $y_i \in K_*$ such that $r_*(y_{i+1}) = y_i$. So $\omega^{-1}((y_i)) = \omega^{-1}((y_1, y_2, \ldots)) = (r_*(y_1), y_1, y_2, \ldots)$. That is, $\omega^{-1}\big((q(x_i), q(\tilde{\sigma}_*(x_i)), \ldots)\big) = (r_* \circ q(x_i), r_* \circ q(\tilde{\sigma}_*(x_i)), \ldots) = (r_* \circ q(x_i), r_* \circ q(x_{i+1}), \ldots) = (r_*(y_i), r_*(y_{i+1}), \ldots) = (r_*(y_i), y_i, \ldots)$ which indeed is an element of Ω, $\forall i \in \mathbb{N}$. So $\hat{q}$ is well-defined and maps continuously onto each coordinate in the inverse limit space Ω.

Consider commutativity and let $(x_i)_{i \in \mathbb{N}} \in \tilde{\Sigma}_*$ then

$$\begin{aligned}
\hat{q} \circ \tilde{\sigma}_*((x_i)) &= \hat{q}((x_{i+1})) \\
&= \big(q(x_{i+1}), q(\tilde{\sigma}_*(x_{i+1})), \ldots\big) \\
&= \big(q(x_{i+1}), q(x_{i+2}), \ldots\big) \\
&= (y_{i+1}, y_{i+2}, \ldots).
\end{aligned}$$

Whereas

$$\begin{aligned}
\omega \circ \hat{q}((x_i)) &= \omega\big((q(x_i), q(\tilde{\sigma}_*(x_i)), \ldots)\big) \\
&= \omega\big((q(x_i), q(x_{i+1}), \ldots)\big) \\
&= \omega\big((y_i, y_{i+1}, \ldots)\big) \\
&= (y_{i+1}, y_{i+2}, \ldots).
\end{aligned}$$

Thus $\hat{q} \circ \tilde{\sigma}_* = \omega \circ \hat{q}$. $\square$

Let $id_{\tilde{\Sigma}_*} : \tilde{\Sigma}_* \to \tilde{\Sigma}_*$ and $id_\Omega : \Omega \to \Omega$ be the identity maps of the respective spaces.

Proposition 6.28. *The composites $\hat{h} \circ \hat{q} = id_{\tilde{\Sigma}_*}$ and $\hat{q} \circ \hat{h} = id_\Omega$. The two solenoids are homeomorphic, $\tilde{\Sigma}_* \cong \Omega$.*

$$\begin{tikzcd}
\tilde{\Sigma}_* \arrow[r, "\hat{q}"] \arrow[dr, "id_{\tilde{\Sigma}_*}"'] & \Omega \arrow[d, "\hat{h}"] \arrow[dr, "id_\Omega"] & \\
& \tilde{\Sigma}_* \arrow[r, "\hat{q}"] & \Omega
\end{tikzcd}$$

Proof. Let $(x_i)_{i\in\mathbb{N}} \in \tilde{\Sigma}_*$ then $\hat{h} \circ \hat{q}((x_i)) = \hat{h}\big((q(x_i), q(\tilde{\sigma}_*(x_i)), \ldots)\big) = \hat{h}\big((y_i, y_{i+1}, \ldots)\big) = (x_i, x_{i+1}, \ldots) \Rightarrow \hat{h} \circ \hat{q} = id_{\tilde{\Sigma}_*}$. Now consider $\hat{q} \circ \hat{h}$ for which the surjection $\hat{h} \Rightarrow \hat{q} \circ \hat{h}((y_i)) = \hat{q}((x_i))$ for some $(y_i)_{i\in\mathbb{N}} \in \Omega$. Then $\hat{q}\big((x_i)\big) = \big(q(x_i), q(\tilde{\sigma}_*(x_i)), \ldots\big) = \big(q(x_i), q(x_{i+1}), \ldots\big) = (y_i, y_{i+1}, \ldots) \Rightarrow \hat{q} \circ \hat{h} = id_\Omega$. It follows that $\hat{h} = \hat{q}^{-1}$ is a left inverse in $\tilde{\Sigma}_*$ and that $\hat{q} = \hat{h}^{-1}$ is a left inverse in Ω. Now each inverse is continuous by dint of $\hat{h}$ being a continuous surjection and $\hat{q}$ being a continuous projection. Thus $\hat{h}$ and $\hat{q}$ are homeomorphisms which give the said homeomorphic spaces. □

Remark 6.29. *As a result of Propositions 6.24 and 6.28 the suspension and the two solenoids are mutually homeomorphic, that is $\mathcal{W}_c \cong \tilde{\Sigma}_* \cong \Omega$.*

6.2.2 A 'Plykin tiling' on the torus

Let a proper substitution ω^2 be defined over $\mathcal{A} = \{0, 1, 2, 3, 4\}$ such that it maps letters to words in the same pattern as that generated by two iterations of the rose map (6.12). That is,

$$\omega^2 : \mathcal{A} \to \mathcal{A}^*, \quad \begin{cases} 0 \mapsto 0123, \\ 1 \mapsto 0421043, \\ 2 \mapsto 0121042123, \\ 3 \mapsto 0121043, \\ 4 \mapsto 042123. \end{cases} \tag{6.16}$$

Then $M_{\omega^2} = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$ which has determinant 1 and rank 5. The matrix is primitive since all its entries $[M_{\omega^2}]_{i,j} > 0$. The PF eigenvalue μ^4 has a left eigenvector $\mathbf{v}_{\omega^2} = [1/\mu, 1, \mu, 1, 1]$. Now let $\mathcal{T}_{\omega^2}$ be a tiling space where a tiling $T \in \mathcal{T}_{\omega^2}$ is built from the set of prototiles $\mathscr{P} = \{0, 1, 2, 3, 4\}$ such that their designated lengths are the components of $\mathbf{v}_{\omega^2}$, namely $|0| = 1/\mu$, $|1| = |3| = |4| = 1$ and $|2| = \mu$. Let $F_{\omega^2} : \mathcal{T}_{\omega^2} \to \mathcal{T}_{\omega^2}$ be the inflation and substitution homeomorphism with stretch factor μ^4 then the translates F_{ω^2} satisfy the Fibonacci relation. Consider the tiling $T \in \mathcal{T}_{\omega^2}$ which is fixed under the substitution ω^2 with origin $0 \in T_0 \setminus T_{-1}$ at the branch point $b \in K_*$. In line with Definition 6.14, the circumferences of the petals in K_* are $|k_0| = 1/\mu$, $|k_1| = |k_3| = |k_4| = 1$, $|k_2| = \mu$ and the rose map r_*^2 has a stretch factor μ^4. For $t \in \mathbb{R}$, let $\bar{p}$ be a continuous surjection defined by

$$\bar{p} : \mathcal{T}_{\omega^2} \to K_*, \ \bar{p}(T) = \begin{cases} [(t, k_j)] & \text{if } 0 \le t \le |k_j|, \\ [(t, k_j k_{j+1})] & \text{if } t = |k_j|, \\ [(t, k_{j+1})] & \text{if } |k_j| \le t \le |k_{j+1}|. \end{cases} \tag{6.17}$$

$$\begin{tikzcd}
\mathcal{T}_{\omega^2} \arrow[d, "\bar{p}"'] & \mathcal{T}_{\omega^2} \arrow[l, "F_{\omega^2}"'] \arrow[d, "\bar{p}"] \\
K_* & K_* \arrow[l, "r_*^2"']
\end{tikzcd}$$

Since T is fixed the maps commute $F_{\omega^2}(p_2) = p_2 \Rightarrow r_*^2(p_2) = \bar{p}(p_2)$.

Remark 6.30. *Since r_*^2 is a map of the rose K_* associated to the proper substitution ω^2, $\varprojlim(K_*, r_*^2) \cong \mathcal{T}_{\omega^2}$. But $\varprojlim(K_*, r_*^2) \cong \varprojlim(K_*, r_*) = \Omega \Rightarrow \Omega \cong \mathcal{T}_{\omega^2}$.*

Recall §5.2 wherein the Plykin attractor Λ_Π on the sphere was 'born', induced by the commuting maps

$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r, "f"] \arrow[d, "\Pi"'] & \mathbb{T}^2 \arrow[d, "\Pi"] \\
S^2 \arrow[r, "f_\Pi"] & S^2
\end{tikzcd}$$

and recall the attractor on the plane $\Lambda_P = \bigcap_{n\in\mathbb{N}} P^n(R)$. Denote the DA map $\tilde{f}_\Pi : \mathbb{T}^2 \to \mathbb{T}^2$ as a lift of the Plykin map f_Π. Let D_i, $i = 1, \ldots, 4$, be the four disjoint discs in $\mathbb{T}^2$ then define the *lifted Plykin attractor* on the torus to be $\tilde{\Lambda}_P := \bigcap_{n\in\mathbb{N}} (\tilde{f}_\Pi)^n(\tilde{N}_\Pi)$, $\tilde{N}_\Pi := \mathbb{T}^2 \setminus \dot{\bigcup}_{i=1}^4 D_i$.

Remark 6.31. *To calculate the Čech[1] (co)homology of $\Omega \cong \mathcal{T}_{\omega^2} \cong \varprojlim(K_*, r_*^2)$, we exploit the continuity of Čech (co)homology. Thus, $\check{H}_1(\varprojlim(K_*, r_*^2)) \cong \varprojlim(H_1(K_*), r_*^2) \cong \mathbb{Z}^5$ and similarly $\check{H}^1(\varprojlim(K_*, r_*^2)) \cong \varinjlim(H_1(K_*), r_*^2) \cong \mathbb{Z}^5$. We note that this concurs with the incidence matrix M_{ω^2} having rank 5 and* $\det(M_{\omega^2}) = 1$.

Theorem 6.32. *The lifted Plykin attractor $\tilde{\Lambda}_P \subseteq \mathbb{T}^2$ is homeomorphic to the tiling space $\mathcal{T}_{\omega^2} \subseteq \mathbb{T}^2$.*

Proof. The solenoid $\Sigma_* = \varprojlim(\mathcal{M}_*, g_*) \cong \Lambda_P \Rightarrow \tilde{\Sigma}_* = \varprojlim(\tilde{\mathcal{M}}_*, \tilde{g}_*) \cong \tilde{\Lambda}_P$. Then $\tilde{\Sigma}_* \cong \Omega \cong \mathcal{T}_{\omega^2}$. Thus $\tilde{\Lambda}_P \cong \mathcal{T}_{\omega^2}$. □

[1]Eduard Čech (1893 - 1960) Czech mathematician.

Chapter 7

Towards classification

In order to capture the classification of attractors and tiling spaces formulated in chapter 8, we need to prepare criteria. Be it the attractor's parameter α or the 'type' of the attractor's source matrix, both these criteria stem from a hyperbolic toral automorphism. So in §7.1 we give core definitions and describe two matrix subgroups of $GL(2,\mathbb{Z})$. In §7.2 we develop our construction method, summarised in the algorithm of §7.3.1, for these two hyperbolic spaces.

7.1 Core definitions and criteria

Definition 7.1. *Let* $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F\left(\overline{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \overline{M\begin{bmatrix} x \\ y \end{bmatrix}}$, *be a hyperbolic toral automorphism determined by a matrix* $M \in GL(2,\mathbb{Z})$. *Let the Perron-Frobenius eigenvalue of* M *be* λ *with expanding eigenvector* $\mathbf{v}^u = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ *whose* slope α *is irrational. Let* λ_s *be the stable eigenvalue of* M.

We quote the following proposition without proof which is given for example in Lemma 2.7 of [24].

Proposition 7.2. *An irrational number* α *is quadratic if and only if* $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ *is an eigenvector of a matrix* M *in* $GL(2,\mathbb{Z})$ *with* $M \neq \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Define $\mathscr{G} := \left\{ g \in GL(2,\mathbb{Z}) \mid \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \text{ is an eigenvector of } g \right\}$.

Lemma 7.3. $\mathscr{G}$ *is a subgroup of* $GL(2,\mathbb{Z})$.

Proof. Let λ_1, λ_2 be eigenvalues of $g_1, g_2 \in \mathscr{G}$. Then $g_2 g_1 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = g_2 \left(\lambda_1 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right) = \lambda_2 \lambda_1 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = g \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ for some $g \in \mathscr{G}$ with eigenvalue $\lambda = \lambda_2 \lambda_1$. The identity $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathscr{G}$ has eigenvalue 1 with eigenvector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Then $\forall g \in \mathscr{G}$ with eigenvalue λ, $ge \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = g \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = e \left(\lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right) = eg \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Let $g \in \mathscr{G}$ have an eigenvalue λ then $\begin{bmatrix} 1 \\ \alpha \end{bmatrix} = g^{-1} g \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = g^{-1} \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda g^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \Rightarrow g^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Thus $\mathscr{G}$ is a subgroup of $GL(2,\mathbb{Z})$. □

Lemma 7.4. *Let* λ *be an eigenvalue of* $g \in \mathscr{G}$. *Then* $\nu : \mathscr{G} \to (\mathbb{R} - \{0\}, \cdot)$, $\nu(g) = \lambda$, *defines a group homomorphism from matrix to scalar multiplication.*

Proof. Let λ_1, λ_2 be eigenvalues of $g_1, g_2 \in \mathscr{G}$. Then $\nu(g_1)\nu(g_2) = \lambda_1 \lambda_2 = \nu(g_1 g_2)$, $\forall g_1 g_2 \in \mathscr{G}$. □

Lemma 7.5. *The homomorphism ν is injective.*

Proof. Let λ_1, λ_2 be eigenvalues of $g_1, g_2 \in \mathscr{G}$ where $g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $g_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ and suppose that $\nu(g_1) = \nu(g_2) \Rightarrow \lambda_1 = \lambda_2$. Then since both matrices admit an eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, $a_1 + \alpha b_1 = \lambda_1,\ a_2 + \alpha b_2 = \lambda_2 \Rightarrow a_1 = a_2,\ b_1 = b_2$. Whereas $c_1 + \alpha d_1 = \alpha\lambda_1,\ c_2 + \alpha d_2 = \alpha\lambda_2 \Rightarrow c_1 = c_2,\ d_1 = d_2$. So $g_1 = g_2$, and ν is injective. $\square$

Theorem 7.6. *The group $\mathscr{G}$ is isomorphic to the direct sum $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Proof. Let τ denote the *trace* of a matrix g. The characteristic equation of $g \in \mathscr{G}$, $\chi_g = 0$, yields an eigenvalue $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 \pm 4}}{2} \Rightarrow \tau \geq 3 \Leftrightarrow g \in SL(2, \mathbb{Z})$. Now let $\tau \to \infty \Rightarrow \lambda \to 0$ (not possible in this context) or $\lambda \to 2\tau/2 = \tau$. Then the ratio between terms $\tau, \tau + 1, \ldots$, is $\frac{\tau+1}{\tau} = 1 + \frac{1}{\tau} \to 1$ as $\tau \to \infty$. So there is a bounded gap between terms. Thus $\nu(\mathscr{G}) \cap (\mathbb{R}^+ - \{0\}, \cdot)$ is a discrete and hence cyclic subgroup of $(\mathbb{R}^+ - \{0\}, \cdot)$. Consider the topological isomorphism $\log : (\mathbb{R}^+ - \{0\}, \cdot) \to (\mathbb{R}, +)$ which maps a discrete subgroup of $(\mathbb{R}^+ - \{0\}, \cdot)$ to a discrete subgroup of $(\mathbb{R}, +)$ which in turn is isomorphic to $(\mathbb{Z}, +)$. Also $\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \cong (\mathbb{Z}_2, +)$. Hence $\mathscr{G} \cong \mathbb{Z} \oplus \mathbb{Z}_2$. $\square$

Recall the involution i (5.3) which fixes the 4 *special points* of $\mathbb{T}^2$. Let these points form the set

$$B := \left\{ \mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right\},$$

where for ease we now suppress the equivalence class 'bar' over $\mathbf{b} \in \mathbb{R}^2/\mathbb{Z}^2$.

Lemma 7.7. *Any toral automorphism $F_M : \mathbb{T}^2 \to \mathbb{T}^2$ defined by a matrix $M \in GL(2, \mathbb{Z})$ will induce a permutation on the 4 special points of $\mathbb{T}^2$ such that $F_M : B \to B$.*

Proof. By definition $\forall\, 0 \leq j \leq 3$, $\mathbf{b}_j \in B$ is such that $\mathbf{b}_j = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$, $\epsilon_i \in \{0, 1/2\}$.

Let $M \in GL(2, \mathbb{Z})$ then $M \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$, $\epsilon_i, \delta_i \in \{0, 1/2\} \bmod 1$. That is, $F_M : B \to B$. □

The set B serves as a *repelling set* of the attractors defined below.

Definition 7.8. *Fix the quadratic irrational α then let $\mathfrak{A}_\alpha \subseteq \mathbb{T}^2$ be an* orientable hyperbolic toral attractor *with 4 complementary domains arising from the points of set B.*

Remark 7.9. *Then $\mathfrak{A}_\alpha$ is homeomorphic to any other attractor $\mathfrak{A}_\alpha$ with parameter α.*

Recall the quotient map $\Pi : \mathbb{T}^2 \to \mathbb{T}^2/i \cong S^2$ (Def. 5.11).

Definition 7.10. *Given any $\mathfrak{A}_\alpha$, let the map Π yield a* non-orientable Plykin attractor *$P\mathfrak{A}_\alpha \subseteq S^2$; since $P\mathfrak{A}_\alpha$ may be projected to $\mathbb{R}^2$ let $P\mathfrak{A}_\alpha$ also be known as a* planar *attractor.*

Let

$$\mathfrak{F} := \{\mathfrak{A}_\alpha \,|\, \alpha \text{ is a quadratic irrational}\};$$

$$P\mathfrak{F} := \{P\mathfrak{A}_\alpha \,|\, \alpha \text{ is a quadratic irrational}\}.$$

When a space $\mathfrak{A}_\alpha \in \mathfrak{F}$ or $P\mathfrak{A}_\alpha \in P\mathfrak{F}$ is the attractor of a map determined by a matrix M we call M the *source* matrix.

Observe that further to Lemma 7.7, in general a matrix $M \in GL(2,\mathbb{Z})$ is such that it is the *parity* of the entries in M which determines the image point of $\mathbf{b}_j \in B$, $0 \le j \le 3$. In light of this consider the epimorphism

$$\text{par} : GL(2,\mathbb{Z}) \twoheadrightarrow GL(2,\mathbb{Z}/2\mathbb{Z}),\ M \mapsto M/2\mathbb{Z}. \tag{7.1}$$

Let $\{\bar{0},\bar{1}\}$ represent the equivalence classes of $\{2\mathbb{Z}, 2\mathbb{Z}+1\}$ under addition modulo 2.

Definition 7.11. *Let $\mathbb{M}$ be a group under matrix multiplication, where $\mathbb{M} = \dot{\bigcup}_{i=0}^{5} \bar{M}_i$ is a union of equivalence classes, each one of which we shall call a matrix* type.

$$\bar{M}_0 = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}},\ \bar{M}_1 = \overline{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}},\ \bar{M}_2 = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}},$$
$$\bar{M}_3 = \overline{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}},\ \bar{M}_4 = \overline{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}},\ \bar{M}_5 = \overline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \bar{I}.$$

Lemma 7.12. *A matrix $M \in GL(2,\mathbb{Z})$ if and only if $M \in \mathbb{M}$.*

Proof. If $M \in GL(2,\mathbb{Z})$, $\det(M) = \pm 1 \in \bar{1}$. Let $M \in \bar{M}_i \subset \mathbb{M}$, $0 \le i \le 5$, then $\det(M) \in \bar{1} \Rightarrow M \in GL(2,\mathbb{Z})$. Conversely assume $M \in GL(2,\mathbb{Z})$ then $\det(M) \in \bar{1} \Rightarrow M \in \bar{M}_i$ for some $\bar{M}_i \subset \mathbb{M}$. Thus $M \in GL(2,\mathbb{Z}) \Leftrightarrow M \in \mathbb{M}$. □

Theorem 7.13. *The group $\mathbb{M}$ is such that $\mathbb{M} = \dot{\bigcup}_{i=0}^{5} \bar{M}_i$ is a partition of $GL(2,\mathbb{Z})$ into 6 equivalence classes.*

Proof. The *kernel*, $\ker(\text{par}) = \{M \in GL(2,\mathbb{Z}) \,|\, \text{par}(M) = \bar{I}\}$. By the *first isomorphism theorem* $GL(2,\mathbb{Z})/\ker(\text{par}) \cong GL(2,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{M}$. Thus $\mathbb{M}$ is a quotient group which partitions $GL(2,\mathbb{Z})$ into 6 equivalence classes. □

7.2 Construction of spaces

7.2.1 Six hyperbolic toral automorphisms

We choose the six maps below so as to give one example of each type of matrix $\bar{M}_i \in \mathbb{M}$, $0 \leq i \leq 5$. Their properties help to develop our analysis of attractors.

$$\mathcal{C}\ (= F_0) : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto A\mathbf{x},\ A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \equiv \overline{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} = \bar{M}_0, \tag{1.10}$$

$$F_1 : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto M_1\mathbf{x},\ M_1 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \equiv \overline{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} = \bar{M}_1, \tag{7.2}$$

$$F_2 : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto M_2\mathbf{x},\ M_2 = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \equiv \overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \bar{M}_2, \tag{7.3}$$

$$F_3 : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto M_3\mathbf{x},\ M_3 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \equiv \overline{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \bar{M}_3, \tag{7.4}$$

$$F_\varphi(= F_4) : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto M_\varphi\mathbf{x},\ M_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \equiv \overline{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} = \bar{M}_4, \tag{1.8}$$

$$F_5 : \mathbb{T}^2 \to \mathbb{T}^2,\ \mathbf{x} \mapsto M_5\mathbf{x},\ M_5 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \equiv \overline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \bar{M}_5 = \bar{I}. \tag{7.5}$$

See Table 7.1 for more details of the four maps F_i, $0 \leq i \leq 3$.

Toral map	$\mathcal{C}\ (= F_0)$	F_1
Matrix	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$
Type	$\bar{M}_0 = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}$	$\bar{M}_1 = \overline{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$
Eigenvalues	$\frac{1}{2}(3 \pm \sqrt{5})$	$2 \pm \sqrt{3}$
Eigenvectors	$\begin{bmatrix} \mu \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \mu \end{bmatrix}$	$\begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix}, \begin{bmatrix} 1-\sqrt{3} \\ 1 \end{bmatrix}$
Slope	$\alpha_0 = \mu^{-1} = \frac{-1+\sqrt{5}}{2}$	$\alpha_1 = \frac{-1+\sqrt{3}}{2}$
Fixed points	$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$	$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$
Periodic points	$\mathbf{b}_1 \mapsto \mathbf{b}_2 \mapsto \mathbf{b}_3 \mapsto \mathbf{b}_1$	$\mathbf{b}_2 \mapsto \mathbf{b}_3 \mapsto \mathbf{b}_2$

Toral map	F_2	F_3
Matrix	$\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$
Type	$\bar{M}_2 = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$	$\bar{M}_3 = \overline{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$
Eigenvalues	$3 \pm 2\sqrt{2}$	$2 \pm \sqrt{3}$
Eigenvectors	$\begin{bmatrix} -1+2\sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -1-2\sqrt{2} \\ 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$
Slope	$\alpha_2 = \frac{1+2\sqrt{2}}{7}$	$\alpha_3 = -1 + \sqrt{3}$
Fixed points	$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$	$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$
Periodic points	$\mathbf{b}_1 \mapsto \mathbf{b}_3 \mapsto \mathbf{b}_1$	$\mathbf{b}_1 \mapsto \mathbf{b}_2 \mapsto \mathbf{b}_1$

Table 7.1: Parameters of 4 hyperbolic toral automorphisms.

7.2.2 Markov partitions

Given Definition 7.1 such that *all entries a_{ij} in M take the same sign* then there exists a Markov generator $\mathcal{R}^*$ for F, the members of which are parallelograms (Theorem 8.4 in [2]). We shall call an 'original' Markov partition, as described on page 91, a *principal partition* $\mathcal{P}$ and a finer subdivided Markov partition its *secondary partition* denoted by $\ddot{\mathcal{P}}$. For each toral map (1.10), (7.2), (7.3) and (7.4) we choose to partition the fundamental domain of $\mathbb{T}^2$ into 3 parallelograms R_i, $i = 1, 2, 3$, labelled A, B, C such that $R_i \cap R_i = \emptyset$ unless $R_i \cap R_i = O \in \mathbb{T}^2$. In a secondary partition the stable leaves of a foliation of $\mathbb{T}^2$ are bisected by a line segment parallel to the unstable leaves of the foliation, thereby for each $i = 1, 2, 3$ dividing R_i into two congruent parallelograms labelled naturally as R_{i_l}, $l = 1, 2$. The bisector of R_i passes through exactly one periodic point $\mathbf{b}_j \in R_i$, $i, j = 1, 2, 3$.

The principal partition $\mathcal{P}$ for the Cat map (1.10) is repeated here in Figure 7.1 for ease of comparison with its secondary partition $\ddot{\mathcal{P}}$. Figures 7.2 to 7.4 show the principal and secondary partitions $\mathcal{P}_i$ and $\ddot{\mathcal{P}}_i$ for the maps (7.2), (7.3) and (7.4) respectively. The fixed points are shown in red.

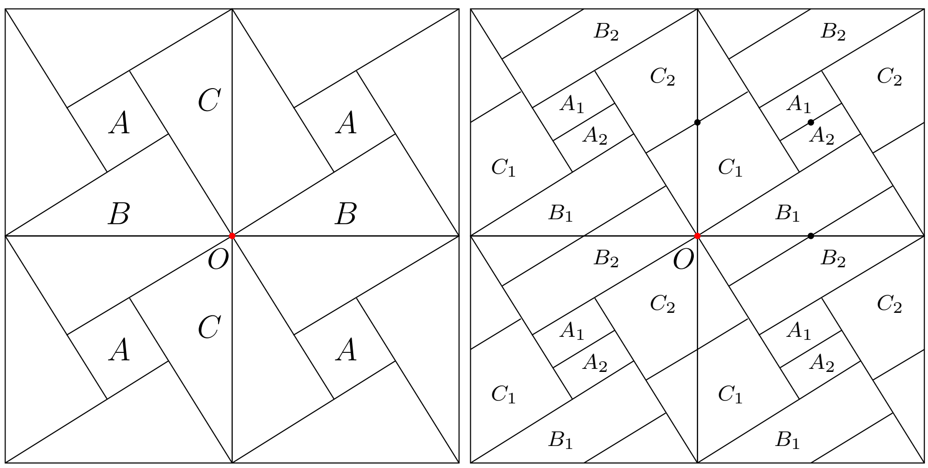


Figure 7.1: Principal and secondary partitions $\mathcal{P}$ and $\ddot{\mathcal{P}}$.

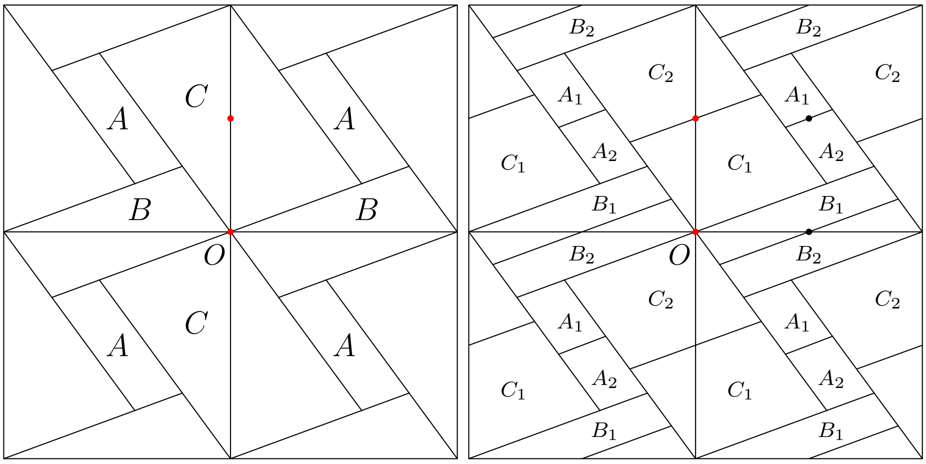


Figure 7.2: Principal and secondary partitions $\mathcal{P}_1$ and $\ddot{\mathcal{P}}_1$.

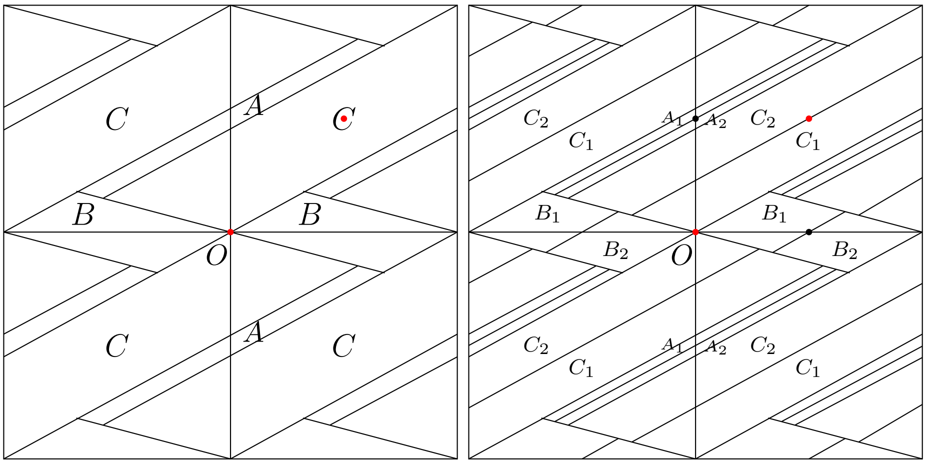


Figure 7.3: Principal and secondary partitions $\mathcal{P}_2$ and $\ddot{\mathcal{P}}_2$.

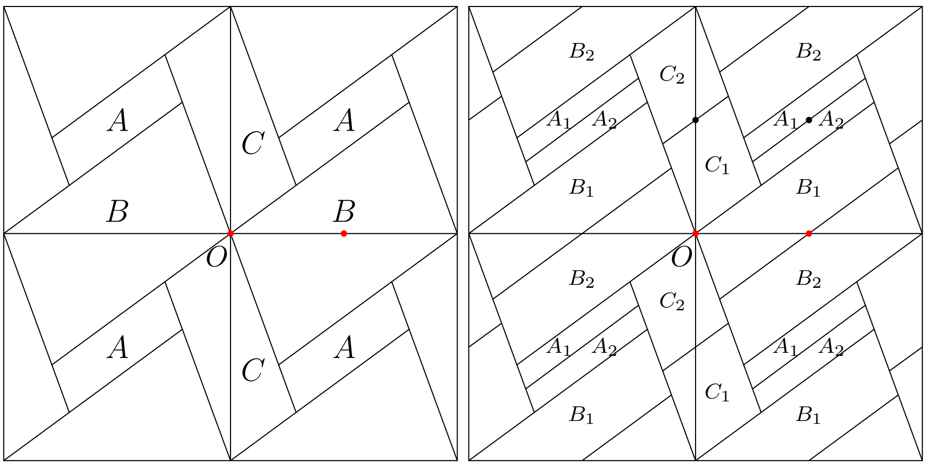


Figure 7.4: Principal and secondary partitions $\mathcal{P}_3$ and $\ddot{\mathcal{P}}_3$.

An observation (Fig. 7.3). In the partition $\mathcal{P}_2$, let p be an expanding eigenline segment through O; let q and s be line segments parallel to the contracting eigenline through points $(1,0)$ and $(0,1)$ respectively. Let $p \cap q = (x_{pq}, y_{pq})$ and $p \cap s = (x_{ps}, y_{ps})$ be coordinates of intersection then $x_{pq} < x_{ps}$. This forces parallelogram A of $\mathcal{P}_2$ in this particular case to straddle two of our chosen representative fundamental domains of $\mathbb{T}^2$. However, choosing a fundamental domain consisting of $A \cup B \cup C$ such that all boundaries are parallel or orthogonal to the eigenlines will overcome this.

7.2.3 A 2θ-space

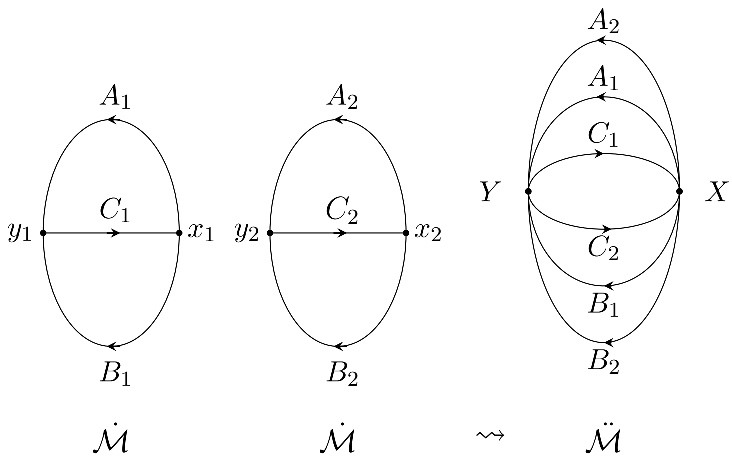


Figure 7.5: From a θ-space to a 2θ-space.

Recall the DA diffeomorphism f (5.2) and the θ-space map g_0 (6.3). For consistency of notation in this setting we now denote the map f as $f_0 = \phi^\tau \circ \mathcal{C}$. Then for a map F_i, $0 \leq i \leq 3$, and matrix M_i with stable eigenvalue λ_s, let a DA diffeomorphism be defined by

$$f_i : \mathbb{T}^2 \to \mathbb{T}^2, \; f_i = \phi^\tau \circ F_i, \tag{7.6}$$

such that for a fixed $\tau > 0$, $e^\tau \lambda_s > 1$. Then with edges labelled $E_1 = A$, $E_2 = B$,

$E_3 = C$, let $\dot{\mathcal{M}}$ denote a generic θ-space derived from a DA diffeomorphism f_i, $0 \leq i \leq 3$. For each $0 \leq i \leq 3$, let $\dot{\vartheta}_i : \dot{\mathcal{M}} \to \dot{\mathcal{M}}$ be an expanding map with stretch factor equal to the PF eigenvalue of the matrix M_i which defines F_i. The mapping of an edge $\dot{\vartheta}_i(E_k) \in \dot{\mathcal{M}}$ is listed in Table 7.2 or 7.3. In tandem with making a finer partitition of a Markov partition, we can derive from $\dot{\mathcal{M}}$ a new generic branched 1-manifold, called a 2θ*-space* denoted by $\ddot{\mathcal{M}}$ (Fig. 7.5). To do so take two disjoint copies of $\dot{\mathcal{M}}$ then identify the labelled branch points x_i, y_i, $i = 1, 2$, as $X = x_1 \sim x_2$ and $Y = y_1 \sim y_2$. Then $\ddot{\mathcal{M}}$ comprises a set of six oriented edges $\{A_l, B_l, C_l \,|\, l = 1, 2\}$. The image element $\ddot{\vartheta}_i(E_{k_l})$, $0 \leq i \leq 3$, $k = 1, 2, 3$, $l = 1, 2$, is found by inspection of the linear transformation $M_i : \mathbb{R}^2 \to \mathbb{R}^2$, $0 \leq i \leq 3$. The definition of a map $\ddot{\vartheta}_i : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$, $0 \leq i \leq 3$, is given in Table 7.2 or 7.3.

7.2.4 Solenoids and suspensions

Definition 7.14. *Associated to each DA map f_i, $0 \leq i \leq 3$, let a solenoid be defined by $\ddot{\Sigma}_i = \varprojlim(\ddot{\mathcal{M}}, \ddot{\vartheta}_i)$ with shift map $\ddot{\sigma}_i : \ddot{\Sigma}_i \to \ddot{\Sigma}_i$, $\ddot{\sigma}_i((x_j)) = (x_{j+1})$, $\forall\, j \in \mathbb{N}$.*

Remark 7.15. *For $i = 0, 2$, $\ddot{\Sigma}_i$, is orientable. For $i = 1, 3$, $\ddot{\vartheta}_i$ is orientation-reversing. However, by inspection of the 2θ-space $\ddot{\mathcal{M}}$, the C edges enter X and the A and B edges leave X in a coherently chosen orientation so $\ddot{\Sigma}_i$, $i = 1, 3$, can be deemed orientable. Thus we may conclude that each solenoid $\ddot{\Sigma}_i$, $0 \leq i \leq 3$, supports a flow ϕ_i^t without rest points (Theorems 1.4 [24], 11 [1]).*

We repeat the combinatorial process of §6.2.1. We set the origin at $C_2 \cap B_1 \in \ddot{\mathcal{M}}$ for each map since for $0 \leq i \leq 3$, $\ddot{\vartheta}_i(C_2) = C_1 \ldots C_2$ and $\ddot{\vartheta}_i(B_1) = B_1 \ldots B_2$. Then for some $n \in \mathbb{N}$ iterate $\ddot{\vartheta}_i^n$ and record the sequence of edges which appear on each map's return to the origin. This produces sets, each consisting of five words, $\ddot{\mathcal{W}}_i := \{w_m \mid w_m = B_1 \ldots C_2,\ 0 \leq m \leq 4\}$, listed in Table 7.2 or 7.3.

DA Map	$f(=f_0)$	f_1
θ map	$g_0 : \mathcal{M}_0 \mapsto \mathcal{M}_0$ $(\dot{\vartheta}_0 : \dot{\mathcal{M}} \to \dot{\mathcal{M}})$	$\dot{\vartheta}_1 : \dot{\mathcal{M}} \mapsto \dot{\mathcal{M}}$
	$A \mapsto B$	$A \mapsto B$
	$B \mapsto BCB$	$B \mapsto BCACB$
	$C \mapsto CAC$	$C \mapsto CACAC$
2θ map	$\ddot{\vartheta}_0 : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$	$\ddot{\vartheta}_1 : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$
	$A_1 \mapsto B_1$	$A_1 \mapsto B_1$
	$B_1 \mapsto B_1C_2B_2$	$B_1 \mapsto B_1C_1A_1C_2B_2$
	$C_1 \mapsto C_1A_2C_2$	$C_1 \mapsto C_1A_2C_1A_1C_2$
	$A_2 \mapsto B_2$	$A_2 \mapsto B_2$
	$B_2 \mapsto B_1C_1B_2$	$B_2 \mapsto B_1C_1A_2C_2B_2$
	$C_2 \mapsto C_1A_1C_2$	$C_2 \mapsto C_1A_2C_2A_1C_2$
Origin	$C_2 \cap B_1$	$C_2 \cap B_1$
Words	$\ddot{\mathcal{W}}_0$:	$\ddot{\mathcal{W}}_1$:
w_0	$B_1C_2B_2C_1A_1C_2$	$B_1C_1A_1C_2B_2C_1A_2C_1A_1C_2$
w_1	$B_1C_1B_2C_1A_2C_2$	$B_1C_1A_2C_2A_1C_2$
w_2	$B_1C_1A_1C_2$	$B_1C_1A_2C_2B_2C_1A_2C_1A_1C_2B_2C_1A_2C_1A_1C_2$
w_3	$B_1C_2B_2C_1A_2C_2$	$B_1C_1A_1C_2B_2C_1A_2C_1A_1C_2B_2C_1A_2C_2A_1C_2$
w_4	$B_1C_1B_2C_1A_2C_2B_2C_1A_1C_2$	$B_1C_1A_2C_2B_2C_1A_2C_1A_1C_2B_2C_1A_2C_2A_1C_2$

Table 7.2: Self-maps of a 2θ-space and 'return map' words.

DA Map	f_2	f_3
θ map	$\dot{\vartheta}_2 : \dot{\mathcal{M}} \mapsto \dot{\mathcal{M}}$	$\dot{\vartheta}_3 : \dot{\mathcal{M}} \mapsto \dot{\mathcal{M}}$
	$A \mapsto CBCBCBC$	$A \mapsto BCB$
	$B \mapsto BAB$	$B \mapsto BCBCB$
	$C \mapsto CBCBCBCBC$	$C \mapsto CAC$
2θ map	$\ddot{\vartheta}_2 : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$	$\ddot{\vartheta}_3 : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$
	$A_1 \mapsto C_1B_2C_1B_1C_2B_1C_2$	$A_1 \mapsto B_1C_2B_2$
	$B_1 \mapsto B_1A_1B_2$	$B_1 \mapsto B_1C_2B_1C_1B_2$
	$C_1 \mapsto C_1B_2C_1B_2C_1B_1C_2B_1C_2$	$C_1 \mapsto C_1A_2C_2$
	$A_2 \mapsto C_1B_2C_1B_2C_2B_1C_2$	$A_2 \mapsto B_1C_1B_2$
	$B_2 \mapsto B_1A_2B_2$	$B_2 \mapsto B_1C_2B_2C_1B_2$
	$C_2 \mapsto C_1B_2C_1B_2C_2B_1C_2B_1C_2$	$C_2 \mapsto C_1A_1C_2$
Origin	$C_2 \cap B_1$	$C_2 \cap B_1$
Words	$\ddot{\mathcal{W}}_2$:	$\ddot{\mathcal{W}}_3$:
w_0	$B_1A_1B_2C_1B_2C_1B_2C_2$	B_1C_2
w_1	B_1C_2	$B_1C_1B_2C_1A_1C_2$
w_2	$B_1A_2B_2C_1B_2C_1B_2C_1B_1C_2$	$B_1C_1B_2C_1A_2C_2$
w_3	$B_1A_2B_2C_1B_2C_1B_2C_2$	$B_1C_2B_2C_1B_2C_1A_2C_2$
w_4	$B_1A_1B_2C_1B_2C_1B_1C_2$	$B_1C_2B_2C_1A_1C_2$

Table 7.3: Self-maps of a 2θ-space and 'return map' words.

An observation. There are two choices for the origin of each map $\ddot{\vartheta}_i$, $0 \leq i \leq 3$. Consider the origin $B_2 \cap C_1 \in \ddot{\mathcal{M}}$ where $\ddot{\vartheta}_i(B_2) = B_1 \dots B_2$ and $\ddot{\vartheta}_i(C_1) = C_1 \dots C_2$. Iterations of $\ddot{\vartheta}_i$ again produce five distinct words, albeit different from those emanating from the alternative origin but of the same structure (not shown). The words B_2C_1 or C_2C_1 do not occur in any $\ddot{\vartheta}_i^n(B_1)$ or $\ddot{\vartheta}_i^n(C_1)$, $n \in \mathbb{N}$, $0 \leq i \leq 3$, so neither $B_2 \cap B_1$ nor $C_2 \cap C_1$ can serve as an origin in $\ddot{\mathcal{M}}$. Having exactly two choices of origin corresponds to two fixed points of the DA map (7.6), that is the two saddle points $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{A}_{\alpha_i}$, $0 \leq i \leq 3$.

Remark 7.16. *Similarly to Proposition 6.24 we may deduce that for* $0 \leq i \leq 3$, *each solenoid* $\ddot{\Sigma}_i$ *is homeomorphic to the respective suspension* $\ddot{\mathcal{W}}_{i_c}$ *of the first return map* ρ_i *of the flow* ϕ_i^t *to* $\tilde{C}_2 \cap \tilde{B}_1$. *For the first projection* $p_{i_1} : \ddot{\Sigma}_i \to \ddot{\mathcal{M}}$, $0 \leq i \leq 3$, 5 *clopen subsets can be described by 'cylinders'. For* $0 \leq m \leq 4$, $0 \leq i \leq 3$,

$$[w_m] := \{x \in \tilde{C}_2 \cap \tilde{B}_1 \,|\, t \geq 0,\ p_{i_1}(\phi_i^t(x)) \textit{ follows the sequence of arcs described by } w_m \textit{ in } \ddot{\mathcal{W}}_i\}.$$

The next Figure 7.6 shows 4 'split open' unstable manifolds in the attractor $\mathfrak{A}_{\alpha_0} := \bigcap_{n \in \mathbb{N}} f_0^n(N)$.

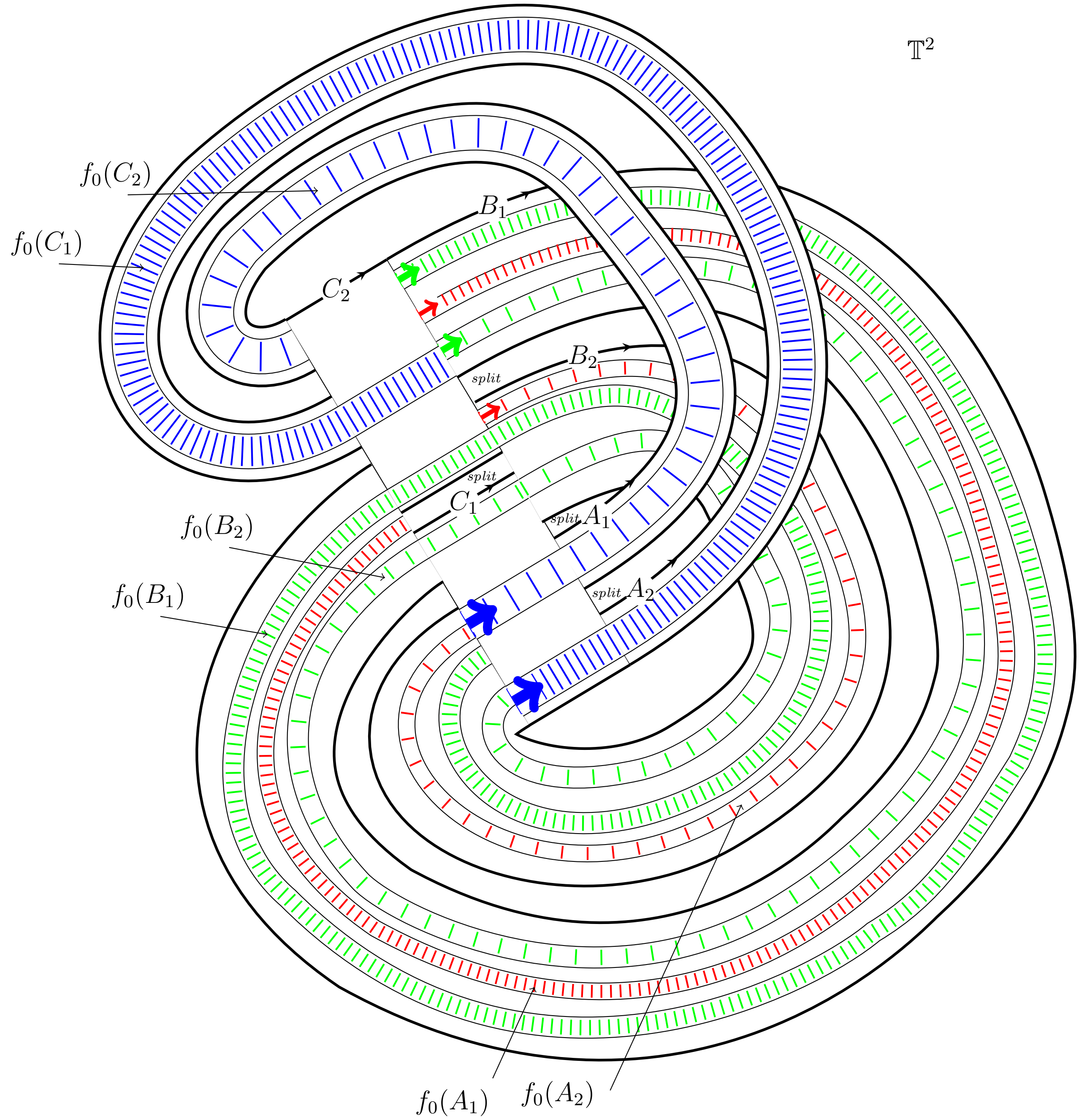


Figure 7.6: The attractor $\mathfrak{A}_{\alpha_0} = \bigcap_{n\in\mathbb{N}} f_0^n(N)$.

7.2.5 Roses

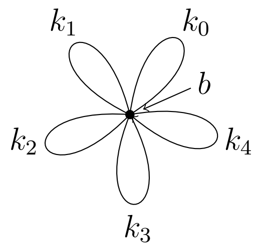


Figure 7.7: A rose $\ddot{K}$ with 5 petals.

Each set of words $\ddot{\mathcal{W}}_i$, $0 \leq i \leq 3$, may be represented by a generic elementary branched 1-manifold denoted by $\ddot{K}$ (Fig. 7.7). Each rose has 5 petals $k_0, \ldots, k_4$ and a single branch point b. Let $\ddot{K}$ be endowed with an expanding map r_i, $0 \leq i \leq 3$ with images $r_i(k_0), \ldots, r_i(k_4)$ where the stretch factor bestowed on r_i is determined by the PF eigenvalue of the matrix M_{r_i} (Tables 7.4 and 7.5).

Toral map	$\mathcal{C}$ $(= F_0)$	F_1
Rose map	$r_0 : \ddot{K} \to \ddot{K}$ $k_0 \mapsto k_0k_1k_2$ $k_1 \mapsto k_3k_4$ $k_2 \mapsto k_3k_2$ $k_3 \mapsto k_0k_4$ $k_4 \mapsto k_3k_4k_1k_2$	$r_1 : \ddot{K} \to \ddot{K}$ $k_0 \mapsto k_0k_1k_2k_1$ $k_1 \mapsto k_3k_1$ $k_2 \mapsto k_3k_2k_1k_2k_1$ $k_3 \mapsto k_0k_1k_2k_1k_4k_1$ $k_4 \mapsto k_3k_2k_1k_4k_1$
Matrix	$M_{r_0} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$	$M_{r_1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 3 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$
det; rank	1; 5	-1; 5
PF e.value	$\mu^2 = \frac{1}{2}(3 + \sqrt{5})$	$2 + \sqrt{3}$

Table 7.4: The rose maps $r_0, r_1 : \ddot{K} \to \ddot{K}$ and their matrices.

Toral map	F_2	F_3
Rose map	$r_2 : \ddot{K} \to \ddot{K}$ $k_0 \mapsto k_0k_1k_2k_1k_2k_1k_3k_1k_1$ $k_1 \mapsto k_0k_1k_1$ $k_2 \mapsto k_4k_1k_2k_1k_2k_1k_2k_1k_0k_1k_1$ $k_3 \mapsto k_4k_1k_2k_1k_2k_1k_3k_1k_1$ $k_4 \mapsto k_0k_1k_2k_1k_2k_1k_0k_1k_1$	$r_3 : \ddot{K} \to \ddot{K}$ $k_0 \mapsto k_0k_1$ $k_1 \mapsto k_0k_2k_3k_4$ $k_2 \mapsto k_0k_2k_3k_1$ $k_3 \mapsto k_0k_1k_3k_3k_1$ $k_4 \mapsto k_0k_1k_3k_4$
Matrix	$M_{r_2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 5 & 2 & 6 & 5 & 5 \\ 2 & 0 & 3 & 2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$	$M_{r_3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$
det; rank	1; 5	-1; 5
PF e.value	$3 + 2\sqrt{2}$	$2 + \sqrt{3}$

Table 7.5: The rose maps $r_2, r_3 : \ddot{K} \to \ddot{K}$ and their matrices.

Definition 7.17. *Associated to each DA map f_i, $0 \leq i \leq 3$, let $\ddot{\Omega}_i = \varprojlim(\ddot{K}, r_i)$ with shift map $\ddot{\omega}_i : \ddot{\Omega}_i \to \ddot{\Omega}_i$, $\ddot{\omega}_i((x_j)) = (x_{j+1})$, $\forall\, j \in \mathbb{N}$, then $(\ddot{K}, r_i)$ is an* elementary presentation *of $\ddot{\Omega}_i$ and $\ddot{\omega}_i$.*

Remark 7.18. *We invoke Propositions 6.24 and 6.28 to conclude that for $0 \leq i \leq 3$ the suspensions and the solenoids are homeomorphic, that is $\ddot{\mathcal{W}}_{i_c} \cong \ddot{\Sigma}_i \cong \ddot{\Omega}_i$.*

7.2.6 Proper substitutions

Definition 7.19. *Associated to each rose map r_i, $0 \leq i \leq 3$, let a* proper substitution *be defined over $\mathcal{A} = \{0, 1, 2, 3, 4\}$, $\ddot{\omega}_i : \mathcal{A} \to \mathcal{A}^*$, $0 \leq i \leq 3$ such that each $\ddot{\omega}_i$ maps letters to words in the pattern generated by r_i^2 respectively.*

See Table 7.6 for the substitution maps and Table 7.7 for their matrices.

Map	$\mathcal{C}\ (= F_0)$	F_1
	$\ddot{\omega}_0 : \mathcal{A} \to \mathcal{A}^*$	$\ddot{\omega}_1 : \mathcal{A} \to \mathcal{A}^*$
	$0 \mapsto 0123432$	$0 \mapsto 0121313212131$
	$1 \mapsto 043412$	$1 \mapsto 01214131$
	$2 \mapsto 0432$	$2 \mapsto 0121413212131 3212131$
	$3 \mapsto 0123412$	$3 \mapsto 01213132121313214131$
	$4 \mapsto 0434123432$	$4 \mapsto 01214132121313214131$

Map	F_2
	$\ddot{\omega}_2 : \mathcal{A} \to \mathcal{A}^*$
	$0 \mapsto 0121213110114121212101101141212121011011412121311011011$
	$1 \mapsto 012121311011011$
	$2 \mapsto 012121011011412121210110114121212101101141212121011011012121311011011$
	$3 \mapsto 0121210110114121212101101141212121011011412121311011011$
	$4 \mapsto 0121213110114121212101101141212121011011012121311011011$

Map	F_3
	$\ddot{\omega}_3 : \mathcal{A} \to \mathcal{A}^*$
	$0 \mapsto 010234$
	$1 \mapsto 010231013310134$
	$2 \mapsto 010231013310234$
	$3 \mapsto 0102340133101331 0234$
	$4 \mapsto 010234013310134$

Table 7.6: Proper substitutions over $\mathcal{A} = \{0, 1, 2, 3, 4\}$.

Map	$\mathcal{C}\ (= F_0)$	F_1	F_2	F_3
Matrix	$M_{\ddot{\omega}_0} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 3 \end{bmatrix}$	$M_{\ddot{\omega}_1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 9 & 9 & 9 \\ 3 & 1 & 5 & 4 & 4 \\ 3 & 1 & 4 & 5 & 4 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$	$M_{\ddot{\omega}_2} = \begin{bmatrix} 8 & 3 & 12 & 9 & 9 \\ 32 & 9 & 40 & 32 & 32 \\ 10 & 2 & 13 & 10 & 10 \\ 2 & 1 & 1 & 1 & 2 \\ 3 & 0 & 3 & 3 & 2 \end{bmatrix}$	$M_{\ddot{\omega}_3} = \begin{bmatrix} 2 & 4 & 4 & 5 & 4 \\ 1 & 5 & 4 & 5 & 4 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 4 & 4 & 6 & 4 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix}$
det; rank	1; 5	1; 5	1; 5	1; 5
e.value	$\mu^4 = \frac{1}{2}(7+3\sqrt{5})$	$7+4\sqrt{3}$	$17+12\sqrt{2}$	$7+4\sqrt{3}$
e.vector	$\left[\frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu^2}, \frac{1}{\mu}, 1\right]$	$\left[\frac{3-\sqrt{3}}{2}, \frac{^-1+\sqrt{3}}{2}, 1, 1, 1\right]$	$\left[1, \frac{^-1+2\sqrt{2}}{7}, \frac{6+2\sqrt{2}}{7}, 1, 1\right]$	$\left[\frac{^-1+\sqrt{3}}{2}, 1, 1, \frac{1+\sqrt{3}}{2}, 1\right]$

Characteristic polynomial	
	$\chi_{M_{\ddot{\omega}_0}} = 1 - 7x + x^2 - x^3 + 7x^4 - x^5$
	$\chi_{M_{\ddot{\omega}_1}} = 1 - 17x + 46x^2 - 46x^3 + 17x^4 - x^5$
	$\chi_{M_{\ddot{\omega}_2}} = 1 - 33x - 34x^2 + 34x^3 + 33x^4 - x^5$
	$\chi_{M_{\ddot{\omega}_3}} = 1 - 17x + 46x^2 - 46x^3 + 17x^4 - x^5$

Table 7.7: Matrix parameters of the proper substitutions in Table 7.6.

7.3 The general case

Given a map F_M from Definition 7.1 let a DA diffeomorphism be defined by

$$\mathfrak{f} : \mathbb{T}^2 \to \mathbb{T}^2, \ \mathfrak{f} = \phi^\tau \circ F_M^n, \tag{7.7}$$

such that for a fixed $\tau > 0$, $e^\tau \lambda_s^n > 1$, $n \in \mathbb{N}$, where λ_s^n is a stable eigenvalue and $\forall\, \mathbf{b}_j \in B$, $0 \leq j \leq 3$, $F_M^n(\mathbf{b}_j) = \mathbf{b}_j$.

Lemma 7.20. *Any DA map $\mathfrak{f}$ yields a toral attractor $\mathfrak{A}_\alpha \in \mathfrak{F}$.*

Proof. Without loss of generality and dropping the subscript M, let F_i be a hyperbolic toral map which satisfies Definition 7.1 then its matrix $M \in \bar{M}_i$ for some $0 \leq i \leq 4$, say $i = k$. Since $\bar{M}_0^3 = \bar{M}_1^2 = \bar{M}_2^2 = \bar{M}_3^2 = \bar{M}_4^3 = \bar{I}$, $F_k^m(\mathbf{b}_j) = \mathbf{b}_j$, $0 \leq j \leq 3$, for an appropriate iterate $m \in \mathbb{N}$. By definition F_k^m has an eigenvector $\mathbf{v}^u = \begin{bmatrix} 1 \\ \alpha_k \end{bmatrix}$ for which α_k is a quadratic irrational. Now consider the DA map $\mathfrak{f}_k = \phi^\tau \circ F_k^m$ then similar to the process adopted in §5.1.2, an attractor with 4 complementary domains arises thus: $\forall\, 0 \leq j \leq 3$ set a source $\mathbf{p}_{0_j}$ to correspond to a fixed point $\mathbf{b}_j$ of F_k^m then let $\mathbf{p}_{0_j} \in V_j \Rightarrow V_j \subset W^u(\mathbf{p}_{0_j})$ which results in the union of four repelling sets $\bigcup_{n\in\mathbb{N}} \mathfrak{f}_k^n(\dot{\bigcup}_{j=0}^3 V_j)$. Let $N := \mathbb{T}^2 \backslash \dot{\bigcup}_{j=0}^3 V_j$ define an attracting region for $\mathfrak{f}_k$ then $\mathfrak{A}_{\alpha k} := \bigcap_{n\in\mathbb{N}} \mathfrak{f}_k^n(N) \in \mathfrak{F}$. Since $\mathfrak{f}_k$ is defined by a non-trivial matrix $M \in \bar{M}_k$ where $\dot{\bigcup}_{k=0}^4 \bar{M}_k \dot{\cup} \bar{M}_5$ exhausts all possible matrix types the lemma holds. □

Remark 7.21. *A DA map may also be derived from a hyperbolic toral automorphism with a periodic orbit(s) (see Def. 5.4).*

Definition 7.22. *Let $(\ddot{\mathcal{M}}, \ddot{\vartheta})$ be a* presentation *of a solenoid $\ddot{\Sigma} = \varprojlim(\ddot{\mathcal{M}}, \ddot{\vartheta})$ with shift map $\ddot{\sigma} : \ddot{\Sigma} \to \ddot{\Sigma}$, $\ddot{\sigma}((x_i)) = (x_{i+1})$, $\forall\, i \in \mathbb{N}$.*

Definition 7.23. *Let $\ddot{\mathcal{W}}_c$ be a* suspension *of the return map of a flow ϕ^t on $\ddot{\Sigma}$ restricted to* 5 *clopen subsets, derived from $\ddot{\mathcal{M}}$, of a Cantor set cross-section of the flow.*

Definition 7.24. *Let $(\ddot{K}, r)$ be an* elementary presentation *of a solenoid $\ddot{\Omega} = \varprojlim(\ddot{K}, r)$ with shift map $\ddot{\omega} : \ddot{\Omega} \to \ddot{\Omega}$, $\ddot{\omega}((x_i)) = (x_{i+1})$, $\forall\, i \in \mathbb{N}$.*

Similar to Remark 6.29, in the general case we propose the following.

Proposition 7.25. *The suspension $\ddot{\mathcal{W}}_c$ and the solenoids $\ddot{\Sigma}$ and $\ddot{\Omega}$ are mutually homeomorphic.*

Proof. By invoking a general argument of Proposition 6.24, we deduce that $\ddot{\Sigma}$ is homeomorphic to $\ddot{\mathcal{W}}_c$ and again by Proposition 6.28 we deduce that $\ddot{\Sigma}$ is homeomorphic to $\ddot{\Omega}$. Thus $\ddot{\mathcal{W}}_c \cong \ddot{\Sigma} \cong \ddot{\Omega}$. $\square$

Recall $\mathbb{D}_\alpha$ of Definition 3.9 on page 56.

Definition 7.26. *Let a* Denjoy continuum *$\mathbb{D}_\alpha^S$ be one which is associated to a set $S = \{s_i \,|\, i \in \mathbb{N}\}$ of points in the disjoint trajectories of a linear flow* (3.1) *which has i sources $s_i \in \mathbb{T}^2 \setminus \mathbb{D}_\alpha^S$. Let the* characteristic *of $\mathbb{D}_\alpha^S$ be denoted $\mathring{\chi}(\mathbb{D}_\alpha^S) = \mathrm{card}(S)$.*

In particular $\mathbb{D}_\alpha^B$ is associated to the set $B \subset \mathbb{T}^2$ defined on page 124 and α is a quadratic irrational. Then $\mathring{\chi}(\mathbb{D}_\alpha^B) = 4$.

Proposition 7.27. *Fix α then the spaces are homeomorphic $\mathfrak{A}_\alpha \cong \mathbb{D}_\alpha^B \cong \ddot{\Sigma}$.*

Proof. Recall the attractor Λ derived from $\mathcal{C}$. The slope $\alpha = \mu^{-1}$. Since the DA attractor Λ is homeomorphic to $\mathbb{D}_\alpha$ it follows that $\mathfrak{A}_{\alpha 0} = \bigcap_{n\in\mathbb{N}} f_0^n(N)$ is homeomorphic to $\mathbb{D}_\alpha^B$. By construction, the 5 clopen subsets of $\ddot{\mathcal{W}}_c$ induce 4 complementary domains so $\ddot{\mathcal{W}}_c$ and $\mathbb{D}_\alpha^B$ are homeomorphic. Now $\ddot{\mathcal{W}}_c$ is homeomorphic to $\ddot{\Sigma}$ (Prop.7.25). Thus the spaces are mutually homeomorphic. $\square$

Definition 7.28. *Let* $\ddot{\omega} : \mathcal{A} \to \mathcal{A}^*$ *be a* proper substitution *over* $\mathcal{A}$ *with* $\mathrm{card}(\mathcal{A}) = 5$. *Let* $\ddot{\omega}$ *be derived from a rose* $\ddot{K}$ *with a set of petals* $\mathcal{K}$ *and* $\mathrm{card}(\mathcal{K}) = 5$. *For all* $i \in \mathcal{A}$, $k_i \in \mathcal{K}$, *let* $\ddot{\omega}(i) \in \mathcal{A}^*$ *take the same pattern of letters as* $r^2(k_i)$ *of the rose map* $r^2 : \ddot{K} \to \ddot{K}$.

Definition 7.29. *With* α *fixed by the toral map* F, *let* $\mathcal{T}_{\alpha(\ddot{\omega})}$ *be a one-dimensional substitution tiling space derived from a proper substitution* $\ddot{\omega}$.

Let an inflation and substitution homeomorphism be $F_{\alpha(\ddot{\omega})} : \mathcal{T}_{\alpha(\ddot{\omega})} \to \mathcal{T}_{\alpha(\ddot{\omega})}$ and let a continuous surjection be $\ddot{p} : \mathcal{T}_{\alpha(\ddot{\omega})} \to \ddot{K}$. Define each map similarly to (1.13) and (6.1) respectively.

Remark 7.30. *Deduce from Remark 6.17: when a rose map, in our case* r^2, *is associated to a proper substitution* $\ddot{\omega}$ *and the tiling space* $\mathcal{T}_{\alpha(\ddot{\omega})}$ *is derived from* $\ddot{\omega}$, *then* $\mathcal{T}_{\alpha(\ddot{\omega})}$ *is homeomorphic to* $\ddot{\Omega} = \varprojlim(\ddot{K}, r)$.

Lemma 7.31. *A one-dimensional substitution tiling space* $\mathcal{T}_{\alpha(\ddot{\omega})}$ *is homeomorphic to an orientable hyperbolic toral attractor* $\mathfrak{A}_\alpha$ *where* α *is equal in both spaces.*

Proof. We know that $\ddot{\Sigma} \cong \ddot{\Omega}$ (Prop. 7.25), $\ddot{\Omega} \cong \mathcal{T}_{\alpha(\ddot{\omega})}$ (Rem. 7.30) and $\ddot{\Sigma} \cong \mathfrak{A}_\alpha$ (Prop. 7.27). Thus $\mathcal{T}_{\alpha(\ddot{\omega})} \cong \mathfrak{A}_\alpha$. □

7.3.1 An algorithm

We give here the 'bare bones' of our method for constructing directly on the torus an attractor with four complementary domains and its homeomorphic tiling space. Because of its mechanical nature, we call it an **algorithm**:

Take a hyperbolic matrix $M \in GL(2, \mathbb{Z})$;

Form a Markov partition $\mathcal{P}$ then a secondary partition $\ddot{\mathcal{P}}$;

Form a branched 1-manifold $\ddot{\mathcal{M}}$ - a 2θ-space;

Derive a self-map $\ddot{\vartheta} : \ddot{\mathcal{M}} \to \ddot{\mathcal{M}}$ using $M : \mathbb{R}^2 \to \mathbb{R}^2$;

The solenoid $\ddot{\Sigma} = \varprojlim(\ddot{\mathcal{M}}, \ddot{\vartheta})$ represents an attractor $\mathfrak{A}_\alpha$.

Choose an origin in $\ddot{\mathcal{M}}$;

Iterate $\ddot{\vartheta}$ and record the 5 return words;

Form an elementary branched 1-manifold $\ddot{K}$ - a rose of 5 petals;

Derive a rose map $r : \ddot{K} \to \ddot{K}$ using $\ddot{\vartheta}$(word sequence);

The solenoid $\ddot{\Omega} = \varprojlim(\ddot{K}, r)$;

Derive a proper substitution $\ddot{\omega} : \mathcal{A} \to \mathcal{A}^*$ using r^2;

Then $\ddot{\Omega}$ is homeomorphic to a tiling space $\mathcal{T}_{\alpha(\ddot{\omega})}$ which is homeomorphic to $\mathfrak{A}_\alpha$ (Lemma 7.31).

Note that by restricting the algorithm to a Markov partition $\mathcal{P}$ with appropriate follow-on we may construct a toral attractor with one complementary domain and its homeomorphic tiling space.

Chapter 8

Classification of spaces

This chapter holds the main results of the thesis. We classify attractors and tiling spaces up to homeomorphism, Theorems 8.4 and 8.19 respectively, and establish the class group of these spaces to be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, Corollaries 8.7 and 8.22 respectively. Furthermore, the structures of the attractors and the tiling spaces are shown to be isomorphic to the permutation groups A_4 or D_4 and to the symmetry group of a square or the rotational symmetry group of a regular tetrahedron $S_r(\mathscr{T})$.

8.1 Classifying attractors

8.1.1 Homeomorphic attractors

Definition 8.1. *[27] If x and y are two numbers such that $x = \dfrac{ay+b}{cy+d}$, where a, b, c, d are integers such that $ad - bc = \pm 1$, then x is said to be* equivalent *to y, (which we write as $x \equiv y$).*

The classification of attractors in Theorem 8.4 appeals to the equivalence of quadratic surds by way of their continued fraction expansion. So consider the *continued fraction expansion* of a real number $x \in (0,1)$ expressed as

$$x = a_0 + 1/(a_1 + 1/(a_2 + 1/(\cdots + 1/(a_{k-1} + 1/x_k))))$$

for each k, where $a_0, \ldots, a_{k-1}$ are integers but x_k is not. For irrational x, the unique expansion is infinite and converges to x as $k \to \infty$. We may also express x in terms of its sequence of *partial quotients*, $x = [a_0, a_1, \ldots]$. Two irrational numbers $x = [a_0, a_1, \ldots, a_i, a_{i+1}, \ldots]$ and $y = [b_0, b_1, \ldots, b_j, b_{j+1}, \ldots]$ are equivalent (Def. 8.1) if and only if their *tails* $[a_i, a_{i+1}, \ldots] = [b_j, b_{j+1}, \ldots]$ agree (Theorem 175 in [27]). . A *periodic continued fraction* is an infinite continued fraction in which $\forall\, i \geq j_0$, $a_i = a_{i+k}$ for a fixed positive k. When α is a quadratic surd, α exhibits a periodic continued fraction expansion (Theorem 177 in [27]).

Definition 8.2. *[5] Let $f, g : X \to Y$ be maps. Then f is* homotopic *to g if there exists a map $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all points $x \in X$.*

Remark 8.3. *Using a method involving inverse limits, the authors of [10] classify Denjoy continua by stating in Theorem 4.6 that for irrationals α and β, $\mathbb{D}_\alpha$ and $\mathbb{D}_\beta$ are homeomorphic if and only if α and β are equivalent. Our focus is on Denjoy continua with* 4 *'blown-up' orbits and so the outline below is more suited to our classification of attractors with four complementary domains.*

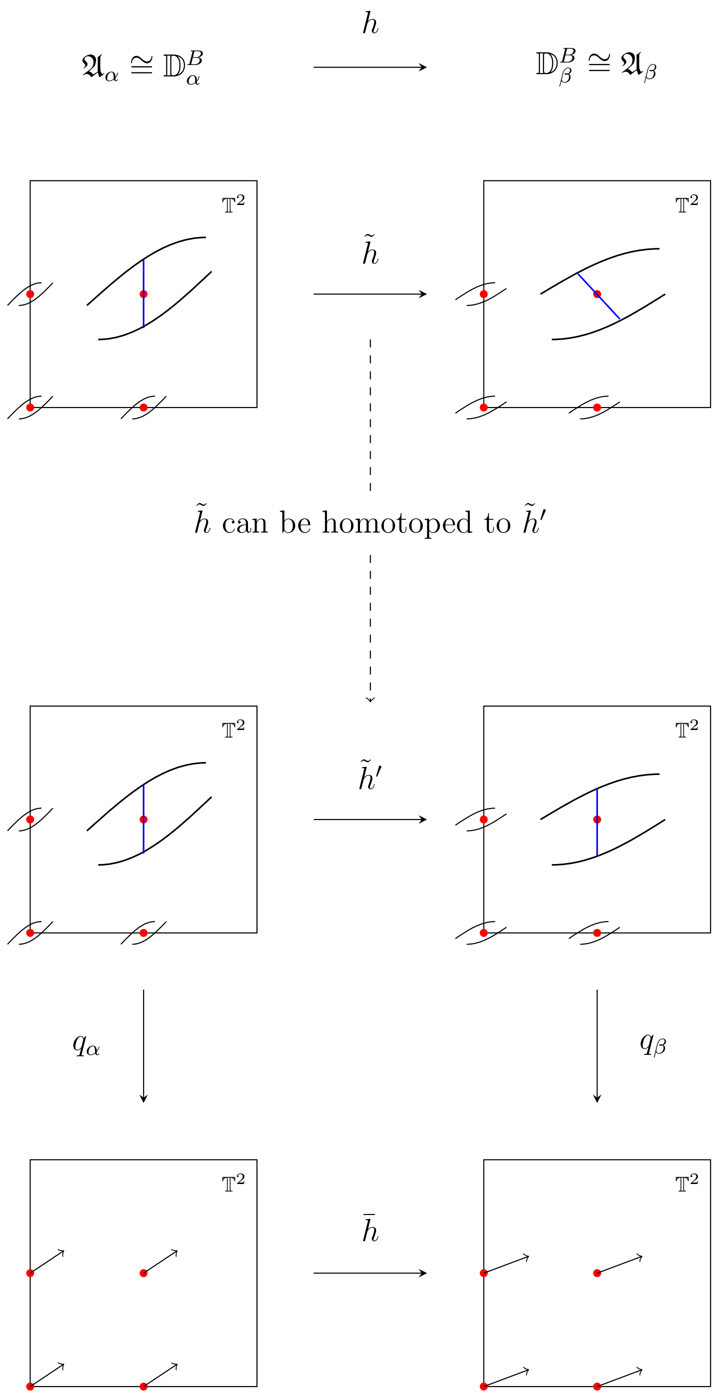
Recall the quotient map q (2.4) in the Denjoy construction and the embedding $\hat{H}$ (3.7) of the suspension $S_{\alpha_c}(D_\alpha)$ of the Denjoy map. Now suspend q, $\hat{H}' : S_{\alpha_c}(q) \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, from which we realise a quotient map $\hat{q} : S_{\alpha_c}(D_\alpha) \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. We know that $S_{\alpha_c}(D_\alpha)$ is homeomorphic to the torus via the homeomorphism $\hat{H}$. Then we obtain a quotient map $q_\alpha := \hat{q} \circ \hat{H}^{-1}$ where $\hat{q}$ is defined in (8.1),

$$\begin{tikzcd} S_{\alpha_c}(D_\alpha) \arrow[r, "\hat{q}"] \arrow[d, "\hat{H}"'] & \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \arrow[d, "id"] \\ \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \arrow[r, "q_\alpha"] & \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \end{tikzcd}$$

$$\hat{q} : S_{\alpha_c}(D_\alpha) \to \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z},\ \hat{q}\left(\widetilde{(x_1, x_2)}\right) = \big(q(x_1), x_2\big). \tag{8.1}$$

Explanatory preamble to Theorem 8.4. Let irrationals α, β parameterise Denjoy continua $\mathbb{D}_\alpha, \mathbb{D}_\beta$ each of which has a single 'blown-up' orbit. We know from work done by [24] that a map $h_1 : \mathbb{D}_\alpha \to \mathbb{D}_\beta$ can be extended to a map on the torus $\tilde{h}_1 : \mathbb{T}^2 \to \mathbb{T}^2$. In order to equip $\mathbb{T}^2$ with the map (8.1) which is well-defined we homotope $\tilde{h}_1$ to a map $\tilde{h}_1' : \mathbb{T}^2 \to \mathbb{T}^2$ (this notion is conveyed in Figure 8.1 with respect to the homotopy of $\tilde{h}$ to $\tilde{h}'$). This produces commuting maps $\bar{h}_1' \circ q_\alpha(\mathbf{x}) = q_\beta \circ \tilde{h}_1'(\mathbf{x})$, $\forall\, \mathbf{x} \in \mathbb{T}^2$.

Now consider Figure 8.1 relating to Denjoy continua $\mathbb{D}^B_\alpha$ and $\mathbb{D}^B_\beta$. For a precise argument on Denjoy continua with one or finitely many orbits see [15], in particular the homeomorphism of Theorem 6.6, $\Sigma_I(\mathbb{D}_\alpha) \cong \Sigma_I(\mathbb{D}_\beta)$, where $\Sigma_I(\mathbb{D}_\alpha)$ denotes the minimal set on the inverse limit space with bonding maps defined by the identity matrix I. Now $\mathfrak{A}_\alpha \cong \mathbb{D}^B_\alpha$ so given a homeomorphism $h : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$, $\tilde{h} : \mathbb{T}^2 \to \mathbb{T}^2$ can be homotoped to $\tilde{h}' : \mathbb{T}^2 \to \mathbb{T}^2$ leading to commuting maps $\bar{h} \circ q_\alpha = q_\beta \circ \tilde{h}'$. That is, $\bar{h}$ maps an *α-foliation* $\mathcal{F}_\alpha$ determined by the decomposition $\left\{\overline{\begin{bmatrix}1\\ \alpha\end{bmatrix} t + \mathbf{x} \,|\, t \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^2}\right\}$ to the *β-foliation* $\mathcal{F}_\beta$ determined by $\left\{\overline{\begin{bmatrix}1\\ \beta\end{bmatrix} t + \mathbf{x} \,|\, t \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^2}\right\}$. Then by Theorem 2.12 in [30], $\bar{h}$ is homotopic to a toral automorphism, say $F_A : \mathbb{T}^2 \to \mathbb{T}^2$ with defining matrix $A \in GL(2, \mathbb{Z})$, where α and β are equivalent such that A maps an α-linear



$\bar{h}$ is homotopic to an automorphism of $\mathbb{T}^2$

Figure 8.1: Supporting diagrams for preamble.

foliation to the β-linear foliation. Conversely, if α and β are equivalent irrationals calculated from matrix A of F_A then the α- and β-linear foliations admit toral flows which are topologically equivalent. A reverse argument to that just outlined yields homeomorphic Denjoy continua $\mathbb{D}_\alpha^B \cong \mathbb{D}_\beta^B$, hence homeomorphic attractors $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta$, if the irrational slopes α and β are equivalent. More formally, consider the next theorem.

Theorem 8.4. *(i) Let two orientable hyperbolic attractors $\mathfrak{A}_\alpha, \mathfrak{A}_\beta \in \mathfrak{F}$ then $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are homeomorphic if and only if α and β are equivalent.*

(ii) Let two non-orientable Plykin attractors $P\mathfrak{A}_\alpha, P\mathfrak{A}_\beta \in P\mathfrak{F}$ then $P\mathfrak{A}_\alpha$ and $P\mathfrak{A}_\beta$ are homeomorphic if and only if α and β are equivalent.

Proof. (i) Taking the argument described in the preamble above says that if $\mathbb{D}_\alpha^B \cong \mathbb{D}_\beta^B \Rightarrow \alpha \equiv \beta$. Let $\mathfrak{A}_\alpha, \mathfrak{A}_\beta \in \mathfrak{F}$. We know that $\mathfrak{A}_\alpha \cong \mathbb{D}_\alpha^B$ and $\mathfrak{A}_\beta \cong \mathbb{D}_\beta^B$ which means that when $\mathbb{D}_\alpha^B$ and $\mathbb{D}_\beta^B$ are homeomorphic then so too are $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$. It follows that homeomorphic attractors $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta \Rightarrow \alpha \equiv \beta$. Conversely, if $\alpha \equiv \beta$ then by Theorem 2.12 in [30] $\exists\, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$ (Def. 8.1) such that $F_A : \mathbb{T}^2 \to \mathbb{T}^2$ maps the α-linear foliation to the β-linear foliation. Then F_A will permute the elements of B (see Lemma 7.7). In so doing, F_A induces a topological equivalence between the toral flows associated to the α- and β-linear foliations which endows a homeomorphism between the attractors. That is $\alpha \equiv \beta \Rightarrow \mathfrak{A}_\alpha \cong \mathfrak{A}_\beta$. Hence $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta \Leftrightarrow \alpha \equiv \beta$.

(ii) Let $P\mathfrak{A}_\alpha, P\mathfrak{A}_\beta \in P\mathfrak{F}$ be one-dimensional non-orientable expanding attractors of diffeomorphisms h_1 and h_2 respectively on a 2-dimensional manifold M. Then by Plykin's Theorem 2.2 in [45], the lifts $\tilde{h}_1$ and $\tilde{h}_2$ induce orientable expanding attractors $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ respectively on a double cover $\tilde{M}$, in our case $\mathbb{T}^2$. Now suppose that $P\mathfrak{A}_\alpha$ and $P\mathfrak{A}_\beta$ are homeomorphic then the lifted attractors $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are homeomorphic and thus by Theorem 8.4(i), $\alpha \equiv \beta$.

Conversely, if $\alpha \equiv \beta$ consider again the map $F_A : \mathbb{T}^2 \to \mathbb{T}^2$ defined by a matrix A which maps the α-foliation to the β-foliation (see preamble). As discussed in (i) above, F_A permutes the points of set B inducing a topological equivalence of flows. Recall the quotient map Π of Definition 5.11. Let $\mathbf{x} \in \mathbb{T}^2$, then $F_A(\pm\mathbf{x}) = \pm F_A(\mathbf{x})$ and $\mathfrak{g} = \Pi \circ F_A : \mathbb{T}^2 \to S^2$ is constant on $\Pi^{-1}(\{y\})$, $y \in S^2$, such that $\mathfrak{g}$ induces a self-homeomorphism $\mathfrak{h} : S^2 \to S^2$ (see Corollary 22.3 in [38]). So $\mathfrak{h} \circ \Pi = \Pi \circ F_A$.

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "F_A"] \arrow[d, "\Pi"'] \arrow[dr, "\mathfrak{g}"] & \mathbb{T}^2 \arrow[d, "\Pi"] \\ S^2 \arrow[r, "\mathfrak{h}"] & S^2 \end{tikzcd}$$

It follows that $\alpha \equiv \beta \Rightarrow \mathfrak{A}_\alpha \cong \mathfrak{A}_\beta \subseteq \mathbb{T}^2$ projects to $P\mathfrak{A}_\alpha \cong P\mathfrak{A}_\beta \subseteq S^2$. $\square$

8.1.2 Class groups

Our Theorem 7.6 on page 124 and its Corollary 8.7 below appeal to a result in [24] that the class group of a Denjoy continuum $\mathbb{D}_\alpha$ is either $\mathbb{Z}_2$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. In our case the class group is $\mathbb{Z} \oplus \mathbb{Z}_2$ which applies to $\mathbb{D}_\alpha^B$ and $\mathfrak{A}_\alpha$.

Definition 8.5. *If $h_i : X \to Y$ $(i = (0,1))$ are homeomorphisms, an* isotopy *joining h_1 to h_2 is a continuous map $H : X \times [0,1] \to Y$ such that $H_0 = h_0, H_1 = h_1$ and $\forall\, t \in [0,1]$, $h_t : X \to Y$ is a homeomorphism where $h_t(x) := H(x,t)$.*

The four pairs of asymptotic path-components of $\mathfrak{A}_\alpha$ can be permuted and it is these induced permutations which determine the isotopy classes of the self-homeomorphisms of $\mathfrak{A}_\alpha$. In §8.1.3 we calculate the permutation groups which arise and in particular Lemma 8.11 acknowledges the role of the matrix type $\bar{M} \in \mathbb{M}$ in the permutations of set B.

Definition 8.6. *Let $\mathfrak{C}_1, \mathfrak{C}_2 \subseteq \mathfrak{A}_\alpha$ be the pair of asymptotic path-components derived from the point $\mathbf{b}_0 \in B \subset \mathbb{T}^2$. Let the restricted class group of all self-homeomorphisms be $\mathfrak{H}_0(\mathfrak{A}_\alpha) = \{\mathfrak{h} : \mathfrak{A}_\alpha \to \mathfrak{A}_\alpha \,|\, \mathfrak{h}(\mathfrak{C}_i) = \mathfrak{C}_i,\ i = 1,2\}$. For $\mathfrak{h} \in \mathfrak{H}_0$, let $[\mathfrak{h}]$ denote the isotopy class of $\mathfrak{h}$, then under composition of maps $[\mathfrak{h}_1] \circ [\mathfrak{h}_2] = [\mathfrak{h}_1 \circ \mathfrak{h}_2]$, let the restricted subgroup be $\mathfrak{K}_0(\mathfrak{A}_\alpha) = \{[\mathfrak{h}] \,|\, \mathfrak{h} \in \mathfrak{H}_0\}$.*

It is valid to restrict considerations to $\mathfrak{K}_0(\mathfrak{A}_\alpha)$ since the subgroups $\mathfrak{K}_j(\mathfrak{A}_\alpha)$, $0 \leq j \leq 3$, associated to points $\mathbf{b}_j$, $0 \leq j \leq 3$, respectively are isomorphic.

Corollary 8.7. *The subgroup $\mathfrak{K}_0(\mathfrak{A}_\alpha)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Proof. Given $\mathfrak{h} : \mathfrak{A}_\alpha \to \mathfrak{A}_\alpha$, the map $\bar{\mathfrak{h}} : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to a toral automorphism defined by a matrix $M \in \mathbb{M}$. Recall the isomorphic groups $\mathscr{G} \cong \mathbb{Z} \oplus \mathbb{Z}_2$ (Theorem 7.6) where $M \in \mathscr{G}$. Then by injectivity, the matrix $M = \nu^{-1}(\lambda)$ is unique and belongs to some type $\bar{M}_i$, $0 \leq i \leq 5$, say $i = t$. It follows that for $k \in \mathbb{Z}$, M^k corresponds to $(k, 0) \in \mathbb{Z} \oplus \mathbb{Z}_2$. Also the involution $i : \mathbb{T}^2 \to \mathbb{T}^2$ does not affect the matrix type so that for $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $-IM = -MI \in \bar{M}_t$ then $-M^k = -IM^k$ which corresponds to $(k, 1) \in \mathbb{Z} \oplus \mathbb{Z}_2$. Hence $\mathfrak{K}_0(\mathfrak{A}_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. □

The standard algebraic groups which occur in §8.1.3, 8.1.4 are found in [31].

8.1.3 Permutation subgroups of S_4

Theorem 8.8. *The group $\mathbb{M}$ is isomorphic to the symmetric group S_3.*

Proof. Using the index set of the points $\{\mathbf{b}_{1 \leq j \leq 3}\} \subset B$ let the permutations on $\{1, 2, 3\}$ form the symmetric group $S_3 = \{(123), (23), (13), (12), (132), e\}$. Neither $\mathbb{M}$ nor S_3 is Abelian[1], both have order 6 and each group has 4 self-inverses. Thus the groups are isomorphic $\mathbb{M} \cong S_3$. □

[1]Niels Henrik Abel (1802 - 1829) Norwegian mathematician.

Let the isomorphism be defined by

$$\iota_M : \mathbb{M} \to S_3, \quad \begin{cases} \bar{M}_0 \to (123) \\ \bar{M}_1 \to (23) \\ \bar{M}_2 \to (13) \\ \bar{M}_3 \to (12) \\ \bar{M}_4 \to (132) \\ \bar{M}_5 = \bar{I} \to e. \end{cases} \tag{8.2}$$

Recall that (abc) is the cycle $a \to b \to c \to a$ and (ab) is $a \to b \to a$ which leaves (c) fixed. These cycles are deemed *even* and *odd* permutations respectively. Calculate the conjugate $\theta(12)\theta^{-1}$ for each $\theta \in S_3$. For example $(123)(12)(132) = (123)(13) = (23)$ then the three conjugacy classes in S_3 are $\{(23), (13), (12)\}$, $\{(123), (132)\}$ and $\{e\}$ as expected which are equivalent to those of $\mathbb{M}$ namely $\{\bar{M}_1, \bar{M}_2, \bar{M}_3\}$, $\{\bar{M}_0, \bar{M}_4\}$ and $\{\bar{M}_5\}$. Denote a generator by $\langle\cdot\rangle$ then the proper cyclic subgroups of $\mathbb{M}$ are $\langle\bar{M}_0\rangle = \langle\bar{M}_4\rangle = \{\bar{M}_0, \bar{M}_4, \bar{M}_5\}$; $\langle\bar{M}_1\rangle = \{\bar{M}_1, \bar{M}_5\}$; $\langle\bar{M}_2\rangle = \{\bar{M}_2, \bar{M}_5\}$ and $\langle\bar{M}_3\rangle = \{\bar{M}_3, \bar{M}_5\}$.

Definition 8.9. *Let $\mathbb{K}$ be a translation group acting on the set B such that $\forall\, k \in \mathbb{K}$ and $\forall\, b \in B$, $k + b \in B$, addition modulo 1.*

Now assign permutations on the index set of all 4 points of B, namely $\{0, 1, 2, 3\}$, according to the group action of $\mathbb{K}$ so that $\mathbf{b}_0 = e$, $\mathbf{b}_1 = (01)(23)$, $\mathbf{b}_2 = (02)(13)$ and $\mathbf{b}_3 = (03)(12)$. Then $\mathbb{K} := \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong V_4$, the *Klein*[2] *4-group*. Now compose the elements of S_3 with $\mathbb{K}$ to give the 24 permutations of the symmetric group S_4. We want to distinguish which matrix type $\bar{M} \in \mathbb{M}$ yields which subgroup(s) of S_4 so with slight abuse we show the elements of $\mathbb{M}$ rather than those of S_3 in our calculations, knowing that $S_3 \cong \mathbb{M}$. We suppress the binary operation $\circ$

[2] Felix Klein (1849 - 1925) German mathematician.

and use juxtaposition $\bar{M}_i \mathbf{b}_j$, $0 \leq i \leq 5$, $0 \leq j \leq 3$, for composition of elements to which we assign the labels $\{p_0, \dots, p_{23}\}$. In Table 8.1 we block the elements of S_4 according to their cyclic order. In Table 8.2 we summarise the subgroups of S_4 which occur; A_4 denotes the *alternating group* of even permutations on four elements and D_4 denotes the *dihedral group* of the symmetries of a square.

Definition 8.10. *Define the set* $A = \bigcup_{i=0,4}(\bar{M}_i \cup \mathbb{K})$ *and* $D = \bigcup_{i=1,2,3}(\bar{M}_i \cup \mathbb{K})$. *Denote the* permutation subgroup *of* S_4 *arising from set* A *by* $\mathbb{B}$ *and those arising from set* D *by the* permutation subgroups $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$.

Lemma 8.11. *The type of matrix* $\bar{M}_i \subset \mathbb{M}$, $0 \leq i \leq 5$, *determines the permutations of the points in* B. *In particular set* A *leads to the subgroup* $\mathbb{B}$ *isomorphic to* A_4 *while set* D *leads to isomorphic subgroups* $\mathbb{B}_i \cong D_4$, $i = 1, 2, 3$.

Proof. The alternating group A_4 is the smallest subgroup of S_4 which owns all the elements arising from set A. Since $\bar{M}_0$ and $\bar{M}_4$ are inverses, $\langle p_{10}, p_4, p_5 \rangle = \langle p_{13}, p_4, p_5 \rangle = \mathbb{B} \cong A_4$. The smallest subgroups which own elements arising from set D are generated by $\langle p_1, p_4, p_5 \rangle = \mathbb{B}_1$, $\langle p_2, p_4, p_5 \rangle = \mathbb{B}_2$ and $\langle p_3, p_4, p_5 \rangle = \mathbb{B}_3$. Then for $i = 1, 2, 3$ each group $\mathbb{B}_i \cong D_4$ by the isomorphism ι_i to be given in (8.4) on page 158. Since A_4 is not isomorphic to D_4 the permutations of $\{\bar{M}_0, \bar{M}_4\} \cup \mathbb{K}$ are not symmetrically congruent to those of $\{\bar{M}_i\} \cup \mathbb{K}$ for any $i = 1, 2, 3$. Thus the type of matrix $M \in \mathbb{M}$ determines the symmetry of the permutations on $\{\mathbf{b}_{0 \leq j \leq 3}\} \subset \mathbb{T}^2$ according to the stated classification. $\square$

identity: $\bar{M}_5\mathbf{b}_0 = \bar{I}e = p_0 = e\bar{I} = \mathbf{b}_0\bar{M}_5$

Cyclic elements of order 2		
$p_1 : \bar{M}_1 e = (23) = e\bar{M}_1$	$p_4 : \bar{I}\,\mathbf{b}_1 = (01)(23) = \mathbf{b}_1\bar{I}$	$p_7 : \bar{M}_1\mathbf{b}_1 = (01) = \mathbf{b}_1\bar{M}_1$
$p_2 : \bar{M}_2 e = (13) = e\bar{M}_2$	$p_5 : \bar{I}\,\mathbf{b}_2 = (02)(13) = \mathbf{b}_2\bar{I}$	$p_8 : \bar{M}_2\mathbf{b}_2 = (02) = \mathbf{b}_2\bar{M}_2$
$p_3 : \bar{M}_3 e = (12) = e\bar{M}_3$	$p_6 : \bar{I}\,\mathbf{b}_3 = (03)(12) = \mathbf{b}_3\bar{I}$	$p_9 : \bar{M}_3\mathbf{b}_3 = (03) = \mathbf{b}_3\bar{M}_3$

Cyclic elements of order 3		
$p_{10} : \bar{M}_0 e = (123) = e\bar{M}_0$	$p_{13} : \bar{M}_4 e = (132) = e\bar{M}_4$	
$p_{11} : \bar{M}_0\mathbf{b}_1 = (021) = \mathbf{b}_2\bar{M}_0$	$p_{14} : \bar{M}_0\mathbf{b}_2 = (032) = \mathbf{b}_3\bar{M}_0$	$p_{16} : \bar{M}_0\mathbf{b}_3 = (013) = \mathbf{b}_1\bar{M}_0$
$p_{12} : \bar{M}_4\mathbf{b}_1 = (031) = \mathbf{b}_3\bar{M}_4$	$p_{15} : \bar{M}_4\mathbf{b}_2 = (012) = \mathbf{b}_1\bar{M}_4$	$p_{17} : \bar{M}_4\mathbf{b}_3 = (023) = \mathbf{b}_2\bar{M}_4$

Cyclic elements of order 4	
$p_{18} : \bar{M}_1\mathbf{b}_2 = (0312) = \mathbf{b}_3\bar{M}_1$	$p_{21} : \bar{M}_1\mathbf{b}_3 = (0213) = \mathbf{b}_2\bar{M}_1$
$p_{19} : \bar{M}_2\mathbf{b}_1 = (0321) = \mathbf{b}_3\bar{M}_2$	$p_{22} : \bar{M}_2\mathbf{b}_3 = (0123) = \mathbf{b}_1\bar{M}_2$
$p_{20} : \bar{M}_3\mathbf{b}_1 = (0231) = \mathbf{b}_2\bar{M}_3$	$p_{23} : \bar{M}_3\mathbf{b}_2 = (0132) = \mathbf{b}_1\bar{M}_3$

Table 8.1: The composition of S_4.

Group	Generator	Subgroup	Order	Isomorphic Group
$\mathbb{M}$	$\langle p_1, p_2 \rangle$	$\{\bar{I}, p_1, p_2, p_3, p_{10}, p_{13}\}$	6	S_3
$(\mathbb{K}, +)$	$\langle p_4, p_5 \rangle$	$\{e, p_4, p_5, p_6\}$	4	V_4

Union	Generator	Subgroup	Order	Iso. Group
$\bar{M}_0 \cup \mathbb{K}$ $\bar{M}_4 \cup \mathbb{K}$	$\langle p_{10}, p_4, p_5 \rangle$ $\shortparallel$ $\langle p_{13}, p_4, p_5 \rangle$	$\mathbb{B}$ $= \{p_0, p_4, p_5, p_6, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}\}$	12	A_4
$\bar{M}_1 \cup \mathbb{K}$	$\langle p_1, p_4, p_5 \rangle$	$\mathbb{B}_1 = \{p_0, p_4, p_5, p_6, p_1, p_7, p_{18}, p_{21}\}$	8	D_4
$\bar{M}_2 \cup \mathbb{K}$	$\langle p_2, p_4, p_5 \rangle$	$\mathbb{B}_2 = \{p_0, p_4, p_5, p_6, p_2, p_8, p_{19}, p_{22}\}$	8	D_4
$\bar{M}_3 \cup \mathbb{K}$	$\langle p_3, p_4, p_5 \rangle$	$\mathbb{B}_3 = \{p_0, p_4, p_5, p_6, p_3, p_9, p_{20}, p_{23}\}$	8	D_4

Table 8.2: Subgroups of S_4.

8.1.4 Symmetry groups

Definition 8.12. *For all* $\mathbf{x} \in \mathbb{T}^2$ *let an* action $*$ *of the group* A_4 *or* D_4 *on* $\mathbb{T}^2$ *be given by* $p * \mathbf{x}$, $\forall p \in A_4$ *or* $p * \mathbf{x}$, $\forall p \in D_4$.

When A_4 acts on B the *orbit* of $\mathbf{b} \in B$ is $A_4(\mathbf{b}) = \{p * \mathbf{b} \in A_4 \,|\, p \in A_4\}$ and similarly the orbit $D_4(\mathbf{b}) = \{p * \mathbf{b} \in D_4 \,|\, p \in D_4\}$. Informally the number of orbits for a group action on a set measures the symmetry in the set, the fewer the orbits the greater the symmetry. So we may deduce that the attractors sourced from matrices belonging to set D show greater symmetry than those sourced from matrices of set A. This turns out to be the case.

Orientable attractors

Definition 8.13. *For* $\mathfrak{A}_\alpha \in \mathfrak{F}$, *let* $S(\mathfrak{A}_\alpha)$ *be the set of permutations of the asymptotic path-components emanating from set* B *which are induced by the self-homeomorphisms of* $\mathfrak{A}_\alpha$. *Then define* $S(\mathfrak{A}_\alpha)$ *to be the* symmetry group *under composition.*

Let $V_\triangle$ be the *set of vertices* of a *regular tetrahedron* $\mathscr{T} \subset \mathbb{R}^3$. For each $j = 0, \ldots, 3$ put $\mathbf{b}_j \in B$ in correspondence with a vertex $v_j \in V_\triangle$. Then let $S_r(\mathscr{T}) = \{e, r_{1.\pi}, r_{2.\pi}, r_{3.\pi}, r_{0.\frac{4\pi}{3}}, r_{0.\frac{2\pi}{3}}, r_{1.\frac{4\pi}{3}}, r_{1.\frac{2\pi}{3}}, r_{2.\frac{4\pi}{3}}, r_{2.\frac{2\pi}{3}}, r_{3.\frac{4\pi}{3}}, r_{3.\frac{2\pi}{3}}\}$ be the *set of rotation symmetries* of $\mathscr{T}$ where e is the identity, $r_{i.\pi}$ is a rotation through π about an axis from the midpoint of an edge $i = 1, 2, 3$ to its opposite edge of $\mathscr{T}$ and $r_{v.\frac{n\pi}{3}}$ is a rotation through $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ about an axis from a vertex $v_j \in V_\triangle$, $j = 0, \ldots, 3$, to the centre of the opposite face of $\mathscr{T}$.

Definition 8.14. *Let* $S_r(\mathscr{T})$ *be the* rotation symmetry group *of a regular tetrahedron* $\mathscr{T}$.

Recall the alternating group A_4 which is a group of even permutations on 4 symbols such that A_4 is orientation-preserving and known to be isomorphic to the group $S_r(\mathscr{T})$. Since the permutation subgroup $\mathbb{B}$ and A_4 are isomorphic, it follows that $\mathbb{B}$ is isomorphic to $S_r(\mathscr{T})$ via the map ι defined by

$$\iota : \mathbb{B} \to S_r(\mathscr{T}), \begin{cases} e \mapsto e, & p_{13} \mapsto r_{0.\frac{2\pi}{3}}, \quad p_{10} \mapsto r_{0.\frac{4\pi}{3}}, \\ p_4 \mapsto r_{1.\pi}, & p_{14} \mapsto r_{1.\frac{2\pi}{3}}, \quad p_{17} \mapsto r_{1.\frac{4\pi}{3}}, \\ p_5 \mapsto r_{2.\pi}, & p_{12} \mapsto r_{2.\frac{2\pi}{3}}, \quad p_{16} \mapsto r_{2.\frac{4\pi}{3}}, \\ p_6 \mapsto r_{3.\pi}, & p_{11} \mapsto r_{3.\frac{2\pi}{3}}, \quad p_{15} \mapsto r_{3.\frac{4\pi}{3}}. \end{cases} \tag{8.3}$$

Let $V_\Box$ be the *set of vertices* of a *square* $\mathscr{S} \subset \mathbb{R}^2$. For each $j = 0, \ldots, 3$ put $\mathbf{b}_j \in B$ in correspondence with a vertex $v_j \in V_\Box$. Then let the *set of symmetries* of $\mathscr{S}$ be $\{e, r_{\frac{\pi}{2}}, r_\pi, r_{\frac{3\pi}{2}}, q_0, q_{\frac{\pi}{4}}, q_{\frac{\pi}{2}}, q_{\frac{3\pi}{4}}\}$ where e is the identity, r_θ is a rotation through angle θ anticlockwise about the centre of $\mathscr{S}$ and q_ψ is a reflection in a diagonal through the centre of $\mathscr{S}$ at an angle ψ to the positive x-axis. Then D_4 is the symmetry group of the square $\mathscr{S}$.

Theorem 8.15. *The symmetry group $S(\mathfrak{A}_\alpha)$ is isomorphic to either the rotation symmetry group of a regular tetrahedron $S_r(\mathscr{T})$ or to the symmetry group of a square D_4. The criterion is the type of the source matrix M: if $M \in \bar{M}_i$, $i = 0, 4$, then $S(\mathfrak{A}_\alpha) \cong S_r(\mathscr{T})$; if $M \in \bar{M}_i$, $i = 1, 2, 3$, then $S(\mathfrak{A}_\alpha) \cong D_4$.*

Proof. Let M be the unique source matrix of an attractor $\mathfrak{A}_\alpha \in \mathfrak{F}$ which is the complement of the repelling set B. Now the points of B are put in correspondence with the vertex sets $V_\triangle \subset \mathscr{T}$ or $V_\Box \subset \mathscr{S}$ which are equipped with group actions $S_r(\mathscr{T})$ and D_4 respectively. This creates the following classification. If $M \in \bar{M}_i$, $i = 0, 4$, then $\bar{M}_i \subset A$ and $\mathbb{B} \cong S_r(\mathscr{T})$ by the map (8.3) whilst if $M \in \bar{M}_i$, $i = 1, 2, 3$, $\bar{M}_i \subset D$ and each $\mathbb{B}_i \cong D_4$ by the three isomorphisms given in (8.4) respectively. Thus $S(\mathfrak{A}_\alpha)$ is isomorphic to either $S_r(\mathscr{T})$ or to D_4 according to the matrix type of M. $\Box$

Define the isomorphisms $\iota_1, \iota_2, \iota_3$ to be given by

$$\iota_1 : \mathbb{B}_1 \to D_4, \begin{cases} e & \mapsto e, \\ p_4 & \mapsto q_0, \\ p_5 & \mapsto r_\pi, \\ p_6 & \mapsto q_{\frac{\pi}{2}}, \\ p_1 & \mapsto q_{\frac{\pi}{4}}, \\ p_7 & \mapsto q_{\frac{3\pi}{4}}, \\ p_{18} & \mapsto r_{\frac{3\pi}{2}}, \\ p_{21} & \mapsto r_{\frac{\pi}{2}}. \end{cases} \iota_2 : \mathbb{B}_2 \to D_4, \begin{cases} e & \mapsto e, \\ p_4 & \mapsto q_0, \\ p_5 & \mapsto r_\pi, \\ p_6 & \mapsto q_{\frac{\pi}{2}}, \\ p_2 & \mapsto q_{\frac{\pi}{4}}, \\ p_8 & \mapsto q_{\frac{3\pi}{4}}, \\ p_{19} & \mapsto r_{\frac{3\pi}{2}}, \\ p_{22} & \mapsto r_{\frac{\pi}{2}}. \end{cases} \iota_3 : \mathbb{B}_3 \to D_4, \begin{cases} e & \mapsto e, \\ p_4 & \mapsto q_0, \\ p_5 & \mapsto r_\pi, \\ p_6 & \mapsto q_{\frac{\pi}{2}}, \\ p_3 & \mapsto q_{\frac{\pi}{4}}, \\ p_9 & \mapsto q_{\frac{3\pi}{4}}, \\ p_{20} & \mapsto r_{\frac{3\pi}{2}}, \\ p_{23} & \mapsto r_{\frac{\pi}{2}}. \end{cases} \tag{8.4}$$

Non-orientable attractors

Definition 8.16. *Let $S(\mathscr{T})$ be the full set of symmetries of $\mathscr{T}$. Then define $S(\mathscr{T})$ to be the* symmetry group.

Definition 8.17. *For $P\mathfrak{A}_\alpha \in P\mathfrak{F}$, let $S(P\mathfrak{A}_\alpha)$ be the set of permutations of the asymptotic path-components emanating from set B which are induced by the self-homeomorphisms of $P\mathfrak{A}_\alpha$. Then define $S(P\mathfrak{A}_\alpha)$ to be the* symmetry group *under composition.*

Theorem 8.18. *The symmetry group $S(P\mathfrak{A}_\alpha)$ is isomorphic to a subgroup of $S(\mathscr{T})$. In particular, if for F_M, $M \in \bar{M}_0 \dot\cup \bar{M}_4$ then $S(P\mathfrak{A}_\alpha) \cong S_r(\mathscr{T})$ while if $M \in \bar{M}_i$, $i = 1, 2, 3$, then $S(P\mathfrak{A}_\alpha) \cong D_4$.*

Proof. The groups S_4 and $S(\mathscr{T})$ are isomorphic via the isomorphism (8.5) and we know by the classification of Lemma 8.11 which subgroups of S_4 arise from which matrix $M \in \bar{M} \subset \mathbb{M}$. So consider again the map $\mathfrak{g} = \Pi \circ F_M$ (see proof Theorem 8.4 (ii)) with $M \in \bar{M}_i$ for some $i = 0, \ldots, 4$. Now the induced map on the sphere preserves the corresponding transverse foliations on the torus which result from the linear transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$. We know that a matrix $M \in \bar{M}_i$, $i = 1, 2$, leads to the permutation subgroup $\mathbb{B} \cong A_4$ and a matrix $M \in \bar{M}_i$, $i = 1, 2, 3$, leads to the permutation subgroup $\mathbb{B}_i$, $i = 1, 2, 3$, respectively. Now $A_4 \cong S_r(\mathscr{T})$ so $\mathbb{B} \cong S_r(\mathscr{T}) \subset S(\mathscr{T})$ and $\forall\, i = 1, 2, 3$, $\mathbb{B}_i \cong D_4 \subset S(\mathscr{T})$. Thus any attractor $P\mathfrak{A}_\alpha$ will have a symmetry group $S(P\mathfrak{A}_\alpha)$ which is isomorphic to a subgroup of $S(\mathscr{T})$ as detailed in the theorem. $\square$

We feel it is of interest to describe the symmetry of these spaces from a geometric perspective. When the group of symmetries act on the 4 vertices of a regular tetrahedron the orbit of each face has 4 elements (rotations). The *stabilizer* subgroup of any *face* f in $\mathscr{T}$ is $\{q \in S(\mathscr{T})\,|\,q * f = f\}$ which has order 6 (reflections). So in total there are $4 \times 6 = 24$ elements. Recall the assignment of set B to the vertex set $V_\triangle$ then let an isomorphism be defined by

$$\iota_4 : S_4 \to S(\mathscr{T}), \begin{cases} \mathbb{B} \mapsto S_r(\mathscr{T}), \\ \{p_1, p_2, p_3\} \mapsto \{q_1, q_2, q_3\}, \\ \{p_7, p_8, p_9\} \mapsto \{q_7, q_8, q_9\}, \\ \{p_{18}, \ldots, p_{23}\} \mapsto \{t_{18}, \ldots t_{23}\}, \end{cases} \tag{8.5}$$

where q is a reflection and t is a composite reflection-rotation. Note that the q and t elements are images of the odd permutations of S_4.

The o-r involution $i : \mathbb{T}^2 \to \mathbb{T}^2$ is geometrically a reflection fixing set B then $\mathfrak{g}(B) = B' \subset S^2$ where the four pairs of asymptotic path-components are being permuted as for the four points in B'. Furthermore if $M \in \bar{M}_0$ the period 3 cycle of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ will now be anti-clockwise and clockwise if $M \in \bar{M}_4$. Since $M \in \bar{M}_0 \dot{\cup} \bar{M}_4$ leads to $\mathbb{B} \cong S_r(\mathscr{T})$ let the even permutations of $\mathbb{B} \subset S_4$ map to the set of rotations now labelled by $\{r\}$. Whereas $M \in \bar{M}_i$, $i = 1, 2, 3$, leads to $\mathbb{B}_i \cong D_4$. So let the odd permutations in S_4 of cycle length 2 map to the 6 reflections labelled 'i' and those of cycle length 4 map to the reflection-rotation elements labelled 'ir'. Then define two isomorphisms $\bar{\iota}$ and $\tilde{\iota}$ by

$$\bar{\iota} : S_4 \to S(G), \begin{cases} \mathbb{B} \mapsto \{r\}, \\ \{p_1, p_2, p_3\} \mapsto \{i_1, i_2, i_3\}, \\ \{p_7, p_8, p_9\} \mapsto \{i_7, i_8, i_9\}, \\ \{p_{18}, \ldots, p_{23}\} \mapsto \{ir_{18}, \ldots ir_{23}\}, \end{cases} \tag{8.6}$$

$$\tilde{\iota} : S(G) \to S(\mathscr{T}), \begin{cases} \{r\} \mapsto S_r(\mathscr{T}), \\ \{i_1, i_2, i_3\} \mapsto \{q_1, q_2, q_3\}, \\ \{i_7, i_8, i_9\} \mapsto \{q_7, q_8, q_9\}, \\ \{ir_{18}, \ldots, ir_{23}\} \mapsto \{t_{18}, \ldots t_{23}\}. \end{cases} \tag{8.7}$$

Clearly, $\iota_4 = \tilde{\iota} \circ \bar{\iota}$. It follows that $S(P\mathfrak{A}_\alpha)$ is isomorphic to a subgroup of $S(\mathscr{T})$ according to the matrix M associated to $P\mathfrak{A}_\alpha$, as previously found.

8.2 Classifying tiling spaces

Let β be fixed by a toral map F' and let $\ddot{\omega}'$ be a proper substitution such that the tiling space $\mathcal{T}_{\beta(\ddot{\omega}')}$ complies with Definition 7.29.

Theorem 8.19. *The two tiling spaces $\mathcal{T}_{\alpha(\ddot{\omega})}$ and $\mathcal{T}_{\beta(\ddot{\omega}')}$ are homeomorphic if and only if α and β are equivalent.*

Proof. The attractors $\mathfrak{A}_\alpha, \mathfrak{A}_\beta \in \mathfrak{F}$ are such that $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta \Leftrightarrow \alpha \equiv \beta$ (Theorem 8.4(i)). But $\mathfrak{A}_\alpha \cong \mathcal{T}_{\alpha(\ddot{\omega})}$ and $\mathfrak{A}_\beta \cong \mathcal{T}_{\beta(\ddot{\omega}')}$ (Lemma 7.31). Thus $\mathcal{T}_{\alpha(\ddot{\omega})} \cong \mathcal{T}_{\beta(\ddot{\omega}')} \Leftrightarrow \alpha \equiv \beta$. $\square$

Lemma 8.20. *For a fixed α, the symmetry and class groups of the tiling space $\mathcal{T}_{\alpha(\ddot{\omega})}$ admit the same classification criteria as for the orientable attractor $\mathfrak{A}_\alpha$.*

Proof. By Lemma 7.31, $\mathcal{T}_{\alpha(\ddot{\omega})}$ is homeomorphic to $\mathfrak{A}_\alpha$ and homeomorphic spaces have isomorphic group structures. $\square$

Lemma 8.20 justifies the statements given below in Corollary 8.22 and Theorem 8.24 which are of the same form as Corollary 8.7 and Theorem 8.15 respectively.

Definition 8.21. *Let $T_1, T_2 \subseteq \mathcal{T}_{\alpha(\ddot{\omega})}$ be the pair of asymptotic path-components derived from the point $\mathbf{b}_0 \in B \subset \mathbb{T}^2$. Let the restricted group of all self-homeomorphisms be $\mathfrak{H}_0(\mathcal{T}_{\alpha(\ddot{\omega})}) = \{F_{\alpha(\ddot{\omega})} : \mathcal{T}_{\alpha(\ddot{\omega})} \to \mathcal{T}_{\alpha(\ddot{\omega})} \,|\, F_{\alpha(\ddot{\omega})}(T_i) = T_i,\, i = 1, 2\}$. Under composition of maps let the restricted subgroup be $\mathfrak{T}_0(\mathcal{T}_{\alpha(\ddot{\omega})}) = \{[F_{\alpha(\ddot{\omega})}] \,|\, F_{\alpha(\ddot{\omega})} \in \mathfrak{H}_0\}$.*

Corollary 8.22. *The subgroup $\mathfrak{T}_0(\mathcal{T}_{\alpha(\ddot{\omega})})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Definition 8.23. *Let $S(\mathcal{T}_{\alpha(\ddot{\omega})})$ be the set of permutations of the pairs of asymptotic tilings of $\mathcal{T}_{\alpha(\ddot{\omega})}$ emanating from set B which are induced by the self-homeomorphisms of $\mathcal{T}_{\alpha(\ddot{\omega})}$. Then define $S(\mathcal{T}_{\alpha(\ddot{\omega})})$ to be the* symmetry group *under composition.*

Theorem 8.24. *The symmetry group $S(\mathcal{T}_{\alpha(\ddot{\omega})})$ is isomorphic to either the rotation symmetry group of a regular tetrahedron $S_r(\mathscr{T})$ or to the symmetry group of a square D_4. The criterion is the type of the source matrix M: if $M \in \bar{M}_i$, $i = 0, 4$, then $S(\mathcal{T}_{\alpha(\ddot{\omega})}) \cong S_r(\mathscr{T})$; if $M \in \bar{M}_i$, $i = 1, 2, 3$, then $S(\mathcal{T}_{\alpha(\ddot{\omega})}) \cong D_4$.*

8.3 Classification of examples

See pages 127 and 128 for details of the maps and their matrices referred to below. We exploit Theorem 175 of Hardy and Wright, page 146, to deduce the equivalence of the eigenvector slopes found in the examples below.

Orientable attractors and tiling spaces

Matrix types $\bar{M}_0$ and $\bar{M}_4$. Consider the lifted Plykin attractor $\tilde{\Lambda}_P$ and the attractor $\mathfrak{A}_{\alpha_0}$ of the Cat diffeomorphism which share the source matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \bar{M}_0$. The attractor $\mathfrak{A}_{\alpha_4}$ of the Fibonacci diffeomorphism comes from the matrix $M_\varphi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \bar{M}_4$. Now the matrix A and M_φ have equal expanding eigenvectors with slopes $\alpha_0 = \alpha_4 = 1/\mu = \frac{-1+\sqrt{5}}{2}$, with continued fraction expansion $[0; 1, 1, 1, \ldots] = [0; \dot{1}]$, so $\alpha_0 \equiv \alpha_4$. Thus the three attractors are homeomorphic (Theorem 8.4(i)). We also know that $\tilde{\Lambda}_P$ is homeomorphic to $\mathcal{T}_{\omega^2}$ (Theorem 6.32) so in fact all six spaces are homeomorphic $\tilde{\Lambda}_P \cong \mathfrak{A}_{\alpha_0} \cong \mathfrak{A}_{\alpha_4} \cong \mathcal{T}_{\omega^2} \cong \mathcal{T}_{\alpha_0(\ddot{\omega}_0)} \cong \mathcal{T}_{\alpha_4(\ddot{\omega}_4)}$ (Lemma 7.31). The permutation subgroup of each space is $\mathbb{B} \cong A_4$ (Lemma 8.11) while their isomorphic symmetry groups $S(\tilde{\Lambda}_P) \cong S(\mathfrak{A}_{\alpha_0}) \cong S(\mathfrak{A}_{\alpha_4}) \cong S(\mathcal{T}_{\omega^2}) \cong S(\mathcal{T}_{\alpha_0(\ddot{\omega}_0)}) \cong S(\mathcal{T}_{\alpha_4(\ddot{\omega}_4)})$ have the structure of $S_r(\mathscr{T})$ (Theorems 8.15, 8.24).

Given a space $\mathfrak{A}_\alpha \in \mathfrak{F}$ with $\mathfrak{K}_0(\mathfrak{A}_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and corresponding source matrix M, $\mathfrak{A}_\alpha$ is homeomorphic to the attractor associated to any map corresponding to M^k ($k \in \mathbb{Z}$). These maps corresponding to $M^k, M^{k'}$ ($k, k' \in \mathbb{Z}$) are typically not topologically conjugate although their attractors are homeomorphic. For example, $\mathfrak{A}_{\alpha_0}$ and $\mathfrak{A}_{\alpha_4}$ are homeomorphic but the underlying DA maps for which they are attractors are not topologically conjugate since, as detailed above, $\mathfrak{A}_{\alpha_0}$ has source matrix A and $\mathfrak{A}_{\alpha_4}$ has source matrix M_φ, with $A = M_\varphi^2$. So we see that the maps for which these spaces are attractors correspond to different elements of $\mathbb{Z} \oplus \mathbb{Z}_2$. Similar reasoning applies to a space $\mathcal{T}_{\alpha(\ddot{\omega})}$ with $\mathfrak{T}_0(\mathcal{T}_{\alpha(\ddot{\omega})}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and corresponding source matrix M.

Remark 8.25. *We know that the tiling spaces $\mathcal{T}_{\omega^2}$ and $\mathcal{T}_{\alpha_0(\ddot{\omega}_0)}$ are derived from the primitive and aperiodic substitutions* (6.16) *and $\ddot{\omega}_0$ respectively. Since the tiling spaces are homeomorphic there exist positive integers m and n such that their corresponding inflation and substitution homeomorphisms $F^m_{\omega^2}$ and $F^n_{\alpha_0(\ddot{\omega}_0)}$ are topologically conjugate (Theorem 2.1 in [9]).*

Proposition 8.26. *There exists a topological conjugacy between the self-homeomorphisms $\tilde{\mathfrak{h}} : \tilde{\Lambda}_P \to \tilde{\Lambda}_P$ and $\mathfrak{h} : \mathfrak{A}_{\alpha_0} \to \mathfrak{A}_{\alpha_0}$.*

$$\begin{tikzcd}
\tilde{\Lambda}_P \arrow[r, "\tilde{\mathfrak{h}}"] \arrow[ddd, bend right, "H"'] & \tilde{\Lambda}_P \arrow[ddd, bend left, "H"] \\
\mathcal{T}_{\omega^2} \arrow[u, "\tilde{g}"] \arrow[r, "F^m_{\omega^2}"] \arrow[d, "f"'] & \mathcal{T}_{\omega^2} \arrow[u, "\tilde{g}"'] \arrow[d, "f"] \\
\mathcal{T}_{\alpha_0(\ddot{\omega}_0)} \arrow[r, "F^n_{\alpha_0(\ddot{\omega}_0)}"] \arrow[d, "g"'] & \mathcal{T}_{\alpha_0(\ddot{\omega}_0)} \arrow[d, "g"] \\
\mathfrak{A}_{\alpha_0} \arrow[r, "\mathfrak{h}"] & \mathfrak{A}_{\alpha_0}
\end{tikzcd}$$

Proof. We know that the spaces $\mathcal{T}_{\omega^2}, \mathcal{T}_{\alpha_0(\ddot{\omega}_0)}, \tilde{\Lambda}_P$ and $\mathfrak{A}_{\alpha_0}$ are mutually homeomorphic. Let the maps $H, f, g, \tilde{g}$ be homeomorphisms between the spaces as shown in the commuting diagram above. By Remark 8.25, let fixed $m, n \in \mathbb{N}$ induce a conjugacy $F^n_{\alpha_0(\ddot{\omega}_0)} \circ f = f \circ F^m_{\omega^2}$ then $(g^{-1}\mathfrak{h}g) \circ f = f \circ (\tilde{g}^{-1}\tilde{\mathfrak{h}}\tilde{g}) \Rightarrow \mathfrak{h}g \circ f = gf \circ (\tilde{g}^{-1}\tilde{\mathfrak{h}}\tilde{g}) \Rightarrow \mathfrak{h}g = gf \circ (\tilde{g}^{-1}\tilde{\mathfrak{h}}\tilde{g}) \circ f^{-1} \Rightarrow \mathfrak{h} = (gf\tilde{g}^{-1}) \circ \tilde{\mathfrak{h}} \circ (\tilde{g}f^{-1}g^{-1}) \Rightarrow \mathfrak{h} = (gf\tilde{g}^{-1}) \circ \tilde{\mathfrak{h}} \circ (gf\tilde{g}^{-1})^{-1}$. That is $\mathfrak{h} = H \circ \tilde{\mathfrak{h}} \circ H^{-1}$ is a topological conjugacy. $\square$

By the topological conjugacy of Proposition 8.26, $[\tilde{\mathfrak{h}}]$, $[\mathfrak{h}]$, and similarly the tiling homeomorphisms $[F^n_{\alpha_0(\ddot{\omega}_0)}]$, $[F^m_{\omega^2}]$, correspond to isomorphic elements of $\mathbb{Z} \oplus \mathbb{Z}_2$.

Matrix types $\bar{M}_1$, $\bar{M}_2$ and $\bar{M}_3$. Observe the continued fraction expansions of the eigenvector slopes

$$\begin{aligned}
\alpha_1 &= \frac{-1+\sqrt{3}}{2} = [0;2,1,2,1,2,1,2,1,2,\ldots] = [0;2,\dot{1},\dot{2}],\\
\alpha_2 &= \frac{1+2\sqrt{2}}{7} = [0;1,1,4,1,4,1,4,1,4,\ldots] = [0;1,\dot{1},\dot{4}],\\
\alpha_3 &= -1+\sqrt{3} = [0;1,2,1,2,1,2,1,2,1,\ldots] = [0;\dot{1},\dot{2}],
\end{aligned}$$

corresponding to the matrices M_1, M_2, M_3 respectively. Since $\alpha_1 \equiv \alpha_3$ the attractor $\mathfrak{A}_{\alpha_1}$ is homeomorphic to $\mathfrak{A}_{\alpha_3}$ whereas the attractor $\mathfrak{A}_{\alpha_2}$ is not homeomorphic to either $\mathfrak{A}_{\alpha_1}$ or $\mathfrak{A}_{\alpha_3}$ since $\alpha_2 \not\equiv \alpha_1$, $\alpha_2 \not\equiv \alpha_3$ (Theorem 8.4(i)). Although our chosen example matrix $M_2 \in \bar{M}_2$ leads to $\mathfrak{A}_{\alpha_2}$ not being homeomorphic to the other two attractors with source matrices belonging to $\bar{M}_1$ and $\bar{M}_3$, there is no obstruction to matrices from types $\bar{M}_1$ and $\bar{M}_2$ realising homeomorphic attractors as the following examples illustrate.

Let $P = \begin{bmatrix} 9 & 2 \\ 23 & 5 \end{bmatrix} \in \bar{M}_1$ and $Q = \begin{bmatrix} 8 & 7 \\ 7 & 6 \end{bmatrix} \in \bar{M}_2$. The eigenvector slopes are $\alpha_P = \frac{-2+5\sqrt{2}}{2} = [2;\dot{1},1,\dot{6}]$ and $\alpha_Q = \frac{-1+5\sqrt{2}}{7} = [0;1,6,\dot{1},1,\dot{6}]$ so $\alpha_P \equiv \alpha_Q$. Thus the attractors $\mathfrak{A}_{\alpha_P}$ and $\mathfrak{A}_{\alpha_Q}$ with source matrices P and Q are homeomorphic. The tiling spaces which are homeomorphic to each other succumb to the same criteria as given for the attractors above but in particular, $\mathfrak{A}_{\alpha_i} \cong \mathcal{T}_{\alpha_j(\ddot{\omega}_j)}$ when $i = j$, $i, j = 1, 2, 3, P, Q$ (Lemma 7.31). The permutation subgroups of the attractors and tiling spaces mentioned in this paragraph are $\mathbb{B}_1 \cong \mathbb{B}_2 \cong \mathbb{B}_3$ (Lemma 8.11) and their isomorphic symmetry groups are $S(\mathfrak{A}_{\alpha_i}) \cong S(\mathcal{T}_{\alpha_i(\ddot{\omega}_i)}) \cong D_4$, $i = 1, 2, 3, P, Q$ (Theorems 8.15, 8.24).

Non-orientable attractors

The classification of the non-orientable attractors mimics that of the orientable attractors by virtue of the matrix type used in their construction. Specifically we appeal to Theorem 8.4 (ii) to deduce that the Plykin attractor Λ_Π, $P\mathfrak{A}_{\alpha_0}$ and $P\mathfrak{A}_{\alpha_4}$ are mutually homeomorphic. Whereas $P\mathfrak{A}_{\alpha_2}$ is not homeomorphic to either $P\mathfrak{A}_{\alpha_1}$ or $P\mathfrak{A}_{\alpha_3}$ which are homeomorphic to each other. However, $P\mathfrak{A}_{\alpha_P}$ is homeomorphic to $P\mathfrak{A}_{\alpha_Q}$.

The symmetry groups of the Plykin attractor $S(\Lambda_\Pi)$, $S(P\mathfrak{A}_{\alpha_0})$ and $S(P\mathfrak{A}_{\alpha_4})$ are isomorphic to $S_r(\mathscr{T})$ while each symmetry group $S(P\mathfrak{A}_{\alpha_i})$, $i = 1, 2, 3, P, Q$, is isomorphic to D_4. (Theorem 8.18).

The classification of self-maps of the non-orientable attractors also replicates those for the orientable attractors in their correspondence to elements of $\mathbb{Z} \oplus \mathbb{Z}_2$ (Corollary 8.7).

Remark 8.27. *The above results demonstrate the implication that homeomorphic spaces yield isomorphic groups but that isomorphic groups do not necessarily yield homeomorphic spaces.*

Remark 8.28. *The symmetry groups are topological invariants. An attractor $\mathfrak{A}_\alpha \in \mathfrak{F}$ whose symmetry group is isomorphic to $\mathbb{B} \cong A_4$ cannot be homeomorphic to an attractor $\mathfrak{A}_\beta \in \mathfrak{F}$ whose symmetry group is isomorphic to one of $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ all of which are isomorphic to D_4. This means that a source matrix $M_\alpha \in \bar{M}_0 \dot{\cup} \bar{M}_4$ will not yield an attractor $\mathfrak{A}_\alpha$ which is homeomorphic to an attractor $\mathfrak{A}_\beta$ sourced from a matrix $M_\beta \in \bar{M}_1 \dot{\cup} \bar{M}_2 \dot{\cup} \bar{M}_3$.*

Chapter 9

Conclusions

Much of chapters 1 to 4 consisted of background and preparatory material. In chapter 5 we described the construction of an orientable expanding hyperbolic toral attractor, from which evolved the non-orientable Plykin attractor on the sphere. Chapter 6 introduced a solenoid as an inverse limit space aligned to a branched 1-manifold. Then began a description of the 'mechanics' needed to build attractors which had one or four complementary domains. Through development of an original construction method in chapter 7, useful criteria were formulated in chapter 8 for the classification of attractors and tiling spaces which was the culmination and main result of this current research.

one complementary domain. Two attractors were derived from branched 1-manifolds of differing design, one from the Cat substitution map γ using the method of Barge and Diamond [8] and the other from taking a geometric perspective of the Cat map $\mathcal{C}$, its Markov partition $\mathcal{P}$ and a method described for example by Robinson [48]. In both cases we knew the source matrix, whose expanding eigenvalue provided the scale factor for the dynamics and whose eigenvector provided the lengths of the 1-cells for the branched 1-manifolds. From this information we found the self-maps of the branched 1-manifolds. A Williams' construction [54] on each branched 1-manifold produced identically structured elementary branched 1-manifolds of roses with four petals where again the self-maps were found by calculation. A proper substitution delivered a substitution tiling space $\mathcal{T}_\gamma$ homeomorphic to the initial attractor Λ (Theorem 6.20).

four complementary domains. In order to lift the Plykin planar attractor Λ_P to the torus we used Yi's construction [57] to produce an orientable double cover equipped with an inverse limit presentation. However in trying to form its rose we had no matrix parameters to work with - combinatorics was the solution. This iterative method located five returns to a nominated origin from which we could build a rose. Each distinct combinatorial sequence provided the 'word' (whose letters were 1-cells) to be assigned to one of the five petals. These words gave the information needed to derive a self-map of the rose over an 'alphabet' of five petals then a proper substitution and a tiling space $\mathcal{T}_{\omega^2}$ homeomorphic to the lifted attractor $\tilde{\Lambda}_P$ with four complementary domains (Theorem 6.32).

the algorithm. Our algorithm given on page 144 will take any hyperbolic toral automorphism with matrix M and produce an orientable attractor homeomorphic to a tiling space. The process includes a Markov principal partition $\mathcal{P}$, its finer secondary partition $\ddot{\mathcal{P}}$, and the linear transformation of M which provides the self-map of a 2θ-space. Then as previously, combinatorial first returns to the origin of the 2θ-space assign the 'words' to the five petals of a generic rose $\ddot{K}$. From here, a proper substitution $\ddot{\omega}$ leads easily to $\mathcal{T}_{\alpha(\ddot{\omega})}$ which is homeomorphic to $\mathfrak{A}_\alpha$ (Lemma 7.31).

The advantages of our process are that it avoids the need for arithmetical calculation, it avoids the need of Yi's 'lifting map' [57] to an orientable branched 1-manifold and it avoids a Williams' construction [54] which may produce superfluous petals in the rose, as happened for K_3 and $\mathcal{M}_2$ (Figs. 6.12 and 6.13).

the classification. Since our algorithm relied heavily on the matrix M of a hyperbolic toral map, we focussed on this defining component M. In chapter 8 we found that this was provident since the properties of M became the main criteria for classifying an attractor $\mathfrak{A}_\alpha \in \mathfrak{F}$ or $P\mathfrak{A}_\alpha \in P\mathfrak{F}$ and indeed for a tiling space $\mathcal{T}_{\alpha(\ddot{\omega})}$. Classification of attractors and tiling spaces was determined up to homeomorphism and up to isomorphism of symmetry and class groups. The groups in question turned out to be the symmetric subgroup $A_4 \subset S_4$, the symmetry group of the square $D_4 \subset S_4$, and the tetrahedral subgroup $S_r(\mathscr{T}) \subset S(\mathscr{T})$ while the self-homeomorphisms of our spaces corresponded to elements of $\mathbb{Z} \oplus \mathbb{Z}_2$. The tetrahedral symmetry correlates nicely with the template of the Plykin attractor in Figure 5.4 on page 84.

Further questions

1. What happens if the 4 'blown up' orbits are positioned at points in the torus other than at the 4 special points of set $B \subset \mathbb{T}^2$? Does this yield an attractor? If so, of what description, can it be classified and by what criteria? Can it be embedded in the plane?

2. Do planar attractors in the set $P\mathfrak{F}$ exhaust all those which can be lifted to $\mathbb{T}^2$? Are there planar attractors not in set $P\mathfrak{F}$ but with 4 complementary domains which can be lifted to a 2-fold covering surface not of genus 1?

3. Given a toral DA attractor of topological dimension one, induced by a particular bump function, what is the Hausdorff measure and dimension of the attractor? In fractal geometry, two attractors would be deemed 'the same' if there is a bi-Lipschitz mapping between them since the Hausdorff dimension is invariant under a bi-Lipschitz[1] transformation (see for example Corollary 2.4 in [21]).

[1]Rudolf O. S. Lipschitz (1832 - 1903) German mathematician.

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