THE INFLUENCE OF A PRIMARY STRESS UPON THE PROPAGATION OF SMALL-AMPLITUDE ELASTIC DISTURBANCES

A Thesis submitted for the degree of

Doctor of Philosophy

by

PHILIP J. MYERS

Department of Mathematics

University of Leicester

University Road

Leicester LE1 7RH

PREFACE

This thesis describes the work done by the author, in the Department of Mathematics at the University of Leicester, during the period 1983 to 1987.

This project was instigated under the supervision of Dr. A.J. Willson whom the author would like to thank for his guidance and encouragement both as an undergraduate and as a postgraduate at Leicester University. In particular the author would like to thank Dr. Willson for his collaboration on the topic in Chapter 5 which is currently awaiting publication, under joint authorship, in the International Journal of Engineering Science. Thanks are also extended to Dr. J.R. Thompson for his very helpful comments and his support during the writing of this thesis.

The author is also indebted to the British Gas Corporation for the award of a research scholarship during the tenure of which part of this work was carried out and to Dr.D.H. Dunford of the On Line Inspection Centre at Cramlington who acted as my liaison with British Gas.

Finally, my thanks go to my parents for their financial support and encouragement and also to Caroline who gave me the incentive to finish what I had started!

THE INFLUENCE OF A PRIMARY STRESS UPON THE PROPAGATION OF SMALL-AMPLITUDE ELASTIC DISTURBANCES.

CONTENTS

CHAPTER 5. A GENERALISATION OF KO'S STRAIN-ENERGY FUNCTION

INTRODUCTION

This thesis considers three problems in the field of elastodynamics.

The first concerns small-amplitude elastic disturbances in an infinite cylinder. The equations describing vibrations in an isotropic cylinder of infinite length and circular cross-section were first formulated by Pochhammer [1] as early as 1876; however, the complicated nature of these equations meant that even modest information was difficult to obtain. Chree [2,3a,3b] also discussed this problem but gave a more general account by including the cases in which the normal section of the cylinder was non-circular and in which the cylinder was composed of anisotropic material. Our approach extends the results of Pochhammer and Chree by utilising a method of successive approximation through which we solve the governing equations to obtain dispersion relations that relate the angular frequency ω and the wave number K ($K=2\pi$/wavelength).

The second investigation is of the propagation of elastic waves in a pre-stressed body, with particular reference to the circular cylinder and the half-space. In this type of problem, the final state of the body may be regarded as a small elastic deformation superimposed on a given finite deformation. The general theory of such deformations has been used by many investigators: Prager [4], Green, Rivlin and Shield [5], Urbanowski [6], Zorski [7], and more recently by Eringen and Suhubi [8, Chapter 4] whose notation we adopt in Chapters 2 and 4. From the equations of motion, boundary conditions and constitutive equations, it is possible to establish a set of linear differential equations which may be solved to give a dispersion relation. In general, however, these linear equations are complicated and little information can be obtained without recourse to considering special cases or specific materials. This work has particular importance when considering the implications for stability.

The final investigation is of a compressible strain-energy function that contains the Ko model as a special case. In order to illustrate various aspects of non-linear behaviour in elastic materials, it is usually necessary to assume an explicit form for the strain-energy function W. In an attempt to describe the behaviour of polyurethane foam rubber Ko [9] proposed a one-parameter strain-energy model which, due to its relative simplicity, has been used by many later investigators. Various other forms for W have been proposed to describe compressible rubberlike materials, principally by Blatz and Ko [10] and Ogden [11]. The proposed model is examined in the light of various *a priori* inequalities, and is then used to obtain solutions to the problem of vibrations in a stressed plate.

We begin in Chapter 1 with a short review of the work carried out by Pochhammer and Chree for vibrations in isotropic, circular cylinders. The method of successive approximation is then applied to cylinders whose radius is small in comparison to the wavelength of the vibration. Dispersion relations are derived for both longitudinal and flexural types of vibrations. Similarly, we set up the fundamental equations describing vibrations in circular cylinders of transversely isotropic composition and also for isotropic cylinders of elliptic cross-section, which we solve to establish the appropriate dispersion relations.

This analysis is extended in Chapter 2 to encompass waves in stressed isotropic cylinders, infinite in length, and having a circular cross-section. We consider only those waves whose wavelength is large compared to the cylinder radius, which is to say that Ka (the product of the wave number K and the cylinder radius a) is taken to be small. The theory is in two parts depending on whether the material is incompressible or compressible. Accordingly, we consider two particular problems: an incompressible cylinder placed under a uniaxial stress (either a compression or an extension) directed along the axis of the cylinder, and a cylinder composed of a compressible material having stresses applied along its axis and also upon its lateral surface. For both of these problems, the governing equations are set up and solved by successive approximation to establish dispersion relations describing both longitudinal and flexural waves. The results are then simplified with the introduction of a modified Young's modulus and a modified Poisson's ratio which take into account the effect of the primary stress. The chapter is concluded with an illustration of our results for three particular cases.

Chapter 3 is concerned with the propagation of surface waves in a pre-stressed half-space. Incompressible and compressible materials are considered in turn, and in both cases our aim is to establish the frequency equation from which we may calculate the wave-velocity in terms of various material constants and the applied stress. The results are again illustrated for specific materials.

We return to the problem of waves in a stressed cylinder in

Chapter 4. The governing equations established in Chapter 2 are now solved completely in terms of Bessel functions. We show that in the limiting cases, when Ka is small and when $Ka \to \infty$, the results calculated in Chapters 2 and 3 respectively are recaptured.

Finally, in Chapter 5, an extension of Ko's strain-energy function, put forward by Dr. A.J. Willson, for isotropic hyperelastic materials is presented. The implications of the Baker-Ericksen inequality, the strengthened tension-extension inequality and the ordered forces inequality are discussed in detail. Particular considerations are given to the configurations of plane stress and plane strain. The dispersion relation governing the propagation of small-amplitude waves in a pre-stressed plate is obtained and limiting solutions are derived for thin plates for both the flexural and longitudinal modes. Numerical results are given for configurations of marginal stability.

CHAPTER 1

SMALL-AMPLITUDE VIBRATIONS IN UNSTRESSED CYLINDERS

§1.1 INTRODUCTION.

In this chapter we present an analysis of the propagation of small-amplitude waves along an elastic cylinder of infinite length. The cylinder material is homogeneous and taken to have uniform density. Our main objective is to derive the dispersion relation linking the velocity of propagation and the wavelength, and to obtain the associated displacement field, for waves whose wavelength is large compared to the cylinder radius. We first obtain results for longitudinal, flexural and torsional modes of vibration in isotropic cylinders of circular cross-section, and then move on to consider vibrations in cylinders composed of a transversely isotropic material and also isotropic cylinders with an elliptic cross-section.

The plan of analysis is as follows. In §1.2 we set up the general elastic equations of motion governing small-amplitude vibrations in terms of a cylindrical polar coordinate system (r,θ,z) and formulate the boundary conditions at the surface of the cylinder. In §1.3 and §1.4 we analyse longitudinal and flexural modes, and review previous investigations. The dispersion relation is then calculated when Ka (the product of the wave number K ($K=2\pi$/wavelength) and the cylinder radius a) becomes small and when Ka is very large. Numerical results for intermediate values of Ka are also presented. §1.5 deals briefly with the torsional mode. For the remaining sections

(§1.6–§1.11) we analyse only those vibrations whose wavelengths are large compared to the cylinder radius. In §1.6 and §1.7 we set up recurrence relations, for both longitudinal and flexural vibrations, that are used to obtain more accurate approximations to the dispersion relation. Circular cylinders of anisotropic composition are dealt with in §1.8 and §1.9, and a discussion of vibrations in an isotropic cylinder of elliptic cross-section (§1.10, §1.11) concludes the chapter.

POCHHAMMER-CHREE EQUATIONS FOR CYLINDRICAL BARS

§1.2 ISOTROPIC CYLINDERS OF CIRCULAR CROSS-SECTION.

Investigations of the vibrations of a long circular cylinder in terms of the general elastic equations were originally carried out by Pochhammer [1] and, independently a few years later, by Chree [2].

The z-axis is taken to coincide with the cylinder axis, and we consider the propagation of an infinite train of sinusoidal waves in which the displacement depends harmonically on both z and the time t. The equations of motion, expressed in cylindrical polar coordinates, are well-known and may be found for example in Love [12, §199] or Kolsky [13, Chapter 3]:

$$\begin{aligned} \rho u_{tt} &= (\lambda+2\mu)\Delta_r+\mu(u_{zz}+u_{\theta\theta}/r^2)-\mu(v_\theta/r^2+v_{r\theta}/r)-\mu w_{rz}, \\ \rho v_{tt} &= (\lambda+2\mu)\Delta_\theta/r+\mu(u_\theta/r^2-u_{r\theta}/r) \\ &\qquad +\mu(v_{rr}+v_r/r-v/r^2+v_{zz})-\mu w_{\theta z}/r, \\ \rho w_{tt} &= (\lambda+2\mu)\Delta_z-\mu(u_{rz}+u_z/r)-\mu v_{\theta z}/r+\mu(w_{rr}+w_r/r+w_{\theta\theta}/r^2), \end{aligned} \tag{1.1}$$

where u, v, w are the components of displacement in the r, θ

and z-directions respectively, ρ is the density, λ and μ are the usual Lamé constants, and Δ is the dilatation given by

$$\Delta = u_r+(u+v_\theta)/r+w_z,$$

with subscripts indicating partial differentiation.

The stress-free boundary conditions on the cylinder surface r=a require the vanishing of the radial components of stress, P_{rr}, $P_{r\theta}$, P_{rz}. These are expressed in terms of the displacements by:

$$P_{rr} = \lambda\Delta+2\mu u_r,$$

$$P_{r\theta} = \mu(u_\theta/r+v_r-v/r), \qquad (1.2)$$

$$P_{rz} = \mu(u_z+w_r).$$

From the resulting eigen-problem, three particular types of vibration may be distinguished: longitudinal, flexural, and torsional vibrations. We examine first longitudinal modes.

§1.3 LONGITUDINAL VIBRATIONS.

A detailed numerical investigation of this problem was first carried out by Bancroft [14]. A further account detailing the dispersion of elastic waves in a circular cylinder for all three types of vibration may be found in Davies [15].

The longitudinal solution is characterised by:

$$u = U(r)\exp\{i(\omega t-Kz)\},$$

$$v = 0,$$

$$w = iW(r)\exp\{i(\omega t-Kz)\}.$$

Now $(1.1)_2$ is satisfied identically whilst $(1.1)_1$ and $(1.1)_3$ reduce to

$$-\rho\omega^2 U = (\lambda+2\mu)(U_{rr}+U_r/r-U/r^2)-\mu K^2 U+K(\lambda+\mu)W_r,$$
$$-\rho\omega^2 W = -K(\lambda+\mu)(U_r+U/r)-K^2(\lambda+2\mu)W+\mu(W_{rr}+W_r/r). \tag{1.3}$$

Solving (1.3) for U and W, see for example [12], gives

$$U = PJ_1(\lambda_1 r)+RJ_1(\lambda_2 r),$$
$$W = -P\lambda_1 J_0(\lambda_1 r)/K+RKJ_0(\lambda_2 r)/\lambda_2, \tag{1.4}$$

where P, R are constants, J_n denotes the Bessel function of order n, and λ_1, λ_2 are given in terms of $X=\rho\omega^2/\mu K^2$, $L=\lambda/\mu$ by

$$\lambda_1^2 = (X-1)K^2, \qquad \lambda_2^2 = [X/(L+2)-1]K^2. \tag{1.5}$$

Note also that U vanishes when r vanishes, appropriately for longitudinal waves, and the cylinder axis remains undisturbed during the vibration.

The boundary conditions $(1.2)_1$, $(1.2)_3$ reduce to

$$(L+2)U_r+LU/r+KLW = 0,$$
$$-KU+W_r = 0, \tag{1.6}$$

which are to be satisfied on the surface r=a. Substituting the values for U and W given by (1.4) into (1.6), and introducing $\psi(x) = xJ_0(x)/J_1(x)$, gives

$$PJ_1(\lambda_1 a)[2\psi(\lambda_1 a)-2]+RJ_1(\lambda_2 a)[(X-2)K^2\psi(\lambda_2 a)/\lambda_2-2] = 0,$$

$$PJ_1(\lambda_1 a)(X-2)-2RJ_1(\lambda_2 a) = 0,$$

Which leads at once to the dispersion relation

$$[X/2-1]^2\psi(\lambda_2 a)+[\psi(\lambda_1 a)-X/2][X/(L+2)-1] = 0. \tag{1.7}$$

The velocity, V_0 say, of longitudinal waves of infinite wavelength in a bar is easily shown to be $\surd(E/\rho)$, where E is Young's modulus (see for instance [13]), or equivalently

$$V_0 = \surd[2\mu(1+\sigma)/\rho], \tag{1.8}$$

where σ is Poisson's ratio ($\sigma=L/2(L+1)$). We have therefore

$$X = 2(V/V_0)^2(1+\sigma),$$

where V is the phase velocity ω/K. Further, by setting $\alpha = (1-2\sigma)/(1-\sigma)$, (1.7) may be rewritten in the form

$$[X/2-1]^2\psi(\lambda_2 a)+[\psi(\lambda_1 a)-X/2][X\alpha/2-1] = 0, \qquad (1.9)$$

where now

$$\lambda_2^2 = (\alpha X/2-1)K^2.$$

So the dispersion relation has been simplified to involve just the dimensionless quantities X, α and Ka. Equation (1.9) has multiple roots each corresponding to a particular mode of vibration. Values of V/V_0 tabulated as functions of σ and Ka are given in Table 1.1 for the fundamental mode, and the results have been plotted in Fig. 1.1 for various values of σ. In Fig. 1.2 we display the first four modes using the value $\sigma = 0.25$; the curves denoted 1, 2, 3 and 4 refer respectively to the fundamental, second, third and fourth roots of the dispersion relation. We display the group velocity curves for longitudinal vibrations in Fig. 1.3

When $Ka\to\infty$, both λ_1^2 and λ_2^2 are negative and we may employ the asymptotic expansion $\psi(ix)\approx x$ (see Bancroft [14]); (1.9) now becomes

$$X^3-8[X^2-(3-\alpha)X+(2-\alpha)] = 0, \qquad (1.10)$$

and apart from a change in notation this is Rayleigh's equation, containing one real root in the range of possible values for the parameter X, that gives the velocity of Rayleigh waves on the surface of a solid isotropic half-space. The value of this root for selected values of Poisson's ratio is presented in Table 1.1 on the line corresponding to $Ka = \infty$.

Approximate solutions to the dispersion relation for the case when Ka is small may be calculated by using power series expansions for the Bessel functions:

Ka	σ = 0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.0	--	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	--	0.999994	0.999975	0.999944	0.999900	0.999843	0.999775	0.999693	0.999600	0.999494	0.999375
0.2	--	0.999975	0.999899	0.999773	0.999596	0.999370	0.999094	0.998769	0.998395	0.997973	0.997502
0.3	--	0.999942	0.999769	0.999482	0.999082	0.998569	0.997946	0.997214	0.996375	0.995432	0.994387
0.4	--	0.999895	0.999582	0.999063	0.998341	0.997421	0.996307	0.995005	0.993521	0.991864	0.990039
0.5	--	0.999832	0.999331	0.998502	0.997354	0.995898	0.994145	0.992110	0.989809	0.987256	0.984471
0.6	--	0.999750	0.999005	0.997779	0.996090	0.993961	0.991418	0.988489	0.985205	0.981598	0.977701
0.7	--	0.999645	0.998592	0.996868	0.994509	0.991560	0.988069	0.984088	0.979673	0.974876	0.969752
0.8	--	0.999512	0.998073	0.995732	0.992558	0.988628	0.984029	0.978847	0.973170	0.967080	0.960655
0.9	--	0.999344	0.997421	0.994321	0.990164	0.985083	0.979215	0.972696	0.965654	0.958204	0.950448
1.0	--	0.999130	0.996598	0.992566	0.987234	0.980817	0.973526	0.965558	0.957083	0.948251	0.939184
1.1	--	0.998853	0.995550	0.990370	0.983642	0.975701	0.966850	0.957353	0.947424	0.937238	0.926927
1.2	--	0.998486	0.994190	0.987592	0.979222	0.969574	0.959060	0.948006	0.936659	0.925201	0.913763
1.3	--	0.997984	0.992386	0.984033	0.973757	0.962250	0.950028	0.937459	0.924794	0.912202	0.899795
1.4	--	0.997268	0.989919	0.979400	0.966969	0.953521	0.939637	0.925678	0.911865	0.898331	0.885152
1.5	--	0.996174	0.986414	0.973271	0.958521	0.943181	0.927802	0.912677	0.897955	0.883712	0.869979
1.6	--	0.994336	0.981208	0.965060	0.948042	0.931062	0.914504	0.898527	0.883188	0.868496	0.854439
1.7	--	0.990769	0.973134	0.954041	0.935209	0.917094	0.899811	0.883369	0.867735	0.852863	0.838705
1.8	--	0.982380	0.960437	0.939550	0.919880	0.901361	0.883901	0.867410	0.851802	0.837005	0.822951
1.9	0.973594	0.962391	0.941656	0.921415	0.902238	0.884139	0.867055	0.850907	0.835618	0.821119	0.807344
2.0	0.933164	0.931824	0.917591	0.900311	0.882828	0.865866	0.849626	0.834147	0.819414	0.805392	0.792037
2.5	0.792636	0.796237	0.795698	0.791409	0.784332	0.775462	0.765569	0.755172	0.744604	0.734069	0.723694
3.0	0.715120	0.717279	0.717628	0.716078	0.712762	0.707966	0.702031	0.695280	0.687990	0.680374	0.672596
3.5	0.670704	0.671378	0.671101	0.669756	0.667329	0.663897	0.659598	0.654599	0.649065	0.643149	0.636981
4.0	0.644549	0.643991	0.642902	0.641172	0.638752	0.635651	0.631926	0.627663	0.622960	0.617920	0.612636
4.5	0.628824	0.627273	0.625432	0.623198	0.620513	0.617358	0.613754	0.609745	0.605393	0.600766	0.595934
5.0	0.619229	0.616878	0.614390	0.611668	0.608648	0.605301	0.601627	0.597649	0.593406	0.588947	0.584321
5.5	0.613325	0.610325	0.607294	0.604135	0.600783	0.597204	0.593384	0.589336	0.585084	0.580662	0.576111
6.0	0.609691	0.606161	0.602673	0.599136	0.595481	0.591668	0.587681	0.583520	0.579203	0.574753	0.570204
6.5	0.607478	0.603509	0.599640	0.595776	0.591852	0.587823	0.583668	0.579382	0.574975	0.570467	0.565882
7.0	0.606168	0.601833	0.597640	0.593498	0.589338	0.585115	0.580802	0.576392	0.571889	0.567308	0.562670
7.5	0.605437	0.600793	0.596326	0.591946	0.587581	0.583184	0.578728	0.574200	0.569603	0.564947	0.560248
8.0	0.605082	0.600174	0.595473	0.590886	0.586343	0.581794	0.577210	0.572575	0.567889	0.563158	0.558399
8.5	0.604971	0.599836	0.594933	0.590168	0.585469	0.580786	0.576088	0.571356	0.566588	0.561788	0.556970
9.0	0.605018	0.599687	0.594607	0.589686	0.584852	0.580051	0.575251	0.570433	0.565591	0.560727	0.555854
9.5	0.605168	0.599664	0.594430	0.589373	0.584418	0.579514	0.574624	0.569728	0.564819	0.559898	0.554973
10.0	0.605382	0.599725	0.594356	0.589177	0.584117	0.579121	0.574151	0.569186	0.564217	0.559243	0.554273
11.0	0.605912	0.599999	0.594399	0.589018	0.583779	0.578627	0.573522	0.568440	0.563369	0.558307	0.553256
12.0	0.606492	0.600371	0.594585	0.589038	0.583655	0.578377	0.573163	0.567987	0.562834	0.557698	0.552583
∞	0.618034	0.609807	0.602132	0.594906	0.588045	0.581481	0.575157	0.569031	0.563070	0.557248	0.551550

Table 1.1. The wave-speed ratio V/V_0, phase velocity V, $V_0=\sqrt{(E/\rho)}$, tabulated for selected values of Ka and σ for longitudinal vibrations in a circular cylinder of radius a.

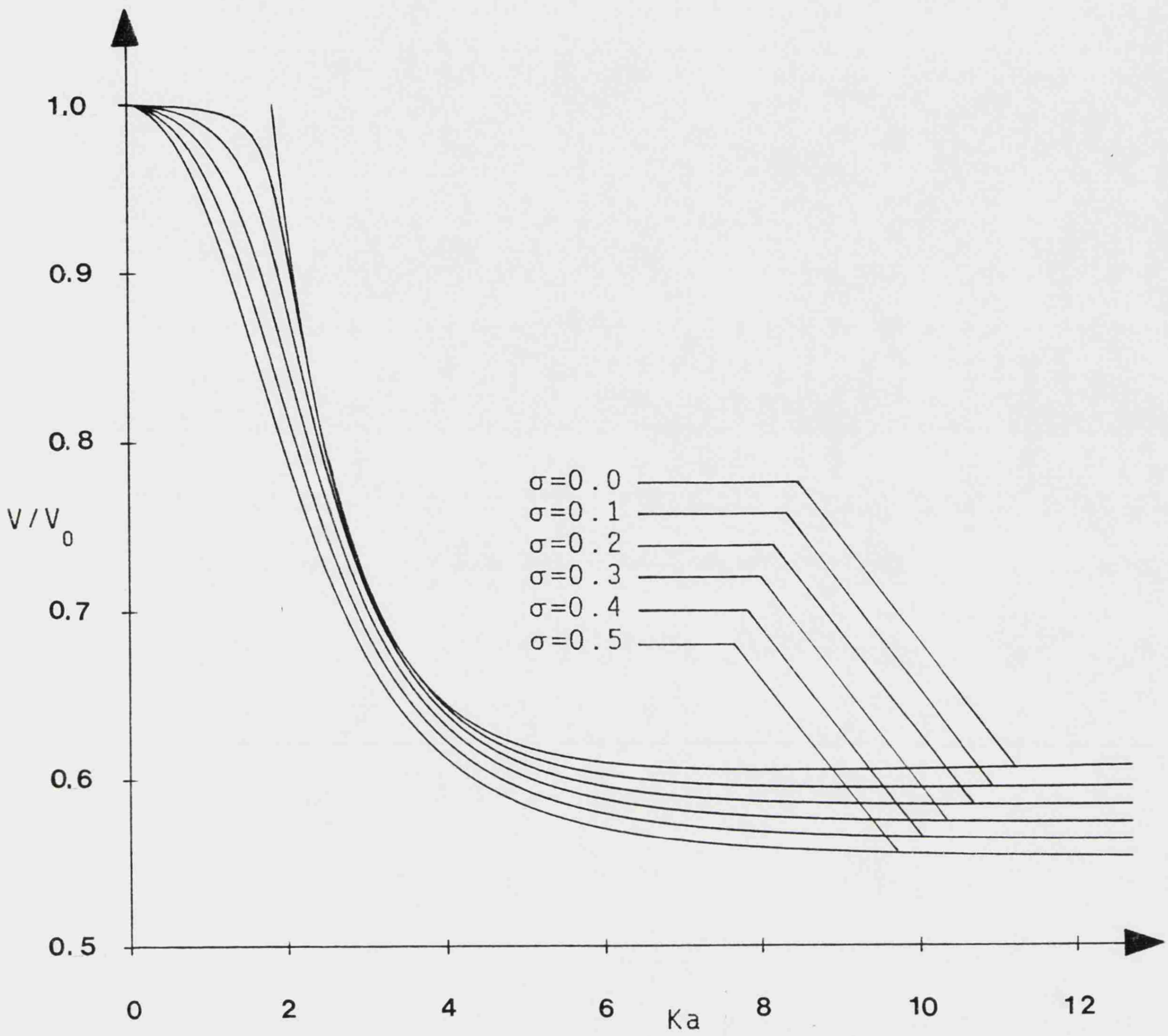

Figure 1.1

Phase velocity V for longitudinal vibrations in circular cylinders of radius a. Curves are plotted for selected values of Poisson's ratio σ where $V_0 = \surd(E/\rho)$.

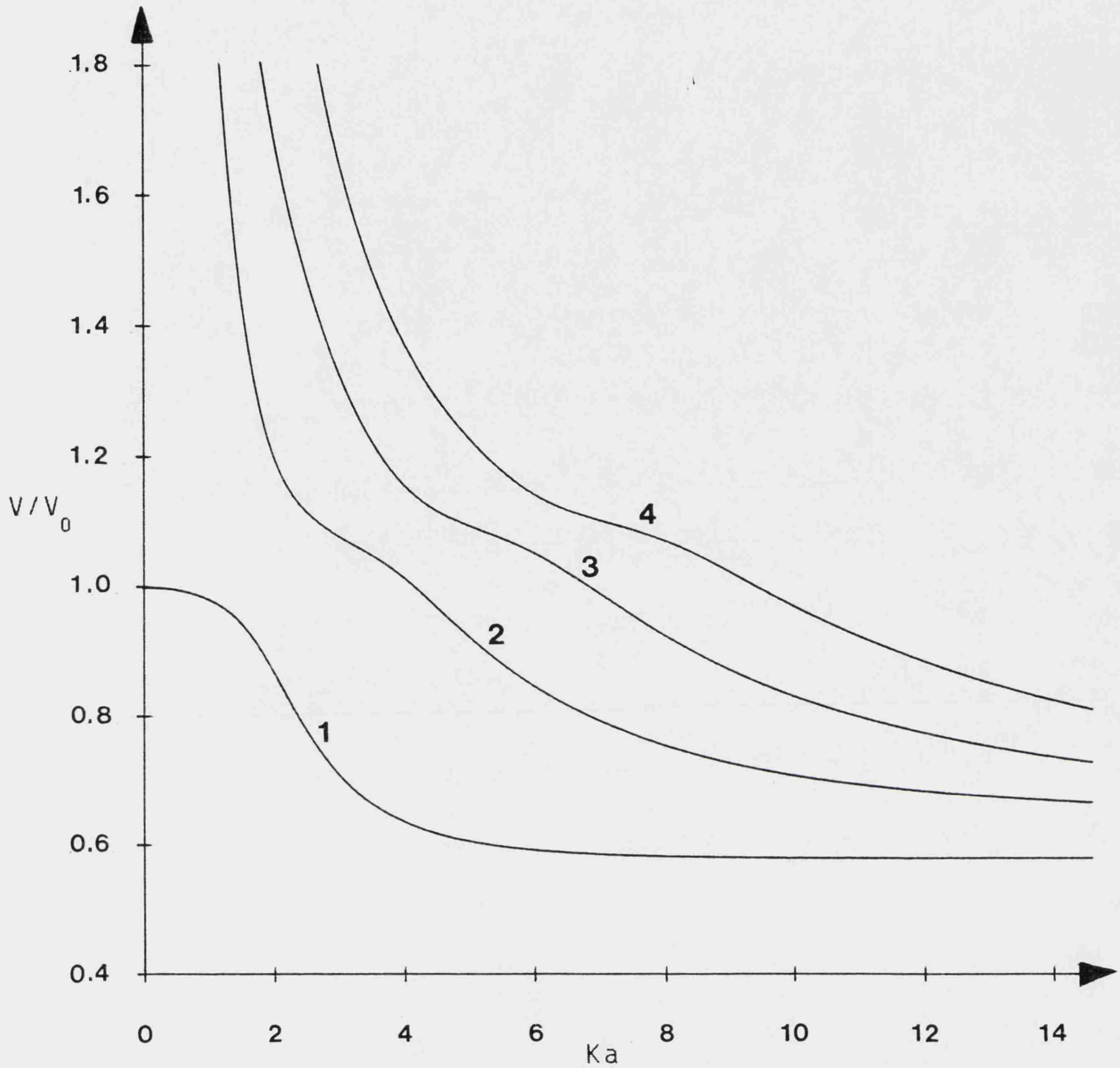

Figure 1.2

The first four longitudinal modes for the phase velocity in a cylinder of radius a are plotted. Here $V_0 = \sqrt{(E/\rho)}$ and $\sigma=0.25$.

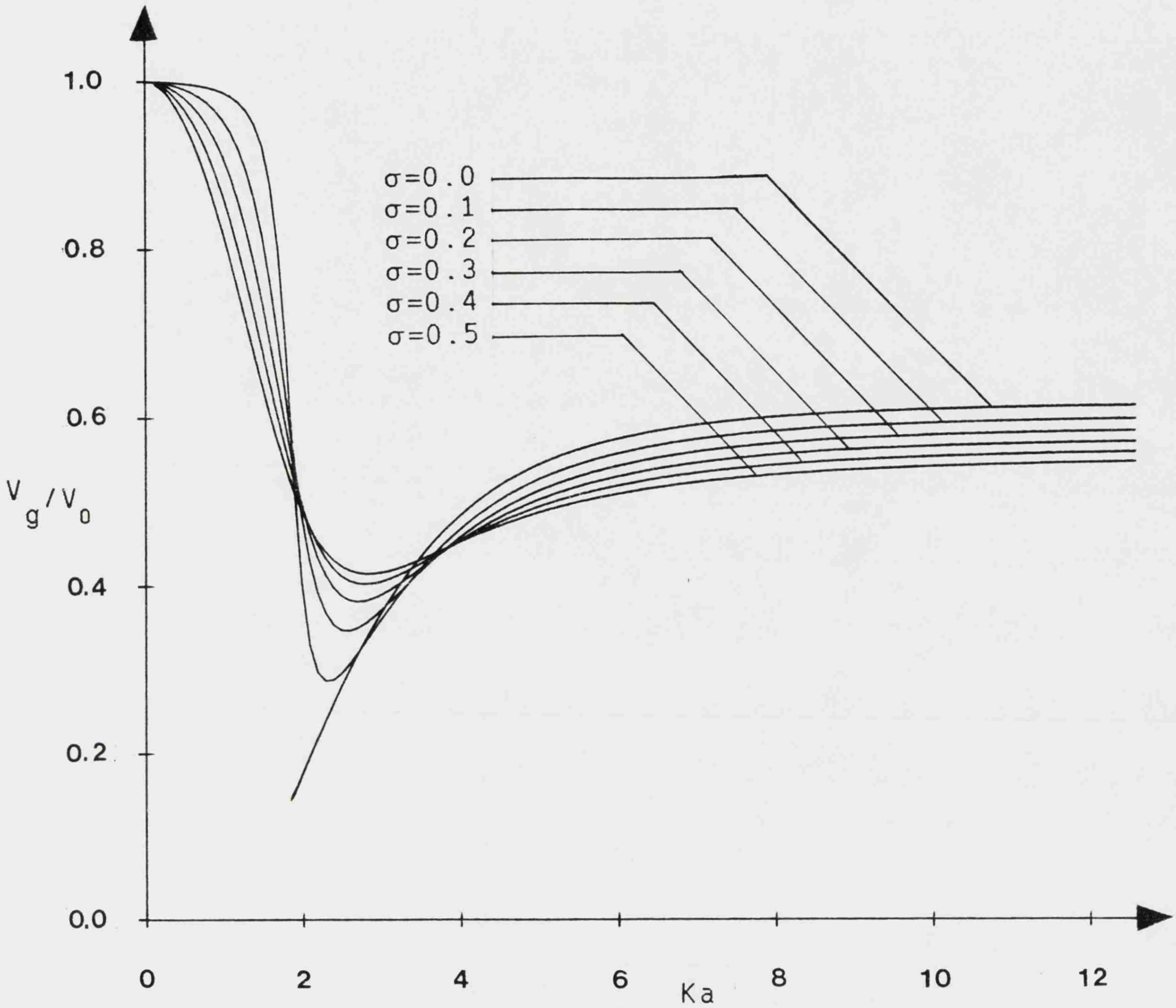

Figure 1.3

The group velocity V_g for longitudinal vibrations in circular cylinders of radius a with $V_0 = \surd(E/\rho)$. Curves are plotted for selected values of Poisson's ratio.

$$J_0(x) = 1-x^2/4+x^4/64-\ldots, \qquad J_1(x) = x/2-x^3/16+\ldots .$$

An approximation to the dispersion relation is found by neglecting $\lambda_1 a$ and $\lambda_2 a$. To this order of working

$$\psi(\lambda_1 a) = \psi(\lambda_2 a) \approx 2,$$

for which (1.9) reduces to

$$X[X(2-\alpha)-(6-4\alpha)] = 0.$$

Ignoring the trivial solution X=0, the speed of propagation of longitudinal waves through a solid cylinder is given to this approximation by

$$V = \surd[2\mu(1+\sigma)/\rho] = \surd(E/\rho).$$

This is in agreement with (1.8), obtained from the elementary treatment in which no dispersion occurs.

The next approximation has

$$\psi(x) \approx 2-x^2/4,$$

and leads from (1.9) to

$$V/V_0 = [1-\sigma(Ka)^2/4]. \tag{1.11}$$

This equation was due originally to Pochhammer [1], and was later derived by Rayleigh [16, §157] using energy considerations alone.

§1.4 FLEXURAL VIBRATIONS.

The anti-symmetric or flexural type of vibration are characterised by the forms

$$\begin{aligned} u &= U(r)\exp[i(\omega t+\theta-Kz)], \\ v &= iV(r)\exp[i(\omega t+\theta-Kz)], \\ w &= iW(r)\exp[i(\omega t+\theta-Kz)], \end{aligned} \tag{1.12}$$

so that we are now contemplating a solution which includes a transverse component of displacement together with a specified $e^{i\theta}$ azimuthal dependence.

The equations of motion (1.1), with (1.12), require

$$-\rho\omega^2 U = (\lambda+2\mu)(U_{rr}+U_r/r)-[(\lambda+3\mu)/r^2+\mu K^2]U$$

$$-[(\lambda+\mu)V_r/r-(\lambda+3\mu)V/r^2]+K(\lambda+\mu)W_r,$$

$$-\rho\omega^2 V = [(\lambda+\mu)U_r/r+(\lambda+3\mu)U/r^2]+\mu(V_{rr}+V_r/r) \qquad (1.13)$$

$$-[(\lambda+3\mu)/r^2+\mu K^2]V+(\lambda+\mu)KW/r,$$

$$-\rho\omega^2 W = -K(\lambda+\mu)(U_r+U/r-V/r)+\mu(W_{rr}+W_r/r-W/r^2)-(\lambda+2\mu)K^2 W,$$

which may be solved to give

$$U = K[PJ_1'(\lambda_2 r)+QJ_1'(\lambda_1 r)+RJ_1(\lambda_1 r)/r],$$

$$V = K[PJ_1(\lambda_2 r)/r+QJ_1(\lambda_1 r)/r+RJ_1'(\lambda_1 r)], \qquad (1.14)$$

$$W = -PK^2 J_1(\lambda_2 r)+Q\lambda_1^2 J_1(\lambda_1 r),$$

where P, Q, R are arbitrary constants and λ_1, λ_2 are defined by (1.5).

If we pause for a moment to consider the real parts of the radial and transverse components of displacement in (1.12), we see that both $u\sin\theta+v\cos\theta$ and w vanish when $r=0$. So points initially lying on the axis of the cylinder move along the axis of x_1. Thus equations (1.12) correspond to motion of the transverse or flexural type. Alternate formulations for flexural waves are possible; for example, we could try

$$u = U(r)\cos\theta.\exp\{i(\omega t-Kz)\},$$

$$v = V(r)\sin\theta.\exp\{i(\omega t-Kz)\},$$

$$w = W(r)\cos\theta.\exp\{i(\omega t-Kz)\},$$

but it is fairly readily seen that the results for this formulation will be in agreement with the findings based on (1.12).

The boundary conditions (1.2), with (1.12), reduce to

$$(L+2)U_r+L(U/r-V/r+KW) = 0,$$

$$U/r-V/r+V_r = 0, \tag{1.15}$$

$$-KU+W_r = 0,$$

on r=a, and these yield three relations between P, Q and R:

$$PJ_1(\lambda_2 a)[(2-X)(Ka)^2+2(2-\psi(\lambda_2 a))]$$

$$+2QJ_1(\lambda_1 a)[(2-\psi(\lambda_1 a))-(\lambda_1 a)^2]-2RJ_1(\lambda_1 a)[2-\psi(\lambda_1 a)] = 0,$$

$$PJ_1(\lambda_2 a)[2\psi(\lambda_2 a)-4]+2QJ_1(\lambda_1 a)[\psi(\lambda_1 a)-2]$$

$$+RJ_1(\lambda_1 a)[2(2-\psi(\lambda_1 a))-(\lambda_1 a)^2] = 0,$$

$$2PJ_1(\lambda_1 a)[\psi(\lambda_2 a)-1]+QJ_1(\lambda_1 a)[\psi(\lambda_1 a)-1](2-X)+RJ_1(\lambda_1 a) = 0,$$

so that for a non-trivial solution

$$\det(D_{ij}) = 0, \tag{1.16}$$

where, after a little tidying up,

$$D_{11} = (X-2), \qquad D_{22} = 2[\psi(\lambda_1 a)-2],$$

$$D_{12} = 2(X-1), \qquad D_{23} = 2[2-\psi(\lambda_1 a)]-(\lambda_1 a)^2,$$

$$D_{13} = (X-1), \qquad D_{31} = 2[\psi(\lambda_2 a)-1],$$

$$D_{21} = 2[\psi(\lambda_2 a)-2], \quad D_{32} = (2-X)[\psi(\lambda_1 a)-1], \quad D_{33} = 1.$$

Equation (1.16) is then the dispersion relation for flexural waves in a circular isotropic cylinder (c.f. Bancroft [14], Hudson [17], Abramson [18]). A detailed numerical investigation of this problem first carried out in [17] claimed incorrectly that (1.16) contained just one real root so that flexural vibrations were propagated in only a single mode. This error was passed on to later work [13], [15], [19] even though Holden [20] in his paper on elastic waves in cylinders and slabs, and later Abramson [18], both refer to the existence of higher modes for flexural vibrations in a cylinder. In Table 1.2 the values of V/V_0 for selected values of Ka and σ are presented. Fig. 1.4 displays the phase velocities of the fundamental mode for a range of values of Poisson's ratio, whilst in Fig. 1.5, for $\sigma = 0.25$, the phase velocity curves are plotted for the first four modes of (1.16).

For vibrations of large wavelength, expansion of the Bessel functions in (1.16) enabled Pochhammer to calculate the approximation

$$\frac{V}{V_0} = \left[1/2 - \frac{(4\sigma^2+15\sigma+10)(Ka)^2}{48(1+\sigma)}\right](Ka). \tag{1.17}$$

Finally, for very small wavelengths, (1.16) may be shown to reduce to the Rayleigh surface wave equation.

§1.5 TORSIONAL VIBRATIONS.

We mention briefly a third possible type of vibration of special interest: the torsional vibration. In this case both u and w vanish, and v is taken to be independent of θ. We have

$$v = V(r)\exp\{i(\omega t-Kz)\}.$$

From $(1.1)_2$, $V(r)$ must be proportional to $J_1(\lambda_1 r)$ with λ_1 defined by (1.5). For V to then satisfy the boundary condition $(1.2)_2$ requires

$$J_2(\lambda_1 a) = 0. \tag{1.18}$$

One solution of (1.18) is $\lambda_1 = 0$. This solution yields the corresponding form V = Ar, A constant, with wave-speed $\sqrt{(\mu/\rho)}$. The positive zeros of J_2, in (1.18), correspond to higher

Ka	σ = 0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.049793	0.049788	0.049783	0.049778	0.049773	0.049769	0.049764	0.049759	0.049755	0.049750	0.049746
0.2	0.098385	0.098346	0.098308	0.098270	0.098233	0.098197	0.098161	0.098125	0.098090	0.098055	0.098020
0.3	0.144746	0.144625	0.144505	0.144388	0.144272	0.144158	0.144046	0.143934	0.143824	0.143714	0.143606
0.4	0.188140	0.187877	0.187620	0.187368	0.187119	0.186874	0.186632	0.186393	0.186156	0.185921	0.185688
0.5	0.228142	0.227684	0.227235	0.226795	0.226361	0.225934	0.225512	0.225096	0.224683	0.224275	0.223869
0.6	0.264605	0.263906	0.263221	0.262549	0.261889	0.261237	0.260595	0.259960	0.259331	0.258708	0.258091
0.7	0.297589	0.296615	0.295662	0.294727	0.293807	0.292901	0.292006	0.291122	0.290247	0.289381	0.288521
0.8	0.327282	0.326012	0.324769	0.323550	0.322351	0.321170	0.320003	0.318850	0.317709	0.316578	0.315456
0.9	0.353941	0.352365	0.350822	0.349309	0.347820	0.346352	0.344902	0.343468	0.342049	0.340642	0.339246
1.0	0.377851	0.375967	0.374123	0.372312	0.370529	0.368771	0.367033	0.365314	0.363611	0.361923	0.360249
1.1	0.399299	0.397111	0.394968	0.392862	0.390788	0.388740	0.386716	0.384712	0.382726	0.380757	0.378804
1.2	0.418554	0.416070	0.413636	0.411242	0.408882	0.406551	0.404245	0.401961	0.399696	0.397450	0.395221
1.3	0.435863	0.433096	0.430382	0.427709	0.425072	0.422465	0.419884	0.417327	0.414790	0.412273	0.409775
1.4	0.451449	0.448412	0.445430	0.442490	0.439587	0.436714	0.433868	0.431045	0.428245	0.425466	0.422707
1.5	0.465509	0.462217	0.458979	0.455785	0.452627	0.449500	0.446398	0.443321	0.440267	0.437236	0.434227
1.6	0.478215	0.474683	0.471205	0.467770	0.464370	0.460999	0.457654	0.454333	0.451036	0.447763	0.444514
1.7	0.489720	0.485963	0.482259	0.478596	0.474966	0.471365	0.467788	0.464235	0.460706	0.457202	0.453724
1.8	0.500154	0.496189	0.492274	0.488397	0.484550	0.480729	0.476933	0.473159	0.469410	0.465687	0.461992
1.9	0.509635	0.505476	0.501364	0.497286	0.493236	0.489208	0.485204	0.481221	0.477263	0.473333	0.469432
2.0	0.518263	0.513925	0.509630	0.505364	0.501123	0.496902	0.492700	0.488521	0.484366	0.480239	0.476145
2.5	0.551346	0.546322	0.541308	0.536294	0.531276	0.526257	0.521244	0.516244	0.511268	0.506323	0.501421
3.0	0.572668	0.567253	0.561806	0.556319	0.550792	0.545234	0.539660	0.534086	0.528528	0.523003	0.517527
3.5	0.586547	0.580967	0.575314	0.569577	0.563760	0.557876	0.551948	0.545998	0.540053	0.534135	0.528266
4.0	0.595522	0.589930	0.584237	0.578422	0.572486	0.566445	0.560327	0.554161	0.547980	0.541812	0.535687
4.5	0.601198	0.595681	0.590052	0.584277	0.578348	0.572281	0.566100	0.559841	0.553541	0.547235	0.540956
5.0	0.604662	0.599245	0.593728	0.588060	0.582222	0.576217	0.570069	0.563810	0.557481	0.551122	0.544770
5.5	0.606674	0.601340	0.595938	0.590402	0.584695	0.578808	0.572754	0.566563	0.560273	0.553925	0.547560
6.0	0.607769	0.602479	0.597168	0.591753	0.586183	0.580436	0.574512	0.568432	0.562228	0.555939	0.549608
6.5	0.608313	0.603025	0.597765	0.592443	0.586994	0.581384	0.575600	0.569651	0.563562	0.557367	0.551104
7.0	0.608546	0.603221	0.597975	0.592709	0.587353	0.581861	0.576209	0.570395	0.564434	0.558350	0.552179
7.5	0.608619	0.603230	0.597964	0.592721	0.587425	0.582024	0.576485	0.570796	0.564961	0.558997	0.552931
8.0	0.608622	0.603149	0.597840	0.592592	0.587327	0.581988	0.576537	0.570955	0.565237	0.559390	0.553434
8.5	0.608605	0.603038	0.597668	0.592394	0.587136	0.581834	0.576448	0.570951	0.565333	0.559594	0.553745
9.0	0.608595	0.602929	0.597489	0.592174	0.586904	0.581618	0.576274	0.570841	0.565306	0.559661	0.553912
9.5	0.608604	0.602838	0.597322	0.591957	0.586662	0.581377	0.576056	0.570669	0.565197	0.559631	0.553970
10.0	0.608636	0.602771	0.597178	0.591758	0.586430	0.581132	0.575820	0.570463	0.565039	0.559536	0.553951
11.0	0.608765	0.602715	0.596971	0.591434	0.586025	0.580682	0.575359	0.570025	0.564656	0.559240	0.553768
12.0	0.608964	0.602748	0.596864	0.591214	0.585717	0.580314	0.574958	0.569616	0.564267	0.558895	0.553492
∞	0.618034	0.609807	0.602132	0.594906	0.588045	0.581481	0.575157	0.569031	0.563070	0.557248	0.551550

Table 1.2. V/V_0 as a function of Ka and σ for flexural vibrations in a circular cylinder of radius a.

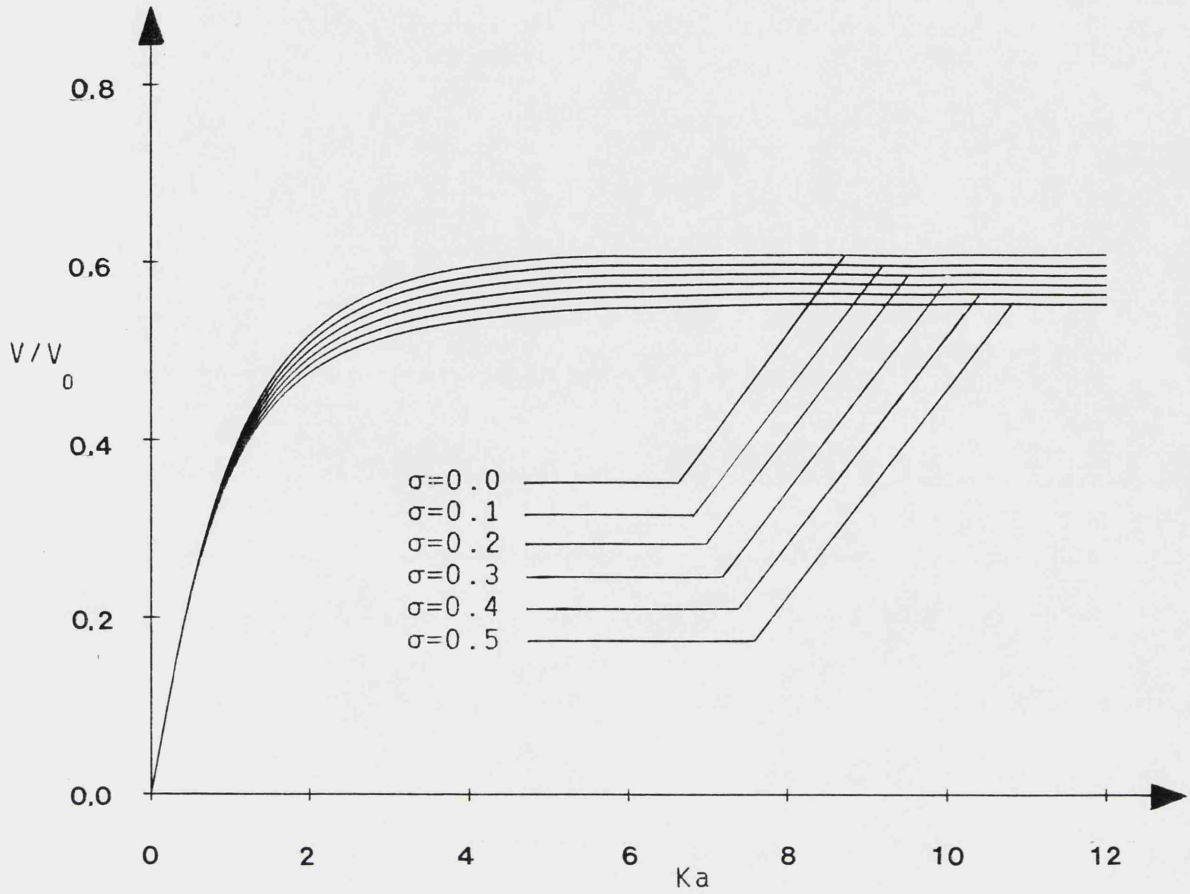

Figure 1.4

For selected values of Poisson's ratio curves of the phase velocity V ($V_0 = \sqrt{(E/\rho)}$) for flexural vibrations in a circular cylinder are plotted.

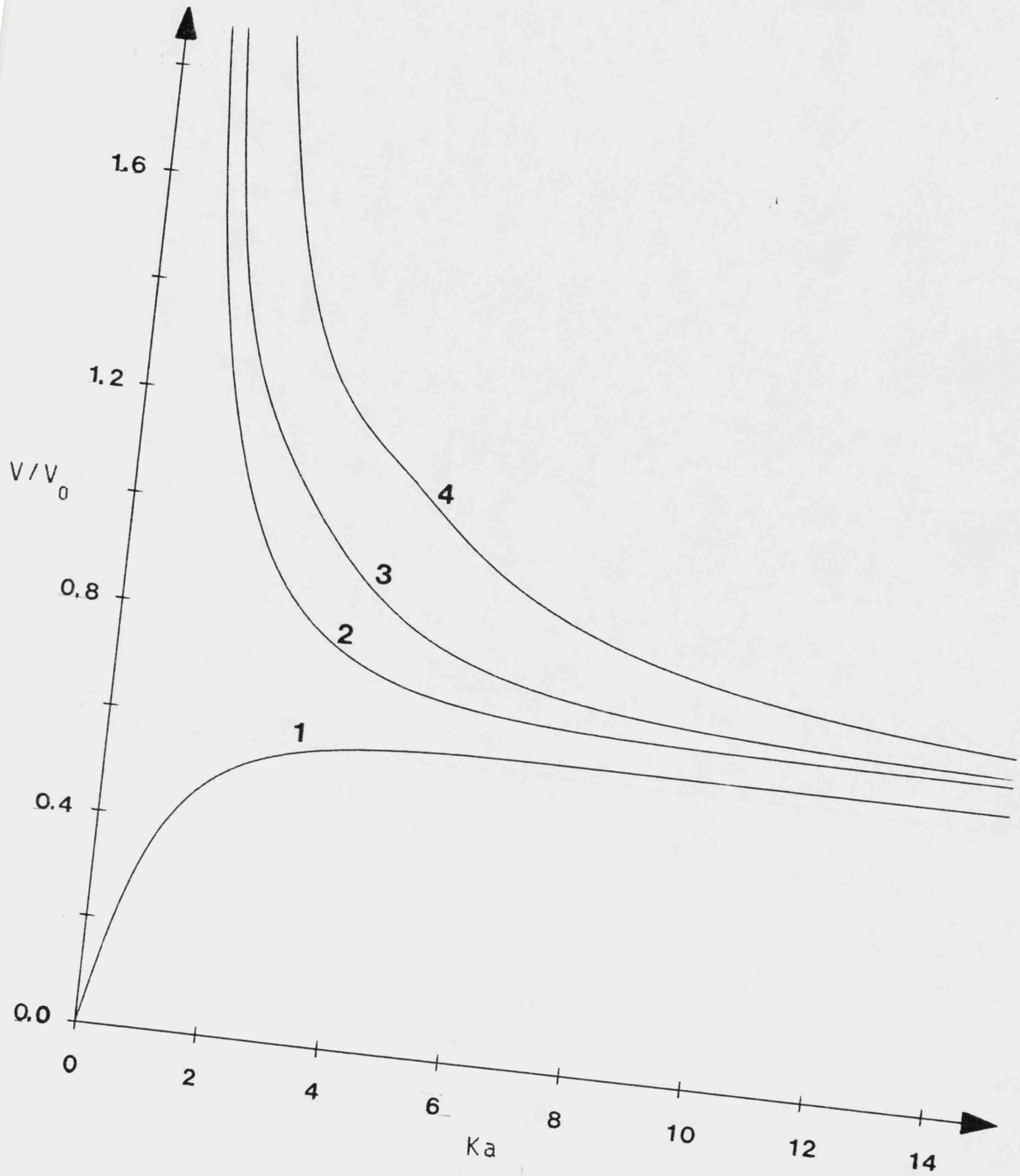

Figure 1.5

Phase velocity curves for the first four flexural modes of vibration in a circular cylinder with σ=0.25.

modes of vibration.

For an further account of torsional vibrations in a cylinder the reader is referred to Eringen and Suhubi [21, §8.10], or Davies [15].

SUCCESSIVE APPROXIMATION ANALYSIS FOR LONG WAVES

§1.6 LONGITUDINAL VIBRATIONS.

In this section we give an alternative treatment of longitudinal vibrations in an isotropic cylinder of circular cross-section, appropriate to the case that Ka is small, from which we shall recover (1.11), and then extend this result by calculating the coefficients of $(Ka)^4$ and $(Ka)^6$ in the dispersion relation.

The procedure is to presuppose a series development for U and W in ascending powers of Kr, and to develop recurrence relations for the coefficients from the equations of motion and the boundary conditions. Accordingly, we write

$$U = A\sum_{n=0}^{\infty}a_n(Kr)^{2n+1}, \qquad W = -A\sum_{n=0}^{\infty}c_n(Kr)^{2n},$$

where A and the coefficients a_n, c_n are constants. Introducing these expressions into (1.3) and equating powers of r^{2n} we find, (for $n\geqslant 0$)

$$\begin{aligned} Xa_n &= 2(n+1)(L+1)c_{n+1}-4(n+1)(n+2)(L+2)a_{n+1}+a_n, \\ Xc_n &= -4(n+1)^2c_{n+1}+(L+2)c_n-2(n+1)(L+1)a_n, \end{aligned} \tag{1.19}$$

with boundary conditions (1.6) now taking the forms

$$\begin{aligned} &\sum_{n=0}^{\infty}[2(n(L+2)+(L+1))a_n-Lc_n](Ka)^{2n} = 0, \\ &\sum_{n=0}^{\infty}[a_n+2(n+1)c_{n+1}](Ka)^{2n} = 0. \end{aligned} \tag{1.20}$$

The convergence of these series is assumed unconditionally since they are just the expansions of the Bessel functions encountered earlier. In effect, equations (1.19) determine a_1, $a_2,\ldots$, c_1, $c_2,\ldots$ in terms of X $(=\rho\omega^2/\mu K^2)$, a_0, c_0. The boundary conditions (1.20) impose two conditions between X and a_0/c_0, and the dispersion relation results from elimination of a_0/c_0 between these latter conditions.

The parameter X is now expressed as an asymptotic series in ascending powers of $(Ka)^2$ $(=\xi$, say), thus

$$X = X^{(0)}+X^{(1)}\xi+X^{(2)}\xi^2+\ldots+X^{(m)}\xi^m+\ldots,$$

with similar expressions for a_n and c_n. Substituting these series forms into (1.19) and equating like powers of ξ we find, after a little rearrangement,

$$\begin{aligned} a_n^{(m)} &= [2n(L+1)c_n^{(m)}+a_{n-1}^{(m)}-\Sigma_{p=0}^{m}X^{(m-p)}a_{n-1}^{(p)}]/4n(n+1)(L+2), \\ c_{n+1}^{(m)} &= [(L+2)c_n^{(m)}-2(n+1)(L+1)a_n^{(m)}-\Sigma_{p=0}^{m}X^{(m-p)}c_n^{(p)}]/4(n+1)^2, \end{aligned} \tag{1.21}$$

for $n\geqslant 1$, $m\geqslant 0$; and for $n=0$

$$X^{(m)} = [(L+2)c_0^{(m)}-2(L+1)a_0^{(m)}-4c_1^{(m)}]/c_0^{(0)}, \tag{1.22}$$

where, without loss of generality, we have taken

$$c_0^{(m)} = \begin{cases} 2(L+1)/L, & m=0, \\ 0, & m\geqslant 1. \end{cases} \tag{1.23}$$

Further, from (1.20), we calculate

$$\begin{aligned} a_0^{(m)} &= [Lc_0^{(m)}-\Sigma_{n=1}^{m}\{2(n(L+2)+(L+1))a_n^{(m-n)}-Lc_n^{(m-n)}\}]/2(L+1), \\ c_1^{(m)} &= -[a_0^{(m)}+\Sigma_{n=1}^{m}\{a_n^{(m-n)}+2(n+1)c_{n+1}^{(m-n)}\}]/2. \end{aligned} \tag{1.24}$$

for $m\geqslant 0$.

As demonstrated in §1.3 by the Pochhammer-Chree analysis, the initial two terms in the dispersion relation are fairly easily obtained; however, the working becomes prohibitive for higher approximations. Using relations (1.21)-(1.24) we propose to extend the approximation to the dispersion relation by a further two terms, demonstrating the relative ease with which new terms may be developed, whilst at the same time building up important details about the associated displacement field.

We begin with m=0; then from (1.24), with (1.23), we have

$$a_0^{(0)} = 1, \qquad c_1^{(0)} = -1/2,$$

so that, from (1.22),

$$X^{(0)} = (3L+2)/(L+1). \tag{1.25}$$

The general situation is that of known $a_n^{(k-n)}$, $c_{n+1}^{(k-n)}$, $X^{(k)}$ for $0 \leqslant n \leqslant k$, $0 \leqslant k \leqslant m-1$, some $m \geqslant 1$. Equations (1.21), with m-n entered for m, furnish $a_n^{(m-n)}$, $c_{n+1}^{(m-n)}$ for $1 \leqslant n \leqslant m$, and equations (1.24) then yield $a_0^{(m)}$ and $c_1^{(m)}$. Finally we use (1.22), with (1.23), to calculate $X^{(m)}$. Initiating this procedure with m=1, we calculate from (1.21)

$$a_1^{(0)} = \frac{-(L^2+4L+2)}{8(L+1)(L+2)}, \qquad c_2^{(0)} = \frac{(3L^2+6L+2)}{32(L+1)(L+2)},$$

then from (1.24)

$$a_0^{(1)} = \frac{(5L^2+12L+6)}{8(L+1)^2(L+2)}, \qquad c_1^{(1)} = \frac{-(2L^3+9L^2+14L+6)}{16(L+1)^2(L+2)},$$

so that (1.22) gives

$$X^{(1)} = \frac{-(3L+2)L^2}{8(L+1)^3}. \tag{1.26}$$

Thus we have easily established the result

$$X = X^{(0)}+X^{(1)}\xi = \left[\frac{3L+2}{L+1}\right]\left[1 - \frac{L^2\xi}{8(L+1)^2}\right],$$

which, expressed in dimensionless form, becomes

$$V/V_0 = [1-\sigma^2(Ka)^2/4],$$

and this is the result obtained by Pochhammer and Chree, previously given by (1.11).

For m=2 the procedure yields successively:

$$a_2^{(0)} = \frac{(3L^4+14L^3+26L^2+18L+4)}{192(L+1)^2(L+2)^2}, \qquad c_3^{(0)} = \frac{-(5L^4+26L^3+40L^2+22L+4)}{1152(L+1)^2(L+2)^2},$$

$$a_1^{(1)} = \frac{-(L^5+5L^4+18L^3+34L^2+25L+6)}{32(L+1)^3(L+2)^2},$$

$$c_2^{(1)} = \frac{-(2L^5-25L^3-49L^2-31L-6)}{128(L+1)^3(L+2)^2},$$

using (1.21), then from (1.24) we calculate

$$a_0^{(2)} = \frac{-(12L^5+23L^4-68L^3-208L^2-170L-44)}{96(L+1)^4(L+2)^2},$$

$$c_2^{(1)} = \frac{(10L^6+44L^5+37L^4-112L^3-260L^2-186L-44)}{192(L+1)^4(L+2)^2},$$

so that (1.22), with (1.23), gives

$$X^{(2)} = \frac{(2L^4-13L^3-56L^2-52L-14)L^2}{96(L+1)^5(L+2)}. \tag{1.27}$$

The procedure of solution to calculate the next term follows in the same way. The calculation is straightforward and we omit details of the coefficients at this stage to present simply the result

$$X^{(3)} = \frac{-(L^7-132L^6-389L^5+514L^4+2816L^3+3324L^2+1572L+264)L^2}{3072(L+1)^7(L+2)^2}. \tag{1.28}$$

We summarise the results (1.25)-(1.28) in dimensionless form with the expression

$$\frac{V}{V_0} = 1 - \frac{\sigma^2(Ka)^2}{4} - \frac{(21\sigma^4+4\sigma^3-29\sigma^2-4\sigma+7)\sigma^2(Ka)^4}{96(1-\sigma^2)} \tag{1.29}$$

$$\frac{-(396\sigma^7-252\sigma^6-824\sigma^5+480\sigma^4+492\sigma^3-254\sigma^2-69\sigma+33)\sigma^2(Ka)^6}{1536(1-\sigma^2)(1-\sigma)}.$$

§1.7 FLEXURAL VIBRATIONS.

For the flexural modes we take the series forms

$$U(r) = A\sum_{n=0}^{\infty} a_n (Kr)^{2n}, \qquad V(r) = A\sum_{n=0}^{\infty} b_n (Kr)^{2n}, \tag{1.30}$$

$$W(r) = A\sum_{n=0}^{\infty} c_n (Kr)^{2n+1},$$

A, a_n, b_n, c_n constants. We proceed in much the same way as in §1.6 to produce recurrence relations from the equations of motion and boundary conditions. Accordingly, from (1.13), we obtain for $n \geqslant 0$,

$$Xa_n = a_n - [4n(n+2)(L+2)+(3L+5)]a_{n+1}$$
$$+[2n(L+1)+(L-1)]b_{n+1} - (2n+1)(L+1)c_n,$$

$$Xb_n = b_n - [2n(L+1)+(3L+5)]a_{n+1} \tag{1.31}$$
$$+[(L-1)-4n(n+2)]b_{n+1} - (L+1)c_n,$$

$$Xc_n = (L+2)c_n + (L+1)[(2n+3)a_{n+1} - b_{n+1}] - 4(n+1)(n+2)c_{n+1},$$

whilst (1.15) provides

$$\sum_{n=0}^{\infty}[(2n(L+2)+(3L+4))a_{n+1} - Lb_{n+1} + Lc_n](Ka)^{2n} = 0,$$

$$\sum_{n=0}^{\infty}[a_{n+1} + (2n+1)b_{n+1}](Ka)^{2n} = 0, \tag{1.32}$$

$$\sum_{n=0}^{\infty}[a_n - (2n+1)c_n](Ka)^{2n} = 0.$$

Assuming asymptotic expansions for X, a_n, b_n and c_n in ascending powers of ξ $(=(Ka)^2)$ and equating coefficients of ξ^n in (1.31), we have, denoting $[16n(n+1)^2(n+2)(L+2)]^{-1}$ by Γ,

$$c_n^{(m)} = [(L+1)[(2n+1)a_n^{(m)} - b_n^{(m)}] + (L+2)c_{n-1}^{(m)}$$
$$- \sum_{p=0}^{m} X^{(m-p)} c_{n-1}^{(p)}]/4n(n+1),$$

$$a_{n+1}^{(m)} = \Gamma[[4(n+1)^2 - (L+3)][a_n^{(m)} - \sum_{p=0}^{m} X^{(m-p)} a_n^{(p)}]$$

$$+[2n(L+1)+(L-1)][b_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}] \qquad (1.33)$$

$$-4n(n+1)(2n+3)(L+1)c_n^{(m)}],$$

$$b_{n+1}^{(m)} = \Gamma[-[2n(L+1)+(3L+5)][a_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]$$

$$+[4n(n+2)(L+2)+(3L+5)][b_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}]$$

$$-4n(n+1)(L+1)c_n^{(m)}]],$$

for $n\geqslant 1$, $m\geqslant 0$. Additionally, from $(1.31)_1$ with $n=0$, we find

$$X^{(m)} = [a_0^{(m)}-(L+1)c_0^{(m)}-(3L+5)a_1^{(m)}+(L-1)b_1^{(m)}]/a_0^{(0)}, \qquad (1.34)$$

where, without loss of generality, we may take

$$a_0^{(m)} = \begin{cases} 1, & m=0, \\ 0, & m\geqslant 1, \end{cases} \qquad (1.35)$$

and, anticipating the result $X^{(0)} = 0$, the difference between $(1.31)_1$ and $(1.31)_2$ with $n=0$ gives $b_0^{(m)} = a_0^{(m)}$.

Similarly, from (1.32), we have

$$c_0^{(m)} = a_0^{(m)}+\sum_{n=1}^{m}\{a_n^{(m-n)}-(2n+1)c_n^{(m-n)}\},$$

$$a_1^{(m)} = -[Lc_0^{(m)}+\sum_{n=1}^{m}\{2((n(L+2)+2L+2)a_{n+1}^{(m-n)} \qquad (1.36)$$

$$+2nLb_{n+1}^{(m-n)}+Lc_n^{(m-n)}\}]/4(L+1),$$

$$b_1^{(m)} = -a_1^{(m)}-\sum_{n=1}^{m}\{a_{n+1}^{(m-n)}+(2n+1)b_{n+1}^{(m-n)}\}.$$

for $m\geqslant 0$.

Now, from (1.36)

$$c_0^{(0)} = 1, \qquad a_1^{(0)} = -b_1^{(0)} = -L/4(L+1),$$

then (1.34) with m set to zero gives, as expected,

$$X^{(0)} = 0. \qquad (1.37)$$

The general situation is that of known $a_{n+1}^{(k-n)}$, $b_{n+1}^{(k-n)}$,

$c_n^{(k-n)}$, $X^{(k)}$ for $0 \leq n \leq k$, $0 \leq k \leq m-1$, some $m \geq 1$. Equations (1.33), with $m-n$ for m, supply $c_n^{(m-n)}$, $a_{n+1}^{(k-n)}$, $b_{n+1}^{(k-n)}$ for $1 \leq n \leq m$. Then (1.36) yield successively $c_0^{(m)}$, $a_1^{(m)}$, $b_1^{(m)}$, and finally, equation (1.34), with (1.35), provides $X^{(m)}$.

We begin with m=1 and using (1.33) we calculate

$$c_1^{(0)} = 1/4, \qquad a_2^{(0)} = \frac{-(9L+5)}{192(L+1)}, \qquad b_2^{(0)} = \frac{(3L-1)}{192(L+1)},$$

and from (1.36)

$$c_0^{(1)} = \frac{-(4L+3)}{4(L+1)}, \qquad a_1^{(1)} = \frac{(24L^2+25L+5)}{96(L+1)^2}, \qquad b_1^{(1)} = \frac{-(24L^2+21L+1)}{96(L+1)^2},$$

so (1.40) provides

$$X^{(1)} = \frac{(3L+2)}{4(L+1)}. \tag{1.38}$$

With m=2 we calculate from (1.33)

$$c_2^{(0)} = 1/64, \qquad a_3^{(0)} = \frac{-(10L+7)}{4608(L+1)}, \qquad b_3^{(0)} = \frac{(2L-1)}{4608(L+1)},$$

$$c_1^{(1)} = \frac{-(15L+11)}{48(L+1)}, \qquad a_2^{(1)} = \frac{(129L^3+456L^2+425L+118)}{2304(L+1)^2(L+2)},$$

$$b_2^{(1)} = \frac{-(75L^3+192L^2+83L-14)}{2304(L+1)^2(L+2)}$$

then with (1.36)

$$c_0^{(2)} = \frac{(102L^2+159L+61)}{96(L+1)^2},$$

$$a_1^{(2)} = \frac{-(1176L^4+4677L^3+5950L^2+2887L+430)}{4608(L+1)^3(L+2)},$$

$$b_1^{(2)} = \frac{(1368L^4+5121L^3+5886L^2+2275L+134)}{4608(L+1)^3(L+2)},$$

so that (1.34) gives

$$X^{(2)} = \frac{-(37L^2+55L+20)}{48(L+1)^2}. \tag{1.39}$$

Finally, for m=3, we omit details of the coefficients and present the result

$$X^{(3)} = \frac{(4335L^3+10122L^2+7729L+1934).}{4608(L+1)^3} \tag{1.40}$$

We summarise (1.37)-(1.40) in dimensionless form by

$$\frac{V}{V_0} = \left[1/2 - \frac{(4\sigma^2+15\sigma+10)(Ka)^2}{48(1+\sigma)} + \frac{(724\sigma^3+2249\sigma^2+2294\sigma+767)(Ka)^4}{4608(1+\sigma)^2} \right] (Ka). \tag{1.41}$$

In Figs. 1.6 and 1.7 we plot the curves marked 1, 2, 3 for longitudinal and flexural waves respectively, where 1 is the full dispersion curve, 2 is the Pochhammer approximation for long waves, and 3 is the curve corresponding to the extended approximation given by (1.29) for longitudinal waves and by (1.41) for flexural waves.

VIBRATIONS IN A TRANSVERSELY ISOTROPIC CYLINDER.

We now look at the problem of small-amplitude vibrations in a circular cylinder composed of a material possessing an axis of symmetry that is parallel to the generators - transversely isotropic material. As before, we consider long waves propagating in the axial direction for both longitudinal and flexural modes. Our investigation is now complicated by the presence of five independent material constants, the stiffnesses, that replace the Lamé constants λ and μ. The stiffnesses (all assumed to be positive) enter the analysis through the following stress-strain relations

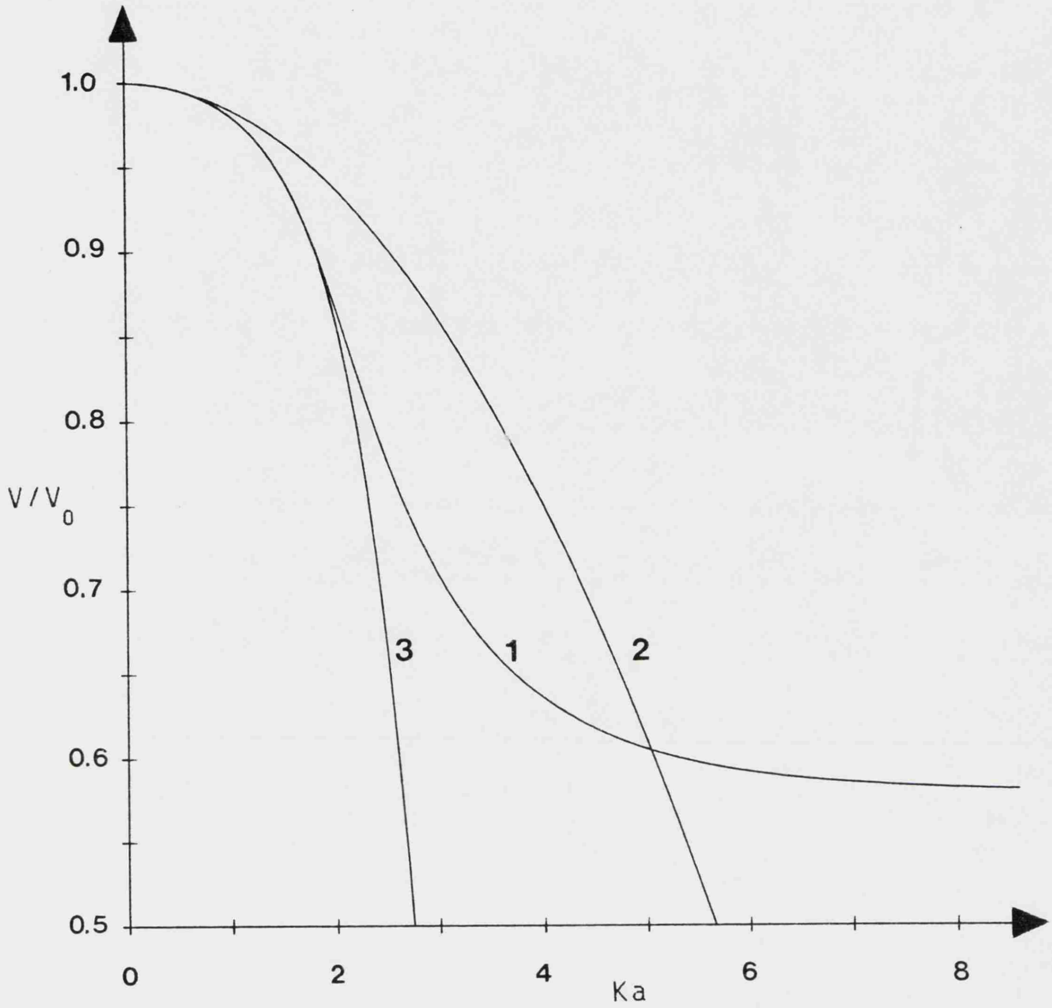

Figure 1.6

Phase velocity curves for longitudinal vibrations in a circular cylinder with $\sigma=0.25$. Curve 1 represents the full dispersion curve calculated from (1.9), curve 2 is Pochhammer's approximation (1.11) and curve 3 is the extended approximation (1.29).

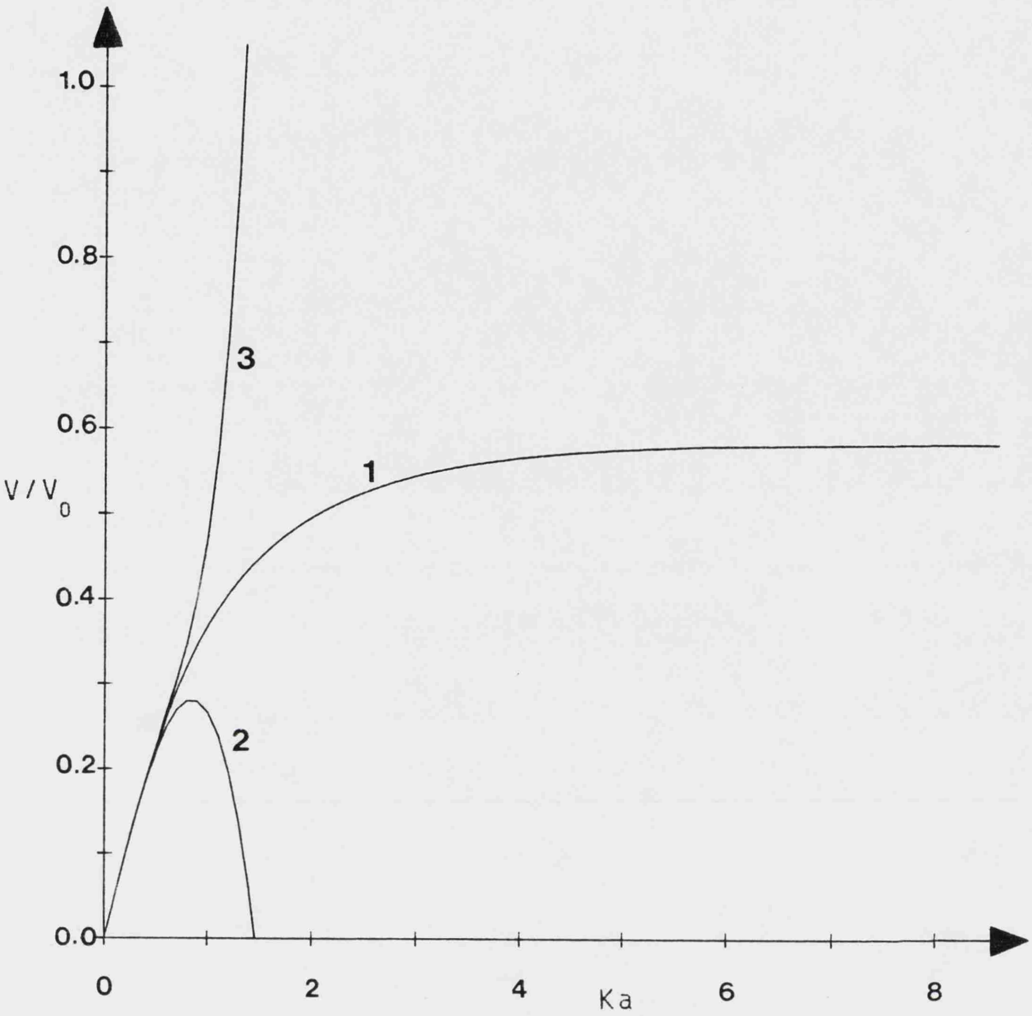

Figure 1.7

Phase velocity curves with $\sigma=0.25$ for flexural vibrations in a circular cylinder, where curve 1 is calculated from (1.16), curve 2 is Pochhammer's result (1.17) and curve 3 is given by equation (1.41).

$$P_{11} = Ae_{11}+Be_{22}+Ce_{33}, \qquad P_{12} = (A-B)e_{12},$$

$$P_{22} = Be_{11}+Ae_{22}+Ce_{33}, \qquad P_{13} = 2Ee_{13}, \qquad (1.42)$$

$$P_{33} = Ce_{11}+Ce_{22}+De_{33}, \qquad P_{23} = 2Ee_{23}.$$

Comparing (1.42) with the isotropic stress-strain relations we see

$$A = D = \lambda+2\mu, \qquad B = C = \lambda, \qquad E = \mu,$$

for the isotropic case.

§1.8 LONGITUDINAL VIBRATIONS.

For convenience we write the displacement field in cartesian coordinates. We have

$$u_1 = AK\sum_{n=0}^{\infty} a_n x_1 (Kr)^{2n} C, \qquad u_2 = AK\sum_{n=0}^{\infty} a_n x_2 (Kr)^{2n} C,$$

$$u_3 = A\sum_{n=0}^{\infty} c_n (Kr)^{2n} S,$$

A constant and $C = \cos(\omega t-Kx_3)$, $S = \sin(\omega t-Kx_3)$.

The stresses are easily calculated from (1.42), and the equations of motion, after comparing coefficients of r^{2n}, give

$$Xa_n = 2(n+1)(C+E)c_{n+1}-4(n+1)(n+2)Aa_{n+1}+Ea_n,$$

$$(1.43)$$

$$Xc_n = -4(n+1)^2 Ec_{n+1}+Dc_n-2(n+1)(C+E)a_n,$$

where now $X = \rho\omega^2/K^2$. On the cylinder surface the stress-free boundary conditions are now

$$\sum_{n=0}^{\infty}[((2n+1)A+B)a_n-Cc_n](Ka)^{2n} = 0,$$

$$(1.44)$$

$$\sum_{n=0}^{\infty}[a_n+2(n+1)c_{n+1}](Ka)^{2n} = 0.$$

We again assume asymptotic series, in ascending powers of $(Ka)^2$, for X and the coefficients a_n, c_n. Equating like powers of $(Ka)^2$, after substituting these series forms into (1.43), give

$$a_n^{(m)} = [2n(C+E)c_n^{(m)}+\alpha_7 a_{n-1}^{(m)} - \sum_{p=0}^{m} X^{(m-p)} a_{n-1}^{(p)}]/4n(n+1)A,$$
$$c_{n+1}^{(m)} = [Dc_n^{(m)}-2(n+1)(C+E)a_n^{(m)} - \sum_{p=0}^{m} X^{(m-p)} c_n^{(p)}]/4(n+1)^2 E, \qquad (1.45)$$

for $n \geqslant 1$, $m \geqslant 0$; and for $n=0$

$$X^{(m)} = [Dc_0^{(m)}-2(C+E)a_0^{(m)}-4Ec_1^{(m)}]/c_0^{(0)}, \qquad (1.46)$$

where we have taken

$$c_0^{(m)} = \begin{cases} (A+B)/C, & m=0, \\ 0, & m \geqslant 1. \end{cases} \qquad (1.47)$$

Similarly, the boundary conditions (1.44) give

$$a_0^{(m)} = [Cc_0^{(m)} - \sum_{n=1}^{m}\{((2n+1)A+B)a_n^{(m-n)}-Cc_n^{(m-n)}\}]/(A+B),$$
$$c_1^{(m)} = -[a_0^{(m)} + \sum_{n=1}^{m}\{a_n^{(m-n)}+2(n+1)c_{n+1}^{(m-n)}\}]/2. \qquad (1.48)$$

for $m \geqslant 0$.

The procedure of solution is now an extension of that given in §1.6 where equations (1.21)-(1.24) are now replaced by (1.45)-(1.48). Following the method prescribed in §1.6 we calculate, from (1.48)

$$a_0^{(0)} = 1, \qquad c_1^{(0)} = -1/2,$$

so that from (1.46), with $m=0$, we have

$$X^{(0)} = D-2C^2/(A+B).$$

We calculate a higher approximation for X from (1.45), and, denoting $Y = [D+C-2C^2/(A+B)]/A$, we find

$$a_1^{(0)} = -Y/8, \qquad c_2^{(0)} = [(C+E)Y-2C^2/(A+B)]/32E,$$

then, from (1.48)

$$a_0^{(1)} = [(3A+B)Y-4C]/8(A+B),$$

$$c_1^{(1)} = -[((3A+B)E+(A+B)C)Y-2C(2E+C)]/16(A+B)E,$$

so that (1.46) now gives

$$X^{(1)} = \left[\frac{-C^2}{2(A+B)^2}\right]\left[D - \frac{2C^2}{A+B}\right]$$

Finally, omitting details of the calculation at this stage, we present the result

$$X^{(2)} = \frac{-C^2 A}{4(A+B)^3}\left[\frac{Y^2(7A+B)}{12} - YC\left[1+\frac{C}{A+B}\right] + \frac{C^2}{3A}\left[1+\frac{3C}{A+B}\right]\right].$$

Substituting the isotropic values for the stiffnesses into the expressions for $X^{(0)}$, $X^{(1)}$ and $X^{(2)}$ we recover (1.25), (1.26) and (1.27) respectively.

§1.9 FLEXURAL VIBRATIONS.

For the displacement field we consider the forms

$$u_1 = A[a_0+K^2\sum_{n=0}^{\infty}(a_{n+1}x_1^2+b_{n+1}x_2^2)(Kr)^{2n}]C,$$

$$u_2 = AK^2\sum_{n=0}^{\infty}(a_{n+1}-b_{n+1})x_1x_2(Kr)^{2n}C,$$

$$u_3 = -AK\sum_{n=0}^{\infty}c_n x_1(Kr)^{2n}S,$$

(see Appendix A.1), from which we may calculate the following recurrence relations

$$\begin{aligned} 2Xa_n &= 2Ea_n-[(8n^2+16+5)A+B]a_{n+1} \\ &\quad +[(2n-1)A+(2n+3)B]b_{n+1}-2(2n+1)(C+E)c_n, \\ 2Xb_n &= 2Eb_n-[(2n+5)A+(2n+1)B]a_{n+1} \\ &\quad -[(4n^2+8n+1)A-(2n+1)(2n+3)]b_{n+1}-2(C+E)c_n, \\ Xc_n &= Dc_n+(C+E)[(2n+3)a_{n+1}-b_{n+1}]-4(n+1)(n+2)Ec_{n+1}, \end{aligned} \tag{1.49}$$

for the equations of motion along with

$$\sum_{n=0}^{\infty}[(2(n+1)A+B)a_{n+1}-Bb_{n+1}+Cc_n](Ka)^{2n} = 0,$$

$$\sum_{n=0}^{\infty}[a_{n+1}+(2n+1)b_{n+1}](Ka)^{2n} = 0, \tag{1.50}$$

$$\sum_{n=0}^{\infty}[a_n-(2n+1)c_n](Ka)^{2n} = 0,$$

for the boundary conditions. Assuming the usual expansions for X, a_n, b_n and c_n in (1.49) we calculate for the coefficients the following relations:

$$c_n^{(m)} = [(C+E)[(2n+1)a_n^{(m)}-b_n^{(m)}]+Dc_{n-1}^{(m)}$$
$$-\sum_{p=0}^{m}X^{(m-p)}c_{n-1}^{(p)}]/4n(n+1)E,$$

$$a_{n+1}^{(m)} = [[(4n^2+8n+1)A-(2n+1)(2n+3)B][Ea_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]$$
$$+[(2n-1)A+(2n+3)B][Eb_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}] \tag{1.51}$$
$$-4n(n+1)(2n+3)(C+E)(A-B)c_n^{(m)}]/16n(n+1)^2(n+2)A(A-B),$$

$$b_{n+1}^{(m)} = [-[(2n+5)A+(2n+1)B][Ea_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]$$
$$+[(8n^2+16n+5)A+B][Eb_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}]$$
$$-4n(n+1)(A-B)(C+E)c_n^{(m)}]/16n(n+1)^2(n+2)A(A-B),$$

for $n\geqslant 1$, $m\geqslant 0$, and

$$X^{(m)} = [2Ea_0^{(m)}-(5A+B)a_1^{(m)}-(A-3B)b_1^{(m)}-2(C+E)c_0^{(m)}]/2, \tag{1.52}$$

for $n=0$, where

$$a_0^{(m)} = b_0^{(m)} = \begin{cases} 1, & m=0, \\ 0, & m\geqslant 1. \end{cases} \tag{1.53}$$

Similarly, from (1.50), we calculate

$$c_0^{(m)} = a_0^{(m)}+\sum_{n=1}^{m}\{a_n^{(m-n)}-(2n+1)c_n^{(m-n)}\},$$

$$a_1^{(m)} = -[Cc_0^{(m)} + \sum_{n=1}^{m}\{2((n+1)A+B)a_{n+1}^{(m-n)} + 2nBb_{n+1}^{(m-n)} + Cc_n^{(m-n)}\}]/2(A+B), \tag{1.54}$$

$$b_1^{(m)} = -a_1^{(m)} - \sum_{n=1}^{m}\{a_{n+1}^{(m-n)} + (2n+1)b_{n+1}^{(m-n)}\},$$

for $m \geqslant 0$.

Equations (1.51)-(1.54) now replace equations (1.33)-(1.36) respectively and are solved using the same procedure set down in §1.7. We begin, therefore, with (1.54) from which we find

$$c_0^{(0)} = 1, \qquad a_1^{(0)} = -b_1^{(0)} = -C/2(A+B),$$

then at once, from (1.52),

$$X^{(0)} = 0,$$

just as we had for the isotropic case.

To the next approximation in X, we obtain, from (1.51)

$$c_1^{(0)} = [D-2C(C+E)/(A+B)]/8E,$$

$$a_2^{(0)} = [(3A-5B)M-10N]/96, \qquad b_2^{(0)} = -[(9A+B)M+2N]/96,$$

where we denote $M = 2Ea_1^{(0)}/A(A-B)$ and $N = 2(C+E)c_1^{(0)}/A$, so that, with (1.54), we have

$$c_0^{(1)} = [2C(3C+E)/(A+B)-3D]/8E,$$

$$a_1^{(1)} = \frac{-1}{48(A+B)}\left[12AC\left[\frac{(A-B)M}{E} - \frac{2N}{C+E}\right] + (3A+B)(A-3B)M - 2N(5A+3B)\right],$$

$$b_1^{(1)} = \frac{1}{48(A+B)}\left[12AC\left[\frac{(A-B)M}{E} - \frac{2N}{C+E}\right] + (3A+B)(5A+B)M - 2N(A-B)\right],$$

then (1.52) provides

$$X^{(1)} = D/4 - C^2/2(A+B).$$

Finally, to the next approximation and following the method outlined in §1.7, we calculate

$$X^{(2)} = \frac{-7[X^{(1)}]^2}{6E} - \frac{X^{(1)}}{12}\left[3 - \frac{2C}{A+B}\right] + \frac{C^2E}{24(A+B)^2};$$

it is then a straightforward matter to recover (1.38) and (1.39) in the isotropic case for the above values of $X^{(1)}$, $X^{(2)}$.

ISOTROPIC CYLINDER OF ELLIPTIC CROSS-SECTION.

We have seen in the previous sections that for circular cylinders the dispersion relation for large wavelengths may be constructed by taking appropriate series expansions for the displacement field. This procedure is now extended to include isotropic cylinders of elliptic cross-section. We take the bounding ellipse to be $x_1^2/a^2+x_2^2/b^2 = 1$ and assume $Ka<<1$, $Kb<<1$ wherever necessary.

The algebraic manipulation in this problem rapidly becomes intractable, so we restrict ourselves to working to the second order in K.

It should be mentioned that the analysis presented in §1.10, §1.11 is only valid, insofar as one cannot claim unconditional convergence of the Taylor series that are introduced.

§1.10 LONGITUDINAL VIBRATIONS.

We seek a solution of the form

$$u_1 = Kx_1[A_0+K^2(A_1x_1^2+A_2x_2^2)+\ldots]C,$$

$$u_2 = Kx_2[B_0+K^2(B_1x_1^2+B_2x_2^2)+\ldots]C,$$

$$u_3 = [C_0+K^2(C_1x_1^2+C_2x_2^2)+K^4(C_3x_1^4+C_4x_1^2x_2^2+C_5x_2^4)+\ldots]S,$$

with $C = \cos(\omega t - Kz)$, $S = \sin(\omega t - Kz)$, as before.

What is attempted here is to determine the coefficients in the above expansions in succession and to a progressively higher degree of approximation in ascending powers of Ka, Kb (briefly powers of K). The approach is a development of that given in §1.6 for the circular cylinder.

First we set out the expressions for the components of stress as far as it is necessary to carry them for present purposes:

$$P_{11} = \mu K[[2A_0 + L(A_0 + B_0 - C_0)] + K^2[(6A_1 + L(3A_1 + B_1 - C_1))x_1^2 + (2A_2 + L(A_2 + 3B_2 - C_2))x_2^2]]C,$$

$$P_{22} = \mu K[[2B_0 + L(B_0 + A_0 - C_0)] + K^2[(2B_1 + L(3A_1 + B_1 - C_1))x_1^2 + (6B_2 + L(A_2 + 3B_2 - C_2))x_2^2]]C,$$

$$P_{33} = \mu K[[-2C_0 + L(A_0 + B_0 - C_0)] + K^2[(-2C_1 + L(3A_1 + B_1 - C_1))x_1^2 + (-2C_2 + L(A_2 + 3B_2 - C_2))x_2^2]]C,$$

$$P_{12} = 2\mu x_1 x_2 K^3 [A_2 + B_1]C,$$

$$P_{13} = \mu x_1 K^2 [[A_0 + 2C_1] + K^2[(A_1 + 4C_3)x_1^2 + (A_2 + 2C_4)x_2^2]]S,$$

$$P_{23} = \mu x_2 K^2 [[B_0 + 2C_2] + K^2[(B_1 + 2C_4)x_1^2 + (B_2 + 4C_5)x_2^2]]S,$$

where $L = \lambda/\mu$. These expressions may then be substituted into the equations of motion $P_{ij,j} = -\rho\omega^2 u_i$ and the stress-free surface conditions $P_{ij}\nu_j = 0$.

The equations of motion identify as zero certain series expressions in which successive terms are homogeneous polynomials in x_1, x_2; it must be the case that the coefficients in these series vanish identically and this forces relations between the coefficients. Most simply, from $p_{3j,j} = -\rho\omega^2 u_3$ one finds by equating to zero the terms independent of x_1, x_2

$$-XC_0 = 2C_1+2C_2+A_0+B_0-2C_0+L(A_0+B_0-C_0). \qquad (1.55)$$

This is regarded as an equation for X $(= \rho\omega^2/\mu K^2)$, that is, as the dispersion relation, when the ratios A_0/C_0, B_0/C_0, C_1/C_0, C_2/C_0 are known. Improvement in X is obtained by iterative development of these ratios in ascending powers of K.

In the same way the surface stresses are series expressions in x_1, x_2, but with the added simplification $x_1^2/a^2+x_2^2/b^2 = 1$. After the parametric substitution $x_1 = a\cos\varphi$, $x_2 = b\sin\varphi$ what emerges is a series in ascending powers of K^2 in which the coefficients are polynomials in $\cos^2\varphi$ (after appropriate application of $\cos^2\varphi+\sin^2\varphi = 1$). The terms in these series must vanish identically, and this in turn requires the identical vanishing of the polynomial coefficients; we find that at each stage of the procedure the number of equations present is equal to the number of coefficients required. Taking first the leading terms in the three components of surface stress one finds

$$2A_0+L(A_0+B_0-C_0) = 0,$$

$$2B_0+L(A_0+B_0-C_0) = 0,$$

$$(B_0+2C_2)+(A_0-B_0+2C_1-2C_2)\cos^2\varphi = 0.$$

Without loss of generality we set $C_0 = 2(L+1)/L$ then it is clear that

$$A_0 = B_0 = -2C_1 = -2C_2 = 1,$$

to zero order in K, and one may now calculate from (1.55)

$$X = (3L+2)/(L+1).$$

The iterative process continues by developing zero order expressions for A_1, A_2, B_1, B_2, C_3, C_4, C_5 and at the same time, first order corrections for A_0, B_0, C_1, C_2 and hence also for X. Terms independent of x_1, x_2 in the equations of motion for i=1,2 give

$$-XA_0 = 12A_1+2B_1+2A_2-2C_1-A_0+2L(3A_1+B_1-C_1),$$

$$-XB_0 = 12B_2+2B_1+2A_2-2C_2-B_0+2L(A_2+3B_2-C_2),$$

while, from inspection of the terms quadratic in x_1, x_2 in the equations of motion for i=3, we find

$$-XC_1 = 3A_1+B_1-2C_1+12C_3+2C_4+L(3A_1+B_1-C_1),$$

$$-XC_2 = A_2+3B_2-2C_2+2C_4+12C_5+L(A_2+3B_2-C_2).$$

(We have two equations and not three since there is no x_1x_2 term).

In the three surface conditions, the terms to first order in K^2 give rise to seven conditions (two linear and one quadratic polynomial in $\cos^2\varphi$), and it is then a matter of straightforward but tedious algebra to compute

$$A_1 = \frac{-(L^2+4L+2)[L(Q^2+2Q+3)+2(Q^2+Q+1)],}{12(Q+1)^2(L+1)^2(L+2)}$$

$$B_1 = \frac{-(L^2+4L+2)[L(Q^2+2Q-1)+2(Q^2+Q-1)],}{4(Q+1)^2(L+1)^2(L+2)}$$

$$C_3 = \frac{[L^3(11Q^2+22Q+3)+4L^2(9Q^2+19Q-1)+16L(2Q^2+5Q-1)+8(Q^2+3Q-1)],}{96(Q+1)^2(L+1)^2(L+2)}$$

$$C_4 = \frac{[L^3(5Q^2+2Q+5)+4L^2(5Q^2-Q+5)+8L(3Q^2-2Q+3)+8(Q^2-Q+1)],}{16(Q+1)^2(L+1)^2(L+2)}$$

$$A_0 = 1 + \frac{[(L+2)Q-L][L^2(3Q^2+4Q+3)+8L(Q^2+Q+1)+4(Q^2+Q+1)](Kb)^2}{8(Q+1)^2(L+1)^2(L+2)},$$

$$C_1 = -1/2 - \frac{\begin{bmatrix} 3L^3(6Q^3+7Q^2-2Q-3)+2L^2(33Q^3+40Q^2-7Q-12) \\ +2L(36Q^3+52Q^2+2Q-6)+8(3Q^3+5Q^2+Q) \end{bmatrix}(Kb)^2}{48(Q+1)^2(L+1)^2(L+2)},$$

where $Q=(a/b)^2$. Expressions for B_2, A_2, C_5, B_0 and C_2 are not given explicitly above as they may be obtained by interchanging the parameters a and b in A_1, B_1, C_3, A_0 and C_1 respectively. From (1.59), it follows at once

$$X = \left[\frac{3L+2}{L+1}\right]\left[1 - \frac{L^2K^2(a^2+b^2)}{16(L+1)^2}\right],$$

from which, with a=b, we recover (1.25) and (1.26) from the results for the circular cylinder.

§1.11 FLEXURAL VIBRATIONS.

For flexural vibrations in an isotropic cylinder of elliptic cross-section we seek a solution of the form

$$u_1 = [A_0+K^2(A_1x_1^2+A_2x_2^2)+K^4(A_3x_1^4+A_4x_1^2x_2^2+A_5x_2^4)+\ldots]C,$$

$$u_2 = K^2x_1x_2[B_0+K^2(B_1x_1^2+B_2x_2^2)+\ldots]C,$$

$$u_3 = -Kx_1[C_0+K^2(C_1x_1^2+C_2x_2^2)+\ldots]S,$$

where again $C = \cos(\omega t-Kx_3)$, $S = \sin(\omega t-Kx_3)$, so that the principal flexure plane is Ox_1x_3.

We compute the stresses so far as they are required:

$$P_{11} = \mu K^2x_1[[4A_1+L(2A_1+B_0+C_0)]+K^2[(8A_3+L(4A_3+B_1+C_1))x_1^2 + (4A_4+L(2A_4+3B_2+C_2))x_2^2]]C,$$

$$P_{22} = \mu K^2 x_1 [[2B_0+L(2A_1+B_0+C_0)]+K^2[(2B_1+L(4A_3+B_1+C_1))x_1^2$$
$$+(6B_2+L(2A_4+3B_2+C_2))x_2^2]]C,$$

$$P_{33} = \mu K^2 x_1 [[2C_0+L(2A_1+B_0+C_0)]+K^2[(2C_1+L(4A_3+B_1+C_1))x_1^2$$
$$+(2C_2+L(2A_4+3B_2+C_2))x_2^2]]C,$$

$$P_{12} = \mu K^2 x_2 [2A_2+B_0+K^2[(2A_4+3B_1)x_1^2+(4A_5+B_2)x_2^2]]C,$$

$$P_{13} = \mu K[A_0-C_0+K^2[(A_1-3C_0)x_1^2+(A_2-C_2)x_2^2]]S,$$

$$P_{23} = \mu K^3 x_1 x_2 [B_0-2C_2]S.$$

As in the previous case for longitudinal vibrations, the equations of motion produce homogeneous polynomials in x_1, x_2 with coefficients that must vanish, and we find from the coefficient independent of x_1, x_2 in $P_{1j,j} = -\rho u_1 \omega^2$ the relation

$$XA_0 = A_0-C_0-4A_1-(2A_2+B_0)-L(2A_1+B_0+C_0), \qquad (1.56)$$

where X is determined after calculating the ratios A_1/A_0, A_2/A_0, B_0/A_0 and C_0/A_0.

The stress-free surface conditions (with the substitution $x_1 = a\cos\varphi$ and $x_2 = b\sin\varphi$) again yield polynomials in $\cos^2\varphi$ and considering only the zero-order terms in K from the components of the boundary conditions, we calculate

$$(2A_2+B_0)+[4A_1-2A_2-B_0+L(2A_1+B_0+C_0)]\cos^2\varphi = 0,$$

$$(2A_2+B_0)+Q[2B_0+L(2A_1+B_0+C_0)] = 0,$$

$$(A_0-C_0) = 0,$$

with $Q=(a/b)^2$ as before. We begin by setting $A_0=1$ so that

$$C_0 = 1, \qquad B_0 = 2A_1 = -2A_2 = -L/2(L+1).$$

At once, from (1.56),

$$X = 0.$$

The next approximation for X is found by calculating zero order values for B_1, B_2, C_1, C_2, A_3, A_4, A_5 and then first order (in K^2) corrections for C_0, B_0, A_1 and A_2.

With i=1 in the equations of motion, and equating to zero the coefficients of the quadratic terms in x_1, x_2, we have

$$XA_1 = A_1-3C_1-24A_3-(2A_4+3B_1)-3L(4A_3+B_1+C_1),$$

$$XA_2 = A_2-C_2-4A_4-(12A_5+3B_2)-L(2A_4+3B_2+C_2),$$

whilst for i=2,3 terms independent of x_1, x_2 give

$$XB_0 = B_0-2C_2-4A_4-6B_1-12B_2-2L(2A_4+3B_2+C_2),$$

$$XC_0 = 2C_0+2A_1+B_0-6C_1-2C_3+L(2A_1+B_0+C_1).$$

From the surface conditions we establish a further seven conditions to first order in K^2 by setting up (as we did in §1.10) two linear and one quadratic equation in $\cos^2\varphi$.

We present the following results for the coefficients:

$$C_2 = \frac{4(2Q+1)+3L(3Q+1)}{12(3Q+1)(L+1)}, \qquad C_3 = \frac{4Q+L(3Q+1)}{4(3Q+1)(L+1)},$$

$$A_3 = \frac{-[9L^2(3Q^3+7Q^2+17Q+5)+4L(10Q^3+25Q^2+56Q+21)+4(3Q^3+9Q^2+19Q+9)]}{192(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$A_4 = \frac{-[L^2(3Q^3+7Q^2+17Q+5)+L(10Q^3+14Q^2+38Q+2)+8(Q^3+Q^2+2Q)]}{32(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$A_5 = \frac{[3L^2(3Q^3+7Q^2+17Q+5)+4L(3Q^3+4Q^2+7Q+2)+4(Q^3-Q^2-7Q-1)]}{192(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$B_2 = \frac{-[3L^2(3Q^3+7Q^2+17Q+5)+2L(5Q^3+11Q^2+35Q+13)+8(3Q+1)]}{48(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$B_3 = \frac{-[3L^2(3Q^3+7Q^2+17Q+5)+4L(3Q^3+8Q^2+19Q+2)+4(Q^3+3Q^2+5Q-1)]}{48(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$C_0 = 1 - \frac{[(3Q+1)L+(2Q+1)](Ka)^2}{(3Q+1)(L+1)},$$

$$B_0 = \frac{-L}{2(L+1)} + \frac{\left[\begin{array}{l} 3L^2(21Q^4+52Q^3+126Q^2+52Q+5)+2L(21Q^4+63Q^3 \\ \qquad +155Q^2+117Q+12)+4(3Q^3+9Q^2+15Q-3)\end{array}\right](Kb)^2}{48(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$A_1 = \frac{-L}{4(L+1)} + \frac{\left[\begin{array}{l} 3L^2(27Q^4+60Q^3+146Q^2+28Q-5)+2L(36Q^4+87Q^3 \\ \qquad +205Q^2+69Q+3)+4(3Q^4+9Q^3+19Q^2+9Q)\end{array}\right](Kb)^2}{96(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

$$A_2 = \frac{-L}{4(L+1)} + \frac{\left[\begin{array}{l} 3L^2(21Q^4+52Q^3+126Q^2+52Q+5)+6L(7Q^4+21Q^3 \\ \qquad +49Q^2+31Q+4)+4(3Q^3+5Q^2+3Q-3)\end{array}\right](Kb)^2}{96(3Q+1)(Q^2+2Q+5)(L+1)^2},$$

so we now find

$$X = \frac{(3L+2)(Ka)^2}{4(L+1)}.$$

A particularly interesting feature of this result is that the wave-speed is unaffected by changes in the length b along the axis transverse to the direction of the vibration. Details of the displacement field, however, are certainly altered.

APPENDIX A.1

In this section we present the analysis which relates the polar displacements u, v, w to their cartesian equivalents u_1, u_2, u_3.

From (1.12), (1.30) we have

$$u = A\sum_{n=0}^{\infty}a_n(Kr)^{2n}.\exp[i(\omega t+\theta-Kz)],$$

$$v = iA\sum_{n=0}^{\infty}b_n(Kr)^{2n}.\exp[i(\omega t+\theta-Kz)], \qquad (A.1.1)$$

$$w = iA\sum_{n=0}^{\infty}c_n(Kr)^{2n+1}.\exp[i(\omega t+\theta-Kz)].$$

Now by replacing i, ω, K by -i, -ω, -K throughout (so that the wave-speed is unaffected by these changes) and then adding these new forms to their previous values in (A.1.1), we calculate

$$u = A\sum_{n=0}^{\infty}a_n(Kr)^{2n}\cos\theta.\exp[i(\omega t-Kz)],$$

$$v = -A\sum_{n=0}^{\infty}b_n(Kr)^{2n}\sin\theta.\exp[i(\omega t-Kz)],$$

$$w = iA\sum_{n=0}^{\infty}c_n(Kr)^{2n+1}\cos\theta.\exp[i(\omega t-Kz)],$$

and taking real parts

$$u = A\sum_{n=0}^{\infty}a_n(Kr)^{2n}\cos\theta.\cos(\omega t-Kz),$$

$$v = -A\sum_{n=0}^{\infty}b_n(Kr)^{2n}\sin\theta.\cos(\omega t-Kz), \qquad (A.1.2)$$

$$w = -A\sum_{n=0}^{\infty}c_n(Kr)^{2n+1}\cos\theta.\sin(\omega t-Kz),$$

this then is a real wave travelling along the Oz-axis.

The transformations from polar to cartesian coordinates are:

$$u_1 = u\cos\theta - v\sin\theta, \qquad x_1 = r\cos\theta,$$

$$u_2 = u\sin\theta + v\cos\theta, \qquad x_2 = r\sin\theta, \qquad (A.1.3)$$

$$u_3 = w, \qquad x_3 = z.$$

It follows then, from (A.1.3) with (A.1.2),

$$u_1 = A[a_0 + K^2 \sum_{n=0}^{\infty} (a_{n+1} x_1^2 + b_{n+1} x_2^2)(Kr)^{2n}]\cos(\omega t - Kx_3),$$

$$u_2 = AK^2 \sum_{n=0}^{\infty} (a_{n+1} - b_{n+1}) x_1 x_2 (Kr)^{2n} \cos(\omega t - Kx_3),$$

$$u_3 = -AK \sum_{n=0}^{\infty} c_n x_1 (Kr)^{2n} \sin(\omega t - Kx_3),$$

provided we have $a_0 = b_0$, and these are the forms used in §1.9.

CHAPTER 2

VIBRATIONS IN SLENDER CYLINDERS UNDER STRESS

§2.1 INTRODUCTION.

In this chapter we present an analysis of the propagation of small-amplitude waves along an elastic cylinder upon which there has been imposed a large primary stress. The cylinder is taken to have circular cross-section and to be infinite in length. Our objective is to obtain the dispersion relation linking frequency and wavelength of the small-amplitude waves (treated as a perturbation), and to obtain details of the associated displacement field, for waves whose wavelength is large compared to the cylinder radius. The material is taken to be homogeneous, isotropic and hyperelastic with strain-energy function W but, at any rate in the first instance, no particular form for W is assumed. Consideration is given to both compressible and incompressible materials, and for each case results are obtained for both longitudinal and flexural modes. We consider, however, only those modes for which the wave-velocity remains finite as $Ka\to 0$ (a the cylinder radius, K the wavenumber). Our results are illustrated in detail for particular types of material.

The plan of analysis is as follows. In §2.2 we consider the steady state of the material produced after the imposition of the primary stress. In §2.3 we consider incompressible materials and obtain the equations of motion governing the vibrations of small-amplitude and the appropriate boundary conditions. In §2.4, §2.5 respectively we analyse longitudinal and flexural waves for incompressible materials and obtain the dispersion relations governing their propagation when Ka is small, together with details of the displacement fields. In §2.6, §2.7, §2.8 we undertake the corresponding analysis for compressible materials. We conclude in §2.9 with a summary and discussion of the principal results achieved so far and follow this in §2.10 with a detailed examination of three illustrative examples.

§2.2 THE EQUILIBRIUM STATE OF PRIMARY STRESS.

In the first instance it is convenient to work with a rectangular cartesian system fixed in space. The typical particle of the elastic medium in its natural stress-free state occupies a position whose coordinates are denoted by X_K, ($K=1,2,3$), and in the general state of the medium the position of the same particle has coordinates x_i, ($i=1,2,3$), where x_i depends upon the X_K and possibly also the time t. These relations may be inverted to give X_K in terms of x_i, t. The elements of the Cauchy deformation tensor c are defined by the relations

$$c_{ij} = X_{K,i}X_{K,j}$$

where suffixes appearing after the comma indicate partial differentiation with respect to the coordinates x_i, and where the double suffix convention has been used. The inverse of c is denoted by c^{-1}, and I_1, I_2, I_3 are the usual three invariants of c^{-1}, that is

$$|c^{-1}-\lambda I| = I_3-I_2\lambda+I_1\lambda^2-\lambda^3,$$

in which I is the identity matrix.

The theory now follows one or other of two lines of development according to whether the material is compressible or incompressible.

It is convenient to begin with the incompressible case. The strain-energy function for the medium, assumed to be hyperelastic, isotropic and homogeneous is denoted by $W(I_1, I_2)$. Then the stresses P_{ij} are given by the equation

$$P_{ij} = 2W_1 c_{ij}^{-1} - \Pi\delta_{ij} - 2W_2 c_{ij}, \qquad (2.1)$$

where δ_{ij} is the Kronecker delta and W_α, $(\alpha=1,2)$, denotes the partial derivative of W with respect to I_α. Here in (2.1) Π denotes an arbitrary hydrostatic pressure and, since there is no change in volume,

$$I_3 = 1, \qquad (2.2)$$

so that the density of the material remains constant.

In the compressible case, however, the strain-energy function W now depends upon all three invariants I_1, I_2, I_3 and the stresses are given by

$$P_{ij} = 2I_3^{-1/2} W_1 c_{ij}^{-1} + 2I_3^{-1/2} (I_2 W_2 + I_3 W_3)\delta_{ij} - 2I_3^{1/2} W_2 c_{ij}, \qquad (2.3)$$

where, of course, $W_3 = \partial W/\partial I_3$.

The steady state achieved after the imposition of an axial traction (taken here to be in the OX_3 direction) along with a lateral force acting on the curved surface of the cylinder is specified by the equations

$$x_1 = \beta X_1, \qquad x_2 = \beta X_2, \qquad x_3 = \gamma X_3, \qquad (2.4)$$

so that β, γ (both positive constants) denote the transverse and axial principal stretches respectively. We shall have $\gamma>1$ in the tensile case, and of course $1>\gamma>0$ in the compressive case.

For an incompressible body, however, we may simplify our calculations by combining the axial and lateral tractions to produce an equivalent stress, τ say, acting only in the axial direction.

Now

$$c = \begin{bmatrix} \beta^{-2} & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & \gamma^{-2} \end{bmatrix}, \qquad c^{-1} = \begin{bmatrix} \beta^{2} & 0 & 0 \\ 0 & \beta^{2} & 0 \\ 0 & 0 & \gamma^{2} \end{bmatrix}, \qquad (2.5)$$

with

$$I_1 = 2\beta^2+\gamma^2, \qquad I_2 = \beta^2(\beta^2+2\gamma^2), \qquad I_3 = \beta^4\gamma^2. \qquad (2.6)$$

We consider now the incompressible and compressible cases in turn. For the incompressible case, from (2.2), (2.6)

$$\beta^4\gamma^2 = 1 \qquad (2.7)$$

and the stress tensor P_{ij} will be diagonal. For an axial principal stress τ, we have from (2.1), (2.5)

$$\tau = 2W_1\gamma^2-\Pi-2W_2\gamma^{-2}, \qquad (2.8)$$

and since the state is to be produced by axial traction alone the transverse principal stress must be zero; thus

$$0 = 2W_1\beta^2-\Pi-2W_2\beta^{-2}. \qquad (2.9)$$

Upon subtracting (2.9) from (2.8) so as to eliminate the unknown hydrostatic pressure Π we obtain

$$\tau = 2W_1(\gamma^2-\beta^2)-2W_2(\gamma^{-2}-\beta^{-2}). \qquad (2.10)$$

In equations (2.8)-(2.10) the partial derivatives W_1, W_2 are to be calculated at the point in (I_1, I_2) space given by

$$I_1 = 2\beta^2+\gamma^2, \qquad I_2 = \beta^2(\beta^2+2\gamma^2). \qquad (2.11)$$

Thus for a given value of τ, equations (2.7), (2.10), (2.11) determine the values of β, γ once the nature of the material is known through the dependence of W upon I_1, I_2.

For the compressible case, on the other hand, from (2.3) and (2.5) we have

$$\begin{aligned} t_3 &= 2I_3^{-1/2}W_1\gamma^2+2I_3^{-1/2}(I_2W_2+I_3W_3)-2I_3^{1/2}W_2\gamma^{-2}, \\ t_1 &= 2I_3^{-1/2}W_1\beta^2+2I_3^{-1/2}(I_2W_2+I_3W_3)-2I_3^{1/2}W_2\beta^{-2}, \end{aligned} \qquad (2.12)$$

t_3, t_1 denoting the axial and lateral principal stresses respectively. For purposes of comparison between the compressible and incompressible case we may assume $\tau = t_3 - t_1$. The partial derivatives of W are to be calculated this time at the point in the three-dimensional (I_1, I_2, I_3) space given by (2.6). Once W is known as a function of I_1, I_2, I_3, for given values of the stresses t_1, t_3, equations (2.6) and (2.12) suffice in principle to determine β and γ. Questions relating to the existence or uniqueness of solutions are best considered when an explicit form is given for $W(I_1, I_2, I_3)$.

Our task now is to derive equations of motion governing small perturbations about the steady state described above. We begin in the next section with the case of incompressible material.

§2.3 SMALL PERTURBATIONS - THE INCOMPRESSIBLE CASE.

Consider the steady-state deformation given by (2.4) and make a small perturbation with cylindrical components u, v, w that depend on the spatial coordinates and on the time t. We shall regard u, v, w and their derivatives as small quantities and shall discard their squares and products.

Expressed in cartesians the deformation is given by

$$x_1 = \beta X_1 + (x_1 u - x_2 v)/r, \qquad x_2 = \beta X_2 + (x_2 u + x_1 v)/r, \tag{2.13}$$

$$x_3 = \gamma X_3 + w,$$

and adopting the familiar device of setting $x_1 = r$, $x_2 = 0$, we calculate

$$\mathbf{c} = \begin{bmatrix} \beta^{-2}(1-2u_r) & \beta^{-2}[(v-u_\theta)/r-v_r] & -[\beta^{-2}u_z+\gamma^{-2}w_r] \\ \beta^{-2}[(v-u_\theta)/r-v_r] & \beta^{-2}[1-2(u+v_\theta)/r] & -[\beta^{-2}v_z+\gamma^{-2}w_\theta/r] \\ -[\beta^{-2}u_z+\gamma^{-2}w_r] & -[\beta^{-2}v_z+\gamma^{-2}w_\theta/r] & \gamma^{-2}[1-2w_z] \end{bmatrix} \quad (2.14)$$

where, since the medium is supposed incompressible

$$u_r+(u+v_\theta)/r+w_z = 0. \quad (2.15)$$

From (2.14) it is a straightforward matter to calculate

$$\mathbf{c}^{-1} = \begin{bmatrix} \beta^{2}(1+2u_r) & -\beta^{2}[(v-u_\theta)/r-v_r] & [\gamma^{2}u_z+\beta^{2}w_r] \\ -\beta^{2}[(v-u_\theta)/r-v_r] & \beta^{2}[1+2(u+v_\theta)/r] & [\gamma^{2}v_z+\beta^{2}w_\theta/r] \\ [\gamma^{2}u_z+\beta^{2}w_r] & [\gamma^{2}v_z+\beta^{2}w_\theta/r] & \gamma^{2}[1+2w_z] \end{bmatrix} \quad (2.16)$$

from which, with the aid of (2.15), we obtain

$$\begin{aligned} I_1 &= 2\beta^2+\gamma^2+2(\gamma^2-\beta^2)w_z, \\ I_2 &= \beta^2(\beta^2+2\gamma^2)+2\beta^2(\gamma^2-\beta^2)w_z. \end{aligned} \quad (2.17)$$

Now as in (2.1)

$$P = 2W_1\mathbf{c}^{-1}-\Pi I-2W_2\mathbf{c}, \quad (2.18)$$

and using (2.9), we write

$$\Pi = \Pi^*+2W_1^0\beta^2-2W_2^0\beta^{-2}, \quad (2.19)$$

where the affix zero indicates that the partial derivatives W_1, W_2 of W have been calculated in the equilibrium state of primary stress; in (2.18) these derivatives have to be calculated at the neighbouring point given by (2.17). The quantity Π^* is the change in hydrostatic pressure due to the perturbation. From (2.18) we have the following expressions

$$P_{rr} = -\Pi^*+2W_1(1+2u_r)\beta^2-2W_2(1-2u_r)\beta^{-2}-2W_1^0\beta^2+2W_2^0\beta^{-2},$$

$$P_{\theta\theta} = -\Pi^*+2W_1[1+2(u+v_\theta)/r]\beta^2-2W_2[1-2(u+v_\theta)/r]\beta^{-2}-2W_1^0\beta^2+2W_2^0\beta^{-2},$$

$$P_{zz} = -\Pi^*+2W_1(1+2w_z)\gamma^2-2W_2(1-2w_z)\gamma^{-2}-2W_1^0\beta^2+2W_2^0\beta^{-2},$$

$$P_{r\theta} = 2W_1[v_r+(u_\theta-v)/r]\beta^2+2W_2[v_r+(u_\theta-v)/r]\beta^{-2},$$

$$P_{rz} = 2W_1(\gamma^2 u_z+\beta^2 w_r)+2W_2(\beta^{-2}u_z+\gamma^{-2}w_r),$$

$$P_{\theta z} = 2W_1(\gamma^2 v_z+\beta^2 w_\theta/r)+2W_2(\beta^{-2}v_z+\gamma^{-2}w_\theta/r).$$

We now expand W_1, W_2 by Taylor series about the equilibrium state and, using (2.17) but retaining only first-order small quantities,

$$\begin{aligned} P_{r\theta} &= 2[W_1^0\beta^2+W_2^0\beta^{-2}][v_r+(u_\theta-v)/r], \\ P_{rz} &= 2(W_1^0\gamma^2+W_2^0\beta^{-2})u_z+2(W_1^0\beta^2+W_2^0\gamma^{-2})w_r, \\ P_{\theta z} &= 2(W_1^0\gamma^2+W_2^0\beta^{-2})v_z+2(W_1^0\beta^2+W_2^0\gamma^{-2})w_\theta/r, \end{aligned} \tag{2.20}$$

and similarly

$$\begin{aligned} P_{rr} &= -\Pi^*+4(W_1^0\beta^2+W_2^0\beta^{-2})u_r+4(\gamma^2-\beta^2)[(W_{11}^0\beta^2+W_{12}^0\beta^4) \\ &\qquad -(W_{12}^0\beta^{-2}+W_{22}^0)]w_z, \\ P_{\theta\theta} &= -\Pi^*+4(W_1^0\beta^2+W_2^0\beta^{-2})(u+v_\theta)/r+4(\gamma^2-\beta^2)[(W_{11}^0\beta^2+W_{12}^0\beta^4) \\ &\qquad -(W_{12}^0\beta^{-2}+W_{22}^0)]w_z, \\ P_{zz} &= \tau-\Pi^*+4(W_1^0\gamma^2+W_2^0\gamma^{-2})w_z+4(\gamma^2-\beta^2)[(W_{11}^0+W_{12}^0\beta^2)\gamma^2 \\ &\qquad -(W_{12}^0+W_{22}^0\beta^2)\gamma^{-2}]w_z, \end{aligned} \tag{2.21}$$

where in the last equation we have used (2.10).

The identity (2.15), together with the arbitrariness inherent in Π^*, allows us to cast (2.20), (2.21) into many different forms. For convenience, however, we choose the form (and, in large part, the notation) that permits comparison in various special cases with the work of Eringen and Suhubi [8]. We define

$$\Phi = 2W_1^0, \quad \Psi = 2W_2^0, \quad A = 2W_{11}^0, \quad B = 2W_{22}^0, \quad F = 2W_{12}^0, \tag{2.22}$$

and

$$\delta_1 = 2\beta^2(\gamma^2-\beta^2)[A+\beta^2(\gamma^2+\beta^2)B+(\gamma^2+2\beta^2)F]-2\beta^2(\Phi+\beta^2\Psi),$$

$$\delta_2 = -2\beta^2(\Phi+\gamma^2\Psi), \qquad \alpha_5 = \beta^2(\Phi+\beta^2\Psi), \tag{2.23}$$

$$\delta_3 = 2(\gamma^2-\beta^2)(\gamma^2 A+2B+3\gamma F)+2\beta^2(\Phi+\beta^2\Psi).$$

We further define

$$p^* = \Pi^*+2(\gamma^2-\beta^2)[\beta^2\Psi+(2+\beta^6)B+(2\gamma+\beta^4)F]w_z. \tag{2.24}$$

The stress components may now be written in the following forms

$$P_{rr} = -p^*+\delta_2(u+v_\theta)/r+\delta_1 w_z, \qquad P_{r\theta} = -\delta_2(v_r+(u_\theta-v)/r)/2,$$

$$P_{\theta\theta} = -p^*+\delta_2 u_r+\delta_1 w_z, \qquad P_{rz} = \alpha_5(u_z+w_r)+\tau u_z, \tag{2.25}$$

$$P_{zz} = \tau-p^*+(\delta_3+2\tau)w_z, \qquad P_{\theta z} = \alpha_5(v_z+w_\theta/r)+\tau v_z.$$

The equations of motion are well-known (see, for example, Filonenko-Borodich [22] or Novozhilov [23]):

$$\rho u_{tt} = P_{rr,r}+P_{r\theta,\theta}/r+P_{rz,z}+(P_{rr}-P_{\theta\theta})/r,$$

$$\rho v_{tt} = P_{r\theta,r}+P_{\theta\theta,\theta}/r+P_{\theta z,z}+2P_{r\theta}/r, \tag{2.26}$$

$$\rho w_{tt} = P_{rz,r}+P_{\theta z,\theta}/r+P_{zz,z}+P_{rz}/r,$$

where ρ is the constant density of the elastic material, and a comma indicates partial differentiation.

The substitution of (2.25) into (2.26) yields the equations of motion:

$$\rho u_{tt} = -p^*_{,r}+[(\alpha_5+\tau)u_{zz}-\delta_2 u_{\theta\theta}/2r^2]+\delta_2(v_{r\theta}+v_\theta/r)/2r$$
$$+(\delta_1+\alpha_5)w_{rz},$$

$$\rho v_{tt} = -p^*_{,\theta}/r+\delta_2(u_{r\theta}-u_\theta/r)/2r+(\alpha_5+\tau)v_{zz} \tag{2.27}$$
$$-\delta_2(v_{rr}+v_r/r-v/r^2)/2+(\delta_1+\alpha_5)w_{\theta z}/r,$$

$$\rho w_{tt} = -p^*_{,z}+(\alpha_5+\tau)(u_{rz}+u_z/r+v_{\theta z}/r)+\alpha_5(w_{rr}+w_r/r+w_{\theta\theta}/r^2)$$

$$+(\delta_3+2\tau)w_{zz}.$$

We also require the boundary conditions that no traction is applied at the deformed curved surface of the cylinder. These are:

$$-p^*+\delta_2(u+v_\theta)/r+\delta_1 w_z = 0,$$

$$v_r+(u_\theta-v)/r = 0, \tag{2.28}$$

$$u_z+w_r = 0,$$

at r=a, where $a = \beta a_0$, and a, a_0 are respectively the radius of the cylinder after the primary stress has been imposed and the radius in the natural state.

Equations (2.27), (2.28) are the basis of our subsequent analysis of wave propagation. So far as we are aware they have not been given previously in this form although for particular materials, and in the particular case in which there is no azimuthal dependence, the corresponding equations are well established (see, for example, Eringen and Suhubi [8]).

§2.4 LONGITUDINAL WAVES IN INCOMPRESSIBLE MATERIAL.

We look for solutions of the equations of motion (2.27) subject to the boundary conditions (2.28) that have v=0 and that are independent of θ. Accordingly, we write

$$u = U(r)\exp\{i(\omega t-Kz)\}, \qquad w = iW(r)\exp\{i(\omega t-Kz)\},$$

$$p^* = P(r)\exp\{i(\omega t-Kz)\}, \tag{2.29}$$

so that the wave travels along the z-axis with phase speed ω/K. We shall specify the functions U, W and P in more detail later; for the moment we substitute (2.29) into (2.27), (2.28). The equations of motion (2.27) become

$$\begin{aligned} \rho\omega^2 U &= P_r+K^2(\alpha_5+\tau)U-K(\delta_1+\alpha_5)W_r, \\ \rho\omega^2 W &= -KP-\alpha_5(W_{rr}+W_r/r)+K^2(\delta_3-\alpha_5+\tau)W, \end{aligned} \tag{2.30}$$

where we have used the incompressibility condition (2.15) in the form

$$U_r+U/r+KW = 0, \tag{2.31}$$

and the boundary conditions (2.28) give

$$\begin{aligned} -P+\delta_2 U/r+K\delta_1 W &= 0, \\ -KU+W_r &= 0, \end{aligned} \tag{2.32}$$

on $r=a$.

We come now to the forms for U, W and P, in the selection of which we are guided by the corresponding expressions for unstressed cylinders. For the group of waves under examination, for which the wave-speed remains finite as $Ka\to 0$, we again seek expansions in ascending powers of Ka. Specifically, we try:

$$U(r) = A\sum_{n=0}^{\infty}a_n(Kr)^{2n+1}, \qquad W(r) = -A\sum_{n=0}^{\infty}c_n(Kr)^{2n},$$
$$P(r) = KA\sum_{n=0}^{\infty}d_n(Kr)^{2n}. \tag{2.33}$$

where A is an arbitrary constant, necessarily small in magnitude, however, because of the linearised derivation of (2.27), (2.28). Equating coefficients of r^{2n} in the incompressibility conditions (2.31) we obtain the relations

$$2(n+1)a_n-c_n = 0, \qquad (n=0,1,2,\ldots). \tag{2.34}$$

Similarly, the equations of motion (2.30) and the boundary conditions (2.32), with c_n eliminated by the use of (2.34), yield respectively:

$$Xa_n = 2(n+1)d_{n+1}+(\alpha_5+\tau)a_n+4(n+1)(n+2)(\delta_1+\alpha_5)a_{n+1},$$

$$2(n+1)Xa_n = d_n+2(n+1)(\delta_3-\alpha_5+\tau)a_n-8(n+1)^2(n+2)\alpha_5 a_{n+1}, \tag{2.35}$$

for $n\geqslant 0$, where $X=\rho\omega^2/K^2$, and

$$\sum_{n=0}^{\infty}[(\delta_2-2(n+1)\delta_1)a_n-d_n](Ka)^{2n} = 0,$$

$$\sum_{n=0}^{\infty}[a_n+4(n+1)(n+2)a_{n+1}](Ka)^{2n} = 0. \tag{2.36}$$

Now equations (2.35) give (for large n)

$$a_{n+1} \sim O(a_n/(n+1)(n+2)), \qquad d_{n+1} \sim O(a_n/(n+1)),$$

so that convergence of (2.33) is assured without restriction on Kr.

The parameter X and the coefficients a_n, d_n are in turn expressed as asymptotic series expansions in ascending powers of $(Ka)^2$, thus

$$X = X^{(0)}+X^{(1)}(Ka)^2+X^{(2)}(Ka)^4+\ldots+X^{(m)}(Ka)^{2m}+\ldots ,$$

and similarly for a_n and d_n. Then from (2.35),

$$d_n^{(m)} = -[(\alpha_5+\tau)a_{n-1}^{(m)}+4n(n+1)(\delta_1+\alpha_5)a_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_{n-1}^{(p)}]/2n,$$

$$a_{n+1}^{(m)} = [d_n^{(m)}+2(n+1)(\delta_3-\alpha_5+\tau)a_n^{(m)} \tag{2.37}$$

$$-2(n+1)\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]/8(n+1)^2(n+2)\alpha_5,$$

for $n\geqslant 1$, $m\geqslant 0$; and additionally from $(2.35)_2$ we have

$$X^{(m)} = [d_0^{(m)}/2+(\delta_3-\alpha_5+\tau)a_0^{(m)}-8\alpha_5 a_1^{(m)}]/a_0^{(0)}, \tag{2.38}$$

where we have taken

$$a_0^{(m)} = \begin{cases} 1, & m=0, \\ 0, & m\geqslant 1. \end{cases}$$

Likewise, equating coefficients of $(Ka)^{2n}$ in the boundary conditions (2.36) we have

$$a_1^{(m)} = -[a_0^{(m)}+\sum_{n=1}^{m}\{a_n^{(m-n)}+4(n+1)(n+2)a_{n+1}^{(m-n)}\}]/8,$$
$$d_0^{(m)} = (\delta_2-2\delta_1)a_0^{(m)}+\sum_{n=1}^{m}\{(\delta_2-2(n+1)\delta_1)a_n^{(m-n)}-d_n^{(m-n)}\}, \tag{2.39}$$

for $m \geq 0$.

The terms in the dispersion relation may now be calculated successively using (2.37)-(2.39).

First, from (2.39) with m=0 we calculate

$$a_1^{(0)} = -1/8, \qquad d_0^{(0)} = \delta_2-2\delta_1.$$

(We may observe that to this approximation, a_1 and, by virtue of (2.34), c_1, are independent of the applied stress τ and independent of the nature of the material.) These values together with (2.38) give

$$X^{(0)} = (\delta_3+\delta_2/2-\delta_1+\tau). \tag{2.40}$$

The general situation now follows familiar lines.

We begin with (2.37) to obtain

$$d_1^{(0)} = (\delta_3+\delta_2/2)/2, \qquad a_2^{(0)} = (\delta_3+\delta_2-\delta_1+\alpha_5)/192\alpha_5,$$

then (2.39), with m=1, give

$$a_1^{(1)} = -(\delta_3+\delta_2-\delta_1)/64\alpha_5, \qquad d_0^{(1)} = -(4\delta_3+3\delta_2-4\delta_1)/8,$$

and so from (2.38) we have

$$X^{(1)} = -(\delta_3+\delta_2/2-\delta_1)/8. \tag{2.41}$$

To the next order of approximation the procedure supplies, from (2.37), setting $Z = (\delta_3+\delta_2-\delta_1)/\alpha_5$:

$$d_2^{(0)} = -(2\delta_3+3\delta_2/2-\delta_1+\delta_1 Z)/32,$$

$$a_3^{(0)} = (3\delta_1-3\delta_2-3\delta_3-\alpha_5-\delta_2 Z/2)/9216\alpha_5,$$

$$d_1^{(1)} = (\delta_2/2+\delta_1 Z)/16,$$

$$a_2^{(1)} = \delta_2(1+Z/2)/1536\alpha_5,$$

then from (2.39), (2.38), setting m=2,

$$d_0^{(2)} = (6\delta_3+2\delta_2-6\delta_1-\delta_2 Z)/96,$$

$$a_1^{(2)} = -(5\delta_1-2\delta_2-5\delta_3+\delta_2 Z)/1536\alpha_5,$$

and

$$X^{(2)} = (\delta_3-\delta_1)/192. \tag{2.42}$$

A further round of calculation reveals

$$X^{(3)} = (\delta_3+\delta_2/2-\delta_1-\alpha_5 Z^2)/3072. \tag{2.43}$$

The significance of the results (2.40)-(2.43) in the general case and for particular materials will be discussed in §2.9 and §2.10.

§2.5 FLEXURAL WAVES IN INCOMPRESSIBLE MATERIAL.

We turn attention now to the more complicated problem of flexural waves in incompressible cylinders. To this end we consider the forms

$$u = U(r)\exp\{i(\omega t+\theta-Kz)\}, \quad v = iV(r)\exp\{i(\omega t+\theta-Kz)\},$$

$$w = iW(r)\exp\{i(\omega t+\theta-Kz)\}, \quad p^* = P(r)\exp\{i(\omega t+\theta-Kz)\}.$$

The incompressibility condition (2.15) now gives

$$U_r+(U-V)/r+KW = 0, \tag{2.44}$$

and the equations of motion (2.27) yield, after some simplification,

$$\rho\omega^2 U = P_r+[(\alpha_5+\tau)K^2-\delta_2/2r^2]U+\delta_2(V_r+V/r)/2r-K(\delta_1+\alpha_5)W_r,$$

$$\rho\omega^2 V = P/r-\delta_2(U_r-U/r)/2r+\delta_2(V_{rr}+V_r/r-V/r^2)/2 +(\alpha_5+\tau)K^2 V-K(\delta_1+\alpha_5)W/r, \tag{2.45}$$

$$\rho\omega^2 W = -KP-\alpha_5(W_{rr}+W_r/r-W/r^2)+K^2(\tau+\delta_3-\alpha_5)W.$$

The boundary conditions (2.28) give

$$-P+\delta_2(U-V)/r+K\delta_1 W = 0,$$

$$V_r+(U-V)/r = 0, \tag{2.46}$$

$$-KU+W_r = 0,$$

on $r=a$.

For the radial dependence U, V, W, P we set

$$U(r) = A\sum_{n=0}^{\infty}a_n(Kr)^{2n}, \qquad V(r) = A\sum_{n=0}^{\infty}b_n(Kr)^{2n},$$

$$W(r) = A\sum_{n=0}^{\infty}c_n(Kr)^{2n+1}, \qquad P(r) = AK\sum_{n=0}^{\infty}d_n(Kr)^{2n+1},$$

where A is an arbitrary constant; in selecting these forms we have again been guided by our knowledge of the solution corresponding to the unstressed cylinder.

Again denoting $\rho\omega^2/K^2$ by X, the governing equations are, from (2.44),

$$(2n+3)a_{n+1}-b_{n+1}+c_n = 0, \qquad (n=0,1,2,\ldots), \tag{2.47}$$

and from (2.45)

$$Xa_n = (\alpha_5+\tau)a_n-(2n+1)(\delta_1+\alpha_5)c_n+(2n+1)d_n-\delta_2[a_{n+1}-(2n+3)b_{n+1}]/2,$$

$$Xb_n = (\alpha_5+\tau)-(\delta_1+\alpha_5)c_n+d_n-(2n+1)\delta_2[a_{n+1}-(2n+3)b_{n+1}]/2, \tag{2.48}$$

$$Xc_n = (\tau+\delta_3-\alpha_5)c_n-d_n-4(n+1)(n+2)\alpha_5 c_{n+1},$$

with (2.46) supplying

$$\sum_{n=0}^{\infty}[\delta_1 c_n-d_n+\delta_2(a_{n+1}-b_{n+1})](Ka)^{2n} = 0,$$

$$\sum_{n=0}^{\infty}[a_n+(2n-1)b_n](Ka)^{2n} = 0, \tag{2.49}$$

$$\sum_{n=0}^{\infty}[a_n-(2n+1)c_n](Ka)^{2n} = 0.$$

For the asymptotic developments in ascending powers of $(Ka)^2$ we find from (2.48), (2.47), after some manipulation,

$$c_n^{(m)} = [-d_{n-1}^{(m)}+(\tau+\delta_3-\alpha_5)c_{n-1}^{(m)}-\sum_{p=0}^{m}X^{(m-p)}c_{n-1}^{(p)}]/4n(n+1)\alpha_5,$$

$$d_n^{(m)} = [4n(n+1)(\delta_1+\alpha_5)c_n^{(m)}-(\alpha_5+\tau)((2n+1)a_n^{(m)}-b_n^{(m)})$$

$$+\sum_{p=0}^{m}X^{(m-p)}((2n+1)a_n^{(p)}-b_n^{(p)})]/4n(n+1), \quad (2.50)$$

$$a_{n+1}^{(m)} = [-(2n+1)d_n^{(m)}+((2n+1)(\delta_1+\alpha_5)-(2n+3)\delta_2/2)c_n^{(m)}$$

$$-(\alpha_5+\tau)a_n^{(m)}+\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]/2(n+1)(n+2)\delta_2,$$

$$b_{n+1}^{(m)} = (2n+3)a_{n+1}^{(m)}+c_n^{(m)}.$$

for $n\geqslant 1$, $m\geqslant 0$, and

$$X^{(m)} = (\alpha_5+\tau)a_0^{(m)}-(\delta_1+\alpha_5)c_0^{(m)}+d_0^{(m)}+\delta_2(3b_1^{(m)}-a_1^{(m)})/2, \quad (2.51)$$

where, without loss of generality, we have set

$$a_0^{(m)} = \begin{cases} 1, & m=0, \\ 0, & m\geqslant 1. \end{cases}$$

Likewise from (2.49), (2.47), we find

$$c_0^{(m)} = a_0^{(m)}+\sum_{n=1}^{m}\{a_n^{(m-n)}-(2n+1)c_n^{(m-n)}\},$$

$$a_1^{(m)} = -[c_0^{(m)}+\sum_{n=1}^{m}\{a_{n+1}^{(m-n)}+(2n+1)b_{n+1}^{(m-n)}\}]/4,$$

$$b_1^{(m)} = c_0^{(m)}+3a_1^{(m)}, \quad (2.52)$$

$$d_0^{(m)} = \delta_1 c_0^{(m)}+\delta_2(a_1^{(m)}-b_1^{(m)})+\sum_{n=1}^{m}\{\delta_1 c_n^{(m-n)}-d_n^{(m-n)}$$

$$+\delta_2(a_{n+1}^{(m-n)}-b_{n+1}^{(m-n)})\},$$

for $m\geqslant 0$.

A first approximation for the dispersion relation may be made by setting m=0 in the relations (2.52) and calculating

$$c_0^{(0)} = 1, \quad a_1^{(0)} = -b_1^{(0)} = -1/4, \quad d_0^{(0)} = \delta_1-\delta_2/2,$$

so that from (2.51) we have

$$X^{(0)} = \tau. \quad (2.53)$$

Thus to a first approximation the speed of flexural waves in a slender cylinder is given by

$$\rho\omega^2/\kappa^2 = \tau,$$

a result independent of the particular material. This is in agreement with the usual elementary argument (see, for instance, Coulson [24]) for waves along a uniform stretched string.

Setting $Y = (\delta_3+\delta_2/2-\delta_1)/\alpha_5$ one finds for a further approximation, from (2.50),

$$c_1^{(0)} = (Y-1)/8, \qquad d_1^{(0)} = ((\delta_1+\alpha_5)Y-\delta_1)/8,$$

$$a_2^{(0)} = (5\delta_2-2\alpha_5-5Y\delta_2)/192\delta_2, \qquad b_2^{(0)} = (\delta_2-10\alpha_5-Y\delta_2)/192\delta_2,$$

then (2.52) give

$$c_0^{(1)} = (1-3Y)/8, \qquad a_1^{(1)} = (2\alpha_5-2\delta_2+5Y\delta_2)/48\delta_2,$$

$$b_1^{(1)} = (2\alpha_5-Y\delta_2)/16\delta_2,$$

$$d_0^{(1)} = [(6\delta_1-\delta_2-2\alpha_5)-Y(18\delta_1-7\delta_2+6\alpha_5)]/48,$$

so that we have

$$X^{(1)} = (\delta_3+\delta_2/2-\delta_1)/4. \qquad (2.54)$$

To the next approximation we calculate

$$X^{(2)} = (1-4Y-7Y^2)\alpha_5/96. \qquad (2.55)$$

§2.6 SMALL PERTURBATIONS - THE COMPRESSIBLE CASE.

In this section we derive the equations of motion and the boundary conditions for elastic cylinders of compressible material. Part of the analysis can be taken over directly from the incompressible case (§2.3) thus equations (2.13), (2.14), (2.16) still hold good, but for the invariants of c^{-1} we now have

$$I_1 = 2\beta^2+\gamma^2+2[\beta^2(u_r+(u+v_\theta)/r)+\gamma^2 w_z],$$

$$I_2 = \beta^2(\beta^2+2\gamma^2)+2\beta^2[(\beta^2+\gamma^2)(u_r+(u+v_\theta)/r)+2\gamma^2 w_z],$$

$$I_3 = \beta^4\gamma^2[1+2(u_r+(u+v_\theta)/r+w_z)].$$

We modify the definitions (2.22) to

$$\Theta = 2I_3^{1/2}W_3, \qquad \Phi = 2I_3^{-1/2}W_1, \qquad \Psi = 2I_3^{-1/2}W_2,$$

$$B_1 = 2I_3^{-1/2}W_{11}, \quad B_2 = 2I_3^{-1/2}W_{22}, \quad B_3 = 2I_3^{-1/2}W_{33}, \tag{2.56}$$

$$C_1 = 2I_3^{-1/2}W_{23}, \quad C_2 = 2I_3^{-1/2}W_{31}, \quad C_3 = 2I_3^{-1/2}W_{12},$$

all evaluated at the equilibrium state given by (2.6).

The relations (2.12) are then

$$\begin{aligned} t_3 &= \gamma^2\Phi+2\beta^2\gamma^2\Psi+\Theta, \\ t_1 &= \beta^2\Phi+\beta^2(\beta^2+\gamma^2)\Psi+\Theta, \end{aligned} \tag{2.57}$$

and, using (2.3), the off-diagonal stress components are:

$$P_{r\theta} = \beta^2(\Phi+\gamma^2\Psi)[v_r+(u_\theta-v)/r],$$

$$P_{\theta z} = \beta^2(\Phi+\beta^2\Psi)(v_z+w_\theta/r)+(t_3-t_1)v_z, \tag{2.58}$$

$$P_{rz} = \beta^2(\Phi+\beta^2\Psi)(u_z+w_r)+(t_3-t_1)u_z,$$

where for the last two of the relations (2.58) we have made use of (2.57). We define for this case

$$\alpha_1 = 2\beta^4[B_1+(\beta^2+\gamma^2)^2B_2+\beta^4\gamma^4B_3+2\beta^2\gamma^2(\beta^2+\gamma^2)C_1+2\beta^2\gamma^2C_2$$
$$+2(\beta^2+\gamma^2)C_3]+\Theta-\beta^2\Phi-\beta^2(\beta^2+\gamma^2)\Psi,$$

$$\alpha_2 = 2\beta^2\gamma^2[B_1+2\beta^2(\beta^2+\gamma^2)B_2+\beta^6\gamma^2B_3+\beta^4(\beta^2+3\gamma^2)C_1$$
$$+\beta^2(\beta^2+\gamma^2)C_2+(3\beta^2+\gamma^2)C_3]+\Theta-\beta^2\Phi-\beta^2(\beta^2-\gamma^2)\Psi, \tag{2.59}$$

$$\alpha_3 = \alpha_2+(\beta^2-\gamma^2)(\Phi+\beta^2\Psi),$$

$$\alpha_4 = 2\gamma^4[B_1+4\beta^4B_2+\beta^8B_3+4\beta^6C_1+2\beta^4C_2+4\beta^2C_3]+\Theta+(2\beta^2-\gamma^2)\Phi+2\beta^4\Psi,$$

$$\alpha_5 = \beta^2(\Phi+\beta^2\Psi), \qquad \alpha_6 = \beta^2(\Phi+\gamma^2\Psi), \qquad \alpha_7 = \gamma^2(\Phi+\beta^2\Psi).$$

Note that α_5 here is defined in a way consistent with (2.23), and that

$$\alpha_2-\alpha_3 = \alpha_7-\alpha_5 = t_3-t_1. \tag{2.60}$$

Now from (2.58), with (2.59),

$$\begin{aligned} P_{r\theta} &= \alpha_6[v_r+(u_\theta-v)/r], \\ P_{\theta z} &= \alpha_5(v_z+w_\theta/r)+(t_3-t_1)v_z, \\ P_{rz} &= \alpha_5(u_z+w_r)+(t_3-t_1)u_z, \end{aligned} \tag{2.61}$$

and similarly

$$\begin{aligned} P_{rr} &= t_1+\alpha_1[u_r+(u+v_\theta)/r]+2(t_1-\Theta)u_r+2\beta^4\Psi(u+v_\theta)/r+\alpha_2w_z, \\ P_{\theta\theta} &= t_1+\alpha_1[u_r+(u+v_\theta)/r]+2\beta^4\Psi u_r+2(t_1-\Theta)(u+v_\theta)/r+\alpha_2w_z, \\ P_{zz} &= t_3+\alpha_3[u_r+(u+v_\theta)/r]+[\alpha_4+2(t_3-t_1)]w_z. \end{aligned} \tag{2.62}$$

To derive the equations of motion we substitute (2.61), (2.62) into (2.26), and denoting

$$R = \alpha_1+2(t_1-\Theta), \qquad S = \alpha_1+2\beta^4\Psi, \qquad T = \alpha_4+2(t_3-t_1), \tag{2.63}$$

we find, making use of (2.60),

$$\begin{aligned} \rho u_{tt} &= R(u_{rr}+u_r/r-u/r^2)+\alpha_6u_{\theta\theta}/r^2+\alpha_7u_{zz}+(\alpha_6+S)v_{r\theta}/r \\ &\qquad -(\alpha_6+R)v_\theta/r^2+(\alpha_2+\alpha_5)w_{rz}, \\ \rho v_{tt} &= (\alpha_6+S)u_{r\theta}/r+(\alpha_6+R)u_\theta/r^2+\alpha_6(v_{rr}+v_r/r-v/r^2) \\ &\qquad +Rv_{\theta\theta}/r^2+\alpha_7v_{zz}+(\alpha_2+\alpha_5)w_{\theta z}/r, \end{aligned} \tag{2.64}$$

$$\rho w_{tt} = (\alpha_2+\alpha_5)(u_{rz}+u_z/r+v_{\theta z}/r)+\alpha_5(w_{rr}+w_r/r+w_{\theta\theta}/r^2)+Tw_{zz},$$

with ρ denoting the density of the material after the primary stress has been applied.

The boundary conditions may be expressed in the form

$$P_{ij}\nu_j = t_1\nu_j \tag{2.65}$$

where $\nu=(1,-u_\theta/r,-u_z)$ is a unit vector normal to the deformed curved surface of the cylinder $X_1^2+X_2^2=a^2$, and a is the cylinder radius after the imposition of the primary stress. Thus using (2.61), (2.62) in (2.65) we have

$$\begin{aligned} &Ru_r+S(u+v_\theta)/r+\alpha_2 w_z = 0, \\ &v_r+(u_\theta-v)/r = 0, \\ &u_z+w_r = 0, \end{aligned} \tag{2.66}$$

on r=a.

It is readily verified that, for cylinders under uniaxial stress, equations (2.64) agree for Ko materials with equations derived by Thompson and Willson [25], and, in the special case in which there is no θ-dependence, with equations obtained by Eringen and Suhubi [8].

§2.7 LONGITUDINAL WAVES IN COMPRESSIBLE MATERIAL.

Following the line established in §2.4 for incompressible materials we look for a solution in the form

$$u = U(r)\exp\{i(\omega t-Kz)\}, \qquad w = iW(r)\exp\{i(\omega t-Kz)\}. \tag{2.67}$$

Again there is no azimuthal θ-dependence nor any transverse component of displacement.

From (2.64), (2.67)

$$\rho\omega^2 U = -R(U_{rr}+U_r/r-U/r^2)+\alpha_7 K^2 U-K(\alpha_2+\alpha_5)W_r,$$
$$\rho\omega^2 W = K(\alpha_2+\alpha_5)(U_r+U/r)-\alpha_5(W_{rr}+W_r/r)+K^2 TW, \tag{2.68}$$

while the boundary conditions (2.66), with (2.67), yield

$$RU_r+SU/r+K\alpha_2 W = 0,$$
$$-KU+W_r = 0, \tag{2.69}$$

on r=a.

We now set

$$U = A\sum_{n=0}^{\infty} a_n(Kr)^{2n+1}, \qquad W = -A\sum_{n=0}^{\infty} c_n(Kr)^{2n}, \tag{2.70}$$

where A is an arbitrary constant. Substitution of (2.70) into (2.68) gives

$$Xa_n = 2(n+1)(\alpha_2+\alpha_5)c_{n+1}-4(n+1)(n+2)Ra_{n+1}+\alpha_7 a_n,$$
$$Xc_n = -4(n+1)^2\alpha_5 c_{n+1}+Tc_n-2(n+1)(\alpha_2+\alpha_5)a_n, \tag{2.71}$$

with X again denoting $\rho\omega^2/K^2$. Similarly from (2.69) we find

$$\sum_{n=0}^{\infty}[((2n+1)R+S)a_n-\alpha_2 c_n](Ka)^{2n} = 0,$$
$$\sum_{n=0}^{\infty}[a_n+2(n+1)c_{n+1}](Ka)^{2n} = 0. \tag{2.72}$$

Again the asymptotic behaviour of the coefficients indicates convergence of (2.70) unconditionally.

In the usual way we seek asymptotic developments for X and the coefficients a_n, c_n in ascending powers of $(Ka)^2$. From the equations of motion (2.71) we calculate

$$a_n^{(m)} = [2n(\alpha_2+\alpha_5)c_n^{(m)}+\alpha_7 a_{n-1}^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_{n-1}^{(p)}]/4n(n+1)R,$$
$$c_{n+1}^{(m)} = [Tc_n^{(m)}-2(n+1)(\alpha_2+\alpha_5)a_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}c_n^{(p)}]/4(n+1)^2\alpha_5, \tag{2.73}$$

for $n\geq 1$, $m\geq 0$; and from $(2.71)_2$ with n=0

$$X^{(m)} = [-4\alpha_5 c_1^{(m)}+Tc_0^{(m)}-2(\alpha_2+\alpha_5)a_0^{(m)}]/c_0^{(0)}, \tag{2.74}$$

taking

$$c_0^{(m)} = \begin{cases} (R+S)/\alpha_2, & m=0, \\ 0, & m\geq 1. \end{cases}$$

Similarly the boundary conditions (2.72) produce

$$\begin{aligned} a_0^{(m)} &= [\alpha_2 c_0^{(m)} - \sum_{n=1}^{m}\{((2n+1)R+S)a_n^{(m-n)} - \alpha_2 c_n^{(m-n)}\}]/(R+S), \\ c_1^{(m)} &= -[a_0^{(m)} + \sum_{n=1}^{m}\{a_n^{(m-n)} + 2(n+1)c_{n+1}^{(m-n)}\}]/2. \end{aligned} \tag{2.75}$$

We find, successively, from (2.75),

$$a_0^{(0)} = 1, \qquad\qquad c_1^{(0)} = -1/2,$$

and these with (2.74) supply the result

$$X^{(0)} = T - 2\alpha_2^2/(R+S). \tag{2.76}$$

In this approximation then, the terms $a_0^{(0)}$ and $c_1^{(0)}$ have the same values that they had in §2.4 for the incompressible case. In fact, because of (2.34) and $(2.75)_2$, the relation

$$a_0^{(0)} = -2c_1^{(0)}$$

will always hold true.

To the next approximation, with $D = [T+\alpha_3-2\alpha_2^2/(R+S)]/R$, we calculate,

$$a_1^{(0)} = -D/8, \qquad c_2^{(0)} = [(\alpha_2+\alpha_5)D-2\alpha_2^2/(R+S)]/32\alpha_5,$$

$$a_0^{(1)} = [(3R+S)D-4\alpha_2]/8(R+S),$$

$$c_1^{(1)} = -[((3R+S)\alpha_5+(R+S)\alpha_2)D-2\alpha_2(2\alpha_5+\alpha_2)]/16(R+S)\alpha_5,$$

so that (2.74) provides

$$X^{(1)} = \frac{-\alpha_2^2}{2(R+S)^2}\left[T \ \frac{-2\alpha_2^2}{(R+S)} - (t_3-t_1)\right]. \tag{2.77}$$

To the next order of approximation, we find,

$$X^{(2)} = \frac{-\alpha_2^2 R}{4(R+S)^3}\left[\frac{D^2(7R+S)}{12} - D\alpha_2\left[1+\frac{\alpha_2}{R+S}\right] + \frac{\alpha_2^2}{3R}\left[1+\frac{3\alpha_2}{R+S}\right]\right]. \tag{2.78}$$

§2.8 FLEXURAL WAVES IN COMPRESSIBLE MATERIAL.

For this case the relevant equations of motion are given in §2.6 by (2.64) and the boundary conditions by (2.66). We write

$$u = U(r)\exp\{i(\omega t+\theta-Kz)\}, \qquad v = iV(r)\exp\{i(\omega t+\theta-Kz)\},$$
$$w = iW(r)\exp\{i(\omega t+\theta-Kz)\}, \tag{2.79}$$

as we did in §2.5 for incompressible materials.

From (2.64), with (2.79), the equations of motion take the form

$$\rho\omega^2 U = -R(U_{rr}+U_r/r-U/r^2)+\alpha_6 U/r^2+\alpha_7 K^2 U+(\alpha_6+S)V_r/r$$
$$-(\alpha_6+R)V/r^2-(\alpha_2+\alpha_5)KW_r,$$
$$\rho\omega^2 V = -(\alpha_6+S)U_r/r-(\alpha_6+R)U/r^2-\alpha_6(V_{rr}+V_r/r-V/r^2) \tag{2.80}$$
$$+RV/r^2+\alpha_7 K^2 V-(\alpha_2+\alpha_5)KW/r,$$
$$\rho\omega^2 W = K(\alpha_2+\alpha_5)(U_r+U/r-V/r)-\alpha_5(W_{rr}+W_r/r-W/r^2)+TK^2 W,$$

and similarly from (2.66), (2.79) the boundary conditions give

$$RU_r+S(U-V)/r+\alpha_2 KW = 0,$$
$$V_r+(U-V)/r = 0, \tag{2.81}$$
$$-KU+W_r = 0,$$

on r=a.

For U, V, W we take the series expansions

$$U(r) = A\sum_{n=0}^{\infty} a_n(Kr)^{2n}, \qquad V(r) = A\sum_{n=0}^{\infty} b_n(Kr)^{2n},$$
$$W(r) = A\sum_{n=0}^{\infty} c_n(Kr)^{2n+1}, \tag{2.82}$$

where **A** is an arbitrary constant.

The substitution of (2.82) into (2.80) gives

$$Xa_n = \alpha_7 a_n - [(2n+1)(2n+3)R-\alpha_6]a_{n+1}$$

$$+[(2n+1)R-(2n+3)\alpha_6]b_{n+1}-(2n+1)(\alpha_2+\alpha_5)c_n,$$

$$Xb_n = \alpha_7 b_n - [(2n+3)R-(2n+1)\alpha_6]a_{n+1} \tag{2.83}$$

$$+[R-(2n+1)(2n+3)\alpha_6]b_{n+1}-(\alpha_2+\alpha_5)c_n,$$

$$Xc_n = Tc_n+(\alpha_2+\alpha_5)[(2n+3)a_{n+1}-b_{n+1}]-4(n+1)(n+2)\alpha_5 c_{n+1},$$

in which we have used the relation $R = S+2\alpha_6$.

Similarly from (2.81), (2.82) we have

$$\sum_{n=0}^{\infty}[(2(n+1)R+S)a_{n+1}-Sb_{n+1}+\alpha_2 c_n](Ka)^{2n} = 0,$$

$$\sum_{n=0}^{\infty}[a_{n+1}+(2n+1)b_{n+1}](Ka)^{2n} = 0, \tag{2.84}$$

$$\sum_{n=0}^{\infty}[a_n-(2n+1)c_n](Ka)^{2n} = 0.$$

The corresponding relations for the coefficients in the asymptotic developments are, from $(2.83)_3$,

$$c_n^{(m)} = [(\alpha_2+\alpha_5)[(2n+1)a_n^{(m)}-b_n^{(m)}]+Tc_{n-1}^{(m)}$$

$$-\sum_{p=0}^{m}X^{(m-p)}c_{n-1}^{(p)}]/4n(n+1)\alpha_5,$$

whilst manipulation of $(2.83)_1$, $(2.83)_2$ yields

$$a_{n+1}^{(m)} = [[(2n+1)(2n+3)\alpha_6-R][\alpha_7 a_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]$$

$$+[(2n+1)R-(2n+3)\alpha_6][\alpha_7 b_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}]$$

$$-4n(n+1)(2n+3)(\alpha_2+\alpha_5)\alpha_6 c_n^{(m)}]/16n(n+1)^2(n+2)R\alpha_6,$$

$$b_{n+1}^{(m)} = [[(2n+1)\alpha_6-(2n+3)R][\alpha_7 a_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}a_n^{(p)}]$$

$$+[(2n+1)(2n+3)R-\alpha_6][\alpha_7 b_n^{(m)}-\sum_{p=0}^{m}X^{(m-p)}b_n^{(p)}]$$

$$-4n(n+1)(\alpha_2+\alpha_5)\alpha_6 c_n^{(m)}]/16n(n+1)^2(n+2)R\alpha_6,$$

these equations holding for $n \geq 1$, $m \geq 0$.

The dispersion relation may be expressed in the form:

$$X^{(m)} = \alpha_7 a_0^{(m)} - (3R-\alpha_6) a_1^{(m)} + (R-3\alpha_6) b_1^{(m)} - (\alpha_2+\alpha_5) c_0^{(m)}, \qquad (2.85)$$

after taking

$$a_0^{(m)} = \begin{cases} 1, & m=0, \\ 0, & m \geq 1, \end{cases}$$

while, from $(2.83)_1$, $(2.83)_2$ with $n=0$, and assuming $X \neq \alpha_7$, we find

$$b_0^{(m)} = a_0^{(m)}.$$

Correspondingly, from the boundary conditions, we have, after a little rearrangement,

$$c_0^{(m)} = a_0^{(m)} + \sum_{n=1}^{m} \{ a_n^{(m-n)} - (2n+1) c_n^{(m-n)} \},$$

$$a_1^{(m)} = -[\alpha_2 c_0^{(m)} + \sum_{n=1}^{m} \{ 2((n+1)R+S) a_{n+1}^{(m-n)} + 2nS b_{n+1}^{(m-n)}$$
$$+ \alpha_2 c_n^{(m-n)} \}]/2(R+S),$$

$$b_1^{(m)} = -a_1^{(m)} - \sum_{n=1}^{m} \{ a_{n+1}^{(m-n)} + (2n+1) b_{n+1}^{(m-n)} \}.$$

We proceed to construct a first order approximation for X using these conditions. With m set to zero we find

$$c_0^{(0)} = 1, \qquad a_1^{(0)} = -b_1^{(0)} = -\alpha_2/2(R+S). \qquad (2.86)$$

Comparing these results with the corresponding results for incompressible materials, we see that the two relations

$$a_1^{(0)} = -b_1^{(0)}, \qquad c_0^{(0)} = a_0^{(0)}$$

are valid for all materials considered here.

Now using (2.86) in (2.85), and with the aid of (2.60), we find

$$X^{(0)} = t_3 - t_1, \qquad (2.87)$$

just as for incompressible materials (with τ interpreted as $t_3 - t_1$).

For the next approximation we find

$$c_1^{(0)} = [T-(t_3-t_1)-2\alpha_2(\alpha_2+\alpha_5)/(R+S)]/8\alpha_5,$$

$$a_2^{(0)} = [(5\alpha_6-R)B-5C]/48, \qquad b_2^{(0)} = [(\alpha_6-5R)B-C]/48,$$

after denoting $B = \alpha_5 a_1^{(0)}/R\alpha_6$ and $C = 2(\alpha_2+\alpha_5)c_1^{(0)}/R$. Using these values we may then calculate

$$c_0^{(1)} = R[\alpha_6 B/\alpha_5 - 3C/2(\alpha_2+\alpha_5)],$$

$$a_1^{(1)} = \frac{-1}{12(R+S)}\left[6R\alpha_2\left[\frac{\alpha_6 B}{\alpha_5} - \frac{C}{(\alpha_2+\alpha_5)}\right]-(2R-\alpha_6)(R-3\alpha_6)B-(4R-3\alpha_6)C\right],$$

$$b_1^{(1)} = \frac{1}{12(R+S)}\left[6R\alpha_2\left[\frac{\alpha_6 B}{\alpha_5} - \frac{C}{(\alpha_2+\alpha_5)}\right]+(2R-\alpha_6)(3R-\alpha_6)B-\alpha_6 C\right],$$

and so, from (2.85),

$$X^{(1)} = \frac{[T-(t_3-t_1)]}{4} - \frac{\alpha_2^2}{2(R+S)}. \tag{2.88}$$

Omitting details of the calculation for the next approximation we present the result

$$X^{(2)} = \frac{-7[X^{(1)}]^2}{6\alpha_5} - \frac{X^{(1)}}{12}\left[3-\frac{2\alpha_2}{(R+S)}\right] + \frac{\alpha_2^2\alpha_5}{24(R+S)^2}. \tag{2.89}$$

An interesting feature of the results formulated for the compressible case is that the equations are closely related to those given for the transversely isotropic case in Chapter 1. By taking

$$R = A, \qquad S = B, \qquad \alpha_2 = \alpha_3 = C,$$

$$T = D, \qquad \alpha_5 = \alpha_7 = E, \qquad t_3 = t_1 = 0,$$

we recover the results established in §1.8, §1.9.

§2.9 DISCUSSION AND SUMMARY.

We are now in a position to discuss and summarise the various dispersion relations governing the propagation of longitudinal and flexural waves in incompressible and compressible materials.

We begin by assembling the equations together for comparison: that is, (2.40)-(2.43) and (2.53)-(2.55) for incompressible cylinders, (2.76)-(2.78) and (2.87)-(2.89) for compressible materials. We use suffices i, c to describe the material, and l, f to describe the type of wave (i incompressible, c compressible, l longitudinal, f flexural). Then

$$\left[\frac{\rho\omega^2}{K^2}\right]_{i,l} = (\Delta+\tau)-\frac{\Delta(Ka)^2}{8}+\frac{(2\Delta-\delta_2)(Ka)^4}{384}-\left[\Delta^2+\Delta(\delta_2-\alpha_5)+\frac{\delta_2^2}{4}\right]\frac{(Ka)^6}{3072\alpha_5},$$

$$\left[\frac{\rho\omega^2}{K^2}\right]_{i,f} = \tau+\frac{\Delta(Ka)^2}{4}+\left[\alpha_5-4\Delta-\frac{7\Delta^2}{\alpha_5}\right]\frac{(Ka)^4}{96},$$

(2.90)

$$\left[\frac{\rho\omega^2}{K^2}\right]_{c,l} = (\Omega+\tau)-\frac{\alpha_2^2\Omega(ka)^2}{2(R+S)^2}-\frac{\alpha_2^2}{4(R+S)^3}\left[\frac{(7R+S)\Omega^2}{12R}+\Omega\alpha_2\left(\frac{R+S-}{6R}\frac{\alpha_2}{R+S}\right)+\frac{\alpha_2^2(S-R)}{12R}\right](Ka)^4,$$

$$\left[\frac{\rho\omega^2}{K^2}\right]_{c,f} = \tau+\frac{\Omega(Ka)^2}{4}+\left[\frac{4\alpha_2^2\alpha_5}{(R+S)^2}-\frac{2[3(R+S)-2\alpha_2]\Omega}{(R+S)}-\frac{7\Omega^2}{\alpha_5}\right]\frac{(Ka)^4}{96},$$

where $\Delta = [\delta_3+\delta_2/2-\delta_1]$, $\Omega = [T-2\alpha_2^2/(R+S)-\tau]$ and $\tau=t_3-t_1$.

First, it is instructive to compare the results (2.90) with those obtained for cylinders free from stress. For an elastic material with Lamé constants λ, μ the dispersion equations for longitudinal and flexural waves in an unstressed elastic cylinder are:

$$\left[\frac{\rho\omega^2}{K^2}\right]_l = \frac{\mu(3L+2)}{(L+1)} - \frac{\mu(3L+2)L^2(Ka)^2}{8(L+1)^3} + \frac{\mu(2L^4-13L^3-56L^2-52L-14)L^2(Ka)^4}{96(L+1)^5(L+2)}$$

$$-\frac{\mu(L^7-132L^6-389L^5+514L^4+2816L^3+3324L^2+1572L+264)L(Ka)^6}{3072(L+1)^7(L+2)},$$

$$\left[\frac{\rho\omega^2}{K^2}\right]_f = \frac{\mu(3L+2)(Ka)^2}{4(L+1)} - \frac{\mu(37L^2+55L+20)(Ka)^4}{48(L+1)^2}, \tag{2.91}$$

where $L=\lambda/\mu$. In the absence of primary stress (so that β and γ are both unity), it is readily seen from the derivation of (2.25) that

$$\delta_1 = \delta_2 = -\delta_3 = -2\mu, \qquad \alpha_5 = \mu, \tag{2.92}$$

and from the derivation of (2.61), (2.62) and (2.63) that

$$R = T = \lambda+2\mu, \qquad \alpha_6 = \alpha_7 = \mu, \qquad S = \alpha_2 = \lambda. \tag{2.93}$$

Using these values for α_5, α_2, R, S, T in $(2.90)_3$, $(2.90)_4$ and setting $\tau=0$, gives

$$\Omega = \mu(3L+2)/(L+1), \qquad \alpha_2^2/(R+S)^2 = L^2/4(L+1)^2$$

and leads directly to a reconciliation with (2.91) to the appropriate order. For incompressible materials we see, from (2.92), that in the absence of primary stress $\Delta = 3\mu$, so that $(2.90)_1$, $(2.90)_2$ give the limiting forms of (2.91) as $L\to\infty$ (the appropriate limit for incompressible materials).

We now restrict our attention to the terms in (2.90) up to and including the coefficients of $(Ka)^2$ which for convenience are rewritten here

$$
\begin{aligned}
(\rho\omega^2/K^2)_{i,1} &= (\delta_3+\delta_2/2-\delta_1+\tau)-(\delta_3+\delta_2/2-\delta_1)(Ka)^2/8,\\
(\rho\omega^2/K^2)_{i,f} &= \tau+(\delta_3+\delta_2/2-\delta_1)(Ka)^2/4,\\
(\rho\omega^2/K^2)_{c,1} &= (T-2\alpha_2^2/(R+S)-\tau)-(T-2\alpha_2^2/(R+S)-\tau)\alpha_2^2(Ka)^2/2(R+S)^2,\\
(\rho\omega^2/K^2)_{c,f} &= \tau+(T-2\alpha_2^2/(R+S)-\tau)(Ka)^2/4.
\end{aligned}
\tag{2.94}
$$

The first observation to make is that the equations (2.94) have a pleasing general symmetry about them; the vital quantities being $(\delta_3-\delta_2/2-\delta_1)$ and $(T-2\alpha_2^2/(R+S))$ modified on occasions by the addition of a term in τ. The coefficients are clearly closely matched. One slightly odd feature is the appearance of the factor $\alpha_2^2/2(R+S)^2$ in the expression for the $(\rho\omega^2/K^2)_{c,1}$ case, a factor not matched elsewhere. A conjecture at this stage, later confirmed, is that this factor arises out of the intrinsic effects of the compressible state of the material.

It is possible, however, to express these results in a form which may be more convenient for comparison with experiment. In the theory of linear elasticity for isotropic materials (see for example Love [12]), there are defined Young's modulus E and Poisson's ratio σ in terms of the Lamé constants λ, μ by the relations

$$E = \mu(3L+2)/(L+1), \qquad \sigma = L/2(L+1),$$

so that (2.91), to $O(Ka)^2$, becomes

$$
\begin{aligned}
(\rho\omega^2/K^2)_1 &= E[1-\sigma^2(Ka)^2/2],\\
(\rho\omega^2/K^2)_f &= E(Ka)^2/4.
\end{aligned}
\tag{2.95}
$$

It might reasonably be hoped that (2.94) could be expressed in a simple form with suitable changes in the definitions of Young's modulus and Poisson's ratio to take account of the primary stress. In the primary state produced by the imposition of the primary stress, the deformation is specified by the equations (see (2.4))

$$x_1 = \beta X_1, \qquad x_2 = \beta X_2, \qquad x_3 = \gamma X_3. \tag{2.96}$$

In parallel, then, with the manner in which E, σ are defined for perturbations of the natural state, let us consider the static deformation

$$x_1 = \beta X_1 + \varepsilon x_1, \qquad x_2 = \beta X_2 + \varepsilon x_2, \qquad x_3 = \gamma X_3 + \eta x_3, \tag{2.97}$$

where ε, η are small and arise from (2.96) by increasing the axial stress by an amount dt. We define a modified Young's modulus E^* and a modified Poisson's ratio σ^* by the relations

$$E^* = dt/\eta, \qquad \sigma^* = -\varepsilon/\eta. \tag{2.98}$$

We now wish to relate E^*, σ^* to quantities already appearing in this chapter. We turn therefore to (2.13), used alike for compressible and incompressible materials, and observe that (2.97) is equivalent to (2.13) provided

$$u = \varepsilon r, \qquad v = 0, \qquad w = \eta z. \tag{2.99}$$

From (2.66), (2.99)

$$(R+S)\varepsilon + \alpha_2 \eta = 0, \tag{2.100}$$

and from $(2.62)_3$

$$dt = 2\alpha_3 \varepsilon + (\alpha_4 + 2\tau)\eta. \tag{2.101}$$

We deduce at once from (2.63), (2.98), (2.100), (2.101)

$$E^* = T - 2\alpha_3\alpha_2/(R+S), \qquad \sigma^* = \alpha_2/(R+S). \tag{2.102}$$

From (2.60), therefore with (2.102),

$$E^* = T - 2\alpha_2^2/(R+S) + 2\sigma^*\tau. \tag{2.103}$$

Thus for compressible materials from (2.94), (2.103)

$$\begin{aligned} (\rho\omega^2/K^2)_{c,l} &= E^* - 2\sigma^*\tau - (\sigma^*)^2[E^* - (2\sigma^*+1)\tau](Ka)^2/2, \\ (\rho\omega^2/K^2)_{c,f} &= \tau + [E^* - (2\sigma^*+1)\tau](Ka)^2/4, \end{aligned} \tag{2.104}$$

to compare with (2.95).

Incompressible materials present a somewhat degenerate case. Of course, we must have $E=3\mu$ and $\sigma=1/2$, and from (2.15), (2.98) $\sigma^*=1/2$ also. But we can still use (2.25), (2.28) with (2.98) for E^* and we find

$$E^* = \delta_3+\delta_2/2-\delta_1+2\tau. \tag{2.105}$$

When (2.105) is substituted into $(2.94)_1$, $(2.94)_2$

$$\begin{aligned}(\rho\omega^2/K^2)_{i,l} &= (E^*-\tau)-(E^*-2\tau)(Ka)^2/8,\\ (\rho\omega^2/K^2)_{i,f} &= \tau+(E^*-2\tau)(Ka)^2/4.\end{aligned} \tag{2.106}$$

It is seen at once that (2.106) agrees with (2.104) when σ^* is set equal to 1/2, the appropriate value for incompressible materials. The advantage of casting the dispersion relations into the forms (2.104), (2.106) is that the quantities appearing in those equations are directly measurable, and so entirely suitable for comparisons between theory and experiment. One cautionary remark, however, must be made: once the primary deformation has been imposed upon the cylinder, there is no longer isotropy with regard to further small displacement. The quantities E^*, σ^* defined above are defined solely in relation to a further incremental rise in axial stress and not for small perturbations in general.

Let us return now to the general notion of dispersion relations and consider in particular the implications for stability for those parts of (2.94) which refer to the flexural mode. We see at once from $(2.94)_4$ that ω^2 may vanish (that is, the onset of instability is imminent) when τ takes a value of $O(Ka)^2$. A little caution is necessary because (2.94) has been derived on the premise that the term in $(Ka)^2$ is a small correction in an expression whose principal term is τ. But when τ is itself of order $(Ka)^2$, $(2.94)_4$ may be expressed

$$\left[\frac{\rho\omega^2}{K^2}\right]_{c,f} = \tau+\mu\left[\frac{3L+2}{4L+4}\right](Ka)^2,$$

correct to terms in $(Ka)^2$. Thus we expect instability when τ corresponds to a compression given by

$$\tau = -\mu\left[\frac{3L+2}{4L+4}\right](Ka)^2.$$

At first sight this result has puzzling practical implications, for it seems to show that for a given negative value of τ, no matter how small, we shall find it possible to find a sufficiently small K so that waves with this wavenumber will be marginally unstable. It would seem then to follow that for all bars and cylinders in compression, instability is inherent. It must be remembered, however, that the results obtained here for wave-propagation relate to cylinders of infinite length. For bars of finite length the boundary conditions at the ends have a role to play. Consider for example a cylinder of length A and suppose that the end-conditions include the restriction that w, the axial displacement, must vanish there. Take equations (2.29), (2.67) and (2.94) in conjunction; we see that we may replace K by -K (leaving other parameters unaltered) and combine solutions to get

$$w = W(r)\sin Kz.\cos(\omega t+\varepsilon).$$

For w to vanish when z=0 imposes no extra condition but for w to vanish when z=A requires K to be a multiple of π/A. Thus the end condition sets a lower limit to K, specifically $K \geqslant \pi/A$, so that we expect instability in this case to arise when

$$\tau = -\mu\left[\frac{3L+2}{4L+4}\right]\left[\frac{\pi a}{A}\right]^2$$

and not until this value of compressive stress is reached. We observe that for instability in the longitudinal mode, (2.94) predicts that much larger values of $|\tau|$ will be required. The effect of the primary stress upon wave-velocity may well be perceived, however, long before instability sets in.

We turn now to another aspect of the dispersion equation. From (2.94) the typical equation can be written in the form

$$\omega^2/K^2 = \alpha^2[1+\xi(Ka)^2], \qquad \alpha>0,$$

before instability arises. Thus, taking the positive root (with ω, K both regarded for the moment as positive), we have

$$\omega = \alpha K[1+\xi(Ka)^2/2]$$

to the order of approximation in which we retain the small quantity $(Ka)^2$. Thus the phase-velocity c is given by

$$c = \omega/K = \alpha[1+\xi(Ka)^2/2],$$

and the group-velocity c_g by

$$c_g = d\omega/dK = \alpha[1+3\xi(Ka)^2/2].$$

Of course, if the correction term in $(Ka)^2$ is not included, both c and c_g are constants, indeed there is no dispersion. The value of ξ, as determined by the theory underlying the derivation of (2.94), gives the correction to c and c_g. In particular, the sign of ξ indicates whether the group velocity increases or decreases as the wavenumber K increases. This has great implications for the manner in which pulses are propagated along the cylinders. The reader is referred to Jeffreys and Jeffreys [26], and an extended account of the propagation of pulses in elastic solids by Davies [15].

§2.10.1 ILLUSTRATIVE EXAMPLES.

We wish now to illustrate the general results obtained above by reference to certain special materials. In the first case we examine the model introduced by Mooney [27] for incompressible materials and for the second case we study the model for compressible materials (intended originally to represent the behaviour of polyurethane foam rubber) given by Ko [9]. Finally, we consider a hyperelastic incompressible material for which the strain-energy W is expressed directly in terms of the principal stretches (rather than the strain invariants). With other workers Ogden [28] has developed the theory for rubberlike materials whose strain-energy may be taken in this form.

§2.10.2 THE MOONEY MODEL.

For the Mooney (or Mooney-Rivlin) model, the material is taken to be incompressible and the strain-energy function W has the particular form

$$W = L(I_1-3)+M(I_2-3), \tag{2.107}$$

where L and M are constants.

In terms of the principal axial stretch γ, we have

$$\beta = \gamma^{-1/2}, \qquad I_2 = 2\gamma+\gamma^{-2}, \tag{2.108}$$

from (2.7), (2.11), and

$$\tau = 2(\gamma^2-\gamma^{-1})(L+\gamma^{-1}M), \tag{2.109}$$

from (2.10), (2.107). The definitions (2.22) are now evaluated; thus

$$\Phi = 2L, \qquad \Psi = 2M, \qquad A = B = F = 0, \tag{2.110}$$

so that the quantities in $(2.90)_1$, $(2.90)_2$ may be found from the substitution of (2.108), (2.110) into (2.23). We obtain (in terms of the axial stress)

$$\delta_1 = -\delta_3 = -2\alpha_5 = -4\gamma^{-1}(L+\gamma^{-1}M), \quad \delta_2 = -4\gamma^{-1}(L+\gamma^2 M),$$
$$\Delta = 6\gamma^{-1}L-2(\gamma-4\gamma^{-2})M. \tag{2.111}$$

The expressions (2.109), (2.111) can then be substituted into $(2.90)_1$, $(2.90)_2$ from which we may determine the wave-speeds (in terms of L, M and γ). Thus, denoting M/L by Γ, the longitudinal mode gives

$$\frac{\rho\omega^2}{LK^2} = 2[2\gamma^{-1}+\gamma^2+3\gamma^{-2}\Gamma]-[3\gamma^{-1}+(4\gamma^{-2}-\gamma)\Gamma]\frac{(Ka)^2}{4}$$
$$+[\gamma^{-1}+\gamma^{-2}\Gamma]\frac{(Ka)^4}{24}-\frac{[1+(9\gamma^{-1}-7\gamma^2)\Gamma+(12\gamma^{-2}-15\gamma+4\gamma^4)\Gamma^2](Ka)^6}{1536(\gamma+\Gamma)}. \tag{2.112}$$

To the order $(Ka)^2$ this particular result was obtained by Suhubi [29]. For the flexural mode

$$\frac{\rho\omega^2}{LK^2} = 2(\gamma-\gamma^{-2})(\gamma+\Gamma)+[3\gamma^{-1}+(4\gamma^{-2}-\gamma)\Gamma]\frac{(Ka)^2}{2}$$
$$-\frac{[74+(194\gamma^{-1}-46\gamma^2)\Gamma+(127\gamma^{-2}-60\gamma+7\gamma^4)\Gamma^2](Ka)^4}{48(\gamma+\Gamma)}. \tag{2.113}$$

The results (2.112), (2.113) are illustrated in Fig. 2.1 where we set Γ=0.4 and plot, for selected values of Ka, the curves of the axial stretch γ against $(c/c_0)^2$ with c denoting the usual wave-velocity ω/K and where $c_0^2 = 2(L+M)/\rho$ so that in the linear approximation c_0 reduces to $\sqrt{(\mu/\rho)}$, the velocity of transverse waves in an unbounded medium. In Figs. 2.2, 2.3 respectively, we display, for selected values of Γ, the longitudinal and flexural modes, for the limiting case when Ka=0. The curves produced for Γ=0 represent the neo-Hookean model.

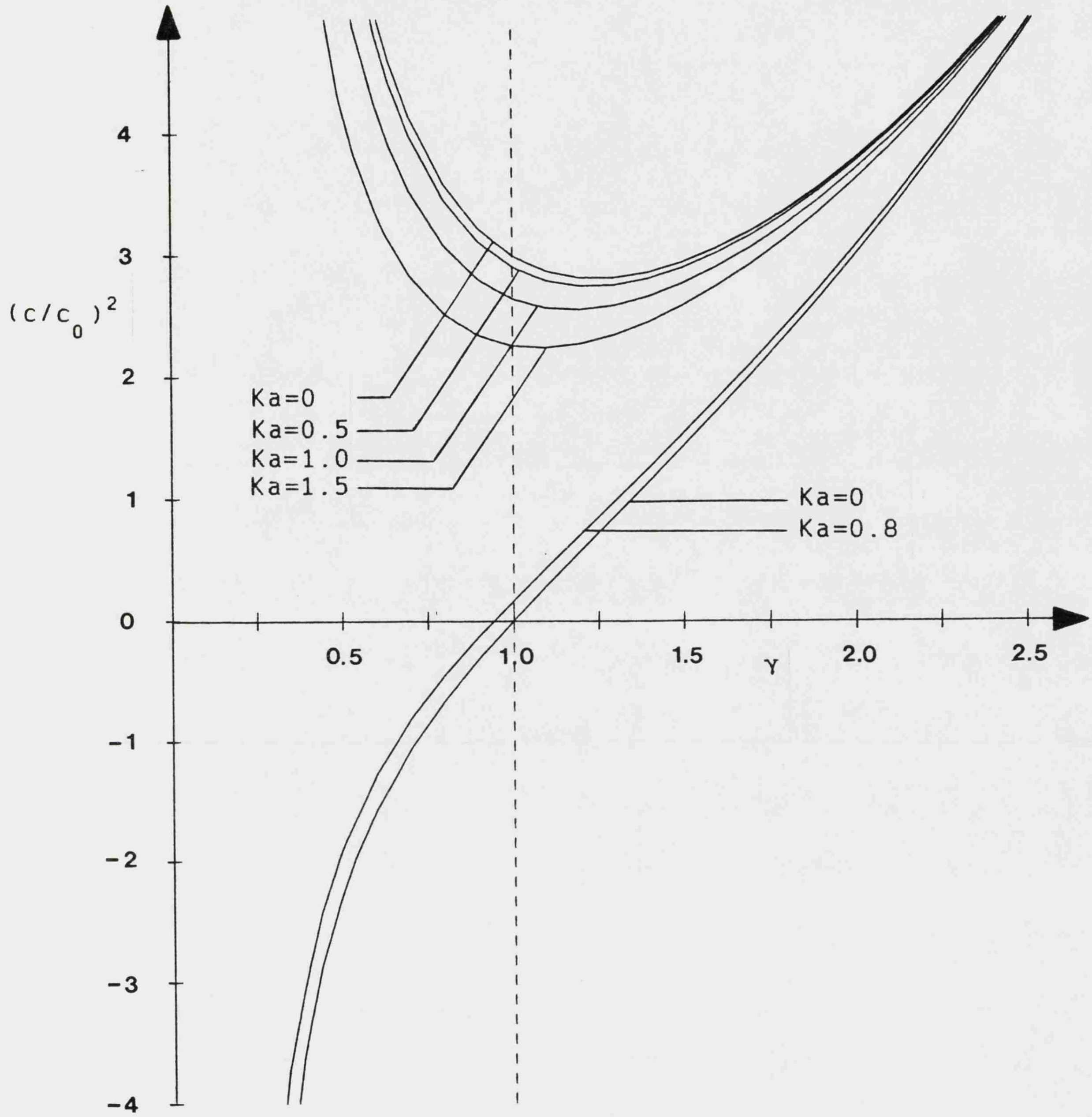

Figure 2.1

Variation of $(c/c_0)^2$ against the axial principal stretch γ for a uniaxially stressed Mooney cylinder with Γ=0.4. The two lower curves correspond to the flexural mode given by (2.113) and the upper set of curves, for the longitudinal mode, are produced from (2.112).

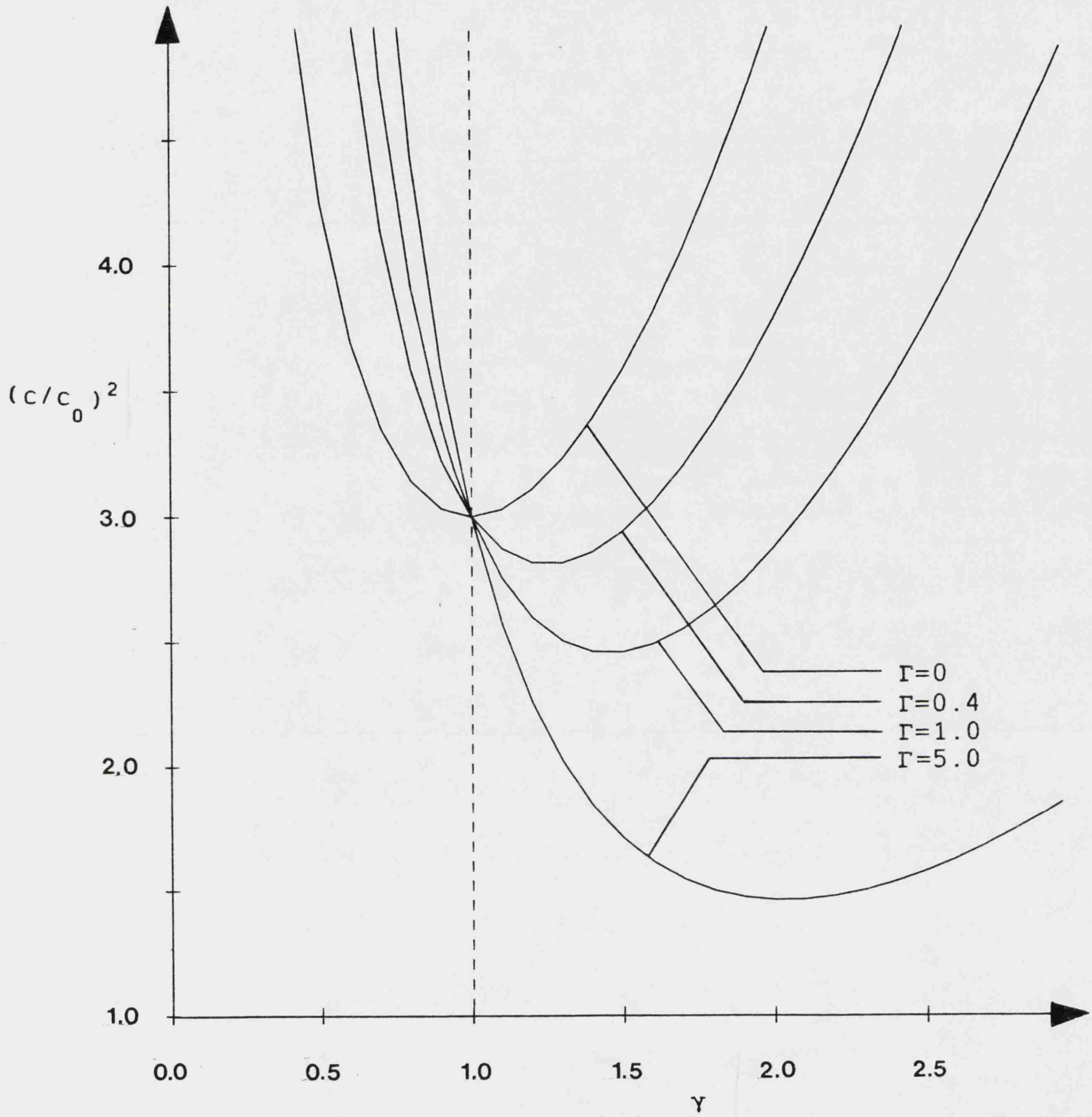

Figure 2.2

For selected values of Γ, the longitudinal curves of $(c/c_0)^2$ against γ are displayed for a uniaxially stressed Mooney cylinder in the limiting case Ka=0.

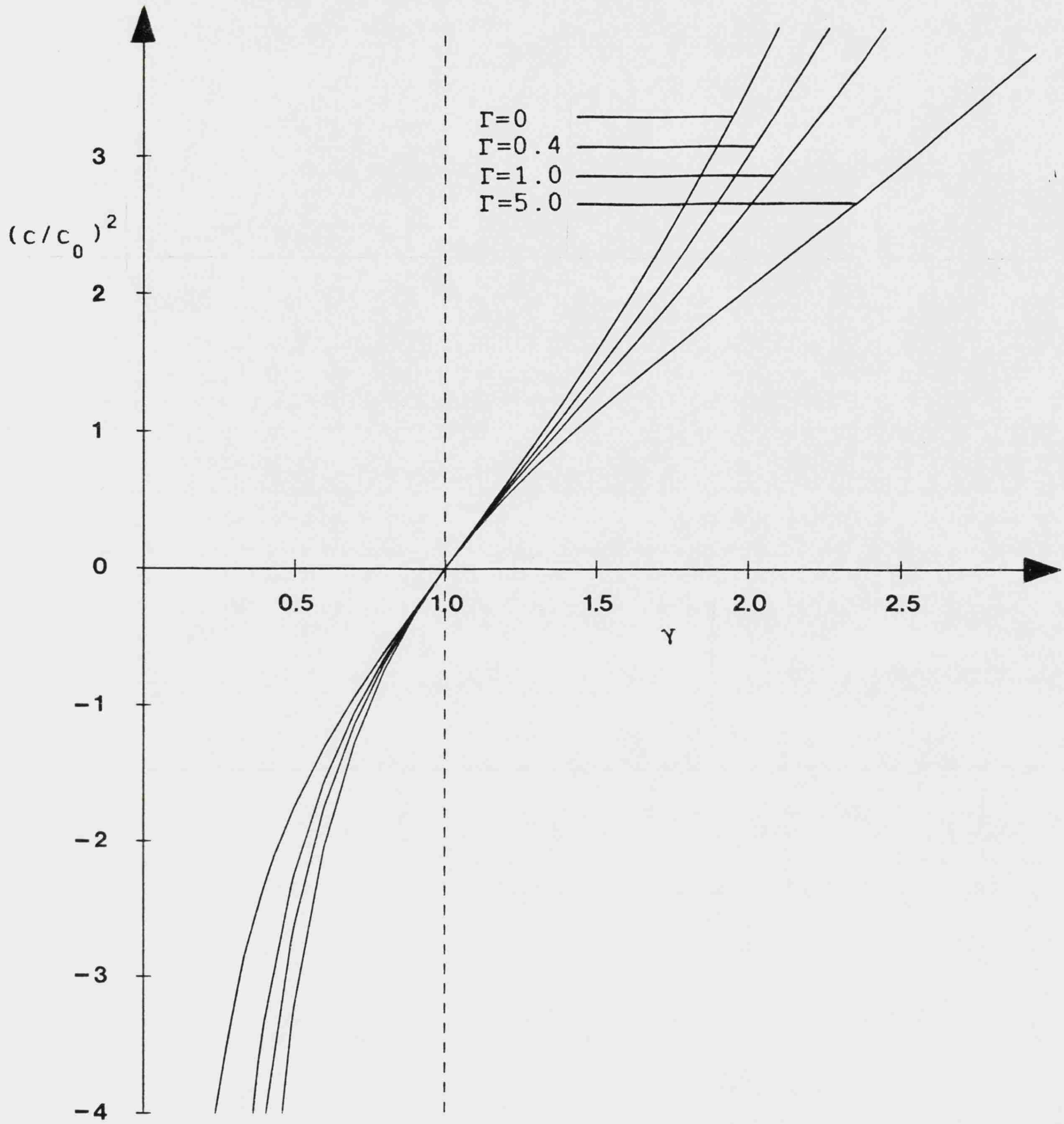

Figure 2.3

For selected values of Γ, the flexural curves of $(c/c_0)^2$ against γ are displayed for a uniaxially stressed Mooney cylinder in the limiting case Ka=0.

§2.10.3 THE KO MODEL.

In [9] Ko proposed as a model for compressible materials the strain-energy function

$$W = \bar{\mu}(I_3^{1/2}+I_2/2I_3), \tag{2.114}$$

$\bar{\mu}$ constant. This is an especially simple model in that the Lamé constants λ, μ governing the linear elastic behaviour are equal (and, in fact, equal to the parameter $\bar{\mu}$ in (2.114)), so that the Poisson's ratio is 1/4.

From (2.56), (2.57), (2.59), (2.63) with (2.114), we find after a little calculation

$$\alpha_2 = \alpha_7 = S = R/3 = \bar{\mu}-t_1, \qquad \alpha_5 = T/3 = \bar{\mu}-t_3. \tag{2.115}$$

For convenience we introduce the standardised stresses T_1, T_3 where

$$T_1 = 1-t_1/\bar{\mu}, \qquad T_3 = 1-t_3/\bar{\mu}, \tag{2.116}$$

so that, from $(2.90)_3$, $(2.90)_4$ with (2.115), (2.116)

$$\left[\frac{\rho\omega^2}{\bar{\mu}K^2}\right]_1 = \frac{(6T_3-T_1)}{2} - \frac{(8T_3-3T_1)(Ka)^2}{64} - \frac{(49T_1^2-268T_1T_3+352T_3^2)(Ka)^4}{9216T_1}, \tag{2.117}$$

and

$$\left[\frac{\rho\omega^2}{\bar{\mu}K^2}\right]_f = (T_1-T_3) + \frac{(8T_3-3T_1)(Ka)^2}{8} - \frac{(63T_1^2-366T_1T_3+527T_3^2)(Ka)^4}{384T_3}. \tag{2.118}$$

When T_1, T_3 are unity (2.117) and (2.118) reduce, as they should, to the forms (2.91) with L there set equal to one. The full expressions (2.117), (2.118) will be used later, for the moment, however, we set Ka=0 and look at the behaviour of these equations for two particular primary deformations. In Fig. 2.4 we display for the longitudinal mode the curves of $(c/c_t)^2$ against the axial stretch γ with $c_t = \surd(\bar{\mu}/\rho)$ denoting the velocity of transverse waves in the linear approximation. The curves plotted correspond to a uniaxial stress; $T_1=1$, $\beta^4\gamma=1$ and an equibiaxial stress; $T_3=1$, $\beta^2\gamma^3=1$. The corresponding curves for the flexural mode are given in Fig. 2.5.

§2.10.4 RUBBERLIKE MATERIALS.

For our third example, we return to isotropic, incompressible materials for which the strain-energy function W is given in terms of the principal stretches $\lambda_1,\lambda_2,\lambda_3$ (where $\lambda_1\lambda_2\lambda_3=1$), measured from the natural state. We write $W = \Sigma(\lambda_1,\lambda_2,\lambda_3)$ and attach suffixes 1,2,3 to Σ to indicate partial derivatives taken with respect to $\lambda_1,\lambda_2,\lambda_3$. Examples of calculations of this kind are given in the work of Ogden and others [28], [30], [31].

One possible approach is to express W in terms of I_1, I_2 and then use $(2.94)_1$, $(2.94)_2$. Often, however, this is a clumsy and time-wasting procedure. A better line is to use (2.106).

First the principal stresses P_i are given in terms of the function Σ and its derivatives by the relation

$$P_i = \lambda_i\Sigma_i-\Pi, \qquad (i=1,2,3),$$

where Π is an arbitrary hydrostatic pressure. So, from a consideration of the primary deformation,

$$\tau = \lambda_3\Sigma_3-\lambda_1\Sigma_1,$$

at $\lambda_1=\lambda_2=\beta$, $\lambda_3=\gamma$. We now make the small perturbation given by (2.97) which has the effect of increasing β, γ, τ to $\beta(1+\varepsilon)$,

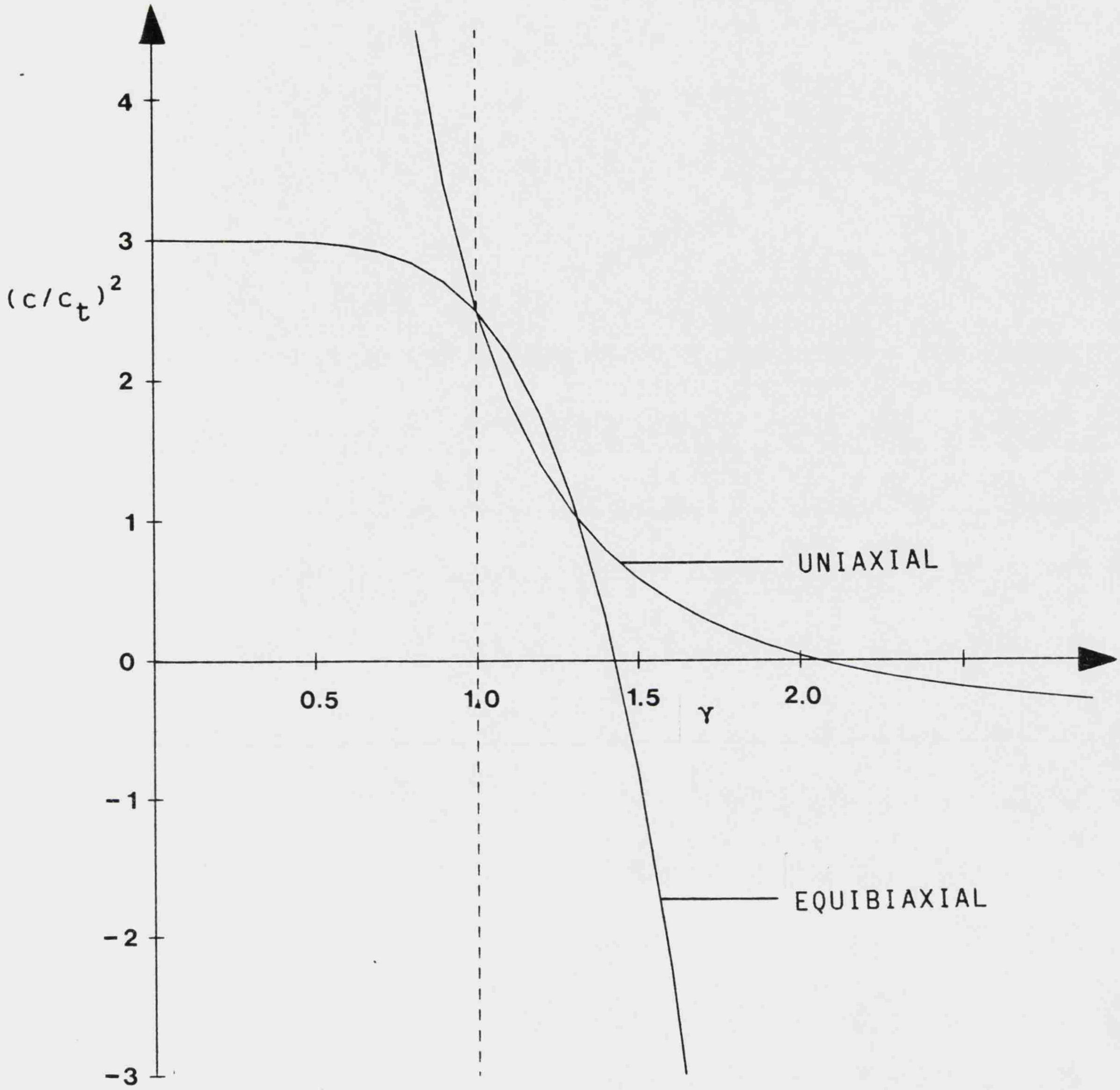

Figure 2.4

Variation of $(c/c_t)^2$ against the axial stretch γ for longitudinal waves in a stressed Ko cylinder for the limiting case Ka=0.

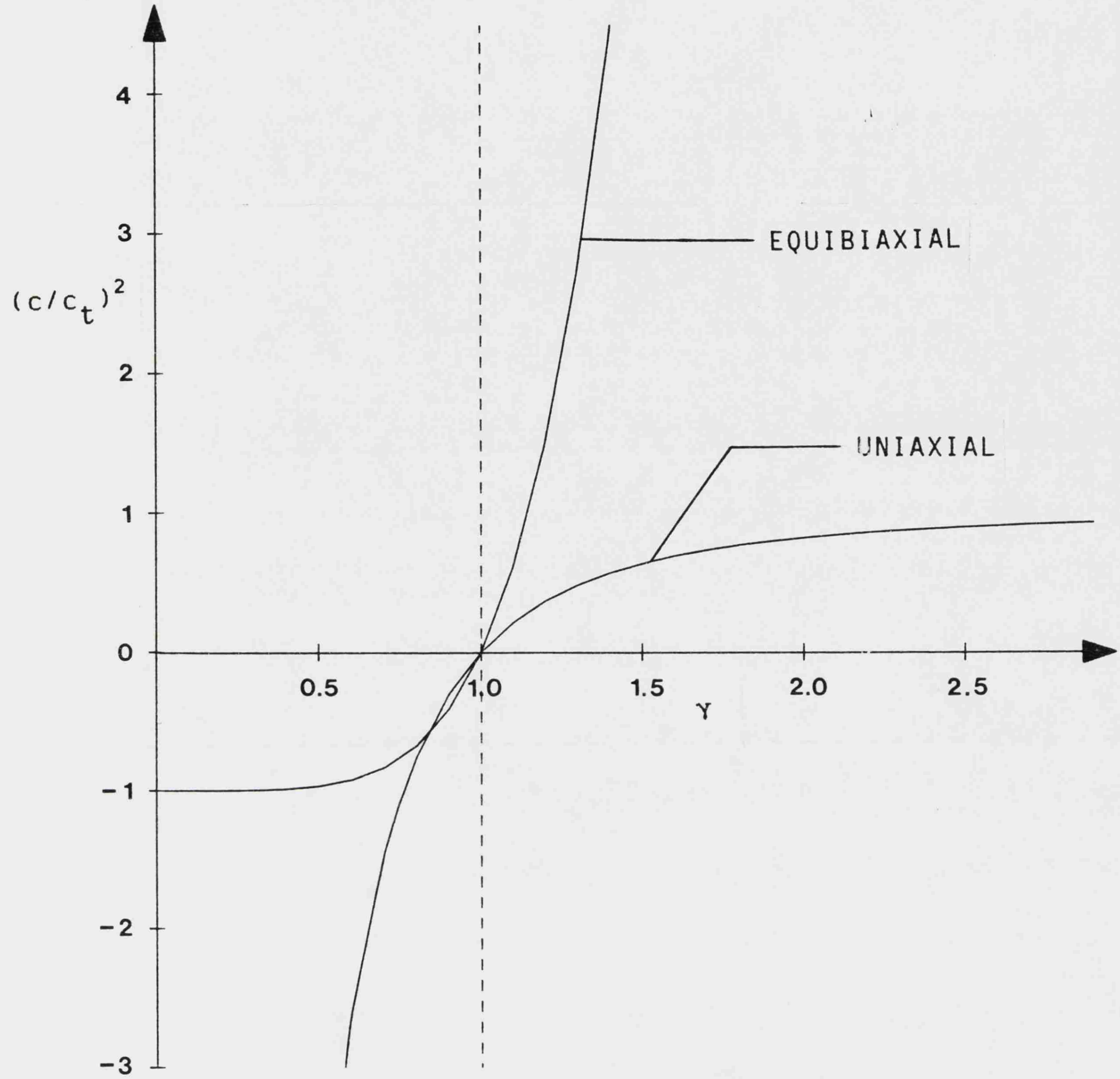

Figure 2.5

Variation of $(c/c_t)^2$ against the axial stretch γ for flexural waves in a stressed Ko cylinder for the limiting case Ka=0.

$\gamma(1+\eta)$, $\tau+dt$ in which ε, η, dt are all small. The incompressibility condition gives $2\varepsilon+\eta=0$, that is, $\sigma^*=1/2$. We can now calculate E^* from the relation $E^*=dt/\eta$ to find

$$E^* = \gamma\Sigma_3+\gamma^{-1/2}\Sigma_1/2-2\gamma^{1/2}\Sigma_{13}+\gamma^2\Sigma_{33}+\gamma^{-1}\Sigma_{11}/2+\gamma^{-1}\Sigma_{12}/2.$$

We can then use (2.106) to give the dispersion relations.

In much of the work published on this model, the function Σ is expressed in the form

$$\Sigma = F(\lambda_1)+F(\lambda_2)+F(\lambda_3)+C,$$

where F is a function of the single variable and C is a constant. In this case

$$\tau = \gamma F'(\gamma)-\gamma^{-1/2}F'(\gamma^{-1/2})$$

and

$$E^* = \gamma F'(\gamma)+\gamma^{-1/2}F'(\gamma^{-1/2})/2+\gamma^2 F''(\gamma)+\gamma^{-1}F''(\gamma^{-1/2})/2.$$

For example, if $\Sigma = L(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)+M(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}-3)$, so that $F(\lambda) = L\lambda^2+M\lambda^{-2}$,

$$\tau = 2(\gamma^2-\gamma^{-1})(L+\gamma^{-1}M),$$

$$E^* = 2(2\gamma^2+\gamma^{-1})L+2(2\gamma^{-2}+\gamma)M,$$

and (2.106) gives

$$(\rho\omega^2/LK^2)_l = 2[2\gamma^{-1}+\gamma^2+3\gamma^{-2}\Gamma]-[3\gamma^{-1}+(4\gamma^{-2}-\gamma)\Gamma](Ka)^2/4,$$

$$(\rho\omega^2/LK^2)_f = 2(\gamma-\gamma^{-2})(\gamma+\Gamma)+[3\gamma^{-1}+(4\gamma^{-2}-\gamma)\Gamma](Ka)^2/2,$$

in agreement with (2.112), (2.113) to $O(Ka)^2$. This is as it should be for with Σ as above, $W = L(I_1-3)+M(I_2-3)$.

This is an especially interesting example because it illustrates the usefulness of equations (2.106) relating our results for the dynamical behaviour of cylinders, in the long wave limit, to data derived from the statical state of stress.

CHAPTER 3

SURFACE WAVES IN A PRE-STRESSED MATERIAL

§3.1 INTRODUCTION.

In a series of papers by Willson [32-37] the properties of small-amplitude surface waves are investigated for a variety of isotropic elastic materials, both incompressible and compressible, which have been placed under different configurations of stress and also for arbitrary directions of propagation. In this chapter we consider a surface wave supported by a semi-infinite medium that has been placed under a large primary deformation, and for which the direction of propagation is along a principal axis. We consider both incompressible and compressible materials in turn and show that Willson's results are recovered as special cases. Our main results are investigated numerically for Mooney and Ko materials.

§3.2 INCOMPRESSIBLE MATERIAL.

Consider a semi-infinite region $x_3 \geqslant 0$ composed of a homogeneous, isotropic, hyperelastic material upon which stresses have been applied along the three axes Ox_1, Ox_2, Ox_3. We suppose that the uniform principal stretches are α, β, γ so that the deformation in the steady state is

$$x_1 = \alpha X_1, \qquad x_2 = \beta X_2, \qquad x_3 = \gamma X_3. \tag{3.1}$$

In this state we denote the non-zero stresses by P_1, P_2 and P_3 so that, using (2.1) with the definitions (2.22),

$$P_1 = \Phi\alpha^2 - \Pi - \Psi\alpha^{-2},$$

$$P_2 = \Phi\beta^2 - \Pi - \Psi\beta^{-2},$$

$$P_3 = \Phi\gamma^2 - \Pi - \Psi\gamma^{-2},$$

and

$$I_1 = \alpha^2+\beta^2+\gamma^2, \quad I_2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2, \quad I_3 = \alpha^2\beta^2\gamma^2. \qquad (3.2)$$

Since the material is incompressible $I_3=1$ so that $\alpha\beta\gamma=1$.

We now consider the perturbed state in which the displacements and stresses differ from the above by only small amounts. The deformation is specified by

$$x_1 = \alpha X_1+u(x_1,x_3,t), \quad x_2 = \beta X_2, \quad x_3 = \gamma X_3+w(x_1,x_3,t), \qquad (3.3)$$

in which we regard u, w as small quantities and in all subsequent calculations we shall neglect their squares and products. Accordingly, for the Cauchy deformation tensor c, we calculate

$$c = \begin{bmatrix} \alpha^{-2}(1-2u_1) & 0 & -\alpha^{-2}u_3-\gamma^{-2}w_1 \\ 0 & \beta^{-2} & 0 \\ -\alpha^{-2}u_3-\gamma^{-2}w_1 & 0 & \gamma^{-2}(1-2w_3) \end{bmatrix},$$

$$c^{-1} = \begin{bmatrix} \alpha^{2}(1+2u_1) & 0 & \alpha^{2}w_1+\gamma^{2}u_3 \\ 0 & \beta^{2} & 0 \\ \alpha^{2}w_1+\gamma^{2}u_3 & 0 & \gamma^{2}(1+2w_3) \end{bmatrix}. \qquad (3.4)$$

The incompressibility condition det c = 1 yields

$$u_1+w_3 = 0, \qquad (3.5)$$

and from (3.4), with (3.5),

$$I_1 = \alpha^2+\beta^2+\gamma^2+2(\gamma^2-\alpha^2)w_3,$$

$$I_2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\beta^2(\gamma^2-\alpha^2)w_3. \qquad (3.6)$$

Assuming that W and its derivatives may be expanded as a

Taylor series about the point (I_1, I_2) given by (3.6), we find for the components of stress

$$P_{11} = P_1 - \Pi + Rw_3, \qquad P_{12} = P_{23} = 0,$$

$$P_{22} = P_2 - \Pi + Sw_3, \qquad P_{13} = Gu_3 + Hw_1, \tag{3.7}$$

$$P_{33} = P_3 - \Pi + Tw_3,$$

where

$$\begin{aligned} R &= 2\alpha^2(\gamma^2-\alpha^2)[A-\beta^6\gamma^4 B+\beta^2(1-\beta^2\gamma^4)F]-2\alpha^2(\Phi+\beta^4\gamma^4\Psi), \\ S &= 2\beta^2(\gamma^2-\alpha^2)[A-\alpha^2\gamma^2 B+(\beta^2-\alpha^4\gamma^4)F], \\ T &= 2\gamma^2(\gamma^2-\alpha^2)[A-\alpha^4\beta^6 B+\beta^2(1-\alpha^4\beta^2)F]+2\gamma^2(\Phi+\alpha^4\beta^4\Psi), \\ G &= \gamma^2(\Phi+\beta^2\Psi), \qquad H = \alpha^2(\Phi+\beta^2\Psi), \end{aligned} \tag{3.8}$$

and Φ, Ψ, A, B, and F are given by (2.22).

The equations of motion become

$$\begin{aligned} -\Pi_1 + (R+H)w_{13} + Gu_{33} &= \rho u_{tt}, \\ -\Pi_3 + Gu_{13} + Hw_{11} + Tw_{33} &= \rho w_{tt}, \end{aligned} \tag{3.9}$$

with ρ the constant density of the medium.

The boundary conditions at the surface $X_3=0$ are

$$P_{ij}v_j = P_3 v_j, \tag{3.10}$$

where $v=(-w_1, 0, 1)$, and, with use of the relation $G-H = P_3-P_1$, this reduces to

$$u_3+w_1 = 0, \qquad -\Pi+Tw_3 = 0. \tag{3.11}$$

The theory is now applied to the study of surface waves. We restrict attention to those monochromatic solutions with harmonic dependence on the spatial coordinates, and seek what are in effect Rayleigh waves. Accordingly we suppose a dependence $\exp[i(\omega t-Kx_1-Kmx_3)]$ for the field quantities, and impose the requirement $\mathrm{Im}(m)<0$ so that the wave is attenuated away from $x_3=0$. The equations of motion (3.9) are then

$$i\Pi+Km[Gm^2-(R+H)-X]w = 0,$$
$$im\Pi+K[(G-T)m^2-H+X]w = 0, \tag{3.12}$$

where we denote $X=\rho\omega^2/K^2$ and have eliminated u with use of (3.5), and for (3.12) to have a non-trivial solution

$$Gm^4-[R-T+H+G+X]m^2+(H-X) = 0. \tag{3.13}$$

This is a quadratic in m^2 whose solutions, m_1^2, m_2^2, say, will each give rise to a single solution m_1, m_2 satisfying $Im(m)<0$ (except when the solution is non-negative, real) and where

$$m_1^2+m_2^2 = (X+R-T+H+G)/G, \qquad m_1^2 m_2^2 = (H-X)/G. \tag{3.14}$$

Since (3.13) has real coefficients, the solutions are complex conjugates (where they are not real) so that the expression $m_1 m_2$ will be negative (a conclusion that also holds when m_1^2, m_2^2 are both negative).

Now, as with Rayleigh waves in an unstressed medium, we seek to satisfy the boundary conditions by selecting a disturbance having two contributions of the type $\exp[i(\omega t-Kx_1-Kmx_3)]$ where ω, K are the same for the two contributions and m takes the values m_1 and m_2 respectively. Hence, noting (3.12), (3.5), we look for solutions of the form

$$u = i[m_1^2 P\exp(-iKm_1x_3)+m_2^2 Q\exp(-iKm_2x_3)]\exp\{i(\omega t-Kx_1)\},$$

$$w = -i[m_1 P\exp(-iKm_1x_3)+m_2 Q\exp(-iKm_2x_3)]\exp\{i(\omega t-Kx_1)\},$$

$$\Pi = K[((G-T)m_1^2-H+X)P\exp(-iKm_1x_3)$$
$$+((G-T)m_2^2-H+X)Q\exp(-iKm_2x_3)]\exp\{i(\omega t-Kx_1)\},$$

where P, Q are arbitrary constants. The boundary conditions (3.11) now become

$$m_1(m_1^2-1)P+m_2(m_2^2-1)Q = 0,$$

$$(Gm_1^2-H+X)P+(Gm_2^2-H+X)Q = 0,$$

and for a non-trivial solution in P, Q we find

$$(m_1-m_2)[(X-H)(m_1^2+m_2^2-1)+Gm_1^2m_2^2+(X-H+G)m_1m_2] = 0.$$

The vanishing of the factor m_1-m_2 corresponds to a trivial solution only and may be disregarded. The remaining factor may then be manipulated with the aid of (3.14) to yield the frequency equation

$$(X+R-T+H-G)[(H-X)/G]^{1/2} = (H-G-X). \qquad (3.15)$$

Of course, when the primary stresses are zero,

$$R = -T = -2\mu, \qquad H = G = \mu,$$

and (3.15) reduces to the well-known result

$$(x-4)(1-x)^{1/2} = -x, \qquad (3.16)$$

where $x = X/\mu$, comparable with Rayleigh's equation (1.10) for the incompressible limit $\lambda\to\infty$. Equation (3.16) has only one physically meaningful solution $x=0.9126...$ (cf. Table 1.1 corresponding to the result for $Ka=\infty$ when $\sigma=0.5$, $V/V_0 = x/\surd 3$).

As an illustrative example we now consider the Mooney model with strain-energy function given by (2.107). From (2.22), (3.8) and denoting M/L by Γ we have

$$R = -4L\alpha^2(1+\beta^4\gamma^4\Gamma), \qquad H = 2L\alpha^2(1+\beta^2\Gamma),$$

$$T = 4L\gamma^2(1+\alpha^4\beta^4\Gamma), \qquad G = 2L\gamma^2(1+\beta^2\Gamma).$$

Using the incompressibility condition $\alpha\beta\gamma=1$ we find

$$T-R = 4L(\alpha^2+\gamma^2)(1+\beta^2\Gamma),$$

so that the frequency equation (3.15) may be written

$$(Y-\alpha^2-3\gamma^2)(\alpha^2-Y)^{1/2} = \gamma(\alpha^2-\gamma^2-Y) \qquad (3.17)$$

with $Y = X/2L(1+\beta^2\Gamma)$. We see immediately that the point at which instability arises, that is, when Y vanishes, (3.17) is independent of the parameter Γ.

As a guide to the behaviour of this model under differing states of primary deformation we briefly explore three specific cases:

(i) uniaxial stress parallel to OX_1;

(ii) uniaxial stress parallel to OX_2;

(iii) equibiaxial stress in the OX_1X_2-plane.

In case (i) we take $P = \mathrm{diag}(P_1,0,0)$ so that $\beta=\gamma$, $\alpha\beta^2=1$ and Υ is now a function of α and Γ alone. Denoting $\alpha^3-\alpha\Upsilon$ by q^2, (3.17) may be expressed in the form

$$q^3+q^2+3q-1 = 0, \tag{3.18}$$

which contains a unique real root at $q=q_0=0.2956$. For the wave-speed we calculate

$$(c/c_0)^2 = (\alpha^2-q_0^2\alpha^{-1})(1+\alpha^{-1}\Gamma)/(1+\Gamma), \tag{3.19}$$

where $c=\omega/K$, $c_0^2= 2(L+M)/\rho$. The results are illustrated in Fig. 3.1 where we plot, for selected values of Γ, the curves of the principal stretch α against $(c/c_0)^2$. The point labeled M, lying on the line $\alpha=1$, corresponds to the unstressed state and at this point $(c/c_0)^2$ takes the value 0.9126. The other point common to all the curves occurs at the marginal point $\alpha=0.4437$.

Similarly, for case (ii), where $P = \mathrm{diag}(0,P_2,0)$ and $\alpha=\gamma$, $\alpha^2\beta=1$, (3.17) may again be reduced to the form (3.18) but now $q^2=1-\Upsilon\alpha^{-2}$ so that

$$(c/c_0)^2 = (1-q_0^2)\alpha^2(1+\alpha^{-4}\Gamma)/(1+\Gamma).$$

The results are displayed in Fig. 3.2.

Finally in case (iii) we set $P = \mathrm{diag}(P_1,P_1,0)$ so $\alpha=\beta$, $\alpha^2\gamma=1$. Again (3.18) holds but now with $q^2=\alpha^6-\alpha^4\Upsilon$; we find that for this particular case

$$(c/c_0)^2 = (\alpha^2-\alpha^4q_0^2)(1+\alpha^2\Gamma)/(1+\Gamma) \tag{3.20}$$

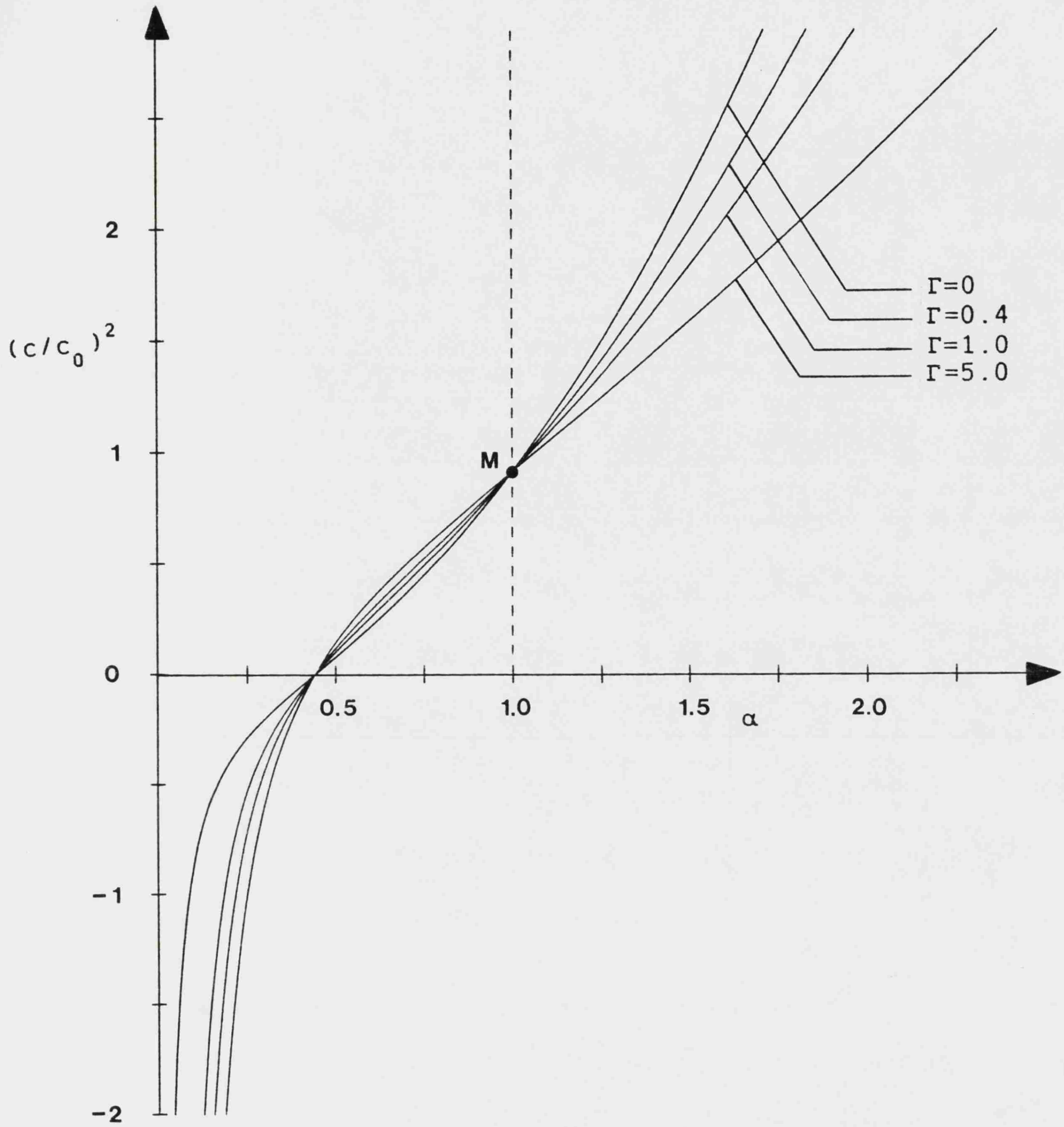

Figure 3.1

Wave-velocity as a function of the principal stretch α for a Mooney half-space uniaxially stressed in the OX_1-direction.

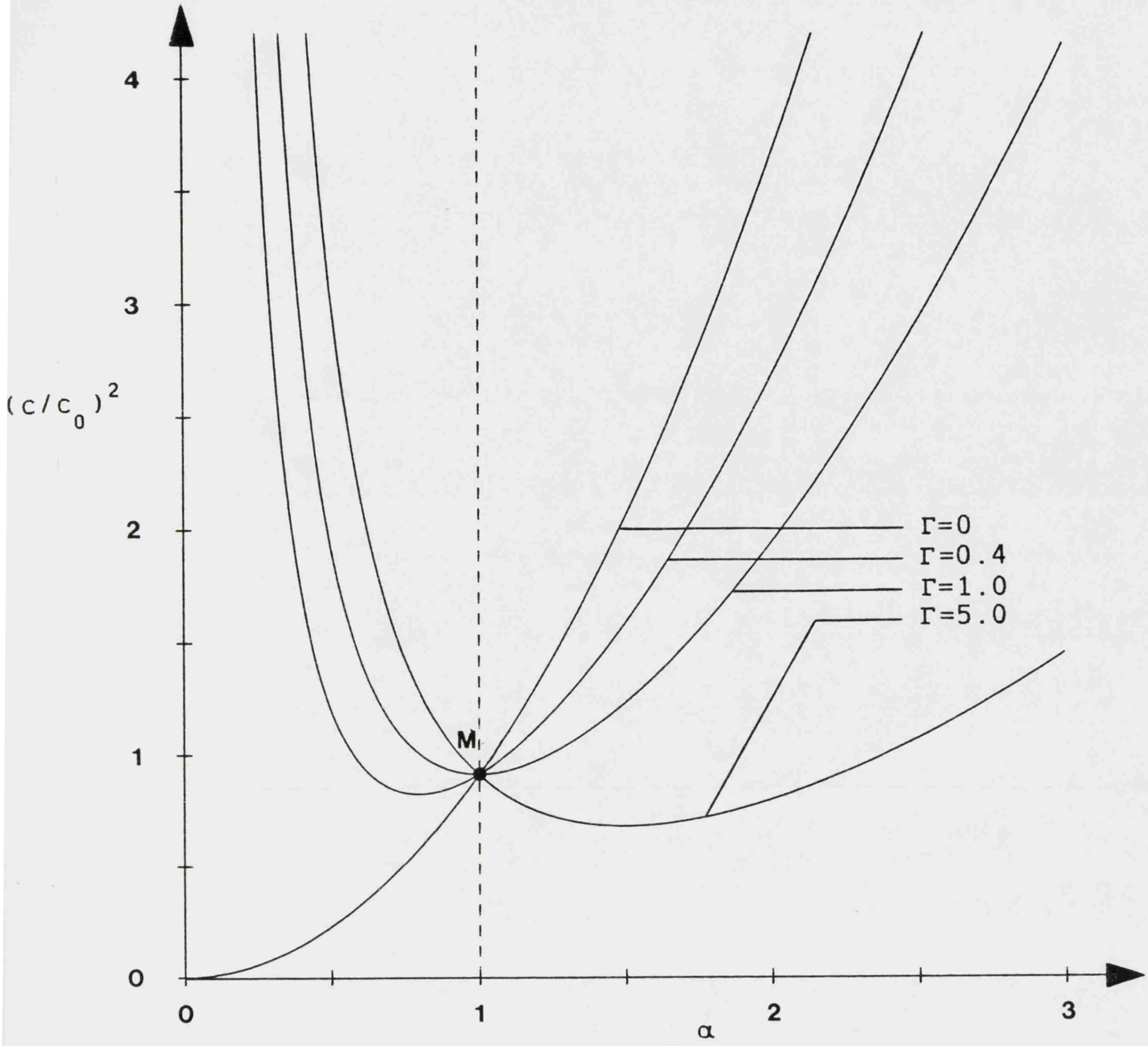

Figure 3.2

Wave-velocity as a function of the principal stretch α for a Mooney half-space uniaxially stressed in the OX_2-direction.

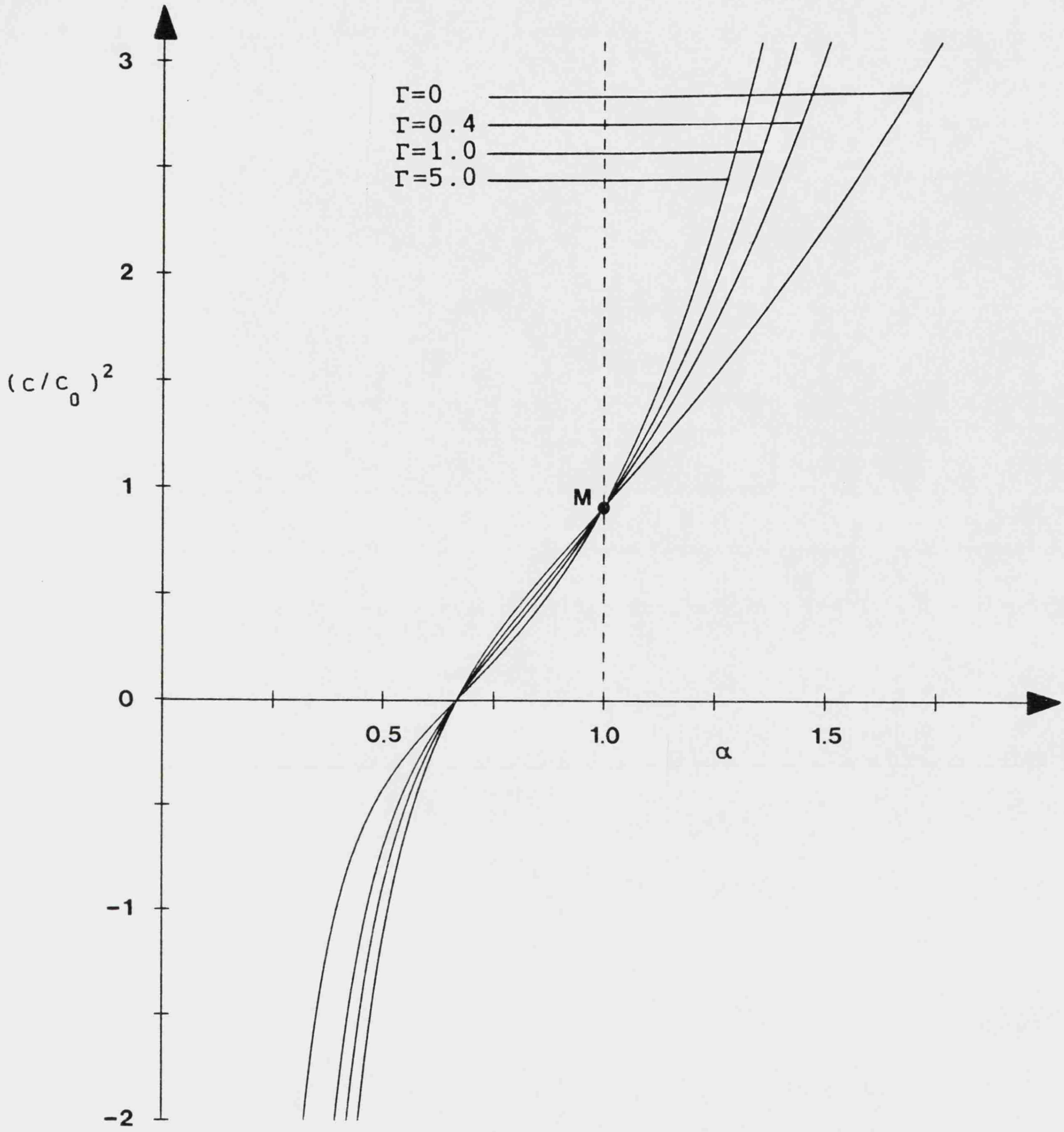

Figure 3.3

Wave-velocity as a function of the principal stretch α for a Mooney half-space under an equibiaxial stress in the OX_1X_2-plane.

and we display the results in Fig. 3.3. For the Neo-Hookean model ($\Gamma=0$), (3.20) is in agreement with the findings of Willson [34, §7].

§3.3 COMPRESSIBLE MATERIAL.

For the general compressible medium, the relations (3.1), (3.2) still hold for the static state and, using (2.3) and the definitions (2.56),

$$P_1 = \alpha^2\Phi+\alpha^2(\beta^2+\gamma^2)\Psi+\Theta,$$

$$P_2 = \beta^2\Phi+\beta^2(\gamma^2+\alpha^2)\Psi+\Theta,$$

$$P_3 = \gamma^2\Phi+\gamma^2(\alpha^2+\beta^2)\Psi+\Theta.$$

The perturbed state is again described by (3.3), (3.4), but now

$$I_1 = \alpha^2+\beta^2+\gamma^2+2(\alpha^2 u_1+\gamma^2 w_3),$$

$$I_2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2[\alpha^2(\beta^2+\gamma^2)u_1+\gamma^2(\alpha^2+\beta^2)w_3],$$

$$I_3 = \alpha^2\beta^2\gamma^2+2\alpha^2\beta^2\gamma^2(u_1+w_3).$$

For the stress components we calculate

$$P_{11} = P_1+Au_1+Bw_3, \qquad P_{12} = P_{23} = 0,$$

$$P_{22} = P_2+Cu_1+Dw_3, \qquad P_{13} = Gu_3+Hw_1,$$

$$P_{33} = P_3+Eu_1+Fw_3,$$

where, using the definitions (2.56),

$$A = 2\alpha^4[B_1+(\beta^2+\gamma^2)^2B_2+\beta^4\gamma^4B_3+2\beta^2\gamma^2(\beta^2+\gamma^2)C_1$$
$$+2\beta^2\gamma^2C_2+2(\beta^2+\gamma^2)C_3]+\alpha^2\Phi+\alpha^2(\beta^2+\gamma^2)\Psi+\Theta,$$

$$B = 2\alpha^2\gamma^2[B_1+(\beta^2+\gamma^2)(\alpha^2+\beta^2)B_2+\alpha^2\beta^4\gamma^2B_3+\beta^2(\beta^2(\alpha^2+\gamma^2)$$

$$+2\alpha^2\gamma^2)C_1+\beta^2(\alpha^2+\gamma^2)C_2+(\alpha^2+2\beta^2+\gamma^2)C_3]-\alpha^2\Phi-\alpha^2(\beta^2-\gamma^2)\Psi+\Theta,$$

$$C = 2\alpha^2\beta^2[B_1+(\alpha^2+\gamma^2)(\beta^2+\gamma^2)B_2+\alpha^2\beta^2\gamma^4B_3+\gamma^2(\gamma^2(\alpha^2+\beta^2)$$

$$+2\alpha^2\beta^2)C_1+\gamma^2(\alpha^2+\beta^2)C_2+(\alpha^2+\beta^2+2\gamma^2)C_3]-\beta^2\Phi+\beta^2(\alpha^2-\gamma^2)\Psi+\Theta,$$

$$D = 2\beta^2\gamma^2[B_1+(\alpha^2+\beta^2)(\alpha^2+\gamma^2)B_2+\alpha^4\beta^2\gamma^2B_3+\alpha^2(\alpha^2(\beta^2+\gamma^2)$$

$$+2\beta^2\gamma^2)C_1+\alpha^2(\beta^2+\gamma^2)C_2+(2\alpha^2+\beta^2+\gamma^2)C_3]-\beta^2\Phi-\beta^2(\alpha^2-\gamma^2)\Psi+\Theta,$$

$$E = 2\alpha^2\gamma^2[B_1+(\alpha^2+\beta^2)(\beta^2+\gamma^2)B_2+\alpha^2\beta^4\gamma^2B_3+\beta^2(\beta^2(\alpha^2+\gamma^2)$$

$$+2\alpha^2\gamma^2)C_1+\beta^2(\alpha^2+\gamma^2)C_2+(\alpha^2+2\beta^2+\gamma^2)C_3]-\gamma^2\Phi+\gamma^2(\alpha^2-\beta^2)\Psi+\Theta,$$

$$F = 2\gamma^4[B_1+(\alpha^2+\beta^2)^2B_2+\alpha^4\beta^4B_3+2\alpha^2\beta^2(\alpha^2+\beta^2)C_1$$

$$+2\alpha^2\beta^2C_2+2(\alpha^2+\beta^2)C_3]+\gamma^2\Phi+\gamma^2(\alpha^2+\beta^2)\Psi+\Theta,$$

$$G = \gamma^2(\Phi+\beta^2\Psi), \qquad H = \alpha^2(\Phi+\beta^2\Psi),$$

and G, H are defined in such a way so as to be consistent with (3.8). The equations of motion become

$$\begin{aligned} Au_{11}+Gu_{33}+(B+H)w_{13} &= \rho u_{tt}, \\ (G+E)u_{13}+Hw_{11}+Fw_{33} &= \rho w_{tt}, \end{aligned} \tag{3.21}$$

where ρ is the density in the deformed state and for the boundary conditions (with use of the identity $P_3-P_1=G-H$) we have

$$w_1+u_3 = 0, \qquad Eu_1+Fw_3 = 0. \tag{3.22}$$

As in the previous section we seek solutions possessing a $\exp[i(\omega t-Kx_1-Kmx_3)]$ dependence, so that, from (3.21), we find

$$FGm^4+[G(H-X)+F(A-X)-(B+H)(G+E)]m^2+(A-X)(H-X) = 0$$

with roots m_1, m_2, say, where

$$m_1^2+m_2^2 = -[G(H-X)+F(A-X)-(B+H)(G+E)]/FG,$$
$$m_1^2 m_2^2 = (A-X)(H-X)/FG. \tag{3.23}$$

The general solution of (3.21) is then

$$u = (B+H)[Pm_1 \exp(-iKm_1 x_3)+Qm_2 \exp(-iKm_2 x_3)]\exp\{i(\omega t-Kx_1)\},$$

$$w = -[(A-X+Gm_1^2)P\exp(-iKm_1 x_3) \tag{3.24}$$
$$+(A-X+Gm_2^2)Q\exp(-iKm_2 x_3)]\exp\{i(\omega t-Kx_1)\},$$

P, Q arbitrary constants. Substituting (3.24) into the boundary conditions (3.22), we must have for a non-trivial solution in P, Q either

$$m_2-m_1 = 0 \tag{3.25}$$

or

$$(X-A)[E(B+H)+F(X-A)-FG(m_1^2+m_2^2)]-EFGm_1^2 m_2^2 \tag{3.26}$$
$$= (B+H)[E^2+F(X-A)]m_1 m_2,$$

where the expression on the right-hand-side of (3.26) has been simplified by the relation B+H = G+E. From (3.26), with (3.23), and rejecting the positive square root of $m_1 m_2$, we calculate

$$X = A \tag{3.27}$$

or

$$(A-X)(H-G-X) = [E^2+F(X-A)][(A-X)(H-X)/FG]^{1/2}, \tag{3.28}$$

assuming B+H is non-zero.

Hayes and Rivlin [38] have shown that the solutions (3.25), (3.27) are degenerate cases, thus (3.28) is the frequency equation for surface waves in a pre-stressed compressible half-space.

When there is no primary stress

$$A = F = \lambda+2\mu, \qquad E = \lambda, \qquad G = H = \mu,$$

λ,μ the usual Lamé constants, and denoting $x=X/\mu$, $a=2\mu/(\lambda+2\mu)$, (3.28) reduces to

$$-x(2-ax) = (2a-4+x)[2(2-ax)(1-x)]^{1/2},$$

which upon squaring and removing the factors $(2-a)$, $(ax-2)$ becomes (1.10), the relation governing Rayleigh waves in an unstresed isotropic half-space.

Willson [32, 33] investigated Rayleigh waves in a semi-infinite medium composed of a particular compressible material under conditions of normal loading and equibiaxial stress. Later [35] he considered an equibiaxially stressed medium composed of Hadamard material. Eringen and Suhubi [8, §4.4] calculated the frequency equation for surface waves in a compressible half-space placed under the biaxial stress $P = \mathrm{diag}(P_1, P_2, 0)$, $P_1 \neq P_2$. The results obtained in these four investigations are encompassed by (3.28).

The result (3.28) is now applied to the Ko model (2.114), so that,

$$A/3 = G = \bar{\mu}/\alpha^3\beta\gamma, \qquad E = F/3 = H = \bar{\mu}/\alpha\beta\gamma^3. \qquad (3.29)$$

For this model we consider the curves on the velocity-stretch diagrams for the following four cases:

(i) uniaxial stress parallel to OX_1;

(ii) normal loading onto the surface $X_3=0$;

(iii) equibiaxial stress in the OX_1X_2-plane;

(iv) equibiaxial stress in the OX_2X_3-plane.

Using the frequency equation (3.28) with (3.29) we plot values of $(c/c_t)^2$, $c_t = \surd(\bar{\mu}/\rho)$, against the principal stretch α.

In case (i) we take $P = \mathrm{diag}(P_1, 0, 0)$, so that $\beta=\gamma$ and $\alpha\beta^4=1$. The curve produced is plotted in Fig. 3.4. K is the point (1,0.8453) in agreement with Rayleigh waves in an unstressed medium having a Poisson's ratio of 1/4 (cf. Tables 1.1, 1.2

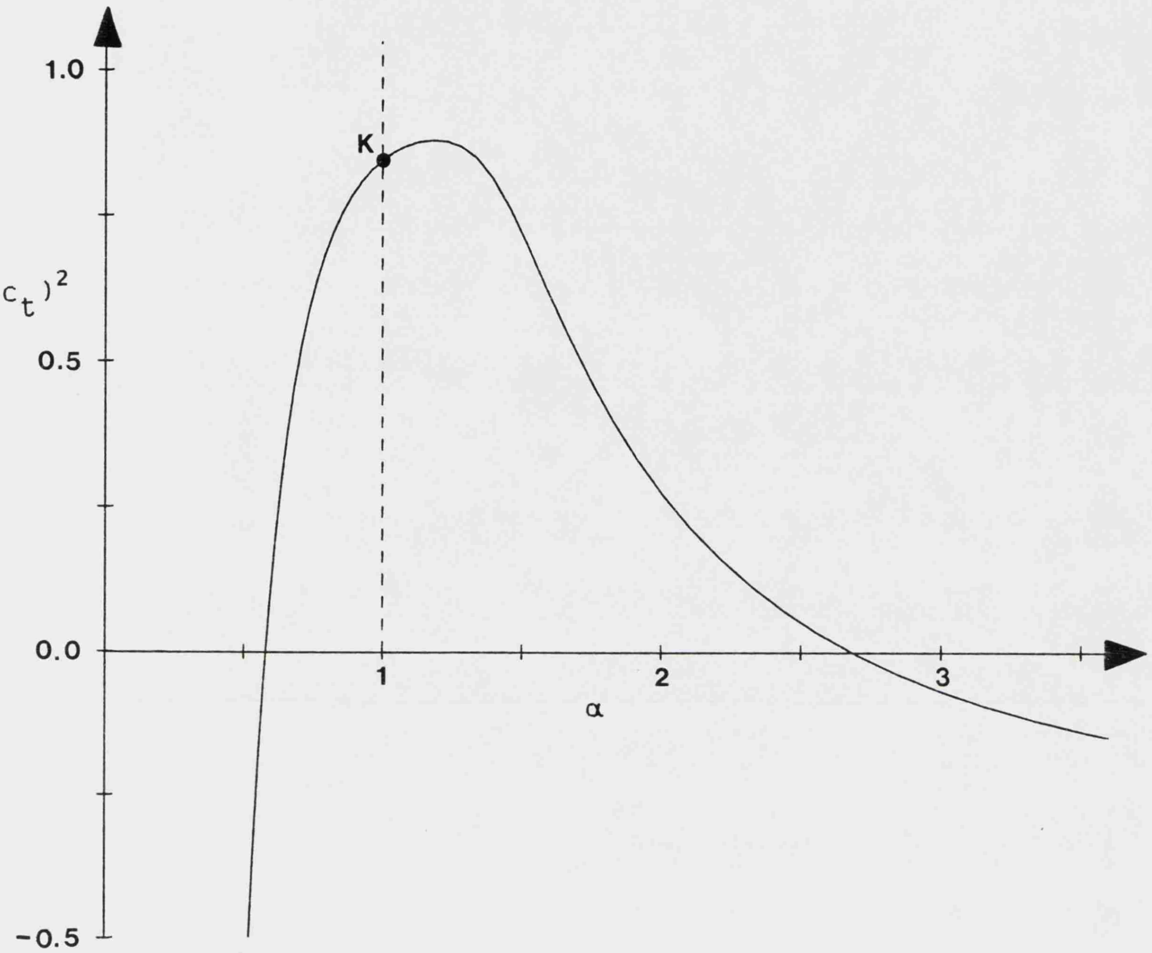

Figure 3.4

Wave-velocity as a function of the principal stretch α for a Ko half-space uniaxially stressed in the OX_1-direction.

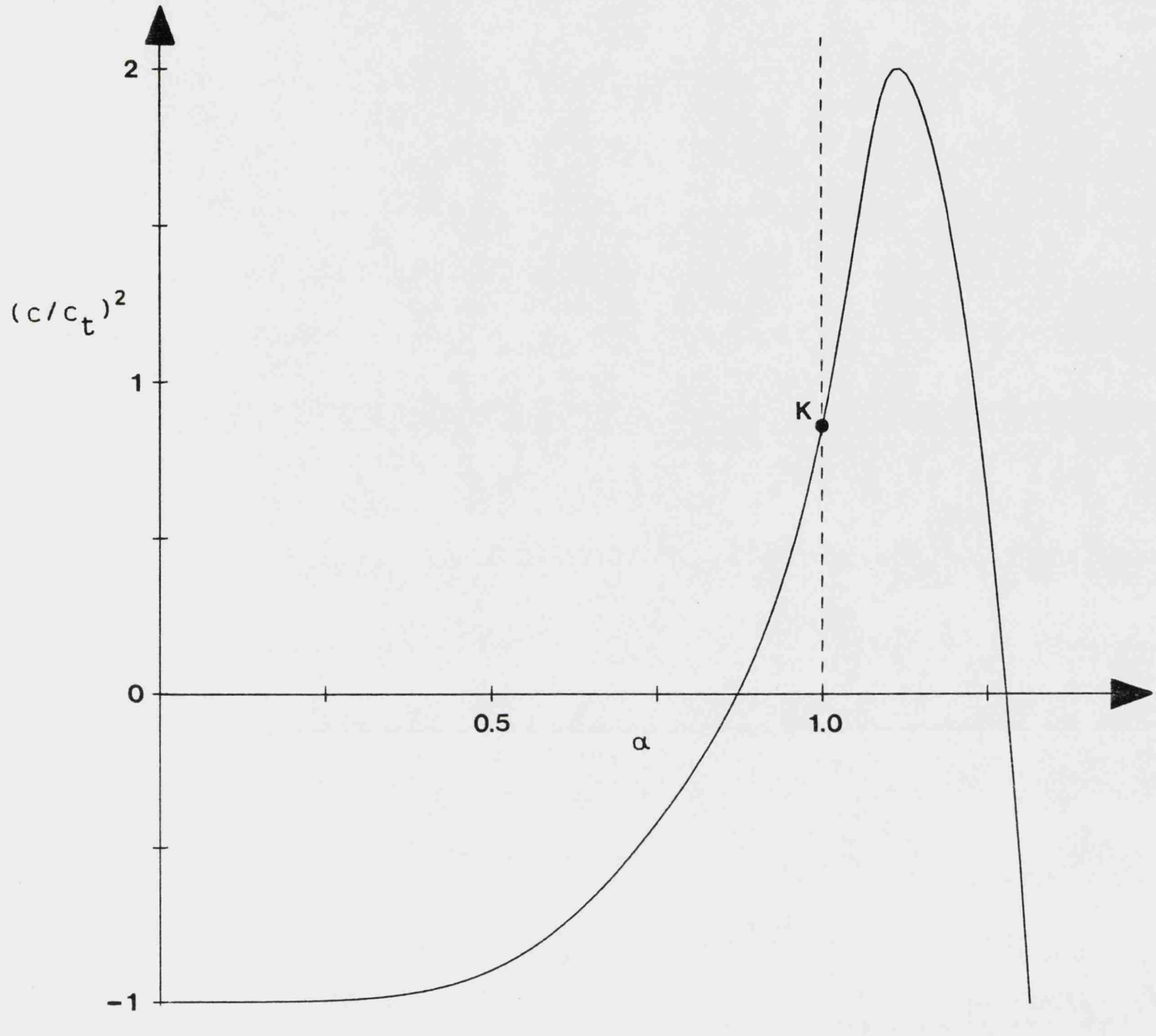

Figure 3.5

Wave-velocity as a function of the principal stretch α for a Ko half-space with a normal load on the face $X_3=0$.

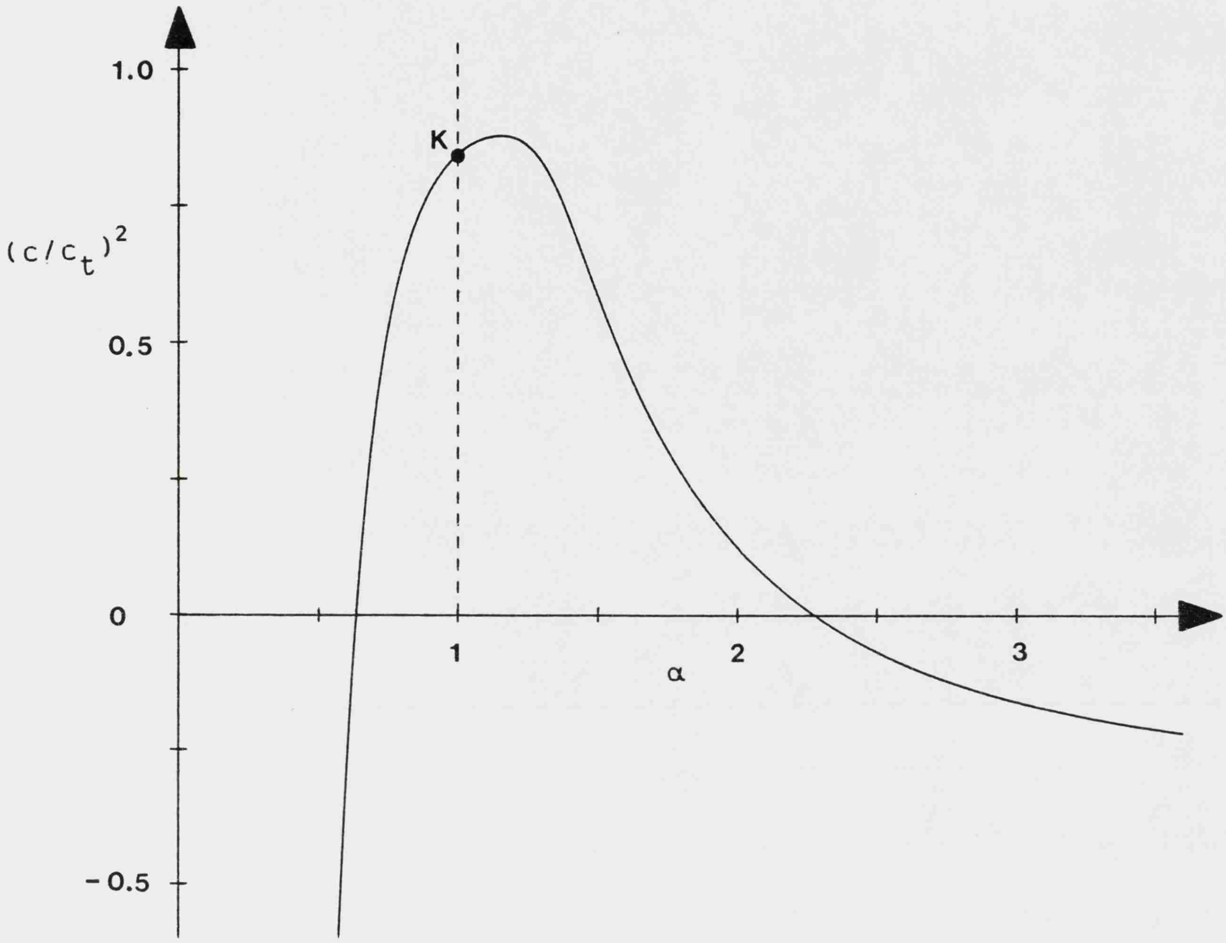

Figure 3.6

Wave-velocity as a function of the principal stretch α for a Ko half-space under an equibiaxial stress in the OX_1X_2-plane.

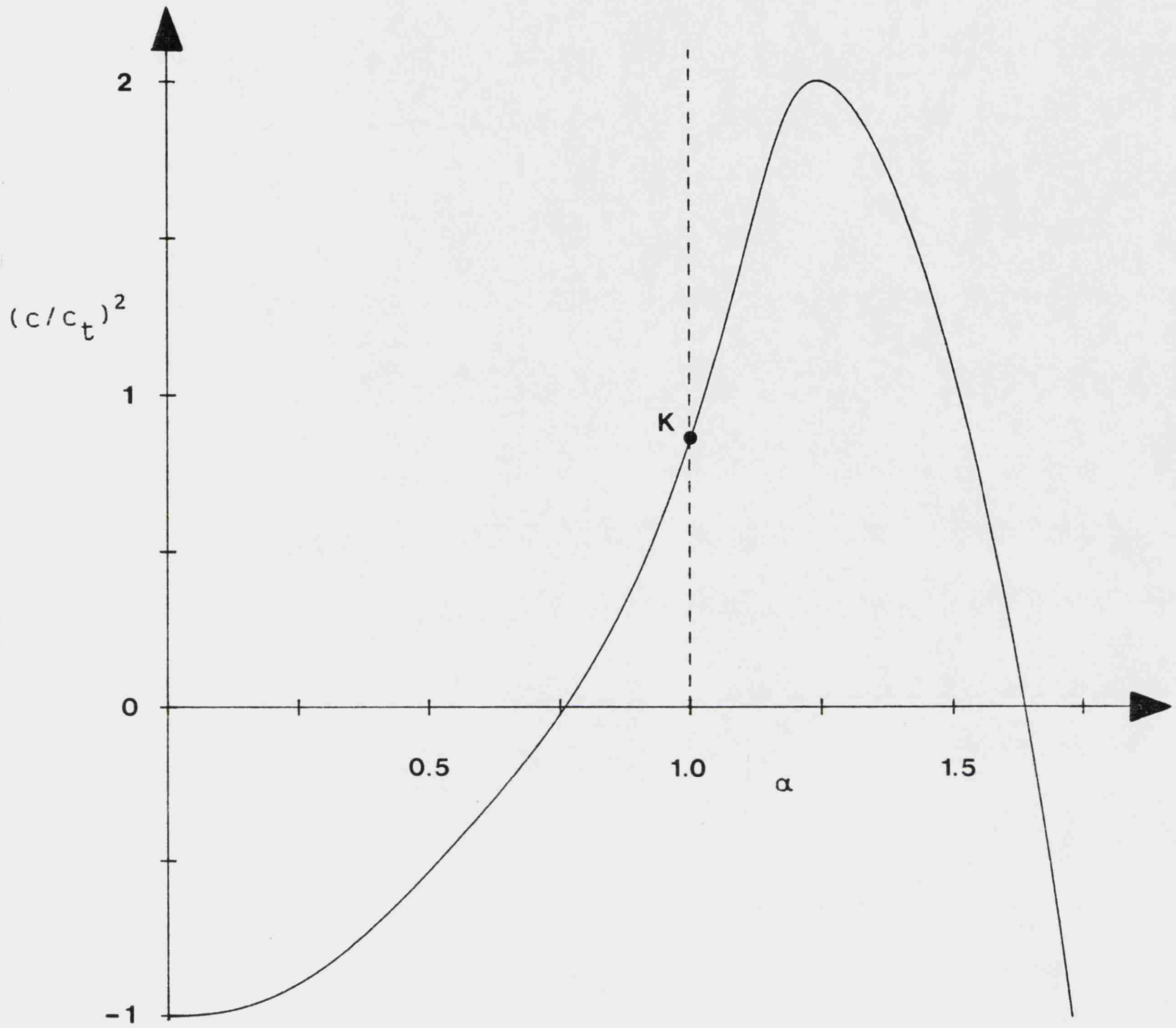

Figure 3.7

Wave-velocity as a function of the principal stretch α for a Ko half-space under an equibiaxial stress in the OX_2X_3-plane.

with V/V_0 taking the value $\sqrt{(2/5)}c/c_t$). The two points at which instability arises are (0.5793,0) and (2.6788,0).

Similarly in Figs. 3.5, 3.6, 3.7 we display the curves corresponding to the cases (ii), (iii) and (iv) respectively. The unstressed point K on the line $\alpha=1$ remains the same for all four graphs and the points of marginal stability are given in Table 3.1.

Table 3.1

Values of the stretch α when the curves in Figs. 3.5, 3.6 and 3.7 cross the α-axis.

Case (ii)	α = 0.8724	α = 1.2793
Case (iii)	α = 0.6345	α = 2.2731
Case (iv)	α = 0.7611	α = 1.6367

CHAPTER 4

VIBRATIONS IN CYLINDERS UNDER STRESS.

§4.1 INTRODUCTION.

In this chapter we return to the problem of small-amplitude elasic vibrations in a pre-stressed cylinder, previously considered in Chapter 2, but we no longer regard Ka as a small quantity. The governing equations set up in Chapter 2, for longitudinal and flexural waves in incompressible and compressible solids, are now solved completely to establish general dispersion relations expressed in terms of Bessel functions. Dispersion relations for longitudinal vibrations in uniaxially stressed cylinders for both compressible and incompressible materials have been calculated by Eringen and Suhubi [8, §4.5], however, no detailed numerical examination of their results has since been carried out. We show that in the limiting cases; when Ka is small (slender cylinders) the results from Chapter 2 are recovered, and when Ka is large we illustrate the recapture of the surface wave limit, obtained in Chapter 3, for the Mooney and Ko models.

§4.2 LONGITUDINAL WAVES IN INCOMPRESSIBLE MATERIAL.

The longitudinal vibrations in an infinite cylinder composed of an incompressible material are governed by the equations of motion (2.30),

$$\rho\omega^2 U = P_r + K^2(\alpha_5+\tau)U - K(\delta_1+\alpha_5)W_r,$$
$$\rho\omega^2 W = -KP - \alpha_5(W_{rr}+W_r/r) + K^2(\delta_3-\alpha_5+\tau)W, \tag{4.1}$$

the incompressibility condition (2.15),

$$U_r + U/r + KW = 0, \tag{4.2}$$

and the boundary conditions (2.32),

$$-KU + W_r = 0,$$
$$-P + \delta_2 U/r + K\delta_1 W = 0, \tag{4.3}$$

satisfied on r=a.

In the same manner in which the equations in Chapter 1 were solved, we calculate, from (4.1), (4.2):

$$U(r) = K[AJ_1(\lambda_1 r) + BJ_1(\lambda_2 r)],$$
$$W(r) = -[A\lambda_1 J_0(\lambda_1 r) + B\lambda_2 J_0(\lambda_2 r)], \tag{4.4}$$
$$P(r) = K[Ab_1 J_0(\lambda_1 r) + Bb_2 J_0(\lambda_2 r)],$$

where A, B are arbitrary constants,

$$b_i = -(\lambda_3^2+\lambda_i^2)(\delta_1+\alpha_5)/\lambda_i, \qquad (i=1,2),$$
$$\lambda_3^2 = \frac{[X-(\alpha_5+\tau)]K^2}{(\delta_1+\alpha_5)}, \qquad \lambda_4^2 = \frac{[X-(\delta_3-\alpha_5+\tau)]K^2}{\alpha_5}, \tag{4.5}$$
$$\lambda_5^2 = \frac{(\delta_1+\alpha_5)K^2}{\alpha_5}, \qquad X = \rho\omega^2/K^2,$$

and λ_1^2, λ_2^2 are the roots of the equation

$$\lambda^4 - (\lambda_4^2+\lambda_5^2)\lambda^2 - \lambda_3^2\lambda_5^2 = 0. \tag{4.6}$$

Substituting (4.4) into (4.3) yields two equations linear in A, B which may be solved to give the longitudinal dispersion relation for incompressible materials:

$$(\lambda_2^2-\lambda_1^2)\delta_2-(K^2-\lambda_1^2)(b_2/\lambda_2+\delta_1)\psi(\lambda_2 a)$$
$$+(K^2-\lambda_2^2)(b_1/\lambda_1+\delta_1)\psi(\lambda_1 a) = 0, \qquad (4.7)$$

where $\psi(x) \doteq xJ_0(x)/J_1(x)$. The above result is in agreement with the form obtained by Eringen and Suhubi [8, §4.5].

To a zeroth order approximation (in Ka) we may set $\psi(x)=2$, then the relation (4.7) may be factorised to give

$$\lambda_3^2(\lambda_1^2-\lambda_2^2)[2(\delta_1+\alpha_5)(K^2-\lambda_4^2)+(\delta_2-2\delta_1)\lambda_5^2] = 0.$$

The factors λ_3^2, $\lambda_1^2-\lambda_2^2$ both correspond to degenerate cases, however, the remaining factor gives the result

$$X = \delta_3+\delta_2/2-\delta_1+\tau,$$

just as we had in (2.40). Higher approximations are, of course, obtainable; however, the calculations involved become exceedingly laborious.

§4.3 FLEXURAL WAVES IN INCOMPRESSIBLE MATERIALS.

For the flexural case the governing equations are given in (2.44)-(2.46) which for ease of reference are rewritten below;

$$U_r+(U-V)/r+kW = 0, \qquad (4.8)$$

for the incompressibility condition,

$$\rho\omega^2 U = P_r+[(\alpha_5+\tau)K^2-\delta_2/2r^2]U+\delta_2(V_r+V/r)/2r-K(\delta_1+\alpha_5)W_r,$$

$$\rho\omega^2 V = P/r-\delta_2(U_r-U/r)/2r+\delta_2(V_{rr}+V_r/r-V/r^2)/2$$
$$+(\alpha_5+\tau)K^2 V-K(\delta_1+\alpha_5)W/r, \qquad (4.9)$$

$$\rho\omega^2 W = -KP-\alpha_5(W_{rr}+W_r/r-W/r^2)+K^2(\tau+\delta_3-\alpha_5)W,$$

for the equations of motion, and

$$-P+\delta_2(U-V)/r+K\delta_1 W = 0,$$

$$V_r+(U-V)/r = 0, \tag{4.10}$$

$$-KU+W_r = 0,$$

for the boundary conditions on r=a.

Solving (4.9), (4.8), we find

$$U(r) = K[AJ_1'(\lambda_1 r)+BJ_1'(\lambda_2 r)+CJ_1(\xi r)/r],$$

$$V(r) = K[AJ_1(\lambda_1 r)/r+BJ_1(\lambda_2 r)/r+CJ_1'(\xi r)], \tag{4.11}$$

$$W(r) = A\lambda_1^2 J_1(\lambda_1 r)+B\lambda_2^2 J_1(\lambda_2 r),$$

$$P(r) = -K[A\lambda_1 b_1 J_1(\lambda_1 r)+B\lambda_2 b_2 J_1(\lambda_2 r)],$$

where A, B, C are constants, and

$$\xi^2 = -2(\delta_1+\alpha_5)\lambda_3^2/\delta_2. \tag{4.12}$$

Inserting (4.11) into the conditions (4.10) and solving for A, B, C, we obtain for the flexural dispersion relation:

$$\det(D_{ij}) = 0, \tag{4.13}$$

where

$$D_{11} = \lambda_2 b_2+\lambda_2^2\delta_1, \qquad D_{22} = 2[\psi(\lambda_1 a)-2],$$

$$D_{12} = \lambda_1 b_1+\lambda_1^2\delta_1, \qquad D_{23} = 2[2-\psi(\xi a)]-(\xi a)^2,$$

$$D_{13} = (\delta_1+\alpha_5)\lambda_3^2, \qquad D_{31} = (K^2-\lambda_2^2)[\psi(\lambda_2 a)-1],$$

$$D_{21} = 2[\psi(\lambda_2 a)-2], \quad D_{32} = (K^2-\lambda_1^2)[\psi(\lambda_1 a)-1], \quad D_{33} = K^2.$$

Setting $\psi=2$ in (4.13) we calculate

$$\lambda_3^2(\lambda_1^2-\lambda_2^2)(X-\tau) = 0,$$

where again the first two factors are degenerate so that to this approximation $X=\tau$, as expected. Higher approximations are not easily obtainable.

In the absence of the initial stress, from (2.92) with (4.5), (4.6), (4.12), we have

$$\lambda_3^2 = -\lambda_4^2 = (1-Z)K^2, \qquad \lambda_5^2 = -K^2,$$

$$\lambda_1^2 = \xi^2 = (Z-1)K^2, \quad \lambda_2^2 = -K^2, \quad b_1 = 0, \quad b_2 = -ZK^2/\lambda_2,$$

where $Z = X/\mu$, μ the modulus of rigidity. The relations (4.7), (4.13) now reduce to the unstressed dispersion relations (1.9), (1.16) after the incompressible limit $L \to \infty$ has been applied.

§4.4 THE MOONEY MODEL.

To illustrate our results we again employ the Mooney model (2.107) so that, from (4.5), (4.6), (4.12), with (2.109), (2.111), we obtain

$$\lambda_3^2 = YK^2, \qquad \lambda_4^2 = -YK^2, \qquad \lambda_5^2 = -K^2,$$

$$\lambda_1^2 = -YK^2, \quad \lambda_2^2 = -K^2, \quad \xi^2 = -(1+\gamma^{-1}\Gamma)YK^2/(1+\gamma^2\Gamma), \qquad (4.14)$$

$$b_1 = 0, \qquad \lambda_2 b_2 = 2(Y-1)(L+\gamma^{-1}M)K^2/\gamma,$$

where

$$Y = \gamma^3 - \gamma(c/c_0)^2(1+\Gamma)/(1+\gamma^{-1}\Gamma). \qquad (4.15)$$

Of course the usual notations $\Gamma = M/L$, $c = \omega/K$, $c_0^2 = 2(L+M)/\rho$ still hold.

When $Ka \to \infty$, we may reasonably expect to recover the surface wave velocity given by (3.19) (with γ replacing α). Utilising the result (3.19) in (4.15) we find that Y is always positive, thus $\lambda_1 a$, $\lambda_2 a$ will be purely imaginary in this particular case, so that, using the asymptotic expansion $\psi(ix) \to x$, (4.7), (4.13), the respective dispersion relations for longitudinal and flexural modes, may be factorised to give the form expressed by (3.18).

The results for longitudinal and flexural modes are displayed in Figs. 4.1, 4.2 respectively. In both figures Γ is fixed at 0.4 and for a range of values for the axial stretch γ we plot the curves of $(c/c_0)^2$ against Ka. An interesting feature to note in Figs. 4.1, 4.2 is that the Mooney model is unstable only for points of compression. As Ka increases in size the curves of constant γ converge rapidly to the wave-velocity predicted by (3.19). In Fig. 4.3, again for Γ=0.4, we compare the results (2.112), (2.113), the respective longitudinal and flexural approximations for small Ka, shown in the figure by the broken curves, with the exact dispersion relation curves.

§4.5 LONGITUDINAL WAVES IN COMPRESSIBLE MATERIAL.

For a compressible cylinder under axial and lateral tractions the governing equations are given by the equations of motion (2.68),

$$\begin{aligned} \rho\omega^2 U &= -R(U_{rr}+U_r/r-U/r^2)+\alpha_7 K^2 U-K(\alpha_2+\alpha_5)W_r, \\ \rho\omega^2 W &= K(\alpha_2+\alpha_5)(U_r+U/r)-\alpha_5(W_{rr}+W_r/r)+K^2 TW, \end{aligned} \tag{4.16}$$

and the boundary conditions (2.69),

$$\begin{aligned} RU_r+SU/r+K\alpha_2 W &= 0, \\ -KU+W_r &= 0, \end{aligned} \tag{4.17}$$

satisfied on r=a.

The equations (4.16) may be solved to give

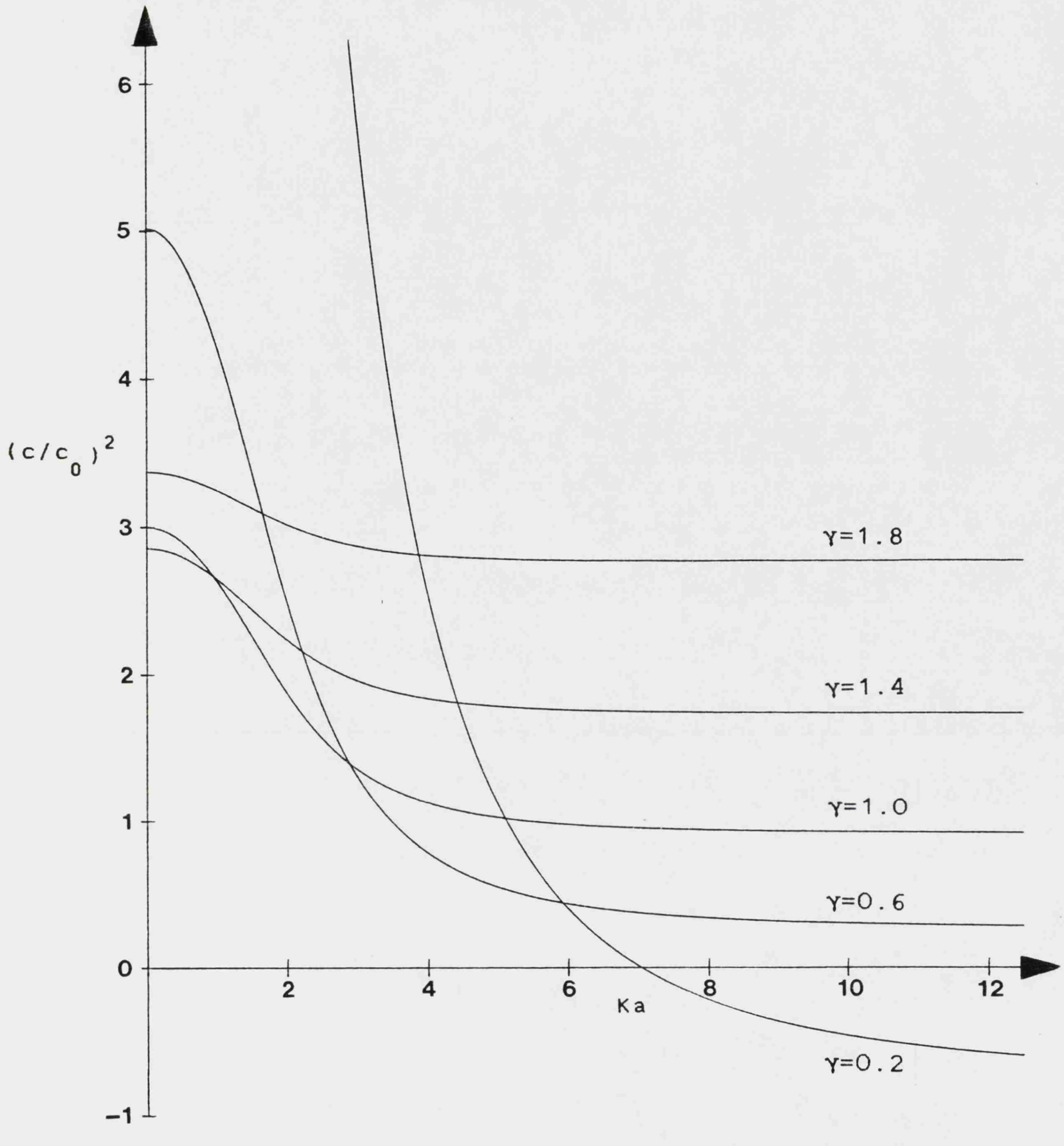

Figure 4.1

Variation of $(c/c_0)^2$ against Ka for longitudinal vibrations in a uniaxially stressed Mooney cylinder with $\Gamma=0.4$. Curves are plotted for selected values of the axial principal stretch γ.

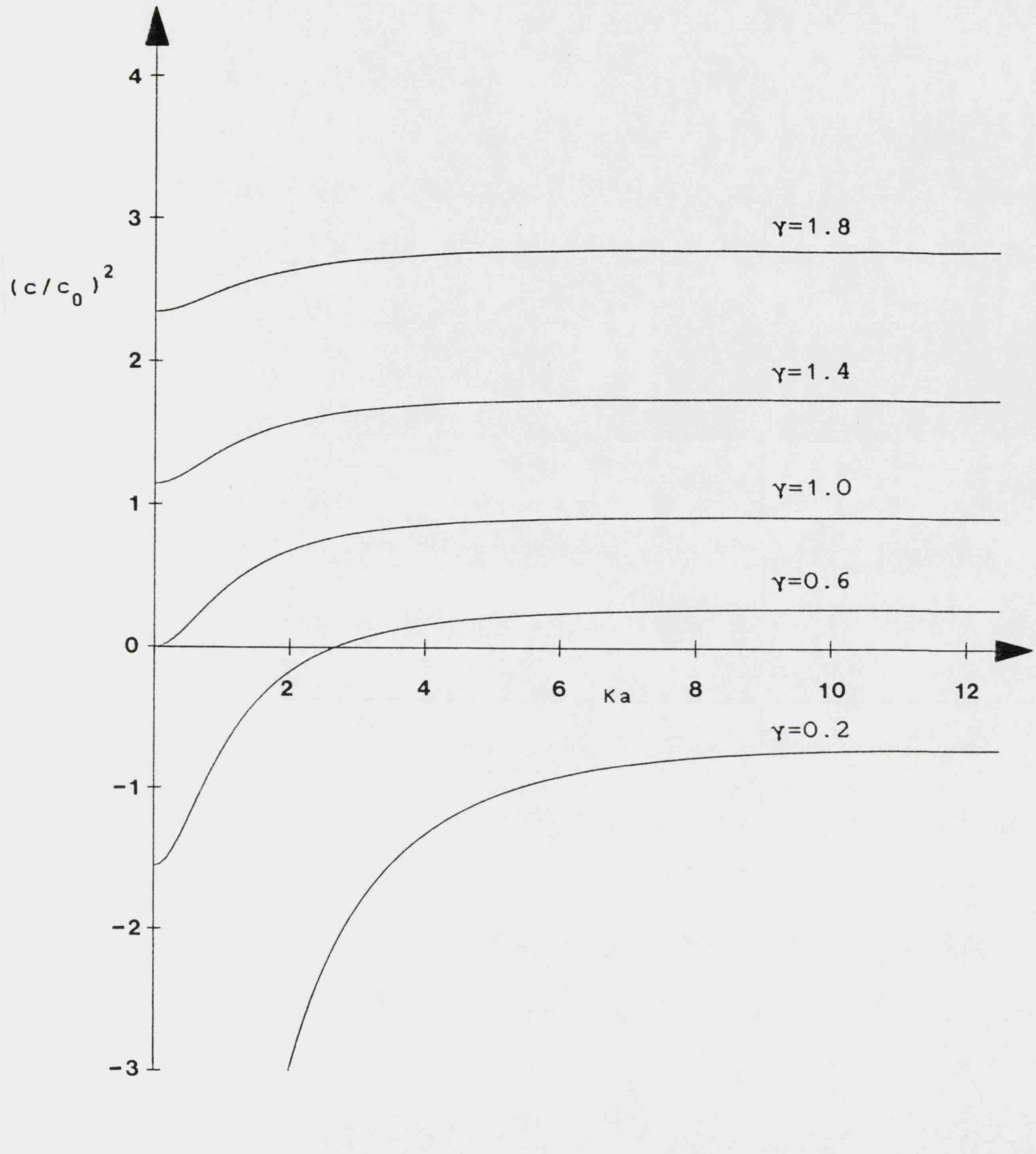

Figure 4.2

Variation of $(c/c_0)^2$ against Ka for flexural vibrations in a uniaxially stressed Mooney cylinder with $\Gamma=0.4$.

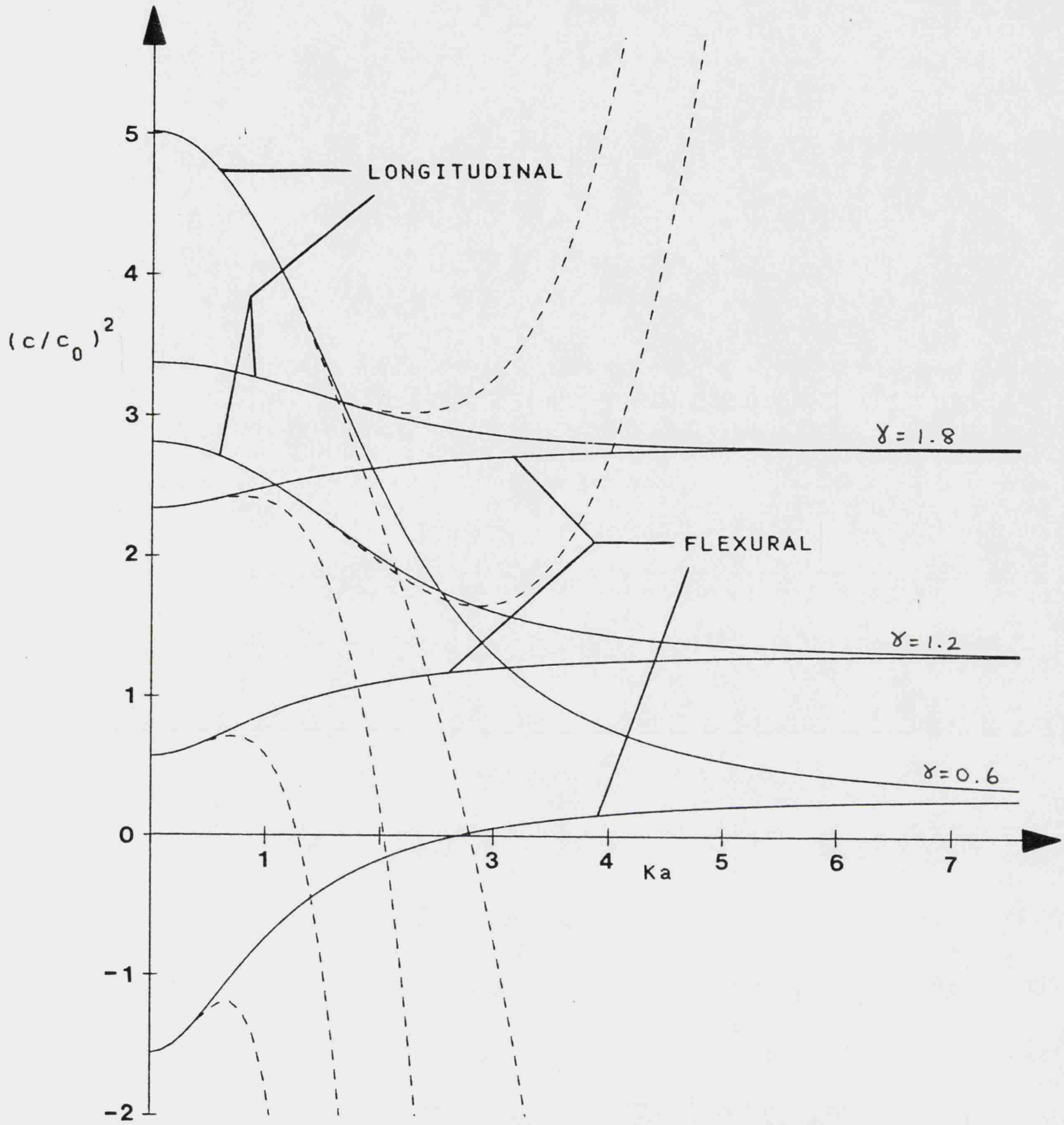

Figure 4.3

For both longitudinal and flexural vibrations we contrast the exact dispersion curves (represented in the figure by solid curves) against their corresponding approximations for slender cylinders (broken curves) for a Mooney cylinder with $\Gamma=0.4$.

$$U(r) = K[AJ_1(\mu_1 r)+BJ_1(\mu_2 r)],$$
$$W(r) = A\mu_1 c_1 J_0(\mu_1 r)+B\mu_2 c_2 J_0(\mu_2 r), \tag{4.18}$$

where A, B are constants,

$$c_i = \frac{(\alpha_2+\alpha_5)K^2}{\alpha_5(\mu_3^2-\mu_i^2)} = \frac{R(\mu_4^2-\mu_i^2)}{(\alpha_2+\alpha_5)\mu_i^2}, \qquad (i=1,2), \tag{4.19}$$

$$\mu_3^2 = (X-T)K^2/\alpha_5, \quad \mu_4^2 = (X-\alpha_7)K^2/R, \quad \mu_5^2 = (\alpha_2+\alpha_5)^2K^2/R\alpha_5,$$

and μ_1^2, μ_2^2 are the roots of

$$\mu^4-(\mu_3^2+\mu_4^2+\mu_5^2)\mu^2+\mu_3^2\mu_4^2 = 0. \tag{4.20}$$

Inserting (4.18) into the conditions (4.17), and eliminating the unknowns A, B, we calculate for the dispersion relation:

$$(K^2+c_2\mu_2^2)(R+\alpha_2 c_1)\psi(\mu_1 a)-(K^2+c_1\mu_1^2)(R+\alpha_2 c_2)\psi(\mu_2 a)$$
$$+2\alpha_6(c_1\mu_1^2-c_2\mu_2^2) = 0, \tag{4.21}$$

and, if we specialise to the case of a uniaxially stressed cylinder, (4.21) agrees, apart from a difference in notation, with the relation obtained by Eringen and Suhubi.

For a slender cylinder, we may set $\psi=2$ in (4.21) to obtain, after a little calculation and neglecting the solution $\mu_1^2 = \mu_2^2$, the result (2.76).

§4.6 FLEXURAL WAVES IN COMPRESSIBLE MATERIAL.

From §2.8, the governing equations are:

$$\rho\omega^2 U = -R(U_{rr}+U_r/r-U/r^2)+\alpha_6 U/r^2+\alpha_7 K^2 U+(\alpha_6+S)V_r/r$$
$$-(\alpha_6+R)V/r^2-(\alpha_2+\alpha_5)KW_r,$$

$$\rho\omega^2 V = -(\alpha_6+S)U_r/r-(\alpha_6+R)U/r^2-\alpha_6(V_{rr}+V_r/r-V/r^2) \qquad (4.22)$$

$$+RV/r^2+\alpha_7 K^2 V-(\alpha_2+\alpha_5)KW/r,$$

$$\rho\omega^2 W = K(\alpha_2+\alpha_5)(U_r+U/r-V/r)-\alpha_5(W_{rr}+W_r/r-W/r^2)+TK^2 W,$$

from the equations of motion (2.80) and,

$$RU_r+S(U-V)/r+\alpha_2 KW = 0,$$

$$V_r+(U-V)/r = 0, \qquad (4.23)$$

$$-KU+W_r = 0,$$

from the boundary conditions (2.81) satisfied on $r=a$.

For the general solutions of (4.22) we find

$$U(r) = K[AJ_1'(\mu_1 r)+BJ_1'(\mu_2 r)+CJ_1(\zeta r)/r],$$

$$V(r) = K[AJ_1(\mu_1 r)/r+BJ_1(\mu_2 r)/r+CJ_1'(\zeta r)], \qquad (4.24)$$

$$W(r) = -[Ac_1\mu_1^2 J_1(\mu_1 r)+Bc_2\mu_2^2 J_1(\mu_2 r)],$$

where

$$\zeta^2 = (X-\alpha_7)K^2/\alpha_6, \qquad (4.25)$$

then (4.24) with (4.23) provide the dispersion relation

$$\det(D_{ij}) = 0, \qquad (4.26)$$

where

$$D_{11} = \alpha_5\mu_2^2+\alpha_2\mu_4^2, \qquad D_{22} = 2[\psi(\mu_1 a)-2],$$

$$D_{12} = \alpha_5\mu_1^2+\alpha_2\mu_4^2, \qquad D_{23} = 2[2-\psi(\zeta a)]-(\zeta a)^2,$$

$$D_{13} = (\alpha_2+\alpha_5)\mu_4^2, \qquad D_{31} = (K^2+c_2\mu_2^2)[\psi(\mu_2 a)-1],$$

$$D_{21} = 2[\psi(\mu_2 a)-2], \quad D_{32} = (K^2+c_1\mu_1^2)[\psi(\mu_1 a)-1], \quad D_{33} = K^2.$$

To a zeroth approximation we readily find the result (2.87) after disregarding the degenerate solutions $\zeta^2 = 0$, $\mu_1^2 = \mu_2^2$.

For an unstressed cylinder we have, using (2.92), (2.93) with (4.19), (4.20), (4.25),

$$\mu_3^2 = [Z-(L+2)]K^2, \quad \mu_4^2 = (Z-1)K^2/(L+2), \quad \mu_5^2 = (L+1)^2K^2/(L+2),$$

$$\mu_1^2 = \zeta^2 = (Z-1)K^2, \quad \mu_2^2 = [Z/(L+2)-1]K^2, \quad c_1 = -1, \quad c_2 = K^2/\lambda_2^2,$$

where $Z = X/\mu$, $L = \lambda/\mu$, λ, μ Lamé constants; and the dispersion relations (4.21), (4.26) reduce to their unstressed forms (1.9), (1.16) respectively.

§4.7 THE KO MODEL.

For the Ko model (2.114) we calculate, from (4.19), (4.25), with (2.115),

$$c_i = \frac{(T_1+T_3)K^2}{T_3(\mu_3^2-\mu_i^2)} = \frac{3T_1(\mu_4^2-\mu_i^2)}{(T_1+T_3)\mu_i^2}, \qquad \zeta^2 = (Y-T_1)K^2/T_1,$$

$$\mu_3^2 = (Y-3T_3)K^2/T_3, \quad \mu_4^2 = (Y-T_1)K^2/3T_1, \quad \mu_5^2 = (T_1+T_3)^2K^2/3T_1T_3, \tag{4.27}$$

with T_1, T_3 defined by (2.116) and where we denote $X/\bar{\mu}$ by Y.

We consider first the marginal ($Y=0$) behaviour of a Ko cylinder placed under a uniaxial stress directed along the axis of the cylinder. From the dispersion relations (4.21), (4.26) with (4.27) we illustrate the marginal states in Fig. 4.4 for both longitudinal and flexural waves. We see that, unlike the Mooney cylinder, we have instability for both compressions and extensions. We see also that as Ka increases the curves approach the surface wave limit predicted in §3.3, shown in the figure by the two short broken lines.

For the non-marginal case, and denoting $c_t = \sqrt{(\bar{\mu}/\rho)}$, we plot in Figs. 4.5, 4.6 the wave-speed ratio $(c/c_t)^2$ against Ka for various values of the axial stretch γ for the longitudinal and flexural modes respectively. Again the values of the curves on

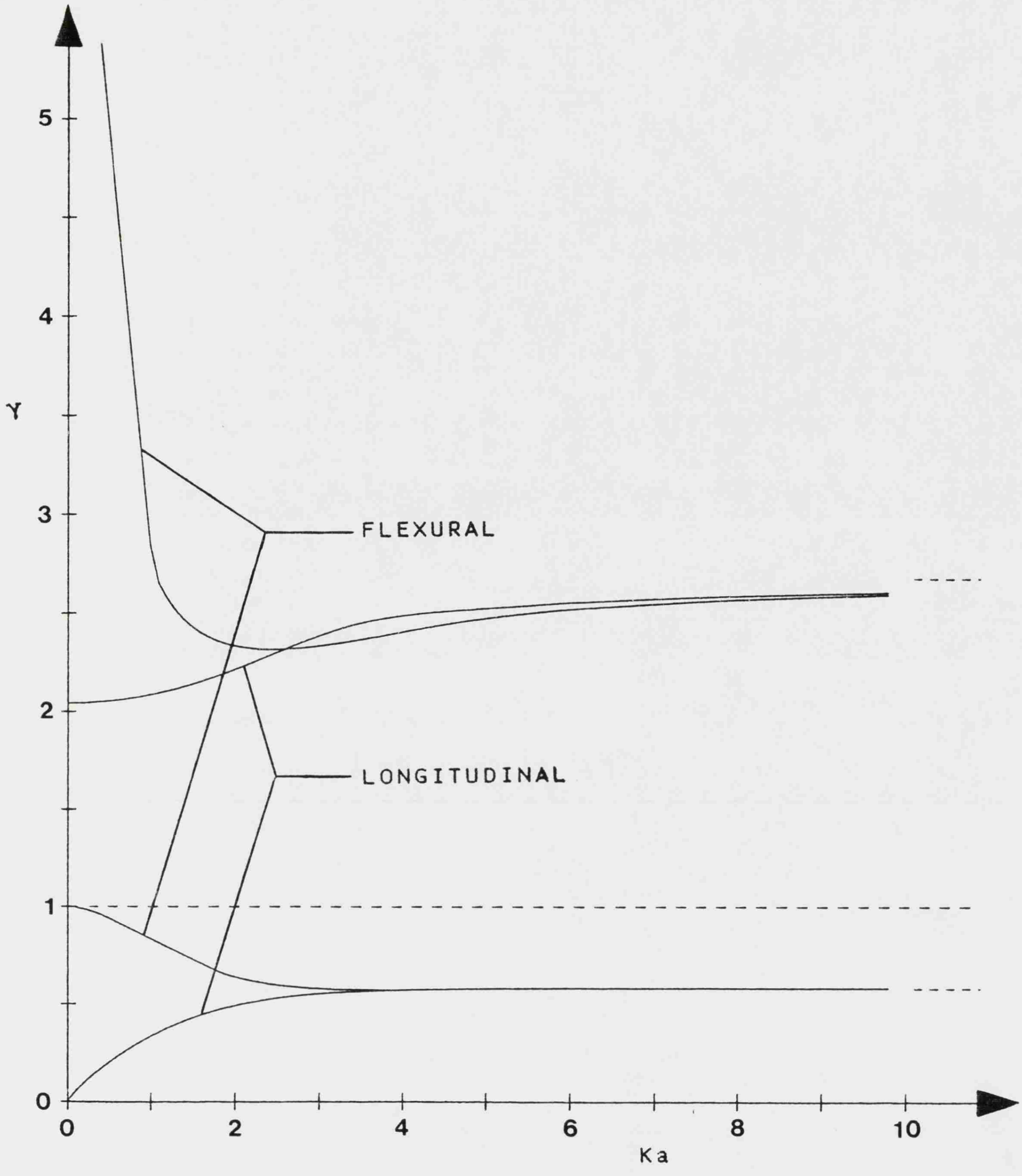

Figure 4.4

Curves of marginal stability for a uniaxially stressed Ko cylinder. For large Ka the curves approach the limits predicted for surface waves, represented in the figure by the short broken lines.

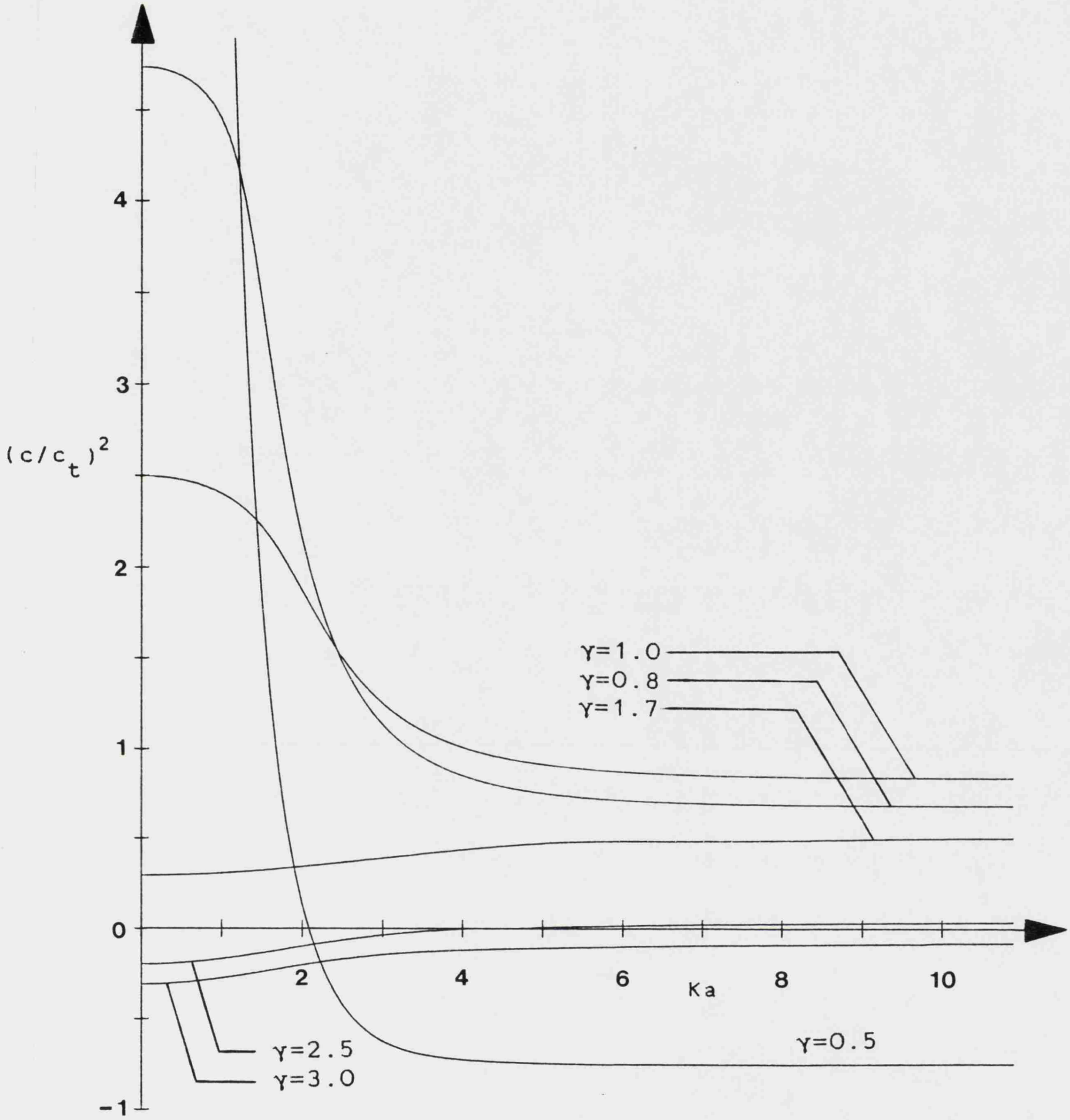

Figure 4.5

Variation of $(c/c_t)^2$ against Ka for longitudinal vibrations in a uniaxially stressed Ko cylinder.

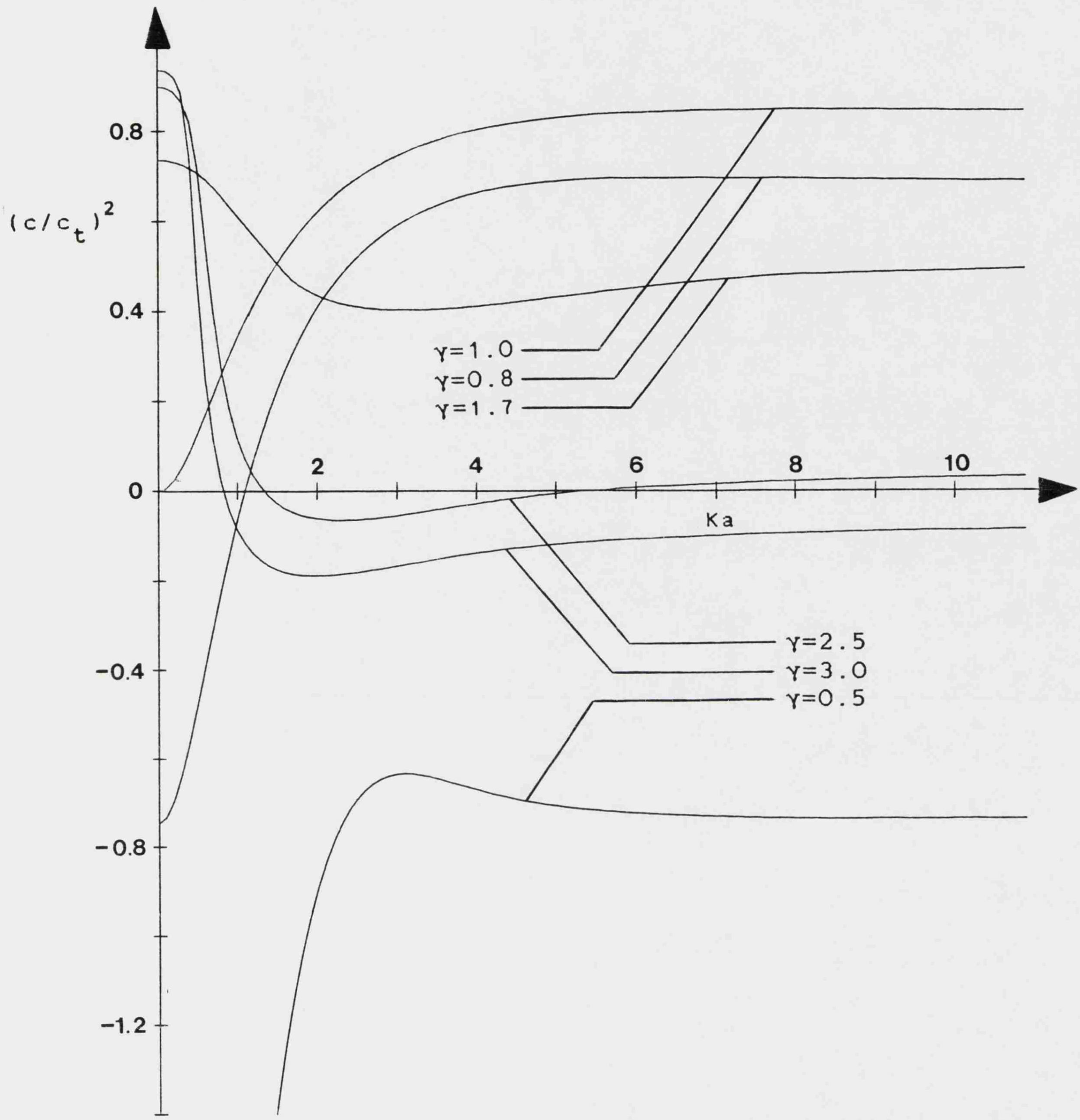

Figure 4.6

Variation of $(c/c_t)^2$ against Ka for flexural vibrations in a uniaxially stressed Ko cylinder.

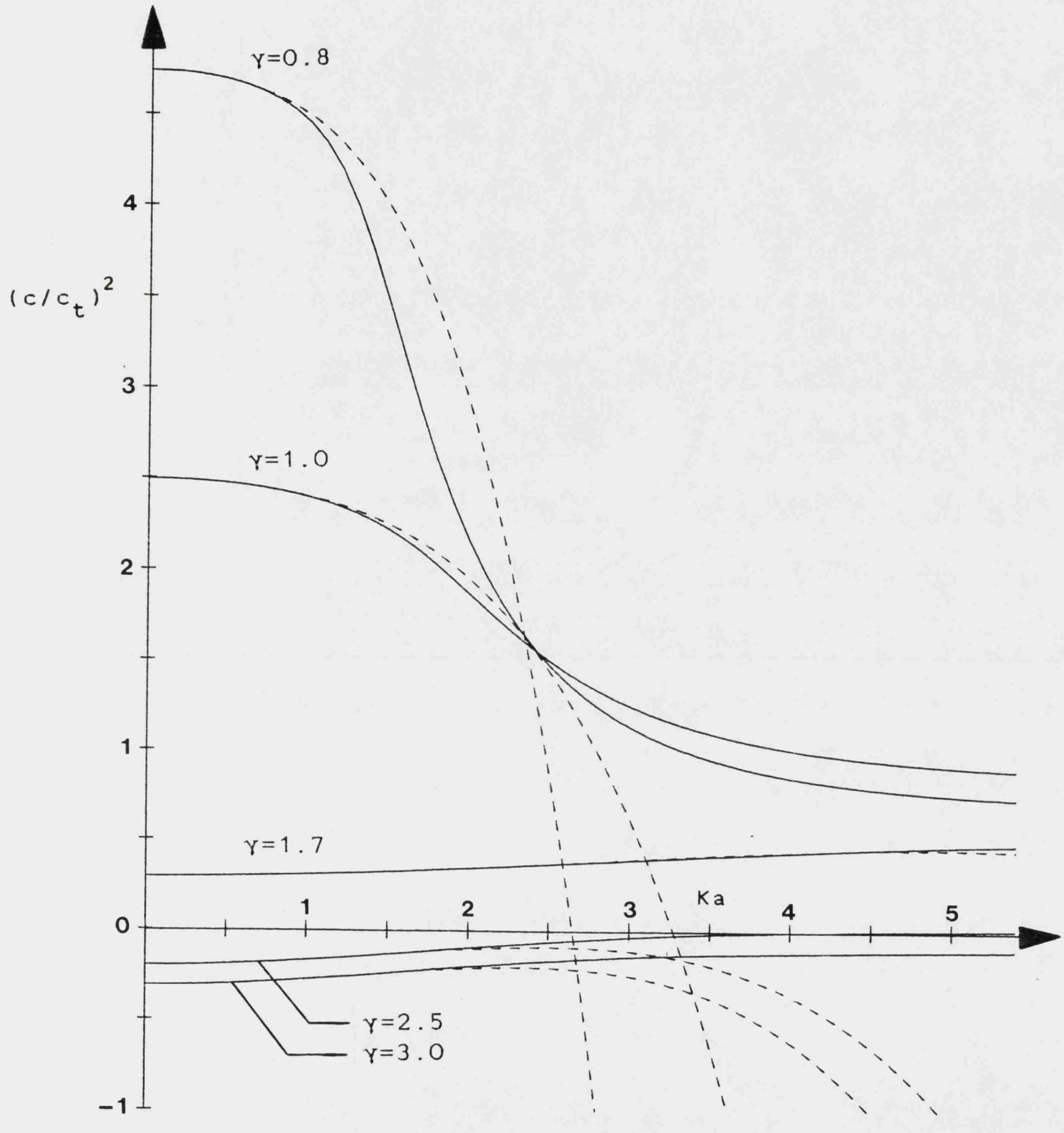

Figure 4.7

For longitudinal vibrations in a uniaxially stressed Ko cylinder we contrast the exact dispersion curves (solid curves) with their corresponding approximations for slender cylinders (broken curves).

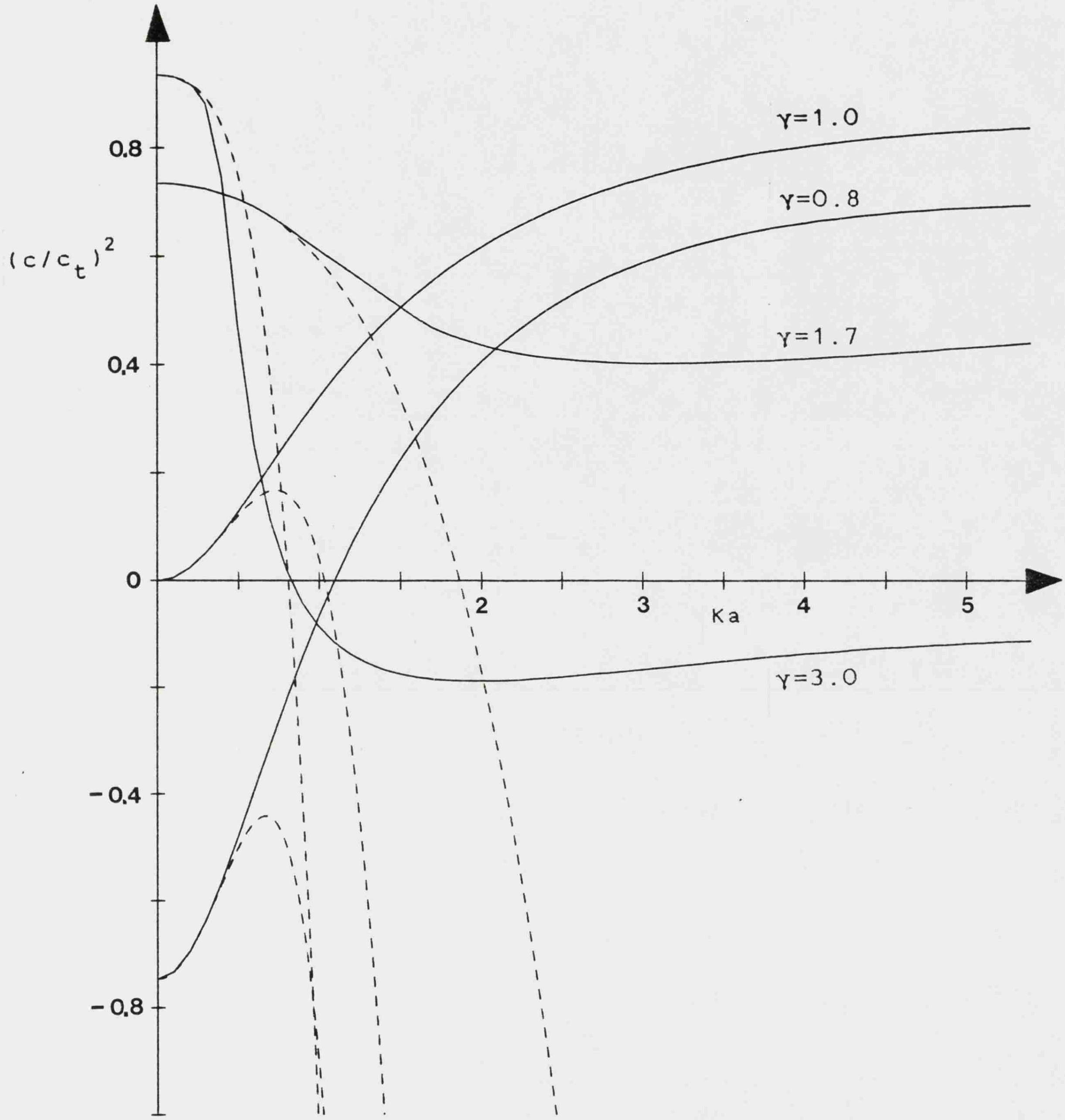

Figure 4.8

For flexural vibrations in a uniaxially stressed Ko cylinder we contrast the exact dispersion curves (solid curves) with their corresponding approximations for slender cylinders (broken curves).

the right-hand-side of the figures are predicted by our results in §3.3.

Finally, we contrast our findings with the results established in §2.10.3 for slender Ko cylinders. In Fig. 4.7, for selected values of γ, we plot the wave-velocity (2.117), represented in the figure by the broken curves, and the curves calculated from the longitudinal dispersion relation (4.21). The corresponding result for the flexural case, using (2.118) and (4.26), are illustrated in Fig. 4.8.

CHAPTER 5

A GENERALISATION OF KO'S STRAIN-ENERGY FUNCTION

§5.1 INTRODUCTION.

In [9] Ko proposed for the elastic strain-energy function

$$W = \bar{\mu}[I_3^{1/2}+I_2/2I_3], \tag{5.1}$$

$\bar{\mu}$ constant; $\bar{\mu}>0$, where I_1, I_2, I_3 are the invariants of the left Cauchy-Green strain-tensor **B**, with a neutral stress-free configuration taken as the reference state. Originally intended to describe the behaviour of polyurethane foam rubber and similar materials, the form (5.1) has been used by many workers in a much wider context for illustrating various theoretical aspects of non-linear behaviour in elastic materials (see for example [8], [25]). When it is applied to deformations in which the elastic strains are small, the usual linear behaviour for isotropic materials is recovered, with Lamé constants λ, μ both equal to $\bar{\mu}$, so that Poisson's ratio σ is 1/4.

The purpose of this chapter is to introduce the model strain-energy function

$$W = \bar{\mu}[I_3^{1/2}+nI_2/2I_3+(n-1)I_1/2], \tag{5.2}$$

$\bar{\mu}$, n constants. The motivation to consider this alternative function lies in theoretical rather than experimental observation. The difficulties inherent in problems of non-linear elastic behaviour severely limit the scope of investigations in terms of a general strain-energy function (see [39] for an example of such an investigation), and a reasonable line of development is to explore fully the potential of such models as are mathematically tractable and have physical relevance. The present model is proposed in this spirit. The following sections show that physically reasonable behaviour is manifested for a greater range of principal stretches than is the case for $n=1$, and this together with the avoidance of the constraint $\sigma=1/4$ may have application.

In §5.2 of this chapter the model form (5.2) is examined in the light of various *a priori* constitutive inequalities that have been proposed by other workers in attempts to guarantee that the predicted behaviour is in accord with natural notions of what is physically reasonable. In §5.3 particular consideration is given to equilibrium configurations of plane strain and plane stress. Finally in §5.4 we turn to the dynamical behaviour of the model and consider the vibrations of a stretched plate composed of elastic material for which (5.2) holds. In particular, the conditions for marginal stability are established in terms of the parameters $\bar{\mu}$, n and the applied principal stretches λ_1, λ_2.

§5.2 BASIC CONSIDERATIONS.

For a clear account of the principles of non-linear elasticity the reader is referred to [40]. For a given deformation the stress-tensor **P** can be found from the equations

$$\mathbf{P} = h_0\mathbf{I}+h_1\mathbf{B}+h_{-1}\mathbf{B}^{-1}, \tag{5.3}$$

where **I** is the identity and h_0, h_1, h_{-1} are the response functions given by

$$h_0 = 2I_3^{-1/2}[I_2W_2+I_3W_3], \quad h_1 = 2I_3^{-1/2}W_1, \quad h_{-1} = -2I_3^{1/2}W_2, \tag{5.4}$$

where $W_1 = \partial W/\partial I_1$ and similarly for W_2, W_3. Now substitution for W from (5.2) into (5.4), combined with (5.3), yields the stress-strain relations

$$\mathbf{P} = \bar{\mu}[\mathbf{I}+I_3^{-1/2}\{(n-1)\mathbf{B}-n\mathbf{B}^{-1}\}].$$

In the reference state $\mathbf{B}=\mathbf{I}$, $I_3=1$ and $\mathbf{P}$ vanishes for all n, so this state is indeed free from stress. For a *small deformation* B may be written as $\mathbf{I}+2\mathbf{e}$, where $\mathbf{e}$ is the usual strain-tensor. Taking $\mathbf{e}$ in its cartesian component form e_{ij} and retaining just first-order terms in these small quantities, we find $I_3=1+2e_{kk}$ (using the summation convention) and obtain

$$P_{ij} = \bar{\mu}[e_{kk}\delta_{ij}+(4n-2)e_{ij}].$$

Thus in the linear small-strain approximation W, as given by (5.2), leads to Lamé constants

$$\lambda = \bar{\mu}, \qquad \mu = (2n-1)\bar{\mu},$$

and Young's modulus E and Poisson's ratio σ are given by

$$E = \bar{\mu}[4n-1-1/2n], \qquad \sigma = 1/4n.$$

Now physical considerations demand that μ, the modulus of rigidity, is positive so (since $\bar{\mu}$ has been assumed positive in (5.2)) we conclude that $n>1/2$. We impose this restriction on n in all that follows.

First consider a homogeneous finite deformation in which the typical particle at (X_1, X_2, X_3) in the reference state moves to (x_1, x_2, x_3) in the current state, where

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \qquad (\lambda_i>0), \qquad (5.5)$$

a fixed system of cartesian axes being used throughout. The quantities λ_i are then the principal stretches. Since

$$B_{ij} = x_{i,K}x_{j,K}, \qquad (5.6)$$

the principal stresses are P_1, P_2 and P_3 where

$$P_i = \bar{\mu}[1+\{(n-1)\lambda_i^2-n\lambda_i^{-2}\}/(\lambda_1\lambda_2\lambda_3)], \qquad (i=1,2,3). \qquad (5.7)$$

In the past, various restrictions have been placed upon W in order to ensure a physically reasonable response. Reviews of these restrictions and the interconnections between them are given in [41], [42]. Recently, however, Dunn [43] has constructed an ingenious example to show that even when several such requirements are met, the response is not always satisfactory.

We begin with the Baker-Ericksen inequality which is satisfied if and only if the greater principal stress is associated with the greater principal stretch, that is

$$(P_i - P_j)(\lambda_i - \lambda_j) > 0 \quad \text{if} \quad \lambda_i \neq \lambda_j.$$

From (5.7) it is easily seen that this is equivalent to the requirement that $(n-1)\lambda^2 - n\lambda^{-2}$ shall be a monotonic increasing function of λ, for $\lambda>0$. If $n\geq 1$ this is certainly satisfied but for $1/2<n<1$ we must have

$$\lambda < \lambda_B(n) = [n/(1-n)]^{1/4}.$$

In Table 5.1 the values of $\lambda_B(n)$ are shown for selected values of n.

Table 5.1

For selected values of n, the values of $\lambda_B(n)$ shown are the greatest values of the principal stretches if the Baker-Ericksen inequality is to hold.

n	=	0.5	0.6	0.7	0.8	0.9
$\lambda_B(n)$	=	1.000	1.107	1.236	1.414	1.732
n	=	0.92	0.94	0.96	0.98	1.00
$\lambda_B(n)$	=	1.842	1.990	2.213	2.646	∞

Of course if the Baker-Ericksen inequality is imposed in the strict sense that it has to be satisfied for all λ_1, λ_2, $\lambda_3>0$ then the consequence is that n must be greater than or equal to unity. However, we adopt the viewpoint here that the inequality is to be interpreted as imposing joint demands upon n and the λ_i. Thus for $1/2<n<1$ we must have $\lambda_i<\lambda_B(n)$ for

$i=1,2,3$, if the Baker-Ericksen inequality is to be satisfied.

Another restriction commonly encountered is the "strengthened tension-extension inequality" which requires that each principal stress shall be a strongly monotone increasing function of its associated principal stretch. Again from (5.7) we see that this is always satisfied if $n \geqslant 1$ but for $1/2<n<1$ it requires that

$$\lambda_i < [3n/(1-n)]^{1/4} = (1.315\ldots)\lambda_B(n).$$

Yet another such inequality is the ordered forces inequality which can be cast into the form

$$(P_1\lambda_2\lambda_3 - P_2\lambda_3\lambda_1) > 0 \quad \text{if} \quad \lambda_1 \neq \lambda_2.$$

From (5.7) an easy calculation shows that this is equivalent to the requirement that

$$Q \equiv n-1+n(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)/(\lambda_1\lambda_2)^3-\lambda_3 > 0$$

for all values of λ_1, λ_2, λ_3 to be used in any calculation of physical relevance. Now the analysis is much more complicated since, even if $n \geqslant 1$, Q is certainly negative for sufficienty large λ_3; but we may argue as follows. In the (λ_1, λ_2) plane, consider the rectangle bounded by the lines $\lambda_1=0$, $\lambda_2=0$, $\lambda_1=1+\delta$, $\lambda_2=1+\delta$, $(\delta \geqslant 0)$. Then since $\partial Q/\partial\lambda_1$ and $\partial Q/\partial\lambda_2$ are both negative, the smallest value of Q (for a fixed value of λ_3) inside the rectangle is achieved at $\lambda_1=\lambda_2=1+\delta$ and there $Q=n-1+3n(1+\delta)^{-4}-\lambda_3$. Thus provided λ_3 does not exceed $n-1+3n(1+\delta)^{-4}$ the ordered forces inequality will hold. Suppose we set

$$\lambda_3 = 1+\delta = n-1+3n(1+\delta)^{-4}$$

and determine δ in terms of n, say $\delta=\delta(n)$. Then if no principal stretch exceeds $\lambda_0(n)=1+\delta(n)$ the ordered forces inequality will be obeyed. In Table 5.2 the values of $\lambda_0(n)$ obtained in this way are shown for selected values of n.

Table 5.2

For selected values of n, the values of $\lambda_0(n)$ shown specify the size of the cube in $(\lambda_1, \lambda_2, \lambda_3)$ space, for the points of which the ordered forces inequality will certainly hold.

n =	0.5	0.6	0.7	0.8	0.9
$\lambda_0(n)$ =	1.000	1.055	1.106	1.154	1.200
n =	1.0	2.0	3.0	4.0	5.0
$\lambda_0(n)$ =	1.246	1.707	2.314	3.126	4.055
n =	6.0	7.0	8.0	9.0	10.0
$\lambda_0(n)$ =	5.028	6.016	7.010	8.007	9.005

It is to be emphasised that this is a *sufficient* condition, that is to say, the ordered forces inequality may well be satisfied at points in the $(\lambda_1, \lambda_2, \lambda_3)$ space outside this cube of edge $\lambda_0(n)$, but inside the cube it will certainly be satisfied. We shall return to this point in §5.3.

Other restrictions upon n and the λ_i may follow from a consideration of specific types of deformation. Consider an elastic bar, obeying (5.2), placed under uniaxial stress T parallel to the axis Ox_1. Then from (5.7) since both P_2, P_3 vanish

$$\lambda_1+n-1 = n\lambda_2^{-4}.$$

For $1/2<n<1$ then, under compression, λ_2 becomes very large as λ_1 approaches $(1-n)$. We must conclude therefore that physical reality breaks down in our model before this stage in the compressive case is reached. Also from (5.7), the stretch-stress relation for bars under uniaxial stress is

$$T = \bar{\mu}[1+\{(n-1)\lambda_1-n\lambda_1^{-3}\}\{(\lambda_1+n-1)/n\}^{1/2}].$$

It is reasonable to require that for elastic deformations of this kind T shall be a monotonic increasing function of λ_1. This is certainly true if $n\geqslant1$ but for $1/2<n<1$ this imposes an upper limit to λ_1, denoted here by $\lambda_u(n)$. Table 5.3 shows how

$\lambda_u(n)$ depends upon n.

Table 5.3

For selected values of n, the values of $\lambda_u(n)$ specify the greatest axial stretch if the axial principal stress is to be a monotone increasing function of the stretch.

n =	0.5	0.6	0.7	0.8	0.9
$\lambda_u(n)$ =	1.000	1.176	1.353	1.576	1.954
n =	0.92	0.94	0.96	0.98	1.00
$\lambda_u(n)$ =	2.081	2.252	2.509	3.003	∞

Now consider the deformation produced by a hydrostatic pressure Π, so that $\lambda_1=\lambda_2=\lambda_3$ (=λ say) in (5.7), where

$$\Pi = \bar{\mu}[n\lambda^{-5}-(n-1)\lambda^{-1}-1].$$

With ρ, ρ_0 denoting the densities in the current and reference states respectively, this result gives the pressure-density relation

$$\Pi = \bar{\mu}[n(\rho/\rho_0)^{5/3}-(n-1)(\rho/\rho_0)^{1/3}-1].$$

It is easily verified that for $\Pi>0$, $d\Pi/d\rho$ will be positive for all $n>1/2$. If we demand that $d\Pi/d\rho$ shall be positive even when Π is negative, then for $n>1$ we must have

$$\lambda < \lambda_H(n) = [5n/(n-1)]^{1/4}.$$

For $n\leqslant 1$, however, this requirement is satisfied for all λ.

The conditions to be imposed upon W are sometimes expressed [44] in terms of the principal wave speeds, that is, the speeds of waves propagated along the axes Ox_1, Ox_2, Ox_3 as perturbations of the primary state specified by (5.5). But Truesdell [44] has established the universal relations

$$c_{11}^2 = \frac{\lambda_1\partial P_1}{\rho\ \partial\lambda_1}, \qquad c_{12}^2 = \frac{\lambda_1^2(P_1-P_2)}{\rho\ (\lambda_1^2-\lambda_2^2)}, \qquad c_{13}^2 = \frac{\lambda_1^2(P_1-P_3)}{\rho\ (\lambda_1^2-\lambda_3^2)},$$

where c_{11}, c_{12}, c_{13} are the speeds of waves travelling parallel to Ox_1, the first being the longitudinal principal wave speed and the others the speeds of the transverse principal waves. Thus the requirement that c_{11}^2 shall be positive is equivalent to the strengthened tension-extension inequality, and the requirements that c_{12}^2 and c_{13}^2 shall be positive is equivalent to the Baker-Ericksen inequality.

It is clear from the results above that the requirements commonly placed upon the strain-energy function W in order to achieve a physically reasonable response, when applied to (5.2), distinguish sharply between the two cases $n \geqslant 1$ and $1/2<n<1$. Thus the original Ko model (5.1) marks the boundary between them. If $n \geqslant 1$ the Baker-Ericksen and the strengthened tension-extension inequalities will always be satisfied (so that the speeds of all principal waves are real), and in uniaxial stress the applied tension is a monotonic increasing function of the axial stretch. It is easily shown too that the ordered forces inequality holds not only for all states of uniaxial tension or compression but also for all small perturbations about these states. Further, for materials under hydrostatic compression $d\Pi/dp$ will always be positive (though this is also true if $1/2<n<1$). Finally, the ordered forces inequality will certainly be satisfied if the biggest principal stretch does not exceed $\lambda_0(n)$, a quantity which increases with n.

§5.3 TWO-DIMENSIONAL CONFIGURATIONS.

In this section we consider two-dimensional homogeneous configurations of plane stress and plane strain, with particular reference to the ordered forces inequality. In view of the results of §5.2 we shall impose upon W in (5.2) the requirement that $n \geqslant 1$. These configurations are of basic interest and importance in their own right; in particular, however, the study of plane stress is an essential preliminary to the analysis of plate vibrations given in §5.4.

Consider first a state of plane stress with regard to planes normal to OX_3; we take the deformation specified in (5.5), therefore, and require P_3 in (5.7) to vanish. Thus $\lambda_1, \lambda_2 > 0$ and

$$\lambda_1 \lambda_2 = n\lambda_3^{-3} - (n-1)\lambda_3 . \tag{5.8}$$

If the ordered forces inequality is to be valid, then

$$(\lambda_2 \lambda_3 P_1 - \lambda_1 \lambda_3 P_2)(\lambda_1 - \lambda_2) > 0 \quad \text{for} \quad \lambda_1 \neq \lambda_2 \tag{5.9}$$

and

$$\begin{aligned} P_1(\lambda_1 - \lambda_3) > 0 \quad \text{for} \quad \lambda_1 \neq \lambda_3, \\ P_2(\lambda_2 - \lambda_3) > 0 \quad \text{for} \quad \lambda_2 \neq \lambda_3. \end{aligned} \tag{5.10}$$

We are assuming in this section that $n \geqslant 1$ so the Baker-Ericksen inequality holds, so (5.10) is certainly valid. Easy manipulation of (5.9) shows that it is equivalent to the requirement that

$$\lambda_3 < n-1+n(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)/(\lambda_1\lambda_2)^3, \tag{5.11}$$

where λ_3 is given in terms of λ_1 and λ_2 (and n) by (5.8). Take first the original Ko model with $n=1$. Then (5.8), (5.11) give $(\lambda_1\lambda_2)^8 < (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)^3$. This inequality is clearly satisfied in the (λ_1, λ_2) plane in the neighbourhood of the point N, where $\lambda_1 = \lambda_2 = 1$. The situation is illustrated in Fig. 5.1. The bounding curve $(\lambda_1\lambda_2)^8 = (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)^3$, indicated in that figure by the label $n=1$ divides the positive quadrant of the (λ_1, λ_2) plane into two parts. The ordered forces inequality will hold in the part including N and bounded by the axes, which are asymptotes to the bounding curve. This curve cuts the line

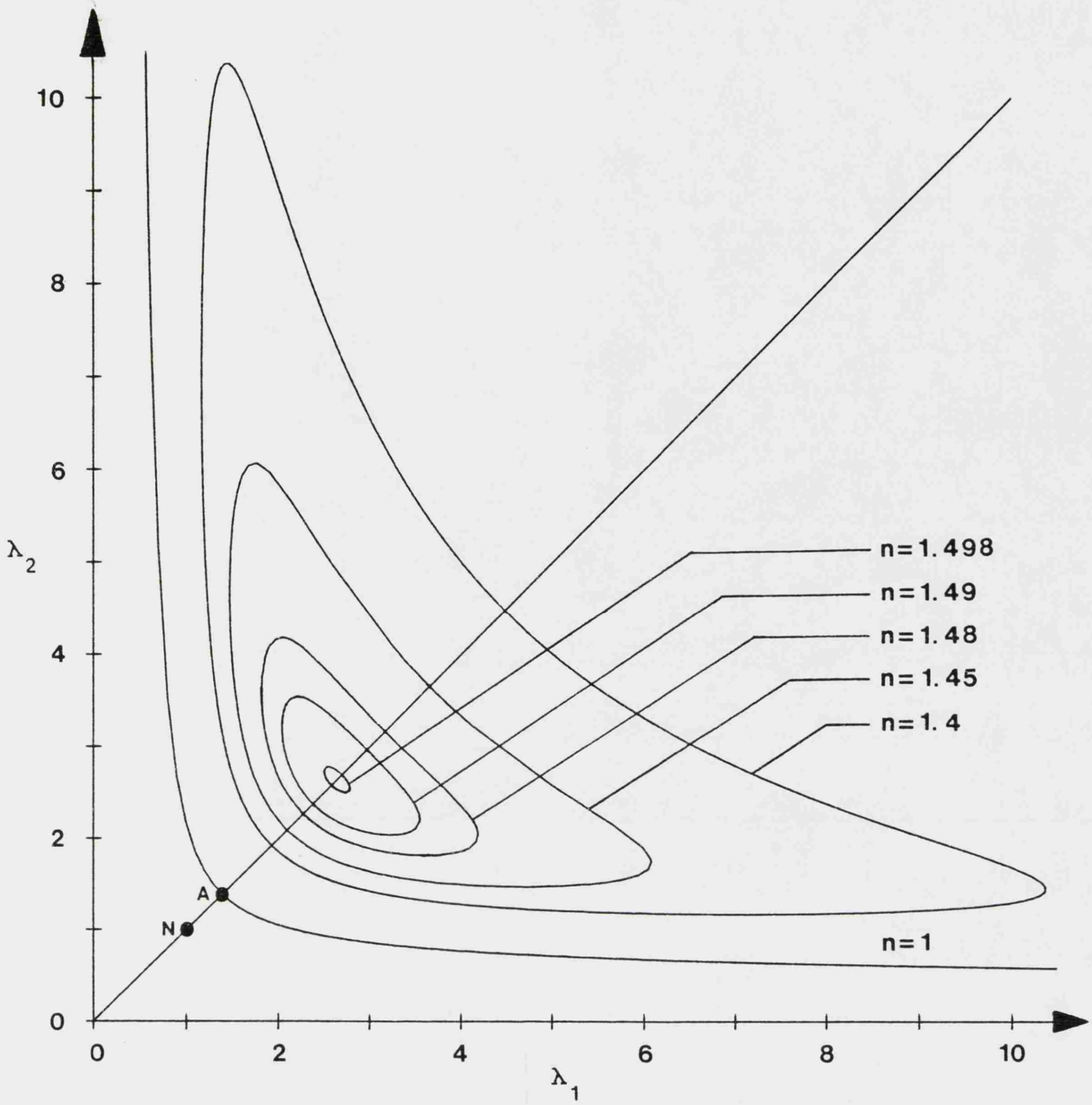

Figure 5.1

Variation of λ_1 against λ_2 for the generalised Ko model in states of plane stress. For selected values of the parameter n the curves shown enclose those states for which the ordered forces inequality is not satisfied. For n>1.498... the ordered forces inequality holds for all states of plane stress.

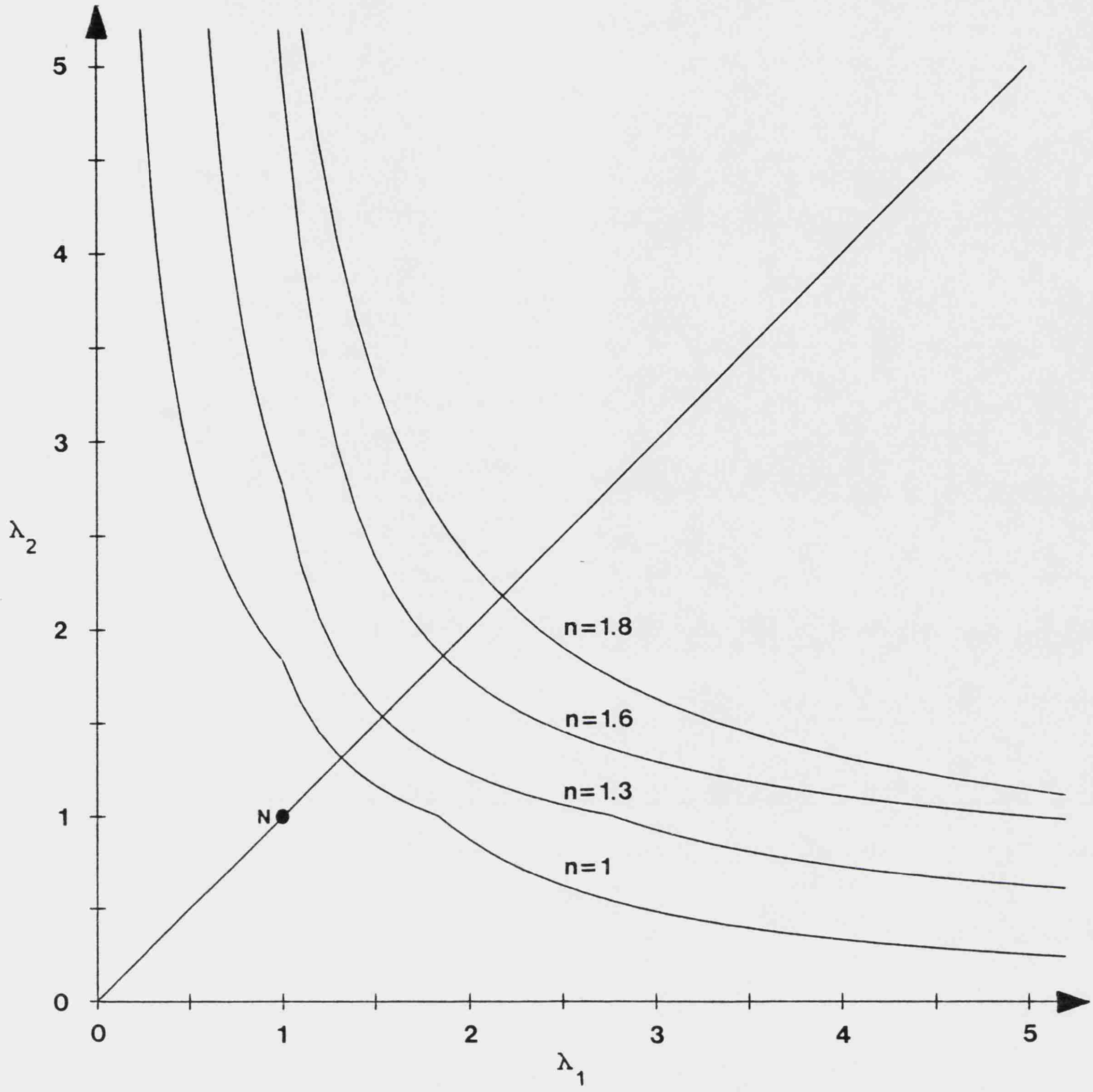

Figure 5.2

Variation of λ_1 against λ_2 for the generalised Ko model in states of plane strain. The ordered forces inequality holds below these curves in the area which contains the neutral point N(1,1).

$\lambda_1=\lambda_2$ at the point A, where $\lambda_1=\lambda_2=1.3904\ldots$.

For $n>1$, however, the bounding curves are closed (see Fig. 5.1, where the curves for n=1.4, 1.45, 1.48, 1.49, 1.498 are shown). The ordered forces inequality holds *outside* these closed curves, that is, it holds in an area containing N. As n approaches 1.498 (approximately) the bounding curve gets smaller and finally disappears. Thus for $n>1.498\ldots$ the ordered forces inequality holds for all homogeneous plane stress configurations.

Now consider states of plane strain with regard to planes normal to OX_3. Thus we set $\lambda_3=1$ in (5.5) and then the ordered forces inequality will require for the in-plane forces

$$(\lambda_2 P_1-\lambda_1 P_2)(\lambda_1-\lambda_2) > 0 \quad \text{for} \quad \lambda_1 \neq \lambda_2, \tag{5.12}$$

and, taking into account the out-of-plane force $\lambda_1\lambda_2 P_3$, it requires also that

$$\begin{aligned} (P_1-\lambda_1 P_3)(\lambda_1-1) > 0 \quad \text{for} \quad \lambda_1 \neq 1, \\ (P_2-\lambda_2 P_3)(\lambda_2-1) > 0 \quad \text{for} \quad \lambda_2 \neq 1. \end{aligned} \tag{5.13}$$

It is easily seen that (5.12) is equivalent to the requirement

$$G \equiv (n-2)(\lambda_1\lambda_2)^3+n(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) > 0 \quad \text{for} \quad \lambda_1,\ \lambda_2 > 0. \tag{5.14}$$

For $n\geqslant 2$, (5.14) is clearly satisfied. For a fixed value of n, with $1\leqslant n<2$, the curve $G=0$ in the (λ_1, λ_2) plane has the coordinate axes as asymptotes and is symmetrical about the line $\lambda_1=\lambda_2$ which it cuts at $\lambda_1=[3n/(2-n)]^{1/4}>1$. For a given value of n, $(1\leqslant n<2)$, at any point on the same side of the curve $G=0$ as N(1,1) the requirement (5.12) will be satisfied. However, this is not the end of the matter for (5.13), relating in part to the out-of-plane force, must also be taken into account. The requirements (5.13) may be cast into the form

$$H_1 \equiv (n-1-\lambda_2)+n(\lambda_1^{-3}+\lambda_1^{-2}+\lambda_1^{-1}) > 0,$$
$$H_2 \equiv (n-1-\lambda_1)+n(\lambda_2^{-3}+\lambda_2^{-2}+\lambda_2^{-1}) > 0. \tag{5.15}$$

Consider first Ko material ($n=1$): the curve $G=0$ is now the bounding curve only in the region $1<\lambda_1<1.839...$, the remaining part of the boundary being taken up by the curves $H_1=0$ and $H_2=0$. Calculation reveals that this pattern, where the overall bounding curve is at some stage governed in turn by one of the three curves $G=0$, $H_1=0$, $H_2=0$, is repeated for all values of n in the range $1<n<1.745...$. For $n>1.745...$ the bounding curve may be constructed solely from the requirements (5.15). The bounding curves for $n=1$, 1.3, 1.6 and 1.8 are shown in Fig. 5.2.

§5.4 WAVES IN A STRESSED PLATE.

Consider a plate of elastic material, with a strain-energy function given by (5.2), unbounded in the Ox_1, Ox_2 directions and having thickness $2H$ in the neutral state. The plate is put under large tensions or compressions directed along the Ox_1, Ox_2 axes. The state now achieved is called the primary state and the stresses in this state are given the affix zero. Taking the principal stretches to be λ_1, λ_2, λ_3, we see that the deformation is given by (5.5), that the stresses P^0_{11}, P^0_{22} due to the applied forces are P_1, P_2 given by (5.7) and that $P^0_{33}=P_3=0$. Of course P^0_{ij} vanishes for $i\neq j$. In this state the upper and lower faces are $x_3 = \pm\lambda_3 H = \pm h$.

The plate is now perturbed, the additional small deformation comprising a two-dimensional wave travelling along the direction of the principal axes Ox_1. The upper and lower faces are kept free from traction. Thus in the perturbed state (the displacement being taken parallel to the plane Ox_1x_3)

$$x_1 = \lambda_1 X_1 + u(x_1, x_3, t),$$

$$x_2 = \lambda_2 X_2, \qquad (5.16)$$

$$x_3 = \lambda_3 X_3 + w(x_1, x_3, t),$$

where u, w have derivatives so small that their squares and products may be neglected. This additional deformation causes small changes in the stresses, so now

$$P_{ij} = P^0_{ij} + P^*_{ij},$$

the contribution P^*_{ij} being due to the perturbation. In particular, from (5.2)-(5.4), (5.6), (5.16) it is readily found that

$$P^*_{11} = a_{11}u_1 + a_{13}w_3, \quad P^*_{33} = a_{31}u_1 + a_{33}w_3, \quad P^*_{13} = \alpha u_3 + \beta w_1,$$

where

$$a_{11} = \Delta[(n-1)\lambda_1^2 + 3n\lambda_1^{-2}], \qquad a_{13} = \Delta[-(n-1)\lambda_1^2 + n\lambda_1^{-2}],$$

$$a_{31} = \Delta[-(n-1)\lambda_3^2 + n\lambda_3^{-2}], \qquad a_{33} = \Delta[(n-1)\lambda_3^2 + 3n\lambda_3^{-2}],$$

$$\alpha = \Delta[(n-1)\lambda_3^2 + n\lambda_1^{-2}], \qquad \beta = \Delta[(n-1)\lambda_1^2 + n\lambda_3^{-2}],$$

$$\Delta = \bar{\mu}(\lambda_1\lambda_2\lambda_3)^{-1}.$$

With the density in the primary state denoted by ρ, the equations of motion yield

$$(\rho u_{tt} - a_{11}u_{11} - \alpha u_{33}) - (\beta + a_{13})w_{13} = 0,$$

$$-(\alpha + a_{31})u_{13} + (\rho w_{tt} - \beta w_{11} - a_{33}w_{33}) = 0. \qquad (5.17)$$

For a solution of (5.17), we try

$$u = A\cosh(Ksx_3)\exp[i(\omega t - Kx_1)],$$

$$w = B\sinh(Ksx_3)\exp[i(\omega t - Kx_1)],$$

and so obtain (with ω^2/K^2 denoted by c^2)

$$(\rho c^2-a_{11}+\alpha s^2)A-(\beta+a_{13})isB = 0,$$
$$-(\alpha+a_{31})isA+(\rho c^2-\beta+a_{33}s^2)B = 0. \tag{5.18}$$

For non-trivial solutions in A, B in (5.18) the parameter s must satisfy

$$(\rho c^2-a_{11}+\alpha s^2)(\rho c^2-\beta+a_{33}s^2)+(\alpha+a_{31})(\beta+a_{13})s^2 = 0. \tag{5.19}$$

This is a quadratic equation in s^2 with roots s_1^2 and s_2^2, say. It follows, therefore, that a solution for the deformation field may be sought by setting

$$u = i(\beta+a_{13})[Ls_1\cosh(Ks_1x_3)+Ms_2\cosh(Ks_2x_3)],$$
$$w = [(\rho c^2-a_{11}+\alpha s_1^2)L\sinh(Ks_1x_3)+(\rho c^2-a_{11}+\alpha s_2^2)M\sinh(Ks_2x_3)], \tag{5.20}$$

in which the factor $\exp[i(\omega t-kx_1)]$ is suppressed and where L, M are constants that must be found from the boundary conditions. The upper and lower faces are free from traction so (using the relation $P_1=\beta-\alpha$) we must have

$$u_3+w_1 = 0, \quad a_{31}u_1+a_{33}w_3 = 0 \quad \text{on} \quad x_3 = \pm h.$$

These conditions yield two simultaneous equations in L, M and for a non-trivial solution

$$\frac{s_2\tanh(Ks_1h)}{s_1\tanh(Ks_2h)} = Q, \tag{5.21}$$

where

$$Q = \frac{[a_{31}s_2^2-(\rho c^2-a_{11})][a_{31}(\beta+a_{13})+a_{33}(\rho c^2-a_{11})+\alpha a_{33}s_1^2]}{[a_{31}s_1^2-(\rho c^2-a_{11})][a_{31}(\beta+a_{13})+a_{33}(\rho c^2-a_{11})+\alpha a_{33}s_2^2]}. \tag{5.22}$$

Equation (5.21) is the dispersion equation for the waves given by (5.20), usually called the longitudinal or extensional waves. By interchanging the hyperbolic functions cosh(x), sinh(x) throughout the analysis so far, we may obtain also the dispersion equation for flexural waves, that is,

$$\frac{s_2 \tanh(Ks_2 h)}{s_1 \tanh(Ks_1 h)} = Q. \tag{5.23}$$

Our initial objective is to examine the conditions for marginal stability by studying (5.21), (5.23) with ρc^2 set equal to zero. From (5.21), (5.23)

$$\frac{\tanh(Ks_1 h)}{\tanh(Ks_2 h)} = \left[\frac{Qs_1}{s_2}\right]^{\pm 1} \tag{5.24}$$

with the upper sign for longitudinal waves, and the lower sign for flexural waves, where from (5.19) for marginal stability s_1^2 and s_2^2 are the roots of the quadratic equation in s^2

$$\alpha a_{33} s^4 - [a_{11} a_{33} + \alpha\beta - (\alpha + a_{31})(\beta + a_{13})] s^2 + \beta a_{11} = 0. \tag{5.25}$$

Consider first, however, the case of a very thin plate for which kh is very small. Assuming for the solutions of principal interest that s_1^2, s_2^2 remain finite as kh tends to zero, and denoting $\Sigma = (a_{31}/a_{33})$, $\Gamma = -(\alpha - \beta + a_{11} - \Sigma a_{31})/3$, (5.24) yields as an approximation for flexural waves

$$\rho c^2 = P_1 - \Gamma(Kh)^2 + \Gamma[-18\Gamma + 5\alpha - 2\Sigma\alpha](Kh)^4/15\alpha,$$

and for the longitudinal waves

$$\rho c^2 = (a_{11} - \Sigma a_{31})\Gamma\Sigma^2(Kh)^2 - \Gamma\Sigma^3[18\Gamma + 5\Sigma a_{31} - 2a_{31}](Kh)^4/15a_{31}.$$

Thus the condition for the flexural mode to be in marginal stability, in the limit as $kh \to 0$, is simply that P_1 shall vanish, that is,

$$\lambda_2 = n\lambda_1^{-4} - (n-1).$$

Similarly the condition for the longitudinal mode in very thin plates to be in marginal stability is

$$a_{31}^2 - a_{11} a_{33} = 0. \tag{5.26}$$

Equation (5.26), together with the condition that P_3 shall vanish, determine pairs of values (λ_1, λ_2) giving marginal stability in the longitudinal mode. The results are presented in Fig. 5.3 for $n = 1, 1.03, 1.10$ and 1.30. The sight-line $\lambda_1 = \lambda_2$ has been included in this and later diagrams; its intersections with the appropriate n-curve give equibiaxial states of marginal stability. For these states all directions of wave-propagation are equivalent. In particular for the Ko model ($n=1$) it is found that for marginal stability in the longitudinal mode

$$\lambda_2 = 27\lambda_1^{-4}, \qquad \lambda_3 = \lambda_1/3.$$

Consider now the case of a very thick plate for which in (5.24) kh is taken to be very large. Without loss of generality s_1, s_2 may be taken to have positive real parts. Then for the modes of interest, the hyperbolic terms in (5.38) may be replaced by unity so that both the flexural and longitudinal modes are governed (to this approximation) by the same dispersion equation

$$Qs_1 = s_2. \tag{5.27}$$

We proceed as before to obtain states of marginal stability by setting ρc^2 in (5.22) to zero, and using (5.25), (5.27). For the Ko model this gives

$$\lambda_2 = (6\pm\sqrt{33})^{3/2}\lambda_1^{-4}, \qquad \lambda_3 = (6\pm\sqrt{33})^{-1/2}\lambda_1. \tag{5.28}$$

The wide disparity of the numerical factors in (5.28) makes for difficulties of presentation. In Fig. 5.4 we display for $n=1$ the values (λ_1, λ_2) obtained by choosing the upper sign in (5.28), in Fig. 5.5 those obtained from the lower sign. We refer to these as the first and second family of solutions respectively, and for other values of n ($n = 1.03, 1.10$ and 1.30 in Figs. 5.4, 5.5) these families arise similarly. Examination of Fig. 5.4 reveals the sensitivity of the solutions to changes in n, in particular for the equibiaxial

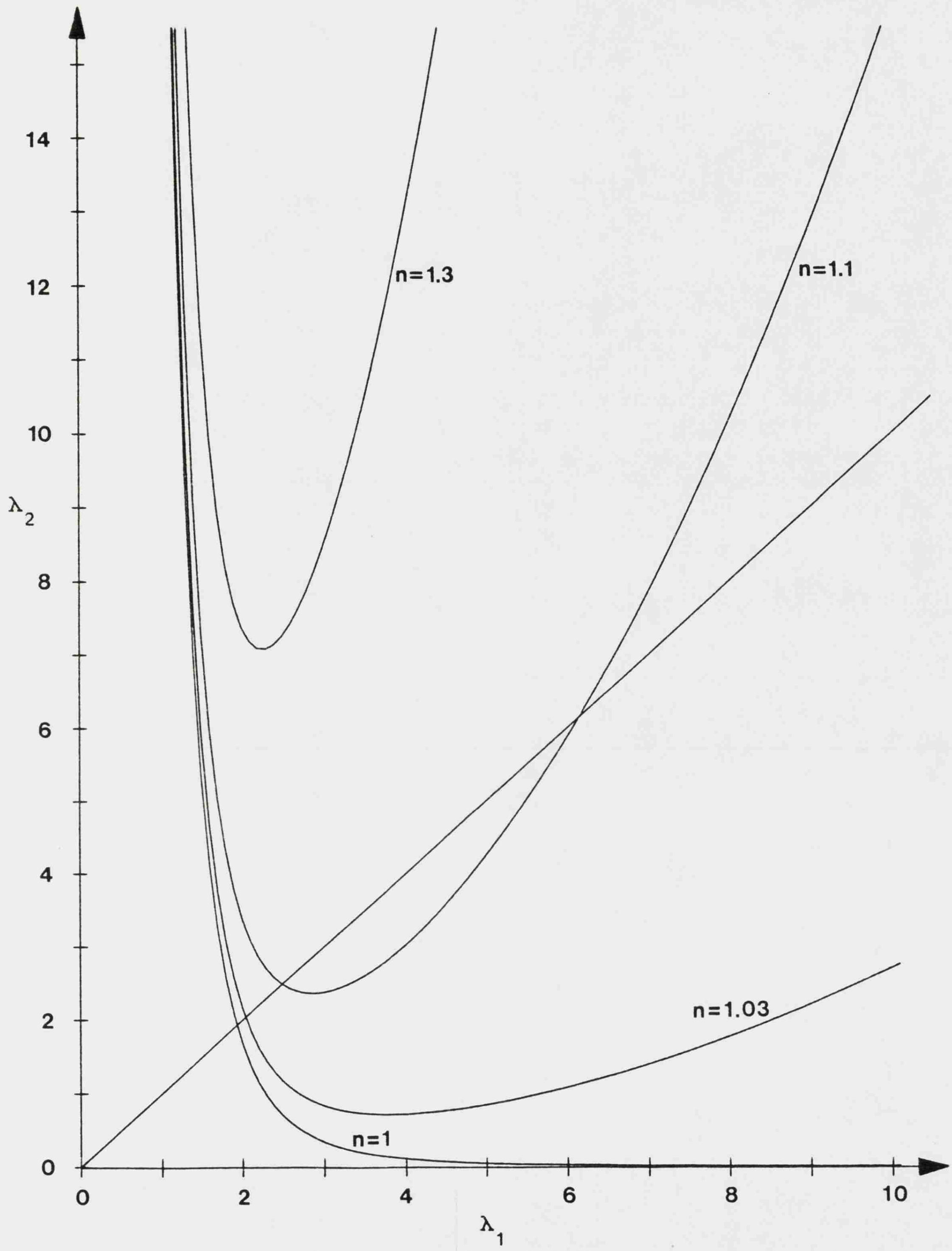

Figure 5.3

For a thin plate ($Kh<<1$) composed of generalised Ko material, the curves show states of marginal stability in the longitudinal mode.

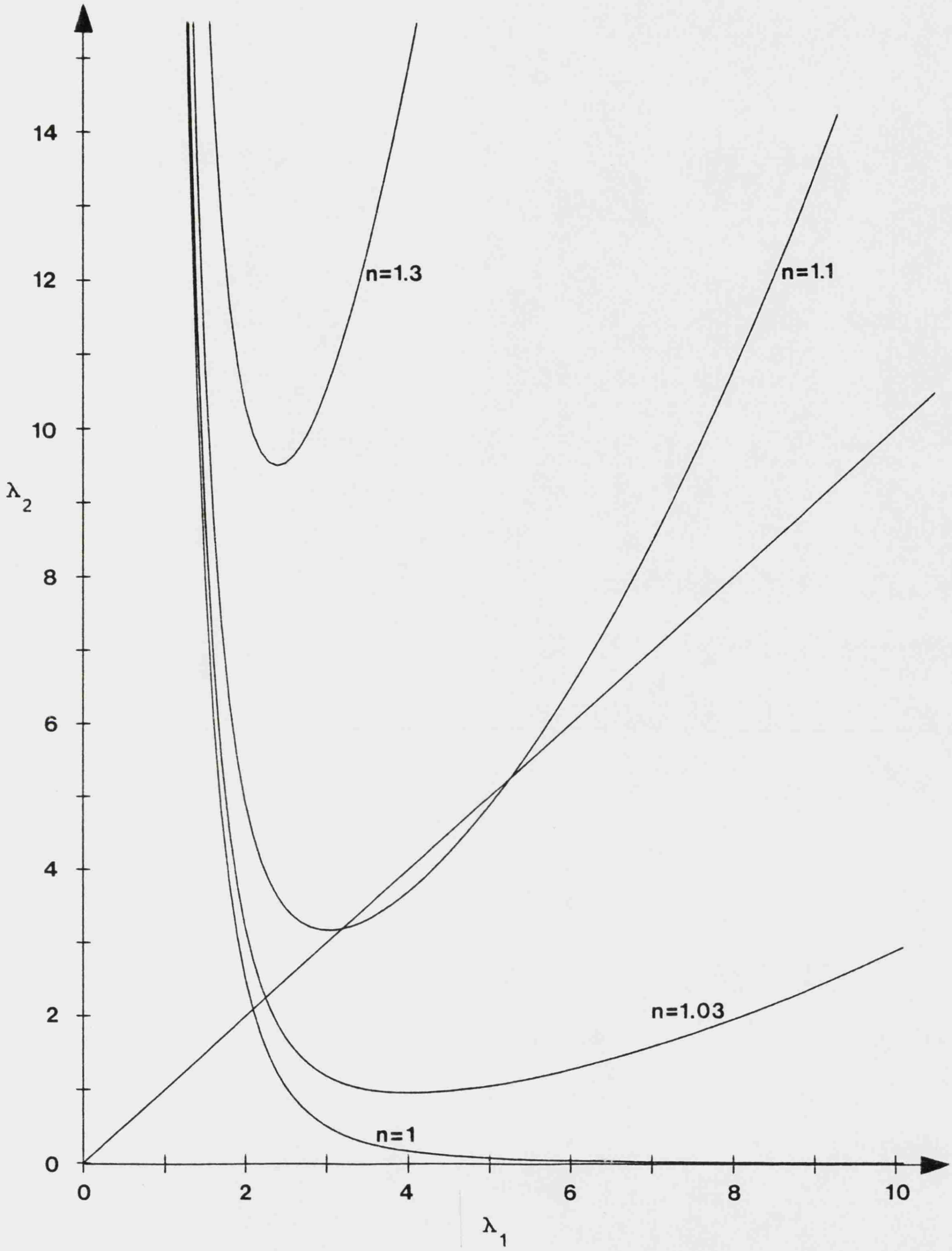

Figure 5.4

Variation of λ_1 against λ_2 for a thick plate (Kh>>1) composed of generalised Ko material. The curves represent the first family of solutions indicating marginal stability in the flexural and longitudinal modes.

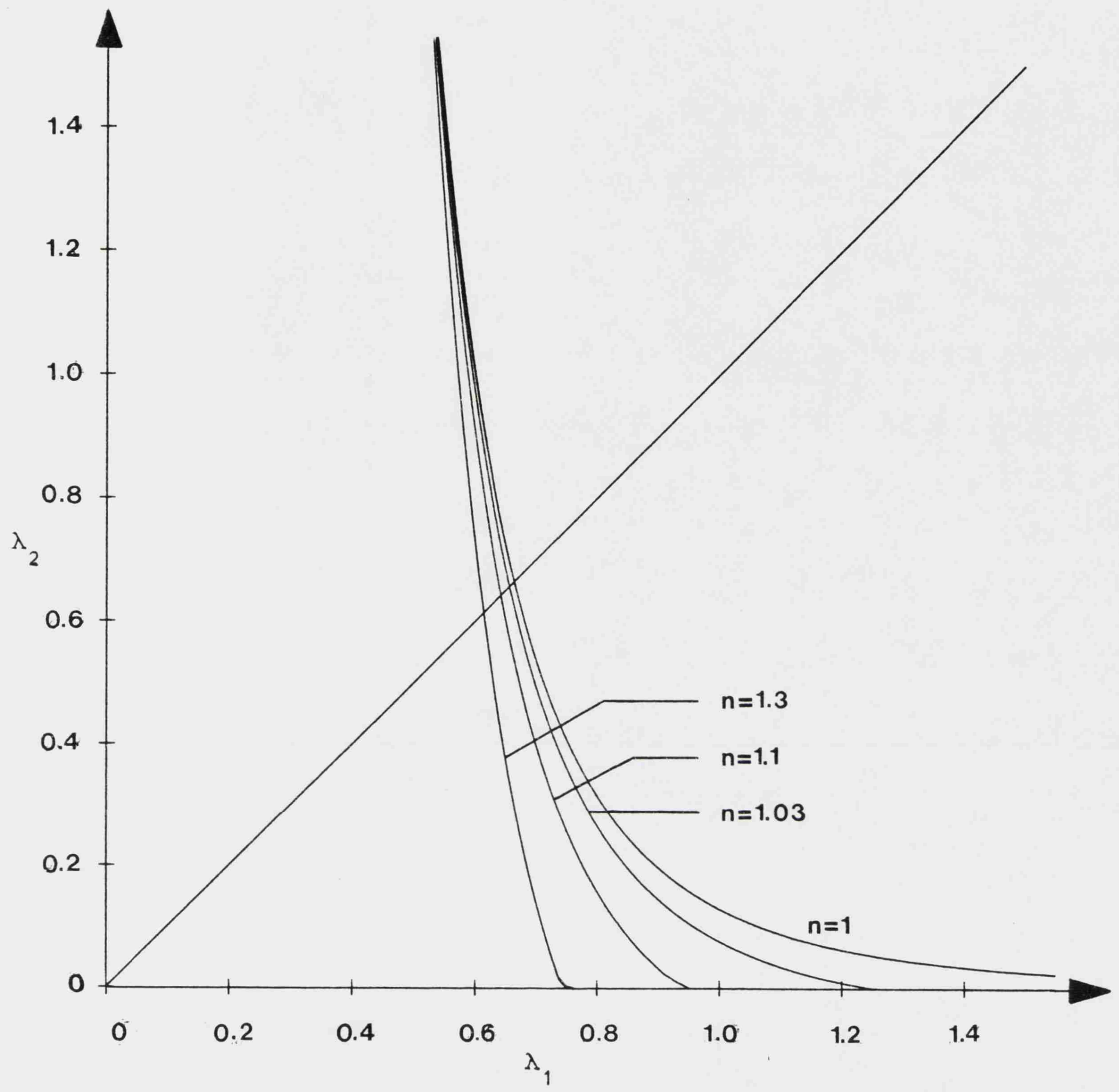

Figure 5.5

Variation of λ_1 against λ_2 for a thick plate (Kh>>1) composed of generalised Ko material. The curves represent the second family of solutions indicating marginal stability in the flexural and longitudinal modes.

case. The behaviour of the second families as shown in Fig. 5.5, however, is quite different. It may be seen that for those solutions arising from extension ($\lambda_1>1$) in the direction of wave-propagation a severe compression ($\lambda_2<1$) has to be imposed in the transverse direction. The remaining solutions correspond to compression in the direction of wave propagation; in particular for the equibiaxial case we see that $\lambda_1=\lambda_2\sim0.65$ for the values of n considered here.

We turn our attention now to consider the marginal behaviour of a plate placed under a uniaxial stress applied in the direction of Ox_1. From (5.24), we display for both longitudinal and flexural waves, the marginal curves given in Figs. 5.6 and 5.7 respectively for selected values of n. The curves representing marginal states of compression ($\lambda_1<1$) are insensitive to changes in the values of n considered here. For marginal states of extension ($\lambda_1>1$) the behaviour is quite different and the curves produced are very sensitive to even small changes in n. On the right-hand-side of Figs. 5.6, 5.7 (Kh large) the curves close up as n increases (n>1) until finally these curves vanish for n=1.02212... ($\lambda_1\approx4.1$) leaving no further points of marginal stability in the flexural case. We still have curves of marginal stability in the longitudinal case but these curves also vanish for n=1.03037... ($\lambda_1\approx3.6$).

The marginal behaviour of a plate under an equibiaxial stress produces curves similar to those found for the uniaxially stressed plate; however, the curves produced are less sensitive to the value of n. Thus in the equibiaxial case for large values of Kh the marginal curves vanish for n=1.10689... ($\lambda_1\approx3.9$) and when Kh=0 the curves for longitudinal waves disappear for n=1.12684... ($\lambda_1\approx3.4$).

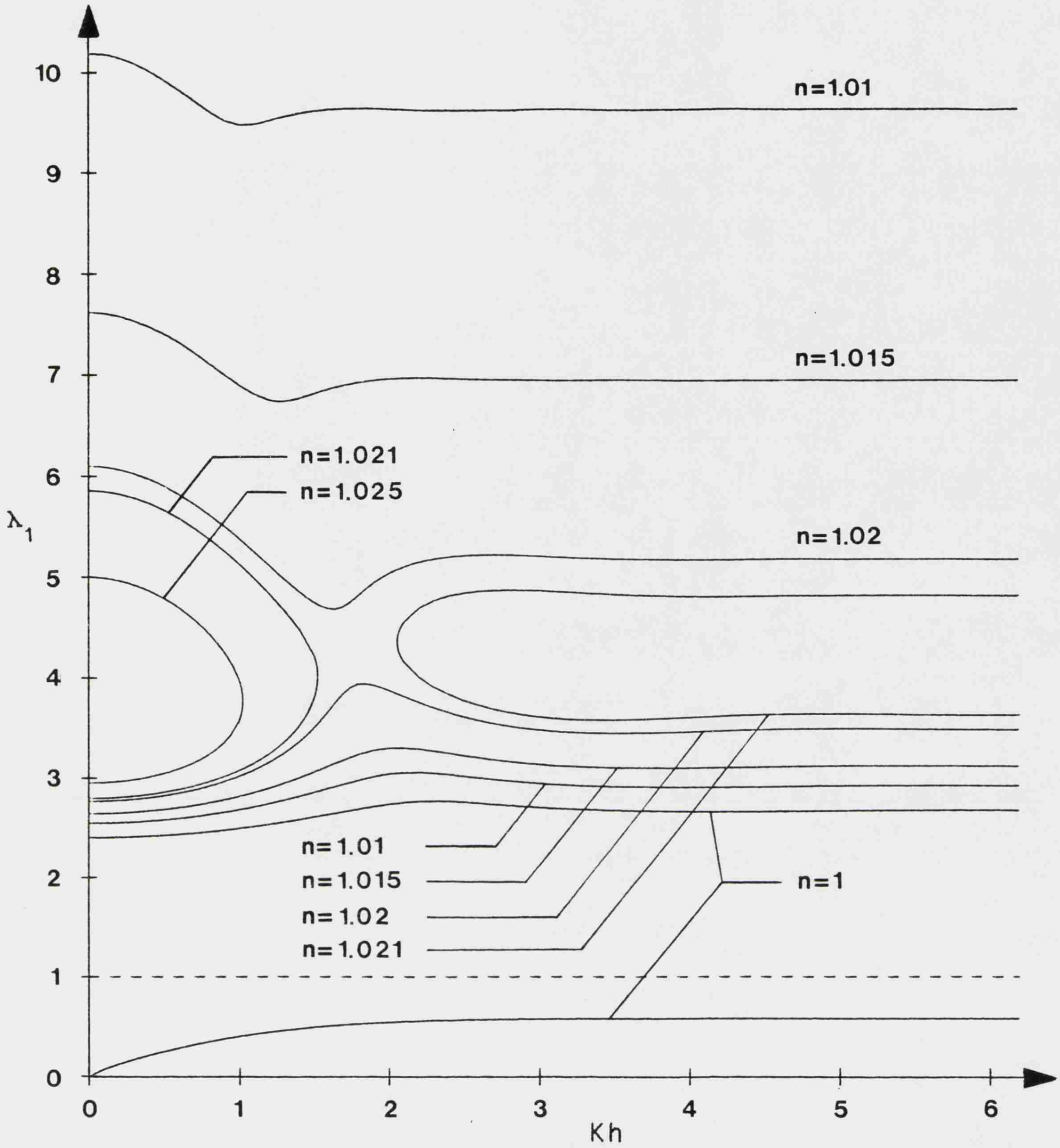

Figure 5.6

For selected values of n the curves shown represent marginal states of stability for longitudinal waves in a uniaxially stressed plate. For an extension ($\lambda_1 > 1$) all curves disappear for n=1.03037...

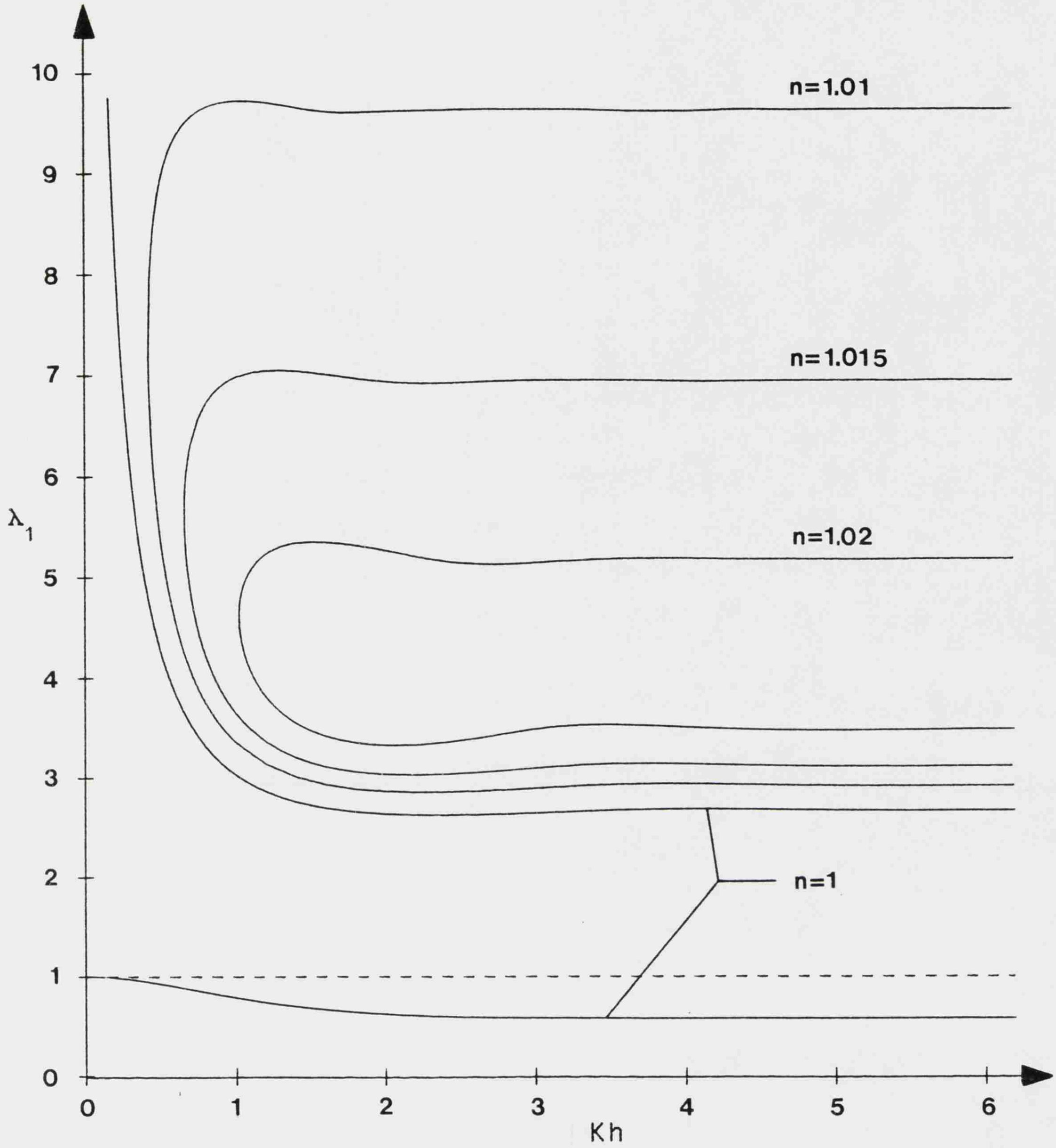

Figure 5.7

Curves of marginal stability for flexural waves in a uniaxially stressed plate composed of generalised Ko material. For $\lambda_1 > 1$ all curves disappear for n=1.02212...

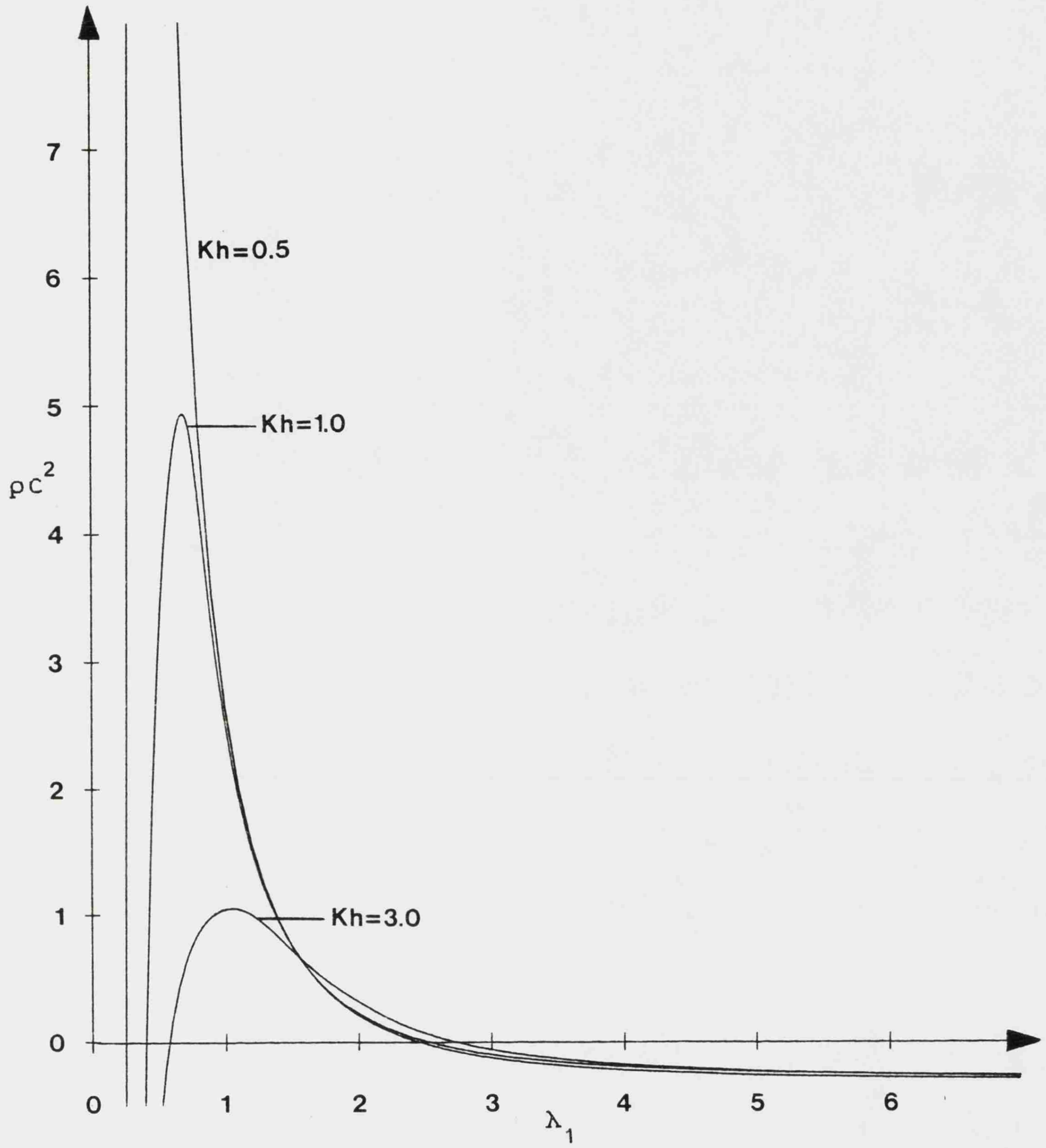

Figure 5.8

Variation of ρc^2 against λ_1 for selected values of Kh in a uniaxially stressed plate composed of Ko material.

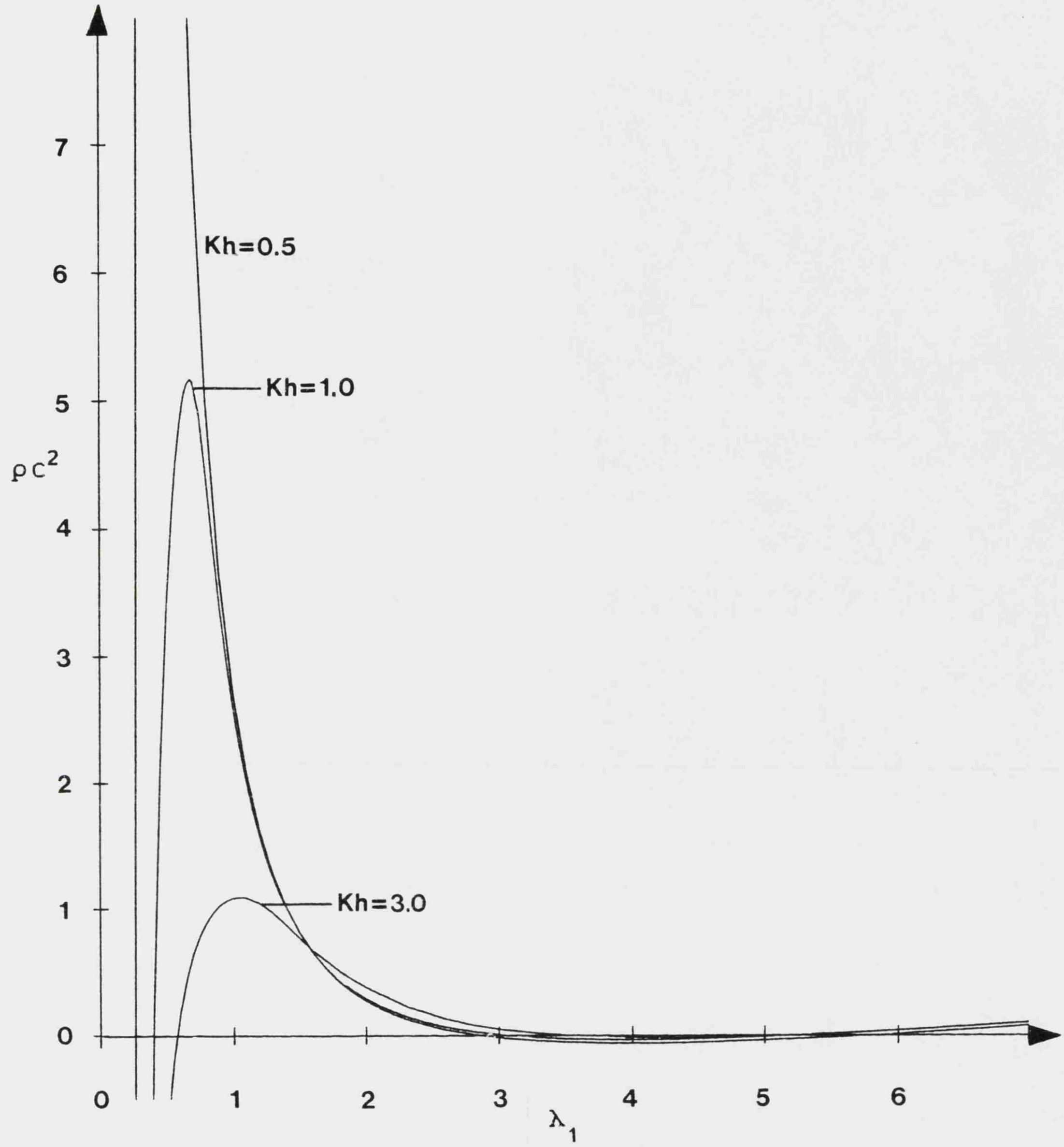

Figure 5.9

Variation of ρc^2 against λ_1 for selected values of Kh in a uniaxially stressed plate composed of generalised Ko material with n=1.02.

Finally, we produce the graphs of ρc^2 against λ_1 for a uniaxially stretched plate and selected values of Kh. We consider the two cases where n=1, n=1.02 corresponding to Figs. 5.8 and 5.9 respectively. The points where the curves cross the λ_1-axis mark the points of marginal stability.

REFERENCES

[1] POCHHAMMER, L., Ueber die fortpflanzungsgeschwindigkeiten kleiner schwingungen in einem unbegrenzten isotropen kreiscylinder. J. Reine Angew. Math., 81, 324 (1876).

[2] CHREE, C., Longitudinal vibrations of a circular bar. Q. J. of Pure and Appl. Maths., **84**, 287 (1886).

[3a] CHREE, C., On longitudinal vibrations. Q. J. of Pure and Appl. Maths., **92**, 317 (1889).

[3b] CHREE, C., On the longitudinal vibrations of aeolotropic bars with one axis of material symmetry. Q. J. of Pure and Appl. Maths., **96**, 340 (1890).

[4] PRAGER, W., The general variational principle of the theory of structural stability. Q. Appl. Math., **4**, 378 (1946).

[5] GREEN, A.E., RIVLIN, R.S. and SHIELD, R.T., General theory of small elastic deformations superposed on finite elastic deformations. Proc. Roy. Soc. Lond., **A211**, 128 (1952).

[6] URBANOWSKI, W., Small deformations superposed on finite deformations of a curvilinearly orthotropic body. Arch. Mech. Stos., **11**, 223 (1959).

[7] ZORSKI, H., On the equations describing small deformation superposed on finite deformation. Proc. Int. Symp. Second-Order Effects, Haifa 1962, (Macmillan, New York), 109 (1964).

[8] ERINGEN, A.C. and SUHUBI, E.S., Elastodynamics, Vol. I Finite Motions, (Academic Press, New York), (1974).

[9] KO, W.L., Application of Finite Elasticity Theory to the behaviour of Rubber-like Materials, Ph. D. Thesis, California Institute of Technology, (1963).

[10] BLATZ, P.J. and KO, W.L., Application of finite elastic theory to the deformation of rubbery materials. Trans. Soc. Rheol., **6**, 223 (1962).

[11] OGDEN, R.W., Large deformation isotropic elasticity II - on the correlation of theory and experiment for compressible rubberlike solids. Proc. Roy. Soc. Lond., **A328**, 567 (1972).

[12] LOVE, A.E.H., A Treatise on the Mathematical Theory of Elasticity, (New York, Dover), (1944).

[13] KOLSKY, H., Stress Waves in Solids, (Oxford University Press, New York), (1953).

[14] BANCROFT, D., The velocity of longitudinal waves in cylindrical bars. Phys. Rev., **59**, 588 (1941).

[15] DAVIES, R.M., Chapter: Stress Waves in Solids, in Surveys in Mechanics, G.K.Batchelor and R.M.Davies, eds., (Cambridge), (1959).

[16] RAYLEIGH, LORD, Theory of Sound, (Macmillan: London), (1894).

[17] HUDSON, G.E., Dispersion of elastic waves in solid circular cylinders. Phys. Rev., **63**, 46 (1943).

[18] ABRAMSON, H.N., Flexural waves in elastic beams of circular cross section. J. Acous. Soc. Amer., **29**, 42 (1957).

[19] VOLTERRA, E., A one-dimensional theory of wave-propagation in elastic rods based on the method of internal constraints. Ing. Arch., **23**, No. 6, 410 (1955).

[20] HOLDEN, A.N., Longitudinal modes of elastic waves in isotropic cylinders and slabs. Bell System Tech. J., **30**, No. 4, 956 (1951).

[21] ERINGEN, A.C. and SUHUBI, E.S., Elastodynamics, Vol. II Linear Theory, (Academic Press, New York), (1974).

[22] FILONENKO-BORODICH, M., Theory of Elasticity, (Dover), (1965).

[23] NOVOZHILOV, V.V., Theory of Elasticity, (Pergammon), (1961).

[24] COULSON, C.A., Waves, (Oliver and Boyd), (1961).

[25] THOMPSON, J.R. and WILLSON, A.J., Vibrations of a thin-walled elastic cylinder under axial stress. Int. J. Engng. Sci., **17**, 725 (1979).

[26] JEFFREYS, H. and JEFFREYS, B., Methods of Mathematical Physics, (Cambridge), (1962).

[27] MOONEY, M., J. Appl. Phys., **11**, (1940).

[28] OGDEN, R.W., Large deformation isotropic elasticity I - on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. Lond., **A** 326, 565 (1972).

[29] SUHUBI, E.S., Small longitudinal vibration of an initially stretched circular cylinder. Int. J. Engng. Sci., **2**, 509 (1965).

[30] OGDEN, R.W. and CHADWICK, P., On the deformation of solid and tubular cylinders of incompressible isotropic elastic material. J. Mech. Phys. Solids, **20**, 77 (1972).

[31] OGDEN, R.W., CHADWICK, P. and HADDON, E.W., Combined axial and torsional shear of a tube of incompressible isotropic elastic material. Qtrly. J. Mech. App. Maths., XXVI, (pt. I), 23 (1973).

[32] WILLSON, A. J., Wave-propagation in uniaxially-stressed elastic media. Pure Appl. Geophys., 93, 5 (1972).

[33] WILLSON, A. J., Wave-propagation in biaxially-stressed elastic media. Pure Appl. Geophys., 95, 48 (1972).

[34] WILLSON, A. J., Surface and plate waves in biaxially-stressed elastic media. Pure Appl. Geophys., 102, 182 (1973).

[35] WILLSON, A. J., Surface waves in restricted Hadamard materials. Pure Appl. Geophys., 110, 1967 (1973).

[36] WILLSON, A. J., Surface waves in uniaxially-stressed Mooney material. Pure Appl. Geophys., 112, 352 (1974).

[37] WILLSON, A. J., The anomalous surface wave in uniaxially-stressed elastic material. Pure Appl. Geophys., 112, 665 (1974).

[38] HAYES, W. and RIVLIN, R. S., Surface waves in deformed elastic materials. Arch. Rational Mech. Anal., 8, 358 (1961)

[39] WILLSON, A.J., Wave propagation in thin pre-stressed elastic plates. Int. J. Engng. Sci., 15, 245 (1977).

[40] ERINGEN, A.C., Nonlinear Theory of Continuous Media, (Mcgraw-Hill, New York), (1962).

[41] TRUESDELL, C. and NOLL, W., The Non-linear Field Theories of Mechanics, Handbuch der Physik (ed. S. FLUGGE) Vol. III/3, (Springer, Berlin), (1965).

[42] WANG, C-C. and TRUESDELL, C., Introduction to Rational Elasticity, (Noordhoff, Leyden), (1973).

[43] DUNN, J.E., Certain a priori inequalities and a peculiar elastic material. Q. Jl. Mech. Appl. Math., 36 (3), 351 (1983).

[44] TRUESDELL, C., General and exact theory of waves in finite elastic strain. Arch. Rational Mech. Anal., 8, 19 (1961).

THE INFLUENCE OF A PRIMARY STRESS UPON THE PROPAGATION OF SMALL-AMPLITUDE ELASTIC DISTURBANCES

by

PHILIP J. MYERS

ABSTRACT

This thesis considers three problems in the field of elastodynamics.

The first concerns small-amplitude elastic disturbances in an infinite cylinder, a problem first investigated by Pochhammer [1] and Chree [2]. Our approach extends the results of Pochhammer and Chree by utilising a method of successive approximation through which the governing equations are solved to produce dispersion relations.

The second investigation, recently considered by Eringen and Suhubi [3], is of the propagation of elastic waves in a pre-stressed body, with particular reference to the circular cylinder and the half-space. The governing equations are again solved via successive approximation to give new and detailed results describing the wave motion.

The final investigation is of a compressible strain-energy function which is an extension of the Ko model. The model is examined in the light of various a priori inequalities, and is then used to obtain solutions to the problem of vibrations in a stressed plate.

[1] POCHHAMMER, L., J. Reine Angew. Math., **81**, (1876).
[2] CHREE, C., Trans. Camb. Phil. Soc., **14**, (1889).
[3] ERINGEN, A.C. and SUHUBI, E.S., Elastodynamics, Vol. I Finite Motions, (Academic Press, New York), (1974).