Cubature H_∞ Information Filter and its Extensions

Kumar Pakki Bharani Chandra[a], Da-Wei Gu[b], Ian Postlethwaite[c]

[a]*College of Engineering, Mathematics and Physical Sciences, University of Exeter, EX4 4QF, UK*
(e-mail: bharanichandra@gmail.com, b.c.k.pakki@exeter.ac.uk).
[b]*Control Research Group, University of Leicester, LE1 7RH, UK*
(e-mail: dag@leicester.ac.uk).
[c]*Newcastle University, Newcastle upon Tyne, NE1 7RU, UK*
(e-mail: ian.postlethwaite@newcastle.ac.uk).

Abstract

State estimation for nonlinear systems with Gaussian or non-Gaussian noises, and with single and multiple sensors, is presented. The key purpose is to propose a derivative free estimator using concepts from the information filter, the H_∞ filter, and the cubature Kalman filter (CKF). The proposed estimator is called the cubature H_∞ information filter (CH_∞IF); it has the capability to deal with highly nonlinear systems like the CKF, like the H_∞ filter it can estimate states with stochastic or deterministic noises, and similar to the information filter it can be easily extended to handle measurements from multiple sensors. A numerically stable square-root CH_∞IF is developed and extended to multiple sensors. The CH_∞IF is implemented to estimate the states of a nonlinear permanent magnet synchronous motor model. Comparisons are made with an extended H_∞ information filter.

Keywords: Nonlinear state estimation, Multi-sensor, Kalman filter, $H_\infty filter$, Information filter

1. Introduction

State estimation for nonlinear systems is an active area of research and is essential for many real-life applications. One of the most preferred estimators for nonlinear systems is the extended Kalman filter (EKF), which is an extended version of the classical Kalman filter [1, 2]. Other notable nonlinear state estimators include Gaussian-mixture filters [3], Quadrature filters [4], Gaussian-Hermite filters [5], Fourier-Hermite filters [6], sliding mode observers [7], central difference filters [8], particle filters [9, 10, 11], unscented Kalman filters (UKFs) [12, 13] and cubature Kalman filters (CKFs) [14].

The EKF formulation is based on the first order Taylor's series approximation of nonlinear state and measurement models (Jacobians), and may not be suitable for *highly* nonlinear systems. For some models, such as piecewise continuous nonlinear systems [6], where it is difficult to obtain the Jacobians, derivative filters like EKF should be avoided. Furthermore, apriori statistical knowledge of process and sensor noise is required for EKF. Deterministic sigma-point filters like UKFs and CKFs, "or particle filters" can be used to estimate the states of the nonlinear system without evaluating the Jacobians. But UKF and CKF have limited capability to deal with non-Gaussian noises. Particle filters can handle non-Gaussian noises, but their performance is dictated by the number of stochastically selected samples or particles. For better accuracy more particles are required and hence they are computationally expensive filters. More recently, H_∞ filters and their variants [15, 16, 17, 18, 19, 20, 21, 22, 23, 24] have been investigated and utilised to deal with non-Gaussian noises. An extended H_∞ filter ($EH_\infty F$) can be used for nonlinear systems with non-Gaussian noise, but they need Jacobians. The CKF and H_∞ filters are combined to handle nonlinear systems with unknown noise statistics [25]; however this estimator cannot directly deal with measurements from multiple sensors. In several real life applications, where the measurements come from different sets of sensors, Kalman filters are seldom used. Alternatively, an algebraic equivalent form of Kalman filter, an information filter is preferred over the standard Kalman filter due to its simpler update stage. For nonlinear state estimation with multiple sensors, an extended information filter (EIF) can be used [26]. However, the EIF is not a derivative free filter and requires Jacobians during the prediction and update stages and hence is not preferred for highly nonlinear systems; and they have limited capability to handle non-Gaussian noise. A few derivative free information filters like unscented information filters [27], cubature information filters [28, 29], etc. have been recently proposed for nonlinear systems with Gaussian noises. H_∞ filters in the information domain have been extended to nonlinear systems, but many of these EH_∞Fs are not suitable for nonlinear systems where the nonlinearity is severe; this is due to the fact that EH_∞Fs are Jacobian based filters. In [30], we presented an earlier version of this work consisting of basic cubature H_∞ information filter. In this paper, we present the cubature H_∞

information filter (CH_∞IF) and its extensions, which have the capability to estimate the states of highly nonlinear systems in the presence of Gaussian or non-Gaussian noises, and can handle measurements from multiple sensors.

The paper is structured as follows. Filtering preliminaries are given in Section 2; the CH_∞IF is derived in Section 3; the square-root extension of the CH_∞IF is presented in Section 4; the applicability of multi-sensor CH_∞IF for state estimation of a permanent magnet synchronous motor is described in Section 5; and conclusions are given in Section 6.

2. Filtering Preliminaries

This section presents the most relevant filtering approaches required for development of the CH_∞IF. The key focus will be on the EH_∞F, the EIF and the CKF. Note that these filters will only be briefly discussed here; for more details see for example [2] and [26] for EH_∞Fs and EIFs, respectively, and [14] for the CKF.

2.1. Extended H_∞ Filter

In the last two decades there has been an increasing interest in robust filters using H_∞ theory and several authors have come up with different forms of so called H_∞ filters [15, 16, 17, 31]. In this section a game theory based H_∞ filter will be discussed which is mainly based on [2, 16, 32].
The nonlinear discrete plant and measurement models are given by

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) + \mathbf{w}_{k-1} \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k \tag{2}$$

where the state vector, control input and the measured outputs are denoted by $\mathbf{x}_k$, $\mathbf{u}_k$ and $\mathbf{z}_k$, respectively. The functions $\mathbf{f}$ and $\mathbf{h}$ are the nonlinear functions of states and control inputs. The plant and measurement noises are represented by $\mathbf{w}_{k-1}$ and $\mathbf{v}_k$. In most of the Kalman filtering approaches these noises are assumed as Gaussian and have zero-mean, whereas in the H_∞ filtering approaches they are not assumed to follow any particular probability distribution. In this section, it will be assumed that $\mathbf{w}_k$ and $\mathbf{v}_k$ can vary randomly or they can be deterministic, and they can also have non-zero mean.
The cost function for the H_∞ filter is of the form [2, 16],

$$\mathbf{J}_\infty = \frac{\sum_{k=0}^{N-1} ||\mathbf{x}_k - \hat{\mathbf{x}}_k||^2_{\mathbf{L}_k}}{||\mathbf{x}_0 - \hat{\mathbf{x}}_0||^2_{\mathbf{P}_0^{-1}} + \sum_{k=0}^{N}(||\mathbf{w}_k||^2_{\mathbf{Q}_k^{-1}} + ||\mathbf{v}_k||^2_{\mathbf{R}_k^{-1}})} \tag{3}$$

where the weighting matrices $\mathbf{P}_0$, $\mathbf{Q}_k$, $\mathbf{R}_k$, and $\mathbf{L}_k$ are symmetric positive definite weighing matrices chosen by the user based on the problem at hand. Note that this cost function is slightly different from the one in [2, 16]. The numerator of (3) is the norm of the state estimation errors, however if one has to estimate the linear combination of states then the numerator of (3) has to be an error norm of a linear combination of states as given in [2, 16]; which can however be absorbed by $\mathbf{L}_k$ in (3).

In the worst case noise and the initial conditions, the aim of the H_∞ filter is minimize the state estimation error in such a way that the performance measure J_∞ is bounded as

$$\sup \mathbf{J}_\infty < \gamma^2 \tag{4}$$

where 'sup' means supremum and the attenuation parameter $\gamma > 0$.

Several solutions to this H_∞ problem are available in [16], [17], [2], etc. However, in this paper the solution given in [32] will be used, as the H_∞ filter structure in [32] closely resembles with Kalman filter. For nonlinear systems, an EH_∞F can be used where the nonlinear functions are replaced by the Jacobians. Like EKF, an EH_∞F can be expressed in two stages (prediction and update) [32].

Prediction stage in the EH_∞F:

The predicted state and predicted auxiliary matrix are:

$$\mathbf{x}_{k|k-1} = \mathbf{f}(\mathbf{x}_{k-1|k-1}, \mathbf{u}_{k-1}) \tag{5}$$
$$\mathbf{P}_{k|k-1} = \nabla\mathbf{f}\mathbf{P}_{k-1|k-1}\nabla\mathbf{f}^T + \mathbf{Q}_k, \tag{6}$$

where $\nabla\mathbf{f}$ is the Jacobian of $\mathbf{f}$ evaluated at $\mathbf{x}_{k-1|k-1}$.

Update stage in the EH_∞F:

The updated state and updated auxiliary matrix are:

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_\infty[\mathbf{z}_k - \mathbf{h}(\mathbf{x}_{k|k-1}, \mathbf{u}_k)] \tag{7}$$
$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \nabla\mathbf{h}^T\mathbf{R}_k^{-1}\nabla\mathbf{h} - \gamma^{-2}\mathbb{I}_n \tag{8}$$

where $\mathbb{I}_n$ is the n^{th} order identity matrix, $\nabla\mathbf{h}$ is the Jacobian of $\mathbf{h}$ evaluated at $\mathbf{x}_{k|k-1}$, and

$$\mathbf{K}_\infty = \mathbf{P}_{k|k-1}\nabla\mathbf{h}^T[\nabla\mathbf{h}\mathbf{P}_{k|k-1}\nabla\mathbf{h}^T + \mathbf{R}_k]^{-1}. \tag{9}$$

2.2. Extended Information Filter

The Kalman filter propagates the state and covariance matrix at various stages. However, in some applications like multi-sensor state estimation, an information filter (an alternate form of Kalman filter), is preferred due to its simpler update stage to fuse the measurements from multiple sensors [26]. For nonlinear systems, an EIF can be used [26] where the information state vector and the information matrix are propagated rather than state vector and covariance matrix. The information matrix is the inverse of the covariance matrix, and the information vector is the product of information matrix and state vector.

Consider the discrete nonlinear process and measurement dynamics in (1) and (2). Unlike the H_∞ filter formulation, in EIF formulation the process and measurement noises, $\mathbf{w}_{k-1}$ and $\mathbf{v}_k$, are assumed as Gaussian, and their corresponding covariance matrices are $\mathbf{Q}_{k-1}$ and $\mathbf{R}_k$. The prediction and update stages of the EIF are given below.

Prediction stage in the EIF:

The predicted information matrix and predicted information vector are

$$\mathcal{I}_{k|k-1} = \mathbf{P}_{k|k-1}^{-1} = \left[\nabla\mathbf{f}\mathcal{I}_{k-1|k-1}^{-1}\nabla\mathbf{f}^T + \mathbf{Q}_{k-1}\right]^{-1} \tag{10}$$
$$\mathfrak{s}_{k|k-1} = \mathcal{I}_{k|k-1}\mathbf{x}_{k|k-1} \tag{11}$$

where

$$\mathbf{x}_{k|k-1} = \mathbf{f}(\mathbf{x}_{k-1|k-1}, \mathbf{u}_{k-1}).$$

Update stage in the EIF:
The updated information vector and updated information matrix are

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \mathbf{i}_k \tag{12}$$
$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \mathbf{I}_k. \tag{13}$$

where

$$\mathbf{i}_k = \nabla\mathbf{h}^T\mathbf{R}_k^{-1}\left[\nu_k + \nabla\mathbf{h}\mathbf{x}_{k|k-1}\right] \tag{14}$$
$$\mathbf{I}_k = \nabla\mathbf{h}^T\mathbf{R}_k^{-1}\nabla\mathbf{h} \tag{15}$$

and

$$\nu_k = \mathbf{z}_k - \mathbf{h}(\mathbf{x}_{k|k-1}, \mathbf{u}_k). \tag{16}$$

The state vector and covariance matrix at various stages can easily be recovered from the information vector and information matrix [33]

$$\mathbf{x}_{k|k} = \mathcal{I}_{k|k}\backslash\mathfrak{s}_{k|k} \tag{17}$$
$$\mathbf{P}_{k|k} = \mathcal{I}_{k|k}\backslash\mathbb{I}_n \tag{18}$$

where $\mathbb{I}_n$ is the n^{th} order identity matrix and '\' is a left-divide operator.

2.3. Cubature Kalman Filter

The CKF based on the cubature rule is a promising tool to estimate the states of nonlinear systems with Gaussian noises [14]. Its efficacy has been demonstrated on various applications and is found to be one of the best methods for state estimation of nonlinear systems with Gaussian noise. It has an improved accuracy over the EKF and the UKF.

Consider the discrete process and measurement models in (1) and (2) where the noises $\mathbf{w}_{k-1}$ and $\mathbf{v}_k$ are assumed to be Gaussian and their corresponding covariances are $\mathbf{Q}_{k-1}$ and $\mathbf{R}_k$. The prediction and update stages for the CKF are given below.

Prediction stage in the CKF:

The predicted state and the predicted covariance matrix are

$$\mathbf{x}_{k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\chi^{*}_{i,k|k-1} \tag{19}$$

$$\mathbf{P}_{k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\chi^{*}_{i,k|k-1}\chi^{*T}_{i,k|k-1} - \mathbf{x}_{k|k-1}\mathbf{x}^{T}_{k|k-1} + \mathbf{Q}_{k-1} \tag{20}$$

where

$$\chi^{*}_{i,k|k-1} = \mathbf{f}(\chi_{i,k-1|k-1}, \mathbf{u}_{k-1}), \tag{21}$$

$$\chi_{i,k-1|k-1} = \sqrt{\mathbf{P}_{k-1|k-1}}\xi_i + \mathbf{x}_{k-1|k-1}, \tag{22}$$

and ξ_i is the $i-th$ element of the following set

$$\sqrt{n}\left[\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\ldots,\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix},\begin{pmatrix}-1\\0\\\vdots\\0\end{pmatrix},\ldots,\begin{pmatrix}0\\\vdots\\0\\-1\end{pmatrix}\right]. \tag{23}$$

Update stage in the CKF:

The updated state and the updated covariance matrix are

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{z}_{k|k-1}) \tag{24}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{P}_{zz,k|k-1}\mathbf{K}^{T}_{k} \tag{25}$$

where

$$\mathbf{K}_k = \mathbf{P}_{xz,k|k-1}\mathbf{P}^{-1}_{zz,k|k-1} \tag{26}$$

$$\mathbf{P}_{xz,k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\chi_{i,k|k-1}\mathbb{Z}^{T}_{i,k|k-1} - \mathbf{x}_{k|k-1}\mathbf{z}^{T}_{k|k-1} \tag{27}$$

$$\mathbf{P}_{zz,k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\mathbb{Z}_{i,k|k-1}\mathbb{Z}^{T}_{i,k|k-1} - \mathbf{z}_{k|k-1}\mathbf{z}^{T}_{k|k-1} + \mathbf{R}_k \tag{28}$$

and

$$\mathbf{z}_{k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\mathbb{Z}_{i,k|k-1} \tag{29}$$

$$\mathbb{Z}_{i,k|k-1} = \mathbf{h}(\chi_{i,k|k-1}, \mathbf{u}_k) \tag{30}$$

$$\chi_{i,k|k-1} = \sqrt{\mathbf{P}_{k|k-1}}\xi_i + \mathbf{x}_{k|k-1}. \tag{31}$$

3. Cubature H_∞ Information Filters

The state estimators discussed in Section 2 have their own merits and demerits. For nonlinear systems, an EH_∞F can estimate the states and has a capability to deal with Gaussian as well as non-Gaussian noises, but it needs the Jacobians evaluated at various stages which are first order approximations of nonlinear functions. The EIF can deal with multi-sensor state estimation with computationally efficient update stage, but it can only deal with Gaussian noise and similar to the EH_∞F, the EIF also needs Jacobians at various stages. The CKF is a derivative free and numerically stable state estimator for nonlinear systems; however, it assumes that the process and measurement noises are Gaussian. In this section, a new filter called the 'cubature H_∞ information filter' (CH_∞IF) will be developed and will be extended to multiple sensors. The CH_∞IF will have the advantages of the three filters discussed in Section 2, such as it is derivative free for nonlinear systems, can handle Gaussian and non-Gaussian noises, and can easily be extended to multi-sensor state estimation.

Consider the discrete process and measurement models given in (1) and (2). In this section, the noises are assumed to be of generic nature and will have the same properties of the EH_∞F noises discussed in Section 2.1, such that they can be stochastic or deterministic and can have non-zero mean, etc. The key idea of the CH_∞IF is to use the prediction step from the CKF in information form, and to obtain the update step by fusing the EH_∞F, the EIF and the CKF. First the extended H_∞ information filter (EH_∞IF) will be derived and then the derivative free CH_∞IF will be developed.

3.1. Extended H_∞ Information Filter

Initialise the information vector and information matrix, $\mathcal{I}_{0|0}$ and $\mathfrak{s}_{0|0}$, for $k = 1$.

Prediction stage in the EH_∞IF:

The predicted information matrix (inverse of the predicted covariance matrix) and the corresponding information vector are

$$\mathcal{I}_{k|k-1} = \left[\nabla\mathbf{f}\mathcal{I}^{-1}_{k-1|k-1}\nabla\mathbf{f}^T + \mathbf{Q}_{k-1}\right]^{-1} \tag{32}$$

$$\mathfrak{s}_{k|k-1} = \mathcal{I}_{k|k-1}\mathbf{x}_{k|k-1} \tag{33}$$

where

$$\mathbf{x}_{k|k-1} = \mathbf{f}(\mathcal{I}_{k|k-1_{k-1|k-1}}\backslash\mathfrak{s}_{k-1|k-1}, \mathbf{u}_{k-1}).$$

Note that the prediction step for the EH_∞IF is the same as that of the one given in the EIF in Section 2.2, except that $\mathbf{Q}_{k-1}$ and $\mathbf{R}_k$ are the weighting matrices rather than noise covariances. The update stage, where the sensor measurements are required, will be different from the traditional EH_∞F or EIF given in Sections 2.1 and 2.2.

Update stage in the EH_∞IF:

The updated information vector and information matrix are

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \mathbf{i}_k \tag{34}$$

$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \mathbf{I}_k. \tag{35}$$

where

$$\mathbf{i}_k = \nabla\mathbf{h}^T\mathbf{R}_k^{-1}\left[\nu_k + \nabla\mathbf{h}\mathbf{x}_{k|k-1}\right] \tag{36}$$

$$\mathbf{I}_k = \nabla\mathbf{h}^T\mathbf{R}_k^{-1}\nabla\mathbf{h} - \gamma^{-2}\mathbb{I}_n \tag{37}$$

and

$$\nu_k = \mathbf{z}_k - \mathbf{h}(\mathbf{x}_{k|k-1}, \mathbf{u}_k). \tag{38}$$

Note that the main difference between the update stages of the EIF and the EH_∞IF is the information matrix contribution $\mathbf{I}_k$.

For multi-sensor state estimation, the measurement comes from different sensors and can be fused to estimate the state vector efficiently in the following way

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \sum_{j=1}^{D}\mathbf{i}_{j,k} \tag{39}$$

$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \sum_{j=1}^{D}\mathbf{I}_{j,k}, \tag{40}$$

where

$$\mathbf{i}_{j,k} = \nabla\mathbf{h}^T_{j,k|k-1}\mathbf{R}^{-1}_{j,k}[\nu_{j,k} + \nabla\mathbf{h}^T_{j,k|k-1}\mathbf{x}_{k|k-1}] \quad (41)$$

$$\mathbf{I}_{j,k} = \nabla\mathbf{h}^T_{j,k|k-1}\mathbf{R}^{-1}_{j,k}\nabla\mathbf{h}^T_{j,k|k-1} - \gamma^{-2}\mathbb{I}_n \quad (42)$$

$$(43)$$

and $\nabla\mathbf{h}^T_{j,k|k-1}$ represents the 'j^{th}' sensor Jacobian of $\mathbf{h}_{j,k|k-1}$, and $\mathbf{R}_{j,k}$ is the corresponding noise.

3.2. Cubature H_∞ Information Filter

This section presents the derivation of the CH_∞IF, and its extension to multi-sensor state estimation. The key idea is to fuse the CKF and the EH_∞IF to form a derivative free state estimator for nonlinear systems. The EH_∞IF can deal with multi-sensor for Gaussian and non-Gaussian noises, but it needs the Jacobians which can then compromise the accuracy of state estimation. The linear error propagation property will be used to derive the CH_∞IF. The derived filter will have the desired properties of the CKF and the EH_∞IF like derivative free estimation, the ability to handle Gaussian and non-Gaussian noises and multi-sensor state estimation. The prediction stage in the CH_∞IF will be the same as that of the CKF given in Section 2.3. However, it will propagate the information vector and the information matrix; for more details, see the prediction stage in Algorithm 1.

The updated information vector and information matrix are given by

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \mathbf{i}_k, \quad (44)$$

$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \mathbf{I}_k. \quad (45)$$

The update stage for the EH_∞IF given in Section 3.1 will be explored to develop the update stage for the CH_∞IF. Consider the linear error propagation property [25, 27, 28, 34, 35]

$$\mathbf{P}_{zz,k|k-1} \simeq \nabla\mathbf{h}\mathbf{P}_{k|k-1}\nabla\mathbf{h}^T, \quad (46)$$

$$\mathbf{P}_{xz,k|k-1} \simeq \mathbf{P}_{k|k-1}\nabla\mathbf{h}^T. \quad (47)$$

Pre-multiplying by $\mathbf{P}^{-1}_{k|k-1}$ on both sides of (47) yields

$$\begin{aligned} \nabla\mathbf{h}^T &= \mathbf{P}^{-1}_{k|k-1}\mathbf{P}_{xz,k|k-1}, \\ &= \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}. \end{aligned} \quad (48)$$

Using (48) in (36) and (37), we get

$$\mathbf{i}_k = \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}^{-1}_k[\nu_k + \mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1}\mathbf{x}_{k|k-1}], \quad (49)$$

$$\mathbf{I}_k = \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}^{-1}_k\mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1} - \gamma^{-2}\mathbb{I}_n. \quad (50)$$

The derivative free information matrix and cross error covariance matrix, $\mathcal{I}_{k|k-1} = \mathbf{P}^{-1}_{k|k-1}$ and $\mathbf{P}_{xz,k|k-1}$, can be obtained from (20) and (27). The information contributions in (49) and (50), along with (44) and (45) represents the update stage of the CH_∞IF.

One of the key advantages of using information filters are their ability to estimate efficiently the states with multiple sensors. There are several applications where multiple sensors are preferred to estimate the states such as the target tracking of a re-entry vehicle, where the aircraft states are estimated using radars located at different altitudes [27], etc. The update step described above for the CH_∞IF can easily be extended for multiple sensors. The structure of the update stage for the multi-sensor CH_∞IF will be the same as that of the EH_∞IF given in Section 3.1, but the information contribution factors $\mathbf{i}_{j,k}$ and $\mathbf{I}_{j,k}$ will be different, and they are derivative free. The updated information vector and the corresponding information matrix for CH_∞IF with multiple sensors are

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \sum_{j=1}^{D}\mathbf{i}_{j,k} \quad (51)$$

$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \sum_{j=1}^{D}\mathbf{I}_{j,k}, \quad (52)$$

where

$$\mathbf{i}_{j,k} = \mathcal{I}_{j,k|k-1}\mathbf{P}_{xz,j,k|k-1}\mathbf{R}^{-1}_{j,k}[\nu_{j,k} + \mathbf{P}^T_{xz,j,k|k-1}\mathcal{I}^T_{j,k|k-1}\mathbf{x}_{k|k-1}] \quad (53)$$

$$\mathbf{I}_{j,k} = \mathcal{I}_{j,k|k-1}\mathbf{P}_{xz,j,k|k-1}\mathbf{R}^{-1}_{j,k}\mathbf{P}^T_{xz,j,k|k-1}\mathcal{I}^T_{j,k|k-1} - \gamma^{-2}\mathbb{I}_n \quad (54)$$

The CH_∞IF and its extension to multiple sensors are summarised in Algorithm 1.

Algorithm 1 Cubature H_∞ Information Filter

Initialise the information vector and information matrix, $\mathcal{I}_{0|0}$ and $\mathfrak{s}_{0|0}$ for $k = 1$.

Prediction

1: Evaluate $\chi_{i,k-1|k-1}, \chi^*_{i,k|k-1}$ and $\mathbf{x}_{k|k-1}$ as

$$\begin{aligned}\chi_{i,k-1|k-1} &= \sqrt{\mathcal{I}_{k-1|k-1}\backslash\mathbb{I}_n}\xi_i + (\mathcal{I}_{k-1|k-1}\backslash\mathfrak{s}_{k-1|k-1}),\\ \chi^*_{i,k|k-1} &= \mathbf{f}(\chi_{i,k-1|k-1}, \mathbf{u}_{k-1})\\ \mathbf{x}_{k|k-1} &= \frac{1}{2n}\sum_{i=1}^{2n}\chi^*_{i,k|k-1}\end{aligned}$$

where '\' is left-divide operator, $\mathbb{I}_n$ is the n^{th} order identity matrix, and ξ_i is given in (23).

2: The predicted information matrix and information vector are:

$$\begin{aligned}\mathcal{I}_{k|k-1} &= \left[\frac{1}{2n}\sum_{i=1}^{2n}\chi^*_{i,k|k-1}\chi^{*T}_{i,k|k-1} - \mathbf{x}_{k|k-1}\mathbf{x}^T_{k|k-1} + \mathbf{Q}_{k-1}\right]\Big\backslash\mathbb{I}_n\\ \mathfrak{s}_{k|k-1} &= \frac{\mathcal{I}_{k|k-1}}{2n}\left[\sum_{i=1}^{2n}\chi^*_{i,k-1|k-1}\right].\end{aligned}$$

Measurement Update

1: Evaluate $\chi_{i,k|k-1}$, $\mathbb{Z}_{i,k|k-1}$ and $\mathbf{z}_{k|k-1}$ as

$$\begin{aligned}\chi_{i,k|k-1} &= \sqrt{\mathcal{I}_{k|k-1}\backslash\mathbb{I}_n}\xi_i + \mathbf{x}_{k|k-1}\\ \mathbb{Z}_{i,k|k-1} &= \mathbf{h}(\chi_{i,k|k-1}, \mathbf{u}_k)\\ \mathbf{z}_{k|k-1} &= \frac{1}{2n}\sum_{i=1}^{2n}\mathbb{Z}_{i,k|k-1}\end{aligned}$$

2: Evaluate

$$\mathbf{P}_{xz,k|k-1} = \frac{1}{2n}\sum_{i=1}^{2n}\chi_{i,k|k-1}\mathbb{Z}^T_{i,k|k-1} - \mathbf{x}_{k|k-1}\mathbf{z}^T_{k|k-1}$$

3: The information contributions are

$$\begin{aligned}\mathbf{i}_k &= \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}[\nu_k + \mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1}\mathbf{x}_{k|k-1}]\\ \mathbf{I}_k &= \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1} - \gamma^{-2}\mathbb{I}_n,\end{aligned}$$

where γ is the attenuation parameter.

4: Finally, the update information vector and information matrix are

$$\begin{aligned}\mathfrak{s}_{k|k} &= \mathfrak{s}_{k|k-1} + \mathbf{i}_k,\\ \mathcal{I}_{k|k} &= \mathcal{I}_{k|k-1} + \mathbf{I}_k\end{aligned}$$

Measurement Update for Multi-sensor State estimation

$$\begin{aligned}\mathfrak{s}_{k|k} &= \mathfrak{s}_{k|k-1} + \sum_{j=1}^{D}\mathbf{i}_{j,k}\\ \mathcal{I}_{k|k} &= \mathcal{I}_{k|k-1} + \sum_{j=1}^{D}\mathbf{I}_{j,k},\end{aligned}$$

where

$$\begin{aligned}\mathbf{I}_{j,k} &= \mathcal{I}_{j,k|k-1}\mathbf{P}_{xz,j,k|k-1}\mathbf{R}^{-1}_{j,k}\mathbf{P}^T_{xz,j,k|k-1}\mathcal{I}^T_{j,k|k-1} - \gamma^{-2}\mathbb{I}_n\\ \mathbf{i}_{j,k} &= \mathcal{I}_{j,k|k-1}\mathbf{P}_{xz,j,k|k-1}\mathbf{R}^{-1}_{j,k}[\nu_{j,k} + \mathbf{P}^T_{xz,j,k|k-1}\mathcal{I}^T_{j,k|k-1}\mathbf{x}_{k|k-1}].\end{aligned}$$

4. Square-root Cubature H_∞ Information Filter

For implementation purposes the square-root version of the filters are preferred due to their enhanced numerical stability [2, 4, 20, 28, 25, 36, 37] and have been implemented since the Apollo mission [38]. In this section, a numerically stable square-root version of the CH_∞IF will be developed. Square-root filters arise when covariance or information matrices are replaced by their square-root factors; these square-root factors are then propagated at various stages. Square-root factors for the information matrix and other matrices are defined such that

$$
\begin{aligned}
\mathcal{I} &= \mathcal{I}_s \mathcal{I}_s^T && (55)\\
\mathbf{P}_{xz} &= \mathbf{P}_{xz,s}\mathbf{P}_{xz,s}^T && (56)\\
\mathbf{R}^{-1} &= \mathbf{R}_{si}\mathbf{R}_{si}^T && (57)\\
\mathbf{Q} &= \mathbf{Q}_s\mathbf{Q}_s^T && (58)
\end{aligned}
$$

where $\mathcal{I}_s = \mathbf{P}^{\frac{-T}{2}}$, $\mathbf{P}_{xz,s} = \mathbf{P}_{xz,s}^{\frac{1}{2}}$, $\mathbf{R}_{si} = R^{\frac{-T}{2}}$, and $\mathbf{Q}_s = \mathbf{Q}^{\frac{1}{2}}$. Note that these square-root factors are not unique and can be calculated using different numerical techniques such as the QR decomposition, etc. [36, 38, 39]. The square-root CH_∞IF can also be written in the prediction and update stages. The prediction stage in the square-root CH_∞IF (SR-CH_∞IF) is the same as that of the square-root CKF [14], but the information vector and the square-root of the information matrix are propagated rather than the state and covariance matrix.

The prediction stage of SR-CH_∞IF has straight forward equations, please see the prediction stage of Algorithm 2 for more details. However, the update stage is not straight forward and the hence the detailed derivation is given in the below Theorem 1. The update stage for the square-root CH_∞IF using J-orthogonal transformation is given in Theorem 1.

Theorem 1. *The updated information vector and information matrix for the SR-CH_∞IF can be obtained from*

$$
\begin{bmatrix} \mathcal{I}_{s,k|k-1}\boldsymbol{P}_{zz,s,k|k-1}\boldsymbol{R}_{si,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1}\mathbb{I}_n \\ \mathbf{z}_{a,k}^T\boldsymbol{R}_{si,k} & \mathfrak{s}_{s,k|k-1}^T & \boldsymbol{0} \end{bmatrix}\Theta_J = \begin{bmatrix} \mathcal{I}_{s,k|k} & \boldsymbol{0} & \boldsymbol{0} \\ \mathfrak{s}_{s,k|k}^T & \boldsymbol{0} & \star \end{bmatrix}, \tag{59}
$$

where $\star$ represent the terms which are irrelevant for the SR-CH_∞IF, $\mathbf{z}_{a,k} = \nu_k + \boldsymbol{P}_{zz,s,k|k-1}^T\mathcal{I}_{s,k|k-1}^T\mathbf{x}_{k|k-1}$ and Θ_J is a J-unitary matrix which satisfies

$$
\Theta_J\Theta_J^T = J \quad \textit{and} \quad J = \begin{bmatrix} \mathbb{I}_n & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathbb{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\mathbb{I}_n \end{bmatrix}.
$$

Proof. Squaring the left hand side (LHS) of (59) yields

$$
\begin{bmatrix} \mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,s,k|k-1}\mathbf{R}_{si,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1}\mathbb{I}_n \\ \mathbf{z}_{a,k}^T\mathbf{R}_{si,k} & \mathfrak{s}_{s,k|k-1}^T & \mathbf{0} \end{bmatrix}\Theta_J\Theta_J^T \begin{bmatrix} \mathbf{R}_{si,k}^T\mathbf{P}_{zz,s,k|k-1}^T\mathcal{I}_{s,k|k-1}^T & \mathbf{R}_{si,k}^T\mathbf{z}_{a,k} \\ \mathcal{I}_{s,k|k-1}^T & \mathfrak{s}_{s,k|k-1} \\ \gamma^{-1}\mathbb{I}_n & \mathbf{0} \end{bmatrix}, \tag{60}
$$

Further, (60) can be written as

$$
\begin{aligned}
&\begin{bmatrix} \mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,s,k|k-1}\mathbf{R}_{si,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1}\mathbb{I}_n \\ \mathbf{z}_{a,k}^T\mathbf{R}_{si,k} & \mathfrak{s}_{s,k|k-1}^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbb{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbb{I}_n \end{bmatrix}\begin{bmatrix} \mathbf{R}_{si,k}^T\mathbf{P}_{zz,s,k|k-1}^T\mathcal{I}_{s,k|k-1}^T & \mathbf{R}_{si,k}^T\mathbf{z}_{a,k} \\ \mathcal{I}_{s,k|k-1}^T & \mathfrak{s}_{s,k|k-1} \\ \gamma^{-1}\mathbb{I}_n & \mathbf{0} \end{bmatrix}, \\
= &\begin{bmatrix} (\mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,s,k|k-1}\mathbf{R}_k^{-1}\mathbf{P}_{zz,s,k|k-1}^T\mathcal{I}_{s,k|k-1}^T + \mathcal{I}_{k|k-1} - \gamma^{-2}\mathbb{I}_n) & (\mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,s,k|k-1}\mathbf{R}_k^{-1}\mathbf{z}_{a,k} + \mathcal{I}_{s,k|k-1}\mathfrak{s}_{s,k|k-1}) \\ \star & \star \end{bmatrix}
\end{aligned} \tag{61}
$$

The covariance matrix in (46) can be factorised as

$$
\begin{aligned}
\mathbf{P}_{zz,k|k-1} &= \nabla\mathbf{h}\mathbf{P}_{k|k-1}\nabla\mathbf{h}^T \\
&= \nabla\mathbf{h}\mathbf{P}_{k|k-1}^{\frac{1}{2}}\mathbf{P}_{k|k-1}^{\frac{T}{2}}\nabla\mathbf{h}^T \\
&= (\nabla\mathbf{h}\mathbf{P}_{s,k|k-1})(\nabla\mathbf{h}\mathbf{P}_{s,k|k-1})^T \\
&= \mathbf{P}_{zz,s,k|k-1}\mathbf{P}_{zz,s,k|k-1}^T.
\end{aligned} \tag{62}
$$

The $\mathbf{P}_{xz,k|k-1}$ in (47) can be represented in terms of $\mathbf{P}_{zz,s,k|k-1}$ and $\mathbf{P}_{s,k|k-1}$ as

$$\begin{aligned} \mathbf{P}_{xz,k|k-1} &= \mathbf{P}_{k|k-1}\nabla\mathbf{h}^T \\ &= \mathbf{P}^{\frac{1}{2}}_{k|k-1}(\nabla\mathbf{h}\mathbf{P}^{\frac{1}{2}}_{k|k-1})^T \\ &= \mathbf{P}_{s,k|k-1}\mathbf{P}^T_{zz,s,k|k-1} \end{aligned} \tag{63}$$

Pre-multiplying the information matrix on both sides of the aforementioned equation yields

$$\begin{aligned} \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1} &= \mathcal{I}_{k|k-1}\mathbf{P}_{s,k|k-1}\mathbf{P}^T_{zz,s,k|k-1} \\ &= \mathcal{I}_{s,k|k-1}\mathcal{I}^T_{s,k|k-1}\mathbf{P}_{s,k|k-1}\mathbf{P}^T_{zz,s,k|k-1} \\ &= \mathcal{I}_{s,k|k-1}\mathbf{P}^T_{zz,s,k|k-1} \end{aligned} \tag{64}$$

Substituting (64) in (61) yields

$$\begin{aligned} &\begin{bmatrix} (\mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1} + \mathcal{I}_{k|k-1} - \gamma^{-2}\mathbb{I}_n) & (\mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{z}_{a,k} + \mathcal{I}_{s,k|k-1}\mathfrak{s}_{s,k|k-1}) \\ \star & \star \end{bmatrix} \\ = &\begin{bmatrix} (\mathcal{I}_{k|k-1} + \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1} - \gamma^{-2}\mathbb{I}_n) & (\mathfrak{s}_{k|k-1} + \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{z}_{a,k}) \\ \star & \star \end{bmatrix}. \end{aligned} \tag{65}$$

Now by squaring the right hand side (RHS) of the update stage in (59), we get

$$\begin{aligned} &\begin{bmatrix} \mathcal{I}_{s,k|k} & \mathbf{0} & \mathbf{0} \\ \mathfrak{s}^T_{s,k|k} & \mathbf{0} & \star \end{bmatrix} \begin{bmatrix} \mathcal{I}^T_{s,k|k} & \mathfrak{s}_{s,k|k} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{bmatrix} \\ = &\begin{bmatrix} \mathcal{I}_{k|k}\mathcal{I}^T_{s,k|k} & \mathcal{I}_{s,k|k}\mathfrak{s}_{s,k|k} \\ \star & \star \end{bmatrix} \\ = &\begin{bmatrix} \mathcal{I}_{s,k|k} & \mathfrak{s}_{k|k} \\ \star & \star \end{bmatrix}. \end{aligned} \tag{66}$$

By equating the corresponding elements of (65) and (66), which corresponds to the LHS and RHS of (59), we get

$$\mathfrak{s}_{k|k} = \mathfrak{s}_{k|k-1} + \mathbf{i}_k \tag{67}$$

$$\mathcal{I}_{k|k} = \mathcal{I}_{k|k-1} + \mathbf{I}_k. \tag{68}$$

where

$$\mathbf{i}_k = \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}[\nu_k + \mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1}\mathbf{x}_{k|k-1}] \tag{69}$$

$$\mathbf{I}_k = \mathcal{I}_{k|k-1}\mathbf{P}_{xz,k|k-1}\mathbf{R}_k^{-1}\mathbf{P}^T_{xz,k|k-1}\mathcal{I}^T_{k|k-1} - \gamma^{-2}\mathbb{I}_n. \tag{70}$$

Note that the information vector and the information matrix given in (67) and (68), and the corresponding information contributions in (69) and (70) for the SR-CH_∞IF are the same as that of the CH_∞IF in (44), (45), (49) and (50). This proves that the update stage of the SR-CH_∞IF given in Theorem 1 is equivalent of the CH_∞IF in Section 3.2. □

The multi-sensor SR-CH_∞IF will be the same as that of the prediction stage for the single sensor SR-CH_∞IF (since the multiple sensors only affect the update stage). The updated information vector and information matrix of the multi-sensor SR-CH_∞IF can be obtained from

$$\begin{bmatrix} \mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,1,s,k|k-1}\mathbf{R}_{si,1,k} & \mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,2,s,k|k-1}\mathbf{R}_{si,2,k} & \dots & \mathcal{I}_{s,k|k-1}\mathbf{P}_{zz,D,s,k|k-1}\mathbf{R}_{si,D,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1}\mathbb{I} \\ \mathbf{z}_{a,1,k} & \mathbf{z}_{a,2,k} & \dots & \mathbf{z}_{a,D,k} & \mathfrak{s}_{k|k-1} & \mathbf{0} \end{bmatrix}\Theta_J = \begin{bmatrix} \mathcal{I}_{s,k|k} & \mathbf{0} & \mathbf{0} \\ \mathfrak{s}_{s,k|k} & \mathbf{0} & \star \end{bmatrix} \tag{71}$$

The update stage for multi-sensor SR-CH_∞IF in (71) can easily be proved along the similar lines of the proof of Theorem 1. The SR-CH_∞IF for single and multi-sensors is given in Algorithm 2.

Apart from the SR-CH_∞IF's applicability for multi-sensor state estimation in the presence of Gaussian and non-Gaussian noises, the proposed numerically stable SR-CH_∞IF can handle ill-conditioned covariance matrices, has double order precision, suitable for efficient real-time implementation, and can easily estimate the states with near-perfect measurements.

Algorithm 2 Square-root Cubature H_∞ Information Filter

Initialise the information vector and square-root information matrix, $\mathcal{I}_{s,0|0}$ and $\mathfrak{s}_{s,0|0}$ for $k = 1$.

Prediction

1: Evaluate $\chi_{i,k-1|k-1}, \chi^*_{i,k|k-1}$ and $\mathbf{x}_{k|k-1}$ as

$$\begin{aligned}
\chi_{i,k-1|k-1} &= (\mathcal{I}_{s,k-1|k-1} \backslash \mathbb{I}_n)\, \xi_i + (\mathcal{I}_{s,k-1|k-1} \backslash \mathfrak{s}_{s,k-1|k-1}),\\
\chi^*_{i,k|k-1} &= \mathbf{f}(\chi_{i,k-1|k-1}, \mathbf{u}_{k-1})\\
\mathbf{x}_{k|k-1} &= \frac{1}{2n} \sum_{i=1}^{2n} \chi^*_{i,k|k-1}
\end{aligned}$$

where '$\backslash$' is left-divide operator, $\mathbb{I}_n$ is the n^{th} order identity matrix, and ξ_i is given in (23).

2: Evaluate the information matrix

$$\begin{aligned}
\mathcal{I}_{s,k|k-1} &= \left[\mathbf{qr}\left(\mathbf{X}^*_{i,k|k-1} \quad \mathbf{Q}_{s,k-1}\right)^T\right]^T \Big\backslash \mathbb{I}_n\\
\mathfrak{s}_{s,k|k-1} &= \mathcal{I}_{s,k|k-1} \mathbf{x}_{k|k-1}
\end{aligned}$$

where

$$\mathbf{X}^*_{i,k|k-1} = \frac{1}{\sqrt{2n}} \left[\chi^*_{1,k|k-1} - \hat{\mathbf{x}}_{k|k-1} \quad \chi^*_{2,k|k-1} - \hat{\mathbf{x}}_{k|k-1} \ldots \quad \chi^*_{2n,k|k-1} - \hat{\mathbf{x}}_{k|k-1}\right].$$

Measurement Update

1: Evaluate $\chi_{i,k|k-1}$, $\mathbb{Z}_{i,k|k-1}$ and $\mathbf{z}_{k|k-1}$ as

$$\begin{aligned}
\chi_{i,k|k-1} &= \mathcal{I}_{s,k|k-1} \backslash \mathbb{I}_n \xi_i + \mathbf{x}_{k|k-1}\\
\mathbb{Z}_{i,k|k-1} &= \mathbf{h}(\chi_{i,k|k-1}, \mathbf{u}_k)\\
\mathbf{z}_{k|k-1} &= \frac{1}{2n} \sum_{i=1}^{2n} \mathbb{Z}_{i,k|k-1}.
\end{aligned}$$

2: Evaluate

$$\mathbf{P}_{zz,s,k|k-1} = \left[\mathbf{qr}(\mathbf{Z}_{i,k|k-1} \quad \mathbf{R}_{si,k})^T\right]^T \Big\backslash$$

where

$$\mathbf{Z}_{i,k|k-1} = \frac{1}{\sqrt{2n}} \left[\mathbb{Z}_{1,k|k-1} - \mathbf{z}_{k|k-1} \quad \mathbb{Z}_{2,k|k-1} - \mathbf{z}_{k|k-1} \ldots \quad \mathbb{Z}_{2n,k|k-1} - \mathbf{z}_{k|k-1}\right].$$

3: For a single sensor, the square-root information matrix and the corresponding information vector can be obtained from

$$\begin{bmatrix} \mathcal{I}_{s,k|k-1} \mathbf{P}_{zz,s,k|k-1} \mathbf{R}_{si,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1} \mathbb{I}_n \\ \mathbf{z}^T_{a,k} \mathbf{R}_{si,k} & \mathfrak{s}^T_{s,k|k-1} & \mathbf{0} \end{bmatrix} \Theta_J = \begin{bmatrix} \mathcal{I}_{s,k|k} & \mathbf{0} & \mathbf{0} \\ \mathfrak{s}^T_{s,k|k} & \mathbf{0} & \star \end{bmatrix},$$

where γ is an attenuation parameter, Θ_J is a J-unitary matrix, $\mathbf{z}_{a,k} = \nu_k + \mathbf{P}^T_{zz,s,k|k-1} \mathcal{I}^T_{s,k|k-1} \mathbf{x}_{k|k-1}$, and $\star$ represent the terms which are irrelevant for the SR-CH_∞IF.

Measurement Update for Multi-sensor State Estimation

1: For multiple sensors, the square-root information matrix and the corresponding information vector can be obtained from

$$\begin{bmatrix} \mathcal{I}_{s,k|k-1} \mathbf{P}_{zz,1,s,k|k-1} \mathbf{R}_{si,1,k} & \mathcal{I}_{s,k|k-1} \mathbf{P}_{zz,2,s,k|k-1} \mathbf{R}_{si,2,k} & \ldots & \mathcal{I}_{s,k|k-1} \mathbf{P}_{zz,D,s,k|k-1} \mathbf{R}_{si,D,k} & \mathcal{I}_{s,k|k-1} & \gamma^{-1} \mathbb{I}_n \\ \mathbf{z}_{a,1,k} & \mathbf{z}_{a,2,k} & \ldots & \mathbf{z}_{a,D,k} & \mathfrak{s}_{k|k-1} & \mathbf{0} \end{bmatrix} \Theta_J = \begin{bmatrix} \mathcal{I}_{s,k|k} & \mathbf{0} & \mathbf{0} \\ \mathfrak{s}_{s,k|k} & \mathbf{0} & \star \end{bmatrix}$$

5. State Estimation of a Permanent Magnet Synchronous Motor

State estimation of a permanent magnet synchronous motor (PMSM) using the proposed multi-sensor SR-CH_∞IF is considered in this section. Two cases are considered; the first case deals with the state estimation of PMSM in the presence of Gaussian noise and the second one deals with non-Gaussian noise. Consider the state equations of a discrete nonlinear PMSM model [2, 28]

$$\begin{bmatrix} i_{1,k+1} \\ i_{2,k+1} \\ \omega_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} i_{1,k} + dt(-\frac{R}{L} i_{1,k} + \frac{\omega_k \lambda}{L} \sin\theta_k + \frac{1}{L} u_{1,k}) \\ i_{2,k} + dt(-\frac{R}{L} i_{2,k} - \frac{\omega_k \lambda}{L} \cos\theta_k + \frac{1}{L} u_{2,k}) \\ \omega_k + dt(-\frac{3\lambda}{2J} i_{1,k} \sin\theta_k + \frac{3\lambda}{2J} i_{2,k} \cos\theta_{4,k} - \frac{F\omega_k}{J}) \\ \theta_k + dt(\omega_k) \end{bmatrix}$$

where i_1 and i_2 are the winding currents, ω is the speed and θ is the rotor position of the PMSM. The inputs to the PMSM model are

$$\begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} = \begin{bmatrix} \sin(0.002\pi k) \\ \cos(0.002\pi k) \end{bmatrix}$$

It is assumed that the measured outputs of the PMSM model are currents

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} i_{1,k} \\ i_{2,k} \end{bmatrix}.$$

The PMSM parameters are chosen as [2]

Winding Inductance	L = 0.003 H
Winding resistance	R = 1.9 Ω
Moment of inertia	J = 0.00018 $kg - m^2$
Coefficient of viscous friction	B = 0.001 Nms
Motor constant	$\lambda = 0.1$

and the sampling time dt is 0.001 s. The key objective of the SR-CH_∞IF is to estimate the speed, ω, and the rotor position, θ, using the current measurements, i_1 and i_2, in the presence of Gaussian and non-Gaussian noises. In the next few sections various comparisons of the proposed SR-CH_∞IF with classical H_∞ filters are detailed. First a comparison and simulation results of the classical extended H_∞ filter and SR-CH_∞IF with a single-sensor measurement are given. Then simulation results for multi-sensor based estimation for low and high Gaussian noises are demonstrated. Finally, the simulations for various estimators in the presence of non-Gaussian noise and near perfect measurements are performed. Note that an attempt has been made to estimate the states using the square-root unscented H_∞ information filter with a set of tuning parameters as suggested in [13], $\alpha = 1 \times 10^{-3}$, $\beta = 2$ and $\kappa = 3 - n = -1$. In a few simulations, it was found that the state estimates of PMSM, using a square-root unscented H_∞ information filter, diverges and hence their responses are not shown here. This divergence is possibly due to the unavailability of positive semi-definite matrices because of the negative κ. However, when the parameter κ is selected as 0, the unscented filters boil down to the cubature filters, but note that the selection of $\kappa = 0$ in unscented filters cannot be justified theoretically or mathematically [14]. This instability of the unscented filters is discussed in [14].

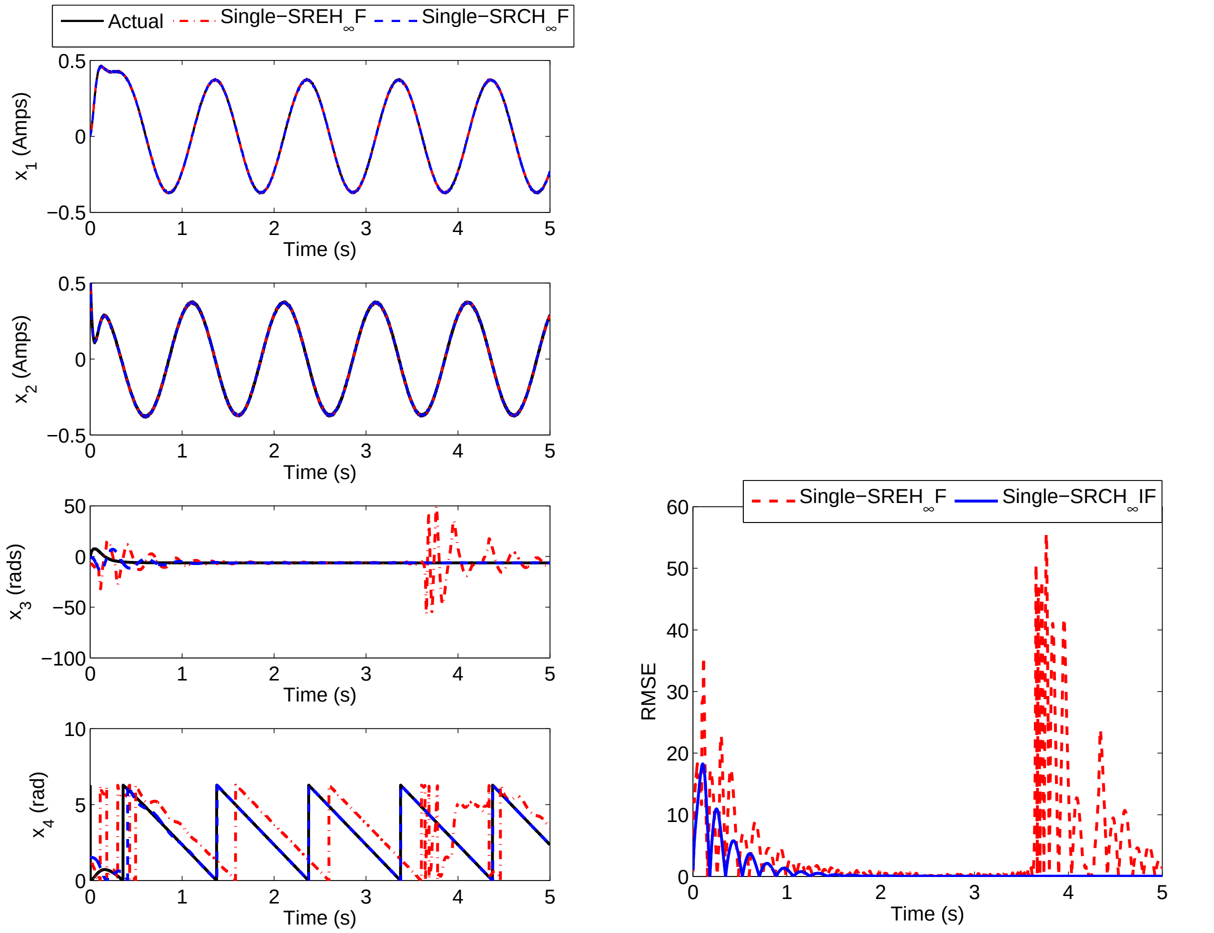
5.1. *PMSM State estimation in the presence of Gaussian noises*

In this section, it is assumed that the process and measurement noises are Gaussian. For all the simulation results given in this paper, the initial conditions for the actual PMSM states and the information vector are randomly selected from $\mathcal{N}([0.1 \quad 0.1 \quad 0.1 \quad 0.1]^T, 0.1 \times I_4)$.

5.1.1. *Single-sensor state estimation*

Two hundred Monte-Carlo simulations are performed using the single-sensor SR-EH_∞F and SR-CH_∞IF. Note that in this section the comparison has been made with the classical square-root extended H_∞ filter and the proposed SR-CH_∞IF. The process and measurement noise covariance matrices for single-sensor state estimation are chosen as

$$\mathbf{Q} = \begin{bmatrix} 6.25 & 0 & 0 & 0 \\ 0 & 6.25 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 \times 10^{-6} \end{bmatrix}, \quad \mathbf{R}_1 = \begin{bmatrix} 2.5 \times 10^{-6} & 0 \\ 0 & 2.5 \times 10^{-6} \end{bmatrix}.$$



(a) States and their estimates using single-sensor SR-EH_∞F and SR-CH_∞IF

(b) RMSE of speed using single-sensor SR-EH_∞F and SR-CH_∞IF

Figure 1: PMSM simulation results in the presence of Gaussian noise using single-sensor measurements

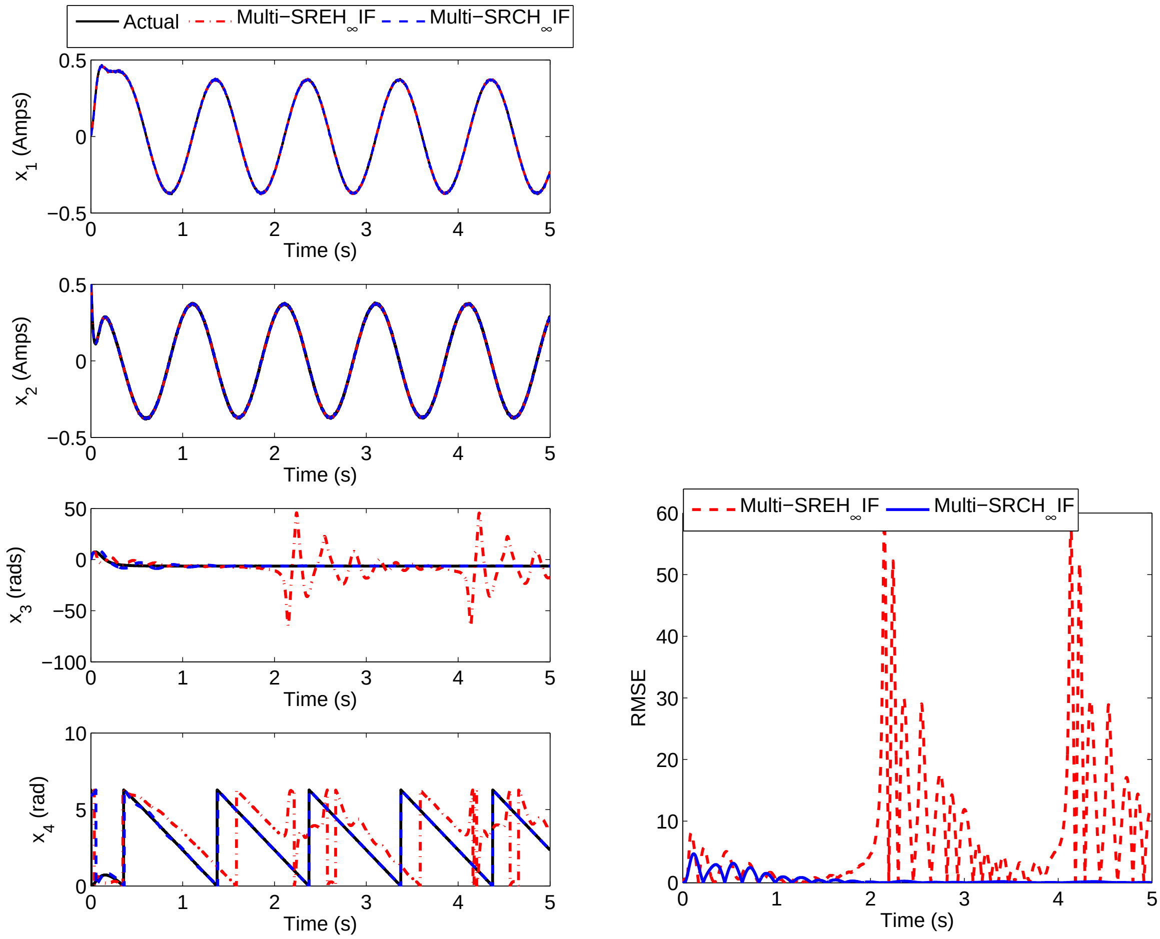
Simulation results of a typical run are shown in Figure 1. The actual states and their estimates using the SR-EH_∞IF and SR-CH_∞IF are shown in Figure 1a. The first two states, which are also the measured outputs, and their estimates using SR-EH_∞F and SR-CH_∞IF almost overlap. The estimated speed and rotor position using the SR-CH_∞IF are very close to the actual states, whereas SR-EH_∞IF has large estimation errors. The root mean square errors (RMSEs) for the speed using SR-EH_∞F and SR-CH_∞IF are shown in Figure 1b, where the SR-EH_∞IF RMSE is larger than the SR-CH_∞IF. The maximum RMSE error is 55.65 for the SR-EH_∞IF and 18.25 for the SR-CH_∞IF, and the average RMSE error is 10.19 for the SR-EH_∞IF and 3.02 for the SR-CH_∞IF.

5.1.2. Multi-sensor state estimation in the presence of low-Gaussian noise

For multi-sensor state estimation, two sets of current sensors are used to estimate the states. The first set is the same as that of the single-sensor based estimation given in Section 5.1.1 and for the second set the measurement noise covariance is chosen as

$$\mathbf{R}_2 = \begin{bmatrix} 5\times 10^{-6} & 0 \\ 0 & 5\times 10^{-6} \end{bmatrix}.$$

Simulation results of state estimation in the presence of low-Gaussian noise using the multi-sensor SR-EH_∞IF and SR-CH_∞IF are shown in Figure 2. The actual states and their estimates using the SR-EH_∞IF and SR-CH_∞IF are shown in Figure 2a. The estimates using the SR-EH_∞IF are more erroneous than the SR-CH_∞IF. The root means square errors (RMSEs) for the speed using SR-EH_∞IF and SR-CH_∞IF are shown in Figure 2b, where the SR-EH_∞IF RMSE is larger than the SR-CH_∞IF. The maximum RMSE error is 58.6 for the SR-EH_∞IF and 4.7 for the SR-CH_∞IF, and the average RMSE error is 12.89 for the SR-EH_∞IF and 1.21 for the SR-CH_∞IF.



(a) States and their estimates using SR-EH_∞IF and SR-CH_∞IF

(b) RMSE of Speed using SR-EH_∞IF and SR-CH_∞IF

Figure 2: PMSM simulation results in the presence of low Gaussian noise

5.1.3. Multi-sensor state estimation in the presence of high-Gaussian noise

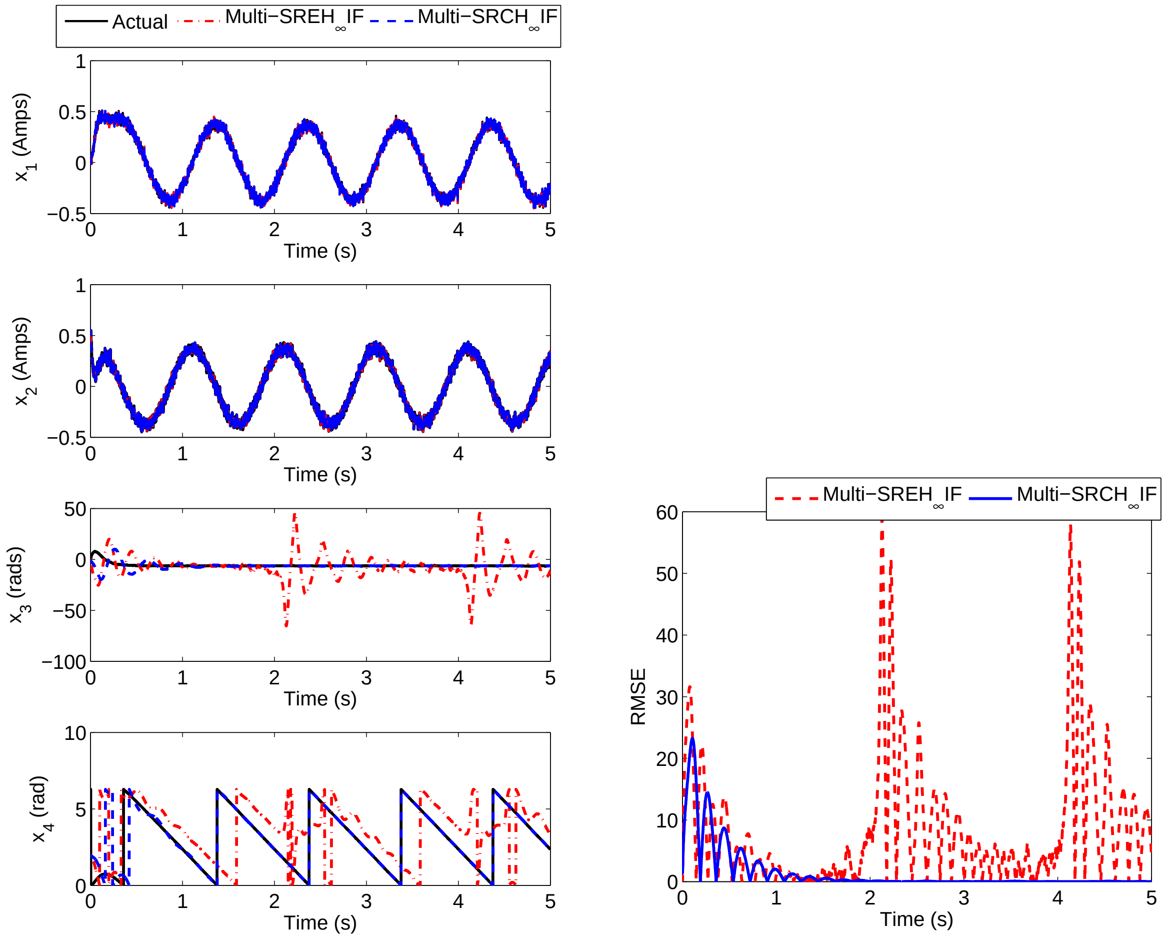
In this section, the simulations are performed with high-Gaussian noise. It is assumed that the sensors and plant model used in this subsection are more prone to noise such that their noise covariance matrices are assumed to be 12 times those given in Section 5.1.2. The process and measurement noise covariance matrices for high-Gaussian noise are chosen as:

$$\mathbf{Q} = \begin{bmatrix} 75 & 0 & 0 & 0 \\ 0 & 75 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.2\times10^{-5} \end{bmatrix}, \quad \mathbf{R}_1 = \begin{bmatrix} 3\times10^{-5} & 0 \\ 0 & 3\times10^{-5} \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} 6\times10^{-5} & 0 \\ 0 & 6\times10^{-5} \end{bmatrix}.$$

Simulation results of a typical simulation run are shown in Figure 3. The actual states and their estimates using the SR-EH_∞IF and SR-CH_∞IF are shown in Figure 3a. Due to the noisy current sensors, the first two states (currents of PMSM) in Figure 3a are noisy. The root means square errors (RMSEs) for the speed using SR-EH_∞IF and SR-CH_∞IF are shown in Figure 3b, where the SR-EH_∞IF RMSE is larger than the SR-CH_∞IF. The maximum RMSE error is 58.95 for the SR-EH_∞IF and 23.25 for the SR-CH_∞IF, and the average RMSE error is 13.59 for the SR-EH_∞IF and 4.09 for the SR-CH_∞IF.

5.2. PMSM Multi-sensor State estimation in the presence of Non-Gaussian noises

In some of the control applications, the process and measurement noise can be approximated by a Rayleigh probability distribution function [40]. Rayleigh noise can be generated using the Matlab command 'raylrnd'. To show the efficacy of the proposed method in the presence of non-Gaussian noise, the simulations are performed for low and high intensity Rayleigh noise.



(a) States and their estimates using SR-EH_∞IF and SR-CH_∞IF

(b) RMSE of Speed using SR-EH_∞IF and SR-CH_∞IF

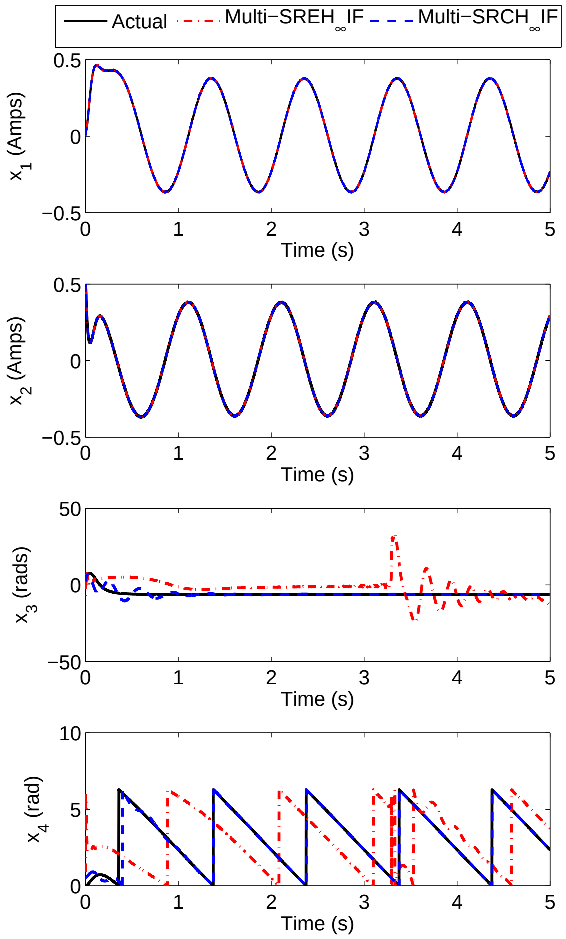
Figure 3: PMSM simulation results in the presence of high Gaussian noise

5.2.1. Multi-sensor state estimation in the presence of low non-Gaussian noise

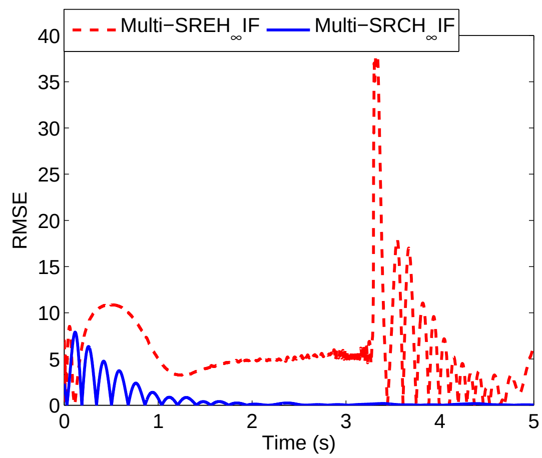
The process and measurement covariance matrices are the same as of low-Gaussian noise in the Section 5.1.2, however, the plant and measurements are corrupted by Rayleigh noise. Results of a typical simulation run using the Rayleigh noise are shown in Figure 4. The actual states and their estimates using the SR-EH_∞IF and SR-CH_∞IF are shown in Figure 4a. In this case also the first two states and their estimates using SR-EH_∞IF and SR-CH_∞IF almost overlap and estimation errors of speed and rotor position using the SR-EH_∞IF are large. The RMSEs for the speed using SR-EH_∞IF and SR-CH_∞IF are shown in Figure 4b. The maximum RMSE error is 39.74 for the SR-EH_∞IF and 8.24 for the SR-CH_∞IF, and the average RMSE error is 12.89 for the SR-EH_∞IF and 1.64 for the SR-CH_∞IF.

5.2.2. Multi-sensor state estimation in the presence of high-non Gaussian noise

The process and measurement covariance matrices are the same as of high-Gaussian noise in the Section 5.1.3, however, the plant and measurements are corrupted by Rayleigh noise. Results of a typical simulation run using the high intensity Rayleigh noise are shown in Figure 5. The actual states and their estimates using the SR-EH_∞IF and SR-CH_∞IF are shown in Figure 5a. The first two states are seen to be noisy due to the high intensity Rayleigh noise added to them. In this case also the first two states and their estimates using SR-EH_∞IF and SR-CH_∞IF almost overlap and estimation errors of speed and rotor position using the SR-EH_∞IF are large. The RMSEs for the speed using SR-EH_∞IF and SR-CH_∞IF are shown in Figure 5b. The maximum RMSE error is 48.37 for the SR-EH_∞IF and 32 for the SR-CH_∞IF, and the average RMSE error is 12.82 for the SR-EH_∞IF and 5.52 for the SR-CH_∞IF.



(a) States and their estimates using SR-EH_∞IF and SR-CH_∞IF



(b) RMSE of speed using SR-EH_∞IF and SR-CH_∞IF

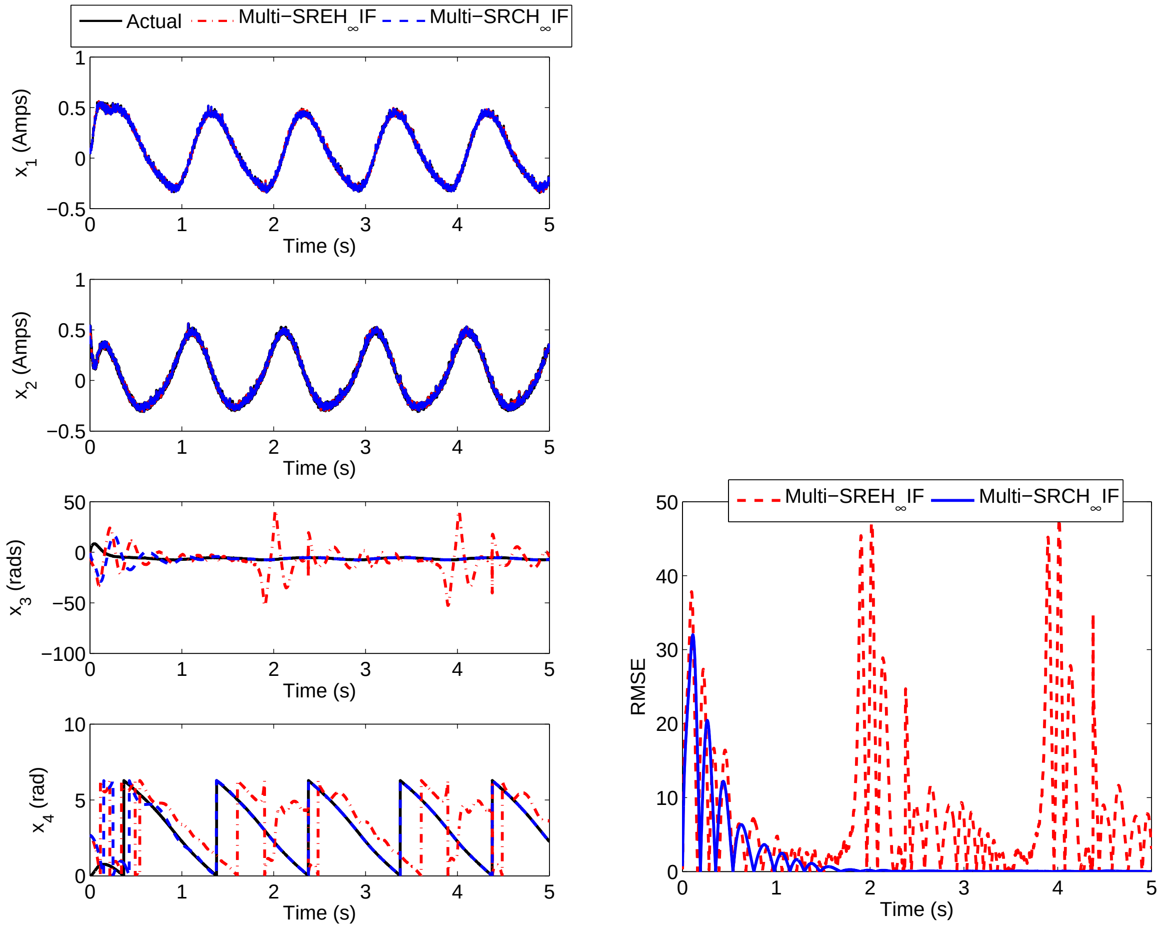
Figure 4: PMSM simulation results in the presence of low non-Gaussian noise

5.3. PMSM multi-sensor state estimation in the presence of near-perfect measurements

This section presents the simulation results with near-perfect plant and measurement models in the presence of Gaussian and non-Gaussian noise. It is assumed that the sensors provide near-perfect measurements and the process models are also almost perfect. The process and measurement noise covariance matrices are chosen as:

$$\mathbf{Q} = \begin{bmatrix} 1\times10^{-20} & 0 & 0 & 0 \\ 0 & 1\times10^{-20} & 0 & 0 \\ 0 & 0 & 1\times10^{-20} & 0 \\ 0 & 0 & 0 & 1\times10^{-20} \end{bmatrix}, \mathbf{R}_1 = \begin{bmatrix} 1\times10^{-20} & 0 \\ 0 & 1\times10^{-20} \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} 1\times10^{-20} & 0 \\ 0 & 1\times10^{-20} \end{bmatrix}.$$

The square-root filter has inherent tendency to effectively estimate the states in the presence of near-perfect measurements, for more details please see [37], [38], [2], [25]. The estimated and actual states with near-perfect measurements are very close for both the Gaussian and Rayleigh noises, and hence the plots of states and their estimates are not shown. The root means square errors (RMSEs) for the speed in the presence of Gaussian and non-Gaussian noise using multi-sensor SR-CH_∞IF are shown in Figure 6. The RMSE errors for both cases are very low. The maximum RMSE error is 3.18 for the Gaussian noise and 3.19 for the Rayleigh noise, and the average RMSE error is 0.5816 for the Gaussian noise and 0.5869 for the non-Gaussian noise respectively. Since the SR-EH_∞IF is a Jacobian based approach, it is sometimes difficult to implement on various applications for example when the Jacobians are ill-conditioned or for piecewise nonlinear systems. However, the proposed SR-CH_∞IF is a Jacobian free approach and hence one can avoid the difficulties due to Jacobians. The simulation results in this section show the efficacy of the proposed SR-CH_∞IF. It can be observed that the proposed filter has a tendency to handle low and high Gaussian noise and as well non-Gaussian noise. Further, the SR-CH_∞IF can be handy in



(a) States and their estimates using SR-EH_∞IF and SR-CH_∞IF

(b) RMSE of Speed using SR-EH_∞IF and SR-CH_∞IF

Figure 5: PMSM simulation results in the presence of high non-Gaussian noise

applications where the sensor-measurements are near-perfect. This shows the robustness of the proposed filter in the presence of various types of noises. The simulations have been performed using 64 bit Matlab on a computer with the processor speed on 3.4 GHz. The average computation time, for the simulations given in this paper, are 3.1 s and 3.6 s for SR-EH_∞IF and SR-CH_∞IF, respectively. However, it can be seen that in all the simulations the

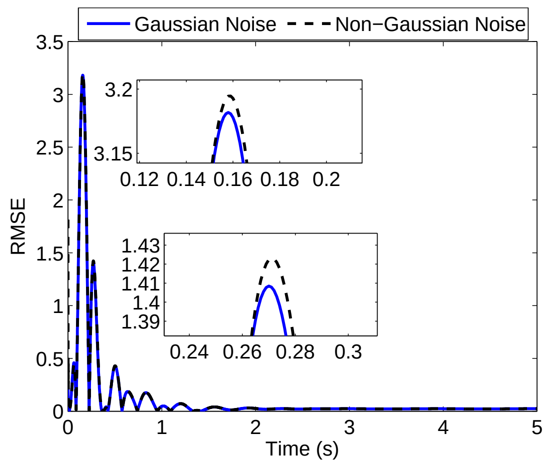


Figure 6: PMSM simulation results in the presence of near-perfect Gaussian and non-Gaussian noise using multi-sensor SR-CH_∞IF

quality of the state estimate using the proposed SR-CH_∞IF is far better than the SR-EH_∞IF.

6. Conclusion

In this paper, we have developed a cubature H_∞ information filter and its square-root version. These proposed estimators are further extended to handle measurements from multiple sensors. The cubature H_∞ information filter is derived from the cubature Kalman filter, the extended H_∞ filter and from the extended information filter. The desirable features of the cubature H_∞ information filter are:

1. It is a derivative free (Jacobians evaluations are not required) state estimator for nonlinear systems.
2. It has computationally an easier measurement update.
3. It has a capability to handle Gaussian and non-Gaussian noises.
4. It is easy to deal with measurements from single sensor and multiple sensors.
5. For numerical accuracy, a square-root version of the cubature H_∞ information filter can be used.

The efficacy of the square-root cubature H_∞ information filter is verified on a simulation example of multi-sensor permanent magnet synchronous motor. The superior performance of the cubature H_∞ information filter over extended H_∞ information filter in the presence of Gaussian and non-Gaussian noises has been demonstrated. The work in this paper can be further explored in various directions, for example in this paper the filter tuning parameters $\mathbf{Q}$ and $\mathbf{R}$ are heuristically selected. A proper mechanism to tune these matrices can be incorporated in the current framework to enhance further over-all robustness and performance. Furthermore it is assumed that there is no time delay due to the sensor dynamics. It would be interesting to extend the current work along the lines of [22].

References

[1] B. Anderson, J. Moore, Optimal Filtering, Prentice-Hall, 1979.

[2] D. Simon, Optimal state estimation: Kalman, H infinity, and nonlinear approaches, John Wiley & Sons, 2006.

[3] K. Ito, K. Xiong, Gaussian filters for nonlinear filtering problems, IEEE Transactions on Automatic Control 45 (5) (2000) 910–927.

[4] I. Arasaratnam, S. Haykin, Square-root quadrature kalman filtering, IEEE Transactions on Signal Processing 54 (6) (2009) 2589–2593.

[5] I. Arasaratnam, S. Haykin, R. J. Elliott, Discrete-time nonlinear filtering algorithms using Gauss–Hermite quadrature, Proceedings of the IEEE 95 (5) (2007) 953–977.

[6] J. Sarmavuori, S. Särkkä, Fourier-hermite kalman filter, IEEE transactions on automatic control 57 (6) (2012) 1511–1515.

[7] C. Edwards, S. K. Spurgeon, R. J. Patton, Sliding mode observers for fault detection and isolation, Automatica 36 (4) (2000) 541–553.

[8] M. Norgaard, N. K. Poulsen, O. Ravn, New developments in state estimation for nonlinear systems, Automatica 36 (11) (2000) 1627–1638.

[9] K. Murphy, Bayesian map learning in dynamic environments, in: Proceedings of the Neural Information Processing Systems, 1999, pp. 1015–1021.

[10] M. S. Arulampalam, S. Maskell, N. Gordon, T. Clapp, A tutorial on particle filters for online nonlinear/non-gaussian bayesian tracking, IEEE Transactions on Signal Processing 50 (2) (2002) 174–188.

[11] A. Doucet, S. Godsill, C. Andrieu, On sequential monte carlo sampling methods for bayesian filtering, Statistics and computing 10 (3) (2000) 197–208.

[12] S. Julier, Process models for the navigation of high speed land vehicles, PhD thesis, University of Oxford (1996).

[13] S. Julier, J. Uhlmann, A new method of the nonlinear transformation of means and covariances in filters and estimators, IEEE Transactions on Automatic Control 45 (3) (2000) 477–482.

[14] I. Arasaratnam, S. Haykin, Cubature Kalman filters, IEEE Transactions on Automatic Control 54 (6) (2009) 1254–1269.

[15] G. A. Einicke, L. B. White, Robust extended kalman filtering, IEEE Transactions on Signal Processing 47 (9) (1999) 2596–2599.

[16] R. N. Banavar, J. L. Speyer, A linear-quadratic game approach to estimation and smoothing, in: American Control Conference, 1991, 1991, pp. 2818–2822.

[17] X. Shen, L. Deng, Game theory approach to discrete H_∞ filter design, IEEE Transactions on Signal Processing 45 (4) (1997) 1092–1095.

[18] P. Shi, X. Luan, C.-L. Liu, Filtering for discrete-time systems with stochastic incomplete measurement and mixed delays, IEEE Transactions on Industrial Electronics 59 (6) (2012) 2732–2739.

[19] M. Aliyu, J. Luttamaguzi, 3-dof-pid nonlinear h-infinity estimators, International Journal of Control 83 (6) (2010) 1136–1144.

[20] M. H. Terra, J. Y. Ishihara, G. Jesus, Information filtering and array algorithms for discrete-time markovian jump linear systems, IEEE Transactions on Automatic Control 54 (1) (2009) 158.

[21] D. Ding, Z. Wang, H. Dong, H. Shu, Distributed h-infinity state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case, Automatica 48 (8) (2012) 1575–1585.

[22] J. Qiu, G. Feng, J. Yang, A new design of delay-dependent robust filtering for discrete-time t–s fuzzy systems with time-varying delay, Fuzzy Systems, IEEE Transactions on 17 (5) (2009) 1044–1058.

[23] J. Qiu, G. Feng, H. Gao, Fuzzy-model-based piecewise static-output-feedback controller design for networked nonlinear systems, Fuzzy Systems, IEEE Transactions on 18 (5) (2010) 919–934.

[24] J. Qiu, G. Feng, H. Gao, Observer-based piecewise affine output feedback controller synthesis of continuous-time t–s fuzzy affine dynamic systems using quantized measurements, Fuzzy Systems, IEEE Transactions on 20 (6) (2012) 1046–1062.

[25] K. P. B. Chandra, D.-W. Gu, I. Postlethwaite, A cubature H_∞ filter and its square-root version, International Journal of Control 87 (4) (2014) 764–776.

[26] A. Mutambara, Decentralized Estimation and Control for Multi-sensor Systems, CRC Press, 1998.

[27] D.-J. Lee, Nonlinear estimation and multiple sensor fusion using unscented information filtering, IEEE Signal Processing Letters 15.

[28] K. P. B. Chandra, D.-W. Gu, I. Postlethwaite, Square root cubature information filter, IEEE Sensors Journal 13 (2) (2013) 750–758.

[29] B. Jia, M. Xin, Multiple sensor estimation using a new fifth-degree cubature information filter, Transactions of the Institute of Measurement and Control 37 (1) (2015) 15–24.

[30] K. P. B. Chandra, D.-W. Gu, I. Postlethwaite, Cubature H_∞ information filter, in: European Control Conference (ECC), 2013, pp. 2639–2644.

[31] I. Yaesh, U. Shaked, Game theory approach to state estimation of linear discrete-time processes and its relation to H_∞ optimal estimation, International Journal of Control 55 (6) (1992) 1443–1452.

[32] F. Yang, Z. Wang, S. Lauria, X. Liu, Mobile robot localization using robust extended H_∞ filtering, Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering 223 (8) (2009) 1067–1080.

[33] R. Eustice, H. Singh, J. J. Leonard, M. Walter, R. Ballard, Visually navigating the rms titanic with slam information filters., in: Robotics: Science and Systems, 2005, pp. 57–64.

[34] G. Sibley, G. Sukhatme, L. Matthies, The iterated sigma point filter with applications to long range stereo, in: Proceddings of Robotics: Science and Systems, 2006.

[35] W. Li, Y. Jia, H-infinity filtering for a class of nonlinear discrete-time systems based on unscented transform, Signal Processing 90 (12) (2010) 3301–3307.

[36] T. Kailath, A. H. Sayed, B. Hassibi, Linear estimation, Vol. 1, Prentice Hall Upper Saddle River, NJ, 2000.

[37] J. Tsyganova, M. Kulikova, State sensitivity evaluation within ud based array covariance filters, IEEE Transactions on Automatic Control 58 (11) (2013) 2944–2950.

[38] M. S. Grewal, A. P. Andrews, Applications of kalman filtering in aerospace 1960 to the present [historical perspectives], IEEE Control Systems magazine 30 (3) (2010) 69–78.

[39] B. Hassibi, A. H. Sayed, T. Kailath, Indefinite-quadratic estimation and control, Studies in Applied and Numerical Mathematics.

[40] G. G. Rigatos, Particle filtering for state estimation in nonlinear industrial systems, IEEE transactions on Instrumentation and measurement 58 (11) (2009) 3885–3900.