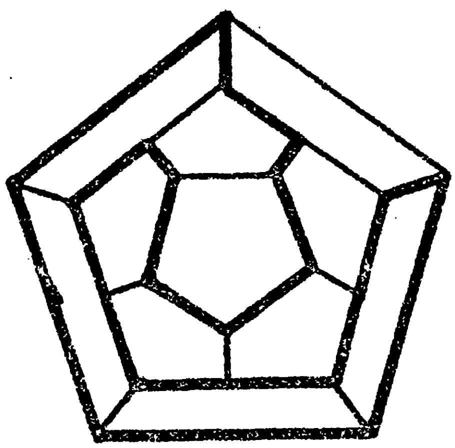
HAMILTONIAN CIRCUITS

IN TRIVALENT PLANAR GRAPHS

by

W. L. Price



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SUMMARY

The author has investigated the properties of Hamiltonian circuits in a class of trivalent planar graphs and he has attempted, with partial success, to establish conditions for the existence of Hamiltonian circuits in such graphs.

Because the Hamiltonian circuits of a trivalent planar graph are related to the four-colourings of the graph some aspects of the four-colour problem are discussed. The author describes a colouring algorithm which extends the early work of Kempe, together with an algorithm based on the Heawood congruences which enables the parity of the number of four-colourings to be determined without necessarily generating all of the four-colourings. It is shown that the number of Hamiltonian circuits has the same parity as the number of four-colourings and that the number of Hamiltonian circuits which pass through any edge of a trivalent planar graph is either even or zero. A proof is given that the latter number is non-zero, for every edge of the graph, whenever the family of four-colourings has either of two stated properties.

The author describes two original algorithms, independent of four-colourings, which generate a family of Hamiltonian circuits in a trivalent planar graph. One algorithm embodies a transformation procedure which enables a family of Hamiltonian circuits to be generated from a given Hamiltonian circuit, while the other generates directly all Hamiltonian circuits which include a chosen edge of the graph.

In a new theorem the author proves the existence of Hamiltonian circuits in any trivalent planar graph whose property is that one or more members of a family of related graphs has odd parity.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the advice and encouragement of Dr. O.P.D. Cutteridge. Thanks are due to Helen Sheppard for typing the script and to Barbara Shore for preparing the diagrams.

CONTENTS

CHAPTER 1

INTRODUCTION

1.0. The first paper on graphs was written by the Swiss mathematician, Leonhard Euler [7]. The Königsberg Bridge problem, the famous unsolved problem of his day, had attracted Euler's interest and in 1736 he proved, using a graph theoretic method, that the problem had no solution. Euler's subsequent work laid the foundations of topological graph theory. A century later graph theory was still largely a 'fringe' subject, associated with certain types of mathematical puzzles and games. In 1852 Francis Guthrie[16] posed the problem which has become known as the four-colour conjecture and which is still the most celebrated of all unsolved problems in graph theory. The concept of a Hamiltonian circuit - a simple circuit which passes through every vertex of a graph - arose as the spin-off of a mathematical game invented in 1859 by Sir William Rowan Hamilton[9]. In the latter half of the nineteenth century, however, graph theory began to find applications within the physical sciences and the subject itself began to profit from the interplay of theory and application. Systematic methods for the analysis of electrical networks, first investigated by Kirchhoff[12] in 1847 and developed by Maxwell[14] in 1892, are built on the foundations of Euler's graph theory. In 1889 Cayley's[5] work on chemical identification problems led to the introduction into graph theory of the 'tree' - a concept now of fundamental importance and central to modern topological network theory. During the present century the subject has developed at an ever increasing rate and the last ten years in particular has seen a large increase in the annual output of published papers dealing with graph theory and its applications: because of their intuitive diagrammatic representation graphs have been found extremely useful in modelling systems which arise in science, engineering, social science and economics.

1.1. The author was introduced to graph theory a generation ago while receiving his education in communication engineering, but only in recent years has he experienced a quickening of interest in the subject. In preparation for research within the field of finite automata the author felt the need to extend his knowledge of graph theory and in the course of his studies he found himself increasingly drawn towards some of the fundamental problems of the subject. Attracted in particular by the nature of Hamiltonian circuits the author conducted some preliminary investigations which led him to discover an important property of the family of Hamiltonian circuits in trivalent planar graphs (the property referred to in Chapter 2 as the property P1) and also to construct a counter-example to the proposition that all such graphs possess Hamiltonian circuits. Only at a later stage did he learn from the literature that this property had been discovered by Tutte[23] and by Smith[21] in 1946, and that the identical counter-example had been obtained by Tutte[23] in 1946. The author was not too discouraged by this revelation for by this time he had formulated a conjecture for which he was unable to find either a proof or a counter-example - a new challenge had thus presented itself. What ought, perhaps, to have dissuaded him from pursuing this conjecture was the realisation that a proof, had it existed, would have implied the solution to the four-colour problem. Undeterred, however, the author continued his research, and his investigations into the properties of Hamiltonian circuits was well under way when he eventually discovered that graphs which constituted counter-examples to his conjecture had already been constructed by Tutte[25,26] and others. Although he has failed to achieve results of major significance the author believes that the new concepts and original algorithms which emerge from his work could well be of value to further research in this field. The author does not claim that his research will find immediate application in engineering but he believes that it has latent potential in the field of circuit theory. The advent of printed circuits has emphasised the importance of planar (as distinct from non-planar) networks and in recent years Chen[6] and others have found uses for Hamiltonian circuits within the context of topological network analysis. Finally, it might be pertinent to remark that a question posed by a final-year undergraduate (of Leicester University) during a recent lecture on electrical network theory, was answered satisfactorily only after a lengthy and interesting discussion between the lecturer concerned and the author on the subject of Hamiltonian circuits in trivalent planar graphs.

1.2. Conjectures concerning the existence of Hamiltonian circuits in trivalent planar graphs are described towards the end of Chapter 2 which first establishes the conceptual framework and terminology necessary to an understanding of these conjectures and of their relationship to the four-colour problem. Chapter 3 provides a brief historical background to the four-colour problem. The treatment does not bring the subject up-to-date but concentrates on the earlier work of Kempe[11], Tait[22] and Heawood[10] whose ideas the author has found to be relevant to his own research. This chapter includes two algorithms, devised by the author, which extend the work of Kempe and Heawood. Chapter 4 begins with a study of the relationships between the family of Hamiltonian circuits and the family of four-colourings of a graph. The author then describes two algorithms which generate a family of Hamiltonian circuits by methods which are independent of four-colour properties. This chapter concludes with a new theorem which proves the existence of Hamiltonian circuits in graphs which possess certain properties. In Chapter 5 the author discusses the conclusions to be drawn from his work and, by posing questions, he suggests a programme of further research.

CHAPTER 2

CONCEPTS AND CONJECTURE

2.0. Conjectures concerning the existence of Hamiltonian circuits in trivalent planar graphs are described towards the end of this chapter. The preliminary discussion introduces the concepts and terminology* relevant to an understanding of these conjectures and their relationship to the classical four-colour conjecture, and paves the way for the more detailed investigations of later chapters.

Graphs.

2.1. In what follows the term 'graph' will be taken to mean a planar graph which is undirected, connected and trivalent** (every vertex is of order 3). The term 'dual' will imply the dual of such a graph. Clearly the dual is planar, undirected and connected and has the property that each face is triangular.

2.2. Consider the graph G_1 and its dual G_1' shown in Fig. 1. The vertices of G_1 are arbitrarily numbered and the faces of G_1' are numbered correspondingly. Each face of G_1, and the corresponding vertex of G_1', is similarly assigned an index letter. It will be observed that G_1' contains a circuit of length 2 because vertices a and c are connected by 2 edges. Consider also the graph G_2 and its dual G_2' shown in Fig. 2.

* Reference 3 provides a good introduction to graph theory.

** The terminology of graph theory is not standardised. Some authorities use the term 'map' rather than 'graph' and 'cubic' rather than 'trivalent'.

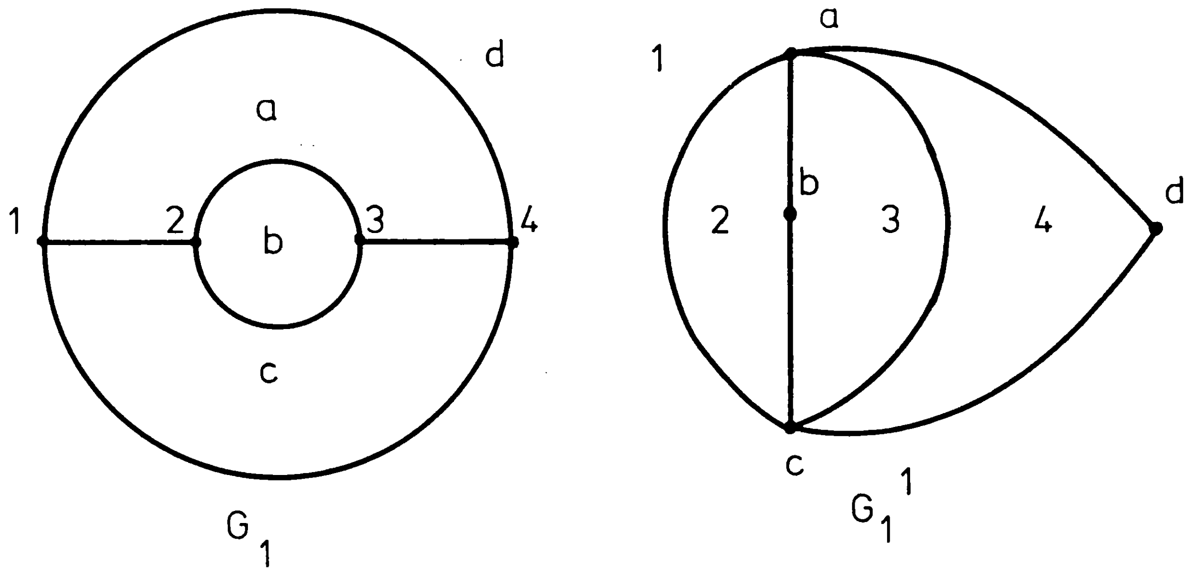

d
a
1
2
b
3
4
c
G 1
a
1
b
2
3
4
d
c
G 1 1

FIG. 1.

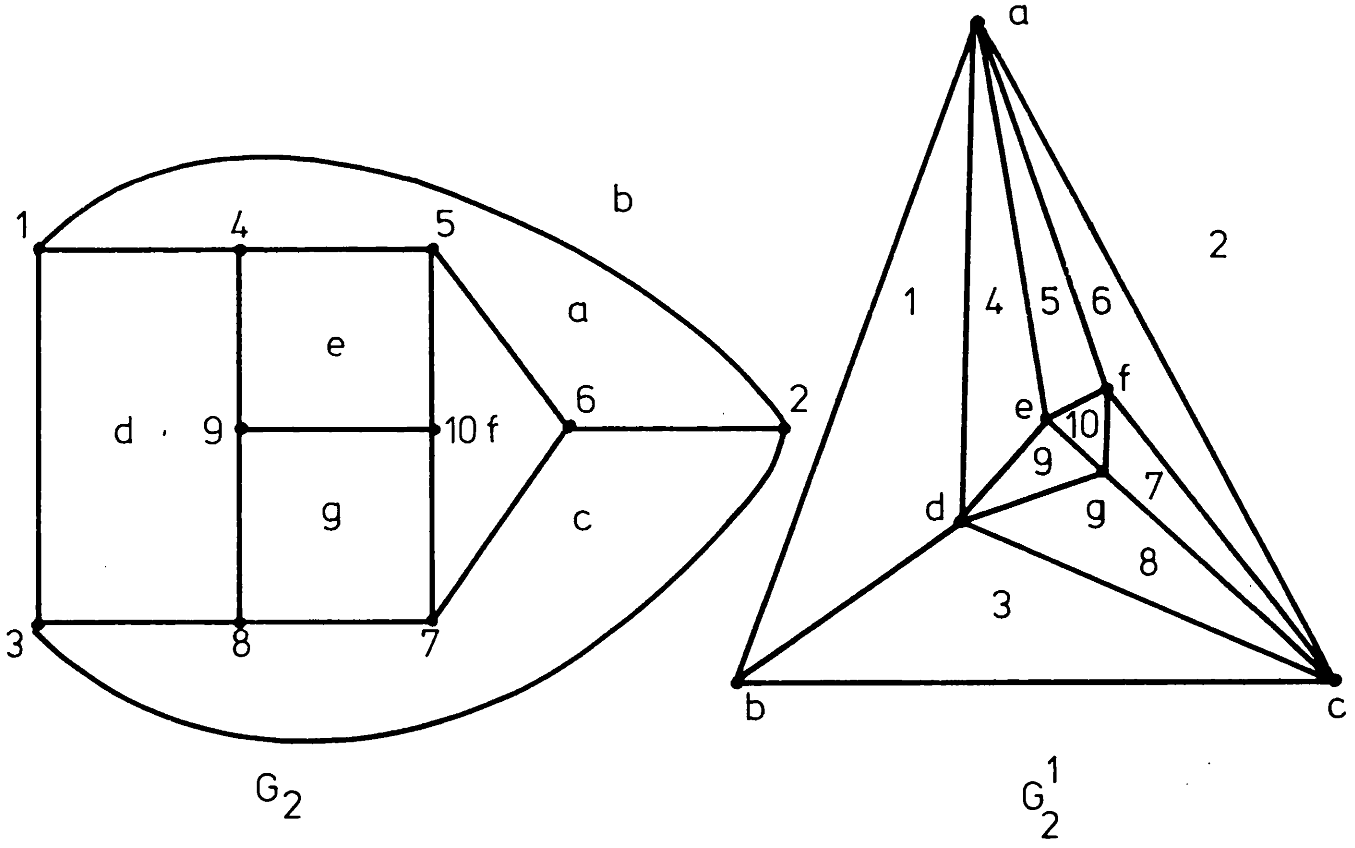

b
1
4
5
a
e
d
9
10 f
6
2
g
c
3
8
7
G2
a
2
1
4
5
6
f
e
10
9
7
d
g
8
3
b
c
G 1 2

FIG. 2

It is apparent that in addition to those circuits which bound the faces of G_2', and which are necessarily of length 3, G_2' contains a circuit of length 3 which is not a face circuit - viz. the circuit acd which encompasses the vertices e, f and g.

2.3. The term prime* graph will be used to denote a graph whose dual contains no circuits of length 3, or less than 3, other than face circuits. The graphs G_3 and G_4 shown, together with their duals, in Fig. 3 and Fig. 4 respectively, are prime graphs. The graph G_2 may be derived by nesting the graph G_4 within the graph G_3 in such a way that the face adc of the dual G_4' is embedded within the face adc of the dual G_3', the corresponding edges of these faces being merged. Thus, the non-prime graph G_2 may be factored* into the prime graphs G_3 and G_4.

Hamiltonian Circuits.

2.4. A Hamiltonian circuit, or H-circuit, in a graph G is a simple circuit which passes once through each vertex of G. An account of the origin of this concept, which stems from the work of Sir William Rowan Hamilton, is given in the next chapter. Fig. 5(a) shows an H-circuit in G_3. Clearly, by symmetry, G_3 possesses 3 H-circuits. To each H-circuit in a graph G there corresponds a particular two-tree in the dual G′ such that the tree edges of the two-tree are in one-to-one correspondence with the non-circuit edges of the H-circuit in G. Such a two-tree will be called an H-tree*, and has the property (not common to all two-trees) that one, and only one, tree edge is associated with each face of the dual. Fig. 5(b) shows the H-tree in G_3' which corresponds to the

* The author's terminology.

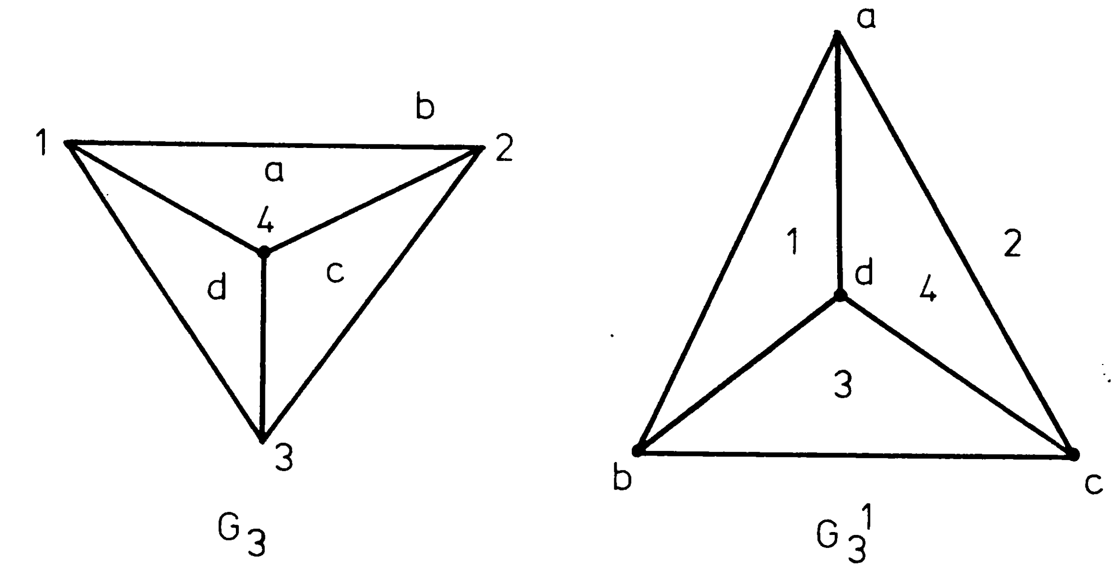


FIG. 3

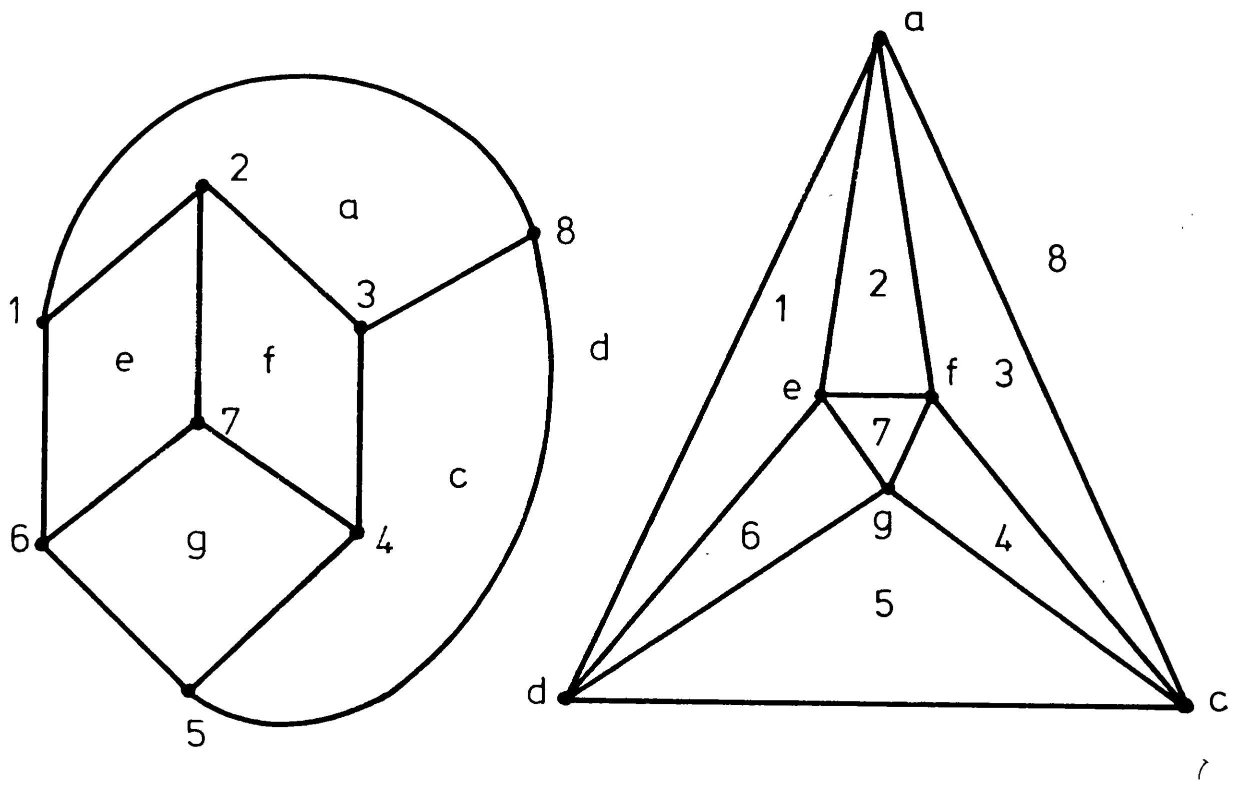
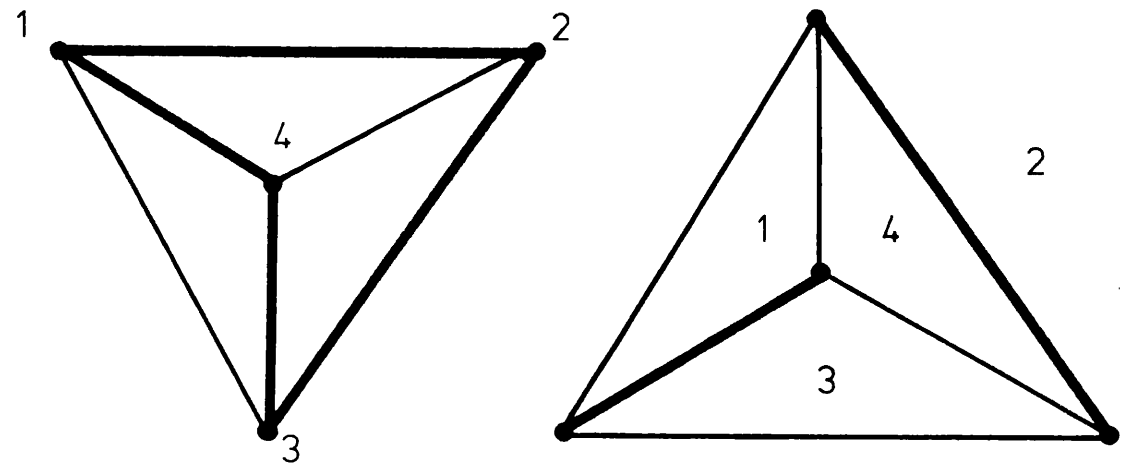
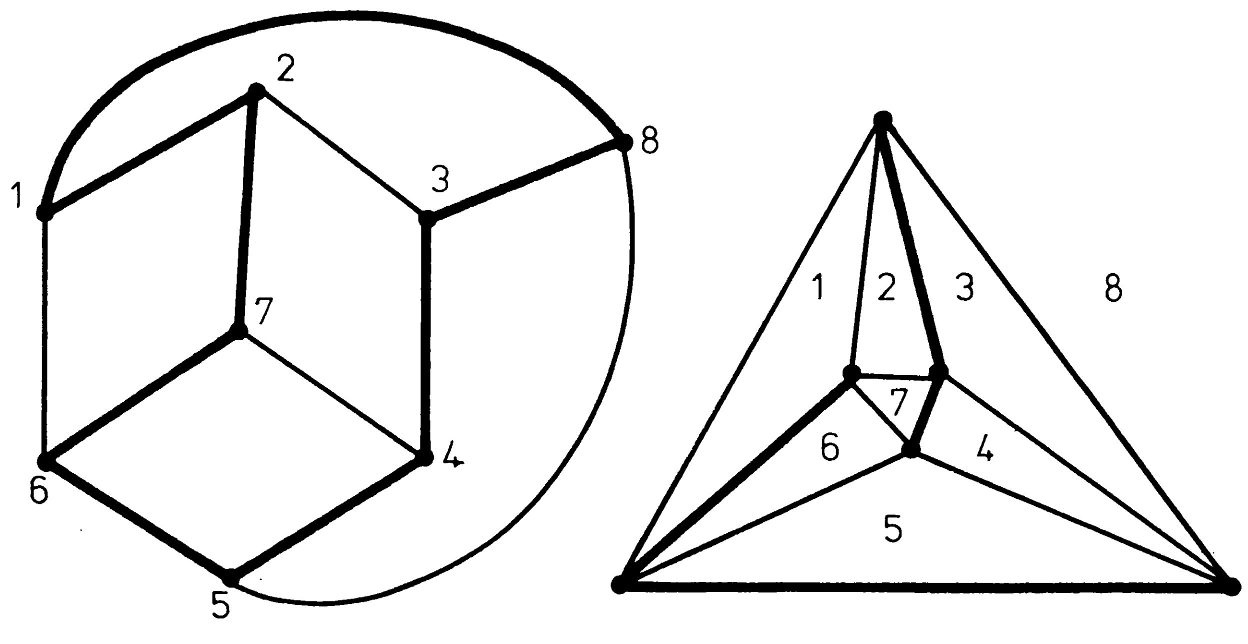


FIG. 4



(a) H- circuit in G_3 (b) H-tree in G_3^1

FIG. 5



(a) H-circuit in G_4 (b) H-tree in G_4^1

FIG. 6.

H-circuit of G_3 shown in Fig. 5(a). In this (somewhat trivial) example each of the two trees comprises a single edge of the dual. Fig. 6 shows an H-circuit in G_4 and the corresponding H-tree in G_4'. From the symmetries of the graph it is apparent that G_4 possesses 6 H-circuits. Less trivially, Fig. 7 shows an H-circuit (one of 25) in a more complex graph G_5, together with the corresponding H-tree in G_5'. In this example each of the two components of the H-tree is branched.

The Family of H-circuits.

2.5. The complete set of H-circuits belonging to a graph G will be called the family* of H-circuits in G. Correspondingly the dual G′ possesses a family of H-trees. A graph G will be said to have odd or even parity* according as the number of H-circuits in the family is odd or even. Thus G_3 has odd parity (the family has three members) and G_4 has even parity (the family comprises 6 H-circuits).

2.6. An important property of the family, later to be proved (and easily verified for the given examples), may be stated as follows:

P1. For any graph G the number of H-circuits which pass through any given edge of G is either even or zero.

From this property, other properties of interest may be derived.

Let T be the total number of H-circuits in the family of G,

T_E be the number of H-circuits which include some edge E of G,

and $\bar{T}_E$ be the number of H-circuits which do not include the edge E.

Then $$\bar{T}_E = T - T_E .$$

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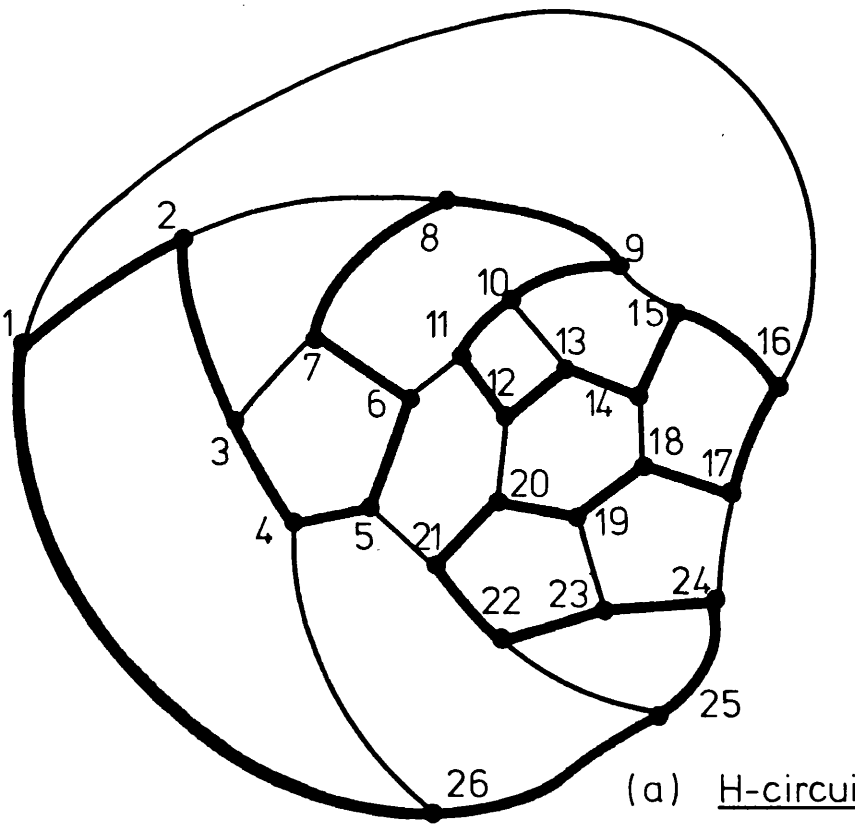
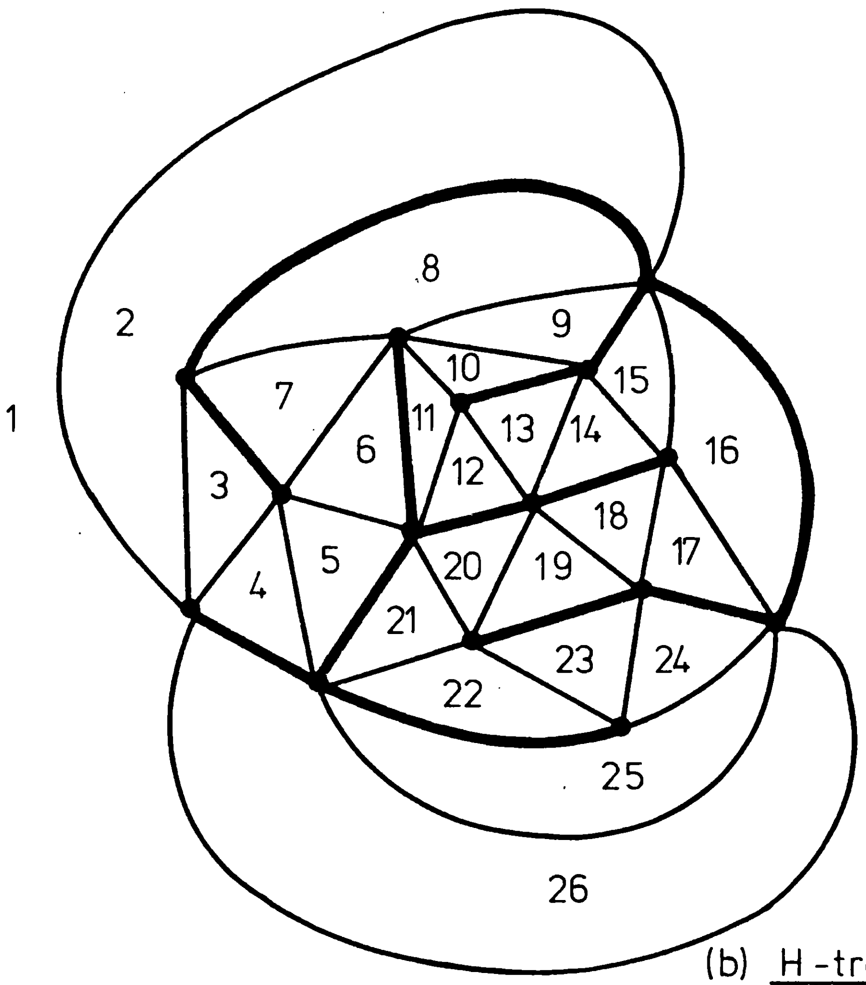
* The author's terminology.



(a) H-circuit in G_5



(b) H-tree in G_5^1

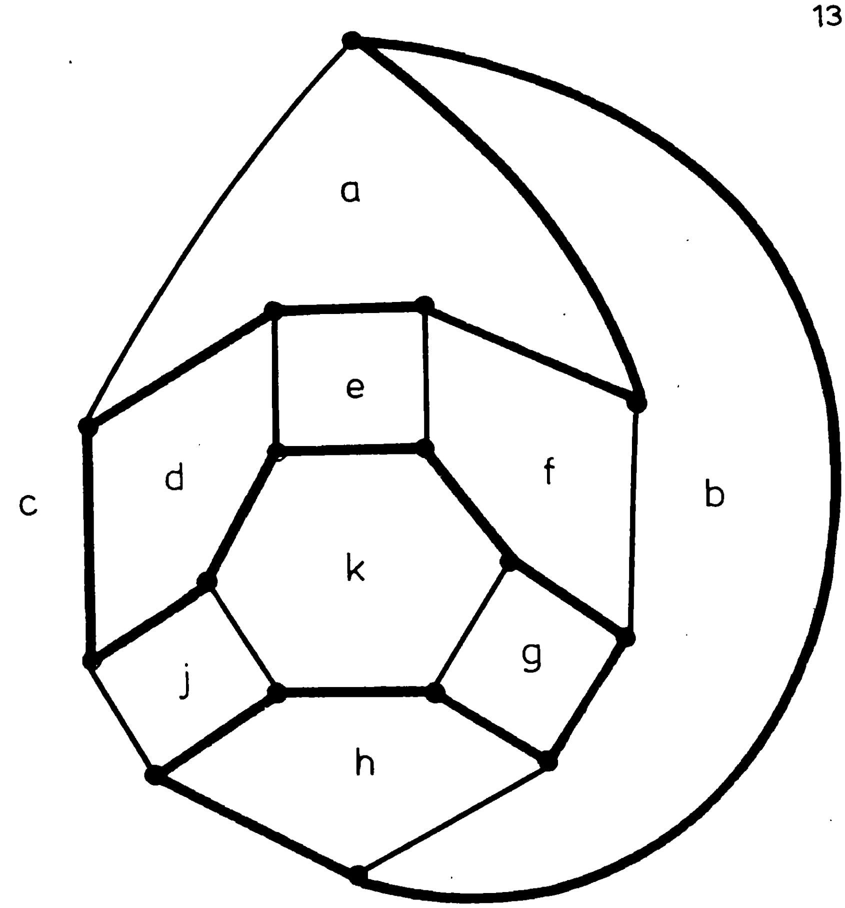
FIG. 7

If G has odd parity then T is odd and therefore $\overline{T}_E$ is odd because T_E is even (by property P1). Hence, for any edge E of G, there exists at least one H-circuit which does not include E. Because each vertex of G is trivalent it follows that if an H-circuit excludes one edge at any vertex v then the H-circuit must pass through the other two edges at v. Hence the property:

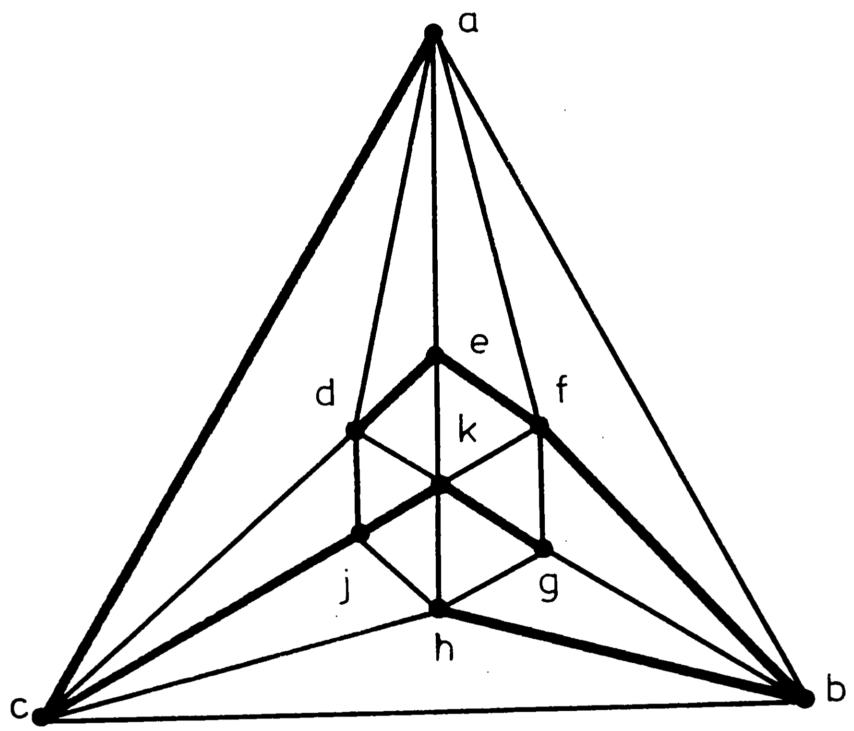
P2. If G has odd parity then the number of H-circuits which pass through any given edge of G is even and non-zero (i.e. $T_E > 0$).

<u>2.7.</u> If G has even parity then T, T_E and $\overline{T}_E$ are all even or zero. It is not possible to prove from property P1 that $T_E > 0$. Fig. 8 shows an H-circuit in a graph G_6, together with the corresponding H-tree in the dual G_6'. From the reflectional and rotational symmetries it is apparent that G_6 has a family of 6 H-circuits and hence G_6 has even parity. A study of the family reveals that $T_E > 0$ for every edge and that $\overline{T}_E > 0$ for all edges except dk, fk and hk. Thus every H-circuit includes these three edges and, correspondingly, there is no H-tree in the dual which has one of these edges as a tree edge. While in general $\overline{T}_E \not> 0$ for every edge of a graph which has even parity, all the graphs so far considered have the property $T_E > 0$ for every edge E.

<u>2.8.</u> Let the dual G_6', of Fig. 8(b), be redrawn as in Fig. 9 so that the face dek becomes the exterior face DEK. The upper-case letters distinguish the re-drawn graph from the original. There is no H-tree for which dk in Fig. 8(b) or DK in Fig. 9 is a tree edge. Now let a new dual G_7' be formed by nesting Fig. 9 into the dek face of Fig. 8(b) in such a way that the vertices D,E,K are identified with vertices e,k,d respectively, so that edge DE merges with edge ek, EK with kd and KD with



(a) H-circuit in G_6



(b) H-tree in G_6^1

FIG. 8

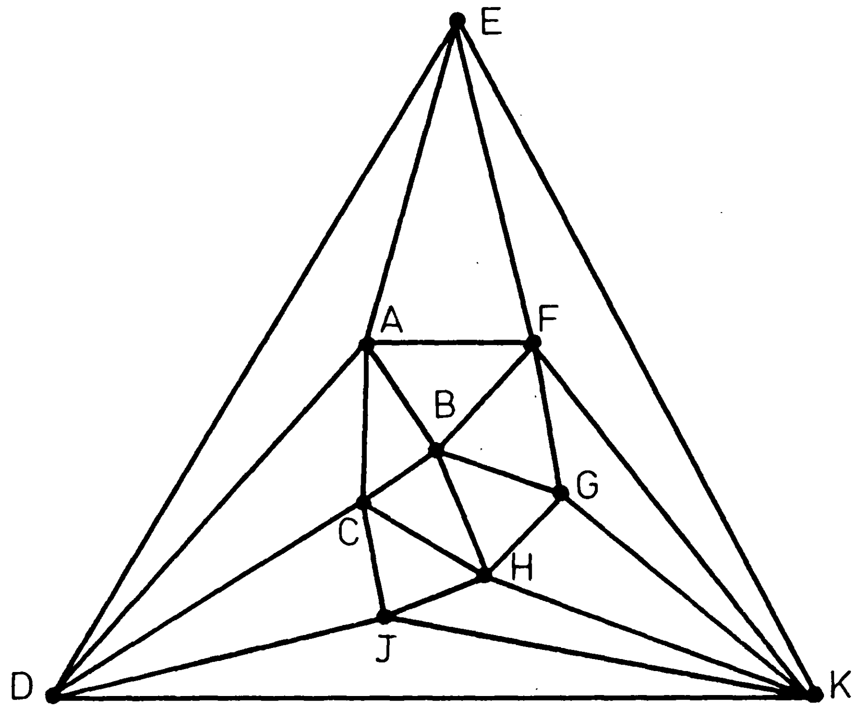


FIG. 9. A re-drawing of G_6^1

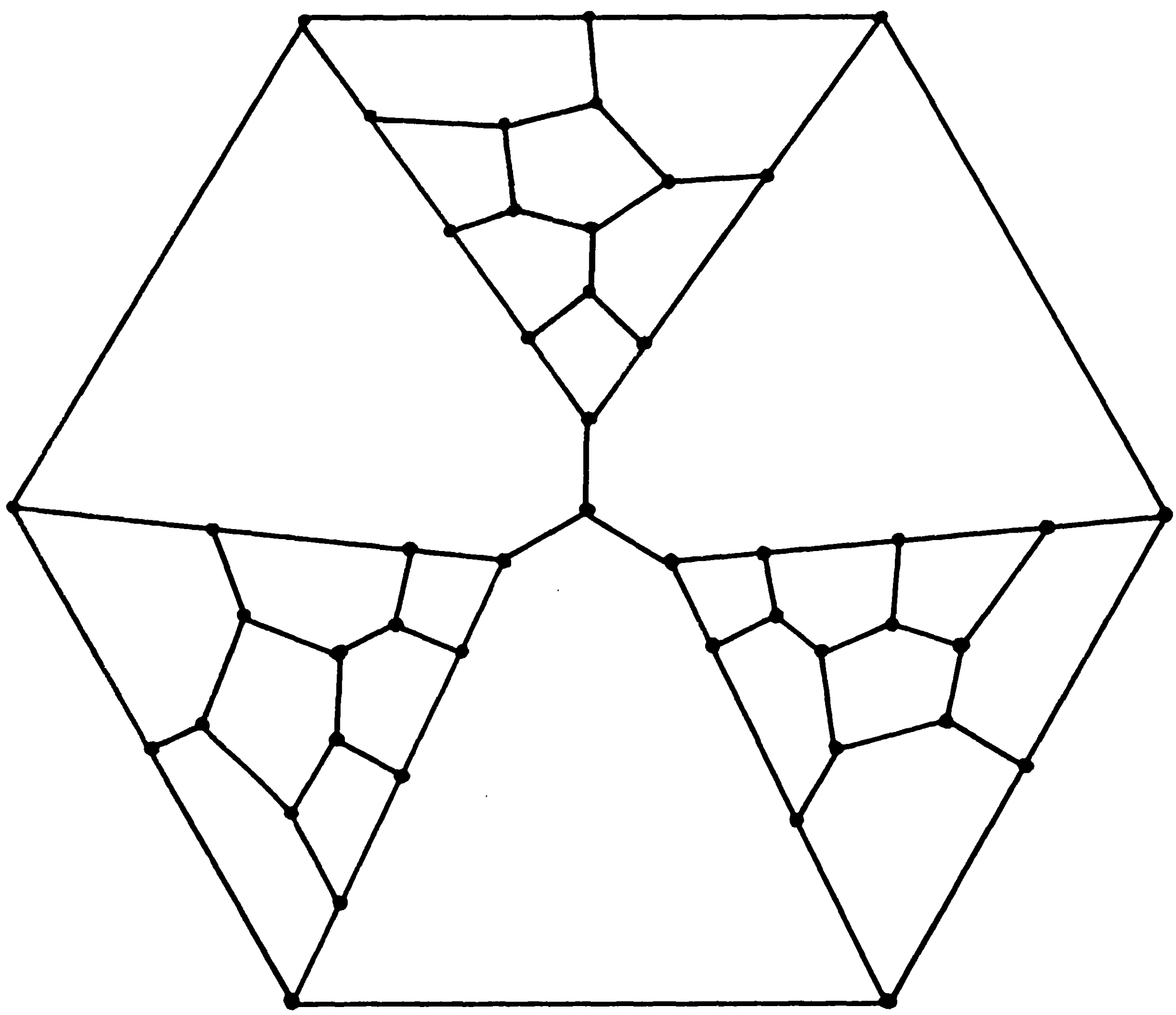


FIG. 10 G_8 - A graph with no H-circuits de. Whereas the graph G_6 is prime the new graph G_7, whose dual is G_7', is non-prime. The H-trees of G_7' are obtained by taking a pair of H-trees, one appropriate to Fig. 8(a) and one appropriate to Fig. 9, which are compatible at the merged edges. This compatibility constraint, coupled with the observations made earlier, imply that the H-trees of G_7' are formed by combining an H-tree for Fig. 8(a) which has ek as a tree edge with an H-tree for Fig. 9 which has DE as a tree edge, the resulting composite H-trees all having the property that the merged edge joining De with Ek is a tree edge. Hence, for the corresponding edge in the graph G_7, $T_E = 0$. The graph G_7 thus has the property that H-circuits exist but that $T_E \ngtr 0$ for all edges of G_7.

<u>2.9</u>. The nesting procedure may be taken one step further to form the graph G_8, shown in Fig. 10, whose dual is a three-fold development of the dual G_6' shown in Fig. 8(b). There are now no compatible groupings of H-trees from G_6' to form composite H-trees in G_8', so that G_8 does not possess H-circuits.

<u>2.10</u>. The graphs G_7 (for which $T > 0$ but $T_E \ngtr 0$ for all edges) and G_8 (for which $T = 0$ and hence $T_E = 0$ for all edges) are non-prime. The author's early experiments led him to conjecture that all prime graphs possess Hamiltonian circuits and he was motivated to search for a counter-example by the realisation that this conjecture, if valid, would imply the solution of the four-colour problem. It is pertinent therefore, at this stage in the discussion, to review the essential properties of the four-colourings of a graph.

Four-colourings of a Graph.

2.11. A graph G is four-colourable* if each face of G may be assigned one of four symbols in such a way that no two faces which possess a common bounding edge have the same symbol. Correspondingly, the dual G' is four-colourable in the sense that no two adjacent vertices (vertices linked by an edge) are assigned the same symbol. The graph G_3 has only the one four-colouring (permutations of the four symbols being excluded) because, as shown in Fig. 11, if three faces meeting at a vertex are arbitrarily assigned the symbols α, β and γ, then the remaining face must be assigned the fourth symbol δ. The graph G_4 has a family of 4 four-colourings, shown in Fig. 12 (one of these requires only 3 of the 4 available symbols).

The Four-colour Conjecture.

2.12. The four-colour conjecture asserts that any planar graph (not necessarily restricted by the conditions of para. 2.1) is face-colourable in not more than four colours. The conjecture has defied proof (or disproof by counter-example) for over a century and remains one of the classical unsolved problems in topological graph theory. The historical background to this conjecture will be reviewed in Chapter 3.

2.13. It is well known[2,3,16] that if the four-colour conjecture could be proved for prime graphs (as defined by paras. 2.1 and 2.3) then the proof

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* For a general planar graph the term 'four-colouring' may refer either to a colouring of the faces or to a colouring of the vertices. Within the context of this thesis the term will imply a face colouring of the graph (vertex-colouring of the dual).

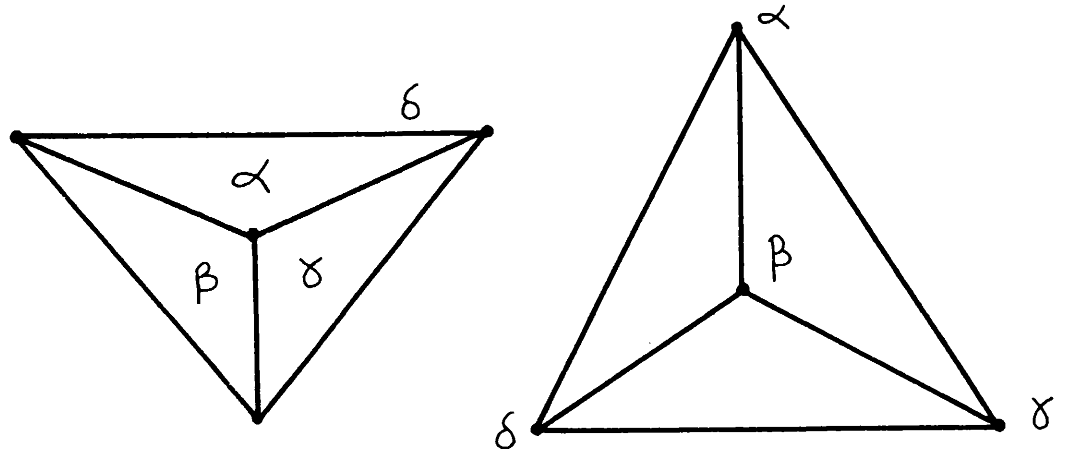

FIG. 11. The four-colouring of G_3 and its dual G_3^1

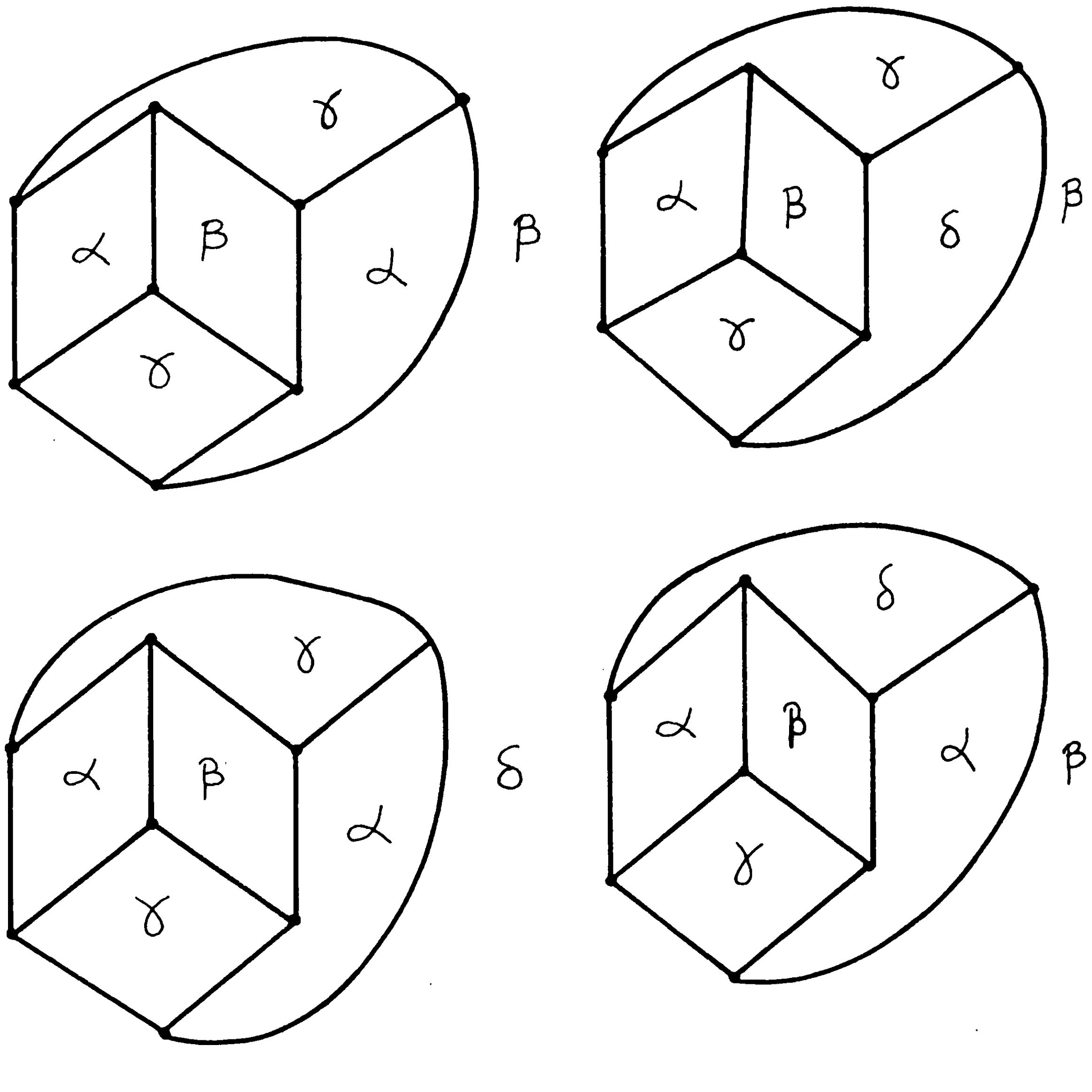

FIG. 12 The family of four-colourings of G_4

for any planar graph would be established. Consider, for example, the non-trivalent graph Γ whose dual Γ' is shown in Fig. 13. To show that Γ is four-colourable it is sufficient to obtain a vertex four-colouring of Γ'. The dual Γ' may be modified by the addition of further edges so as to make each face triangular. The modification can be achieved in a variety of ways and one such choice of additional edges results in the dual shown in Fig. 14. This will be recognised as G_2', which is non-prime and which has prime factors G_3' and G_4' that are four-colourable. By an appropriate permutation of the colourings of the three external vertices of G_4' any one of the four-colourings of G_4' can be embedded into the four-colouring of G_3' in such a way that the merged vertices are compatible in colour. One of the resultant four-colourings of G_2 is shown in Fig. 15. Finally, the additional edges may be removed thus leaving a four-colouring of the original dual Γ'.

The Relationship between Hamiltonian Circuits and Four-colourings of a Graph.

2.14. A graph which possesses an H-circuit is four-colourable. In order to prove this statement it is convenient to consider the H-tree in the dual G' which corresponds to the given H-circuit in G. Because the set of edges which constitute each of the two components form a tree, and hence have no closed circuits, it is always possible to assign different colours to the adjacent vertices of a tree edge by using two symbols (α and β) to colour alternate vertices of one component and the other two symbols (γ and δ) to colour alternate vertices of the other component. Because one, and only one, tree edge is associated with each face of the dual it follows that the two vertices connected by each

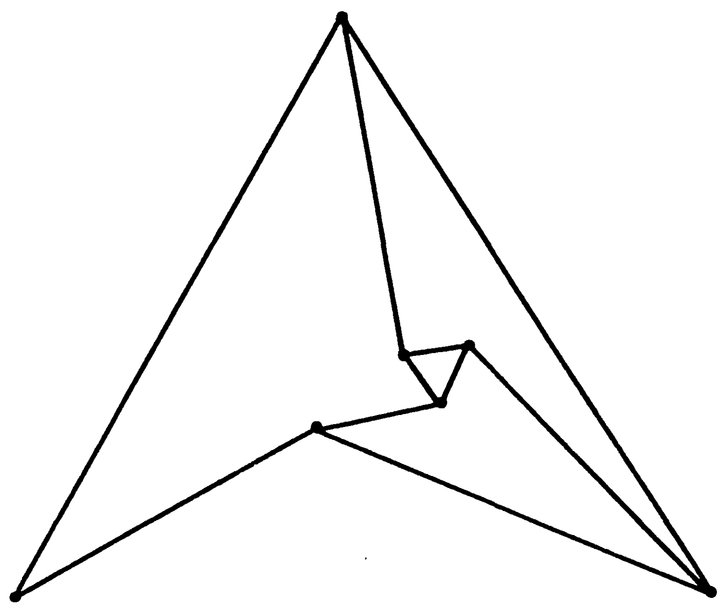

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FIG. 13. The dual, Γ^1, of a non-trivalent planar graph Γ

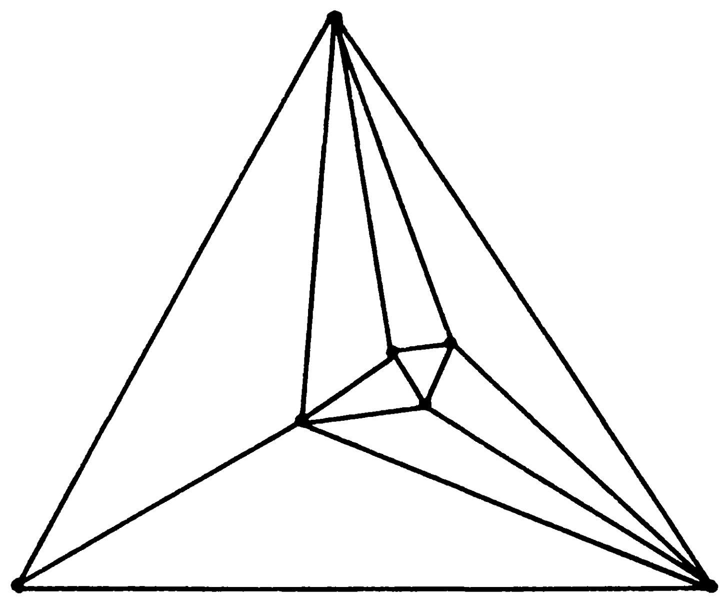
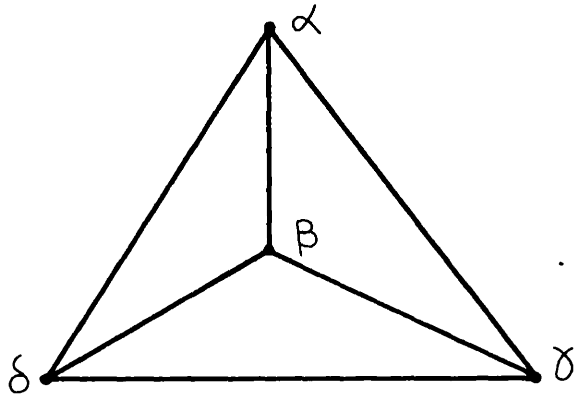
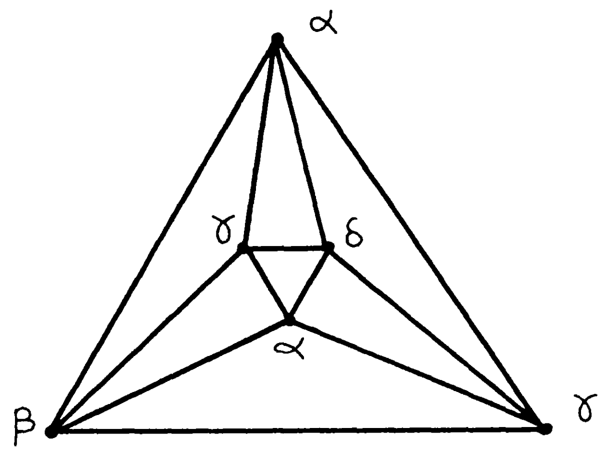


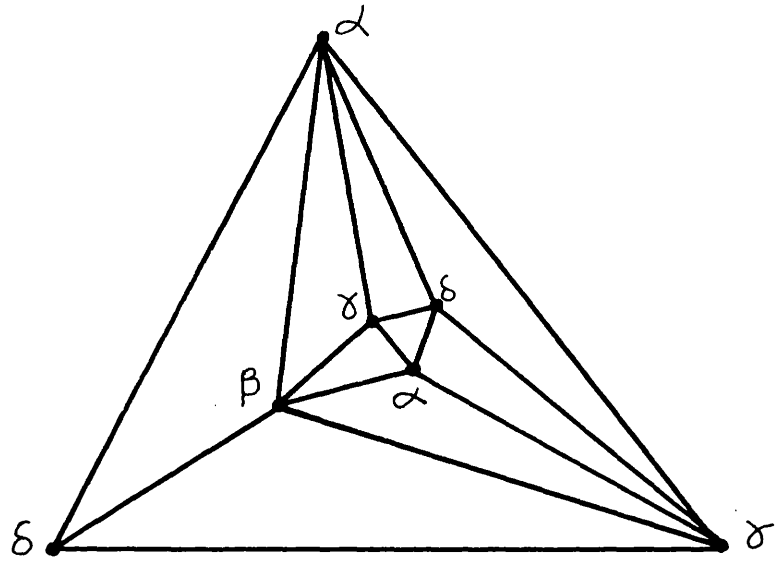
FIG. 14. Γ^1 modified by the addition of edges (G_2^1)



(a) The four-colouring of G_3^1



(b) A four-colouring of G_4^1



(c) The resultant four-colouring of G_2^1

FIG. 15

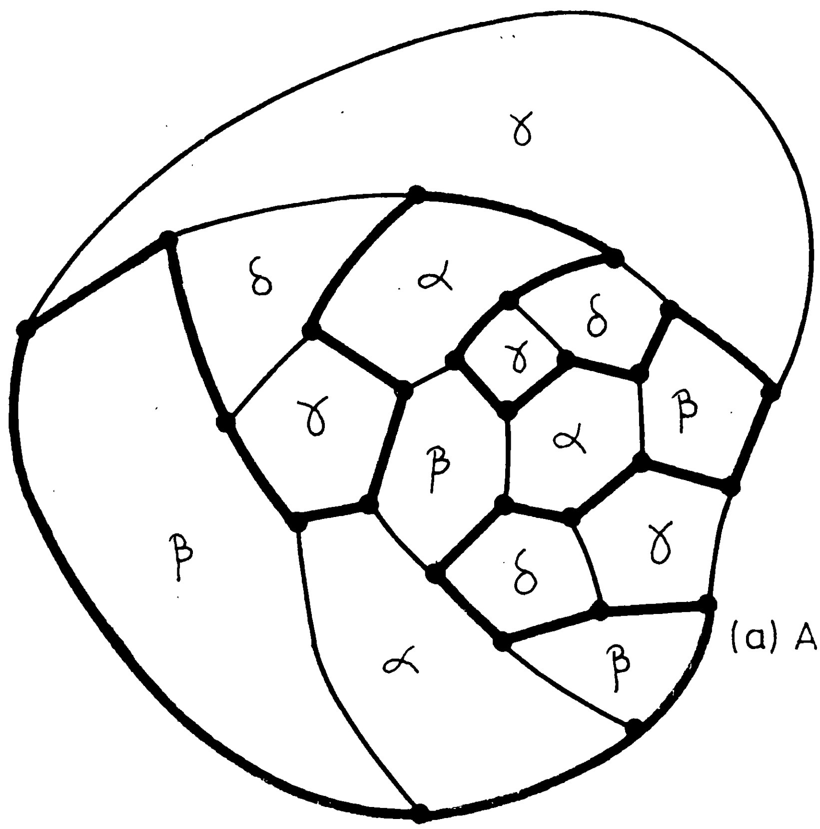
non-tree edge of G' are also assigned different colours, one from the set $\{\alpha, \beta\}$ and one from the set $\{\gamma, \delta\}$. This procedure is illustrated in Fig. 16 which shows a four-colouring of G_5 based on the H-circuit given in Fig. 7.

2.15. It is not necessarily true that a graph which is four-colourable possesses an H-circuit. The graph G_8, for example, does not possess an H-circuit but it is four-colourable because its three prime-factors, each of which is the graph G_6, are four-colourable. One of the four-colourings of G_8 is shown in Fig. 17.

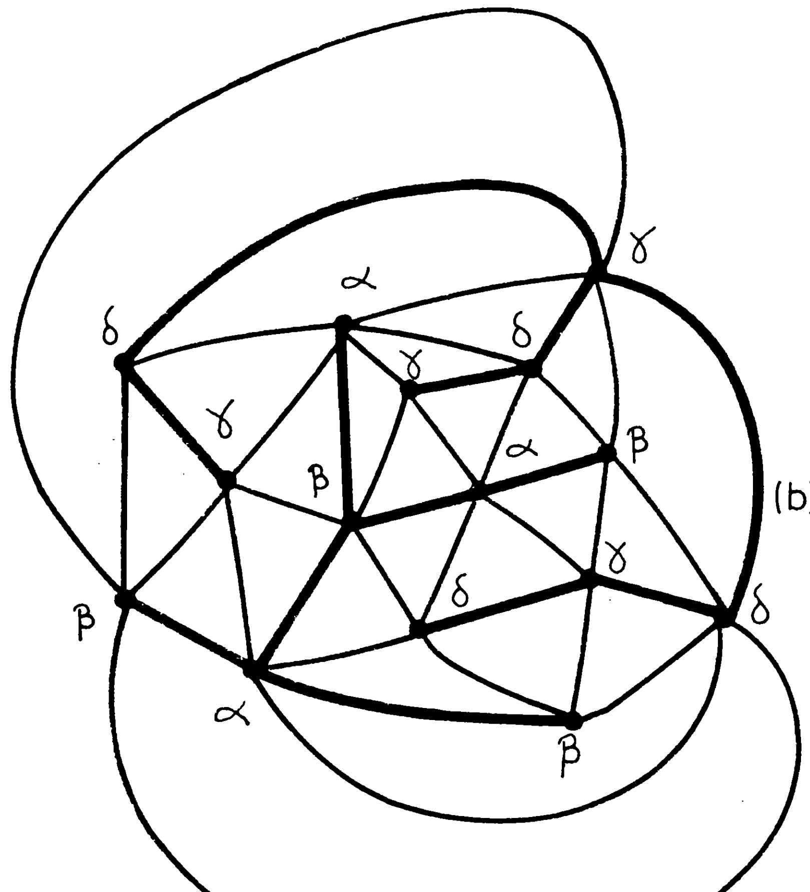
2.16. While the number of H-circuits of a graph is not, in general, equal to the number of four-colourings it will be observed from the given examples that these numbers are either both odd or both even. For example G_3 has 3 H-circuits and 1 four-colouring ; G_4 has 6 H-circuits and 4 four-colourings. It will be proved in Chapter 4 that this relationship is true for all graphs and hence one may refer unambiguously to the parity of a graph in the context either of the family of H-circuits or of the family of four-colourings.

Conjecture.

2.17. The preceding paragraphs have shown that the existence of a Hamiltonian circuit in a graph implies the existence of a four-colouring but that the converse is not necessarily true (unless the number of four-colourings is odd). In this sense a conjecture concerning the existence of H-circuits is stronger than a conjecture concerning the existence of four-colourings. In 1884 Tait conjectured that the graph of every



(a) A face colouring of G_5



(b) A vertex colouring of G_5'

FIG. 16

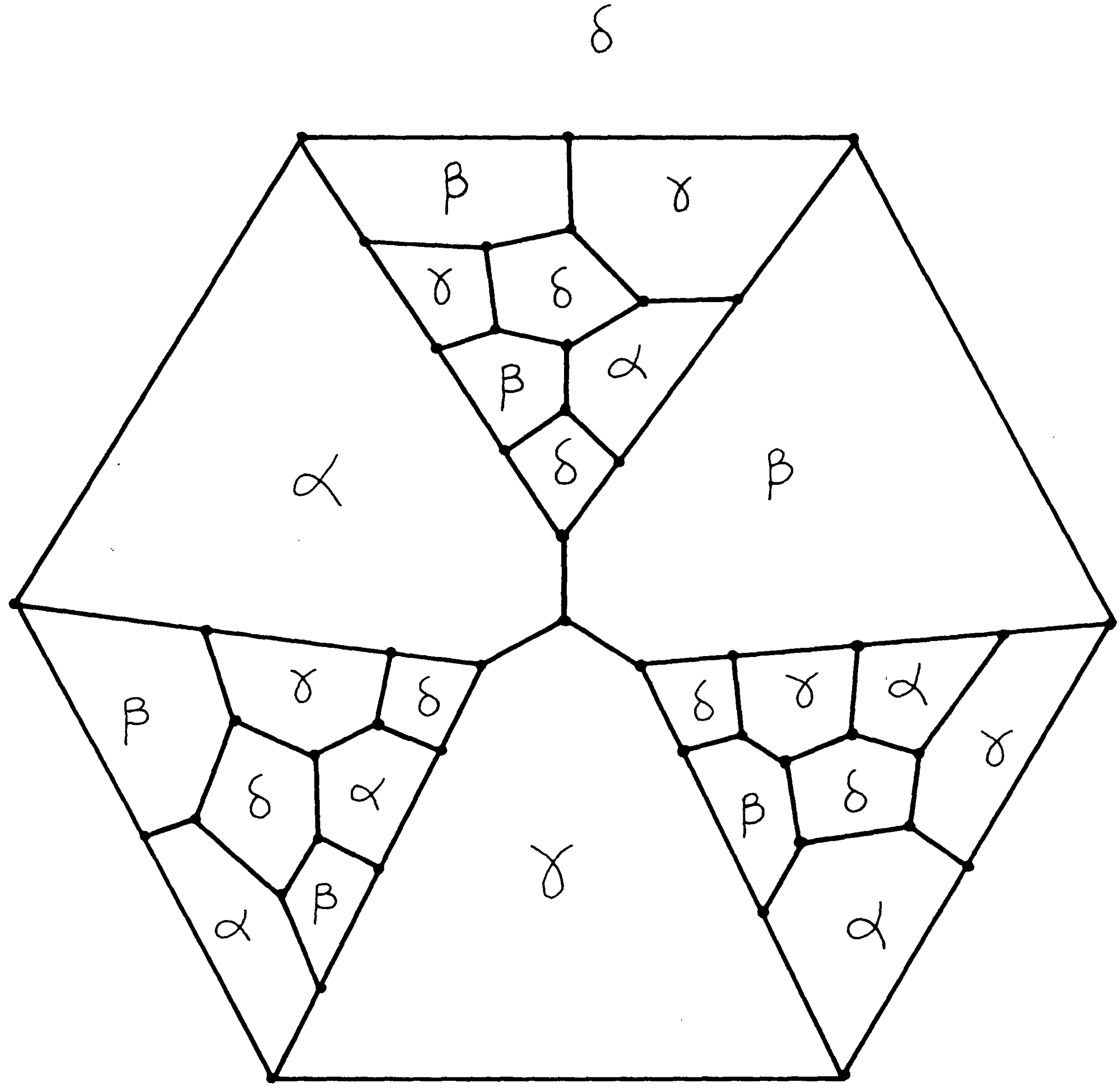


FIG. 17. A four - colouring of G_8

convex polyhedron possesses a Hamiltonian circuit. Because a graph is a skeleton of a convex polyhedron if and only if it is planar and 3-connected*, Tait's conjecture implies that every 3-connected planar graph has an H-circuit. In 1946 Tutte[23] found a counter-example to Tait's conjecture, this counter-example being identical to the graph G_8 of Fig. 10 (which was obtained independently by the author using a method essentially similar to that employed by Tutte).

2.18. The observations made in paragraph 2.10 led the author to conjecture that any prime graph possesses a family of Hamiltonian circuits such that each edge of the graph is included in at least two circuits (i.e. $T_E > 0$ for every edge). Before he was able to devise a counter-example, however, the author discovered that Tutte[25,26], Lederberg[13], and Kozyrev and Grinberg[20] had already constructed prime graphs which are non-Hamiltonian (i.e. which have no Hamiltonian circuits). Two of these, the graph of Kozyrev and Grinberg (1968) and one of Tutte's graphs (1972), are shown in Fig. 18. Although these graphs are non-Hamiltonian all are four-colourable.

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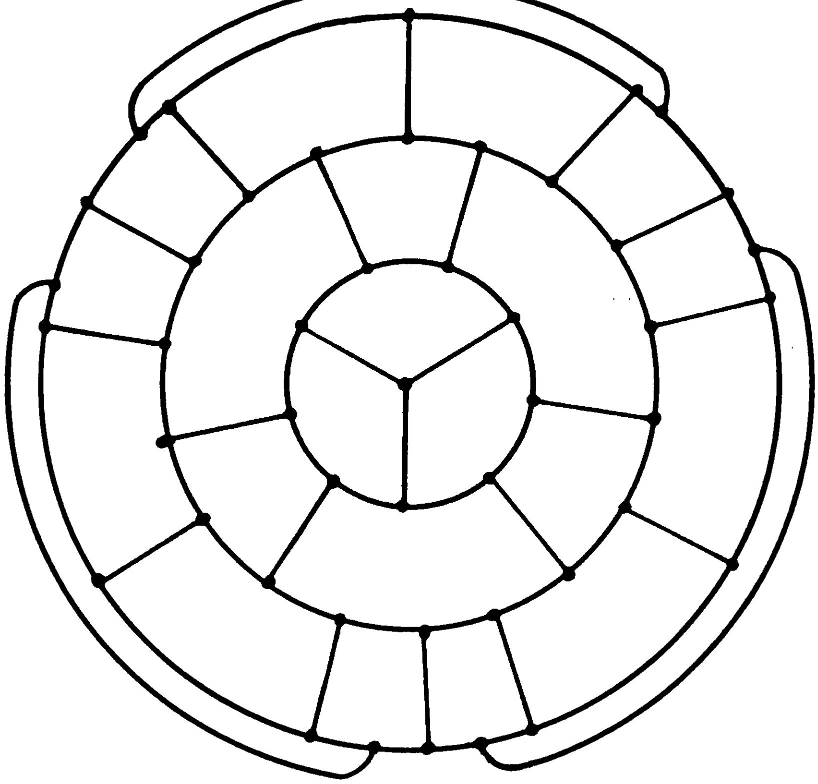
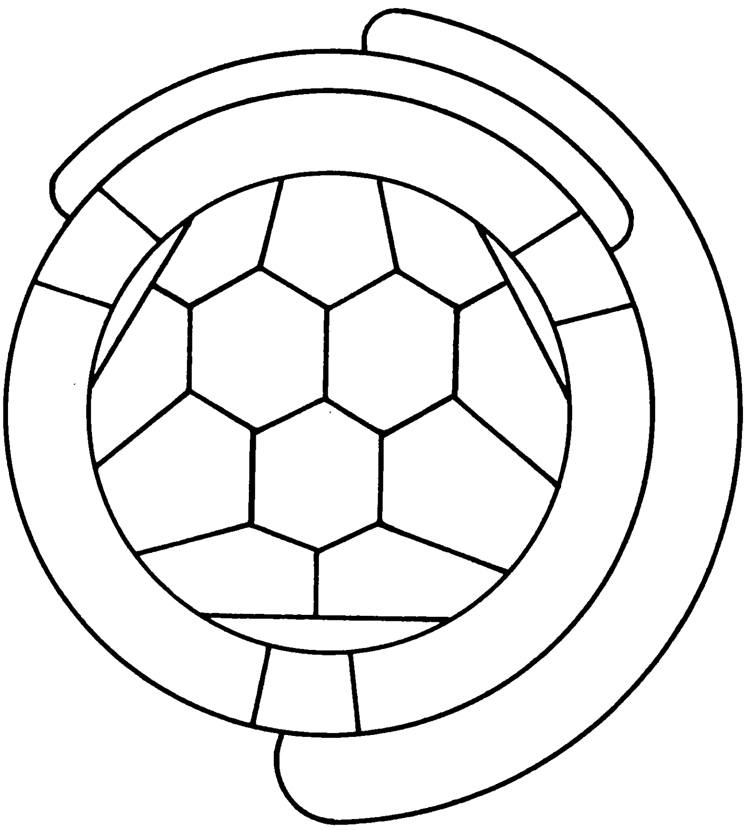
* A graph is k-connected if every pair of distinct vertices v and w are joined by at least k chains which have no common vertices except v and w. See, for example, reference 3.



(a) Graph of Kozyrev and Grinberg



(a) A Tutte graph

FIG. 18. Non-Hamiltonian graphs

CHAPTER 3

THE FOUR-COLOUR PROBLEM

3.0. The author's main concern has been to investigate the properties of families of Hamiltonian circuits in prime graphs, a specific goal being the attempt to establish conditions for the existence of Hamiltonian circuits in such graphs. Because of the link between H-circuits and four-colourings it seemed reasonable to begin the investigation by seeking potentially useful results from within the well established field of study associated with the four-colour problem. This chapter reviews the historical background to the four-colour conjecture, discusses some of the earlier contributions to the subject which the author has found to be relevant and helpful, and describes two algorithms which represent the author's attempts to develop these ideas.

The Four-colour Conjecture.

3.1. Accounts of the historical origins of the four-colour problem and the early attempts at a solution are given by Ball[1] and May[15]. The first known written statement of the conjecture appeared in a letter[16] from Augustus De Morgan, Professor of Mathematics at University College, London, to his friend Sir William Rowan Hamilton at Trinity College, Dublin. In this letter, dated October 23rd, 1852, De Morgan wrote

> "A student of mine asked me today to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be anyhow divided, and the compartments differently coloured, so that figures with any portion of a common boundary

line are differently coloured - four colours may be wanted but no more. Query : cannot a necessity for five or more be invented?..."

The student was Frederick Guthrie, whose brother, Francis, also at one time a student of De Morgan, had first posed the problem and claimed to have a solution. The four-colour problem is sometimes known as Guthrie's problem, although Francis Guthrie, who later became Professor of Mathematics at the South African University, Cape Town, never published anything on the subject.

3.2. During the quarter-century following De Morgan's communication the problem aroused little interest and did not appear in print until 1878 when Cayley[4] published an inquiry as to whether the conjecture had been proven. In 1879 Kempe[11] published a 'proof' of the conjecture. In 1890 Heawood[10] pointed out the error in Kempe's proof and was able to establish, using Kempe's approach to the problem, that a planar graph can always be coloured with five colours. Although subsequent attempts to find a planar graph which could not be coloured using only four colours have been in vain, progress has been made on characterising some of the properties which must be possessed by a counter-example to the four-colour conjecture if one exists. In 1922 Franklin[8] proved that a planar graph requiring five colours must have at least 26 faces and later researchers have extended this number. In 1969 Ore and Stemple[17] proved that a counter-example would need to have at least 39 faces.

3.3. In 1859 Sir William Rowan Hamilton invented a peculiar puzzle which he named "Around the World". The puzzle[9] comprised a regular dodecahedron made of wood, the twenty vertices being labelled with the names of various important cities, and the objective was to find a cyclic route which passes once through each vertex. Because the dodecahedron was cumbersome Hamilton also produced a version of the puzzle in which the solid was replaced by a planar graph isomorphic to the graph formed by the edges of the dodecahedron. This graph, together with one of its circuits is shown on the title page. Hamilton sold his idea to a game manufacturer in Dublin for about twenty-five guineas but the puzzle was not a commercial success.

3.4. Tait[22] showed, in 1880, that the existence of a Hamiltonian circuit in a trivalent planar graph implies a four-colouring of the graph. In 1891 Peterson[18] proved that either the vertices of a trivalent planar graph can be toured by a Hamiltonian circuit or there exists a collection of mutually exclusive subcircuits such that each vertex is on one subcircuit. It follows, using a result due to Tait (given later in this chapter), that if all the subcircuits are of even length then the graph is four-colourable. In 1931 Whitney[27] proved that the dual of a prime graph always has a Hamiltonian circuit.

Kempe's 'Proof'.

3.5. Kempe's[11] attempt at a proof of the four-colour conjecture illustrates both the simplicity of concept and the apparent proximity of a solution - two features which have attracted many mathematicians, both professional and amateur, to the problem. The essence of Kempe's approach is set out below but, for convenience, the argument is restricted to prime graphs (as defined in paras. 2.1 and 2.3) and considers a vertex-colouring of the dual rather than a face-colouring of the graph.

3.6. A well-known formula, first derived by Euler[16], relates the numbers of vertices, edges and faces of any planar graph. Euler's formula states that

$$v + f = e + 2 \qquad (1)$$

where v, f and e are, respectively, the numbers of vertices, faces and edges. Each face of a dual is triangular. Hence, because each edge is associated with two faces,

$$3f = 2e \qquad (2)$$

The order of a vertex is the number of edges incident at that vertex. Let ϕ_k be the number of vertices of order k. The dual of a prime graph has no vertices of order < 3 (para. 2.3)* and hence, because each edge is associated with two vertices,

$$v = \phi_4 + \phi_5 + \phi_6 + \phi_7 + \phi_8 + \phi_9 + \ldots \qquad (3),$$

$$\text{and} \quad 2e = 4\phi_4 + 5\phi_5 + 6\phi_6 + 7\phi_7 + 8\phi_8 + 9\phi_9 + \ldots \qquad (4).$$

From equations (1), (2), (3) and (4) one obtains

$$2\phi_4 + \phi_5 = 12 + \phi_7 + 2\phi_8 + 3\phi_9 + \ldots \qquad (5)$$

Because each of the ϕ's is a positive integer or zero it follows from equation (5) that ϕ_4 and ϕ_5 cannot both be zero. Thus, the dual of a prime graph always* contains vertices of order 4 or 5.

3.7. Kempe's method proceeds by induction. Assume that all prime duals with n, or fewer, vertices are four-colourable and consider the

* With the sole exception of the prime graph G_3.

vertex colouring of a dual with n + 1 vertices. By the previous argument this dual must contain vertices of order 4 or 5. First suppose that the dual has a vertex, V, of order 4. If V is merged with any one of its four neighbours, by contracting the edge which joins them, one obtains a dual with n vertices which, by assumption, is four-colourable. The problem therefore is to assign a colour to V given a four-colouring of the other n vertices of the dual.

3.8. If the four vertices neighbouring V are coloured* using only two colours, as shown in Fig. 19(a), or three colours (Fig. 19(b)), then clearly V can be coloured appropriately. Assume then that the neighbouring vertices are differently coloured a, b, c and d. Suppose that the vertices coloured a and c (Fig. 19(c)) are linked by a chain of edges which passes through vertices alternately coloured a and c. This ac chain, together with the two edges which form a return path through V, form a closed circuit. A new four-colouring is obtained by interchanging the colourings of those vertices within this circuit which were originally coloured b and d, the colourings of the vertices outside the circuit being unaltered. The vertex adjacent to V which was originally coloured b is now coloured d (choosing, arbitrarily, the 'inside' of the ac circuit to be that region which contains this vertex), thus releasing b for the colouring of V. If the vertices coloured a and c are not linked by an ac chain then a bd chain must link the neighbour vertices coloured b and d (Fig. 19(d)). A similar argument releases c for the colouring of V. Hence, in all possible circumstances, a vertex of order 4 can be coloured.

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* In this chapter the symbols a, b, c and d will denote colourings rather than distinctive labellings of the vertices.

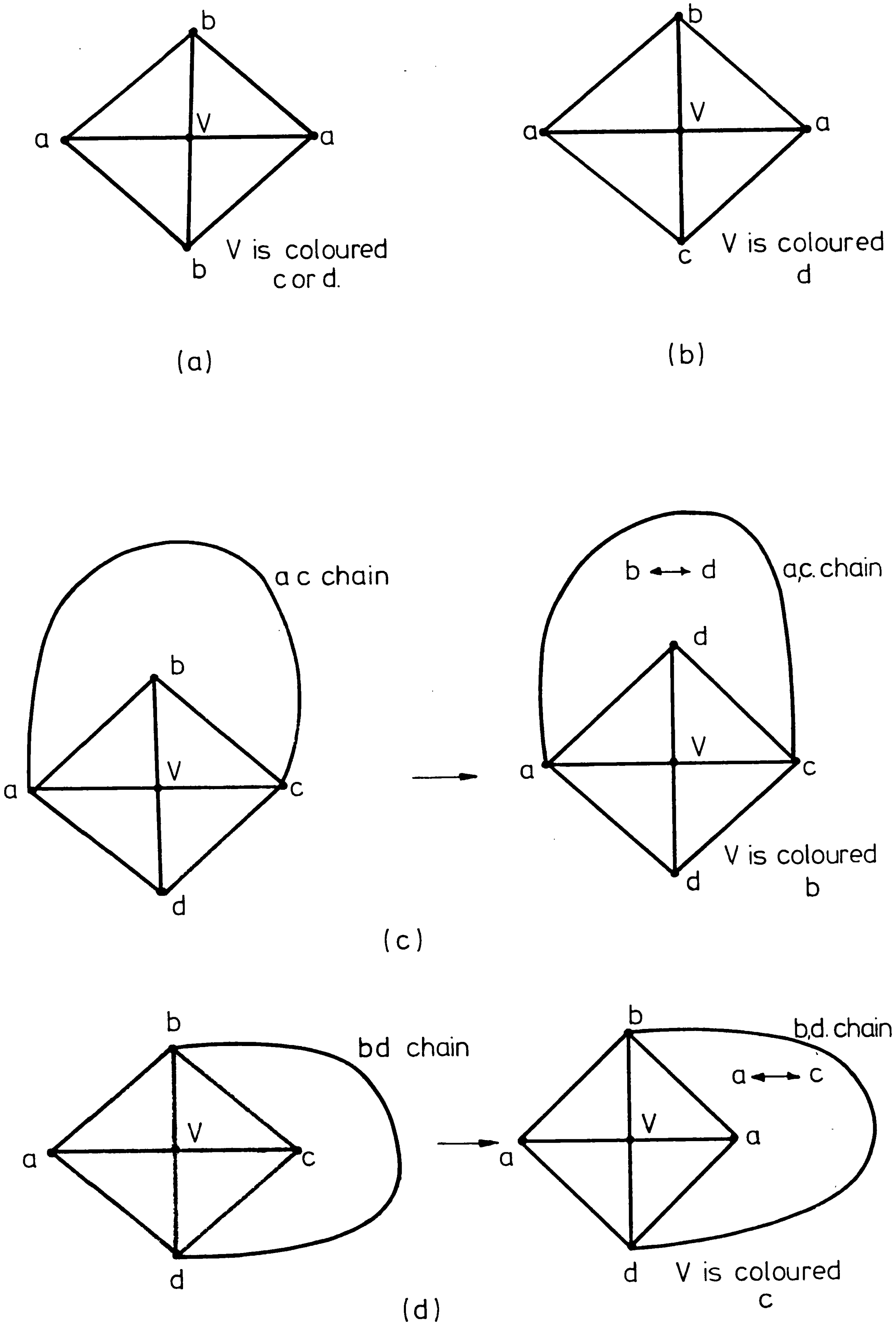

FIG. 19. Colouring a Vertex of order 4

3.9. If the dual with n + 1 vertices does not contain a vertex of order 4 then it must contain a vertex V of order 5. The problem is to assign a colour to V given a four-colouring of the other n vertices of the dual. If the five neighbour vertices use only three of the four colours then V is assigned the fourth colour (Fig. 20(a)). If the neighbouring vertices use all four colours then, without loss of generality, one may suppose them to be coloured as in Fig. 20(b). If a bc chain exists then, by the previous colour reversal argument, the neighbour vertex coloured d may be re-coloured a thus releasing d for the vertex V. A similar argument applies if a bd chain exists (Fig. 20(c)). If neither a bc chain nor a bd chain exists then there must be both ad and ac chains. Suppose, as in Fig. 20(d), that these chains do not intersect, then the neighbour vertices coloured b may be re-coloured c and d respectively thus releasing b for the colouring of V. If, however, the ad and ac chains intersect, as for example in Fig. 20(e), then it might not be possible to interchange the colours within each independently. Following an interchange within the ad circuit the ac chain might be broken. In this case a bd chain exists but if this bd chain intersects the ad chain the argument is not advanced because the new situation (Fig. 20(f)) is essentially similar to that which preceded it (Fig. 20(e)). (A similar situation might arise if the initial interchange is within the ac circuit).

3.10. It was Kempe's failure to appreciate fully this latter possibility which led him to believe that he had proved the four-colour conjecture because, but for this special case, he had shown that any dual with n + 1 vertices could be four-coloured if all duals with n vertices are four-colourable and the induction seemed complete. Kempe's 'proof' went

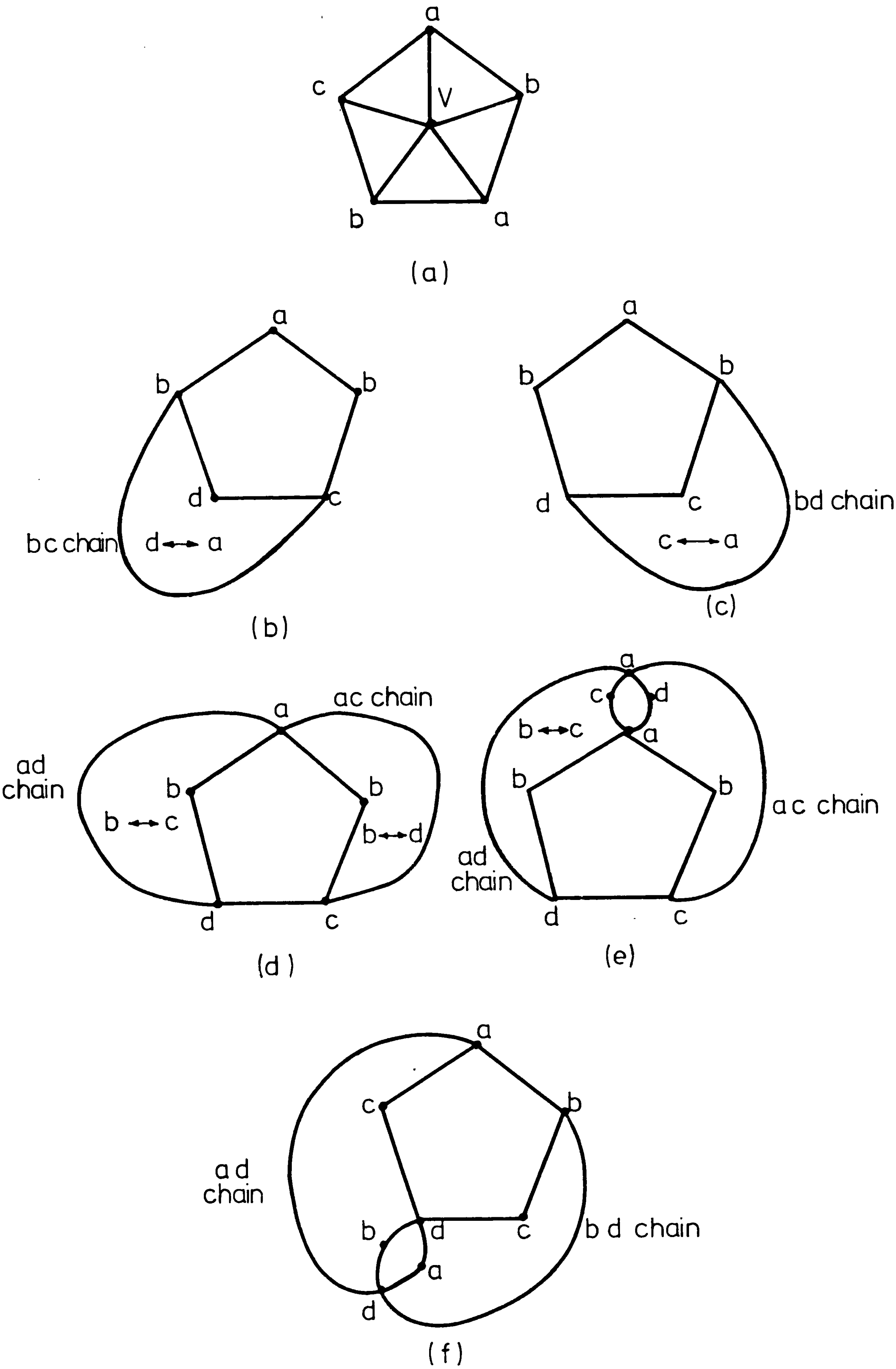

FIG. 20. Colouring a Vertex of order 5

unchallenged for eleven years until the error was pointed out by Heawood[10], who, by using an obvious extension of Kempe's method, proved that planar graphs are five-colourable.

A Colouring Algorithm.

3.11. The author has devised and explored the following algorithm, suggested by the special case referred to above, for the colouring of a vertex of order 5. The algorithm involves an iterative sequence of interchanges and in order to define precisely the interchange operation it is necessary to consider the possibility of more than one chain, as illustrated in Fig. 21. The interchange is to be effected within the innermost circuit - i.e. that circuit formed by an ac chain, together with the two corresponding edges through V, which is not bridged internally by some part of an ac chain. (The inside of the circuit is that region which contains just one of the vertices neighbouring V; the outside contains two such vertices.). Clearly the innermost circuit, so defined, is unique.

3.12. The procedure is illustrated by Fig. 22, where it is assumed that the starting situation, Fig. 22(a), has both ad and ac circuits which might, or might not, intersect. Initially one may interchange either within the (innermost) ac circuit or within the (innermost) ad circuit: suppose that the ac circuit is chosen. Following an interchange within the ac circuit either an ad chain exists, in which case an interchange within it releases b for the colouring of V, or a bc chain exists, in which case the situation is that shown in Fig. 22(b). This completes one iteration. The next iteration involves an interchange within the bc circuit, following which either an ac chain exists, and the algorithm

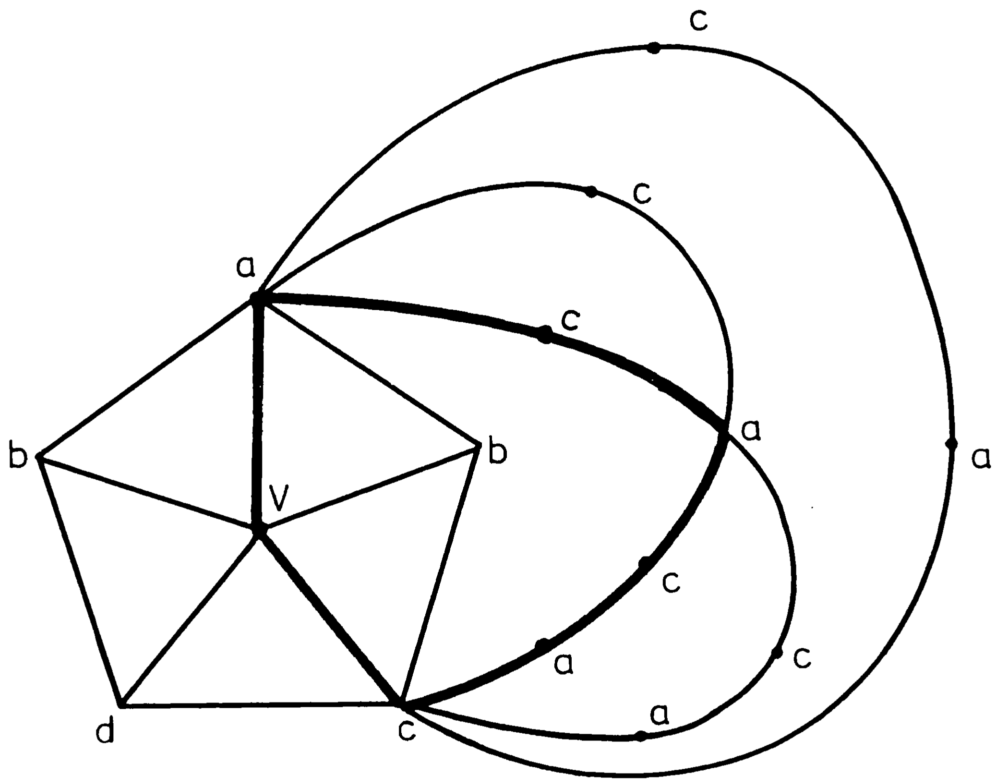

FIG. 21 Innermost circuit

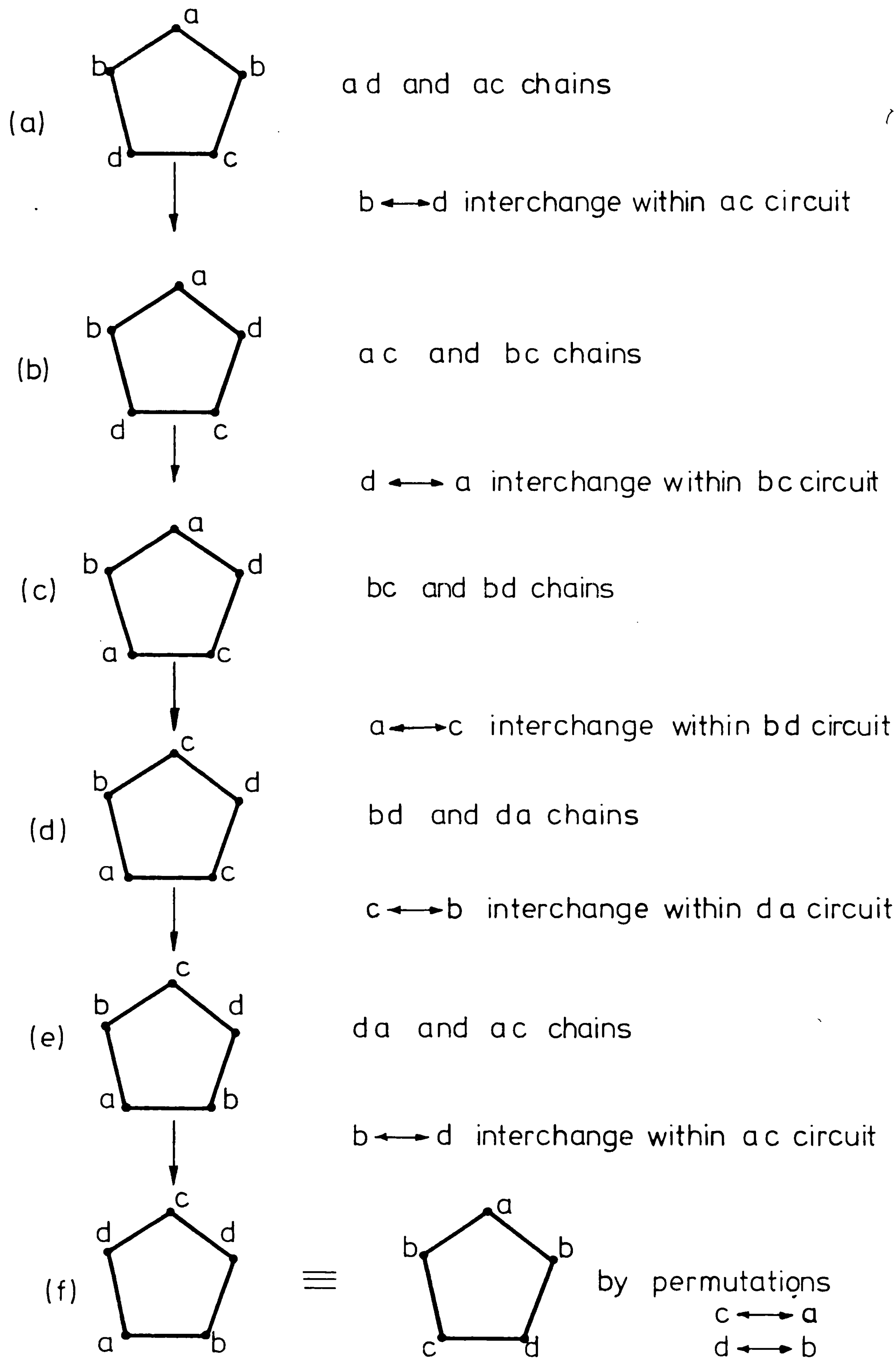

FIG. 22. Colouring algorithm

terminates in a solution (the release of d for the colouring of V), or a bd chain exists and the algorithm proceeds by a further iteration. Because the total number of four-colourings of the dual is finite ($< 4^n$) either the algorithm terminates after a finite number of steps or it closes on itself to restore the original four-colouring of Fig. 22(a). In the latter case had the initial interchange been within the ad circuit instead of the ac circuit the cycle of iterations would have been traversed in the opposite direction, the algorithm being reversible.

3.13. Suppose that the algorithm progresses through five iterations from the situation of Fig. 22(a) to that of Fig. 22(f). In order to compare these four-colourings it is convenient to permute the colours of Fig. 22 (f) by the interchanges c $\leftrightarrow$ a and d $\leftrightarrow$ b throughout the dual (this permutation does not constitute a new four-colouring). It is apparent that the colouring of Fig. 22(f) can not be identical to that of Fig. 22(a) because the colourings of two of V's neighbours are reversed. However a further sequence of five interchanges will reverse these yet again. Consequently in order that the algorithm should close on itself, rather than halt at a solution, the number of iterations in the cycle must be an integral multiple of 10. The author has attempted to construct a dual for which the algorithm does cycle in this manner but so far he has failed to find one. Such a dual, if it exists, would not necessarily provide a counter-example to the four-colour conjecture (it would merely indicate a failure of this particular colouring algorithm) but it would have interesting properties, probably exhibiting some form of symmetry based on the number 10.

Conjectures Equivalent to the Four-colour Conjecture.

3.14. Tait[22] has shown that if a trivalent planar graph is face-colourable in four colours then the edges of the graph may be coloured using only three colours in such a way that the three edges which meet at any vertex are differently coloured. Conversely, if the edges of the graph are three-colourable, in the above sense, then the faces are four-colourable. Consequently the conjecture that the edges of a trivalent planar graph are three-colourable is equivalent to the four-colour conjecture. Correspondingly in the dual, if the vertices are four-colourable then the edges are three-colourable, the three edges bounding each triangular face being differently coloured.

3.15. A Tait three-colouring of the dual is obtained immediately from a given four-colouring of the vertices by the following rule:-

$$\text{An edge which links vertices coloured} \begin{cases} \text{a b} & \text{or} \quad \text{c d} \\ \text{a c} & \text{or} \quad \text{b d} \\ \text{a d} & \text{or} \quad \text{b c} \end{cases}$$

$$\text{is coloured} \begin{cases} 1 \\ 2 \\ 3 \end{cases} .$$

Because the three vertices associated with any triangular face of the dual are differently coloured it immediately follows that no two edges of the face can be assigned the same edge colouring, and the condition for a three-colouring is satisfied. The procedure is illustrated by Fig. 23 which shows a four-colouring of G_6' together with the corresponding Tait three-colouring. A trivalent planar graph which possesses a Hamiltonian circuit, or a collection of mutually exclusive subcircuits

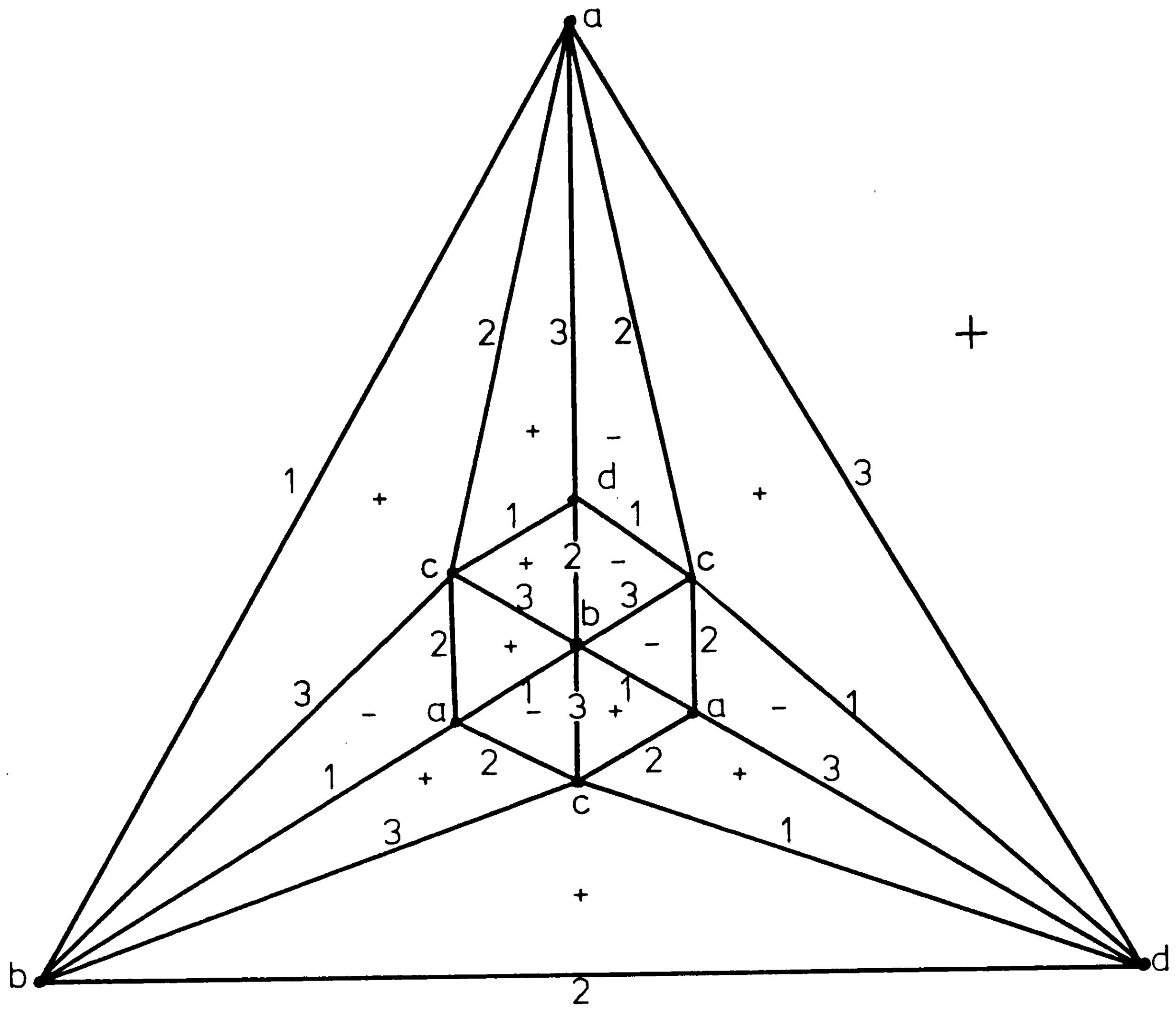

FIG. 23. Tait and Heawood colourings of G'_6

all of even length (see para. 3.4) is always Tait three-colourable and hence four-colourable, for one may associate the edge colours 1 and 2 with alternate edges of the H-circuit (which is always of even length) or of each of the subcircuits, the non- circuit edges being assigned the colour 3.

3.16. Heawood[10] obtained an equivalent conjecture which involves a two-colouring of the vertices of a trivalent planar graph, or, correspondingly, a two-colouring of the faces of the dual. Suppose that a Tait colouring of the dual is given and consider, for each face, a cyclic tour of the three bounding edges such that the edge colourings are encountered in the order $1 \to 2 \to 3$. A face is assigned the value +1 or -1 according as the tour is described in the clockwise or anticlockwise direction. This procedure is exemplified in Fig. 23 which shows the Heawood two-colouring as well as the Tait three-colouring and vertex four-colouring. The consistency of the Heawood and Tait colourings requires, for every vertex of the dual, that the summation of the face values around a vertex shall be zero modulo 3. Conversely, if the faces of a dual can be assigned values x_i (+1 or -1) such that around each vertex $\Sigma x_i \equiv 0 \pmod 3$, then the dual has a Tait colouring and hence a four-colouring. The sign convention, as defined above, is arbitrary and a reversal of the signs of all the x_i does not constitute a new Heawood colouring. Consequently, without loss of generality, one of the faces may be arbitrarily assigned the fixed value +1.

3.17. Because there exists a one-one relationship between a Heawood colouring and a four-colouring, the family of four-colourings of a graph may be obtained as the solution set of a system of congruence equations and Heawood[10,16] has investigated the properties of such a system. The Heawood formulation is well suited to the construction of a computer algorithm for the determination of the solution set but the extent of the search makes it impracticable to use a computer for other than relatively simple graphs. Given that f is the number of faces of a trivalent planar graph then, by Euler's formula, the number of vertices is 2f - 4. The dual has 2f - 4 faces, one of which may be assigned the fixed value +1, and the determination of the solution set of the congruence equations requires a search through 2^{2f-5} combinations of values - a number which exceeds one million for a graph with as few as thirteen faces.

A Parity Algorithm.

3.18. The author has devised a procedure which, by reducing the field of search from 2^{2f-5} combinations to 2^{f-3} combinations, allows the solution set of the Heawood congruence equations to be obtained more economically. The author's primary interest is not, however, in the full solution set but rather in the parity (i.e. whether the number of four-colourings is odd or even) and the procedure has been developed into an algorithm for determining the parity of a given prime graph. The method will be illustrated by means of an example.

3.19. Whitney[27] has shown that the dual of any prime graph (as defined in paras. 2.1 and 2.3) has a Hamiltonian circuit. Given a prime dual the algorithm begins by choosing (arbitrarily) some Hamiltonian circuit of the dual and assigning symbols x_i to those faces lying on one side of the circuit and symbols y_i to those faces lying on the other side. If the given graph has f faces then the dual has f vertices and 2f - 4 faces, f - 2 faces on each side of the Hamiltonian circuit. Each of the f vertices of the dual gives rise to a congruence equation but the equations are not independent and it is sufficient to consider only f - 2 equations provided that the two omitted relate to a pair of vertices (any pair) which are adjacent on the Hamiltonian circuit. The resultant set of congruences comprises f - 2 equations relating the f - 2 x values and the f - 2 y values. The matrix of coefficients for these equations may be partitioned in the form

$$[X\ Y]\ ,$$

where X and Y are the square matrices of order f - 2 formed by the coefficients of the x_i and y_i respectively. By using a Hamiltonian circuit to partition the coefficients it is always possible so to arrange the equations that either X or Y is a triangular matrix with each of the elements of the leading diagonal equal to unity*. Hence both X and Y are non-singular and the equations can be solved for the x_i in terms of the y_i, or vice-versa.

3.20. The procedure is illustrated in Fig. 24 which shows the dual G_4' (Fig. 4) re-drawn so as to emphasise an arbitrarily chosen Hamiltonian circuit. The set of congruence equations arising from the vertices 1, 2, 3 and 4, arranged so that X is a triangular matrix, is

vertex

$$\left.\begin{array}{lllllllllll}
4 & x_1 + x_2 + x_3 & & + y_1 & & & & = 0 \pmod 3 \\
1 & & x_2 + x_3 + x_4 & & & + y_3 & & = 0 \\
3 & & & x_3 + x_4 + y_1 + y_2 & & & & = 0 \\
2 & & & x_4 & + y_2 & + y_3 & + y_4 & = 0
\end{array}\right\} \qquad (1)$$

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* A general proof that a Hamiltonian partitioning has this property is given in the Appendix.

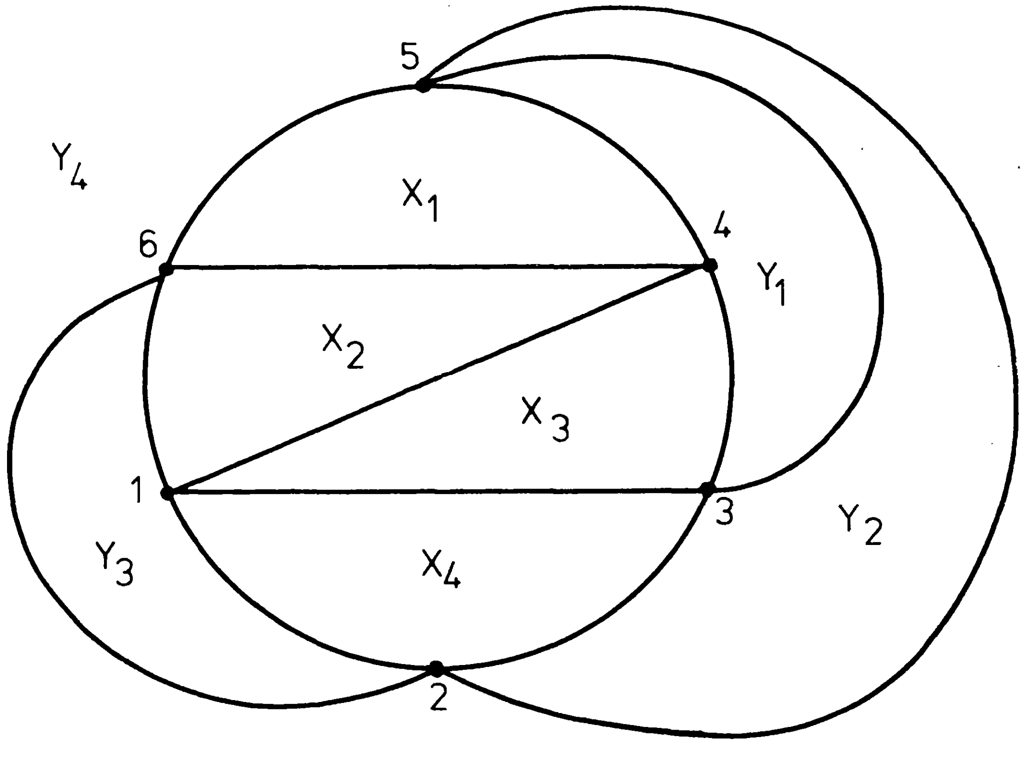


FIG. 24. Hamiltonian partitioning of G_4^1

(If the equations are written in the vertex order 1, 3, 2 and 4 then Y is a triangular matrix).

Solving these equations for the x_i in terms of the y_i gives

$$\left.\begin{aligned} x_1 &= -(y_1 + y_2 + y_4) \\ x_2 &= (y_1 + y_2 - y_3) \\ x_3 &= -(y_1 - y_3 - y_4) \\ x_4 &= -(y_2 + y_3 + y_4) \end{aligned}\right\} \qquad (2)$$

It will be observed that these values automatically satisfy the congruence equations which arise from vertices 5 and 6. Because each of the x_i and y_i in the Heawood solution set has the value +1 or -1 the full set of solutions is obtained from the equations

$$\left.\begin{aligned} y_1 + y_2 + y_4 &= \pm 1 \pmod 3 \\ y_1 + y_2 - y_3 &= \pm 1 \\ y_1 - y_3 - y_4 &= \pm 1 \\ y_2 + y_3 + y_4 &= \pm 1 \end{aligned}\right\} \qquad (3)$$

Because one of the y_i may be arbitrarily assigned the value +1 the number of variables involved in the reduced set of congruences, (3), is three, as compared with seven in the original set of congruences, (1). Taking $y_1 = +1$, a search through the eight possible combinations of y_2, y_3 and y_4 reveals the set of four solutions

$$(y_1, y_2, y_3, y_4) = \left\{\begin{array}{l} (1, 1, 1, -1) \\ (1, -1, 1, -1) \\ (1, -1, 1, 1) \\ (1, -1, -1, 1) \end{array}\right\},$$

the corresponding values of the x_i being obtained from (2), and hence

G_4 has a family of 4 four-colourings. In general terms the reduction procedure removes f - 2 variables from the Heawood congruence equations, thus reducing the search field from 2^{2f-5} to 2^{f-3} combinations.

3.21. The set of equations (3) can be combined into a single equation by multiplying together the expressions on the left-hand side and equating the result to ± 1, the resultant equation being simplified by setting $y_i^2 = 1$ for all i (since $y_i = \pm 1$) and by combining like terms using modulo 3 arithmetic. For the given example the result is

$$y_1\, y_2 + y_1\, y_3 - y_2\, y_4 + y_3\, y_4 + y_1\, y_2\, y_3\, y_4 = \pm 1$$

or, after setting $y_1 = +1$,

$$y_2 + y_3 - y_2\, y_4 + y_3\, y_4 + y_2\, y_3\, y_4 = \pm 1 \tag{4}$$

A search algorithm based on equation (4) rather than the set of equations (3) requires only one test instead of, in general, f - 2 tests for each possible combination of the y_i.

3.22. The form of equation exemplified by (4) involves products of the variables but is necessarily linear in each of the variables considered separately and is well suited to an algorithm for determining the parity of a solution set. One of the variables, y_2 say, is selected and the equation written in the form $A + y_2 . B = \pm 1$, where A and B are functions of the variables other than y_2. For any particular combination of these other variables each of A and B has the value 0, +1 or -1 (the arithmetic being modulo 3), hence the possible situations are:-

a) $\underline{A = 0,\ B = 0}$. In this case there is no value of y_2 which satisfies $A + y_2 . B = \pm 1$. Consequently this particular combination of variables is not a member of the solution set.

b) <u>$A = 0$, $B = \pm 1$.</u> In this case $A + y_2 . B = \pm 1$ either if $y_2 = +1$ or if $y_2 = -1$, irrespective of the values of the other variables (providing $A = 0$ and $B = \pm 1$). Hence the situation gives rise to two solutions.

c) <u>$A = \pm 1$, $B = 0$.</u> Again $A + y_2 . B = \pm 1$ either if $y_2 = +1$ or if $y_2 = -1$. This situation also gives rise to two solutions.

d) <u>$A = \pm 1$, $B = \pm 1$.</u> Because $1 + 1 = -1 \pmod 3$ and $-1 - 1 = +1 \pmod 3$ it follows that for each of the four possible combinations of values of A and B there is a unique value of y_2 (+1 or -1) such that $A + y_2 . B = \pm 1$. This situation gives rise to one solution.

Because cases a, b and c each contribute an even, or zero, number of solutions to the solution set of (4) it follows that the parity of the solutions contributed by category d situations is the same as that of the total solution set. The equations $A = \pm 1$ and $B = \pm 1$ may be combined into a single equation $A. B = \pm 1$ which, after multiplying out, is similar in form to (4) but has one fewer variable. The process is repeated, eliminating one variable at each step, until either A or B is identically zero or only a single variable remains and the constants A and B are both ± 1. Because the parity is preserved at each iteration the parity of the original equation (4), and hence of the associated graph, is odd if the final values of A and B are both ± 1; otherwise the parity is even.

<u>3.23</u>. The first step in applying this algorithm to equation (4)

$$y_2 + y_3 - y_2 y_4 + y_3 y_4 + y_2 y_3 y_4 = \pm 1 \tag{4}$$

is to express the equation in the form

$$(y_3 + y_3 y_4) + y_2 (1 - y_4 + y_3 y_4) = \pm 1.$$

Thus, $A = (y_3 + y_3 y_4)$ and $B = (1 - y_4 + y_3 y_4)$
and the reduced equation is

$$(y_3 + y_3 y_4)(1 - y_4 + y_3 y_4) = \pm 1 ,$$

or, on multiplying out,

$$1 + y_4 = \pm 1 \qquad (5)$$

The second step amounts to expressing equation (5) in the form (linear in y_3),

$$(1 + y_4) + y_3 . \ 0 = \pm 1 .$$

Thus $A = (1 + y_4)$ and $B = 0$. Because $B = 0$ the algorithm terminates and the parity of the solution set of equation (4) (and hence the parity of the graph G_4) is even.

3.24. The algorithm described above enables the parity of any prime graph to be determined without the necessity of obtaining the full solution set - i.e. the number of four-colourings, but the procedure is tedious for all but the simplest graphs. The author's research in this area was motivated by the hope that some insight might be gained into those properties of a graph which determine the parity, but it would appear that, in general, there is no simple topological formula for parity and any parity algorithm must involve a tedious combinatorial process.

3.25. An interesting result concerning the non-existence of four-colourings in a prime graph is obtainable using the methods of the preceding paragraphs. For a given prime dual, the procedure described in paragraphs 3.19 - 3.21 leads to an equation of the form

$$\begin{aligned}
&c_o + c_2\, y_2 + c_3\, y_3 + \cdots\cdots \\
&\quad + c_{23}\, y_2\, y_3 + c_{24}\, y_2\, y_4 + \cdots\cdots\cdots \\
&\quad + c_{234}\, y_2\, y_3\, y_4 + c_{235}\, y_2\, y_3\, y_5 + \cdots\cdots \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad = \pm 1 \qquad (6)
\end{aligned}$$

A necessary and sufficient condition that this equation has no solutions, and hence the graph has no four-colourings, is that each of the coefficients on the left-hand size is zero (mod 3). Clearly the condition is sufficient. To show that the condition is necessary let the left-hand side be expressed in the form

$$A + y_2 . B$$

where $\quad A = c_o + c_3\, y_3 + \cdots\cdots + c_{34}\, y_3\, y_4 + \cdots\cdots + c_{345}\, y_3\, y_4\, y_5 + \cdots,$

$$B = c_2 + c_{23}\, y_3 + \cdots\cdots + c_{234}\, y_3\, y_4 + \cdots\cdots + c_{2345}\, y_3\, y_4\, y_5 + \cdots$$

Given that equation (6) has no solutions it follows that $A + B = 0$ and $A - B = 0$ and hence, for all combinations of the y_i, $i > 2$, $A = 0$ and $B = 0$. Now A and B are both similar in form to the expression on the left of (6), but with one fewer variable. The above procedure may be repeated, eliminating one variable at each step, the ultimate result being that each of the coefficients in (6) is identically zero. This result must, of course, apply irrespective of whichever Hamiltonian circuit in the dual is used to partition the original set of Heawood congruences and therefore implies some relationship between the coefficients in these congruence equations. A further understanding of this relationship might offer insight into the topological properties of a counter-example to the four-colour conjecture, if one exists.

Conclusion.

3.26. The Tait and Heawood conjectures are but two of many possible variations[19] on Guthrie's original conjecture. The literature on the four-colour problem is very extensive but, surprisingly in view of the long history of the problem, not until 1967 were the significant results of research in this field brought together in the form of a text-book[16] devoted solely to the subject. Although the problem has so far defied solution the attempts to solve it have generated new concepts and tools which have greatly enriched the field of graph theory. This chapter has high-lighted just a few aspects of the problem, emphasising those early ideas which the author has found helpful and have led him to develop algorithms which extend the work of Kempe and Heawood. The hope that these developments might result in a simple topological parity formula has not materialised. Such a formula would have had a bearing on later work because, as will be shown in the next chapter, the parity of the family of four-colourings is the same as that of the family of Hamiltonian circuits.

CHAPTER 4

HAMILTONIAN CIRCUITS

4.0. This chapter, which forms the principal part of the thesis, gives an account of the author's research into the properties of families of Hamiltonian circuits and of the problems encountered in the search for existence theorems. The chapter begins with a study of the relationships between the family of H-circuits of a graph and the family of four-colourings. Subsequent sections describe two algorithms, both independent of four-colourings, for the generation of a family of H-circuits. The first of these is essentially topological in nature while the second involves a combinatorial process. The chapter concludes with an existence theorem based on the parity properties of graphs and their families.

Four-colourings and Hamiltonian Circuits

4.1. The upper part of Fig. 25 shows a four-colouring, C_1, together with the corresponding Tait colouring, of a prime dual G'_9. It will be observed that the edges coloured 2 together form an H-tree. Likewise the edges coloured 3 constitute an H-tree. To each of these H-trees there corresponds an H-circuit in the graph G_9. The edges coloured 1 do not form an H-tree because four of these edges make a closed circuit. Viewed from the standpoint of the vertex four-colouring these four edges constitute a cd circuit within which, using Kempe's principle of interchange, one may transpose the vertex colours a and b to obtain another four-colouring C_2, shown in the lower part of Fig. 25. From the standpoint of the Tait colouring this amounts to an interchange of

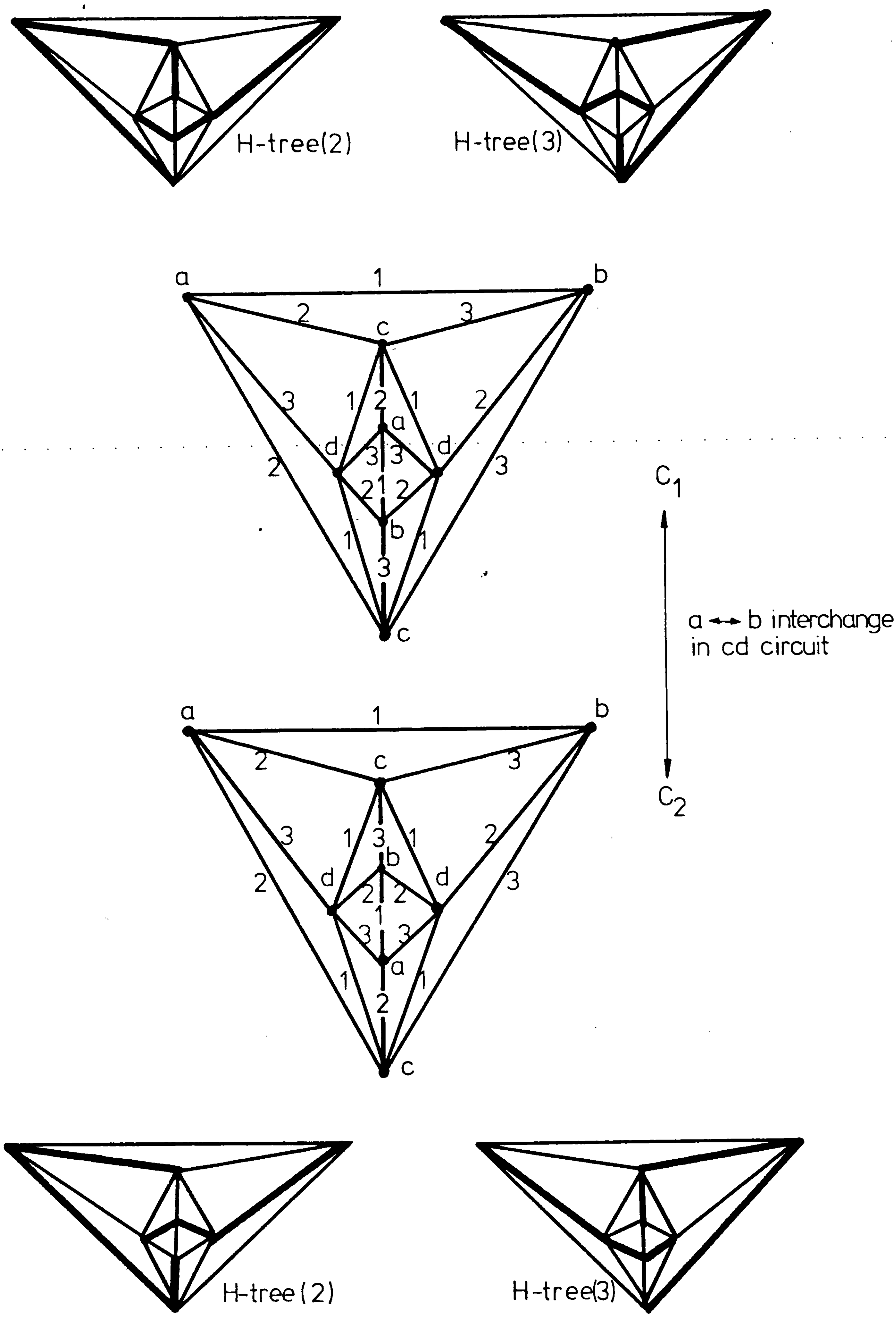


FIG. 25. Colouring of $G_q^{'}$

the edge colourings 2 and 3 within the circuit of edges coloured 1. The interchange operation will be said to <u>transform</u> the colouring C_1 into the colouring C_2 (or vice-versa, the operation being reversible). Associated with C_2 are two more H-trees, formed by the edges coloured 2 and the edges coloured 3. The dual G_9' has one other four-colouring, C_3, not linked with C_1 or C_2. This colouring is shown in Fig. 26. Associated with C_3 are three H-trees formed by the edges coloured 1, 2 and 3 respectively, there being no circuits.

<u>4.2</u>. The relationship between the family of three four-colourings of G_9' and the family of seven H-trees may conveniently be depicted by the <u>transformation diagram</u> of Fig. 27. In this diagram each circle represents a four-colouring and a line joining two circles represents a transformation between colourings (i.e. a circuit interchange), the number associated with the line being the Tait colouring of the edges which form the circuit. Each dot represents an H-tree and the line joining the dot to a circle relates the H-tree to its parent four-colouring. The line is labelled with the Tait colouring of the edges which form the H-tree. The transformation diagram for G_9' consists of two separate <u>components</u>.

<u>4.3</u>. The transformation diagram for any dual may be constructed in a similar manner but in order that the diagram shall be unique it is necessary to formulate certain rules of procedure:-

a) The diagram is valid with respect to some chosen face of the dual. Without loss of generality it is convenient to choose the infinite face as the <u>basis</u> of the transformation because it is always possible to draw the dual in such a way that any arbitrarily chosen face becomes the infinite face.

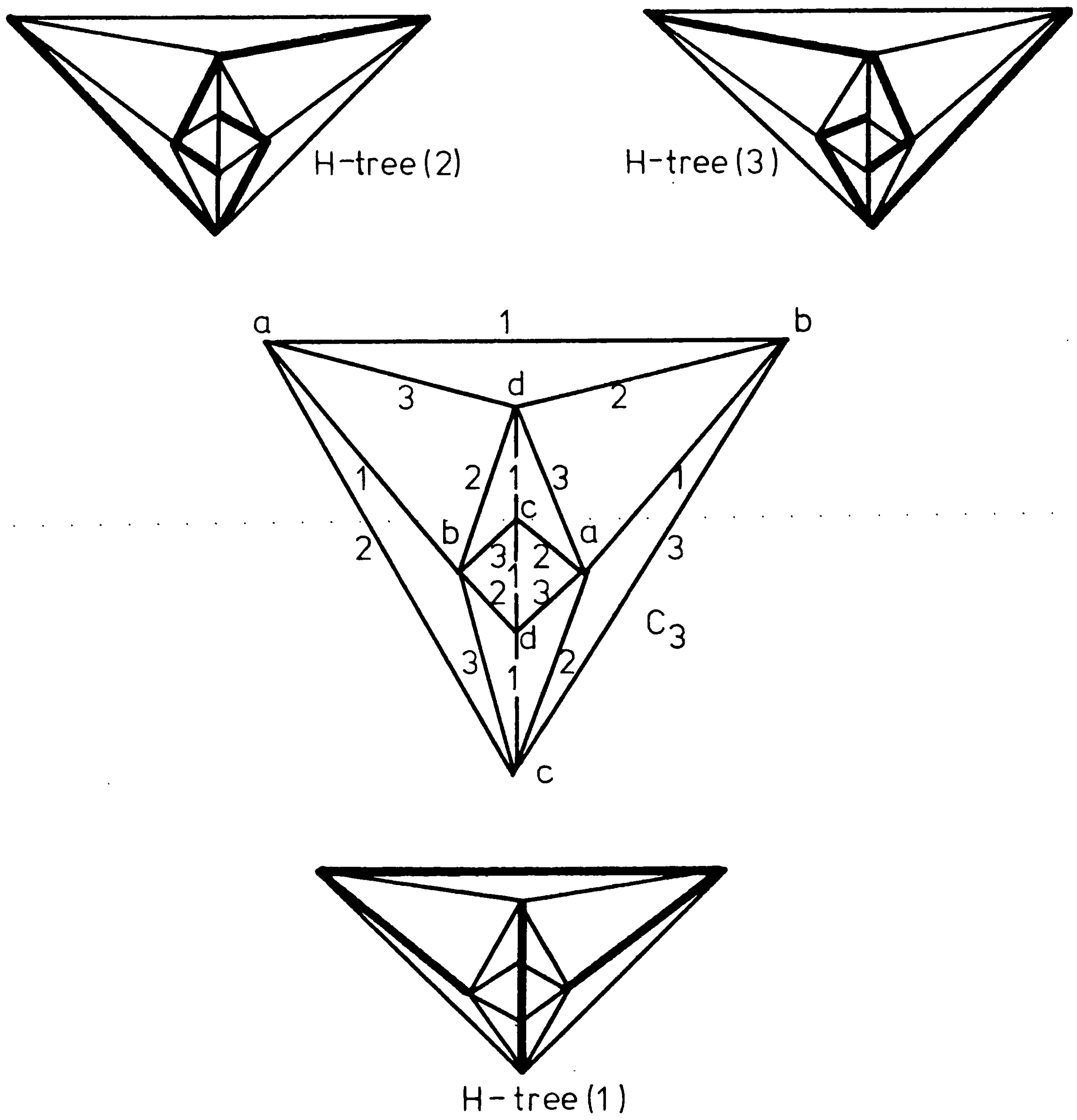


FIG. 26 A colouring of G_g^1.

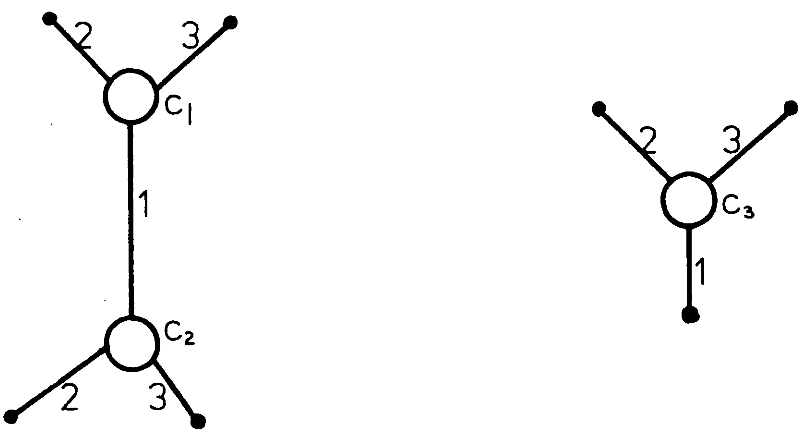


FIG. 27 Transformation diagram for G_g^1

b) The <u>outside</u> region of any circuit within which a colour interchange is effected is defined as that region which contains the infinite face of the dual. Thus an interchange <u>within</u> a circuit leaves unaltered the vertex and edge colourings associated with the infinite face. If the transformation diagram contains more than one component the vertices and edges of the infinite face are assigned the same colourings for each component. These colourings are the invariants of the transformation.

c) For a given four-colouring the edges of the dual which have a particular Tait colouring may, in general, form more than one circuit. The transformation is defined as applying simultaneously to all these circuits - i.e. all vertex (and edge) colourings which are eligible for interchange are so changed.

Each component of a transformation diagram constructed in accordance with the foregoing rules is characterised by:

a set of circles (four-colourings), each having three <u>links</u> with other circles or dots (one link for each of the three Tait colourings);

a set of dots (H-trees), each having one link with a circle;

and a set of links, each bearing a number (Tait colouring) which relates to a specific edge of the infinite face of the dual.

A change in basis (making some other face of the dual the infinite face) leaves the topological structure of the diagram unaltered but, in general, this change results in a different numbering of the links.

<u>4.4</u>. From the properties of the transformation diagram one derives certain general relationships between the four-colourings and the H-trees of any dual G′. For the <u>sub-family</u> of four-colourings represented by

any component of the diagram let

c be the number of four-colourings,

h_k ($k = 1$, 2 or 3) be the number of H-trees with Tait colour k,

and $h = h_1 + h_2 + h_3$ be the total number of H-trees.

Consider any link labelled k. Either this link connects two four-colourings or it joins a four-colouring to an H-tree included in the number h_k. Every four-colouring of the component and every H-tree included in h_k is associated with one such link. Hence h_k and c are either both even or both odd. It follows that h_1, h_2, h_3, h and c all have the same parity. Because each H-tree in G' is in one-one correspondence with an H-circuit in G, the following property holds for any graph G:

Parity property. For each separate sub-family, and therefore for the family as a whole, the parity of the H-circuits is the same as the parity of the four-colourings.

4.5. By definition h_k is the number of H-trees which include, as a tree edge, that edge of the infinite face of the dual which has the (invariant) Tait colouring, k. The number of H-trees which exclude this edge, $h - h_k$, is necessarily even (or zero) because h and h_k have the same parity. A change in basis results, in general, in a change in the values of h_1, h_2 and h_3, but the parities of the h_k are unaffected because h and c are invariant. Hence, for each sub-family, the number of H-trees which exclude any given edge of G' is even (or zero). Thus the following property holds for any graph G:

Closure property. For each separate sub-family, and therefore for the family as a whole, the number of H-circuits which include any given edge of G is either even or zero.

Suppose that some H-circuit is given. This alone does not satisfy the closure property, therefore a second H-circuit exists. Because this is distinct from the first the two H-circuits will have some, but not all, of their edges in common. These two circuits together fail to satisfy the closure property and therefore at least one more H-circuit is required to ensure closure - i.e. to form a complete sub-family. Thus, if a graph has one H-circuit it must have at least three. In its global form - i.e. as applied to the family as a whole - the closure property (referred to in para. 2.6 as the property P1) was established by Tutte[23] in 1946. In the same year a further proof of this property was communicated privately to Berge by C.A.B. Smith[21].

4.6. The author's investigations into the properties of transformation diagrams were conducted in the hope that an answer might be found to the following question:

Given that a prime graph G has a (non-empty) family of H-circuits, in what circumstances can it be proved that the number of H-circuits which pass through any given edge of G is non-zero?

The question can be answered for those graphs whose transformation diagrams exhibit one or both of two specific features:

a) Suppose that a sub-family (represented by a component of the transformation diagram) has odd parity. Then $h_k > 0$ ($k = 1$, 2 or 3), and therefore $h - h_k > 0$. Hence, by the argument of paragraph 4.5, H-circuits pass through any given edge of any graph which has at least one sub-family of odd parity irrespective of the global parity of the graph.

b) Suppose that the transformation diagram includes a four-colouring which has links with two H-trees. Because the H-trees have different Tait colourings it follows that the tree edges of one are distinct from the tree edges of the other; hence the two corresponding H-circuits together include every edge of the graph.

Each component of the transformation diagram shown in Fig. 27 exhibits one, or both, of these properties. The same is true of the transformation diagram for G_5' (vide Fig. 7) which is given in Fig. 29. Neither property applies to the transformation diagram for G_6' shown in Fig. 28. This diagram has only one component, of even parity, with one H-tree associated with each four-colouring. Nevertheless, because the H-trees do not all have the same colouring it follows that H-trees exist which exclude any given edge of the infinite face of G_6'. By changing the basis it can be shown that the same property holds for every face; hence the family of H-circuits together include every edge of the graph G_6.

4.7. If a component of the transformation diagram of a graph does not possess the property (b) above, then each four-colouring of the component must be included in a closed circuit of linkages with other four-colourings. If the parity of this component is even then it is possible, for some basis, that all of its H-trees will have the same colouring. Such a component, considered in isolation, does not support the hypothesis that every edge of the parent graph is included in at least two H-circuits. The prime dual G_{10}', shown together with its transformation diagram in Fig. 30, has been constructed to illustrate this property. Every edge of G_{10} is included in at least two H-circuits because the larger component of the transformation diagram exhibits property (b), but this cannot be proved by applying the transformation procedure to the four-colouring C1 (Fig. 30).

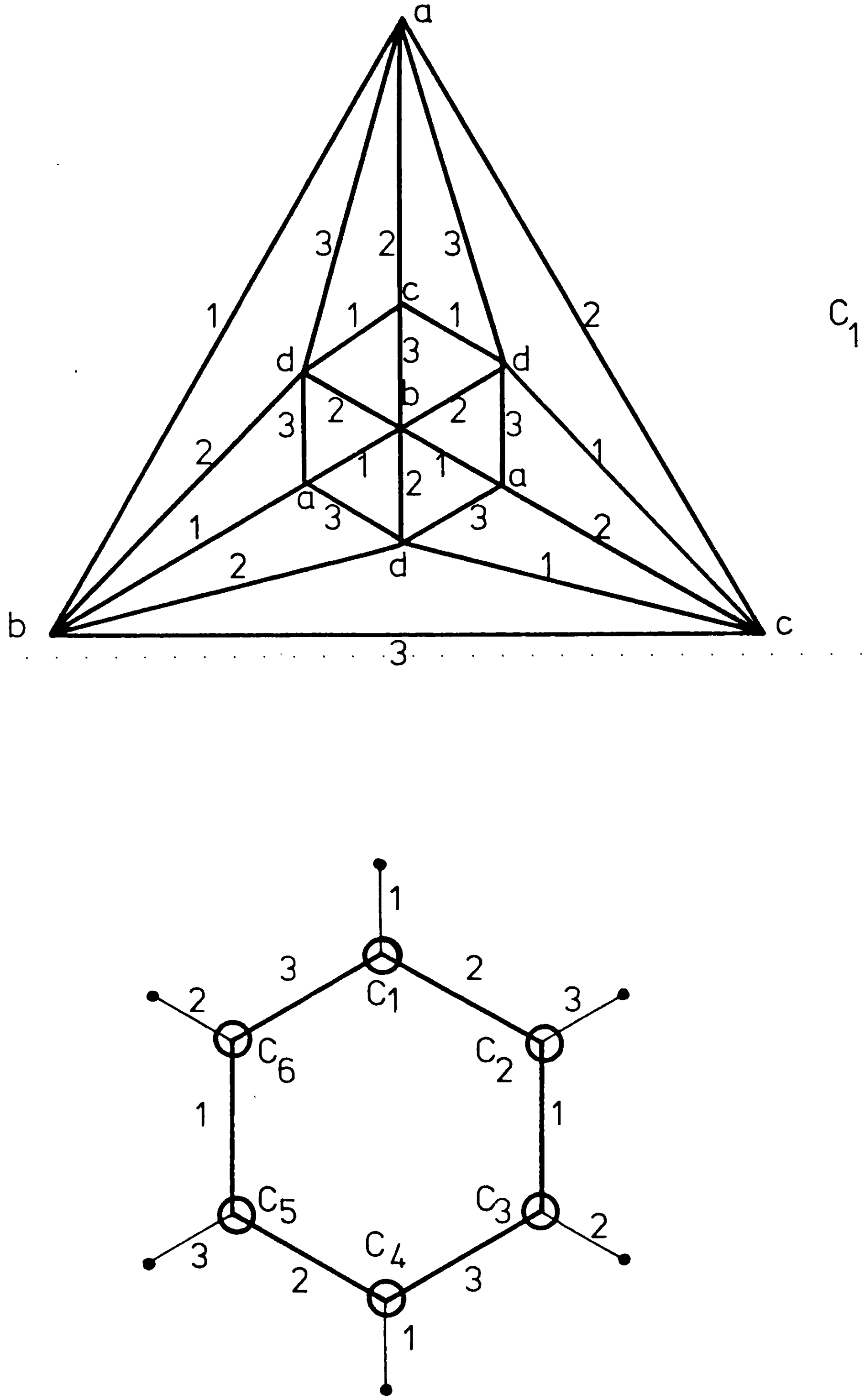


FIG. 28. Transformation diagram for G_6^1

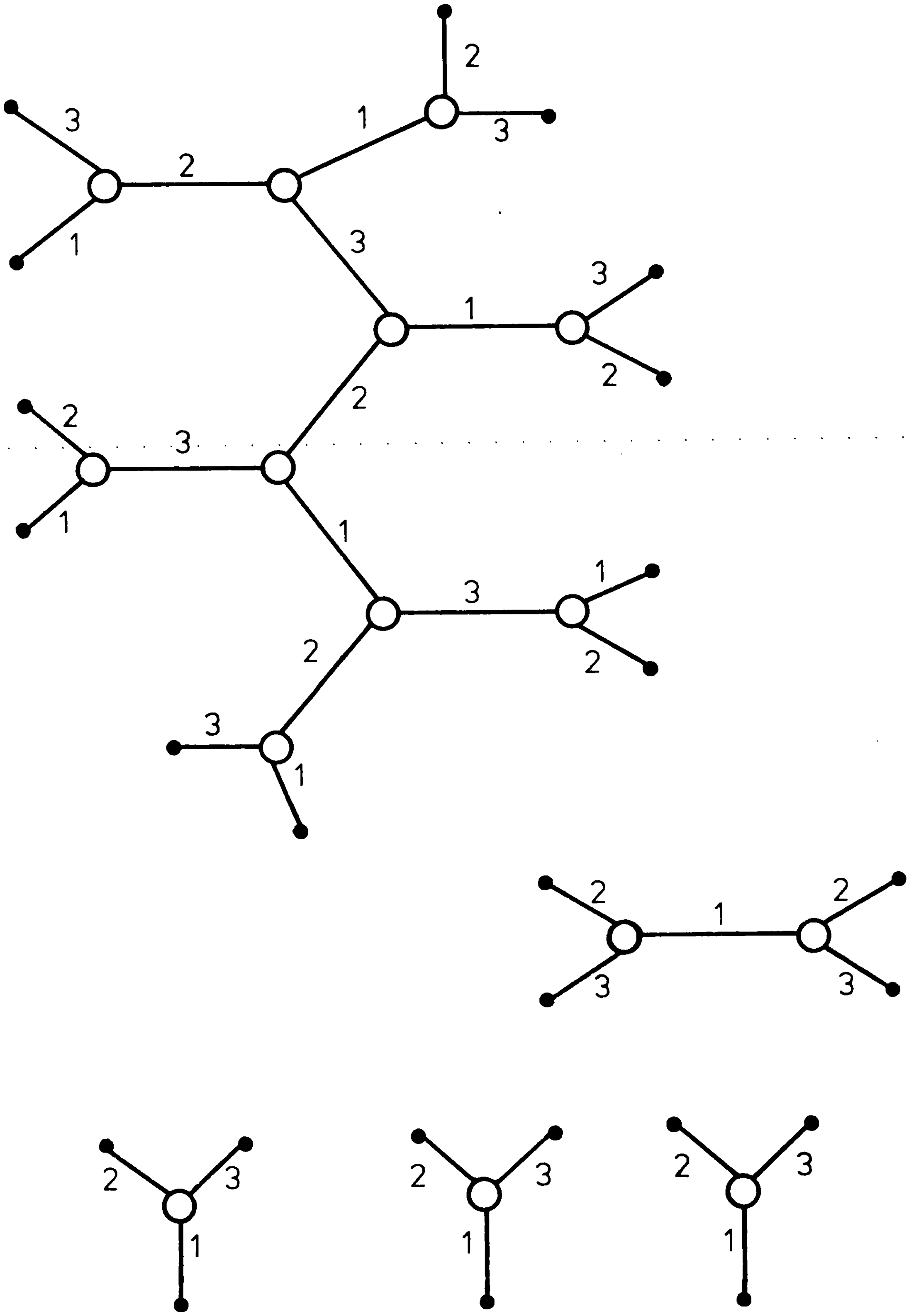
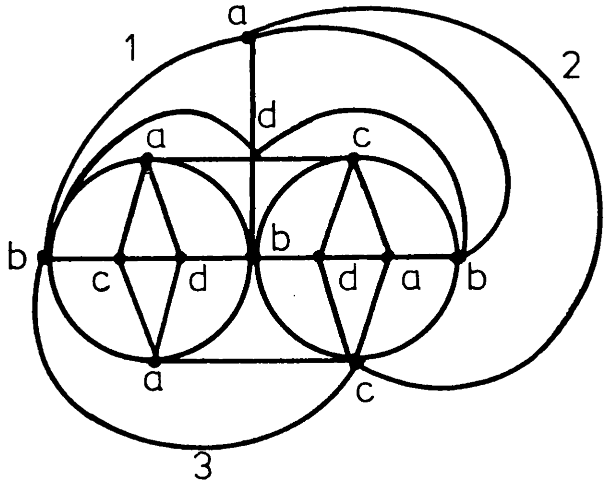


FIG. 29. Transformation diagram for G_5^1



C_1 of $G_{10}^{\prime}$

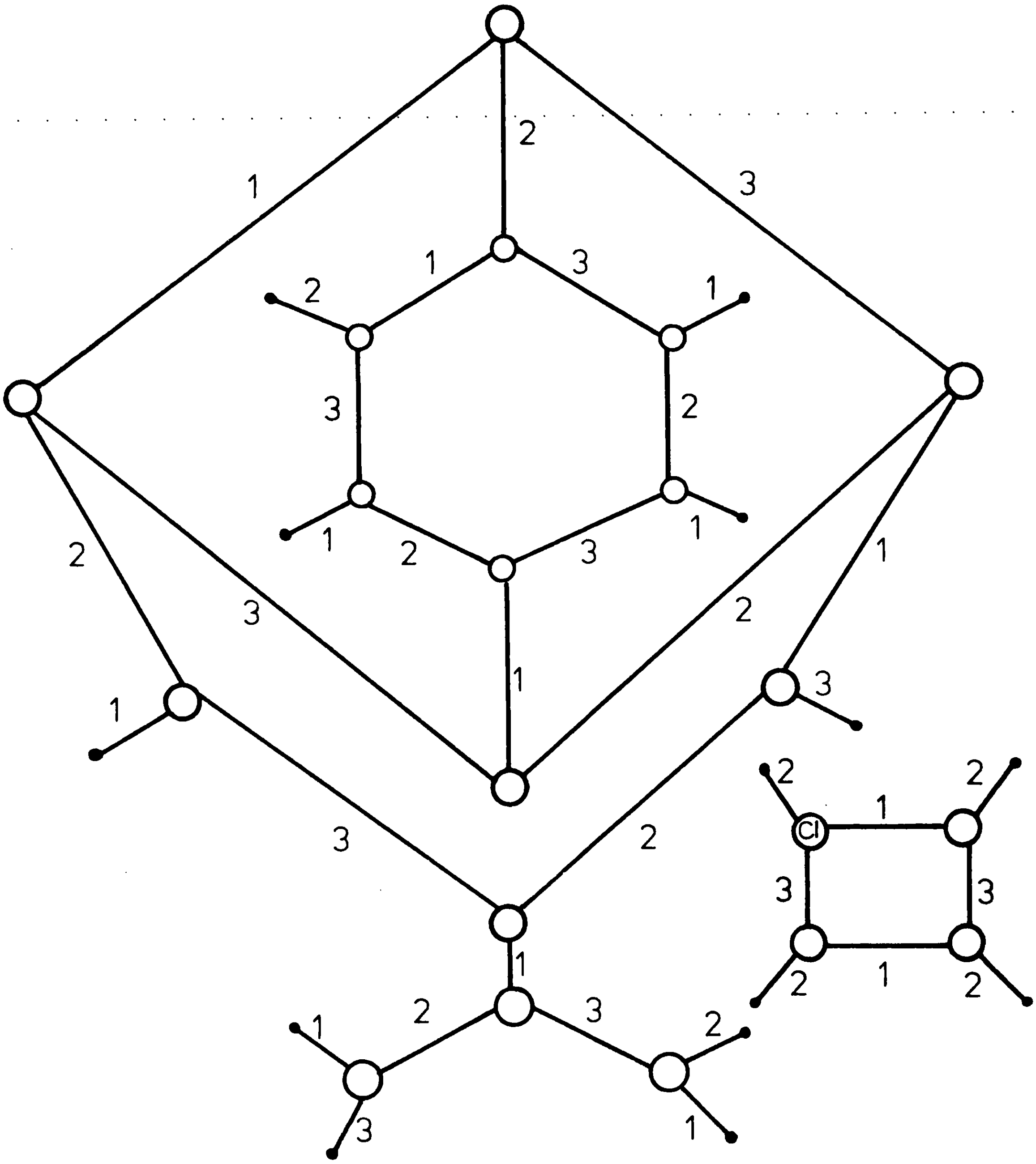


FIG. 30. Transformation diagram for $G_{10}^{\prime}$

4.8. Transformation diagrams have interesting properties but they fail to provide a complete answer to the question posed in paragraph 4.6. In particular, two subsidiary questions still require an answer.

a) What properties are possessed by the transformation diagram of a prime graph which are not necessarily found in the diagram of a non-prime graph? The properties developed in this section in no way depend upon the graph being prime.

b) Is there some global closure property, embracing the whole family, which goes beyond the sub-family closure property described in paragraph 4.5? The example provided by the graph G_{10} shows that unless one can establish some relationship between the sub-families little progress can be made towards answering the main question.

Because his primary interest is in the family of H-circuits rather than in the four-colourings of a graph, the author has devised a transformation which, by abandoning the dependence on four-colourings, provides a comprehensive link between the H-circuits of the family.

A Transformation Algorithm.

4.9. Given a graph G, together with one of its H-circuits, the procedure to be described generates a family of H-circuits. The essential feature of the procedure is an algorithm which, by means of an iterative sequence of elementary operations, transforms the given H-circuit into another H-circuit. This algorithm will be explained in the context of an illustrative example.

4.10. In Fig. 31(a) the graph G_9 (whose dual appears in Figs. 25 and 26) is drawn in such a way as to emphasise a particular H-circuit. It is helpful to imagine that the bold lines of the figure, representing the circuit edges, result from the superposition of twelve bold markers* on the otherwise uniformly feint edges of the graph. Suppose that the marker presently on the edge (2,3) - i.e. the edge connecting vertices 2 and 3 - is moved to lie on the edge (2,6), as shown in Fig. 31(b), as though it were pivoted at vertex 2. This constitutes the first step in the algorithm (the arbitrary nature of this first move will be discussed later). The resultant pattern of marked edges forms a closed circuit of nine edges together with a tail of three edges. Vertex 6, where the tail joins the circuit, is the only vertex at which three marked edges meet. Two of the three markers incident at vertex 6 lie on the circuit, one being the marker on the edge (6,2) which featured in the first move, while the other, on the edges (6,7), is the next to be moved. At the second step the algorithm requires that the marker on (6,7) be moved, as though pivoted at vertex 7, to the vacant edge (1,7). The markers now occupy the positions shown in Fig. 31(c).

4.11. The algorithm may be generalised as follows. Prior to each step, except the first, the markers form a circuit together with a tail. The vertex t, at the end of the tail, is associated with one marker. All three edges incident at the vertex j, the junction of tail and circuit, have markers. Every other vertex has two marked edges. Of the two markers incident at vertex j which lie on the circuit one was moved at the previous step while the other, on the edge (v,j) where v is a vertex adjacent to j, is the subject of the next move. Because vertex

* The author has found it convenient to use matchsticks as movable markers.

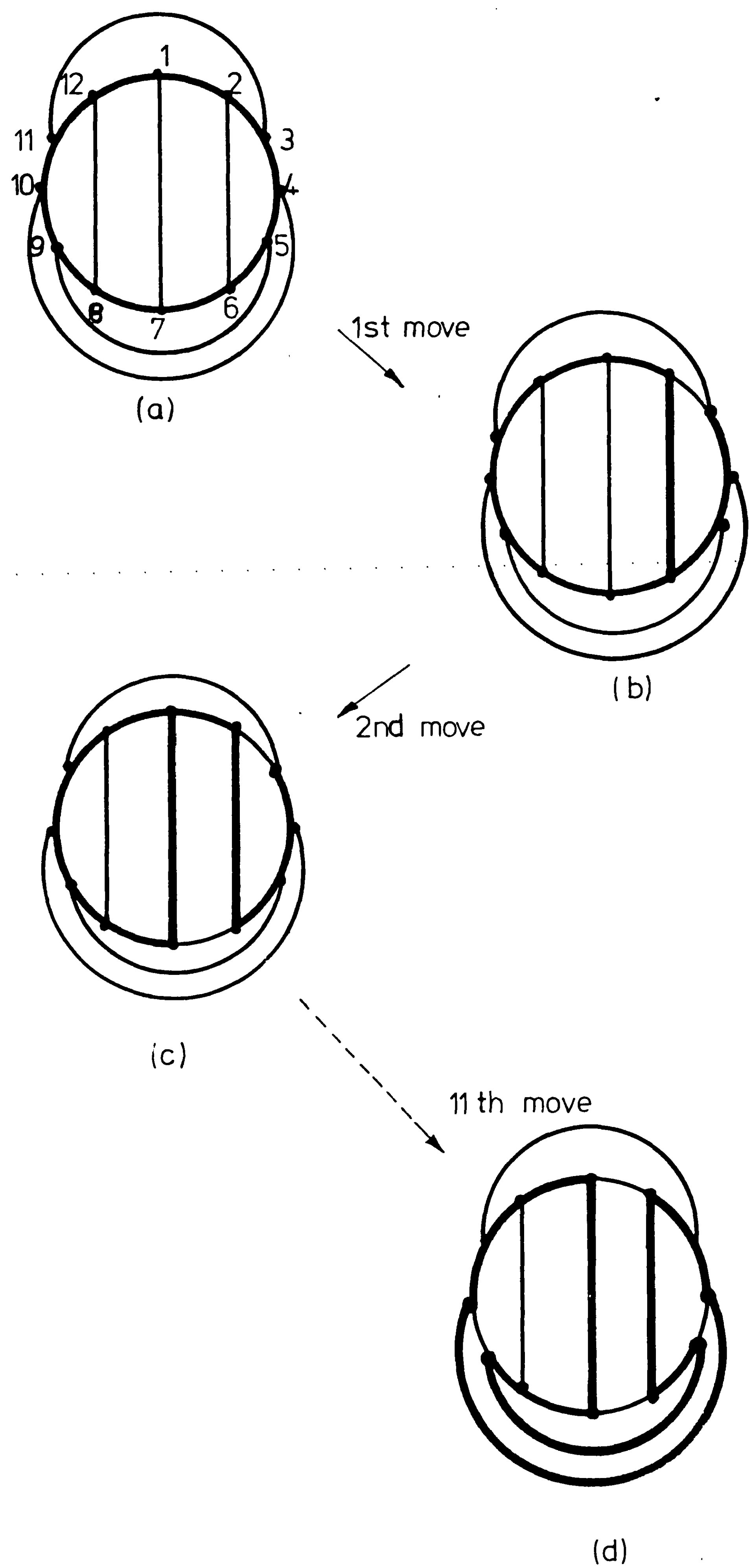


FIG. 31. Transformation algorithm for G_9

v is distinct from both j and t and therefore has two marked edges, there is necessarily one, and only one, unmarked edge (v,w) incident at v, where w is a vertex adjacent to v. The algorithm requires that the marker on the edge (v,j) shall be moved to the edge (v,w). The situation following the move is dependent on whether or not the vertices w and t are distinct. If w is distinct from t then w now has three marked edges and becomes the junction, while t still has one marked edge. All vertices other than w and t have two incident markers. Hence the markers form a circuit together with a tail and the algorithm proceeds by a further iteration. If the vertex w is identical with t then, following the move, every vertex has two associated markers; there is no longer a tail. Therefore the algorithm terminates with the markers forming an H-circuit.

4.12. The sequence of operations defined by the algorithm is unique (except for the first step) and reversible. When applied in the reverse direction the algorithm restores the original H-circuit. In the forward direction the algorithm must terminate in an H-circuit, distinct from the original, after a finite number of iterations (the number of possible arrangements of the markers being finite). For the illustrative example, starting with the H-circuit of Fig. 31(a) and the given first move, the algorithm terminates, after eleven iterations, in the H-circuit shown in Fig. 31(d).

4.13. Two further properties of the transformation need to be considered. Firstly, it is apparent that the algorithm can never require the movement of the marker which denotes the extreme edge of the tail. Consequently this edge is common to both the initial H-circuit and the H-circuit which results from the transformation. Secondly, any one of the markers on the edges which form the given H-circuit may be chosen for the first move and the selected marker can be placed in either of two positions according to which end of the edge is regarded as the pivotal vertex. Thus, by an appropriate choice of initial move, any one of a number of different H-circuits can be generated from the one given H-circuit.

4.14. These two properties, together with the closure property, make possible the generation of a family of H-circuits. Suppose that, at some stage in the generation procedure, the set of H-circuits generated by the transformation do not satisfy the closure property. Then there exists (at least) one edge E of the graph which is included in an odd number of H-circuits from the set. The transformation algorithm may be applied to each of these H-circuits in turn, choosing for the first move that marker which is on the circuit edge next to E when viewed in an anticlockwise sense (to make the move unique). This ensures that E is the extreme tail edge and hence that E is common to the initial and final H-circuits. This procedure must result in the generation of an even number of distinct H-circuits which include the edge E. The procedure is repeated for other edges until the closure property is satisfied. The family of G_9 comprises seven H-circuits : from the H-circuit given in Fig. 31(a) each of the other six can be generated by an appropriate choice of first move.

4.15. In the absence of a truly global closure property there is no guarantee that the transformation procedure described above will generate all members of the family of H-circuits of a graph, but for each of the many graphs investigated by the author the procedure does generate the complete family. Although this transformation is essentially more powerful than that based on four-colourings the author has so far been unable to determine the circumstances in which the generated family has the property of including any edge of a graph in at least two H-circuits but he believes that a deeper study of the properties of the transformation might reveal the necessary conditions.

A Generation Algorithm

4.16. Although the transformation procedures described in previous paragraphs offer insights into the relationships between the members of the family of H-circuits of a graph, their deficiencies as family generators have led the author to seek a different approach which necessarily generates the complete family. The algorithm now to be described generates directly the sub-set $\{H_E\}$ of $\{H\}$, where $\{H\}$ denotes the set of all H-circuits of a graph G - i.e. the complete family - and $\{H_E\}$ represents those members of the family which includes a given edge E of G. The formal specification of the algorithm will follow a preliminary discussion of the underlying principle.

4.17. To each H-circuit in G which passes through the edge E there corresponds an H-tree in G' which excludes the edge E' (where E' in G' corresponds to E in G). In the context of this algorithm it is convenient to assign a direction to each of the tree edges of the H-tree, the directions being dependent on the chosen edge E'. The two vertices in G' which are connected by E' are the <u>root vertices</u> of the directed two-tree and the tree edges are directed towards the roots. In Fig. 32 an H-tree of G_9' is drawn so as to show the edge directions appropriate to

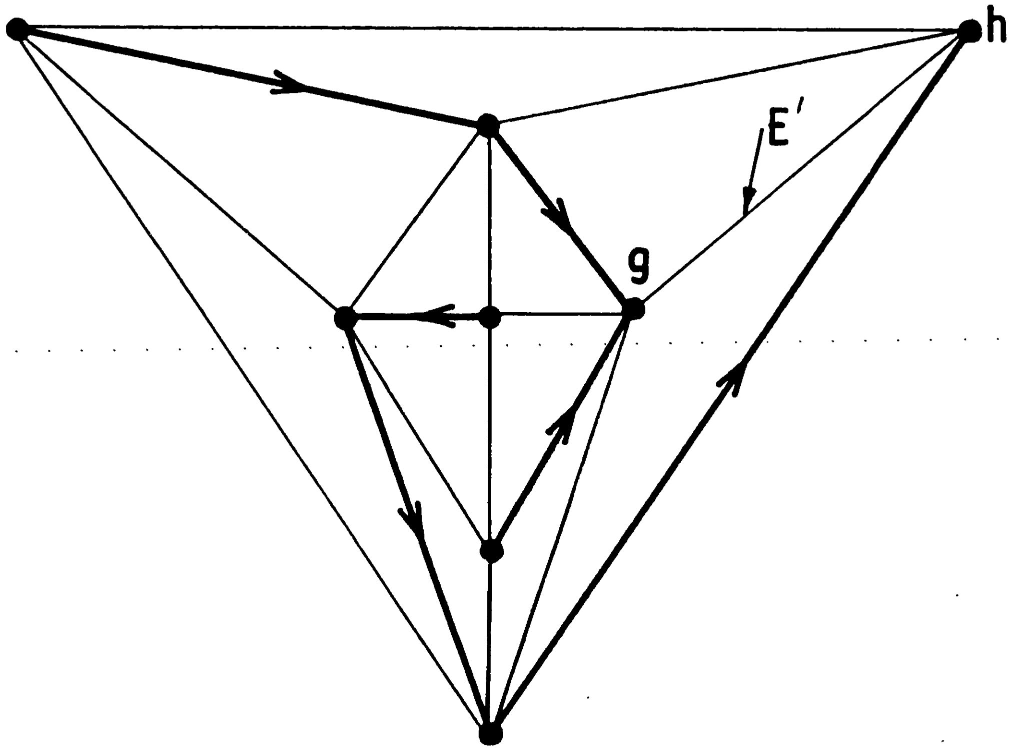


FIG. 32. Directed edges of an H-tree in G_g'

the choice of g and h as root vertices. It is evident that from each vertex of G'_9, other than the root vertices, there emanates exactly one outwardly directed edge. This property clearly extends to any H-tree by virtue of the properties of the tree structure.

4.18. Assume that G (and hence G′) is given and that some edge E of G (and hence E′ of G′) is specified. Suppose that a sub-set of the edges of G′ be assigned directions in such a way as to satisfy each of the two conditions which follow:

Condition 1. Let the two vertices connected by E′ be the root vertices and from each vertex of G′ other than the root vertices let one, and only one, of the incident edges be chosen as an outwardly directed edge. The total number of directed edges must equal the number of non-root vertices - i.e. if an edge is chosen to be a directed edge for both of the vertices with which it is associated, then this edge must be counted twice (once in each direction).

If condition 1 is satisfied then, starting from any vertex other than a root vertex, and following only directed edges one proceeds by a unique path which either terminates in a root vertex or enters a directed circuit. (The chosen edges cannot form a circuit other than a directed circuit for this would imply that at least one vertex possesses more than one outwardly directed edge).

Condition 2. Let the directed edges be so chosen that not more than one directed edge bounds each triangular face of G′.

The next step is to prove that if condition 2 is satisfied then directed circuits cannot arise. Consider a circuit formed by a set of directed edges satisfying condition 1. Because the two root vertices are connected by the edge E', which is not a directed edge and hence is not on the circuit, it follows that the root vertices both lie on the same side of the circuit. Consider that region, R, of G' which is bounded by the circuit and which does not contain the root vertices.

Let n be the number of edges on the circuit, and let v, f, e be respectively the number of vertices, faces and edges within R. Applying Euler's relation to the region R gives

$$v + f = e + 1 ,$$

and because each face is triangular it follows that

$$3f = n + 2e .$$

Hence, eliminating e,

$$2v + n = f + 2 .$$

Now the number of directed edges arising within R is v (one for each internal vertex) and each of these is associated with two faces. The number of directed edges on the circuit is n and each of these is associated with one face within R. But $2v + n = f + 2 > f$, hence it cannot be true that each of the faces is bounded by only one directed edge and condition 2 is violated. Therefore if both conditions 1 and 2 are satisfied it follows that circuits cannot arise and the directed edges constitute an H-tree in G' to which there corresponds an H-circuit in G within the set $\{H_E\}$.

4.19. The foregoing principle suggests the following algorithmic procedure which is set out in three steps.

1. Let each triangular face of G' be arbitrarily assigned an index number (represented by p, q, r .....) and each vertex an index letter (represented by a, b, c, ... k, ...). Let each edge be denoted by an un-ordered number pair (p, q) where p and q are the index numbers of the faces which are separated by that edge.

2. For each vertex, k, other than the two root vertices (those associated with the specified edge E'), form the vertex sum

$$s_k = [(p, q) + (q, r) + ... + (z, p)]$$

where each term corresponds to an edge incident on that vertex.

3. Form the product function

$$f_{E'} = s_a * s_b * s_c ....$$

of all these vertex sums where the product operator * is defined as follows.

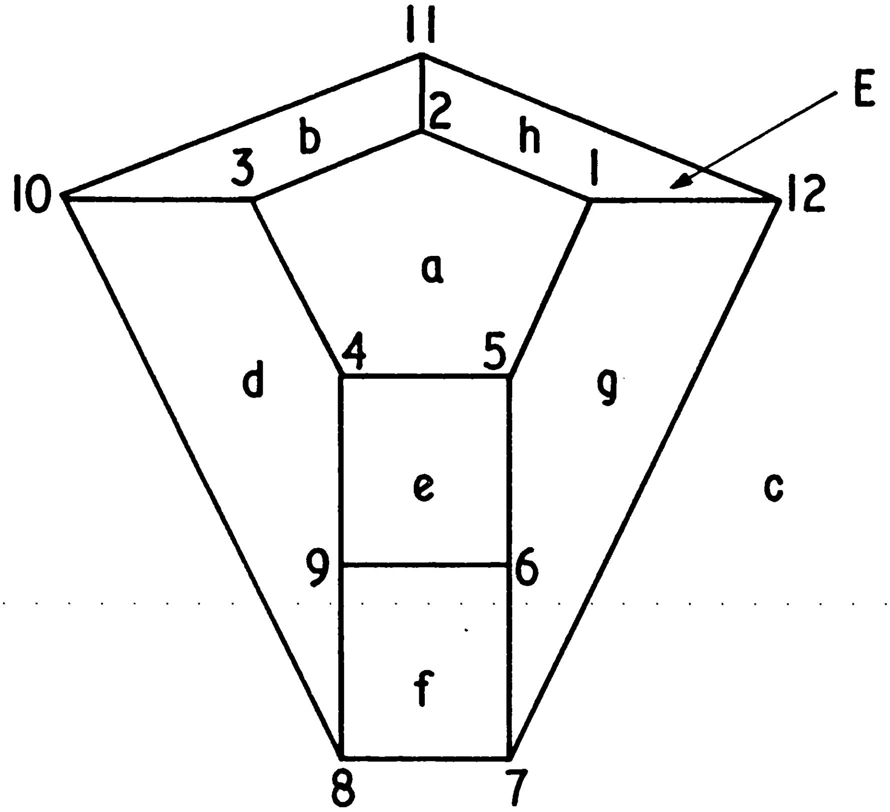
The product $s_a * s_b$ is the sum of terms such as (p, q) (r, s) where (p, q) is any term in s_a and (r, s) any term in s_b such that the index numbers p, q, r, s are all different. Thus, if s_a includes the term (1, 2) and s_b includes the terms (2, 5) and (3, 4), the product $s_a * s_b$ will include the term (1, 2) (3, 4) but not (1, 2) (2, 5). No significance attaches to the ordering of the product terms, thus (1, 2) (3, 4) = (3, 4) (1, 2) = (4, 3) (1, 2) etc.

Similarly the product $s_a * s_b * s_c$ is the sum of terms such as (p, q) (r, s) (t, u), formed by taking one term from each of s_a, s_b and s_c, such that p, q, r, s, t, u are all different. The extension to higher order products is obvious. The associative law holds for the product operation so that the vertex sums may be multiplied successively in any order and $f_{E'}$ is unique. If $f_{E'} \neq 0$ each term in $f_{E'}$, is formed by taking one term (representing an edge) from each of the vertex sums, thus satisfying condition 1. Furthermore the definition of the product operation ensures that, in each term of $f_{E'}$, no face index number appears more than once. Hence each face is associated with one, and only one, directed edge and condition 2 is satisfied. It follows that the set of edges represented by the components of any term of $f_{E'}$ form an H-tree in G' such that E' is not a tree edge. Thus $f_{E'}$ represents the whole family of such H-trees and hence the algorithm generates the set $\{H_E\}$ in G.

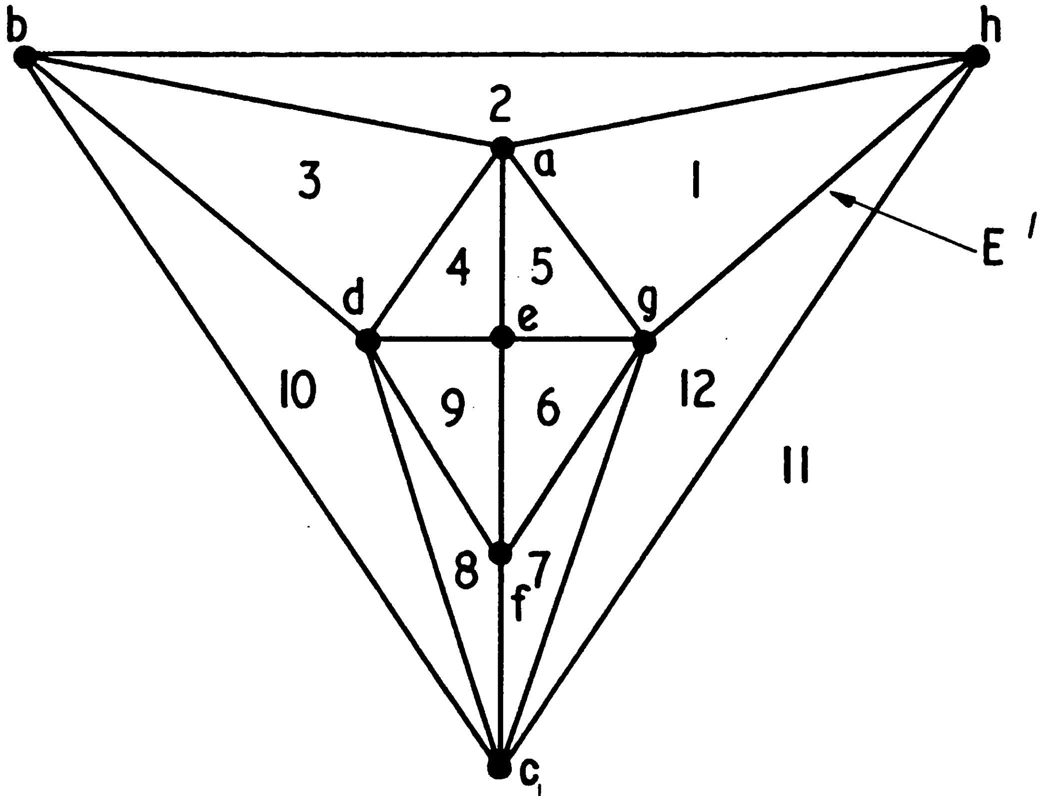
4.20. Although the algorithm has been described in terms of the dual G', the generating function f_E can be obtained directly from G and the terms of f_E interpreted as the H-circuits in G which include the edge E. The components of each term of f_E now represent non-circuit edges - i.e. if these are deleted from G the remaining edges form a Hamiltonian circuit through E. The procedure will be illustrated by means of a specific example. The graph G_9 (the subject of previous examples) is shown, together with its dual, in Fig. 33.

Vertices in G_9 (corresponding to faces in G_9') are arbitrarily assigned index numbers and faces in G_9 are arbitrarily assigned index letters. The aim is to form the set $\{H_E\}$ where E is the edge linking vertices 1 and 12 in G_9. Observe that E separates the faces g and h which correspond to root vertices in G_9'.

First form the face sums for faces a, b, c, d, e, f. Thus



(a) G_9



(b) G_9^1

FIG.33. Generation algorithm for G_9

$$s_a = [(1,2) + (2,3) + (3,4) + (4,5) + (5,1)] \ ,$$
$$s_b = [(2,3) + (3,10) + (10,11) + (11,2)] \ ,$$
etc.

Then
$$\begin{aligned} s_a * s_b = [&(1,2)(3,10) + (1,2)(10,11) \\ &+ (2,3)(10,11) \\ &+ (3,4)(10,11) + (3,4)(11,2) \\ &+ (4,5)(2,3) + (4,5)(3,10) + (4,5)(10,11) + (4,5)(11,2) \\ &+ (5,1)(2,3) + (5,1)(3,10) + (5,1)(10,11) + (5,1)(11,2)] \end{aligned}$$

The subsequent steps in forming the product function are straightforward (though tedious!) and one finally obtains

$$\begin{aligned} f_E &= s_a * s_b * s_c * s_d * s_e * s_f \\ &= [(1,2)(3,10)(4,5)(8,9)(6,7)(11,12) \\ &\quad + (2,3)(1,5)(8,10)(4,9)(6,7)(11,12) \\ &\quad + (1,2)(3,10)(4,9)(5,6)(7,8)(11,12) \\ &\quad + (1,2)(3,4)(5,6)(8,9)(10,11)(7,12)] \end{aligned}$$

The first term in f_E represents the H-circuit obtained by deleting the edges which join vertices 1 and 2, 3 and 10, 4 and 5, etc. in G. This circuit is shown in Fig. 34(e). Similarly the other terms of f_E represent the other three members of $\{H_E\}$ shown in Fig. 34(g), (34(f) and 34(d).

<u>4.21</u>. The procedure is modified if the aim is to obtain $\{H\}$, the entire family of H-circuits. Clearly every member of $\{H\}$ passes through two of the three edges incident at any vertex. Choose any vertex, vertex 1 for example, and form the <u>partial</u> product function

$$P = s_b * s_c * s_d * s_e * s_f$$

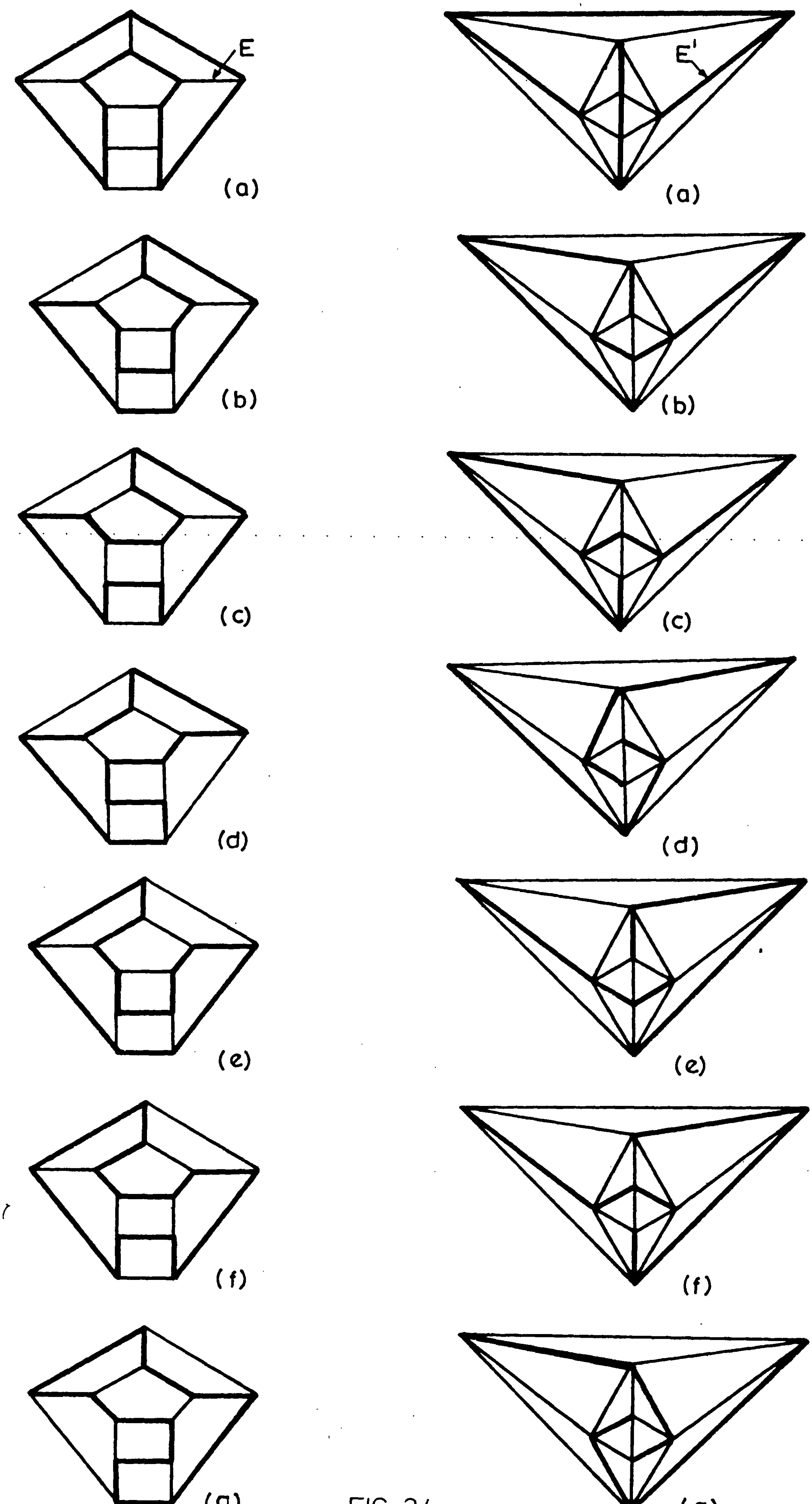


FIG. 34

H- CIRCUITS IN G_9

H- TREES IN G'_9

which excludes the terms s_a, s_g, s_h corresponding to the three faces incident at vertex 1. Then $P * s_a$ generates the H-circuits which pass through edge E, separating g and h as previously shown. Similarly $P * s_g$ generates the H-circuits which pass through the edge which separates a and h, and $P * s_h$ generates the H-circuits which pass through the edge which seperates a and g. Thus the function $(s_a + s_g + s_h) * P$ generates each member of $\{H\}$. It will be observed that the determination of P, which forms the bulk of the procedure, needs to be performed once only, so that the formation of $\{H\}$ involves little more work than that required to obtain $\{H_E\}$.

Application of this procedure to G_9 in the example results in the family of seven H-circuits shown in Fig. 34.

<u>4.22</u>. The generating function, f_E, and its partial products possess a simple parity property. This property leads immediately to a proof that the number of H-circuits which pass through any edge of G is even. (In this section the term 'even number' will be taken to mean an even integer or zero). Consider the face sum, s, for any face of G. Let σ_k be the number of terms in s which contain the vertex number k, and $\sigma_{k\ell}$ be the number of terms which contain both k and ℓ.

Thus, if $s = [(1,2) + (2,3) + (3,4) + (4,5) + (5,1)]$,

then $\sigma_1 = 2$, $\sigma_2 = 2$, $\sigma_6 = 0$ etc.

and $\sigma_{12} = 1$, $\sigma_{13} = 0$ etc.

It is evident from the definition of s that $\sigma_k = 2$ or 0 according as k is, or is not, a vertex on the boundary of the face which generates s. The value of $\sigma_{k\ell}$ is 1 if k and ℓ are joined by an edge which bounds the face - otherwise, $\sigma_{k\ell} = 0$.

Let $\sigma_{\bar{k}}$ be the number of terms in s which do not contain the index number k, and $\sigma_{\overline{k\ell}}$ be the number of terms which contain neither k nor ℓ.

Then $$\sigma_{\bar{k}} = S - \sigma_k \text{ and } \sigma_{\overline{k\ell}} = S - \sigma_k - \sigma_\ell + \sigma_{k\ell}$$

where S is the total number of terms in s.

Consider next a product t of any number of face sums

$$t = s_a * s_b * s_c \;\cdots\cdots$$

i.e. t is either a partial product of f_E or $t = f_E$.

Let $$\tau_k \, , \; \tau_{k\ell} \, , \; \tau_{\bar{k}} \, , \; \tau_{\overline{k\ell}}$$

be defined for t in the same way as

$$\sigma_k \, , \; \sigma_{k\ell} \, , \; \sigma_{\bar{k}} \, . \; \sigma_{\overline{k\ell}}$$

are defined for s. Thus, in the product $s_a * s_b$ of the example given in paragraph 4.20,

$$\tau_1 = 6, \; \tau_{12} = 4, \; \tau_{23} = 5, \; \tau_{\bar{1}} = 7, \; \tau_{\overline{12}} = 3.$$

Now suppose that a new partial product t' is obtained by forming the product of t with some face sum s, giving $t'' = t * s$.

From the definition of the product operator, it follows that the number of terms in t' which contain some arbitrary index number p is given by

$$\tau'_p = \sum_k \sigma_{pk} \, \tau_{\overline{pk}} + \sum_k \tau_{pk} \, \sigma_{\overline{pk}}$$

where the summation ranges over all index numbers k except k = p.

Hence

$$\tau'_p = \sum_k \sigma_{pk} \left[T - \tau_p - \tau_k + \tau_{pk}\right] + \sum_k \tau_{pk} \left[S - \sigma_p - \sigma_k + \sigma_{pk}\right]$$

where T is the total number of terms in t.

But

$$\sum_k \sigma_{pk} = \sigma_p \text{ and } \sum_k \tau_{pk} = \tau_p \, ,$$

therefore,

$$\tau'_p = \sigma_p T + \tau_p S - 2\sigma_p \tau_p + 2\sum_k \sigma_{pk} \tau_{pk}$$

$$- \sum_k \sigma_{pk} \tau_k - \sum_k \tau_{pk} \sigma_k .$$

Now, σ_k is even for all k. If it is assumed that τ_k is even for all k it follows that τ'_p is even whatever the parity of σ_{pk} and τ_{pk}. Because p was chosen arbitrarily τ'_k is even for all k. Hence, by induction, τ_k is even for any partial product and, in particular, for the complete generating function f_E.

An extension of the argument shows that, for any partial product,

$$\tau_k, \tau_{k\ell m}, \tau_{k\ell mno}, \ldots\ldots \text{ etc.}$$

are all even (where $\tau_{k\ell m}$ is the number of terms which contain all of k, ℓ and m etc.)

whereas

$$\tau_{k\ell}, \tau_{k\ell mn}, \ldots\ldots\ldots\ldots \text{ etc.}$$

may be either odd or even.

4.23. The results obtained in the previous paragraph lead directly to a proof of the closure property. Each term in the function f_E contains every vertex index number. In particular each term contains the index number 1. But τ_1 is even and therefore the number of terms in f_E is even, for every choice of the edge E. Hence,

> the number of H-circuits which pass through any edge of a trivalent planar graph is either even or zero.

The following parity property is also of interest. Suppose that, for some dual G′, a partial product P (para. 4.21) excludes the terms corresponding to the vertices a, b and c which define a triangular face of G′.

Let this face be numbered 1 and let the adjacent face separated by the edge ab be numbered 2. Every term of the partial product necessary excludes 1 (because this face number appears only in s_a, s_b and s_c) and also excludes one other face number (not necessarily 2). For each term in P which excludes 2 there exists an H-tree in G which includes ab as a tree edge (each of these H-trees corresponds to a term in both P * s_a and P * s_b). Hence the number, $\tau_{\bar{2}}$, of such terms has the same parity as the complete family of H-trees. Because τ_2 is even it follows that $\tau_{\bar{2}}$ has the same parity as the total number of terms in P.

Hence,

> the parity of a graph G is equal to the parity of the number of terms in the partial product of all the face sums excepting those which correspond to the three faces incident at some chosen vertex of G.

4.24. The author had hoped to establish the necessary and sufficient conditions such that for every edge E of a prime graph G, $f_E \neq 0$. The failure to obtain these conditions is a consequence of the author's present inability to impose on the combinatorial procedure the constraints implicit in the topological structure of the graph - in particular the constraint which results from the assumption that the graph is prime. When the topological and combinatorial problems are better understood it might be possible to use this algorithm to establish existence theorems. In the meantime the author has had to be content with such results as can be obtained from parity relationships. Parity properties, in particular the closure property which states that f_E is even, can be made to yield existence proofs only if it is assumed that some property of the graph or its families has odd parity. Thus, for example, it is easily proved that $f_E \neq 0$ for those graphs which have an odd number of four-colourings, or which have a sub-family of odd parity. In order, therefore, that this chapter shall close on a more positive and optimistic note a further example of such an existence proof is given. As far as the author is aware the existence theorem which follows has not previously appeared in the literature.

An Existence Theorem.

4.25. Consider some edge E' of a dual G'. Let the vertices connected by E' be labelled c and d and let the other two vertices associated with the faces separated by E' be a and b, as shown in Fig. 35(a). Now suppose that E' be switched so that it connects a and b instead of c and d. This change results in a different dual, ${}_CG'_{E'}$, (Fig. 35(b)), which will be called the conjugate of G' with respect to E'. To this dual there corresponds a graph ${}_CG_E$ which is the conjugate of G with respect to E (where the edge E in G corresponds to E' in G'). Suppose that the edge E' in G' is contracted so that vertices c and d merge into the single vertex v. This change results in yet another dual, ${}_RG'_{E'}$, which will be called the reduction of G' with respect to the edge E'. Next consider any H-tree in G' which includes E' as a tree edge (Fig. 35(c)). To this H-tree there corresponds an H-tree in ${}_RG'_{E'}$ which differs from the original H-tree only in that one tree edge has been eliminated. The H-tree in ${}_RG'_{E'}$ is such that neither av nor vb are tree edges (Fig. 35(d)). Similarly any H-tree in the conjugate ${}_CG'_{E'}$ which has both cb and db as tree edges (Fig. 35(e)) reduces to give an H-tree in ${}_RG'_{E'}$, again with

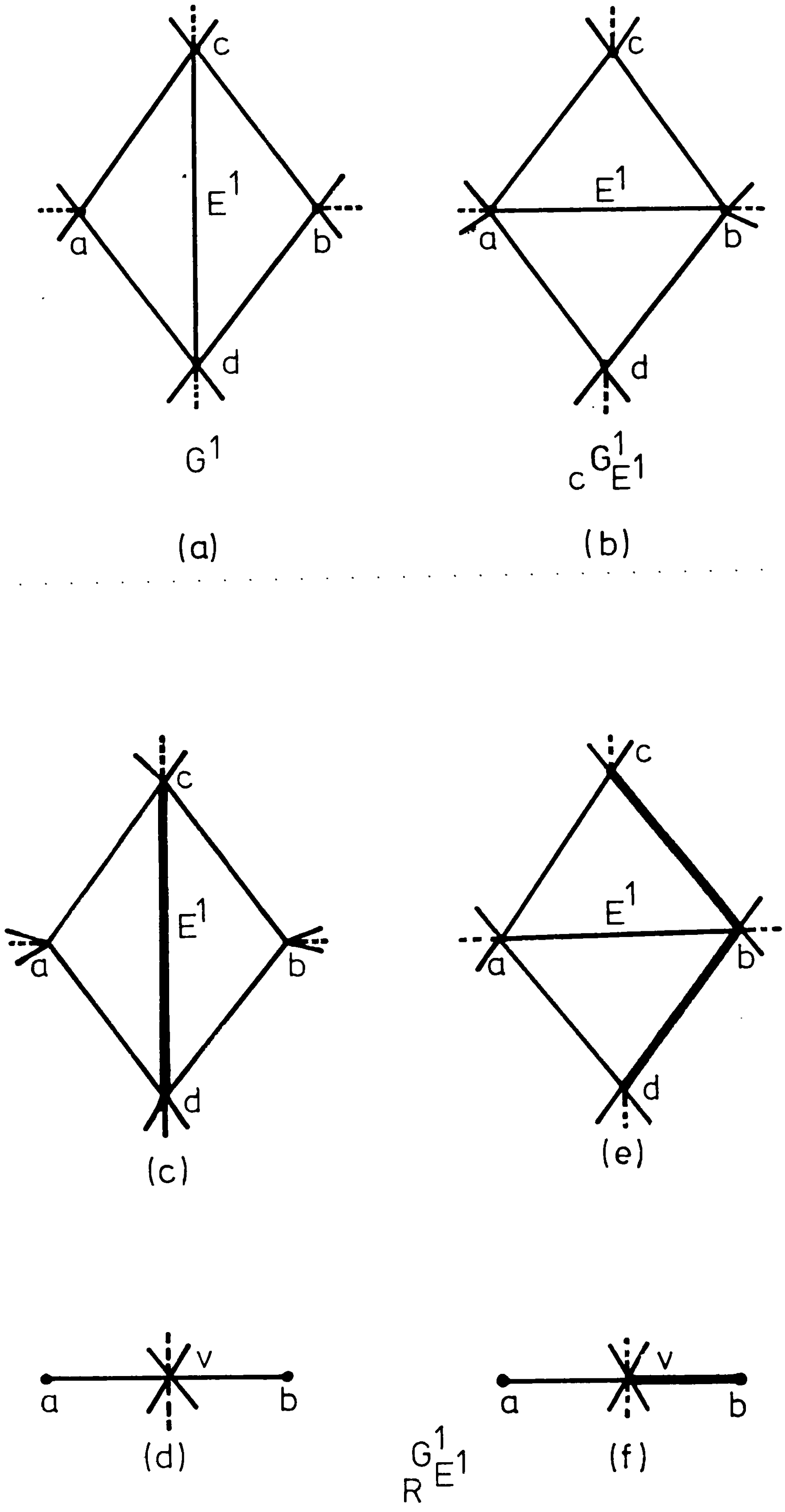


FIG 35. An existence theorem

the elimination of one tree edge, the edges cb and db having merged (Fig. 35(f)). For this H-tree vb is a tree edge but av is not. Now the closure property applied to ${}_{R}G'_{E'}$ requires that the number of H-trees for which av is not a tree edge shall be even (or zero). Therefore the number of such H-trees for which vb is a tree edge (Fig. 35(f)) has the same parity as the number for which vb is not a tree edge (Fig. 35 (d)). If each of these numbers is odd it follows that the parity of G' is odd and also that an odd number of H-trees of the conjugate ${}_{C}G'_{E'}$ have the configuration shown in Fig. 35(e). Hence the theorem,

<u>Theorem</u>. If a graph G has odd parity then the conjugate graph with respect to any edge E possesses H-circuits which pass through E (this edge being switched in the conjugate).

The theorem has two important corollaries :

<u>Corollary 1</u>. If some edge E of a graph G gives rise to a conjugate graph having odd parity then G possesses H-circuits which include E.

<u>Corollary 2</u>. If at least one of the conjugate graphs of a given graph G has odd parity, then G possesses a family of H-circuits.

This theorem does not require G to be prime. The simplest prime graph (as far as the author is aware) which does <u>not</u> have at least one conjugate of odd parity is Hamilton's graph (title page). For this graph the conjugates are all identical (by symmetry) and have even parity.

CHAPTER 5

CONCLUSION

5.0. In conclusion the author recapitulates his achievements and comments on what he had hoped to achieve when he began this research. Suggestions for further research are offered in the form of questions.

The Four-colour Problem

5.1. The colouring algorithm described in paragraph 3.11 has no direct bearing on the properties of Hamiltonian circuits but, because the four-colour problem is of interest in its own right, the author has devoted some of his time to an exploration of the algorithm. In particular he has attempted to construct a dual for which the algorithm does not terminate but cycles. The experimental procedure used by the author involves the building-up of the dual step by step, starting with an incomplete dual (not all of the faces being triangular). At each iteration of the algorithm new edges and, if necessary, new vertices (suitably coloured) are added to the dual in such a way as to bring about a colour chain which forces the algorithm to proceed to a further iteration. In all such experiments the author found that it was not possible to prevent the algorithm terminating without the frequent addition of new vertices and he feels intuitively that if an upper limit is set to the permitted number of vertices then the algorithm will eventually terminate however the dual is constructed. But intuition does not constitute proof and the following question remains open:

Question 1. Does there exist a prime dual having no vertices of order 4, together with a four-colouring of all vertices except for one vertex, v, of order 5, such that the colouring algorithm applied to v fails to terminate?

Such a dual, if it exists, would not necessarily constitute a counter-example to the four-colour conjecture but it might have interesting properties: in particular, because the cycle length is a multiple of 10, it would probably exhibit some form of symmetry. The author's experiments suggest the further question:

Question 2. If the answer to question 1 is affirmative, what is the minimum number of vertices which a dual having this property must possess?

Any dual which forms a counter-example to the four-colour conjecture must have at least this minimum number of vertices.

5.2. The algorithm described in paragraph 3.18, which the author devised as a means of determining the four-colour parity of a graph, leads also to a result which is directly relevant to the four-colour conjecture. In paragraph 3.25 the author has shown that a necessary and sufficient condition for a graph to have no four-colourings is that each of the coefficients in a certain equation based on the Heawood congruences is zero (mod 3). This result suggests the composite question:

Question 3. What topological properties of a graph are implied by the author's condition? Can it be proved that no graph possesses these properties? If no proof is possible can a graph having these properties be constructed?

<u>The Generation and Properties of Hamiltonian Circuits.</u>

<u>5.3.</u> In the first part of Chapter 4 the author has shown that the relationships between the four-colourings and the Hamiltonian circuits of a graph can conveniently be represented by means of a transformation diagram, and he has proved that each component of a transformation diagram exhibits the parity property (para. 4.4) and the closure property (para. 4.5). This latter result raises the question:

<u>Question 4.</u> Is there some global closure property, stronger than the sub-family closure property, which necessarily embraces the complete family of Hamiltonian circuits?

The author has also shown (para. 4.6) that if the transformation diagram of some graph possesses either of two specific properties then every edge of the graph is Hamiltonian - i.e. is included in a non-empty set of Hamiltonian circuits. The absence of these properties does not necessarily imply that some edges of the graph are non-Hamiltonian. From a given Hamiltonian circuit a sub-family of H-circuits can be generated by means of the four-colour transformation procedure, but, in general, this procedure will not generate the complete family - the transformation diagram consists of a collection of unrelated components. Because the properties of the complete family cannot be inferred from the properties of any one sub-family (para. 4.7) it is not in general possible, solely by means of the transformation procedure (and one given H-circuit), to determine whether or not any given edge of the graph is Hamiltonian.

<u>5.4.</u> Because it is independent of four-colourings the transformation algorithm of paragraph 4.9 is more powerful : in all the experiments which the author has conducted it has proved possible to generate the

complete family of Hamiltonian circuits from any given H-circuit. The author had hoped that this transformation might reveal the necessary and sufficient conditions for every edge of a prime graph to be Hamiltonian. Experiments with graphs which include non-Hamiltonian edges (e.g. G_7) have given the author some insight into the problem and he feels that a deeper understanding of the properties of the transformation might lead to an answer to the question:

Question 5. Given a prime graph together with a Hamiltonian circuit and some edge E not on the circuit, in what circumstances can it be proved that, with a suitable choice of initial move, the author's transformation will generate a Hamiltonian circuit which includes E?

5.5. Although the transformation procedure helps to establish relationships between the members of the family of Hamiltonian circuits, no proof can be offered that it necessarily generates all members of the family from a given H-circuit. A direct and certain means of generating the complete family is afforded by the generation algorithm described in paragraph 4.19. Because this algorithm generates directly all the Hamiltonian circuits which include some given edge of the graph it would seem to be potentially capable of determining the necessary and sufficient conditions for each edge to be Hamiltonian. Those general properties of the generating function which the author has obtained (para.4.22) refer to the parity of occurrence of the face index numbers but in deriving these results no account has been taken of the fact that the index numbers always enter the expression in pairs, each pair corresponding to an edge of the graph. Furthermore no use has been made of the assumption that the graph is prime. The author has not yet found a satisfactory way of imposing these additional constraints on the combinatorial process to obtain stronger properties of the generating function. The outstanding question, therefore, is:

Question 6. Can the generation algorithm be made to yield necessary and sufficient conditions a) for a prime graph to be Hamiltonian, b) for every edge of a prime graph to be Hamiltonian?

Parity Relationships and Existence Proofs.

5.6. The only existence proofs to emerge from the author's work are those based on parity relationships. The author has shown (para. 4.6) that if a graph has an odd number of Hamiltonian circuits, or a sub-family of odd parity, then every edge of the graph is Hamiltonian. Because the parity of the family of four-colourings is the same as the parity of the family of H-circuits (para. 4.4) the global parity of a graph is one of its most important properties. The global parity may be found either from the parity algorithm (para. 3.18) or from the generation algorithm (para. 4.23). The author had hoped that these algorithms might reveal some relatively simple topological formula for the parity of a graph but he has not found such a formula. Hence the question:

Question 7. Is there some topological formula, simpler than the author's parity algorithm, which determines the parity of a prime graph?

(The parity of a non-prime graph is equal to the product of the parities of its prime factors).

5.7. The author has shown that by involving the conjugate and reduced graphs the closure property can be made to yield further existence theorems (para. 4.25). In particular he has proved the existence of Hamiltonian circuits in any graph which has at least one conjugate graph of odd parity. The example of Hamilton's graph shows that this is a sufficient but not necessary condition for the existence of Hamiltonian circuits. The author believes that prime graphs which have no conjugate graphs of odd parity (e.g. the non-Hamiltonian graphs of Fig. 18) are very rare and exhibit special symmetries. Only within this class of graphs will a counter-example to the four-colour conjecture (if one exists) be found.

Finale

5.8. The author has shared in the disappointment and frustrations of the many who have entered this field only to find that their problems - especially those which have implications for the four-colour conjecture - are apparently intractable. The four-colour problem is presently analogous to the rainbow - fascinating, elusive, but seeming to have a definite conclusion not too far away. Perhaps the end will for ever lie just over the horizon but so long as it is (almost) in prospect someone will continue the quest for the proverbial crock of gold.

APPENDIX

HAMILTONIAN PARTITIONING OF THE FACES OF A DUAL

A.0. The procedure described in paragraph 3.19 uses a Hamiltonian circuit to partition the faces of the dual of a prime graph, symbols x_i being given to faces lying on one side of the circuit and symbols y_i to those faces lying on the other side. By means of the following algorithm the indexing of the faces and the ordering of the congruence equations is accomplished in such a way that the matrix, X, of the coefficient of the x_i in the equations, is a triangular matrix having each of the elements of the leading diagonal equal to unity.

(Alternatively, the matrix Y may be so structured).

The Indexing Algorithm.

A.1. Assume that a prime dual is given, together with one of its Hamiltonian circuits. Choose any pair of vertices which are adjacent on the circuit and let these be assigned index letters v and w. The matrix X is a square matrix of order n, where n + 2 is the number of vertices of the dual. Each column of this matrix corresponds to one of the n faces within the circuit and each row corresponds to one of the vertices other than v and w. The matrix is generated column by column, one face index number and one vertex index number being assigned at each step. Prior to the k^{th} step ($1 \leq k \leq n$) index numbers have been assigned to k - 1 faces and to k - 1 vertices, and k - 1 columns of the matrix X have been completed. Each vertex of every face already indexed has been assigned an index number or letter. Of the faces within the circuit and not yet indexed at least one has precisely one vertex which is not yet indexed. Choose any one of these faces (if there are more than one) and assign the index number k to this face and to its un-indexed vertex. The k^{th} column of the matrix is completed by entering 1 in the k^{th} row and in the other rows (if any) which correspond to those index numbers associated with the two previously indexed vertices. All other entries in the k^{th} column of the matrix are 0's.

A.2. This procedure is illustrated by the example shown in Fig. 36. Prior to the first step the only vertices indexed are those labelled v and w. Only one face within the circuit is associated with both v and w, and the first step of the algorithm assigns the index number 1 to this face and to the third vertex of this face. The first column of the X matrix is completed by entering 1 as the first element and 0 for each of the other elements because no other numbered vertices are associated with the first face. At the second step either of the two faces adjacent to the previously indexed face may be chosen - in the example the face which includes vertex w is selected arbitrarily. The second column of the matrix has 1's in the first two rows and 0's elsewhere because face number 2 is not associated with any vertices other than those previously indexed (1, 2 and w). The procedure finally results in the matrix :

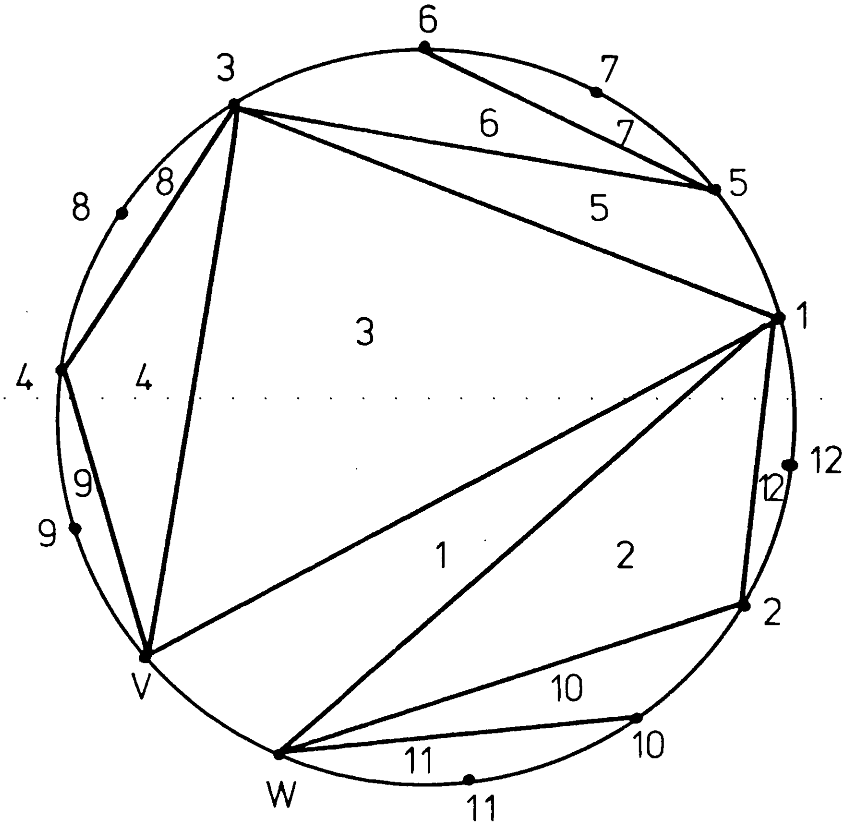


FIG. 36 Hamiltonian partitioning

$$
\begin{array}{cc}
\begin{array}{r} \text{vertex} \\ \text{index} \end{array} \begin{array}{r} \text{face} \\ \text{index} \end{array} & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

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