A classification of the point spectrum of constant length substitution tiling spaces and general fixed point theorems for tilings

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Abstract We examine the point spectrum of the various tiling spaces that result from different choices of tile lengths for substitution of constant length on a two or a three letter alphabet. In some cases we establish insensitivity to changes in length. In a wide range of cases, we establish that the typical choice of length leads to trivial point spectrum.

We also consider a problem related to tilings of the integers and their connection to fixed point theorems. Using this connection, we prove a generalization of the Banach Contraction Principle.

Dedication I have truly lived a blessed life filled with many loving and supporting people. This dissertation work, therefore, is dedicated to all the people who have brought so much into my life.
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Contents

List of Figures

List of Tables

List of Abbreviations and Symbols

card	cardinality
gcd	greatest common divisor
inf	infimum
min	minimum
mod	modulo
PF	Perron-Frobenius

$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	the set of positive integers including zero
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integers
σ_{pp}	the point spectrum
M_θ	the incidence matrix of a substitution theta
$\mathcal{T}_\theta$	the substitution tiling space
$\mathcal{R}$	the integer module generated by recurrence vectors
$\equiv$	congruent

Introduction

This thesis consists of two parts. The first part comprises Chapters 1-3 and is mainly concerned with the translation dynamics of substitution tiling spaces associated to primitive, aperiodic substitutions of constant length. Before giving a synopsis of each chapter we describe the content of our research which we consider to be original.

More precisely the main theme of the first part is the classification of the point spectrum of one-dimensional substitution tiling spaces associated to primitive, aperiodic substitutions of constant length on a two or a three letter alphabet $\mathcal{A}$.

The primary focus of the paper [7] by Clark and Sadun is to what extent the point spectrum of the tiling spaces associated to substitutions depend on the changes in the tile lengths. The substitutions of constant length considered there were primarily of interest as examples of substitutions for which the dynamics is sensitive to changes in the tile length. The ultimate result of this thesis is that our classification of the point spectrum in a two letter or a three letter case includes classes of substitutions that are sensitive to changes in tile lengths and other classes of substitutions that are independent of these changes. Although the classes of substitutions that are insensitive to changes in lengths are limited, they present interesting cases to be considered.

The substitution tiling space has a canonical choice for lengths of prototiles based on the left Perron-Frobenius eigenvector of the incidence matrix M_θ. In the case of a constant length substitution the canonical choice would give all tiles the same length. Our plan is to investigate the effect on the translation dynamics of changing the tile lengths from those of the canonical choices.

One of the motivating factors in studying and understanding the dynamical spectrum or the discrete spectrum of the translation dynamics of tiling spaces is that it has been established that the dynamical spectrum is related to diffraction spectrum of materials with corresponding patterns (see [2]). Therefore, it is of interest to identify the patterns whose spectrum remain unaffected by changes in tile lengths.

Similar topics have been investigated also in the papers [8], [9] and [20], but none of them have addressed the special cases we consider. More recently, these ideas have been used to study the homeomorphism group of tiling spaces in [18] and there has been a connection established with number theory in [21].

We prepare the groundwork in Chapter 2 where we use a flow under the ceiling function to provide an alternative description of the substitution tiling space to the description given by Barge and Diamond in [3]. We introduce Theorem 2.62 to illustrate the point spectrum of the substitution tiling space where all the tiles have the same length.

In Chapter 3, a detailed exposition of how the point spectrum is affected by the change of the tile lengths via studying two major cases, the two and three letter alphabet $\mathcal{A}$. In the two letter case, examining the effect of changing tile lengths on the the point spectrum of a substitution tiling space was investigated by Clark and Sadun in [7] under specific conditions, whereas in my thesis a general conclusion for the point spectrum is established independent of [7]. This will be extended to the examination of the three letter case, where the classification of the point spectrum is based on the different possibilities of the roots (specifically the roots of the quadratic monic polynomial) associated to the characteristic polynomial of the incidence matrix M_θ. This yields two major cases: complex roots and real roots.

The second part of this thesis comprises Chapter 4. There we address another type of tiling problem related to tilings of the integers which can specifically have a bearing on fixed point theorems. In particular, we prove a generalization of the Banach Contraction Principle. The proof involves establishing a connection between tilings of the integers and fixed points of functions.

Chapter 1. In general, by a dynamical system we mean a pair (X, T), where X is just a set (called a phase space), while T is a group of self-transformations on X. Usually, the space X is endowed with some kind of structure, and the acting transformations respect this structure. The theory of dynamical systems is interdisciplinary, as it involves research methods from many other branches of mathematics. There has been an intense development of the theory over the past few decades and it grew into a science successfully competing for applications in the practical sciences with statistical and even with numerical techniques. And so, if X is a measure space (most often a probability space), then the transformations are assumed to be measurable and preserve the measure by preimage (sometimes one requires only that they are nonsingular). The branch investigating measure dynamical systems is called ergodic theory. If X is assumed to be a topological space (usually compact), then T is required to be continuous. These systems are the subject of topological dynamics. In this chapter we study tilings of Euclidean space from the point of view of dynamical systems theory, and in particular, symbolic dynamics. The dynamical systems described here involve the full tiling space $\mathcal{T}_{\mathscr{P}}$ (Def.1.11), the tilings we study are tilings (Def.1.9) of $\mathbb{R}^d$ by translations of a finite number of basic tile types called "prototiles". All the tilings considered in this chapter satisfy the locally finiteness condition (Def.1.16). The tiling topology is based on the tiling metric (Lemma 1.17) which is complete and compact (Lemmas 1.18 and 1.20). As tiling spaces (Def.1.22) play a key role in the next chapters, we give a notion of tiling spaces and refer to a special kind of tiling space called the orbit closure of a tiling. The link between repetitivity (Def.1.30) and almost-periodicity (Def.1.34) within a tiling space is discussed in the Proposition 1.40. Proposition 1.45 states that a dynamical system $(\mathcal{O}(x), T)$ is minimal if and only if x is repetitive. And this follows from combining Proposition 1.40 and "Gottschalk's Theorem" (Theorem 1.43).

Chapter 2. We consider the symbolic dynamics in the classical one-dimensional case and introduce the notion in the context of shift spaces generated by a primitive substitution rule (Lemma 2.20). Then we show that the shift space is minimal. We distinguish between two kinds of dynamical systems (substitution subshifts), those which arise from a primitive, aperiodic substitution of constant length on two letters (as an example, discrete or continuous substitution) and the ones that arise from a primitive, aperiodic substitution of constant lengths on three letters. The one-dimensional tilings are described here (Def. 2.22) and in particular one-dimensional substitution tilings. Informally a suspension (Def. 2.25) construction turns a map into a flow. By this process, we prove that the suspension of the full shift space and the full tiling space support flows which are topologically conjugate (Prop. 2.27). Then by lemma 2.29, we show that there is a homeomorphism between the suspension of the substitution subshift and the associated substitution tiling space.

In preparation for determining the point spectrum (Def. 2.56) of the substitution tiling space $\mathcal{T}_\theta$, with all the tile lengths the same, associated to a primitive, aperiodic substitution θ of constant length l, we introduce the notions of the height of the substitution (Def. 2.45), the n-adic system $(Z(l), \tau)$ with the τ addition by 1 (Def. 2.35) and the system $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ (Def. 2.50). First, we describe in detail how to determine the point spectrum of the constant c suspension flow of τ addition by one on the suspension space $\{Z(l)\}_c$ (Prop. 2.59). Second, we give a brief description (Prop. 2.60) of how to determine the point spectrum of the system.

Then we use the trace relation (Def. 2.42) to determine the point spectrum of the constant c suspension flow of the shift map S on the suspension subshift (Theorem 2.61). Finally, we conclude in Theorem 2.62 the point spectrum of the substitution tiling space with all the tiles having the same length.

Chapter 3. This chapter describes our main results for a classification of the point spectrums of substitution tiling spaces. Clark and Sadun give in Theorem 2.3 of [7] a condition (3.1) for determining the point spectrum in terms of the tile length vector, recurrence vector (Def. 3.1) and the incidence matrix of the substitution. The set $\mathcal{R}$ of integer module generated by recurrence vectors (Def. 3.5) is used to give a new criteria for the existence of the point spectrum (Theorem 3.6). Theorem 3.7 gives partial spectral information in terms of the eigenvalues of the incidence matrix and the ratios of lengths of the tiles because it is applied for the class of substitutions whose incidence matrices have non zero eigenvalues and when there exists a full recurrence vector (Def. 3.2). So we present a Corollary 3.8 which we think is useful for determining the point spectrum in the case of substitutions with zero eigenvalues. Establishing the typical case (Def. 3.9) associated to a typical choice of length vectors is helpful in determining the sensitivity of the point spectrum to changes in tile lengths. It will be vital for us to determine the integer module generated by recurrence vectors, in the case of a two letter alphabet $\mathcal{A}$ we establish that $\mathcal{R} = \mathbb{Z}^2$. The situation is different in the case of a three letter alphabet $\mathcal{A}$, there are cases where $\mathcal{R} = \mathbb{Z}^3$ and other cases such as $\mathcal{R} \neq \mathbb{Z}^3$. We describe the relationship between the height of the substitution and the existence of full recurrence vectors with $\mathcal{R}$. We list relevant examples which show these relationships.

We examine the extent to which the point spectrum of a substitution tiling space associated to a primitive, aperiodic substitution of constant length on two letters is affected by changing the tile lengths (Theorem 3.26).

In the case of a substitution on three letters, our classification of the point spectrum is based on different possibilities of the roots of the quadratic monic polynomial $q(x)$ of the characteristic polynomial of its incidence matrix M_θ. In the first place, either we have the case of complex roots or the case of real roots. In the case of complex conjugate roots, the point spectrum will have sensitivity to changes in the tile lengths (Prop. 3.28). We investigate the point spectrum in the case of real roots through dividing it into four major cases each one of them is divided into subcases due to the height of the substitution.

Case I: If the roots r_1 and r_2 of $q(x)$ are of magnitude greater than or equal 1, that is, $|r_1|$ and $|r_2| \geq 1$. That is the case where we have eigenvalues of the incidence matrix with magnitude greater than or equal one.

If the height equals one or two, we show that the point spectrum is sensitive to changes in tile lengths (Propositions 3.31, 3.32).

Case II: If r_1 and r_2 are the roots of $q(x)$ such that $r_1 = 0$ and $r_2 \in \mathbb{Z} - \{0\}$.

If the height equals one, then in either case $\mathcal{R} = \mathbb{Z}^3$ or $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k,\ell \in \mathbb{Z} \right\}$, we conclude that the point spectrum is trivial. Thus the point spectrum will have sensitivity to changes in tile lengths (Prop. 3.34, 3.36). While in the case of a height two substitution we establish that the point spectrum is insensitive to changes in tile lengths (Prop. 3.37).

Case III: If $q(x)$ has 0 as a double root, that is, $r_1 = r_2 = 0$.

In this case, where zero occurs as a double root, the height two substitution subcase simply does not exist and so we give our examination of the point spectrum only for the height one substitutions (Prop. 3.38). The point spectrum will be insensitive to changes in tile lengths.

Case IV. If the roots r_1 and r_2 of $q(x)$ are such that $|r_1| > 1$ and $0 < |r_2| < 1$. This is the case where $q(x)$ has a Pisot root. Although we cannot handle the general case, we can give a conclusion due to some special cases. Again the only subcase that we have is the height-one substitution, we show that the point spectrum will have sensitivity to changes in the tile lengths (Prop. 3.43).

Chapter 4. The Banach contraction principle states that every contraction on a complete metric space has a unique fixed point. As a generalization of it, the following conjecture was considered in [17].

Conjecture I. Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{d(f^m(x), f^m(y)) : m \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x,y \in X$ and some $K \in (0,1)$, where $\mathbb{J}$ is a subset of positive integers. Then f has a fixed point.

Rather than requiring that a mapping be a contraction, we consider the following generalization of Conjecture I and provide a tiling proof for our result.

Conjecture II. Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{\alpha(x,y)d(f^m(x),f^m(y)) : m \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x,y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$, where $\mathbb{J}$ is a subset of positive integers. Then f has a fixed point.

However, relevant to this thesis, Stein in [31] established conjecture I for the class of strongly continuous mappings and $\mathbb{J} = \{1,2,..,n\}$. In [17], the authors showed that conjecture I is true if $\mathbb{J} = \{1,2\}$.
Because the process of constructing an analytical proof of our main result is complicated, we follow the same method devised by Stein in [17] which involves tilings of the integers while in the proof of our results that lead to the main result we use analytical methods (Theorems 4.3, 4.4, 4.5). To use the tiling methodology, we define a good collection of tiles (Def.4.6) and present some rules defined in [17] that yield to the proof of our main fixed point result (Theorem 4.7).

Papers:

1. D. Abuzaid, Fixed point theorems and tiling problems, Filomat, (2015) (in press).

Chapter 1

Some Notions in Tiling Theory and Tiling Dynamical System

In this chapter, we study tilings of Euclidean space. The tilings we study are tilings of $\mathbb{R}^d$ by translations of a finite number of basic tile types called "*prototiles*".
Throughout this chapter, a dynamical system will be a pair (X,T), where X is a compact metric space (the phase space) and T is a continuous action of a group, usually $\mathbb{R}^d$. Some basic notions and fundamental results of tiling theory are presented. Necessary notations and the terminology used in the sequel are also fixed. None of the results in this chapter are original and most of the material is taken from [15], [25] and [26].

1.1 Basic definitions

Definition 1.1. A *topological group* is a triple $(G,\cdot,\tau)$, where $(G,\cdot)$ is a group and (G,τ) is a topological space, such that the following conditions hold:

1. The operation $\cdot$ on G is a continuous function from $G\times G$ into G (the topology on $G\times G$ is the product topology determined by τ).
2. The function $f : G \to G$ defined by $f(a) = a^{-1}$, for each $a \in G$ is continuous.

An example of a topological group is $(\mathbb{R}^d,+,\tau)$, where $+$ denotes ordinary addition of $\mathbb{R}^d$ and τ is the Euclidean topology.

Definition 1.2. Let X be a topological space. Let $(G,\cdot)$ be a topological group. An *action* of G on X is a continuous map $\alpha : G \times X \to X$ such that

1. $\alpha(e,x) = x$ for all $x \in X$, where e is the identity element of the group G.
2. $\alpha(g_1, \alpha(g_2,x)) = \alpha(g_1 \cdot g_2,\ x)$ for all $x \in X$ and $g_1, g_2 \in G$.

Note that for any given action α and any $g_0 \in G$, the function $\alpha^{g_0} : X \to X$, where $\alpha_{g_0}(x) = \alpha(g_0, x)$, is a homeomorphism with inverse $\alpha^{(-g_0)}$.

Definition 1.3. Let f and g be two actions of the group G on the topological spaces X and Y respectively. A *topological semi-conjugacy from f to g* is a surjective continuous map $h : X \to Y$ such that $g^t \circ h = h \circ f^t$ for all $t \in G$. If h is a homeomorphism, it is called a *topological conjugacy* and f and g are said to be *topologically conjugate*.

Definition 1.4. Let α be a topological group action of $(G,\cdot)$ on X. The set $M \subseteq X$ is *minimal* if it is non-empty, closed, invariant under the action and there are no proper closed and action invariant subsets of M.

Definition 1.5. For an action α of G on X and $x \in X$, the *orbit* of x is denoted by

$$\mathcal{O}(x) = \{\alpha(g,x) \mid g \in G\}.$$

The following result is well known.

Theorem 1.6. *Let α be a topological group action of $(G,\cdot)$ on X. Then $M \subseteq X$ is minimal if and only if M is non-empty and for all $x \in M$, $\overline{\mathcal{O}(x)} = M$, where $\overline{\mathcal{O}(x)}$ denotes the closure of $\mathcal{O}(x)$.*

By an application of Zorn's Lemma, one obtains the following result.

Theorem 1.7. *Let α be a topological group action of $(G,\cdot)$ on X. Any non-empty, closed, invariant subset of X contains a minimal set M.*

1.2 Tilings

We now begin to introduce a structure on tilings of $\mathbb{R}^d$ that allows us to use dynamical systems to study their properties.

Definition 1.8. A subset D of $\mathbb{R}^d, d \geq 1$, is called a *tile* if it is homeomorphic to a d-dimensional closed ball.

Example 1.1. Tiles in $\mathbb{R}$ are closed intervals and in $\mathbb{R}^2$ are often closed polygons.

Definition 1.9. A *tiling* x of $\mathbb{R}^d$ is a collection $\{D_i\}$ of tiles such that any two tiles have pairwise disjoint interiors and their union is $\mathbb{R}^d$, (that is, $\{D_i\}$ covers $\mathbb{R}^d$).

Definition 1.10. We say that two tiles D_1 and D_2 are *equivalent* if one is a translation of the other. It is denoted by $D_1 \sim D_2$. Equivalence class representatives are called *prototiles.*

Definition 1.11. Let $\mathscr{P}$ be a finite collection of prototiles (finite set of inequivalent tiles) and $\mathcal{T}_{\mathscr{P}}$ be the collection of all possible tilings of $\mathbb{R}^d$ formed from translations of elements of $\mathscr{P}$. $\mathcal{T}_{\mathscr{P}}$ is usually called the *full tiling space*.

Remark 1.12. *Geometry is generally concerned with properties of objects that are invariant under congruence and dynamics is concerned with group actions.*

In this chapter, we will be interested in how groups of rigid motions act on sets of tilings. Of central interest will be the action of $\mathbb{R}^d$ by translation.

Definition 1.13. Let $T^t x \in \mathcal{T}_{\mathscr{P}},\ t \in \mathbb{R}^d$ and $x \in \mathcal{T}_{\mathscr{P}}$, be the tiling of $\mathbb{R}^d$ in which each tile $D \in x$ has been shifted by the vector $-t$, that is, $T^t x = \{D - t \mid D \in x\}$. This *action of* $\mathbb{R}^d$ *on* $\mathcal{T}_{\mathscr{P}}$ is denoted by T.

The *orbit* of a tiling $x \in \mathcal{T}_{\mathscr{P}}$ is the set $\mathcal{O}(x) = \{T^t x \mid t \in \mathbb{R}^d\}$ of translates of x.

Definition 1.14. Let $\mathscr{P}$ be a set of prototiles. A $\mathscr{P}$*-patch* p_x is a finite subset of a tiling $x \in \mathcal{T}_{\mathscr{P}}$ such that the union of tiles in p_x is connected. This union of tiles in p_x is called the *support* of p_x and is written as $\mathrm{supp}(p_x)$.

Definition 1.15. The notion of equivalence extends to patches, and a set of equivalence class representatives of patches is denoted by $\mathscr{P}^*$. The subset of patches of n tiles, called the *n-patches*, is denoted by $\mathscr{P}^{(n)} \subseteq \mathscr{P}^*$.

Definition 1.16. A tiling space $\mathcal{T}_{\mathscr{P}}$ is *locally finite* if $\mathscr{P}^{(2)}$ is finite; equivalently, $\mathscr{P}^{(n)}$ is finite for each n.

From now on, whenever we write $\mathscr{P}$, $\mathscr{P}^*$or $\mathcal{T}_{\mathscr{P}}$ it will always implicitly include a choice of a finite $\mathscr{P}^{(2)}$. For the case of polygonal prototiles in $\mathbb{R}^2$, we will assume that all tiles meet edge-to-edge. This will guarantee achieving local finiteness condition.

Example 1.2. Let S be the set consisting of a single 1×1 square prototile. Without any local finiteness condition, fault lines exist in the tilings $\mathcal{T}_S$ with a continuum of possible displacements. The edge-to-edge condition guarantees that every $x \in \mathcal{T}_S$ is a translation of a single periodic tiling.

1.3 The tiling topology

We shall show that locally finite tiling spaces have useful topological properties. The tiling topology is based on a simple idea: two tilings are said to be close if after a small translation they agree on a large ball around the origin.

For $K \subseteq \mathbb{R}^d$ bounded and $x \in \mathcal{T}_{\mathscr{P}}$, let $x[[K]]$ denote the set of all sub-patches $p_x \subseteq x$ such that $K \subseteq \operatorname{supp}(p_x)$. The smallest such patch is represented by $x[K]$.
Given $r > 0$ and $t \in \mathbb{R}^d$, let

$$B_r = \{t \in \mathbb{R}^d \mid ||t|| < r\},$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^d$.

Lemma 1.17. *For $x, y \in \mathcal{T}_{\mathscr{P}}$, define*

$$\begin{aligned} d(x,y) = \inf(\{\sqrt{2}/2\} \quad \cup \quad &\{0 < r < \sqrt{2}/2 \mid \exists\ p_x \in x[[B_{1/r}]],\ p_y \in y[[B_{1/r}]], \\ &\text{with } \ T^t p_x = p_y \ \text{ for some } \ ||t|| \leq r\}). \end{aligned} \tag{1.1}$$

Then d defines a metric on $\mathcal{T}_{\mathscr{P}}$

Proof. It is clear from the definition that $0 \leq d(x,y) \leq \frac{\sqrt{2}}{2}$ for all $x, y \in \mathcal{T}_{\mathscr{P}}$.
We now show that $d(x,y) = 0$ if and only if $x = y$. Assume that $x \neq y$, then there are two possibilities:

1. The two tilings have the origin placed at different locations within a prototile. Then in order that they agree on a patch centered at the origin one of them must be translated by some vector. One thus takes a vector of this type of smallest magnitude. So, in this case, $d(x,y) > 0$.

2. The two tilings x, y have the origin placed at the same location relative to some prototile. Then, since x is not the same as y, there must be some $r > 0$ such that

the tilings x and y have different prototiles at a distance $\frac{1}{r}$ from the origin. Thus the two tilings would not agree on a ball of radius $\geq \frac{1}{r}$. As a result, again we must have $d(x,y) > 0$. Consequently, we have $d(x,y) = 0$ implies $x = y$.

Conversely, if $x = y$, then the two tilings x, y agree on a ball of radius $\frac{1}{r}$ for each $r > 0$. This implies that the infimum of r's is 0. Thus $d(x,y) = 0$.

Obviously, $d(x,y) = d(y,x)$ for all $x, y \in \mathcal{T}_{\mathscr{P}}$.

Finally, we will prove the triangle inequality. Let $0 < d(x,y) = a' \leq d(y,z) = b'$ with $a' + b' < \sqrt{2}/2$. Let $0 < \varepsilon < \sqrt{2}/2 - (a' + b')$ and set $a = a' + \varepsilon/2$ and $b = b' + \varepsilon/2$.

Since $d(x,y) = a'$, there exist $p_x \in x[[B_{1/a'}]], p_y \in y[[B_{1/a'}]]$ with $T^t p_x = p_y$ for some $||t|| \leq a'$. But $a' < a$, therefore $1/a < 1/a'$ and so $B_{1/a} \subseteq B_{1/a'}$. This implies that there exist $p_x \in x[[B_{1/a}]], p_y \in y[[B_{1/a}]]$, with $T^t p_x = p_y$ for some $||t|| < a$.
Similarly, since $d(y,z) = b'$, there exist $p_y^{'} \in y[[B_{1/b}]], p_z^{'} \in z[[B_{1/b}]]$ with $T^{-s} p_z^{'} = p_y^{'}$ for some $||s|| < b$.

Let $y_0 = p_y \cap p_y^{'}$, $x_0 = T^{-t} y_0 \subseteq p_x$ and $z_0 = T^s y_0 \subseteq p_z$. Then $T^{-(t+s)} z_0 = x_0$, where $||t+s|| \leq ||t|| + ||s|| < a + b$.

Note that

$$\begin{aligned} a + b &= a' + b' + \varepsilon/2 + \varepsilon/2 = (a' + b') + \varepsilon \\ &< (a' + b') + \sqrt{2}/2 - (a' + b') = \sqrt{2}/2, \end{aligned}$$

this implies $0 < a \leq b < \sqrt{2}/2$. Also $\frac{1}{a+b} \leq \frac{1}{b} - a$ if and only if $b(a+b) \leq 1$.

Let $c = a + b$. Then

$$0 < \frac{1}{c} = \frac{1}{a+b} \leq \frac{1}{b} - a.$$

We claim that

$$B_{1/c} \subseteq (B_{1/b} + t).$$

Indeed, let $h \in B_{1/c}$. Then $||h|| < 1/c$. Now

$$||h - t|| \leq ||h|| + ||t|| < \frac{1}{c} + a \leq \frac{1}{b} - a + a = \frac{1}{b}.$$

This implies that

$$h \in (B_{1/b} + t) \text{ and so } B_{1/c} \subseteq (B_{1/b} + t).$$

Now $p_y, p_y^{'} \in y[[B_{1/b}]]$ and thus

$$x_0 \in x[[B_{1/b} + t]] \subseteq x[[B_{1/c}]].$$

Also $z_0 \in Z[[B_{1/c}]]$. Since $T^{-(t+s)}z_0 = x_0$, where $||t+s|| < a+b$, we have

$$d(x,z) \leq a+b = a' + \varepsilon/2 + b' + \varepsilon/2 = d(x,y) + d(y,z) + \varepsilon,$$

where $\varepsilon > 0$ is chosen arbitrarily small. Consequently, we have

$$d(x,z) \leq d(x,y) + d(y,z).$$

Hence d is a metric on $\mathcal{T}_{\mathscr{P}}$. □

1.4 Completeness and Compactness

Lemma 1.18. *The tiling metric d is complete.*

Proof. Let (x_n) be a Cauchy sequence of tilings. Assume that $d(x_{n+1}, x_n) > 0$ and let $s_n = d(x_{n+1}, x_n) + \frac{1}{2^n}$. We construct a subsequence (s_{n_k}) of (s_n) as follows:

1. Since (x_n) is Cauchy, we can choose $N > 1$ such that $d(x_N, x_{N+1}) < \frac{1}{2^2}$. Set $n_1 = N$. Then we have
$$s_{n_1} = d(x_N, x_{N+1}) + \frac{1}{2^N} < \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}.$$

2. Assume that $n_1, n_2, \ldots, n_k$ have been chosen such that

   (a) $n_1 < n_2 < \ldots < n_k$.

   (b) $s_{n_i} < \frac{1}{2^i}$, $i = 1, 2, \ldots, k$.

We shall show that $s_{n_k} < \frac{1}{2^k}$ for all $k \geq 1$. Choose $M > n_k$ and $M > k+2$ such that $d(x_M, x_{M+1}) < \frac{1}{2^{k+2}}$.

Let $n_{k+1} = M$. Then we have

$$n_1 < n_2 < \ldots < n_k < n_{k+1}$$

and

$$
\begin{aligned}
s_{n_{k+1}} &= d(x_{n_{k+1}+1}, x_{n_{k+1}}) + \frac{1}{2^{n_{k+1}}} \\
&= d(x_{M+1}, x_M) + \frac{1}{2^M} \\
&< \frac{1}{2^{k+2}} + \frac{1}{2^{k+2}} = \frac{1}{2^{k+1}}.
\end{aligned}
$$

So, by the Principle of Mathematical Induction,

$$s_{n_k} < \frac{1}{2^k} \quad \text{for all } \ k \geq 1.$$

We claim that (s_{n_k}) is decreasing. If not, then $d(x_{n_{k+1}}, x_{n_k})$ does not converge to zero and so (x_{n_k}) is not Cauchy. Since $s_{n_k} < \frac{1}{2^k}$ for all $k \geq 1$ and $\sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$, by the Comparison Test, $\sum_{k=1}^{\infty} s_{n_k} < \infty$.

For convenience, now we shall use the subscript n instead of n_k. Let $d(x_1, x_2) = d_1$. Then by the definition of the metric, there exist two sub-patches p_{x_1}, p'_{x_2} match around the origin in a ball of radius $\frac{1}{d_1}$, that is, there exist $p_{x_1} \in x_1[[B_{1/d_1}]]$ and $p'_{x_2} \in x_2[[B_{1/d_1}]]$ such that

$$T^{t_1} p_{x_1} = p'_{x_2}, \; ||t_1|| \leq d_1 \; (t_1 \in \mathbb{R}^d).$$

Since $d_1 < s_1$, it follows that $B_{1/s_1} \subseteq B_{1/d_1} \subseteq \operatorname{supp}(p_{x_1})$ and so $p_{x_1} \in x_1[[B_{1/s_1}]], \; p'_{x_2} \in x_2[[B_{1/s_1}]]$ such that

$$T^{t_1} p_{x_1} = p'_{x_2}, \;\; ||t_1|| < s_1.$$

Now let $d(x_2, x_3) = d_2$. Then there exist $p_{x_2} \in x_2[[B_{1/d_2}]], \; p_{x_3} \in x_3[[B_{1/d_2}]]$ such that

$$T^{t_2} p x_2{}' = p'_{x_3}, ||t_2|| < d_2 \;\; (t_2 \in \mathbb{R}^d).$$

Since $d_2 < s_2$, we have $p_{x_2} \in x_2[[B_{1/s_2}]], \; p_{x_3} \in x_3[[B_{1/s_2}]]$ such that

$$T^{t_2} p_{x_2} = p_{x_3}, \; ||t_2|| < s_2.$$

Continuing in this manner, for each n, we can find $t_n \in \mathbb{R}^d$ with $||t_n|| < s_n$ such that

$$T^{t_n} p_{x_n} \subseteq p_{x_{n+1}}.$$

Since (s_n) is decreasing,

$$B_{1/s_1} \subseteq B_{1/s_2} \subseteq \ldots \subseteq B_{1/s_n} \subseteq \ldots.$$

Note that $B_{1/s_1} \subseteq B_{1/s_2}$, $p'_{x_2} \in x_2[[B_{1/s_1}]]$, $p_{x_2} \in x_2[[B_{1/s_2}]]$. This implies $T^{t_1} p_{x_1} \subseteq p_{x_2}$ and so on.

Let

$$r_n = \sum_{k=n}^{\infty} t_k.$$

Then

$$r_n = r_{n+1} + t_n$$

and so

$$T^{r_n} p_{x_n} = T^{r_{n+1}} T^{t_n} p_{x_n} \subseteq T^{r_{n+1}} p_{x_{n+1}}.$$

This means that $T^{r_n} p_{x_n}$ is an increasing sequence of patches. Now define a tiling

$$x = \bigcup_n T^{r_n} p_{x_n}.$$

Notice that

$$d(x, x_n) \leq \max\{||r_n||, s_n\}.$$

Since

$$||r_n|| = ||\sum_{k=n}^{\infty} t_k|| \leq \sum_{k=n}^{\infty} ||t_k|| < \sum_{k=n}^{\infty} s_k,$$

$||r_n|| \to 0$ as $n \to \infty$ and also $s_n \to 0$ as $n \to \infty$. As a result, we have $d(x, x_n) \to 0$ as $n \to \infty$, that is, the subsequence (x_{n_k}) converges to x.

Since (x_n) is a Cauchy sequence and its subsequence (x_{n_k}) converges to x, it follows that the sequence (x_n) itself converges to x. Hence the tiling metric space is complete. $\square$

Lemma 1.19. *Let $\mathcal{T}_{\mathscr{P}}$ be a locally finite tiling space. Then $\mathcal{T}_{\mathscr{P}}$ is totally bounded.*

Proof. Let $\varepsilon > 0$. Choose n such that all patches containing a ball of radius $\frac{1}{\varepsilon}$ centered at the origin are contained in some of $\mathscr{P}^{(n)}$. By local finiteness, $\mathscr{P}^{(n)}$ is finite. Assume that we have m patches of n tiles, say $p_1^{(n)}, p_2^{(n)}, \cdots, p_m^{(n)}$ and choose some tilings $x_1, x_2, \cdots, x_m$ that have these patches around the origin. Any patch containing the ball $B(0, \frac{1}{\varepsilon})$ will match one of $p_m^{(n)}$ after translation of x_m by vector of magnitude less than ε. The corresponding tiling y_m will be within ε of x_m. So $d(x_m, y_m) < \varepsilon$. But $B(x_m, \varepsilon)$ is an ε-ball in the tiling topology, therefore the m ε-balls form a covering of the tiling space, that is,

$$\mathcal{T}_{\mathscr{P}} = \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

and hence $\mathcal{T}_{\mathscr{P}}$ is totally bounded. $\square$

Theorem 1.20. *Let $\mathcal{T}_{\mathscr{P}}$ be a locally finite tiling space. Then $\mathcal{T}_{\mathscr{P}}$ is compact in the tiling metric d.*

Proof. It is well-known that a metric space is compact if and only if it is complete and totally bounded. By Lemma 1.18, $\mathcal{T}_{\mathscr{P}}$ is complete and by Lemma 1.19 $\mathcal{T}_{\mathscr{P}}$ is totally bounded. Consequently, $\mathcal{T}_{\mathscr{P}}$ is compact. $\square$

Theorem 1.21. *Let $\mathcal{T}_{\mathscr{P}}$ be a locally finite tiling spaces. The action T of $\mathbb{R}^d$ by translation on $\mathcal{T}_{\mathscr{P}}$ is continuous.*

Proof. Let $(t,x) \in \mathbb{R}^d \times \mathcal{T}_{\mathscr{P}}$, and $\varepsilon > 0$. Define the metric D on $\mathbb{R}^d \times \mathcal{T}_{\mathscr{P}}$ as follows:

$$D[(t,x),(t',y)] = \max\{\mathrm{dist}(t,t'),d(x,y)\},$$

where dist is the usual metric on $\mathbb{R}^d$ and d is the tiling metric on $\mathcal{T}_{\mathscr{P}}$. If $||t'|| > ||t||$, choose $\delta < \frac{1}{2}\left(\frac{1}{\frac{1}{\varepsilon}+||t'||}\right)$ such that $D[(t,x),(t',y)] < \delta$. Then $||t-t'|| < \delta$ and $d(x,y) < \delta$. So $\exists\, p_x \in x[[B_{1/\delta}]], p_y \in y[[B_{1/\delta}]]$ with $T^s p_x = p_y$ for some $||s|| \leq \delta$. Note that

$$||t'|| \leq ||t|| + ||t-t'||$$

and so

$$||t'|| < ||t|| + \delta.$$

Thus $t' = t + u$, where $||u|| < \delta$. If $T^s p_x = p_y$ and $t' = t+u$, then

$$T^t(T^s p_x) = T^t p_y$$

implies

$$T^{u+s} T^t p_x = T^{t'} p_y$$

Since $p_y \subseteq y$ is a patch of y containing a ball of radius $\frac{1}{\delta} > 2\left(\frac{1}{\varepsilon} + ||t'||\right)$ around the origin, this implies that $T^{t'} p_y \subseteq y$ is a patch of y containing a ball of radius $\frac{1}{\varepsilon}$ around the origin after translation by t'. Also $p_x \subseteq x$ is a patch of x containing a ball centered at the origin of radius $\frac{1}{\delta} > 2\left(\frac{1}{\varepsilon} + ||t'||\right)$. Thus $T^t p_x \subseteq x$ is a patch containing a ball centered at the origin of radius $2\left(\frac{1}{\varepsilon} + ||t||\right)$ after translation by vector t with $||t|| < ||t'||$. And so containing a ball centered at the origin of radius $\frac{1}{\varepsilon}$.

$$||u+s|| \leq ||u|| + ||s|| < 2\delta < \frac{\varepsilon}{1+\varepsilon||t'||} < \varepsilon.$$

So

$$T^{u+s}(T^t p_x) = T^{t'} p_y$$

for some $||s+u|| < \varepsilon$. Thus

$$d(T^t p_x, T^{t'} p_y) < \varepsilon,$$

and hence $T : \mathbb{R}^d \times \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}}$ is continuous. $\square$

1.5 Tiling spaces

Symbolic dynamics investigates a special kind of dynamical system called a *symbolic dynamical system.* The classical case is 1-dimensional, but we discuss here the general d-dimensional case.

Definition 1.22. Let $\mathcal{T}_{\mathscr{P}}$ be a full d-dimensional tiling space and let T represent the translation action of $\mathbb{R}^d$. A *tiling space* $\mathcal{T}$ is a closed T-invariant subset $\mathcal{T} \subseteq \mathcal{T}_{\mathscr{P}}$. We call the pair $(\mathcal{T}, T)$ a *tiling dynamical system.*

Definition 1.23. Let $\mathcal{T}_{\mathscr{P}}$ be a full tiling space and let $\mathcal{F} \subseteq \mathscr{P}^*$. Let $\mathcal{T}_{\setminus \mathcal{F}} \subseteq \mathcal{T}_{\mathscr{P}}$ be the set of all tilings $x \in \mathcal{T}_{\mathscr{P}}$ such that no patch p_x in x is equivalent to any patch in $\mathcal{F}$. We call a set $\mathcal{F}$ a set of *forbidden patches*.

Note that we will use the notation p in general for any patch in $\mathscr{P}^*$.

Lemma 1.24. *For any $\mathcal{F} \subseteq \mathscr{P}^*$, the set $\mathcal{T}_{\setminus \mathcal{F}}$ is a tiling space.*

Proof. First we show that $\mathcal{T}_{\setminus \mathcal{F}}$ is T-invariant. Let x be any tiling $\mathcal{T}_{\setminus \mathcal{F}}$, then x contains no patch in $\mathcal{F}$. We know that the patches that occur in any translation of x are the same patches that occur in x. Thus for any vector $t \in \mathbb{R}^d$, the translation $T^t x$ is also in $\mathcal{T}_{\setminus \mathcal{F}}$, that is, $T^t x \in \mathcal{T}_{\setminus \mathcal{F}}$ and so $\mathcal{T}_{\setminus \mathcal{F}}$ is T-invariant.

It remains to show that $\mathcal{T}_{\setminus \mathcal{F}}$ is closed, it is enough to show that $\mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\setminus \mathcal{F}}$ is open. Let $y \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\setminus \mathcal{F}}$ be a tiling. By Definition 1.23, there is a patch p of $\mathcal{F}$ such that p occurs in y. Since p is compact, every point in p is within a finite distance from the origin. Let d_1 be the distance between p and the origin and let d_2 denote the diameter of p (note that p is compact if and only if it is closed and bounded). Set $r_1 = d_1 + d_2$ and $r_2 = r_1 + \frac{1}{r_1}$. Then p is contained in a ball of radius r_2 centered at the origin. If we shift y by $\frac{1}{r_2} < \frac{1}{r_1}$ and thus p will be in the shifted version of the tiling. So any tiling z within $\frac{1}{r_2}$ of y contains a copy of the patch p and therefore will be in $\mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\backslash\mathcal{F}}$. This means that there exists

$$B_{1/r_2}(y) = \left\{ z \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\backslash\mathcal{F}} \mid d(z,y) < \frac{1}{r_2} \right\}$$

such that $y \in B_{1/r_2}(y) \subseteq \mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\backslash\mathcal{F}}$. Hence $\mathcal{T}_{\mathscr{P}} - \mathcal{T}_{\backslash\mathcal{F}}$ is open and so $\mathcal{T}_{\backslash\mathcal{F}}$ is closed. $\square$

Proposition 1.25. *Let $\mathcal{T} \subseteq \mathcal{T}_{\mathscr{P}}$ be a tiling space. Then for every tiling $y \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}$, there exists a patch of y that doesn't occur in $\mathcal{T}$.*

Proof. Assume that there exists a tiling $y \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}$, such that each patch of y does occur in a tiling in $\mathcal{T}$. If p_n is a patch of y determined by $[-n,n]^d$ occurring in a tiling in $\mathcal{T}$, then by T-invariance of $\mathcal{T}$ there exists a tiling $x_n \in \mathcal{T}$ which matches y exactly on $[-n,n]^d$. This implies $d(x_n,y) \leq \frac{1}{n}$ for all n and hence $x_n \to y$. Now y is a limit point of a sequence (x_n) of tilings in $\mathcal{T}$. Since $\mathcal{T}$ is closed, then $y \in \mathcal{T}$ which is a contradiction. Hence for every tiling $y \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}$, there exists a patch of y that doesn't occur in any tiling in $\mathcal{T}$. $\square$

Remark 1.26. *Every tiling space $\mathcal{T} \subseteq \mathcal{T}_{\mathscr{P}}$ is defined by a set $\mathcal{F}$ of forbidden patches.*
To see this, let $\mathcal{F} = \{p_y | \, y \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}\}$, where p_y is a patch of y that does not occur in $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}_{\backslash\mathcal{F}}$.
Indeed, if $x \in \mathcal{T}$, then no patch in x is equivalent to any patch p_y in $\mathcal{F}$. Consequently $x \in \mathcal{T}_{\backslash\mathcal{F}}$ and so $\mathcal{T} \subseteq \mathcal{T}_{\backslash\mathcal{F}}$.
On the other hand, if there exists an $x \in \mathcal{T}_{\backslash\mathcal{F}}$ such that $x \notin \mathcal{T}$. Then $x \in \mathcal{T}_{\mathscr{P}} - \mathcal{T}$ and by Proposition 1.25, x would have a corresponding patch $p_x \in \mathcal{F}$ which is a contradiction and so $\mathcal{T}_{\backslash\mathcal{F}} \subseteq \mathcal{T}$. Hence $\mathcal{T} = \mathcal{T}_{\backslash\mathcal{F}}$.

Definition 1.27. A tiling space $\mathcal{T} \subseteq \mathcal{T}_{\mathscr{P}}$ is called a *finite type tiling space* if there exists a finite $\mathcal{F} \subseteq \mathscr{P}^*$ so that $\mathcal{T} = \mathcal{T}_{\backslash\mathcal{F}}$.

Consider the following question: Suppose we are given a set $\mathscr{P}$ of prototiles and a set $\mathcal{F} \subseteq \mathscr{P}^*$of forbidden patches. Is $\mathcal{T}_{\backslash\mathcal{F}} \neq \emptyset$?

The following result gives a positive answer to the above question.

Theorem 1.28. *Let $\mathscr{P}$ be a collection of prototiles with a local finiteness condition $\mathscr{P}^{(2)}$ and let $\mathcal{F} \subseteq \mathscr{P}^*$ be a set of forbidden patches. Define $\mathscr{P}^+ \subseteq \mathscr{P}^*$ to be the set of patches that do not contain any forbidden sub-patches. Then $\mathcal{T}_{\backslash\mathcal{F}} \neq \emptyset$ if and only if for each $r > 0$ there is a patch $p \in \mathscr{P}^+$ with $B_r \subseteq \operatorname{supp}(p)$.*

1.6 Applications of topological dynamics

Recall that for given n linearly independent vectors $b_1, \cdots, b_n \in \mathbb{R}^m$, the lattice generated by them is defined as $L(b_1, b_2, \cdots, b_n) = \{\sum x_i b_i | x_i \in \mathbb{Z}\}$. We refer to $b_1, \cdots, b_n$ as a basis of the lattice. Equivalently, if we define B as the $m \times n$ matrix whose columns are $b_1, \cdots, b_n$, then the lattice generated by B is $L(B) = L(b_1, b_2, \cdots, b_n) = \{Bx | x \in \mathbb{Z}^n\}$. We say that the rank of the lattice is n and its dimension is m. If $n = m$, the lattice is called a *full-rank* lattice.
It is easy to see that, L is a lattice if and only if L is a discrete subgroup of $(\mathbb{R}^n, +)$.

For matrix $B, P(B) = \{Bx | x \in [0,1)^n\}$ is the *fundamental parallelepiped* of B.
For a full rank lattice $L(B), P(B)$ tiles $\mathbb{R}^n$ in the parttern $L(B)$, in the sense that $\mathbb{R}^n = \{P(B) + x \mid x \in L(B)\}$.

Definition 1.29. A tiling x of $\mathbb{R}^d$ is called a *periodic tiling* if its isotropy group $\Gamma_x = \{t \in \mathbb{R}^d \mid T^t x = x\}$ is a lattice of full rank, that is a subgroup of $\mathbb{R}^d$ with d linear independent generators. A tiling x is called *aperiodic* if $\Gamma_x = \{0\}$.

Definition 1.30. A tiling x is called *repetitive* if for any patch p_x in x there is an $r > 0$ such that for any $t \in \mathbb{R}^d$ there is a translation $T^s p_x$ of p_x in x such that

$$\mathrm{supp}(T^s p_x) \subseteq B_r + t.$$

Remark 1.31.

1. *In other words, a tiling x is repetitive if there is an* $r > 0$ *such that any ball of radius r always contains a copy of the patch.*

2. *All periodic tilings are repetitive since one can choose the r in the definition of repetitive to be larger than the diameter of a fundamental parallelepiped and then any patch will have a period within r of any point. So we can think of repetitivity as a generalization of periodicity.*

Definition 1.32. Let (X,T) be a dynamical system. Let $U \subseteq X$ be open and let $x \in X$. Define the *return set* of x to U to be

$$R(x,U) = \{t \in \mathbb{R}^d \mid T^t x \in U\}.$$

Definition 1.33. A set $R \subseteq \mathbb{R}^d$ is called *relatively dense* if there is an $r > 0$ such that every r-ball in $\mathbb{R}^d$ intersects R.

Definition 1.34. A point $x \in X$ is called *almost periodic* if $R(x,U)$ is relatively dense for every open $U \subseteq X$ with $R(x,U) \neq \emptyset$.

Definition 1.35. For a tiling space $\mathcal{T} \subseteq \mathcal{T}_{\mathscr{P}}$, let $p \in \mathscr{P}^*$ and $R = \text{supp}(p)$. Set $\mathcal{T}(p) = \{x \in \mathcal{T} : x[R] = p\}$. For $\varepsilon > 0$, define the *cylinder set*

$$U_{(p,\varepsilon)} = T^{B_\varepsilon}\mathcal{T}(p) = \{T^t x \mid x \in \mathcal{T}(p), t \in B_\varepsilon\}.$$

Lemma 1.36. *$U_{(p,\varepsilon)}$ is open.*

Proof. Let $z \in U_{(p,\varepsilon)}$. Then there exists $x \in \mathcal{T}(p)$ and $t \in B_\varepsilon$ such that $z = T^t x$. Since p is compact, all points in $R = \text{supp}(p)$ are within a finite distance, say ω, from the origin. Let

$$\delta = \frac{1}{2}\min\left\{\frac{1}{\omega+\varepsilon}, \varepsilon\right\}.$$

Then the ball of radius $\frac{1}{\delta}$ centered at the origin contains R and all points within ε distance from R. So the ball (in the tiling metric) $B_\delta(z)$ consists of all tilings z' that agree with z on a ball of radius $\frac{1}{\delta}$ after small translation less than $\delta < \varepsilon$. Therefore, $B_\delta(z) \subseteq U_{(p,\varepsilon)}$ because $z' \in \mathcal{T}(p)$ and $\delta < \varepsilon$. Since z is an arbitrary point of $U_{(p,\varepsilon)}$, we conclude that $U_{(p,\varepsilon)}$ is open. $\square$

Remark 1.37. *Without loss of generality, we can assume by translating that the support of each patch p in $\mathscr{P}^*$ contains the largest possible ball B_r around the origin.*

Lemma 1.38. *The collection of cylinder sets $\{U_{(p,\varepsilon)} \mid p \in \mathscr{P}^*, \varepsilon > 0\}$ forms a basis for the tiling topology.*

Proof. Let $U \subseteq \mathcal{T}$ be any open set containing x. Then there exists $B_d(x,\varepsilon)$ such that $x \in B_d(x,\varepsilon) \subseteq U$.

For the tiling x, x will contain a patch p centered at the origin of radius greater than $\varepsilon + 1 + \frac{1}{\varepsilon}$. Consider a cylinder set $U_{(p,\varepsilon)} = \{T^t z \mid z \in \mathcal{T}(p), \ ||t|| < \varepsilon\}$ containing $x = T^{-t}(T^t x)$. Any translation $(T^t z)$ of $z \in \mathcal{T}(p)$ by amount $||t|| < \varepsilon$ will agree with x on a ball of radius $\frac{1}{\varepsilon}$ around the origin, and then $d(T^t z, x) < \varepsilon$. Thus for all elements $w = T^t z \in U_{(p,\varepsilon)}$, $d(w,x) < \varepsilon$ and $x \in U_{(p,\varepsilon)}$. So

$$x \in U_{(p,\varepsilon)} \subseteq B_d(x,\varepsilon) \subseteq U.$$

Hence $\{U_{(p,\varepsilon)} \mid p \in \mathscr{P}^*, \varepsilon > 0\}$ forms a basis for the tiling topology. $\square$

Note that in Definition 1.34, one can replace the open sets U by the sets of the basis given in the above lemma.

Lemma 1.39. *If $R \subseteq \mathbb{R}^d$ is relatively dense, then for any $s \in \mathbb{R}^d$ the set $s+R = \{s+r \mid r \in R\}$ is relatively dense.*

The proof is self evident.

Proposition 1.40. *Let $(\mathcal{T}, T)$ be a tiling dynamical system. Then a tiling $x \in \mathcal{T}$ is repetitive if and only if x is an almost periodic point.*

Proof. Assume that x is repetitive. Let $U \subseteq \mathcal{T}$ be an open set with $R(x,U) \neq \emptyset$. Thus there exists $t_0 \in \mathbb{R}^d$ such that $T^{t_0}x \in U$. Since T^{t_0} is a homeomorphism, there exists $\varepsilon > 0$ such that $T^{t_0}(B(x,\varepsilon)) \subseteq U$.
Let p_x be a patch in x containing a ball $B(0, \frac{1}{\varepsilon})$ around the origin. By repetitivity, there is an $r(p_x) > 0$ such that for all $t \in \mathbb{R}^d$ there is a translated copy $(T^s p_x)$ of p_x in $B_r + t$. Thus any translation by size less than ε belongs to $B(x,\varepsilon)$, say $T^s x \in B(x,\varepsilon)$ where $||s|| < \varepsilon$.

Now

$$B(0, \frac{1}{\varepsilon}) \subseteq p_x$$

implies

$$T^s B(0, \frac{1}{\varepsilon}) \subseteq T^s p_x$$

and so

$$B(s, \frac{1}{\varepsilon}) \subseteq T^s p_x \subseteq B_r + t.$$

Thus in particular, $s \in B_r + t$ so $R(x, B(x,\varepsilon)) \cap (B_r + t) \neq \emptyset$. Therefore, $R(x, B(x,\varepsilon))$ is relatively dense. By Lemma 1.39,

$$t_0 + R(x, B(x,\varepsilon)) \quad \text{is relatively dense.}$$

But

$$R(x,U) \supseteq t_0 + R(x, B(x,\varepsilon)).$$

This implies $R(x,U)$ is also relatively dense. Hence x is almost periodic.

Conversely, assume that x is almost periodic tiling. Let p_x be a patch in x around the origin of size $\frac{1}{\varepsilon}$ and let $B(x,\varepsilon) = U_\varepsilon$ be an open ball in the tiling metric with $R(x,U_\varepsilon) \neq \emptyset$.

By almost periodicity, there exists $r > 0$ such that for all $t \in \mathbb{R}^d$, $(B_r + t) \cap R(x,U_\varepsilon) \neq \emptyset$. Then there exists $s \in \mathbb{R}^d$ such that $s \in B_r + t$ with $T^s x \in U_\varepsilon$.
So that $T^s x$ and x agree on a ball around the origin of size $\frac{1}{\varepsilon}$ after translation by some $u \in \mathbb{R}^d$ with $||u|| \leq \varepsilon$. Thus there exists $s' = s + u \in (B_{r+2\varepsilon} + t)$.
Now let $r' = r + 2\varepsilon$. This implies that x has a translated copy $(T^{s'} p_x)$ of p_x contained in $B_{r'} + t$. Since this is true for all $t \in \mathbb{R}^d$, x is repetitive. $\square$

Lemma 1.41. *Let $(\mathcal{T},T)$ be a tiling dynamical system and $M \subseteq \mathcal{T}$ is T-invariant. Then $\overline{M}$ is T-invariant.*

Proof. Assume that M is T-invariant, and let $x \in \overline{M}$. Then there exists a sequence of points x_n in M such that $x_n \to x \in \mathcal{T}$. By continuity of the translation action,

$$T^t x_n \to T^t x$$

But M is T-invariant, then $T^t x_n \in M$ for all n. So we have a sequence of points $T^t(x_n)$ of M that converges to $T^t x$. Then $T^t(x) \in \overline{M}$. This implies $\overline{M}$ is T-invariant. $\square$

Remark 1.42. *Since the orbit closure $\overline{\mathcal{O}(x)} \subseteq \mathcal{T}_{\mathcal{P}}$ is a closed T-invariant, it follows that $(\overline{\mathcal{O}(x)},T)$ is a tiling dynamical system.*

Theorem 1.43. *(Gottschalk's Theorem). A dynamical system $(\overline{\mathcal{O}(x)},T)$ is minimal if and only if x is almost periodic.*

Proof. Suppose that $(\overline{\mathcal{O}(x)},T)$ is minimal and x is not almost periodic. Then there exists an open set $V \subset \overline{\mathcal{O}(x)}$ such that $R(x,V) \neq \emptyset$ but $R(x,V)$ is not relatively dense. Thus for every $n \in \mathbb{N}$, there exists $t_n \in \mathbb{R}^d$ such that

$$(B_n + t_n) \cap R(x,V) = \emptyset. \tag{1.2}$$

Since $(\overline{\mathcal{O}(x)},T)$ is compact, the sequence $T^{\mathbf{t_n}} x$ has a convergent subsequence $T^{t_{n_i}} x \to y$ as $i \to \infty$, for some $y \in \overline{\mathcal{O}(x)}$.

Since $\overline{\mathcal{O}(x)}$ is minimal, $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)}$. Since $R(x,V) \neq \emptyset$, there exists $t_0 \in \mathbb{R}^d$ such that $T^{t_0}x \in V \subseteq \overline{\mathcal{O}(x)}$. Thus $T^{\mathbf{t}_0}x \in \overline{\mathcal{O}(y)}$ and then $V \cap \mathcal{O}(y) \neq \emptyset$ which implies, there exists $t \in \mathbb{R}^d$ such that $T^t y \in V$.

By continuity, there exists an open set U such that $y \in U$ and $T^t(U) \subseteq V$. Since $T^{t_{n_i}}x \to y$, there exists a positive integer $N \in \mathbb{N}$ such that

$$T^{t_{n_i}}x \in U,\ \forall\, i \geq N.$$

Thus $\forall\, i \geq N, T^{t_{n_i}+t}x \in V$ and so, $\forall\, i \geq N, t_{n_i}+t \in R(x,V)$. If $n_i > ||t||$, then $t_{n_i}+t \in R(x,V) \cap (B_n + t_{n_i})$. This contradicts (1.2).

Conversely, suppose that x is almost periodic. Then $R(x,U)$ is relatively dense for every open set $U \subseteq \overline{\mathcal{O}(x)}$ with $R(x,U) \neq \emptyset$. Let $U = B(x,\varepsilon)$, then there exists $r > 0$ such that $\forall\, t \in \mathbb{R}^d$,

$$(B_r + t) \cap R(x,U) \neq \emptyset. \tag{1.3}$$

Assume that $\overline{\mathcal{O}(x)}$ is not minimal. Then there exists $y \in \overline{\mathcal{O}(x)}$ such that

$$\overline{\mathcal{O}(y)} \subset \overline{\mathcal{O}(x)}.$$

Let $Y = \overline{\mathcal{O}(y)}$. Note that $x \notin Y$, since otherwise $\overline{\mathcal{O}(x)} \subset Y$ and thus $\overline{\mathcal{O}(x)} = Y$. Let $\operatorname{dist}(x,Y) = 2\varepsilon$. Recall, $T : \mathbb{R}^d \times \overline{\mathcal{O}(x)} \to \overline{\mathcal{O}(x)}$ is continuous.

Thus $\exists\, \delta > 0$ such that if $d(z,y) < \delta$, then $\forall\, t \in B_r, d(T^t z, T^t y) < \varepsilon$. Since $\overline{\mathcal{O}(x)}$, and $\overline{B}_r$ are compact, $\overline{B}_r \times \overline{\mathcal{O}(x)}$ is compact and so T is uniformly continuous on $\overline{B}_r \times \overline{\mathcal{O}(x)}$. Thus $\forall t \in \overline{B}_r$, if $d(z,y) < \delta$, then

$$d(T^t z, T^t y) < \varepsilon. \tag{1.4}$$

Since $y \in \overline{\mathcal{O}(x)}$, there exists $s \in \mathbb{R}^d$ such that

$$T^s x \in B(y,\delta).$$

For some $t \in B_r + s, T^t x \in U$. Thus for some $u \in B_r$,

$$T^u(T^s x) = T^{u+s}x \in U.$$

So by (1.4)

$$d(T^u(T^s x)), T^u(y)) < \varepsilon.$$

As a result,

$$\begin{aligned} d(T^u(y),x) &\leq d(T^u(y),T^u(T^s(x))) \\ &+ d(T^u(T^s(x)),x) \\ &< \varepsilon+\varepsilon = 2\varepsilon, \end{aligned}$$

which is a contradiction. Thus $(\overline{\mathcal{O}(x)},T)$ is minimal. □

Remark 1.44. *It follows from the minimality of the dynamical system $(\overline{\mathcal{O}(x)},T)$ that $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)}$ for all $y \in \overline{\mathcal{O}(x)}$. In this case it follows from Gottschalk's theorem that y is almost periodic too.*

Combining Proposition 1.40 and Theorem 1.43 we obtain the following important result.

Proposition 1.45. *The dynamical system $(\overline{\mathcal{O}(x)},T)$ is minimal if and only if x is repetitive.*

Definition 1.46. Two repetitive tilings x,y are said to be *locally isomorphic* if $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$. Dynamically, local isomorphism means x and y belong to the same minimal tiling dynamical system.

Proposition 1.47. *If a tiling x is periodic, then $\mathcal{O}(x) = \overline{\mathcal{O}(x)}$.*

Proof. Assume that x is periodic. Let $y \in \overline{\mathcal{O}(x)}$, then there exists a sequence of points in $\mathcal{O}(x)$, say $T^{t_n}x$, that converges to y.

By periodicity, each $t_n \in \mathbb{R}^d$ is equivalent (modulo the lattice) to a point P_n in the fundamental parallelepiped. That is, $t_n = P_n + l_n$, where l_n is a point in the lattice. Since the sequence P_n is in the fundamental domain which is compact, then P_n has a convergent subsequence $P_{n_i} \to t$ in the parallelepiped. Since each l_n is in the lattice of periods, $T^{l_n}x = x$. Apply the group action property to conclude that for each

$$T^{(P_{n_i}+l_{n_i})}x = T^{P_{n_i}}x.$$

And then by continuity of the translation action on $\mathbb{R}^d$,

$$\begin{aligned} \lim_{i\to\infty} T^{t_{n_i}}x &= \lim_{i\to\infty} T^{(P_{n_i}+l_{n_i})}x \\ &= \lim_{i\to\infty} T^{P_{n_i}}(T^{l_{n_i}}x) \\ &= \lim_{i\to\infty} T^{P_{n_i}}x = T^t x \in O(x). \end{aligned}$$

Since the sequence $T^{t_n}x \to y$, then $T^{t_{n_i}}x \to y$.

$$\lim_{i\to\infty} T^{t_{n_i}}x = T^t x = y.$$

This implies $y \in \mathcal{O}(x)$ and hence $\mathcal{O}(x) = \overline{\mathcal{O}(x)}$. $\square$

Chapter 2

Symbolic Dynamics

In this chapter, we introduce a notion of symbolic dynamics in the classical one-dimensional case, in particular, we will consider substitution subshifts and their associated tiling spaces. We will conclude the chapter by considering the dynamical spectrum of the tiling spaces associated to substitutions of constant length.
The definitions in §2.1 and §2.2 are adapted from [2], [10], [13] and [24].

2.1 The shift spaces and associated dynamical systems

Definition 2.1. Over a finite alphabet $\mathcal{A}$, we define the *full one-sided shift* $\mathcal{A}^{\mathbb{N}_0}$ by $\{(u_i)_{i\in\mathbb{N}_0} \mid u_i \in \mathcal{A},\ \forall\, i \in \mathbb{N}_0\}$ and the *two-sided shift* $\mathcal{A}^{\mathbb{Z}}$ by $\{(u_i)_{i\in\mathbb{Z}} \mid u_i \in \mathcal{A},\ \forall\, i \in \mathbb{Z}\}$. The *shift map* S acts on both spaces and is defined by $S((u_i)) = (u_{i+1})$ where (u_i) is an infinite or bi-infinite sequence. This shift is continuous in both cases and possesses a continuous inverse on $\mathcal{A}^{\mathbb{Z}}$. For our purpose, we are interested in bi-infinite sequences. Let the metric $\boldsymbol{d}$ on $\mathcal{A}^{\mathbb{Z}}$ be defined by:

$$\text{for } u,v \in \mathcal{A}^{\mathbb{Z}},\ \boldsymbol{d}(u,v) = \begin{cases} 0 & if\ u = v, \\ 2^{-k} & if\ u \neq v,\ k = \inf\{i \in \mathbb{N}_0,\ u_i \neq v_i \text{ or } u_{-i} \neq v_{-i}\}. \end{cases} \tag{2.1}$$

This metric induces the product topology and thus turns $\mathcal{A}^{\mathbb{Z}}$ into a compact metric space. Hence we get the invertible dynamical system $(\mathcal{A}^{\mathbb{Z}}, S)$.

Definition 2.2. A *word* is a finite sequence of letters from the n-letter alphabet $\mathcal{A}$. The *length* of a finite word w, denoted by $|w|$, is the number of letters $a \in \mathcal{A}$ that occur in w.

Definition 2.3. Let $u_{[k,l]}$ with $k,l \in \mathbb{Z}$ and $k \leq l$ be the finite subword of $u \in \mathcal{A}^{\mathbb{Z}}$ from position k to l (with $u_{[k,k]} = u_k$). The *cylinder sets* for a finite word w of length $m \geq 1$ are defined to be

$$[w]_k = \left\{ u \in \mathcal{A}^{\mathbb{Z}} \mid u_{[k,k+m-1]} = w \right\} \quad \text{with } k \in \mathbb{Z}.$$

The family of all such cylinder sets forms a basis for the product topology.

Remark 2.4. *If we consider the set of all non-empty finite words in $\mathcal{A}$ denoted by $\mathcal{A}^*$, then the product topology extends in a natural way to $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{Z}}$. Indeed, let $\mathcal{B}$ be a new alphabet obtained by adding a further letter to $\mathcal{A}$, then words in $\mathcal{A}^*$ can be considered as sequences in $\mathcal{B}^{\mathbb{Z}}$, by extending them by the new letter in $\mathcal{B}$. The set $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{Z}}$ is thus metric and compact, as a closed subset of $\mathcal{B}^{\mathbb{Z}}$. It is convenient to let $\mathbf{d}$ be a metric on $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{Z}}$ similarly defined as equation (2.1).*

Definition 2.5. A *shift space (subshift)* is a closed shift invariant subset of a full shift $\mathcal{A}^{\mathbb{Z}}$. The pair (X,S) consisting of a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, together with the shift map S forms a *symbolic dynamical system*. Given $u \in \mathcal{A}^{\mathbb{Z}}$, the symbolic dynamical system associated with u is the system $(\overline{\mathcal{O}(u)}, S)$, where $\overline{\mathcal{O}(u)} \subseteq \mathcal{A}^{\mathbb{Z}}$ is the closure of the orbit of the sequence u under the shift action; the *orbit* $\mathcal{O}(u)$ is the set $\left\{ S^j(u) \mid j \in \mathbb{Z} \right\}$.

Definition 2.6. A sequence $u \in \mathcal{A}^{\mathbb{Z}}$ is *shift-periodic* if $S^k(u) = u$ for some $k \geq 1$ and we say that u has period k under S. The subshift is *aperiodic* if it contains no periodic elements under the shift action.

Now we will consider the notions of almost periodic bi-infinite sequences and repetitive ones. These notions are slightly different from the ones given in Chapter 1. In other words, when talking about almost periodic sequences we are talking about relatively dense sets in $\mathbb{Z}$. Also, the definition for repetitive sequences is given as follows:

Definition 2.7. A bi-infinite sequence $u \in \mathcal{A}^{\mathbb{Z}}$ is *repetitive* if every finite word occurring in u occurs in an infinite number of positions with bounded gaps. In other words, for every word w in u there exists $s > 0$ such that for every $j \in \mathbb{Z}, w$ is a subword of $u_j \cdots u_{j+s-1}$.

The following result presents the equivalence of repetitive and almost periodic sequence in $\mathcal{A}^{\mathbb{Z}}$.

Proposition 2.8. *A sequence $u \in \mathcal{A}^{\mathbb{Z}}$ is repetitive if and only if it is almost periodic.*

For the proof, one can follow an argument analogous to the argument applied to tiling in Chapter 1.

The following proposition states formally the connection between the minimality of the system $(\overline{\mathcal{O}(u)}, S)$ and the repetitivity of $u \in \mathcal{A}^{\mathbb{Z}}$.

Proposition 2.9. *The system* $(\overline{\mathcal{O}(u)}, S)$ *is minimal if and only if* u *is repetitive.*

This follows from Gottschalk's theorem [15] and Proposition 2.8.

2.2 Substitutions and symbolic dynamical Systems

In this section we present the notion of substitution and construct shift spaces via a primitive substitution rule.

Definition 2.10. With a finite alphabet $\mathcal{A} = \{a_i \mid 1 \leq i \leq n\}$ and the set $\mathcal{A}^*$ of all non-empty finite words over $\mathcal{A}$, define a *substitution* to be a function $\theta : \mathcal{A} \to \mathcal{A}^*$.

The map extends to a morphism of $\mathcal{A}^*$ by concatenation. That is, $\theta(ww') = \theta(w)\theta(w')$, where $w, w' \in \mathcal{A}^*$. It also extends in a natural way to a map defined over $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$. Namely, for $u = (u_i)_{i\in\mathbb{Z}}$,

$$\theta(\cdots u_{-2}u_{-1} \cdot u_0u_1 \cdots) = \cdots \theta(u_{-2})\theta(u_{-1}) \cdot \theta(u_0)(\theta(u_1) \cdots .$$

Definition 2.11. Let θ be a substitution on a finite n-letter alphabet $\mathcal{A}$, θ is called of *constant length* if $|\theta(a)| = l$ for every letter $a \in \mathcal{A}$.
A finite word is called *legal* for θ, if it occurs as a subword of $\theta^m(a_i)$ for some $1 \leq i \leq n$ and some $m \in \mathbb{N}$.

Remark 2.12. *Legal words have the property that they are mapped to legal words under the substitution.*

Definition 2.13. A substitution θ defined over the n-letter alphabet $\mathcal{A} = \{a_i \mid 1 \leq i \leq n\}$, is *primitive* if there exists a positive integer k such that, for every $a_i, a_j \in \mathcal{A}$, the letter a_i occurs in $\theta^k(a_j)$.
The *incidence matrix* of a substitution θ defined over the n-letter alphabet is the $n \times n$ matrix M_θ such that each of its entries (i, j) is $|\theta(a_j)|_{a_i}$, that is, the number of occurrences of a_i in $\theta(a_j)$.
M_θ is *primitive* if $\exists\ k > 0$ such that $M_\theta^k > 0$, that is, has positive entries. Clearly, a substitution is primitive if its incidence matrix is primitive.

Now we will define a shift space constructed via the primitive substitution rule θ as follows:

Definition 2.14. Let θ be a substitution defined over the n-letter alphabet $\mathcal{A} = \{a_1, a_2, \cdots, a_n\}$. Define the subset $\Sigma_\theta \subseteq \mathcal{A}^{\mathbb{Z}}$ by

$$\Sigma_\theta = \{(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \mid \forall\, j \in \mathbb{Z},\ k \in \mathbb{N} \,:\, u_j \cdots u_{j+k} \text{ is a legal word}\}. \tag{2.2}$$

Note that primitivity of a substitution θ allows that Σ_θ can be written as follows:

$$\begin{aligned}\Sigma_\theta = \{&(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \mid \forall\, j \in \mathbb{Z},\ k \in \mathbb{N}\ \exists\, m \in \mathbb{N} \ni \\ &u_j \cdots u_{j+k} \text{ is a subword of } \theta^m(a_i) \text{ for all } 1 \le i \le n\}.\end{aligned} \tag{2.3}$$

Definition 2.15. A bi-infinite sequence $u \in \Sigma_\theta$ is called a *periodic point* of a substitution θ if $\theta^k(u) = u$ for some $k \in \mathbb{N}$. If $k = 1$, then u is called a *fixed point* of θ.

An *n-word of θ* is any one of the words $\theta^n(a_i)$ for $a_i \in \mathcal{A}$.

The next lemma shows the existence of bi-infinite fixed points in the case of a primitive substitution. We do not include the proof of the lemma here as it can be found in Lemma 4.3, [2].

Lemma 2.16. *If θ is a primitive substitution on a finite n-letter alphabet $\mathcal{A}$ with $n \ge 2$, there exists some $k \in \mathbb{N}$ and some $u \in \Sigma_\theta$ such that u is a fixed point of θ^k. In other words, the substitution θ has at least one periodic point.*

Note that Lemma 2.16 still holds for one-sided fixed points.

We give an example that supports the previous lemma (Example 1, [29]).

Example 2.1. Consider the Thue-Morse substitution θ on an alphabet $\mathcal{A} = \{a, b\}$ defined by $\theta(a) = ab, \theta(b) = ba$. Obviously θ is primitive. We can choose $b \cdot a$ as a legal two letter word such that $\theta^2(a)$ starts with a and $\theta^2(b)$ ends with b. Thus $\theta^2(b \cdot a) = baab \cdot abba$ contains $b \cdot a$, and generally $\theta^{n+2}(b \cdot a)$ contains $\theta^n(b \cdot a)$. Consequently, the fixed point sequence can be obtained from $b \cdot a$ as an iteration limit of the sequence $\theta^{n+2}(b \cdot a)$ of finite words of increasing length. Thus $u = \cdots abba\ baab \cdot abba\ baab \cdots$ is a fixed point of θ^2, that is $\theta^2(u) = u$.

Proposition 2.17. *Given a substitution θ on the n-letter alphabet $\mathcal{A} = \{a_i \mid 1 \le i \le n\}$, then Σ_θ is a subshift of the full shift $\mathcal{A}^{\mathbb{Z}}$.*

Proof. Let $S : \Sigma_\theta \to \Sigma_\theta$ be the shift map defined on Σ_θ. Firstly, we will prove that Σ_θ is shift invariant. Let $u = (u_i)_{i\in\mathbb{Z}} \in \Sigma_\theta$ and $t \in \mathbb{Z}$, then any finite string $u_j u_{j+1} \cdots u_{j+k}$ in

$S^t(u)$, for some $j \in \mathbb{Z}$, $k \in \mathbb{N}$, is in u located somewhere in a different position. Since $u = (u_i)_{i\in\mathbb{Z}} \in \Sigma_\theta$, then $\theta^m(a_i)$ contains $u_j u_{j+1} \cdots u_{j+k}$ as a sub word for some $m \in \mathbb{N}$, $i \in \{1,2,\cdots,n\}$. Thus $S^t(u) \in \Sigma_\theta$ and Σ_θ is shift invariant.

Secondly, let $\{(w_i^t)_{i\in\mathbb{Z}}\}_{t\in\mathbb{N}}$ be a sequence of points in Σ_θ such that $(w_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ be a limit point, that is, $\{(w_i^t)_{i\in\mathbb{Z}}\}_{t\in\mathbb{N}} \to (w_i)_{i\in\mathbb{Z}}$ as $t \to \infty$, and let $w_j w_{j+1} \cdots w_{j+k}$ be any finite string that occurs in $w = (w_i)_{i\in\mathbb{Z}}$, where $j \in \mathbb{Z}$, $k \in \mathbb{N}$. Choose $\varepsilon < \frac{1}{2^N}$ where $N = \max\{|j|, |j+k|\}$, then there exists a positive integer $l_0 \in \mathbb{N}$ such that

$$\boldsymbol{d}((w_i^{l_0})_{i\in\mathbb{Z}}, (w_i)_{i\in\mathbb{Z}}) < \frac{1}{2^N}.$$

Thus

$$w_j^{l_0} w_{j+1}^{l_0} \cdots w_{j+k}^{l_0} = w_j w_{j+1} \cdots w_{j+k}.$$

But $(w_i^{l_0})_{i\in\mathbb{Z}} \in \Sigma_\theta$, which implies that $w_j^{l_0} w_{j+1}^{l_0} \cdots w_{j+k}^{l_0}$ is a subword of $\theta^m(a_i)$ for some $m \in \mathbb{N}$ and $i \in \{1,2,\cdots,n\}$, Therefore $w_j w_{j+1} \cdots w_{j+k}$ is a subword of $\theta^m(a_i)$, that is, $(w_i)_{i\in\mathbb{Z}} \in \Sigma_\theta$. Hence Σ_θ is closed. □

Corollary 2.18. *If u is a point in the subshift Σ_θ, where θ is a substitution on the n-letter alphabet $\mathcal{A}$, then $\{S^j(u) \mid j \in \mathbb{Z}\} \subseteq \Sigma_\theta$.*

Remark 2.19. *The substitution subshift Σ_θ is invariant under θ.*

Consider the subshift Σ_θ, where θ is a primitive substitution. The following lemma gives an alternative description of Σ_θ in terms of the closure of the shift orbit of one of its periodic points.

Lemma 2.20. *Let $u \in \Sigma_\theta$ be one of the periodic points of a primitive substitution θ on the n-letter alphabet $\mathcal{A}$. Then*

$$\Sigma_\theta = \overline{\{S^j(u) \mid j \in \mathbb{Z}\}}.$$

Proof. As a result of Proposition 2.17 and Corollary 2.18,

$$\overline{\{S^j(u) \mid j \in \mathbb{Z}\}} \subseteq \Sigma_\theta.$$

So it suffices to show that

$$\Sigma_\theta \subseteq \overline{\{S^j(u) \mid j \in \mathbb{Z}\}}.$$

Let $w \in \Sigma_\theta$ and $w' = w_{-N} \cdots w_0 \cdots w_N$ be a finite subword of w. It follows that there exists a basic open set $[w']_{k=-N}^{N}$ containing w.

By primitivity, there exists $m \in \mathbb{N}$ such that w' is a subword of $\theta^m(a_i)$ for all $a_i \in \mathcal{A}$ and thus w' is a subword of $\theta^l(a_i)$ for all $a_i \in \mathcal{A}$ and for all $l \geq m$. Let $u = (u_i)_{i \in \mathbb{Z}}$ be any θ-periodic point, then $u = \theta^p(u)$ for some $p > 0$. If we choose $k \in \mathbb{N}$ such that $pk > m$, we will have

$$u = \theta^{pk}(u)$$

$$u = \cdots \theta^{pk}(u_{-1}) \cdot \theta^{pk}(u_0)\theta^{pk}(u_1) \cdots$$

Now w' is a subword of $\theta^{pk}(u_0)$ and consequently a subword of u. Choosing k' appropriately, we get

$$S^{k'}(u) = \cdots w_{-N-1}w_{-N} \cdots w_0 \cdots w_N w_{N+1} \cdots$$

which implies

$$S^{k'}(u) \in [w']_{k=-N}^{N},$$

that is

$$[w']_{k=-N}^{N} \cap \{S^j(u) \mid j \in \mathbb{Z}\} \neq \emptyset$$

and so

$$w \in \overline{\{S^j(u) \mid j \in \mathbb{Z}\}}.$$

□

Note that the previous result is independent of the choice of periodic point.

Lemma 2.21. *Let $u \in \Sigma_\theta$ be a bi-infinite periodic point of a primitive substitution θ on the finite alphabet $\mathcal{A} = \{a_i \mid 1 \leq i \leq n\}$. Then the shift space $\overline{\{S^j(u) \mid j \in \mathbb{Z}\}}$ is minimal.*

Proof. By Proposition 2.9, it sufficies to show that u is repetitive. Let w be any finite word occurring in $u = (u_i)_{i \in \mathbb{Z}}$. Since u is a θ periodic point, then $u = \theta^p(u)$ for some $p > 0$. By primitivity of a substitution θ, there exists $m \in \mathbb{N}$ such that w is a subword of $\theta^m(a_i)$ for all $a_i \in \mathcal{A}$.

Choosing $k \in \mathbb{N}$ such that $pk > m$ implies

$$\begin{aligned} u &= \theta^{pk}(u) \\ &= \cdots \theta^{pk}(u_{-1}) \cdot \theta^{pk}(u_0)\theta^{pk}(u_1) \cdots \end{aligned}$$

Now each $\theta^{pk}(u_i)$ contains w as a subword where $u_i = a_i$ for some $a_i \in \mathcal{A}$. Since $\mathcal{A}$ has a finite number of letters and each $\theta^{pk}(u_i)$ has a finite length, we may choose

$$L' = \max\{|\theta^{pk}(a_i)| \mid a_i \in \mathcal{A}\}.$$

If we let $L = 2 \times L'$, then every word occurring in u of length L contains W as a subword. Hence u is repetitive. $\square$

As a result of Lemma 2.21, one can conclude the minimality of the substitution subshift Σ_θ.

2.3 One-dimensional tilings

The content and definitions in §2.3 are adapted from [3], [5] and [7]. In Chapter 1, we studied the $\mathbb{R}^d$-dimensional tilings. In this section we continue our discussion but on one-dimensional tilings in particular.

Definition 2.22. Given a collection $\mathscr{P} = \{P_1, \cdots, P_n\}$ where each P_i is a closed interval, a *tiling* x of $\mathbb{R}$ by $\mathscr{P}$ is a collection of closed intervals $\{x_i\}_{i\in\mathbb{Z}}$ satisfying:

1. $\cup_{i\in\mathbb{Z}} x_i = \mathbb{R}$,
2. for each $i \in \mathbb{Z}$, each tile x_i is the translate of some $P_i \in \mathscr{P}$,
3. $x_i \cap x_{i+1}$ is a singleton for each $i \in \mathbb{Z}$.

Definition 2.23. A continuous flow or simply a flow is a continuous action of the group $\mathbb{R}$ on a compact metric space X.

Recall that if we let $\mathcal{T}_{\mathscr{P}}$ the full tiling space of all possible tilings of $\mathbb{R}$ by prototiles from the set $\mathscr{P}$ with the tiling metric as defined in Lemma 1.17, then the continuous $\mathbb{R}$-action $\phi : \mathbb{R} \times \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}}$ on a tiling x defines a flow, given by $\phi(t, x) = x - t$.
Also the orbit of x is the set $\mathcal{O}(x) = \{x - t \mid t \in \mathbb{R}\}$ of all translates of x.

The most commonly studied class of tiling spaces is the class of orbit closures. Thus we will give the following definition.

Definition 2.24. A *tiling space* $\mathcal{T}$ of $x \in \mathcal{T}_{\mathscr{P}}$ is the orbit closure $\overline{\mathcal{O}(x)}$.

2.3.1 Substitution tiling spaces

A substitution tiling space $\mathcal{T}_\theta$ arising from a primitive, aperiodic substitution θ was described by Barge and Diamond in [3]. Whereas the substitution rule θ induces an inflation map on a tiling, characterized by expansion and translation, this map extends to a homeomorphism $F_\theta : \mathcal{T}_\theta \to \mathcal{T}_\theta$. For such a substitution tiling space, the precise construction has a canonical choice of tile length, that is, the lengths are picked according to the left Perron-Frobenius eigenvector of the incidence matrix M_θ of the substitution.
One could also choose different lengths for the tiles, this will not affect the topology of the resulting tiling space. But as we shall see that may or may not affect the dynamics of the translation action.
In the particular case of a substitution of constant length, in the canonical construction of the associated substitution tiling space all of the tiles have the same length. Typically this length would be 1 but one may normalize the left Perron-Frobenius eigenvector in different ways.

2.3.2 The suspension construction

Suspension is a construction which turns a map into a flow. We base the following definition on [5]. Throughout, the notation $[(...,...)]$ indicates the equivalence class.

Definition 2.25. Given a homeomorphism $f : X \to X$ on a compact metric space X, and a function $c : X \to \mathbb{R}^+$ bounded away from 0, consider the quotient space (suspension space) $X_c = \{(x,t) \in X \times \mathbb{R}^+ \mid 0 \leq t \leq c(x)\}/\sim$. The *suspension of f with ceiling function c* is the flow defined on $\mathbb{R} \times X_c$, such that

1. for $t \geq 0$, $\sim$ is the equivalence relation defined by:

$$(x, c(x)) \sim (f(x), 0), \tag{2.4}$$

and the positive semi-flow $\phi^t : X_c \to X_c$ given by $\phi^t[(x,s)] = [(f^n(x), s')]$, where n and s' satisfy:

$$\sum_{i=0}^{n-1} c(f^i(x)) + s' = s + t \qquad 0 \leq s' \leq c(f^n(x)),$$

2. for $t \leq 0$, $\sim$ is the equivalence relation defined by:

$$(x, c(x)) \sim (f^{-1}(x), 0), \tag{2.5}$$

and the negative semi-flow $\phi^t : X_c \to X_c$ given by $\phi^t[(x,s)] = [(f^{-n}(x), s')]$, where n and s' satisfy:

$$\sum_{-(n-1)}^{i=0} c(f^i(x)) + s' = s + |t|, \qquad 0 \le s' \le c(f^{-n}(x)).$$

A suspension flow is called *a flow under a function.*

Remark 2.26.

1. *Obviously, in the case of positive (negative) semi-orbit, we will have positive (negative) semi-flow.*

2. *The positive semi-flow is defined on $[0,\infty) \times X_c$ and the negative one is defined on $(-\infty, 0] \times X_c$. They agree on their overlap $\{0\} \times X_c$ and so by the pasting lemma it yields a continuous flow on $\mathbb{R} \times X_c$.*

3. *The suspension space X_c is called an ordinary suspension if $c(x) = \mathfrak{c}$ for all $x \in X$. In this case, the suspension flow is the natural additive flow $\phi^t : X_c \to X_c$ given by $\phi^t[(x,s)] = [(x, s+t)]$.*

2.3.3 Shift and tiling spaces

Let $\mathcal{A}^{\mathbb{Z}}$ be the full shift over the n-letter alphabet $\mathcal{A} = \{a_j \mid 1 \le j \le n\}$. Define the symbolic cylinder $[a] = \{((u_i)_{i\in\mathbb{Z}}) \in \mathcal{A}^{\mathbb{Z}} \mid u_0 = a \in \mathcal{A}\}$. Then $\mathcal{A}^{\mathbb{Z}}$ is partitioned into non-empty disjoint clopen subsets $[a_j]$, $1 \le j \le n$. That is

$$\mathcal{A}^{\mathbb{Z}} = [a_1] \cup [a_2] \cup \cdots \cup [a_n].$$

Given the shift map $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, in order to cater for the suspension of a map involving cylinders with different lengths we define the ceiling function $c : \mathcal{A}^{\mathbb{Z}} \to (0,\infty)$ by $c((u_i)_{i\in\mathbb{Z}}) = l_j = l(u_0)$, where l_j is the length associated to $(u_i)_{i\in\mathbb{Z}} \in [a_j]$. As noticed, the ceiling function c is constant for each one of the cylinder sets $[a_j]$, $1 \le j \le n$, thus associate to each cylinder set $[a_j]$ the fixed length $l_j \in (0,\infty)$. This fixed length is called *the height of the cylinder*.

Consider the quotient space

$$\mathcal{A}_c^{\mathbb{Z}} = \{((u_i)_{i\in\mathbb{Z}}, t) \in \mathcal{A}^{\mathbb{Z}} \times \mathbb{R}^+ \mid 0 \le t \le c((u_i)_{i\in\mathbb{Z}})\} / \sim,$$

where $\sim$ is the equivalence relation generated by

$$((u_i)_{i\in\mathbb{Z}}, c((u_i)_{i\in\mathbb{Z}})) \sim (S((u_i)_{i\in\mathbb{Z}}), 0),$$

Note that for each $i \in \mathbb{Z}$, $S((u_i)_{i\in\mathbb{Z}}) = (u_{i+1})_{i\in\mathbb{Z}} \in [a_j]$ for some $1 \leq j \leq n$. The construction of $\mathcal{A}_c^{\mathbb{Z}}$ determines that for each $[((u_i)_{i\in\mathbb{Z}}, t)] \in \mathcal{A}_c^{\mathbb{Z}}, (u_i)_{i\in\mathbb{Z}} \in [a_j]$ and $0 \leq t \leq l_j$ for some $1 \leq j \leq n$. Thus $((u_i)_{i\in\mathbb{Z}}, t) \in [a_j] \times [0, l_j]$ and consequently $\mathcal{A}_c^{\mathbb{Z}}$ can be partitioned as follows:

$$\mathcal{A}_c^{\mathbb{Z}} = ([a_1] \times [0, l_1)) \cup \cdots \cup ([a_n] \times [0, l_n)).$$

The suspension of S under the ceiling function c is the flow defined on $\mathbb{R} \times A_c^{\mathbb{Z}}$ such that

1. for $t \geq 0$, the positive semi-flow $\Phi^t : \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{A}_c^{\mathbb{Z}}$ given by

$$\Phi^t[((u_i)_{i\in\mathbb{Z}}, S)] = [(S^n((u_i)_{i\in\mathbb{Z}}), s')],$$

where n and s' satisfy:

$$\sum_{i=0}^{n-1} c(S^i((u_i)_{i\in\mathbb{Z}})) + s' = s + t, \quad 0 \leq s' \leq c(S^n((u_i)_{i\in\mathbb{Z}})).$$

That is

$$c(u_i)_{i\in\mathbb{Z}} + c((u_{i+1})_{i\in\mathbb{Z}} + \cdots + c((u_{i+(n-1)})_{i\in\mathbb{Z}}) + s' = s + t$$

which implies

$$l[u_0] + l[u_1] + \cdots + l[u_{n-1}] + s' = s + t, \quad 0 \leq s' \leq l[u_n],$$

2. for $t \leq 0$, the negative semi-flow $\Phi^t : \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{A}_c^{\mathbb{Z}}$ given by

$$\Phi^t[((u_i)_{i\in\mathbb{Z}}, s)] = [(S^{-n}((u_i)_{i\in\mathbb{Z}}), s')],$$

where n and s' satisfy:

$$l[u_0] + l[u_{-1}] + \cdots + l[u_{-(n-1)}] + s' = s + |t|, \quad 0 \leq s' \leq l[u_{-n}].$$

Now consider a full tiling space $\mathcal{T}_{\mathscr{P}}$ made from a finite set of prototiles $\mathscr{P} = \{P_1, P_2, \cdots, P_n\}$ such that card $(\mathscr{P}) = \text{card}(\mathcal{A}) = n$. Then associate with each $a_j \in \mathcal{A}$ a closed interval $P_j = [0, l_j]$ of length l_j, where $l_j = l(u_0), u_0 = a_j$, and define the map $\tau : \mathcal{A}^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}}$

by

$$\tau((u_i)) = (x_i),\ \forall\, i \in \mathbb{Z} \tag{2.6}$$

such that the left endpoint of the tile x_0 corresponding to u_0 is at $0 \in \mathbb{R}$ and each x_i is a translate of the prototile associated with u_i.
The following proposition illustrates the conjugacy between the respective $\mathbb{R}$ actions on the suspension $\mathcal{A}_c^{\mathbb{Z}}$ and the full tiling space $\mathcal{T}_{\mathscr{P}}$.

Proposition 2.27. *The two continuous flows $(\mathcal{A}_c^{\mathbb{Z}}, \Phi)$ and $(\mathcal{T}_{\mathscr{P}}, \phi)$ are topologically conjugate. That is, there exists a homeomorphism that conjugates the respective $\mathbb{R}$ actions.*

$$\begin{tikzcd}
\mathcal{A}_c^{\mathbb{Z}} \arrow[r, "\Phi^t"] \arrow[d, "\tau_c"'] & \mathcal{A}_c^{\mathbb{Z}} \arrow[d, "\tau_c"] \\
\mathcal{T}_{\mathscr{P}} \arrow[r, "\phi^t"] & \mathcal{T}_{\mathscr{P}}
\end{tikzcd}$$

Proof. Given $\Phi : \mathbb{R} \times A_c^{\mathbb{Z}} \to A_c^{\mathbb{Z}}$ and $\phi : \mathbb{R} \times \mathcal{T}_{\mathscr{P}} \to \mathcal{T}_{\mathscr{P}}$. Define the map $\tau_c : \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}}$ by:

$$\tau_c[((u_i)_{i\in\mathbb{Z}}, s)] = \phi^s(\tau((u_i)_{i\in\mathbb{Z}})) \tag{2.7}$$

By construction, the map (2.6) defines a one-one correspondence between a sequence $(u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and a tiling $(x_i)_{i\in\mathbb{Z}} \in \mathcal{T}_{\mathscr{P}}$ so the composite map τ_c is a bijection.
First, we shall give the proof of continuity for the map τ_c. Consider the following subset $\tilde{S} \subseteq \mathcal{A}^{\mathbb{Z}} \times \mathbb{R}^+$, where

$$\tilde{S} = \{((u_i)_{i\in\mathbb{Z}}, t) \in \mathcal{A}^{\mathbb{Z}} \times \mathbb{R}^+ \mid 0 \le t \le c((u_i)_{i\in\mathbb{Z}})\},$$

and define the map $\tilde{h} : \tilde{S} \to \mathcal{T}_{\mathscr{P}}$ by:

$$\tilde{h}((u_i)_{i\in\mathbb{Z}}, t) = \phi^t(\tau(u_i)_{i\in\mathbb{Z}}) = \phi^t(x_i)_{i\in\mathbb{Z}}.$$

That means that the image of any pair $((u_i)_{i\in\mathbb{Z}}, t)$ is a tiling $x = (x_i)_{i\in\mathbb{Z}}$ having the origin located at t units to the right of the left end point of an x_0-tile.

The map $\tilde{h}$ is constant on each set $P^{-1}[((u_i), t)]$ for all $[((u_i)_{i\in\mathbb{Z}}, t)] \in \mathcal{A}_c^{\mathbb{Z}}$, where $P : \tilde{S} \to \mathcal{A}_c^{\mathbb{Z}}$ is the quotient map. As P is a quotient map, then it suffices to show that $\tilde{h}$ is continuous.

Let $\varepsilon > 0$ and $((u_i)_{i\in\mathbb{Z}},t) \in \tilde{S}$. Define the metric D on $\tilde{S}$ as follows:

$$D[((u_i)_{i\in\mathbb{Z}},t),((v_i)_{i\in\mathbb{Z}},s)] = \max\{\mathrm{dist}(t,s),\ \boldsymbol{d}((u_i)_{i\in\mathbb{Z}},(v_i)_{i\in\mathbb{Z}})\},$$

where dist is the usual metric on $\mathbb{R}$ and $\boldsymbol{d}$ is the metric defined on $\mathcal{A}^{\mathbb{Z}}$.

We now construct $\delta > 0$ such that if $D[((u_i)_{i\in\mathbb{Z}},t),((v_i)_{i\in\mathbb{Z}},s)] < \delta$, then $d(\tilde{h}(u,t),\tilde{h}(v,s)) < \varepsilon$, where d is the tiling metric. Choose N such that $(N-2)l_{\min} > \frac{1}{\varepsilon}$, where $l_{\min}$ is the minimum length of the tiles, then choose $\delta < \min\{\frac{1}{2^{N-2}},\varepsilon\}$. Notice that

$$\boldsymbol{d}(u,v) < \delta < \frac{1}{2^{N-2}},\ u = (u_i)_{i\in\mathbb{Z}},\ v = (v_i)_{i\in\mathbb{Z}}.$$

So

$$[u_{-N},u_N] = [v_{-N},v_N].$$

That means the sequences u and v are matched from $-N$ to N. Since

$$\sum_{i=0}^{N} l(x_i) \geq \sum_{i=0}^{N} l_{\min} = Nl_{\min} > (N-2)l_{\min} > \frac{1}{\varepsilon},$$

the tilings $\tilde{h}(u,t)$ and $\tilde{h}(v,t)$ match around the origin in a ball of radius $(N-2)l_{\min} > \frac{1}{\varepsilon}$. By translating the tiling $\tilde{h}(v,s)$ by $t-s$, the two tilings $\tilde{h}(u,t)$ and $\tilde{h}(v,s)$ agree on a ball centered at the origin of radius $(N-2)l_{\min} > \frac{1}{\varepsilon}$ for some $|t-s| < \delta \leq \varepsilon$. Thus

$$d(\tilde{h}(u,t),\tilde{h}(v,s)) < \varepsilon.$$

Hence τ_c is continuous and consequently a homeomorphism.

It remains to show that $\tau_c \circ \Phi^t = \phi^t \circ \tau_c$, $\forall\, t \in \mathbb{R}$. Consider first that $t \geq 0$ and $[((u_i)_{i\in\mathbb{Z}},s)] \in \mathcal{A}_c^{\mathbb{Z}}$, then

$$\begin{aligned}
\tau_c \circ \Phi^t[((u_i)_{i\in\mathbb{Z}},s)] &= \tau_c[(S^n((u_i)_{i\in\mathbb{Z}}),s')] \\
&= \tau_c[((u_{i+n})_{i\in\mathbb{Z}}),s')] \\
&= \phi^{s'}(\tau((u_{i+n})_{i\in\mathbb{Z}})) \\
&= \phi^{s'}((x_{i+n})_{i\in\mathbb{Z}}) \\
&= (x_{i+n})_{i\in\mathbb{Z}} - s'.
\end{aligned}$$

On the other hand

$$\begin{aligned}\phi^t \circ \tau_c[((u_i)_{i\in\mathbb{Z}}, s)] &= \phi^t \phi^s(\tau((u_i)_{i\in\mathbb{Z}})) \\ &= \phi^{t+s}((x_i)_{i\in\mathbb{Z}}) \\ &= (x_i)_{i\in\mathbb{Z}} - (t+s).\end{aligned}$$

Since

$$s' = (t+s) - \sum_{i=0}^{n-1} l[u_i],$$

then

$$\phi^t \circ \tau_c[((u_i)_{i\in\mathbb{Z}}, s)] = (x_{i+n})_{i\in\mathbb{Z}} - s'.$$

Similarly, in the case when $t \leq 0$, it can be shown that

$$\tau_c \circ \Phi^t = \phi^t \circ \tau_c.$$

Hence

$$\tau_c \circ \Phi^t = \phi^t \circ \tau_c \quad \forall\, t \in \mathbb{R}$$

and thus the flows are topologically conjugate. □

Remark 2.28. *From Proposition 2.27, the map $\tau_c : \mathcal{A}_c^{\mathbb{Z}} \to \mathcal{T}_{\mathscr{P}}$ is continuous and bijective. By restricting the suspension on the full shift to a suspension on its substitution subshift Σ_θ, the following map is obtained*

$$\tau_c\,|_{\{\Sigma_\theta\}_c} : \{\Sigma_\theta\}_c \to \mathcal{T}_{\mathscr{P}} \tag{2.8}$$

which is continuous and one-one but not onto.

Lemma 2.29. *Given a substitution θ on the n-letter alphabet $\mathcal{A} = \{a_j \mid 1 \leq j \leq n\}$. There is a natural homeomorphism between the suspension of the subshift $\{\Sigma_\theta\}_c$ and the associated substitution tiling space.*

Proof. The construction of the substitution tiling space based on the collection of intervals $\mathscr{P} = \{P_1, P_2, \cdots, P_n\}$ with respective lengths $l_1, l_1, \cdots, l_n$, where the ceiling function c which takes the value l_j for the cylinder $[a_j]$, $1 \leq j \leq n$. By the map (2.7), each element $[((u_i)_{i\in\mathbb{Z}}, s)] \in \{\Sigma_\theta\}_c$ is mapped to a tiling $x = \phi^s(\tau(u_i))$, where τ is given by (2.6). That is, tiling x that has origin located at s units to the right of the left point of a x_0-tile and with tiles that then follow the pattern of the sequence $(u_i)_{i\in\mathbb{Z}}$ to the right and left of this

tile. Such a tiling x is called a *substitution tiling*.
Recall that Σ_θ is minimal and so the suspension $\{\Sigma_\theta\}_c$ is minimal as well. In order to construct the tiling space we can choose a substitution tiling that corresponds to any specific element $[((u_i)_{i\in\mathbb{Z}}, s)] \in \{\Sigma_\theta\}_c$ and then take the closure of the translation orbit of this tiling. It follows that we get a tiling space associated to the substitution called the *substitution tiling space* and denoted by $\mathcal{T}_\theta$.

Define the map $h : \{\Sigma_\theta\}_c \to \mathcal{T}_\theta$ by

$$h[((u_i)_{i\in\mathbb{Z}}, t)] = \phi^t(\tau(u_i)_{i\in\mathbb{Z}}),$$

where

$$\tau((u_i)_{i\in\mathbb{Z}}) = (x_i)_{i\in\mathbb{Z}}.$$

Since the map (2.8) is continuous, and $\mathcal{T}_\theta$ is a subspace of $\mathcal{T}_{\mathscr{P}}$ such that $\bar{h}(\{\Sigma_\theta\}_c) = \mathcal{T}_\theta$, where

$$\bar{h} = \tau_c \mid_{\{\Sigma_\theta\}_c}.$$

Then $h : \{\Sigma_\theta\}_c \to \mathcal{T}_\theta$ is a continuous bijective map, and thus h is a homeomorphism. □

Remark 2.30.

1. *As being explained in the proof, flows under a function provide an alternative description of tiling spaces.*

2. *The substitution tiling space $\mathcal{T}_\theta$ is nothing but the suspension above the subshift where we are using a ceiling function that depends on cylinder sets. Above each cylinder we assign a constant value (the height of the cylinder) but the value may not be the same for all cylinders.*

3. *Regardless of which choices of height we make, there will be a natural homeomorphism between the suspension $\{\Sigma_\theta\}_c$ and the associated substitution tiling space.*

4. *In our case, we can choose values of the height of the cylinders so that the homeomorphism h conjugates the suspension flow and the flow on $\mathcal{T}_\theta$ when we make the choice of lengths $L_1 = L_2 = \ldots = L_n$ for the tiles.*

2.4 The spectrum and dynamical systems

In this section, we will distinguish between two kinds of dynamical systems. The first one corresponds to substitutions of constant length on a two letter alphabet $\mathcal{A}$. The second dynamical system corresponds to substitutions of constant length on the n-letter alphabet $\mathcal{A}$.

2.4.1 Discrete and continuous substitutions

The notions of discrete or continuous substitutions on a two letter alphabet $\mathcal{A}$ and their corresponding dynamical systems are discussed briefly, as well as the notion of the n-adic system. The link between the system that arises from a discrete substitution and the n-adic system is also presented. Most of the material in §2.4.1 is included in the book [4] and [10].

Definition 2.31. Let θ be a substitution of constant length l on a two letter alphabet $\mathcal{A} = \{0,1\}$ and let $\theta(0) = a = a_0 \cdots a_{l-1}$ and $\theta(1) = b = b_0 \cdots b_{l-1}$. Then θ is called *finite* if any one of the following condition holds:

1. $a_i = b_i$ for all $i \in \mathbb{N}_0$.
2. $a_i = 0$ for all $i \in \mathbb{N}_0$ or $b_i = 1$ for all $i \in \mathbb{N}_0$.
3. $a_i = 1$ for all $i \in \mathbb{N}_0$ and $b_i = 0$ for all $i \in \mathbb{N}_0$.
4. l is odd and $a = \tilde{b} = 0101 \cdots 010$ or $1010 \cdots 101$. Where $\tilde{b}$ the mirror of b is obtained by replacing all the zeros in b by ones and all the ones by zeros.

θ is called *continuous* if it is not finite and $a = \tilde{b}$.

θ is called *discrete* if it is neither finite nor continuous.

Remark 2.32. *If the substitution θ is discrete, then both $I_1 = \{i \mid a_i = b_i\}$ and $J_1 = \{i \mid a_i \neq b_i\}$ are non-empty, that is $\theta(0)$ and $\theta(1)$ agree at some but not every place. We remark that in the original paper [10] they mistakenly claim that the converse implication holds.*

Lemma 2.33. *For substitutions on two letters, if the substitution is discrete or continuous, then it is primitive and aperiodic. If on the other hand the substitution is periodic, then the associated shift space is also periodic.*

Proof. This follows Coven and Keane, [10] except for primitivity. Additionally, they showed that the dynamical systems that correspond to the substitution subshifts Σ_θ and Σ_σ, where θ is discrete and σ is continuous, cannot be conjugate to each other.

To prove the primitivity, assume that a substitution θ (discrete or continuous) is not primitive. Then $\theta(0)$ and $\theta(1)$ cannot both contain 0's and 1's. Without loss of generality assume that $\theta(0)$ is the one which does not contain both 0's and 1's. Then either $\theta(0)$ is all 0's or $\theta(0)$ is all 1's. If $\theta(0)$ is all 0's, then the substitution will be a finite substitution which is a contradiction. If $\theta(0)$ is all 1's, then $\theta(1)$ must contain 0's and 1's otherwise it becomes a finite substitution. Thus $\theta^2(0)$ and $\theta^2(1)$ both contain 0's and 1's which is a contradiction to the hypothesis that θ is not primitive. $\square$

n-adic integers $Z(n)$ and n-adic system

We follow the definitions and description given on pages 19-21 of [4].

Definition 2.34. Let n be some positive integer such that $n \geq 2$, then a sequence of integers $(x_m) = (x_0, x_1, .., x_m, ...)$ satisfying $x_m \equiv x_{m-1} (\bmod n^m)$ for all $m \geq 1$, determines an *n-adic integer.*
The set of all n-adic integers will be denoted by $Z(n)$.

Note that to distinguish ordinary integers from n-adic integers, ordinary integers will be called rational integers. Each rational integer x is associated with an n-adic integer determined by the sequence $(x, x, ..., x, ...)$. Thus we may assume that the set of rational integers is a subset of all n-adic integers. The sum of two n-adic integers determined by the sequences (x_m) *and* (y_m) is the n-adic integer determined by the sequence $(x_m + y_m)$. It can be shown that every n-adic integer is determined by some canonical sequence, a sequence each term of which satisfies the conditions $x_m \equiv x_{m-1} (\bmod n^m)$ for all $m \geq 1$ and $x_m \equiv \bar{x}_m (\bmod n^{m+1}), 0 \leq \bar{x}_m < n^{m+1}$, and two distinct canonical sequences determine distinct n-adic integers. Thus the n-adic integers are in $1-1$ correspondence with the canonical sequences. Since every canonical sequence has the form $(z_0, z_0 + z_1 n, z_0 + z_1 n + z_2 n^2, ...)$, where $0 \leq z_i < n$ and on the other hand every sequence of this type is a canonical sequence, then we can alternatively define $Z(n)$ as follows:

Definition 2.35. For $n \geq 2$, let $Z(n)$ be the *additive group of n-adic integers* defined by

$$Z(n) = \{\sum_{i=0}^{\infty} z_i n^i \mid z_i = 0, 1, \cdots, n-1\}.$$

$Z(n)$ is a compact abelian group with $\tau : Z(n) \to Z(n)$ defined by $\tau(z) = z+1$. The system $(Z(n), \tau)$ is called the *n-adic system.*

Note that $(Z(n), \tau)$ forms a minimal dynamical system.

Remark 2.36. *We can identify $Z(n)$ with the cantor set $\{0,1,...,n-1\}^{\mathbb{Z}_0^+}$ and endow it with the product topology.*

As in [16] and [22], a substitution θ of constant length l on a two letter alphabet $\mathcal{A} = \{0,1\}$ defines a map λ^2 (which corresponds to the substitution θ^2). This map has either one, two, or four fixed points. Also the minimal substitution subshifts associated to discrete or continuous substitutions are $\Sigma_\theta = \overline{\mathcal{O}(w')}$, where $w' \in \Sigma_\theta$ is any θ^2-fixed point. Note that w' is a θ-periodic point.

In [10], Coven and Keane gave an explicit construction of the semi-conjugacy between the subshift arising from a discrete substitution of constant length l on a two letter alphabet $\mathcal{A}$ and the group of l-adic integers.
The following result describes this semi-conjugacy and additionally shows that it is a measure theoretic isomorphism.

Remark 2.37. *Let (X,ϕ) and (Y,ψ) be two systems with invariant measures μ and ν respectively. A semi-conjugacy $\pi : (X,\phi) \to (Y,\psi)$ is a measure-theoretic isomorphism if there is an invariant subset Y' of Y such that $\nu(Y') = 1$ and π is one-one on $\pi^{-1}(Y')$.*

Theorem 2.38. *Consider the discrete substitution θ of constant length l on a two letter alphabet $\mathcal{A}$ and define the map $\pi : (\Sigma_\theta, S) \to (Z(l), \tau)$ by $\pi(w) = \lim_{i\to\infty} \tau^{k_i}(0)$, where $\{k_i\}$ is chosen so that $S^{k_i}(w') \to w$ for a θ^2-fixed point w'. Then π is a semi- conjugacy taking w' to 0. Furthermore, the map π is a measure- theoretic isomorphism.*

Proof. See for the proof (Theorem 1 and Corollary 1, [10]). □

2.4.2 Substitutions of constant length on the n-letter alphabet $\mathcal{A}$.

In this section, we discuss the trace relation on substitutions of constant length on the n-letter alphabet $\mathcal{A}$. We shall see in Section §2.4.4 that how this is related to the point spectrum of the translation action on the associated tiling spaces. The following definitions and lemmas are adapted from [12].

Definition 2.39. Let (X,T) be a minimal dynamical system. A *cyclic partition* of X is a partition $\{X_i\}_{i=0}^{m-1}$ of X into disjoint subsets such that $X_{i+1} = TX_i$ for $0 \le i < m-1$ and $TX_{m-1} = X_0$.

Let $n \ge 1$. A T^n*-invariant partition* of X is a partition of X whose elements are all closed and T^n-invariant.

A T^n*-minimal partition of* X *is a partition of* X *whose elements are all* T^n*-minimal.*

Lemma 2.40. *Consider the minimal dynamical system* (X,T) *and let n be a positive integer. Then there exists a cyclic* T^n*-minimal partition. This partition is unique up to cyclic permutations of its members.*

Definition 2.41. The number of elements of a cyclic T^n-minimal partition will be denoted by $\gamma(n)$ for each $n \ge 1$. The equivalence relation whose classes are the members of the cyclic T^n-minimal partition will be denoted by Λ_n.

Definition 2.42. The *trace relation* Λ of a minimal dynamical system is defined by $\Lambda = \bigcap_{n\ge 1} \Lambda_n$.

Lemma 2.43. *The trace relation* $\Lambda = \bigcap_{n:n=\gamma(n)} \Lambda_n$.

This follows from the fact that the function γ satisfies the following Properties:

1. $1 \le \gamma(n) \le n$ and $\gamma(n)$ divides n.
2. $\Lambda_{\gamma(n)} = \Lambda_n$ and thus $\gamma(\gamma(n)) = \gamma(n)$.
3. If m divides n then $\Lambda_m \supset \Lambda_n$; moreover if $\gamma(n) = n$ then $\gamma(m) = m$.
4. If $(m,n) = 1$ then $\Lambda(mn) = \Lambda_m \cap \Lambda_n$ and $\gamma(mn) = \gamma(m)\gamma(n)$.

Lemma 2.44. *Consider the dynamical system* (Σ_θ, S) *associated to the minimal substitution subshift* Σ_θ *obtained from a primitive substitution* θ *of constant length l on a finite alphabet* $\mathcal{A}$*. Then either* $\gamma(l^n) = l^n$ *for all* $n \ge 1$ *or* θ *is periodic.*

Definition 2.45. Consider the system (Σ_θ, S), where Σ_θ is the minimal substitution subshift obtained from a primitive substitution of constant length l on the n-letter alphabet $\mathcal{A}$. Let w be any sequence in Σ_θ, then the number $h(\theta) = \max\{r \ge 1 \mid (r,l) = 1, r \text{ divides } \gcd\{a \mid w_a = w_0\}\}$ is called the *height* of θ.

Remark 2.46.

1. *The height $h(\theta)$ is independent of the choice of the sequence $w \in \Sigma_\theta$.*

2. $1 \leq h(\theta) \leq n$.

3. *Let k be an integer, let $P_k = \{a \mid w_{a+k} = w_k\}$ and $g_k = \gcd P_k$. Then*

$$\{r \geq 1 \mid (r,l) = 1, r \text{ divides } g_0\} = \{r \geq 1 \mid (r,l) = 1, r \text{ divides } g_k\}.$$

4. *If $h(\theta) = n$, then θ is periodic.*

   If θ is one-one, then θ is periodic if and only if $h(\theta) = n$.

Remark 2.47. *By using the criteria given in Remark 2.46(4), it follows that all our examples throughout are aperiodic.*

The following results are desirable in establishing the height of primitive, aperiodic substitutions of constant length on a two letter alphabet, in particular discrete or continuous, or on a three letter alphabet.

Lemma 2.48. *All primitive, aperiodic substitutions of constant length on a two letter alphabet $\mathcal{A}$ have height one.*

Proof. Let θ be a primitive, aperiodic substitution of constant length on a two letter alphabet $\mathcal{A}$. By Remark 2.46, the substitution θ must have height one otherwise it will be a periodic substitution which is a contradiction to the fact that the substitution θ is aperiodic. □

Lemma 2.49. *All primitive, aperiodic substitutions of constant length on a three letter alphabet $\mathcal{A}$ either have height one or have height two.*

Proof. If a substitution θ is primitive, aperiodic of constant length, then by Remark 2.46, θ must have height one or height two otherwise it will be a periodic substitution which is a contradiction to the fact that θ is aperiodic. □

Definition 2.50. Let θ be a primitive, aperiodic substitution of constant length l and height $h(\theta) = h$. Define the system $(\mathbb{Z}_h, \tau_h)$, where $\mathbb{Z}_h = \{[\mathbf{k}], k = 0, \cdots, h-1\}$ is a cyclic group of order h and the map $\tau_h : \mathbb{Z}_h \to \mathbb{Z}_h$ defined by

$$\tau_h([\mathbf{k}]) = [\mathbf{k}+1],\ k = 0, \cdots,\ h-1.$$

Note that the system $(\mathbb{Z}_h, \tau_h)$ forms a minimal dynamical system. Since $(h,l) = 1$, then the product system $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ is minimal.

The following theorem (Theorem 13, [12]) shows that the systems $(\Sigma_\theta/\Lambda, S_\Lambda)$ and $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ are conjugate to each others.

Theorem 2.51. [12] *Consider the substitution dynamical system (Σ_θ, S) obtained from a primitive, aperiodic substitution θ of constant length l with height $h(\theta) = h$. Define the trace relation Λ on Σ_θ. Then there is a conjugacy*

$$\Psi : (\Sigma_\theta/\Lambda, S_\Lambda) \to (Z(l) \times \mathbb{Z}_h, \tau \times \tau_h).$$

As an immediate consequence of Theorem 2.51, we will establish Theorem 2.53 which is the analogue of Theorem 2.38 but is more general. Specifically, it applies to any primitive, aperiodic substitution of constant length on the n-letter alphabet $\mathcal{A}$ with any height.

Definition 2.52. [5] Consider the dynamical system (X, f), where X is a compact metric space. Then f is said to be *equicontinuous* if the family $\{f^n(x) : n \in \mathbb{Z}\}$ is equicontinuous, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x,y) < \delta$ implies that

$$d_X(f^n(x), f^n(y)) < \varepsilon$$

for all $n \in \mathbb{Z}$.

Theorem 2.53. *Let (Σ_θ, S) be a substitution dynamical system, where θ is a primitive, aperiodic substitution of constant length l and height $h(\theta) = h$. Then there is a semi-conjugacy*

$$\pi : (\Sigma_\theta, S) \to (Z(l) \times \mathbb{Z}_h, \tau \times \tau_h).$$

Proof.

$$\begin{tikzcd}
(\Sigma_\theta, s) \arrow[r, "\tilde{\pi}"] \arrow[dr, "\pi"] & (\Sigma_\theta/\Lambda, S_\Lambda) \arrow[d] \\
 & (Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)
\end{tikzcd}$$

The proof follows from the fact that there is a semi-conjugacy say $\tilde{\pi} : (\Sigma_\theta, S) \to (\Sigma_\theta/\Lambda, S_\Lambda)$, and by Theorem 2.51 there is a conjugacy

$$\Psi : (\Sigma_\theta/\Lambda, S_\Lambda) \to (Z(l) \times \mathbb{Z}_h, \tau \times \tau_h).$$

Thus the composite

$$\pi = \Psi \circ \tilde{\pi} : (\Sigma_\theta, S) \to (Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$$

forms a semi-conjugacy. $\square$

Remark 2.54. *Since the trace relation $\Lambda = \bigcap_{n \geq 1} \Lambda_n$ is the intersection of all closed invariant equivalence relations Λ_n, $n \geq 1$ on the subshift Σ_θ, then $(\Sigma_\theta/\Lambda, S_\Lambda)$ is the largest factor for which S_Λ is equicontinuous (Chapter V, [32]).*

2.4.3 Suspension space and point spectrum

In this section we study some basic properties of the point spectrum σ_{pp} defined in 2.56 of general constant $\mathfrak{c}$ suspension flow.

Definition 2.55. [33] Let $\phi : \mathbb{R} \times X \to X$ be a continuous $\mathbb{R}$-action on a compact metric space X and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. An *eigenfunction* of ϕ is a Borel measurable function $f : X \to S^1$ for which there exists $r \in \mathbb{R}$ such that

$$f(t \cdot x) = e^{2\pi i r t} f(x) \quad \text{for all } t \in \mathbb{R},\ x \in X,$$

where $t \cdot x$ denotes $\phi(t, x)$.

r is called the *eigenvalue* of ϕ corresponding to the eigenfunction f.

Definition 2.56. The *point spectrum* denoted by σ_{pp} is the set of all eigenvalues of a continuous $\mathbb{R}$-action ϕ on a compact metric space X.

We point out that the point spectrum σ_{pp} forms a countable group under addition operation and the set of all eigenfunctions forms a group with point-wise multiplication operation.

Remark 2.57. *By (Theorem 2.3, [7]), all eigenfunctions of the flows we shall consider are continuous. Hence we shall restrict our attention to continuous eigenfunctions and their eigenvalues.*

Construction of eigenfunctions (eigenvalues) of the constant $\mathfrak{c}$ suspension flow

Given a homeomorphism $f : X \to X$ on a compact metric space X with a ceiling function $c : X \to \mathbb{R}^+$, let the quotient space (suspension space) be $X_c = X \times [0, \mathfrak{c}]/\sim$, where $c(x) =$

$\mathfrak{c}$ for all $x \in X$ and $\sim$ is the equivalence relation is generated by $(x, \mathfrak{c}) \sim (f(x), 0)$. The space X_c can be partitioned into a finite number of disjoint clopen subsets called cylinder sets, each one is of constant height $\mathfrak{c}$.

Consider the suspension flow $\phi^t : X_c \to X_c$ that works in a natural additive way, that is, for all $t \in \mathbb{R}$ and $[(x,s)] \in X_c$

$$\phi^t[(x,s)] = [(x, s+t)].$$

Let $S^1 = [0,1]/\sim$, where $\sim$ is the equivalence relation define by identifying 0 and 1. Define the function $f_n : X_c \to S^1$ by

$$f_n[(x,s)] = e^{2\pi i \frac{1}{\mathfrak{c}} ns} \quad \text{for all } n \in \mathbb{Z}.$$

It can be shown that f_n is well-defined and for all $t \in \mathbb{R}$, $(x,s) \in X_c$

$$\begin{aligned} f_n(t \cdot [(x,s)]) &= f_n[(x, s+t)] \\ &= e^{2\pi i \frac{1}{\mathfrak{c}} n(s+t)} \\ &= e^{2\pi i \frac{1}{\mathfrak{c}} nt} f_n[(x,s)]. \end{aligned}$$

Thus f_n is an eigenfunction with corresponding eigenvalue $r_n = \frac{1}{\mathfrak{c}} n$ for all $n \in \mathbb{Z}$.

Note that for $n = 1$, the eigenfunction $f_1[(x,s)] = e^{2\pi i \frac{1}{\mathfrak{c}} s}$ will generate the subgroup.

$$G_f = \{f_n[(x,s)] \mid n \in \mathbb{Z}\},$$

furthermore, $r_1 = \frac{1}{\mathfrak{c}}$ will generate the subgroup

$$G_r = \{\frac{1}{\mathfrak{c}} n \mid n \in \mathbb{Z}\}.$$

G_f and G_r are both infinite cyclic groups generated by f_1 and r_1 respectively, thus they are isomorphic to the additive group of integers $(\mathbb{Z}, +)$.

In general, any suspension space with the constant height $\mathfrak{c}$ will have a subgroup of eigenfunctions (eigenvalues) that is isomorphic to the additive group of integers $(\mathbb{Z}, +)$.

2.4.4 The n-adic system and point spectrum

In preparation for studying the point spectrum of the suspension of Σ_θ, we dicuss the trace relation and explain how it is used to determine the point spectrum via studying the point spectrum of the suspension of n-adic systems.
Our aim is to obtain the group of all eigenvalues of the constant $\mathfrak{c}$ suspension flow of τ-addition by 1 on $Z(n)$. Up to this point we have determined a subgroup of these eigenvalues, but we have not determined the entire group.

Definition 2.58. Define the additive group of the ring $\mathbb{Z}[\frac{1}{n}]$ to be $\{\frac{m}{n^k} \mid m \in \mathbb{Z}, k \in \mathbb{N}\}$. $\mathbb{Z}[\frac{1}{n}]$ is an abelian group under the operation of addition of numbers.

The following is my own interpretation of some established result.

Proposition 2.59. *Given the n-adic system $(Z(n),\tau)$ with a ceiling function c such that $c(z) = \mathfrak{c}$ for all $z \in Z(n)$, consider the suspension space $\{Z(n)\}_c = Z(n) \times [0,\mathfrak{c}]/\sim$, where the equivalence relation $\sim$ is generated by $(z,\mathfrak{c}) \sim (\tau(z),0)$. The constant $\mathfrak{c}$ suspension flow of τ-addition by* 1 *will have point spectrum σ_{pp} isomorphic to the additive group $(\mathbb{Z}[\frac{1}{n}],+)$.*

Proof. The subgroup $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{n}]$ corresponds to the group of eigenvalues $\{r_n = \frac{1}{\mathfrak{c}}n \mid n \in \mathbb{Z}\}$ with associated eigenfunctions $\{f_n[(z,s)] = e^{2\pi i \frac{1}{\mathfrak{c}}ns} \,|n \in \mathbb{Z}\}$ as considered previously.

To construct the isomorphism between the additive group $(\sigma_{pp},+)$ and the additive group $(\mathbb{Z}[\frac{1}{n}],+)$, our objective is to obtain the elements in σ_{pp} that correspond to the elements in $\mathbb{Z}[\frac{1}{n}]$ which are of the form $\frac{m}{n^k}, m \in \mathbb{Z}, k \in \mathbb{N}$. In order to establish this objective, we will divide the discussion into the following stages:

First stage: We will obtain the elements in σ_{pp} that correspond to $\frac{1}{n}, \frac{2}{n}, \cdots$.

Second stage: We will obtain the elements in σ_{pp} that correspond to $\frac{1}{n^2}, \frac{2}{n^2}, \cdots$.

k'th stage: We will obtain the elements in σ_{pp} that correspond to $\frac{1}{n^k}, \frac{2}{n^k}, \cdots$.

As for the first stage, observe that the element $\frac{1}{n} \in \mathbb{Z}[\frac{1}{n}]$ is uniquely determined by the property $\frac{1}{n} \times n = 1$.
We now identify an eigenfunction g such that plays the same role that $\frac{1}{n}$ does in $\mathbb{Z}[\frac{1}{n}]$. That is, $f_1 = g \cdots g \quad (n-\text{times})$.

Let us consider the following diagram and associated description:

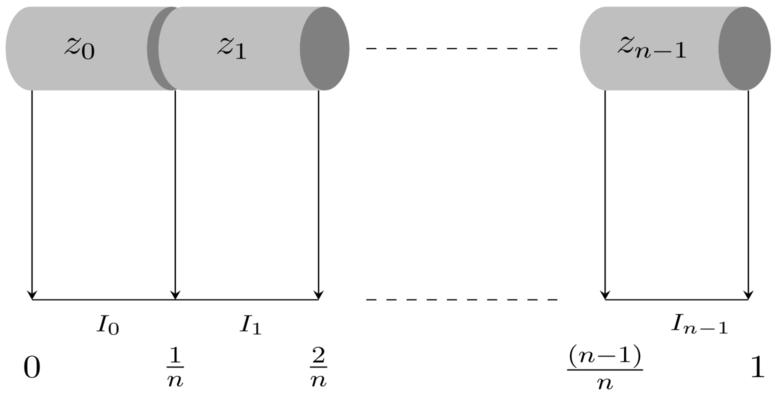

FIGURE 2.1: Illustration of the third step in the construction of an eigenfunction.

1(a) Partition $\{Z(n)\}_c$ into n clopen subsets (cylinders)

$$Z_j = \left\{ \sum_{t=0}^{\infty} z_t n^t \mid z_0 = j \right\}, \ j = 0, 1, \cdots, n-1.$$

2(a) Divide S^1 into n equal subintervals

$$I_j = \left[\frac{j}{n}, \frac{j+1}{n} \right], \ j = 0, 1, \cdots, n-1.$$

3(a) Align these cylinders in a way that they respect the τ-additive flow structure such that the bottom of the cylinder Z_0 and the top of the cylinder Z_{n-1} are being identified to $0 \in S^1$ and $1 \in S^1$ respectively, that is, assigned to the same point. Each one of the cylinders Z_j in succession is mapped to the corresponding subinterval $I_j, j = 0, \cdots, n-1$, of S^1.

4(a) Rescale time such that $r\mathfrak{c} = \frac{1}{n}$, where $\mathfrak{c}$ is the common value for the height of the cylinders Z_j.

5(a) Over each cylinder, define the map

$$g[(z,s)] = e^{2\pi i \frac{1}{n}(jn^0 + \frac{s}{\mathfrak{c}})},$$

where

$$z = \sum_{t=0}^{\infty} z_t n^t \in Z_j.$$

$$g[(z,s)] = \begin{cases} e^{2\pi i \frac{s}{cn}} & \text{if } z_0 = 0 \\ e^{2\pi i \frac{s}{cn} + \frac{1}{n}} & \text{if } z_0 = 1 \\ \vdots & \vdots \\ e^{2\pi i \frac{s}{cn} + \frac{n-1}{n}} & \text{if } z_0 = n-1. \end{cases}$$

One can easily check that the piecewise function g forms an eigenfunction with associated eigenvalue $\frac{1}{cn}$. Once we get an eigenfunction (eigenvalue) corresponding to $\frac{1}{n}$, we can automatically get eigenfunctions (eigenvalues) corresponding to $\frac{2}{n}, \frac{3}{n}, \cdots$ by multiplying g by itself less than m-times, $m \in \mathbb{Z}$.

As for the second stage, look at the next diagram with associated description:

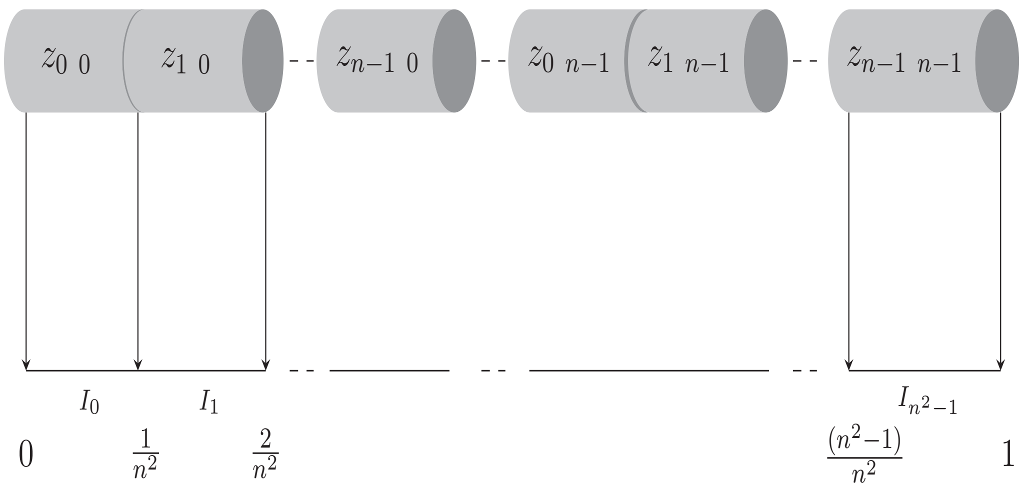

FIGURE 2.2: Illustration of the second step in the construction of an eigenfunction.

1(b) Partition $\{Z(n)\}_c$ into n^2 clopen subsets (cylinders)

$$Z_{j_1 j_2} = \left\{ \sum_{t=0}^{\infty} z_t n^t \mid z_0 = j_1, z_1 = j_2 \right\}, \ \forall\ j_1, j_2 \in \{0, 1, \cdots, n-1\}$$

and consider the circle S^1, which is divided into n^2 equal subintervals

$$I_j = \left[\frac{j}{n^2}, \frac{j+1}{n^2} \right], \ j = 0, 1, \cdots, n^2 - 1.$$

2(b) Align the cylinders $Z_{j_1 j_2}$ in a way that respect the τ-additive flow structure such that the bottom of the cylinder Z_{00} and the top of the cylinder $Z_{n-1\,n-1}$ are being identified to the same point 0 and 1 respectively. Also, map each one of the cylinders in succession to corresponding subinterval of S^1.

3(b) Rescale time such that $r\mathfrak{c} = \frac{1}{n^2}$, again $\mathfrak{c}$ is the common value for the height of the cylinders $Z_{j_1 j_2}$.

4(b) For each cylinder $Z_{j_1 j_2}$, define the map

$$h[(z,s)] = e^{2\pi i \frac{1}{n^2}(j_1 n^0 + j_2 n^1 + \frac{s}{\mathfrak{c}})},$$

where

$$z = \sum_{t=0}^{\infty} z_t n^t \in Z_{j_1 j_2}.$$

One can check that h is an eigenfunction with associated eigenvalue $\frac{1}{\mathfrak{c}n^2}$ such that h satisfies $g = h \cdots h$ (n times). Then automatically obtain eigenfunctions (eigenvalues) correspond to $\frac{2}{n^2}, \frac{3}{n^2}, \cdots$.

As for the k'th stage, follow the following steps:

1(c) Partition the quotient space $\{Z(n)\}_c$ into n^k clopen subsets (cylinders)

$$Z_{j_1,\cdots,j_k} = \left\{ \sum_{t=0}^{\infty} z_t n^t \mid z_0 = j_1, \cdots, z_{k-1} = j_k \right\},$$

$\forall\, j_1, \cdots, j_k \in \{0, 1, \cdots, n-1\}$.

Devide the circle S^1 into n^k equal subintervals

$$I_j = \left[\frac{j}{n^k}, \frac{j+1}{n^k} \right],\ j = 0, 1, \cdots, n^k - 1.$$

2(c) Align the cylinders $Z_{j_1 j_2 \cdots j_k}$ in a way that respect the τ-additive flow structure such that the bottom of the cylinder $Z_{00\cdots 0}$ and the top of the cylinder $Z_{n-1\cdots n-1}$ are being identified to the same point 0 and 1 respectively. Also, map each one of the cylinders in succession to corresponding subinterval of S^1.

3(c) Rescale time such that $r\mathfrak{c} = \frac{1}{n^k}$, where $\mathfrak{c}$ is the common value for the height of the cylinders $Z_{j_1 \cdots j_k}$.

4(c) For each cylinder $Z_{j_1\cdots j_k}$, define the map

$$\beta[(z,s)] = e^{2\pi i \frac{1}{n^k}(j_1 n^0 + j_2 n^1 + \cdots + j_k n^{k-1} + \frac{s}{\mathfrak{c}})},$$

where

$$z = \sum_{t=0}^{\infty} z_t n^t \in Z_{j_1\cdots j_k}.$$

Again, one can check that β is an eigenfunction with associated eigenvalue $\frac{1}{\mathfrak{c}n^k}$. Thus one can obtain the eigenfunctions with associated eigenvalues $\frac{1}{\mathfrak{c}n^k}m$ that correspond to $\frac{m}{n^k} \in \mathbb{Z}[\frac{1}{n}]$, where $m \in \mathbb{Z},\ k \in \mathbb{N}$.
This set of eigenvalues $\left\{\frac{1}{\mathfrak{c}n^k}m \mid m \in \mathbb{Z}, k \in \mathbb{N}\right\}$ forms an additive group, which leads to the question: Is this the group of all eigenvalues? The answer is yes, as will be explained in the following argument.
One can show that, given any two distinct points in the suspension space of n-adic integers $\{Z(n)\}_c$, we can separate them with eigenfunctions associated to the eigenvalues that are described earlier. That is, if we have two different points in the suspension space, we can find eigenfunctions that will assign them to different values on the circle. Therefore, since the eigenfunctions separate all points, then it must be the complete set of all eigenvalues and eigenfunctions (Chapter 3, [33]).
For any group of eigenvalues with associated eigenfunctions, there is a corresponding equicontinuous factor. In our case, one could note that the induced semi-conjugacy from the suspension space $\{Z(n)\}_c$ onto the corresponding equicontinuous factor is a topological conjugacy of flows.

To prove the isomorphism, define the map $\phi : \{\frac{1}{\mathfrak{c}n^k}m \mid m \in \mathbb{Z},\ k \in \mathbb{N}\} \to \mathbb{Z}[\frac{1}{n}]$ by $\phi(\frac{m}{\mathfrak{c}n^k}) = \frac{m}{n^k}$. It is easy to check that ϕ is one-one and onto and a homomorphism. So ϕ is an isomorphism and thus the additive group of eigenvalues $(\sigma_{pp}, +)$ of the constant $\mathfrak{c}$ suspension flow of τ-addition by 1 on $Z(n)$ and the additive group $(\mathbb{Z}[\frac{1}{n}], +)$ are isomorphic to each other. □

As a result of the previous proposition, the constant $\mathfrak{c}$ suspension flow of τ-addition by one on $Z(n)$ will have eigenvalues of the form $\frac{1}{\mathfrak{c}} \frac{m}{n^k}$ for some $m \in \mathbb{Z}, k \in \mathbb{N}$ and thus the point spectrum $\sigma_{pp} = \frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{n}]$, that is, a scalar multiple of $\mathbb{Z}[\frac{1}{n}]$.

The following proposition is useful for determining the point spectrum of the dynamical system $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ defined in Definition 2.50 where h is the height of a primitive, aperiodic substitution θ of length l.

Proposition 2.60. *Consider the system $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ where $\mathbb{Z}_h$ is the cyclic group of order h. Then the constant $\mathfrak{c}$ suspension flow of $(\tau \times \tau_h)$ will have point spectrum $\sigma_{pp} = \frac{1}{h}\frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{l}]$.*

Proof. A way of determining the eigenvalues of the constant $\mathfrak{c}$ suspension flow of $(\tau \times \tau_h)$ on $(Z(l) \times \mathbb{Z}_h)$ is to construct the eigenfunctions as we did before: at the first stage we have $h \cdot l$ cylinders and at each stage we subdivide those cylinders. Then we can show that the eigenvalues correspond to these eigenfunctions are of the form $\frac{1}{h}\frac{1}{\mathfrak{c}}\frac{m}{l^k}$ for some $m \in \mathbb{Z}$, $k \in \mathbb{N}$ and hence the point spectrum $\sigma_{pp} = \frac{1}{h}\frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{l}]$. $\square$

Lemma 2.61. *Consider the suspension space $\{\Sigma_\theta\}_c$, where θ is a primitive, aperiodic substitution of constant length l and height $h(\theta) = h$. Let Λ be the trace relation given in Definition 2.43, then the constant $\mathfrak{c}$ suspension flow of S on Σ_θ will have point spectrum*

$$\sigma_{pp} = \frac{1}{h}\frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{l}].$$

Proof. By Remark 2.54, the system $(\Sigma_\theta/\Lambda, S_\Lambda)$ is the largest equicontinuous factor of (Σ_θ, S) and thus has the same eigenvalues as (Σ_θ, S). And by Theorem 2.51, the two systems $(\Sigma_\theta/\Lambda, S_\Lambda)$ and $(Z(l) \times \mathbb{Z}_h, \tau \times \tau_h)$ are conjugate to each other. Thus the constant c suspension flow of $\tau \times \tau_h$ and S have the same eigenvalues.
Hence, the point spectrum σ_{pp} of the constant $\mathfrak{c}$ suspension flow of S on Σ_θ is equal to $\frac{1}{h}\frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{l}]$. $\square$

We follow in part Remark 2.30(4), Lemma 2.61 and discussion presented at p.135 in [33] to state the following important result:

Theorem 2.62. *If a substitution θ on the n-letter alphabet $\mathcal{A}$ is primitive, aperiodic, of constant length l and height $h(\theta) = h$, then the substitution tiling space $\mathcal{T}_\theta$ with all the tile lengths $L_1 = L_2 = \cdots = L_n = \mathfrak{c}$ has point spectrum $\sigma_{pp} = \frac{1}{h}\frac{1}{\mathfrak{c}}\mathbb{Z}[\frac{1}{l}]$.*

Chapter 3

Point Spectrum of Substitution Tiling Spaces on a two or three letter alphabet

In this chapter we shall discuss and study how the point spectrum σ_{pp} of the substitution tiling space $\mathcal{T}_\theta$ is affected by changing the tile lengths from those of the canonical choices, where θ is a primitive, aperiodic substitution of constant length l. The study will cover two major cases: the two letter and the three letter alphabet $\mathcal{A}$.

In the first section §3.1 we introduce some definitions, propositions, and theorems associated to the main concepts in our study. In the subsequent sections §3.2 and §3.3, we give a careful study of the nature of the point spectrum in the case of a two letter and a three letter alphabet $\mathcal{A}$ respectively.

3.1 Definitions, propositions and theorems

Definition 3.1. [7] Let $\mathcal{A} = \{a_1, a_2 \cdots, a_n\}$ be a finite alphabet of n-letters. For a word $w = w_0 \cdots w_k$ from the alphabet $\mathcal{A}$ of the full shift $\mathcal{A}^{\mathbb{Z}}$, the *population vector* $v = (v_1, \cdots, v_n)^T$ of the finite word w gives the number of the occurrences v_i of the letter a_i in w.

A *recurrence word* is a finite word w in $u \in \mathcal{A}^{\mathbb{Z}}$, $w = u_r u_{r+1} \cdots u_s$ satisfying the condition $u_{s+1} = u_r$.

A *recurrence vector* is the population vector for a recurrence word.

Definition 3.2. [7] A recurrence vector v of the word w from the finite alphabet $\mathcal{A}$ of the substitution subshift Σ_θ is called *full* if the vectors $\{M_\theta^k v\}$, with k ranging from 0 to $n-1$, are linearly independent; where M_θ is the incidence matrix associated to the substitution θ.

Remark 3.3. *Every primitive, aperiodic substitution on a two letter alphabet $\mathcal{A} = \{0,1\}$ admits a full recurrence vector, namely $(0,1)^T$ or $(1,0)^T$, because if we have neither $(0,1)^T$ nor $(1,0)^T$, this means that we will not have either 00 or 11. This implies that the sequences in the θ-subshift are of the form $\cdots 0101 \cdots$ which are periodic sequences. By primitivity of the substitution, these recurrence vectors must be full.*

The following theorem (Theorem 3.4, [7]) is vital for determining the point spectrum of the tiling space $\mathcal{T}_\theta$ associated to the substitution θ with $n \times n$ incidence matrix M_θ over an n-letter alphabet. Let $L = (L_1, ..., L_n)$ denote the tile length vector.

Theorem 3.4. *Let θ be a primitive, aperiodic substitution on a finite n-letter alphabet $\mathcal{A}$. The number K is in the point spectrum of the substitution tiling space $\mathcal{T}_\theta$ if and only if, for every recurrence vector v,*

$$KLM_\theta^m v \to 0 \,(\mathrm{mod} 1) \text{ as } m \to \infty. \tag{3.1}$$

Since (3.1) is an integer linear condition on v, (3.1) holds for a set of vectors V if and only if it holds for the $\mathbb{Z}$-module generated by V. With this in mind, we introduce the following definition.

Definition 3.5. Let $\mathcal{R}$ denote the $\mathbb{Z}$*-module generated by the set of recurrence vectors.*

We can make changes in the method of the convergence in (3.1) to give the following theorem.

Theorem 3.6. *The number K is in the point spectrum of the substitution tiling space $\mathcal{T}_\theta$ if and only if, for every vector $v \in \mathcal{R}$,*

$$KLM_\theta^m v \to 0 \,(\mathrm{mod} 1) \text{ as } m \to \infty. \tag{3.2}$$

An important result for the investigation of the substitutions of constant length on two letters or three letters is Theorem 2.4 in [7]. We give the following

Theorem 3.7. *Let θ be a primitive, aperiodic substitution on a finite n-letter alphabet $\mathcal{A}$. Suppose that all the eigenvalues of the incidence matrix M_θ are of magnitude 1 or greater, and there exists a full recurrence vector. If the ratio of any two tile lengths is irrational, then there is trivial point spectrum. If the ratio of tile lengths are all rational, then the point spectrum is contained in $\mathbb{Q}/L_1$.*

In order to simplify the notation for the following proof, let $M_\theta = M$.

Proof. Let K be in the point spectrum, and consider the sequence of real numbers $t_m = KLM^m v$, where v is a fixed full recurrence vector. Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ be the characteristic polynomial of M. Note that the a_i are all integers since M is an integer matrix. Since $p(M) = 0$, then

$$M^n + a_{n-1}M^{n-1} + \cdots + a_0 = 0$$

$$\begin{aligned}
M^n &= -(a_{n-1}M^{n-1} + \cdots + a_0) \\
M^{m+n} &= -(a_{n-1}M^{(m+n)-1} + \cdots + a_0 M^m) \\
KLM^{m+n}v &= -KLv(a_{n-1}M^{(m+n)-1} + \cdots + a_0 M^m) \\
t_{m+n} &= -\sum_{k=0}^{n-1} a_k t_{m+k}. \qquad (3.3)
\end{aligned}$$

Hence t_m satisfy the recurrence relation (3.3). By Theorem 3.4, the t_m converge to zero (mod 1). That is, we can write

$$t_m = i_m + \varepsilon_m, \qquad (3.4)$$

where the i_m are integers, and the ε_m converge to zero as real numbers. By substituting the division (3.4) into the recursion (3.3), we get:

$$i_{m+n} + \varepsilon_{m+n} = -\sum_{k=0}^{n-1} a_k(i_{m+k} + \varepsilon_{m+k})$$

$$i_{m+n} + \varepsilon_{m+n} = -\left((a_0 i_m + \cdots + a_{n-1} i_{(m+n)-1}) + (a_0\varepsilon_m + \cdots + a_{n-1}\varepsilon_{(m+n)-1})\right) \qquad (3.5)$$

As mentioned, for sufficiently large m the sequence ε_m converges to zero, so we can take ε_m bounded by $\frac{1}{\sum_{i=0}^{n-1} a_i}$, which implies that each individual $\varepsilon_m < \frac{1}{\sum_{i=0}^{n-1} a_i}$. Then

$$\begin{aligned}
a_0\varepsilon_m + a_1\varepsilon_{m+1} + \cdots + a_{n-1}\varepsilon_{(m+n)-1} &\leq |a_0\varepsilon_m + a_1\varepsilon_{m+1} + \cdots + a_{n-1}\ \varepsilon_{(m+n)-1}| \\
&< \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{\sum_{i=0}^{n-1} |a_i|} \\
&= 1
\end{aligned}$$

Note that the left hand side of equation (3.5) is broken up into the integer part i_{m+n} and the fractional part ε_{m+n} which is not an integer. The right hand side by our calculation has as its integer part $a_0 i_m + \cdots + a_{n-1}\ i_{(m+n)-1}$ and as its fractional part $a_0\varepsilon_m + \cdots + a_{n-1}\varepsilon_{(m+n)-1}$. Hence

$$\varepsilon_{m+n} = -\sum_{k=0}^{n-1} a_k \varepsilon_{m+k}$$

and

$$i_{m+n} = -\sum_{k=0}^{n-1} a_k i_{m+k}.$$

Thus both the i and the ε must separately satisfy the recursion (3.3). For the sake of simplicity we shall assume M_θ is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$. In this case any solution to the recursion relation (3.3) is a linear combination of powers of eigenvalues of M_θ, and the case where M is not diagonalizable can be handled by considering instead a polynomial in m times eigenvalues to the m^{th}power. Now ε_m is the solution of the recurrence relation (3.3), let $\varepsilon_m = c_1\lambda_1^m + c_2\lambda_2^m + \cdots + c_n\lambda_n^m$. From the hypothesis, the eigenvalues are all of magnitude 1 or greater. This implies that the linear combination converges to zero only if it is identically zero, therefore ε_m must be identically zero for all sufficiently large values of m. The non-diagonalizable case is handled similarly.

The vector $M^m v$ is an integer column vector and so each t_m is an integer linear combination of the elements of the vector KL. From a sequence of n consecutive terms t_m, one can obtain the system of n linear equations in n variables $KL_1, KL_2, \cdots, KL_n$. The coefficient matrix A of this system consists of the n row vectors $M^m v$ with m ranging from 0 to $n-1$. Since v is full, then the n row vectors of this integer coefficient matrix are linearly independent which implies that A is invertible and so the linear system has a unique solution. Since the t_m are integers (for m large enough), the components of KL must then all be rational. Thus if it is defined for all i and j,

$$\frac{Kl_i}{Kl_j} = \frac{L_i}{L_j}$$

is also rational. So, if for some i and $j, L_i/L_j$ is irrational we must have $K = 0$. Hence there is trivial point spectrum $\sigma_{pp} = \{0\}$. Suppose on the other hand that for all i and $j, L_i/L_j$ is rational. Then for each i, $L_i = q_1 L_1$ for some $q_1 \in \mathbb{Q}$. By the observation that KL must be rational for each component, we have $KL_i = KL_1 q_1$ is a rational number. Let $KL_1 q_1 = q_2$ for some $q_2 \in \mathbb{Q}$. Then $K = q/L_1$ for some $q = q_2/q_1 \in \mathbb{Q}$. Hence $K \in \mathbb{Q}/L_1$. $\square$

Although Theorem 3.7 applies to a wide variety of substitutions of constant length, there are substitutions of constant length with zero eigenvalues for which the following corollary can be useful.

Corollary 3.8. *Let θ be a primitive, aperiodic substitution with the incidence matrix M_θ. Suppose that all the eigenvalues of M_θ are of magnitude equal or greater than* 1 *or equal to* 0*. If K is in the point spectrum, then the numbers $t_m = KLM_\theta^m v$ have to be integers for n sufficiently large, where $v \in \mathcal{R}$.*

If we review the proof of Theorem 3.7, one can realize that the conclusion $t_m = KLM_\theta^m v \in \mathbb{Z}$ for sufficiently large m does not depend on the assumption that 0 is not an eigenvalue or the existence of a full recurrence vector.

Definition 3.9. If there is a set of length vectors with full measure in the set of all possible length vectors for which $\sigma_{pp} = \{0\}$, then we say that the *typical case* is $\sigma_{pp} = \{0\}$.

Remark 3.10. *When the typical case is $\sigma_{pp} = \{0\}$, then there will be exceptional choices for L for which σ_{pp} is larger for example when $L = (1,1,1)$ or (c,c,c).*

Proposition 3.11. *Under the same hypothesis as in Theorem 3.7, the typical case is $\sigma_{pp} = \{0\}$.*

Proof. Although similar arguments apply in general, we shall consider here the three letter case as this is the case we shall need.
Consider the space of length vectors $\{L = (L_1, L_2, L_3) | \, L \in \mathbb{R}^{+3}\}$. According to Theorem 3.7, we examine the nature of a set of $\mathbb{R}^{+3}$ that satisfies: for some i, j, $L_i/L_j \notin \mathbb{Q}$.

Let
$$Z = \left\{ (L_1, L_2, L_3) \in \mathbb{R}^{+3} | \, \exists \, i, j, \; \frac{L_i}{L_j} \notin \mathbb{Q} \right\}.$$
One of the ways of thinking of Z is as built up out of its intersection with the hyperplanes perpendicular to one of the coordinate axes. For the sake of definiteness, let us partition Z using the hyperplanes with fixed final coordinate.

Then for a given value $c \in (0, \infty)$, we have

$$Z_c = \left\{ (L_1, L_2, c) \in \mathbb{R}^{+3} | \, \frac{L_1}{L_2} \notin \mathbb{Q} \text{ or } \frac{L_1}{c} \notin \mathbb{Q} \text{ or } \frac{L_2}{c} \notin \mathbb{Q} \right\}.$$

Thus when $c = 1$,we have

$$Z_1 = \left\{ (L_1, L_2, 1) \in \mathbb{R}^{+3} | \, \frac{L_1}{L_2} \notin \mathbb{Q} \text{ or } L_1 \notin \mathbb{Q} \text{ or } L_2 \notin \mathbb{Q} \right\}.$$

For a given $c \in (0, \infty)$, letting

$$R_c = \left\{ (L_1, L_2, L_3) \in \mathbb{R}^{+3} | \, L_3 = c \right\},$$

we have for example

$$R_1 \setminus Z_1 = \left\{ (L_1, L_2, 1) | \, L_1, L_2 \in \mathbb{Q}^+ \right\}.$$

As this is a countable set, we have that Z_1 has full measure in R_1. A similar result holds for all c, and so we see by integration that all of Z has full measure. Thus we have sensitivity to changes in lengths and the typical case is $\sigma_{pp} = \{0\}$. $\square$

Remark 3.12. *In future propositions, when establishing the typical case under similar circumstances, a similar proof will apply but we shall omit it without comment.*

As we shall be making repeated use of Theorem 3.6, it will be vital for us to determine the $\mathbb{Z}$-module of recurrence vectors $\mathcal{R}$ associated to substitutions. We shall begin with the two letter case where we have a very general result. And then we shall discuss some special cases in the three letter case.

Lemma 3.13. *Given a primitive, aperiodic substitution on a two letter alphabet $\mathcal{A} = \{0, 1\}$, then $\mathcal{R} = \mathbb{Z}^2$.*

Proof. By Remark 3.3, we can assume without loss of generality that the substitution θ admits the full recurrence vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Also, we have 010 or 101 otherwise we will have sequences which end either with only 1's or 0's which are not possible in primitive substitutions. In either case this will lead to the recurrence vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and hence $\mathcal{R}$ will include

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which implies

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \subset \mathcal{R}$$

and thus $\mathcal{R} = \mathbb{Z}^2$.

$\square$

Proposition 3.14. *Let θ be a primitive, aperiodic substitution of constant length on a three letter alphabet $\mathcal{A} = \{0,1,2\}$. If the word ii occurs in the θ-subshift, where $i \in \{0,1,2\}$, then the substitution has height $h(\theta) = 1$.*

Proof. Let $i \in \mathcal{A}$ be a symbol such that i is repeated, which implies that i occurs in an even and odd index of some sequence in the θ-subshift. Assume that i occurs in k and $k+1$ index, where k is odd and $k+1$ is even. Then $P_k = \{a \mid w_{a+k} = w_k\}$ contains the number 1 and so $g_k = \gcd P_k = 1$. $\square$

Proposition 3.15. *Let θ be a primitive, aperiodic substitution of constant length on a three letter alphabet $\mathcal{A} = \{0,1,2\}$. If the height of the substitution $h(\theta) = 1$ and there exist at least two repeated symbols, then $\mathcal{R} = \mathbb{Z}^3$.*

Proof. Assume that the symbols 00 and 11 occur in the θ-subshift which implies that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are in $\mathcal{R}$. By minimality of the substitution subshift, the word $2 \cdots 2$ occurs somewhere (with no 2's in between). Thus we will have the following recurrence vector $\begin{pmatrix} m \\ n \\ 1 \end{pmatrix}$ for some $m,n \in \mathbb{N}_0$. Then

$$\begin{pmatrix} m \\ n \\ 1 \end{pmatrix} - m \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{R}.$$

Hence $\mathcal{R} = \mathbb{Z}^3$. $\square$

Remark 3.16. *Among the height one substitutions of constant length, we can find ones with no repeated symbols such that $\mathcal{R} = \mathbb{Z}^3$ as the following example shows.*

Example 3.1. $\theta(0) = 101010$, $\theta(1) = 101020$ and $\theta(2) = 101210$.

The incidence matrix is $M_\theta = \begin{pmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ and the height $h(\theta) = 1$ since the length $l = 6$ is even. Words occurring in the θ-subshift are 101, 020, 201012 and 10201 revealing that among recurrence vectors are $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

$\begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, then

$\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \in \mathcal{R}$, $\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \in \mathcal{R}$ and

$\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \in \mathcal{R}$. Hence $\mathcal{R} = \mathbb{Z}^3$.

Lemma 3.17. *Let θ be a primitive, aperiodic substitution of constant length on a three letter alphabet $A = \{0,1,2\}$. If the height of the substitution $h(\theta) = 2$, then $\mathcal{R} = \left\{ \begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell \in \mathbb{Z} \right\}$ up to a permutation of coordinates. Whenever $\mathcal{R}$ has this form, there exists no full recurrence vector.*

Proof. Let $(w_i)_{i\in\mathbb{Z}}$ be a sequence in the substitution subshift Σ_θ such that $(w_i)_{i\in\mathbb{Z}}$ be a θ-fixed point, then for each $a_i \in \mathcal{A}$, $a_i \in \{0,1,2\}$, a_i occurs in w_k for only even k or only odd k, otherwise $P_k = \{a \mid w_{a+k} = w_k\}$ contains an odd number in violation that $h(\theta) = 2$. Observe that there is a surjection between the finite alphabet $\mathcal{A}$ and the set of spots $\{$even , odd$\}$, thus one of the symbols must take one of the spots and the other two symbols share the other spot. Which means that one of the symbols must occur every other symbol. Without loss of generality, assume that 0 occurs every other symbol such that it takes the even spot. Under these assumptions the recurrence vector of any recurrence word occurring in the θ-subshift will have the form $\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix}$ for some $k,l \in \mathbb{N}_0$. The words 010 and 020 will occur in the θ-subshift revealing that among the recurrence vectors are

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The integer span of these two vectors is

$$\left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k,\ell \in \mathbb{Z} \right\},$$

which forms a subgroup of $\mathbb{Z}^3$, thus $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k,\ell \in \mathbb{Z} \right\}$.

We shall now show that in this situation the substitution admits no full recurrence vector.
Let $v = k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \ell \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ be a recurrence vector, then $M_\theta v$ and $M_\theta^2 v$ are recurrence vectors.
Consider the equation $c_1 v + c_2 M_\theta v + c_3 M_\theta^2 v = 0$, where

$$M_\theta v = \begin{pmatrix} k_1+\ell_1 \\ k_1 \\ \ell_1 \end{pmatrix} \quad \text{and} \quad M_\theta^2 v = \begin{pmatrix} k_2+\ell_2 \\ k_2 \\ \ell_2 \end{pmatrix} \text{ for some } k_1,\ell_1,k_2,\ell_2 \in \mathbb{N}_0.$$

This implies

$$c_1 \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} + c_2 \begin{pmatrix} k_1+\ell_1 \\ k_1 \\ \ell_1 \end{pmatrix} + c_3 \begin{pmatrix} k_2+\ell_2 \\ k_2 \\ \ell_2 \end{pmatrix} = 0.$$

Then the determinant

$$\begin{vmatrix} k+\ell & k_1+\ell_1 & k_2+\ell_2 \\ k & k_1 & k_2 \\ \ell & \ell_1 & \ell_2 \end{vmatrix} = 0.$$

Hence the set $\{v, M_\theta v, M_\theta^2 v\}$ is not linearly independent and thus there exists no full recurrence vector. $\square$

Remark 3.18. *One can find height one substitutions such that*

$$\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\} \text{ (see Example 3.3 below).}$$

Now I shall investigate a large class of examples of substitutions of constant length on a three letter alphabet with height equal to one or two.
By the previous Lemma 3.17 and the previous Remark 3.18, we can assume without loss of generality that we have sequences of the form $0-0$ and I refer to all such substitution subshifts as being of type $0-0$.

Lemma 3.19. *If θ is a primitive, aperiodic substitution of constant odd length l such that the substitution subshift of type $0-0$, then either (i) or (ii) holds*

(i) $\theta(0) = 0-0\cdots-0$ and so the incidence matrix M_θ is of the form

$$M_\theta = \begin{pmatrix} \frac{l+1}{2} & \frac{l-1}{2} & \frac{l-1}{2} \\ r & s & j \\ \frac{l-1}{2}-r & \frac{l+1}{2}-s & \frac{l+1}{2}-j \end{pmatrix} \quad \text{for some } r,s,j \in \mathbb{N}_0. \tag{3.6}$$

(ii) $\theta(0) = -0-0\cdots0-$ and so the incidence matrix M_θ is of the form

$$M_\theta = \begin{pmatrix} \frac{l-1}{2} & \frac{l+1}{2} & \frac{l+1}{2} \\ r & s & j \\ \frac{l+1}{2}-r & \frac{l-1}{2}-s & \frac{l-1}{2}-j \end{pmatrix} \quad \text{for some } r,s,j \in \mathbb{N}_0. \tag{3.7}$$

Proof. Observe that this kind of substitution has height $h(\theta) = 2$. Since the sequences in the subshift are of the form $0-0$ and the substitution has odd length, then $\theta(0)$ either starts and ends with 0 or starts and ends with something different from 0. For the first case, consider the words 010 and 020 that occur somewhere in the subshift and then apply the substitution. One can notice that $\theta(1)$ must start and end with something different from 0 otherwise we will not have a substitution of type $0-0$. Similarly, $\theta(2)$ starts and ends with something different from 0. Clearly, M_θ is of the form (3.6) and can be easily checked that $(-1,1,1)$ is a left eigenvector associated to the eigenvalue equal to 1.
For the second case, if we consider the words 010 and 020 that occur somewhere in the subshift and apply the substitution, then $\theta(1)$ must start and end with 0 otherwise we will not have a sequence of the form $0-0$. Similarly $\theta(2)$ must start and end with 0. Clearly,

M_θ is of the form (3.8) and can be easily checked that $(-1,1,1)$ is a left eigenvector associated to the eigenvalue equal to -1. $\square$

Note that when the vector $(-1,1,1)$ is paired with any vector of the form $\begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix}$ for some $k,\ \ell \in \mathbb{Z}$, we will get the zero vector.

Lemma 3.20. *If θ is a primitive, aperiodic substitution of constant even length l such that the substitution subshift is of type $0-0$, then $(-1,1,1)$ occurs as a left eigenvector associated to the eigenvalue zero.*

Proof. Observe that the substitution θ has height 1 ($h(\theta)$ cannot divide the length, so $h(\theta)$ cannot be 2). Independent of the nature of the substitution, the incidence matrix M_θ will be of the form

$$M_\theta = \begin{pmatrix} \frac{l}{2} & \frac{l}{2} & \frac{l}{2} \\ r & s & j \\ \frac{l}{2}-r & \frac{l}{2}-s & \frac{l}{2}-j \end{pmatrix} \quad \text{for some } r,\ s,\ j \in \mathbb{N}_0. \tag{3.8}$$

It can be easily checked that $(-1,1,1)$ is the left eigenvector associated to the eigenvalue 0. $\square$

Remark 3.21.

1. *A substitution subshift of type 0-0 either has odd length with height equals two or has even length with height equals one.*

2. *All height two substitutions are of type 0-0, but not all of the height one substitutions are of type 0-0 (Example 3.10).*

3. *In all the substitution subshifts of type $0-0$ with height equal one or two, $\mathcal{R}$ has the form*

$$\left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k,\ell \in \mathbb{Z} \right\} \text{ and there exists no full recurrence vector.}$$

One can easily check the following proposition.

Proposition 3.22. *Let θ be a height one substitution with even length l and $\mathcal{R} = \mathbb{Z}^3$. If its incidence matrix M_θ is of the form* $\begin{pmatrix} \frac{l}{2} & \frac{l}{2} & \frac{l}{2} \\ r & r & r \\ \frac{l}{2}-r & \frac{l}{2}-r & \frac{l}{2}-r \end{pmatrix}$, *where $0 < r < \frac{l}{2}$, then there exists no full recurrence vector.*

As mentioned before, in all of the height two substitutions $\mathcal{R} \neq \mathbb{Z}^3$ such that $\mathcal{R}$ has the form

$$\left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$$

and there exists no full recurrence vector (Example 3.12). On the other hand, in the height one substitutions one could have either $\mathcal{R} = \mathbb{Z}^3$ (Example 3.2) or $\mathcal{R} \neq \mathbb{Z}^3$, and if $\mathcal{R} \neq \mathbb{Z}^3$, we can have either

$$\left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$$

(Example 3.3), or another form

$$\left\{ \begin{pmatrix} m \\ n \\ n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\} \text{(Example 3.4).}$$

We also provide an example to demonstrate that even in the height one case when $\mathcal{R} = \mathbb{Z}^3$ we do not necessarily have a full recurrence vector (Example 3.5).

Remark 3.23. *We cannot say in general that in the case of height one substitutions when $\mathcal{R} \neq \mathbb{Z}^3$, then there exists no full recurrence vector. But this is the case for examples we know.*

Listed below are the examples showing the connection between height and $\mathcal{R}$ and the connection between $\mathcal{R}$ and the existence of full recurrence vectors.

From now on and without further comment, in all given examples we shall assume that θ denotes a primitive, aperiodic substitution on a finite alphabet $\mathcal{A}$.

Example 3.2. $\theta(0) = 0120,\ \theta(1) = 1120\ \text{ and }\ \theta(2) = 0012.$

The incidence matrix is $M_\theta = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Since we have two successive 0's, then the substitution has height $h(\theta) = 1$. Moreover, we have also two successive 1's which implies that $\mathcal{R} = \mathbb{Z}^3$. A word occurring in the θ-subshift is 00, revealing that among the recurrence vectors is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which one can check directly is full.

Example 3.3. $\theta(0) = 0102,\ \theta(1) = 0201\ \text{ and }\ \theta(2) = 0101.$

This substitution is a $0-0$ type substitution of even length $l = 4$, then it has height $h(\theta) = 1$ and $\mathcal{R}$ has the form $\begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix}$, $k,\ \ell \in \mathbb{Z}$. Therefore the substitution has no full recurrence vector.

Example 3.4. $\theta(0) = 001200,\ \theta(1) = 012000\ \text{ and }\ \theta(2) = 000120.$

Here the substitution has length $l = 6$ and incidence matrix $M_\theta = \begin{pmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Since we have successive 0's, then the substitution has height $h(\theta) = 1$. Examination of the words occurring in the θ-subshift reveal that the recurrence vectors are all of the form $\begin{pmatrix} m \\ n \\ n \end{pmatrix}$ for some $m, n \in \mathbb{N}_0$. Thus $\mathcal{R} = \left\{ \begin{pmatrix} m \\ n \\ n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}$.

Example 3.5. $\theta(0) = 0210,\ \theta(1) = 0120\ \text{ and }\ \theta(2) = 0012.$

This substitution has even length $l = 4$ and so height $h(\theta) = 1$. The words occurring in the θ-subshift are $00, 1001$, and 10021, showing that among recurrence vectors are

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Now

$$\begin{pmatrix}2\\1\\0\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \in \mathcal{R},$$

and

$$\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \in \mathcal{R}.$$

Then clearly $\mathcal{R} = \mathbb{Z}^3$. The incidence matrix of the substitution is $M_\theta = \begin{pmatrix}2 & 2 & 2\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$ so by Proposition 3.22 the substitution has no full recurrence vector.

Example	**Length** l	**Height** $\mathbf{h}(\theta)$	$\mathcal{R}$	**Full recurrence vector**
3.12	3	2	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell \in \mathbb{Z}\right\}$	no full recurrence vector
3.2	4	1	$\mathbb{Z}^3$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$
3.10	3	1	$\mathbb{Z}^3$	$\begin{pmatrix}1\\2\\0\end{pmatrix}$
3.3	4	1	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell \in \mathbb{Z}\right\}$	no full recurrence vector
3.4	6	1	$\left\{\begin{pmatrix}m\\n\\n\end{pmatrix} \mid k,\ell \in \mathbb{Z}\right\}$	no full recurrence vector
3.5	4	1	$\mathbb{Z}^3$	no full recurrence vector

TABLE 3.1: Connection between $\mathcal{R}$ and the heights and the existence of full recurrence vectors.

3.2 Point spectrum in the case of a two letter alphabet $\mathcal{A}$

In this section we examine the dependence of the point spectrum σ_{pp} of the substitution tiling space $\mathcal{T}_\theta$ on the changes of tile lengths $L = (L_1, L_2)$, where θ is a primitive, aperiodic substitution of constant length on two letters.
The following Theorem 3.24 which was introduced by Clark and Sadun in [7] gives more spectral information than Theorem 3.7, but only when the alphabet $\mathcal{A}$ has two letters and the substitution is of constant length.

Theorem 3.24. *Suppose that we have a primitive, aperiodic substitution θ on a two letter alphabet $\mathcal{A} = \{0,1\}$ of constant length l, and $\theta(i)$ contains l_i 0's and $l - l_i$ 1's, where $i \in \{0,1\}$. Suppose further $1 \leq l_0, l_1 \leq l-1$ and $l_0 \neq l_1$, let z be the greatest common factor of l and $l_0 - l_1$. Then the point spectrum σ_{pp} depends as follows on the ratio L_1/L_2 :*

1. if $L_1 = L_2$, then there is a positive integer N such that

$$N\mathbb{Z}[\frac{1}{l}] \subset NL_1\sigma_{pp} \subset \mathbb{Z}[\frac{1}{l}];$$

2. if $L_1/L_2 \in \mathbb{Q} - \{1\}$, then there exist positive integers N_1 and N_2 such that

$$N_1\mathbb{Z}[\frac{1}{z}] \subset N_2L_1\sigma_{pp} \subset [\frac{1}{z}];$$

3. if $L_1/L_2 \notin \mathbb{Q}$, then $\sigma_{pp} = \{0\}$.

Obviously, Theorem 3.24 applies to any primitive, aperiodic substitution of constant length l on a two letter alphabet $\mathcal{A} = \{0,1\}$, when the two additional conditions are satisfied:

1. $l_0 \neq l_1$
2. $1 \leq l_0, l_1 \leq l-1$.

Remark 3.25. *Let θ be a primitive, aperiodic substitution of constant length on two letters $\{0,1\}$, where $\theta(i)$ contains l_i 0's and $l - l_i$ 1's, then the incidence matrix is $M_\theta = \begin{pmatrix} l_0 & l_1 \\ l-l_0 & l-l_1 \end{pmatrix}$ with eigenvalues l and $l_0 - l_1$.*

The point spectrum σ_{pp} and sensitivity

Now we will establish a theorem which completes the analysis of sensitivity to changes in tile lengths in the case of a two letter alphabet $\mathcal{A}$ through the explanation of how the point spectrum σ_{pp} is affected by these changes, independent of Theorem 3.24.

Theorem 3.26. *Let θ be a primitive, aperiodic substitution of constant length l on the alphabet $\mathcal{A} = \{0,1\}$. Letting l_i be the number of 0's in $\theta(i)$, we have the following cases:*
(a) $l_0 = l_1$. Here, the point spectrum σ_{pp} is insensitive to changes in length and is always a scalar multiple of $\mathbb{Z}[\frac{1}{l}]$.
(b) $l_0 \neq l_1$. Here, the point spectrum σ_{pp} is sensitive to changes in tile length $L = (L_1, L_2)$ and depends as follows on the ratio L_1/L_2 :
(1) if $L_1 = L_2 = c$, then $\sigma_{pp} = \frac{1}{c}\mathbb{Z}[\frac{1}{l}]$, where $c \in \mathbb{R}^+$,
(2) if $L_1/L_2 \in \mathbb{Q}$, then $\sigma_{pp} \subseteq \mathbb{Q}/L_1$,
(3) if $L_1/L_2 \notin \mathbb{Q}$, then $\sigma_{pp} = \{0\}$.

Proof. (a) If we have $l_0 = l_1$, then the general length vector $L = (L_1, L_2)$ can be decomposed into the linear combination of the two left eigenvectors v_1, v_2 as follows:

$$L = c_1 v_1 + c_2 v_2,$$

where v_1, v_2 are associated to the eigenvalues of the incidence matrix of the substitution $\theta, r_1 = l$ and $r_2 = 0$. For any $v \in \mathcal{R}$

$$LM^n v = c_1 r_1^n (1,1) v$$

implies

$$\begin{aligned} KLM^n v &= Kc_1 l^n (1,1) v \\ &= Kc_1 l^n (1,1) v. \end{aligned}$$

For sufficiently large n and for all $v \in \mathcal{R}$, $KLM^n v \to 0$ (mod 1) if and only if K is of the form $\frac{1}{c_1}\frac{m}{l^r}$ for some $m \in \mathbb{Z}, r \in \mathbb{N}$. That is, $K \in \frac{1}{c_1}Z[\frac{1}{l}]$. Thus the point spectrum is always a scalar multiple of $\mathbb{Z}[\frac{1}{l}]$.
Observe that in this case c_1 cannot be zero for us to have a legitimate length vector.

(b) Recall that in the case of a two letter alphabet $\mathcal{A}$ the height $h(\theta) = 1$, by applying Theorem 2.62, if $L_1 = L_2 = c$, then the point spectrum is $\sigma_{pp} = \frac{1}{c}\mathbb{Z}[\frac{1}{l}]$.
Since $l_0 \neq l_1$, then both eigenvalues l and $l_0 - l_1$ have magnitude one or greater. By

Remark 3.3, we know there exists a full recurrence vector and so by applying Theorem 3.7, we get that, if $L_1/L_2 \in \mathbb{Q}$, then σ_{pp} is contained in $\mathbb{Q}/L_1$. If $L_1/L_2 \notin \mathbb{Q}$, then there is trivial point spectrum, that is, $\sigma_{pp} = \{0\}$. Hence if $l_0 \neq l_1$, we have sensitivity to changes in tile lengths $L = (L_1, L_2)$. □

Observe that our description of σ_{pp} in the case that $L_1 = L_2 = c$ is more precise than in that of Theorem 3.24.

Remark 3.27. *By Lemma 2.33, Theorem 3.26 applies to all discrete and continuous substitutions.*

3.3 Examining the nature of the point spectrum when $\mathcal{A}$ is a three letter alphabet

Let θ be a primitive, aperiodic substitution on three letters of constant length l. and let M_θ be the incidence matrix with characteristic polynomial $P(x)$. By the Gauss Lemma, $P(x)$ can be factorized as follows:

$$P(x) = (x-l)q(x), \text{ where } q(x) = x^2 + bx + c$$

is a quadratic, monic polynomial with integer coefficients. $q(x)$ is either going to have a pair of complex conjugate roots or real roots. We will study the nature of the point spectrum based on the nature of the roots r_1 and r_2 as demonstrated in §3.3.1 and §3.3.2.

3.3.1 The quadratic polynomial $q(x)$ has complex roots

In this subsection we will refer to $q(x)$ when its roots are complex numbers that are not real.

Note that if the incidence matrix M_θ for the substitution θ has complex roots in addition to l, then the sequences in the θ-substitution subshift cannot be of the form $0-0$ because the associated matrices M_θ have eigenvalue ± 1 or 0. Thus we cannot have complex roots in the case of substitution subshifts of type $0-0$, and hence all substitutions are of height one in this case.

Proposition 3.28. *Let θ be a height one substitution of constant length l such that its incidence matrix M_θ has complex conjugate eigenvalues. If there exists a full recurrence vector, then the point spectrum is sensitive to changes in the tile lengths and typically $\sigma_{pp} = \{0\}$.*

Proof. Let $q(x) = x^2 + bx + c$, where $b, c \in \mathbb{Z}$, $q(x)$ can be factorized over $\mathbb{C}$ as follows: $q(x) = (x - r_1)(x - r_2)$, where r_1 and r_2 are the complex conjugate roots of $q(x)$ such that $r_1 r_2 = |r_1|^2 = |r_2|^2 = c \in \mathbb{Z}^+ - \{0\}$, otherwise we will not have complex roots. Then $|r_i| \geq 1$, where $i = 1, 2$. Hence, as long as we have a full recurrence vector we can apply Theorem 3.7 and thus we will have sensitivity to changes in length vectors. □

Remark 3.29. *We cannot say in general that all the substitutions with complex roots have full recurrence vectors, but the examples we identify have full recurrence vectors.*

Example 3.6. $\theta(0) = 2022,\ \theta(1) = 0211\ \text{ and }\ \theta(2) = 1212.$

The substitution is of length l = 4 and its incidence matrix is $M_\theta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 2 \end{pmatrix}$, with eigenvalues $4, \frac{1+7i}{2}$ and $\frac{1-7i}{2}$. Since we have two successive 2's and 1's, we have the height $h = 1$ and also $\mathcal{R} = \mathbb{Z}^3$ (see Propositions 3.14 and 3.15). A word occurring in the θ-subshift is 11, revealing that among the recurrence vectors is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which one can check directly is full. As a result, Proposition 3.28 applies to θ. Thus for a typical choice of length, the point spectrum will be $\{0\}$, and hence we will have sensitivity to changes in length. Also by Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{4}]$.

3.3.2 The quadratic polynomial $q(x)$ has real roots

Studying the nature of the point spectrum when the quadratic polynomial $q(x)$ has real roots will be divided into cases, each case is divided into two subcases according to the height. To be more precise, height one substitutions such that $\mathcal{R} = \mathbb{Z}^3$ or $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$ are studied when it is possible.

Case I: If the roots r_1 and r_2 of $q(x)$ are of magnitude greater than or equal 1, that is, $|r_1|$ and $|r_2| \geq 1$.

(i) The height of the substitution $h(\theta) = 1$.

Recall that for any height one substitution we can have either $\mathcal{R} = \mathbb{Z}^3$ or $\mathcal{R} \neq \mathbb{Z}^3$. Bear in mind that we cannot have a height one substitution such that the $\mathbb{Z}$-module generated by recurrence vectors $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$, where all the eigenvalues of its incidence matrix M_θ are all greater than or equal 1 in magnitude, because in this case M_θ will have 0 as an eigenvalue. As a result of this, we will review the case of height one substitutions with all eigenvalues of M_θ having magnitude grater than or equal to one, where $\mathcal{R} = \mathbb{Z}^3$. In such a case, the existence of the full recurrence vector plays a very important role to determine the sensitivity of σ_{pp} to changes in tile lengths, which means that if there exists a full recurrence vector, Theorem 3.7 applies to θ and we get sensitivity to changes in length.

Remark 3.30. *One thing left to determine is whether there must be a full recurrence vector in this case when $\mathcal{R} = \mathbb{Z}^3$.*

Proposition 3.31. *If θ is a height one substitution of constant length l, with $\mathcal{R} = \mathbb{Z}^3$ and eigenvalues associated to the incidence matrix M_θ are all of magnitude greater than or equal to one, then if there exists a full recurrence vector we will have sensitivity to changes in tile lengths.*

For the proof, one can apply Theorem 3.7.

Example 3.7. $\theta(0) = 2022,\ \theta(1) = 0202$ and $\theta(2) = 0101$.

The incidence matrix is $M_\theta = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$, with associated eigenvalues $4, -1$ and -2.

Since the length of the substitution $l = 4$ is even, then the height $h(\theta) = 1$ and so by Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{4}]$. Words occurring in the θ-subshifts are $22, 020$, and 010, the associated recurrence vectors are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Now

$$\begin{pmatrix}1\\0\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then

$$\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\subseteq\mathcal{R}$$

and hence $\mathcal{R}=\mathbb{Z}^3$. One can check directly that $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is a full recurrence vector. As a result, Theorem 3.7 applies to θ, and for a typical choice of length, the point spectrum will be $\{0\}$. Thus we will have sensitivity to changes in tile lengths.

Example 3.8. $\theta(0)=0012010,\ \theta(1)=2010102\ \text{ and }\ \theta(2)=1020202.$

Here the substitution θ is of length $l=7$, its incidence matrix is $M_\theta=\begin{pmatrix}4&3&3\\2&2&1\\1&2&3\end{pmatrix}$ with associated eigenvalues $7,1$ and 1. By Proposition 3.14, the substitution has height $h(\theta)=1$ because we have two successive 0's. Words occurring in the θ-subshift are $00,010,$ and 202 and their associated recurrence vectors are $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$.

Then $\begin{pmatrix}1\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$.

Thus

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\subseteq\mathcal{R}$$

and hence $\mathcal{R}=\mathbb{Z}^3$. The constant $\mathfrak{c}=1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{7}]$. one can check directly that the recurrence vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is full. As a result, Proposition 3.31 applies to θ and so we will get sensitivity to changes in tile lengths.

(ii) The height of the substitution $h(\theta) = 2$.

Proposition 3.32. *If θ is a height two substitution of constant length l and eigenvalues are all of magnitude one or greater, then for a typical choice of length vectors $\sigma_{pp} = \{0\}$.*

Proof. The general length vector $L = (L_1, L_2, L_3)$ can be expressed as follows:

$$L = c_1(1,1,1) + c_2(-1,1,1) + c_3 v_3,$$

where $(1,1,1), (-1,1,1)$ and v_3 are left eigenvectors associated to the eigenvalues $l, r_1 = \pm 1$, and r_2 respectively. Without loss of generality, assume that $r_1 = 1$, then

$$LM^n = c_1 l^n (1,1,1) + c_2(-1,1,1) + c_3 r_2^n v_3.$$

For any $v \in \mathcal{R}, v = \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix}$ for some $k, \ell \in \mathbb{Z}$, we have

$$LM^n v = c_1 l^n (1,1,1)v + c_2(-1,1,1)v + c_3 r_2^n v_3 v$$

$$LM^n v = c_1 l^n (2k+2l) + 0 + c_3 r_2^n m,$$

where $m = v_3 v$, $m \in \mathbb{Z}$.

$$KLM^n v = K(c_1 l^n (2k+2l) + c_3 r_2^n m).$$

Let $\frac{c_1}{c_3} \notin \mathbb{Q}$, we will prove that $K = 0$. Assume the $K \neq 0$, then by Corollary 3.8, for sufficiently large n

$$t_n = K(c_1 l^n (2k+2l) + c_3 r_2^n m) = \nu \in \mathbb{Z}$$

and

$$t_{n+1} = K(c l^{n+1} (2k+2l) + c_3 r_2^{n+1} m) = \mu \in \mathbb{Z}.$$

Therefore,

$$\frac{t_{n+1}}{t_n} = \frac{c_1 l^{n+1}(2k+2l) + c_3 r_2^{n+1} m}{c_1 l^n (2k+2l) + c_3 r_2^n m} = \frac{\mu}{\nu} \in \mathbb{Q},$$

which implies

$$c_1 l^{n+1}(2k+2l) + c_3 r_2^{n+1} m = \frac{\mu}{\nu}(c_1 l^n (2k+2l) + c_3 r_2^n m)$$

and so
$$\frac{c_1}{c_3}\left(l^{n+1}(2k+2l)-\frac{\mu}{\nu}l^n(2k+2l)\right)=\frac{\mu}{\nu}r_2^n m-r_2^{n+1}m.$$
Let
$$l^{n+1}(2k+2l)-\frac{\mu}{\nu}l^n(2k+2l)=q_1\in\mathbb{Q}$$
$$\frac{\mu}{\nu}r_2^n m-r_2^{n+1}m=q_2\in\mathbb{Q}.$$
Then
$$\frac{c_1}{c_3}q_1=q_2,$$
which implies that
$$\frac{c_1}{c_3}=\frac{q_2}{q_1}\in\mathbb{Q},$$
which is a contradiction. Thus $K=0$, that is, we will have trivial point spectrum $\sigma_{pp}=\{0\}$. Hence, in the case of a height two substitution with eigenvalues greater than or equal to 1 in magnitude, we will get sensitivity to changes in length. Observe that by Theorem 2.62, the constant $\mathfrak{c}=1$ suspension flow will have eigenvalues $\frac{1}{2}\mathbb{Z}[\frac{1}{l}]$. □

Example 3.9. $\theta(0)=01020,\ \theta(1)=10101\ \text{ and }\ \theta(2)=10202.$

This substitution is of length $l=5$, height $h(\theta)=2$, and with incidence matrix $M_\theta=\begin{pmatrix}3&2&2\\1&2&1\\1&1&2\end{pmatrix}$ with associated eigenvalues $5,1$ and 1. The left eigenvectors associated to these eigenvalues are $(1,1,1),(-1,1,1)$ and $(-1,2,0)$ respectively. By applying Proposition 3.32, for a typical choice of length vectors we get trivial point spectrum $\sigma_{pp}=\{0\}$ and thus sensitivity to changes in tile lengths.

Remark 3.33. *If we have repeated roots, it does not affect the outcome.*

Case II: If r_1 and r_2 are the roots of $q(x)$ such that $r_1=0$ and $r_2\in\mathbb{Z}-\{0\}$.

(i) The height of the substitution $h(\theta)=1$.

- Firstly, we consider the case where $\mathcal{R}=\mathbb{Z}^3$.

Proposition 3.34. *Let θ be a height one substitution of constant length l, with $\mathcal{R}=\mathbb{Z}^3$ and eigenvalues l, $r_1=0$ and $r_2\in\mathbb{Z}-\{0\}$. Then for a typical choice of length vectors $\sigma_{pp}=\{0\}$.*

Proof. The left eigenvectors associated to the eigenvalues $l, r_2 \in \mathbb{Z} - \{0\}$ and $r_1 = 0$, are $v_1 = (1,1,1), v_2$ and v_3 respectively, and the general length vector $L = (L_1, L_2, L_3)$ can be expressed as follows:

$$L = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$LM^n = c_1 l^n (1,1,1) + c_2 r_2^n v_2 + 0.$$

For any vector $v \in \mathcal{R}$,

$$LM^n v = c_1 l^n (1,1,1) v + c_2 r_2^n v_2 v$$

$$KLM^n v = K(c_1 l^n (1,1,1) + c_2 r_2^n v_2) v,$$

where v_2 can be chosen to have a 1 in one of its coordinates. Without loss of generality, let $v_2 = (1,x,y)$. For K to be in σ_{pp}, it is sufficient to show that $KLM^n v \to 0 \pmod 1$ as $n \to \infty$ for the following bases vectors in $\mathcal{R}$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

For $v = e_1$, we have

$$KLM^n v = K(c_1 l^n + c_2 r_2^n).$$

Assume that $K \neq 0$ and $\frac{c_1}{c_2} \notin \mathbb{Q}$, then by Corollary 3.8 for sufficiently large n

$$t_n = K(c_1 l^n + c_2 r_2^n) = \nu \in \mathbb{Z}$$

and

$$t_{n+1} = K(c_1 l^{n+1} + c_2 r_2^{n+1}) = \mu \in \mathbb{Z}.$$

Then

$$\frac{t_{n+1}}{t_n} = \frac{c_1 l^{n+1} + c_2 r_2^{n+1}}{c_1 l^n + c_2 r_2^n} = \frac{\mu}{\nu} \in \mathbb{Q},$$

which implies

$$\frac{\frac{c_1}{c_2} l^{n+1} + r_2^{n+1}}{\frac{c_1}{c_2} l^n + r_2^n} = \frac{\mu}{\nu}$$

and so

$$\frac{c_1}{c_2}\left(\frac{\mu}{\nu} l^n - l^{n+1}\right) = r_2^{n+1} - \frac{\mu}{\nu} r_2^n.$$

Let

$$\frac{\mu}{\nu} l^n - l^{n+1} = q_1 \in \mathbb{Q}$$

and

$$r_2^{n+1} - \frac{\mu}{\nu} r_2^n = q_2 \in \mathbb{Q},$$

then

$$\frac{c_1}{c_2}q_1 = q_2$$

which implies that

$$\frac{c_1}{c_2} = \frac{q_2}{q_1} \in \mathbb{Q},$$

which is a contradiction. Thus $K = 0$.

Observe that, any potential K will have to work for all of the v's in $\mathcal{R}$ at the same time. If for one of the v's the only possibility for K is 0, then it suffices to conclude that the point spectrum must be trivial, that is, $\sigma_{pp} = \{0\}$. Hence we will have sensitivity to changes of length. $\square$

Remark 3.35. *By Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{l}]$, where l is the length of the substitution.*

Example 3.10. $\theta(0) = 001,\ \theta(1) = 201\ \text{ and }\ \theta(2) = 102.$

Here the incidence matrix is $M_\theta = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ with associated eigenvalues $3, 1$ and 0.

The height $h(\theta) = 1$ because we have two successive 0's. Moreover, we have also two successive 1's which implies that $\mathcal{R} = \mathbb{Z}^3$. A word occurring in the θ-subshift is 0110, revealing that among the recurrence vectors is $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, which one can check directly is full.

By Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{l}]$, where $l = 3$ is the length of the substitution and by the previous Proposition 3.34, for a typical choice of length, the point spectrum $\sigma_{pp} = \{0\}$.

- Secondly, we consider the case $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$.

Proposition 3.36. *If θ is a height one substitution of constant length l such that $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$, and eigenvalues l, $r_1 = 0$ and $r_2 \in \mathbb{Z} - \{0\}$, then for a typical choice of length vectors $\sigma_{pp} = \{0\}$.*

Proof. Recall from before that this substitution has even length l and of type 0-0, and so the incidence matrix is of the form (3.8) with 0 as one of its eigenvalues. Let the eigenvalues of M_θ be $l, r_2 \in \mathbb{Z} - \{0\}$ and $r_1 = 0$ with associated left eigenvectors $v_1 = (1,1,1), v_2$ and $v_3 = (-1,1,1)$ respectively. Then

$$L = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$L = c_1(1,1,1) + c_2 v_2 + c_3(-1,1,1).$$

For any $v = \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \in \mathcal{R}$, where $k, \ell \in \mathbb{Z}$

$$LM^n v = (c_1 l^n (1,1,1) + c_2 r_2^n v_2 + 0) v$$

and

$$KLM^n v = K(c_1 l^n (2k+2l) + c_2 r_2^n m),$$

where $m = v_2 v \in \mathbb{Z}$. Assume that $K \neq 0$ and $\frac{c_1}{c_2} \notin \mathbb{Q}$. Then by Corollary 3.8, for sufficiently large n,

$$t_n = K(c_1 l^n (2k+2\ell) + c_2 r_2^n m) = \nu \in \mathbb{Z}$$

and

$$t_{n+1} = K(c_1 l^{n+1} (2k+2\ell) + c_1 r_2^{n+1} m) = \mu \in \mathbb{Z}.$$

So

$$\frac{t_{n+1}}{t_n} = \frac{c_1 l^{n+1}(2k+2\ell) + c_2 r_2^{n+1} m}{c_1 l^n (2k+2\ell) + c_2 r_2^n m} = \frac{\mu}{\nu} \in \mathbb{Q},$$

which implies

$$\frac{c_1}{c_2}(l^{n+1}(2k+2l) + r_2^{n+1} m) = \frac{\mu}{\nu}(\frac{c_1}{c_2} l^n (2k+2\ell) + r_2^n m)$$

and so

$$\frac{c_1}{c_2}(l^{n+1}(2k+2\ell) - \frac{\mu}{\nu} l^n (2k+2\ell)) = \frac{\mu}{\nu} r_2^n m - r_2^{n+1} m.$$

Let

$$l^{n+1}(2k+2\ell) - \frac{\mu}{\nu} l^n (2k+2\ell) = q_1 \in \mathbb{Q}$$

and

$$\frac{\mu}{\nu} r_2^n m - r_2^{n+1} m = q_2 \in \mathbb{Q}.$$

Then

$$\frac{c_1}{c_2} q_1 = q_2,$$

which implies

$$\frac{c_1}{c_2} = \frac{q_2}{q_1} \in \mathbb{Q},$$

which is a contradiction. Hence $K = 0$ and $\sigma_{pp} = \{0\}$. $\square$

Example 3.11. $\theta(0) = 0102,\ \theta(1) = 0201$ and $\theta(2) = 0101$.

Here the substitution θ is of type 0-0 with even length $l = 4$, height $h(\theta) = 1$, and thus $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$. The incidence matrix of the substitution is $M_\theta = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ with eigenvalues $4, -1$ and 0, and associated left eigenvectors $(1,1,1)$, $(-1,-1,4)$ and $(-1,1,1)$ respectively. By the above Proposition 3.36, we will have sensitivity to changes in tile lengths.
Note that by Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}\left[\frac{1}{4}\right]$.

(ii) The height of the substitution $h(\theta) = 2$.

Proposition 3.37. *If θ is a height two substitution of constant length l and incidence matrix M_θ with associated eigenvalues l, $r_1 = 0$ and $r_2 \in \mathbb{Z} - \{0\}$, then we will have insensitivity to changes in lengths and the point spectrum σ_{pp} is always a scalar multiple of $\frac{1}{2}\mathbb{Z}\left[\frac{1}{l}\right]$.*

Proof. As mentioned before, any substitution θ of height $h(\theta) = 2$ is of type $0-0$ with odd length l and $\mathcal{R} = \left\{ \begin{pmatrix} k+\ell \\ k \\ \ell \end{pmatrix} \mid k, \ell \in \mathbb{Z} \right\}$. Among the eigenvalues of the incidence matrix M_θ is $+1$ or -1 with associated left eigenvector $(-1,1,1)$. Without loss of generality, assume that $+1$ is an eigenvalue, let $v_1 = (1,1,1), v_2 = (-1,1,1)$ and v_3 be the left eigenvectors associated to eigenvalues $l, r_2 = 1$ and $r_1 = 0$ respectively.
Expressing the general length vector $L = (L_1, L_2, L_3)$ as a linear combination of the three left eigenvectors of M_θ, direct calculation shows that $c_1 \neq 0$ for L to have positive entries.
Now we see that for any $v \in \mathcal{R}$,

$$
\begin{aligned}
KLM^n v &= K(c_1(1,1,1)+c_2(-1,1,1)+c_3 v_3)M^n v \\
&= K(c_1 l^n(1,1,1)+c_2(-1,1,1)+0)v \\
&= K(c_1 l^n(1,1,1)+c_2(-1,1,1))v \\
&= Kc_1 l^n(2k+2\ell) \\
&= 2Kc_1 l^n(k+\ell).
\end{aligned}
$$

For sufficiently large n and for all $v \in \mathcal{R}$, the expression $2Kc_1 l^n(k+\ell)$ will converge to 0 (mod 1) if and only if K is of the form $\frac{1}{c_1}\,\frac{m}{2\cdot l^r}$ for some $m \in \mathbb{Z}, r \in \mathbb{N}$. Hence the point spectrum will be a scalar multiple of $\frac{1}{2}\,\mathbb{Z}\left[\frac{1}{l}\right]$ and we see that we have a very rigid structure in this case. Which means that we have insensitivity to changes in the tile lengths. $\square$

Example 3.12. $\theta(0) = 010,\ \theta(1) = 201$ and $\theta(2) = 102$.

This substitution θ is of type 0-0, has odd length l=3 and thus the height $h(\theta) = 2$. The incidence matrix is $M_\theta = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ with eigenvalues $3, 1$ and 0, and associated left eigenvectors $(1,1,1), (-1,1,1)$ and $(1,-2,1)$. By the previous Proposition 3.37, the point spectrum σ_{pp} is a scalar multiple of $\frac{1}{2}\mathbb{Z}[\frac{1}{3}]$ and thus we will have insensitivity to changes in tile lengths.

Case III. If $q(x)$ has 0 as a double root, that is, $r_1 = r_2 = 0$.

The subcase (ii), where the substitution has height equal to two, simply does not exist because in such case the incidence matrix M_θ will always have ± 1 as an eigenvalue. Thus we only have the following subcase:

(i) The height of the substitution $h(\theta) = 1$.

Proposition 3.38. *Let θ be a substitution of constant length l and incidence matrix M_θ with eigenvalues $l, 0$ and 0, with associated left eigenvectors $(1,1,1), v_2$ and v_3. Then the point spectrum σ_{pp} is always a scalar multiple of $\mathbb{Z}\left[\frac{1}{l}\right]$.*

Proof. Expressing the general length vector $L = (L_1, L_2, L_3)$ as a linear combination of the three generalized left eigenvectors $(1,1,1), v_2$ and v_3 of M_θ associated to the eigenvalues, we see that for any $v \in \mathcal{R}$

$$\begin{aligned} KLM^n v &= K(c_1(1,1,1) + c_2 v_2 + c_3 v_3) M^n v \\ &= K(c_1 l^n (1,1,1) + 0 + 0) v \\ &= K c_1 l^n (1,1,1) v. \end{aligned}$$

We know that K is in σ_{pp} if and only if $KLM^n v \to 0 \pmod 1$ as $n \to \infty$. Since σ_{pp} is countable (as can be seen from Theorem 3.1 in [33]), then we cannot have $c_1 = 0$ since that would lead to an uncountable σ_{pp}.
Let $(1,1,1)v = m_v^{'} \in \mathbb{Z}$ and $\gamma = \gcd\{m_v^{'} \mid m_v^{'} > 0\}$. Then for sufficiently large n and for all $v \in \mathcal{R}$, $KLM^n v \to 0 \pmod 1$ if and only if K is of the form $\frac{1}{c_1} \frac{1}{\gamma} \frac{m}{l^r}$ for some $m \in \mathbb{Z}, r \in \mathbb{N}$. Thus the point spectrum σ_{pp} is always a scalar multiple of $\mathbb{Z}\left[\frac{1}{l}\right]$ and so we have a very rigid structure. □

Remark 3.39.

1. *As can be noticed, the argument is independent of the nature of $\mathcal{R}$.*
2. *This proposition also applies to Example 3.3.*

Example 3.13. $\theta(0) = 0102$, $\theta(1) = 0201$ and $\theta(2) = 0102$.

Here the substitution θ is a height one substitution of type 0-0 with even length l=4 and incidence matrix $M_\theta = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with eigenvalues $4, 0$ and 0, and associated left eigenvectors $(1,1,1), (-1,1,1)$ and $(1,\ 0,-2)$. By applying Proposition 3.38, we see that the point spectrum will be a scalar multiple of $\mathbb{Z}\left[\frac{1}{4}\right]$ and so we will have complete insensitivity to changes in tile lengths.

Example 3.14. $\theta(0) = 012$, $\theta(1) = 201$ and $\theta(2) = 120$.

The incidence matrix is $M_\theta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with eigenvalues $3, 0$ and 0, and associated left eigenvectors $(1,1,1), (-1,0,1)$ and $(-1,1,0)$. The substitution has height $h(\theta) = 1$

since we have two successive 0's. Indeed, we have also two successive 1's and 2's, thus $\mathcal{R} = \mathbb{Z}^3$ by Proposition 3.15. By Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}\left[\frac{1}{3}\right]$, and by Proposition 3.38, the point spectrum will be a scalar multiple of $\mathbb{Z}\left[\frac{1}{3}\right]$ and so we will have complete insensitivity to changes in tile lengths.

Case IV. If the roots r_1 and r_2 of $q(x)$ are such that $|r_1| > 1$ and $0 < |r_2| < 1$.

We will give some definitions and theorems that we shall need.

Definition 3.40. A real number $\lambda > 1$ is called a *Pisot number* if and only if it is an algebraic integer and all its Galois conjugates (other than λ) are of modulus less than 1.

Pisot numbers have the following characterization:
$\lambda > 1$ is a Pisot number if and only if $\lambda^n \to 0 \pmod 1$ as $n \to \infty$.

The following characterization of Pisot numbers can be found in [11].

Theorem 3.41. *Suppose that $\lambda > 1$ is an algebraic number (over the field of rational numbers $\mathbb{Q}$). The following are equivalent*

(i) λ is a Pisot number
(ii) There exists non-zero real x such that

$$\lim_{n\to\infty} \lambda^n x = 0 \pmod 1.$$

Moreover, any x satisfying (ii) belongs to $\mathbb{Q}[\lambda]$, the field extension of $\mathbb{Q}$ by λ.

Let the set X_λ be defined by

$$X_\lambda = \left\{ x \in \mathbb{R} \mid \lim_{n\to\infty} \lambda^n x = 0 \pmod 1 \right\},$$

where λ is a Pisot number. The following theorem gives a characterization of the set X_λ.

Theorem 3.42. *[23] Suppose $\lambda > 1$ is Pisot. Let p' be the derivative of the monic irreducible polynomial of λ over $\mathbb{Z}$, and $\mathbb{Z}[\lambda]^* = \frac{1}{p'(\lambda)}\mathbb{Z}[\lambda]$. Then $x \in X_\lambda$ if and only if $\lambda^n x \in \mathbb{Z}[\lambda]^*$ for some $n \geq 0$. $\mathbb{Z}[\lambda]^*$ is called the dual lattice of λ.*

Now we are going to examine the nature of the point spectrum when $q(x)$ has a Pisot root. Although we cannot handle the general case, we are able to give conclusions in some special cases.

Note that if $q(x)$ has a root r such that $0 < |r| < 1$, then it must be quadratic irrational (by the rational root theorem) and the other root $\bar{r}$, which is the conjugate of r, satisfies $|\bar{r}| > 1$ since $r\bar{r} = c \in \mathbb{Z} - \{0\}$, where c is the constant term in $q(x) = x^2 + bx + c$.
In this case, we cannot have a height two substitution, because $+1$ or -1 occurs as a root of $q(x)$. Also, we cannot have a height one substitution of type 0-0, because in this situation 0 occurs as a root of $q(x)$. Hence the only case that we are going to examine is the following case:

(i) The height of the substitution $h(\theta) = 1$.

Proposition 3.43. *Let θ be a substitution of constant length l such that its incidence matrix M_θ has eigenvalues l, r_1 and r_2 with $|r_1| > 1$ and $0 < |r_2| < 1$, that is, r_1 or $-r_1$ is a Pisot number. Expressing the general length vector $L = (L_1, L_2, L_3)$ by*

$$L = s(1,1,1) + c_1 v_1 + c_2 v_2,$$

where $(1,1,1)$, v_1 and v_2 be the associated left eigenvectors, then we have the following conclusions for σ_{pp} of the substitution tiling space $\mathcal{T}_\theta$:

1. *if $c_1 = 0$, then $\sigma_{pp} = d\,\mathbb{Z}\left[\frac{1}{l}\right]$ $for\ some$ $d \in \mathbb{R} - \{0\}$,*
2. *if $c_2 = 0$ and $\frac{s}{c_1} \notin \mathbb{Q}[r_1]$, then $\sigma_{pp} = \{0\}$,*
3. *if $s = 0$, then $\sigma_{pp} \subseteq \frac{1}{c_1}\mathbb{Q}[r_1]$ and contains a scalar multiple of $\mathbb{Z}[r_1]$.*

Thus we have sensitivity to changes in tile lengths.

Proof. For any $v \in \mathcal{R}$, we have

$$KLM^n v = K(sl^n(1,1,1) + c_1 r_1^n v_1 + c_2 r_2^n v_2)v.$$

Here the entries of v_i's, $i = 1,2$, are no longer rational numbers typically. It can be shown that these entries can be chosen to be in the quadratic field $\mathbb{Q}[r_1]$.

1. Let $c_1 = 0$, then we can conclude that $s \neq 0$ by an argument similar to the one given in Proposition 3.38. Then for sufficiently large n,
   $KLM^n v = K(sl^n(1,1,1) + c_2 r_2^n v_2)v \to Ksl^n(1,1,1)v$, (because $|r_2| < 1$, then $r_2^n \to 0$ as $n \to \infty$).
   Let $(1,1,1)v = m_v' \in \mathbb{Z}$ and $\gamma = \gcd\{m_v' \mid m_v' > 0\}$. Then $K \in \sigma_{pp}$ if and only if for

sufficiently large n and for all $v \in \mathcal{R}$, $Ksl^n(1,1,1)v \to 0(mod 1)$ if and only if K is of the form $\frac{1}{s\gamma}\frac{m}{l^r}$ for some $m \in \mathbb{Z}, r \in \mathbb{N}_0$, that is $K \in \frac{1}{s\gamma}\mathbb{Z}[\frac{1}{l}]$. Therefore, the point spectrum is a scalar multiple of $\mathbb{Z}[\frac{1}{l}]$.

2. Let $c_2 = 0$ and $\frac{s}{c_1} \notin \mathbb{Q}[r_1]$, then we conclude that $c_1 \neq 0$ because technically speaking $\frac{s}{c_1}$ does not make any sense if $c_1 = 0$. We will prove in this special case that we will have trivial point spectrum. Assume that $K \neq 0$. Since we do not have any contribution from the eigenvector with associated small eigenvalue in this case, we may apply the same arguments as in the proof of Corollary 3.8 to conclude that for sufficiently large n,

$$t_n = K(sl^n(1,1,1)v + c_1 r_1^n v_1 v) = \nu \in \mathbb{Z}$$

and

$$t_{n+1} = K(sl^{n+1}(1,1,1)v + c_1 r_1^n v_1 v) = \mu \in \mathbb{Z}.$$

So

$$\frac{t_{n+1}}{t_n} = \frac{sl^{n+1}(1,1,1)v + c_1 r_1^{n+1} v_1 v}{sl^n(1,1,1)r + c_1 r_1^n v_1 v} = \frac{\mu}{\nu} \in \mathbb{Q},$$

which implies

$$sl^{n+1}(1,1,1)v + c_1 r_1^{n+1} v_1 v = \frac{\mu}{\nu}(sl^n(1,1,1)v + c_1 r_1^n v_1 v).$$

Note that $(1,1,1)v \in \mathbb{Z}$ and $v_1 v \in \mathbb{Q}[r_1]$. Let $(1,1,1)v = d_1$ and $v_1 v = d_2$, then

$$\frac{s}{c_1}(l^{n+1}d_1 - \frac{\mu}{\nu} l^n d_1) = \frac{\mu}{\nu} r_1^n d_2 - r_1^{n+1} d_2.$$

Now

$$l^{n+1}d_1 - \frac{\mu}{\nu} l^n d_1 = q_1 \in \mathbb{Q}[r_1]$$

and

$$\frac{\mu}{\nu} r_1^n d_2 - r_1^{n+1} d_2 = q_2 \in \mathbb{Q}[r_1].$$

Thus

$$\frac{s}{c_1} q_1 = q_2.$$

So $\frac{s}{c_1} = \frac{q_2}{q_1} \in \mathbb{Q}[r_1]$ which is a contradiction. Hence $K = 0$ and $\sigma_{pp} = \{0\}$.

3. Let $s = 0$, then by a similar reasoning to 1. $c_1 \neq 0$. For $v \in \mathcal{R}$,

$$KLM^n v = K(c_1 r_1^n v_1 + c_2 r_2^n v_2)v.$$

For sufficiently large n,
$$KLM^n v \to Kc_1 r_1^n v_1 v$$
but
$$Kc_1 r_1^n v_1 v \to 0 (mod 1)$$
if and only if
$$Kc_1 v_1 v \in X_{r_1}$$
if and only if $\exists\ n_0 \in \mathbb{N}_0$ such that
$$r_1^{n_0} Kc_1 v_1 v \in \mathbb{Z}[r_1]^*$$
if and only if
$$r_1^{n_0} Kc_1 v_1 v = (a_1 + b_1 r_1) \frac{1}{q'(r_1)},$$
where $q'(r_1)$ is the derivative of the monic irreducible polynomial of r_1 over $\mathbb{Z}$

if and only if
$$K = \frac{(a_1 + b_1 r_1)}{q'(r_1) v_1 v c_1 r_1^{n_0}},$$
where a_1 , $b_1 \in \mathbb{Z}$. That is, $K \in \frac{1}{c_1}\mathbb{Q}[r_1]$ and hence $\sigma_{pp} \subseteq \frac{1}{c_1}\mathbb{Q}[r_1]$.

Now we will show that $\sigma_{pp} \neq \{0\}$. As mentioned before, for K to be in σ_{pp} we must show that there exists $n_0 \in \mathbb{N}_0$ such that for all $v \in \mathcal{R}$
$$r_1^{n_0} Kc_1 v_1 v\, q'(r_1) \in \mathbb{Z}[r_1].$$

It can be demonstrated that there exists a rational number $\rho \in \mathbb{Q} - \{0\}$ such that all the entries of v_1 are in $\rho\mathbb{Z}[r_1]$ which implies that for all $v \in \mathcal{R}$, $v_1 v \in \rho\mathbb{Z}[r_1]$. Therefore,
$$r_1^{n_0} Kc_1 v_1 v\, q'(r_1) \in \rho K c_1 \mathbb{Z}[r_1].$$

If $K \in \frac{1}{c_1\rho}\mathbb{Z}[r_1]$, then $r_1^{n_0} Kc_1 v_1 v\, q'(r_1) \in \mathbb{Z}[r_1]$ and hence $\frac{1}{c_1\rho}\mathbb{Z}[r_1] \subseteq \sigma pp$. □

Remark 3.44.

1. *Examining the point spectrum σ_{pp} in some special cases shows that it is sensitive to changes in tile lengths. Nevertheless, we do not have techniques to handle the general case when $sc_1c_2 \neq 0$.*

2. *Note that some of the special cases may not exist in the sense that it will depend on the precise nature of the v_i's, when $i = 1,2$. For example, if v_1 has some positive and some negative entries, and if we say s and c_2 equal to 0, then no value of c_1 will make a legal length vector. So there are some examples where some special cases actually do not occur.*

3. *We do not know any example with Pisot root when $\mathcal{R} \neq \mathbb{Z}^3$.*

Example 3.15. $\theta(0) = 011010121,\ \theta(1) = 010101212$ and $\theta(2) = 122020212$.

Here the substitution has length $l = 9$, its incidence matrix is $M_\theta = \begin{pmatrix} 3 & 3 & 2 \\ 5 & 4 & 2 \\ 1 & 2 & 5 \end{pmatrix}$ with eigenvalues 9, $r_1 = \frac{1}{2}(3+\sqrt{13})$ and $r_2 = \frac{1}{2}(3-\sqrt{13})$ with associated left eigenvectors $(1,1,1)$, $v_1 = (\frac{1}{12}(-47+11\sqrt{13}), \frac{1}{6}(13-4\sqrt{13}), 1)$ and $v_2 = (\frac{1}{12}(-47-11\sqrt{13}), \frac{1}{6}(13+4\sqrt{13}), 1)$. As can be noticed, the root r_1 is a unit Pisot root.

Since we have two successive 1's the height $h(\theta) = 1$ and so by Theorem 2.62, the constant $\mathfrak{c} = 1$ suspension flow will have eigenvalues $\mathbb{Z}[\frac{1}{9}]$. Successive 1's and 2's occur in the θ-subshift, therefore $\mathcal{R} = \mathbb{Z}^3$.

Expressing the length vector $L = (L_1, L_2, L_3)$ by $L = s(1,1,1) + c_1v_1 + c_2v_2$ and applying Proposition 3.43, we see that:

1. if $c_1 = 0$, then σ_{pp} will be a scalar multiple of $\mathbb{Z}\left[\frac{1}{9}\right]$,

2. if $c_2 = 0$ *and* $\frac{s}{c_1} \notin \mathbb{Q}[r_1]$, then $\sigma_{pp} = 0$,

3. if $s = 0$, then $\sigma_{pp} \subseteq \frac{1}{c_1}\mathbb{Q}[r_1]$. For this particular $v_1, v_1 \in \frac{1}{6}\mathbb{Z}[r_1]$, which implies that for all $v \in \mathcal{R}$,
$$v_1 v \in \frac{1}{6}\mathbb{Z}[r_1].$$
Therefore
$$r_1^{n_0} K c_1 v_1 v\, q'(r_1) \in \frac{1}{6} K c_1 \mathbb{Z}[r_1].$$
If $K \in \frac{6}{c_1}\mathbb{Z}[r_1]$, then
$$r_1^{n_0} K c_1 v_1 v\, q'(r_1) \in \mathbb{Z}[r_1]$$
and hence $\frac{6}{c_1}\mathbb{Z}[r_1] \subseteq \sigma_{pp}$.

Example	l	$\mathbf{h}(\theta)$	$\mathcal{R}$	**Full recurrence vector**	$\mathbf{r_1}$	$\mathbf{r_2}$	σ_{pp}
3.6	4	1	$\mathbb{Z}^3$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\frac{1+7i}{2}$	$\frac{1-7i}{2}$	typical$\{0\}$
3.7	4	1	$\mathbb{Z}^3$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	-1	-2	typical$\{0\}$
3.8	7	1	$\mathbb{Z}^3$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	1	1	typical $\{0\}$
3.9	5	2	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell\in\mathbb{Z}\right\}$	none	1	1	typical $\{0\}$
3.10	3	1	$\mathbb{Z}^3$	$\begin{pmatrix}1\\2\\0\end{pmatrix}$	1	0	typical $\{0\}$
3.11	4	1	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell\in\mathbb{Z}\right\}$	none	0	1	typical $\{0\}$
3.12	3	2	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell\in\mathbb{Z}\right\}$	none	0	1	scalar multiple of $\frac{1}{2}\mathbb{Z}[\frac{1}{3}]$
3.4	6	1	$\left\{\begin{pmatrix}m\\n\\n\end{pmatrix} \mid m,n\in\mathbb{Z}\right\}$	none	0	0	scalar multiple of $Z[\frac{1}{6}]$
3.13	4	1	$\left\{\begin{pmatrix}k+\ell\\k\\\ell\end{pmatrix} \mid k,\ell\in\mathbb{Z}\right\}$	none	0	0	scalar multiple of $Z[\frac{1}{3}]$
3.14	4	1	$\mathbb{Z}^3$	none	0	0	scalar multiple of $Z[\frac{1}{4}]$
3.15	9	1	$\mathbb{Z}^3$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\frac{1}{2}(3+\sqrt{13})$	$\frac{1}{2}(3-\sqrt{13})$	sensitive

TABLE 3.2: Classification of point spectrum σ_{pp}.

Chapter 4

Fixed Point Theorems and Tiling Problems

In this chapter we consider a different type of tiling problem related to tilings of the integers. We show that a general fixed point conjecture (see Conjecture II below) is true for $\mathbb{J} = \{1,2\}$ and also give a tiling proof of our result. The result of this chapter will appear in [1].

4.1 Fixed points conjectures

The well-known Banach contraction principle states that every contraction from a complete metric space into itself has a unique fixed point. It has played a fundamental role in various areas of pure and applied sciences. During the last 50 years, it has been generalized and extended in many ways by a number of authors. In [17] the following interesting conjecture, connected with Banach's fixed point theorem, was considered.

Conjecture I. Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{d(f^n(x), f^n(y)) \mid n \in \mathbb{J}\} \leq Kd(x,y) \tag{4.1}$$

for all $x, y \in X$ and some $K \in (0,1)$, where $\mathbb{J}$ is a subset of positive integers. Then f has a fixed point.

Remark 4.1.

1. *The condition (4.1) does not imply the continuity of f.*

2. *If $\mathbb{J} = \{1\}$ in the condition (4.1), then f is a contraction on X.*

3. *If f^k is a contraction, then the condition (4.1) holds.*

We also note that the case $\mathbb{J} = \{1\}$ corresponds to the Banach contraction principle and the case $\mathbb{J} = \{k\}$, where $k \in \mathbb{N}$, to a result in [6]. Conjecture I is not true when $\mathbb{J} = \mathbb{N}$, as shown in [17] and [31].

Example 4.1. *[31]* Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define a mapping $f : X \to X$ by

$$f(x) = \sqrt{x^2 + 1}$$

for all $x \in X$. Then $f^n(x) = \sqrt{x^2 + n}$ for all $x \in X$, and for all $x, y \in X$ with $x < y$, we can find $K \in (0, 1)$ such that

$$\inf\{|f^n(x) - f^n(y) \mid n \in \mathbb{N}\} \leq K|x - y|.$$

However, it is clear that f has no fixed points.

Definition 4.2. Let $f : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$. We say that f is

1. *α-admissible* [30] if $x, y \in X$ and $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$;

2. *triangular α-admissible* [19] if

   (a) $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$, $x, y \in X$;

   (b) $\begin{cases} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{cases}$ implies $\alpha(x, z) \geq 1$, $x, y, z \in X$.

In this chapter, we consider the following generalization:

Conjecture II. Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{\alpha(x,y)d(f^m(x), f^m(y)) \mid m \in \mathbb{J}\} \leq Kd(x,y) \tag{4.2}$$

for all $x,y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$, where $\mathbb{J}$ is a subset of positive integers. Then f has a fixed point.

Example 4.2. Let $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} \sqrt{x^2+1} & \text{if } x \geq 0 \\ 2x & \text{otherwise,} \end{cases}$$

and $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^n(x) = \sqrt{x^2+n}$ for $x \in [0,\infty)$ and $f^n(x) = 2^n x$ otherwise. Note that for all $x,y \in X$, we can find $K \in (0,1)$ such that

$$\inf\{\alpha(x,y)|f^n(x) - f^n(y) \mid n \in \mathbb{N}\} \leq K|x-y|.$$

So f satisfies (4.2). However, f does not satisfy (4.1). To see this, let $x = -1$ and $y = 0$, then

$$\inf\{|f^n(-1) - f^n(0) \mid n \in \mathbb{N}\} = \inf\{2^n + \sqrt{n} \mid n \in \mathbb{N}\} > K|-1-0|$$

for all $K \in [0,1)$. It is clear that f has no fixed points.

Taking $\alpha(x,y) = 1$ for all $x,y \in X$, it follows that if Conjecture I holds then Conjecture II holds as well. We note that Conjecture II is not true for infinite $\mathbb{J}$. One is led to conjecture whether Conjectures I and II are true if $\mathbb{J}$ is finite. In [31], Stein established Conjecture I for the class of strongly continuous mappings and $\mathbb{J} = \{1,2,..,n\}$. In [17], the authors showed that Conjecture I is true if $\mathbb{J} = \{1,2\}$ without any additional assumption on f. In this chapter we show that Conjecture II is true for $\mathbb{J} = \{1,2\}$. We also give a tiling proof of our result.

4.2 Fixed point theorems

Theorem 4.3. *Let (X,d) be a metric space and let $f : X \to X$ satisfy the following condition:*

$$\inf\{\alpha(x,y)d(f^m(x),f^m(y)) : m \in \mathbb{N}\} \leq Kd(x,y)$$

for all $x,y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$. Suppose that
(i) f is $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and $m \in \mathbb{N}$ such that $f^m(x_0) = x_0$.
Then x_0 is a fixed point of f.

Proof. Let $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and $f^m(x_0) = x_0$. Define the sequence (x_i) in X by $x_{i+1} = f(x_i)$ for $i \in \mathbb{N} \cup \{0\}$. Then it follows from $\alpha-$admissibility of f that $\alpha(x_0,x_1) \geq 1$ which implies $\alpha(x_1,x_2) = \alpha(f(x_0), f(x_1)) \geq 1$ and thus, by induction, $\alpha(x_i,x_{i+1}) \geq 1$ for all i. Choose L such that $K < L < 1$. Now for each $i \in \{0,1,\ldots,m-1\}$, there is $m_i \in \mathbb{N}$ such that

$$\alpha(x_i,x_{i+1})d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq Ld(x_i,x_{i+1})$$

and so

$$d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq \alpha(x_i,x_{i+1})d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq Ld(x_i,x_{i+1}).$$

Since $f^m(x_0) = x_0$, following arguments as in Lemma 1 of [17], we can find a sequence (k_i) in $\{0,1,\ldots,m-1\}$ such that

$$d(f^{k_i}(x_0), f^{k_i+1}(x_0)) \leq Ld(x_{k_{i-1}}, x_{k_{i-1}+1}).$$

Since $k_i \in \{0,1,\ldots,m-1\}$, we can find i and j in $\mathbb{N}$ such that $k_{i+j} = k_i$. Thus

$$\begin{aligned} d(f^{k_i}(x_0), f^{k_i+1}(x_0)) &= d(f^{k_{i+j}}(x_0), f^{k_{i+j}+1}(x_0)) \\ &\leq L^j d(x_{k_i}, x_{k_{i+1}}) \\ &= L^j d(f^{k_i}(x_0), f^{k_i+1}(x_0)). \end{aligned}$$

Since $L < 1$, we have $d(f^{k_i}(x_0), f^{k_i+1}(x_0)) = 0$ and so $f^{k_i}(x_0) = f^{k_i+1}(x_0) = f(f^{k_i}(x_0))$. That is, $f^{k_i}(x_0)$ is a fixed point of f. But $m - k_i > 0$ and $f^{m-k_i}(f^{k_i}(x_0)) = f^{k_i}(x_0)$, that is, $f^{k_i}(x_0)$ is a fixed point of f^{m-k_i}. This implies that $f^m(x_0) = f^{k_i}(x_0)$. But $f^m(x_0) = x_0$.

Therefore $f^m(x_0) = f^{k_i}(x_0) = x_0$. Hence x_0 is a fixed point of f. $\square$

Theorem 4.4. *Let (X,d) be a metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{\alpha(x,y)d(f^m(x), f^m(y)) : m \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x,y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$, where $\mathbb{J}$ is a finite subset of positive integers. Suppose that
(i) f is triangular $\alpha-$admissible;
(ii) there exists $x,z \in X$ such that $\alpha(x, f(x)) \geq 1$, $\alpha(z, f(z)) \geq 1$, $\alpha(z,x) \geq 1$ and for any $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that $d(z, f^{i+N}(x)) < \varepsilon$ for any $i \in \{0\} \cup \mathbb{J}$.
Then f has a fixed point.

Proof. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{1+K}$. Choose $N = N(\delta)$ as mentioned in the hypothesis such that $d(z, f^{i+N}(x)) < \delta$ for any $i \in \{0\} \cup \mathbb{J}$. By (i) and (ii), $\alpha(x, f^N(x)) \geq 1$ and $\alpha(z,x) \geq 1$ and so $\alpha(z, f^N(x)) \geq 1$. Also there exists $m \in \mathbb{J}$ such that

$$\alpha(z, f^N(x))d(f^m(z), f^m(f^N(x))) \leq Kd(z, f^N(x)).$$

and so

$$\begin{aligned} d(f^m(z), f^m(f^N(x))) &\leq \alpha(z, f^N(x))d(f^m(z), f^m(f^N(x))) \\ &\leq Kd(z, f^N(x)) < K\delta. \end{aligned}$$

As a result, we have

$$\begin{aligned} d(z, f^m(z)) &\leq d(z, f^{m+N}(x)) + d(f^{m+N}(x), f^m(z)) \\ &\leq \delta + K\delta = \varepsilon. \end{aligned}$$

Since $\mathbb{J}$ is finite, there exists $m \in \mathbb{J}$ such that $f^m(z) = z$. By Theorem 4.3 , z is a fixed point of f. $\square$

Theorem 4.5. *Let (X,d) be a complete metric space and let $f : X \to X$ be a continuous mapping satisfying*

$$\inf\{\alpha(x,y)d(f^m(x), f^m(y)) \mid m \in \mathbb{N}\} \leq Kd(x,y)$$

for all $x, y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$. Suppose that
(i) f is $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and an increasing sequence (k_i) of integers such that
(a) for all $i \in \mathbb{N}$, $d(f^{k_i}(x_0), f^{k_{i-1}}(x_0)) \leq CL^{k_{i-1}}$ for some $0 < L < 1$ and $C > 0$;
(b) there is a positive integer m such that $k_i - k_{i-1} = m$ for infinitely many i.
Then f has a fixed point.

Proof. It follows from (a) that the sequence $(f^{k_i}(x_0))$ is Cauchy and so converges to x (say) by the completeness of X. The continuity of f further implies that the limit $\lim_{i\to\infty} f^m(f^{k_i}(x_0))$ exists. By virtue of (b), there is a subsequence (i_n) such that $f^m(f^{k_{i_n}}(x_0)) = f^{k_{i_n+1}}(x_0)$. Thus $(f^m(f^{k_i}(x_0)))$ and $(f^{k_i}(x_0))$ have a common subsequence and so have the same limits. As a result, we have

$$f^m(x) = f^m(\lim_{i\to\infty} f^{k_i}(x_0)) = \lim_{i\to\infty} f^m(f^{k_i}(x_0)) = \lim_{i\to\infty} f^{k_i}(x_0) = x.$$

Hence f has a periodic point and the result now follows from Theorem 4.3. □

4.3 On a tiling problem and tiling proof of a fixed point theorem

Let (X,d) be a complete metric space, let $x_0 \in X$, and let $f : X \to X$ be triangular α-admissible with $\alpha(x_0, f(x_0)) \geq 1$ and satisfy the following condition:

$$\min\{\alpha(x,y)d(f^n(x), f^n(y)) \mid n \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x, y \in X$ and some $K \in (0,1)$, where $\mathbb{J} = \{1,2,...,N\}$.

Let $x(q, q+k)$ denote a tile in one dimension of length k that starts from q. Our aim is to have a usable bound for the term $d(f^q(x_0), f^{q+k}(x_0))$, which implies the sequence of iterates is Cauchy and its limit is the fixed point. The idea here is to be able to tile that segment of the line that goes from q to $q+k$ with $x(q, q+k)$, a tile that starts from q and is of length k. In order to obtain a collection of tiles whose metric analog is a Cauchy sequence, following [17] we have the following notion.

Definition 4.6. Given $x_0 \in X$, a set of tiles $\mathcal{E}$ is called a *good collection of tiles for* x_0 if and only if there exist $C > 0$ and $0 < L < 1$ such that for all tiles $x(q, q+k)$ in $\mathcal{E}$,

$$d(f^q(x_0), f^{q+k}(x_0)) \leq CL^q.$$

Our tiling problem affects fixed point theorems and consists of an initial good collection of tiles, set of rules which enable us to enlarge the collection, and a goal showing that the good collection can be enlarged according to rules such that it includes a pre-determined sub-collection of tiles. For instance, our objective is to enlarge the original good collection such that it contains all but finitely many adjacent tiles of the same length. If a tile of length of 4 starts from 5 and covers 5 – 6, 6 – 7, 7 – 8, 8 – 9, then the next adjacent tiles of length 4 starts at 9 and covers 9 – 10, 10 – 11, 11 – 12, 12 – 13. Assuming that the good collection of tiles consists of adjacent tiles of length 4, starting at 5, which cover all but a finite portion of the real line corresponds to showing that the sequence $(f^5(x_0), f^9(x_0), f^{13}(x_0), \ldots)$ is Cauchy and thus converges. If f satisfies the assumption of results of previous section, then f has a periodic point and thus has a fixed point. We present some rules defined in [17] which lead to a tiling proof of our fixed point theorem.

Rule 1. *Suppose that we have a good collection $\mathcal{E}$ of tiles. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If $x(q, q+k)$ lies in $\mathcal{E}$, then at least one of the tiles $x(q+1, q+k+1), x(q+2, q+k+2), \ldots, x(q+N, q+k+N)$ lies in $\mathcal{E}'$.*

Proof. If $x(q, q+k)$ lies in $\mathcal{E}$, then $d(f^q(x_0), f^{q+k}(x_0)) \leq CL^q$. Since $\alpha(x_0, f(x_0)) \geq 1$, f is triangular α-admissible, $\alpha(f(x_0), f^2(x_0)) \geq 1$ and so $\alpha(x_0, f^2(x_0)) \geq 1$. By induction, we have $\alpha(x_0, f^k(x_0)) \geq 1$ for all $k \in \mathbb{N}$.

Since f is α-admissible, we have, by induction, $\alpha(f^q(x_0), f^{q+k}(x_0)) \geq 1$. By the assumption on f, there exists $j_1 \in \{1, \ldots, N\}$ such that

$$\alpha(f^q(x_0), f^{q+k}(x_0))d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) \leq Kd(f^q(x_0), f^{q+k}(x_0))$$

and so

$$\begin{aligned} d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) &\leq \alpha(f^q(x_0), f^{q+k}(x_0))d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) \\ &\leq Kd(f^q(x_0), f^{q+k}(x_0)) \leq KCL^q. \end{aligned}$$

Continuing in this way, we can find a sequence $\{j_n \mid n = 1, 2, 3, \ldots\}$ such that $d(f^{q+j_n}(x_0), f^{q+k+j_n}(x_0)) \leq K^n CL^q$ and $1 \leq j_{n+1} - j_n \leq N$. Thus $n \leq j_n \leq nN$ which implies $n \geq$

$\frac{j_n}{N}$ and $K^n \le K^{\frac{j_n}{N}}$. So $d(f^{q+j_n}(x_0), f^{q+k+j_n}(x_0)) \le CR^{q+j_n}$, where $R = \max\{K^{\frac{1}{N}}, L\}$. Consequently, the collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all tiles of the from $x(q+i_n, q+i_n+k)$ with constants of the collections $\mathcal{E}'$, $C > 0$ and $0 < R < 1$. □

Rule 2. *Suppose that we have a good collection $\mathcal{E}$ of tiles. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If x_1 and x_2 are adjacent tiles in $\mathcal{E}$ with x_1 preceding x_2, then $\mathcal{E}'$ contains the tile that begins at the start of x_1 and ends at the end of x_2.*

Proof. If $x(q, q+k)$ lies in $\mathcal{E}$, then $d(f^q(x_0), f^{q+k}(x_0)) \le CL^q$. Suppose that $i < n < p$ and that $x_1 = x(i, i+n)$ and $x_2 = x(i+n, i+n+p)$. Then

$$\begin{aligned} d(f^i(x_0), f^{i+n+p}(x_0)) &\le d(f^i(x_0), f^{i+n}(x_0)) + d(f^{i+n}(x_0), f^{i+n+p}(x_0)) \\ &\le CL^i + CL^{i+n} = C(1+L^n)L^i \le 2CL^i. \end{aligned}$$

The collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all sum of two adjacent tiles in $\mathcal{E}$ with constants of the collections $\mathcal{E}'$, $2C > 0$ and $0 < L < 1$. □

Rule 3. *Suppose that we have a good collection $\mathcal{E}$ of tiles and that $q \in \mathbb{N}$ is fixed. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If $\mathcal{E}$ contains two tiles which either begin or end at the same point, and the longer tile is of length less than or equal to q, then $\mathcal{E}'$ contains the difference of the shorter and longer tiles.*

Proof. Suppose $i < n < p$. We consider the following two cases:

Case 1: If the tiles $x(i, i+p)$ and $x(i, i+n)$ belong to $\mathcal{E}$, then

$$\begin{aligned} d(f^{i+n}(x_0), f^{i+p}(x_0)) &\le d(f^{i+n}(x_0), f^i(x_0)) + d(f^i(x_0), f^{i+p}(x_0)) \\ &\le CL^i + CL^i = 2CL^i = \frac{2C}{L^{n-i}}L^n \le \frac{2C}{L^q}L^{i+n}. \end{aligned}$$

Here we assume that the longer tile is of length less than or equal to q.

Case 2: If the tiles $x(i, i+p)$ and $x(i+n, i+p)$ belong to $\mathcal{E}$, then

$$\begin{aligned} d(f^i(x_0), f^{i+n}(x_0)) &\le d(f^i(x_0), f^{i+p}(x_0)) + d(f^{i+p}(x_0), f^{i+n}(x_0)) \\ &\le CL^i + CL^{i+n} = C(1+L^n)L^i \le 2CL^i \le \frac{2C}{L^q}L^i. \end{aligned}$$

The collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all differences of tiles of length less than or equal to q in $\mathcal{E}$ which begin or end at the same point with constants of the collections $\mathcal{E}'$, $\frac{2C}{L^q} > 0$ and $0 < L < 1$. $\square$

Applying the above rules, we are able to prove the following fixed point theorem. Note that any finite collection of tiles is a good collection for any constant $L < 1$, by choosing the constant C sufficiently large.

Theorem 4.7. *Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{\alpha(x,y)d(f^m(x), f^m(y)) \mid m = 1,2\} \leq Kd(x,y)$$

for all $x,y \in X$, some $K \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$. Suppose that
(i) f is continuous and triangular $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$.
Then the sequence $(f^q(x_0))$ is Cauchy and f has a fixed point (which is the limit of the sequence).

Proof. We follow [17]. Let $\mathcal{E}_0$ be the good collection consisting of tiles $x(0,1)$ and $x(0,2)$. Apply Rule 1 to $\mathcal{E}_0$ to get a good collection $\mathcal{E}_1$. Observe that if $x(q,q+1)$, with $q \geq 1$, does not lie in $\mathcal{E}_1$, then both $x(q-1,q)$ and $x(q+1,q+2)$ lie in $\mathcal{E}_1$. Similarly, if $x(q,q+2)$, with $q \geq 1$, does not lie in $\mathcal{E}_1$, then both $x(q-1,q+1)$ and $x(q+1,q+3)$ lie in $\mathcal{E}_1$. Since all tiles in $\mathcal{E}_1$ are of length less than or equal to 2, we obtain a good collection $\mathcal{E}_2$ by applying Rule 3 to $\mathcal{E}_1$. We claim that $\mathcal{E}_2$ includes the tile $x(q,q+1)$ for $q \geq 2$. If $x(q,q+1)$ lies in $\mathcal{E}_1$, then we are done since $\mathcal{E}_1 \subset \mathcal{E}_2$. If $x(q,q+1)$ does not lie in $\mathcal{E}_1$, then both $x(q-1,q)$ and $x(q+1,q+2)$ lie in $\mathcal{E}_1$. If $x(q-1,q+1)$ lies in $\mathcal{E}_1$, then applying Rule 3 to tiles $x(q-1,q+1)$ and $x(q-1,q)$ both belonging to $\mathcal{E}_1$ to get the tile $x(q,q+1)$ lies in $\mathcal{E}_2$. If $x(q-1,q+1)$ does not lie in $\mathcal{E}_1$, then applying Rule 3 to tiles $x(q,q+2)$ and $x(q+1,q+2)$ both belonging to $\mathcal{E}_1$ to get the tile $x(q,q+2)$ lies in $\mathcal{E}_1$ and $x(q,q+1)$ lies in $\mathcal{E}_2$. This implies that $d(f^q(x_0), f^{q+1}(x_0)) \leq CL^q$ for $q \geq 2$. Suppose $p = q+k$ for $k \geq 1$. Then

$$\begin{aligned}
d(f^q(x_0), f^p(x_0)) &\leq d(f^q(x_0), f^{q+1}(x_0)) + d(f^{q+1}(x_0), f^{q+2}(x_0)) + \\
&\quad ... + d(f^{q+k-1}(x_0), f^{q+k}(x_0)) \\
&\leq CL^q + CL^{q+1} + ... + CL^{q+k-1} \\
&\leq \sum_{n=q}^{\infty} CL^n.
\end{aligned}$$

Since $0 < L < 1$, the sequence $(f^q(x_0))$ is Cauchy and so converges to $x \in X$. Since f is continuous, $(f^{q+1}(x_0))$ converges to $f(x)$.
Since $d(x, f(x)) = \lim_{n\to\infty} d(f^q(x_0), f^{q+1}(x_0)) = 0$, this implies that x is a fixed point of f. □

Taking $\alpha(x,y) = 1$ for all $x, y \in X$, we get the following corollary. Note that in this case we do not require the continuity of f instead we apply Theorem 4.4. Indeed, $\{f^q(x_0)\}$ is a Cauchy sequence as above and so converges to $x \in X$. Thus for any $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that $d(f^{i+N}(x_0), x) < \varepsilon$ for all $i \in \{0\} \cup \mathbb{J}$. So, by Theorem 4.4, f has a fixed point. For uniqueness, choose L such that $K < L < 1$. If $x = f(x)$ and $y = f(y)$ with $x \neq y$. Then there exist $m \in \{1,2\}$ such that $d(f^m(x), f^m(y)) \leq Ld(x,y)$. This implies $d(x,y) \leq Ld(x,y)$. This is a contradiction since $L < 1$.

Corollary 4.8. *[17] Let (X,d) be a complete metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{d(f^m(x), f^m(y)) \mid m = 1, 2\} \leq Kd(x,y)$$

for all $x, y \in X$ and some $K \in (0,1)$. Then f has a unique fixed point.

We end the chapter with the following problem.

Problem *Let (X,d) be a complete metric space, let $x_0 \in X$, and let $f : X \to X$ be α-admissible with $\alpha(x_0, f(x_0)) \geq 1$ and satisfy the following condition:*

$$\min\{\alpha(x,y)d(f^n(x), f^n(y)) \mid n \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x, y \in X$ and some $K \in (0,1)$, where $\mathbb{J} = \{1, 2, ..., N\}$. Does f have a fixed point?

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