Cohomology of Tiling Spaces: Beyond Primitive Substitutions

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ABSTRACT

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Beyond Primitive Substitutions**

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This thesis explores the combinatorial and topological properties of tiling spaces associated to 1-dimensional symbolic systems of aperiodic type and their associated algebraic invariants. We develop a framework for studying systems which are more general than primitive substitutions, naturally partitioned into two instances: in the first instance we allow systems associated to sequences of substitutions of primitive type from a finite family of substitutions (called mixed substitutions); in the second instance we concentrate on systems associated to a single substitution, but where we entirely remove the condition of primitivity.

We generalise the notion of a Barge-Diamond complex, in the one-dimensional case, to any mixed system of symbolic substitutions. This gives a way of describing the associated tiling space as an inverse limit of complexes. We give an effective method for calculating the Čech cohomology of the tiling space via an exact sequence relating the associated sequence of substitution matrices and certain subcomplexes appearing in the approximants. As an application, we show that there exists a system of substitutions on two letters which exhibit an uncountable collection of minimal tiling spaces with distinct isomorphism classes of Čech cohomology.

In considering non-primitive substitutions, we naturally divide this set of substitutions into two cases: the minimal substitutions and the non-minimal substitutions. We provide a detailed method for replacing any non-primitive but minimal substitution with a topologically conjugate primitive substitution, and a more simple method for replacing the substitution with a primitive substitution whose tiling space is orbit equivalent. We show that an Anderson-Putnam complex with a collaring of some appropriately large radius suffices to provide a model of the tiling space as an inverse limit with a single map. We apply these methods to effectively calculate the Čech cohomology of any substitution which does not admit a periodic point in its subshift. Using its set of closed invariant subspaces, we provide a pair of invariants which are each strictly finer than the usual Čech cohomology for a substitution tiling space.

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DEDICATION

To the memories of David and Susan Rust. I owe all that I am to them.

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INTRODUCTION

The study of aperiodic tilings of the plane has a rich history which emerged from the worlds of computer science and mathematical logic when Berger proved the undecidability of the domino problem in the 1960s [13]. It is now also a topic of general interest to the study of dynamical systems, topology, Diophantine approximation, ergodic theory, computer graphics, mathematical physics and even virology [19, 56, 14, 4, 44, 12, 59]. Once it was discovered that sets of tiles exist which can only tile the plane aperiodically, a flurry of new discoveries quickly followed, culminating in the celebrated discovery of the famous *Penrose tilings* [51] and in the discovery of quasicrystals [57] for which Shechtman was awarded the Nobel Prize in Chemistry in 2011. Various methods for constructing aperiodic tilings have been developed including the *substitution*, *cut-and-project*, and *matching rule* methods. In this work we consider only the substitution method and other closely related methods.

The idea of a *substitution* is a natural one; whether on an alphabet comprising a finite set of symbols (in which case we call it a symbolic substitution), or on a finite set of translation-classes of tiles in $\mathbb{R}^n$ (in which case we call it a tiling substitution). We take a collection of objects and impose a rule which replaces each member of that collection with some non-empty configuration of copies of elements from the same collection. Under mild assumptions, given such a rule, we may iterate the process and so in the limit consider a substitution of an infinite union of elements of the collection which 'cover' the space in which they live (whether that be the integers $\mathbb{Z}$ in the symbolic case, or the real space $\mathbb{R}^n$ in the tiling case). For our purposes, we principally consider the symbolic setting, but keep the tiling setting in mind also, as the interplay between the two narratives is useful.

In the symbolic case in one dimension, there is then a notion of a bi-infinite 'limit' sequence of this iterated process. Such a limiting process is in general not unique, and so we are led to consider the entire collection of all bi-infinite sequences which are admitted by such a process of repeated substitution. Such a collection is referred to as the *subshift* of the substitution. The fundamental theme of symbolic dynamics is to study properties of the individual elements of the subshift by identifying corresponding dynamical properties of the subshift under the natural shift action and vice versa. There is then an associated *tiling space* to the substitution given by suspending the subshift and again, properties of this space reveal properties of the individual elements of the space (which may be viewed as tilings of the real line). The advantage to taking such a suspension and forming the tiling space is that the topology of the tiling space retains much of the dynamical information of the subshift, and so we can apply techniques from topology to study dynamical properties of the subshift. In particular, we can calculate invariants from algebraic topology of these spaces.

Mixed Substitutions

In some specialised settings, we understand the topology and dynamics of the associated tiling space and subshift very well. Given this, an observation which allows us to generalise this framework is that different substitutions acting on the same alphabet can be composed—this allows us to form an infinite sequence of prescribed compositions of different substitutions. The dynamical, topological and combinatorial properties of sequences defined by such *S-adic systems* have been well studied for many years (see [28] and the references within). They have been classically defined as those sequences which appear as a limit word of a sequence of substitutions $w = \lim_{n\to\infty}((\phi_1 \circ \cdots \circ \phi_n)(a))$ for some letter in a finite alphabet a, and the substitutions ϕ_i belonging to some finite family of substitutions S. In [36], Gähler and Maloney extended this class of sequences to an analogous class of tilings of $\mathbb{R}^n$ which generalise the now well-studied *substitution tilings*—these are referred to as *mixed substitution* or *multisubstitution* tiling systems. It is of general interest to be able to calculate the Čech cohomology and other invariants of tiling spaces associated to such aperiodic tilings (see [7, 9, 19, 32, 33, 35, 55, 60]) and so it seems a worthy goal to build machinery to do that in the mixed substitution setting.

Using techniques developed by authors such as Kellendonk [43], Anderson & Putnam [2], Gähler [34], and Barge & Sadun [10], a general method for calculating the Čech cohomology of tiling spaces associated to mixed substitution systems was developed by Gähler and Maloney. In particular, they showed with an example that the topology of the associated tiling spaces of mixed substitution systems can be dependent on the order in which the substitutions are applied, and not just on the family of substitutions being considered.

This example provided a set of 1-dimensional mixed substitution tiling spaces over a fixed collection of two substitutions, some of whose first Čech cohomology groups differ, and in fact have ranks varying depending on the choice of the sequence in which the substitutions are applied. This certainly hints that the family of mixed substitution tiling spaces has a richer structure than the classical case of tiling spaces associated to singular substitutions.

One naturally asks how much more wildly this class of spaces can behave. We already have some partial results. For instance it is well known that the Sturmian sequences can all be generated as limits of a system of two substitutions on two letters. It is also known that the tiling spaces associated to the Sturmian sequences w_α and w_β are homeomorphic if and only if the generating slopes α and β have continued fraction representations whose tails agree after some finite number of shifts [31, 11]. In particular, this tells us that there are an uncountable number of distinct homeomorphism-types of mixed substitution tiling spaces—in contrast with the case of singular substitutions. Unfortunately, these spaces cannot be distinguished by their Čech cohomology which are all isomorphic to the direct sum of two copies of the integers $\mathbb{Z}^2$.

The main theorem of Chapter 2 is an improvement on this result. We show, using a theorem of Goodearl and Rushing [38], that an uncountable family exists which can be distinguished by their cohomology. This has the (perhaps surprising) consequence that there exist tiling spaces Ω for which the first Čech cohomology group $\check{H}^1(\Omega)$ cannot be written in the form

$$A \oplus (\mathbb{Z}\,[1/n_1] \oplus \cdots \oplus \mathbb{Z}\,[1/n_k])$$

for some finitely generated abelian group A and natural numbers n_i, $1 \leq i \leq k$, because there are only countably many distinct isomorphism classes of such groups. Moreover, it appears that these pathological cohomology groups are in some sense typical. Nevertheless, almost every currently determined cohomology group of a tiling space is of the above form.

In order to prove this result, we leverage a construction by Barge and Diamond [7] of the so-called BD (Barge-Diamond) complex of a substitution. The BD-complex is a CW complex associated to a single substitution, built from combinatorial data and with the property that a suitably chosen continuous map on this complex, induced by the substitution, has an inverse limit which is homeomorphic to the tiling space. A cellular map can then be defined which is homotopic to this induced map, and which also acts simplicially on a particular subcomplex of the BD-complex and maps this subcomplex into itself. A relative cohomology approach can then be used to produce an exact sequence which allows for the straightforward computation of the Čech cohomology of the tiling space in terms of the cohomology of a simplicial complex, and the direct limit of the transpose of the transition matrix associated to the substitution. These constructions and results were later generalised to tilings of

$\mathbb{R}^n$ for all positive dimensions by Barge, Diamond, Hunton and Sadun in [9].

The original construction of these complexes, and the induced substitution maps between them, was only developed for single substitutions and so it is necessary to generalise their work to the case of mixed substitution systems. We essentially mirror the theory developed by Gähler and Maloney [36] for the Anderson-Putnam complex, but instead for the BD-complex, and only in one dimension—though it should be noted that an extension to higher dimensions would not be difficult, given [9].

Non-primitive Substitutions

Even in considering stationary systems of substitutions, where the sequence of substitutions is constant, our understanding of the topology of the associated tiling spaces is incomplete. In the literature, the view is often taken that the class of all possible substitutions, with all their various behaviours, is too large to consider as a whole. Instead it is almost always the case that one imposes a condition of primitivity (or less commonly of *irreducibility*) on the substitutions being studied. There does exist a sparse set of results that relaxes this standard assumption [15, 16, 17, 20, 48], however most concentrate on the measure theoretic and ergodic properties of the associated subshifts and tiling spaces, and very few results exist which probe the topology and dynamics of these spaces in the non-primitive setting.

In Chapter 3 the substitutions which we study fall into two distinct classes—those whose subshifts are minimal, in the sense that the orbit of every element of the subshift is dense under the shift action, and those whose subshifts are non-minimal, in which case there exists more than a single non-empty, closed, shift-invariant subspace of the subshift.

The method of using return words to recode a substitution was introduced by Durand in his Ph.D. Thesis and published in [23]. We provide a detailed method for replacing any non-primitive but minimal substitution with a primitive substitution whose subshift is topologically conjugate using the method of return words. Such a result has historically been considered a part of the folklore in substitutional dynamics since the work of Durand. However, there exist subtleties in the method which only become apparent with the full details, and the proof that the method provides a substitution with the necessary properties is surprisingly involved.

An intermediary step in the construction of the topologically conjugate primitive substitution is a primitive substitution whose tiling space is orbit equivalent to the original. Such a substitution is sufficient to study the topology of the associated tiling space, and is afforded the advantage of being much more simple to calculate than the conjugate substitution (and often acts on a much smaller alphabet). We apply these methods to provide examples which show that the topology of the tiling space of a minimal substitution is not necessarily as restricted by the size of its alphabet as in the case that a substitution is primitive.

In the second half of the chapter, we probe the topological structure of the tiling space associated to a substitution which does not admit a minimal subshift. We show that an Anderson-Putnam complex with a collaring of some appropriately large radius suffices to provide a simple inverse limit model of the tiling space in terms of a single bonding map induced by the substitution. We apply these methods to effectively calculate the Čech cohomology of any substitution which does not admit a periodic point in its subshift—no further condition on the substitution is assumed.

A structure theorem is proved for the tiling space in terms of its closed shift invariant subspaces. We show that for an aperiodic substitution, there exist only finitely many such closed invariant subspaces and provide a method for identifying each via a one-to-one correspondence with certain subcomplexes of the collared Anderson-Putnam complex. We use this result to then describe each such subspace as an inverse limit of the corresponding subcomplex under a map induced by the substitution, and so describe a method for calculating the Čech cohomology of each closed invariant subspace. We further apply these methods to allow for a calculation of the cohomology of the quotient of the tiling space by any of these subspaces. These cohomology groups of the quotient are identified algebraically with a direct limit of induced substitution maps acting on the cohomology of a corresponding quotient complex of the collared Anderson-Putnam complex.

Using its set of closed invariant subspaces, we provide a pair of invariants which are strictly finer than the usual Čech cohomology groups for a substitution tiling space. We show that these diagrams of cohomology groups are functorial in nature. We provide examples where these invariants can distinguish between substitutional tiling spaces whose Čech cohomology groups are isomorphic.

Grout

One difficulty in performing Čech cohomology calculations for substitutions beyond the most simple examples is that the complexes and matrices involved can become rather large, even in the case of one-dimension. General purpose software exists to lighten this burden somewhat, but the tools are incomplete, fragmented, user-unfriendly or else publicly unavailable. To this end, we describe in the final chapter of this thesis the implementation of a program developed by the author and a collaborator in an attempt to provide the community with an easy-to-use, freely available program for performing many of the calculations one is interested in for symbolic substitutions.

The program, Grout, is a GUI fronted program that can compute combinatorial properties and topological invariants of recognisable and primitive symbolic substitutions on finite alphabets and their associated tiling spaces. We review the necessary theory from the study of aperiodic 1-dimensional tilings and provide pseudocode highlighting the algorithms that we have implemented into the GUI. Grout is written using C++ and its standard library.

Grout is able to check some simple properties such as if a substitution is constant or primitive. We describe how Grout is able to determine the first n values of the word-complexity function of the subshift of a primitive substitution, as well as the implementation of an algorithm for estimating the natural scaling factor, tile lengths and tile frequencies of the geometric realisation of a primitive symbolic substitution. Grout is also able to check the recognisability of primitive substitutions in a deterministic manner. Our recognisability check appears to be the first complete implementation of such a check in the literature, and we describe in detail the methods used by Grout to perform such a task.

The primary function of Grout is a collection of methods for calculating the Čech cohomology $\check{H}^1$ of the tiling space associated to a primitive recognisable substitution. Grout implements three different methods for calculating the cohomology of tiling spaces associated to symbolic substitutions on finite alphabets.

1. The method of Barge-Diamond complexes as introduced in [7]
2. The method of Anderson-Putnam complexes as introduced in [2]
3. The method of forming an equivalent *left proper* substitution as outlined in [26]

All three outputs are algebraically equivalent—that is, they represent isomorphic groups—but it is not always obvious that this is the case given the presentations. This disparity between presentations of results for the equivalent methods was one of the major motivating factors for developing Grout. These cohomology groups are extremely laborious to calculate by hand unless special criteria are met.

It is hoped that the use of this program will make testing conjectures in tiling theory and symbolic substitutional dynamics more efficient, as well as allowing for the confirmation of hand calculations and comparison of different methods of calculation (especially methods of calculating cohomology). Analysis of large data sets which can be potentially generated by the Grout source code, and the recognition of underlying patterns in the data may also aid to further the theory.

The GUI front end for Grout is powered by Qt [1]. Grout has been designed with user experience in mind and includes many ease-of-use properties such as the ability to save and load examples, and convenient methods of sharing examples with other users via short strings that encode a substitution. There is also an option to export all of the data that has been calculated to a pre formatted LaTeX file including all the TikZ code for the considered complexes. This should be useful for those needing to typeset such diagrams in the future by fully automating the generation of diagrams in TikZ. Most of the drawn complexes presented in this document were generated by Grout, with only slight modifications.

1. PRELIMINARIES

Here we introduce the basic definitions and results which will be used throughout this work. For an introduction to the topological theory of aperiodic tilings, substitutions and their cohomology, we refer the reader to the books [56] and [4] where most of these definitions and conventions appear.

1.1 Substitutions and Subshifts

Let $\mathcal{A}$ be a finite alphabet and for natural numbers n, let $\mathcal{A}^n$ be the set of words of length n using symbols from $\mathcal{A}$. We denote the length of the word $u = u_1 \dots u_l$ by $|u| = l$. By convention, $\mathcal{A}^0 = \{\epsilon\}$ where ϵ is the empty word and $|\epsilon| = 0$. Denote the union of the positive-length words by $\mathcal{A}^+ = \bigcup_{n\geq 1} \mathcal{A}^n$. If the empty word ϵ is also included, then we denote the union $\mathcal{A}^+ \cup \{\epsilon\}$ by $\mathcal{A}^*$. This set $\mathcal{A}^*$ forms a free monoid under concatenation of words.

Definition 1.1.1. A *substitution* ϕ on $\mathcal{A}$ is a function $\phi\colon \mathcal{A} \to \mathcal{A}^+$. We can extend the substitution ϕ in a natural way to a morphism $\phi\colon \mathcal{A}^* \to \mathcal{A}^*$ given, for a word $u = u_1 \dots u_n \in \mathcal{A}^n$, by setting $\phi(u) = \phi(u_1) \dots \phi(u_n)$.

Example 1.1.2. Let $\mathcal{A} = \{a, b\}$ and let $\phi\colon \mathcal{A} \to \mathcal{A}^+$ be the substitution defined by $\phi\colon a \mapsto b$, $b \mapsto ba$. By iterating the substitution, we get an infinite sequence of words

$$a \mapsto b \mapsto ba \mapsto bab \mapsto babba \mapsto babbabab \mapsto babbababbabba \mapsto \cdots$$

This substitution is called the *Fibonacci substitution*.

The symbol w_i denotes the label assigned to the ith component of the bi-infinite sequence $w \in \mathcal{A}^{\mathbb{Z}}$. We may further extend the above definition of a substitution to bi-infinite sequences $\phi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. For a bi-infinite sequence $w \in \mathcal{A}^{\mathbb{Z}}$, with $w = \dots w_{-2}w_{-1} \cdot w_0w_1w_2 \dots$ we set

$$\phi(w) = \dots \phi(w_{-2})\phi(w_{-1}) \cdot \phi(w_0)\phi(w_1)\phi(w_2) \dots$$

with the dot $\cdot$ representing the separator of the (-1)st and 0th component of the respective sequences.

For a substitution $\phi\colon \mathcal{A} \to \mathcal{A}^+$ on an alphabet $\mathcal{A} = \{a^1, a^2, \dots, a^l\}$, there is an associated substitution matrix M_ϕ of dimension $l \times l$ given by setting m_{ij}, the i,j entry of M_ϕ, to be the number of times that the letter a^i appears in the word $\phi(a^j)$.

Definition 1.1.3. A substitution ϕ is called *primitive* if there exists a positive natural number p such that the matrix M_ϕ^p has strictly positive entries. Equivalently, if there exists a positive natural number p such that for all $a, a' \in \mathcal{A}$ the letter a' appears in the word $\phi^p(a)$.

Example 1.1.4. The Fibonacci substitution ϕ has substitution matrix $M_\phi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We see that $M_\phi^2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and so ϕ is primitive.

For words $u, v \in \mathcal{A}^*$, we write $u \subset v$ to mean u is a subword of v, and $u \subsetneq v$ to mean u is a proper subword of v. For a bi-infinite word $w \in \mathcal{A}^{\mathbb{Z}}$, we similarly write $u \subset w$ to mean u is a subword of w.

Definition 1.1.5. Let $\phi\colon \mathcal{A} \to \mathcal{A}^+$ be a substitution. We say a word $u \in \mathcal{A}^*$ is *admitted* by the substitution ϕ if there exists a letter $a \in \mathcal{A}$ and a natural number $k \geq 0$ such that $u \subset \phi^k(a)$ and denote by $\mathcal{L}^n \subset \mathcal{A}^n$ the set of all words of length n which are admitted by ϕ. Our convention is that the empty word ϵ is admitted by all substitutions. We form the *language* of ϕ by taking the set of all admitted words $\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}^n$.

We say a bi-infinite sequence $w \in \mathcal{A}^{\mathbb{Z}}$ is *admitted* by ϕ if every subword of w is admitted by ϕ and denote by X_ϕ the set of all bi-infinite sequences admitted by ϕ. The set X_ϕ has a natural (metric) topology inherited from the product topology on $\mathcal{A}^{\mathbb{Z}}$ and a natural shift map $\sigma\colon X_\phi \to X_\phi$ given by $\sigma(w)_i = w_{i+1}$. We call the pair (X_ϕ, σ) the *subshift* associated to ϕ and we will often abbreviate the pair to just X_ϕ when the context is clear.

Example 1.1.6. For the Fibonacci substitution ϕ, we have the length-2 admitted words being $\mathcal{L}^2 = \{ab, ba, bb\}$ as all three words appear as subwords of $\phi^3(b)$ and it is not hard to see that, as no new two letter words appear as subwords of $\phi^4(b)$ and ϕ is primitive, then the two-letter word aa will never appear as a subword of any word of the form $\phi^n(a^i)$ for $a^i \in \mathcal{A}$.

We note that the pointed word $b.b$ appears as a subword of the pointed word $\phi^2(b).\phi^2(b)$ and so we get a nested sequence of inclusions

$$b.b \subset \phi^2(b).\phi^2(b) \subset \phi^4(b).\phi^4(b) \subset \phi^6(b).\phi^6(b) \subset \dots$$

which has a bi-infinite limit word w that by construction is admitted by ϕ. It can be shown that X_ϕ is equal to the closure of $\{\sigma^n(w) \mid n \in \mathbb{Z}\}$ as a subspace of $\mathcal{A}^{\mathbb{Z}}$.

We say ϕ is a *periodic* substitution if X_ϕ is finite, and ϕ is *aperiodic* otherwise. We say ϕ is *strongly aperiodic* if X_ϕ contains no σ-periodic points (equivalently, X_ϕ contains no periodic closed invariant subspaces). If ϕ is aperiodic and primitive, then X_ϕ is strongly aperiodic and topologically a Cantor set (in particular X_ϕ is non-empty) and σ is a minimal action on X_ϕ - that is, the only closed shift-invariant subspaces of X_ϕ are the empty set $\emptyset$ and the subshift itself X_ϕ. Equivalently, the orbit of every point under σ is dense in X_ϕ. As every primitive substitution admits a sequence w such that under some power p we have $\phi^p(w) = w$, this means that an alternative definition of X_ϕ in the primitive case is $X_\phi = \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}\}}$ for some w fixed under a power of ϕ.

1.2 Tiling Spaces

Definition 1.2.1. Let ϕ be a substitution on the alphabet $\mathcal{A}$ with associated subshift X_ϕ. The *tiling space* associated to ϕ is the quotient space

$$\Omega_\phi = (X_\phi \times [0,1])/\sim$$

where $\sim$ is generated by the relation $(w, 0) \sim (\sigma(w), 1)$.

For a bi-infinite sequence $w = \ldots w_{-2}w_{-1} \cdot w_0w_1w_2 \ldots \in X_\phi$ and real number $t \in [0, 1)$, one should think of a point in Ω_ϕ as being a partition or tiling of $\mathbb{R}$ with unit-length intervals (called tiles) coloured by the symbols of $\mathcal{A}$. A point $T = (w, t) \in \Omega_\phi$ denotes the tiling with a unit w_0-tile at the origin, and where t describes the point in the w_0-tile over which the origin lies. There is then a unit w_{-1}-tile and a unit w_1-tile to the left and right respectively of the w_0-tile, and so on. Two tilings $T, T' \in \Omega_\phi$ are considered ϵ-*close* in this topology if, after a translate by a distance at most ϵ, the tiles around the origin in $T' - \epsilon$ within a ball of radius $1/\epsilon$ lie exactly on top of the tiles around the origin in T within a ball of the same radius and share the same labels.

If ϕ is primitive and aperiodic, then Ω_ϕ is a compact connected metric space which fibers over the circle with Cantor set fibers. The natural translation action $T \mapsto T+t$ for $t \in \mathbb{R}$ equips Ω_ϕ with a continuous $\mathbb{R}$ action which is minimal whenever ϕ is primitive. In this respect, tiling spaces are closely related to the more well-known spaces, the solenoids. To some degree, tiling spaces may be thought of informally as non-homogeneous solenoids. We note that there exist non-primitive substitutions with associated tiling spaces whose translation action is minimal, so primitivity is only a sufficient condition for minimality. This will be explored in Chapter 3.

Definition 1.2.2. Let $w = \ldots w_{-2}w_{-1} \cdot w_0w_1w_2 \ldots$ be a bi-infinite sequence in X_ϕ

and let $t \in [0, 1)$, so that (w, t) is an element of the tiling space Ω_ϕ. We define a map on the tiling spaces which we call $\phi\colon \Omega_\phi \to \Omega_\phi$, given by

$$\phi(w, t) = (\sigma^{\lfloor \tilde{t} \rfloor}(\phi(w)), \tilde{t} - \lfloor \tilde{t} \rfloor)$$

where $\tilde{t} = |\phi(w_0)| \cdot t$ and $\lfloor - \rfloor$ is the floor function.

This map is continuous. Intuitively, we take a unit tiling in Ω_ϕ with a prescribed origin and partition each tile of type a uniformly with respect to the substituted word $\phi(a)$ into tiles of length $\frac{1}{|\phi(a)|}$. We then expand each tile away from the origin so that each new tile is again of unit length, and with the origin lying proportionally above the tile it appears in after partitioning the original tiling.

Definition 1.2.3. A substitution ϕ is said to be *recognisable* if the map $\phi\colon \Omega_\phi \to \Omega_\phi$ is injective.

It is a result of Mossé [49] that a primitive substitution is aperiodic if and only if it is recognisable.

Definition 1.2.4. A substitution ϕ has the *unique composition property* if for any $w \in X_\phi$, there is a unique $w' \in X_\phi$ and $0 \leq n < |\phi(w'_0)|$ such that $\sigma^n(\phi(w')) = w$. Equivalently, there is a unique way of partitioning the symbols in $w = \dots w_{-1}w_0w_1\dots$ into words which are substituted letters (up to a small translation)

$$\dots \phi(w'_{-1})\phi(w'_0)\phi(w'_1)\dots = w$$

We note that recognisability of a substitution is equivalent to it having the unique composition property.

If ϕ is recognisable, then as the substitution map on the tiling space is always surjective, and Ω_ϕ is both compact and Hausdorff, the map $\phi\colon \Omega_\phi \to \Omega_\phi$ is a homeomorphism of the tiling space.

We note that Ω_ϕ is not a cell complex in the aperiodic case. For various reasons then, one wishes to model the tiling space in a more familiar way. For our purposes, that means writing Ω_ϕ as an inverse limit of cell complexes. A good reference text for the topology of inverse limits is Hatcher [41].

Definition 1.2.5. For $i \geq 0$, let $f_i\colon X_{i+1} \to X_i$ be a sequence of continuous maps between topological spaces X_i. The inverse limit $\varprojlim(X_i, f_i)$ is defined by

$$\varprojlim(X_i, f_i) = \{(x_0, x_1, \dots) \mid f_i(x_{i+1}) = x_i\}$$

equipped with the subspace topology inherited from the product topology on $\prod_{i=0}^{\infty} X_i$.

Let $\pi_i \colon \varprojlim(X_i, f_i) \to X_i$ be the projection map onto the ith coordinate. The inverse limit together with the sequence of maps π_i satisfies the universal property of a limit in the category of topological spaces Top. That is, if there exist maps $p_i \colon Y \to X_i$ with the property that $f_i \circ p_{i+1} = p_i$, then there exists a unique $\pi \colon \varprojlim(X_i, f_i) \to Y$ such that $p_i \circ \pi = \pi_i$.

There are several methods for constructing inverse limit models of Ω_ϕ in the literature. In this work we consider two methods. The first is attributed to Barge and Diamond [7], and the second is attributed to Anderson and Putnam [2].

1.2.1 The Barge-Diamond Complex

The first inverse limit model for a substitution tiling space which we will consider is built from the *Barge-Diamond complex* which was introduced by Barge and Diamond [7].

Definition 1.2.6. Let $\mathcal{A}$ be a finite alphabet with primitive substitution ϕ. Let $\epsilon = \min_{a \in \mathcal{A}} \left\{ \frac{1}{2|\phi(a)|} \right\}$ be a small positive real number. For $a \in \mathcal{A}$, let

$$e_a = [\epsilon, 1 - \epsilon] \times \{a\}$$

and for $ab \in \mathcal{L}^2$, let

$$e_{ab} = [-\epsilon, \epsilon] \times \{ab\}.$$

The *Barge-Diamond complex* for the substitution ϕ is denoted by K_ϕ and is defined to be

$$K_\phi = \left(\bigcup_{a \in \mathcal{A}} e_a \cup \bigcup_{ab \in \mathcal{L}^2} e_{ab} \right) \Big/ \sim$$

where for all $a, b \in \mathcal{A}$,

$$(1 - \epsilon, a) \sim (-\epsilon, ab) \quad \text{and} \quad (\epsilon, a) \sim (\epsilon, ba).$$

We also define the *subcomplex of vertex edges* S_ϕ of K_ϕ by

$$S_\phi = \bigcup_{ab \in \mathcal{L}^2} e_{ab} \Big/ \sim$$

where for all $a, b, c \in \mathcal{A}$,

$$(-\epsilon, ab) \sim (-\epsilon, ac) \quad \text{and} \quad (\epsilon, ba) \sim (\epsilon, ca).$$

The other edges e_a in K_ϕ are called *tile edges*.

Remark 1.2.7. This is a slight modification (in terms of geometry, not topology) of the usual Barge-Diamond complex for a unit-length substitution tiling, rather than for tilings with natural tile lengths which are not necessarily of unit-length, mainly in the choice of ϵ and lengths of edges. For our purposes, this definition is more convenient, especially later when substitutions may not be primitive, and when we consider systems of substitutions.

We also remark that the subcomplex S_ϕ need not be connected. Such an example was given by Barge and Diamond in [7] where $\mathcal{A} = \{a, b, c, d\}$ is an alphabet on four letters and ϕ is a single substitution given by

$$\phi\colon \begin{cases} a \mapsto abcda & c \mapsto cdbc \\ b \mapsto ab & d \mapsto db \end{cases}$$

which admits pairs $\mathcal{L}^2_\phi = \{aa, ab, ba, bc, da, db\} \cup \{cd\}$ where the vertex edge e_{cd} is disjoint from the rest of the vertex edges appearing in S_ϕ.

Example 1.2.8. As a simple example, we present the BD-complex for the Fibonacci substitution ϕ on the alphabet $\mathcal{A} = \{a, b\}$ given by

$$\phi\colon \begin{cases} a \mapsto b \\ b \mapsto ba \end{cases}$$

As mentioned previously, we have $\mathcal{L}^2 = \{ab, ba, bb\}$. So S_ϕ has all but one of the possible edges that can appear as a subcomplex of vertex edges for a substitution on two letters. See Figure 1.1 for the associated BD-complex. We have labelled the tile edges e_x and the vertex edges e_{xy} by their indices for better readability.

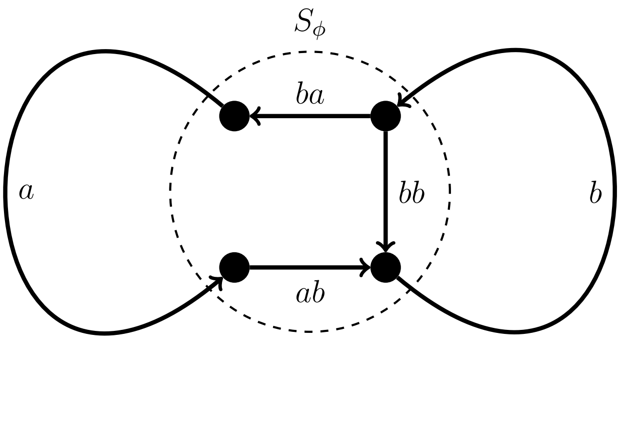


Fig. 1.1: The Barge-Diamond complex K_ϕ with the subcomplex of vertex cells S_ϕ circled accordingly.

Definition 1.2.9. Let ϕ be a substitution system and let (w, t) be a tiling in the tiling space Ω_ϕ with $t \in [0,1)$. Suppose $w = \ldots w_{-1} \cdot w_0 w_1 \ldots$. We define a surjective continuous map $p\colon \Omega_\phi \to K_\phi$ by

$$p(w,t) = \begin{cases} (t, w_{-1}w_0) & \text{if } t \in [0,\epsilon] \\ (t, w_0) & \text{if } t \in [\epsilon, 1-\epsilon] \\ (t-1, w_0w_1) & \text{if } t \in [1-\epsilon, 1) \end{cases}$$

There is a unique continuous map $f\colon K_\phi \to K_\phi$ such that $f \circ p = p \circ \phi$. This map is induced in the obvious way by the substitution on edges.

Remark 1.2.10. The map f is not cellular with respect to the tile and vertex edges of the BD-complex because the edges are, in general, expanded as they are substituted; for instance vertex edges e_{ab} are not just mapped onto some other vertex edge e_{cd}, but also overlap into the adjacent tile edges e_c and e_d. In [7], Barge and Diamond accounted for this by introducing a cellular map g which is homotopic to f. We do something similar in Chapter 2.

Theorem 1.2.11 (Barge-Diamond [7]). *For a primitive, recognisable substitution ϕ, there is a homeomorphism*

$$\Omega_\phi \cong \varprojlim (K_\phi, f)$$

between the mixed substitution tiling space and the inverse limit of the associated inverse system of induced substitution maps on the Barge-Diamond complexes.

1.2.2 The Anderson-Putnam Complex

The second inverse limit model which we consider for a substitution tiling space is built from the *Anderson-Putnam complex* which was introduced in Anderson and Putnam's seminal paper [2].

Definition 1.2.12. Let ϕ be a substitution on the alphabet $\mathcal{A}$. Let u be a word admitted by ϕ. We say that u uniquely extends in X_ϕ if there is a unique pair of letter $l, r \in \mathcal{A}$ such that the word lur is admitted by ϕ.

If, for every a such that $\phi^k(a)$ appears as a subword of X_ϕ, we have that $\phi^k(a)$ uniquely extends in X_ϕ, then we say ϕ *forces the border* at level k.

Example 1.2.13. The Fibonacci substitution does not force the border because $\phi^k(b)$ always begins with the letter b, and both of the words $a\phi^k(b)$ and $b\phi^k(b)$ are admitted by ϕ.

The substitution ϕ on the alphabet $\mathcal{A} = \{0, 1\}$ given by $\phi\colon 0 \mapsto 001,\ 1 \mapsto 01$ does force the border. This can be seen by noting that the words 000 and 11 are not admitted by ϕ. Hence, as $\phi^2(0) = 00100101$ and $\phi^2(1) = 00101$, these words can both only extend to the left by the letter 1 and to the right by the letter 0. It follows that ϕ forces the border at level 2.

By $l(u)$ and $r(u)$ we denote the leftmost and rightmost letters of the word u respectively.

Definition 1.2.14. Let ϕ be a substitution on the alphabet $\mathcal{A}$ and define a new alphabet $\mathcal{A}_1$ whose letters are letters $a \in \mathcal{A}$ but indexed by the possible three letter words admitted by ϕ whose central letter is a. So $\mathcal{A}_1 = \{a_u \mid a \in \mathcal{A}, u = a_l a a_r \in \mathcal{L}^3\}$.

We define the *collared substitution* ϕ_1 on $\mathcal{A}_1$ which is induced by ϕ in the following way. Suppose that $\phi(a) = b_1 \dots b_k$ and that $r(\phi(a_l)) = c$ and $l(\phi(a_r)) = d$. We define

$$\phi_1(a_{a_l a a_r}) = (b_1)_{cb_1b_2}(b_2)_{b_1b_2b_3} \dots (b_k)_{b_{k-1}b_k d}.$$

Example 1.2.15. Let ϕ be the Fibonacci substitution. Then $\mathcal{A}_1 = \{b_{aba}, b_{abb}, a_{bab}, b_{bba}\}$ and the collared substitution ϕ_1 is given on $\mathcal{A}_1$ by

$$\begin{array}{lcl} b_{aba} & \mapsto & b_{bba}a_{bab} \\ b_{abb} & \mapsto & b_{bba}a_{bab} \\ a_{bab} & \mapsto & b_{abb} \\ b_{bba} & \mapsto & b_{aba}a_{bab} \end{array}$$

or can be more neatly re-written as $\phi\colon 0 \mapsto 32,\ 1 \mapsto 32,\ 2 \mapsto 1,\ 3 \mapsto 02$.

For primitive substitutions ϕ, the collared substitution ϕ_1 always forces the border [2]. The forgetful map $X_{\phi_1} \to X_\phi$ on subshifts given by mapping $a_u \mapsto a$ is a topological conjugacy, and so the tiling space for ϕ_1 is also conjugate to the tiling space for ϕ.

Definition 1.2.16. Let ϕ be a substitution on the alphabet $\mathcal{A}$ and let Ω_ϕ be the associated tiling space. Use the convention that a point $T \in \Omega_s ub$ is written coordinatewise as (w, t), $w \in X_\phi$ and $t \in [0, 1)$. We define the *Anderson-Putnam complex* Γ of ϕ to be $\Omega/\!\sim$ where $\sim$ is the equivalence relation given by taking the transitive closure of the relation $(w, t) \sim (w', t')$ if $t = t' \in (0, 1)$ and $w_0 = w'_0$ or $t = t' = 0$ and $w_{-1} = w'_{-1}$ or $w_0 = w'_0$.

We define the *collared Anderson-Putnam complex* Γ_1 to be the Anderson-Putnam complex associated to the collared substitution ϕ_1.

Let $p\colon \Omega_\phi \to \Gamma$ be the natural quotient map. We define a map $f\colon \Gamma \to \Gamma$ to be the unique map which makes the following square commute

$$\begin{tikzcd}
\Omega_\phi & \Omega_\phi \arrow[l, "\phi"'] \arrow[d, "p"'] \\
\Gamma & \Gamma \arrow[l, "f"']
\arrow[from=1-1, to=2-1, "p"']
\end{tikzcd}$$

Let $p_1 \colon \Omega_{\phi_1} \to \Gamma_1$ be the natural quotient map. We similarly define a map $f_1 \colon \Gamma_1 \to \Gamma_1$ to be the unique map such that $p_1 \circ \phi_1 = f_1 \circ p_1$.

The space Γ has the homeomorphism type of a graph, where edges have a natural orientation and are labelled by letters of the alphabet. The end of an a edge meets the beginning of a b edge if the word ab is admitted by the substitution. The collared AP-complex Γ_1 can be thought of similarly as an oriented graph with edges labelled by collared letters in the alphabet $\mathcal{A}_1$. The maps f and f_1 are the obvious ones induced by substitution and collared substitution on the edges.

Theorem 1.2.17 ([2])**.** *Let ϕ be a primitive, recognisable substitution which forces the border. The natural map $h \colon \Omega_\phi \to \varprojlim(\Gamma, f)$ given by*

$$h(x) = (p(x), p(\phi^{-1}(x)), p(\phi^{-2}(x)), \ldots)$$

is a homeomorphism.

If we remove the condition that ϕ forces the border, then we need to use the collared complex Γ_1, as the collared substitution ϕ_1 always forces the border. As there exists a topological conjugacy $t \colon \Omega_\phi \to \Omega_{\phi_1}$ we get

Theorem 1.2.18. *Let ϕ be a primitive, recognisable substitution. The natural map $h \colon \Omega_{\phi_1} \to \varprojlim(\Gamma_1, f_1)$ given by*

$$h(x) = (p_1(x), p_1(\phi_1^{-1}(x)), p_1(\phi_1^{-2}(x)), \ldots)$$

is a homeomorphism. Hence $h \circ t \colon \Omega_\phi \to \varprojlim(\Gamma_1, f_1)$ is a homeomorphism.

1.2.3 Čech Cohomology

In order to talk about the Čech cohomology of a tiling space, we first need to introduce the important notion of a direct limit of groups [46].

Definition 1.2.19. For $i \geq 0$, let $h_i \colon G_i \to G_{i+1}$ be a sequence of homomorphisms between groups G_i and for $j \geq i$ let[1] $h_{ij} = h_j \circ h_{j-1} \circ \ldots \circ h_i$. The direct limit $\varinjlim(G_i, h_i)$ is defined by

$$\varinjlim(G_i, h_i) = \bigsqcup_{i \geq 0} G_i \Big/ \sim$$

where $g_i \in G_i$ and $g_j \in G_j$ are related by $\sim$ if and only if there exists $k \geq i, j$ such that $h_{ik}(g_i) = h_{jk}(g_j)$. The group operation on $\varinjlim(G_i, h_i)$ is defined by $[g_i][g_j] = [h_{ik}(g_i)h_{jk}(g_j)]$ where $k = \max\{i, j\}$.

It is an exercise to show that $\sim$ is an equivalence relation and that the group operation on $\varinjlim(G_i, h_i)$ is well-defined and satisfies the axioms of a group.

Let $\pi_i \colon G_i \to \varinjlim(G_i, h_i)$ be the map $g_i \mapsto [g_i]$ which is a homomorphism. The direct limit together with the sequence of maps π_i satisfies the universal property of a colimit in the category of groups Grp. That is, if there exist maps $p_i \colon G_i \to G$ with the property that $p_{i+1} \circ h_i = p_i$, then there exists a unique $\pi \colon \varinjlim(G_i, h_i) \to G$ such that $\pi \circ \pi_i = p_i$.

Important invariants of tiling spaces are the associated Čech cohomology groups. We refer the reader to [18] for an introduction to Čech cohomology. Although the definition of the Čech cohomology $\check{H}^\bullet(X)$ of a topological space X is rather involved, for our purposes we can make use of properties of the Čech cohomology which are suitable to perform calculations and develop the majority of the theory for tiling spaces. With this in mind, we mention the necessary results related to Čech cohomology which will be used throughout this work. We will always assume that singular and Čech cohomology is taken with integer coefficients.

Theorem 1.2.20. *Let X be a topological space. The Čech cohomology with integer coefficients $\check{H}^n$ in degree $n \geq 0$ is a contravariant functor $Top \to Ab$ from the category of topological spaces to the category of abelian groups satisfying the Eilenberg-Steenrod axioms[30, 58] and such that if (X_i, f_i) is a sequence of continuous maps $f_i \colon X_{i+1} \to X_i$ on compact Hausdorff topological spaces X_i, then*

$$\check{H}^n \varprojlim(X_i, f_i) \cong \varinjlim(\check{H}^n(X_i), f_i^*).$$

The isomorphism in the above theorem will be referred to as the *continuity* of Čech cohomology.

Let H^n be the singular cohomology with integer coefficients in degree n. We remark that as a consequence of the Eilenberg-Steenrod axioms, if X is a CW-complex, then $\check{H}^n(X)$ is naturally isomorphic to $H^n(X)$. So for CW-complexes, the Čech theory

[1] In the case $i = j$ then $h_{ii} = \mathrm{Id}_{G_i}$.

and the singular theory coincide. Coupled together with the continuity of Čech cohomology, we deduce a proposition [18] which will be used continually throughout this work.

Proposition 1.2.21. *Let X be a topological space and suppose that there exists a sequence f_i of continuous maps $f_i\colon X_{i+1} \to X_i$ on CW-complexes X_i such that $X \cong \varprojlim(X_i, f_i)$, then*

$$\check{H}^n(X) \cong \varinjlim(H^n(X_i), f_i^*).$$

Example 1.2.22. For each $i \geq 0$, let $X_i = S^1$ and $f_i\colon X_{i+1} \to X_i$ be the doubling map $\times 2\colon e^{ix} \mapsto e^{2ix}$ on the circle. The inverse limit $X = \varprojlim(S^1, \times 2)$ is well-known to be homeomorphic to the dyadic solenoid. We find that the induced map $\times 2^*$ on first degree cohomology of the circle acts as the doubling map on the integers. So by the above proposition we have $\check{H}^1(X) \cong \varinjlim(\mathbb{Z}, \times 2) \cong \mathbb{Z}[1/2]$ where $\mathbb{Z}[1/2]$ is the group of dyadic integers $\mathbb{Z}[1/2] = \{a2^n \mid a, n \in \mathbb{Z}\}$ under addition.

Example 1.2.23. From our previous example, we know that the BD-complex K_ϕ of the Fibonacci substitution ϕ is homotopy equivalent to the wedge of two circles, and it is not hard to check that the induced map in first degree cohomology of the map $f\colon K_\phi \to K_\phi$ acts like the matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on the generators $(1, 0)$ and $(0, 1)$ of the group $\mathbb{Z}^2 = H^1(K_\phi)$. As M is unimodular, its action on $\mathbb{Z}^2$ is an isomorphism, and we know that the tiling space associated to ϕ is homeomorphic to the inverse limit $\varprojlim(K_\phi, f)$, then by the above proposition, we have $\check{H}^1(\Omega_\phi) \cong \varinjlim(\mathbb{Z}^2, M) \cong \mathbb{Z}^2$.

2. MIXED SUBSTITUTIONS

The material of this chapter appears in [54], largely unchanged from how it is presented here.

This chapter is primarily concerned with 1-dimensional tiling systems of mixed substitution type. A mixed substitution system is a finite family of substitutions on a shared finite alphabet, together with an infinite sequence prescribing the order in which to apply substitutions from this family. We will explore the combinatorial and topological properties of such systems via their tiling spaces.

In Section 2.1, we introduce notation and basic definitions relating to the 3-adic numbers. We provide an overview of the Goodearl-Rushing result [38] and, to keep the work as self-contained as possible, a reproduction of the proof of this result. Briefly, the result shows that the set of direct limits over $\mathbb{Z}^2$ of arbitrary sequences of matrices of the form $\left(\begin{smallmatrix} 1 & i \\ 0 & 3 \end{smallmatrix}\right)$, $i \in \{0, 1, 2\}$ satisfies the property that only countably many sequences correspond to any particular isomorphism class of groups. Notation is also introduced which will be used in the eventual proof of Theorem 2.4.7.

In Section 2.2, we introduce the definitions and main results relating to mixed substitution systems and prove key properties of their associated tiling spaces.

In Section 2.3, we construct, for a mixed substitution system, a sequence of BD-complexes and induced maps between them. We show that the inverse limit of this sequence of maps is homeomorphic to the associated tiling space. We construct a sequence of homotopic cellular maps which can be used to effectively compute the first Čech cohomology of the tiling space via an exact sequence. It was remarked in [36] that it might be possible to construct a 'universal' BD-complex for a family of mixed substitutions which satisfy some suitable property termed *self-correction*. The universal BD-complex associated to a self-correcting mixed substitution system should satisfy the property that at each stage of the inverse limit representation of the tiling space, the approximant is the same, without affecting the cohomology of the limit. We provide one possible candidate for this property and show that such a universal BD-complex can be constructed which behaves well in this sense if the mixed substitution system is self-correcting according to this definition.

In Section 2.4, we use the Goodearl-Rushing result to define a family of mixed substitution systems we call the *mixed Chacon tilings*, with the properties necessary to prove the main result of this chapter.

Theorem 2.4.7. *There exists a family of minimal mixed substitution tiling spaces exhibiting an uncountable collection of distinct isomorphism classes of first Čech cohomology groups.*

2.1 Background Algebra and Analysis

2.1.1 The 3-adic Numbers

We briefly review the notation surrounding the 3-adic numbers and 3-adic integers, which will be used throughout the statement and proof of the Goodearl-Rushing result. We refer the reader to [39] for a gentle introduction to the theory of p-adic numbers. Let $\mathbb{Q}$ be the set of rational numbers on which we place the 3-*adic metric*. The metric is given by first defining the 3-*adic valuation* $v_3\colon \mathbb{Q} \to \mathbb{Q} \cup \{\infty\}$ and using this to define a 3-*adic absolute value* $|\cdot|_3\colon \mathbb{Q} \to \mathbb{R}$.

Definition 2.1.1. For a rational number x, if x is non-zero write it in the form $x = 3^n \frac{a}{b}$ where n, a, b are integers, with a, b not divisible by 3. The 3-*adic valuation* $v_3\colon \mathbb{Q} \to \mathbb{Q} \cup \{\infty\}$ is given on x by

$$v_3(x) = \begin{cases} \infty, & \text{if } x = 0 \\ n, & \text{if } x \neq 0 \end{cases}.$$

The 3-*adic absolute value* $|\cdot|_3\colon \mathbb{Q} \to \mathbb{R}$ on x is then given by

$$|x|_3 = 3^{-v_3(x)}$$

where $3^{-\infty}$ is defined to be 0.

Given a pair of rational numbers x and y, the 3-*adic metric* $d_3\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ is given by

$$d_3(x, y) = |x - y|_3$$

which is quickly verified to be a metric.

We can consider the completion of $\mathbb{Q}$ under the 3-adic metric, known as the *field of 3-adic numbers* and denoted $\mathbb{Q}_3$. The addition and multiplication operations on $\mathbb{Q}_3$ are inherited from $\mathbb{Q}$ using the property that these operations on $\mathbb{Q}$ are uniformly continuous with respect to the 3-adic metric, and so extend to the completion.

Let $\mathbb{Z}_3$ be the set of 3-adic integers, seen as a subset of $\mathbb{Q}_3$, given by the completion of the integers $\mathbb{Z}$ under the 3-adic metric. The set $\mathbb{Z}_3$ forms a subring of $\mathbb{Q}_3$. The elements of $\mathbb{Z}_3$ are represented by sequences of digits $\epsilon_k \in \{0,1,2\}$ where ϵ_k is the kth digit of the 3-adic integer α with unique standard expansion $\alpha = \sum_{k=0}^{\infty} \epsilon_k 3^k$. We denote the *$n$-partial expansion* of α by $\alpha_n = \sum_{k=0}^{n} \epsilon_k 3^k$.

2.1.2 Uncountability of Isomorphism Classes of Direct Limits

For the benefit of the reader, this section is a self-contained proof of the result that will be needed in Section 2.4. The proof was originally given in [38]. It is fairly technical, and the material may be safely skipped, with the exception of the statement of Theorem 2.1.3, and the definitions of the groups G_α and the equivalence relation $\sim$ on $\mathbb{Z}_3$.

The Goodearl-Rushing Method

Let $B_i = \begin{pmatrix} 1 & i \\ 0 & 3 \end{pmatrix}$ for $i \in \{0,1,2\}$. We will be considering direct systems consisting of sequences of these matrices. Let us fix a 3-adic integer $\alpha \in \mathbb{Z}_3$ which has digits ϵ_n for $n \geq 0$ and let $G_\alpha = \varinjlim(B_{\epsilon_n})$, the direct limit of the matrices B_{ϵ_n} acting on the group $\mathbb{Z}^2$.

Let V be a 2-dimensional vector space over $\mathbb{Q}$. As the matrices B_i are invertible over $\mathbb{Q}$, we can rewrite the above direct limit as a sequence of inclusions of rank-2 free abelian subgroups of V such that the direct limit is then a union of these subgroups. We wish to do this in such a way that we keep track of the generating elements of the subgroups.

Let $\{w_0, z_0\}$ be a $\mathbb{Q}$-basis of V and define $A_{\alpha,0} = \langle w_0, z_0 \rangle$ to be the free abelian subgroup of V generated by these elements. Set

$$\begin{aligned} w_n &= w_0 \\ z_n &= 3^{-n}(z_0 - \alpha_{n-1} w_0) \end{aligned} \tag{2.1}$$

and similarly define $A_{\alpha,n} = \langle w_n, z_n \rangle$. It can be shown using the fact that $\alpha_n - \alpha_{n-1} = 3^n \epsilon_n$ that

$$z_n = 3z_{n+1} + \epsilon_n w_{n+1} \tag{2.2}$$

and so in particular $A_{\alpha,n} \subset A_{\alpha,n+1}$. Let $i_n \colon A_{\alpha,n} \to A_{\alpha,n+1}$ be the inclusion map.

For fixed generators $a, b \in \mathbb{Z}^2$ define the group isomorphism $g_n \colon \mathbb{Z}^2 \to A_{\alpha,n}$ by $g_n(ka + lb) = kw_n + lz_n$. It is easy to see using the relation in (2.2) that $g_{n+1} \circ B_{\epsilon_n} =$

$i_n \circ g_n$, meaning the diagram

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, "B_{\epsilon_0}"] \arrow[d, "g_0"'] & \mathbb{Z}^2 \arrow[r, "B_{\epsilon_1}"] \arrow[d, "g_1"'] & \mathbb{Z}^2 \arrow[r, "B_{\epsilon_2}"] \arrow[d, "g_2"'] & \cdots \\ A_{\alpha,0} \arrow[r, "i_0"'] & A_{\alpha,1} \arrow[r, "i_1"'] & A_{\alpha,2} \arrow[r, "i_2"'] & \cdots \end{tikzcd}$$

commutes, and so we may conclude that

$$G_\alpha = \varinjlim(B_{\epsilon_n}) \cong \varinjlim(A_{\alpha,n}, i_n) \cong \bigcup_{n\in\mathbb{N}} A_{\alpha,n}.$$

We will write A_α to denote $\bigcup A_{\alpha,n}$.

Proposition 2.1.2. *For any $\alpha \in \mathbb{Z}_3$, the group A_α is not finitely generated.*

Proof. Consider the projection homomorphism $V \to \mathbb{Q}\colon w_0 \mapsto 0, z_0 \mapsto 1$. This projection restricted to the subgroup $\bigcup A_{\alpha,n}$ has image equal to the additive group of triadic integers $\mathbb{Z}[\frac{1}{3}] = \{a3^{-n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$ which is not finitely generated. It follows that A_α is infinitely generated. □

Determining Isomorphisms

We now ask the question, given $\alpha = \ldots\epsilon_2\epsilon_1\epsilon_0$ and $\alpha' = \ldots\epsilon'_2\epsilon'_1\epsilon'_0$, when precisely is A_α isomorphic to $A_{\alpha'}$?

Theorem 2.1.3 (Goodearl-Rushing)**.** *Let $\sim$ be the equivalence relation on $\mathbb{Z}_3$ given by $\alpha \sim \alpha'$ if and only if $G_\alpha \cong G_{\alpha'}$. The equivalence classes of $\sim$ are all countable.*

Proof. Let us suppose that $G_\alpha \cong G_{\alpha'}$ for a particular pair $\alpha, \alpha' \in \mathbb{Z}_3$, then $A_\alpha \cong A_{\alpha'}$, and let $\varphi\colon A_\alpha \to A_{\alpha'}$ be a group isomorphism. We note that $A_\alpha \otimes_{\mathbb{Z}} \mathbb{Q} \cong A_{\alpha'} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V$ and so in particular φ extends uniquely to an automorphism $\tilde{\varphi}\colon V \to V$ of vector spaces. Let us represent $\tilde{\varphi}$ by its associated ($\mathbb{Q}$-invertible) matrix $\frac{1}{t}\begin{pmatrix} r_w & r_z \\ s_w & s_z \end{pmatrix}$ with respect to the basis $\{w_0, z_0\}$ of V, for integers r_w, r_z, s_w, s_z, t.

We see that

$$\tilde{\varphi}(w_0) = \frac{r_w}{t}w_0 + \frac{s_w}{t}z_0 \tag{2.3}$$

$$\tilde{\varphi}(z_0) = \frac{r_z}{t}w_0 + \frac{s_z}{t}z_0 \tag{2.4}$$

and using (2.1) we get for all $n \geq 0$ that

$$\tilde{\varphi}(z_n) \quad = \quad 3^{-n}\left(\frac{r_z}{t} - \alpha_{n-1}\frac{r_w}{t}\right) w_0 + 3^{-n}\left(\frac{s_z}{t} - \alpha_{n-1}\frac{s_w}{t}\right) z_0. \tag{2.5}$$

Given that $\tilde{\varphi}(A_\alpha) = A_{\alpha'}$, there must exist a natural number $p \geq 0$ such that $\tilde{\varphi}(w_0) \in A_{\alpha',p}$ and $\tilde{\varphi}(z_0) \in A_{\alpha',p}$. Let $k(n)$ be the minimal natural number such that $\tilde{\varphi}(w_n)$ and $\tilde{\varphi}(z_n)$ are elements of $A_{\alpha',k(n)}$. Recalling (2.2) we can also see that $\tilde{\varphi}(w_n)$ and $\tilde{\varphi}(z_n)$ are in $A_{\alpha',k(n+1)}$ and so in particular $k(n) \leq k(n+1)$ and so $(k(i))_{i\geq 0}$ is a non-decreasing sequence.

Supposing $(k(i))_{i\geq 0}$ was bounded by some natural number k, we would have that $\tilde{\varphi}(w_n)$ and $\tilde{\varphi}(z_n)$ are in $A_{\alpha',k}$ for all $n \geq 0$ and so $\tilde{\varphi}(A_\alpha)$ is a subgroup of $A_{\alpha',k}$. But note, $A_{\alpha',k}$ is finitely generated, and as $\tilde{\varphi}$ is a bijection, this would mean a finitely generated abelian group contained an infinitely generated subgroup. This can clearly not be the case and so we conclude that $(k(i))_{i\geq 0}$ must be unbounded.

As we know $\tilde{\varphi}(z_n) \in A_{\alpha',k(n)}$, we then have integers a_n, b_n such that

$$\tilde{\varphi}(z_n) = a_n w'_{k(n)} + b_n z'_{k(n)}. \tag{2.6}$$

Suppose b_n is divisible by 3 and so $b_n = 3c_n$ for some integer c_n, then we find by substituting for w'_n and z'_n using (2.2) that

$$\tilde{\varphi}(z_n) = a_n w'_{k(n)-1} + c_n(z'_{k(n)-1} - \epsilon'_{k(n)-1} w'_{k(n)-1})$$

which would imply that $\tilde{\varphi}(z_n) \in A_{\alpha',k(n)-1}$, contradicting the minimality of $k(n)$. It follows that b_n is not divisible by 3.

Comparing (2.5) and (2.6) we find that

$$3^{-n}\left(\frac{r_z}{t} - \alpha_{n-1}\frac{r_w}{t}\right) w_0 + 3^{-n}\left(\frac{s_z}{t} - \alpha_{n-1}\frac{s_w}{t}\right) z_0 = a_n w'_{k(n)} + b_n z'_{k(n)}$$

which, after substituting for $w'_{k(n)}$ and $z'_{k(n)}$ using (2.1), becomes

$$3^{-n}\left(\frac{r_z}{t} - \alpha_{n-1}\frac{r_w}{t}\right) w_0 + 3^{-n}\left(\frac{s_z}{t} - \alpha_{n-1}\frac{s_w}{t}\right) z_0 = a_n w_0 + b_n 3^{-k(n)}(z_0 - \alpha'_{k(n)-1} w_0).$$

Picking out w_0 and z_0 components gives us

$$\begin{aligned} 3^{-n}(r_z - \alpha_{n-1} r_w)/t &= 3^{-k(n)}(3^{k(n)} a_n - \alpha'_{k(n)-1} b_n) \\ 3^{-n}(s_z - \alpha_{n-1} s_w)/t &= 3^{-k(n)} b_n. \end{aligned} \tag{2.7}$$

Cross-multiplying the equations in (2.7) gives

$$(r_z - \alpha_{n-1} r_w) b_n = (s_z - \alpha_{n-1} s_w)(3^{k(n)} a_n - \alpha'_{k(n)-1} b_n).$$

As it is not a multiple of 3 we can divide through by b_n to give

$$r_z - \alpha_{n-1} r_w = (s_z - \alpha_{n-1} s_w)(3^{k(n)}(a_n/b_n) - \alpha'_{k(n)-1}). \tag{2.8}$$

We can now take the limit of the sequence of equations given in (2.8) as $n \to \infty$ in the 3-adic metric on $\mathbb{Q}_3$. Without loss of generality suppose $a_n \neq 0$ and write $a_n = 3^{q(n)} d_n$ for an integer d_n not divisible by 3 and $q(n) \geq 0$ some natural number. Then it is clear that

$$|3^{k(n)} a_n/b_n|_3 = |3^{k(n)+q(n)} d_n/b_n|_3 = 3^{-(k(n)+q(n))}$$

and so, because $k(n)$ is an unbounded, non-decreasing sequence of natural numbers, as n tends to ∞, this valuation must tend to 0. So

$$\lim_{n \to \infty} d_3(3^{k(n)} a_n/b_n, 0) = 0.$$

It follows that $\lim\limits_{n \to \infty} 3^{k(n)} a_n/b_n = 0$.

Taking the limit of (2.8) then tells us that

$$\begin{aligned} r_z - \alpha r_w &= (\alpha s_w - s_z)\alpha' \\ \implies \qquad \alpha' &= \frac{r_z - \alpha r_w}{\alpha s_w - s_z} \end{aligned}$$

and so we may finally conclude that for any given $\alpha \in \mathbb{Z}_3$, there are at most a countable number of distinct 3-adic integers $\alpha' \in \mathbb{Z}_3$ such that $G_\alpha \cong G_{\alpha'}$, given by varying r_w, r_z, s_w, s_z over the integers in the above equation. It follows that the $\sim$-equivalence classes on $\mathbb{Z}_3$ are all countable. □

Remark 2.1.4. One should note that the above numerical relationship between α and α' is only a necessary condition. One should also note that it is a more general relation than just α and α' being 'tail equivalent' in their 3-adic expansions - tail equivalence corresponds to the case $s_w = 0, s_z = -1, r_w = 3^k$ for some non-negative k (possibly with the roles of α and α' switched depending on where their tails coincide).

2.2 Mixed Substitutions

2.2.1 Definitions

We now shift our attention to one-dimensional mixed substitution tilings. We would like to be able to compute the Čech cohomology of the inverse limits of certain 1-dimensional CW complexes in the style of Barge-Diamond [7], under maps induced by sequences of symbolic substitutions. In order to do this, we need to develop the theory of mixed substitution tiling spaces in this setting. Much of this theory, with regard to the Anderson-Putnam complexes [2], was originally formulated by Gähler and Maloney [36], and our notation will be closely based on theirs.

Let $F = \{\phi_0, \phi_1, \ldots, \phi_k\}$ be a finite set of substitutions on $\mathcal{A}$. Consider an infinite sequence $s = (s_0, s_1, s_2, \ldots) \in \{0, 1, \ldots, k\}^{\mathbb{N}}$. For a fixed alphabet $\mathcal{A}$, we call a pair (F, s) a *mixed substitution system.* For natural numbers $k \leq l$, let $M_{s[k,l]} = M_{s_k} M_{s_{k+1}} \cdots M_{s_{l-1}} M_{s_l}$ be the associated substitution matrix of the substitution $\phi_{s[k,l]} = \phi_{s_k}\phi_{s_{k+1}} \cdots \phi_{s_{l-1}}\phi_{s_l}$.

Definition 2.2.1. The mixed substitution system (F, s) is called *weakly primitive* if for all natural numbers n, there exists a positive natural number k such that the matrix $M_{s[n,n+k]}$ has strictly positive entries.

The mixed substitution system (F, s) is called *strongly primitive* if there exists a positive natural number k such that for all natural numbers n, the matrix $M_{s[n,n+k]}$ has strictly positive entries.

Remark 2.2.2. Strongly primitive substitution systems are referred to simply as *primitive* in [36] and *bounded primitive* in [50]. There are still other conventions for this notation [28]. We take the convention that whenever we say a system is primitive, we will mean weakly primitive in the sense defined above.

To a mixed substitution system (F, s) we will associate a topological space $\Omega_{F,s}$ called the *continuous hull* or *tiling space* of the mixed substitution system (F, s).

Definition 2.2.3. We say that a word $u \in \mathcal{A}^*$ is *admitted* by the mixed substitution system (F, s) if there exists a letter $a \in \mathcal{A}$ and a natural number $k \geq 0$ such that $u \subset \phi_{s[0,k]}(a)$ and denote by $\mathcal{L}^n_{F,s} \subset \mathcal{A}^n$ the set of all words of length n which are admitted by (F, s). We form the *language* of (F, s) by taking the set of all admitted words $\mathcal{L}_{F,s} = \bigcup_{n \geq 0} \mathcal{L}^n_{F,s}$.

We say a bi-infinite sequence $w \in \mathcal{A}^{\mathbb{Z}}$ is *admitted* by (F, s) if every subword of w is admitted by (F, s) and denote by $X_{F,s}$ the set of all bi-infinite sequences admitted by (F, s). The set $X_{F,s}$ has a natural (metric) topology inherited from the product topology on $\mathcal{A}^{\mathbb{Z}}$ and a natural shift map $\sigma\colon X_{F,s} \to X_{F,s}$ given by $\sigma(w)_i = w_{i+1}$. We call the pair $(X_{F,s}, \sigma)$ the *subshift* associated to (F, s) and we will often abbreviate the pair to just $X_{F,s}$ when the context is clear.

Remark 2.2.4. It is worth remarking that a pair of theorems of Durand provides a close link between the strength of primitivity of a mixed substitution system and the growth rate of its repetitivity function. We do not explain the terms in these statements but instead refer the reader to [24, 25].

Theorem 2.2.5 (Durand)**.** *A sequence is repetitive (equivalently its shift is uniformly recurrent) if and only if it is admitted by a weakly primitive mixed substitution system (with $|F|$ possibly infinite).*

Theorem 2.2.6 (Durand)**.** *A sequence is linearly repetitive (equivalently its shift is linearly recurrent) if and only if it is admitted by a strongly primitive and proper mixed substitution system.*

Definition 2.2.7. Let (F, s) be a substitution on the alphabet $\mathcal{A}$ with associated subshift $X_{F,s}$. The *tiling space* associated to (F, s) is the quotient space

$$\Omega_{F,s} = (X_{F,s} \times [0, 1])/\sim$$

where $(w, 0) \sim (\sigma(w), 1)$.

Remark 2.2.8. We note that the Gähler-Maloney approach [36] retains geometric data associated to tilings in the form of tile lengths and expanding factors of substitutions—this is especially important if one wishes to generalise to tilings in arbitrary dimensions, or study the natural dynamical and ergodic properties of the tiling space. However, this means that it is more difficult to handle systems with substitutions whose expanding factors do not exist, as in the case when the individual substitutions are not primitive, or whose length vectors (given by the left Perron-Frobenius eigenvector) do not coincide. By only considering the combinatorial data in dimension-one, we are able to handle these cases rather easily and, as will be seen, only require primitivity of the mixed substitution system, and not the individual substitutions—many of our examples fall into these cases, including the example used to prove Theorem 2.4.7 and the well-known pair of substitutions which generate the Sturmian sequences. We will look at a generalised example of the Sturmian substitutions in Example 2.3.13.

2.2.2 Properties of the Tiling Space

Proposition 2.2.9. *If (F, s) is weakly primitive, then $\Omega_{F,s}$ is non-empty.*

Proof. Without loss of generality, assume $|\mathcal{A}| > 1$. As (F, s) is weakly primitive, suppose $k(n)$ is such that $M_{s[n,n+k(n)]}$ has strictly positive entries for $n \geq 0$. Let $n_0 = n$, and $n_{i+1} = n_i + k(n_i) + 1$. This means that the matrix

$$M_{s[n_0,n_3-1]} = M_{s[n_0,n_0+k(n_0)]} M_{s[n_1,n_1+k(n_1)]} M_{s[n_2,n_2+k(n_2)]}$$

will have all entries at least as large as 3. From this we see that for some letter $a \in \mathcal{A}$ there must be a copy of the letter a appearing in the interior[1] of the word $\phi_{s[n_0,n_3-1]}(a)$, since the leftmost and rightmost letters account for at most two as. Let $n = 0$. By considering the sequence of words

$$\phi_{s[0,n_3-1]}(a), \phi_{s[0,n_6-1]}(a), \ldots, \phi_{s[0,n_{3i}-1]}(a), \ldots$$

we see that the ith word appears as a subword of the $(i+1)$st word. As a appears in the interior of $\phi_{s[0,n_3-1]}(a)$, this sequence of words expands in both directions, where here we assign an 'origin' to the interior element a. In the limit of this (not necessarily unique) sequence of increasing containments of words, we are left with a bi-infinite sequence which by construction is admitted by (F, s). It follows that $X_{F,s}$ is non-empty, and then so is $\Omega_{F,s}$. □

The next result is a consequence of the right to left implication of Theorem 2.2.5 but we provide an elementary proof for completeness.

Proposition 2.2.10. *If (F, s) is weakly primitive, then the translation action on the tiling space $\Omega_{F,s}$ is minimal.*

Proof. Let u be a word which is admitted by (F, s) and let $n \geq 0$ and $a \in \mathcal{A}$ be such that u appears as a substring of the word $\phi_{s[0,n]}(a)$. Let $w \in X_{F,s}$ be a limit of applying the sequence of substitutions $(\phi_{s[0,i]})_{i\geq 0}$ to the letter $a \in \mathcal{A}$. We will show that there are bounded gaps between subsequent occurrences of the word u in the bi-infinite sequence w.

As the letter a appears in $\phi_{s[0,k(0)]}(b)$ for all $b \in \mathcal{A}$, the word u must then appear as a subword of the words $\phi_{s[0,n+k(0)]}(b)$ for all $b \in \mathcal{A}$. Let L be the maximum of the lengths of the words $\phi_{s[0,n+k(0)]}(b)$ over all $b \in \mathcal{A}$. The bi-infinite sequence $w \in X_{F,s}$ can be decomposed into a concatenation of these words $\phi_{s[0,n+k(0)]}(b)$ and so it follows that the word u appears in w with gap at most $2L$.

It follows that for all bi-infinite sequences $w' \in X_{F,s}$, and for all $\epsilon > 0$, there exists a $k \geq 0$ such that $d(\sigma^k(w), w') < \epsilon$ and so w' belongs to the closure of the shift

[1] The interior of the word $a_1 a_2 \ldots a_{i-1} a_i$ is the word $a_2 \ldots a_{i-1}$.

orbit of w. So $(X_{F,s}, \sigma)$ is a minimal dynamical system. This readily implies that the translation action inherited by $X_{F,s}$ is minimal. $\square$

In particular, for a weakly primitive system (F, s) and for any bi-infinite sequence $w \in X_{F,s}$, we have $X_{F,s} = \overline{\{\sigma^k(w) \mid k \in \mathbb{Z}\}}$ and $\Omega_{F,s} = \overline{\{(\sigma^{\lfloor t \rfloor}(w), t - \lfloor t \rfloor) \mid t \in \mathbb{R}\}}$.

Definition 2.2.11. Let $w = \dots w_{-2}w_{-1} \cdot w_0w_1w_2 \dots$ be a bi-infinite sequence in $X_{F,\sigma^{i+1}(s)}$ and let $t \in [0, 1)$, so that (w, t) is an element of the tiling space $\Omega_{F,\sigma^{i+1}(s)}$. We define a map between tiling spaces which we call $\phi_{s_i} : \Omega_{F,\sigma^{i+1}(s)} \to \Omega_{F,\sigma^i(s)}$, given by

$$\phi_{s_i}(w, t) = (\sigma^{\lfloor \tilde{t} \rfloor}(\phi_{s_i}(w)), \tilde{t} - \lfloor \tilde{t} \rfloor)$$

where $\tilde{t} = |\phi_{s_i}(w_0)| \cdot t$, and where $|\phi_{s_i}(a)|$ is the length of the substituted word $\phi_{s_i}(a)$.

This map is continuous. Intuitively, we take a unit tiling in $\Omega_{F,\sigma^{i+1}(s)}$ with a prescribed origin and partition each tile of type a uniformly with respect to the substituted word $\phi_{s_i}(a)$ into tiles of length $\frac{1}{|\phi_{s_i}(a)|}$. We then expand each tile away from the origin so that each new tile is again of unit length, and with the origin lying proportionally above the tile it appears in after partitioning the original tiling.

Definition 2.2.12. A mixed substitution tiling system (F, s) is said to be *recognisable* if for every $i \geq 0$ the map $\phi_{s_i} : \Omega_{F,\sigma^{i+1}(s)} \to \Omega_{F,\sigma^i(s)}$ is injective.

Definition 2.2.13. A mixed substitution tiling system (F, s) has the *unique composition property* if for any $i \geq 0$ and $w \in X_{F,\sigma^i(s)}$, there is a unique $w' \in X_{F,\sigma^{i+1}(s)}$ and $0 \leq n < |\phi_{s_i}((w')_0)|$ such that $\sigma^n(\phi_{s_i}(w')) = w$. Equivalently, there is a unique way of partitioning the symbols in $w = \dots w_{-1}w_0w_1 \dots$ into words which are substituted letters (up to a small translation) $\dots \phi_{s_i}(w'_{-1})\phi_{s_i}(w'_0)\phi_{s_i}(w'_1) \dots = w$.

We note that recognisability of a substitution is equivalent to it having the unique composition property.

2.3 Inverse Limits and Cohomology

2.3.1 The Barge-Diamond Complex for Mixed Substitutions

Recall from Chapter 1 that in [7] Barge and Diamond introduced a cell complex for 1-dimensional substitution tilings which we refer to as the BD-complex. This was later extended by Barge, Diamond, Hunton and Sadun in [9] to arbitrary dimensions and allowed for symmetry groups beyond translations. Intuitively, we think of their complex as being the Anderson-Putnam complex of a collared version of the tiling, but where we collar points instead of tiles. These collared points then retain transition information between tiles and so induce border forcing—a term originally coined by Kellendonk [43] and utilised by Anderson and Putnam [2] in their seminal paper. Under a suitable choice of map on the BD-complex induced by the substitution, they produced an inverse system whose limit is homeomorphic to the relevant tiling space. The complex has the advantage of being more manageable than the Anderson-Putnam collared complex for the computation of cohomology groups, as well as giving conceptually insightful information about where the generators of cohomology are coming from with regard to the tiling.

The appearance of an exact sequence coming from considering the relative cohomology groups of their complex, and a certain subcomplex of 'vertex edges', allows for the cohomology of a substitution tiling space to be built from relatively easy to compute pieces—most notably for us, one of these pieces is the direct limit of the transpose of the original substitution matrix.

In order to apply their technique to our setting, we need to extend their method to more general sequences of substitutions, much like Gähler and Maloney did in [36] for the Anderson-Putnam complex in their treatment of mixed substitution systems.

Definition 2.3.1. Let $\mathcal{A}$ be a finite alphabet and let (F, s) be a primitive substitution system over $\mathcal{A}$. Let $\epsilon = \min\limits_{a \in \mathcal{A}, \phi \in F} \left\{ \dfrac{1}{2|\phi(a)|} \right\}$ be a small positive real number. For $a \in \mathcal{A}$, let

$$e_a = [\epsilon, 1-\epsilon] \times \{a\}$$

and for $ab \in \mathcal{L}^2_{F,s}$, let

$$e_{ab} = [-\epsilon, \epsilon] \times \{ab\}.$$

The *Barge-Diamond complex* for the mixed substitution system (F, s) is denoted by $K_{F,s}$ and is defined to be

$$K_{F,s} = \left(\bigcup_{a \in \mathcal{A}} e_a \cup \bigcup_{ab \in \mathcal{L}^2_{F,s}} e_{ab} \right) \Big/ \sim$$

where for all $a, b \in \mathcal{A}$,

$$(1-\epsilon, a) \sim (-\epsilon, ab) \quad \text{and} \quad (\epsilon, a) \sim (\epsilon, ba).$$

We also define the *subcomplex of vertex edges* $S_{F,s}$ of $K_{F,s}$ by

$$S_{F,s} = \bigcup_{ab \in \mathcal{L}^2_{F,s}} e_{ab} \Big/ \sim$$

where for all $a, b, c \in \mathcal{A}$,

$$(-\epsilon, ab) \sim (-\epsilon, ac) \quad \text{and} \quad (\epsilon, ba) \sim (\epsilon, ca).$$

The other edges e_a in $K_{F,s}$ are called *tile edges*.

Remark 2.3.2. This is only a slight modification of the usual Barge-Diamond complex for a unit-length substitution tiling, mainly in the choice of ϵ and lengths of edges—we necessarily lose geometric information about tile lengths because we do not necessarily have a compatible Perron-Frobenius eigenvalue, as our transition matrices may not necessarily be individually primitive or have coinciding spectra.

From the construction, a vertex edge e_{ab} is only included in $K_{F,s}$ if the two-letter word ab is admitted by (F, s). This means that the BD-complex for a system (F, s) will be dependent on s. In particular, the sequence of complexes $(K_{F,\sigma^i(s)})_{i\geq 0}$ may not be constant, as would be the case in the classical setting of a single substitution where $F = \{\phi\}$.

Example 2.3.3. As an example for where the BD-complexes for $(F, \sigma^i(s))$ and $(F, \sigma^j(s))$ can differ, consider the set of substitutions $F = \{\phi_0, \phi_1\}$ on the alphabet $\mathcal{A} = \{a, b\}$ given by

$$\phi_0 \colon \begin{cases} a & \mapsto & b \\ b & \mapsto & ba \end{cases}, \quad \phi_1 \colon \begin{cases} a & \mapsto & ab \\ b & \mapsto & ba \end{cases}.$$

Here, ϕ_0 is the Fibonacci substitution and ϕ_1 is the Thue-Morse substitution. Let s be the sequence $s = (1, 0, 0, 0 \ldots)$, so $s_0 = 1$ and $s_i = 0$ for all $i \geq 1$.

It can be easily checked that $\mathcal{L}^2_{F,s} = \{aa, ab, ba, bb\}$ and $\mathcal{L}^2_{F,\sigma(s)} = \{ab, ba, bb\}$. So $K_{F,s}$ and $K_{F,\sigma(s)}$ have a different number of vertex edges. After that, $K_{F\sigma^i(s)}$ will be equal to $K_{F,\sigma(s)}$ for all positive i. See Figure 2.1 for the associated BD-complexes. We have labelled the tile edges e_x and the vertex edges e_{xy} by their indices for better readability.

Definition 2.3.4. Let (F, s) be a mixed substitution system and let (w, t) be a tiling in the tiling space $\Omega_{F,\sigma^i(s)}$ with $t \in [0, 1)$. Suppose $w = \ldots w_{-1} \cdot w_0 w_1 \ldots$. We define a surjective continuous map $p_i \colon \Omega_{F,\sigma^i(s)} \to K_{F,\sigma^i(s)}$ by

$$p_i(w, t) = \begin{cases} (t, w_{-1}w_0) & \text{if } t \in [0, \epsilon] \\ (t, w_0) & \text{if } t \in [\epsilon, 1-\epsilon] \\ (t-1, w_0 w_1) & \text{if } t \in [1-\epsilon, 1) \end{cases}$$

There is a unique continuous map $f_i \colon K_{F,\sigma^{i+1}(s)} \to K_{F,\sigma^i(s)}$ such that $f_i \circ p_{i+1} = p_i \circ \phi_{s_i}$. This map is induced in the obvious way by the substitution on edges.

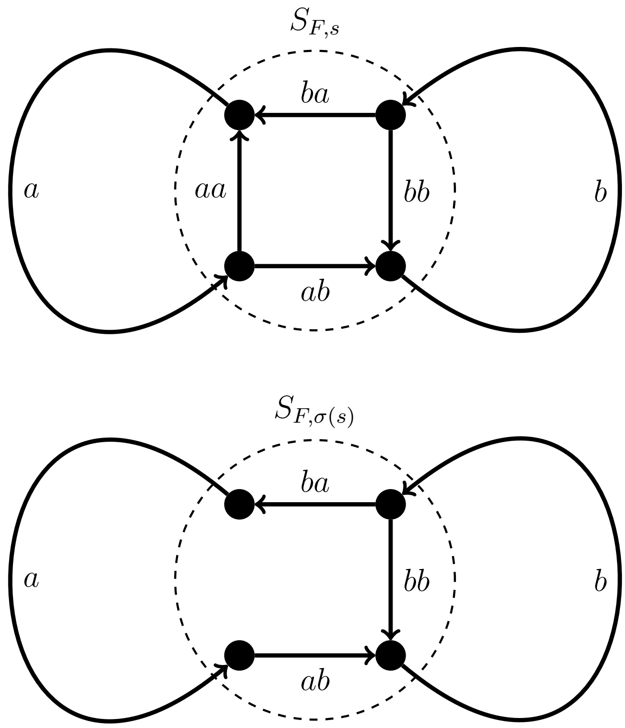


Fig. 2.1: The Barge-Diamond complexes $K_{F,s}$ and $K_{F,\sigma(s)}$ with their subcomplexes of vertex cells $S_{F,s}$ and $S_{F,\sigma(s)}$ circled accordingly.

Remark 2.3.5. The maps f_i are not cellular with respect to the tile and vertex edges of the BD-complexes because the edges are, in general, expanded as they are substituted; for instance vertex edges e_{ab} are not just mapped onto some other vertex edge e_{cd}, but also overlap into the adjacent tile edges e_c and e_d. We account for this later by introducing a cellular map g_i which is homotopic to f_i.

To reduce notation, we assume primitivity and recognisability always hold from this point. Also for notational convenience, let $K_i = K_{F,\sigma^i(s)}$, $S_i = S_{F,\sigma^i(s)}$, $\Omega_i = \Omega_{F,\sigma^i(s)}$, and set $\Omega = \Omega_0 = \Omega_{F,s}$.

Theorem 2.3.6. *For a primitive, recognisable mixed substitution system* (F, s)*, there is a homeomorphism*

$$\Omega \cong \varprojlim(K_i, f_i)$$

between the mixed substitution tiling space and the inverse limit of the associated inverse system of induced substitution maps on the Barge-Diamond complexes.

The proof is essentially identical to the one given by Barge and Diamond in [7] for the case of a single substitution.

Proof. From the definition of the maps f_i, we have commuting diagrams

$$\begin{tikzcd} \Omega_i \arrow[d, "p_i"'] & \Omega_{i+1} \arrow[l, "\phi_{s_i}"'] \arrow[d, "p_{i+1}"] \\ K_i & K_{i+1} \arrow[l, "f_i"'] \end{tikzcd}$$

for each $i \geq 0$. We note that the maps ϕ_{s_i} are homeomorphisms by recognisability and compactness. These commutative diagrams induce a map $p\colon \Omega \to \varprojlim(K_i, f_i)$ given by

$$p(x) = (p_0(x), p_1(\phi_{s_0}^{-1}(x)), p_2(\phi_{s_1}^{-1}(\phi_{s_0}^{-1}(x))), \ldots, p_{i+1}(\phi_{s[0,i]}^{-1}(x)), \ldots).$$

As each p_i is surjective, so then is p.

Let $V = \{(\epsilon, a)\}_{a\in\mathcal{A}} \cup \{(1-\epsilon, a)\}_{a\in\mathcal{A}}$ and choose a point $y = (y_0, y_1, \ldots) \in \varprojlim f_i$. If there exists an $n \geq 0$ such that $y_n \in V$, then since ϵ was chosen small enough, y_{n-1} will be in $e_a \setminus V$ for some symbol $a \in \mathcal{A}$. So, if (w,t) is in $p^{-1}(y)$, then the 0th tile of $\phi_{s[0,n-1]}^{-1}(w,t)$ is determined, and also the placement of the origin in the interior of this tile. So a patch of length $|\phi_{s[0,n]}(a)|$ around the origin in (w,t) is also determined, and by primitivity if $y_n \in V$ for arbitrarily large n, then the length of this determined patch increases without bound. It follows that the entire tiling (w,t) is determined.

If y_n is not in V for arbitrarily large n then there is some $N \geq 0$ such that for all $n \geq N$, either $y_n \in e_{ab}$ for a pair $a, b \in \mathcal{A}$ or $y_n \in e_a \setminus V$ for some $a \in \mathcal{A}$. In the first case, the 0th and (-1)st tiles of $\phi_{s[0,n-1]}^{-1}(w,t)$ are determined, and the position of the origin within one of these tiles. In the second case, the 0th tile is determined and the position of the origin in this tile. Following the above argument this allows us to conclude that arbitrarily large patches around the origin are determined and these patches eventually cover the entire real line, hence (w,t) is fully determined. So, the map p is injective and so also bijective. By usual compactness arguments, we conclude that p is a homeomorphism. $\square$

Recall that $l(u)$ and $r(u)$ are the leftmost and rightmost letters of the word u respectively.

Definition 2.3.7. If $\phi_{s_i}(a) = a_1 a_2 \ldots a_k$, we define a cellular map $g_i\colon K_{i+1} \to K_i$ between consecutive BD-complexes on tile edges by

$$g_i(e_a) := e_{a_1} \cup e_{a_1 a_2} \cup e_{a_2} \cup \cdots \cup e_{a_{k-1} a_k} \cup e_{a_k}$$

in an orientation preserving, and uniformly expanding way, and on vertex edges by

$$g_i(e_{ab}) := e_{a_k b_1}$$

where $r(\phi_{s_i}(a)) = a_k$ and $l(\phi_{s_i}(b)) = b_1$.

The maps f_i and g_i are homotopic. The maps g_i also satisfy the property that $g_i(S_{i+1}) \subset S_i$ and $g_i|_{S_{i+1}}$ is simplicial.

Theorem 2.3.8. *There is an isomorphism of groups*

$$\check{H}^1(\Omega) \cong \varinjlim (H^1(K_i), g_i^*)$$

between the first Čech cohomology of the mixed substitution tiling space and the direct limit of induced maps g_i^* acting on the first cohomology groups of the Barge-Diamond complexes.

Proof. Since f_i and g_i are homotopic, Čech cohomology is isomorphic to singular cohomology on CW-complexes, Čech cohomology is a continuous functor, and Ω is homeomorphic to $\varprojlim(K_i, f_i)$, we get

$$\varinjlim(H^1(K_i), g_i^*) = \varinjlim(H^1(K_i), f_i^*) \cong \check{H}^1(\varprojlim(K_i, f_i)) \cong \check{H}^1(\Omega).$$

□

Theorem 2.3.9. *Let $|\mathcal{A}| = l$ and $\Xi = \varprojlim(S_i, g_i)$. There is an exact sequence*

$$0 \to \tilde{H}^0(\Xi) \to \varinjlim(\mathbb{Z}^l, M_{s_i}^T) \to \check{H}^1(\Omega) \to \check{H}^1(\Xi) \to 0.$$

Proof. Consider the sequence of pairs $(K_i, S_i)_{i \geq 0}$. To each pair there is associated a long exact sequence in reduced singular cohomology

$$\cdots \to \tilde{H}^{n-1}(S_i) \to \tilde{H}^n(K_i, S_i) \to \tilde{H}^n(K_i) \to \tilde{H}^n(S_i) \to \tilde{H}^{n+1}(K_i, S_i) \to \cdots$$

which is trivial outside of degree 0 and 1. Moreover the spaces K_i and K_i/S_i are connected, so $\tilde{H}^0(K_i) = 0$ and $\tilde{H}^0(K_i, S_i) = 0$. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}^0(S_{i+1}) \arrow[r] & H^1(K_{i+1}, S_{i+1}) \arrow[r] & H^1(K_{i+1}) \arrow[r] & H^1(S_{i+1}) \arrow[r] & 0 \\
0 \arrow[r] & \tilde{H}^0(S_i) \arrow[r] \arrow[u, "g_i^*"] & H^1(K_i, S_i) \arrow[r] \arrow[u, "g_i^*"] & H^1(K_i) \arrow[r] \arrow[u, "g_i^*"] & H^1(S_i) \arrow[r] \arrow[u, "g_i^*"] & 0
\end{tikzcd}$$

whose rows are the exact sequences of the pairs (K_i, S_i), and with vertical homomorphisms induced by the maps g_i. Taking the direct limit along each column of this diagram produces an exact sequence

$$0 \to \varinjlim(\tilde{H}^0(S_i), g_i^*) \to \varinjlim(H^1(K_i, S_i), g_i^*) \to \varinjlim(H^1(K_i), g_i^*) \to \varinjlim(H^1(S_i), g_i^*) \to 0.$$

Note that S_i is a closed subcomplex of the CW complex K_i, so (K_i, S_i) is a good pair, and we can identify the relative cohomology group $H^n(K_i, S_i)$ with the cohomology of the quotient $H^n(K_i/S_i)$. For $n = 1$ this is the first cohomology of a wedge of l circles giving $H^1(K_i, S_i) \cong \mathbb{Z}^l$. Moreover the induced map g_i^* on the relative cohomology acts as the transpose of the substitution matrix $M_{s_i}^T$ on the direct sum of l copies of the integers.

Putting this together with the fact that Čech cohomology is isomorphic to singular cohomology for CW complexes, Čech cohomology is continuous, and Theorem 2.3.8 completes the proof. □

Remark 2.3.10. We can say more than this if there is an appropriate notion of an eventual range, as there is in the classical case of a single substitution when $F = \{\phi\}$. In this case, $g_i^{k+1}(S_i) \subset g_i^k(S_i)$ for all $k \geq 0$ and as $g_i|_{S_i}$ is simplicial, and S_i is a finite simplicial complex, this intersection must stabilise to some eventual range $S_{ER} = \bigcap_k g_i^k(S_i)$. Since g_i restricts to a simplicial map on S_{ER}, the inverse limit of the maps $g_i|_{S_{i+1}}$ is just the inverse limit of a map which permutes simplices in the eventual range. The inverse limit of a sequence of homeomorphisms is homeomorphic to any space appearing in that limit, so $\varprojlim(S_i, g_i)$ is homeomorphic to the eventual range S_{ER}, and then the cohomology of this inverse limit is readily determined as the simplicial cohomology of the eventual range.

In our case where (F, s) may not be a trivial system with s constant, we can still determine the inverse limit of S_i under the maps g_i using a similar simplicial analysis, but an eventual range does not always exist.

Lemma 2.3.11. *Let C be a finite simplicial complex of dimension n and let*

$$f_i \colon C_{i+1} \to C_i$$

be a sequence of simplicial maps between a family $\{C_i\}_{i\geq 0}$ of subcomplexes of C. The inverse limit space $\varprojlim(C_i, f_i)$ has the homeomorphism type of a finite simplicial complex of dimension at most n. □

Corollary 2.3.12. *Let m be the rank of the first singular cohomology of Ξ. If Ξ is connected, then*

$$\check{H}^1(\Omega) \cong \varinjlim(\mathbb{Z}^l, M_{s_i}^T) \oplus \mathbb{Z}^m.$$

Proof. Each of the S_i is a subcomplex of the complex $S_F = \left(\bigcup_{ab \in \mathcal{A}^2} e_{ab}\right)/\!\sim$ which is the complex built from all possible vertex edges e_{ab} with $ab \in \mathcal{A}^2$. The map g_i restricted to S_{i+1} is a simplicial map for each i, and so the inverse limit Ξ is a one-dimensional simplicial complex by Lemma 2.3.11. The Čech cohomology of a simplicial complex is isomorphic to the singular cohomology, giving $\check{H}^1(\Xi) \cong \mathbb{Z}^m$. If Ξ is connected, then $\tilde{H}^0(\Xi) = 0$, and since $\mathbb{Z}^m$ is free abelian, the short exact sequence of Theorem 2.3.9 splits to give the result. □

If Ξ is not connected then the direct limit of transpose matrices must be quotiented by $k - 1$ copies of the integers $\mathbb{Z}$ where k is the number of connected components in Ξ. The exact sequence of Theorem 2.3.9 tells you how this group sits inside the direct limit.

Example 2.3.13. As a basic example, we can use this result to provide a short proof that $\check{H}^1(\Omega) \cong \mathbb{Z}^d$ for a tiling space Ω associated to an *Arnoux-Rauzy* sequence on d-letters. The Arnoux-Rauzy sequences are a special class of sequences, introduced by Arnoux and Rauzy in [3], belonging to the family of episturmian sequences[2], which generalise Sturmian sequences (the case where $d = 2$). For an alphabet $\mathcal{A} = \{a_1, \dots, a_d\}$ on d letters, the Arnoux-Rauzy substitutions are given by the d substitutions

$$\mu_i \colon \begin{cases} a_i & \mapsto & a_i \\ a_j & \mapsto & a_j a_i, \quad j \neq i \end{cases}$$

for each $i \in \{1, \dots, d\}$. The Arnoux-Rauzy sequences are then the sequences that are admitted by the mixed substitution systems (F, s) where $F = \{\mu_1, \dots, \mu_d\}$ and s contains an infinite number of terms of each type (to enforce primitivity). So (F, s) is primitive, even though the individual substitutions are not, and is well known to have the unique composition property.

If we consider the map g_{μ_i}, for fixed i, acting on the subcomplex of vertex edges (at any level in the inverse limit), the image will always be a subset of the union of the edges $\bigcup_{j \in \{1,\dots,d\}} e_{ij}$. We can see this by noting that $g_{\mu_i}(e_{kj}) = e_{ij}$ for any k. This subcomplex is contractible, hence the image of $g^*_{\mu_i}|_{H^1(S_i)}$ is trivial, and so we can conclude that $H^1(\Xi)$ is the limit of a sequence of zero maps, which is trivial. By Corollary 2.3.12, it follows that $\check{H}^1(\Omega_{F,s})$ is isomorphic to $\varinjlim(\mathbb{Z}^d, M^T_{\mu_i})$, but note that each of the transition matrices is invertible over the integers $\mathbb{Z}$ and so actually the direct limit of matrices is just isomorphic to $\mathbb{Z}^d$. This completes the proof.

In [36], it was shown that the maximum rank of $\check{H}^1$ for a mixed substitution on d letters is $d^2 - d + 1$. We show that the same result can be reached with a basic

[2] The set of d-episturmian sequences is the unique set of sequences on d letters which are closed under reversal and have at most one right special factor of length k for each $k \geq 1$.

combinatorial argument using BD-complexes. Let $\mathrm{rk}(G)$ be the rank of the group G.

Proposition 2.3.14 (Gähler-Maloney)**.** *For a primitive, recognisable mixed substitution system (F, s) on an alphabet $|\mathcal{A}| = d$, the rank of the Čech cohomology group, $\mathrm{rk}(\check{H}^1(\Omega_{F,s}))$ is bounded above by $d^2 - d + 1$.*

Proof. The BD-complexes K_i of (F, s) have $2d$ vertices (given by each end of the tile edges e_a for $a \in \mathcal{A}$), and at most $d^2 + d$ edges (d^2 vertex edges of the form e_{ab} for $a, b \in \mathcal{A}$ and d tile edges e_a). So the Euler characteristic is bounded below by $V - E = 2d - (d^2 + d) = d - d^2$, giving $\chi \geq d - d^2$. By definition of the Euler characteristic, we have $\chi = \mathrm{rk}(H^0(K_i)) - \mathrm{rk}(H^1(K_i))$, and so as K_i is necessarily connected, $\mathrm{rk}(H^1(K_i)) \leq d^2 - d + 1$. Taking the direct limit of g_i^* acting on $H^1(K_i)$ then tells us that $\mathrm{rk}(\check{H}^1(\Omega)) \leq d^2 - d + 1$, because the rank of the limit cannot exceed the bound of the ranks of the approximants. □

Gähler and Maloney also showed in [36], with a family of examples, that this bound is tight.

Definition 2.3.15. Let (F, s) be a mixed substitution system. If, for infinitely many $i \geq 0$ there exists a letter $a \in \mathcal{A}$ such that for all $b \in \mathcal{A}$ there exists a word u_b such that $\phi_{s_i}(b) = au_b$ (resp. $\phi_{s_i}(b) = u_b a$), then we say (F, s) is *left proper* (resp. *right proper*).

Proposition 2.3.16. *Let $|\mathcal{A}| = l$. Let (F, s) be a left or right proper primitive recognisable substitution system. $\check{H}^1(\Omega_{F,s}) \cong \varinjlim(\mathbb{Z}^l, M_{s_i}^T)$.*

Proof. Without loss of generality, suppose (F, s) is left proper, and let $(i_1, i_2, \ldots)$ be a sequence of natural numbers with the property that there is a sequence of letters $a_1, a_2, \ldots \in \mathcal{A}$ such that for all $b \in \mathcal{A}$ we have $\phi_{s_i}(b) = a_i u$ for some word u.

The image of the vertex edge e_{xy} under the substitution map g_{i_j} is then a vertex edge of the form e_{za_j} for every $xy \in \mathcal{L}^2$. The complex given by taking the union of any collection of such edges e_{za_j} is contractible, and so on the level of cohomology, the induced homomorphism $g_{i_j}^* \colon H^n(S_{i_j}) \to H^n(S_{i_j+1})$ is the zero morphism. As the sequence $(i_j)_{j\geq 0}$ is infinite, the sequence of homomorphisms $g_k^* \colon H^n(S_k) \to H^n(S_{k+1})$ contains an infinite subsequence of zero morphisms and so $\varinjlim(H^n(S_i), g_n^*)$ is trivial for $n = 0, 1$. By Theorem 2.3.9, there is then an isomorphism $\varinjlim(\mathbb{Z}^l, M_{s_i}^T) \to \check{H}^1(\Omega_{F,s})$. □

2.3.2 The Universal Barge-Diamond Complex

A problem to contend with in the construction of the BD-complex for a mixed substitution system (F, s) is that there are a potentially large set of complexes and maps which can appear in the inverse limit. It may be helpful to instead build an inverse system whose approximants are all the same, and where the family of maps appearing in the system are only as large as the family of substitutions F. This was achieved, dependant on a combinatorial condition, for the Anderson-Putnam complexes which appear in [36] where the so-called *universal Anderson-Putnam complex* was introduced. We are also able to achieve this depending on a compatibility condition of the sequence s of substitutions in the system. This 'self-correcting' condition is similar to the one introduced in [36].

First, let us define the complex that will be our candidate universal BD-complex.

Definition 2.3.17. Let $\mathcal{A}$ be a finite alphabet and let $F = \{\phi_0, \phi_1, \ldots, \phi_k\}$ be a set of substitutions on $\mathcal{A}$. Let $\epsilon = \min\limits_{a\in\mathcal{A},\phi\in F} \left\{ \frac{1}{2|\phi(a)|} \right\}$ be a small positive real number. For $a \in \mathcal{A}$, let

$$e_a = [\epsilon, 1-\epsilon] \times \{a\}$$

and for $ab \in \mathcal{A}^2$, let

$$e_{ab} = [-\epsilon, \epsilon] \times \{ab\}.$$

The *universal Barge-Diamond complex* for F is denoted by K_F and is defined to be

$$K_F = \left(\bigcup_{a\in\mathcal{A}} e_a \cup \bigcup_{ab\in\mathcal{A}^2} e_{ab} \right) \Big/ \sim$$

where for all $a, b \in \mathcal{A}$,

$$(1-\epsilon, a) \sim (-\epsilon, ab) \quad \text{and} \quad (\epsilon, a) \sim (\epsilon, ba).$$

We also define the *subcomplex of vertex edges* S_F of K_F by

$$S_F = \bigcup_{ab\in\mathcal{A}^2} e_{ab} \Big/ \sim$$

where for all $a, b, c \in \mathcal{A}$,

$$(-\epsilon, ab) \sim (-\epsilon, ac) \quad \text{and} \quad (\epsilon, ba) \sim (\epsilon, ca).$$

See Figure 2.2 for the universal BD-complex for a set of substitutions F on an alphabet on three letters, $\mathcal{A} = \{a, b, c\}$.

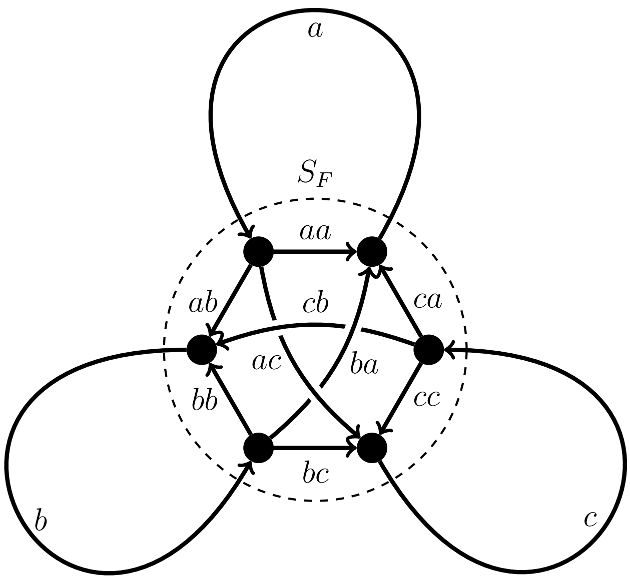


Fig. 2.2: The universal Barge-Diamond complex K_F for $\mathcal{A} = \{a, b, c\}$.

Definition 2.3.18. For each $\phi \in F$, if $\phi(a) = a_1 a_2 \dots a_k$, we define a map $g_\phi \colon K_F \to K_F$ on the universal BD-complex for F by

$$g_\phi(e_a) := e_{a_1} \cup e_{a_1 a_2} \cup \cdots \cup e_{a_{k-1} a_k} \cup e_{a_k}$$

in an orientation preserving and uniformly expanding way, and on vertex edges by

$$g_\phi(e_{ab}) := e_{a_k b_1}$$

where $r(\phi(a)) = a_k$ and $l(\phi(b)) = b_1$.

Definition 2.3.19. Let (F, s) be a mixed substitution system on the alphabet $\mathcal{A}$. Let $\mathcal{A}^2$ be the set of two-letter words in symbols from $\mathcal{A}$. If for every $i \geq 0$, there exists an $N \geq 1$ such that for all $ab \in \mathcal{A}^2$ there is $cd \in \mathcal{L}^2_{F,\sigma^i(s)}$ such that

$$r(\phi_{s[i,i+N]}(a)) l(\phi_{s[i,i+N]}(b)) = cd,$$

then we say (F, s) is *self-correcting*.

Remark 2.3.20. The self-correcting property has been introduced for similar reasons as to why the property was introduced in [36]. Note that the definitions are not the same, and our definition is tailored to work specifically in the Barge-Diamond setting. Self-correcting substitution systems are sufficient to allow us to add cells to the complexes appearing in the inverse limit representation of the tiling space of the system, and not change the cohomology of the inverse limit. In particular, we can use the same universal BD-complex at each level of the inverse system if (F, s) is self-correcting.

Example 2.3.21. Let $\mathcal{A} = \{a, b\}$ be an alphabet on two letters and let $F = \{\phi_0\}$ be the single substitution given by

$$\phi_0\colon \begin{cases} a & \mapsto & b \\ b & \mapsto & ba \end{cases},$$

the Fibonacci substitution. There is only one possible sequence of substitutions to consider which is the constant sequence $s = (0, 0, \ldots)$. We see that

$$\begin{array}{rclrcl} r(\phi_0(a))l(\phi_0(a)) & = & bb & r(\phi_0(b))l(\phi_0(a)) & = & ab \\ r(\phi_0(a))l(\phi_0(b)) & = & bb & r(\phi_0(b))l(\phi_0(b)) & = & ab \end{array}$$

which are all admitted two-letter strings in $\mathcal{L}^2_{\phi_0} = \mathcal{L}^2_{F,\sigma^i(s)} = \{ab, ba, bb\}$ for all $i \geq 0$, and so (F, s) is self-correcting.

Example 2.3.22. The non-degenerate mixed Chacon substitution systems which appear in Section 2.4 are all automatically self-correcting because their set of admitted two-letter words is complete. That is, $\mathcal{L}^2_{F,\sigma^i(s)} = \mathcal{A}^2$ for all $i \geq 0$.

Example 2.3.23. The mixed substitution system associated to an Arnoux-Rauzy sequence on an alphabet with d letters, as introduced in Example 2.3.13, can be seen to be self-correcting. Let $m \geq 0$ be fixed and suppose $s_m = i$. We have that $r(\mu_i(a_j a_k))l(\mu_i(a_j a_k)) = a_i a_j$ which is clearly admitted by $(F, \sigma^m(s))$ because by primitivity, a_j appears somewhere in the sequences appearing in $X_{F,\sigma^{m+1}(s)}$, and preceded by some letter $a_{j'}$. The substituted word $\mu_i(a_{j'} a_j)$ is $a_i a_j a_i$ if $j' = i$, or $a_{j'} a_i a_j a_i$ if $j' \neq i$, both of which contain $a_i a_j$. So all two-letter words are corrected by $(F, \sigma^m(s))$ after one substitution for all $m \geq 0$, hence (F, s) is self-correcting.

Example 2.3.24. Let $F = \{\phi_0\}$ be the single substitution given by

$$\phi_0\colon \begin{cases} a & \mapsto & aaba \\ b & \mapsto & bab \end{cases}.$$

Again, there is only one possible sequence of substitutions to consider which is the constant sequence $s = (0, 0, \ldots)$. This time, we see that

$$\begin{array}{rclrcl} r(\phi_0(a))l(\phi_0(a)) & = & aa & r(\phi_0(b))l(\phi_0(a)) & = & ba \\ r(\phi_0(a))l(\phi_0(b)) & = & ab & r(\phi_0(b))l(\phi_0(b)) & = & bb \end{array}$$

but the only admitted two-letter strings for ϕ_0 are $\mathcal{L}^2_{\phi_0} = \mathcal{L}^2_{F,\sigma^i(s)} = \{aa, ab, ba\}$ for all $i \geq 0$. Since the transition of bb is fixed under substitution, this pair will never be corrected to an admitted pair, and so (F, s) is **not** self-correcting because bb is not admitted by $\{\phi_0\}$.

Theorem 2.3.25. *For a self-correcting mixed substitution system (F, s), there is an equality*

$$\varprojlim(K_i, g_{s_i}) = \varprojlim(K_F, g_{\phi_{s_i}})$$

of the inverse limits of induced cellular substitution maps on the usual Barge-Diamond complexes K_i and on the universal Barge-Diamond complex K_F, seen as subsets of $\prod_{i\geq 0} K_F$.

Proof. It is clear that $\varprojlim(K_i, g_{\phi_{s_i}})$ is a subset of $\varprojlim(K_F, g_{\phi_{s_i}})$ because for every i, $K_i \subset K_F$. In order to show the other inclusion, we make use of the self-correcting property of (F, s).

Pick a point $x = (x_0, x_1, x_2, \ldots) \in \varprojlim(K_F, g_{\phi_{s_i}})$. If x_i is a point in K_i for every $i \geq 0$, then $x \in \varprojlim(K_i, g_{\phi_{s_i}})$ and we are done. If there is some $i \geq 0$ such that $x_i \notin K_i$ then x_i must be in the interior of a vertex edge e_{ab} of S_F which is not a vertex edge appearing in S_i. Let $N \geq 1$ be such that for all $ab \in \mathcal{A}^2$, there are $c, d \in \mathcal{A}$ such that

$$r(\phi_{s[i,i+N]}(a))l(\phi_{s[i,i+N]}(b)) = cd$$

and with $cd \in \mathcal{L}_{F,\sigma^{i+N}(s)}$. The integer N exists because (F, s) is self-correcting. Note that for all $z \in S_F$, we have $g_{\phi_{s[i,i+N]}}(z) \in S_i$ and so there exists no x_{i+N} such that $g_{\phi_{s[i,i+N]}}(x_{i+N}) = x_i$. From the definition of the inverse limit then, such an $x = (x_0, x_1, x_2, \ldots)$ with some x_i in $K_F \setminus K_i$ cannot exist. It follows that for all $i \geq 0$, we must have $x_i \in K_i$ and so $x \in \varprojlim(K_i, g_{\phi_{s_i}})$. Hence $\varprojlim(K_i, g_{\phi_{s_i}}) = \varprojlim(K_F, g_{\phi_{s_i}})$. □

Remark 2.3.26. If (F, s) is not self-correcting, then it is interesting to ask what the inverse limit on the universal BD-complex actually is. In general, the inverse limit will still be a mixed substitution tiling space, but it may be a non-minimal space if (F, s) is not self-correcting, ie the system of substitutions will not necessarily be weakly primitive (by the above, this non-minimal space will contain $\Omega_{F,s}$ as a closed minimal subspace). Non-minimal substitutions will be studied in Chapter 3.

If one takes as an example the single substitution $\phi\colon a \mapsto aaba$, $b \mapsto bab$ from above, then the non-admitted two-letter word bb does not get corrected under substitution, so ϕ is not self-correcting. For the system $F = \{\phi\}$, the inverse limit $\varprojlim(K_F, g_\phi)$ is really the tiling space of the non-minimal substitution $\phi\colon a \mapsto aaba$, $b \mapsto bab$, $c \mapsto bb$.

For a general stationary system $F = \{\phi\}$, if the substitution system does not self-correct, then for ever pair xy which is not corrected, one needs to include a dummy letter X_{xy} into the alphabet which is mapped $X_{xy} \mapsto xy$ in order to describe a substitution whose tiling space is the inverse limit of the universal BD-complex. For non-stationary systems, a similar but more complicated operation can be done.

Corollary 2.3.27. *For a primitive recognisable self-correcting mixed substitution system (F, s), there is an isomorphism*

$$\check{H}^1(\Omega) \cong \varinjlim(H^1(K_F), g^*_{\phi_{s_i}})$$

between the Čech cohomology of the mixed substitution tiling space and the direct limit of the induced homomorphisms on the cohomology of the universal Barge-Diamond complex for F.

Proof. Making use of Theorem 2.3.8 it suffices to show that there is an isomorphism of direct limits

$$\varinjlim(H^1(K_F), (g_{\phi_{s_i}})^*) \cong \varinjlim(H^1(K_i), (g_{\phi_{s_i}})^*)$$

which follows from Theorem 2.3.25 and the continuity and functoriality of Čech cohomology. □

2.4 Uncountability of Set of Cohomology Groups

The machinery is now in place to be able to introduce our new example and prove the main result of this chapter, Theorem 2.4.7.

2.4.1 The Mixed Chacon Substitution System

Let $\mathcal{A} = \{a, b\}$ and let $F = \{\psi_0, \psi_1, \psi_2\}$ be the set of substitutions $\psi_i \colon \mathcal{A} \to \mathcal{A}^+$ given by

$$\psi_0\colon \begin{cases} a & \mapsto & aabba \\ b & \mapsto & b \end{cases}, \quad \psi_1\colon \begin{cases} a & \mapsto & aab \\ b & \mapsto & bba \end{cases}, \quad \psi_2\colon \begin{cases} a & \mapsto & a \\ b & \mapsto & bbaab \end{cases}.$$

We call (F, s) a *mixed Chacon substitution system* because ψ_0 and ψ_2 are each mutually locally derivable to the classical Chacon substitution. The substitution ψ_1 is not strictly necessary to achieve the final result[3], but it seems natural to include ψ_1 in the system for aesthetic reasons.

There are associated substitution matrices

$$M_0 = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \; M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

[3] In fact we only need any two of the three in order to still be able to use the uncountability result of Goodearl and Rushing.

Let $L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which has inverse given by $L^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and note the identities

$$LM_0^T L^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = B_0 \tag{2.9}$$

$$LM_1^T L^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = B_1 \tag{2.10}$$

$$LM_2^T L^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = B_2 \tag{2.11}$$

Let $\alpha \in \mathbb{Z}_3$ be a 3-adic integer with digits $\ldots \epsilon_2\epsilon_1\epsilon_0$. We only wish to consider a specific family of such 3-adic integers. Let $s_\alpha = (s_0, s_1, s_2, \ldots)$ be the associated sequence of digits appearing in α, so $s_n = \epsilon_n$.

Definition 2.4.1. A sequence $s = (s_0, s_1, s_2, \ldots) \in \{0, 1, 2\}^{\mathbb{N}}$ is *degenerate* if there exists a natural number N such that either $s_n = 0$ for all $n \geq N$ or $s_n = 2$ for all $n \geq N$. That is, the sequence s is eventually constant 0s or constant 2s. We say a 3-adic integer α is *degenerate* if its associated sequence of digits s_α is degenerate.

Remark 2.4.2. Non-degeneracy is only a technical condition which forces weak primitivity of the relevant mixed substitution sequences, allowing us to use the previous results on primitive sequences of substitutions. A similar condition can also be defined to enforce strong primitivity, but in this case it is not necessary. Removing these spurious cases helps to simplify the proof[4].

Proposition 2.4.3. *Let $F = \{\psi_0, \psi_1, \psi_2\}$. For a 3-adic integer $\alpha \in \mathbb{Z}_3$, the associated system of mixed substitutions (F, s_α) is weakly primitive if and only if α is non-degenerate.*

Proof. The necessity of non-degeneracy of α for primitivity of (F, s_α) is clear, as all positive powers of M_0 and M_2 contain a zero-entry.

For sufficiency, let $n \geq 0$ be given. It is easy to verify that for $i, j \in \{0, 1, 2\}$, all products M_iM_j have strictly positive entries except for M_0M_0 and M_2M_2. If $M_{s_n} = M_1$ then we are done. If $M_{s_n} = M_0$ then by the non-degeneracy of α, there exists a least $k \geq 1$ such that $M_{s_{n+k}} = M_1$ or M_2. In both cases the matrix $M_{s[n+k-1,n+k]}$ has strictly positive entries and so, since the matrix $M_{s[n,n+k-2]}$ has strictly positive diagonal, we conclude that $M_{s[n,n+k]}$ has strictly positive entries. This similarly holds if $M_{s_n} = M_2$. So (F, s_α) is weakly primitive. $\square$

[4] In fact this only removes, up to homeomorphism, a single tiling space from the family of spaces we are studying—the one associated to a constant sequence of substitutions $s = (0, 0, 0, \ldots)$, which is the usual Chacon tiling space and is well studied.

Proposition 2.4.4. *There are an uncountable number of non-degenerate* 3*-adic integers.*

Proof. There are only a countable number of degenerate 3-adic integers given by those sequences with some finite initial string followed by a constant tail of 0s or 2s, since the set of finite initial strings is countable and the union of two countable sets is countable. The complement of a countable subset of an uncountable set is uncountable and so the non-degenerate 3-adic integers form an uncountable set. □

2.4.2 Calculating Cohomology for the Mixed Chacon Tilings

Lemma 2.4.5. *If α is a non-degenerate* 3*-adic integer then (F, s_α) is recognisable.*

Proof. We use the equivalence between recognisability and the unique composition property. Suppose $s_i = 1$, then for any $w \in X_{F,\sigma^i(s_\alpha)}$ we note that the symbol a can only appear in a string of length $1, 2$ or 3. We can partition the symbols appearing in the sequence w according to the rule that:

- If a symbol a appears in the string bab then we know the patch extends to $b(bab)ba$ and a belongs to a substituted word $\psi_{s_i}(b)\psi_{s_i}(b)$.
- If a symbol a appears in the string $baab$ then we know the patch extends to $aa(baab)$ and a belongs to the substituted word $\psi_{s_i}(a)\psi_{s_i}(a)$.
- If a symbol a appears in the string $baaab$ then we know the patch extends to $b(baaab)$ and a belongs to a substituted word $\psi_{s_i}(b)\psi_{s_i}(a)$.

As the symbol a appears in any substituted word, this is enough to see that unique composition holds in this case.

If $s_i = 0$ then we note that the symbol b either appears[5] in a string of length $1, 2$ or 3. We can partition the symbols appearing in the sequence w according to the rule that:

- If a symbol b appears in a string aba then we know the patch extends to $aabb(aba)abba$ and b belongs to the substituted word $\psi_{s_i}(a)\psi_{s_i}(b)\psi_{s_i}(a)$.
- If a symbol b appears in a string $aabba$ then we know the patch extends to $aabba$ and b belongs to the substituted word $\psi_{s_i}(a)$.

[5] It is not obvious that length 4 cannot occur. This is true because, of the three substitutions ψ_i, only ψ_0 can produce a string of 4 bs, but only when a similar string has been produced in an earlier substituted word. By induction this never happens.

- If a symbol b appears in a string $babba$ then we know the patch extends to $aab(babba)abba$ and b belongs to the substituted word $\psi_{s_i}(a)\psi_{s_i}(b)\psi_{s_i}(b)\psi_{s_i}(a)$.
- If a symbol b appears in a string $abbba$ then we know the patch extends to $aabb(abbba)abba$ and b belongs to the substituted word

$$\psi_{s_i}(a)\psi_{s_i}(b)\psi_{s_i}(b)\psi_{s_i}(b)\psi_{s_i}(b)\psi_{s_i}(a).$$

As the symbol b appears in any substituted word, this is enough to see that unique composition holds in this case. Without loss of generality, if $s_i = 2$ unique composition also holds, based on the above case $s_i = 0$ but with the roles of a and b reversed. $\square$

We are now in a position to prove our main result. Recall from Section 2.1.2, and the proof of the Goodearl-Rushing result, that for $\alpha = \ldots \epsilon_2\epsilon_1\epsilon_0$ a 3-adic integer, $G_\alpha := \varinjlim(B_{\epsilon_n})$.

Theorem 2.4.6. *If α is a non-degenerate 3-adic integer then*

$$\check{H}^1(\Omega_{F,s_\alpha}) \cong G_\alpha \oplus \mathbb{Z}.$$

Proof. Let (F, s_α) be a mixed substitution system for non-degenerate $\alpha \in \mathbb{Z}_3$. By Proposition 2.4.3 and Lemma 2.4.5, (F, s_α) is a primitive, recognisable mixed substitution system and so Theorem 2.3.6 applies. This means we can calculate the first Čech cohomology of the space Ω_{F,s_α} using Theorem 2.3.9. It is easy to show that for every i,

$$\mathcal{L}^2_{F,\sigma^i(s_\alpha)} = \{aa, ab, ba, bb\}$$

and so the BD-complex for $(F, \sigma^i(s_\alpha))$ is given by the two tile edges for the tiles a and b and all four possible vertex edges as shown in Figure 2.3.

We see that the subcomplex $S_{F,\sigma^i(s_\alpha)}$ is homeomorphic to a circle for all i and the induced maps g_i either fix $S_{F,\sigma^i(s_\alpha)}$ (if $s_i = 0, 2$) or reflect $S_{F,\sigma^i(s_\alpha)}$ (if $s_i = 1$). So Ξ is topologically a circle and then $\tilde{H}^0(\Xi) = 0$ and $\check{H}^1(\Xi) = \mathbb{Z}$. The exact sequence from 2.3.9 then becomes a split short exact sequence

$$0 \to \varinjlim(\mathbb{Z}^2, M^T_{\epsilon_n}) \to \check{H}^1(\Omega_{F,s_\alpha}) \to \mathbb{Z} \to 0.$$

So the first Čech cohomology $\check{H}^1(\Omega_{F,s_\alpha})$ of the tiling space is given by

$$\check{H}^1(\Omega_{F,s_\alpha}) \cong \varinjlim(M^T_{\epsilon_n}) \oplus \mathbb{Z}.$$

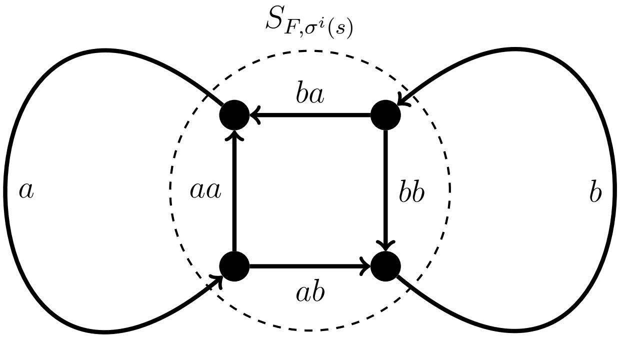


Fig. 2.3: The Barge-Diamond complex for $(F, \sigma^i(s_\alpha))$.

Using equations (2.9)–(2.11) we see that the diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "M_{\epsilon_0}^T"] \arrow[d, "L"', "\cong"] & \mathbb{Z}^2 \arrow[r, "M_{\epsilon_1}^T"] \arrow[d, "L"', "\cong"] & \mathbb{Z}^2 \arrow[r, "M_{\epsilon_2}^T"] \arrow[d, "L"', "\cong"] & \cdots \\
\mathbb{Z}^2 \arrow[r, "B_{\epsilon_0}"'] & \mathbb{Z}^2 \arrow[r, "B_{\epsilon_1}"'] & \mathbb{Z}^2 \arrow[r, "B_{\epsilon_2}"'] & \cdots
\end{tikzcd}$$

commutes, giving $\varinjlim(M_{\epsilon_n}^T) \cong \varinjlim(B_{\epsilon_n})$, which by definition is G_α. We conclude that

$$\check{H}^1(\Omega_{F,s_\alpha}) \cong \varinjlim(M_{\epsilon_n}^T) \oplus \mathbb{Z} \cong G_\alpha \oplus \mathbb{Z}.$$

$\square$

Theorem 2.4.7. *There exists a family of minimal mixed substitution tiling spaces exhibiting an uncountable collection of distinct isomorphism classes of first Čech cohomology groups.*

Proof. By setting up the usual diagram of split exact sequences and performing a diagram chase, it is easy to show that two abelian groups A and B are isomorphic if and only if $A \oplus \mathbb{Z}$ and $B \oplus \mathbb{Z}$ are isomorphic. So if $G_\alpha \oplus \mathbb{Z} \cong G_{\alpha'} \oplus \mathbb{Z}$ then $G_\alpha \cong G_{\alpha'}$. Then, by the definition of $\sim$-equivalent 3-adic integers, we see that for $\alpha, \alpha' \in \mathbb{Z}_3$, the groups $\check{H}^1(\Omega_{F,s_\alpha})$ and $\check{H}^1(\Omega_{F,s_{\alpha'}})$ are isomorphic if and only if $G_\alpha \oplus \mathbb{Z}$ and $G_{\alpha'} \oplus \mathbb{Z}$ are isomorphic, which is if and only if $\alpha \sim \alpha'$.

By Theorem 2.1.3 these equivalence classes are all countable. Recall that (assuming a countable version of the axiom of choice) a countable disjoint union of countable sets is countable. As there are an uncountable number of non-degenerate 3-adic integers by Proposition 2.4.4, and the $\sim$-equivalence classes partition this set into countable subsets, it follows that there are an uncountable number of distinct isomorphism classes of first Čech cohomology groups $\check{H}^1(\Omega_{F,s_\alpha})$ for non-degenerate 3-adic integers α. □

2.4.3 *Discussion*

We think it is important to highlight that cohomology almost certainly does not fully distinguish the mixed Chacon tiling spaces up to homeomorphism—consider all those periodic (primitive) sequences s_α. The respective groups G_α fit into the extension problem $0 \to \mathbb{Z} \to G_\alpha \to \mathbb{Z}[1/3] \to 0$. This extension problem admits only finitely many solutions up to isomorphism, but it is unlikely that these *substitutive* mixed Chacon tiling spaces only comprise finitely many homeomorphism classes. What seems more likely is that the natural order structure on these groups could be a strong enough invariant to distinguish these spaces up to homeomorphism.

One should note that there is nothing particularly special about the set of substitutions chosen to exhibit Theorem 2.4.7. The important step, in order to take advantage of the Goodearl-Rushing result, was to find a set of non-negative matrices which conjugate to the matrices B_i (or the p-adic generalisation of these matrices) via a mutual conjugating matrix L and to then show that there are enough primitive and recognisable sequences of substitutions exhibiting these as substitution matrices. Many other such examples exist (with varying levels of tame or wild behaviours). The mixed Chacon systems happened to be one of the earliest and also simplest found. We encourage the study of other examples in order to begin to understand the full range of behaviours that can be adopted by the Čech cohomology groups of tiling spaces.

Even in cases where the Goodearl-Rushing method cannot be applied, we expect that the uncountability of isomorphism classes of cohomology is the generic case for families of mixed substitution systems whenever the substitutions appearing in F have substitution matrices which are not too closely related (that is, they do not share some power). Qualifying and proving this statement in the general case appears to be a difficult problem.

We remark that as a consequence of Theorem 2.4.7, for *most* 3-adic integers α, the cohomology groups $\check{H}^1(\Omega_{F,s_\alpha})$ of the mixed Chacon tiling spaces cannot be written in the form

$$A \oplus (\mathbb{Z}[1/n_1] \oplus \cdots \oplus \mathbb{Z}[1/n_k]) \tag{2.12}$$

for some finitely generated abelian group A and natural numbers n_i, $1 \leq i \leq k$. Of interest is that these cohomology groups all appear as embedded subgroups of $\mathbb{Q}^2 \oplus \mathbb{Z}$, with G_α embedded as a full rank subgroup of the first summand. In particular they all have rank 3 as abelian groups, so the number of different isomorphism classes of cohomology is being governed by the complexity of the subgroup structure of the group $\mathbb{Q}^2$, rather than a rank phenomenon.

If we call groups which cannot be written in the above form (2.12) *pathological*, it is interesting to ask how typical these pathological cohomology groups appear for general tilings. Thanks are given to Greg Maloney for pointing out to the author that there indeed exist explicit examples of single primitive (even Pisot) symbolic substitutions for which the cohomology is provably pathological. Finding pathological cohomology groups for tiling spaces was one of the original motivating problems for this work, and so it is encouraging that it appears to be a generic behaviour.

3. NON-PRIMITIVE SUBSTITUTIONS

The material presented in this chapter is based on joint work with Greg Maloney. These results appear in [47].

The goal of this chapter is to study one-dimensional tiling spaces arising from non-primitive substitution rules, in terms of the topology, dynamics, and cohomology. This study naturally divides into two cases: the case where the tiling space is minimal, and the case where it is non-minimal. The minimal case is treated in Section 3.1, the main result of which is the following theorem.

Theorem 3.1.1. *Let ϕ be a minimal substitution with non-empty minimal subshift X_ϕ. There exists an alphabet $\mathcal{Z}$ and a primitive substitution θ on $\mathcal{Z}$ such that X_θ is topologically conjugate to X_ϕ.*

This is similar to, but slightly stronger than, a result from the section on Open problems and perspectives (Section 6.2) of [23].

The non-minimal case is treated in Section 3.3. The main result of this section is the following theorem.

Theorem 3.3.9. *Let ϕ be a strongly aperiodic substitution. There exists a complex Γ and a map $f\colon \Gamma \to \Gamma$ such that there is a homeomorphism $h\colon \Omega_\phi \to \varprojlim(\Gamma, f)$.*

The rest of the chapter is devoted to building a structure theorem of non-minimal tiling spaces in terms of their closed shift-invariant subspaces. In particular, we identify a correspondence between such subspaces and subcomplexes of the complex Γ above. The subspaces are found to be homeomorphic to an inverse limit of self-maps acting on the corresponding subcomplex of Γ.

Examples are given throughout the chapter to justify the level of care that needs to be taken in building the machinery, and to give an exposition of how the machinery is put into practice when performing calculations.

3.0.1 Subshifts and Tiling Spaces

Let ϕ be a substitution on the alphabet $\mathcal{A}$ with associated subshift X_ϕ.

Recall from Chapter 1 that a word $u \in \mathcal{A}^*$ is *admitted* by ϕ if there exists a letter $a \in \mathcal{A}$ and a natural number $k \geq 0$ such that $u \subset \phi^k(a)$. The *language* $\mathcal{L}_\phi$ is the set of all admitted words for the substitution ϕ. We remark that it is not necessarily the case that every word in the language of a substitution appears as the subword of a sequence in the subshift - for example ab is in the language of the substitution $\phi\colon a \mapsto ab, b \mapsto b$, but the subshift for this substitution is the single periodic sequence $\dots bbb \dots$ which does not contain ab as a subword.

We say a word u is *legal* if it appears as a subword of a sequence of the subshift for the substitution ϕ. Then the set $\hat{\mathcal{L}}_\phi$ of legal words for ϕ is a subset of the language $\mathcal{L}_\phi$. If $\hat{\mathcal{L}}_\phi = \mathcal{L}_\phi$ then we say that ϕ is an *admissible* substitution. Every primitive substitution is admissible. Some of the results of this chapter would be simplified if we chose to focus only on admissible substitutions, however this will not be an assumption that we make.

Let L be a non-empty subset of the subshift X_ϕ. If, for every point s in L, it is true that $L = \overline{\{\sigma^i(s)\}}_{i\in\mathbb{Z}}$, the orbit closure of s, then L is called a *minimal component* of X_ϕ. If the subshift X_ϕ is a minimal component of itself, then ϕ is called a minimal substitution and X_ϕ is called a *minimal* subshift, otherwise ϕ and X_ϕ are called *non-minimal*.

For a primitive substitution, any admitted word is also legal. If u and v are words, let us use the notation $|v|_u$ to denote the number of occurrences of u as a subword of v. A subshift is called *linearly recurrent* if there exists a natural number $C \in \mathbb{N}$ such that, for all legal words u and v, if $|v| > C|u|$, then $|v|_u \geq 1$.

One fact that will play an important role in this section is the following, which was proved in [22].

Theorem 3.0.1. *Let ϕ be a substitution on $\mathcal{A}$. The subshift X_ϕ is minimal if and only if it is linearly recurrent.*

Recall that we say ϕ is *recognisable* if the induced substitution map on the tiling space $\phi\colon \Omega_\phi \to \Omega_\phi$ is injective. As with subshifts, there is a notion of minimality and minimal components for tiling spaces. We call $\Lambda \subset \Omega_\phi$ a *minimal component* of Ω_ϕ if $\Lambda = (L \times I)/\sim$ for some minimal component L of the subshift X_ϕ, and we say that Ω_ϕ is a *minimal tiling space* if it is a minimal component of itself. In the section on non-minimal substitutions this notion of minimality will be extended to any compact dynamical system, but for now this definition is more convenient.

There are many properties of primitive substitutions which one is likely to take for granted, and so we take this opportunity to explicitly spell out some of these properties and how such properties can fail in the general case (giving both minimal and non-minimal examples where appropriate).

The following results can be found in various places in the literature. We refer the reader to [56] for a concise resource of proofs for most of these results.

Proposition 3.0.2. *Let $\mathcal{A}$ be an alphabet on d letters. If $\phi\colon \mathcal{A} \to \mathcal{A}^+$ is primitive, then:*

1. *X_ϕ is non-empty*
2. *$X_{\phi^k} = X_\phi$ for all $k \geq 1$*
3. *$|\phi^k(a)| \to \infty$ as $k \to \infty$ for all $a \in \mathcal{A}$*
4. *$\sigma\colon X_\phi \to X_\phi$ is minimal. In particular Ω_ϕ is connected*
5. *ϕ is aperiodic if and only if ϕ is strongly aperiodic*
6. *$\operatorname{rk} \check{H}^1(\Omega) \leq d^2 - d + 1$ (see [36] and Theorem 2.3.14)*
7. *Ω_ϕ has at most d^2 asymptotic orbits (see [8])*
8. *If ϕ is recognisable then ϕ is aperiodic*

Proposition 3.0.3. *Counterexamples to the above listed properties in the absence of primitivity are given by:*

1. *Let $\mathcal{A} = \{a, b\}$. If $\phi\colon a \mapsto b$, $b \mapsto a$ then X_ϕ is empty*
2. *Let $\mathcal{A} = \{0, \overline{0}, 1, X\}$. If $\phi\colon 0 \mapsto \overline{0}\overline{0}1\overline{0}$, $\overline{0} \mapsto 0010$, $1 \mapsto 1$, $X \mapsto 0\overline{0}$ then $0\overline{0} \in \hat{\mathcal{L}}_\phi$ but $0\overline{0} \notin \hat{\mathcal{L}}_{\phi^2}$ and so $X_{\phi^2} \subsetneq X_\phi$*
3. *Let $\mathcal{A} = \{a, b, c\}$. If $\phi\colon a \mapsto aaca$, $b \mapsto b$, $c \mapsto bb$ then $|\phi^k(b)| \to 1$ and $|\phi^k(c)| \to 2$ as $k \to \infty$. For a non-minimal case, see the above example for point 2 and the letter 1*
4. *See the counterexample for point 2 for a connected example. The substitution $a \mapsto ab$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$ has a tiling space Ω with two connected components*
5. *Let $\mathcal{A} = \{a, b, c, d\}$. If $\phi\colon a \mapsto ab$, $b \mapsto a$, $c \mapsto cc$, $d \mapsto ca$ then*

$$X_\phi = X_{Fib} \sqcup \bigcup_{n \in \mathbb{Z}} \{\sigma^n(\ldots ccc.abaab\ldots)\} \sqcup \{\ldots cc.cc\ldots\}$$

where Fib is the Fibonacci substitution given by restricting ϕ to the subalphabet $\{a, b\}$. The substitution ϕ is not strongly aperiodic because it contains the point $\ldots cc.cc\ldots$ which is fixed under σ. The substitution ϕ is aperiodic because X_{Fib} is infinite

6. *A minimal counterexample will be given in Section 3.2*

7. *A minimal counterexample will be given in Section 3.2*

8. *Let $\mathcal{A} = \{a, b\}$. If $a \mapsto ab$, $b \mapsto b$ then $X_\phi = \{\dots bb.bb \dots\}$ and Ω_ϕ is homeomorphic to a circle, with the induced substitution map $\phi \colon \Omega_\phi \to \Omega_\phi$ acting as the identity, hence is injective. It follows that ϕ is recognisable, but not aperiodic*

3.1 The Minimal Case

Suppose that $\phi \colon \mathcal{A} \to \mathcal{A}^+$ is a minimal substitution, which means the subshift X_ϕ is then linearly recurrent by Theorem 3.0.1. The main result of this section is the following.

Theorem 3.1.1. *Let ϕ be a minimal substitution with non-empty minimal subshift X_ϕ. There exists an alphabet $\mathcal{Z}$ and a primitive substitution θ on $\mathcal{Z}$ such that X_θ is topologically conjugate to X_ϕ.*

The idea of the theorem is that non-primitive substitutions are 'pathological' and primitive ones are 'well behaved', and the theorem makes it possible to replace a non-primitive substitution with a primitive one if the substitution is minimal. This is similar to, but slightly stronger than, a result from the section on Open problems and perspectives (Section 6.2) of [23]. There are three reasons for presenting this result here. Firstly, the result of [23] does not appear to be well known, but is basic enough that it seems worthwhile to draw attention to it. Secondly, the proof appearing in [23] is only a sketch, using the Chacon substitution[1] as an illustrative example, whereas a complete proof appears here. Thirdly, the result presented here is slightly broader than that of [23]. Specifically, the result in [23] deals with a minimal sequence that is a fixed point of a substitution and is generated from some one-letter seed; the result here deals with minimal substitution subshifts consisting of an entire family of bi-infinite sequences, none of which need be generated from any finite seed. Example 3.1.2 is an example of such a substitution subshift that contains no sequence generated by a finite seed.

Example 3.1.2. Let

$$\phi \colon \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases} .$$

[1] This incarnation of the Chacon substitution is not to be confused with the mixed Chacon substitutions of Chapter 2, which are loosely related but more general.

Then X_ϕ is periodic—it contains only the constant sequence $\dots bbb\dots$. This sequence does not contain any instance of the letter a, which is the only letter of which the images under ϕ^n grow without bound.

3.1.1 Periodicity and Aperiodicity

The lemmas below divide the class of substitutions with minimal subshifts into two subclasses, depending upon whether or not there is any legal letter, the length of which grows without bound under ϕ. In particular, Lemma 3.1.4 shows that, in the absence of such a legal letter, the subshift must be periodic, as in Example 3.1.2.

These lemmas involve a partition of the alphabet into two subsets. For a substitution $\phi\colon \mathcal{A} \to \mathcal{A}^+$, define

$$\mathcal{A}_\infty := \{b \in \mathcal{A} : |\phi^n(b)| \to \infty\}$$
$$\mathcal{A}_1 := \{a \in \mathcal{A} : \exists M \text{ such that } |\phi^n(a)| \le M \text{ for all } n \in \mathbb{N}\}.$$

Then $\mathcal{A}$ is the disjoint union of $\mathcal{A}_1$ and $\mathcal{A}_\infty$, and if X_ϕ is non-empty then $\mathcal{A}_\infty$ is non-empty. Note also that, for every $b \in \mathcal{A}_\infty$, $\phi(b)$ must contain at least one letter in $\mathcal{A}_\infty$, whereas for every $a \in \mathcal{A}_1$, $\phi(a)$ contains only letters in $\mathcal{A}_1$.

Lemma 3.1.3. *Let ϕ be a minimal substitution on $\mathcal{A}$. If there exists $b \in \mathcal{A}_\infty$ such that at least two subletters of $\phi(b)$ are elements of $\mathcal{A}_\infty$, then there exists a bi-infinite sequence $w \in X_\phi$ that contains a legal letter from $\mathcal{A}_\infty$, and that is fixed under some power N of ϕ—i.e., $\phi^N(w) = w$.*

Proof. Let us denote by S the set of all pointed words u in the language of ϕ that have the form

$$u = b_{-k-1}a_{-k}a_{-k+1}\dots a_{-1}.a_0\dots a_m b_{m+1}, \tag{3.1}$$

where $k \ge 0$, $m \ge -1$, $a_i \in \mathcal{A}_1$ for $-k \le i \le m$, and $b_{-k-1}, b_{m+1} \in \mathcal{A}_\infty$. By pointed words, we mean that the words $a.bc$ and $ab.c$ are considered as different elements of S, as the first has $k = 0, m = 1$ and the second has $k = 1, m = 0$. Note that these words all have finite length, and there are only finitely many of them because X_ϕ is linearly recurrent. Moreover the hypothesis that $\phi(b)$ contains at least two subletters from $\mathcal{A}_\infty$ implies that S is non-empty.

Define a map $f\colon S \to S$ as follows. For a sequence u of the form in 3.1, the subsequences $\phi(b_{-k-1})$ and $\phi(b_{m+1})$ are words of $v = \phi(u)$, occurring at the beginning and the end respectively. Each of these words contains at least one letter from $\mathcal{A}_\infty$.

Let b^- be the last such letter occurring in $\phi(b_{-k-1})$, and let b^+ be the first such letter occurring in $\phi(b_{m+1})$. Then v contains a subsequence of the form $v_{m_1} \dots v_{m_2}$, where $m_1 < 0 \le m_2$, $v_{m_1} = b^-$, $v_{m_2} = b^+$, and $v_i \in \mathcal{A}_1$ for all $m_1 < i < m_2$. Moreover, $v_{m_1} \dots v_{m_2} \in S$. Therefore let us define $f(u) = v_{m_1} \dots v_{m_2}$.

As the set S is finite, it contains some sequence u that is sent to itself under f^N for some $N \in \mathbb{N}$. But this means that u is a subsequence of $\phi^N(u)$, which in turn must be a subsequence of $\phi^{2N}(u)$, and so on. Because u begins with a letter of $\mathcal{A}_\infty$ at a negative index and ends with a letter of $\mathcal{A}_\infty$ at a non-negative index, these sequences grow to cover all indices; that is, given $n \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that $\phi^{iN}(u)$ has an entry at index n. Let w_n denote this entry, and let w denote the sequence, the nth entry of which is the letter w_n obtained in this way. The fact that $\phi^{iN}(u)$ is a subsequence of $\phi^{(i+1)N}(u)$ means that this is well defined, and the construction of w means in particular that $\phi^N(w) = w$. The result follows from the fact that u is a word of w, and u contains letters from $\mathcal{A}_\infty$ in at least two positions. $\square$

Lemma 3.1.4. *Let ϕ be a minimal substitution on $\mathcal{A}$. If, for all $b \in \mathcal{A}_\infty$, $\phi(b)$ has a letter of $\mathcal{A}_\infty$ at only one index, then X_ϕ is periodic.*

Proof. It will suffice to show that X_ϕ contains a single periodic sequence w, because minimality will then imply that all elements of X_ϕ are translates of w.

Let us pick out two special subsets of $\mathcal{A}_1$. Define

$$\mathcal{A}_0 := \{a \in \mathcal{A} : |\phi(a)| = 1\}$$

and, noting that ϕ sends $\mathcal{A}_0$ to itself,

$$\mathcal{A}_{00} := \bigcap_{n=0}^{\infty} \phi^n(\mathcal{A}_0).$$

Of these two alphabets, $\mathcal{A}_{00}$ is the one of interest here; it will turn out that all sequences in X_ϕ are made from it. Indeed, given any $a \in \mathcal{A}_1$, there exists $N \in \mathbb{N}$ such that $\phi^N(a)$ consists entirely of letters from $\mathcal{A}_0$. Then by the definition of $\mathcal{A}_{00}$, we can apply ϕ a few more times to arrive in $\mathcal{A}_{00}$; that is, there exists $M \ge N$ such that $\phi^M(a)$ consists entirely of letters from $\mathcal{A}_{00}$. Taking the maximum of all such M over all elements of $\mathcal{A}_1$ yields $N_0 \in \mathbb{N}$ such that, for all $a \in \mathcal{A}_1$, $\phi^{N_0}(a)$ consists entirely of letters from $\mathcal{A}_{00}$.

Now let us construct a periodic sequence x in X_ϕ. Define a function $g\colon \mathcal{A}_\infty \to \mathcal{A}_\infty$ by letting $g(b)$ be the unique letter from $\mathcal{A}_\infty$ that is contained in $\phi(b)$. Then, as $\mathcal{A}_\infty$ is finite, there exists some $b \in \mathcal{A}_\infty$ and some $N \in \mathbb{N}$ such that $g^N(b) = b$. By replacing N with a multiple if necessary, we may suppose that $N > N_0$. Because ϕ permutes $\mathcal{A}_{00}$, we may suppose, after possibly replacing N with another multiple, that $\phi^N(a) = a$ for all $a \in \mathcal{A}_{00}$.

Note that b appears exactly once in $\phi^N(b)$, and no other letter in $\mathcal{A}_\infty$ appears in it at all, so the first and the last letters of $\phi^N(b)$ cannot both be from $\mathcal{A}_\infty$. Let us suppose without much loss of generality that the first letter of $\phi^N(b)$ is not from $\mathcal{A}_\infty$.

Write $\phi^N(b) = a_1 \dots a_r$, and say $a_i = b$ for some $i > 1$. Then

$$\begin{aligned}\phi^{2N}(b) &= \phi^N(a_1)\dots\phi^N(a_{i-1})\phi^N(b)\dots\phi^N(a_r)\\ &= \phi^N(a_1)\dots\phi^N(a_{i-1})a_1\dots a_{i-1}b\dots a_r\dots\phi^N(a_r).\end{aligned}$$

Note that, for $j < i$, $\phi^N(a_j) \in \mathcal{A}_{00}^*$, so $\phi^{kN}(a_j) = \phi^N(a_j)$ for any $k \in \mathbb{N}$. Thus

$$\begin{aligned}\phi^{3N}(b) &= \phi^{2N}(a_1)\dots\phi^{2N}(a_{i-1})\phi^N(a_1)\dots\phi^N(a_{i-1})\phi^N(b)\dots\\ &= \phi^N(a_1)\dots\phi^N(a_{i-1})\phi^N(a_1)\dots\phi^N(a_{i-1})a_1\dots a_{i-1}b\dots a_r\dots\end{aligned}$$

Continuing in this fashion, we see that $\phi^{(k+1)N}(b)$ begins with the word

$$\phi^N(a_1)\dots\phi^N(a_{i-1})$$

repeated k times. Therefore X_ϕ contains the periodic sequence w constructed by concatenating infinitely many copies of this word, as required. □

In the absence of the hypothesis that X_ϕ be minimal, the proof of Lemma 3.1.4 can be repeated with minor modifications to prove the following.

Lemma 3.1.5. *Suppose that $\mathcal{A}_\infty$ contains a letter b with the property that, for all $n \geq 0$, there is exactly one subletter of $\phi^n(b)$ which is a member of $\mathcal{A}_\infty$. Then X_ϕ contains a periodic sequence, the entries of which all lie in $\mathcal{A}_1$.*

3.1.2 A New Substitution

Any periodic minimal subshift is equal to a primitive substitution subshift of constant length (say, the substitution that sends each legal letter to the same sequence u with the property that $\dots u.uu\dots$ is in the subshift), so in the periodic case the conclusion of Theorem 3.1.1 is immediately true. Therefore we may suppose henceforth that X_ϕ is aperiodic, and hence, by Lemmas 3.1.3 and 3.1.4, that X_ϕ contains a bi-infinite sequence w that is invariant under ϕ^N and that contains a legal letter

$b \in \mathcal{A}_\infty$. Moreover, the proof of Lemma 3.1.3 shows that $\phi^N(b)$ contains b, and any legal word u appears in $\phi^{k_w N}(b)$ for some $k_w \in \mathbb{N}$. By passing to a multiple of N if necessary and observing that X_ϕ is linearly recurrent, we may suppose further that $\phi^N(b)$ contains at least two copies of the letter b.

Let us introduce some notation that will be useful in the proof of Theorem 3.1.1. Given $a \in \mathcal{A}$, let $\overline{a}$ denote any letter of $\mathcal{A}$ except a. Given a word u and some $m \geq 0$, let u^m denote the sequence obtained by concatenating m copies of u (so if $m = 0$ this is the empty word ϵ). Let u^* denote any word obtained by concatenating 0 or more copies of u, and let u^{m+*} denote any sequence obtained by concatenating at least m copies of u. Words of the form $b\overline{b}^*$ will play an important role in the argument that follows, where b is the legal letter in $\mathcal{A}_\infty$; these are words consisting of a b, followed by any number of letters different from b.

Define $\mathcal{B} := \{v \text{ of the form } b\overline{b}^* : vb \text{ is legal}\}$, the set of all words v beginning with b followed by letters which are distinct from b and such that vb is a legal word. In the language of [23], these are the *return words to* b.

Enumerate the elements of $\mathcal{B} \setminus \{b\}$: $\mathcal{B} \setminus \{b\} = \{v_1, \ldots, v_k\}$. If $b \in \mathcal{B}$, then write $v_0 = b$.

We can break $\phi^N(b)$ into block form:

$$\phi^N(b) = uv_{01} \ldots v_{0r_0},$$

where u has the form $\overline{b}^*$ and, for $1 \leq j \leq r_0$, v_{0j} has the form $b\overline{b}^*$. Moreover, as $\phi^N(b)$ contains b in at least two distinct places, we know that $r_0 > 1$. And, as b is legal, so is $\phi^N(b)$, so if $j < r_0$ then the sequence $v_{0j}b$ is legal, and so $v_{0j} \in \mathcal{B}$. v_{0r_0} need not be in $\mathcal{B}$.

For each $i \geq 1$, we can write

$$\phi^N(v_i) = \phi^N(b)w_i v_{i1} \ldots v_{ir_i},$$

where $r_i \geq 0$, w_i has the form $\overline{b}^*$, and, for $1 \leq j \leq r_i$, v_{ij} has the form $b\overline{b}^*$. If $r_i > 0$, then for all $j < r_i$, the sequence $v_{ij}b$ appears in $\phi^N(v_i)$, and hence is legal, so $v_{ij} \in \mathcal{B}$. v_{ir_i} need not be in $\mathcal{B}$, but $v_i b$ is legal, and hence $\phi^N(v_i b)$ is legal, and this sequence contains $v_{ir_i}ub$. Therefore, if $r_i > 0$, then $v_{ir_i}u \in \mathcal{B}$; let us denote this sequence by v'_{ir_i}.

Further, although the sequence v_{0r_0} from above need not be in $\mathcal{B}$, for all i with $r_i > 0$ it is true that $v_{0r_0}w_i \in \mathcal{B}$, and for all i with $r_i = 0$ it is true that $v_{0r_0}w_i u \in \mathcal{B}$. Let us denote by w'_i the sequence $v_{0r_0}w_i$ if $r_i > 0$ or $v_{0r_0}w_i u$ if $r_i = 0$. Also $v_{0r_0}u \in \mathcal{B}$; let us denote this sequence by v'_{0r_0}.

Let $\mathcal{C}$ be a new alphabet, disjoint from $\mathcal{A}$ and $\mathcal{B}$, but with the same number of elements as $\mathcal{B}$, and let $\alpha\colon \mathcal{B} \to \mathcal{C}$ be a set bijection. α extends naturally to a map $\mathcal{B}^+ \to \mathcal{C}^+$. For $v \in \mathcal{B}$, let $\tilde{v}$ denote $\alpha(v)$. Define a substitution $\psi\colon \mathcal{C} \to \mathcal{C}^+$ by

$$\psi(\tilde{v}_0) = \tilde{v}_{01} \dots \tilde{v}_{0r_0-1}\tilde{v}'_{0r_0}$$

if $v_0 = b \in \mathcal{B}$, and

$$\psi(\tilde{v}_i) = \begin{cases} \tilde{v}_{01} \dots \tilde{v}_{0r_0-1}\tilde{w}'_i\tilde{v}_{i1} \dots \tilde{v}_{ir_i-1}\tilde{v}'_{ir_i} & \text{if } r_i > 0 \\ \tilde{v}_{01} \dots \tilde{v}_{0r_0-1}\tilde{w}'_i & \text{if } r_i = 0 \end{cases}$$

for all $i > 0$.

Lemma 3.1.6. *The substitution $\psi\colon \mathcal{C} \to \mathcal{C}^+$ defined above is primitive.*

Proof. For all $v \in \mathcal{B}$ there exists $n_v \in \mathbb{N}$ such that vb is a word of $\phi^{n_v N}(b)$. But the hypothesis that b is a word of $\phi^N(b)$ means that, for all $k \leq l$, $\phi^{kN}(b)$ is a word of $\phi^{lN}(b)$. Thus, picking $l = \max_{v\in\mathcal{B}} n_v$ means that, for all $v \in \mathcal{B}$, vb is a word of $\phi^{lN}(b)$. Because all of the words $\{vb : v \in \mathcal{B}\}$ can be found in $\phi^{lN}(b)$, and because any two of these can have overlap in at most their first or last letters, it is possible to find all of the elements of $\mathcal{B}$ as words of $\phi^{lN}(b)$, no two of which share any common indices.

Moreover, for all $w \in \mathcal{B}$, $\phi^N(w)$ starts with uv_{01} and b is a word of v_{01}, so $\phi^{(l+1)N}(w)$ contains every $v \in \mathcal{B}$ within the block $\phi^{lN}(v_{01})$ that begins at index $|\phi^{lN}(u)|$.

Then for all $w \in \mathcal{B}$, $\psi(\tilde{w})$ starts with $\tilde{v}_{01}$, so $\psi^{l+1}(\tilde{w})$ contains $\tilde{v}$ for all $v \in \mathcal{B}$. Therefore ψ is primitive. $\square$

3.1.3 Topological Conjugacy

The new substitution ψ is related to ϕ (specifically, they give rise to homeomorphic tiling spaces—see Section 3.0.1), but it does not necessarily give rise to a topologically conjugate subshift. For this the following result, proved in [23, Proposition 3.1] and paraphrased here, will be useful.

Proposition 3.1.7. *Let $\psi\colon \mathcal{C} \to \mathcal{C}^+$ be a primitive substitution and let g be a map from $\mathcal{C}$ to $\mathcal{A}^+$. Let $X_g \subset \mathcal{A}^{\mathbb{Z}}$ denote the subshift generated by $g(X_\psi)$—that is, $X_g := \{\sigma^n_{\mathcal{A}}(g(x)) : x \in X_\psi, n \in \mathbb{Z}\}$. Then there exists an alphabet $\mathcal{Z}$, a primitive substitution $\theta\colon \mathcal{Z} \to \mathcal{Z}^+$, and a map $h\colon \mathcal{Z} \to \mathcal{A}$ such that $h(X_\theta) = X_g$.*

We can apply this result to the current setting by using the substitution $\psi\colon \mathcal{C} \to \mathcal{C}^+$ defined above, which was shown to be primitive in Lemma 3.1.6, and the map $g\colon \mathcal{C} \to$

$\mathcal{A}^+$ defined by $g(\tilde{v}_i) = v_i$, where $v_i \in \mathcal{A}^+$ is viewed as a sequence possibly consisting of more than one letter. Then the subshift X_g from the statement of Proposition 3.1.7 is exactly the original substitution subshift X_ϕ. Therefore Proposition 3.1.7 guarantees the existence of a factor map—in fact, a one-block code [45]—from a primitive substitution subshift X_θ to the given minimal substitution subshift X_ϕ.

If we look at how $\mathcal{Z}$ and θ are defined in the proof of Proposition 3.1.7 in [23], then it becomes clear that the factor map h is in fact a topological conjugacy—i.e., it has an inverse that is also a factor map. Indeed, $\mathcal{Z}$ is the set of all pairs $(\tilde{v}, k)$, where $v \in \mathcal{B}$ and $1 \leq k \leq |v|$. Every sequence $w \in X_\phi$ can be represented uniquely as a concatenation of return words $v \in \mathcal{B}$ (with the origin possibly contained in the interior of such a word). Then there is a map $p : X_\phi \to \mathcal{Z}^{\mathbb{Z}}$ defined in the following way on a sequence $w \in X_\phi$: If w_j falls at position k in the return word v_i, then $p(w)_j = (\tilde{v}_i, k)$. This is a sliding block code with block size equal to $\max_{v \in \mathcal{B}} |v|$, and the one-block code h is its inverse. The usefulness of Proposition 3.1.7 is in showing that $p(X_\phi)$ is in fact a primitive substitution subshift, which completes the proof of Theorem 3.1.1.

3.2 Examples and Applications

The primitive substitution subshift X_θ is topologically conjugate to the original minimal subshift X_ϕ, which is a very strong condition, but this comes at a price: if we follow the recipe from [23, Proposition 3.1] strictly, then the new alphabet $\mathcal{Z}$ may be quite large—see Proposition 3.2.1, below, for an example in which $|\mathcal{A}| = 2$, $|\mathcal{C}| = 3$ and $|\mathcal{Z}| = 9$. For some computational purposes, particularly purposes involving tiling spaces, the substitution $\psi\colon \mathcal{C} \to \mathcal{C}^+$ can be just as good as θ, and typically uses a smaller alphabet.

Consider the substitutions $\phi\colon \mathcal{A} \to \mathcal{A}^+$ and $\psi\colon \mathcal{C} \to \mathcal{C}^+$ from Theorem 3.1.1, and the map $\alpha\colon \mathcal{C} \to \mathcal{B}^+ \subset \mathcal{A}^+$. Then the tiling spaces Ω_ϕ and Ω_ψ are homeomorphic via the map

$$\begin{aligned} f\colon \Omega_\psi &\to \Omega_\phi \\ (w, t) &\mapsto (\sigma^{\lfloor \tilde{t} \rfloor}(\alpha(w)), \tilde{t} - \lfloor \tilde{t} \rfloor), \end{aligned}$$

where $\tilde{t} = |\alpha(w_0)| \cdot t$.

This means that, for practical purposes, we can use Ω_ψ to compute the topological invariants of Ω_ϕ. This is the approach in the following examples and applications, which illustrate the construction outlined in Section 3.1. The first example illustrates some of the greater ‘freedom’ in behaviour exhibited by minimal non-primitive substitutions on small alphabets.

Recall from [36] (and also proved using our Barge-Diamond techniques in Proposition 2.3.14) that, if Ω_ϕ is the tiling space associated to a primitive substitution ϕ on an alphabet $\mathcal{A}$ with k letters, the rank of the first Čech cohomology $\check{H}^1$ of Ω_ϕ is bounded above by $k^2 - k + 1$ and this bound is tight. Recall from [8] that X_ϕ has at most k^2 asymptotic orbits (equivalently, Ω_ϕ has at most k^2 asymptotic arc components) and this bound is tight. These results both fail spectacularly if we allow for non-primitive minimal substitutions—this result suggests that the alphabet size is not as much of a limiting factor with respect to the topological and dynamical properties of a substitution.

Proposition 3.2.1. *Let $\mathcal{A} = \{a, b\}$ be an alphabet on only two letters. For all $n \geq 2$ there exists a minimal substitution $\phi_n \colon \mathcal{A} \to \mathcal{A}^+$ such that $\check{H}^1(\Omega_{\phi_n})$ has rank n and X_{ϕ_n} has at least n asmyptotic orbits.*

We construct ϕ_n explicitly and use the methods from Section 3.1 to prove the claim.

Proof. We define our family of substitutions ϕ_n by

$$\phi_n \colon \begin{cases} a & \mapsto & ab\,ab^2 \,\dots ab^{n-1}\,ab^n\,a \\ b & \mapsto & b \end{cases}$$

We leave confirmation of minimality of the substitution ϕ_n to the reader. The decomposition $\mathcal{A} = \mathcal{A}_\infty \sqcup \mathcal{A}_1 = \{a\} \sqcup \{b\}$ is quickly found and, as ϕ_n satisfies the hypotheses of Lemma 3.1.3 we know that a can be used as the seed letter for our return words. As per the proof of Lemma 3.1.3, the set $S = \{ab^i.b^ja \mid i+j \leq n,\ 0 \leq i, j\}$ has a fixed point under $f^1(= \mathrm{Id}_S)$, and $\phi_n(a)$ contains at least two distinct copies of a so we can choose $N = 1$.

The return words to a are $\mathcal{B}_n = \{ab^i \mid 1 \leq i \leq n\}$. Let $v_i = ab^i$. The word $\phi(a)$ can be written as $uv_{01} \dots v_{0r_0}$ with $u = \epsilon$ the empty word, $r_0 = n+1$, $v_{0i} = ab^i = v_i$ for $1 \leq i \leq n$ and $v_{0r_0} = a$. For each $1 \leq i \leq n$ we can write $\phi(v_i) = \phi(a)w_i$ with $w_i = b^i$ and we note that $r_i = 0$ for each $i \geq 1$. So, $w_i' = v_{0r_0}w_iu = ab^i = v_i$.

We form $\mathcal{C}_n = \{\tilde{v} \mid v \in \mathcal{B}_n\} = \{\tilde{v}_i \mid 1 \leq i \leq n\}$ and define, for each $1 \leq i \leq n$ the substitution $\psi_n \colon \mathcal{C}_n \to \mathcal{C}_n^+$ by

$$\begin{aligned} \psi_n(\tilde{v}_i) &= \tilde{v}_{01}\tilde{v}_{02} \dots \tilde{v}_{0n}\tilde{w}_i' \\ &= \tilde{v}_1\tilde{v}_2 \dots \tilde{v}_n\tilde{v}_i. \end{aligned}$$

This can more succinctly be written on the alphabet $\{1, \ldots, n\}$ as

$$\psi_n \colon i \mapsto 12 \ldots ni.$$

The reader is invited to verify, following the proof of [23, Proposition 3.1], that the substitution θ defined by

$$\begin{array}{lllllllll}
\theta(A) & = & AB & \theta(L) & = & AB & \theta(W) & = & AB \\
\theta(B) & = & LMNWXYZAB & \theta(M) & = & LMN & \theta(X) & = & LMN \\
 & & & \theta(N) & = & WXYZLMN & \theta(Y) & = & WXYZ \\
 & & & & & & \theta(Z) & = & WXYZ
\end{array}$$

produces a subshift that is topologically conjugate to ϕ_3, where the conjugacy $h \colon \{A, B, L, M, N, W, X, Y, Z\} \to \{a, b\}$ is given by $h(A) = h(L) = h(W) = a$ and $h(B) = h(M) = h(N) = h(X) = h(Y) = h(Z) = b$. (Of course, it is clear that a smaller alphabet can be used; this is what is obtained when the recipe is followed without modification.)

By Lemma 3.1.6 and the discussion in Section 3.0.1, ψ_n is a primitive substitution whose tiling space Ω_{ψ_n} is homeomorphic to Ω_{ϕ_n}. By the aperiodicity of ϕ_n and a result of Mossé [49], ψ_n is recognisable. We notice that ψ_n is also a left-proper substitution and so by Proposition 2.3.16, the first Čech cohomology of Ω_{ψ_n} (and hence of Ω_{ϕ_n}) is given by the direct limit of the transpose of the incidence matrix of ψ_n acting on the group $\mathbb{Z}^n$.

The incidence matrix of ψ_n is the symmetric matrix $M_n = \mathbf{1}_n + I_n$ where $\mathbf{1}_n$ is the $n \times n$ matrix of all 1s, and I_n is the $n \times n$ identity matrix. It is easy to check that M_n has full rank and so

$$\mathrm{rk}\check{H}^1(\Omega_{\phi_n}) = \mathrm{rk}\check{H}^1(\Omega_{\psi_n}) = \mathrm{rk}\varinjlim(M_n) = n.$$

To prove the claim about asymptotic orbits, we note that there exists a right infinite sequence v such that for every $i \in \{1, 2, \ldots, n\}$ there exists a left infinite sequence u_i (found by repeated substitution on the sequence $i.1$) such that $u_i i.1v$ is a point in X_{ψ_n}. By construction then, $u_i i.1v = u_j j.1v$ if and only if $i = j$, and the bi-infinite sequences $u_i i.1v$ and $u_j j.1v$ agree on all components right of the origin for all pairs i, j. It follows that each pair i, j leads to a right asymptotic pair of orbits in X_{ψ_n} and so there exist at least n asymptotic orbits.

Equivalently then, Ω_{ψ_n} has at least n asymptotic arc components. These are preserved under homeomorphism and so Ω_{ϕ_n} also has at least n asymptotic arc components. Equivalently, X_{ϕ_n} has at least n asymptotic orbits. $\square$

The following example illustrates how N can be greater than 1 (One can check that the substitution is related to the Thue-Morse substitution $0 \mapsto 01,\ 1 \mapsto 10$ by mapping the letter c to the empty word ϵ).

Example 3.2.2. Let $\phi\colon a \mapsto acb,\ b \mapsto bca,\ c \mapsto c$.

The decomposition $\mathcal{A} = \mathcal{A}_\infty \sqcup \mathcal{A}_1 = \{a, b\} \sqcup \{c\}$ is quickly found and, as ϕ satisfies the hypotheses of Lemma 3.1.3 we know that a can be used as the seed letter for our return words. As per the proof of Lemma 3.1.3, the set

$$S = \{x.cy, xc.y \mid x, y \in \mathcal{A}_\infty\}$$

has a fixed point under $f^2 (= \mathrm{Id}_S)$, and $\phi^2(a)$ contains at least two distinct copies of a so we can choose $N = 2$.

The return words to a are $\mathcal{B} = \{ac, acbc, acbcbc\}$. Let $v_i = a(cb)^{i-1}c$, $i = 1, 2, 3$. The word $\phi^N(a) = \phi^2(a) = acbcbca$ can be written as $uv_{01} \ldots v_{0r_0}$ with $u = \epsilon$ the empty word, $r_0 = 2$, $v_{01} = acbcbc = v_3$, and $v_{0r_0} = v_{02} = a$. We can write:

$\phi^2(v_1) = \phi^2(a)w_1$ with

$w_1 = c$, $r_1 = 0$

$\phi^2(v_2) = \phi^2(a)w_2v_{21}v_{22}$ with

$w_2 = cbc$, $v_{21} = ac = v_1$, $v_{22} = acbc = v_2$, $r_2 = 2$

$\phi^2(v_3) = \phi^2(a)w_3v_{31}v_{32}v_{33}v_{34}$ with

$w_3 = cbc$, $v_{31} = ac = v_1$, $v_{32} = acbcbc = v_3$, $v_{33} = ac = v_1$, $v_{34} = acbc = v_2$, $r_3 = 4$.

As $u = \epsilon$, $v'_{ir_i} = v_{ir_i}$. As $r_i = 0$ only if $i = 1$, we have:

$w'_1 = v_{0r_0}w_1u = ac = v_1$

$w'_2 = v_{0r_0}w_2 = acbc = v_2$

$w'_3 = v_{0r_0}w_3 = acbc = v_2$

$\phi^2(v_1) = \phi^2(a)w_1$ with

$w_1 = c$, $r_1 = 0$

$\phi^2(v_2) = \phi^2(a)w_2v_{21}v_{22}$ with

$w_2 = cbc$, $v_{21} = ac = v_1$, $v_{22} = acbc = v_2$, $r_2 = 2$

$\phi^2(v_3) = \phi^2(a)w_3v_{31}v_{32}v_{33}v_{34}$ with

$w_3 = cbc$, $v_{31} = ac = v_1$, $v_{32} = acbcbc = v_3$, $v_{33} = ac = v_1$, $v_{34} = acbc = v_2$, $r_3 = 4$.

As $u = \epsilon$, $v'_{ir_i} = v_{ir_i}$. As $r_i = 0$ only if $i = 1$, we have

$w'_1 = v_{0r_0} w_1 u = ac = v_1$

$w'_2 = v_{0r_0} w_2 = acbc = v_2$

$w'_3 = v_{0r_0} w_3 = acbc = v_2$.

We form $\mathcal{C} = \{\tilde{v} \mid v \in \mathcal{B}\} = \{\tilde{v_1}, \tilde{v_2}, \tilde{v_3}\}$ and define the substitution $\psi \colon \mathcal{C} \to \mathcal{C}^*$ by

$$\begin{aligned}
\psi(\tilde{v}_1) &= \tilde{v}_{01}\tilde{w}'_1 \\
&= \tilde{v}_3\tilde{v}_1 \\
\psi(\tilde{v}_2) &= \tilde{v}_{01}\tilde{w}'_2\tilde{v}_{21}\tilde{v}'_{22} \\
&= \tilde{v}_3\tilde{v}_2\tilde{v}_1\tilde{v}_2. \\
\psi(\tilde{v}_3) &= \tilde{v}_{01}\tilde{w}'_3\tilde{v}_{31}\tilde{v}_{32}\tilde{v}_{33}\tilde{v}'_{34} \\
&= \tilde{v}_3\tilde{v}_2\tilde{v}_1\tilde{v}_3\tilde{v}_1\tilde{v}_2.
\end{aligned}$$

This can more succinctly be written on the alphabet $\{1, 2, 3\}$ as

$$\begin{aligned}
\psi(1) &= 31 \\
\psi(2) &= 3212 \\
\psi(3) &= 321312
\end{aligned}$$

The following example illustrates where u may be non-trivial. It is also an example of a topologically equivalent primitive substitution on fewer letters than the original minimal substitution (one can check that the associated tiling space is homeomorphic to the tiling space of the Fibonacci substitution). We omit much of the writing and just give a list of notation so that the reader may confirm their own calculations.

Example 3.2.3. Let $\phi \colon a \mapsto bc,\ b \mapsto b,\ c \mapsto ca$.

- $\mathcal{A} = \mathcal{A}_\infty \sqcup \mathcal{A}_1 = \{a, c\} \sqcup \{b\}$
- Seed letter - a
- $S = \{a.bc, c.bc, ab.c, cb.c\}$
- f^2 has a fixed point
- $\phi^4(a)$ contains at least two distinct copies of $a \implies N = 4$
- $\mathcal{B} = \{abcbc, abc\}$, $v_1 = abcbc$, $v_2 = abc$
- $\phi^N(a) = \phi^4(a) = bcabcbca$
- $u = bc$, $v_{01} = abcbc = v_1$, $v_{0r_0} = v_{02} = a$, $r_0 = 2$

- $\phi^4(v_1) = \phi^4(a)w_1v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}$
- $w_1 = bc$, $v_{11} = v_1$, $v_{12} = v_2$, $v_{13} = v_1$, $v_{14} = v_1$, $v_{15} = v_2$, $v_{16} = abc$, $r_1 = 6$
- $\phi^4(v_2) = \phi^4(a)w_2v_{21}v_{22}v_{23}$
- $w_2 = bc$, $v_{21} = v_1$, $v_{22} = v_2$, $v_{23} = abc$, $r_2 = 3$
- $v'_{16} = v_{16}u = abcbc = v_1$
- $v'_{23} = v_{23}u = abcbc = v_1$
- $w'_1 = v_{0r_0}w_1 = abc = v_2$
- $w'_2 = v_{0r_0}w_2 = abc = v_2$

$$\begin{aligned} \psi(1) &= 12121121 \\ \psi(2) &= 12121 \end{aligned}$$

We encourage the reader to try the example $\phi\colon a \mapsto acb$, $b \mapsto adb$, $c \mapsto dd$, $d \mapsto d$ where the function f is not a bijection; and to also try the example of the non-primitive Chacon substitution $\phi\colon a \mapsto aaba$, $b \mapsto b$ where $\mathcal{B}$ contains the single letter return word a and so v_0 needs to be treated.

3.3 The Non-Minimal Case

In this section, we now turn our attention to the case of those substitutions which give rise to non-minimal subshifts. We call such substitutions *non-minimal substitutions.*

Definition 3.3.1. We say the word $u \in \mathcal{A}^*$ is a *bounded word* for ϕ if $u \in \hat{\mathcal{L}}_\phi$ is a legal word for ϕ and $\lim_{n\to\infty} |\phi^n(u)| < \infty$. If u is a legal word for ϕ but u is not bounded for ϕ, then we say u is *expanding* for ϕ. Let B be the set of bounded words for the substitution ϕ. If B is finite, we say ϕ is *tame*. If B is infinite, we say ϕ is *wild.*

Example 3.3.2. The substitution $\phi\colon a \mapsto bab$, $b \mapsto b$ is wild, as the periodic sequence $\dots bb.bb \dots$ is an element of the subshift and the words b^n are all bounded for ϕ.

Example 3.3.3. The substitution $\phi'\colon a \mapsto bab$, $b \mapsto bbb$ is tame, as $|\phi(u)| = 3|u|$ for all words u.

Note that the subshifts X_ϕ and $X_{\phi'}$ for these two examples are the same, so tameness is only a property of a substitution and not its associated subshift.

Let $n \geq 0$. If $a \in \mathcal{A}$ is a letter admitted by ϕ and the word $u = a_{-n} \dots a_{-1} a a_1 \dots a_n$ is legal, then we say the formal pair (a, u) is an *n-collared letter* and denote each such pair by the new letter a_u.

We define a new alphabet $\mathcal{A}_n$ for ϕ, where $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_n$ is the set of all n-collared letters for the substitution ϕ. There is a forgetful map $f_{n,m} \colon \mathcal{A}_n \to \mathcal{A}_m$ where, if

$$u = a_{-n} \dots a_{-m} \dots a_{-1} a a_1 \dots a_m \dots a_n$$

and

$$v = a_{-m} \dots a_{-1} a a_1 \dots a_m$$

then we define $f_{n,m}(a_u) = a_v$. We can extend this forgetful map to $\mathcal{A}_i^*$ and $\mathcal{A}_i^{\mathbb{Z}}$.

If u is the word $c_1 c_2 \dots c_l$, then for those i and n where it is well-defined, let $c_i(n)$ be the subword $c_{i-n} \dots c_i \dots c_{i+n}$. Suppose $\phi(a) = b_1 \dots b_k$ and let a_u be an n-collared letter. Note that $b_1 \dots b_k$ is a subword of $\phi(u)$, so we can define $b_i(n)$ for each $1 \leq i \leq k$. There is an induced substitution on $\mathcal{A}_n$ defined by

$$\phi_n(a_u) = (b_1)_{b_1(n)} \dots (b_k)_{b_k(n)}.$$

This induced substitution commutes with the forgetful maps. That is,

$$\phi_m \circ f_{n,m} = f_{n,m} \circ \phi_n.$$

Note also that $f_{n,m} \colon X_{\phi_n} \to X_{\phi_m}$ is a topological conjugacy between subshifts, and in particular X_{ϕ_n} is conjugate to X_ϕ for all n.

Lemma 3.3.4. *Let ϕ be a tame substitution. Let N be one greater than the maximum length of any bounded word for ϕ, $N = \max_{u \in B} |u| + 1$. The substitution $\phi_N \colon \mathcal{A}_N \to \mathcal{A}_N^+$ forces the border at some level k.*

Proof. Let k be such that, for every expanding letter $a \in \mathcal{A}$, we have $|\phi^k(a)| > N$. Let $a_u \in \mathcal{A}_N$ be an N-collared letter such that $\phi_N^k(a_u)$ appears as a subword of $w \in X_{\phi_N}$. As ϕ is tame, there exists a letter $l = w_{i-j}$ to the left of a and a letter $r = w_{i+j'}$ to the right of a that are both expanding letters. Further, the indices j and j' can be chosen so that $j, j' < N$.

So, $u = w_{i-N} \dots l \dots a \dots r \dots w_{i+N}$ where $w_i = a$ is the central letter of the word and then

$$\phi^k(u) = \phi^k(w_{i-N}) \dots \phi^k(l) \dots \phi^k(a) \dots \phi^k(r) \dots \phi^k(w_{i+N})$$

Let $u_l = \phi^k(l)$ and $u_r = \phi^k(r)$. As $|\phi^k(l)| > N$ and $|\phi^k(r)| > N$, and we know all tiles within N places of a in u, we can determine all N-collared tiles out until at least the rightmost N-collared letter of $\phi^k(l)$ to the left, and at least the leftmost N-collared letter of $\phi^k(r)$ to the right. These tiles lie outside of $\phi_N^k(a_u)$ and so $\phi_N^k(a_u)$ uniquely extends in X_{ϕ_N}. It follows that ϕ_N forces the border at level k. $\square$

Let $\mathcal{A}_1$ be the set of bounded letters for a substitution ϕ and let $\mathcal{A}_\infty = \mathcal{A} \setminus \mathcal{A}_1$ be the set of expanding letters for ϕ.

Lemma 3.3.5. *If the leftmost and rightmost letter of $\phi(a)$ are elements of $\mathcal{A}_\infty$ for all $a \in \mathcal{A}_\infty$, then ϕ is tame.*

Proof. Suppose that ϕ is wild, so the set

$$B = \{u \in \hat{\mathcal{L}}_\phi \mid \lim_{n\to\infty} |\phi^n(u)| < \infty\}$$

of bounded words for ϕ is infinite. For a bounded letter $b \in \mathcal{A}_1$, let k_b be the limit $k_b = \lim_{n\to\infty} |\phi^n(b)|$ and let $k = \sup_{b\in\mathcal{A}}\{k_b \mid b \in B\}$, the length of the longest bounded word which is an iterated substitute of a single bounded letter. For an expanding letter a, let l_a be the length of the longest bounded subword of $\phi(a)$ and let

$$l = \sup_{a\in\mathcal{A}_\infty,\, n\geq 0} \{|\phi^n(u)| \mid u \in B, |u| \leq l_a, u \subset \phi(a)\}$$

which is a finite set and so has a finite supremum.

Let $v \in B$ be a bounded word of length $|v| \geq \max\{k, l\} + 1$ which exists because ϕ is wild. As v is legal, there exists a sequence $w \in X_\phi$ for which v is a subword, and as all legal words are in the language of ϕ, there exists a letter $a \in \mathcal{A}$ and a minimal power n such that $v \subset \phi^n(a)$ and v is not a subword of $\phi^{n-1}(a')$ for any letter $a' \in \mathcal{A}$. By our choice of v, the letter a cannot be a bounded letter and so $\lim_{n\to\infty} |\phi^n(a)| = \infty$ and a is expanding. Partition $\phi^n(a)$ into words of the form $\phi^{n-1}(a_i)$ where $\phi(a) = a_0 \dots a_l$.

As n is minimal, v must be a subword of the concatenation of several words of the form $\phi^{n-1}(a_i) \dots \phi^{n-1}(a_j)$ and in particular v is not contained in any single word of the form $\phi^{n-1}(a_i)$ because $|v| > l$ and so 'overlaps' at least two of these words. Replace, if needed, the letters $a_i \dots a_j$ with some minimal subset that still has this property, so that w intersects either the leftmost or rightmost letter of every word $\phi^{n-1}(a_i), \dots, \phi^{n-1}(a_j)$. By our assumption on the leftmost and rightmost letters of expanding letters, the letters $a_i, \dots, a_j$ can therefore not be expanding, as otherwise v would contain expanding letters itself and hence not be bounded. This means that $a_i, \dots, a_j$ must all be bounded letters. However, v was chosen to be longer than the length of any iterated substitution of bounded words appearing in $\phi(a)$ for any expanding letter a. This gives a contradiction.

□

Let $\mathcal{A}_{right} \subset \mathcal{A}_\infty$ be the set of expanding letters such that for every $a \in \mathcal{A}_{right}$ the rightmost letter of $\phi(a)$ is a bounded letter, and define $\mathcal{A}_{left}$ similarly. If ϕ is wild, then either $\mathcal{A}_{right}$ or $\mathcal{A}_{left}$ is non-empty by Lemma 3.3.5.

Lemma 3.3.6. *Let ϕ be a wild substitution. Either there exists a letter $a \in \mathcal{A}_{right}$ and an increasing sequence of integers N_i such that the rightmost expanding letter appearing in $\phi^{N_i}(a)$ is also in $\mathcal{A}_{right}$ for all $i \geq 1$ or there exists a letter $a \in \mathcal{A}_{left}$ and an increasing sequence of integers N_i such that the leftmost expanding letter appearing in $\phi^{N_i}(a)$ is also in $\mathcal{A}_{left}$ for all $i \geq 1$.*

Proof. First, suppose that there is no $a \in \mathcal{A}_{left}$ and increasing sequence of integers N_i such that the leftmost expanding letter appearing in $\phi^{N_i}(a)$ is also in $\mathcal{A}_{left}$ for all $i \geq 1$. By this assumption, there exists an N such that the leftmost expanding letter in $\phi^{N+k}(a)$ is never in $\mathcal{A}_{left}$ for any expanding letter a. Let U_{left} be the set of bounded words that appear at the start of any word of the form $\phi^n(a)$ for any expanding letter a. This set is finite because $\phi^{N+k}(a)$ will be a word of the form ubv where u is bounded and the leftmost letter of $\phi(b)$ is expanding and also not in $\mathcal{A}_{left}$. Let $k_{left} = \max\{|u| \mid u \in U_{left}\}$.

Suppose further that there is no $a \in \mathcal{A}_{right}$ and increasing sequence of integers N_i such that the leftmost expanding letter appearing in $\phi^{N_i}(a)$ is also in $\mathcal{A}_{right}$ for all $i \geq 1$. Then we can similarly form U_{right}, the set of bounded words that appear at the end of any word of the form $\phi^N(a)$ for $a \in \mathcal{A}_{right}$. Let $k_{right} = \max\{|u| \mid u \in U_{right}\}$.

It is easy to see that the only legal bounded words for ϕ are either bounded words appearing as subwords contained in the interior of $\phi(a)$ for an expanding a, or words of the form u_1u_2 for $u_1 \in U_{right}$ and $u_2 \in U_{left}$. It follows that every bounded word has length at most $\max\{k_{left} + k_{right}, |\phi(a)| \mid a \in \mathcal{A}\}$ and so ϕ is tame. □

Recall that if X_ϕ contains no σ-periodic points, then we say ϕ is *strongly aperiodic.*

Theorem 3.3.7. *Let ϕ be a substitution on the alphabet $\mathcal{A}$. If ϕ is strongly aperiodic, then ϕ is tame.*

Proof. We prove the contrapositive. Suppose that ϕ is wild. By Lemma 3.3.6, we may assume without loss of generality that there exists a letter $a \in \mathcal{A}_{right}$ and an increasing sequence of integers N_i such that the rightmost expanding letter of $\phi^{N_i}(a)$ is also in $\mathcal{A}_{right}$. Note that the rightmost expanding letter of $\phi^{N_k}(a)$ must also have the same property as a for the shifted sequence of integers $M_i = N_{i-k}$. So, by possibly choosing a different $a \in \mathcal{A}_{right}$ we may further assume without loss of generality that there is a power N so that the rightmost expanding letter of $\phi^N(a)$ is a. So, let $\phi^N(a) = vau$ where u is a bounded word. Then by induction, we have

$$\phi^{(k+1)N}(a) = \phi^{kN}(v) \ldots \phi^N(v) vau \phi^N(u) \ldots \phi^{kN}(u).$$

Now, as u is a bounded word, there exists a K such that $|\phi^{(K+1)N}(u)| = |\phi^{KN}(u)|$ and as there are only finitely many words of this length, by possibly replacing ϕ with a power, we can choose K such that $\phi^{(K+1)N}(u) = \phi^{KN}(u)$. So for all $j \geq K$, the word $(\phi^{KN}(u))^j$ appears as a subword of $\phi^n(a)$ for some n. As such, the periodic sequence

$$\ldots \phi^{KN}(u) \phi^{KN}(u) \phi^{KN}(u) \ldots$$

is admitted by ϕ. This means that the subshift X_ϕ contains a periodic point. □

Let ϕ be a substitution on the alphabet $\mathcal{A}$ and let Ω be the associated tiling space. Use the convention that a point $T \in \Omega$ is written coordinate-wise as (w, t), $w \in X_\phi$ and $t \in [0, 1)$. Recall that we define the *Anderson-Putnam complex* Γ of ϕ to be $\Omega/{\sim}$ where $\sim$ is the equivalence relation given by taking the transitive closure of the relation $(w, t) \sim (w', t')$ if $t = t' \in (0, 1)$ and $w_0 = w'_0$ or $t = t' = 0$ and $w_{-1} = w'_{-1}$ or $w_0 = w'_0$.

Definition 3.3.8. We define the *n-collared Anderson Putnam complex* Γ_n to be the Anderson-Putnam complex associated to the n-collared substitution ϕ_n.

Let $p_n \colon \Omega \to \Gamma_n$ be the natural quotient map. We define a map $f_n \colon \Gamma_n \to \Gamma_n$ to be the unique map which makes the following square commute

$$\begin{tikzcd} \Omega \arrow[d, "p_n"'] & \Omega \arrow[l, "\phi"'] \arrow[d, "p_n"'] \\ \Gamma_n & \Gamma_n \arrow[l, "f_n"'] \end{tikzcd}$$

For a strongly aperiodic substitution ϕ, let $N_\phi = \max_{u \in B} |u| + 1$ be one greater than the length of the longest bounded word for ϕ. This natural number exists by Theorem 3.3.7. The following theorem removes the hypothesis of primitivity from the classic Anderson-Putnam theorem [2] if we allow ourselves to collar letters out to a sufficient radius.

Theorem 3.3.9. *Let ϕ be a strongly aperiodic recognisable substitution. The natural map $h\colon \Omega \to \varprojlim(\Gamma_{N_\phi}, f_{N_\phi})$ given by*

$$h(x) = (p_{N_\phi}(x), p_{N_\phi}(\phi^{-1}(x)), p_{N_\phi}(\phi^{-2}(x)), \ldots)$$

is a homeomorphism.

Proof. Recognisability of ϕ means that $\phi\colon \Omega \to \Omega$ has a well-defined inverse and so h is well-defined. By the choice of N_ϕ and Lemma 3.3.4, the N_ϕ-collared substitution ϕ_{N_ϕ} forces the border at level k. Hence, a point in the inverse limit describes a unique tiling of the line, and so h is both injective and surjective. As h is a continuous bijection from a compact space to a Hausdorff space, h is a homeomorphism. □

We may further reduce the list of hypotheses for this theorem by making use of a result of Bezuglyi, Kwiatowski and Medynets [15].

Theorem 3.3.10 (Bezuglyi-Kwiatowski-Medynets)**.** *If ϕ is strongly aperiodic, then ϕ is recognisable.*

Corollary 3.3.11. *Let ϕ be a strongly aperiodic substitution. The map $h\colon \Omega \to \varprojlim(\Gamma_{N_\phi}, f_{N_\phi})$ is a homeomorphism.*

Remark 3.3.12. We remark that there exist recognisable substitutions which are not strongly aperiodic or even aperiodic. Take as an example the substitution $\phi\colon a \mapsto ab,\ b \mapsto b$ whose induced substitution on the tiling space is just the identity map on a circle, and so is injective, hence ϕ is recognisable even though ϕ is a periodic substitution. This is perhaps surprising to a reader who is used to primitive substitutions, where recognisability, aperiodicity and strong aperiodicity are all equivalent.

3.4 Closed Invariant Subspaces of Non-minimal Tiling Spaces

3.4.1 Invariant Subspaces

Let Ω be a compact metric space and let G act continuously on the right of Ω via $\rho\colon \Omega \times G \to \Omega$ and let $\rho_\tau\colon \Omega \to \Omega$ be given by $x \mapsto \rho(x, \tau)$. We will normally only consider $G = \mathbb{R}$ or $G = \mathbb{Z}$, but the following machinery is suitable to be applied in the general case (in particular, tiling spaces in higher dimensions which have actions of higher dimensional Euclidean groups).

If Λ is a closed subspace of Ω such that $\rho_\tau(\Lambda) = \Lambda$ for all $\tau \in G$, we call Λ a *closed invariant subspace* with respect to the action, or *CIS* for short. The set of CISs $\mathcal{C}$

forms a lattice under inclusion of subspaces. The least elements of $\mathcal{C}$ that are not empty are the minimal sets of the action on Ω. The unique maximal element of $\mathcal{C}$ is the whole space Ω. We note without much further comment, but find it interesting, that $\mathcal{C}^C = \{\Omega \setminus \Lambda \mid \Lambda \in \mathcal{C}\}$ is a topology on the set Ω (in general, more coarse than the original topology induced by the metric on Ω). This topology is indiscrete if and only if (Ω, ρ) is minimal. Any continuous map between dynamical systems which maps orbits to orbits will also induce a continuous map between the spaces endowed with the topology $\mathcal{C}^C$, and so the homeomorphism type of the topological space $(\Omega, \mathcal{C}^C)$ is an orbit equivalence invariant of the dynamical system (Ω, ρ).

Lemma 3.4.1. *Let $f\colon \Omega \to \Omega'$ be a continuous map which maps G-orbits to G-orbits. If Λ is a CIS of Ω with respect to the action of G on Ω, then $f(\Lambda)$ is a CIS of Ω' with respect to the action of G on Ω'.*

Proof. Let Λ be a CIS of Ω. As Ω is compact and Λ is a closed subspace, Λ is compact, so the image of Λ under a continuous map is compact. As Ω' is Hausdorff, a compact subspace of Ω' must be closed, and so $f(\Lambda)$ is a closed subspace of Ω'.

Let $\mathcal{O}_x = \{\rho_\tau(x) \mid \tau \in G\}$ be the orbit of a point $x \in \Omega$ under the G-action. From the definition of a CIS, if $x \in \Lambda$ then $\rho_\tau(x) \in \Lambda$ for all τ and so Λ contains $\mathcal{O}_x$ for all points $x \in \Lambda$. If $y \in f(\Lambda)$, then there exists an $x \in \Lambda$ such that $f(x) = y$. The image of an orbit under f is also an orbit, and so as $f(\mathcal{O}_x) \subset f(\Lambda)$, and as y is a point on that orbit, we find that $f(\mathcal{O}_x) = \mathcal{O}_y$ and $\mathcal{O}_y \subset f(\Lambda)$. Hence $f(\Lambda)$ is invariant under the action of G, and so forms a CIS. $\square$

Let Ω be a compact metric space on which the group G acts on the right and let $\mathcal{C}$ be the set of CISs for Ω.

Definition 3.4.2. The *inclusion diagram* D_Ω for Ω is a diagram whose objects are the elements of $\mathcal{C}$ and whose arrows $i_{jk}\colon \Lambda_j \to \Lambda_k$ are given by inclusion for every pair j, k such that $\Lambda_j \subset \Lambda_k$.

The *inclusion cohomology diagram* of Ω, denoted $\check{H}^*(D_\Omega)$, is given by the diagram of groups induced by applying the Čech cohomology functor to D_Ω.

Definition 3.4.3. The *quotient diagram* D^Ω for Ω is a diagram with objects Ω/Λ for every $\Lambda \in \mathcal{C}$ and an arrow $q_{jk}\colon \Omega/\Lambda_j \to \Omega/\Lambda_k$ given by the quotient map for every pair j, k such that $\Lambda_j \subset \Lambda_k$.

The *quotient cohomology diagram* of Ω, denoted $\check{H}^*(D^\Omega)$, is given by the diagram of groups induced by applying the Čech cohomology functor to D^Ω.

Remark 3.4.4. Note that all of the arrows appearing in D_Ω and D^Ω commute with the G-action induced on the objects Λ and Ω/Λ for all $\Lambda \in \mathcal{C}$ (The action is well defined on quotients because either an orbit is mapped injectively onto a subspace of Ω/Λ or it is mapped to the point $[\Lambda] \in \Omega/\Lambda$). So, the inclusion and quotient diagrams both admit commuting G-actions.

If D and E are diagrams of groups, we say a collection of homomorphisms $f = \{f_i \colon G \to H \mid G \in D, H \in E\}$ is a *map of diagrams* if the diagram $D \sqcup_f E$ commutes, where the objects of $D \sqcup_f E$ are given by the disjoint union of the objects in D and E and the homomorphisms of $D \sqcup_f E$ are given by the union of the homomorphisms in D and E together with the homomorphisms in f.

Definition 3.4.5. Let $f \colon \Omega \to \Omega'$ be an orbit-preserving map. We define the *induced map on inclusion cohomology diagrams* $f^* \colon \check{H}^*(D_{\Omega'}) \to \check{H}^*(D_\Omega)$ by

$$f^* = \{f|_\Lambda^* \colon \check{H}^*(\Lambda') \to \check{H}^*(\Lambda) \mid f(\Lambda) = \Lambda'\}.$$

Lemma 3.4.1 tells us that this induced map of diagrams of groups f^* is non-empty (for all non-empty Ω').

Lemma 3.4.6. *The induced map on inclusion cohomology diagrams f^* is a map of diagrams of groups.*

Proof. Suppose $f|_\Lambda^*, f|_{\Lambda'}^* \in f^*$ and suppose without loss of generality that $\Lambda \subset \Lambda'$ with inclusion map $i \colon \Lambda \to \Lambda'$. Then we must have $f(\Lambda) \subset f(\Lambda')$ and an inclusion map $j \colon f(\Lambda) \to f(\Lambda')$. If $x \in \Lambda$ then $f|_{\Lambda'}(i(x)) = f|_{\Lambda'}(x) = f(x)$ and $j(f|_\Lambda(x)) = j(f(x)) = f(x)$. So,

$$f|_{\Lambda'} \circ i = j \circ f|_\Lambda$$

and then by applying the Čech cohomology functor we get

$$i^* \circ f|_{\Lambda'}^* = f|_\Lambda^* \circ j^*$$

as required. □

For a CIS Λ of Ω, let $q_\Lambda \colon \Omega \to \Omega/\Lambda$ be the corresponding quotient map. For an orbit-preserving map $g \colon \Omega \to \Omega'$, if $g(\Lambda) = \Lambda'$, then there is a unique map $g_\Lambda \colon \Omega/\Lambda \to \Omega'/\Lambda'$ such that

$$g_\Lambda \circ q_\Lambda = q_{\Lambda'} \circ g$$

Definition 3.4.7. Let $g \colon \Omega \to \Omega'$ be an orbit-preserving map. We define the *induced map on quotient cohomology diagrams* $g^* \colon \check{H}^*(D^{\Omega'}) \to \check{H}^*(D^\Omega)$ by

$$g^* = \{g_\Lambda^* \colon \check{H}^*(\Omega'/\Lambda') \to \check{H}^*(\Omega/\Lambda) \mid g(\Lambda) = \Lambda'\}$$

Lemma 3.4.1 tells us that this induced map g^* is non-empty.

Lemma 3.4.8. *The induced map on quotient cohomology diagrams g^* is a map of diagrams of groups.*

The proof is very similar to the proof of Lemma 3.4.6

Theorem 3.4.9. *Inclusion and quotient cohomology diagrams, together with their induced maps are contravariant functors from the category of G-actions on compact metric spaces and orbit-preserving maps to the category of diagrams of abelian groups and homomorphisms.*

Proof. Let $\Omega \xrightarrow{f} \Omega' \xrightarrow{g} \Omega''$ be a pair of orbit preserving maps. Let Λ be a CIS of Ω. The map of diagrams of groups $f^* \circ g^*$ is the set of all compositions $f|_\Lambda^* \circ g|_{f(\Lambda)}^*$ which by functoriality of cohomology is equal to $(g \circ f)|_\Lambda^*$. The map of diagrams of groups $(g \circ f)^*$ is the set of all maps $(g \circ f)|_\Lambda^*$ for CISs Λ of Ω and so $f^* \circ g^* = (g \circ f)^*$ as required.

A similar argument shows the functoriality of the quotient cohomology diagram. $\square$

Corollary 3.4.10. *Both $\check{H}^*(D_\Omega)$ and $\check{H}^*(D^\Omega)$ are at least as strong an invariant of tiling spaces (up to orbit-equivalence) as Čech cohomology.*

We will see in the next section that examples exist where $\check{H}^*(D_\Omega)$ and $\check{H}^*(D^\Omega)$ can distinguish pairs of spaces whose cohomology coincides. So they are in fact strictly stronger invariants than Čech cohomology on its own.

3.4.2 Invariant Subspaces of Substitution Tiling Spaces

From now on, we assume that ϕ is a strongly aperiodic substitution. Let Ω be the associated tiling space and let $\rho\colon \Omega \times \mathbb{R} \to \Omega$ be the associated flow on Ω given by

$$\rho((w,t),\tau) = (\sigma^{\lfloor t+\tau \rfloor}(w), t+\tau \mod 1).$$

Note that orbits in this setting are precisely the path components of the tiling space. So, even though the previous machinery has been defined for dynamical systems, for tiling spaces the dynamical and topological setting coincide. We could have just as easily considered the set of closed unions of path components, rather than closed invariant subspaces.

Lemma 3.4.11. *Let $\mathcal{C}$ be the set of CISs for a substitution ϕ on the alphabet $\mathcal{A}$. The set $\mathcal{C}$ is finite.*

To reduce notation, we identify without further comment the tilings $T \in \Omega_{\phi_N}$ and $f_{N,0}(T) \in \Omega_\phi$ where $f_{N,0}$ is the induced forgetful map which removes collaring information on a collared letter $a_v \in \mathcal{A}_N$.

Proof. Let $f_N \colon \Gamma_N \to \Gamma_N$ be the induced substitution map on the N-collared AP-complex for ϕ and suppose N is chosen large enough that $\Omega \cong \varprojlim(\Gamma_N, f_N)$, which exists by strong aperiodicity of ϕ and Theorem 3.3.9. Let $\Lambda \in \mathcal{C}$ be a CIS of Ω. As Λ is invariant under translation ρ, the image of Λ under the quotient map $p_N \colon \Omega \to \Gamma_N$ must be a subcomplex of Γ_N.

Now, suppose $\Lambda' \in \mathcal{C}$ and that $p_N(\Lambda) = p_N(\Lambda')$. We want to show that Λ and Λ' must be the same subspace. Suppose for contradiction and without loss of generality, that $\Lambda' \setminus \Lambda$ is non-empty. Let T be a tiling found in Λ' but not Λ. By construction then, T contains a patch of tiles labelled by the word $u \in \mathcal{A}^*$ which does not appear in any tiling in Λ. Given that ϕ_N forces the border, there exists a least natural number n and a non-empty set of N-collared letters $\mathcal{A}_u = \{a_{v_1}, \dots, a_{v_m}\} \subset \mathcal{A}_N$ such that $u \subset \phi^n(v_i)$ for each a_{v_i}. As v_i is legal by definition, a_{v_i} is a legal letter.

Recall that $\phi_N \colon \Omega \to \Omega$ is a homeomorphism by recognisability, and this function maps orbits to orbits, and so $\phi_N^{-n}(\Lambda)$ and $\phi_N^{-n}(\Lambda')$ are CISs of Ω. By construction $\phi_N^{-n}(T)$ is in $\phi_N^{-n}(\Lambda')$ but not $\phi_N^{-n}(\Lambda)$. The tiling $\phi_N^{-n}(T)$ contains a tile $a_{v_i} \in \mathcal{A}_u$ and so there exists a $t \in \mathbb{R}$ so that $T_0 = \phi_N^{-n}(T) - t$ has a tile $a_{v_i} \in \mathcal{A}_u$ which contains the origin in its interior. As Λ and Λ' are CISs, $T_0 \in \phi_N^{-n}(\Lambda')$ and $T_0 \notin \phi_N^{-n}(\Lambda)$. The image of T_0 under the quotient map p_N lies in the interior of the edge of the N-collared AP-complex Γ_N which is labelled by the N-collared letter a_{v_i}. If $p_N(\phi_N^{-n}(\Lambda))$ intersected this edge, then $\phi_N^{-n}(\Lambda)$ would contain a tiling which contained an a_{v_i} tile, but then Λ would contain a tiling which contained a patch labelled by the word u. This contradicts the choice of u not being a patch in any tiling in Λ.

It follows that if $p_N(\Lambda) = p_N(\Lambda')$ for CISs Λ, Λ', then $\Lambda = \Lambda'$. Hence, a CIS is fully determined by the associated subcomplex of the AP-complex which it maps to under the quotient map. There are only finitely many subcomplexes of any AP-complex and so there can only be finitely many CISs in $\mathcal{C}$ of Ω. □

Remark 3.4.12. It is important to note that the choice of N large enough to induce border forcing is key in the proof of the above Lemma. If N is not chosen large enough, then the quotient map $p_N \colon \Omega \to \Gamma_N$ may send distinct CISs to the same subcomplex of Γ_N.

As an example, consider the substitution $\phi\colon a \mapsto aba,\ b \mapsto bbab,\ c \mapsto aa$ whose tiling space has exactly one non-empty proper CIS Λ corresponding to the tilings which do not contain the patch labelled by the word aa (the tiling space associated to the same substitution restricted to the subalphabet $\{a, b\}$). However, if N is chosen to be $N = 0$, and so $\phi_0 = \phi$ does not force the border, then $\Gamma_0 = \Gamma$ is just the wedge of two circles; one given by the edge labelled a and the other given by the edge labelled b. The image of the CIS Λ is the whole AP-complex Γ, which is also the image of the entire tiling space Ω, and so we cannot distinguish these CISs by their image under the map p_0 in this case, as border forcing fails.

We will also see that the substitutions in Proposition 3.5.2 would also fail if N is chosen too small (that is, if $N = 0$). For both substitutions, one would not be able to distinguish the subspaces Λ_4 and $\Lambda_5 = \Omega$ by their images under the quotient map.

For any $M \geq 1$, it is not hard to find examples where for any choice $N \geq M$ we are able to distinguish all CISs by their images under p_N, but there exist CISs Λ, Λ' such that if $N < M$ then $p_N(\Lambda) = p_N(\Lambda')$.

Theorem 3.4.13. *Let $f_N \colon \Gamma_N \to \Gamma_N$ be the induced substitution map on the N-collared AP-complex for ϕ. There exists an integer n so that for all $\Lambda \in \mathcal{C}$, there exists a subcomplex $\Gamma_\Lambda \subset \Gamma_N$ such that $f_N^n(\Gamma_\Lambda) = \Gamma_\Lambda$ and $\varprojlim(\Gamma_\Lambda, f_N^n) = \Lambda$.*

Proof. As ϕ is recognisable, the substitution acts as a homeomorphism on Ω and so the substitution permutes CISs of the tiling space. By Lemma 3.4.11, $\mathcal{C}$ is finite. As such, an integer n can be chosen so that $\phi^n(\Lambda) = \Lambda$ for all $\Lambda \in \mathcal{C}$.

Let $p_N \colon \Omega \to \Gamma_N$ be the quotient map from the tiling space to the N-collared AP-complex. Let $p = p_N|_\Lambda$, be the restriction of the quotient map to Λ. As Λ is a CIS, the image Γ_Λ of p is a subcomplex of Γ_N. Recall that $p_N \circ \phi = f_N \circ p_N$, and so

$$p \circ \phi^n = f_N^n \circ p. \tag{3.2}$$

Let $h_\Lambda \colon \Lambda \to \varprojlim(\Gamma_\Lambda, f_N^n)$ be defined by

$$h_\Lambda(x) = (p(x), p(\phi^{-n}(x)), p(\phi^{-2n}(x)), \ldots)$$

which is well defined by 3.2. As h_Λ is a telescoped version of h with modified domain and codomain, it is clearly injective, so it only remains to show that h_Λ is surjective onto the inverse limit.

A point in the inverse limit corresponds to a unique tiling in the tiling space as ϕ_N forces the border. Suppose $(x_0, x_1, x_2, \ldots) \in \varprojlim(\Gamma_\Lambda, f_N^n)$ was not in the image of h_Λ, then there exists some i for which the patch described by the finite subsequence of points $(x_0, x_1, \ldots, x_i)$ does not appear in a tiling in Λ. But this means that the shifted sequence $(x_i, x_{i+1}, \ldots)$ is also not in the image of h_Λ, as the shift is a homeomorphism, and so the point $x_i \in \Gamma_\Lambda$ must not describe the label w_0 of the tile at the origin of any tiling in Λ. This is impossible by how the Anderson-Putnam complex and the quotient map p are defined, as Γ_Λ is the image of Λ under p. It follows that no such point in the inverse limit exists and h_Λ is surjective. $\square$

3.4.3 Identifying Closed Invariant Subspaces

Let K be a subcomplex of Γ_N and let $EV(K) = \bigcup_{i\geq 0}(f_N^n)^i(K)$ be the *eventual range* of K. The eventual range of a subcomplex is itself a subcomplex. The set of eventual ranges $EV = \{EV(K) \mid K \text{ is a subcomplex of } \Gamma_N\}$ therefore forms a finite set. As has been shown, every CIS in $\mathcal{C}$ corresponds to a unique subcomplex in EV.

Proposition 3.4.14.

$$|\mathcal{C}| \leq |EV|$$

We show by example that this expression may be a strict inequality in general.

Example 3.4.15. The Chacon substitution $\phi\colon a \mapsto aaba, b \mapsto b$ is minimal and so $|\mathcal{C}| = 1$. The Chacon substitution is recognisable and tame, and we find that the set of bounded words for ϕ is $B = \{b\}$, so $N = 2$. It has a 2-collared AP-complex with exactly two subcomplexes which are eventual ranges. These are given by the single edge $[b_{babaa}]$ and the entire complex Γ_2. It follows that $|EV| = 2$ and so $|EV| > |\mathcal{C}|$. Obviously, the image of Ω under $p_2\colon \Omega \to \Gamma_2$ is the whole of Γ_2, and so the edge $[b_{babaa}]$ does not correspond to any CIS of ϕ.

Note that the expression in Proposition 3.4.14 is an equality if and only if the set of bounded words B is empty, as an inverse sequence of collections of edges which are only labelled by bounded letters can only correspond to an invariant subspace whose tilings have bounded words of arbitrary length, hence ϕ_N would in such a case be wild. By Theorem 3.3.7 and our assumption that ϕ is strongly aperiodic, this cannot happen. Hence, we can refine Proposition 3.4.14. Let EV_∞ be the set of eventual ranges of all possible finite unions of edges labelled by N-collared letter whose length grows without bound under the substitution ϕ_N. That is, if $a_1, \ldots, a_k \in (\mathcal{A}_N)_\infty$ let $K_{a_1,\ldots,a_k} = \bigcup_{i\geq 0}(f_N^n)^i([a_1] \cup \ldots \cup [a_k])$, then

$$EV_\infty = \{K_{a_1,\ldots,a_k} \mid \{a_1, \ldots, a_k\} \subset (\mathcal{A}_N)_\infty\}.$$

Proposition 3.4.16. *There is a one-to-one correspondence*

$$EV_\infty \longleftrightarrow \mathcal{C}.$$

Moreover, for every $\Gamma \in EV_\infty$, there is a unique CIS $\Lambda \in \mathcal{C}$ such that $\Gamma = \Gamma_\Lambda$, and so $\varprojlim(\Gamma, f_N^n) \cong \Lambda$.

In general, each subcomplex in EV_∞ can be found by substituting a finite collection of edges under f_N^n corresponding to an element of the alphabet $(\mathcal{A}_N)_\infty$, the expanding N-collared letters.

This gives us a general method for identifying CISs when ϕ is strongly aperiodic: Find all N-collared letters in $(\mathcal{A}_N)_\infty$; Substitute the letter $a \in (\mathcal{A}_N)_\infty$ and for each $i \geq 0$ record the set of letters appearing in the substituted word $\phi_N^i(a)$; When this collection of sets of letters no longer changes, move to the next letter $b \in (\mathcal{A}_N)_\infty$ and repeat until all letters in $(\mathcal{A}_N)_\infty$ have been exhausted; Find a natural number n such that for every $a \in (\mathcal{A}_N)_\infty$ the letters appearing in $\phi_N^{kn}(a)$ are the same for all k, which exists by the above arguments; the sets of letters appearing in the words $\phi_N^n(a)$ correspond to a unique subcomplex of Γ_N for each a. Every subcomplex of Γ_N in EV_∞ then is a finite union of subcomplexes 'born' from such iterated expanding letters, which in turn each correspond to a CIS of Ω. Every CIS is identified in such a way.

Altogether, this gives us the (crude) bound

Proposition 3.4.17.

$$|EV_\infty| = |\mathcal{C}| \leq 2^{|(\mathcal{A}_N)_\infty|}$$

3.4.4 Cohomology of Quotients

Given that we are considering closed subspaces of tiling spaces, the exact sequence in relative cohomology is a valid method for calculating the cohomology of Ω in terms of $\check{H}^i(\Lambda)$ and $\check{H}^i(\Omega/\Lambda)$ for some CIS Λ of Ω. Indeed, this is one method of calculation which is effective and we give an example of such a calculation in Example 3.5.1. So, as well as fitting into the quotient cohomology diagram of a non-minimal tiling space, the groups $\check{H}^i(\Omega/\Lambda)$ are also useful in calculating the cohomology of the entire space Ω in terms of smaller, more manageable pieces.

With this in mind, we identify the cohomology of Ω/Λ with the direct limit of cohomology groups of the quotient complex $\Gamma^\Lambda = \Gamma_N/\Gamma_\Lambda$. The associated inverse limit of quotient complexes is in fact homoemorphic to the quotient space Ω/Λ. However, the proof is more involved than what follows, and we do not need this result to calculate cohomology. We refer the reader to our paper [47] for a detailed proof of the existence of this homeomorphism.

Let Λ be a CIS of Ω and let Γ_Λ be the associated subcomplex of Γ_N as identified in the previous section which, to some power n of the substitution ϕ_N, is a fixed subcomplex. Let $q\colon \Gamma_N \to \Gamma^\Lambda$ be the quotient map. Let $f_\Lambda\colon \Gamma^\Lambda \to \Gamma^\Lambda$ be the unique continuous map such that $q \circ f_N^n = f_\Lambda \circ q$.

Theorem 3.4.18. *There is an isomorphism*

$$\check{H}^i(\Omega/\Lambda) \cong \varinjlim(\tilde{H}^i(\Gamma^\Lambda), f_\Lambda^*).$$

Proof. Consider the commuting diagram

$$\begin{tikzcd}
\Lambda \arrow[rr, hook] \arrow[dr, "\phi_N^n|_\Lambda"'] \arrow[dd] & & \Omega \arrow[rr, two heads] \arrow[dr, "\phi_N^n"'] \arrow[dd] & & \Omega/\Lambda \arrow[dr, "\phi_N^n"'] \arrow[dd] & \\
& \Lambda \arrow[rr, hook] \arrow[dd, "p_N"] & & \Omega \arrow[rr, two heads] \arrow[dd, "p_N"] & & \Omega/\Lambda \arrow[dd, "p_N"] \\
\Gamma_\Lambda \arrow[rr, hook] \arrow[dr, "f_N^n|_{\Gamma_\Lambda}"'] & & \Gamma_N \arrow[rr, two heads] \arrow[dr, "f_N^n"'] & & \Gamma^\Lambda \arrow[dr, "f_\Lambda"'] & \\
& \Gamma_\Lambda \arrow[rr, hook] & & \Gamma_N \arrow[rr, two heads] & & \Gamma^\Lambda
\end{tikzcd}$$

where the rows are the cofibration sequences associated to the inclusions of each of the closed subspaces Λ and Γ_Λ into Ω and Γ_N respectively. The upper diagonal maps are (nth powers of) the substitution homeomorphisms on each of the spaces Λ and Ω and the induced homeomorphism on the quotient Ω/Λ. The lower diagonal maps are the bonding maps of the associated inverse sequences of complexes.

By passing to the long exact sequence in reduced Čech cohomology associated to a cofibration sequence, and taking direct limits along the induced bonding homomorphisms, this gives us a map between LESs as follows (bonding maps of direct limits are implicit by context).

$$\begin{tikzcd}
\check{H}^{i-1}(\Omega) \arrow[r] & \check{H}^{i-1}(\Lambda) \arrow[r] & \check{H}^i(\Omega/\Lambda) \arrow[r] & \check{H}^i(\Omega) \arrow[r] & \check{H}^i(\Lambda) \\
\varinjlim(\tilde{H}^{i-1}(\Gamma_N)) \arrow[r] \arrow[u, "\alpha_{i-1}"] & \varinjlim(\tilde{H}^{i-1}(\Gamma_\Lambda)) \arrow[r] \arrow[u, "\beta_{i-1}"] & \varinjlim(\tilde{H}^{i-1}(\Gamma^\Lambda)) \arrow[r] \arrow[u, "\gamma_i"] & \varinjlim(\tilde{H}^i(\Gamma_N)) \arrow[r] \arrow[u, "\alpha_i"] & \varinjlim(\tilde{H}^i(\Gamma_\Lambda)) \arrow[u, "\beta_i"]
\end{tikzcd}$$

The homomorphisms α_{i-1} and α_i are isomorphisms by Theorem 3.3.9, and the homomorphisms β_{i-1} and β_i are isomorphisms by Theorem 3.4.13. By the Five Lemma, this implies that γ_i is an isomorphism. $\square$

3.5 Examples

Example 3.5.1. We define the *Fibonacci substitution with one handle* to be given by

$$\phi\colon 0 \mapsto 001,\ 1 \mapsto 01,\ 2 \mapsto 021$$

By substituting 1-collared letters (and noting that $B = \emptyset$), we find that there are two non-empty invariant subcomplexes Γ_{Λ_1} and Γ_{Λ_2}, both fixed under ϕ_1, corresponding to the collections of 1-collared letters

$$\Gamma_{\Lambda_1} = \cup\{[0_{001}], [1_{010}], [0_{100}], [0_{101}]\}$$

and

$$\Gamma_{\Lambda_2} = \cup\{[0_{001}], [1_{010}], [2_{021}], [0_{100}], [0_{101}], [0_{102}], [1_{210}]\}.$$

The 1-collared AP-complex appears in Figure 3.5.1. An oriented edge from ab to bc denotes an edge labelled by the the letter b_{abc} in the alphabet $\mathcal{A}_1$ of 1-collared letters.

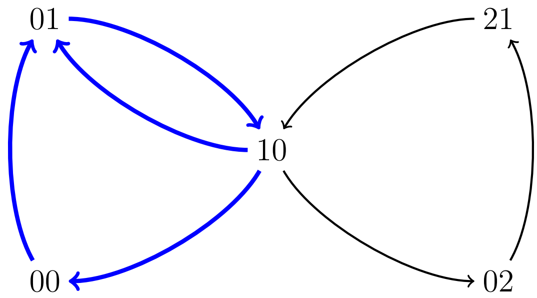


Fig. 3.1: The 1-collared AP-complex for the Fibonacci substitution with one handle, with the subcomplex Γ_{Λ_1} coloured blue.

The subcomplex Γ_{Λ_1} in blue corresponds to a CIS given by considering the restriction of the substitution to the subalphabet $\{0, 1\}$ which is (a re-encoding of) the Fibonacci substitution which is connected and has first cohomology $\check{H}^1(\Omega_{Fib}) \cong \mathbb{Z}^2$. The subcomplex Γ_{Λ_2} corresponds to the CIS which is the entire tiling space, which is connected and has first cohomology $\check{H}^1(\Omega) \cong \varinjlim \left(\mathbb{Z}^3, \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right)\right) \cong \mathbb{Z}^3$, where the unimodular matrix $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ is found by choosing appropriate generators of $H^1(\Gamma_1)$. The only other CIS is the emptyset.

So $\check{H}^*(D_\Omega)$ is given by the diagrams

$$\begin{aligned} \check{H}^0(D_\Omega)\colon &\ \mathbb{Z} \to \mathbb{Z} \to 0 \\ \check{H}^1(D_\Omega)\colon &\ \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0 \end{aligned}$$

We can use Theorem 3.4.18 to see, as $\Gamma_1/\Gamma_{\Lambda_1}$ is a circle and ϕ_1 acts on this quotient complex by a map which is homotopic to the identity, that $\check{H}^i(\Omega/\Lambda_1) \cong \varinjlim(H^i(S^1), \mathrm{Id}) \cong \mathbb{Z}$ for $i = 0, 1$.

So $\check{H}^*(D^\Omega)$ is given by the diagrams

$$\begin{aligned} \check{H}^0(D^\Omega)\colon &\ \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \\ \check{H}^1(D^\Omega)\colon &\ 0 \to \mathbb{Z} \to \mathbb{Z}^3 \end{aligned}$$

Alternatively, we could have used the fact that Λ_1 is a closed connected subspace of Ω and so we get an exact sequence in reduced Čech cohomology

$$0 \to \check{H}^1(\Omega/\Lambda) \to \check{H}^1(\Omega) \to \check{H}^1(\Lambda) \to 0$$

which splits (as $\check{H}^1(\Lambda) \cong \mathbb{Z}^2$) to give $\check{H}^1(\Omega) \cong \check{H}^1(\Omega/\Lambda) \oplus \mathbb{Z}^2$. As above, we can identify $\check{H}^1(\Omega/\Lambda)$ with $H^1(S^1)$ and so $\check{H}^1(\Omega) \cong \mathbb{Z}^3$.

This distinguishes Ω from the tiling space associated to the Tribonacci substitution which has $\check{H}^0(\Omega_{\mathrm{Trib}}) \cong \mathbb{Z}$ and $\check{H}^1(\Omega_{\mathrm{Trib}}) \cong \mathbb{Z}^3$ but no proper, non-empty CISs. So the diagrams $\check{H}^*(D_{\Omega_{\mathrm{Trib}}})$ and $\check{H}^*(D^{\Omega_{\mathrm{Trib}}})$ have a different shape and so cannot be isomorphic to the diagrams for ϕ.

Consider the following two substitutions.

'Two Tribonaccis with a bridge':

$$\phi_1\colon 0 \mapsto 0201,\ 1 \mapsto 001,\ 2 \mapsto 0,\ \overline{0} \mapsto \overline{0201},\ \overline{1} \mapsto \overline{001},\ \overline{2} \mapsto \overline{0},\ X \mapsto 1\overline{0}$$

and 'Quadribonacci and Fibonacci with a bridge':

$$\phi_2\colon 0 \mapsto 0201,\ 1 \mapsto 0301,\ 2 \mapsto 001,\ 3 \mapsto 0,\ \overline{0} \mapsto \overline{001},\ \overline{1} \mapsto \overline{01},\ X \mapsto 1\overline{0}$$

Proposition 3.5.2. *$\check{H}^*(\Omega_{\phi_1})$ is isomorphic to $\check{H}^*(\Omega_{\phi_2})$ but they have degree 1 inclusion cohomology diagrams*

$$\check{H}^1(D_{\Omega_1}): \begin{tikzcd} & \mathbb{Z}^6 \arrow[d] & \\ & \mathbb{Z}^6 \arrow[dl] \arrow[dr] & \\ \mathbb{Z}^3 \arrow[dr] & & \mathbb{Z}^3 \arrow[dl] \\ & 0 & \end{tikzcd} \qquad \check{H}^1(D_{\Omega_2}): \begin{tikzcd} & \mathbb{Z}^6 \arrow[d] & \\ & \mathbb{Z}^6 \arrow[dl] \arrow[dr] & \\ \mathbb{Z}^4 \arrow[dr] & & \mathbb{Z}^2 \arrow[dl] \\ & 0 & \end{tikzcd}$$

and degree 1 *quotient cohomology diagrams*

$$\check{H}^1(D^{\Omega_1}): \begin{tikzcd} & 0 \arrow[d] & \\ & \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathbb{Z}^3 \arrow[dr] & & \mathbb{Z}^3 \arrow[dl] \\ & \mathbb{Z}^6 & \end{tikzcd} \qquad \check{H}^1(D^{\Omega_2}): \begin{tikzcd} & 0 \arrow[d] & \\ & \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathbb{Z}^2 \arrow[dr] & & \mathbb{Z}^4 \arrow[dl] \\ & \mathbb{Z}^6 & \end{tikzcd}$$

Proof. For both substitutions $N = 1$. For ϕ_1, the 1-collared alphabet is given by

$$\begin{aligned} \mathcal{A}_1 \;\; = \;\; & \{0_{001}, 0_{002}, 1_{010}, 2_{020}, 0_{100}, 0_{102}, 0_{201}\} \cup \{1_{01\bar{0}}, \bar{0}_{\overline{102}}\} \cup \\ & \{\bar{0}_{\overline{001}}, \bar{0}_{\overline{002}}, \bar{1}_{\overline{010}}, \bar{2}_{\overline{020}}, \bar{0}_{\overline{100}}, \bar{0}_{\overline{102}}, \bar{0}_{\overline{201}}\} \end{aligned}$$

The 1-collared AP-complex for ϕ_1 is given in Figure 3.2.

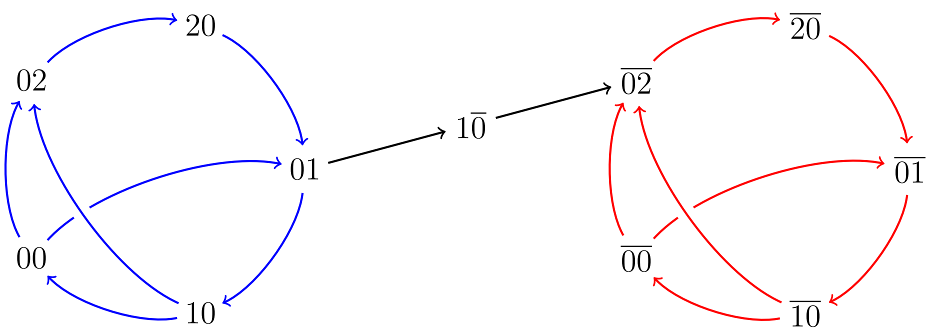


Fig. 3.2: The 1-collared AP-complex for the ‘Two Tribonaccis with a bridge’ substitution.

We note that Ω_1 has five CISs:

- $\Lambda_1 = \emptyset$, the empty set
- Λ_2, given by restricting the substitution to the alphabet $\{0, 1, 2\}$
- Λ_3, given by restricting the substitution to the alphabet $\{\bar{0}, \bar{1}, \bar{2}\}$

- $\Lambda_4 = \Lambda_2 \sqcup \Lambda_3$, the union of the disjoint CISs above, given by restricting the substitution to the alphabet $\{0,1,2,\overline{0},\overline{1},\overline{2}\}$

- $\Lambda_5 = \Omega_1$, the full tiling space

We note that a choice of 1-cycles generating the homology $H_1(\Gamma_1)$ of the AP-complex can be given by the oriented sum of edges

$$\begin{aligned}
\gamma_1 &= [2_{020}] + [0_{201}] + [1_{010}] + [0_{100}] + [0_{002}] \\
\gamma_2 &= [2_{020}] + [0_{201}] + [1_{010}] + [0_{102}] \\
\gamma_3 &= [0_{100}] + [0_{001}] + [1_{010}] \\
\gamma_4 &= [\overline{2}_{\overline{020}}] + [\overline{0}_{\overline{201}}] + [\overline{1}_{\overline{010}}] + [\overline{0}_{\overline{100}}] + [\overline{0}_{\overline{002}}] \\
\gamma_5 &= [\overline{2}_{\overline{020}}] + [\overline{0}_{\overline{201}}] + [\overline{1}_{\overline{010}}] + [\overline{0}_{\overline{102}}] \\
\gamma_6 &= [\overline{0}_{\overline{100}}] + [\overline{0}_{\overline{001}}] + [\overline{1}_{\overline{010}}]
\end{aligned}$$

and the substitution acts on these 1-cycles by the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

hence the cohomology of Ω_1 is then given by the direct limit of the transpose of M. Write $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ in block matrix form. Note that M_1, M_2 and M are all unimodular. Using Theorem 3.4.13, we can identify each CIS with an inverse limit of the substitution (in this case to the first power) acting on a particular subcomplex of Γ_1. These subcomplexes can be found using the method outlined in the above discussion. We have chosen our generators γ_i in such a way that the homology of each subcomplex is generated by some subset of these 1-cycles. We calculate

$$\begin{aligned}
\check{H}^1(\Lambda_1) &= \check{H}^1(\emptyset) &&= 0 \\
\check{H}^1(\Lambda_2) &= \varinjlim(\mathbb{Z}^3_{\langle\gamma_1,\gamma_2,\gamma_3\rangle}, M_1^T) &&= \mathbb{Z}^3 \\
\check{H}^1(\Lambda_3) &= \varinjlim(\mathbb{Z}^3_{\langle\gamma_4,\gamma_5,\gamma_6\rangle}, M_2^T) &&= \mathbb{Z}^3 \\
\check{H}^1(\Lambda_4) &= \check{H}^1(\Lambda_2) \oplus \check{H}^1(\Lambda_3) &&= \mathbb{Z}^6 \\
\check{H}^1(\Lambda_5) &= \varinjlim(\mathbb{Z}^3_{\langle\gamma_1,\dots,\gamma_6\rangle}, M^T) &&= \mathbb{Z}^6
\end{aligned}$$

From here, we can use the exact sequence in reduced Čech cohomology to find the cohomology groups of each of the quotients, except for the quotient Ω/Λ_4, where the exact sequence does not reduce nicely to a split short exact sequence, since

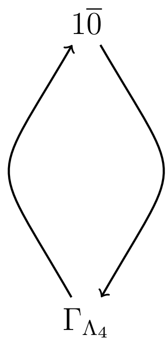


Fig. 3.3: The quotient complex $\Gamma_1/\Gamma_{\Lambda_4}$ for the 'Two Tribonaccis with a bridge' substitution.

$\tilde{H}^0(\Lambda_4) \cong \mathbb{Z}$, as Λ_4 is composed of exact two connected components.

Hence, to find the cohomology of the quotient Ω/Λ in this case, we use Theorem 3.4.18, and identify the quotient complex $\Gamma_1/\Gamma_{\Lambda_4}$, given in Figure 3.3. This quotient complex is a circle. The induced substitution acts on $\Gamma_1/\Gamma_{\Lambda_4}$ by a map which is homotopic to the identity, and so $\check{H}^1(\Omega/\Lambda_4) \cong \varinjlim(H^1(S^1), \mathrm{Id}^*) \cong \mathbb{Z}$.

In the case of the second considered substitution, we give the 1-collared AP-complex for ϕ_2 in Figure 3.4. The calculation of cohomology for ϕ_2 is similar to ϕ_1, except we can choose six generating 1-cycles of $H_1(\Gamma_1)$ in such a way that the matrix M acting on H_1 has a block diagonal structure with M_1 and M_2 of ranks 4 and 2 respectively, and with all three of M_1, M_2, M still being unimodular. We leave the details to the reader.

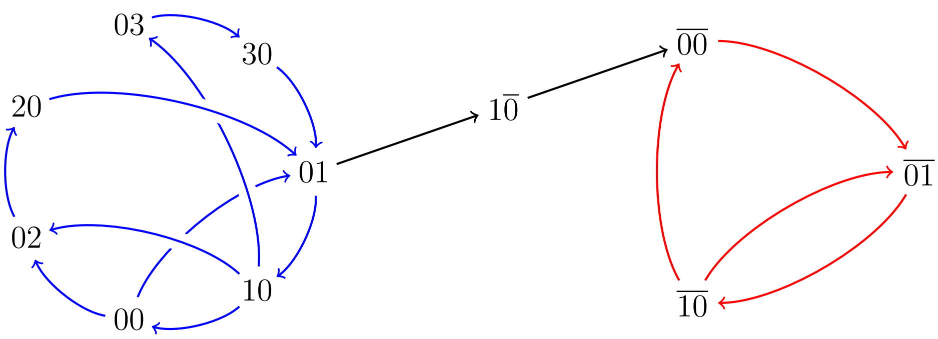


Fig. 3.4: The 1-collared AP-complex for the 'Quadribonacci and Fibonacci with a bridge' substitution.

□

Hence, $\check{H}^1(D_\Omega)$ and $\check{H}^1(D^\Omega)$ can distinguish tiling spaces which have the same cohomology and the same shape of diagrams of CISs.

3.5.1 Discussion

Barge-Diamond Complexes for Non-primitive Substitution

We provide some informal discussion for likely useful avenues of future study building on the work outlined in this Chapter.

One may ask why we have been using collared Anderson-Putnam complexes and not Barge-Diamond complexes in this section. It is a valid question and some thought has been given to its answer. Mostly, (a slightly modified version of) the BD-complex is a suitable replacement for the N-collared AP-complex and most of the previous section would hold with very little changed. However, the advantages afforded to the Barge-Diamond method are less apparent when there exist bounded letters in the alphabet. When all letters are expanding and the substitution is strongly aperiodic, a very similar argument to the original proof presented by Barge and Diamond [7] will carry through, and one can then apply the usual method of replacing the induced substitution on the BD-complex with a homotopic map which is simplicial on the vertex-edges.

When there exist bounded words in the subshift, the usual BD-complex with an ϵ-ball collaring at each point[2] does not suffice to get the necessary homeomorphism to the inverse limit (for broadly the same reasons that the 1-collaring does not suffice to induce border-forcing when B is non-empty).

Instead, the approach that one could take is to collar points with a ball of radius $N - 1 + \epsilon$ at each point—this is equivalent to replacing the substitution with its $(N - 1)$-collared substitution and then using the ϵ-ball collaring on this collared substitution (and so we are using the usual BD-complex $K_{\phi_{N-1}}$ for the collared substitution ϕ_{N-1}). This has the advantage of needing to collar out one fewer times than in the AP-complex approach. Moreover, we can still replace the induced substitution map with a homotopic map which acts simplicially on the distinguished subcomplex of transition edges. Unlike in the minimal case, it is not necessarily true that $\tilde{H}^0(K_{\phi_{N-1}})$ is trivial, as Ω_ϕ may have multiple connected components and so extension problems coming from the Barge-Diamond exact sequence will in general be more difficult.

To illustrate this alternative method, we briefly go over an example calculation using the Chacon substitution.

[2] See [9] for an explanation of what it means to *collar points* in the tiling, instead of collaring tiles.

Example 3.5.3. Let ϕ be given by $\phi\colon a \mapsto aaba,\ b \mapsto b$, the Chacon substitution on the alphabet $\{a, b\}$. Let

$$1 = a_{aaa},\ 2 = a_{aab},\ 3 = b_{aba},\ 4 = a_{bab},\ 5 = a_{baa}.$$

The 1-collared substitution is given by

$$\phi_1\colon 1 \mapsto 1235,\ 2 \mapsto 1234,\ 3 \mapsto 3,\ 4 \mapsto 5234, \colon\ 5 \mapsto 5325$$

and the BD-complex is given in Figure 3.5.

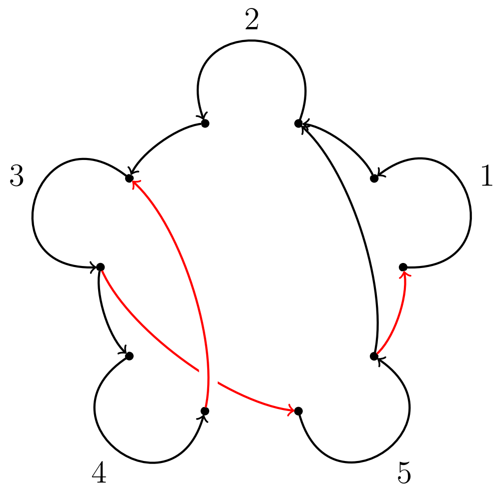


Fig. 3.5: The Barge-Diamond complex K_{ϕ_1} for the 1-collared Chacon substitution with the subcomplex of transition edges in the eventual range coloured red

The eventual range of the map g acting on the subcomplex S of transition edges is the collection $\{e_{35}, e_{43}, e_{51}\}$ coloured in red. The substitution acts on this eventual range like the identity. Note that S has exactly three connected components, all of which are contractible. It follows that the Barge-Diamond exact sequence for this substitution is given by

$$0 \to \mathbb{Z}^2 \to \varinjlim \left(\mathbb{Z}^5, \begin{pmatrix} 1&1&1&0&1 \\ 1&1&1&1&0 \\ 0&0&1&0&0 \\ 0&1&1&1&1 \\ 0&1&1&0&2 \end{pmatrix} \right) \to \check{H}^1(\Omega) \to 0 \to 0$$

Experience with examples seems to suggest that it is often the case that the eventual range of S under the induced substitution will often have multiple connected components whenever $N > 1$ and especially when ϕ is not minimal, and so we seem to lose the advantage normally afforded to us with Barge-Diamond calculation where it is often the case that the exact sequence splits. In fact, it is probably more efficient in the above example to directly find generators of the cohomology of the entire complex K_{ϕ_1} (where in this case there are only three generators) and to calculate the induced substitution on $H^1(K_{\phi_1})$ in order to calculate $\check{H}^1(\Omega)$. If we do that, we find that $\check{H}^1(\Omega) \cong \varinjlim \left(\mathbb{Z}^3, \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 1 \end{smallmatrix}\right)\right)$.

Extensions of Substitutions by Other Substitutions

So far, our only examples of non-minimal substitutions that have been presented have been relatively tame—the tiling spaces have all been a finite collection of minimal tiling spaces which are possibly connected by a finite number of path components which asymptotically approach some sub-collection of the minimal sets. In particular, by quotienting out by the disjoint union of the minimal sets, we are left with a space homeomorphic to a cell complex. While these spaces are interesting, and serve as good test cases for our machinery, the range of possible behaviours for non-minimal substitutions is much more varied.

For instance, we could break the asymptotic behaviour in the above described examples, and instead have new path components which approach minimal sets proximally, instead of asymptotically. Such an example is given by

$$\phi\colon 0 \mapsto 001,\ 1 \mapsto 01,\ 2 \mapsto X021X,\ X \mapsto X$$

whose proximal path component is the orbit of the word

$$\ldots X00100101X001X021X01X00101X\ldots$$

where the sparse appearances of the symbols X, which become more rare the further one travels from the single 2, serves to break the asymptotic nature of the handle. One might call this substitution the *Fibonacci with one proximal handle substitution*. Note that the inclusion and quotient cohomology diagrams of this substitution and the Fibonacci with one handle substitution are isomorphic, while they are most certainly not homeomorphic tiling spaces (consider the complement of small neighbourhoods of the minimal set Ω_{Fib} in each example).

Example 3.5.4. Another interesting examples is given by

$$\phi\colon 0 \mapsto 001,\ 1 \mapsto 01,\ a \mapsto aaba,\ b \mapsto b,\ X \mapsto 1aXa0$$

which has a Fibonacci minimal CIS, a Chacon minimal CIS, two single path components associated to the sequences

$$\ldots 0010010100101aabaaabaaba\ldots$$
$$\ldots aabaaabaaba0010010100100\ldots$$

which are asymptotic to the two distinct minimal sets in either direction, and a final path component associated to the sequence

$$\ldots 00101aabaaababaaba01aaba1aXa0aaba001aabaaababaaba00100101\ldots$$

which is proximal to all of the other CISs in both directions and asymptotic to none.

To support this direction of exploring more varied behaviour, we introduce a curious family of examples where the quotients of the tiling space by the CISs are of particular interest, and where there is a natural factor map onto the minimal set of the tiling space. In particular the complement $\Omega \setminus \Lambda_{\text{min}}$ will often have uncountably many path components, where Λ_{min} is the disjoint union of the minimal CISs. One might think of this construction as 'extending' one substitution by another in a proximal fashion. Similar examples appear naturally in the literature [6].

Suppose that ϕ and ψ are primitive substitutions on $\mathcal{A}$ and $\mathcal{B}$ respectively and suppose that $|\mathcal{B}| \leq |\mathcal{A}|$. Let $i\colon \mathcal{B} \to \mathcal{A}$ be an injection. Assume that for each $b \in \mathcal{B}$, if $\psi(b) = b_1 \ldots b_n$, then there exists an interior subsequence $(a_{k_1}, \ldots, a_{k_n})$ of $\phi(i(b)) = a_1 \ldots a_m$ of the form $(i(b_1), \ldots, i(b_n))$ (if not, take a high enough power of ϕ so that there is). Here by interior, we mean that $a_{k_1} \neq a_1$ and $a_{k_n} \neq a_m$.

Definition 3.5.5. Let ϕ and ψ be as above and choose an injection $i\colon \mathcal{B} \to \mathcal{A}$ and a set of subsequences $S = \{s_b = (a_{k_1}, \ldots, a_{k_n}) \mid b \in \mathcal{B}\}$ of $\phi(i(b))$ as above.

Define a new substitution $[\phi, \psi]_S$ on the alphabet $\mathcal{A} \sqcup \mathcal{B}$ by $[\phi, \psi]_S(a) = \phi(a)$ for all $a \in \mathcal{A}$ and for $b \in \mathcal{B}$ by $[\phi, \psi]_S(b) = \phi(i(b))$ except replace the occurrence of a_{k_j} with b_j.

There is a natural factor map $\Omega_{[\phi,\psi]_S} \to \Omega_\phi$ given by mapping the letters $b \in \mathcal{B}$ to $i(b)$.

Example 3.5.6. If $\phi\colon 0 \mapsto 00100101,\ 1 \mapsto 00101$ and $\psi\colon a \mapsto aa$, then we could choose the injection $a \mapsto 0$ and then choose as the subsequence of $\phi(i(a)) = 0_{(1)}0_{(2)}1_{(3)}0_{(4)}0_{(5)}1_{(6)}0_{(7)}1_{(8)}$ the sequence $(0_{(4)}, 0_{(5)})$ so $S = \{(0_{(4)}, 0_{(5)})\}$. Then our extended substitution $[\phi, \psi]_S$ is given by

$$[\phi, \psi]_S\colon 0 \mapsto 00100101,\ 1 \mapsto 00101,\ a \mapsto 001aa101.$$

Example 3.5.7. Let $\psi = \mathrm{Id}$ be the substitution on the alphabet $\{x\}$ given by $\mathrm{Id}(x) = x$, and let $i\colon \{x\} \to \mathcal{A}$ be given by $i(x) = a$ for some $a \in \mathcal{A}$. As ϕ is primitive by assumption, let a_{k_1} be an occurrence of the letter a in the interior of the word $\phi^n(a)$ for some positive natural n. Let $S = \{(a_{k_1})\}$.

The substitution $[\phi, \mathrm{Id}]_S$ is just the substitution ϕ with a single handle. That is, the tiling space for $[\phi, \mathrm{Id}]_S$ is just the tiling space for ϕ with a single extra one-dimensional path component which asymptotically approaches the minimal component in both directions. The image of the handle under the factor map onto Ω_ϕ is precisely the orbit of the limit word $\lim_{j\to\infty} \phi^{jn}(a)$ expanded about the interior letter a_{k_1} appearing in $\phi^n(a)$. By iterating this method, we can add as many handles as we like.

Example 3.5.8. The substitution

$$[\mathrm{TM}, \mathrm{PD}]\colon 0 \mapsto 01101001,\ 1 \mapsto 10010110,\ a \mapsto 011ab001,\ b \mapsto 10a1a110$$

is an extension of the (cube of the) Thue-Morse substitution $\mathrm{TM}\colon 0 \mapsto 01,\ 1 \mapsto 10$ by the period doubling substitution $\mathrm{PD}\colon a \mapsto ab,\ b \mapsto aa$.

In general, the substitution tiling space $\Omega_{[\phi,\psi]_S}$ has exactly one non-empty proper CIS which is exactly the tiling space Ω_ϕ given by restriction of the substitution to the subalphabet $\mathcal{A}$.

There is a close relationship between the quotient complex Γ_{Ω_ϕ} and the AP-complex Γ_ψ of the substitution ψ. Let $f\colon \Gamma_{\Omega_\phi} \to \Gamma_{\Omega_\phi}$ and $g\colon \Gamma_\psi \to \Gamma_\psi$ be the respective bonding maps. It would appear that more often than not there is a map $h\colon \Gamma_{\Omega_\phi} \to \Gamma_\psi$ which conjugates these bonding maps up to homotopy, that is $g \circ h \simeq h \circ f$. This would seem to suggest a close relationship between the spaces $\Omega_{[\phi,\psi]_S}/\Omega_\phi$ and Ω_ψ, perhaps up to shape equivalence[3].

Question 3.5.9. What is the relationship between $\Omega_{[\phi,\psi]_S}/\Omega_\phi$ and Ω_ψ. What conditions on ϕ and ψ are needed for this relationship to hold?

We note that in Example 3.5.6 above, ψ is not aperiodic, and we suspect that $\Omega_{[\phi,\psi]_S}/\Omega_\phi$ is shape equivalent to the dyadic solenoid—in fact $\Gamma_1/\Gamma_{\Omega_\phi}$ is a circle, and the induced substitution $[\phi,\psi]_S$ on this quotient complex is homotopic to the doubling map, so by Theorem 3.4.18 the first cohomology of the quotient is given by $\check{H}^1(\Omega_{[\phi,\psi]_S}/\Omega_\phi) \cong \mathbb{Z}[1/2]$. As Ω_ψ in this case is a circle with $\check{H}^1(\Omega_\psi) \cong \mathbb{Z}$, it would appear that the relationship hinted at above relies at least on the recognisability of ψ.

If S is chosen differently, then the quotient complex Γ_{Ω_ϕ} can be different. For example if $S' = \{(0_{(2)}, 0_{(7)})\}$ then Γ_{Ω_ϕ} is homotopy equivalent to a wedge of two circles. However, the induced map on cohomology acts like the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and so we still find that $\check{H}^1(\Omega_{[\phi,\psi]_{S'}}/\Omega_\phi) \cong \mathbb{Z}[1/2]$.

[3] For an introduction and overview of the role of shape theory in the study of tiling spaces, we refer the reader to [19]

Example 3.5.10. As mentioned, similar examples to these extended substitutions appear naturally in the literature. In [6], Barge and Diamond outline a method for associating, to a primitive aperiodic substiution ϕ, a new substitution $\tilde{\phi}$ which is non-minimal, and in general takes the form of a substitution built in the above manner. They show that the homeomorphism type of the tiling space $\Omega_{\tilde{\phi}}$ is a homeomorphism invariant of the tiling space Ω_ϕ, and so the cohomology $\check{H}^i(\Omega_{\tilde{\phi}})$ is also a topological invariant for Ω_ϕ. The method for forming the substitution $\tilde{\phi}$ from the so-called *balanced pairs of words associated to pairs of asymptotic composants* is involved, and it would be cumbersome to reproduce the construction here. We instead refer the reader to their paper [6].

Using this construction, it can be shown that given the Fibonacci substitution $\phi_{Fib}\colon 0 \mapsto 001,\ 1 \mapsto 01$, the associated substitution $\tilde{\phi}_{Fib}$ is given by $\tilde{\phi}_{Fib}\colon a \mapsto aab,\ b \mapsto ab,\ c \mapsto acab$. The tiling space of this substitution is orbit equivalent to a Fibonacci with one handle substitution $[\phi_{Fib}, \mathrm{Id}]_S$ (the equivalence is given by the single c tile absorbing the a tile to its right).

Example 3.5.11. Considering the substitutions

$$\begin{array}{llll} \phi_1\colon & a \mapsto cab & b \mapsto ac & c \mapsto a \\ \phi_2\colon & a \mapsto bbac & b \mapsto a & c \mapsto b. \end{array}$$

It is an exercise for the reader to check that we have cohomology groups $\check{H}^1(\Omega_{\phi_1}) \cong \check{H}^1(\Omega_{\phi_2}) \cong \mathbb{Z}^5$. So, cohomology does not distinguish the tiling spaces of these two substitutions. It is also the case that several other invariants of primitive substitution tiling spaces fail to distinguish these substitutions. We can instead form the two new substitutions $\tilde{\phi}_1, \tilde{\phi}_2$. We omit the specific presentations of these substitutions owing to their extremely large size—$\tilde{\phi}_1$ has an alphabet on 19 letters, $\tilde{\phi}_2$ has an alphabet on 87 letters.

Using the results of this chapter, we can calculate that $\operatorname{rk} \check{H}^1(\Omega_{\tilde{\phi}_1}) = 17$ and $68 \leq \operatorname{rk} \check{H}^1(\Omega_{\tilde{\phi}_2}) \leq 74$ and so by the result of Barge and Diamond, these invariants distinguish the substitutions ϕ_1 and ϕ_2. Hence we have $\Omega_{\phi_1} \not\cong \Omega_{\phi_2}$.

Acknowledgement. The author would like to thank Scott Balchin for offering to write a computer program to help determine the substitutions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ after it became apparent that hand calculations would not be feasible in a reasonable amount of time.

The importance of the choice of the set of subsequences S in the construction of $[\phi, \psi]_S$ is not immediately apparent. It seems unlikely that the resulting tiling space is independent of the choice of S. By taking powers of ϕ, one can generate infinitely many distinct such choices. By construction, the inclusion and quotient cohomology diagrams of these spaces will all be very similar (if not identical), and so a stronger invariant is likely needed to distinguish such substitutions topologically.

Question 3.5.12. Does there exist a pair of substitution ϕ, ψ and sets of subsequences S, S' such that $\Omega_{[\phi,\psi]_S}$ and $\Omega_{[\phi,\psi]_{S'}}$ are not homeomorphic? If such behaviour is typical, what tools are needed to topologically or dynamically distinguish such pairs of spaces in general?

4. GROUT

Supplementary Resources

Grout is available to download for Windows and Mac OSX along with the supporting documentation at the following URL

`www2.le.ac.uk/departments/mathematics/extranet/staff-material/staff-profiles/scott-balchin`

4.1 Outline

The material of this chapter appears in [5], largely unchanged from how it is presented here.

Grout is a program developed by the author and Scott Balchin for exploring various notions related to 1-dimensional symbolic substitutions and their tiling spaces. In this chapter, we explain the functions of Grout, the properties of substitutions that Grout has been built to calculate, and provide explanations for how Grout has been coded to calculate such properties using a deterministic algorithm. Of particular note is the algorithm designed to check for recognisability of a primitive substitution which, to our knowledge, is the first time such an algorithm has been written down in full and implemented.

It is hoped that the use of this program will make testing conjectures in tiling theory and symbolic substitutional dynamics more efficient, as well as allowing for the confirmation of hand calculations and comparison of different methods of calculation (especially methods of calculating cohomology). Analysis of large data sets which can be potentially generated by the Grout source code, and the recognition of underlying patterns in the data may also aid to further the theory.

The GUI front end for Grout is powered by Qt[1]. Grout has been designed with user experience in mind and includes many ease-of-use properties such as the ability to save and load examples, and convenient methods of sharing examples with other users via short strings that encode a substitution. There is also an option to export all of the data that has been calculated to a pre formatted LaTeX file including all the

TikZ code for the considered complexes. This should be useful for those needing to typeset such diagrams in the future by fully automating the generation of diagrams in TikZ.

In Section 4.2 we introduce the relevant tiling theory along with pseudocode for most of the non-trivial components of Grout that have been implemented. Section 4.3 will cover specifically those methods implemented to compute cohomology for tiling spaces. Throughout, we give instances of these methods being applied to the Fibonacci substitution. In Section 4.4 we give a range of other examples of calculations from Grout.

The primary function of Grout is a collection of methods for calculating the Čech cohomology $\check{H}^1$ of the tiling space associated to a primitive recognisable substitution. Grout implements three different methods for calculating the cohomology of tiling spaces associated to symbolic substitutions on finite alphabets.

1. The method of Barge-Diamond complexes as introduced in [7]
2. The method of Anderson-Putnam complexes as introduced in [2]
3. The method of forming an equivalent *left proper* substitution as outlined in [26]

All three outputs are algebraically equivalent—that is, they represent isomorphic groups—but it is not always obvious that this is the case given the presentations. This disparity between presentations of results for the equivalent methods was one of the major motivating factors for developing Grout. These cohomology groups are extremely laborious to calculate by hand for large alphabets unless special criteria are met.

4.2 Grout and its Functions

4.2.1 Substitution Structure

We begin by outlining how we encode a substitution rule into Grout and how we implement the substitution rule. In general, we have done most of the implementation by string manipulation methods.

We use a class *sub* which has as its element a vector of strings. We always assume that our alphabet is ordered $a, b, c, \ldots$. The first entry of a *sub* class vector is $\phi(a)$, the second is $\phi(b)$ and so on. To validate the input we check that the number of unique characters appearing in all of the $\phi(x)$ is equal to the length of the alphabet, which is the length of the vector. The GUI also employs the use of regular expressions to prevent illegal characters from being entered.

Example 4.2.1 (Fibonacci Substitution). The Fibonacci tiling is given by a substitution rule on the alphabet $\mathcal{A} = \{a, b\}$ and is defined as

$$\phi\colon \begin{cases} a & \mapsto & b \\ b & \mapsto & ba \end{cases}$$

The Fibonacci substitution is our main example used throughout this chapter. See Section 4.4 for a selection of outputs for other common examples of substitutions.

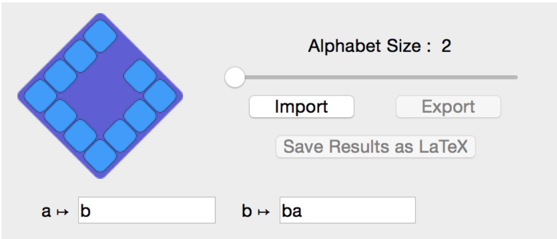


Fig. 4.1: The Fibonacci substitution entered into Grout

Next we implement a way to perform an iteration of ϕ on a string. We do not include any checks to validate that the string can be iterated on, as all strings that will be passed to this function will be created by the program itself, and therefore valid.

Algorithm 1 Substitution functions

```
function iterate(string rhs)
    result = empty string
    for each character x in rhs do
        append φ(x) to result
    output result
```

4.2.2 *Substitution Matrices and their Properties*

Recall that for a substitution ϕ on an l-letter alphabet $\mathcal{A}$ there is an associated *substitution matrix* M_ϕ of dimension $l \times l$ given by setting m_{ij} to be the number of times that the letter a^i appears in $\phi(a^j)$.

We will not give the algorithm for constructing the substitution matrix, the definition can be taken as a pseudo-algorithm. We implement a square matrix class to work with the substitution matrix. The first property that we will be checking for the substitution matrix is primitivity.

Recall that a substitution $\phi\colon \mathcal{A} \to \mathcal{A}^+$ is called primitive if there exists a positive natural number p such that the matrix M_ϕ^p has strictly positive entries. Such a matrix M is also called *primitive*.

To check this condition on M_ϕ, we count the number of zeros in the matrix and if the number of zeros is 0 then we can conclude that the substitution is primitive. If not, then square the matrix and recount the number of zeros. If the number of zeros does not change then we can conclude that the substitution is not primitive. This means that this check always halts.

Algorithm 2 Primitivity check

```
function primitive
    matrix = substitution matrix of φ
    while true do {
        a = number of zeros in matrix
        matrix = matrix × matrix
        b = number of zeros in matrix
        if a=b and a!=0
            output false
        if b=0
            output true
    }
```

We will be checking primitivity for all substitutions before we do calculations on them as if the substitution is not primitive many of the methods will not work, or will return false positive results[1]. Grout will always display whether a given substitution is primitive or not, it can also output the substitution matrix if asked to do so.

The next thing that we can do with the substitution matrix is give the tile frequencies and tile lengths of the substitution. This requires us to compute the eigenvalues of the matrix. We have implemented the QR method for computing the eigenvalues (for example see [37]). This gives us approximations to the real eigenvalues, and for the complex ones we simply give the conjugate pairs by their absolute values, and we give the results to two decimal places. The eigenvalues of a substitution matrix may be printed out by ticking the eigenvalues box. We refer the reader to [45] for a text on the Perron-Frobenius theory of primitive matrices.

Proposition 4.2.2 (Perron-Frobenius)**.** *Let M be a primitive matrix.*

i There is a positive real number λ_{PF}, called the Perron-Frobenius eigenvalue*, such that λ_{PF} is a simple eigenvalue of M and any other eigenvalue λ is such that $|\lambda| < \lambda_{PF}$.*

[1] We remark that one could use methods from Chapter 3 to extend some of the functions of Grout to non-primitive substitutions.

ii There exist left and right eigenvectors, called the left and right Perron-Frobenius eigenvectors*,* $\mathbf{l}_{PF}$ *and* $\mathbf{r}_{PF}$ *associated to* λ_{PF} *whose entries are all positive and which are unique up to scaling.*

Given the above theorem, it is natural to ask what information is contained in the PF eigenvalue and eigenvectors of M_ϕ for a primitive substitution ϕ.

If we were to assign a length to the tiles labelled by each letter, then we would hope for such a length assignment to behave well with the given substitution. The left PF eigenvector offers a natural choice of length assignments. If we assign to the letter a^i the length $(\mathbf{l}_{PF})_i$, the ith component of the left PF eigenvector, then we can replace our combinatorial substitution by a geometric substitution. This geometric substitution expands the tile with label a^i by a factor of λ_{PF} and then partitions this new interval into tiles of lengths and labels given according to the combinatorial substitution. In order to give a unique output, Grout normalises the left PF eigenvector so that the smallest entry is 1.

The information contained in the right PF eigenvector is also useful. The right PF eigenvector, once normalised so that the sum of the entries is 1, gives the relative frequencies of each of the letters appearing in any particular bi-infinite sequence which is admitted by a primitive ϕ. That is, if $|u|_i$ is the number of times the letter a^i appears in the word u, and letting $w_{[-k,k]} = w_{-k} \dots w_{-1}w_0w_1 \dots w_k$, then

$$\lim_{k\to\infty} |w_{[-k,k]}|_i / |w_{[-k,k]}| = (\mathbf{r}_{PF})_i$$

for any $w \in X_\phi$.

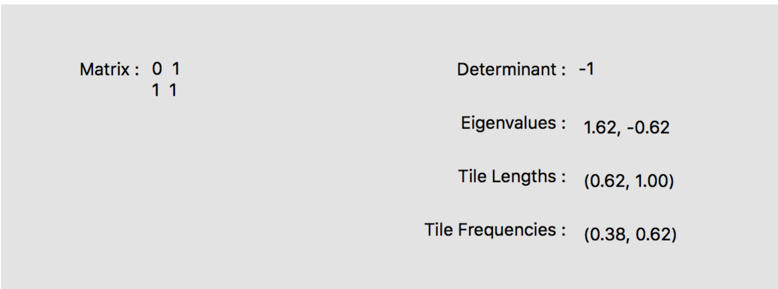


Fig. 4.2: The results for the matrix calculations for the Fibonacci substitutions

4.2.3 Enumerating n-Letter Words

Now that we have introduced the basic structure of the substitutions, and discussed the problem of primitivity and other matrix related calculations, we will discuss our first main function in Grout.

Definition 4.2.3. Given a substitution $\phi\colon \mathcal{A} \to \mathcal{A}^+$, we define the *complexity function* at n to be the number of unique n-letter words admitted by ϕ. We denote this function by $p_\phi(n)$, and so $p_\phi(n) = |\mathcal{L}_n|$.

The complexity function of a tiling is a useful invariant [42]. One is usually interested in either a deterministic formula for p_ϕ or information about the growth rate of p_ϕ such as polynomial degree; Grout can be used to at least give circumstantial evidence for these, though has no means of calculating either (this appears to be a very difficult problem in general).

Of particular interest are the number of 2 and 3 letter words, as we will be using them later to compute cohomology. Our function will not only enumerate the number of n-letter words, but will also print out these words if required. The algorithm uses C++ sets as a data structure to store the n-letter words as it is automatically ordered and does now allow repetitions which leads to fast computation. We start by generating a length m admitted seed word w such that $m \geq n$, and count all unique n-letter words appearing as subwords of the seed. We then apply ϕ to the seed and add all new n-letter words to the result. At each stage we count the size before and after adding the new words. If the size does not change we can stop, as no new n-letter words will be generated after a step without any new n-letter words. It follows that the value $p_\phi(n)$ is computable in finite time for any fixed $n \geq 1$.

Algorithm 3 Finding all n-letter words

```
function nlw(int n)
    result = empty ordered set
    seed = 'a'
    while seed length < n do
        seed = iterate(seed)

    difference = 1
    while difference != 0 do {
        a = cardinality of result
        seed = iterate(seed)
        for each n length word w in seed do
            append w to result
        b = cardinality of result
        difference = b-a
    }
    output result
```

Example 4.2.4. It is well known that the complexity for the Fibonacci substitution satisfies $p_\phi(n) = n + 1$, and we can verify this for any value of n by computing its complexity in Grout.

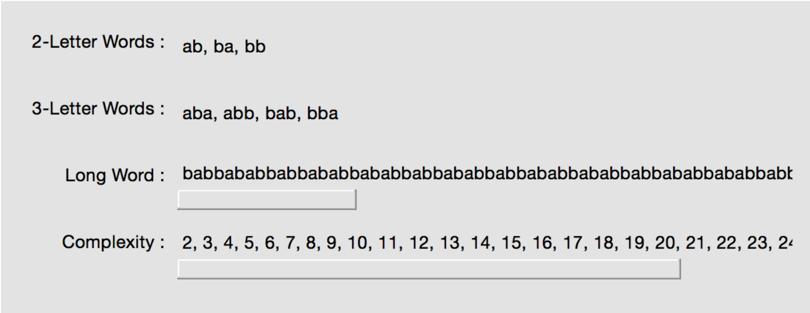


Fig. 4.3: The words results for the Fibonacci substitution displayed in Grout

4.2.4 *Barge-Diamond and Anderson-Putnam Complexes*

Grout has the ability to output two simplicial complexes as PDFs (provided that the user has PDFLaTeX installed). The first of these is the *Barge-Diamond complex* [7].

For convenience to the reader, we recall the (combinatorial) construction of the Barge-Diamond complex K_ϕ for a substitution ϕ on an alphabet $\mathcal{A} = \{a^1, \ldots a^l\}$. To construct the BD-complex, we have two vertices for each a^i, an *in* node v_i^+ and an *out* node v_i^-. We draw an edge from v_i^+ to v_i^- for all i (the tile edges). Then for all two letter words $a^i a^j \in \mathcal{L}^2$ admitted by ϕ, we draw an edge from v_i^- to v_j^+ (the vertex edges).

Example 4.2.5. As we have seen previously, the only two letter words admitted by the Fibonacci substitution are ab, ba and bb, this gives us the following Barge-Diamond complex output in Grout.

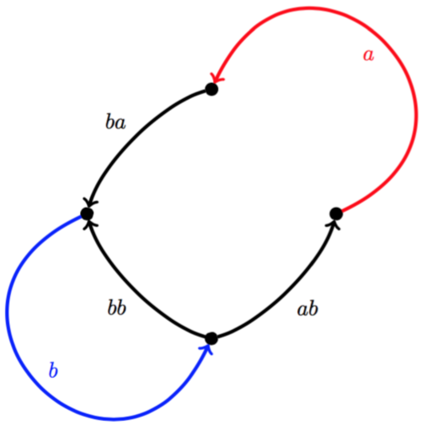


Fig. 4.4: The Barge-Diamond complex for the Fibonacci substitution.

The other complex that we consider for a substitution is (a variant of) the 1-collared Anderson-Putnam complex [2]. For brevity we will shorten this to just *the AP-*

complex for the remainder. This particular definition is based on what Gähler and Maloney call the *Modified Anderson-Putnam complex* in [36]. The AP-complex is constructed by making use of both the two and three letter words.

Definition 4.2.6. Let $\mathcal{A} = \{a^1, \ldots, a^l\}$ be an alphabet with a substitution $\phi\colon \mathcal{A} \to \mathcal{A}^+$, then we construct the (modified) *Anderson-Putnam complex* $\hat{\Gamma}$ of ϕ as follows. We have a vertex v_{ij} for each two letter word $a^i a^j \in \mathcal{L}^2$ admitted by ϕ. We draw an edge from v_{ij} to v_{jk} if and only if the three letter word $a^i a^j a^k$ is admitted by ϕ.

Remark 4.2.7. One should note that this modified AP-complex is slightly different to the definition originally introduced by Anderson and Putnam. In particular, the original definition distinguishes between different occurrences of a two letter word $a^i a^j$ if the occurrences of three letter words containing as a subword $a^i a^j$ do not overlap on some admitted four letter word. For example, if the language of a substitution included the two letter word ab, the three letter words xab, yab, abw, abz, and the four letter words $xabw, yabz$ but the words $xabz$ and $yabw$ did not belong to $\mathcal{L}$, then the original definition of the AP-complex would have two instances of vertices with the label ab, say $(ab)_1$ and $(ab)_2$. In our definition, these vertices are identified, so that $(ab)_1 \sim (ab)_2 \sim ab$. An example of such a substitution is given by $\phi\colon a \mapsto bc,\ b \mapsto baab,\ c \mapsto caac$ where we label exactly one vertex with the label aa, but the original definition would require we include two distinct vertices labelled $(aa)_1$ and $(aa)_2$.

In our discussion of cohomology calculated via AP-complexes in Section 4.3.2, we use this version of the AP-complex to describe the performed calculations, and Grout implements this particular method. It would have been possible to use the original definition, or one of the many variant AP-complexes that have been defined in the literature. There are at least three such variants discussed in [36], of varying complexities and situations in which they can be used.

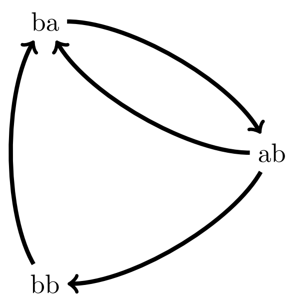


Fig. 4.5: The Modified Anderson-Putnam complex for the Fibonacci substitution

4.2.5 Recognisability

Recall that we say ϕ is recognisable (equivalently, ϕ has the unique composition property) if for every $w \in X_\phi$ there is a unique way of writing w as a substituted bi-infinite sequence which is admitted by ϕ. That is, there exists a unique bi-infinite sequence $w' = \dots w'_{-1} w'_0 w'_1 \dots \in X_\phi$ and a finite shift n of at most $|\phi(w'_0)|$ such that $w = \sigma^n(\phi(w'))$.

Equivalently, we say ϕ is recognisable if there exists a natural number $K \geq 1$ such that for all admitted words $v \in \mathcal{L}$ with $|v| > 2K$, there exist unique words x, y of length $|x|, |y| \leq K$ and a unique admitted word $u \in \mathcal{L}$ such that $v = x\phi(u)y$.

As has been emphasised in previous chapters, recognisability is an important property of a substitution as many of the tools used to study the topology of the associated tiling space rely on recognisability as a hypothesis, much like primitivity. Recall that recognisability of a primitive substitution is equivalent to aperiodicity of the subshift X_ϕ [49], and we make use of this result to decide recognisability. The algorithm designed to determine if a given substitution is recognisable relies on finding a fixed letter and return words to that fixed letter.

Definition 4.2.8. Given a substitution ϕ on an alphabet $\mathcal{A}$, the letter a is said to be *fixed* (on the left) of *order* k if there exists some integer k such that $\phi^k(a) = au$ for some word u. Every substitution has at least one fixed letter and the value of k for such a letter is bounded by the size of the alphabet.

Let a be a letter fixed by ϕ on the left. Recall that a *return word* to a is a word v such that $v = au$ for some (possibly empty) word $u \in (\mathcal{A} \setminus \{a\})^*$, and va is an admitted word of the substitution.

Recall that if ϕ is primitive then, due to the minimality of the substitution, the set of return words to any letter is finite. This is a consequence of the linear recurrence of the subshift X_ϕ [22].

We will use these return words to determine whether a substitution is recognisable or not. The following proposition appears in [40].

Proposition 4.2.9. *Let ϕ be a primitive substitution on $\mathcal{A}$ and let a be a fixed letter. Let $\mathcal{R}$ be the set of all return words to a. So $\mathcal{R} = \{v \mid v = au,\ aua \in \mathcal{L},\ u \in (\mathcal{A} \setminus \{a\})^*\}$. The substitution ϕ is not recognisable if and only if, for all $v, v' \in \mathcal{R}$, there exists a $p \geq 1$ such that $\phi^p(vv') = \phi^p(v'v)$.*

As $\mathcal{R}$ is finite, and together with the next proposition which appears in [21] and [29], this gives us a finite deterministic check for recognisability.

Algorithm 4 Finding all return words to fixed letter f

function returnwords(character f)
    result = empty ordered set
    length = 2
    while new return words are being added **do** {
        nwords = nlw(length)
        for all words w in nwords **do** {
            if last character of w = first character of w= f **and** w has no other f appearing **do**
                append w to result
        }
        length = length + 1
    }
    output result

Proposition 4.2.10. *Let ϕ be a substitution on $\mathcal{A}$ and let $|\mathcal{A}| = n$. For words $u, w \in \mathcal{A}^+$, there exists a $p \geq 1$ such that $\phi^p(u) = \phi^p(w)$ if and only if $\phi^n(u) = \phi^n(w)$.*

That is, if some iterated substitution of u and v are ever equal, then their iterates must become equal by the nth iteration of the substitution at the latest, where n is the size of the alphabet. In the algorithm, k is taken to be the k from the definition of the fixed letter, and n is the size of the alphabet.

Algorithm 5 Recognisability check

function recognisable
    rwords = returnwords(f)
    for each word w in rwords **do**
        for each word $v \neq w$ in rwords **do**
            if$(\phi^{k\times n}(w + v) = \phi^{k\times n}(v + w))$
                output true
    output false

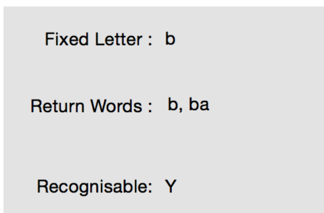


Fig. 4.6: The recognisability results for the Fibonacci substitution

4.3 Cohomology of Tiling Spaces in Grout

4.3.1 Via Barge-Diamond

Let ϕ be a primitive, recognisable substitution on the alphabet $\mathcal{A}$. Let K_ϕ be the Barge-Diamond complex of ϕ and let S_ϕ be the subcomplex of K_ϕ formed by the vertex edges, all those edges labelled with two letter admitted words $a^i a^j$.

Let $\tilde{\phi}\colon S_\phi \to S_\phi$ be a graph morphism defined in the following way on vertices. Let $l(i)$ and $r(i)$ be such that $\phi(a^i) = a^{l(i)} u a^{r(i)}$ for some possibly empty word u, and where in the case that u is empty, we may have $\phi(a^i) = a^{l(i)} = a^{r(i)}$. Define $\tilde{\phi}(v_i^+) = v_{l(i)}^+$ and $\tilde{\phi}(v_i^-) = v_{r(i)}^-$. Note that if $a^i a^j$ is admitted by ϕ, then $a^{r(i)} a^{l(j)}$ is also admitted by ϕ, and so $\tilde{\phi}$ is a well defined graph morphism on S_ϕ. As $\tilde{\phi}(S_\phi) \subset S_\phi$, we can define the eventual range $ER = \bigcap_{m \geq 0} \tilde{\phi}^m(S_\phi)$ (which stabilises after finitely many substitutions).

For this method of computation we make use of the following result attributed to Barge and Diamond [7], and which is a special case of Theorem 2.3.9 from Chapter 2.

Proposition 4.3.1. *There is a short exact sequence*

$$0 \to \tilde{H}^0(ER) \to \varinjlim M_\phi^T \to \check{H}^1(\Omega_\phi) \to H^1(ER) \to 0.$$

The eventual range ER is a (possibly disconnected) graph, and so $\tilde{H}^0(ER)$ and $H^1(ER)$ are finitely generated free abelian groups of some ranks k and l respectively. Hence we have

Corollary 4.3.2.

$$\check{H}^1(\Omega_\phi) \cong \varinjlim M_\phi^T / \mathbb{Z}^k \oplus \mathbb{Z}^l.$$

Grout displays the Čech cohomology using the Barge-Diamond method in the above form of Corollary 4.3.2.

These results fail in general if ϕ is not primitive or recognisable, and so Grout only performs this calculation after checking these two conditions.

The only involved part of this calculation is finding the eventual range of the Barge-Diamond complex. After we have this we can simply find the number of connected components and use the Euler characteristic to find the reduced cohomology in rank zero and one. Therefore, we now give the algorithm that we use to find the eventual range. We denote by $w[0]$ the first letter of the word ab and $w[1]$ the second letter of ab.

Algorithm 6 Eventual range of the Barge-Diamond complex

```
function eventual range
    twoletter = nlw(2)
    difference = 1
    while difference != 0 do {
        temp = empty ordered set
        for each word w in twoletter do
            append last character of iterate(w[0]) + first character of iterate(w[1]) to temp
        difference = cardinality of twoletter - cardinality of temp
        twoletter = temp
    }
    output twoletter
```

4.3.2 Via Anderson-Putnam

Let ϕ be a primitive, recognisable substitution on the alphabet $\mathcal{A}$. Let Γ be the AP-complex of ϕ. Recall that Anderson and Putnam showed in [2] that the Čech cohomology of Ω_ϕ is determined by the direct limit of an induced map acting on the cohomology of Γ. Using the modified AP-complex $\hat{\Gamma}$, Gähler and Maloney showed that this complex, as defined in Section 4.2.4, can be used in place of the originally defined AP-complex. We define the map acting on the AP-complex $\hat{\Gamma}$ in the following way.

Again, let $l(i)$ and $r(i)$ be such that $\phi(a^i) = a^{l(i)} u a^{r(i)}$ for some possible empty word u, and where in the case that u is empty, we may have $\phi(a^i) = a^{(l(i)} = a^{r(i)}$. Let E be an edge with label $a^i a^j a^k$ and define $L = |\phi(a^j)|$. Suppose $\phi(a^j) = a_1 a_2 \dots a_L$. Define a continuous map called the *collared substitution* $\tilde{\phi} \colon \tilde{\Gamma} \to \tilde{\Gamma}$ by mapping the edge E to the ordered collection of edges with labels

$$[a^{r(i)} a_1 a_2][a_1 a_2 a_3] \cdots [a_{L-2} a_{L-1} a_L][a_{L-1} a_L a^{l(k)}]$$

in an orientation preserving way and at normalised speed. This map is well defined and continuous, hence an induced map $\tilde{\phi}^* \colon H^1(\hat{\Gamma}) \to H^1(\hat{\Gamma})$ on cohomology exists. We use the following result from [36] (Or [2] if $\hat{\Gamma}$ is replaced with the original definition of the AP-complex Γ).

Proposition 4.3.3.

$$\check{H}^1(\Omega_\phi) \cong \varinjlim (H^1(\hat{\Gamma}), \tilde{\phi}^*)$$

Let $R = \mathrm{rk} H^1(\hat{\Gamma})$, the rank of the cohomology of $\hat{\Gamma}$. Grout finds an explicit generating set of cocycles for the cohomology of $\hat{\Gamma}$ and then outputs the induced map $\tilde{\phi}^*$ as the associated $R \times R$-matrix M_{AP}, which should be interpreted as acting on $\mathbb{Z}^R$

with respect to this generating set. So $\check{H}^1(\Omega_\phi) \cong \varinjlim M_{AP}$.

The algorithm for this computation begins by constructing the boundary matrix for the AP-complex. To do this we take all of the admitted three letter words abc and we use the convention that the boundary of this edge is $bc - ab$. Using this, we construct the associated $m \times n$ boundary matrix B where m is the number of two letter words and n is the number of three letter words. We then use standard methods from linear algebra to find a maximal set of linearly independent n dimensional vectors g such that $Bg = 0$, searching over the set of all vectors of 0s and 1s. By construction, this set generates the kernel of the boundary map inside the simplicial 1-chain group of $\hat{\Gamma}$. This gives us a generating set of cycle vectors for the first homology of $\hat{\Gamma}$.

We then apply the collared substitution to each of these generating vectors, giving us a new set of image vectors. Using Gaussian elimination, we find the coordinates of these image vectors in terms of the generating vectors. This induced map on homology can be represented as a square matrix. The transpose of this matrix M_{AP} then represents the induced map on cohomology, and M_{AP} is the output for the cohomology calculation via the Anderson-Putnam method. It should be noted that this algorithm is not efficient in the case where the substitution has many three letter words, as the dimension m of the 1-chain complex is the dominant limiting factor when finding linearly independent generating cycles. The time complexity increases exponentially with respect to m.

4.3.3 Via Properisation

For this method of computation we make use of a technique involving return words, as outlined in [26], for replacing a primitive substitution with an equivalent *pre-left proper* primitive substitution. One may then use the fact that if ϕ is a recognisable pre-left proper primitive substitution, then $\check{H}^1(\Omega_\phi) \cong \varinjlim M_\phi^T$ (a special case of Proposition 2.3.16 for stationary systems).

We begin by defining what it means to be proper.

Definition 4.3.4.

A substitution is *left proper* if there exists a letter $a \in \mathcal{A}$ such that the leftmost letter of $\phi(b)$ is a for all $b \in \mathcal{A}$. That is, $\phi(b) = au_b$ for some $u_b \in \mathcal{A}^*$.

A substitution is *right proper* if there exists a letter $a \in \mathcal{A}$ such that the rightmost letter of $\phi(b)$ is a for all $b \in \mathcal{A}$. That is, $\phi(b) = u_b a$ for some $u_b \in \mathcal{A}^*$.

A substitution is *fully proper*[2] if it is both left and right proper.

[2] We will often abbreviate this to just *proper*.

A substitution is *pre-left proper* if some power of the substitution is left proper. Similarly for *pre-right proper* and *pre-fully proper*.

The following algorithm produces what we call the *pre-left properisation* of a substitution, so-called because there exists a finite power of the new substitution which is left proper. As per usual k will be the one from the definition of the fixed letter f.

Note that if v is a return word to the fixed letter a, then $va \in \mathcal{L}$ and so $\phi^k(va)$ must also be admitted by ϕ. But $\phi^k(va) = \phi^k(v)\phi^k(a)$, and both of $\phi^k(v)$ and $\phi^k(a)$ begin with the fixed letter a, hence $\phi^k(v)$ is an exact composition of return words to a. So, if we apply ϕ^k to a return word, then the result is a composition of return words. We will denote by ψ this newly constructed substitution rule on the new alphabet $\mathcal{R}$ of return words.

Algorithm 7 Pre-left properisation

function preprop
    rwords = returnwords(f)
    ψ = empty substitution with alphabet size being the cardinality of rwords
    for all words w in rwords **do** {
        temp = $\phi^k(w)$
        decomposition = decompose temp into return words $w_{i_1} \ldots w_{i_m}$
        $\psi(w)$ = decomposition
    }
    output ψ

This algorithm gives us a new substitution on possibly more letters than with what we began, and we may take a power of this substitution to get a left proper one. That such a power exists is clear. Indeed, every return word $v \in \mathcal{R}$ begins with the fixed letter a and, by primitivity, $\phi^i(a)$ contains at least two copies of the letter a for large enough i, so $\phi^i(a) = v_0 a u$ for some return word $v_0 \in \mathcal{R}$ and some other word u. But then $\psi^i(v)$ must begin with v_0. It follows that ψ^i is a left-proper substitution with leftmost letter v_0.

We may also form an equivalent fully proper substitution on $\mathcal{R}$ by composing ψ^i with its *right conjugate*. The right conjugate $\phi^{(R)}$ of a left proper substitution ϕ is given by setting $\phi^{(R)}(b) = u_b a$ where a is the fixed letter such that $\phi(b) = a u_b$ for all $b \in \mathcal{A}$. The right conjugate is a right proper substitution, and the composition of a left proper and right proper substitution is both left and right proper, hence fully proper. It is easy to show (see [27]) that $X_{\phi\circ\phi^{(R)}}$ and X_ϕ are topologically conjugate subshifts. In fact, a word is admitted by $\phi \circ \phi^{(R)}$ if and only if it is admitted by ϕ, so $X_{\phi\circ\phi^{(R)}}$ and X_ϕ are equal. Hence, $\check{H}^1(\Omega_{\phi\circ\phi^{(R)}}) \cong \check{H}^1(\Omega_\phi)$.

We make use of the following which has been paraphrased from results appearing in the work of Durand, Host and Skau in [26].

Proposition 4.3.5. *Let ϕ be a primitive substitution on $\mathcal{A}$ and let ψ be the pre-left properisation of ψ. The tiling space Ω_ψ is homeomorphic to Ω_ϕ.*

Hence we get the corollary

Corollary 4.3.6.

$$\check{H}^1(\Omega_\psi) \cong \check{H}^1(\Omega_\phi)$$

As ψ^i is left proper, $\check{H}^1(\Omega_{\psi^i}) \cong \varinjlim M^T_{\psi^i} \cong \varinjlim (M^i_\psi)^T \cong \varinjlim M^T_\psi$. Hence $\check{H}^1(\Omega_\phi) \cong \varinjlim M^T_\psi$.

Grout outputs the pre-left properisation ψ, the left properisation ψ^i, the full properisation $\psi^i \circ (\psi^i)^{(R)}$, and owing to the above, Grout also outputs the matrix M^T_ψ in the cohomology section.

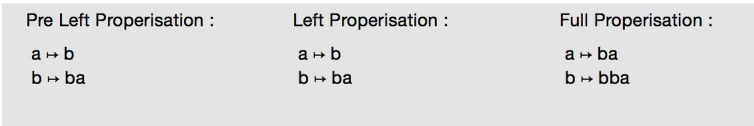


Fig. 4.7: The properisation results of the Fibonacci substitution

Remark 4.3.7. If the substitution is already proper, the properisation algorithm may return a different proper version of this substitution. This may seem like it is a feature which has no use, but by iterating this process, we find that the sequence of substitutions is eventually periodic, first proved by Durand [23]. It was therefore decided to leave this feature intact, in order to study such sequences of properisations.

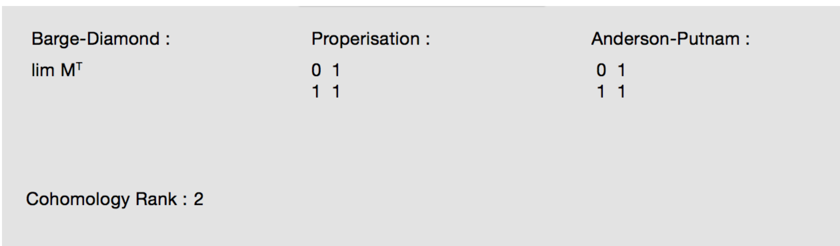


Fig. 4.8: The cohomology results of the Fibonacci substitution

4.4 Example Outputs

In this section we provide the LaTeX output of cohomological calculations from Grout for a selection of both well-known and not so well-known substitutions appearing in the literature.

1. The *Thue-Morse substitution* is one of the most well-studied substitutions in symbolic dynamics [52], possibly only superseded by the Fibonacci substitution in the attention it has received. It would be remiss to include a list of example outputs which did not include the results for this substitution.

2. The *Tribonacci substitution* is an example of a unimodular irreducible Pisot substitution, first described by Rauzy in his seminal paper [53] introducing the so-called Rauzy fractals, and still actively studied for its interest to symbolic dynamicists and fractal geometers.

3. The *Disconnected Subcomplex substitution* was first described by Barge and Diamond [7] as an example of a substitution whose BD-complex has a disconnected subcomplex of edges labelled by two letter words. This is reflected in the cohomology calculation via the BD-complex, where a non-trivial quotient appears according to the formula for the cohomology described in Corollary 4.3.2.

4. The Fibonacci and Tribonacci substitutions are the first and second in an infinite family of primitive recognisable substitutions which take the form

$$\begin{array}{rcl} a_j & \mapsto & a_1 a_{j+1}, \quad \text{if } 1 \leq j < n \\ a_n & \mapsto & a_1 \end{array}$$

   for an alphabet $\{a_1, \ldots a_n\}$ on n letters. One might call these substitutions the *n-ibonacci substitutions*. We have chosen to show the output for the Hexibonacci substitution where $n = 6$, as the associated Barge-Diamond complex is particularly pleasing.

Thue-Morse

$$\begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & ba \end{array}$$

Substitution Matrix :

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Full Properisation :

$$\begin{aligned} a &\mapsto cacbacab \\ b &\mapsto cbabcacbabcbacab \\ c &\mapsto cbabcacbacabcacbabcbacab \end{aligned}$$

Barge-Diamond Cohomology Group : $\varinjlim M^T \oplus \mathbb{Z}^1$

Properisation Cohomology Matrix :

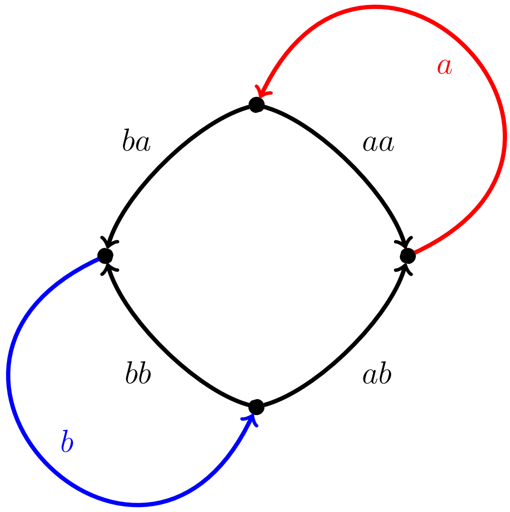
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Anderson-Putnman Cohomology Matrix :

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

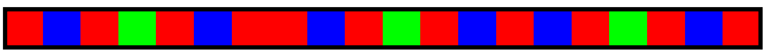
Cohomology Rank : 2

Barge-Diamond Complex



Tribonacci

$$\begin{aligned} a &\mapsto ab \\ b &\mapsto ac \\ c &\mapsto a \end{aligned}$$



Substitution Matrix :

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Fixed Letter : a

Return Words : a, ab, ac

Recognisable : Yes

Full Properisation :

$$\begin{array}{ccc} a & \mapsto & bc \\ b & \mapsto & babc \\ c & \mapsto & bbc \end{array}$$

Barge-Diamond Cohomology Group : $\varinjlim M^T$

Properisation Cohomology Matrix :

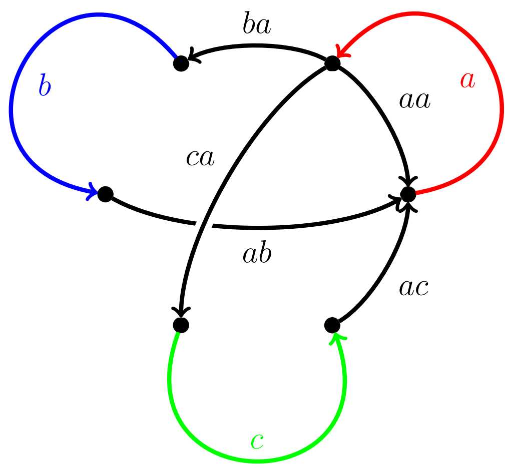
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Anderson-Putnman Cohomology Matrix :

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

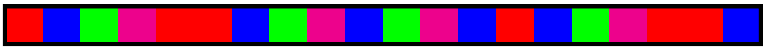
Cohomology Rank : 3

Barge-Diamond Complex



Disconnected Subcomplex

$$\begin{array}{lcl} a & \mapsto & abcda \\ b & \mapsto & ab \\ c & \mapsto & cdbc \\ d & \mapsto & db \end{array}$$



Substitution Matrix :

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Fixed Letter : a

Return Words : a, ab, abcd, abcdbcdb

Recognisable : Yes

Full Properisation :

$$\begin{array}{lcl} a & \mapsto & cacad \\ b & \mapsto & cacabcad \\ c & \mapsto & cacaddbcad \\ d & \mapsto & cacaddbcaddbcabcad \end{array}$$

Barge-Diamond Cohomology Group : $\varinjlim M^T/\mathbb{Z}^1$

Properisation Cohomology Matrix :

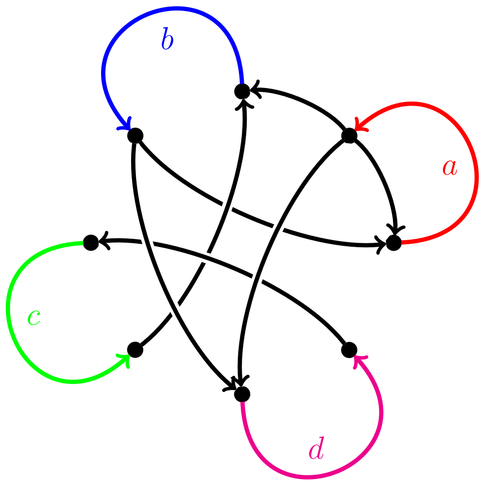
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Anderson-Putnman Cohomology Matrix :

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

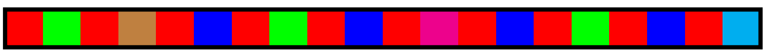
Cohomology Rank : 3

Barge-Diamond Complex



Hexibonacci

$$\begin{aligned} a &\mapsto ab \\ b &\mapsto ac \\ c &\mapsto ad \\ d &\mapsto ae \\ e &\mapsto af \\ f &\mapsto a \end{aligned}$$



Substitution Matrix :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fixed Letter : a

Return Words : a, ab, ac, ad, ae, af

Recognisable : Yes

Full Properisation :

$$\begin{array}{rcl} a & \mapsto & bc \\ b & \mapsto & bdbc \\ c & \mapsto & bebc \\ d & \mapsto & bfbc \\ e & \mapsto & babc \\ f & \mapsto & bbc \end{array}$$

Barge-Diamond Cohomology Group : $\varinjlim M^T$

Properisation Cohomology Matrix :

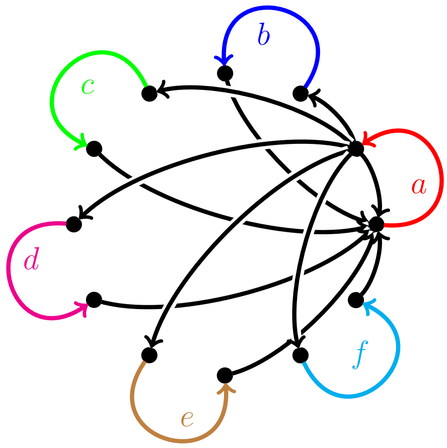
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Anderson-Putnman Cohomology Matrix :

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Cohomology Rank : 6

Barge-Diamond Complex :



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