OPTIMUM SHAPE PROBLEMS FOR DISTRIBUTED PARAMETER SYSTEMS

A Thesis submitted by

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for the Degree of

DOCTOR OF PHILOSOPHY

in the

FACULTY OF SCIENCE

of the

UNIVERSITY OF LEICESTER

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University of Leicester

1977.

STATEMENT

This thesis is based on work conducted by the author in the Department of Mathematics of the University of Leicester mainly during the period between October 1973 and November 1976.

All the work recorded in this thesis is original unless otherwise acknowledged in the text or references. None of the work has been submitted for another degree in this or any other university.

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September 1977. J.M. Edwards.

THESIS
5215565
14.3.78

ACKNOWLEDGEMENTS

I wish to express my most sincere thanks to my supervisor, Professor T.V. Davies, for his constant help and guidance during the past four years.

My thanks are due to Miss. C.E. Pick for her patience in typing this thesis.

I would like to thank my husband, Dr. C.A.G. Brooks, for his support and encouragement throughout.

A Science Research Council Grant is gratefully acknowledged.

SUMMARY

In this thesis the variation of a functional defined on a variable domain has been studied and applied to the problem of finding the optimum shape of the domain in which some performance criterion has an extremum. The method most frequently used is one due to Gelfand and Fomin. It is applied to problems governed by first and second order partial differential equations, unsteady one dimensionsal gas movements and the problem of minimum drag on a body with axial symmetry in Stokes' flow.

CONTENTS

INTRODUCTION

INTRODUCTION

Distributed parameter system theory refers to those systems whose governing equations are partial differential equations, defined over a domain S, and whose controls are either distributed over S or on parts of the boundary of S. The study of distributed parameter systems was initiated by Butkovskii and Lerner[1-4]. In this thesis the definition of distributed parameter systems has been extended to include continuum problems where the shape of the boundary is control since there are problems in which the shape of the domain is unknown and needs to be determined in order to minimise or maximise some performance criterion. For example the problem of designing the most efficient body for extracting the energy from incident sea waves has recently been discussed by Salter[5]. This problem may be interpreted as the problem of finding the optimum shape of a floating body which minimises the reflection and transmission of the incident wave. Some problems have the boundary of the domain depending on time. Such a problem in which the system is governed by a parabolic equation of the heat conduction type has been considered by Degtyarev [6] and its necessary conditions for opimality obtained.

The earliest reference to variable domain problems appears to be in Forsyth's "Calculus of Variations,"[7] (Chapters ix, x and xi). In their text book "Calculus of Variations"[8] (Chapter 7) Gelfand and Fomin discuss the theory of the first variation of a functional, $J\left[u\right] = \int\cdots\int_R F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})\, dx_1, \ldots, dx_n$, where the independent variables $x_1, \ldots, x_n$, and hence the domain, vary as well as the function u and its derivatives. Neither Forsyth nor Gelfand-

Fomin gives examples of their theory.

In Chapter One the Gelfand - Fomin theorem is extended to m unknown functions and at the end of the chapter two simple examples are given to illustrate the Gelfand - Fomin theorem. In Chapters Two and Three first and second order hyperbolic partial differential equation examples of the extension of the Gelfand - Fomin theorem are discussed.

In Chapter Four a boundary control problem from unsteady one-dimensional gas movements, in which a semi - infinite gas domain is bounded at one end by a moving piston, is discussed using standard characteristic theory. Various problems arise in which the piston movement may be regarded as a control and the one considered is that of determining the piston curve in order to minimise a given functional. In Chapter Five the same problem is resolved using the Gelfand - Fomin theorem, with identical results.

In Chapter Six the Gelfand - Fomin theorem is applied to the problem of minimum drag on a body with axial symmetry in Stokes' flow. Three papers by Pironneau[9-11] have already appeared on this problem but Pironneau's method is not the same as that considered in this thesis. In Chapters Seven and Eight the equations determined in Chapter Six for finding the body of minimum drag in Stokes' flow are discussed firstly by considering the shape near the end point and secondly by a singularity solution.

CHAPTER ONE

CHAPTER ONE

Variation of a Functional Defined on a Variable Domain.

In Section 37 of their book "Calculus of Variations" Gelfand and Fomin derive the first variation of an r-tuple integral where not only the dependent variable and its derivatives vary but also the independent variables, and hence the region of integration, vary. In this chapter this method is extended to m dependent variables, since the theorem is required later in this extended form.

Consider the system

$$J(z_1, z_2, \ldots, z_m) = \int_R \cdots \int F(x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_m, z_{1_{x_1}}, \ldots z_{1_{x_n}}, \ldots, z_{m_{x_1}}, \ldots, z_{m_{x_n}})\, dx_1, \ldots dx_n \qquad (1.1)$$

where R is the simply connected domain of the independent variables $x_1, x_2, \ldots x_n$, and $z_1, z_2, \ldots, z_m$ are functions of $x_1, x_2, \ldots, x_n$, defined and continuous, with continuous first and second derivatives, in R. The integral F is assumed to have continuous first and second derivatives with respect to all its arguments in R.

For simplicity vector notation is used with

$$\underline{x} = (x_1, x_2, \ldots, x_n) \quad ; \quad \underline{z} = (z_1, z_2, \ldots, z_m) \quad ;$$

$$\underline{dx} = (dx_1, dx_2, \ldots, dx_n) \quad ; \quad \nabla\underline{z} = \left(\frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_1}{\partial x_n}, \ldots, \frac{\partial z_m}{\partial x_1}, \ldots, \frac{\partial z_m}{\partial x_n}\right).$$

So equation (1.1) can conveniently be written in the form

$$J[\underline{x}(\underline{z})] = \int_R F(\underline{x}, \underline{z}, \nabla\underline{z})\, d\underline{x} \quad . \qquad (1.2)$$

Consider the family of continuous transformations

$$\left.\begin{array}{ll} x^*_s = \Phi_s(\underline{x},\underline{z},\nabla\underline{z},\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m) \ , & s = 1,2,\ldots,n; \\ z^*_k = \Psi_k(\underline{x},\underline{z},\nabla\underline{z},\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m) \ , & k = 1,2,\ldots,m; \end{array}\right\} \quad (1.3)$$

depending on m parameters $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$, where Φ_s and Ψ_k are differentiable with respect to $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$ and the values $\varepsilon_1 = 0$, $\varepsilon_2 = 0,\ldots,$ $\varepsilon_m = 0$ correspond to the identity transformations so that

$$\left.\begin{array}{ll} x_s \equiv \Phi_s(\underline{x},\underline{z},\nabla\underline{z},0,\ldots,0) \ , & s = 1,2,\ldots,n; \\ z_k \equiv \Psi_k(\underline{x},\underline{z},\nabla\underline{z},0,\ldots,0) \ , & k = 1,2,\ldots,m; \end{array}\right\} \quad (1.4)$$

Now $z_k(x_1,x_2,\ldots,x_n) = C_k =$ constant, $k = 1,2,\ldots,m$, may be thought of as a surface σ_k in the n+1 space E_{n+1} with respect to the coordinates $x_1,x_2,\ldots,x_n,z_k$, and the transformations (1.3) map $\sigma_1, \sigma_2,\ldots,\sigma_m$ into $\sigma_1^*, \sigma_2^*,\ldots,\sigma_m^*$ in the new space E^*_{n+1} with the coordinates $x^*_1,x^*_2,\ldots,x^*_n,z^*_k$. Similarly the functional $J[\underline{z}(\underline{x})]$ in (1.2) transforms into

$$J[\underline{z}^*(\underline{x}^*)] = \int_{R^*} F(\underline{x}^*,\underline{z}^*,\nabla^*\underline{z}^*)d\underline{x}^* \ , \quad (1.5)$$

where $\nabla^*\underline{z}^* = \left(\frac{\partial z^*_1}{\partial x^*_1},\ldots,\frac{\partial z^*_1}{\partial x^*_n},\ldots,\frac{\partial z^*_m}{\partial x^*_1},\ldots,\frac{\partial x^*_m}{\partial x^*_n}\right)$ and R* is the new transformed domain.

The object now is to calculate the terms of order $\underline{\varepsilon} = (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m)$ (that is the principal linear part, δJ, relative to $\underline{\varepsilon}$) of the difference

$$\Delta J = J[\underline{z}^*(\underline{x}^*)] - J[\underline{z}(\underline{x})] \ , \quad (1.6)$$

$\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_m$ being regarded as infinitesimal quantities. Because of the identity relations (1.4) coupled with the continuity of the transformations (1.3) it follows by Taylor's theorem that when $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$ are sufficiently small

$$x^*_s = x_s + \varepsilon_1 \left.\frac{\partial \Phi^*_s(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\varepsilon_1}\right|_{\underline{\varepsilon}=0} + \ldots + \varepsilon_m \left.\frac{\partial \Phi^*_s(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\varepsilon_m}\right|_{\underline{\varepsilon}=0} + O(\underline{\varepsilon}^2)$$

$$s = 1,2,\ldots,n$$

$$z^*_k = z_k + \varepsilon_1 \left.\frac{\partial \Psi^*_k(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\varepsilon_1}\right|_{\underline{\varepsilon}=0} + \ldots + \varepsilon_m \left.\frac{\partial \Psi^*_k(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\varepsilon_m}\right|_{\underline{\varepsilon}=0} + O(\underline{\varepsilon}^2)$$

$$k = 1,2,\ldots m.$$

These transformations can be written more simply in the form

$$\left.\begin{aligned} x^*_s &= x_s + \sum_{l=1}^{m} \varepsilon_l\, \phi_s^{(l)}(\underline{x},\underline{z},\nabla\underline{z}) + O(\underline{\varepsilon}^2)\ ,\quad s = 1,2,\ldots,n\ , \\ z^*_k &= z_k + \sum_{l=1}^{m} \varepsilon_l\, \psi_k^{(l)}(\underline{x},\underline{z},\nabla\underline{z}) + O(\underline{\varepsilon}^2)\ ,\quad k = 1,2,\ldots,m\ , \end{aligned}\right\} \qquad (1.7)$$

where $\phi_s^{(l)}(\underline{x},\underline{z},\nabla\underline{z}) = \left.\dfrac{\partial\, \Phi_s(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\, \varepsilon_l}\right|_{\underline{\varepsilon}=0}$; $l = 1,2,\ldots,m;$

$$\psi_k^{(l)}(\underline{x},\underline{z},\nabla\underline{z}) = \left.\frac{\partial \Psi_k(\underline{x},\underline{z},\nabla\underline{z},\underline{\varepsilon})}{\partial\, \varepsilon_l}\right|_{\underline{\varepsilon}=0} \ ; \quad l = 1,2\ldots,m.$$

For a given surface σ_k, $(k = 1,2,\ldots,m)$,

with equation $C_k = z_k(\underline{x})$, (1.7) leads to the increments

$$\begin{aligned} \Delta x_s = x^*_s - x_s &= \sum_{l=1}^{m} \varepsilon_l\, \phi_s^{(l)}(\underline{x}) + O(\underline{\varepsilon}^2) \\ &= \delta x_s + O(\underline{\varepsilon}^2)\ ,\quad s = 1,2,\ldots,n\ ; \end{aligned} \qquad (1.8)$$

$$\Delta z_k = z^*_k(\underline{x}^*) - z_k(\underline{x}) = \sum_{l=1}^{m} \varepsilon_l\, \psi^{(l)}(\underline{x}) + O(\underline{\varepsilon}^2)$$

$$= \delta z_k + O(\underline{\varepsilon}^2) \quad , \quad k = 1,2,\ldots,m \; ;(1.9)$$

where the arguments $\underline{z}$ and $\nabla\underline{z}$ have been replaced by $\underline{z}(\underline{x})$ and $\nabla\underline{z}(\underline{x})$. Thus (1.9) gives the change in value of z_k in going from a point $[\underline{x}, z_1(\underline{x}), \ldots, z_{k-1}(\underline{x}), z_k(\underline{x}), z_{k+1}(\underline{x}), \ldots, z_m(\underline{x})]$ to a point $[\underline{x}^*, z_1(\underline{x}^*), \ldots, z_{k-1}(\underline{x}^*), z_k^*(\underline{x}^*), z_{k+1}(\underline{x}^*), \ldots, z_m(\underline{x}^*)]$, $s = 1,2,\ldots,n$. The variations δx_s and δz_k corresponding to (1.7) are defined as the principal linear parts (relative to $\underline{\varepsilon}$) of the increments in the right hand sides of equations (1.8) and (1.9), that is

$$\delta x_s = \sum_{l=1}^{m} \varepsilon_l \phi_s^{(l)}(\underline{x}) \quad , \quad s = 1,2,\ldots,n \quad ; \qquad (1.10)$$

$$\delta z_k = \sum_{l=1}^{m} \varepsilon_l \psi_k^{(l)}(\underline{x}) \quad , \quad k = 1,2,\ldots,m \quad . \qquad (1.11)$$

Consider the increment

$$\overline{\Delta z}_k = z_k^*(\underline{x}) - z_k(\underline{x}) \quad , \quad k = 1,2,\ldots,m \quad ,$$

that is the change in z_k in going from the point $[\underline{x}, z_1(\underline{x}), \ldots, z_{k-1}(\underline{x}), z_k(\underline{x}), z_{k+1}(\underline{x}), \ldots, z_m(\underline{x})]$ on the surface σ_k to the point $[\underline{x}, z_1(\underline{x}), \ldots, z_{k-1}(\underline{x}), z_k^*(\underline{x}), z_{k+1}(x), \ldots, z_m(\underline{x})]$ on the surface σ_k^* with the <u>same</u> $\underline{x}$-coordinate.

The notation

$$\begin{aligned} \overline{\Delta z}_k &= z_k^*(\underline{x}) - z_k(x) \\ &= \sum_{l=1}^{m} \varepsilon_l \overline{\psi}^{(l)} + O(\underline{\varepsilon}^2) \\ &= \overline{\delta z}_k + O(\underline{\varepsilon}^2) \qquad , \; k = 1, 2, \ldots, m \; , \end{aligned} \qquad (1.12)$$

is used to find the relationship between $\overline{\delta z}_k$ and δz_k. Now

$$\begin{aligned} \Delta z_k &= z_k^*(\underline{x}^*) - z_k(\underline{x}) \\ &= [z_k^*(\underline{x}^*) - z_k^*(\underline{x})] + [z_k^*(\underline{x}) - z_k(\underline{x})] \\ &= \left\{ \sum_{s=1}^{n} \frac{\partial z_k^*}{\partial x_s}(x_s^* - x_s) + O(\underline{\varepsilon}^2) \right\} + [\overline{\delta z}_k + O(\underline{\varepsilon}^2)] \end{aligned}$$

$$= \sum_{s=1}^{n} \frac{\partial z_k^*}{\partial x_s} \delta x_s + \overline{\delta z}_k + O(\underline{\varepsilon}^2), \quad k = 1, 2, \ldots, m.$$

Since $\frac{\partial z_k^*}{\partial x_s}$ and $\frac{\partial z_k}{\partial x_s}$ differ only by a quantity of order $\underline{\varepsilon}$ this equation may be written as

$$\delta z_k = \sum_{s=1}^{n} \frac{\partial z_k}{\partial x_s} \delta x_s + \overline{\delta z}_k \qquad , k = 1, 2, \ldots, m, \tag{1.13}$$

where δz_k is the principal linear part of Δz_k (relative to $\underline{\varepsilon}$).

An alternative form of (1.13) is, using (1.10), (1.11) and (1.12),

$$\sum_{l=1}^{m} \varepsilon_l \psi_k^{(l)}(\underline{x}) = \sum_{s=1}^{n}\sum_{l=1}^{m} \frac{\partial z_k}{\partial x_s} \varepsilon_l \phi_s^{(l)}(\underline{x}) + \sum_{l=1}^{m} \varepsilon_l \overline{\psi}_k^{(l)}(\underline{x}), \tag{1.14}$$

$$s = 1,2,..,n;\ k = 1,2,..m.$$

Consider the expression for the increment, $\frac{\partial}{\partial x_s}(\Delta z_k)$, of the gradient $\nabla \underline{z}$, that is, $\frac{\partial(\Delta z_k)}{\partial x_s} = \frac{\partial z_k^*(\underline{x}^*)}{\partial x_s^*} - \frac{\partial z_k(\underline{x})}{\partial x_s}$, or more precisely its principal linear part (relative to $\underline{\varepsilon}$) $\frac{\partial(\delta z_k)}{\partial x_s}$, $s = 1,2,\ldots n$; $k = 1, 2, \ldots m$. It can be derived from (1.7) that

$$\frac{\partial x_i^*}{\partial x_s} = \delta_{si} + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s} + O(\underline{\varepsilon}^2), \quad i,s = 1,2,\ldots n \quad ; \tag{1.15}$$

where δ_{si} is the Kronecker delta. It now follows from the chain rule that

$$\frac{\partial}{\partial x_s} = \sum_{i=1}^{n} \frac{\partial x_i^*}{\partial x_s} \frac{\partial}{\partial x_i^*}$$

$$= \sum_{i=1}^{n} \left\{ \delta_{si} + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s} + O(\underline{\varepsilon}^2) \right\} \frac{\partial}{\partial x_i^*}$$

$$= \frac{\partial}{\partial x_s^*} + \sum_{i=1}^{n}\sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s} \frac{\partial}{\partial x_i^*} + O(\underline{\varepsilon}^2),$$

hence,

$$\frac{\partial}{\partial x_s} - \frac{\partial}{\partial x_s^*} = \sum_{i=1}^{n}\sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s} \frac{\partial}{\partial x_i^*} + O(\underline{\varepsilon}^2). \tag{1.16}$$

The increment in $\frac{\partial z_k}{\partial x_s}$ is given by

$$\Delta\left(\frac{\partial z_k}{\partial x_s}\right) = \frac{\partial z_k^*(\underline{x}^*)}{\partial x_s^*} - \frac{\partial z_k(\underline{x})}{\partial x_s}$$

$$= \frac{\partial}{\partial x_s^*}\left\{ z_k^*(\underline{x}^*) - z_k(\underline{x}^*) \right\} + \frac{\partial}{\partial x_s}\left\{ z_k(\underline{x}^*) - z_k(\underline{x}) \right\}$$

$$+\left(\frac{\partial}{\partial x_s^*} - \frac{\partial}{\partial x_s}\right) z_k(\underline{x}^*) \;, \quad k = 1,2,..,m \quad (1.17)$$

Analysing the three terms on the right hand side of equation (1.17) separately gives (a) from (1.12).

$$z_k^*(\underline{x}) - z_k(\underline{x}) = \sum_{l=1}^{m} \varepsilon_l \bar{\psi}_k^{(l)}(\underline{x}) + O(\underline{\varepsilon}^2) \;, \qquad k = 1,2,...m.$$

hence, using (1.16)

$$\frac{\partial}{\partial x_s^*}\left\{z_k^*(\underline{x}^*) - z_k(\underline{x}^*)\right\} = \left\{\frac{\partial}{\partial x_{sm}} - \sum_{i=1}^{n}\sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_i^{(l)}(x)}{\partial x_s}\frac{\partial}{\partial x_i^*} + O(\underline{\varepsilon}^2)\right\} \times \left\{\sum_{l=1} \varepsilon_l \psi_k^{(l)}(\underline{x}^*) + O(\underline{\varepsilon}^2)\right\}$$

$$= \sum_{l=1}^{m} \varepsilon_l \frac{\partial \bar{\psi}_k^{(l)}(x^*)}{\partial x_s} + O(\underline{\varepsilon}^2), \quad k = 1,2,...m, (1.18)$$

(b) $$\frac{\partial}{\partial x_s}\left\{z_k(\underline{x}^*) - z_k(\underline{x})\right\} = \frac{\partial}{\partial x_s}\left\{\sum_{i=1}^{n} \frac{\partial z_k(x)}{\partial x_i}(x_i^* - x_i) + O(\underline{\varepsilon}^2)\right\}$$

and using (1.8) this equation becomes

$$\frac{\partial}{\partial x_s}\left\{z_k(\underline{x}^*) - z_k(\underline{x})\right\} = \sum_{l=1}^{m}\sum_{i=1}^{n} \varepsilon_l \frac{\partial z_k(x)}{\partial x_i} \phi_i^{(l)}(\underline{x}) + O(\underline{\varepsilon}^2) \;, \quad (1.19)$$

$$k = 1,2,..., m;$$

(c) for the final term on the right hand side of equation (1.17)

$$\left(\frac{\partial}{\partial x_s^*} - \frac{\partial}{\partial x_s}\right) z_k(\underline{x}^*) = \left(\frac{\partial}{\partial x_s^*} - \frac{\partial}{\partial x_s}\right) z_k(\underline{x}) + O(\underline{\varepsilon}^2), \quad k = 1, 2, ...m,$$

and applying (1.16)

$$\left(\frac{\partial}{\partial x_s^*} - \frac{\partial}{\partial x_s}\right) z_k(\underline{x}^*) = - \sum_{l=1}^{m}\sum_{i=1}^{n} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s}\frac{\partial z_k(\underline{x})}{\partial x_i^*} + O(\underline{\varepsilon}^2)$$

$$= - \sum_{l=1}^{m}\sum_{i=1}^{n} \varepsilon_l \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s}\frac{\partial z_k(\underline{x})}{\partial x_i} \;, \quad k = 1,2,...m, (1.20)$$

since $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_i^*}$ differ by a term of order ε. Adding together equations (1.18), (1.19) and (1.20) and using (1.17) gives

$$\Delta\left(\frac{\partial z_k}{\partial x_s}\right) = \sum_{l=1}^{m} \varepsilon_l \left\{\frac{\partial \bar{\psi}^{(l)}(\underline{x})}{\partial x_s} + \frac{\partial}{\partial x_s}\left[\sum_{i=1}^{n} \frac{\partial z_k(\underline{x})}{\partial x_s}\phi_i^{(l)}(\underline{x})\right] - \sum_{i=1}^{n} \frac{\partial \phi_i^{(l)}(\underline{x})}{\partial x_s}\frac{\partial z_k(\underline{x})}{\partial x_i}\right\} + O(\underline{\varepsilon}^2)$$

$$= \sum_{l=1}^{m} \varepsilon_l \left\{ \frac{\partial \overline{\psi}_k^{(l)}(\underline{x})}{\partial x_s} + \sum_{i=1}^{n} \frac{\partial^2 z_k(\underline{x})}{\partial x_s \, \partial x_i} \, \phi_i^{(l)}(\underline{x}) \right\} + O(\underline{\varepsilon}^2) \quad , \qquad (1.21)$$

$$k = 1,2,\ldots m.$$

Finally using the defininition of $\overline{\psi}_k(\underline{x})$ in (1.12) and ϕ_i in (1.8) gives

$$\Delta\left(\frac{\partial z_k}{\partial x_s}\right) = \overline{\delta z}_{k x_s} + \sum_{i=1}^{n} \frac{\partial^2 z_k(\underline{x})}{\partial x_s \partial x_i} \, \delta x_i + O(\underline{\varepsilon}^2), \qquad k = 1,2,\ldots m, \quad (1.22)$$

and the principal linear part, $\delta z_{k_{x_s}}$, of $\Delta\left(\frac{\delta z_k}{\delta x_s}\right)$ is given by

$$\delta z_{k x_s} = \overline{\delta z}_{k x_s} + \sum_{i=1}^{n} \frac{\partial^2 z_k(\underline{x})}{\partial x_s \partial x_i} \, \delta x_i \quad , \qquad k = 1,2,\ldots,m \quad . \quad (1.23)$$

Consider now the increment ΔJ defined in (1.6). The following result will be established:

$$\Delta J = \sum_{l=1}^{m} \sum_{k=1}^{m} \left\{ \varepsilon_l \int_R \left[F_{z_k} - \sum_{s=1}^{n} \frac{\partial}{\partial x_s} F_{z_{k x_s}} \right] \overline{\psi}_k^{(l)} \, d\underline{x} \right\}$$
$$+ \sum_{l=1}^{m} \varepsilon_l \int_R \sum_{s=1}^{n} \frac{\partial}{\partial x_s} \left[\sum_{k=1}^{m} F_{z_{k x_s}} \overline{\psi}_k^{(l)} + F \phi_s^{(l)} \right] d\underline{x}, \qquad (1.24)$$

where $\overline{\psi}^{(l)}$ is given in terms of $\psi^{(l)}$ in (1.14) .

The proof of equation (1.24) is as follows: by definition (1.6)

$$\Delta J = \int_{R^*} F(\underline{x}^*, \underline{z}^*, \nabla^* \underline{x}^*) \, d\underline{x}^* - \int_R F(\underline{x}, \underline{z}, \nabla \underline{z}) \, d\underline{x}$$
$$= \int_R \left\{ F(\underline{x}^*, \underline{z}^*, \nabla^* \underline{z}^*) \, \frac{\partial(x_1^*, x_2^*, \ldots, x_n^*)}{\partial(x_1, x_2, \ldots, x_n)} - F(\underline{x}, \underline{z}, \nabla \underline{z}) \right\} d\underline{x} \qquad (1.25)$$

From the definition of a Jacobian, and (1.8),

$$\frac{\partial(x_1^*, x_2^*, \ldots, x_n^*)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} 1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_1^{(l)}}{\partial x_1} , & \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_2^{(l)}}{\partial x_1} , & \ldots\ldots\ldots\ldots , & \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_n^{(l)}}{\partial x_1} \\ \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_1^{(l)}}{\partial x_2} , & 1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_2^{(l)}}{\partial x_2} , \ \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_3^{(l)}}{\partial x_2} & \ldots\ldots\ldots , & \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi^{(l)}}{\partial x_2} \\ \vdots & & & \\ \sum_{l=1}^{m} \frac{\partial \phi_1^{(l)}}{\partial x_n} , & \ldots\ldots\ldots\ldots , \ \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_{n-1}^{(l)}}{\partial x_n} , & & 1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_n^{(l)}}{\partial x_n} \end{vmatrix}$$

hence

$$\frac{\partial(x_1^*, x_2^*, \ldots, x_n^*)}{\partial(x_1, x_2, \ldots, x_n)} = \left(1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_1^{(l)}}{\partial x_s}\right)\left(1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_2^{(l)}}{\partial x_2}\right) \cdots \left(1 + \sum_{l=1}^{m} \varepsilon_l \frac{\partial \phi_n^{(l)}}{\partial x_n}\right)$$
$$+ O(\underline{\varepsilon}^2)$$

$$= 1 + \sum_{l=1}^{m} \sum_{s=1}^{n} \varepsilon_l \frac{\partial \phi_s^{(l)}}{\partial x_s} + O(\underline{\varepsilon}^2)$$

Thus from (1.25)

$$\Delta J = \int_R \Big\{ F(\underline{x}^*,\underline{z}^*,\nabla\underline{z}^*) \Big[1 + \sum_{l=1}^{m}\sum_{s=1}^{n} \varepsilon_l \frac{\partial\phi_s^{(l)}}{\partial x_s} + O(\underline{\varepsilon}^2)\Big] - F(\underline{x},\underline{z},\nabla\underline{z}) \Big\}\, d\underline{x} \quad . \tag{1.26}$$

Taylor's theorem is now used to expand the first term in the integrand of (1.26) remembering the notation

$$x_s^* = x_s + \delta x_s \ , \ z_k^* = z_k + \delta z_k \ , \ \frac{\partial z_k^*}{\partial x_s^*} = \frac{\partial z_k}{\partial x_s} + \frac{\partial(\delta z_k)}{\partial x_s} \ ;$$

$$\Delta J = \int_R \Big\{ \Big[F(\underline{x},\underline{z},\nabla\underline{z}) + \sum_{s=1}^{n} F_{x_s}\delta x_s + \sum_{k=1}^{m} \delta z_k F_{z_k} + \sum_{s=1}^{n}\sum_{k=1}^{m} \delta z_{k_{x_s}} F_{z_{k_{x_s}}} \Big] \times \Big[1 + \sum_{l=1}^{m}\sum_{s=1}^{n} \varepsilon_l \frac{\partial\phi_s^{(l)}}{\partial x_s}\Big] - F(\underline{x},\underline{z},\nabla\underline{z}) \Big\}\, dx$$

$$\Delta J = \int_R \Big\{ \sum_{s=1}^{n} F_{x_s}\delta x_s + \sum_{k=1}^{m} \delta z_k F_{z_k} + \sum_{s=1}^{n}\sum_{k=1}^{m} \delta z_{k_{x_s}} F_{z_{k_{x_s}}} + \sum_{l=1}^{m}\sum_{s=1}^{n} \varepsilon_l F(\underline{x},\underline{z},\nabla\underline{z}) \frac{\partial\phi_s^{(l)}}{\partial x_s} \Big\}\, d\underline{x} \ . \tag{1.27}$$

Equation (1.8) is used to replace $\sum_{l=1}^{m} \varepsilon_l \phi_s^{(l)}$ by δx_s in the final term of the integrand of (1.27) and using (1.13) and (1.22) this gives, correct to the first order in $\underline{\varepsilon}$,

$$\delta J = \int_R \Big\{ \sum_{s=1}^{n} F_{x_s}\delta x_s + \sum_{k=1}^{m} \Big(\overline{\delta z}_k + \sum_{s=1}^{n} \frac{\partial z_k}{\partial x_s}\delta x_s\Big) F_{z_k} + \sum_{s=1}^{n}\sum_{k=1}^{m} \Big(\overline{\delta z}_{k_{x_s}} + \sum_{i=1}^{n} \frac{\partial^2 z_k}{\partial x_s \partial x_i}\Big) F_{z_{k_{x_s}}} + F(\underline{x},\underline{z},\nabla\underline{z}) \sum_{s=1}^{n} \frac{\partial(\delta x_s)}{\partial x_s} \Big\}\, d\underline{x} \ , \tag{1.28}$$

where δJ is the prinicpal linear part of ΔJ, relative to $\underline{\varepsilon}$. This is now expressed in the form $G(\underline{x})\overline{\delta \underline{z}} + \mathrm{div}(\ldots)$

$$\overline{\delta z}_{k_{x_s}} F_{z_{k_{x_s}}} = \frac{\partial}{\partial x_s}\Big\{ \overline{\delta z}_k F_{z_{x_s}} \Big\} - \overline{\delta z}_k \frac{\partial}{\partial x_s} F_{z_{k_{x_s}}}$$

and thus equation (1.28) can be rearranged into the form

$$\delta J = \int_R \sum_{k=1}^{m} \overline{\delta z}_k \Big\{ F_{z_k} - \sum_{s=1}^{n} \frac{\partial}{\partial x_s} F_{z_{k_{x_s}}} \Big\}\, d\underline{x} + \int_R \Big\{ \sum_{s=1}^{n} F_{x_s}\delta x_s + F\frac{\partial}{\partial x_s}(\delta x_s) + \sum_{k=1}^{m} F_{z_k} \frac{\partial z_k}{\partial x_s}\delta x_s +$$

$$+ \sum_{k=1}^{m} \sum_{i=1}^{n} F_{z_{k x_s}} \frac{\partial^2 z}{\partial x_s \, \partial x_i} \delta x_i \Big\} \, d\underline{x}$$

$$+ \int_R \sum_{s=1}^{n} \sum_{k=1}^{m} \frac{\partial}{\partial x_s} \left\{ \overline{\delta z}_k \, F_{z_{k x_s}} \right\} d\underline{x}$$

hence

$$\delta J = \int_R \sum_{k=1}^{m} \overline{\delta z}_k \left\{ F_{z_k} - \sum_{s=1}^{n} \frac{\partial}{\partial x_s} F_{z_{k x_s}} \right\} d\underline{x} + \int_R \sum_{s=1}^{n} \sum_{k=1}^{m} \frac{\partial}{\partial x_s} \left\{ F \delta x_s + \overline{\delta z}_k \, F_{z_{k x_s}} \right\} d\underline{x} \quad . \qquad (1.29)$$

This expression is the same as that quoted in (1.24) since

$$\overline{\delta z}_k = \sum_{l=1}^{m} \varepsilon_l \, \overline{\psi}_k^{(l)}(\underline{x}) \qquad k = 1, 2, \ldots, m \qquad \text{and}$$

$$\delta x_s = \sum_{l=1}^{m} \varepsilon_l \, \phi_s^{(l)}(\underline{x}) \quad , \; s = 1, 2, \ldots, n.$$

Two simple examples will now be discussed to illustrate the Gelfand-Fomin theorem.

Case 1. $m = 1, n = 1$.

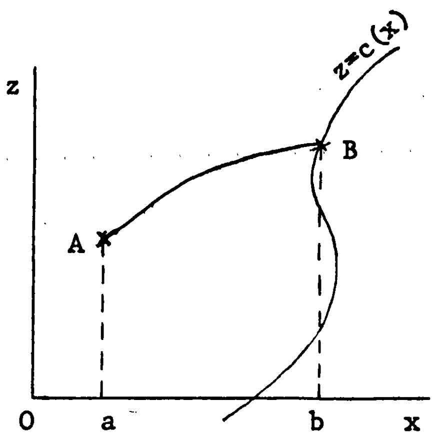


Figure 1.1

The problem is to find the shape of the curve connecting the fixed point A and the curve $z = c(x)$ which minimises

$$J(z) = \int_a^b F(x, z(x), z'(x)) \, dx$$

where the point $(a, z(a))$ is fixed but the value of b may vary.

From equation (1.29)

$$\delta J = \int_a^b \overline{\delta z}\left\{F_z - \frac{\partial}{\partial x_s} F_{z_x}\right\} dx + \int_a^b \frac{d}{dx}\left\{F\,\delta x + \overline{\delta z}\,F_{z_x}\right\} dx,$$

$$\delta J = \int_a^b \overline{\delta z}\left\{F_z - \frac{\partial}{\partial x_s} F_{z_x}\right\} dx + \left[F\,\delta x + \overline{\delta z}\,F_{z_x}\right]_a^b .$$

From equation (1.13)

$$\overline{\delta z} = \delta z - \frac{dz}{dx}\,\delta x$$

At the end B since $z = c(x)$, and $\delta z = c'(x)\,\delta x$, $\overline{\delta z} = c'(x) - \frac{dz}{dx}$, at $x=b$.

At $x = a$, δx and δz are zero since A is a fixed point so,

$$\delta J = \int_a^b \overline{\delta z}\left\{F_z - \frac{\partial}{\partial x} F_{z_x}\right\} dx + \delta x\left\{F + \left[c'(x) - z'(x)\right] F_{z_x}\right\}_{x=b} .$$

For a minimum δJ is zero, so as $\overline{\delta z}$ and δx are arbitary variations

$$F_z - \frac{\partial}{\partial x} F_{z_x} = 0 \quad , \qquad (x,z) \in z = c(x) \quad , \qquad (1.30)$$

$$F + \left[c'(x) - z'(x)\right] F_{z_x} = 0 \quad \text{at } x = b \quad . \qquad (1.31)$$

Equations (1.30) and (1.31) are the same as those that are derived when this problem is solved by the Euler Variational method. (1.31) is the well-known transversality condition.

Case II . $n = 2$.

In this example the performance index

$$J = \iint_S F(x,y,z,z_x,z_y)\,dxdy \qquad (1.32)$$

is minimised over the domain S as the position of the curve C which bounds S varies. z is required to take prescribed values on C so that on C there is the condition

$$z = g(x,y) \quad , \qquad (x,y) \in C$$

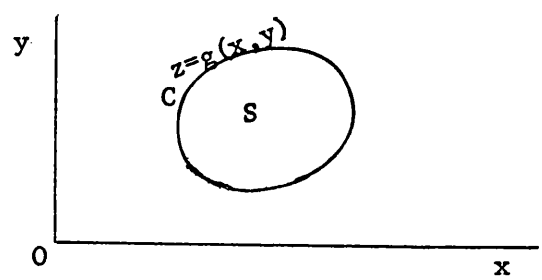


Figure 1.2

From equation (1.29)

$$\delta J = \iint_S \overline{\delta z} \left\{ F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} \right\} dx\, dy$$
$$+ \iint_S \left\{ \frac{\partial}{\partial x} \left[F\delta x + \overline{\delta z}\, F_{z_x} \right] + \frac{\partial}{\partial y} \left[F\delta y + \overline{\delta z}\, F z_y \right] \right\} dx\, dy$$

Applying Stokes' theorem in two dimensions to the second integrand, δJ becomes

$$\delta J = \iint_S \overline{\delta z} \left\{ F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} \right\} dx\, dy$$
$$+ \oint_c \left\{ \left[F\delta x + \overline{\delta z} \frac{\partial F}{\partial z_x} \right] dy - \left[F\delta y + \overline{\delta z} \frac{\partial F}{\partial z_y} \right] dx \right\}. \quad (1.33)$$

From the equation (1.31)

$$\overline{\delta z} = \delta z - \frac{\partial z}{\partial x} \delta x - \frac{\partial z}{\partial y} \delta y \,.$$

As $z = g(x,y)$, $\delta z = \frac{\partial g}{\partial x} \delta x + \frac{\partial g}{\partial y} \delta y$ so

$$\overline{\delta z} = \delta x \left(\frac{\partial g}{\partial x} - \frac{\partial z}{\partial x} \right) + \delta y \left(\frac{\partial g}{\partial y} - \frac{\partial z}{\partial y} \right)$$

and (1.33) may be written as

$$\delta J = \iint_S \overline{\delta z} \left\{ F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} \right\} dx\, dy$$
$$+ \oint_c \left\{ \delta x \left[F dy + (g_x - z_x) \frac{\partial}{\partial z_x} F\, dy - (g_x - z_x) \frac{\partial}{\partial z_y} F\, dx \right] + \right.$$
$$\left. + \delta y \left[(g_y - z_y) \frac{\partial F}{\partial z_x} dy - F\, dx - (g_y - z_y) \frac{\partial F}{\partial z_y} dx \right] \right\} .$$

For a minimum of J in (1.32) δJ must be zero. Since $\overline{\delta z}, \delta x$ and δy are arbitary variations

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0 \qquad , \qquad (1.34)$$

$$F y'(x) + (g_x - z_x) \frac{\partial F}{\partial z_x} y'(x) - (g_x - z_x) \frac{\partial F}{\partial z_y} = 0 \qquad \text{,on C} \quad (1.35)$$

$$(g_y - z_y) \frac{\partial F}{\partial z_x} y'(x) - F - (g_y - z_y) \frac{\partial F}{\partial z_y} = 0 \qquad \text{,on C} \quad (1.36)$$

The conditions (1.35) and (1.36) are not independent since if (1.36) is multiplied by $y'(x)$ and added to (1.35) then

$$y'(x) \frac{\partial F}{\partial z_x} \left\{ (g_x - z_x) + (g_y - z_y) y'(x) \right\} - \frac{\partial F}{\partial z_y} \left\{ (g_x - z_x) + (g_y - z_y) y'(x) \right\} = 0$$

The term

$$\left\{ g_x - z_x + (g_y - z_y)\, y'(x) \right\}$$

is the differential along the curve C of the function g - z, therefore since z = g on C this vanishes. Hence one and only one transversality condition remains.

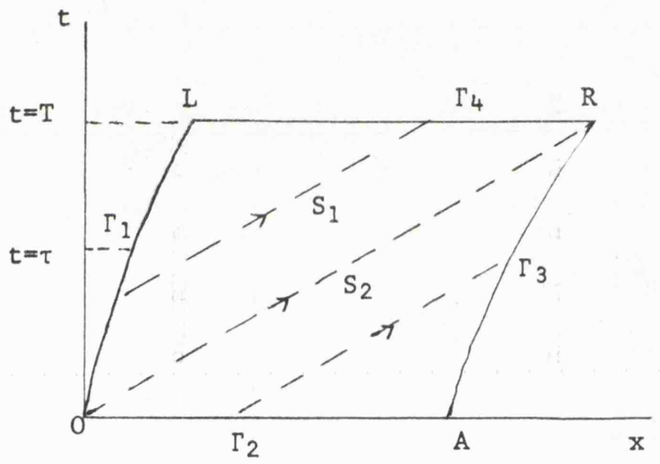
CHAPTER TWO

CHAPTER TWO

A First Order Hyperbolic Partial Differential Equation Example of the of the Use of the Gelfand Fomin Theorem.

In order to acquire experience in the handling of the Gelfand-Fomin Theorem the following simple hyperbolic partial differential equation problem is considered.

Let S be the domain in the (x,t) plane indicated in the diagram; S is bounded by the closed curve OARL and the various parts of the boundary need to be discussed.



$S = S_1 \cup S_2$

Figure 2.1

In the first place OA is a portion of $t = 0$, O being the origin of coordinates and A a given fixed point; LR is a portion of the line $t = T$. It is convenient to label the four portions of the boundary as Γ_1, Γ_2, Γ_3, Γ_4, as shown in figure 2.1. In particular it is assumed that the equation of the curve OL is expressed in the form

$$x = \alpha(\tau)\ , \quad t = \tau \qquad\qquad 0 \leqslant \tau \leqslant T \qquad , \qquad\qquad (2.1)$$

τ being a time parameter, with $\alpha(0) = 0$.

It is assumed that a function $\emptyset(x,t)$ is defined for all $(x,t) \in S$

and $\emptyset$ satisfies in S the quasi-linear partial differential equation

$$\frac{\partial \emptyset}{\partial t} = g(x,t,\emptyset,\emptyset_x) \equiv -A(x,t,\emptyset)\emptyset_x + B(x,t,\emptyset), \quad (x,t)\epsilon S, \qquad (2.2)$$

where A and B are functions of x,t and $\emptyset$. The ordinary differential equation of the family of characteristics for equation (2.2) is

$$\frac{dx}{dt} = A(x,t,\emptyset) \quad , \qquad (2.3)$$

and certain restrictions will be placed on A as follows. In the first place it is postulated that

$$A(x,t,\emptyset) > 0 \qquad \text{for all} \quad (x,t)\ \epsilon S , \qquad (2.4)$$

further, it is assumed that the funtion A is such that, travelling along the characteristics with x increasing, each characteristic commencing at any point of OA or OL will travel into the domain S and will eventually meet either LR or AR in a single point. This implies that the slope of OL at the point τ must be greater than the slope of the characteristic at τ, that is

$$A \left\{ \alpha(\tau), \tau, \emptyset|_\tau \right\} > \alpha'(\tau) , \qquad 0 < \tau < T . \qquad (2.5)$$

It is assumed that the particular characteristic of the family (2.3) which commences at O ultimately intersects the line $t = T$ at the point R, thus all the characteristics commencing along OL will meet LR and the characteristics commencing along OA will meet AR. The characteristic OR divides S into two parts S_1 and S_2.

Also it is assumed that the boundary conditions upon $\emptyset$ on the portions Γ_1 and Γ_2 are as follows:

$$M(x,t,\emptyset) \equiv 0 \quad , \ (x,t)\ \epsilon\ \Gamma_1 , \qquad (2.6)$$

$$N(x,\emptyset) \equiv 0 \quad , \ (x,t)\ \epsilon\ \Gamma_2 . \qquad (2.7)$$

The control problem can now be stated. It is postulated that the position of the curve OL has to be found, subject to (2.5) being satisfied, in order to minimise the performance criterion I defined by

$$I = \iint_{S_1} P(x,\ t,\ \emptyset,\ \emptyset_x\ )dx\ dt + \int_{LR} Q(x,T,\emptyset)\ dx + \int_{\tau=0}^{\tau=T} f(\tau,\alpha\ \alpha',\alpha'')d\tau \quad ,(2.8)$$

the functions P, Q and f being prescribed; in other words the function $\alpha(\tau)$ which was introduced in (2.1) must be determined. It is clear from characteristic theory that any variations in the position of the curve OL, such that $\alpha(0) = 0$, will influence the value of $\emptyset$ in S_1 only, the value of $\emptyset$ in S_2 being unaffected by such variations. It is for this reason that the double integral in (2.8) is taken over the domain S_1 only and not over the whole domain S.

Consider now in place of I a new functional J given by

$$J = \iint_{S_1} \{P + \lambda(g - \emptyset_t)\}\ dx\ dt + \int_{LR} Q\ dx + \int_{\tau=0}^{T} f\ d\tau \ ; \qquad (2.9)$$

where λ is a Lagrange multiplier depending on x and t. By introducing a Hamiltonian H defined by

$$H = P(x,t,\emptyset,\emptyset_x\ ) + \lambda g(x,t,\emptyset,\emptyset_x\ )\ , \qquad (2.10)$$

J in (2.9) can be written in the form

$$J = \iint_{S_1} (H - \lambda\emptyset_t)\ dx\ dt + \int_{arcLR} Q\ dx + \int_{\tau=0}^{T} f\ d\tau \quad . \qquad (2.11)$$

The value of the increment, δJ, in J when a variation occurs in the location of OL is now investigated. The variation in the position of OL can be done by adding to $\alpha(\tau)$ and increment $\delta\alpha(\tau)$ at the same time τ.

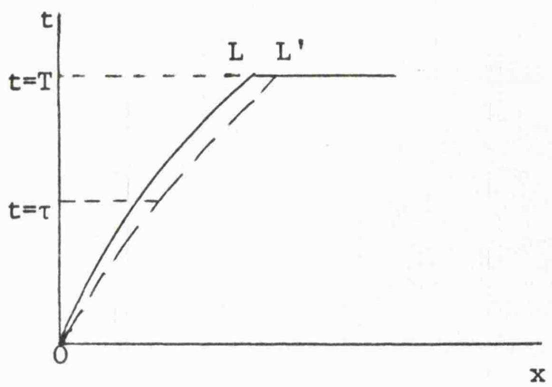


Figure 2.2

The curve OL i.e. $x = \alpha(\tau)$, $t = \tau$, (2.12)

will be regarded as the curve which provides the minimum of I in (2.8) and the varied curve is OL', namely

$$x = \alpha(\tau) + \delta\alpha(\tau) \quad , t = \tau, \quad 0 < \tau < T \tag{2.13}$$

$\alpha(\tau)$ and $\delta\alpha(\tau)$ are assumed to be continuous functions satisfying

$$\alpha(0) = 0, \qquad \delta\alpha(0) = 0 \quad . \tag{2.14}$$

The postulate (2.14) implies that no variation occurs at the origin so that the characteristic OR is unaltered in position. The new value of ϕ on OL' will follow from the boundary condition (2.6) but the value of ϕ on Γ_2 , see (2.7), remains unchanged in the variation and likewise on the characteristic OR

$$\delta\phi = 0 \ , \quad (x,t) \in \quad \text{characteristic OR} \quad . \tag{2.15}$$

Specialising the Gelfand - Fomin result to the two dimensional space S_1 in the (x,t) plane this result can be stated as follows : with

$$\chi_1(\phi) = \int_{S_1}\int F(x,t,\phi,\phi_x,\phi_t)\, dx\, dt \quad , \tag{2.16}$$

$$F(x,t,\phi,\phi_x,\phi_t) = (H - \lambda\phi_t) \quad , \tag{2.17}$$

the increment $\delta\chi_1$ is given, from (1.29), by

$$\delta\chi_1 = \int_{S_1}\int \overline{\delta\phi} \left\{ F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t} \right\} dx\, dt$$

$$+ \int_{S_1}\int \left\{ \frac{\partial}{\partial x} (F\, \delta x + \overline{\delta\phi}\, F_{\phi_x}) + \frac{\partial}{\partial t} (F\delta t + \overline{\delta\phi}\, F_{\phi_t}) \right\} dx\, dt \tag{2.18}$$

where $\delta\phi$, from (1.13), the increment in the function ϕ, is related to $\overline{\delta\phi}$ by

$$\delta\phi = \overline{\delta\phi} + \frac{\partial\phi}{\partial x}\, \delta x + \frac{\partial\phi}{\partial t}\, \delta t \quad , \tag{2.19}$$

and δx, δt are the increments in x and t arising from the variation in the domain S_1 .

Using Stokes' Theorem in [2] , (2.18) can be written in the form

$$\delta X_1 = \int_{S_1}\int \overline{\delta\phi} \left\{F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right\} dx\, dt + \int_{OR+RL+LO} \left\{(F\delta x + \overline{\delta\phi} F_{\phi_x}) \, dt - (F\, \delta t + \overline{\delta\phi}\, F_{\phi_t}) \, dx\right\} \quad . \tag{2.20}$$

The variation of the line integral

$$X_2 = \int_{LR} Q\, (x, T, \phi)\, dx \quad , \tag{2.21}$$

can also be discussed using the Gelfand - Formin result. Thus using (1.29) gives

$$\delta X_2 = \int_{LR} \overline{\delta\phi}\left\{Q_\phi - \frac{\partial}{\partial x} Q_{\phi_x}\right\} dx + \int_{LR} \frac{\partial}{\partial x}\left\{Q\, \delta x + \overline{\delta\phi}_{\phi_x}\right\} dx.$$

Q_{ϕ_x} is zero since Q is independent of ϕ_x , hence

$$\delta X_2 = \int_{LR} \overline{\delta\phi}\, Q_\phi\, dx + \int_{LR} \frac{\partial}{\partial x} (Q\delta x) dx$$

$$= \int_{LR} \overline{\delta\phi}\, Q_\phi\, dx + \left[Q\, \delta x\right]_{x=x_L}^{x=x_R}$$

At $x = x_R$, $\delta x = 0$, so finally

$$\delta X_2 = \int_{LR} \overline{\delta\phi} Q_\phi dx - Q(x_L, T, \phi_L)\, \delta x\Big|_{x=x_L} \tag{2.22}$$

δJ in (2.11) can now be calculated using (2.20) and (2.22). Thus

$$\delta J = \int_{S_1}\int \overline{\delta\phi}\left\{F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right\} dx\, dt + \int_{OR+RL+LO}\left\{(F\delta x + \delta\phi\, F_{\phi_x})\, dt - (F\delta t + \overline{\delta\phi}\, F_{\phi_t})\, dx\right\}$$
$$+ \int_{LR} \overline{\delta\phi}\, Q_\phi\, dx - Q\, (x_L, T, \phi_L)\, \delta x\Big|_{x=x_L} + \int_{\tau=0}^{T}\left\{f_\alpha \delta\alpha + f_{\alpha'}\delta\alpha' + f_{\alpha''}\, \delta\alpha''\right\} d\tau \tag{2.23}$$

At any point on OR x, t and ϕ remain unaltered by a variation of the position of the curve OL and so δx, δt and $\delta\phi$ are zero at such a point, which means, from (2.19), that $\overline{\delta\phi}$ is zero on OR. Therefore there is no contribution to δJ from the integral along OR. On RL $t = T$ therefore dt and δt are zero. τ is unchanged by the variation of OL so on OL $\delta\tau = \delta t = 0$. So

$$\delta J = \int_{S_1}\int \overline{\delta\phi}\left\{F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right\} dx\, dt + \int_{RL} -\overline{\delta\phi}\, F_{\phi_t}\, dx + \int_{LR} \delta\phi\, Q_\phi\, dx$$
$$+ \int_{LO} \left\{ (F\,\delta x + \overline{\delta\phi}\, F_{\phi_x}) dt - \overline{\delta\phi}\, F_{\phi_t}\, dx \right\} - Q(x_L, T, Q_L)\, \delta x \Big|_{x=x_L}$$
$$+ \int_{\tau=0}^{T} \left\{ f_\alpha \delta\alpha + f_{\alpha'} \delta\alpha' + f_{\alpha''} \delta\alpha'' \right\} d\tau \quad . \qquad (2.24)$$

$$\delta J = \int_{S_1}\int \overline{\delta\phi}\left\{F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right\} dx\, dt + \int_{\text{arc } LR} \overline{\delta\phi}(Q_\phi + F_{\phi_t}) dx$$
$$- \int_{\text{arc } OL} \left\{ (F\,\delta x + \overline{\delta\phi}\, F_{\phi_x}) dt - \overline{\delta\phi}\, F_{\phi_t}\, dx \right\} - Q(x_L, T, \phi_L)\, \delta x \Big|_{x=x_L}$$
$$+ \int_{\tau=0}^{T} \left\{ f_\alpha\, \delta\alpha + f_{\alpha'}\, \delta\alpha' + f_{\alpha''}\, \delta\alpha'' \right\} d\tau \quad . \qquad (2.25)$$

On OL the boundary condition must be satisfied, and so

$$M(\alpha(\tau), \tau, \phi) = 0 \quad , \quad (x,t) \in OL \; ;$$

and in the varied state the boundary condition to be satisfied is

$$M(\alpha(\tau) + \delta\alpha(\tau), \tau, \phi + \delta\phi) = 0 \quad , (x,t) \in OL' \quad . \qquad (2.26)$$

Expanding (2.27) by Taylor's theorem

$$M(\alpha(\tau), \tau, \phi) + \frac{\partial M}{\partial \alpha}\delta\alpha + \frac{\partial M}{\partial \phi}\delta\phi + \ldots\ldots = 0, \; (x,t) \in OL' \quad ,$$

where $\frac{\partial M}{\partial \alpha} = \frac{\partial M}{\partial x}\Big|_{x=\alpha}$.

OL is the curve that minimises J and so for a minimum

$$\frac{\partial M}{\partial \alpha}\delta\alpha + \frac{\partial M}{\partial \phi}\delta\phi = 0 \quad , \quad (x,t) \in OL \; , \qquad (2.27)$$

and so

$$\delta\phi = -\frac{\delta\alpha M_\alpha}{M_\phi} \quad , \text{ and}$$

$$\overline{\delta\phi} = -\left(\frac{M_\alpha}{M_\phi} + \frac{\partial\phi}{\partial x}\right)\delta\alpha \quad , (x,t) \in OL \; ; \qquad (2.28)$$

thus

$$\delta J = \int_{S_1}\int \overline{\delta\phi}\left\{F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right\} dx\, dt + \int_{LR} \overline{\delta\phi}\, (\phi_\phi + F_{\phi_t})\, dx$$
$$- \int_{OL} \delta\alpha \left\{ F - F_{\phi_x}\left(\frac{M_\alpha}{M_\phi} + \phi_x\right) dt + F_{\phi_t}\left(\frac{M_\alpha}{M_\phi} + \phi_x\right) dx \right\}$$

$$- Q\,(x_L,\,T,\,Q_L\,)\;\delta x\Big|_{x=x_L} + \int_{\tau=0}^{T}\Big\{ f_\alpha\delta\alpha + f_{\alpha'}\,\delta\alpha' + f_{\alpha''}\delta\alpha''\Big\}\,d\tau \quad (2.29)$$

$$\delta J = \int_{S1}\int_{T}\overline{\delta\emptyset}\Big\{F_\phi - \frac{\partial}{\partial x}F_{\phi_x} - \frac{\partial}{\partial t}F_{\phi_t}\Big\}\,dx\,dt + \int_{LR}\overline{\delta\emptyset}\,(Q_\phi + F_{\phi_t})\,dx$$
$$- \int_{\tau=0}\delta\alpha\Big\{F - F_{\phi_x}\big(\frac{M_\alpha}{M_\phi} + \emptyset_x\big) + F_{\phi_t}\big(\frac{M_\alpha}{M_\phi} + \emptyset_x\big)\frac{d\alpha(\tau)}{d\tau}\Big\}\,d\tau$$
$$- Q(x_L,\,T,\,\emptyset_L)\;\delta x\Big|_{x=x_L} + \int_{\tau=0}^{T}\Big\{ f_\alpha\delta\alpha + f_{\alpha'}\delta\alpha + f_{\alpha''}\delta\alpha''\Big\}\,d\tau \quad (2.30)$$

since on the arc OL $x = \alpha(\tau)$, $t = \tau$, and dx has been replaced by $\alpha'(\tau)d\tau$.

Now integrating $f_{\alpha'}\delta\alpha'$ and $f_{\alpha''}\delta\alpha''$ by parts gives

$$\int_{\tau=0}^{T} f_{\alpha'}\delta\alpha' d\tau = \delta\alpha\Big|_{\tau=0}^{T} f_{\alpha'} - \int_{\tau=0}^{T}\delta\alpha\,\frac{\partial f_{\alpha'}}{\partial\tau}\,d\tau\;, \quad (2.31)$$

$$\int_{\tau=0}^{T} f_{\alpha''}\,\delta\alpha''\,d\tau = \delta\alpha'\Big|_{\tau=0}^{T} f_{\alpha''} - \int_{\tau=0}^{T}\delta\alpha'\,\frac{\partial F_{\alpha''}}{\partial\tau}d\tau$$
$$= \delta\alpha'\Big|_{\tau=0}^{T} f_{\alpha''} - \delta\alpha\Big|_{\tau=0}^{T}\frac{\partial f_{\alpha''}}{\partial\tau} + \int_{\tau=0}^{T}\delta\alpha\,\frac{\partial^2 f_{\alpha''}}{\partial\tau^2}\,d\tau\;. \quad (2.32)$$

Since there is no variation in the curve OL at the origin $\delta\alpha$ is zero at $\tau = 0$, and so δJ may now be written as

$$\delta J = \int_{S1}\int_{T}\overline{\delta\emptyset}\Big\{F_\phi - \frac{\partial}{\partial x}F_{\phi_x} - \frac{\partial}{\partial t}F_{\phi_t}\Big\}dx\,dt + \int_{arcLR}\overline{\delta\emptyset}(Q_\phi + F_{\phi_t})\,dx$$
$$+ \int_{\tau=0}\delta\alpha\Big\{f - \frac{\partial f_{\alpha'}}{\partial\tau} + \frac{\partial^2 f_{\alpha''}}{\partial\tau^2} - F + \Big[F_{\phi_x} - F_{\phi_t}\,\alpha'(\tau)\Big]\Big[\frac{M_\alpha}{M_\phi} + \emptyset_x\Big]\Big\}\,d\tau$$
$$+ \Big[\delta\alpha\big(F_\alpha - \frac{\partial f_{\alpha''}}{\partial\tau} - Q(x,T,\emptyset)\big)\Big]_{\tau=T} + \Big[\delta\alpha'\,f_{\alpha''}\Big]_{\tau=0}^{\tau=T}\;. \quad (2.33)$$

As $\overline{\delta\emptyset}$ and $\delta\alpha$ are arbitary variations it follows that for I to be a minimum δJ is zero so

$$F_\phi - \frac{\partial}{\partial x}F_{\phi_x} - \frac{\partial}{\partial t}F_{\phi_t} = 0\;,\qquad (x,t)\in S_1\;, \quad (2.34)$$

$$Q_\phi + F_{\phi_t} = 0\;,\qquad (x,t)\in LR\;, \quad (2.35)$$

$$f - \frac{\partial f_{\alpha'}}{\partial\tau} + \frac{\partial^2 f_{\alpha''}}{\partial\tau^2} - F + \Big[F_{\phi_x} - F_{\phi_t}\,\alpha'(\tau)\Big]\Big[\frac{M_\alpha}{M_\phi} + \emptyset_x\Big] = 0,$$
$$(x,t)\in OL\;; \quad (2.36)$$

Since $\delta\alpha(\neq 0)$ and $\delta\alpha'$ $(\neq 0)$ are independent variations at $\tau = T$

$$f_\alpha - \frac{\partial f_{\alpha''}}{\partial \tau} - Q(x,T,\emptyset) = 0 \;, \qquad \text{at } \tau = T \qquad (2.37)$$

$$f_{\alpha''} = 0 \qquad (2.37\text{ a})$$

At $\tau = 0$ either $\alpha'(0)$ is given or $f_{\alpha''} = 0$. (2.37 b)

From (2.17) $F(x,t,\emptyset,\emptyset_x,\emptyset_t) = H(x,t,\emptyset,\emptyset_x) - \lambda\emptyset_t$

and from (2.10) $H(x,t,\emptyset,\emptyset_x) = P(x,t,\emptyset,\emptyset_x) + \lambda g(x,t,\emptyset,\emptyset_x)$

and so

$$F_\phi = P_\phi + \lambda g_\phi \;,$$

$$F_{\phi_x} = P_{\phi_x} + \lambda g_{\phi_x} \;,$$

$$F_{\phi_t} = -\lambda \;.$$

Using the above (2.34), (2.35) and (2.37) can be rewritten as:

$$P_\phi - \lambda g_\phi - \frac{\partial}{\partial x}\left[P_{\phi_x} + \lambda g_{\phi_x}\right] + \frac{\partial \lambda}{\partial t} = 0 \;, \quad (x,t) \in S_1 \;, \qquad (2.38)$$

$$Q_\phi - \lambda = 0 \;, \qquad (x,t) \in LR \;, \qquad (2.39)$$

$$f_\alpha - \frac{\partial f}{\partial \tau}_{\alpha'} + \frac{\partial^2 f}{\partial \tau^2}_{\alpha''} - P - \lambda(g - \emptyset_t) + \left[P_{\phi_x} + \lambda g_{\phi_x} + \lambda\alpha'(\tau)\right]\left[\frac{M_\alpha}{M_\phi} + \emptyset_x\right] = 0,$$
$$(x,t) \in OL \;. \qquad (2.40)$$

Equation (2.40) is the transversality condition and from it the value of $\alpha(\tau)$ which minimises J may be found. Equation (2.38) is the co-state equation.

A simple example of the above theory will now be discussed. In this example the state equation is given by

$$\frac{\partial \emptyset}{\partial t} + a\frac{\partial \emptyset}{\partial x} = 0 \;, \qquad a > 0 \;, \qquad (2.41)$$

i.e. $g(x,t,\emptyset,\emptyset_x) = -\, a\dfrac{\partial \emptyset}{\partial x}$. (2.42)

The performance criterion is defined as

$$I = \int_{S_1}\int \tfrac{1}{2}\phi^2 \, dx \, dt + \int_{\tau=0}^{T} (\tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha'^2) \, d\tau \quad , \qquad (2.43)$$

i.e. $$P(x,t,\phi,\phi_x) = \tfrac{1}{2}\phi^2 \quad , \qquad (2.44)$$

$$Q(x,T,\phi) = 0 \quad , \qquad (2.45)$$

$$f(\alpha,\alpha',\alpha'',\tau) = \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha'^2 \quad . \qquad (2.46)$$

$$J = \int_{S_1}\int \left\{\tfrac{1}{2}\phi^2 - \lambda(a\phi_x + \phi_t) \, dx\right\} dt + \int_{\tau=0}^{T} (\tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha'^2) d\tau . \qquad (2.47)$$

The boundary condition on OL is

$$M(x,\phi,t) = \phi(x,t) - \phi_0(x,t) = 0 \quad x = \alpha(\tau), \quad t = \tau . \qquad (2.48)$$

The ordinary differential equations of the family of characteristics of equation (2.41) are

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{d\phi}{dt} = 0$$

which imply

$x - a\,t$ = constant and ϕ = constant, on the characteristics and so

$$\phi(x,t) = \chi(x - a\,t) \qquad (2.49)$$

where χ is an arbitary function.

At $x = \alpha(\tau)$, $t = \tau$, $\phi(x,t) = \phi_0(x,t)$ so

$$\chi(\alpha(\tau) - a\tau) = \phi_0(\alpha(\tau),\tau) . \qquad (2.50)$$

From equations (2.38), (2.41) and (2.44)

$$\phi(x,t) - \frac{\partial}{\partial x}(-a\lambda) + \frac{\partial\lambda}{\partial t} = 0$$

and using (2.49),

$$a\frac{\partial\lambda}{\partial x} + \frac{\partial\lambda}{\partial t} = -\chi(x - a\,t) \quad . \qquad (2.51)$$

To solve equation (2.51) put

$$\xi = x - a\,t \quad , \qquad \eta = t \quad ,$$

then

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\frac{a\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

and so (2.51) becomes

$$\frac{a\partial\lambda}{\partial\xi} - \frac{a\partial\lambda}{\partial\xi} + \frac{\partial\lambda}{\partial\eta} = -\chi(\xi\ )$$

$$\frac{\partial\lambda}{\partial\eta} = -\chi(\xi) \qquad (2.52)$$

so $\lambda = -\eta\chi(\xi) + \chi_1(\eta)$, (2.53)

where χ_1 is an arbitary function. Therefore

$$\lambda = -t\,\chi(x - at) + \chi_1(x - at).$$

From (2.39) λ is zero at $t = T$ since Q is zero so

$$T\,\chi(x - aT) = \chi_1\,(x - aT)$$

and

$$\lambda = (T - t)\,\chi(x - at). \qquad (2.54)$$

M_α and M_ϕ can be found from (2.48)

$$M_\alpha = \left.\frac{\partial M}{\partial x}\right|_{x=\alpha} , \quad \text{and}$$

$$\frac{\partial M}{\partial x} = -\frac{\partial \phi_o}{\partial x} , \quad \text{so}$$

$$M_\alpha = -\left.\frac{\partial \phi_o}{\partial x}\right|_{x=\alpha} .$$

$$M_\phi = 1,$$

and so in this problem equation (2.40) becomes:

$$\alpha(\tau) - \alpha''(\tau) - \tfrac{1}{2}\,\phi^2(x,t) + \lambda\left\{\left\{\frac{\partial\phi}{\partial x} - \frac{\partial\phi_o}{\partial x}\right\}_{\substack{x=\alpha(\tau)\\ t=\tau}}\alpha'(\tau) - a\right\} = 0,\ x = \alpha(\tau),\ t = \tau. \qquad (2.55)$$

From (2.49) $\phi(x,t) = \chi\,(x - at)$

therefore $\dfrac{\partial\phi}{\partial x} = \chi'(x - at)$

$$\left.\frac{\partial \phi}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} = \chi'(\alpha(\tau) - a\tau). \tag{2.56}$$

Put $\alpha(\tau) - a\tau = w(\tau)$, then

$$\chi\{w(\tau)\} = \phi_o(\alpha(\tau),\tau) \text{ from (2.50)},$$

so differentiating with respect to τ,

$$\chi'\{w(\tau)\}\frac{dw}{d\tau} = \left.\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} \alpha'(\tau) + \frac{\partial \phi_o}{\partial \tau},$$

or

$$[\alpha'(\tau) - a]\,\chi'(\alpha(\tau) - a\tau) = \left.\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} \alpha'(\tau) + \frac{\partial \phi_o}{\partial \tau}. \tag{2.57}$$

Using (2.57), (2.56) can be written as

$$\left.\frac{\partial \phi}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} = \frac{\left.\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} \alpha'(\tau) + \frac{\partial \phi_o}{\partial \tau}}{\alpha'(\tau) - a}$$

so

$$\left.\left\{\frac{\partial \phi}{\partial x} - \frac{\partial \phi_o}{\partial x}\right\}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} = \frac{\left.\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}} \alpha'(\tau) + \frac{\partial \phi_o}{\partial \tau}}{\alpha'(\tau) - a} - \left.\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}}$$

$$= \frac{\left.\frac{\partial \phi_o}{\partial \tau} + a\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}}}{\alpha'(\tau) - a}. \tag{2.58}$$

From (2.54) $\lambda = (T - t)\chi(\alpha(\tau) - a\tau)$ on OL and as

$$\phi_o(\alpha(\tau),\tau) = \chi(\alpha(\tau) - a\tau)$$

$$\lambda = (T - \tau)\,\phi_o(\alpha(\tau),\tau), \quad \text{on OL.} \tag{2.59}$$

$\frac{1}{2}\phi^2(x,t)$ becomes $\frac{1}{2}\phi_o^2(\alpha(\tau),\tau)$ on OL and so using this, (2.59) and (2.58), (2.57) can be written as

$$\alpha(\tau) - \alpha''(\tau) - \tfrac{1}{2}\phi_o^2(\alpha(\tau),\tau) + (T-\tau)\,\phi_o(\alpha(\tau),\tau)\frac{\left\{\left.\frac{\partial \phi_o}{\partial \tau} + a\frac{\partial \phi_o}{\partial x}\right|_{\substack{x=\alpha(\tau)\\ t=\tau}}\right\}}{\alpha'(\tau) - a} \times$$

$$\times \quad (\alpha'(\tau) - a) \quad = 0$$

$$\alpha(\tau) - \alpha''(\tau) - \tfrac{1}{2}\phi_0^2(\alpha(\tau), \tau) + (T - \tau)\, \phi_0(\alpha(\tau), \tau) \left\{ \frac{\partial \phi_0}{\partial \tau} + \frac{a\,\partial \phi_0}{\partial x} \bigg|_{\substack{x=\alpha(\tau)\\ t=\tau}} \right\} = 0 \tag{2.60}$$

The solution for $\alpha(\tau)$ which minimises I may be found from equation (2.60), together with the boundary conditions (2.14) and (2.37).

Take the particular case where $\phi_0(x,t) = x^{\frac{1}{2}}$. Here $\phi_0(\alpha(\tau), \tau) = \alpha^{\frac{1}{2}}(\tau)$, so (2.60) becomes

$$\alpha(\tau) - \alpha''(\tau) - \tfrac{1}{2}\alpha(\tau) + (T - \tau)\, \alpha^{\frac{1}{2}}(\tau) .\ \tfrac{1}{2}\, a\, \alpha^{-\frac{1}{2}}(\tau) = 0$$

$$\alpha''(\tau) - \tfrac{1}{2}\left\{ \alpha(\tau) + a(T - \tau) \right\} = 0 \quad . \tag{2.61}$$

Putting $\psi(\tau) = \alpha(\tau) + a(T - \tau)$ (2.62)

then $\psi''(\tau) = \alpha''(\tau)$

and (2.61) becomes

$$\psi''(\tau) - \tfrac{1}{2}\psi(\tau) = 0 \quad . \tag{2.63}$$

The boundary conditions on (2.63) are:

from (2.16),

$$\alpha(0) = 0, \text{ i.e. } \psi = aT \quad , \tau = 0;$$

from (2.37),

$$\alpha'(T) = 0, \text{ i.e. } \psi'(T) = -a \quad , \tau = T.$$

Using these conditions (2.63) may be solved for $\psi(\tau)$ and hence the value of $\alpha(\tau)$ which minimises I may be found from (2.61).

CHAPTER THREE

CHAPTER THREE

A Second Order Hyperbolic Partial Differential Equation Example of the Use of the Gelfand-Fomin Theorem.

Let S be the domain in the (x,t) plane indicated in figure (3.1); S is bounded by the closed curve OARL with AR being a portion of the line $x = l$ and LR a portion of the line $t = T$. It is assumed that the equation of OL may be expressed in the form

$$x = \alpha(\tau) \quad , \; t = \tau \quad , \; 0 \leq \tau \leq T \quad , \qquad (3.1)$$

τ being a time parameter, with $\alpha(0) = 0$.

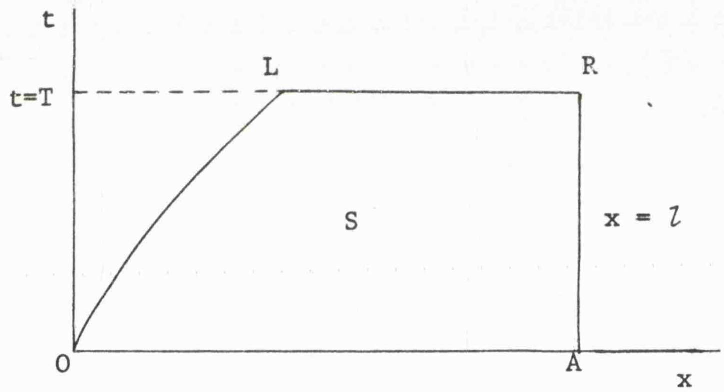


Figure 3.1

The shape of the curve OL is unknown initially, that is $\alpha(\tau)$ is an unknown function of τ, and later it is attempted to find the curve OL in order to minimise a particular performance criterion. With $\alpha(0) = 0$ the curve OL always passes through the origin.

A function $\phi(x,t)$ is defined for all $(x,t) \in S$ and $\phi(x,t)$ satisfies in S the second order partial differential equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \quad , \qquad (3.2)$$

where c is a constant.

Putting $$\frac{\partial \phi}{\partial t} = \frac{c\partial \psi}{\partial x} \quad , \tag{3.3}$$

then $$\frac{c\partial \phi}{\partial x} = \frac{\partial \psi}{\partial t} \quad . \tag{3.4}$$

The boundary conditions on OA are

$$\phi(x,0) = \phi_0(x) \quad , \quad \psi(x,0) = \psi_0(x) \quad ; \tag{3.5}$$

on AR,

$$\phi(l,t) = 0 \quad ; \tag{3.6}$$

and on OL,

$$M(\phi,\psi,\ \alpha(\tau),\ \alpha'(\tau),\tau) = 0 \quad . \tag{3.7}$$

The ordinary differential equations for the families of characteristics for equation (3.2) are:

$$C+:\quad dx - c\,dt = 0 \qquad \text{i.e. } x - ct = \text{constant} = \xi \quad ; \tag{3.8}$$

$$C-:\quad dx + c\,dt = 0 \qquad \text{i.e. } x + ct = \text{constant} = \eta \quad . \tag{3.9}$$

It is assumed that a moving point on a C+ characteristic commencing at any point on OL or OA will travel with increasing time into the domain S and will eventually meet either LR or AR in a single point. This implies that the slope of OL at the point $t = \tau$ must be greater than the slope of the characteristic at that point, that is,

$$c > \alpha'(\tau) \quad , \qquad 0 < \tau < T. \tag{3.10}$$

It is also assumed that each C- characteristic commencing at any point on OA or AR will travel (with $dt > 0$) into the domain S and will eventually meet either OL or LR in a single point.

From (3.8) and (3.9)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad , \quad \frac{\partial}{\partial t} = -\frac{c\partial}{\partial \xi} + \frac{c\partial}{\partial \eta}$$

and equations (3.3) and (3.4) become

$$c\left(-\frac{\partial\phi}{\partial\xi} + \frac{\partial\phi}{\partial\eta}\right) = c\left(\frac{\partial\psi}{\partial\xi} + \frac{\partial\psi}{\partial\eta}\right) \quad .$$

and

$$c\left(\frac{\partial\phi}{\partial\xi} + \frac{\partial\phi}{\partial\eta}\right) = c\left(-\frac{\partial\psi}{\partial\xi} + \frac{\partial\psi}{\partial\eta}\right) \quad ,$$

giving on addition and subtraction

$$2\,\frac{\partial\phi}{\partial\eta} = 2\frac{\partial\psi}{\partial\eta} \quad , \quad 2\,\frac{\partial\phi}{\partial\xi} = -2\frac{\partial\psi}{\partial\xi} \quad .$$

It follows from the above equations that

$$\frac{\partial}{\partial\eta}(\phi - \psi) = 0 \,, \quad \frac{\partial}{\partial\xi}(\phi + \psi) = 0 \quad ,$$

hence

(a) $\phi - \psi$ is constant along the ξ = constant characteristic ;

(b) $\phi + \psi$ is constant along the η = constant characteristic.

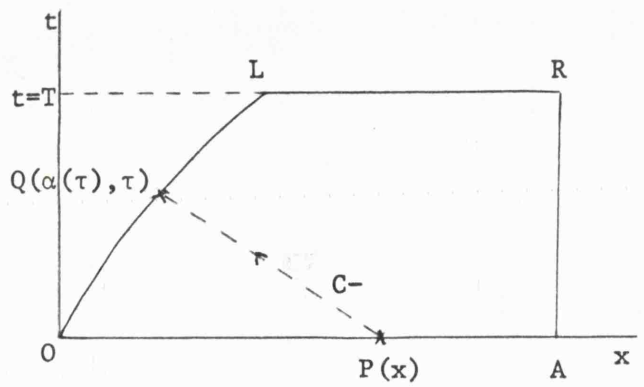


Figure 3.2

Accordingly if PQ is a C- characteristic then

$$\phi_Q + \psi_Q = \phi_p + \psi_p$$

where ϕ_Q, ψ_Q, ϕ_p and ψ_p denote the values of ϕ and ψ at the points P and Q (see Figure (3.2) and this equation can be written in the form

$$\phi(\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) = \phi_o(x) + \psi_o(x) \quad , \tag{3.11}$$

where (x,0) are the coordinates of the point P. Since x + ct is constant along the C- characteristics this becomes

$$\emptyset\ (\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) = \emptyset_o\ (\alpha(\tau) + c\tau\ ) + \psi_o(\alpha(\tau) + c\tau\ )$$

hence

$$N\ \equiv \emptyset(\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) - \emptyset_o\ (\alpha(\tau) + c\tau) - \psi_o(\alpha(\tau) + c\tau) = 0 \quad (3.12)$$

is true for all τ in the range $0<\tau<T$ and is valid on OL. Accordingly there are two conditions to be satisfied on OL, namely (3.7) and (3.12).

The controllable area of the domain S must now be determined when the curve OL varies in position. Consider first the case where the C+ characteristic through the origin meets the line AR in a point H. The C+ characteristic through any point Q in the triangle OAH will originate on the line OA and the C- characteristic through this point will originate on either OA or AR and so the values of $\emptyset$ and ψ at Q will not be affected by any variation of the position of the curve OL.

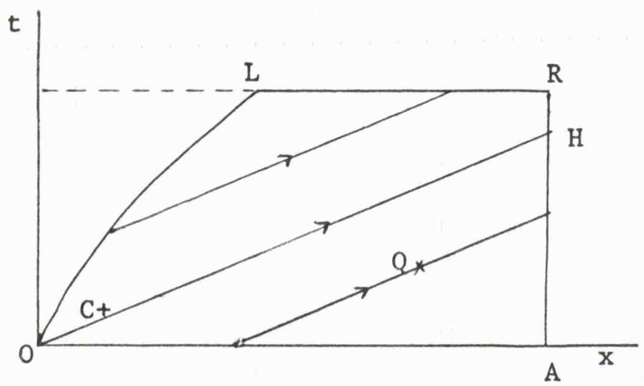


Figure 3.3

Hence the domain OAH is uncontrollable. At any point, B, in the domain OHRL the C+ characterisitic will originate on OL and so the values of $\emptyset$ and ψ at B will alter with a variation of OL. Hence the domain OHRL will be regarded as controllable.

Consider next the case where the C+ characteristic through the origin meets the line LR in a point K. It can be shown by a similar

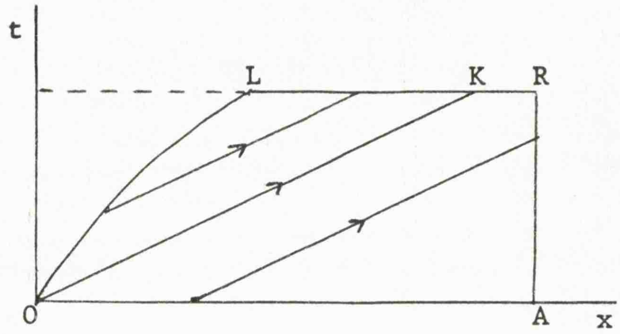


Figure 3.4

argument that the domain OARK is uncontrollable and that the domain OKL is controllable.

The latter case will now be discussed more fully. The position of the line AR will be taken to be such that the C- characteristic through K originates on OA and not AR. The control problem is to minimise a performance index I given by

$$I = \int\int_{S_1} P(\emptyset, \psi, x, t)\, dx\, dt + \int_{LK} Q(\emptyset, \psi, x, T) dx + \int_{\tau=0}^{T} f(\alpha, \alpha', \alpha'', \tau) d\tau, \tag{3.13}$$

where S_1 is the domain OKL, as the position of the curve OL varies.

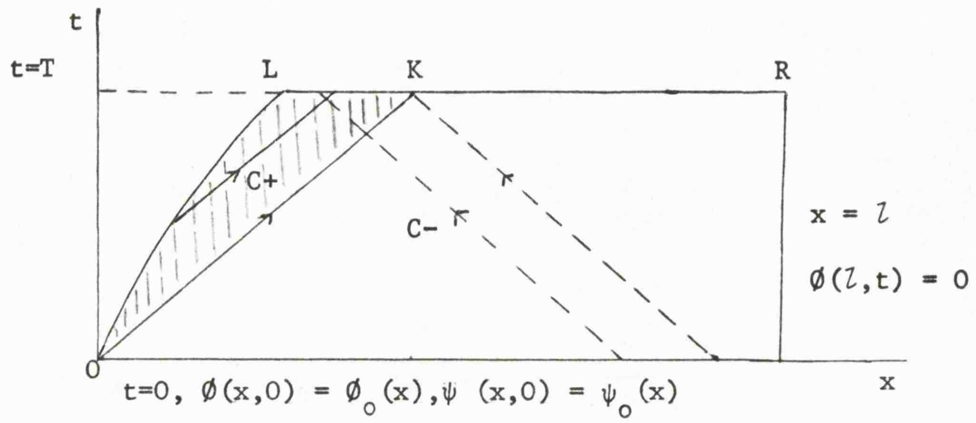


Figure 3.5

In physical terms this can be interpreted as a string of length l being fixed at one end, A. The string is moved, with the free end O describing the curve OL after time T. The control problem will determine the optimum path for O to follow to minimise a given performance criterion. If $\phi(x,t)$ represents the position of the string at a point x at time t, then, if the string is to be as

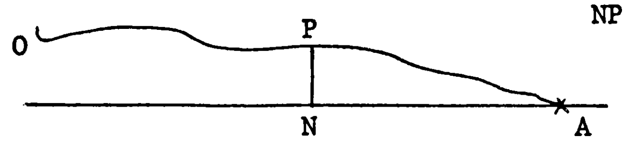


Figure 3.6

close as possible to some prescribed shape $\Phi(x)$ at time T, the performance index will be $P \equiv 0$, $Q \equiv \{\phi(x,T) - \Phi(x)\}^2$. ψ is related to the velocity by $c\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial t}$ and so for the velocity also to be as near as possible to a prescribed velocity $\Psi(x)$ at time T, Q becomes

$$Q \equiv \{\phi(x,t) - \Phi(x)\}^2 + \{\psi(x,t) - \Psi(x)\}^2 .$$

Consider now instead of I given in (3.13) the new functional J given by

$$J = \int\int_{S_1} \{P + \lambda(\phi_t - c\psi_x) + \mu(\psi_t - c\phi_x)\} \,dx\,dt$$
$$+ \int_{LK} Q(\phi,\psi,x,T)\,dx + \int_0^T f(\alpha,\alpha',\alpha'',\tau)\,d\tau \quad , \quad (3.14)$$

where λ and μ are Lagrange multipliers depending on x and t. The increment, δJ, in J as the position of the curve OL varies must now be found. The variation of position may be achieved by adding to $\alpha(\tau)$ the increment $\delta\alpha(\tau)$ at the same time τ. The curve OL i.e. $x = \alpha(\tau)$, $t = \tau$, $(0<\tau<T)$, will be regarded as the curve which provides the minimum for I in (3.13) and the varied curve will be OL', namely

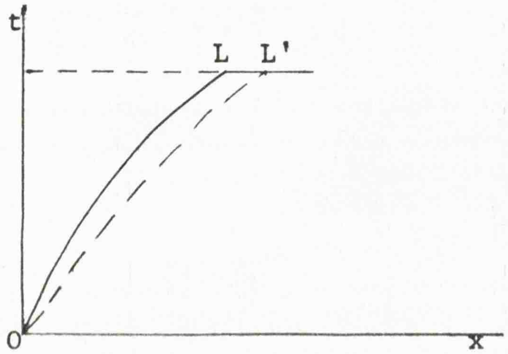


Figure 3.7

$$x = \alpha(\tau) + \delta\alpha(\tau), \qquad t = \tau \ , \qquad (0<\tau<T).$$

The functions $\alpha(\tau)$ and $\delta\alpha(\tau)$ are assumed continuous and satisfying

$$\alpha(0) = 0 \ , \qquad \delta\alpha(0) = 0 \ . \tag{3.15}$$

The extension of the Gelfand-Fomin result may now be used to find δJ.

Let

$$F(\emptyset,\psi,\emptyset_x,\psi_x,\emptyset_t,\psi_t,x,t) = P + \lambda(\emptyset_t - c\psi_x) + \mu(\psi_t - c\emptyset_x), \tag{3.16}$$

and $\chi_1 = \int\int_{S_1} F \, dx\, dt$, then

$$\delta\chi_1 = \int\int_{S_1} \left\{ \overline{\delta\emptyset}\left[F_\phi - \frac{\partial}{\partial x}F_{\phi_x} - \frac{\partial}{\partial t}F_{\phi_t}\right] + \overline{\delta\psi}\left[F_\psi - \frac{\partial}{\partial x}F_{\psi_x} - \frac{\partial}{\partial t}F_{\psi_t}\right]\right\} dx\, dt$$

$$+ \int\int_{S_1}\left\{\frac{\partial}{\partial x}\left[F\delta x + \overline{\delta\emptyset}F_{\phi_x} + \overline{\delta\psi}F_{\psi_x}\right] + \frac{\partial}{\partial t}\left[F\delta t + \overline{\delta\emptyset}F_{\phi_t} + \overline{\delta\psi}F_{\psi_t}\right]\right\} dx\, dt, \tag{3.17}$$

where $\delta\emptyset$ and $\delta\psi$, the increments in $\emptyset$ and ψ, are related to $\overline{\delta\emptyset}$ and $\overline{\delta\psi}$ by

$$\delta\emptyset = \overline{\delta\emptyset} + \frac{\partial\emptyset}{\partial x}\delta x + \frac{\partial\emptyset}{\partial t}\delta t \qquad ; \tag{3.18}$$

$$\delta\psi = \overline{\delta\psi} + \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial t}\delta t \qquad . \tag{3.19}$$

Applying Stoke's theorem in [2] to the second integral in (3.17) gives

$$\delta\chi_1 = \int\int_{S_1}\left\{\overline{\delta\emptyset}\left[F_\phi - \frac{\partial}{\partial x}F_{\phi_x} - \frac{\partial}{\partial t}F_{\phi_t}\right] + \overline{\delta\psi}\left[F_\psi - \frac{\partial}{\partial x}F_{\psi_x} - \frac{\partial}{\partial t}F_{\psi_t}\right]\right\} dx\, dt$$

$$+ \int_{OK+KL+LO} \left\{ \left[F\delta x + \overline{\delta\emptyset}F_{\phi_x} + \overline{\delta\psi}\, F_{\psi_x}\right] dt - \left[F\delta t + \overline{\delta\emptyset}F_{\phi_t} + \overline{\delta\psi}F_{\psi_t}\right]\right\} dx \; . \tag{3.20}$$

Let $\chi_2 = \int_{LK} Q(\emptyset,\psi,x,T)dx$, then

$$\delta\chi_2 = \int_{LK} \left\{ \overline{\delta\emptyset} \left[Q_\phi - \frac{\partial}{\partial x} Q_{\phi_x}\right] + \left[\overline{\delta\psi}\, Q_\psi - \frac{\partial}{\partial x} Q_{\psi_x}\right] \right\} dx$$
$$+ \int_{LK} \frac{\partial}{\partial x} \left[Q\delta x + \overline{\delta\emptyset}\, Q_{\phi_x} + \overline{\delta\psi}\, Q_{\psi_x}\right] dx \; .$$

Q is independent of $\emptyset_x$ and ψ_x so

$$\delta\chi_2 = \int_{LK} \left\{\overline{\delta\emptyset}\, Q_\phi + \overline{\delta\psi}Q_\psi\right\}dx + Q\delta x \Big|_{x=x_L}^{x=x_K}$$

and since δx is zero at the point K

$$\delta\chi_2 = \int_{LK} \left\{\overline{\delta\emptyset}\, Q_\phi + \overline{\delta\psi}Q_\psi\right\} dx + Q\delta x\Big|_{x=x_L} . \tag{3.21}$$

δJ may now be written down from equations (3.20),(3.21) and the variation in $f(\alpha,\alpha',\alpha'',\tau)$.

$$\delta J = \int\int_{S_1} \left\{ \overline{\delta\emptyset} \left[F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right] + \left[\overline{\delta\psi}\; F_\psi - \frac{\partial}{\partial x} F_{\psi_x} - \frac{\partial}{\partial t} F_{\psi_t}\right]\right\} dx\, dt$$
$$+ \int_{OK+KL+LO} \left\{ \left[F\delta x + \overline{\delta\emptyset}F_{\phi_x} + \overline{\delta\psi}\, F_{\psi_x}\right] dt - \left[F\delta t + \overline{\delta\emptyset}F_{\phi_t} + \overline{\delta\psi}\, F_{\psi_t}\right] dx \right\}$$
$$+ \int_{LK} \left\{ \overline{\delta\emptyset}\, Q_\phi + \overline{\delta\psi}\, Q_\psi \right\} dx + Q\delta x\Big|_{x=x_L}$$
$$+ \int_0^T \left\{ f_\alpha \delta\alpha + f_{\alpha'}\delta\alpha' + f_{\alpha''}\delta\alpha''\right\} d\tau . \tag{3.22}$$

At any point on OK $x,t,\emptyset$ and ψ remain unaltered by a variation of the position of curve OL and so $\delta x,\delta t,\delta\emptyset$ and $\delta\psi$ are zero at such a point, which means, from (3.18) and (3.19), that $\overline{\delta\emptyset}$ and $\overline{\delta\psi}$ are zero on OK. Therefore there is no contribution to δJ from the integral along OK. On OL $x = \alpha(\tau)$, $t = \tau$ and as τ is unaltered $\delta t = \delta\tau = 0$. Since $t = T$ on LK, dt and δt are zero on LK. δJ can therefore be written as

$$\delta J = \int\int_{S_1} \left\{ \overline{\delta\emptyset} \left[F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t}\right] + \overline{\delta\psi} \left[F_\psi - \frac{\partial}{\partial x} F_{\psi_x} - \frac{\partial}{\partial t} F_{\psi_t}\right]\right\} dx\, dt$$
$$+ \int_{LO} \left\{ \left[F\delta\alpha + \overline{\delta\emptyset}F_{\phi_x} + \overline{\delta\psi}\, F_{\psi_x}\right] d\tau - \left[\overline{\delta\emptyset}F_{\phi_t} + \overline{\delta\psi}F_{\psi_t}\right] d\alpha \right\}$$
$$+ \int_{LK} \left\{ \overline{\delta\emptyset} \left[Q_\phi + F_{\phi_t}\right] + \overline{\delta\psi}\left[Q_{\psi_t} + F_{\psi_t}\right]\right\} dx + Q\delta x \Big|_{x=x_L}$$
$$+ \int_0^T \left\{ f_\alpha \delta\alpha + f_{\alpha'}\delta\alpha' + f_{\alpha''}\delta\alpha''\right\} d\tau \; . \tag{3.23}$$

On OL the boundary condition $M \equiv 0$ must be satisfied so

$$M(\emptyset,\psi,\alpha(\tau),\alpha'(\tau),\tau) = 0 \quad .$$

On OL', i.e. $x = \alpha(\tau) + \delta\alpha(\tau)$, $t = \tau$,

$M \equiv 0$ must also be satisfied so

$$M(\emptyset + \delta\emptyset, \psi + \delta\psi, \alpha(\tau) + \delta\alpha(\tau), \alpha'(\tau) + \delta\alpha'(\tau),\tau) = 0 \ .$$

Expanding this by Taylor's theorem gives

$$M(\emptyset,\psi,\alpha(\tau),\alpha'(\tau),\tau) + \delta\emptyset M_\phi + \delta\psi M_\psi + \delta\alpha M_\alpha + \delta\alpha' M_{\alpha'} = 0$$

where $M = M(\emptyset, \psi,\alpha(\tau),\alpha'(\tau),\tau)$.

It follows from the two equations $M(\emptyset,\psi,\alpha(\tau),\alpha'(\tau),\tau) = 0$

and $M(\emptyset + \delta\emptyset, \psi + \delta\psi, \alpha + \delta\alpha, \alpha' + \delta\alpha', \tau) = 0$ that

$$\delta\emptyset M_\phi + \delta\psi M_\psi + \delta\alpha M_\alpha + \delta\alpha' M_{\alpha'} = 0, \quad \text{on OL.} \tag{3.24}$$

Equation (3.12) gives a second relationship between $\emptyset$ and ψ for all values of τ on OL and in a similar way it follows that

$$\delta\emptyset N_\phi + \delta\psi N_\psi + \delta\alpha N_\alpha = 0, \quad \text{on OL.} \tag{3.25}$$

Eliminating first $\delta\psi$ and then $\delta\emptyset$ from (3.24) and (3.25) gives

$$\delta\emptyset = \frac{\delta\alpha(M_\alpha N_\psi - N_\alpha M_\psi) + \delta\alpha' M_{\alpha'} N_\psi}{M_\phi N_\psi - N_{o,} M_\psi} \quad ;$$

$$\delta\psi = \frac{\delta\alpha(M_\alpha N_\phi - N_\alpha M_\phi) + \delta\alpha' M_{\alpha'} N_\phi}{M_\psi N_\phi - N_\psi M_\phi} \quad .$$

For convenience let

$$\frac{M_\alpha N_\psi - N_\alpha M_\psi}{M_\phi N_\psi - N_\phi M_\psi} = A_1 \quad ;$$

$$\frac{M_{\alpha'} N_\psi}{M_\phi N_\psi - N_\phi M_\psi} = B_1 \quad ;$$

$$\frac{M_\alpha N_\phi - N_\alpha M_\phi}{M_\psi N_\phi - N_\psi M_\phi} = A_2 \quad ; \qquad (3.26)$$

$$\frac{M_{\alpha'} N_\phi}{M_\psi N_\phi - N_\psi M_\phi} = B_2 \quad ;$$

then

$$\delta\emptyset = \delta\alpha A_1 + \delta\alpha' B_1 \quad ; \qquad \delta\psi = \delta\alpha A_2 + \delta\alpha' B_2 \; ;$$

and from (3.18) and (3.19)

$$\overline{\delta\emptyset} = \delta\alpha A_1 + \delta\alpha' B_1 - \frac{\partial\emptyset}{\partial x}\delta x - \frac{\partial\emptyset}{\partial t}\delta t \quad , \qquad (3.27)$$

$$\overline{\delta\psi} = \delta\alpha A_2 + \delta\alpha' B_2 - \frac{\partial\psi}{\partial x}\delta x - \frac{\partial\psi}{\partial t}\delta t \quad . \qquad (3.28)$$

Since $x = \alpha(\tau)$ and $t = \tau$ on OL $\delta x = \delta\alpha$ and $\delta t = \delta\tau$ and since τ is unchanged by any variation in OL (i.e. $\delta\tau$ is zero), then

$$\overline{\delta\emptyset} = \delta\alpha \left\{ A_1 - \frac{\partial\emptyset}{\partial\alpha} \right\} + \delta\alpha' B_1 \quad , \text{ on OL;}$$

$$\overline{\delta\psi} = \delta\alpha \left\{ A_2 - \frac{\partial\psi}{\partial\alpha} \right\} + \delta\alpha' B_2 \quad , \text{ on OL;}$$

where

$$\frac{\partial\emptyset}{\partial\alpha} = \frac{\partial\emptyset}{\partial x}\bigg|_{x=\alpha(\tau)} \quad , \qquad \frac{\partial\psi}{\partial\alpha} = \frac{\partial\psi}{\partial x}\bigg|_{x=\alpha(\tau)} \quad .$$

Using integration by parts in the final integral of (3.23) and writing $d\alpha$ as $\alpha'(\tau)d\tau$, δJ may now be written as

$$\begin{aligned}\delta J = & \int_{S_T}\!\!\int \left\{ \overline{\delta\emptyset} \left[F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t} \right] + \overline{\delta\psi} \left[F_\psi - \frac{\partial}{\partial x} F_{\psi_x} - \frac{\partial}{\partial t} F_{\psi_t} \right] \right\} dx\, dt \\ & + \int_0 \Big\{ \delta\alpha \Big[f_\alpha - \frac{df}{d\tau}{}_{\alpha'} + \frac{d^2 f}{d\tau^2}{}_{\alpha''} - F - F_{\phi_x}\left(A_1 - \frac{\partial\emptyset}{\partial\alpha}\right) - F_{\psi_x}\left(A_2 - \frac{\partial\psi}{\partial\alpha}\right) + \\ & + F_{\phi_t}\left(A_1 - \frac{\partial\emptyset}{\partial\alpha}\right)\alpha'(\tau) + F_{\psi_t}\left(A_2 - \frac{\partial\psi}{\partial\alpha}\right)\alpha'(\tau) \Big] - \\ & - \delta\alpha' \left[F_{\phi_x} B_1 + F_{\psi_x} B_2 - (F_{\phi_t} B_1 + F_{\psi_t} B_2)\,\alpha'(\tau) \right] \Big\} d\tau \\ & + \int_{LK} \left\{ \overline{\delta\emptyset} \left[Q_\phi + F_{\phi_t} \right] + \overline{\delta\psi} \left[Q_\psi + F_{\psi_t} \right] \right\} dx + Q\delta x \bigg|_{x=x_L} \\ & + \delta\alpha \left[f_\alpha - \frac{df_{\alpha''}}{d\tau} \right]_{\tau=0}^{T} + \delta\alpha' f_{\alpha''} \bigg|_{\tau=0}^{T} \quad . \qquad (3.29)\end{aligned}$$

Integrating $\delta\alpha' \left[F_{\phi_x} B_1 + F_{\psi_x} B_2 - (F_{\phi_t} B_1 + F_{\psi_t} B_2)\,\alpha'(\tau) \right]$

by parts gives

$$\delta\alpha \left\{ (F_{\phi_x} - F_{\phi_t}\alpha'(\tau))\, B_1 + (F_{\psi_x} - F_{\psi_t}\alpha'(\tau))\, B_2 \right\} \Big|_{\tau=0}^{T}$$
$$- \int_0^T \delta\alpha \left\{ \frac{d}{d\tau} \left[(F_{\phi_x} - F_{\phi_t}\alpha'(\tau))\, B_1 + (F_{\psi_x} - F_{\psi_t}\alpha'(\tau))\, B_2 \right] \right\} d\tau,$$

and from (3.15) $\delta\alpha(0) = 0$. Accordingly

$$\delta J = \int_{S_1}\int \left\{ \overline{\delta\phi} \left[F_\phi - \frac{\partial}{\partial x} F_{\phi_x} - \frac{\partial}{\partial t} F_{\phi_t} \right] + \overline{\delta\psi} \left[F_\psi - \frac{\partial}{\partial x} F_{\psi_x} - \frac{\partial}{\partial t} F_{\psi_t} \right] \right\} dx dt$$
$$\int_0^T \delta\alpha \left\{ f_\alpha - \frac{df_{\alpha'}}{d\tau} + \frac{d^2 f_{\alpha''}}{d\tau^2} - F - (F_{\phi_x} - F_{\phi_t}\alpha'(\tau))\left(A_1 - \frac{\partial\phi}{\partial\alpha}\right) - \right.$$
$$- (F_{\psi_x} - F_{\psi_t}\alpha'(\tau))\left(A_2 - \frac{\partial\psi}{\partial\alpha}\right) +$$
$$\left. + \frac{d}{d\tau} \left[(F_{\phi_x} - F_{\phi_t}\alpha'(\tau))B_1 + (F_{\psi_x} - F_{\psi_t}\alpha'(\tau))\, B_2 \right] \right\} d\tau$$
$$+ \int_{KL} \left\{ \overline{\delta\phi}\,(Q_\phi + F_{\phi_t}) + \overline{\delta\psi}\,(Q_\psi + F_{\psi_t}) \right\} dx + Q\,\delta x \Big|_{x=x_L}$$
$$+ \delta\alpha \left\{ f_{\alpha'} - \frac{df_{\alpha''}}{d\tau} - (F_{\phi_x} - F_{\phi_t}\alpha'(\tau))\, B_1 - (F_{\psi_x} - F_{\psi_t}\alpha'(\tau))\, B_2 \right\} \Big|_{\tau=T}$$
$$+ \delta\alpha' f_{\alpha''} \Big|_{\tau=0}^{T} \quad . \qquad (3.30)$$

Substituting for F from (3.16) gives

$$\delta J = \int_T\int_{S_1} \left\{ \overline{\delta\phi} \left[P_\phi + \frac{c\partial\mu}{\partial x} - \frac{\partial\lambda}{\partial t} \right] + \overline{\delta\psi} \left[P_\psi + \frac{\partial\lambda}{\partial x} - \frac{\partial\mu}{\partial t} \right] \right\} dx\, dt$$
$$+ \int_0^T \delta\alpha \left\{ f_\alpha - \frac{df_{\alpha'}}{d\tau} + \frac{d^2 f_{\alpha''}}{d\tau^2} - P - \lambda\left[\frac{\partial\phi}{\partial t} - c\frac{\partial\psi}{\partial x}\right] - \mu\left[\frac{\partial\psi}{\partial t} - \frac{c\partial\phi}{\partial x}\right] + \right.$$
$$+ (c\mu + \lambda\alpha'(\tau))\left(A_1 - \frac{\partial\phi}{\partial\alpha}\right) + (c\lambda + \mu\alpha'(\tau))\left(A_2 - \frac{\partial\psi}{\partial\alpha}\right) -$$
$$\left. - \frac{d}{d\tau} \left[(c\mu + \lambda\alpha'(\tau))B_1 + (c\lambda + \mu\alpha'(\tau))B_2 \right] \right\} d\tau$$
$$+ \int_{LK} \left\{ \overline{\delta\phi}\,(Q_\phi + \lambda) + \overline{\delta\psi}\,(Q_\psi + \mu) \right\} dx$$
$$+ \delta\alpha \left\{ f_{\alpha'} - \frac{df_{\alpha''}}{d\tau} + (c\mu + \lambda\alpha'(\tau))\, B_1 + (c\lambda + \mu\alpha'(\tau))\, B_2 + Q \right\} \Big|_{\tau=T}$$
$$+ \delta\alpha' f_{\alpha''} \Big|_{\tau=0}^{T} \quad . \qquad (3.31)$$

For a minimum of I in (3.13) δJ must be zero. Since $\overline{\delta\phi}$ and $\overline{\delta\psi}$ are non zero and unrelated in S_1 and on LK and since $\delta\alpha$ and $\delta\alpha'$

are arbitary variations then when δJ is zero

$$\frac{\partial P}{\partial \phi} + c\frac{\partial \mu}{\partial x} - \frac{\partial \lambda}{\partial t} = 0 , \qquad (x,t) \in S_1 , \qquad (3.32)$$

$$\frac{\partial P}{\partial \psi} + c\frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial t} = 0, \qquad (x,t) \in S_1 , \qquad (3.33)$$

$$f_\alpha - \frac{d f_{\alpha'}}{d\tau} + \frac{d^2 f_{\alpha''}}{d\tau^2} - P + (c\mu + \lambda\alpha'(\tau))(A_1 - \frac{\partial \phi}{\partial \alpha}) + (c\lambda + \mu\alpha'(\tau))(A_2 - \frac{\partial \psi}{\partial \alpha}) -$$
$$- \frac{d}{d\tau}\left[(c\mu + \lambda\alpha'(\tau)) B_1 + (c\lambda + \mu\alpha'(\tau)) B_2 \right] = 0 , \quad (x,t) \in OL, \quad (3.34)$$

$$Q_\phi + \lambda = 0 , \qquad (x,t) \in LK, \qquad (3.35)$$

$$Q_\psi + \mu = 0 , \qquad (x,t) \in LK, \qquad (3.36)$$

$$f_{\alpha'} - \frac{d f_{\alpha''}}{d\tau} + P + (c\mu + \lambda\alpha'(\tau)) B_1 + (c\lambda + \mu\alpha'(\tau)) B_2 + Q = 0, \quad \tau = T \quad (3.37)$$

$$f_{\alpha''} = 0 , \qquad \tau = T , \qquad (3.38)$$

at $\tau = 0$ either $\alpha'(0)$ is given or $f_{\alpha''} = 0$. (3.39)

As an example to illustrate the above theory the case, described earlier, of the string being required to be as close as possible to a prescribed shape $\Phi(x)$ at time T will now be discussed. Here

$$P \equiv 0 , \qquad Q \equiv \{ \phi(x,T) - \Phi(x) \}^2$$

and f will be taken to be

$$f \equiv \tfrac{1}{2}\alpha^2(\tau) + \tfrac{1}{2}\alpha'^2(\tau)$$

and the initial and boundary conditions are

$$\phi_0 \equiv 0, \qquad \psi_0 \equiv 0 ,$$

$$M \equiv \phi - \alpha(\tau) = 0 , \qquad \phi(l,t) \equiv 0 .$$

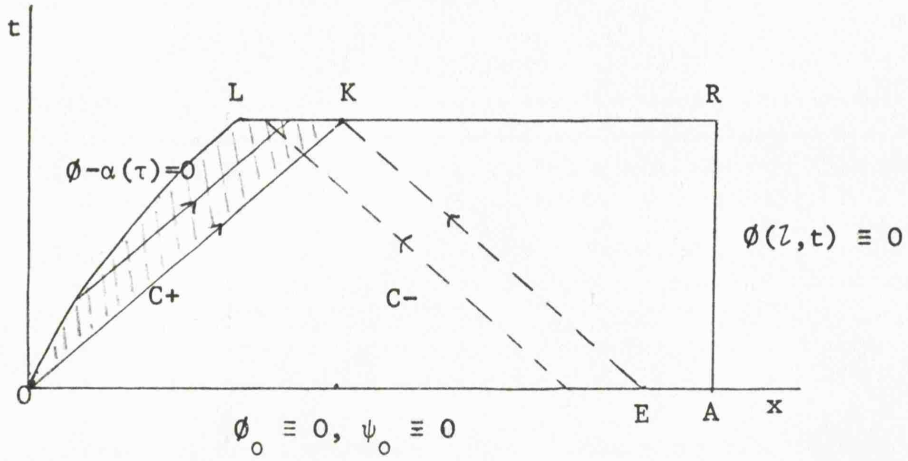


Figure 3.8

It is assumed also that the line AR (see Figure (3.8)) is such that the C- characteristic through K meets the x - axis in a point E such that $x_E < x_A$.

The state equations are the same as (3.3) and (3.4) namely

$$\frac{\partial \emptyset}{\partial t} = \frac{c\partial \psi}{\partial x} ,$$

$$c\frac{\partial \emptyset}{\partial x} = \frac{\partial \psi}{\partial t} .$$

Equations (3.32) and (3.33) become

$$c\frac{\partial \mu}{\partial x} - \frac{\partial \lambda}{\partial t} = 0 , \qquad (3.40)$$

$$c\frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial t} = 0 . \qquad (3.41)$$

Differentiating (3.40) with respect to t and (3.41) with respect to x gives

$$c\frac{\partial^2 \mu}{\partial x \partial t} - \frac{\partial^2 \lambda}{\partial t^2} = 0,$$

$$c\frac{\partial^2 \lambda}{\partial x^2} - \frac{\partial^2 \mu}{\partial t \partial x} = 0 ,$$

and so

$$c^2\frac{\partial^2 \lambda}{\partial x^2} - \frac{\partial^2 \lambda}{\partial t^2} = 0. \qquad (3.42)$$

The solution to (3.42) is

$$\lambda(x,t) = A(x - ct) + B(x + ct) \quad , \tag{3.43}$$

where A and B are arbitrary functions.

From (3.40)

$$\frac{\partial\lambda}{\partial t} = c\,\frac{\partial\mu}{\partial x}$$

so

$$c\,\frac{\partial\mu}{\partial x} = -\,c\,A'(x - ct) + cB'(x + ct)$$

hence

$$\mu(x,t) = -\,A(x - ct) + B(x + ct) \tag{3.44}$$

From equation (3.26) it can be seen that, since Q is independent of ψ in this case, μ is zero on OK, that is when $t = T$, so

$$\mu(x,T) = -\,A(x - cT) + B(x + cT) = 0$$

therefore

$$A(x - cT) \equiv B(x + cT) \qquad \text{for all } x.$$

Put $x + cT = \xi$, then

$$A(\xi - 2cT) \equiv B(\xi) \qquad \text{for all } \xi \text{ , and so}$$

$$\lambda(x,t) = A(x - ct) + A(x + ct - 2cT) \quad , \tag{3.45}$$

$$\mu(x,t) = -\,A(x - ct) + A(x + ct - 2cT) \; . \tag{3.46}$$

Equation (3.35) gives ,with $Q = \emptyset(x.T) - \phi(x)\ ^2$,

$$\lambda = -\,2\left\{ \emptyset(x,T) - \Phi(x) \right\} \quad , \qquad (x,t)\in LK. \tag{3.47}$$

It has already been seen that $\emptyset - \psi$ is constant along the C+ characteristics so

$$\emptyset(x,t) - \psi(x,t) = \emptyset(\alpha(\tau),\tau) - \psi(\alpha(\tau),\tau) \,. \tag{3.48}$$

$\phi + \psi$ is constant along the C- characteristics, so, from (3.12),

$$\phi(\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) = \phi_0\,(\alpha(\tau) + c\tau) + \psi_0(\alpha(\tau) + c\tau). \qquad (3.49)$$

Since the boundary condition on OL is $\phi - \alpha(\tau) = 0$, (3.49) may be written as

$$\psi(\alpha(\tau),\tau) = \phi_0\,(\alpha(\tau) + c\,\tau) + \psi_0(\alpha(\tau) + c\,\tau) - \alpha(\tau)$$

and using this and the boundary condition (3.48) may be written as

$$\phi(x,t) - \psi(x,t) = 2\alpha(\tau) - \phi_0\,(\alpha(\tau) + c\tau) - \psi_0(\alpha(\tau) + c\tau)$$

hence

$$\phi(x,t) - \psi(x,t) = 2\alpha(\tau) \quad , \qquad (3.50)$$

since ϕ_0 and ψ_0 are assumed to vanish identically on OA. Proceeding along the C - characteristic through (x,t), which characteristic intersects the x - axis at the point $(x_0, 0)$, then

$$\phi(x,t) + \psi(x,t) = \phi_0\,(x_0) + \psi_0(x_0)$$

hence

$$\phi(x,t) + \psi(x,t) = 0. \qquad (3.51)$$

Adding (3.50) and (3.51) gives

$$2\phi(x,t) = 2\alpha(\tau)\ , \qquad (3.52)$$

for all (x,t) in OKL, and so using (3.47),

$$-\lambda(x,T) + 2\Phi(x) = 2\alpha(\tau), \qquad (3.53)$$

where τ is defined by $\alpha(\tau) - c\tau = x - cT$.

From (3.45) $\lambda(x,T) = 2A(x - cT)$, so

$$-2A(x - cT) + 2\,\Phi(x) = 2\alpha(\tau), \qquad \alpha(\tau) - c\tau = x - cT,$$

$$A\{\alpha(\tau) - c\tau\} \equiv \quad -\alpha(\tau) + \Phi\{\alpha(\tau) - c\tau + cT\}\ , \text{ for all } \tau \ . \qquad (3.54)$$

From (3.12) a condition on OL is

$$N \equiv \emptyset(\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) - \emptyset_o(\alpha(\tau) + c\tau) - \psi_o(\alpha(\tau) + c\tau) = 0$$

and since $\emptyset_o$ and ψ_o are assumed to be zero OA

$$N \equiv \emptyset(\alpha(\tau),\tau) + \psi(\alpha(\tau),\tau) = 0 .$$

The other condition on OL is

$$M \equiv \emptyset - \alpha(\tau) = 0.$$

Using the defininitions in (3.26)

$$A_1 = -1 \quad ;$$
$$B_1 = 0 \quad ;$$
$$A_2 = 1 \quad ;$$
$$B_2 = 0 \quad .$$

$\frac{\partial\emptyset}{\partial\alpha}$ and $\frac{\partial\psi}{\partial\alpha}$ in equation (3.34) must now be determined. From (3.52)

$$\emptyset(x,t) = \alpha(\tau), \text{ and } \alpha(\tau) - c\tau = x - cT.$$

so

$$\frac{\partial\emptyset}{\partial x} = \alpha'(\tau)\frac{\partial\tau}{\partial x}, \text{ and } \left[\alpha'(\tau) - c\right]\frac{\partial\tau}{\partial x} = 1 ,$$

$$\frac{\partial\emptyset}{\partial x} = \frac{\alpha'(\tau)}{\alpha'(\tau) - c} .$$

$$\frac{\partial\emptyset}{\partial\alpha} = \frac{\partial\emptyset}{\partial x}\bigg|_{x = \alpha(\tau)} = \frac{\alpha'(\tau)}{\alpha'(\tau) - c} .$$

From (3.51)

$$\psi(x,t) = -\alpha(\tau) \quad \text{and} \quad \alpha(\tau) - c\tau = x - cT ,$$

so

$$\frac{\partial\psi}{\partial x} = -\alpha'(\tau)\frac{\partial\tau}{\partial x} \quad \text{and} \left[\alpha'(\tau) - c\right]\frac{\partial\tau}{\partial x} = 1 ,$$

hence $\frac{\partial\psi}{\partial\alpha} = \frac{-\alpha'(\tau)}{\alpha'(\tau) - c}.$

Since $f = \frac{1}{2}\alpha^2(\tau) + \frac{1}{2}\alpha'^2(\tau)$

$$f_\alpha = \alpha \quad , \quad \frac{df_{\alpha'}}{d\tau} = \alpha'' \quad , \quad \frac{d^2 f_{\alpha''}}{d\tau^2} = 0 .$$

The transversality condition (3.34) can now be written as

$$\alpha(\tau) - \alpha''(\tau) + \left[c\mu + \lambda\alpha'(\tau)\right]\left\{-1 - \frac{\alpha'(\tau)}{\alpha'(\tau) - c}\right\} + $$
$$+ \left[c\lambda + \mu\alpha'(\tau)\right]\left\{1 + \frac{\alpha'(\tau)}{\alpha'(\tau) - c}\right\} = 0 ,$$

$$\left[\alpha(\tau) - \alpha''(\tau)\right]\left[\alpha'(\tau) - c\right] - \left[c\mu + \lambda\,\alpha'(\tau) - c\lambda - \mu\alpha'(\tau)\right]\left[2\alpha'(\tau) + c\right] = 0 \quad (3.55)$$

From (3.45), (3.46) and (3.54)

$$\lambda(\alpha(\tau),\tau) = A\left\{\alpha(\tau) - c\tau\right\} + A\left\{\alpha(\tau) + c\tau - 2cT\right\}$$
$$= -\alpha(\tau) + \Phi\left\{\alpha(\tau) - c\tau + cT\right\} + A\left\{\alpha(\tau) + c\tau - 2cT\right\} ;$$
$$\mu(\alpha(\tau),\tau) = -A\left\{\alpha(\tau) - c\tau\right\} + A\left\{\alpha(\tau) + c\tau - 2cT\right\}$$
$$= \alpha(\tau) - \Phi\left\{\alpha(\tau) - c\tau + cT\right\} + A\left\{\alpha(\tau) + c\tau - 2cT\right\} .$$

Thus replacing λ and μ in (3.55) gives

$$\left[\alpha(\tau) - \alpha''(\tau)\right]\left[\alpha'(\tau) - c\right] - \left[\alpha'(\tau) - c\right]\left[-2\alpha(\tau) + 2\Phi\left\{\alpha(\tau) - c\tau + cT\right\}\right]\left[2\alpha'(\tau) - c\right] = 0$$

$$\alpha''(\tau) - \alpha(\tau) - 4\alpha'(\tau)\alpha(\tau) + 4\Phi\left\{\alpha(\tau) - c\tau + cT\right\}\alpha'(\tau) + 2c\alpha(\tau) -$$
$$- 2c\Phi\left\{\alpha(\tau) - c\tau + cT\right\} = 0$$

$$\alpha''(\tau) + 4\alpha'(\tau)\left[\Phi\left\{\alpha(\tau) - c\tau + cT\right\} - \alpha(\tau)\right] + 2c\alpha(\tau) - 2c\Phi\left\{\alpha(\tau) - c\tau + cT\right\} = 0 \quad (3.56)$$

$\alpha(\tau)$ may be determined from equation (3.56) together with the boundary condition at $\tau = T$ obtained from equation (3.37) .

CHAPTER FOUR

CHAPTER FOUR

A Boundary Control Problem in Unsteady One Dimensional Gas Movements.

This chapter is concerned with the one dimensional movement of a gas in a semi-infinite tube of uniform section, the gas being bounded by a moving piston. At time $t = 0$ the piston is at the origin $x = 0$ and the gas in $x > 0$ is in a state of rest with uniform density ρ_o and uniform sound speed c_o $(c_o^2 = \kappa\gamma\rho_o^{\gamma-1})$. For $t > 0$ the piston is moved away from the gas so that at the time $t = \tau$ its displacement is

$$x = \alpha(\tau) \ , \quad \alpha(0) = 0 \quad , \quad \alpha(\tau) > 0 \quad , \tag{4.1}$$

where τ is a time parameter. A wave of rarefaction is formed at $t = 0$ and this travels in the direction $x > 0$ so that the leading edge of the rarefaction wave is at $x > c_o t$ at time t.

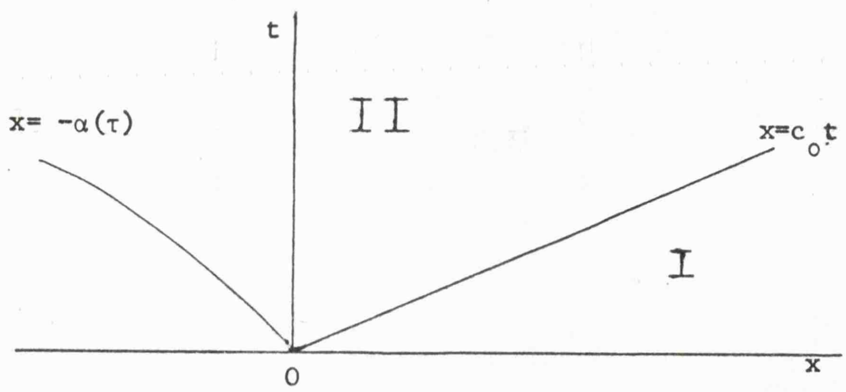


Figure 4.1

For $x > c_o t$ (Region I, see figure 4.1) the gas remains undisturbed. In Region II the gas moves in the x direction with speed $u(x,t)$, density $\rho(x,t)$ and pressure $p(x,t)$ and the governing equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \ , \tag{4.2}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \qquad . \tag{4.3}$$

It is assumed that the adiabatic condition

$$p = \kappa \rho^{\gamma} \tag{4.4}$$

is satisfied with $\gamma = c_p / c_v$. Equations (4.2) and (4.3) can be rewritten in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2}{\gamma-1} c \frac{\partial c}{\partial x} = 0 \qquad ; \tag{4.5}$$

$$\frac{2}{\gamma-1}\frac{\partial c}{\partial t} + \frac{2}{\gamma-1} u \frac{\partial c}{\partial x} + c \frac{\partial u}{\partial x} = 0 \qquad ; \tag{4.6}$$

where c, the local velocity of sound, is defined by

$$c^2 = \kappa \gamma \rho^{\gamma-1} \qquad . \tag{4.7}$$

In the problem the piston movement will be looked upon as the control, and the piston movement must be determined such that, for example,

$$I = \tfrac{1}{2} \int_{x=-\alpha(\tau)}^{x=c_o T} \left[\phi(x) \{ u(x,T) - u^*(x) \}^2 + \psi(x) \{ c(x,T) - c^*(x) \}^2 \right] dx + \tfrac{1}{2} \int_{\tau=0}^{T} \{ a\alpha^2(\tau) + b\alpha'^2(\tau) + c\alpha''^2(\tau) \} d\tau \quad , \tag{4.8}$$

with $\phi > 0, \psi > 0 \ \forall\ x$, is a minimum, in other words the piston control is found so that $u(x,T)$ is as close as possible to a prescribed function $u^*(x)$ and $c(x,T)$ is as close as possible to a prescribed function $c^*(x)$, with the minimum expenditure of control energy. In general however the problem is taken to be that of minimising a general function of the form:

$$I = \int_{x=-\alpha(\tau)}^{x=c_o T} f\{ x, u(x,T), c(x,T) \} dx + \int_{\tau=0}^{T} F\{ \alpha(\tau), \alpha'(\tau), \alpha''(\tau) \} d\tau \quad , \tag{4.9}$$

where f and F are prescribed functions.

The method of handling this problem is as follows: From (4.4) and (4.5)

$$\left\{ \frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x} \right\} \left(u + \frac{2}{\gamma-1} c \right) = 0, \tag{4.10}$$

$$\left\{ \frac{\partial}{\partial t} + (u - c) \frac{\partial}{\partial x} \right\} \left(u - \frac{2}{\gamma-1} c \right) = 0, \tag{4.11}$$

hence

$(u + \frac{2}{\gamma-1} c)$ remains constant along the C+ characteristics given by

$$\frac{dx}{dt} = u + c \ , \tag{4.12}$$

$(u - \frac{2}{\gamma-1} c)$ remains constant along the C- characteristics given by

$$\frac{dx}{dt} = u - c \ . \tag{4.13}$$

The different regions in the (x,t) space will be distinguished as follows. In Region I, namely $x > c_o t$, $t > 0$ the gas is at rest with $u = 0$, $\rho = \rho_o$, $c = c_o$, thus in Region I the C+ and the C- characteristics are families of straight lines $x \pm c_o t$ = constant and Region I will be bounded by $x = c_o t$.

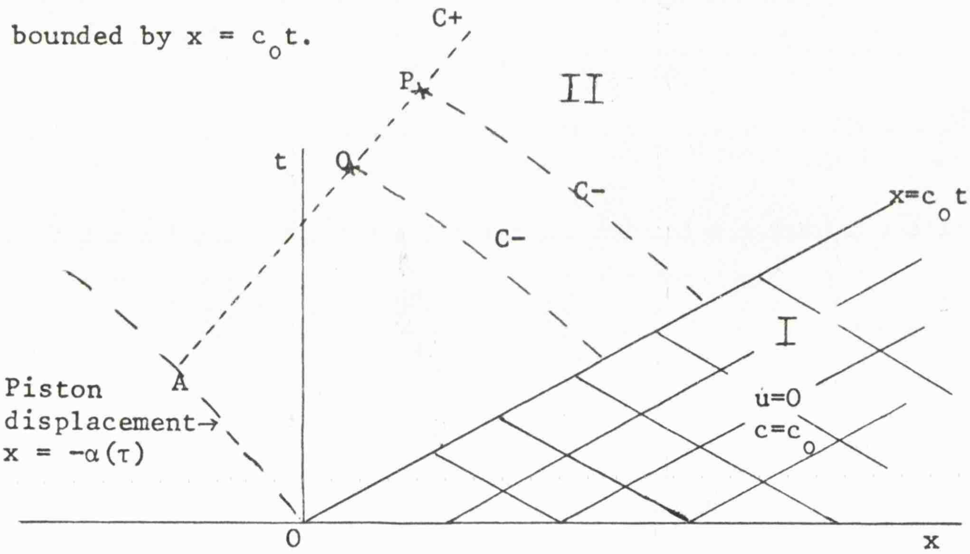


Figure 4.2.

The Region II is on the other side of the line $x = c_o t$ from Region I. Region II called a Simple Wave Region (Courant and Friedrichs[12]) and in this Region it can be proved that the C+ characteristics are straight lines. For if P and Q are any two points in II lying on the same C+ curve which starts at A thus from (4.12)

$$u_p - \frac{2}{\gamma-1} c_p = - \frac{2}{\gamma-1} c_o \ , \tag{4.15}$$

$$u_Q - \frac{2}{\gamma-1} c_Q = - \frac{2}{\gamma-1} c_o \ . \tag{4.16}$$

From (4.15) and (4.16) it is deduced that $u_p = u_Q$, $c_p = c_Q$; hence the slope of the C+ characteristics at P, namely $\frac{1}{(u_p + c_p)}$ is the same as the slope of the C+ characteristic at Q, namely $\dfrac{1}{(u_Q + c_Q)}$, hence the C+ characteristic is a straight line. The C- characteristics in Region II remain as general curves satisfying (4.13). Continuing with the theory it is deduced that if $A\{-\alpha(\tau),\tau\}$ lies on the piston displacement curve then from the above theory

$$c_p = c_A \quad , \quad u_p = u_A \quad . \tag{4.17}$$

Travelling on the C- characteristic through A back to Region I

$$u_A - \frac{2}{\gamma-1} c_A = - \frac{2}{\gamma-1} c_o$$

or

$$c_A = c_o + \frac{\gamma-1}{2} u_A \quad . \tag{4.18}$$

Thus the slope of the C+ characteristics through A will be

$$\frac{1}{u_A + c_A} = \frac{1}{c_o + \frac{\gamma+1}{2} u_A} = \frac{1}{c_o - \frac{\gamma+1}{2}\alpha'(\tau)} , \tag{4.19}$$

where $\alpha'(\tau) = \dfrac{d\alpha}{d\tau}$. The equation of the straight line C+ characteristic through A will be

$$t - \tau = \frac{1}{c_o - \frac{\gamma+1}{2} \alpha'(\tau)} \left(x + \alpha(\tau) \right) . \tag{4.20}$$

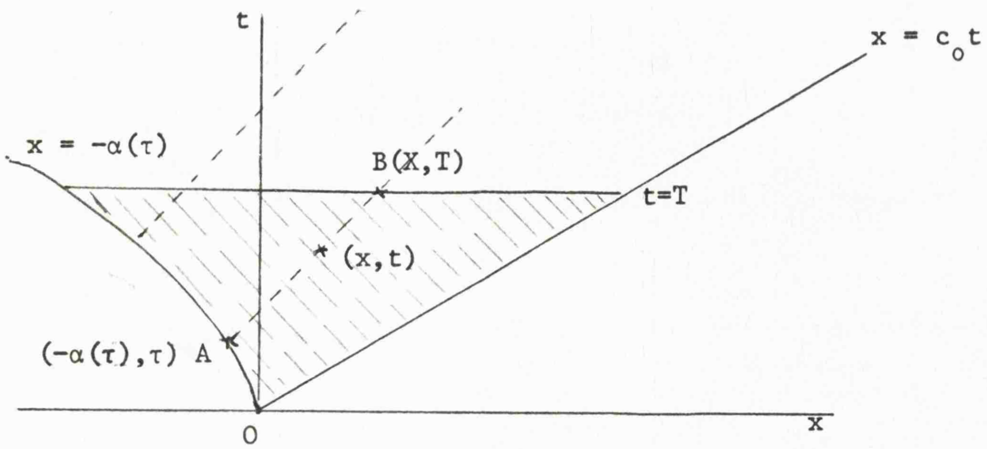


Figure 4.3

Suppose the C+ characteristic (4.20) meets $t = T$ at the point B whose co-ordinates are (X,T), then from (4.20), X will be given by

$$X = -\alpha(\tau) + (T-\tau)\left[c_o - \frac{\gamma+1}{2}\alpha'(\tau)\right] \quad . \tag{4.21}$$

Furthermore, using (4.17), the values of u_B and c_B are as follows:

$$u(X,T) = u_B = u_A = -\alpha'(\tau), \tag{4.22}$$

$$c(X,T) = c_B = c_A = c_o - \frac{\gamma-1}{2}\alpha'(\tau) \quad , \tag{4.23}$$

using (4.18). The above theory relating to Region II is valid providing the speed of the piston does not become excessive and this limitation is discussed as follows. Equation (4.18) can be written in the form

$$c_A = c_o - \frac{\gamma-1}{2}\alpha'(\tau) \ ,$$

noting that $c_A = 0$ if $\alpha'(\tau) = \frac{2c_o}{\gamma-1}$; the vanishing of c implies the vanishing of the density ρ , thus if the piston speed becomes equal to $\frac{2c_o}{\gamma-1}$, the density of the gas in contact with the piston will be zero. If the piston speed now exceeds $\frac{2c_o}{\gamma-1}$ the piston will lose contact with the gas and a vacuum will form between the piston and the gas. In this event clearly no control of the gas movement is possible. Thus in the above problem it will be assumed that $0 < \alpha'(\tau) < \frac{2c_o}{\gamma-1}$.

The substitution (4.21)[13] is now used to change from the variable X into the new variable τ. Now from (4.21)

$$\begin{aligned} dX &= \left\{ -\alpha'(\tau) - c_o + \frac{\gamma+1}{2}\alpha'(\tau) + (T-\tau)\left(-\frac{\gamma+1}{2}\alpha''(\tau)\right)\right\} d\tau \\ &= -\left\{ c_o - \frac{\gamma-1}{2}\alpha'(\tau) + \frac{\gamma+1}{2}(T-\tau)\,\alpha''(\tau)\right\} d\tau . \end{aligned} \tag{4.24}$$

It is deduced from (4.21) that $X = c_oT$ will correspond to $\tau = 0$ provided that $\alpha'(0) = 0$ and $X = -\alpha(T)$ will correspond to $\tau = T$. Hence (4.9) can be written in the form

$$I = \int_{\tau=0}^{\tau=T} f\left\{-\alpha(\tau) + (T-\tau)\left[c_o - \frac{\gamma+1}{2}\alpha'(\tau)\right], -\alpha'(\tau), c_o - \frac{\gamma-1}{2}\alpha'(\tau)\right\}\times$$
$$\times\left\{c_o - \frac{\gamma-1}{2}\alpha'(\tau) + \frac{\gamma+1}{2}(T-\tau)\alpha''(\tau)\right\} d\tau \quad +$$
$$+ \int_{\tau=0}^{T} F\left\{\alpha(\tau), \alpha'(\tau), \alpha''(\tau)\right\} d\tau \quad . \qquad (4.25)$$

or

$$I = \int_{\tau=0}^{T} g\{\tau, \alpha(\tau), \alpha'(\tau), \alpha''(\tau)\} d\tau \quad . \qquad (4.26)$$

where

$$g(\tau,\alpha(\tau),\alpha'(\tau),\alpha''(\tau)) = \left\{c_o - \frac{\gamma-1}{2}\alpha'(\tau) + \frac{\gamma+1}{2}(T-\tau)\alpha''(\tau)\right\} \times$$
$$\times f\left\{-\alpha(\tau) + (T-\tau)\left[c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right], -\alpha'(\tau), c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right\}$$
$$+ F\{\alpha(\tau), \alpha'(\tau), \alpha''(\tau)\} \quad . \qquad (4.27)$$

Thus the original problem has been transformed into one of finding the function $\alpha(\tau)$ which will provide the minimum of the funcitonal I in (4.26) which is the classical Euler problem in the calculus of variations In order to study the boundary conditions the problem is tackled as follows:

Consider the function

$$J(\varepsilon) = \int_0^T g\{\tau, \alpha(\tau) + \varepsilon\eta(\tau), \alpha'(\tau) + \varepsilon\eta'(\tau), \alpha''(\tau) + \varepsilon\eta''(\tau)\} d\tau$$

where $y = \alpha(\tau)$ is the function which gives the minimum of I in (4.26)

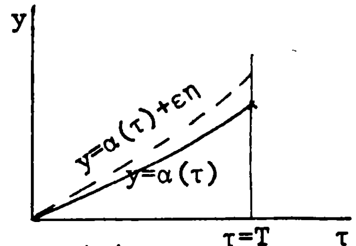


Figure 4.4

and where $y = \alpha(\tau) + \varepsilon\eta(\tau)$ is a neighbouring function. $J(0) = I$ and the necessary condition for a minimum is $J'(0) = 0$.

$$J'(0) = \int_0^T \{\eta(\tau)g_\alpha + \eta'(\tau)g_{\alpha'} + \eta''(\tau)g_{\alpha''}\} d\tau$$
$$= \int_0^T \left[\eta(\tau)g_\alpha + \eta'(\tau)\left\{g_{\alpha'} - \frac{d}{d\tau}g_{\alpha''}\right\}\right] d\tau + \left[\eta'(\tau)g_{\alpha''}\right]_0^T$$
$$= \int_0^T \eta(\tau)\left\{g_\alpha - \frac{d}{d\tau}\left[g_{\alpha'} - \frac{d}{d\tau}g_{\alpha''}\right]\right\}d\tau + \left[\eta'(\tau)g_{\alpha''} + \eta(\tau)\left\{g_{\alpha'} - \frac{d}{d\tau}g_{\alpha''}\right\}\right]_0^T .$$

Thus the necessary conditions for $J'(0) = 0$ with arbitrary $\eta(\tau)$ are

$$g_{\alpha} - \frac{d}{d\tau} g_{\alpha'} + \frac{d^2}{d\tau^2} g_{\alpha''} = 0 \qquad , \qquad (4.28)$$

and

$$\left[\eta'(\tau) g_{\alpha''} + \eta(\tau) \left\{ g_{\alpha'} - \frac{d}{d\tau} g_{\alpha''} \right\}\right]_0^T = 0 \qquad . \qquad (4.29)$$

Consider first the differential equation (4.28).

Writing

$$\chi(\tau,\alpha(\tau),\alpha'(\tau)) \equiv f\left\{ -\alpha(\tau) + (T-\tau)\left[c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right] , -\alpha'(\tau), c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) \right\} , \qquad (4.30)$$

then (4.27) can be written in the form

$$g(\tau,\alpha,\alpha',\alpha'') = \left\{ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau) \right\} \chi\,(\tau,\alpha(\tau),\alpha''(\tau)) + F(\alpha(\tau),\alpha'(\tau),\alpha''(\tau)) \quad . \qquad (4.31)$$

From (4.31) it is deduced that

$$g_{\alpha'} = -\frac{(\gamma-1)}{2}\chi + \left\{ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau) \right\} \chi_{\alpha'} + F_{\alpha'} , \qquad (4.32)$$

$$g_{\alpha''} = \frac{(\gamma+1)}{2}(T-\tau)\,\chi(\tau,\alpha(\tau),\alpha'(\tau)) + F_{\alpha''} \qquad , \qquad (4.33)$$

$$g_{\alpha} = \left\{ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\alpha+1)}{2}(T-\tau)\alpha''(\tau) \right\} \chi_{\alpha} + F_{\alpha} \; ; \qquad (4.34)$$

thus equation (4.28) can be written as follows:

$$\frac{d^2}{d\tau^2}\left\{ \frac{(\gamma+1)}{2}(T-\tau)\,\chi(\tau,\alpha(\tau),\alpha'(\tau)) + F_{\alpha''} \right\}$$
$$+ \frac{d}{d\tau}\left\{ \frac{(\gamma-1)}{2}\chi - \left[c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau) \right]\chi_{\alpha'} - F_{\alpha'} \right\}$$
$$+ \left\{ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau) \right\} \chi_{\alpha} + F_{\alpha} = 0. \qquad (4.35)$$

Consider now the boundary conditions for the problem. Two of the boundary conditions upon $\alpha(\tau)$ have already been noted and these are as follows:

$$\alpha(0) = 0 \qquad , \; \alpha'(0) = 0 \qquad . \qquad (4.36)$$

The conditions (4.36) imply that $\eta(0) = 0$ and $\eta'(0) = 0$ and thus (4.29) can now be written in the form

$$\eta'(T)\ g_{\alpha''}\Big|_{\tau=T} + \eta(T)\left\{ g_{\alpha'} - \frac{d}{d\tau}\, g_{\alpha''} \right\}_{\tau=T} = 0 \quad . \tag{4.37}$$

Since $\eta(\tau)$ is an arbitary variation it follows that the coefficients of $\eta(\tau)$ and $\eta'(\tau)$ in (4.37) must both be zero, hence

$$g_{\alpha''} = 0 \quad , \quad \tau = T \quad ; \tag{4.38}$$

$$g_{\alpha'} - \frac{d}{d\tau}\ g_{\alpha''} = 0 \ , \qquad \tau = T \quad . \tag{4.39}$$

These conditions allied with the two conditions upon $\alpha(\tau)$ in (4.36) provide the appropriate conditions for the unique solution of $\alpha(\tau)$ in the problem.

CHAPTER FIVE

CHAPTER FIVE

The Application of the Gelfand - Fomin Theorem in the Unsteady One Dimensional Gas Problem.

The unsteady one dimensional gas problem is now discussed using the Gelfand - Fomin theorem. The notation is the same as that used in Chapter Four.

The governing equations of the gas are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2}{\gamma-1} c\frac{\partial c}{\partial x} = 0 \quad , \tag{5.1}$$

$$\frac{2}{\gamma-1}\frac{\partial c}{\partial t} + \frac{2}{\gamma-1} u\frac{\partial c}{\partial x} + c\frac{\partial u}{\partial x} = 0 \quad . \tag{5.2}$$

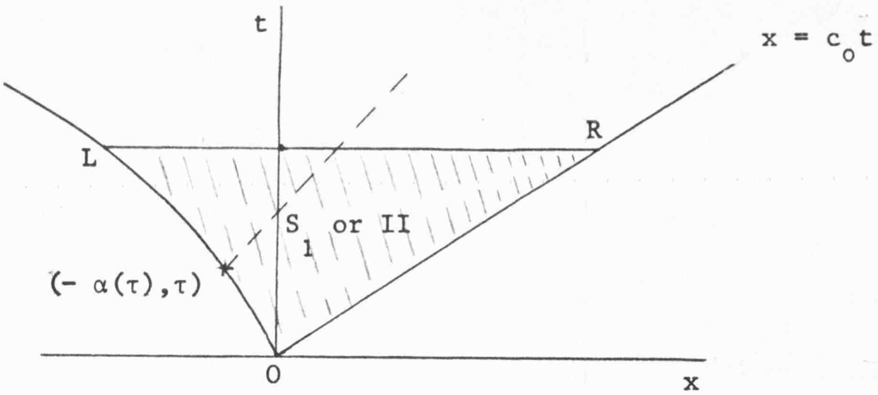


Figure 5.1

As before the performance index I to be minimised is given by

$$I = \int_{LR} f\{x,u(x,T),c(x,T)\}\,dx + \int_{\tau=0}^{T} F\{\alpha(\tau),\alpha'(\tau)\alpha''(\tau),\tau\}d\tau \quad . \tag{5.3}$$

Consider instead of I the new functional J where

$$J = \iint_{S_1}\left\{\xi(x,t)\left[u_t + uu_x + \frac{2}{\gamma-1}cc_x\right] + \eta(x,t)\left[\frac{2}{\gamma-1}c_t + \frac{2}{\gamma-1}uc_x + cu_x\right]\right\}dxdt$$
$$+ \int_{LR} f\{x,u,c\}\,dx + \int_{\tau=0}^{T} F\{\alpha(\tau),\alpha'(\tau),\alpha''(\tau),\tau\}\,d\tau \quad , \tag{5.4}$$

ξ and η being Lagrange multipliers depending on x and t. Let

$$\Phi = \xi(x,t)\left[u_t + uu_x + \frac{2}{\gamma-1}cc_x\right] + \eta(x,t)\left[\frac{2}{\gamma-1}c_t + \frac{2}{\gamma-1}uc_x + cu_x\right], \quad (5.5)$$

and

$$J_1 = \int\int_{S_1} \Phi \, dx\, dt. \quad (5.6)$$

Applying the Gelfand - Fomin theorem to J_1 the variation in J_1, that is δJ_1, is given by

$$\delta J_1 = \int\int_{S_1}\left\{\overline{\delta u}\left[\Phi_u - \frac{\partial}{\partial x}\Phi_{u_x} - \frac{\partial}{\partial t}\Phi_{u_t}\right] + \overline{\delta c}\left[\Phi_c - \frac{\partial}{\partial x}\Phi_{c_x} - \frac{\partial}{\partial t}\Phi_{c_t}\right]\right\}dx\,dt$$
$$+\int\int_{S_1}\left\{\frac{\partial}{\partial x}\left[\Phi\delta x + \overline{\delta u}\,\Phi_{u_x} + \overline{\delta c}\,\Phi_{c_x}\right] + \frac{\partial}{\partial t}\left[\Phi\delta t + \overline{\delta u}\,\Phi_{u_t} + \overline{\delta c}\,\Phi_{c_t}\right]\right\}dx\,dt$$

and using Stokes' theorem on the second integral this becomes

$$\delta J_1 = \int\int_{S_1}\left\{\overline{\delta u}\left[\Phi_u - \frac{\partial}{\partial t}\Phi_{u_x} - \frac{\partial}{\partial t}\Phi_{u_t}\right] + \overline{\delta c}\left[\Phi_c - \frac{\partial}{\partial x}\Phi_{c_x} - \frac{\partial}{\partial t}\Phi_{c_t}\right]\right\}dx\,dt$$
$$+ \int_{OR+RL+LO}\left\{\left[\Phi\delta x + \overline{\delta u}\,\Phi_{u_x} + \overline{\delta c}\,\Phi_{c_x}\right]dt - \left[\Phi\delta t + \overline{\delta u}\,\Phi_{u_t} + \overline{\delta c}\,\Phi_{c_t}\right]dx\right\}$$
$$(5.7)$$

It is known from the characteristic theory that x, c, t and u remain unaltered on OR and so there is no contribution to δJ from the integral along OR. On LR, that is $t = T$, δt and dt are zero so the integration along RL becomes

$$\int_{LR}\left\{\overline{\delta u}\,\Phi_{u_t} + \overline{\delta c}\,\Phi_{c_t}\right\}dx. \quad (5.8)$$

On LO $x = -\alpha(\tau)$, $t = \tau$ and the value of τ at a point on LO is unaltered by the variation of position of LO so $\delta t = 0$ and $\delta x = -\delta\alpha(\tau)$.

$\overline{\delta u}$ and $\overline{\delta c}$ are defined by

$$\overline{\delta u} = \delta u - \frac{\partial u}{\partial x}\delta x - \frac{\partial u}{\partial t}\delta t \quad ;$$
$$\overline{\delta c} = \delta c - \frac{\partial c}{\partial x}\delta x - \frac{\partial c}{\partial t}\delta t \quad .$$

On LO these become

$$\overline{\delta u} = \delta u + \frac{\partial u}{\partial \alpha}\partial\alpha \quad , \qquad \overline{\delta c} = \delta c + \frac{\partial c}{\partial \alpha}\delta\alpha$$

where $\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\Big|_{x = -\alpha(\tau)}$, $\frac{\partial c}{\partial \alpha} \quad \frac{\partial c}{\partial x}\Big|_{x=-\alpha(\tau)}$

From equation (4.24) the boundary condition on OL is

$$u(x,\tau) + \alpha'(\tau) = 0, \quad x = -\alpha(\tau) \quad ;$$

and the varied conditions are

$\delta\tau = 0, \quad \delta x = -\delta\alpha \quad , \quad \delta u = -\delta\alpha'$

and on OL

$$c(x,\tau) = c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)$$

so $\delta c = -\frac{(\gamma-1)}{2}\ \delta\alpha'(\tau)$.

Therefore on OL $\overline{\delta u}$ and $\overline{\delta c}$ may be written as

$$\overline{\delta u} = -\ \delta\alpha' + \frac{\partial u}{\partial \alpha}\ \delta\alpha$$
$$\overline{\delta c} = -\frac{(\gamma-1)}{2}\delta\alpha' + \frac{\partial c}{\partial \alpha}\ \delta\alpha$$

and the integration along LO may be written as

$$-\int_{LO}\Big\{\Big[\Phi\delta\alpha + (\delta\alpha' - \frac{\partial u}{\partial \alpha}\ \delta\alpha)\Phi_{u_x} + (\frac{(\gamma-1)}{2}\delta\alpha' - \frac{\partial c}{\partial \alpha}\ \delta\alpha)\Phi_{c_x}\Big]\ d\tau$$
$$+\Big[(\delta\alpha' - \frac{\partial u}{\partial \alpha}\delta\alpha)\Phi_{u_t} + (\frac{(\gamma-1)}{2}\delta\alpha' - \frac{\partial c}{\partial \alpha}\ \delta\alpha)\Phi_{c_t}\Big]\ d\alpha\Big\}$$

or

$$\int_{OL}\Big\{\Phi\delta\alpha + (\delta\alpha' - \frac{\partial u}{\partial \alpha}\ \delta\alpha)\Phi_{u_x} + (\frac{(\gamma-1)}{2}\delta\alpha' - \frac{\partial c}{\partial \alpha}\ \delta\alpha)\Phi_{c_x} +$$
$$+ (\delta\alpha' - \frac{\partial u}{\partial \alpha}\ \delta\alpha)\Phi_{u_t}\ \alpha'(\tau) + (\frac{(\gamma-1)}{2}\delta\alpha' - \frac{\partial c}{\partial \alpha}\ \delta\alpha)\Phi_{c_t}\ \alpha'(\tau)\Big\}\ d\tau$$
$$= \int_{OL}\Big\{\delta\alpha\Big[\Phi - \frac{\partial u}{\partial \alpha}\ (\Phi_{u_x} + \Phi_{u_t}\ \alpha'(\tau)) - \frac{\partial c}{\partial \alpha}\ (\Phi_{c_x} + \Phi_{c_t}\ \alpha'(\tau))\Big]$$
$$+ \delta\alpha'\Big[\Phi_{u_x} + \Phi_{u_t}\ \alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\ \alpha'(\tau))\Big]\Big\}\ d\tau \quad .(5.9)$$

Integrating $\int_{OL}\delta\alpha'\Big[\Phi_{u_x} + \Phi_{u_t}\ \alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\ \alpha'(\tau))\Big]\ d\tau$

by parts gives

$$\delta\alpha\Big[\Phi_{u_x} + \Phi_{u_t}\ \alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\ \alpha'(\tau))\Big]_{\tau=0}^{\tau=T}$$
$$-\int_{OL}\frac{\delta\alpha\partial}{\partial\tau}\Big\{\Phi_{u_x} + \Phi_{u_t}\ \alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\ \alpha'(\tau)\ )\Big\}d\tau,$$

and (5.9) may be written as

$$\int_{OL} \delta\alpha\Big\{\Phi - \frac{\partial u}{\partial \alpha}(\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau)) - \frac{\partial c}{\partial \alpha}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))$$

$$- \frac{\partial}{\partial \tau}\Big[\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\Big]\Big\} d\tau$$

$$+ \delta\alpha\Big[\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\Big]_{\tau=T} \qquad (5.10)$$

since $\delta\alpha = 0$ at $\tau = 0$.

(5.7) may now be written as

$$\delta J_1 = \int_{S_1}\int \Big\{ \overline{\delta u}\Big[\Phi_u - \frac{\partial}{\partial x}\Phi_{u_x} - \frac{\partial}{\partial t}\Phi_{u_t}\Big] + \overline{\delta c}\Big[\Phi_c - \frac{\partial}{\partial x}\Phi_{c_x} - \frac{\partial}{\partial t}\Phi_{c_t}\Big]\Big\} dx\, dt$$

$$+ \int_{LR} \Big\{\overline{\delta u}\,\Phi_{u_t} + \overline{\delta c}\,\Phi_{c_t}\Big\} dx$$

$$+ \int_0^{\tau} \delta\alpha\Big\{\Phi - \frac{\partial u}{\partial \alpha}(\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau)) - \frac{\partial c}{\partial \alpha}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau)) -$$

$$- \frac{\partial}{\partial \tau}\Big[\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\Big]\Big\} d\tau$$

$$+ \delta\alpha\Big[\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\Big]_{\tau=T}. \qquad (5.11)$$

Let $\int_{LR} f\,\{x,u,c\}$ be J_2, then by the Gelfand - Fomin theorem

$$\delta J_2 = \int_{LR} \Big\{ \overline{\delta u}\Big[f_u - \frac{\partial}{\partial x} f_{u_x}\Big] + \overline{\delta c}\Big[f_c - \frac{\partial}{\partial x} f_{c_x}\Big]\Big\} dx$$

$$+ \int_{LR} \frac{\partial}{\partial x}(f\delta x + \overline{\delta u} f_{u_x} + \overline{\delta c}\, f_{c_x})\, dx$$

and since f is independent of u_x and c_x

$$\delta J_2 = \int_{LR} \{\overline{\delta u}\, f_u + \overline{\delta c}\, f_c\} dx + \int_{LR} \frac{\partial}{\partial x}(f\delta x)\, dx$$

$$= \int_{LR}\{\overline{\delta u}\, f_u + \overline{\delta c}\, f_c\} dx - f\delta x\Big|_{x=x_L}^{x=x_R} \qquad (5.12)$$

and at $x = x_R$ δx is zero.

If $J_3 = \int_{\tau=0}^{T} F\{\alpha(\tau),\alpha'(\tau),\alpha''(\tau),\tau\}\, d\tau$, then

$$\delta J_3 = \int_{\tau=0}^{T}\{F_\alpha\,\delta\alpha + F_{\alpha'}\,\delta\alpha' + F_{\alpha''}\,\delta\alpha''\}\, d\tau$$

and integrating $F_{\alpha'}\delta\alpha'$ and $F_{\alpha''}\delta\alpha''$ by parts this becomes, as in previous examples,

$$\delta J_3 = \int_0^T \delta\alpha\Big\{F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2}\Big\} d\tau + \delta\alpha\Big[F_{\alpha'} - \frac{dF_{\alpha''}}{d\tau}\Big]_{\tau=T} + F_{\alpha''}\,\delta\alpha'\Big|_{\tau=T}.$$

(5.13)

δJ, the total variation of J, is the sum of (5.11), (5.12) and (5.13), so

$$\delta J = \int_{S_1}\!\!\int\left\{\overline{\delta u}\left[\Phi_u - \frac{\partial}{\partial x}\Phi_{u_x} - \frac{\partial}{\partial t}\Phi_{u_t}\right] + \overline{\delta c}\left[\Phi_c - \frac{\partial}{\partial x}\Phi_{c_x} - \frac{\partial}{\partial t}\Phi_{c_t}\right]\right\} dx\, dt$$
$$+ \int_{LR}\left\{ \overline{\delta u}(\Phi_{u_t} + f_u) + \overline{\delta c}(\Phi_{c_t} + f_c)\right\} dx$$
$$+ \int_0^T \delta\alpha\left\{F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} + \Phi - \frac{\partial u}{\partial\alpha}(\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau)) - \right.$$
$$\left. - \frac{\partial c}{\partial\alpha}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau)) - \frac{\partial}{\partial\tau}\left[\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\right]\right\} d\tau$$
$$+ \delta\alpha\left[F_{\alpha'} - \frac{dF_{\alpha''}}{d\tau} + \Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau)) - f\right]_{\tau=T}$$
$$+ F_{\alpha''}\,\delta\alpha'\Big|_{\tau=T} \qquad (5.14)$$

For a minimum of I in (5.3) δJ must be zero and since $\overline{\delta u}$, $\overline{\delta c}$, $\delta\alpha$ and $\delta\alpha'$ are independent arbitary variations this implies that

$$\Phi_u - \frac{\partial}{\partial x}\Phi_{u_x} - \frac{\partial}{\partial t}\Phi_{u_t} = 0, \qquad (x,t) \in S_1, \qquad (5.15)$$

$$\Phi_c - \frac{\partial}{\partial x}\Phi_{c_x} - \frac{\partial}{\partial t}\Phi_{c_t} = 0, \qquad (x,t) \in S_1, \qquad (5.16)$$

$$\Phi_{u_t} + f_u = 0, \qquad (x,t) \in LR, \qquad (5.17)$$

$$\Phi_{c_t} + f_c = 0, \qquad (x,t) \in LR, \qquad (5.18)$$

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F''_\alpha}{d\tau^2} + \Phi - \frac{\partial u}{\partial\alpha}(\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau)) - \frac{\partial c}{\partial\alpha}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))$$
$$- \frac{\partial}{\partial\tau}\left\{\Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau))\right\} = 0,$$
$$(x,t) \in OL, \qquad (5.19)$$

$$F_{\alpha'} - \frac{dF_{\alpha''}}{d\tau} + f + \Phi_{u_x} + \Phi_{u_t}\alpha'(\tau) + \frac{(\gamma-1)}{2}(\Phi_{c_x} + \Phi_{c_t}\alpha'(\tau)) = 0, \quad \tau = T, (5.20)$$

$$F_{\alpha''}\,\delta\alpha' = 0, \qquad \tau = T. \qquad (5.21)$$

Substituting the value for Φ from (5.5) into (5.15) and (5.16) gives

$$\frac{3-\gamma}{\gamma-1}\,\eta\frac{\partial c}{\partial x} - u\frac{\partial\xi}{\partial x} - c\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial t} = 0,$$

and $\frac{\gamma-3}{\gamma-1}\,\eta\frac{\partial u}{\partial x} - \frac{2}{\gamma-1}c\frac{\partial\xi}{\partial x} - \frac{2}{\gamma-1}u\frac{\partial\eta}{\partial x} - \frac{2}{\gamma-1}\frac{\partial\eta}{\partial t} = 0$;

and these may be written as

$$-\frac{(3-\gamma)}{\gamma-1}\eta\frac{\partial c}{\partial x} + u\frac{\partial\xi}{\partial x} + c\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial t} = 0 \quad ; \qquad (5.22)$$

$$\frac{3-\gamma}{2}\eta\frac{\partial u}{\partial x} + c\frac{\partial \xi}{\partial x} + u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0 \quad . \tag{5.23}$$

Adding (5.22) and (5.23) gives

$$(u + c)\frac{\partial}{\partial x}(\xi + \eta) + \frac{\partial}{\partial t}(\xi + \eta) = \frac{(\gamma-3)}{2}\eta\frac{\partial}{\partial x}\left\{u - \frac{2c}{\gamma-1}\right\}, \tag{5.24}$$

and subtracting (5.23) from (5.22) gives

$$(u - c)\frac{\partial}{\partial x}(\xi - \eta) + \frac{\partial}{\partial t}(\xi - \eta) = \frac{3-\gamma}{2}\eta\frac{\partial}{\partial x}\left\{u + \frac{2}{\gamma-1}c\right\} . \tag{5.25}$$

It is known from the characteristic theory of equations (5.1) and (5.2), [(4.13)] , that

$$u - \frac{2}{\gamma-1}c = -\frac{2}{\gamma-1}c_o \qquad \text{for all} \quad (x,t) \in S_1$$

hence (5.24) becomes

$$\left\{(u + c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right\}(\xi + \eta) = 0 \tag{5.26}$$

and this can be interpreted as $(\xi + \eta)$ is constant along $\frac{dx}{dt} = u + c$, the C+ characteristic.

Substituting for Φ in (5.17) and (5.18) gives

$$f_u + \xi = 0 \quad , \qquad (x,t) \in LR \quad ; \tag{5.27}$$

$$f_c + \frac{2}{\gamma-1}\eta = 0 \; , \qquad (x,t) \in LR \quad . \tag{5.28}$$

Since $(\xi + \eta)$ is constant along the C+ characteristic

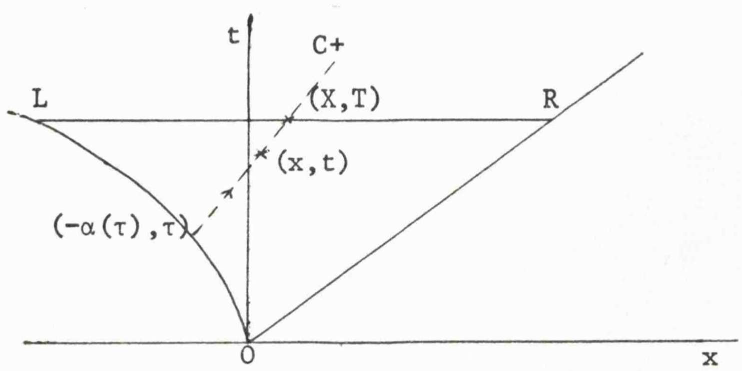


Figure 5.2

$$\xi(x,t) + \eta(x,t) = \xi(-\alpha(\tau),\tau) + \eta(-\alpha(\tau),\tau)$$

$$= \xi(X,T) + \eta(X,T),$$

and from (5.27) and (5.28)

$$\xi(X,T) + \eta(X,T) = -\left\{f_u + \frac{\gamma-1}{2} f_c\right\}_{\substack{t=T \\ x=X}}$$

so $\xi(x,t) + \eta(x,t) = \xi(-\alpha(\tau),\tau) + \eta(-\alpha(\tau),\tau) = -\left\{f_u + \frac{\gamma-1}{2} f_c\right\}_{\substack{t=T \\ x=X}}$. (5.29)

Substituting for Φ from (5.5) in (5.19) gives

$$F_{\dot\alpha} - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} - \frac{\partial u}{\partial \alpha}(\xi u + \eta c + \xi\alpha'(\tau)) - \frac{\partial c}{\partial \alpha}\left(\frac{2}{\gamma-1}\xi c + \frac{2}{\gamma-1}\eta u + \frac{2}{\gamma-1}\eta\alpha'(\tau)\right)$$
$$- \frac{\partial}{\partial\tau}\left\{\xi u + \eta c + \frac{(\gamma-1)}{2}\left(\frac{2}{\gamma-1}\xi c + \frac{2}{\gamma-1}\eta u\right) + \alpha'(\tau)\left(\xi + \frac{(\gamma-1)}{2}\cdot\frac{2}{(\gamma-1)}\eta\right)\right\} = 0$$
$$(x,t) \in OL, \qquad (5.30)$$

and $u + \alpha'(\tau) = 0$ on OL so (5.30) becomes

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} - \frac{\partial u}{\partial \alpha}\eta c - \frac{\partial c}{\partial \alpha}\cdot\frac{2}{\gamma-1}\xi c - \frac{\partial}{\partial\tau}\left\{(\eta + \xi)c\right\} = 0. \quad (5.31)$$

$\frac{\partial u}{\partial \alpha}$ and $\frac{\partial c}{\partial \alpha}$ must now be determined.

Since $u(x,t) = -\alpha'(\tau)$ and, from (4.18),

$c(x,t) = c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)$ then

$$\frac{\partial u}{\partial x} = -\alpha''(\tau)\frac{\partial\tau}{\partial x} \text{ and } \frac{\partial c}{\partial x} = c_o - \frac{(\gamma-1)}{2}\alpha''(\tau)\frac{\partial\tau}{\partial x}.$$

From (4.20) τ is related to x by the equation

$$x = -\alpha(\tau) + (t-\tau)\left\{c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right\}$$

so $\frac{\partial x}{\partial\tau} = -\alpha'(\tau) - \left\{c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right\} - \frac{(\gamma+1)}{2}\alpha''(\tau)(t-\tau)$

and since $t = \tau$ on OL, $\frac{\partial x}{\partial\tau}$ on OL becomes

$$\frac{\partial x}{\partial\tau} = -\alpha'(\tau) - c_o + \frac{(\gamma+1)}{2}\alpha'(\tau), \qquad (x,t) \in OL$$

and $\frac{\partial\tau}{\partial x} = \dfrac{1}{\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o}$ on OL.

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\bigg|_{\substack{x=-\alpha(\tau)\\ t=\tau}} \qquad \frac{\partial u}{\partial c} = \frac{\partial u}{\partial x}\bigg|_{\substack{x=-\alpha\ \tau\\ t=\tau}}$$

$$\text{so } \frac{\partial u}{\partial \alpha} = \frac{-\alpha''(\tau)}{\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o}, \quad \frac{\partial c}{\partial \alpha} = \frac{-\frac{(\gamma-1)}{2}\alpha''(\tau)}{\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o} \quad \text{. on OL}$$

(5.31) may now be written as

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} + \frac{\eta c\alpha''(\tau)}{\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o} + \frac{\xi c\alpha''(\tau)}{\frac{(\gamma-1)}{2}\alpha'(\tau) - c} - \frac{\partial}{\partial\tau}\left\{c\ (\xi + \eta)\right\} = 0$$

and since $c = c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)$ and $(\xi + \eta) = -\left\{f_u + \frac{(\gamma-1)}{2}f\right\}_{\substack{t=T\\ x=X}}$

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} + \alpha''\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}} + \frac{\partial}{\partial\tau}\left\{\left(c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right)\left(f_u + \frac{(\gamma-1)}{2}f_c\right)\right\}_{\substack{t=T\\ x=X}} = 0$$

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} + \alpha''\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}} - \frac{(\gamma-1)}{2}\alpha''(\tau)\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}}$$
$$+ \left(c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right)\frac{\partial}{\partial\tau}\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}} = 0$$

$$F_\alpha - \frac{dF_{\alpha'}}{d\tau} + \frac{d^2F_{\alpha''}}{d\tau^2} = \frac{(\gamma-3)}{2}\alpha''\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}} - \left\{c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right\} \times$$
$$\times \frac{\partial}{\partial\tau}\left\{f_u + \frac{(\gamma-1)}{2}f_c\right\}_{\substack{t=T\\ x=X}} = 0 \ . \quad (5.32)$$

where $X = -\alpha(\tau) + (T - \tau)\left\{c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right\}$

When the value for Φ from (5.5) is substituted in the boundary condition (5.20) that becomes

$$F_{\alpha'} - \frac{dF_{\alpha''}}{d\tau} + f - \left(c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right)\left(f_u + \frac{(\gamma-1)}{2}f_c\right)_{\substack{t=T\\ x=X}} = 0 \qquad \tau = T \ .(5.33)$$

Equation (5.32) is the transversality condition corresponding to equation (4.35) in the previous chapter. It will now be shown that these two equations are identical.

Equation (4.35) is given by

$$\frac{d^2}{d\tau^2}\left\{\frac{(\gamma+1)}{2}(T-\tau)\ \chi(\tau,\alpha(\tau),\alpha'(\tau)) + F_{\alpha''}\right\}$$
$$+ \frac{d}{d\tau}\left\{\frac{(\gamma-1)}{2}\chi - \left[c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau)\right]\chi_{\alpha'} - F_{\alpha'}\right\}$$
$$+ \left\{c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau)\right\}\chi_\alpha + F_\alpha = 0 \ . \quad (5.34)$$

From (4.30)

$$\chi \equiv f\left\{-\alpha(\tau) + (T-\tau)\left[c_o - \frac{(\gamma+1)}{2}\alpha'(\tau)\right],\ -\alpha'(\tau), c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right\} .$$

so

$$\frac{d}{d\tau}\left\{\frac{(\gamma+1)}{2}(T-\tau)\chi\right\} = -\frac{(\gamma+1)}{2}\chi + \frac{(\gamma+1)}{2}(T-\tau)\left\{\left[-\alpha'(\tau) - c_o + \frac{(\gamma+1)}{2}\alpha'(\tau)\right.\right.$$
$$\left.\left. - (T-\tau)\frac{(\gamma+1)}{2}\alpha''(\tau)\right] f_x - \alpha''\left[f_u + \frac{(\gamma-1)}{2} f_c\right]\right\}$$

and

$$\frac{d^2}{d\tau^2}\left\{\frac{(\gamma+1)}{2}(T-\tau)\chi\right\} = -(\gamma+1)\left\{\left[\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o - (T-\tau)\frac{(\gamma+1)}{2}\alpha''(\tau)\right] f_x\right.,$$
$$\left. - \alpha''(\tau)\left[f_u + \frac{(\gamma-1)}{2}f_c\right]\right\}$$
$$+ \frac{(\gamma+1)}{2}(T-\tau)\left\{\left[\frac{(\gamma-1)}{2}\alpha''(\tau) + \frac{(\gamma+1)}{2}\alpha''(\tau) - (T-\tau)\frac{(\gamma+1)}{2}\alpha'''(\tau)\right] f_x\right.$$
$$- \alpha'''(\tau)\left[f_u + \frac{(\gamma-1)}{2}f_c\right] + \left[\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o - (T-\tau)\frac{(\gamma+1)}{2}\alpha'(\tau)\right]\frac{\partial F_x}{\partial\tau}$$
$$\left. - \alpha''(\tau)\frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)}{2}f_c\right]\right\} \tag{5.35}$$

$$\frac{d}{d\tau}\left\{\frac{(\gamma-1)}{2}\chi\right\} = \frac{(\gamma-1)}{2}\left\{\left[\frac{(\gamma-1)}{2}\alpha'(\tau) - c_o - (T-\tau)\frac{(\gamma+1)}{2}\alpha''(\tau)\right] f_x\right.$$
$$\left. - \alpha''(\tau)\left[f_u + \frac{(\gamma-1)}{2} f_c\right]\right\} . \tag{5.36}$$

$$\chi_{\alpha'} = -\frac{(\gamma+1)}{2}(T-\tau)f_x - f_u - \frac{(\gamma-1)}{2}f_c \tag{5.37}$$

so

$$\frac{d}{d\tau}\left\{\left[c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}\alpha''(\tau)\right]\chi_{\alpha'}\right\} =$$
$$\frac{d}{d\tau}\left\{\left[c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}\alpha''(\tau)\right]\left[-\frac{(\gamma+1)}{2}(T-\tau)f_x - f_u - \frac{(\gamma-1)}{2}f_c\right]\right\}$$
$$= \left\{-\frac{(\gamma-1)}{2}\alpha''(\tau) - \frac{(\gamma+1)}{2}\alpha''(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha'''(\tau)\right\}\left\{-\frac{(\gamma+1)}{2}(T-\tau)f_x - f_u - \frac{(\gamma-1)}{2}f_c\right\}$$
$$+ \left\{c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T+\tau)\alpha''(\tau)\right\}\left\{\frac{(\gamma+1)}{2}f_x - \frac{(\gamma+1)}{2}(T-\tau)\frac{\partial f_x}{\partial\tau}\right.$$
$$\left. - \frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)}{2}f_c\right]\right\} . \tag{5.38}$$

$$\chi_\alpha = -f_x . \tag{5.39}$$

Using (5.35), (5.36), (5.37), (5.38) and (5.59), (5.34) may be written as

$$-(\gamma+1)\left\{\left[\frac{(\gamma-1)}{2}\alpha' - c_o - (T-\tau)\frac{(\gamma+1)}{2}\alpha''\right] f_x - \alpha''\left[f_u + \frac{(\gamma-1)}{2} f_c\right]\right\}$$
$$+ \frac{(\gamma+1)}{2}(T-\tau)\left\{\left[\frac{(\gamma-1)}{2}\alpha'' + \frac{(\gamma+1)}{2}\alpha'' - (T-\tau)\frac{(\gamma+1)}{2}\alpha'''\right]f_x - \alpha'''\left[f_u + \frac{(\gamma-1)}{2}f_c\right] +\right.$$
$$\left. + \left[\frac{(\gamma-1)}{2}\alpha' - c_o - (T-\tau)\frac{(\gamma+1)}{2}\alpha''\right]\frac{\partial f_x}{\partial\tau} - \alpha''\frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)}{2}f_c\right]\right\}$$

$$+ \frac{(\gamma-1)}{2}\left\{\left[\frac{(\gamma-1)\alpha'}{2} - c_o - (T-\tau)\frac{(\gamma+1)\alpha''}{2}\right] f_x - \alpha''\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\right\}$$

$$-\left\{c_o - \frac{(\gamma-1)\alpha'}{2} + \frac{(\gamma+1)}{2}(T-\tau)\alpha''\right\}\left\{\frac{(\gamma+1)f_x}{2} - \frac{(\gamma+1)(T-\tau)}{2}\frac{\partial f_x}{\partial\tau} - \frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\right\}$$

$$-\left\{c_o - \frac{(\gamma-1)\alpha'}{2} + \frac{(\gamma+1)}{2}(T-\tau)\alpha''\right\}f_x + \frac{d^2F_{\alpha''}}{d\tau^2} - \frac{dF_{\alpha'}}{d\tau} + F_\alpha = 0$$

which simplifies to

$$f_x\left[\frac{(\gamma-1)\alpha'}{2} - c_o - (T-\tau)\frac{(\gamma+1)\alpha''}{2}\right]\left[-\gamma - 1 + \frac{(\gamma-1)}{2} + \frac{(\gamma+1)}{2} + 1\right]$$

$$+ f_x\left[\frac{(\gamma-1)\alpha''}{2} + \frac{(\gamma+1)\alpha''}{2} - (T-\tau)\frac{(\gamma+1)\alpha''}{2}\right]\left[\frac{(\gamma+1)}{2}(T-\tau) - \frac{(\gamma+1)}{2}(T-\tau)\right]$$

$$+\frac{\partial f_x}{\partial\tau}\left[\frac{(\gamma-1)\alpha'}{2} - c_o - (T-\tau)\frac{(\gamma+1)\alpha''}{2}\right]\left[\frac{(\gamma+1)}{2}(T-\tau) - \frac{(\gamma+1)}{2}(T-\tau)\right]$$

$$+\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\left[-\frac{(\gamma-1)\alpha''}{2} - \frac{(\gamma+1)\alpha''}{2} + \frac{(\gamma+1)}{2}(T-\tau)\alpha''' + (\gamma+1)\alpha''\right.$$

$$\left. - \frac{(\gamma+1)}{2}(T-\tau)\alpha''' - \frac{(\gamma-1)}{2}\alpha''\right]$$

$$+\frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\left[-\frac{(\gamma+1)}{2}(T-\tau)\alpha'' + c_o - \frac{(\gamma-1)\alpha'}{2} + \frac{(\gamma+1)}{2}(T-\tau)\alpha''\right]$$

$$+\frac{d^2F_{\alpha''}}{d\tau^2} - \frac{dF_{\alpha'}}{d\tau} + F_\alpha = 0$$

or

$$\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\alpha''\left[\gamma+1 - \frac{(\gamma-1)}{2} - \frac{(\gamma+1)}{2} - \frac{(\gamma-1)}{2}\right] + \frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\times$$

$$\times\left[c_o - \frac{(\gamma-1)\alpha'}{2}\right]$$

$$+\frac{d^2F_{\alpha''}}{d\tau^2} - \frac{dF_{\alpha'}}{d\tau} + F_\alpha = 0$$

$$\left[f_u + \frac{(\gamma-1)f_c}{2}\right]\left[\frac{(3-\gamma)\alpha''}{2}\right] + \left[c_o - \frac{(\gamma-1)\alpha''}{2}\right]\frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)f_c}{2}\right]$$

$$+\frac{d^2F_{\alpha''}}{d\tau^2} - \frac{dF_{\alpha'}}{d\tau} + F_\alpha = 0 .$$

Finally this becomes

$$\frac{d^2F_{\alpha''}}{d\tau^2} - \frac{dF_{\alpha'}}{d\tau} + F_\alpha = \frac{(\gamma-3)}{2}\alpha''\left[f_u + \frac{(\gamma-1)f_c}{2}\right] - \left[c_o - \frac{(\gamma-1)\alpha''}{2}\right]\frac{\partial}{\partial\tau}\left[f_u + \frac{(\gamma-1)f_c}{2}\right]$$

which is identical to (5.32)

The boundary conditions in Chapter Four are

$$g_{\alpha''} = 0 \quad , \quad \tau = T \quad , \tag{5.40}$$

$$g_{\alpha'} - \frac{d}{d\tau}g_{\alpha''} = 0 \quad , \quad \tau = T \quad . \tag{5.41}$$

From equation (4.31)

$$g(\tau\ ,\alpha(\tau),\alpha'(\tau)\ \alpha''(\tau)\ ) = \left\{ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau) + \frac{(\gamma+1)}{2}(T-\tau)\alpha''(\tau) \right\} \chi(\tau,\alpha(\tau),\alpha'(\tau)\ )$$
$$+ F(\alpha(\tau),\alpha'(\tau),\alpha\ ''(\tau)\ ) \qquad ,$$

and, from (4.30),

$$\chi \equiv f\left\{ - \alpha(\tau) + (T-\tau)\left[c_o - \frac{(\gamma+1)}{2}\alpha'\right], - \alpha'(\tau),\ c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right\}.$$

$$g_{\alpha'} = - \frac{(\gamma-1)}{2}f + \left\{ c_o - \frac{(\gamma-1)}{2}\alpha' + \frac{(\gamma+1)}{2}(T-\tau)\alpha'' \right\} f_{\alpha'} + F_{\alpha'} \qquad (5.42)$$

$$f_{\alpha'} = \left\{ -\frac{(\gamma+1)}{2}(T-\tau)\ f_x - f_u - \frac{(\gamma-1)}{2}f_c \right\} \qquad (5.43)$$

$$g_{\alpha''} = \frac{(\gamma+1)}{2}(T-\tau)\ f\ + F_{\alpha''}$$

$$\frac{d}{d\tau}\ g_{\alpha''}\ = -\frac{(\gamma+1)f}{2}\ + \frac{(\gamma+1)}{2}(T-\tau)\ f_{\alpha'}\ + \frac{d}{d\tau}\ F_{\alpha''} \quad . \qquad (5.44)$$

The left hand side of (5.41) may be written down from (5.42), (5.43) and (5.44)

$$g_{\alpha'}\ - \frac{d}{d\tau}\ g_{\alpha''}\ = \left\{ c_o - \frac{(\gamma-1)}{2}\alpha' + \frac{(\gamma+1)}{2}(T-\tau)\alpha'' \right\} \left\{ - \frac{(\gamma+1)}{2}(T-\tau)f_x - f_u\ \ \frac{(\gamma-1)}{2}f_c \right\} +$$
$$+ F_{\alpha'} + f - \frac{(\gamma+1)}{2}(T-\tau)\ \times$$
$$\times \left\{ \left[-c_o + \frac{(\gamma-1)}{2}\alpha' - (T-\tau)\frac{(\gamma+1)}{2}\alpha'' \right] f_x - \alpha'' f_u - \frac{(\gamma-1)}{2}\alpha'' f_c \right\}$$
$$- \frac{dF}{d\tau}\ \alpha'' \qquad , \qquad \tau\ = T$$

$$= F_{\alpha'} - \frac{d}{d\tau}\ F_{\alpha''} - \left(c_o - \frac{(\gamma-1)}{2}\alpha'(\tau)\right)\left(f_u + \frac{(\gamma-1)}{2}f_c\right) + f, \tau\ = T$$

which is the same as the boundary condition (5.33).

CHAPTER SIX

CHAPTER SIX

The Problem of Minimum Drag on a Body with Axial Symmetry in Stokes' Flow.

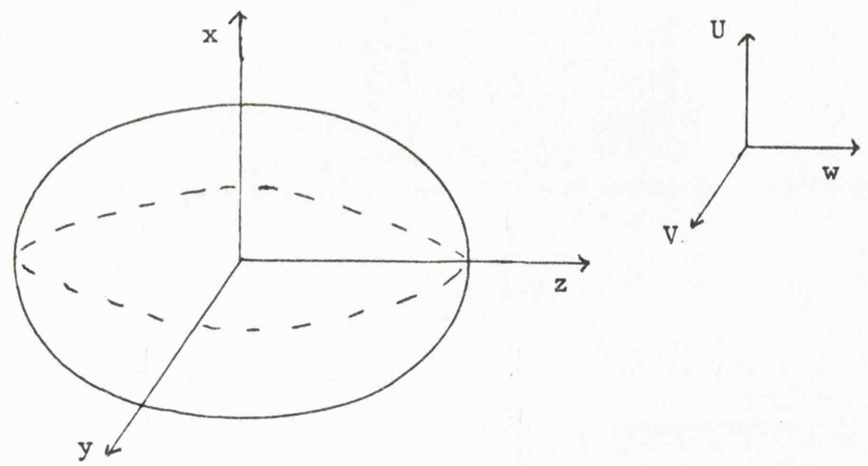


Figure 6.1

Consider an axially symmetric body with its axis of symmetry in the z direction immersed in a stream of viscous liquid in which the flow at infinity is of magnitude W and in the direction Oz. The liquid is assumed to be moving sufficiently slowly at infinity so that Stokes' approximation is valid and the equations of motion are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 U = 0 \quad , \tag{6.1}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,\nabla^2\,V = 0 \quad , \tag{6.2}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2\,w = 0 \quad ; \tag{6.3}$$

and the equation of continuity is

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial w}{\partial z} = 0 \; . \tag{6.4}$$

The problem posed is that of finding the shape of the axially symmetric body of either given internal volume or given surface area which provides the minimum resistance or drag. It is convenient to use cylindrical polar coordinates, writing $x = r\cos\theta$, $y = r\sin\theta$ with $u(r,z)$ as the radial velocity. The equations of motion can then be written in the form,

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) = 0 \quad , \tag{6.4}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = 0 \quad , \tag{6.5}$$

and the equation of continuity as

$$\frac{\partial}{\partial z}(rw) + \frac{\partial}{\partial r}(ru) = 0 \quad . \tag{6.6}$$

The vorticity vector η is given by $\eta = \nabla \times \underline{V}$, where $\underline{V}$ is the velocity vector.

$$\eta = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u & o & w \end{vmatrix}$$

$$= \hat{\theta}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \quad . \tag{6.7}$$

Using (6.7) in (6.4) gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial}{\partial z}\left(\eta + \frac{\partial w}{\partial r}\right) + \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right\} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial \eta}{\partial z} + \frac{\partial^2 w}{\partial z \partial r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right\} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(ur)\right)\right\} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial z}(wr) + \frac{1}{r}\frac{\partial}{\partial r}(ur)\right)\right\} = 0$$

and using the equation of continuity this becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\frac{\partial \eta}{\partial z} = 0 \quad . \tag{6.8}$$

Similarly using (6.7) in (6.5), and the equation of continuity, gives

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial \eta}{\partial r} + \nu\frac{\eta}{r} = 0 \quad . \tag{6.9}$$

To minimise the drag on the body consider the minimisation of the rate of dissipation of energy, I, within the liquid, where

$$I = \nu \iiint_D \left\{ 2U_x^2 + 2V_y^2 + 2w_z^2 + (w_y + V_z)^2 + (V_z + w_x)^2 + (V_x + U_y)^2 \right\} dxdydz$$

Subtracting from this the expression $2\nu \iiint_D \{ U_x + V_y + w_z \}^2 \; dxdydz$,

which is zero when there is no variation in density, gives

$$I = \nu \iiint_D \left\{ (w_y - V_z)^2 + (U_z - w_x)^2 + (V_x - U_y)^2 \right\} dxdydz$$

$$- 4\nu \iiint_D \left\{ V_y w_z - V_z w_y + w_z U_x + w_x U_z + U_x V_y - U_y V_x \right\} dx\,dy\,dz.$$

The first term is the square of the components of the vorticity function

$$\eta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U & V & w \end{vmatrix}$$

and after partial integration the second term becomes

$$- 4\nu \iiint_D \left\{ \frac{\partial}{\partial y}(Vw_z) - \frac{\partial}{\partial z}(Vw_y) + \frac{\partial}{\partial x}(Uw_z) - \frac{\partial}{\partial z}(Uw_x) + \frac{\partial}{\partial x}(UV_y) - \frac{\partial}{\partial y}(UV_x) \right\} dx\,dy\,dz$$

which when the divergence theorem is applied is zero as U and V are zero on the body and at infinity. I may therefore be written as

$$I = \iint_S \nu\, \eta^2 r \; dz\, dr, \qquad (6.10)$$

where S is the domain in the (z,r) plane exterior to the body and the problem is then the determination of C_1 so that I is minimised, where C_1 is the curve of the body in the (r,z) plane.

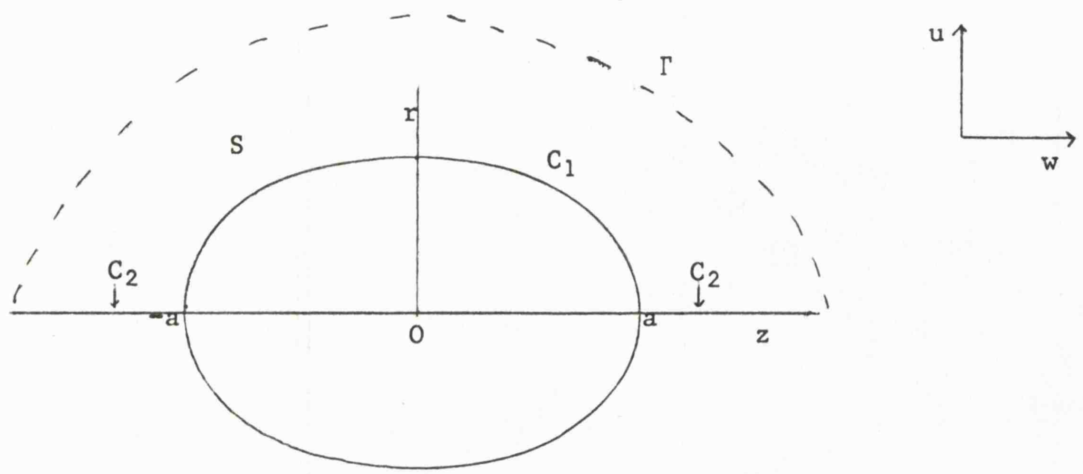


Figure 6.2.

Γ is the boundary at infinity and C_2 is the line, exterior to the body, $r = 0$.

It is assumed that the end point at $-a$ and a are fixed. To ensure that the problem is not trivial an additional constraint is postulated. This constraint is that either the internal volume of the axially symmetric body or the arc length of the body is prescribed. If the shape of the body is given by

$$z = \sigma \quad , \; r = \alpha(\sigma) \tag{6.11}$$

then the volume of the body is

$$\pi \int_{-a}^{a} \alpha^2(\sigma) \; d\sigma$$

and the arc length is

$$\int_{-a}^{a} \{1 + \alpha'^2(\sigma)\}^{1/2} \; d\sigma \; .$$

The following performance criterion is now set up:

$$\begin{aligned} J = \iint_S \Big\{ & \nu r\eta^2 + \lambda_1 (\tfrac{1}{\rho} p_r - \nu\eta_z) + \lambda_2 \; (\tfrac{1}{\rho} p_z + \nu\eta_r + \nu \tfrac{\eta}{r}) \\ & + \lambda_3 (\eta - u_z + w_r) + \lambda_4 (u_r + \tfrac{u}{r} + w_z) \Big\} \; dz \, dr \\ & + \int_{-a}^{a} f \; (\alpha,\alpha',\sigma) d\sigma \; . \end{aligned} \tag{6.12}$$

where λ_1, λ_2, λ_3 and λ_4 are Lagrange multipliers depending on r and z and contain the r contribution to the volume element $r dz dr$. Put

$$\begin{aligned} \chi = \; & \nu r \, \eta^2 + \lambda_1 (\tfrac{1}{\rho} p_r - \nu\eta_z) + \lambda_2 (\tfrac{1}{\rho} p_z + \nu\eta_r + \nu \tfrac{\eta}{r}) \\ & + \lambda_3 (\eta - u_z + w_r) + \lambda_4 (u_r + \tfrac{u}{r} + w_z) \; . \end{aligned} \tag{6.13}$$

then,

$$J = \iint_S \chi(z,r,u,w,\eta,p) \; dz \; dr + \int_{-a}^{a} f(\alpha,\alpha',\sigma) d\sigma \; . \tag{6.14}$$

The minimisation of J is now considered. The Gelfand - Fomin theorem is used to find δJ, that is the variation in J caused by a variation in the position of the curve C_1 .

$$\begin{aligned}\delta J = \int_S\!\!\int \Big\{ & \overline{\delta u}\left[\chi_u - \frac{\partial}{\partial z}\chi_{u_z} - \frac{\partial}{\partial r}\chi_{u_r}\right] + \overline{\delta w}\left[\chi_w - \frac{\partial\chi}{\partial z}{}_{w_z} - \frac{\partial}{\partial r}\chi_{w_r}\right] \\ & + \overline{\delta p}\left[\chi_p - \frac{\partial}{\partial z}\chi_{p_z} - \frac{\partial}{\partial r}\chi_{p_r}\right] + \overline{\delta\eta}\left[\chi_\eta - \frac{\partial}{\partial z}\chi_{\eta_z} - \frac{\partial}{\partial r}\chi_{\eta_r}\right]\Big\}\, dz\, dr \\ & + \int_S\!\!\int \Big\{ \frac{\partial}{\partial z}\left[\chi\,\delta z + \overline{\delta u}\chi_{u_z} + \overline{\delta w}\chi_{w_z} + \overline{\delta p}\chi_{p_z} + \overline{\delta\eta}\,\chi_{\eta_z}\right] \\ & + \frac{\partial}{\partial r}\left[\chi\delta r + \overline{\delta u}\chi_{u_r} + \overline{\delta w}\chi_{w_r} + \overline{\delta p}\chi_{p_r} + \overline{\delta\eta}\chi_{\eta_r}\right]\Big\}\, dz\, dr \\ & + \int_{-\alpha}^{\alpha} \Big\{ f_\alpha \delta\alpha + f_{\alpha'}\delta\alpha' \Big\}\, d\sigma\,, \qquad (6.15)\end{aligned}$$

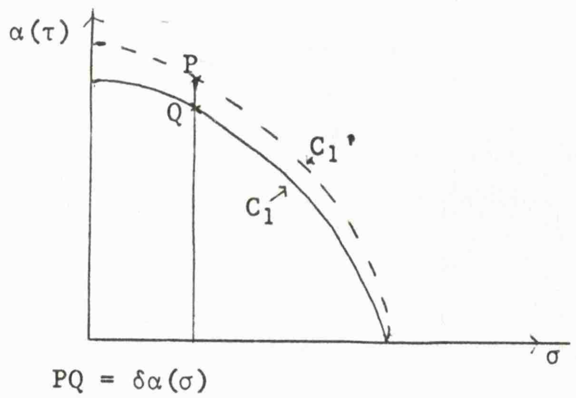
where δu, δw, δp and $\delta\eta$, the increments in u, w, p and η are related to $\overline{\delta u}$, $\overline{\delta w}$, $\overline{\delta p}$ and $\overline{\delta\eta}$ by

$$\delta u = \overline{\delta u} + \frac{\partial u}{\partial z}\delta z + \frac{\partial u}{\partial r}\delta r\,, \quad \delta w = \overline{\delta w} + \frac{\partial w}{\partial z}\delta z + \frac{\partial w}{\partial r}\delta r\,, \qquad (6.16)$$

$$\delta p = \overline{\delta p} + \frac{\partial p}{\partial z}\delta z + \frac{\partial p}{\partial r}\delta r\,, \quad \delta\eta = \overline{\delta\eta} + \frac{\partial \eta}{\partial z}\delta z + \frac{\partial \eta}{\partial r}\delta r\,.$$

Using Stokes' theorem in two dimensions on the second term of the right hand side of (6.15) gives

$$\begin{aligned}\delta J = \int_S\!\!\int \Big\{ & \overline{\delta u}\left[\chi_u - \frac{\partial}{\partial z}\chi_{u_z} - \frac{\partial}{\partial r}\chi_{u_r}\right] + \overline{\delta w}\left[\chi_w - \frac{\partial}{\partial z}\chi_{w_z} - \frac{\partial}{\partial r}\chi_{w_r}\right] \\ & + \overline{\delta p}\left[\chi_p - \frac{\partial}{\partial z}\chi_{p_z} - \frac{\partial}{\partial r}\chi_{p_r}\right] + \overline{\delta\eta}\left[\chi_\eta - \frac{\partial}{\partial z}\chi_{\eta_z} - \frac{\partial}{\partial r}\chi_{\eta_r}\right]\Big\}\, dz\, dr \\ & + \int_{\Gamma\cup C_1\cup C_2} \Big\{ \left[\chi\delta z + \overline{\delta u}\chi_{u_z} + \overline{\delta w}\chi_{w_z} + \overline{\delta p}\chi_{p_z} + \overline{\delta\eta}\chi_{\eta_z}\right] dr \;+ \\ & + \left[\chi\delta r + \overline{\delta u}\chi_{u_r} + \overline{\delta w}\chi_{w_r} + \overline{\delta p}\chi_{p_r} + \overline{\delta\eta}\chi_{\eta_r}\right]\Big\}\, dz \\ & + \int_{-\alpha}^{\alpha} \Big\{ f_\alpha\, \delta\alpha + f_{\alpha'}\, \delta\alpha' \Big\}\, d\sigma. \qquad (6.17)\end{aligned}$$



PQ = $\delta\alpha(\sigma)$

Figure 6.3

On C_1 $\delta r = \delta\alpha(\sigma)$, $\delta z = \delta\sigma$ and $\delta\sigma = 0$. u and w are zero on the body at all times and so δu and δw are also zero.

Hence, using (6.16)

$$\overline{\delta u} = -\frac{\partial u}{\partial \alpha}\delta\alpha \quad , \quad \overline{\delta w} = -\frac{\partial w}{\partial \alpha}\delta\alpha \quad . \tag{6.18}$$

Integrating $f_{\alpha'}\,\delta\alpha'$ by parts gives

$$f_{\alpha'}\,\delta\alpha \Big|_{-a}^{a} - \int_{-a}^{a} \delta\alpha \frac{d}{d\sigma} f_{\alpha'}\,d\sigma \ , \tag{6.19}$$

and the first term disappears since $\delta\alpha$ is zero at $-a$ and a. So the total integral over C_1 may be written as

$$\int_{-a}^{a} \Big\{ \Big[-u_\alpha \delta\alpha \chi_{u_z} - w_\alpha \delta\alpha \chi_{w_z} + \overline{\delta p}\chi_{p_z} + \overline{\delta\eta}\chi_{\eta_z} \Big] d\alpha$$
$$- \Big[-f_\alpha \delta\alpha + \delta\alpha \frac{df_{\alpha'}}{d\sigma} + \chi\delta\alpha - u_\alpha \delta\alpha \chi_{u_r} - w_\alpha \delta\alpha \chi_{w_r} + \overline{\delta p}\chi_{p_r} + \overline{\delta\eta}\chi_{\eta_r} \Big] d\sigma \Big\} \tag{6.20}$$

On C_2, which is the line $r = 0$, dr and δr are zero and the condition $u = 0$ must be satisfied so δu is zero. The contribution to δJ from the integration along C_2 becomes

$$\int_{C_2} - \Big\{ \overline{\delta u}\chi_{u_r} + \overline{\delta w}\chi_{w_r} + \overline{\delta p}\chi_{p_r} + \overline{\delta\eta}\chi_{\eta_r} \Big\} dz \quad . \tag{6.21}$$

On Γ, which lies at infinity, the conditions are $u = 0$, $w = W$, hence δu and δw are zero and at infinity δr and δz may be taken to be zero so the integration along Γ becomes

$$\int_{\Gamma} - \Big\{ \Big[\delta p\chi_{p_z} + \delta\eta\chi_{\eta_z} \Big] dr - \Big[\delta p\chi_{p_r} + \delta\eta\,\chi_{\eta_r} \Big] \Big\} dz \quad . \tag{6.22}$$

Using (6.19) to (6.22) δJ may be written as

$$\delta J = \iint_S \Big\{ \overline{\delta u} \Big[\chi_u - \frac{\partial}{\partial z}\chi_{u_z} - \frac{\partial}{\partial r}\chi_{u_r} \Big] + \overline{\delta w} \Big[\chi_w - \frac{\partial}{\partial z}\chi_{w_z} - \frac{\partial}{\partial r}\chi_{w_r} \Big]$$
$$+ \overline{\delta p} \Big[\chi_p - \frac{\partial}{\partial z}\chi_{p_z} - \frac{\partial}{\partial r}\chi_{p_r} \Big] + \overline{\delta\eta} \Big[\chi_\eta - \frac{\partial}{\partial z}\chi_{\eta_z} - \frac{\partial}{\partial r}\chi_{\eta_r} \Big] \Big\} dz\,dr$$
$$+ \int_{-a}^{a} \Big\{ - \delta\alpha \Big[(u_\alpha \chi_{u_z} + w_\alpha \chi_{w_z}) \, d\alpha$$
$$+ \Big(- f_\alpha + \frac{df_{\alpha'}}{d\sigma} - \chi + u_\alpha \chi_{u_r} + w_\alpha \chi_{w_r} \Big) d\sigma \Big]$$
$$+ \overline{\delta p} \Big[\chi_{p_z} d\alpha - \chi_{p_r} d\sigma \Big] + \overline{\delta\eta} \Big[\chi_{\eta_z} d\alpha - \chi_{\eta_r} d\sigma \Big] \Big\}$$

$$- \int_{C_2} \left\{ \overline{\delta u} \chi_{u_r} + \overline{\delta w} \chi_{w_r} + \overline{\delta p} \chi_{p_r} + \overline{\delta \eta} \chi_{\eta_r} \right\} dz$$

$$- \int_{\Gamma} \left\{ \delta p \left[\chi_{p_z} dr - \chi_{p_r} dz \right] + \delta \eta \left[\chi_{\eta_z} dr - \chi_{\eta_r} dz \right] \right\} . \qquad (6.23)$$

The performance criterion J is minimised when δJ is zero and so for a minimum:

$$\chi_u - \frac{\partial}{\partial z} \chi_{u_z} - \frac{\partial}{\partial r} \chi_{u_r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.24)$$

$$\chi_w - \frac{\partial}{\partial z} \chi_{w_z} - \frac{\partial}{\partial r} \chi_{w_r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.25)$$

$$\chi_p - \frac{\partial}{\partial z} \chi_{p_z} - \frac{\partial}{\partial r} \chi_{p_r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.26)$$

$$\chi_\eta - \frac{\partial}{\partial z} \chi_{\eta_z} - \frac{\partial}{\partial r} \chi_{\eta_r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.27)$$

$$\left[u_\alpha \chi_{u_z} + w_\alpha \chi_{w_z} \right] d\alpha + \left[\chi - u_\alpha \chi_{u_r} - w_\alpha \chi_{w_r} + f_\alpha - \frac{d}{d\sigma} f_{\alpha'} \right] d\sigma = 0, (z,r) \in C_1 \qquad (6.28)$$

$$\chi_{p_z} d\alpha - \chi_{p_r} d\sigma = 0 \quad , \quad (z,r) \in C_1 , \qquad (6.29)$$

$$\chi_{\eta_z} d\alpha - \chi_{\eta_r} d\sigma = 0 \quad , \quad (z,r) \in C_1 , \qquad (6.30)$$

$$\chi_{u_r} = 0 \quad , \quad (z,r) \in C_2 , \qquad (6.31)$$

$$\chi_{w_r} = 0 \quad , \quad (z,r) \in C_2 , \qquad (6.32)$$

$$\chi_{p_r} = 0 \quad , \quad (z,r) \in C_2 , \qquad (6.33)$$

$$\chi_{\eta_r} = 0 \quad , \quad (z,r) \in C_2 , \qquad (6.34)$$

$$\chi_{p_r} dr - \chi_{p_r} dz = 0 \quad , \quad (z,r) \in \Gamma \ , \qquad (6.35)$$

$$\chi_{\eta_z} dr \quad \chi_{\eta_r} dz = 0 \quad , \quad (z,r) \in \Gamma \ . \qquad (6.36)$$

Substituting for χ from (6.13) become:

$$\frac{\partial \lambda_3}{\partial z} - \frac{\partial \lambda_4}{\partial r} + \frac{\lambda_4}{r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.37)$$

$$\frac{\partial \lambda_4}{\partial z} + \frac{\partial \lambda_3}{\partial r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.38)$$

$$\frac{\partial \lambda_2}{\partial z} + \frac{\partial \lambda_1}{\partial r} = 0 \quad , \quad (z,r) \in S \ , \qquad (6.39)$$

$$2\nu r \eta + \lambda_3 + \nu \left(\frac{\partial \lambda_1}{\partial z} - \frac{\partial \lambda_2}{\partial r} + \frac{\lambda_2}{r} \right) = 0 \ , \ (z,r) \in S \ , \qquad (6.40)$$

$$\left[-\lambda_3 u_\alpha + \lambda_4 w_\alpha \right] \alpha'(\sigma) + \nu r \eta^2 - \lambda_4 u_\alpha - \lambda_3 w_\alpha + f_\alpha - \frac{d}{d\sigma} f_{\alpha'} = 0, \ (z,r) \in C_1 \ ,$$

$$(6.41)$$

where $d\alpha$ has been replaced by $\frac{d\alpha}{d\sigma}d\sigma$ and the equation has been divided through by $d\sigma$;

$$\lambda_2 dr - \lambda_1 \; dz = 0 \qquad , \qquad (z,r) \in C_1 \; , \; \Gamma \; , \qquad (6.42)$$

$$\lambda_1 dr - \lambda_2 \; dz = 0 \qquad , \qquad (z,r) \in C_1 \; , \; \Gamma \; , \qquad (6.43)$$

$$\lambda_4 = 0 \qquad , \qquad (z,r) \in C_2 \; , \qquad (6.44)$$

$$\lambda_3 = 0 \qquad , \qquad (z,r) \in C_2 \; , \qquad (6.45)$$

$$\lambda_1 = 0 \qquad , \qquad (z,r) \in C_2 \; , \qquad (6.46)$$

$$\lambda_2 = 0 \qquad , \qquad (z,r) \in C_2 \; , \qquad (6.47)$$

One method of resolving the above problem is as follows. The known stream function ψ, where ψ is defined, from (6.6), by

$$\frac{\partial\psi}{\partial r} = - wr \quad , \quad \frac{\partial\psi}{\partial z} = ur$$

and vorticity function η for the flow past a sphere can be used to calculate u_α, w_α, λ_3 and λ_4. (The methods for these calculations are the same as those used in Chapter 7 for different values of ψ and η (7.21) to (7.31) .) When these are substituted into the transversality condition (6.41) the resulting differential equation $\alpha(\sigma)$ could be solved numerically and the subsequent value for $\alpha(\sigma)$ used as the initial value in the next step on an iteration method. This has not been successfully pursued.

CHAPTER SEVEN

CHAPTER SEVEN

A Study near the Leading Point of the Shape of the Axially Symmetric Body of Minimum Drag in Stokes' Flow.

The equations of the system are:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} - \nu\frac{\partial \eta}{\partial z} = 0 \quad , \tag{7.1}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial \eta}{\partial r} + \frac{\nu\eta}{r} = 0 \quad , \tag{7.2}$$

$$\eta - \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \quad , \tag{7.3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad . \tag{7.4}$$

The elimination of p from (7.1) and (7.2) leads to

$$\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\eta}{r}\right) = 0 \quad . \tag{7.5}$$

It can be seen from (7.4) that the Stokes' stream function, ψ, can be defined by

$$wr = -\frac{\partial \psi}{\partial r} \quad , \quad ur = \frac{\partial \psi}{\partial z} \tag{7.6}$$

and so equation (7.3) can be written as

$$\eta = \frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \psi}{\partial r} \quad . \tag{7.7}$$

Consider first the leading point of the body, $z = -a$, $r = 0$.

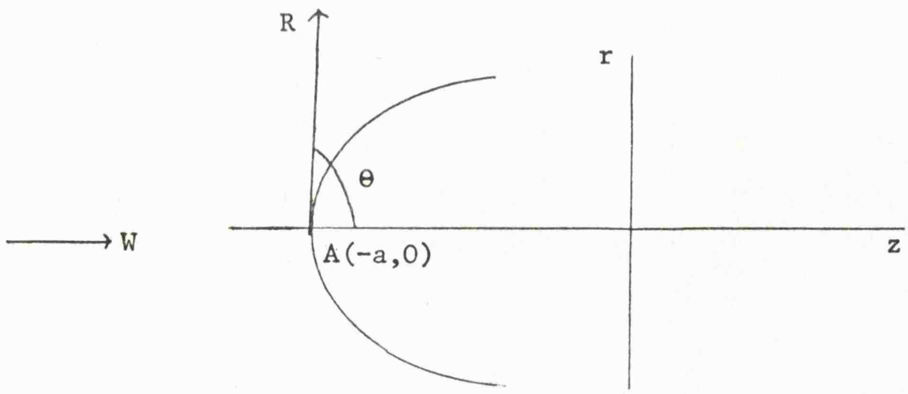


Figure 7.1

It will be assumed that in the neighbourhood of $z = -a$, $r = 0$ the body has a conical shape with a semi-angle θ_0 The coordinates are transformed with

$$z + a = R\cos\theta \quad , \quad r = R\sin\theta \quad . \tag{7.8}$$

so equation (7.7) becomes

$$\eta = \frac{1}{R\sin\theta}\left\{\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{R}\frac{\partial\psi}{\partial\theta}\right\} , \tag{7.9}$$

and equation (7.5) becomes

$$\frac{\partial^2\eta}{\partial R^2} + \frac{1}{R}\frac{\partial\eta}{\partial R} + \frac{1}{R^2}\frac{\partial^2\eta}{\partial\theta^2} + \left(\sin\theta\frac{\partial}{\partial R} + \frac{\cos\theta}{R}\frac{\partial}{\partial\theta}\right)\frac{\eta}{R\sin\theta} = 0 \ . \tag{7.10}$$

The flow in the conical region must satisfy

$$\psi = 0 \quad \text{for } \theta = \pi \quad ; \tag{7.11}$$

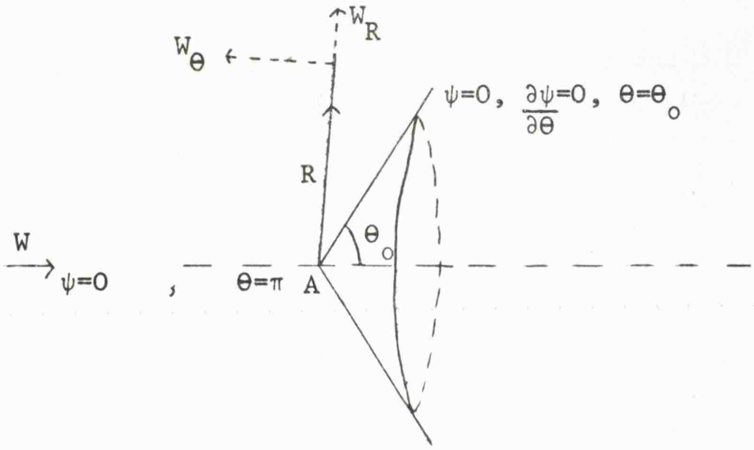


Figure 7.2

and in addition since the radial and transverse components of velocity are

$$W_R = \frac{-1}{R^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad , \quad W_\theta = \frac{1}{R\sin\theta}\frac{\partial\psi}{\partial\theta} \quad ,$$

it follows that the viscous conditions

$$W_R = 0 \quad , \quad W_\theta = 0 \qquad \text{on } \theta = \theta_0$$

lead to

$$\psi = 0 \ , \quad \frac{\partial\psi}{\partial\theta} = 0 \quad , \quad \theta = \theta_0 \quad , \tag{7.12}$$

where $\theta = \theta_0$ is the angle of the conical body near A. Solutions for (7.9) and (7.10) must now be determined, satisfying conditions (7.11) and (7.12) for sufficiently small R.

A solution for η of (7.10) is sought which depends on θ only, and for small R the function $\eta = \eta(\theta)$ will satisfy

$$\frac{d^2\eta}{d\theta^2} + \cot\theta \frac{d\eta}{d\theta} - \eta \operatorname{cosec}^2\theta = 0,$$

$$\frac{d}{d\theta}\left\{ \frac{d\eta}{d\theta} + \eta\cot\theta \right\} = 0 ,$$

$$\frac{d\eta}{d\theta} + \eta\cot\theta = - C ,$$

$$\frac{d}{d\theta}\left\{ \eta\sin\theta \right\} = - C\sin\theta ,$$

$$\eta(\theta) = \frac{C\cos\theta + D}{\sin\theta} , \qquad (7.13)$$

where C and D are arbitrary constants. It is clear from (7.9) and (7.13) that ψ will be of the form

$$\psi = R^3 f(\theta)$$

and $f(\theta)$ will satisfy

$$\frac{d^2 f}{d\theta^2} - \cot\theta \frac{df}{d\theta} + 6f = C\cos\theta + D . \qquad (7.14)$$

A particular integral for f is $\frac{1}{6}C\cos\theta + \frac{1}{6}D$. To find the complementary function put $f(\theta) = \sin\theta\, F(\theta)$, then (7.14) becomes

$$F''(\theta) + \cot\theta\, F'(\theta) + F(\theta)\left\{ 6 - \frac{1}{\sin^2\theta} \right\} = 0.$$

This is the differential equation satisfied by the Associated Legendre polynomial $P_2^1(\cos\theta)$, hence

$$F(\theta) = 3\sin\theta\cos\theta$$

$$f(\theta) = 3\sin^2\theta\cos\theta$$

the second solution possessing a log singularity at $\theta = \pi$. The complete solution for (7.14) is thus

$$f(\theta) = \tfrac{1}{6}\left\{C\cos\theta + D\right\} + A\sin^2\theta\cos\theta \quad ,$$

where A is an arbitrary constant. ψ therefore may be written as

$$\psi = R^3\left\{\tfrac{1}{6}\left[C\cos\theta + D\right] + A\sin^2\theta\cos\theta\right\} .$$

To satisfy $\psi = 0$ on $\theta = \pi$ D must equal C so

$$\psi = R^3\left\{\tfrac{1}{6}C(1+\cos\theta) + A\sin^2\theta\cos\theta\right\} .$$

For ψ to satisfy conditions (7.12)

$$\left.\begin{aligned}&\tfrac{1}{6}C(1+\cos\theta_0) + A\sin^2\theta_0\cos\theta_0 = 0\\ &\tfrac{1}{6}C(1-\sin\theta_0) + A(-\sin^3\theta_0 + 2\cos^2\theta_0\sin\theta_0) = 0\end{aligned}\right\}$$

and these conditions imply that

$$(1+\cos\theta_0)(-\sin^3\theta_0 + 2\cos^2\theta_0\sin\theta_0) + \cos\theta_0\sin^3\theta_0 = 0 \quad ,$$

$$\sin\theta_0(1+\cos\theta_0)\left\{-\sin^2\theta_0 + 2\cos^2\theta_0 + \cos\theta_0(1-\cos\theta_0)\right\} = 0,$$

$$\sin\theta_0(1+\cos\theta_0)\left\{2\cos^2\theta_0 + \cos\theta_0 - 1\right\} = 0 ,$$

$$\sin\theta_0(1+\cos\theta_0)(2\cos\theta_0 - 1)(\cos\theta_0 + 1) = 0 \quad .$$

The solutions $\sin\theta_0 = 0$ and $\cos\theta_0 = -1$ are clearly not acceptable and the required solution is

$$\cos\theta_0 = \tfrac{1}{2} \text{ , or } \theta_0 = \pi/3 .$$

Thus the cone at A has a semi-vertical angle of 60°. This agrees with a result of Sir James Lighthill quoted, without reference, by Pironneau[9]. Using this value for θ_0 in conditions (7.12) gives a value for A of $-\tfrac{2}{3}C$, hence

$$\begin{aligned}\psi &= \tfrac{1}{6}CR^3\left\{(1+\cos\theta) - 4\sin^2\theta\cos\theta\right\} ,\\ &= \tfrac{1}{6}CR^3(1+\cos\theta)\left\{1 - 4\cos\theta(1-\cos\theta)\right\}\\ &= \tfrac{1}{6}CR^3(1+\cos\theta)(1-2\cos\theta)^2 \quad , \pi/3 \leqslant \theta \leqslant \pi . \qquad (7.15)\end{aligned}$$

As C is equal to D from (7.13) η may be written as

$$\eta = C \frac{1 + \cos\theta}{\sin\theta} \quad , \qquad \pi/3 \lesssim \theta \lesssim \pi \; . \tag{7.16}$$

and it is noted that $\eta \to 0$ as $\theta \to \pi$.

A similar study will now be made of the Lagrange multipliers near the leading point. The equations governing the Lagrange multipliers are (6.37) to (6.40), namely,

$$\frac{\partial\lambda_3}{\partial z} - \frac{\partial\lambda_4}{\partial r} + \frac{\lambda_4}{r} = 0 \quad , \tag{7.17}$$

$$\frac{\partial\lambda_4}{\partial z} + \frac{\partial\lambda_3}{\partial r} = 0 \quad , \tag{7.18}$$

$$\frac{\partial\lambda_2}{\partial z} + \frac{\partial\lambda_1}{\partial r} = 0 \quad , \tag{7.19}$$

$$2\nu r\eta + \lambda_3 + \nu\left(\frac{\partial\lambda_1}{\partial z} - \frac{\partial\lambda_2}{\partial r} + \frac{\lambda_2}{r}\right) = 0 \; . \tag{7.20}$$

It will now be established that for the present problem $\lambda_1 = 0$, $\lambda_2 = 0$. In the first place it is noted that when λ_1 and λ_2 vanish equation (7.20) gives

$$\lambda_3 = -2\nu r\eta \quad . \tag{7.21}$$

Eliminating λ_4 between equations (7.17) and (7.18) gives

$$\frac{\partial^2\lambda_3}{\partial z^2} + \frac{\partial^2\lambda_3}{\partial r^2} - \frac{1}{r}\frac{\partial\lambda_3}{\partial r} = 0 \tag{7.22}$$

and when $-2\nu r\eta$ is substituted for λ_3 in (7.22) the resulting equation is

$$\frac{\partial^2\eta}{\partial z^2} + \frac{\partial^2\eta}{\partial r^2} + \frac{1}{r}\frac{\partial\eta}{\partial r} - \frac{\eta}{r^2} = 0$$

which is exactly the same equation in η as that found from the state equations [(7.5)] . This proves (7.21) coupled with $\lambda_1 = 0$, $\lambda_2 = 0$ is a consistent solution. Since λ_1 and λ_2 are zero on the boundaries, [equations (6.42), (6.43), (6.46), (6.47)] , this solution is also consistent with the boundary conditions. When $\lambda_3 = -2\nu r\eta$, equation (7.20) becomes

$$\frac{\partial\lambda_1}{\partial z} - \frac{\partial\lambda_2}{\partial r} + \frac{\lambda_2}{r} = 0 \; .$$

In order to establish the uniqueness of the solution for λ_3 consider

$$\frac{\partial \lambda_2}{\partial z} + \frac{\partial \lambda_1}{\partial r} = 0 \quad ,$$

$$\frac{\partial \lambda_1}{\partial z} - \frac{\partial \lambda_2}{\partial r} + \frac{\lambda_2}{r} = 0 \quad .$$

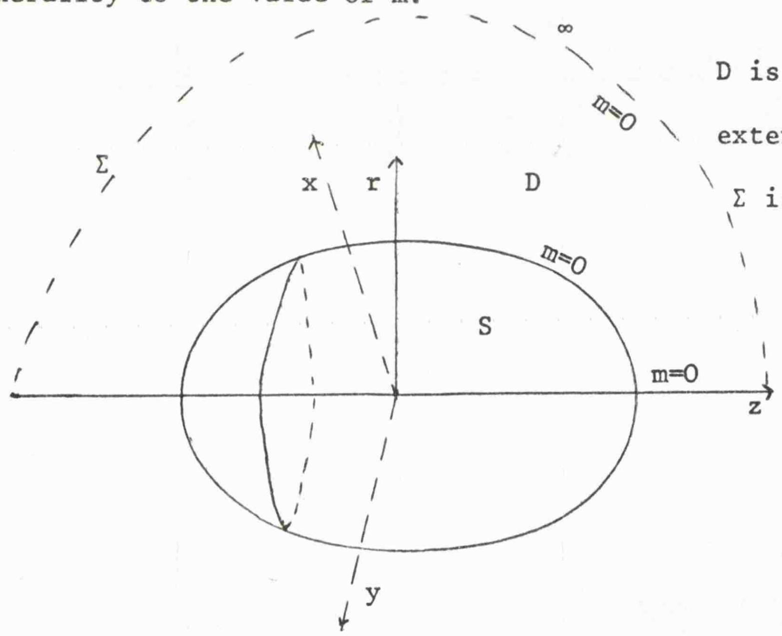
From the first may be written

$$\lambda_1 = \frac{\partial m}{\partial z} \quad , \qquad \lambda_2 = -\frac{\partial m}{\partial r} \quad ; \qquad (7.23)$$

and substituting these into the second gives

$$\frac{\partial^2 m}{\partial z^2} + \frac{\partial^2 m}{\partial r^2} - \frac{1}{r}\frac{\partial m}{\partial r} = 0 \quad . \qquad (7.24)$$

Since λ_1 and λ_2 are zero on the boundaries, m is a constant on the boundaries and this constant may be taken to be zero without any loss of generality to the value of m.



D is the domain exterior to the body S.

Σ is the boundary of D.

Figure 7.3

If $\underline{R}$, $\underline{\emptyset}$ and $\underline{\Psi}$ are continuous functions defined in E_3 with

$\underline{R} = \underline{\emptyset}\, \nabla\, \underline{\Psi}$, then

$$\iiint_D \operatorname{div} \underline{R}\; dx\, dy\, dz = \iint_\Sigma \underline{R}\; d\Sigma$$

so $$\iiint_D \left\{ \underline{\emptyset}\, \nabla^2 \underline{\Psi} + (\nabla \underline{\emptyset}\, \nabla \underline{\Psi}) \right\} dx\, dy\, dz = \iint_\Sigma \underline{\emptyset}\, \frac{\partial \underline{\Psi}}{\partial n}\, d\Sigma$$

$$\iiint_D \left\{ \underline{\emptyset}\, \nabla^2 \underline{\Psi} + \underline{\emptyset}_x \underline{\Psi}_x + \underline{\emptyset}_y \underline{\Psi}_y + \underline{\emptyset}_z \underline{\Psi}_z \right\} dx\, dy\, dz = \iint_\Sigma \underline{\emptyset}\, \frac{\partial \underline{\Psi}}{\partial n}\, d\Sigma .$$

Putting $\underline{\phi} = \underline{\Psi}$ this becomes

$$\iiint_D \left\{ \underline{\phi}\nabla^2\underline{\phi} + \underline{\phi}_x^2 + \underline{\phi}_y^2 + \underline{\phi}_z^2 \right\} dx\, dy\, dz = \iint_\Sigma \underline{\phi}\, \frac{\partial \underline{\phi}}{\partial n}\, d\Sigma.$$

Since this is true for any functions, $\underline{\phi}$, in E_3 m satisfies

$$\iiint_D \left\{ m\nabla^2 m + m_x^2 + m_y^2 + m_z^2 \right\} dx\, dy\, dz = \iint_\Sigma m\, \frac{\partial m}{\partial n}\, d\Sigma. \tag{7.25}$$

The right hand side of (7.25) is zero since m is zero on the boundaries. When the left hand side is transformed to cylindrical polar coordinates (7.25) may be written as

$$\iint_S \left\{ m(m_{rr} + \frac{1}{r}m_r + m_{zz}) + m_r^2 + m_z^2 \right\} r\, dz\, dr = 0$$

and using (7.24) this becomes

$$\int_S\int \left\{ m\, \frac{2}{r}\, m_r + m_r^2 + m_z^2 \right\} r\, dz\, dr = 0$$

$$\int_S\int \left\{ \frac{1}{r}\, (m^2)_r + m_r^2 + m_z^2 \right\} r\, dz\, dr = 0.$$

From this it can be seen that

$$m_r \equiv 0 \quad , \quad m_z \equiv 0 \ ,$$

which means that m is a constant and since m is zero on the boundaries $m \equiv 0$ everywhere. From (7.23)λ_1 and λ_2 are zero everywhere and so $\lambda_3 = -2\nu m\eta$ is the unique solution for λ_3.

From (7.16) it is known that near the leading point

$$\eta = \frac{C(1+\cos\theta)}{\sin\theta} = C\left\{ \frac{[(z+a)^2 + r^2]^{1/2} + (z+a)}{r} \right\}$$

and so writing $\mathfrak{z} = z + a$ the value for λ_3 near the leading point is given by

$$\lambda_3 = -\,2\nu C\left[\mathfrak{z} + (\mathfrak{z}^2 + r^2)^{1/2} \right] \ . \tag{7.26}$$

The value for λ_4 near the leading point may now be found using equations (7.17) and (7.18). From (7.18) it can be seen that

$$\frac{\partial\lambda_4}{\partial\mathfrak{z}} = 2\nu C\, \frac{\partial}{\partial r}\left[\mathfrak{z} + (\mathfrak{z}^2 + r^2)^{1/2} \right] \qquad ,$$

$$= \frac{2\nu Cr}{(z^2 + r^2)^{1/2}} \quad ,$$

$$\lambda_4 = 2\nu Cr \log\left[z + (z^2 + r^2)^{1/2}\right] + h(r) ,$$

where h(r) is an arbitrary function of r. From (7.17)

$$\frac{\partial\lambda_3}{\partial z} = \frac{\partial}{\partial r}\left\{2\nu Cr \log\left[z + (z^2 + r^2)^{1/2}\right] + h(r)\right\} - 2\nu C \log\left[z + (z^2 + r^2)^{1/2}\right] - \frac{h(r)}{r}$$

$$\frac{\partial\lambda_3}{\partial z} = \frac{2\nu Cr^2}{(z^2 + r^2)^{1/2}\left[z + (z^2 + r^2)^{1/2}\right]} + h'(r) - \frac{h(r)}{r}, \qquad (7.27)$$

and since, from (7.26),

$$\frac{\partial\lambda_3}{\partial z} = - 2\nu C \left\{1 + \frac{z}{(z^2 + r^2)^{1/2}}\right\}$$

(7.27) may be written as

$$h'(r) - \frac{h(r)}{r} + 2\nu C\left\{\frac{r^2 + (z^2+r^2)^{1/2}\left[z + (z^2+r^2)^{1/2}\right] + z^2 + z(z^2+r^2)^{1/2}}{(z^2+r^2)^{1/2}\left[z + (z^2+r^2)^{1/2}\right]}\right\} = 0$$

$$h'(r) - \frac{h(r)}{r} + 2\nu C\left\{\frac{2(z^2+r^2)^{1/2}\left[z + (z^2+ r^2)^{1/2}\right]}{(z^2+r^2)^{1/2}\left[z + (z^2+r^2)^{1/2}\right]}\right\} = 0$$

$$h'(r) - \frac{h(r)}{r} + 4\nu C = 0$$

$$h(r) = - 4\nu Cr \log r$$

and so

$$\lambda_4 = 2\nu Cr \log\left\{\frac{z + (z^2 + r^2)^{1/2}}{r^2}\right\} . \qquad (7.28)$$

The shape of the body, $\alpha(\sigma)$, near the leading point may be found from the transversality condition, that is equation (6.41):

$$\alpha'(\sigma)\left[- \lambda_3 u_\alpha + \lambda_4 w_\alpha\right] - \left[\lambda_4 u_\alpha + \lambda_3 w_\alpha\right] + \nu r\eta^2 + f_\alpha - \frac{df_{\alpha'}}{d\sigma} = 0$$

$$(z,r) \in C_1 .$$

To find the solution for $\alpha(\sigma)$ the values of λ_3, λ_4, u_α, w_α and η must be known as functions of r and z. Values for λ_3,λ_4 and η have already been determined in the neighbourhood of the end point and values for u_α and w_α will now be found so that the shape of the body near the end point may be investigated.

The stream function ψ in the neighbourhood of the leading point is known, [(7.15)], to be

$$\psi = \tfrac{1}{6}CR^3 \left\{ (1 + \cos\theta) - 4 \sin^2\theta \cos\theta \right\}$$

and u and w are related to ψ by (7.6), that is

$$wr = -\frac{\partial\psi}{\partial r} \quad , \quad ur = \frac{\partial\psi}{\partial z} \quad .$$

Since $R\cos\theta = z + a = z$ and $R\sin\theta = r$

$$\psi = \frac{C}{6} \left[z^2 + r^2 \right]^{3/2} \left\{ 1 + \frac{z}{(z^2 + r^2)^{1/2}} - \frac{4r^2 z}{(z^2 + r^2)(z^2 + r^2)^{1/2}} \right\}$$

$$= \frac{C}{6} (z^2 + r^2) \left\{ (z^2 + r^2)^{1/2} + z - \frac{4r^2 z}{z^2 + r^2} \right\} . \qquad (7.29)$$

$$\frac{\partial\psi}{\partial r} = \frac{Cr}{3} \left\{ (z^2 + r^2)^{1/2} + z - \frac{4r^2 z}{z^2 + r^2} \right\} + \frac{C}{6} (z^2 + r^2) \left\{ \frac{r}{(z^2+r^2)^{1/2}} - \frac{8rz}{z^2+r^2} + \frac{8r^3 z}{(z^2+r^2)^2} \right\}$$

$$\text{so } w = -\frac{C}{3} \left\{ (z^2 + r^2)^{1/2} + z - \frac{4r^2 z}{z^2 + r^2} \right\} - \frac{C}{6} (z^2+r^2) \left\{ \frac{1}{(z^2+r^2)^{1/2}} - \frac{8z}{z^2+r^2} + \frac{8r^2 z}{(z^2+r^2)^2} \right\}$$

$$= -\frac{C}{6} \left\{ 3(z^2 + r^2)^{1/2} - 6z \right\} .$$

$$\frac{\partial w}{\partial r} = -\frac{C}{6} \left\{ \frac{3r}{(z^2 + r^2)^{1/2}} \right\}$$

and, since $w_\alpha = \left. \frac{\partial w}{\partial r} \right|_{\substack{r = \alpha(\sigma) \\ z = \sigma}}$,

$$w_\alpha = -\frac{C}{6} \left\{ \frac{3\alpha(\sigma)}{[(\sigma+a)^2 + \alpha^2(\sigma)]^{1/2}} \right\} . \qquad (7.30)$$

$$\frac{\partial\psi}{\partial z} = \frac{Cz}{3} \left\{ (z^2 + r^2)^{1/2} + z - \frac{4r^2 z}{z^2 + r^2} \right\} + \frac{C(z^2+r^2)}{6} \left\{ \frac{z}{(z^2+r^2)^{1/2}} + 1 - \frac{4r^2}{z^2+r^2} + \frac{8r^2 z^2}{(z^2+r^2)^2} \right\}$$

$$\text{so } u = \frac{Cz}{3r} \left\{ (z^2 + r^2)^{1/2} + z - \frac{4r^2 z}{z^2+r^2} \right\} + \frac{C}{6r} \left\{ z(z^2 + r^2)^{1/2} + (z^2 + r^2) - 4r^2 + \frac{8r^2 z}{z^2+r^2} \right\}$$

$$= \frac{C}{2} \left\{ \frac{z(z^2+r^2)^{1/2} + z^2 - r^2}{r} \right\} .$$

$$\frac{\partial u}{\partial r} = -\frac{C}{2} \left\{ \frac{z(z^2+r^2)^{1/2} + z^2 - r^2}{r^2} \right\} + \frac{C}{2} \left\{ \frac{zr}{(z^2+r^2)^{1/2} r} - 2 \right\}$$

$$= -\frac{C}{2r^2} \left\{ z(z^2+r^2) + z^2 - r^2 \frac{zr^2}{(z^2+r^2)^{1/2}} + 2r^2 \right\}$$

$$= -\frac{C}{2r^2} \left\{ z^2 + r^2 + \frac{z}{(z^2+r^2)^{1/2}} (z^2 + r^2 - r^2) \right\}$$

$$= -\frac{C}{2r^2} \left\{ z^2 + r^2 + \frac{z^3}{(z^2+r^2)^{1/2}} \right\} .$$

Since $u_\alpha = \left. \frac{\partial u}{\partial r} \right|_{\substack{r=\alpha(\sigma) \\ z=\sigma}}$

$$u_\alpha = -\frac{C}{2\alpha^2(\sigma)}\left\{ \left[(\sigma+a)^2+\alpha^2(\sigma)\right] + \frac{(\sigma+a)^3}{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}}\right\}. \qquad (7.31)$$

The values for λ_3, λ_4 and η near the leading point are:

$$\lambda_3 = -2\nu C\left[(z^2+r^2)^{1/2}+z\right]$$

$$= -2\nu C\left[\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}+(\sigma+a)\right] \quad ;$$

$$\lambda_4 = 2\nu Cr\log\left\{\frac{z+(z^2+r^2)^{1/2}}{r^2}\right\}$$

$$= 2\nu C\alpha(\sigma)\log\left\{\frac{(\sigma+a)+\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}}{\alpha^2(\sigma)}\right\} ;$$

$$\eta = \frac{C(1+\cos\theta)}{\sin\theta}$$

$$= \frac{C}{r}\left[z+(z^2+r^2)^{1/2}\right]$$

$$= \frac{C}{\alpha(\sigma)}\left[(\sigma+a)+\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}\right] .$$

The postulated constraint on the system will be taken to be that of constant arc length and so $f(\alpha(\sigma), \alpha'(\sigma), \sigma)$ in this case is

$$f(\alpha(\sigma), \alpha'(\sigma), \sigma) = \mu\left[1+\alpha'^2(\sigma)\right]^{1/2} ,$$

where μ is a constant. In this case

$$f_\alpha = 0 \quad ; \quad f_{\alpha'} = \frac{\mu\alpha'(\sigma)}{\left[1+\alpha'^2(\sigma)\right]^{1/2}} \quad ; \quad \frac{df_{\alpha'}}{d\sigma} = \frac{\mu\alpha''(\sigma)}{\left[1+\alpha'^2(\sigma)\right]^{3/2}} .$$

The transversality condition may now be written down as

$$\frac{\mu\,\alpha''(\sigma)}{\left[1+\alpha'^2(\sigma)\right]^{3/2}} + \alpha'(\sigma)\left\{\frac{\nu C^2}{\alpha^2(\sigma)}\left[\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{3/2}+(\sigma+a)^3+(\sigma+a)\left[(\sigma+a)^2+\alpha^2(\alpha)\right]\right.\right.$$

$$\left. + \frac{(\sigma+a)^4}{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}}\right] +$$

$$\left. + \frac{\nu C^2\alpha^2(\sigma)}{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}} \log\left\{\frac{(\sigma+a)+\left[(\sigma+a)^2+\sigma^2(\sigma)\right]^{1/2}}{\alpha^2(\sigma)}\right\}\right\}$$

$$- \frac{\nu C^2}{\alpha(\sigma)}\left\{(\sigma+a)^2+\alpha^2(\sigma)+\frac{(\sigma+a)^3}{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}}\right\}\log\left\{\frac{(\sigma+a)+\left[(\sigma+a)^2+\sigma^2(\sigma)\right]^{1/2}}{\alpha^2(\sigma)}\right\}$$

$$+ \nu C^2\alpha(\sigma)\left\{\frac{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}+(\sigma+a)}{\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}}\right\}$$

$$- \frac{\nu C^2}{\alpha(\sigma)}\left\{2(\sigma+a)^2+\alpha^2(\sigma)+2\left[(\sigma+a)^2+\alpha^2(\sigma)\right]^{1/2}(\sigma+a)\right\} = 0, \qquad (7.32)$$

which simplifies to

$$\frac{\mu\,\alpha''(\sigma)}{[1+\alpha'^2(\sigma)]^{3/2}} + \nu C^2 \left\{ \frac{\alpha'(\sigma)\alpha^3(\sigma) - (\sigma+a)^3 - [(\sigma+a)^2 + \alpha^2(\sigma)]^{3/2}}{\alpha(\sigma)\,[(\sigma+a)^2 + \alpha^2(\sigma)]^{1/2}} \right\} \times$$

$$\times \log \left\{ \frac{(\sigma+a) + [(\sigma+a)^2 + \alpha^2(\sigma)]^{1/2}}{\alpha^2(\sigma)} \right\}$$

$$+ \frac{\nu C^2}{[(\sigma+a)^2 + \alpha^2(\sigma)]^{1/2}} \left\{ \left[(\sigma+a) + [(\sigma+a)^2 + \alpha^2(\sigma)]^{1/2} \right] \left[\frac{\alpha'(\sigma)}{\alpha^2(\sigma)} \left[[(\sigma+a)^2 + \alpha^2(\sigma)]^{3/2} + (\sigma+a)^3 \right] \right. \right.$$

$$\left. \left. - \frac{2(\sigma+a)}{\alpha(\sigma)} \left[(\sigma+a)^2 + \alpha^2(\sigma) \right]^{1/2} \right] + \alpha(\sigma)(\sigma+a) \right\} = 0. \qquad (7.32)$$

The solution for $\alpha(\sigma)$ from (7.32) gives the shape, near the leading point, of the body of minimum drag. It is likely that this equation can be resolved numerically but this has not been pursued and instead a method to obtain an approximate solution for $\alpha(\sigma)$ has been studied as follows.

It has already been shown that at the leading point there is a semi-vertical angle of 60°, that is $\alpha'(\sigma) = \sqrt{3}$ at the point $(-a,0)$ and so $\alpha(\sigma) = \sqrt{3}\,(\sigma+a)$. The substitution $(\sigma+a) = \frac{\alpha(\sigma)}{\sqrt{3}}$ is made in equation (7.32) to get an approximate form of the transversality condition, namely:

$$\frac{\mu\,\alpha''(\sigma)}{[1+\alpha'^2(\sigma)]^{3/2}} + \frac{\sqrt{3}}{2}\nu C^2\alpha(\sigma)\left\{ [\sqrt{3} - \alpha'(\sigma)] \log\left[\frac{\sqrt{3}\alpha(\sigma)}{3}\right] + 3\alpha'(\sigma) - \sqrt{3} \right\} = 0. \qquad (7.33)$$

An iteration method is now used taking the known value of $\alpha'(\sigma)$ at the leading point, that is $\alpha'(\sigma) = \sqrt{3}$, as the initial value for $\alpha'(\sigma)$. Equation (7.33) then becomes;

$$\frac{\mu\,\alpha''(\sigma)}{[1+3]^{3/2}} + \frac{\sqrt{3}\nu C^2\alpha(\sigma)}{2} \,.\, 2\sqrt{3} = 0 \quad ,$$

$$\alpha''(\sigma) + \frac{24\,\nu C^2\alpha(\sigma)}{\mu} = 0. \qquad (7.34)$$

Let $\frac{24\nu C^2}{\mu} = m^2$, then the solution to (7.34) is

$$\alpha(\sigma) = A\cos m(\sigma+a) + B\sin m(\sigma+a) \quad .$$

$\alpha(\sigma)$ tends to zero as σ tends to $-$ a therefore A = 0 and

$$\alpha(\sigma) = B \sin m\,(\sigma+a).$$

$$\alpha'(\sigma) = m\,B \cos\,(\sigma+a)$$

$$\alpha'(\sigma) = \sqrt{3} \quad \text{at} \quad \sigma = -a\,, \text{ so}$$

$$\sqrt{3} = m\,B$$

$$\alpha(\sigma) = \frac{\sqrt{3}}{m} \sin m\,(\sigma+a).$$

The symmetry condition $\alpha'(0) = 0$ can be satisfied by an appropriate choice of m as follows:

$$\alpha'(\sigma) = 0 \quad \text{at } \sigma = 0, \text{ so}$$

$$\cos m\,a = 0,$$

$$ma = \frac{\pi}{2}\,.$$

$$\alpha(\sigma) = \frac{2\sqrt{3}}{\pi}\,a \quad \sin \frac{\pi}{2a}\,(\sigma+a). \qquad (7.35)$$

This value for $\alpha(\sigma)$ gives an approximation to the shape of minimum drag between $\sigma = -$ a and $\sigma = 0$.

CHAPTER EIGHT

CHAPTER EIGHT

Singularity Solutions of the Stream Function and Lagrange Multipliers.

The governing equations of the system are

$$\frac{1}{\rho}\frac{\partial p}{\partial r} - \nu\frac{\partial \eta}{\partial z} = 0 \quad , \tag{8.1}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial \eta}{\partial r} + \frac{\nu\eta}{r} = 0 \quad , \tag{8.2}$$

$$\eta - \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \quad , \tag{8.3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad , \tag{8.4}$$

$$\frac{\partial \lambda_3}{\partial z} - \frac{\partial \lambda_4}{\partial r} + \frac{\lambda_4}{r} = 0 \quad , \tag{8.5}$$

$$\frac{\partial \lambda_4}{\partial z} + \frac{\partial \lambda_3}{\partial r} = 0 \quad , \tag{8.6}$$

$$\lambda_3 - 2\nu r\eta = 0 \quad . \tag{8.7}$$

Eliminating p between (8.1) and (8.2) gives

$$\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2} = 0 \tag{8.8}$$

Substituting (8.7) in (8.5) gives

$$\frac{\partial}{\partial r}\left(\frac{\lambda_4}{r}\right) = \frac{1}{r}\frac{\partial}{\partial z}\left(-2\nu r\eta\right)$$

and using (8.1) this becomes

$$\frac{\partial}{\partial r}\left(\frac{\lambda_4}{r}\right) = -\frac{2}{\rho}\frac{\partial p}{\partial r} \quad .$$

Substituting (8.7) in (8.6) gives

$$\frac{\partial \lambda_4}{\partial z} = 2\nu\frac{\partial}{\partial r}(r\eta)$$

and using (8.2) this may be written as

$$\frac{\partial \lambda_4}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{2rp}{\rho}\right) \quad .$$

Therefore

$$\frac{\partial}{\partial r}\left(\frac{\lambda_4}{r} + \frac{2p}{\rho}\right) = 0 \quad , \qquad \frac{\partial}{\partial z}\left(\lambda_4 + \frac{2rp}{\rho}\right) = 0$$

hence $\frac{\lambda_4}{r} + \frac{2p}{\rho} = A$. (8.9)

where A is an arbitrary constant.

If a function χ is introduced such that

$$\eta = \frac{\partial \chi}{\partial r} \tag{8.10}$$

then (8.8) becomes

$$\frac{\partial^3 \chi}{\partial r \partial z^2} + \frac{\partial^3 \chi}{\partial r^3} + \frac{1}{r}\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \chi}{\partial r} = 0$$

that is $\frac{\partial}{\partial r}\left\{ \frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} \right\} = 0$

therefore $\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} = 0.$ (8.11)

This is Laplace's Equation in cylindrical coordinates and it has a basic solution

$$\chi = \frac{1}{\tilde{\omega}} \quad , \tilde{\omega}^2 = (z-\xi)^2 + r^2 \tag{8.12}$$

corresponding to a source singularity at $(\xi, 0)$.

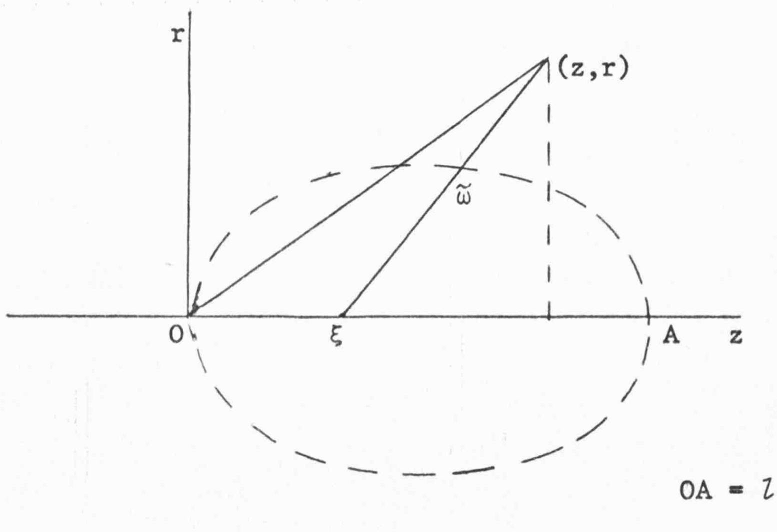


Figure 8.1

From (8.12) it follows that a more general solution for χ can be constructed by distributing source singularities along the z-axis from the leading point of the body (chosen to be the origin) to the tail of the body ($z = l$) this solution being of the form

$$\chi(z,r) = \int_0^l \frac{a(\xi)d\xi}{\tilde{\omega}} = \int_0^l \frac{a(\xi)d\xi}{\sqrt{r^2 + (z-\xi)^2}} \qquad (8.13)$$

where $a(\xi)$ is an unknown source density and is a function of ξ only. It now follows from (8.10) and (8.13) that the vorticity η is given in terms of a by the equation

$$\eta = \int_0^l a(\xi) \frac{\partial}{\partial r} \left(\frac{1}{\tilde{\omega}} \right) d\xi \quad . \qquad (8.14)$$

The singularities $\frac{\partial}{\partial r}\left(\frac{1}{\tilde{\omega}}\right)$ are dipoles pointing in the r direction.

Next the expression for the pressure p in terms of a is considered. Equation (8.1) together with solution (8.14) gives

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \nu \int_0^l a(\xi) \frac{\partial^2}{\partial z \partial r} \left(\frac{1}{\tilde{\omega}} \right) d\xi \quad , \qquad (8.15)$$

and from (8.2)

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial z} &= -\frac{\nu}{r}\frac{\partial}{\partial r} \left\{ r \int_0^l a(\xi) \frac{\partial}{\partial r} \left(\frac{1}{\tilde{\omega}}\right) d\xi \right\} \\
&= -\frac{\nu}{r} \int_0^l a(\xi) \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left(\frac{1}{\tilde{\omega}}\right) \right\} d\xi \\
&= -\frac{\nu}{r} \int_0^l a(\xi) \left\{ r \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \right\} \left(\frac{1}{\tilde{\omega}}\right) d\xi \\
&= -\nu \int_0^l a(\xi) \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right\} \left(\frac{1}{\tilde{\omega}}\right) d\xi
\end{aligned}$$

and since $\frac{1}{\tilde{\omega}}$ satisfies Laplace's equation

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \int_0^l a(\xi) \frac{\partial^2}{\partial z^2} \left(\frac{1}{\tilde{\omega}}\right) d\xi \quad . \qquad (8.16)$$

It can be deduced from (8.15) and (8.16) that, apart from an arbitrary constant,

$$\frac{p}{\rho} = \nu \int_0^l a(\xi) \frac{\partial}{\partial z} \left(\frac{1}{\tilde{\omega}}\right) d\xi \quad . \qquad (8.17)$$

It now follows from (8.7) and (8.14) that

$$\lambda_3 = -2\nu r \int_0^l a(\xi) \frac{\partial}{\partial r} \left(\frac{1}{\tilde{\omega}}\right) d\xi \quad , \qquad (8.18)$$

and from (8.9) that

$$\lambda_4 = Ar - 2\nu r \int_0^l a(\xi) \frac{\partial}{\partial z}\left(\frac{1}{\tilde{\omega}}\right) d\xi \quad . \tag{8.19}$$

It is now necessary to deduce from (8.14) the stream function ψ .

It has already been seen that equation (8.4) gives

$$ru = \frac{\partial \psi}{\partial z} \quad , \quad rw = -\frac{\partial \psi}{\partial r} \tag{8.20}$$

so that from (8.3)

$$r\eta = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} \quad . \tag{8.21}$$

Putting $\psi = r\Psi$, (8.22)

then

$$\eta = \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{1}{r^2}\Psi \quad , \tag{8.23}$$

and writing

$$\Psi = \frac{\partial \Phi}{\partial r} \tag{8.24}$$

together with (8.10) gives

$$\frac{\partial X}{\partial r} = \frac{\partial^3 \Phi}{\partial z^2 \partial r} + \frac{\partial^3 \Phi}{\partial r^3} + \frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r} \tag{8.25}$$

so that

$$X = \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} \equiv \nabla^2 \Phi \quad , \tag{8.26}$$

where ∇^2 is the three dimensional Laplacian. A solution must now be found for Φ from

$$\nabla^2 \Phi = \int_0^l \frac{a(\xi) d\xi}{\tilde{\omega}} \quad , \quad \tilde{\omega}^2 = (z-\xi)^2 + r^2 \quad . \tag{8.27}$$

Consider the function

$$\Phi = \int_0^l a(\xi) d\xi \left\{ \alpha\tilde{\omega} + \beta \log \tilde{\omega} + \frac{\gamma}{\tilde{\omega}} \right\} d\xi \quad ,$$

where α, β, γ are constants; it is easily shown that

$$\nabla^2 \tilde{\omega} = \frac{2}{\tilde{\omega}} \quad ,$$

$$\nabla^2 \log \tilde{\omega} = \frac{1}{\tilde{\omega}^2} ,$$

$$\nabla^2 \frac{1}{\tilde{\omega}} = 0 \quad .$$

Thus the particular solution Φ_1 for Φ from (8.27) corresponds

$\alpha = \frac{1}{2}, \beta = 0, \gamma = 0$ and hence

$$\Phi_1 = \tfrac{1}{2} \int_0^l \tilde{\omega}\, a(\xi) d\xi \tag{8.28}$$

and to this particular solution a complementary function of the form

$$\Phi_2 = \int_0^l \frac{b(\xi)}{\tilde{\omega}} d\xi \quad . \tag{8.29}$$

can be added, where $b(\xi)$ is an arbitrary function of ξ, since $\nabla^2\Phi$ vanishes. This gives a solution for Φ of the form

$$\Phi = \tfrac{1}{2} \int_0^l \tilde{\omega}\, a(\xi) d\xi + \int_0^l \frac{b(\xi)}{\tilde{\omega}} d\xi \quad . \tag{8.30}$$

The function Ψ defined in (8.24) is then

$$\begin{aligned} \Psi &= \tfrac{1}{2} \int_0^l a(\xi) \frac{\partial \tilde{\omega}}{\partial r} d\xi + \int_0^l b(\xi) \frac{\partial}{\partial r}\left(\frac{1}{\tilde{\omega}}\right) d\xi \\ &= \tfrac{1}{2} \int_0^l r\, \frac{a(\xi)}{\tilde{\omega}} d\xi + \int_0^l b(\xi) \frac{\partial}{\partial r}\left(\frac{1}{\tilde{\omega}}\right) d\xi \end{aligned} \tag{8.31}$$

and the stream function ψ in (8.22) becomes

$$\psi = \tfrac{1}{2} r^2 \int_0^l \frac{a(\xi) d\xi}{\tilde{\omega}} + r \int_0^l b(\xi) \frac{\partial}{\partial r}\left(\frac{1}{\tilde{\omega}}\right) d\xi \quad . \tag{8.32}$$

In (8.32) $a(\xi)$ and $b(\xi)$ are two arbitrary functions of ξ, the former having entered originally in (8.13).

The complete stream function can now be constructed. Corresponding to the uniform stream at infinity

$$u = u_o = 0 \quad , \quad w = w_o = W \, .$$

there is a stream function ψ_o such that

$$\frac{\partial \psi_o}{\partial z} = 0 \quad , \quad \frac{\partial \psi_o}{\partial r} = -rW$$

hence

$$\psi_o = -\tfrac{1}{2} r^2 W \, . \tag{8.33}$$

Thus the total stream function ψ^* will be

$$\psi^* = -\tfrac{1}{2} r^2 W + \psi , \tag{8.34}$$

where ψ is given in (8.32). For large values of r the first integral in (8.32) gives $\psi \simeq C_o r$ where C_o is a constant and thus the conditions at infinity, namely u tends to zero and w tends to W will be satisfied by ψ^*.

The boundary conditions on the surface of the body are that the total velocity is zero, in other words

$$u = 0 \quad , \quad w = 0 \quad , \text{ on the body,} \tag{8.35}$$

and in terms of the stream function ψ^* this can be written as

$$\psi^* = 0, \quad \frac{\partial\psi^*}{\partial n} = 0, \quad \text{on the body ,} \tag{8.36}$$

where $\hat{n}$ is the normal derivative. Alternatively the boundary conditions may be used in the more convenient form

$$\psi^* = 0 \; , \quad \frac{\partial\psi^*}{\partial r} = 0 \quad , \text{ on the body ,} \tag{8.37}$$

and using (8.34) it follows that

$$\psi = \tfrac{1}{2} Wr^2 \quad , \quad \text{on the body } r = \alpha(\sigma) \; , \tag{8.38}$$

$$\frac{\partial\psi}{\partial r} = Wr \quad , \text{ on the body } r = \alpha(\sigma) \; . \tag{8.39}$$

(8.32) may be written in the form

$$\psi = \tfrac{1}{2} r^2 \int_0^l \frac{a(\xi)}{\varpi} d\xi - r^2 \int_0^l \frac{b(\xi)d\xi}{\varpi^3} \tag{8.40}$$

hence (8.38) becomes

$$\tfrac{1}{2}\int_0^l \frac{a(\xi)\; d\xi}{\{\alpha^2(\sigma) + (\sigma-\xi)^2\}^{1/2}} - \int_0^l \frac{b(\xi)\; d\xi}{\{\alpha^2(\sigma) + (\sigma-\xi)^2\}^{3/2}} = \tfrac{1}{2}W . \tag{8.41}$$

Likewise

$$\frac{\partial\psi}{\partial r} = r\int_0^l a(\xi) \left(\frac{1}{\varpi} - \frac{\tfrac{1}{2}r^2}{\varpi^3} \right) d\xi - r \int_0^l b(\xi)\left\{ \frac{2}{\varpi^3} - \frac{3r^2}{\varpi^5} \right\} d\xi$$

and thus (8.39) becomes

$$\int_0^l \frac{\{\tfrac{1}{2}\alpha^2(\sigma) + (\sigma-\xi)^2\}}{\{\alpha^2(\sigma) + (\sigma-\xi)^2\}^{3/2}} a(\xi)d\xi - \int_0^l \frac{\{2(\sigma-\xi)^2 - \alpha^2(\sigma)\}}{\{\alpha^2(\sigma) + (\sigma-\xi)^2\}^{5/2}} b(\xi)\; d\xi = W . \tag{8.42}$$

Equations (8.41) and (8.42) provide two coupled integral equations between the unknowns $\alpha(\sigma)$, $a(\xi)$ and $b(\xi)$ and the third relation between these three functions is the transversality condition, namely,

$$\frac{\mu\alpha''(\sigma)}{[1+\alpha'^2(\sigma)]^{3/2}} + \alpha'(\sigma)\left[\lambda_3 u_\alpha - \lambda_4 w_\alpha\right] + \left[\lambda_4 u_\alpha + \lambda_3 w_\alpha\right] - \nu r\eta^2 = 0 \,,$$

$$\text{on } r = \alpha(\sigma) \quad . \qquad (8.43)$$

A certain degree of simplification can be effected in (8.42) because when W is eliminated on the right hand side of (8.41) and (8.42) this gives

$$\int_0^l \frac{\frac{1}{2}\alpha^2(\sigma) + \frac{1}{2}(\sigma-\xi)^2 - \frac{1}{2}\alpha^2(\sigma) - \frac{1}{2}(\sigma-\xi)^2}{\tilde{\omega}^3}\, a(\xi)d\xi$$

$$- \int_0^l \frac{(\sigma-\xi)^2 - \frac{1}{2}\alpha^2(\sigma) - (\sigma-\xi)^2 - \alpha^2(\sigma)}{\tilde{\omega}^5}\, b(\xi)d\xi = 0$$

hence

$$\int_0^l - \frac{\frac{1}{2}\alpha^2(\sigma)\, a(\xi)d\xi}{\tilde{\omega}^3} + \int_0^l \frac{\frac{3}{2}\alpha^2(\sigma)\, b(\xi)d\xi}{\tilde{\omega}^5} = 0 \quad .$$

Since $\alpha(\sigma) \neq 0$ it follows that

$$\int_0^l \frac{a(\xi)d\xi}{\tilde{\omega}^3} - 6\int_0^l \frac{b(\xi)d\xi}{\tilde{\omega}^5} = 0 \,. \qquad (8.44)$$

This equation can replace (8.42) and (8.41) can be written in the form

$$\int_0^l \frac{a(\xi)d\xi}{\tilde{\omega}} - 2\int_0^l \frac{b(\xi)d\xi}{\tilde{\omega}^3} = W \,, \qquad (8.45)$$

where in (8.44) and (8.45)

$$\tilde{\omega}^2 = \alpha^2(\sigma) + (\sigma-\xi)^2 \quad . \qquad (8.46)$$

The resolution of the solution by this method of distributed singularities has not been completed analytically due to the complexity of the problem (although it is possible that the methods described by Landweber[14] and Hocking[15] can be used in getting approximate solutions) but it is likely that the problem from this point onwards can be resolved numerically.

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J.M. EDWARDS Ph.D. THESIS 1977

SUMMARY

In this thesis the variation of a functional defined on a variable domain has been studied and applied to the problem of finding the optimum shape of the domain in which some performance criterion has an extremum. The method most frequently used is one due to Gelfand and Fomin. It is applied to problems governed by first and second order partial differential equations, unsteady one dimensionsal gas movements and the problem of minimum drag on a body with axial symmetry in Stokes' flow.