A STUDY OF ARTIFICIAL SATELLITE RESONANCE ORBITS DUE TO LUNISOLAR PERTURBATIONS

A THESIS SUBMITTED FOR

THE DEGREE OF DOCTOR OF PHILOSOPHY

by

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This thesis is affectionately dedicated to the University of Leicester's CDC Cyber 72 computer with whom I have spent many pleasant evenings.

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CONTENTS

CHAPTER 1

THE GENERAL MOTION OF AN ARTIFICIAL SATELLITE

1.1 Elliptic Motion

If the Earth was a sphere with a radially symmetric density distribution, had no atmosphere, was situated in a Newtonian universe, and was isolated from other bodies in the solar system, the orbit of an artificial satellite would be an ellipse of constant size and shape in a plane whose direction remained fixed relative to the stars. Five parameters, called orbital elements, are required to characterise the size, shape and orientation of a satellite's orbit. A sixth orbital element defines the angular position of the satellite in its orbit.

Two angles specify the orientation of the orbital plane, namely the inclination, I, of the orbital plane to the celestial equator and the longitude of the ascending node, Ω , measured eastwards from the vernal equinox, Υ, to the ascending node, N, (Figure 1.1). Three further parameters define the size and shape of the orbit and its orientation in the orbital plane. The size is specified by the semi-major axis, a, and the shape by the eccentricity, e, (Figure 1.2). The fifth orbital element is the argument of perigee, ω , the angle between the ascending node and the perigee of the orbit, measured in the plane of the satellite's motion (Figure 1.1). The last parameter is the mean anomaly, M, defined by

$$M = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} (t - T) \tag{1.1}$$

where μ is the Keplerian constant GM_E, G is the gravitational constant, M_E the mass of the Earth, t the time and T the time of the most recent passage of the satellite through perigee.

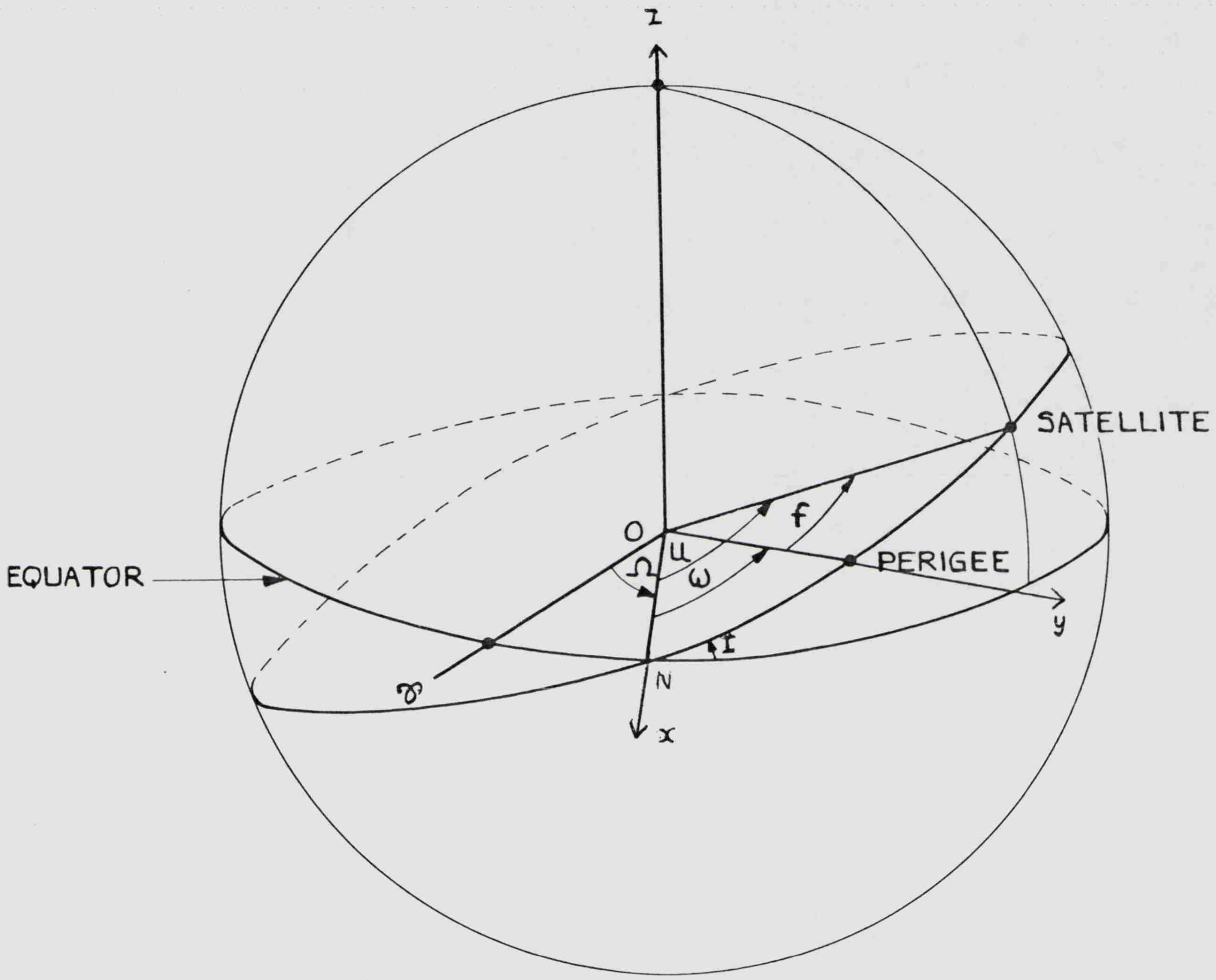


Figure 1.1

Projection of an artificial satellite

orbit on the celestial sphere

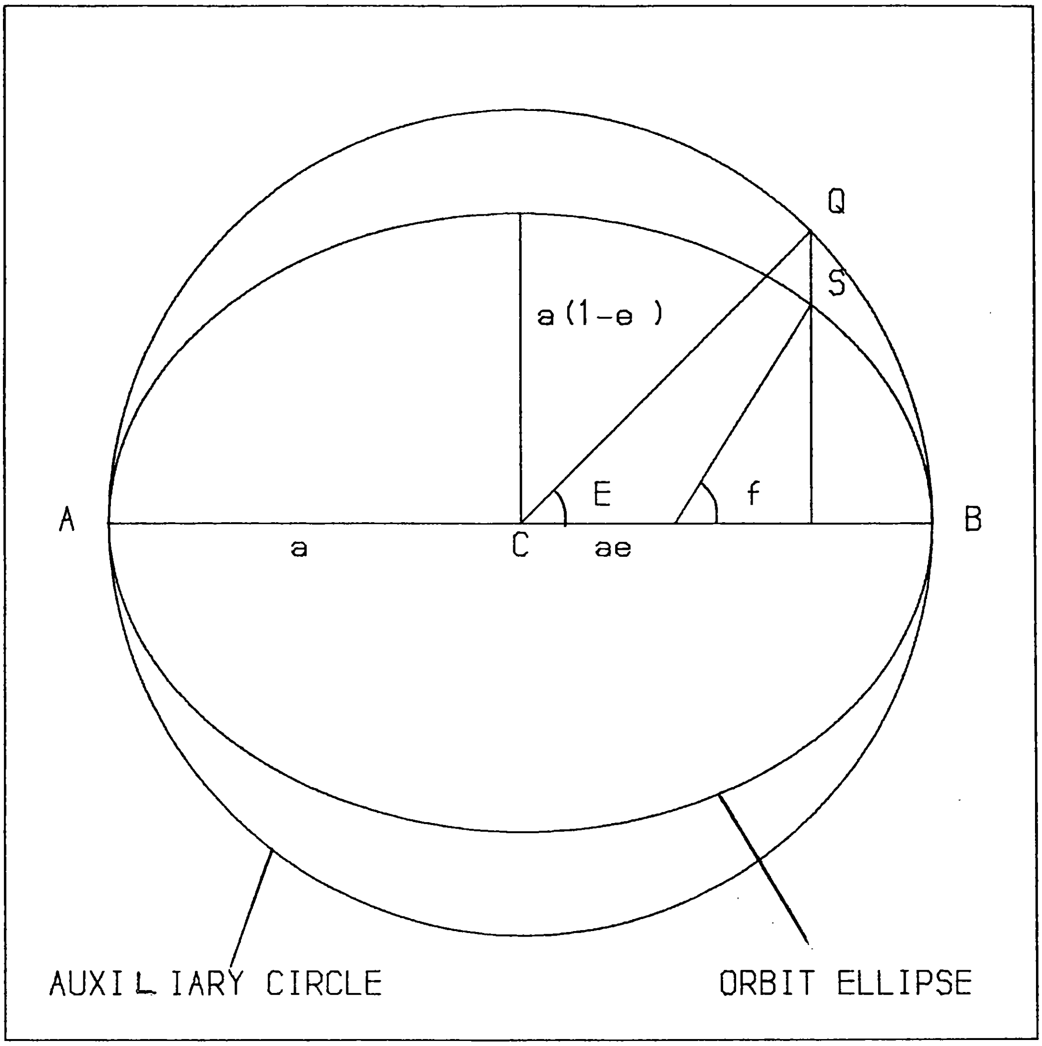


Figure 1.2

Definition of a, e, f and E

In addition to the mean anomaly, two other variables are often used to specify the angular position of a satellite in its orbit. The first of these alternative variables is the true anomaly, f, defined as the angle subtended at the Earth's centre by the satellite's radius vector, $\underline{r}$, and the direction of perigee (Figure 1.2). The true anomaly is related to r, a and e by the equation

$$r = \frac{a(1 - e^2)}{(1 + e \cos f)} \tag{1.2}$$

The second alternative variable to M is the eccentric anomaly, E, the angle between the vector $\underline{CQ}$ and the major axis AB of the orbit (Figure 1.2). It can be shown that M, f and E are related to each other by the set of equations (Brouwer and Clemence, 1961):

$$M = E - e \sin E \tag{1.3}$$

$$r \cos f = a(\cos E - e) \tag{1.4}$$

$$r \sin f = a(1 - e^2)^{\frac{1}{2}} \sin E \tag{1.5}$$

$$r^2 \frac{df}{dM} = a^2(1 - e^2)^{\frac{1}{2}} \tag{1.6}$$

$$r = a(1 - e \cos E) \tag{1.7}$$

1.2 Osculating Motion and the Lagrangian Planetary Equations

In reality the Earth is not a sphere with a radially symmetric density distribution; it does have an atmosphere and it is not situated in a Newtonian universe isolated from other bodies in the solar system. Consequently, the actual path of an Earth artificial satellite will be somewhat different from an ellipse when these perturbing influences are taken into account. Suppose at some epoch, t_o, the disturbing action of these perturbations ceases to exercise any effect on the satellite, the orbit of the satellite would be an ellipse with constant elements a_o, e_o, I_o, ω_o, Ω_o and M_o. If the perturbations had ceased at an epoch, t_1, just subsequent to t_o, the orbit would have been an ellipse with a different set of elements a_1, e_1, I_1, ω_1, Ω_1 and M_1, say; and so, too, for epochs t_2, t_3 ... t_n. Clearly, the perturbed orbit of an artificial satellite can be considered to be an osculating ellipse whose orbital elements are functions of time; the position and velocity vectors of the osculating orbit being equal to those of the true orbit. The concept of an osculating orbit forms the basis for the derivation of the Lagrangian planetary equations - a set of equations fundamental to celestial mechanics. A brief discussion of these equations will now be given.

For an unperturbed satellite, the vector equation of motion relative to an inertial frame is

$$\ddot{\underline{r}} + \frac{\mu \underline{r}}{r^3} = \underline{0} \tag{1.8}$$

If perturbing forces act on a satellite, the right-hand side of equation (1.8) will not be zero. The perturbed equation of motion is of the form

$$\ddot{\underline{r}} + \mu \underline{r} / r^3 = \underline{F} \tag{1.9}$$

where F is the perturbing force per unit mass. In general, F will be a function of the positional co-ordinates (x, y, z) and velocity components (U_x, U_y, U_z) of the satellite and a set of parameters peculiar to each individual perturbing force. The force, $\underline{F}$, can be divided into two basic types - conservative and non-conservative. If there exists a scalar potential function $\emptyset(x, y, z)$ such that

$$\underline{F} = \underline{\nabla}\phi(x,y,z) = \left(\hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}\right) \tag{1.10}$$

then the force, $\underline{F}$, is said to be conservative; $\hat{i}$, $\hat{j}$ and $\hat{k}$ are a set of unit vectors parallel to the Cartesian axes x, y and z centred at the Earth (Figure 1.1). The scalar potential function $\phi(x,y,z)$, known commonly as the disturbing function, is only dependent upon the satellite's positional co-ordinates and the parameters of the perturbing force (s). For a non-conservative force, $\underline{F}$, no scalar potential function exists which satisfies equation (1.10). In the non-conservative case, $\underline{F}$ is a function of the positional co-ordinates (x, y, z) and velocity components (U_x, U_y, U_z), and the parameters of the perturbing force (s).

Let the orbital elements of the osculating orbit, now functions of time, be α_1, α_2, α_3, α_4, α_5 and α_6, corresponding to a, e, I, ω, Ω and M, respectively. The position vector, $\underline{r}$, of the satellite can be expressed as a function of α_j (j = 1,2 ... 6) and the time (Smart, 1965), viz

$$\underline{r} = \underline{r}(\alpha, t) \tag{1.11}$$

On differentiating equation (1.11) with respect to time

$$\frac{d\underline{r}}{dt} = \left(\frac{\partial\underline{r}}{\partial t}\right) + \sum_{j=1}^{6} \frac{\partial\underline{r}}{\partial\alpha_j} \cdot \dot{\alpha}_j \tag{1.12}$$

Since the velocity vectors of the osculating orbit and the actual orbit are the same at any instant

$$\frac{d\underline{r}}{dt} = \left(\frac{\partial\underline{r}}{\partial t}\right) \tag{1.13}$$

Using equation (1.13), (1.12) becomes

$$\sum_{j=1}^{6} \frac{\partial \underline{r}}{\partial \alpha_j} \cdot \dot{\alpha}_j = \underline{0} \qquad (1.14)$$

If (1.13) is then differentiated with respect to time

$$\frac{d^2\underline{r}}{dt^2} = \frac{\partial^2\underline{r}}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial^2\underline{r}}{\partial \alpha \, \partial t} \cdot \dot{\alpha}_j \qquad (1.15)$$

If the equation of motion of the osculating orbit (1.8) and the corresponding equation (1. 9) of perturbed motion for a conservative force are used to eliminate $\frac{\partial^2\underline{r}}{\partial t^2}$ and $\frac{d^2\underline{r}}{dt^2}$ from (1.15), then

$$\sum_{j=1}^{6} \frac{\partial^2\underline{r}}{\partial \alpha \, \partial t} \cdot \dot{\alpha}_j = \underline{\nabla\phi} \qquad (1.16)$$

When $\underline{\nabla\phi}$ is expressed as a function of α_j ($j = 1,2 \ldots 6$), the vector equations (1.14) and (1.16) can be solved for the six $\dot{\alpha}_j$ (Smart, 1953). The expressions for $\dot{a}$, $\dot{e}$, $\dot{I}$, $\dot{\omega}$, $\dot{\Omega}$ and $\dot{M}$ are found to be

$$\frac{da}{dt} = 2\left(\frac{a}{\mu}\right)^{\frac{1}{2}} \frac{\partial\phi}{\partial M} \qquad (1.17)$$

$$\frac{de}{dt} = \frac{1}{(\mu a)^{\frac{1}{2}} e} \left(\frac{\partial\phi}{\partial M}(1 - e^2) - (1 - e^2)^{\frac{1}{2}} \frac{\partial\phi}{\partial\omega}\right) \qquad (1.18)$$

$$\frac{dI}{dt} = \frac{1}{(\mu a)^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} \left(\cot I \frac{\partial\phi}{\partial\omega} - \operatorname{cosec} I \frac{\partial\phi}{\partial\Omega}\right) \qquad (1.19)$$

$$\frac{d\omega}{dt} = \frac{(1 - e^2)^{\frac{1}{2}}}{(\mu a)^{\frac{1}{2}}} \left(\frac{1}{e} \cdot \frac{\partial\phi}{\partial e} - \frac{\cot I}{(1-e^2)} \cdot \frac{\partial\phi}{\partial I}\right) \qquad (1.20)$$

$$\frac{d\Omega}{dt} = \frac{1}{(\mu a)^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}} \sin I} \cdot \frac{\partial\phi}{\partial I} \qquad (1.21)$$

$$\frac{dM}{dt} = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} - \frac{(1-e^2)}{e(\mu a)^{\frac{1}{2}}} \cdot \frac{\partial\phi}{\partial e} - 2\left(\frac{a}{\mu}\right)^{\frac{1}{2}} \cdot \left(\frac{\partial\phi}{\partial a}\right) \qquad (1.22)$$

The derivative $\left(\frac{\partial \emptyset}{\partial a}\right)$ in equation (1.22) refers to differentiation of $\emptyset$ explicitly with respect to 'a': not through the dependence of M on a. The six equations (1.17) - (1.22) are the Lagrangian planetary equations for a conservative perturbing force.

If F_r, F_T and F_N are the magnitudes of the radial, transverse and normal components of the perturbing force, $\underline{F}$, relative to the orbital plane (Figure 1.3), the Lagrangian planetary equations can be written in the alternative form (Roy, 1965).

$$\frac{da}{dt} = \frac{2}{(1-e^2)^{\frac{1}{2}}}\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\left[F_r\, e\sin f + \frac{a(1-e^2)}{r}F_T\right] \tag{1.23}$$

$$\frac{de}{dt} = (1-e^2)^{\frac{1}{2}}\left(\frac{a}{\mu}\right)^{\frac{1}{2}}\left[F_r \sin f + F_T\,(\cos E + \cos f)\right] \tag{1.24}$$

$$\frac{dI}{dt} = \frac{r\cos(\omega+f)}{(\mu a)^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} \cdot F_N \tag{1.25}$$

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{\frac{1}{2}}}{e}\left(\frac{a}{\mu}\right)^{\frac{1}{2}}\left[F_T\left(1+\frac{r}{a(1-e^2)}\right)\sin f - F_r\cos f\right]$$

$$- \frac{r\sin(\omega+f)\cot I}{(\mu a)^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} \cdot F_N \tag{1.26}$$

$$\frac{d\Omega}{dt} = \frac{r\sin(\omega+f)}{(\mu a)^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}\sin I} \cdot F_N \tag{1.27}$$

$$\frac{dM}{dt} = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} + \frac{1}{(\mu a)^{\frac{1}{2}}}\left[\frac{a(1-e^2)}{e}\cos f \,.\, F_r - 2r\,F_r\right.$$

$$\left. - \frac{r\sin f}{e}\left(1+\frac{a(1-e^2)}{r}\right)F_T\right] \tag{1.28}$$

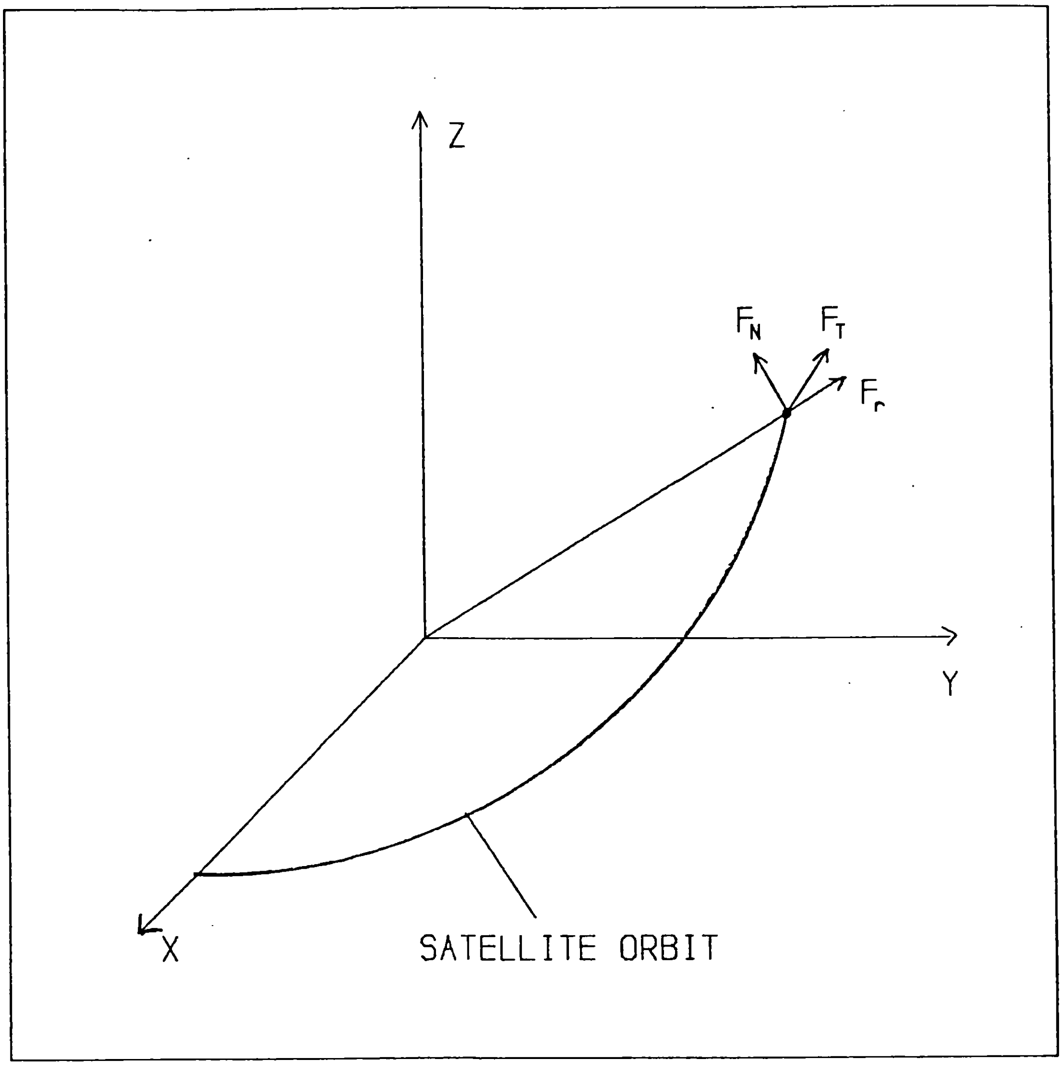


Figure 1.3

Definition of perturbing force components,

F_r, F_T and F_N

Equations (1.23) - (1.28) are the Gaussian form of the Lagrangian planetary equations: this form is applicable to both conservative and non-conservative forces.

If the exact form of $\underline{F}$ or ϕ as a function of the positional co-ordinates and velocity components of the satellite and the parameters of the perturbing force is known, $\underline{F}$ or ϕ can then be expanded as a function of the osculating elements, a, e, I, ω, Ω and M with the aid of the equations of elliptic motion. When $\underline{F}$ or ϕ is in this form, the appropriate set of Lagrangian planetary equations can be used to obtain $\dot{a}$, $\dot{e}$, etc., as a function of the six osculating elliptic elements. The resulting six differential equations, when integrated, will give the time variation of the elliptic elements produced by the perturbing force, $\underline{F}$, or the disturbing function, ϕ.

1.3 Types of Orbital Changes

The exact form of the expansion of $\underline{F}$ or ϕ in terms of the osculating orbital elements and the parameters of the disturbing forces for the various perturbations affecting the motion of an artificial satellite has been derived by a number of authors (Figure 1.4). However, in all cases, the expansions can be expressed in terms of a Poisson series, viz. (Deprit and Rom, 1969)

$$\sum_j A_j B_j(a,e,I) \begin{matrix}\sin\\ \cos\end{matrix} \{\gamma_j\omega + \zeta_j\Omega + \eta_j M + U_j\} \tag{1.29}$$

where A_j is a constant dependent on the perturbing parameters; B_j is a function of a, e and I; γ_j, η_j and ζ_j are integer constants; and U_j is a function of certain of the perturbing parameters. In general, U_j is found to vary linearly with time. Such a Poisson series can produce five types of changes in a satellite's orbital elements;

FORCE ACTING ON SATELLITE	AUTHOR(S)
ZONAL HARMONICS	Cook (1966)
TESSERAL HARMONICS	Kaula (1961), Allan (1965)
LUNISOLAR GRAVITY	Kaula (1962), Allan (1969), Giacaglia (1974)
AIR DRAG	King-Hele (1964, 1966) King-Hele and Scott (1969), King-Hele and Walker (1972, 1976)
SOLAR RADIATION PRESSURE	Hughes (1977)
RELATIVITY	Rubincam (1975)
TIDAL: Body and Ocean	Kaula (1969), Musen and Estes (1972), Musen and Felsentreger (1974), Lambeck et al. (1974)

Figure 1.4 Authors of Various Perturbing Force Expansions

they are

(i) Secular

(ii) Long-Period

(iii) Short-Period

(iv) Resonant

(v) Interactive

Let us examine these in turn:

(i) <u>SECULAR</u>

A secular change in an orbital element causes a continuous increase, or decrease, in its magnitude with time. For a conservative perturbing force, such a change is produced by the set of terms, ϕ_{sec}, in the Poisson series (1.29) which are independent of ω, Ω, M and U_j, viz.

$$\phi_{sec} = \sum_j A_j B_j \, (a,e,I) \tag{1.30}$$

From the set of Lagrangian planetary equations (1.17) - (1.22) it is seen that, if a, e and I are assumed to be approximately constant, then the terms in ϕ_{sec} will produce linear and, hence, secular changes in a satellite's argument of perigee, ω, its nodal longitude, Ω, and its Mean anomaly, M. The parameters a, e and I do not suffer secular changes when a satellite is acted upon by conservative perturbing forces.

The situation for secular changes produced by non-conservative perturbing forces is somewhat different. If terms in $\underline{F}$ are of the form shown in (1.30), then they will not automatically produce secular changes in ω, Ω, and M, as they did for the case of a conservative perturbing force. This is because in the Gaussian-Lagrangian planetary equations certain coefficients contain the periodic functions f, E and M. In order for a term in $\underline{F}$ to produce a secular change in an orbital element, it must combine with these periodic coefficients, as well as the non-periodic coefficients, to form terms of the form, $\emptyset_{sec}$. The Gaussian-Lagrangian planetary equations for which this is true will give secular variations in its appropriate orbital element.

(ii) LONG-PERIOD

It has already been seen that ω and Ω suffer secular changes when a satellite is acted upon by conservative perturbing forces. Consequently, the terms, $\emptyset_{long}$ in the Poisson series (1.29) of the form

$$\emptyset_{long} = \sum_{j} A_j B_j(a,e,I) \begin{matrix} \sin \\ \cos \end{matrix} \left(\Upsilon_j \, \omega + \zeta_j \Omega + U_j \right) \qquad (1.31)$$

resulting from a conservative perturbing force, will be slowly varying periodic functions producing long-period changes in the orbital elements e, I, ω, Ω and M. Such changes are known as long-period perturbations - the periods usually being of the order of several weeks.

For a non-conservative perturbing force, long-period changes in an orbital element will occur if its time rate of change obtained from the appropriate Gaussian-Lagrangian planetary equation contains terms of the type $\emptyset_{long}$. The orbital elements suffering long-period changes will be dependent on the exact form of $\underline{F}$. If $\underline{F}$ contains only a normal component, F_N, such that $F_N \sin(\omega + f)$ equals a series of the type (1.31), then long-period changes will be produced in the orbital elements Ω and ω (equations 1.23 - 1.28).

(iii) SHORT-PERIOD

A short-period perturbation in a satellite's elliptic elements is one which has a period equal to, or less than, the satellite's orbital period. Such changes for a conservative perturbing force are produced by those terms, $\emptyset_{short}$, in the Poisson series (1.29) which are dependent on the Mean anomaly. Here, M - the Mean anomaly - has a period approximately equal to the satellite's orbital period: $\emptyset_{short}$ is therefore of the form

$$\emptyset_{short} = \sum_{j} A_j B_j(a,e,I) \begin{matrix} \sin \\ \cos \end{matrix} \left(\gamma_j \omega + \zeta_j \Omega + \eta_j M + U_j \right) \tag{1.32}$$

From the conservative form of the Lagrangian planetary equations, i.e. equations (1.17) - (1.22), it is seen that, if $\emptyset$ contains terms of the type $\emptyset_{short}$, then all six orbital elements can suffer short-period perturbations.

In the case of a non-conservative perturbing force, short-period changes in an orbital element will occur if its time rate of change contains a Poisson series of the type (1.32). A non-conservative perturbation will, in general, also produce short-period changes in all the orbital elements.

(iv) RESONANT

Under certain circumstances, when acted upon by a conservative perturbing force the semi-major axis, a, the eccentricity, e, and the inclination, I, of a satellite's orbit are such that they cause the arguments, ψ , of particular terms in the Poisson series (1.29) to become approximately constant, i.e.

$$\dot{\psi} = \gamma_j \dot{\omega} + \zeta_j \dot{\Omega} + \eta_j \dot{M} + \dot{U}_j \sim 0 \qquad (1.33)$$

The periods of these so-called resonance terms are therefore very large - a typical period being of the order of several years. The orbital elements suffering a resonant change will be dependent on which angular variables exist in ψ. For example, if ψ contains only Ω, then just I, ω, Ω and M can be affected.

In the case of a non-conservative perturbing force, a resonant change will occur in a particular orbital element if its rate of change, obtained from the appropriate Gaussian-Lagrangian planetary equation, contains a term for which $\dot{\psi} \sim 0$. The nature of resonance orbits and the effect of resonant perturbations on a satellite's orbital elements are discussed in greater detail in Chapter 2; special emphasis being given to lunisolar gravity and solar radiation pressure resonance orbits.

(v) INTERACTIVE

An interactive orbital change is a combination of the previous four types. All artificial satellites at any instant are subjected not to one, but to a number of perturbations, each affecting its motion. Since the effect of a given perturbation is dependent upon the position and velocity of the satellite - which are affected by the other disturbing forces, and vice versa - all the perturbations acting on the satellite tend to interact with each other. This is reflected in the expression for the time variations of its orbital elements. Such expressions are found to contain not only terms with perturbing parameters from a single disturbing force, but combinations of two, or more, such sets of perturbing parameters arising from other perturbations. The latter set of perturbations are known as interactive perturbations.

1.4 The Perturbations Acting on an Artificial Satellite

The main perturbations affecting the motion of an artificial satellite are:

(a) the radially asymmetric nature of the Earth's gravitational field;

(b) the gravitational attraction of the Sun and Moon;

(c) solar radiation pressure;

(d) the Earth's atmosphere.

In addition, a satellite is subjected to a number of minor perturbations which include:

(e) general relativistic effects;

(f) oceanic and body Earth tides.

We will examine these in turn.

(a) The radially asymmetric nature of the Earth gravitational field

Newton (1687) showed that, if the Earth was a sphere of constant density, or a sphere with a radially symmetric density distribution, then the Earth could be treated as a point mass. If this was true, and no other forces acted on a satellite, its orbit would be an ellipse having constant orbital elements with the Earth's centre at one focus.

In reality, the Earth is not a sphere and its density distribution is not radially symmetric. Consequently, the Earth's gravitational potential, U, at an exterior point (r, θ, ϕ) is dependent not only on its radial distance, r, from the Earth's centre of gravity but also on its geocentric latitude, θ, and its longitude, ϕ, measured eastwards from Greenwich along the celestial equator. If the non-sphericity and inhomogeneity of the Earth is taken into account, its gravitational potential, $U(r,\theta,\phi)$, can be shown to satisfy Laplace's equation (Tisserand, 1892).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\emptyset^2} = 0 \tag{1.34}$$

The solution of (1.34) for the Earth's gravitational potential can be expressed as a double infinite series of spherical harmonics, (Fitzpatrick, 1970), viz.

$$U = \frac{GM_E}{r}\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R_E}{r}\right)^n P_n(\sin\theta) + \sum_{n=2}^{\infty}\left(\frac{R_E}{r}\right)^n \sum_{m=1}^{n} P_n^{\,m}(\sin\theta)\right.$$

$$\left. \times\left[C_{n,m}\cos m\emptyset + S_{n,m}\sin m\emptyset\right]\right] \tag{1.35}$$

where R_E is the mean equatorial radius of the Earth; P_n $(\sin\theta)$ - the Legendre polynomial of degree n and argument θ; $P_n^{\,m}$ $(\sin\theta)$ - the associated Legendre function of degree n, order m and argument θ. J_n is the zonal harmonic coefficient of degree n. Lastly, $C_{n,m}$ and $S_{n,m}$ are the tesseral harmonic coefficients of degree n and order m.

The first term in equation (1.35) is the gravitational potential the Earth would have if it was a homogeneous sphere: the two subsequent sets of terms represent the latitudinal and longitudinal dependence of the Earth's gravitational field, resulting from its non-sphericity and inhomogeneity.

The zonal harmonic coefficients, J_n, are the parameters which characterise the longitudinally averaged shape and density distribution of the Earth. In particular, the even zonals reflect symmetries about the equator, whilst the odd zonals reflect the asymmetries. For example, if only J_2 was present in the Earth's geopotential, a polar cross-section through the Earth would be elliptical in shape; similarly, if only J_3 was present, the same cross-section would be 'pear shaped'; and if only J_4 was present, the cross-section would be cube shaped (see Figure 1.5). However, if all the zonal harmonics are included, then to a first approximation, the equation for the polar cross-section of the Earth's shape, averaged over all longitudes, is given by $r = R_E[1 - \Delta(\theta) + \Delta(0) - \Delta'(\theta)]$ where

$$\Delta(\theta) = \sum_{n=2}^{\infty} J_n P_n(\sin\theta) - \frac{1}{2} n_o^2 \frac{R_E^3}{\mu} \cos^2\theta$$

$$\Delta'(\theta) = 3J_2P_2(\sin\theta)\,\Delta''(\theta) - \Delta''^2(\theta) + \frac{n_o^2 R_E^3}{\mu}\,\Delta''(\theta)$$

$$\Delta''(\theta) = J_2P_2(\sin\theta) - J_2P_2(0) + \frac{n_o^2 R_E^3}{2\,\mu} - \frac{n_o^2 R_E^3}{2\,\mu}\cos^2\theta$$

$$+ \text{ terms of order } (n_o^6 J_2,\ J_2^2 n_o^2,\ \{J_n^2, J_n n_o^2, J_n J_2;\ n > 2\})$$

where n_o is the angular velocity of the Earth; r - the distance from the Earth's centre of a point on the surface at a latitude, θ. It should be pointed out that the 'shape of the Earth' being discussed here is the shape of the mean sea-level surface (continued under the land). Such a surface is often called the geoid, (figure 1.10).

In general, the changes in the Earth's gravitational field caused by the even zonal harmonics lead to two major perturbations in a satellite's orbit. First, the longitude of the ascending node, Ω, precesses secularly in a direction opposite to the Earth's rotation (if the inclination of the orbit is $< 90^o$), and in the same direction (if I is $> 90^o$). The rate of change of Ω is given by (King-Hele, 1958)

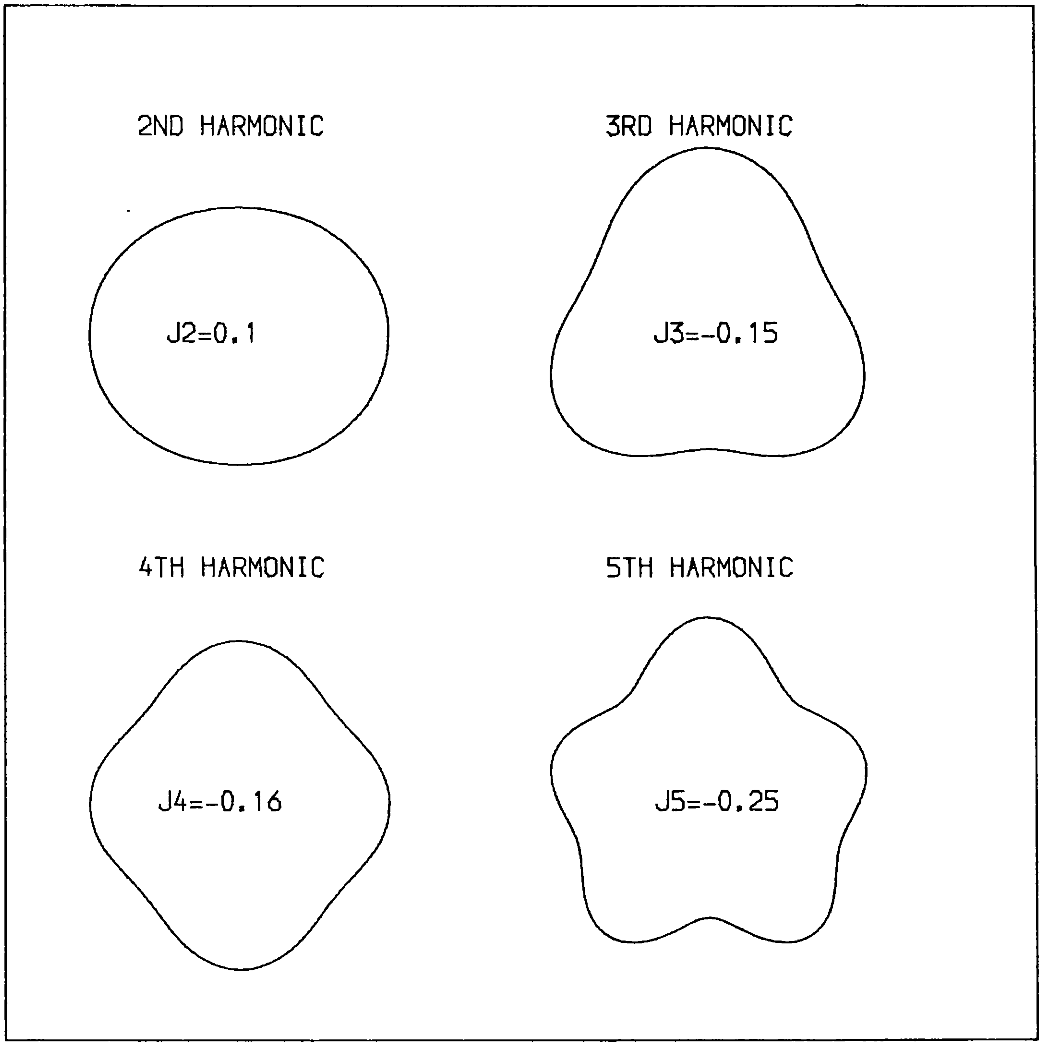


Figure 1.5

Form of 2nd-5th zonal harmonics

$$\frac{d\Omega}{dt} \simeq -\frac{3J_2R_E^2}{2a^2(1-e^2)^2}\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \cos I \tag{1.36}$$

Secondly, the major axis of the orbit rotates in the orbital plane, so that the argument of perigee, ω , changes at a rate given by

$$\frac{d\omega}{dt} \simeq \frac{3J_2R_E^2}{4a^2(1-e^2)}\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} (5\cos^2 I-1) \tag{1.37}$$

For a close (perigee height < 1600 Km) Earth satellite, the changes in Ω and ω can be of the order of several degrees per day. If numerical values for the constants, μ, R_E and J_2 are substituted into (1.36) and (1.37), the rates of change of Ω and ω are found to be

$$\frac{d\Omega}{dt} \simeq -9.97\left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} \cos I \qquad \text{deg/day} \tag{1.38}$$

and

$$\frac{d\omega}{dt} \simeq 4.98\left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} (5\cos^2 I-1) \qquad \text{deg/day} \tag{1.39}$$

$|\dot{\Omega}|$ is approximately zero for satellites in near polar orbits (i.e. $I \simeq 90^\circ$): $\dot{\omega}$ is zero when $\cos I = \pm 1/\sqrt{5}$ or $I =$ 63.4° or 116.6°. The orbital inclination of 63.4° is often called the 'critical inclination'. If the satellite's inclination is less than 63.4° the perigee advances round the orbit in the same direction as the satellite's motion; for I greater than 63.4° it moves backwards. The equations for $\dot{\Omega}$ and $\dot{\omega}$ are not exact because the effects of the other even zonals J_4, J_6, etc., have been neglected. However, the J_2 terms are by far the most important, since J_2 is approximately 1000 times greater than subsequent J_n. Recently determined values for J_2 - J_{20} are given in Table 1.1. The odd zonals also produce changes in Ω and ω , but their effects are small, especially for satellites with near-circular orbits

TABLE 1.1

Values of Zonal Harmonic Coefficients

(J_2, J_4 ... from Wagner (1973); J_3, J_5 ... from King-Hele and Cook (1974).)

$10^9 J_2$	1082635 ± 11	$10^9 J_3$	-2531 ± 7
$10^9 J_4$	-1600 ± 12	$10^9 J_5$	-246 ± 9
$10^9 J_6$	530 ± 26	$10^9 J_7$	-326 ± 11
$10^9 J_8$	-200 ± 29	$10^9 J_9$	-94 ± 12
$10^9 J_{10}$	-224 ± 45	$10^9 J_{11}$	159 ± 16
$10^9 J_{12}$	-208 ± 17	$10^9 J_{13}$	-131 ± 22
$10^9 J_{14}$	-166 ± 25	$10^9 J_{15}$	26 ± 24
$10^9 J_{16}$	3 ± 55	$10^9 J_{17}$	-258 ± 19
$10^9 J_{18}$	-86 ± 56	-	
$10^9 J_{20}$	-85 ± 61	-	

(e < 0.03). The main effect of the odd zonal harmonics is to cause long-period changes in the eccentricity and inclination of a satellite's orbit.

If a satellite's orbital eccentricity is small, the usual set of orbital elements a, e, I, ω, Ω and M are unsuitable for the study of its motion. This situation arises because the perigee of the orbit, and, hence, ω and M, becomes ill-defined as e tends to zero. The result is that singularities occur in the Lagrangian planetary equations for e, ω and M when e = 0. For studies of near-circular orbits, the variables a, e cos ω, e sin ω, I, ω + M and Ω are found to be more suitable (Cook, 1966). If the eccentricity is small (which is true for most satellites launched to date), the long-period changes in e, ω and I, caused by the odd zonal harmonics, can be obtained from the subsidiary variables, x_1, x_2, x_3 and y_2, defined by (Hughes and Meadows, 1977)

$$x_1 = \sqrt{\mu a} \qquad x_2 = \sqrt{x_1}\, e \sin \omega$$

$$x_3 = x_1 \left\{1 - (x_2^2 + y_2^2)/x_1\right\}^{\frac{1}{2}} \cos I \tag{1.40}$$

$$y_2 = \sqrt{x_1}\, e \cos \omega$$

The changes in x_1, x_2, x_3 and y_2 are given by

$$x_1 = \text{constant} \tag{1.41}$$

$$x_2 = P/A + (\overset{o}{x}{}_2^{*} - P/A) \cos\left\{ y_2^{*}/(\overset{o}{x}{}_2^{*} - P/A)\right\} \tag{1.42}$$

$$y_2 = (\overset{o}{x}{}_2^{*} - P/A) \sin \left\{ y_2^{*}/(\overset{o}{x}{}_2^{*} - P/A)\right\} \tag{1.43}$$

$$x_3 = x_1 \cos I_o = \text{constant} \tag{1.44}$$

where

$$P = \left(\frac{\mu^3}{a^5}\right)^{\frac{1}{4}} \sum_{n=3}^{\infty} J_n \left(\frac{R_E}{a}\right)^n \frac{(n-1)}{n(n+1)} P_n^{\,1} (\cos I_o)\, P_n^{\,1} (0) \tag{1.45}$$

$$A = \frac{3J_2}{4} \left(\frac{R_E}{a}\right)^2 \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} (5 \cos^2 I_o - 1) \tag{1.46}$$

and

$$y_2^{\,*} = \overset{o}{y}_2^{\,*} - (P - A\overset{o}{x}_2^{\,*})t \tag{1.47}$$

I_o, $\overset{o}{x}_2^{\,*}$ and $\overset{o}{y}_2^{\,*}$ are the constants of integration obtained from the initial conditions. The equations giving the long-period zonal harmonic perturbations in the eccentricity, argument of perigee and inclination can easily be obtained from (1.40) - (1.45). The results are

$$e = \frac{1}{(\mu a)^{\frac{1}{4}}} \left[(P/A)^2 + \frac{2P}{A} (\overset{o}{x}_2^{\,*} - P/A) \cos\left\{ \overset{o}{y}_2^{\,*}/(\overset{o}{x}_2^{\,*} - P/A) - At \right\} \right]^{\frac{1}{2}} \tag{1.48}$$

$$\omega = \tan^{-1} \left[\frac{P/A + (\overset{o}{x}_2^{\,*} - P/A) \cos\left\{ \overset{o}{y}_2^{\,*}/(\overset{o}{x}_2^{\,*} - P/A) - At \right\}}{(\overset{o}{x}_2^{\,*} - P/A) \sin\left\{ \overset{o}{y}_2^{\,*}/(\overset{o}{x}_2^{\,*} - P/A) - At \right\}} \right] \tag{1.49}$$

$$I = \cos^{-1} \left[\cos I_o \left(1 - \frac{1}{(\mu a)^{\frac{1}{2}}} \left[(P/A)^2 + \frac{2P}{A} (\overset{o}{x}_2^{\,*} - P/A) \cos\left\{ \overset{o}{y}_2^{\,*}/(\overset{o}{x}_2^{\,*} - P/A) - At\right\} + (\overset{o}{x}_2^{\,*} - P/A)^2 \right] \right)^{-\frac{1}{2}} \right] \tag{1.50}$$

Since A is the secular rate of change of ω caused by the J_2 harmonic, and $P_n^{\,1}(0) = 0$ for even n, the variations in e and I for near circular orbits are long-period changes produced by the odd zonal harmonics in the geopotential. The corresponding equations giving the zonal harmonic perturbations in the elements $\omega + M$ and Ω, when e is small, have been given by Hughes and Meadows (1977).

On eliminating $y_2^*/(\overset{o}{x}_2^* - P/A)$ from equations (1.42) and (1.43), x_2 and y_2 are found to satisfy the equation

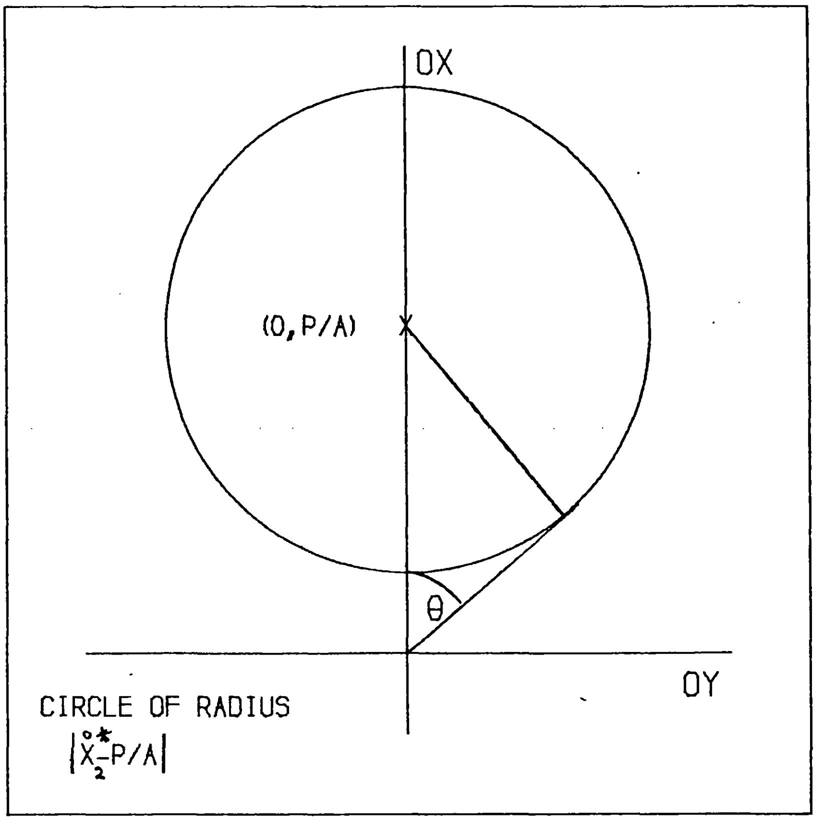
$$(x_2 - P/A)^2 + y_2^2 = (\overset{o}{x}_2^* - P/A)^2 \tag{1.51}$$

which is the equation of a circle in the (x_2, y_2) plane (see figure 1.6), having a radius of $|\overset{o}{x}_2^* - P/A|$ and centre at the point (P/A, 0). The vector Z from the origin, 0, to the point (x_2, y_2) has a modulus of $(\mu a)^{\frac{1}{4}} e$ and an argument, ω, measured anticlockwise relative to the axis Oy_2. Consequently, if $|\overset{o}{x}_2^* - P/A| < P/A$ (figure 1.6a), the argument of perigee, ω, is restricted to the values, $\pi/2 - \Theta < \omega < \pi/2 + \Theta$ (when $P/A > 0$), and to the values $3\pi/2 - \Theta < \omega < 3\pi/2 + \Theta$ (when $P/A < 0$). The angle Θ is given by

$$\Theta = \sin^{-1} (A\,|\overset{o}{x}_2^* - P/A|\,/P) \tag{1.52}$$

For $|\overset{o}{x}_2^* - P/A| < P/A$, the orbital eccentricity, e, varies between the limits $(|P/A| \pm |\overset{o}{x}_2^* - P/A|)/(\mu a)^{\frac{1}{4}}$. Similarly, if $|\overset{o}{x}_2^* - P/A| > P/A$ (figure 16b), the argument of perigee takes all possible values, and the eccentricity varies between the limits $(|\overset{o}{x}_2^* - P/A| \pm |P/A|)/(\mu a)^{\frac{1}{4}}$.

The equation giving the zonal harmonic variations in the orbital elements a, e, I, ω, Ω and M for large, or moderate, eccentricities have been obtained by a number of authors (Brouwer, 1959; Groves, 1960; Merson, 1961). However, since most satellites launched to date have small orbital eccentricities ($e < 0.03$), a solution valid for near-circular orbits will be generally applicable. Recently, the perturbation theories of Hori and Deprit (Hori, 1966; Deprit, 1969) have been used to obtain highly accurate solutions for the motion of a satellite acted upon by the zonal harmonics (Aksnes, 1970; Deprit and Rom, 1969; Kutuzov, 1975; Kinoshita, 1976). Such theories are based on the dynamical methods of Hamilton, Jacobi,



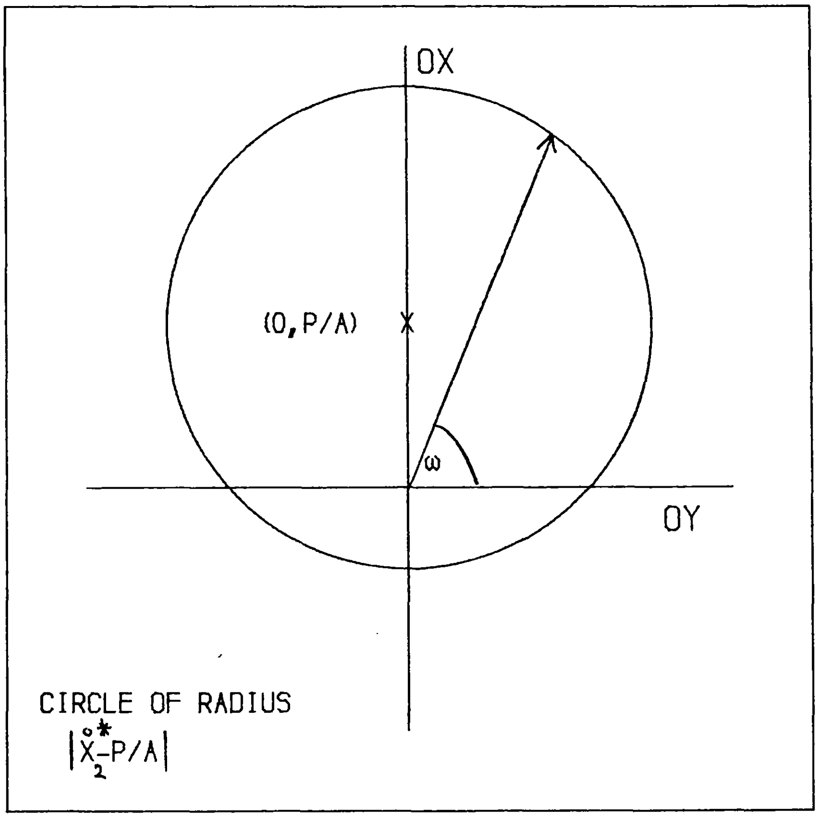


Figure 1.6

Variation of e and ω for a near circular satellite orbit perturbed by the zonal harmonics

Delaunay and Lie (Lie, 1888; von-Zeipel, 1916; Smart, 1953), and can be applied to satellites having small or moderate orbital eccentricities ($e < 0.1$). The solutions obtained by these methods all contain a singularity for satellites with orbital inclinations equal to the 'critical inclination' of 63.4^{o}. Thus the equations (1.42) and (1.43) contain a singularity if $A = 0$ (i.e. if $I \simeq 63.4^{o}$). Opinion as to the nature of the singularity at the 'critical inclination' is divided into two schools. First, there are those who believe it results from the physical nature of the motion for a satellite with an orbital inclination of 63.4^{o}. Message et al. (1962) suggest that the singularity occurs because of a resonance between the satellite's orbital period measured relative to its node and its orbital period relative to its perigee. The second school of thought maintains that the 'critical inclination' is due entirely to the method of mathematical analysis adopted, rather than to any physical effect (Lubowe, 1969(1)). Lubowe has compared, by numerical integration, the orbital changes caused by the zonal harmonics in the geopotential for satellites of differing orbital inclination. He concludes that there is no noticeable difference between the 'critical inclination' and any other inclination. More recently, Deprit (1977) has used the dynamical methods of Hamilton and Lie to show that it is possible to generate two new 'critical inclinations' away from 63.4^{o} when a satellite's orbital eccentricity is small.

The Earth's gravitational field, in addition to suffering latitudinal variations, also undergoes variations with longitude (Izsak, 1961; Kaula, 1963). The gravitational potential, U_{long}, representing the longitudinal variations, is given by the second set of terms in equation (1.35), viz.

$$U_{long} = \sum_{n=2}^{\infty} \sum_{m=1}^{n} \frac{GM_E}{r} \left(\frac{R_E}{r}\right)^n P_n^{\ m}(\sin \Theta) \times [\, C_{n,m} \cos m\phi + S_{n,m} \sin m\phi \,] \tag{1.53}$$

The tesseral harmonics, like the zonal harmonics, characterise the Earth's shape and density distribution. The tesseral harmonics, however, not only reflect latitudinal variations, but longitudinal variations as well. In other words, the tesseral harmonics characterise the longitudinal variations in the geopotential as a function of latitude. The suffix n in equation (1.53) may be regarded as specifying the latitudinal variations, and the suffix m - the meridional variations. For example, if all the tesseral harmonic coefficients were zero except those of the second order ($m = 2$, $n = 2, 3 \ldots$), then a cut along the equator (or any other latitude) would reveal an approximately elliptical cross-section. Similarly, if only the fifteenth-order tesseral harmonic coefficients were present in equation (1.53), a cross-section of the equator would reveal a 15-petalled shape with maxima at 24° intervals in longitude.

The variations of gravity with longitude usually produce only very small orbital perturbations, because the satellite samples all longitudes impartially and the perturbation effects tend to cancel out. The so-called 'longitudinal Earth gravity' resonance orbits are an exception to this rule. If a satellite is in an orbit which makes β revolutions whilst the Earth rotates α times relative to the precessing satellite, viz.

$$\alpha\,(\dot{\omega} + \dot{M}) \simeq \beta\,(n_o - \dot{\Omega})$$

(where n_o is the angular velocity of the Earth's rotation) the satellite is said to be in a $\beta : \alpha$ resonance. The harmonics of order β in the geopotential can be regarded as having 'humps' every $(360/\beta)^{\circ}$ in longitude. Their effect on a resonant orbit will be the same for each consecutive set of α rotations of the Earth; so their influence will build up day after day, while the effects of the other tesseral harmonics tend to cancel out.

The changes in the orbital elements of a satellite experiencing a 'longitudinal Earth gravity' resonance of order β provides a good method for determining the tesseral harmonic coefficients of that order. This method has been used by King-Hele, Walker and Gooding (1975(1), 1975(11)) to determine the tesseral harmonic coefficients of order 15. A satellite in a 15th-order resonance is particularly well suited for the determination of tesseral harmonic coefficients. It has an orbital period near 95 mins, which corresponds to a satellite height of approximately 500 km, where air drag is appreciable. Consequently, a number of satellites each year experience 15th-order resonance as their orbit decays under the influence of air drag. In principle, tesseral harmonic coefficients of order higher or lower than 15 can be obtained by this method. In practice, a number of difficulties arise when orders other than 15 are being considered. A satellite in a higher order resonance will be subjected to a larger drag effect owing to its lower orbit, and, hence, will pass through resonance more quickly. This makes it difficult to obtain accurate observations in sufficiently large numbers, before the satellite passes through the resonance. On the other hand, a satellite in a lower-order resonance will pass through the resonance more slowly. Consequently, very few satellites can be found which are experiencing resonances of order $<$ 15.

The orbital inclination, I, of a resonant satellite is the most useful orbital parameter for the determination of the tesseral harmonic coefficients. First, if the resonant terms in equation (1.53) are expanded as functions of the satellite's orbital elements, their effect on I is of zero order in the eccentricity, e (Allan, 1967(1), 1973). As a result, changes in I are important for satellites with near-circular orbits - such satellites form the majority of satellites launched to date. Second, the effect of the zonal harmonics on a satellite's orbital inclination is small, and can be accurately removed. Lastly, accurate values of the inclination can be obtained from the observational data. The standard deviation in I is usually of the order of 0.001°, compared with a resonant change in the region of 0.02°.

The rate of change of I for a satellite in a near-circular orbit due to the effect of the 15th-order resonant terms may be written as (Allan, 1973)

$$\frac{dI}{dt} = Q\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}\left(\frac{R_E}{a}\right)^{15}\{\bar{C}_{15}\sin\Phi - \bar{S}_{15}\cos\Phi\}$$

$$+ \text{ terms in } (\bar{C}_{30}, S_{30})^{\cos}_{\sin} 2\Phi; \quad (\bar{C}_{45}, \bar{S}_{45})^{\cos}_{\sin} 3\Phi$$

$$+ \text{ terms of order } 10e\ (\bar{C}_{n,15}, \bar{S}_{n,15})^{\cos}_{\sin}(\Phi \pm \omega) \text{ etc.} \tag{1.54}$$

where

$$Q = 0.5877\,(15 - \cos I)(1 + \cos I)\sin^{13} I \tag{1.55}$$

The resonance angle, Φ, is defined by

$$\Phi = (\omega + M) + 15(\Omega - \gamma)$$

where γ is the sidereal angle. Exact 15th-order resonance occurs when $\dot{\Phi} = 0$. The quantities $\bar{C}_{15}$, $\bar{S}_{15}$, $\bar{C}_{30}$ etc. are the lumped harmonic coefficients, and are related to the individual tesseral harmonic coefficients by linear expressions of the form

$$\bar{C}_{15} = \frac{C_{15,15}}{N_{15,15}} + \frac{Q_{17}C_{17,15}}{N_{17,15}} + \frac{Q_{19}C_{19,15}}{N_{19,15}} + \frac{Q_{21}C_{21,15}}{N_{21,15}} + \ldots$$

where the Q_n's are coefficients dependent on the inclination I (Allan, 1973), and the $N_{n,m}$'s are normalising factors defined by

$$N_{n,m} = \left\{ 2(n-m)!\ (2n+1)/(n+m)! \right\}^{\frac{1}{2}}$$

Similar expressions exist for $\bar{S}_{15}$, $\bar{C}_{30}$ and $\bar{S}_{30}$, etc., which are linear functions of the tesseral harmonic coefficients $S_{n,15}$, $C_{n,15}$ and $S_{n,30}$ etc. By fitting the theoretical changes in I to the observed variations, the values of $\bar{C}_{15}$, $\bar{S}_{15}$, etc., can be determined for a particular satellite. Values of these lumped coefficients obtained from a number of satellites at a variety of inclinations, all of which are suffering 15th-order resonance, give a set of simultaneous linear equations in the unknown tesseral harmonic coefficients. These equations can then be solved to obtain values for the coefficients of order 15 and degree 15, 17, 19 ...; order 30 and degree 30, 32 ... 34, etc. Since the expected magnitude of a typical $C_{n,m}$ or $S_{n,m}$ coefficient is, according to Kaula's rule-of-thumb law, of the order of $10^{-5}/n^2 N_{n,m}$ (Kaula, 1966), the 30th, 45th and higher-order terms are likely to be considerably smaller than those of the 15th order. The values of the tesseral harmonic coefficients of order 15 and degree up to 33 have been obtained by King-Hele, Walker and Gooding (1975(1)) from the analysis of 11 resonant orbits of inclinations of between 30° and 90°. Their results are given in Table (1.2). From the changes in the orbital eccentricity of a satellite experiencing 15th-order resonance, the tesseral harmonics of order 15 and even degree ($n = 16, 18, 20, 22 \ldots$) can be obtained (Allan, 1967(11)). The values of the tesseral harmonic coefficients of order 15 and even degree up to 22 obtained by King-Hele, Walker and Gooding (1975(11)) are listed in Table (1.3).

TABLE 1.2

Values of Tesseral Harmonic Coefficients of Order 15 and Odd Degree *

(From King-Hele, Walker and Gooding (1975(1)).)

n	$10^9\ \bar{C}_{n,15}$	$10^9\ \bar{S}_{n,15}$
15	-23.5 ± 0.8	-7.7 ± 0.8
17	6.3 ± 1.5	5.6 ± 1.5
19	-25.1 ± 2.5	-7.3 ± 2.3
21	27.8 ± 3.6	-0.7 ± 3.4
23	17.1 ± 4.1	13.9 ± 4.8
25	- 1.1 ± 3.0	8.5 ± 4.2
27	10.0 ± 3.3	6.7 ± 2.7
29	- 9.4 ± 3.5	0.1 ± 4.7
31	10.1 ± 5.4	3.8 ± 5.6
33	1.1 ± 5.7	3.1 ± 5.8

* $\bar{C}_{n,15}$ and $\bar{S}_{n,15}$ are the normalised harmonic coefficients, related to $C_{n,15}$ and $S_{n,15}$ by the equation

$$\bar{C}_{n,15} = C_{n,15}/N_{n,15} \qquad \bar{S}_{n,15} = S_{n,15}/N_{n,15}$$

where $N_{n,15}$ is the normalising factor given by the equation

$$N_{n,15} = \left\{ 2(n-15)!\ (2n+1)/(n+15)! \right\}^{\frac{1}{2}}$$

TABLE 1.3

Values of Tesseral Harmonic Coefficients of Order 15 and Even Degrees

(From King-Hele, Walker and Gooding (1975(11)).)

n	$10^9 \bar{C}_{n,15}$	$10^9 \bar{S}_{n,15}$
16	-13.7 ± 1.3	-18.5 ± 2.7
18	-42.3 ± 1.8	-34.7 ± 3.4
20	10.5 ± 3.1	29.8 ± 5.2
22	-8.6 ± 3.8	20.2 ± 7.4

More recently, the 31:2 and 29:2 resonances have been studied (Hiller and King-Hele, 1977; Walker, 1977). Values for the lumped harmonic coefficients were obtained, but more orbits of satellites experiencing these resonances need to be analysed before anything can be said about the individual tesseral harmonic coefficients.

(b) The gravitational attraction of the Sun and Moon

For an artificial satellite in a near-circular orbit ($e < 0.03$), having an altitude of less than 1600 km ($a/R_E < 1.25$), the effect of lunisolar gravity perturbations on the orbit is small. The change in the perigee height usually amounts to less than 0.2 km for a satellite in such an orbit, the corresponding zonal harmonic perturbation being of the order of 20 km. All six orbital elements suffer changes; their variations being in general a combination of two, or more, of the five types discussed in Section (1.3). The semi-major axis undergoes only short-period fluctuations, and only ω, Ω and M suffer secular changes due to lunisolar gravity perturbations.

The vector equation of motion for a satellite perturbed by lunisolar gravity is (Roy, 1965)

$$\ddot{\underline{r}} + \mu \underline{r}/r^3 = G \sum_{i=1}^{2} M_i^* \{ (\underline{R}_i - \underline{r})/\Delta_i^3 - \underline{R}_i/R_i^3 \} \qquad (1.56)$$

where $i = 1$ refers to the Moon and $i = 2$ refers to the Sun; M_i^* is the mass of the disturbing body; $\underline{R}_i$ is the position vector of the disturbing body from the Earth; Δ_i is the distance of the satellite from the disturbing body. The corresponding expression for the lunar or solar gravity disturbing potential, U_i, is

$$U_i = GM_i^* (1/\Delta_i - \underline{r} \cdot \underline{R}_i/R_i^3) \qquad (1.57)$$

Now Δ_i can be written as

$$\Delta_i = R_i \left[1 + \left(\frac{r}{R_i}\right)^2 - 2\left(\frac{r}{R_i}\right)^2 \cos \delta_i\right]^{\frac{1}{2}} \tag{1.58}$$

where δ_i is the angle subtended at the Earth's centre by the satellite and the disturbing body. Since Δ_i^{-1} is the generating function for the Legendre polynomials of argument δ_i (Spiegel, 1974), and $\underline{r} \cdot \underline{R}_i = r\,R_i \cos \delta_i$, equation (1.57) becomes

$$U_i = \frac{GM_i^*}{R_i} \sum_{n=2}^{\infty} P_n(\cos \delta_i)\left(\frac{r}{R_i}\right)^n \tag{1.59}$$

The n = 0 term has been omitted from (1.59) because it is independent of the satellite's co-ordinates and will therefore producc no changes in its orbital elements. For a close Earth satellite the ratios r/R_1 and r/R_2 are approximately 2×10^{-2} and 5×10^{-4}, respectively. The ratio, σ, of the lunar and solar gravity disturbing functions, U_1 and U_2, at a point in its orbit is therefore approximately given by

$$\sigma = \frac{U_1}{U_2} \simeq \frac{M_1^*}{M_2^*} \left(\frac{R_2}{R_1}\right)^3 \frac{P_2(\cos \delta_1)}{P_2(\cos \delta_2)} \tag{1.60}$$

which, after the substitution of numerical values for M_1^*, M_2^*, R_1 and R_2, becomes

$$\sigma \simeq 1.85 \; \frac{P_2(\cos \delta_1)}{P_2(\cos \delta_2)} \tag{1.61}$$

Consequently, if the angular displacements of the Sun and Moon from the line passing through the centre of the Earth and the satellite are comparable (i.e. $|\cos \delta_1| = |\cos \delta_2|$), the gravitational effect of the Moon on a close Earth satellite is approximately twice that of the Sun.

If equation (1.59) is expanded as a function of the orbital elements of the satellite and the disturbing body (Kaula, 1962; Allan, 1969(11)), the Lagrangian planetary equations can then be used to obtain expressions for the time derivatives of the orbital elements of a satellite perturbed by the gravitational influence of the disturbing body. A number of authors (Kozai, 1959; Cook, 1962; Smith, 1962; Gooding, 1966) have used this method to obtain the lunisolar gravity perturbations in the orbital elements over one revolution. However, none of the above-mentioned authors gave explicit analytical expressions for the general time variations in the orbital elements produced by the gravitational influence of the Sun and the Moon. This is, perhaps, not surprising considering the enormous complexity of such a task (Fisher, 1972). More recently Cook (1973) and Giacaglia (1974) have adopted a semi-analytical approach, obtaining analytical expressions for the time derivatives of the orbital elements and integrating them numerically by machine. As yet no-one has published a complete analytical approach to this problem.

(c) Solar radiation pressure perturbations

The perturbations due to solar radiation pressure are small for satellites of normal construction, but can be large for balloon-type satellites, which have an enormously large area-to-mass ratio of the order of 10 cm^2/gm. (A typical satellite of normal construction will have an area-to-mass ratio of about 0.04 cm^2/gm.) For example, an oscillation in perigee height of 500 km was produced in the orbit of the balloon satellite Echo 1 by solar radiation pressure perturbations. The period of the cycle - about 10 months - was the synodic period of the perigee, that is the time it took to make one rotation of the Earth relative to the Sun.

Solar radiation pressure perturbations are of two types: first, those resulting from the direct effect of the solar flux on the satellite; and, secondly, those due to sunlight being reflected or re-radiated back onto the satellite from the Earth's atmosphere. The analysis of the effect of solar radiation pressure perturbations on artificial satellite orbits is one of the most difficult problems in celestial mechanics. Let us examine each of the two types of solar radiation pressure perturbations, beginning with the simpler of the two - the direct perturbations.

In order to compute the direct effect of solar radiation pressure on a satellite orbit it is necessary to make a number of assumptions - some of which are not entirely correct. First, we assume that the Sun's output of energy is known, and is constant. Second, the solar radiation flux varies inversely as the square of the distance from the Sun's centre and is incident on the satellite in a direction parallel to the line joining the centres of the Sun and the satellite. Third, the area-to-mass of the satellite can be represented by some mean value. Fourth, the properties of the reflection occurring at the satellite are fully known, or can be represented by some known average value. Lastly, the entry and exit of the satellite into the shadow cast by the Earth can be represented by some simple model. These are just a few of the more important assumptions that need to be made about the direct solar radiation pressure perturbing function. If these assumptions are accepted, then the vector equation of motion of a satellite perturbed by direct solar radiation pressure is (Hughes, 1977)

$$\ddot{\underline{r}} + \mu \underline{r}/r^3 = \frac{- S_o \overline{Aa}^{*2}(2 - \epsilon)\, \sigma \underline{\Delta}}{c m_s \Delta^3} \qquad (1.62)$$

where S_o is the solar radiation flux at a distance, a^*, from the Sun's centre; a^* - the semi-major axis of the Earth's orbit; ϵ - the fraction of the Sun's radiation absorbed by the satellite; $\bar{A}$ - the average cross-sectional area of the satellite exposed to the Sun's radiation; c - the speed of light; m_s - the mass of the satellite; finally, σ is the 'shadow' parameter, which takes a value of 1 if the satellite is in sunlight and a value of 0 if it is in shadow. The corresponding expression for the solar radiation pressure disturbing potential, Φ_{rad}, is

$$\Phi_{rad} = \frac{- S_o \bar{A} a^{*2} (2 - \epsilon)\sigma}{c m_s \Delta} \tag{1.63}$$

This model, although perhaps the best available, still suffers from a number of limitations and difficulties. In practice, the area-to-mass ratio may not be well known; this is complicated by the fact that most satellites are irregularly shaped and rotating in space. The area-to-mass ratio in such cases will be an extremely complicated function of time. Similar remarks also apply to a satellite's reflective properties - which may be poorly determined, perhaps varying over the satellite's surface and changing with time. In the case of a satellite such as the laser ranging satellite, Lageos, which is a sphere of uniform texture with accurately known values for A/m_s and ϵ, the effect of direct solar radiation pressure on its orbit will be well known.

Perhaps the most difficult and uncertain problem concerning direct solar radiation pressure perturbation is the 'shadow effect'. Two different types of approach have been used to allow for the 'shadow effect'; both of which rely on the assumption that $\sigma = 1$, when the satellite is in sunlight, and $\sigma = 0$ when in shadow. The first method adopted by Bryant (1961), Escobal (1962) and Aksnes (1976) consists of initially determining whether, or not, the satellite passes into the Earth's shadow during the course of an orbit. If it does, then the time of entry into the shadow and the time of exit from the shadow are computed. Once these times are known, the expansion of Φ_{rad} in terms of the orbital elements of the satellite and the Sun, together with the Lagrangian planetary equations, can be used to obtain the perturbing effects of direct solar radiation pressure on the orbit. The integration of the resulting differential equation is carried out only during the times the satellite is in sunlight.

The second method approximates the step function

$$\sigma = \begin{cases} 1 & \text{sunlight} \\ 0 & \text{shadow} \end{cases}$$

at any given time by some known function or Fourier series (Lala and Sehnal, 1969; Ferraz-Mello, 1964), which is then expanded in terms of the orbital elements of the satellite and the Sun. If the resulting series for the 'shadow function', σ , is inserted into the expansion of Φ_{rad}, the changes in the satellite's orbital elements due to direct solar radiation pressure can be obtained. In this case, the integration of the resulting differential equations is carried out over the whole time interval - the shadow function, σ, having allowed for the passage of the satellite into and out of the Earth's shadow. The 'shadow effect' is complicated by the fact that σ is not a simple step function which can only take values of 0 or 1. In reality, the Earth's shadow is not cylindrical in shape as a simple step function would suggest: it is, in fact, conical, with central regions of umbra for which $\sigma = 1$ and peripheral regions of penumbra where σ lies between 0 and 1.

The problems concerning the perturbing effect of direct solar radiation pressure on an Earth satellite are well known, but comparatively little attention has been given to the indirect effect of solar radiation scattered or reflected from the Earth. This lack of attention is probably due in part to the fact that all estimates of the latter effect suggest its magnitude is smaller than that of direct radiation, and therefore of less importance. It is also a more difficult problem, defying a simple, but realistic, analytical solution.

The Earth albedo radiation can be divided into 3 components - the infrared, the diffuse and the specular. The infrared is largely a re-emission of radiation received at other wavelengths; while the diffuse and specular components are reflections in the optical of the incident sunlight. The reflected radiation as a fraction of the incident radiation is called the albedo, and is composed of diffuse and specular radiation. If equilibrium is assumed (i.e. the Earth returns as much radiation into space as it receives), then

$$\alpha_D + \alpha_S + \alpha_{IR} = 1$$

where α is the albedo, and the suffixes D, S and IR indicate the diffuse, specular and infrared components, respectively.

Since the optical albedo ($\alpha_D + \alpha_S$) is about 0.4 (Allen, 1962), therefore $\alpha_{IR} = 0.6$, implying that equal consideration should be given to both the optical and infrared perturbations. It is difficult to say what fraction of the optical albedo is diffuse and what fraction is specular. However, since diffuse reflection is produced by the continents, the clouds and snow fields, while only very calm seas or lakes tend to produce specular reflection, a value of $\alpha_S = 0.04$ would seem reasonable (Smith, 1966).

The Earth's albedo is, of course, a variable in both time and position; but, in order to obtain an analytical solution for the effect of albedo radiation on a satellite orbit, certain assumptions have to be made about the albedo which depart from reality and may lead to inaccurate results. This, in essence, is the difficulty concerning albedo radiation perturbations - how can the disturbing function be simplified so that the problem remains solvable without the model becoming unrealistic? Let us now consider the various models proposed to calculate the perturbations in a satellite's orbital elements due to optical and infrared albedo radiation, beginning with the infrared models.

For the infrared albedo perturbations the simplest model that could be introduced is a uniform flux acting radially from the centre of the Earth and varying according to the inverse square law. Such a force is indistinguishable from the central force term in the gravitational and produces no perturbations in a satellite's orbit. The next simplest model (Wyatt, 1963) is one where every point on the Earth's surface emits radiation according to Lamberts law. Wyatt showed by symmetry arguments that the transverse component of the perturbing force is zero and on integrating the radial components over the visible cap of the Earth obtains an identical result to that obtained for the very simplest model. Two other models were proposed by Wyatt, both introducing latitudinal variations in the infrared radiations of the form (1) $\cos \delta$, and (2) $C_1 + C_2 \cos \delta$, where δ is the lattitude and C_1 and C_2 are constants. In order to simplify the problem, Wyatt assumed that the photons move radially outwards from the Earth's centre and that the obliquity of the ecliptic is zero. For these two models Wyatt found no secular or long-period variations in either the anomalistic period or the eccentricity. As yet no-one has attempted to introduce a simple longitudinal variation into the infrared albedo perturbations. The results of Wyatt do, however, tend to suggest that simple models which assume smooth variations in the radiation with respect to position all produce little, or no, perturbing effect on a satellite orbit. It may therefore be that any long-term infrared perturbations will be due to asymmetries and irregularities in the radiation, and that any simple model will automatically indicate little, or no, effect.

In order to calculate the effect of optical albedo radiation on a satellite orbit it is necessary to compute the perturbing forces arising from the reflection of sunlight from the Earth according to Lambert's law. This problem has been studied by a number of authors (Cunningham, 1962, 1963; Levin, 1962; Sehnal, 1963, 1965; Wyatt, 1963; Baker, 1965; Lautman,(1977(1)). However, in all cases simplifications had to be made to the basic problem so that the models remained solvable.

The first study that derived expressions for the perturbations by diffuse reflection was Sehnal (1963), who obtained the changes in the period and eccentricity for a uniform albedo model. He made the additional assumptions that the force on the satellite is radial, the Sun lies in the orbit plane, and the force is zero when the satellite is above the dark side of the Earth. For a satellite of the Echo 1 type in a near-circular orbit at a height of 1600 km, Sehnal found a change of 5×10^{-3} in the period per orbit. Wyatt (1963) has also considered diffuse albedo perturbation and proposed three simple models. The first model is the same as Sehnal's. The second assumes that the force varies as the cosine of the zenith angle of the Sun. In the third, Wyatt tries to account for the fact that the Earth phase function is more peaked than is predicted by diffuse Lambert reflection. The most important result of Wyatt's study is that all three models tend to suggest perturbations due to diffuse albedo radiation are about 8 to 10 percent of those produced by direct sunlight.

The models of Cunningham, Levin, Baker and Lautmann are considerably more complex - the perturbing forces being calculated from the diffuse reflection of sunlight off the Earth according to Lambert's law. Yet, in all cases, simplifications had to be made (e.g. the assumption of a constant albedo). Perhaps the largest source of error in any of the models mentioned so far is the lack of latitudinal variation. Much of the albedo variation is due to changing cloud cover, which has well known variations with latitude. In addition, there are the polar ice caps, which can add to other seasonal variations in the albedo.(Lautman,(1977(11)) has taken these facts into account, and has modified his original model to include a latitudinal variation in the albedo of the form $C_1 + C_2 \sin^2 \delta$, where C_1 and C_2 are constants.

Specular reflection has been discussed extensively by Wyatt. Wyatt argues that the size of the solar image on the surface of the Earth for near satellites is only a few kilometres in radius; and, hence, any waves or ripples on the surface of the sea will tend to smear out the radiation. If this is true, the specular component of the reflected radiation will be lost in the irregularities of the diffuse radiation. Assuming $\Upsilon_s = 0.02$, Wyatt has calculated the perturbations in the eccentricity and the period of a satellite produced by specular reflection, and found them to be negligible.

(d) <u>The Earth's atmosphere</u>

The Earth's atmosphere exerts a drag on an artificial satellite in a direction opposite to that of the satellite's motion. Such a drag force is due to the continual collisions of air molecules, atoms and ions with the satellite. Since the density of the Earth's atmosphere decreases rapidly with height (figure 1.7), a satellite in an elliptical orbit is affected most by air drag at those points in its

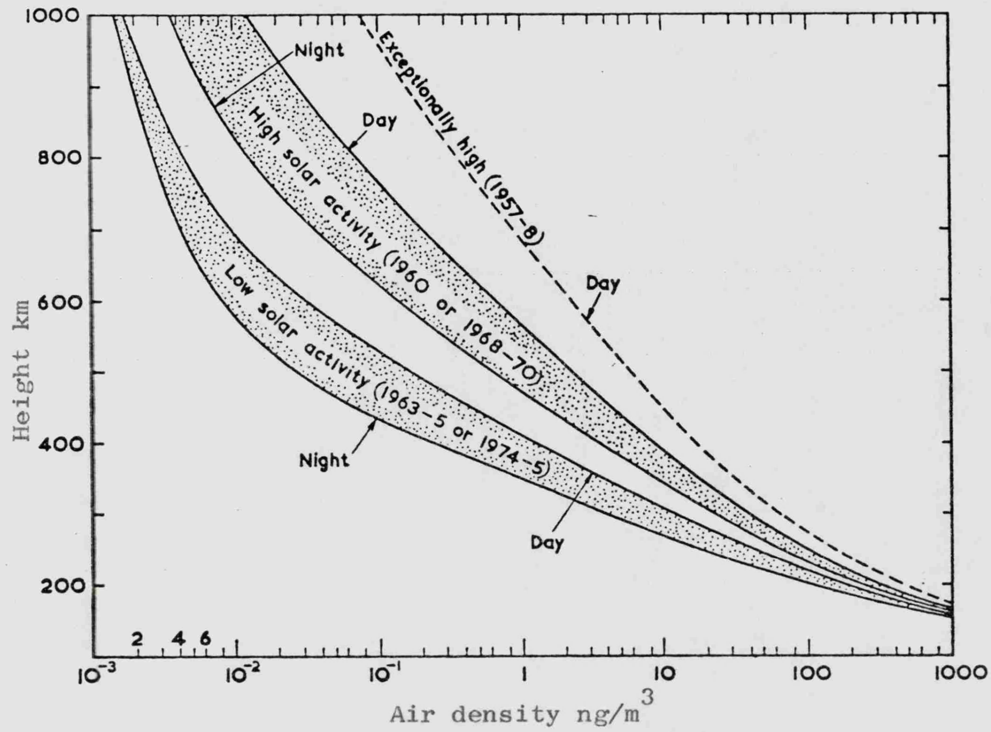


Figure 1.7

Density versus height from 150 to 1000 km for

high and low solar activity

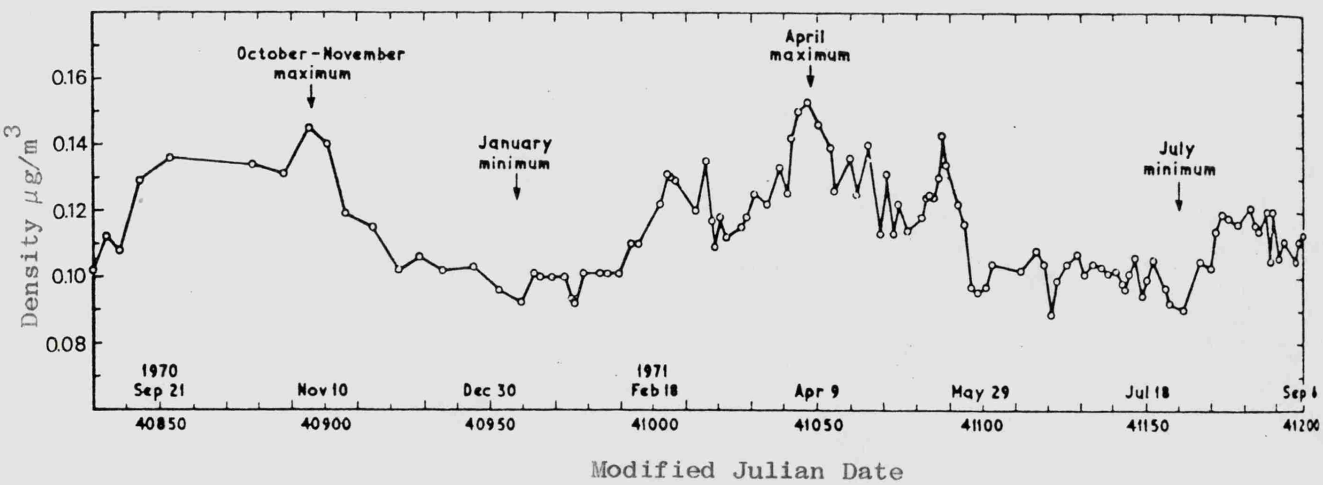


Figure 1.8

Variation of air density at 230 km height, corrected for solar activity and day to night effects, showing semi-annual variation. From analysis of the orbit of 1970-65D. (Walker, 1974) orbit which are closest to the Earth. Therefore, to a first approximation, the effect of air drag is to retard the satellite as it passes its perigee. The result is that its subsequent apogee height is reduced, whilst its perigee height remains virtually unaffected. The orbit therefore contracts and becomes less elliptical (i.e. the semi-major axis, a, and the eccentricity, e, decrease secularly). If the Earth's atmosphere were non-rotating and spherical, only the semi-major axis and the eccentricity of a satellite's orbit would be affected by air drag perturbations. In reality, the Earth's atmosphere is neither stationary nor spherical: it rotates at the rate of 1 revolution per day and is oblate in shape with an ellipticity of about 0.00335. As a result of this atmospheric rotation, the satellite is subjected to small lateral forces which slightly alter the orientation of the orbital plane, leading to small secular changes in the inclination, I, and small periodic changes in the longitude of the ascending node, Ω (King-Hele, 1964; 1966). The effect of the atmospheric oblateness is to produce small periodic changes in the orbit's perigee, (Cook, 1961).

If a satellite is moving with a speed, V, relative to the ambient air in a region of atmosphere of density, ρ, then the air drag force, F, per unit mass acting on the satellite is given by (King-Hele, 1964)

$$F = \frac{C_D \bar{A} \rho V^2}{2m_s} \qquad (1.64)$$

where C_D is the aerodynamic drag coefficient; $\bar{A}$ the average cross-sectional area of the satellite; m_s is its mass. In general, a satellite will be subjected to lift forces, as well as drag forces. Both types of forces will change with time if the satellite is spinning and tumbling as it passes with varying velocity through regions of different density. In the absence of precise knowledge of the satellite's attitude and the atmospheric density at any instant in time, it is not possible to predict the satellite's exact path. Consequently, approximations have to be made concerning lift forces, the variation of atmospheric density and the choice of a suitable value for C_D. For practical purposes, it can be assumed that lift forces will average out as the satellite's attitude changes. Indeed, Cook (1964) and Fiddes (1975) have shown that lift generally has only a very small effect, except for satellites of peculiar shapes with specific variations of incidence (e.g. a flat plate, or a satellite which "flips over" at perigee). The variation in density is usually chosen to be a simple exponential change with height of the form (King-Hele, 1964)

$$\rho = \rho_{po} \exp\{(r_{po} - r)/H\} \tag{1.65}$$

where r is the distance of the satellite from the Earth's centre; ρ_{po} - the density at the initial perigee point, r_{po}; H - the scale height. A number of refinements are often made to this basic model; for example, the introduction of a scale height which varies linearly with r. The day-to-night variation in the air density can also be incorporated into the density model given by equation (1.65).

The choice of suitable values of C_D for satellites of differing shapes presents a great problem. This difficulty is due in part, firstly, to the lack of a good aerodynamic theory and, secondly, to the geometry and rotation of the satellite. However, Cook (1960) has evaluated the drag coefficients of bodies of various shapes at varying angles to the airflow, and derived mean values for rotating bodies. In all cases, the value obtained for C_D was about 2.2 ± 0.1. The problems concerning the evaluation of $\bar{A}$ are the same as those discussed in the section on solar radiation pressure perturbations.

If F is resolved into the components, F_r, F_T and F_N, the Gaussian form of the Lagrangian planetary equations can be used to obtain the variations in the semi-major axis, a, the eccentricity, e, and the inclination, I, over one revolution. The results are (King-Hele, 1964)

$$\Delta a = -\left(\frac{\bar{A}}{m_s}\right) C_D \, X \int_0^{2\pi} \rho \, a^2 \, \frac{(1+e\cos E)^{3/2}}{(1-e\cos E)^{1/2}} \, dE \tag{1.66}$$

$$\Delta e = -\left(\frac{\bar{A}}{m_s}\right) C_D \, X \int_0^{2\pi} \rho \, a \, (1-e^2) \, \frac{(1+e\cos E)^{1/2}}{(1-e\cos E)^{1/2}} \, \cos E \, dE \tag{1.67}$$

$$\Delta I = -\left(\frac{\bar{A}}{2m_s}\right) X^{\frac{1}{2}} \, C_D \int_0^{2\pi} \frac{\rho \, w r^{5/2} \sin I}{\mu^{1/2} (1-e^2)^{1/2}} \, (1+e\cos E)^{1/2} \cos^2(\omega + f) \, dE \tag{1.68}$$

where X is a function of w, V_{po}, r_{po} and I_o defined by

$$X = \left(1 - \frac{w r_{po}}{V_{po}} \cos I_o\right)^2 \tag{1.69}$$

where w is the angular velocity of the Earth's atmosphere; V_{po} - the velocity of the satellite at the initial perigee point, r_{po}; I_o - the initial orbital inclination. The set of equations (1.66) to (1.69) are of fundamental importance in air drag studies: they form the basis for the development of the theory used in the determination of air density and atmospheric rotation rates at various heights from the analysis of satellite orbits (King-Hele, 1964; 1966; King-Hele and Scott, 1969; King-Hele and Walker, 1972; King-Hele, 1976). The application of this theory to the Earth's atmosphere at heights above 150 km has yielded a number of important results concerning variations in wind speed and air density.

In the first place, studies of the changes in the orbital inclination of satellites have shown that the Earth's atmospheric rotation rate is not a constant, but a function of both time and height. King-Hele and Walker (1976) have used a number of earlier orbital determinations (e.g. Hiller, 1974; Walker, 1975) and some new determinations to obtain the variation of wind speed with height and time (figure 1.9). The results obtained indicate that the rotation rate averaged over all local times increases from near 1.0 rev/day at 150 km height to 1.3 near 350 km (corresponding to an average west-to-east wind of 120 m/s), and then decreases to 1.0 at 400 km and, probably, to 0.8 at greater heights. As regards time variations, the maximum west-to-east winds occur in the evening hours 18-24 h local time: these evening winds increase to a maximum of about 150 m/s at heights near 350 km and decline to zero at 600 km. In the morning, 4-12 h local time, the winds are east-to-west, with speeds of 50-100 m/s above 200 km.

The second important geophysical application of air drag perturbations is the determination of atmospheric densities at various heights and times from the analysis of decaying satellite orbits. From such studies a number of important results are obtained.

Firstly, the atmospheric density decreases rapidly with height. From figure 1.7 it can be seen that the density can change by a factor of up to 10^6 in the height range 150-1000 km. Secondly, the atmospheric density undergoes a number of variations with time: for example, the air density is affected by solar activity, the density being lowest at times of low solar activity and highest at times of high solar activity (see figure 1.7). Geomagnetic storms caused by the solar wind also produce changes in the air density: the density may increase by a factor of up to 6 at heights near 600 km. Even at

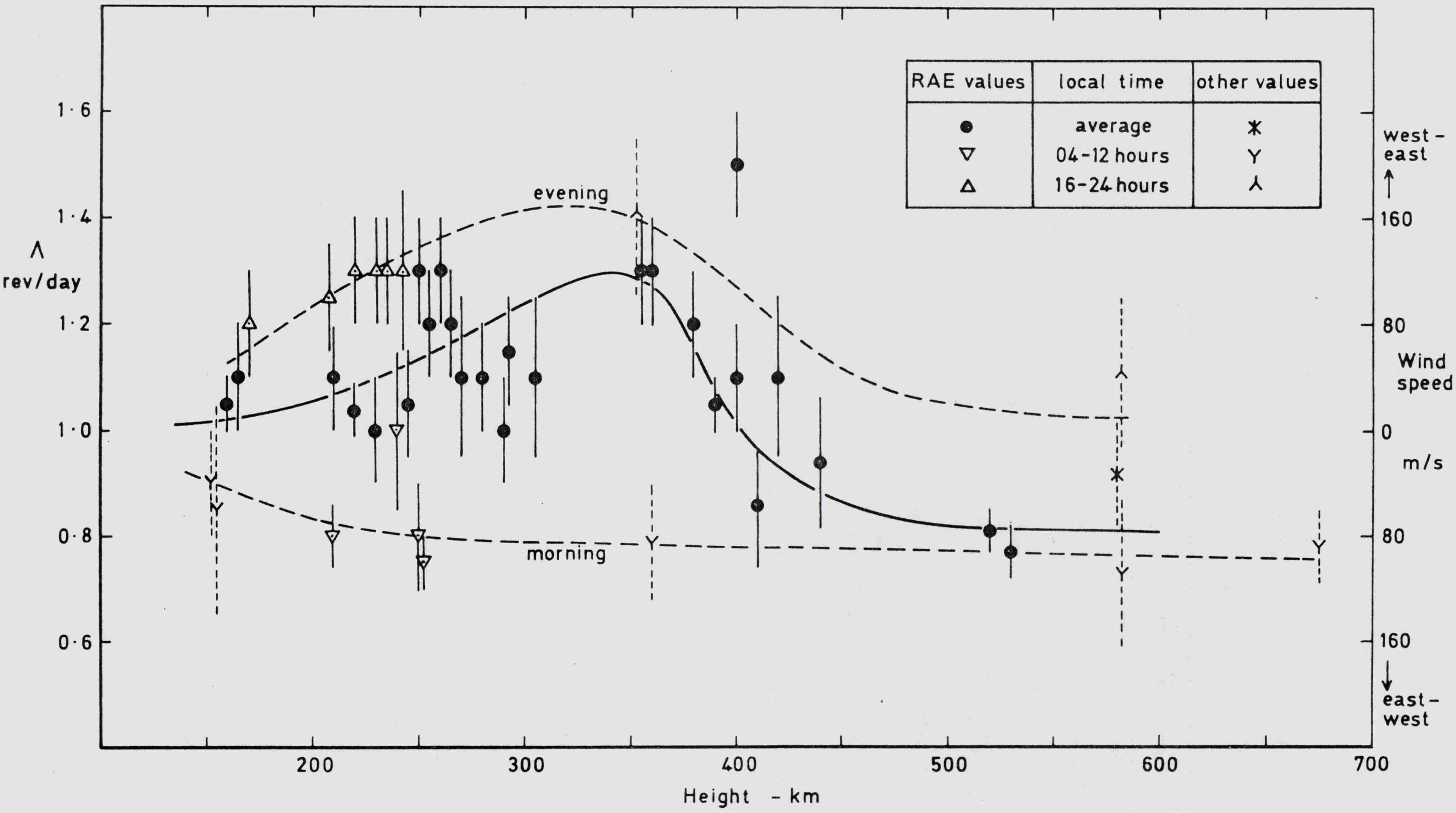


Figure 1.9

Atmosphere rotation rates for heights between 200 km and 600 km (King-Hele and Walker, 1976) heights near 200 km, where the atmosphere is relatively insensitive to solar activity, the density can increase by a factor of nearly 2 during a geomagnetic storm. However, the most baffling variation in the air density is the semi-annual variation (Cook, 1969), (figure 1.8), so called because the density exhibits a minimum in mid-January every year, rises to a maximum in April, decreases in May, suffers a deeper minima in July and then rises to another maxima in late October, usually higher than that in April (Jacchia, Slowey and Campbell, 1969). This effect occurs at heights in the range 100-1000 km, the maximum density exceeding the minimum by a factor of about 1.5 at 200 km, increasing to 2.5 at 500 km and then decreasing to about 2 at an altitude of 1000 km. The amplitude of the semi-diurnal variation is not directly dependent on solar activity, but varies from year to year. Opinion as to the exact nature of its long-term amplitude variation is somewhat divided. Voiskovskii et al. (1973) maintain that the strength of the semi-annual variation is irregular. It has also been suggested that a three-year recurrence period exists (King-Hele and Walker, 1969; Cook, 1972). The cause of the semi-annual variation is not known for certain. Cook (1969) has discussed a number of proposed theories, but perhaps the most likely is that it arises from a seasonal variation at heights below 100 km, increasing in amplitude as it rises into more rarefied air (Volland, 1969a, 1969b).

(e) General relativistic effects

It is now known through the work of Einstein that the universe is non-Newtonian and, hence, the inverse square law of gravitation as formulated by Newton within the framework of Euclidian geometry is only an approximation. Consequently, the actual orbit of an artificial satellite about a spherically symmetric body will not be an ellipse.

Although Einstein and others developed the general theory of relativity in terms of non-Euclidian geometry, an equivalent description of a satellite's motion about a spherically symmetric body can be obtained by retaining Euclidian geometry and modifying the law of gravitation. On this approach, an artificial satellite orbiting the Earth will be acted upon by a gravitational force, $\underline{F}_{grav}$, such that (McVittie, 1962; Krause, 1962)

$$\underline{F}_{grav} = - GM_E\, \underline{r}/r^3 - 3GM_E\, h^2 \underline{r}/c^2 r^5 + \text{terms of order } J_2,\ J_3 \ldots J_n \text{ etc.} \tag{1.70}$$

where c is the speed of light and h - the angular momentum of the satellite. The first term in equation (1.70) is the usual inverse square force; the second term being the main relativistic perturbing force. Other relativistic correction terms exist, but are of negligible importance, the largest having a size approximately 1/1000th of that of the main relativistic term. The corresponding disturbing potential, U_{rel}, of the main relativistic term is given by

$$U_{rel} = \frac{GM_E\, h^2}{c^2 r^3} \tag{1.71}$$

Rubincam (1975) has expanded equation (1.71) in terms of a satellite's elliptic elements, and used the Lagrangian planetary equations to obtain expressions for their time variation due to the main relativistic perturbing force. For the six orbital elements, Rubincam found that the inclination, I, and the longitude of the ascending node, Ω, were unaffected. The argument of perigee, ω, and the Mean anomaly, M, suffer secular, as well as short-period, perturbations. The remaining two elements, a, and e, suffer only short-period changes. For example, the secular rate of change in the argument of perigee, $\left[\frac{d\omega}{dt}\right]_{sec}$ is such that

$$\left[\frac{d\omega}{dt}\right]_{sec} = \frac{3(GM_E)^{3/2}}{c^2 a^{5/2}} (1 - e^2)^{-1} \tag{1.72}$$

The value of $\left[\frac{d\omega}{dt}\right]_{sec}$ for a typical satellite, such as Nimbus 6 which has an orbit with a semi-major axis of 7476 km and an eccentricity of 0.0007, is found to be about 8 x 10^{-6} deg/day, compared with a corresponding zonal harmonic change of more than 2 deg/day. Such changes are exceedingly small, and as yet cannot be separated from the other perturbations acting on a satellite. However, with an improved knowledge of satellite perturbations, and particularly those due to radiation and tidal effects, it may become possible to measure the relativistic forces acting on laser ranging satellites such as LAGEOS and STARLETTE.

(f) Oceanic and body Earth tides

Due to the gravitational attraction of the Sun and the Moon on the Earth, its shape and density distributions are not constant, but are, instead, periodic functions of time. The amount of tidal deformation from the mean at a particular point on the Earth's surface is dependent on the positions of the Sun and the Moon and on the elastic and fluid properties of the Earth. As a result of such tidal deformations, the geopotential is itself time dependent, which therefore introduces additional perturbations in the orbit of an artificial satellite. Perturbations of this kind are known as tidal perturbations, and can be divided into two types - oceanic and body. The oceanic perturbations are, as their name suggests, produced by the tidal deformations of the Earth's seas and oceans. The body perturbations result from the tidal deformations of the solid Earth.

If the Earth (assumed spherical) was perfectly elastic in its solid parts and perfectly fluid in its seas and oceans, the tidal effects of the Sun and the Moon would deform the Earth at a particular point by an amount equal to $R_E^{\,2}(U_1+U_2)/GM_E$, where U_2 and U_1 are the disturbing potentials of the Sun and Moon, respectively (see Section 1.4b), evaluated at the point under consideration. Since the Earth is neither perfectly elastic, nor perfectly fluid, the amplitudes are reduced by factors as large as the Love numbers K_2, K_3 ... K_n. The situation is further complicated by the friction which accompanies the deformation. This, together with the rotation of the Earth, causes the tidal bulge to be carried forward, resulting in the tide being high, not when the Sun or Moon is overhead, but at some later time. In order to include this effect in the model, it is necessary to introduce a second set of tidal parameters, ϵ_2, ϵ_3 ... ϵ_n, known as the phase lags. These will be defined shortly. It is these two sets of parameters - the Love numbers and the phase lags - which characterise the elastic and fluid properties of the Earth. An accurate knowledge of such parameters is of great importance in the understanding of the Earth's interior and the motion of its seas and oceans. The analysis of artificial satellite orbits provides a useful method for the determination of the K's and ϵ's.

The tidal deformation due to the Sun and the Moon at a particular point on the Earth's surface is now given by

$$\frac{M_1^*}{M_E}\,\frac{R_E^{\,3}}{R_1^{\,2}}\sum_{n=2}^{\infty}K_n\left(\frac{R_E}{R_1}\right)^{n-1}P_n(\cos{}^n\delta_1)+\frac{M_2^*}{M_E}\,\frac{R_E^{\,3}}{R_2^{\,2}}\sum_{n=2}^{\infty}K_n\left(\frac{R_E}{R_2}\right)^{n-1}P_n(\cos{}^n\delta_2) \tag{1.73}$$

where the angles ${}^n\delta_1$ and ${}^n\delta_2$ are the angles subtended at the centre of the Earth between the point under consideration and the nth fictitious Moon and the nth fictitious Sun, respectively. The nth fictitious Moon has an orbit with the same values of a, e, I and ω, but differing Ω

and M. The values of the longitude of the ascending node, ${}^{n}\Omega_1$ and the mean anomaly, ${}^{n}M_1$, for the nth fictitious Moon are related to those of the real Moon by the equations

$$\begin{aligned} {}^{n}\Omega_1 &= \Omega_1 + \epsilon_n \\ {}^{n}M_1 &= M_1 - \epsilon_n \end{aligned} \tag{1.74}$$

The corresponding expression for the disturbing potential, U_{TID}, acting on an artificial satellite due to the lunisolar tides is given by

$$U_{TID} = \sum_{i=1}^{2} \frac{GM_i^* R_E^2}{r\, R_i} \sum_{n=2}^{\infty} K_n \frac{R_E^{2n-1}}{(rR_i)^n} P_n(\cos {}^{n}\delta_i) \tag{1.75}$$

where r is the geocentric distance of the satellite and ${}^{n}\delta_1$ and ${}^{n}\delta_2$ are the geocentric angles of the radius vector of the satellite to the directions of the nth fictitious Moon and the nth fictitious Sun, respectively.

A number of authors have proposed other tidal models, all of which rely on the same basic approach used in obtaining the model described by equation (1.75). For example, Kozai (1965) assumes all the phase lags to be equal, i.e. $\epsilon_2 = \epsilon_3 = \ldots \epsilon_n$; whilst Musen and Estes (1972) put all the phase lags equal to zero. In a later model, Musen and Felsentreger (1974) still have $\epsilon_2 = \epsilon_3 = \ldots \epsilon_n = 0$, but now take into account the oblateness of the Earth. The most complicated model to date is the one proposed by Kaula (1969), which includes the effects of latitudinal variations in a tide's amplitude and phase, i.e. $K_n = K_n(\theta)$ and $\epsilon_n = \epsilon_n(\theta)$.

Inconsistent values have been obtained for the Love numbers and phase lags when such models are applied to specific satellite orbits. Kozai (1968) analysed the tidal perturbations in the inclinations of three satellites with inclinations in the range 33^{o} - 50^{o}, obtaining values for the K_2 Love number varying between 0.23 and 0.33 and phase lags from 0^{o} to 9^{o}. Newton (1968) has analysed the tidal perturbations in the inclination and node of four polar satellites. Once again the results were not consistent, K_2 being in the range 0.28 to 0.44, while ϵ_2 varies between 0^{o} and 2 - 5^{o}.

More recently, Lambeck et al. (1973) and Cazenave et al. (1977) have pointed out that the values obtained for K_2 and ϵ_2 by Kozai and Newton are not in agreement with the expected values of $K_2 \approx 0.3$ and $\epsilon_2 < 0 - 5^{o}$ for a solid Earth. This implies that the oceanic tides have a significant effect on a satellite orbit. A better method of evaluating tidal perturbations, they suggest, would be to have two separate models - one for the oceanic tides and one for the body tides, instead of a unified model incorporating both oceanic and body tides (Lambeck et al., 1974). This approach has the advantage that it becomes possible to distinguish between the effects of oceanic tides and body tides on a satellite orbit. Information can then be readily obtained concerning the Earth's elastic properties, as distinct from its fluid properties, and vice-versa.

1.5 The Present State of Knowledge

The perturbed motion of an artificial satellite presents mathematicians with a dynamical problem of great difficulty that defies a simple solution. This is due in part to the relatively large number of perturbing forces acting on a satellite, and, in part, to the uncertainties that arise in the development of mathematical models which adequately represent the physical nature of the perturbing forces. Although substantial progress has been made in the subject's twenty-year history - particularly in the determination of geophysical data from changes in the orbits of satellites - a number of unsolved problems still remain.

Prior to 1957, very little was known about the Earth's shape, except for the polar flattening, which was thought to be about 1 part in 297.1 (Jeffreys, 1952). This situation was drastically altered by the launchings of the first artificial satellites. Analysis of the orbit of Sputnik 2 and of other satellites (Merson and King-Hele, 1958; King-Hele and Merson, 1959) showed that the accepted value of the flattening was appreciably in error. The value now established is 1 part in 298.25, so that the equatorial diameter exceeds the polar diameter by about 42.77 km, which is about 170 m less than was previously thought. Such a revision was not only important for artificial satellite orbit theory but also for geophysics and geodesy. It was important for geophysics, because it showed that the flattening was significantly different from the hydrostatic value for a liquid Earth of 1 part in 299.7 (Khan, 1973); and for geodesy, because some measurements at that time were accurate to within 5 m, and an error of over 100 m in the basic spheroid was unacceptable. Since then, additional harmonics, both zonal and tesseral, have been determined with ever increasing accuracy, enabling geoid maps to be drawn (figure 1.11) with accuracies of 2 or 3 m (Richardson and Lerch, 1974). As a result of such determinations together with the recent dynamical methods of Deprit and Hori, analytical theories have now been developed for the zonal harmonics perturbations on satellites, which are equal in accuracy to those obtained by numerical integration (Kinoshita, 1976).

Two other significant advances also need to be mentioned. Firstly, the determination of air densities and atmospheric winds for various heights and times have greatly improved our knowledge of the

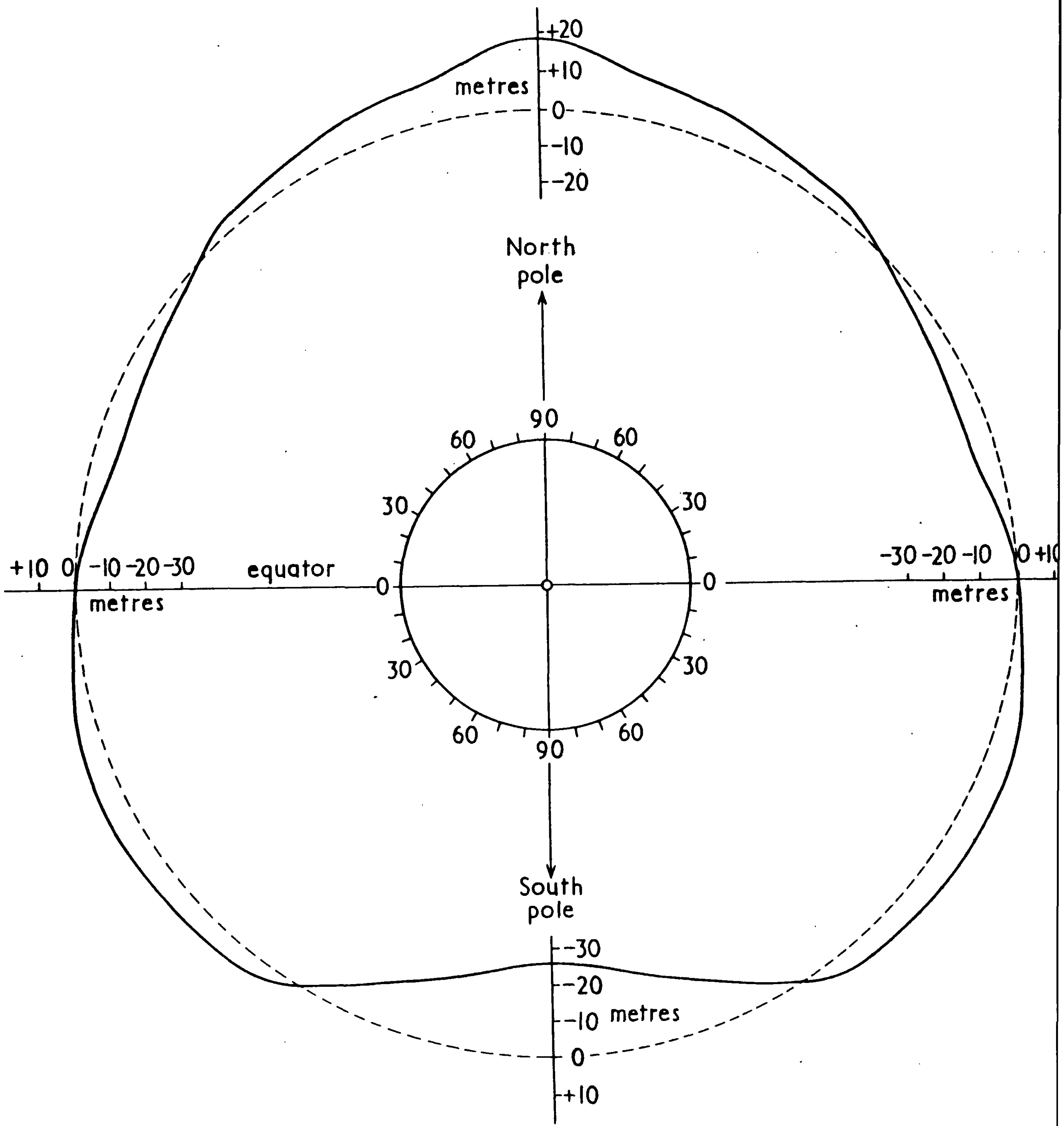


Figure 1.10

Heights of meridional geoid section (solid line) relative to a spheroid of flattening 1/298.25 (broken line) (with odd harmonics of King-Hele and Cook (1974) and even harmonics of Wagner (1973))

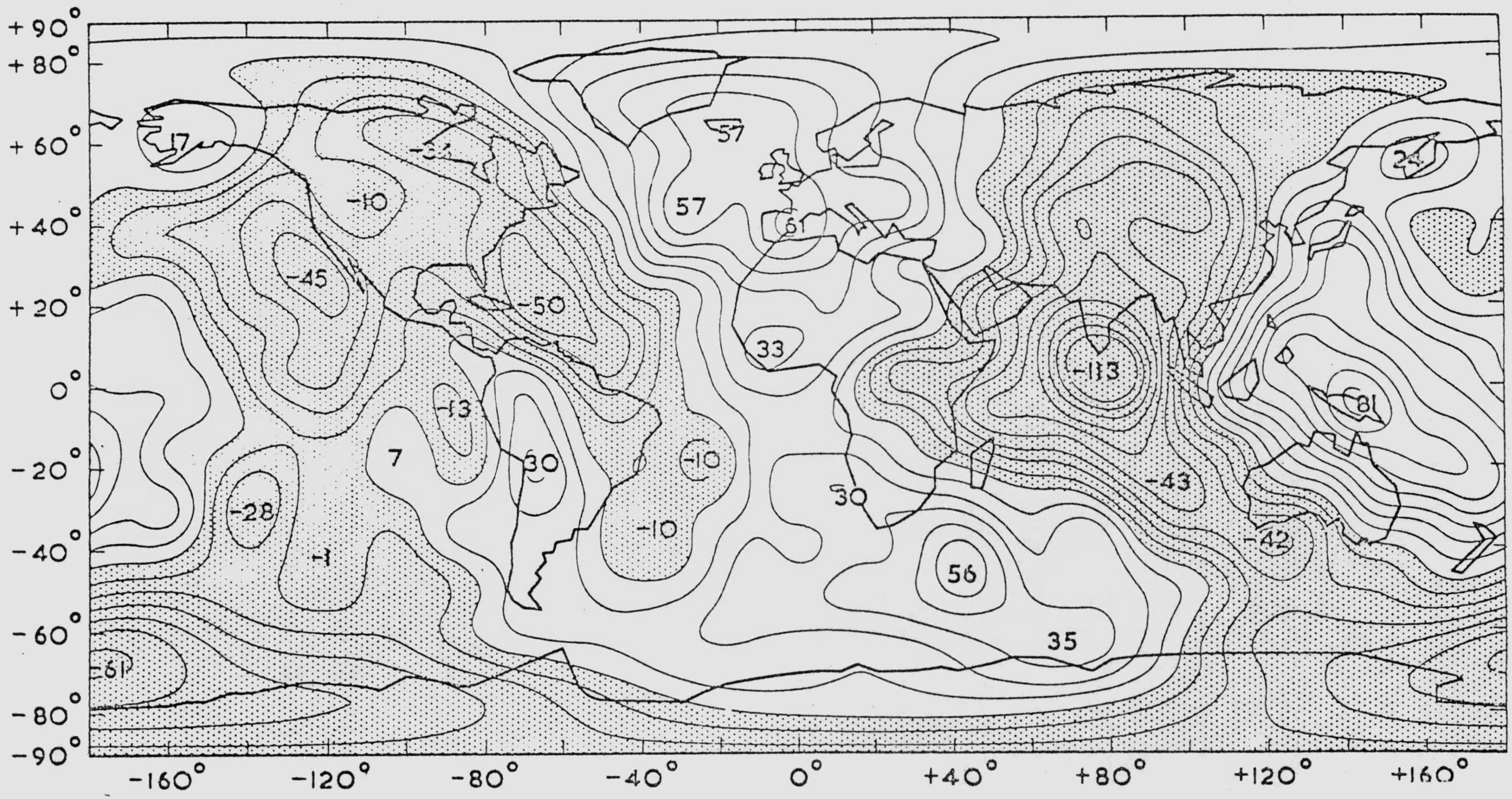


Figure 1.11

Geoid of Smithsonian Standard Earth II. Contours at 10 m intervals relative to a spheroid of flattening 1/298.25.

Earth's atmosphere at heights between 150 km and 600 km. Secondly, the recent work of Deprit (1977) on the problem of the 'critical inclination' in satellite theory has finally settled the arguments on this issue, which have lasted over twenty years (Message et al., 1962; Lubowe, 1969(1), Lubowe, 1969(11); Garfinkel, 1969, 1970).

Despite these great improvements, numerous problems and deficiencies still exist. To illustrate this point let us mention four particular examples. Firstly, no completely analytical theory has been developed for the effect of lunisolar gravity on a satellite orbit, although a number of good numerical and semi-analytical models exist. Secondly, no satisfactory mathematical model exists which adequately represents the physical nature of albedo radiation effects. Thirdly, improvements need to be made to the theoretical representation of tidal effects, if satellites are to be used for accurate determinations of Love numbers and phase lags. Fourthly, a purely analytical theory for the combined effect of air drag and gravity on a satellite orbit has not been attempted since the efforts of Brouwer and Hori (1961). The theory obtained by them is only valid for satellites with virtually circular orbits.

It is hoped that the next twenty years of artificial satellite orbit research will provide solutions to these outstanding problems.

CHAPTER 2

LUNISOLAR GRAVITY AND DIRECT SOLAR RADIATION PRESSURE RESONANCE ORBITS

2.1 The Nature of the Resonance Orbits

An artificial satellite orbiting the Earth is subject to a number of periodic perturbing forces. For example, the satellite's orbital motion causes it to experience the same longitudinally averaged geopotential every revolution; the rotation of the Earth and the satellite's own motion combine to produce periodic variations in the longitudinally dependent part of the geopotential acting on the satellite; the perturbing influences of the Sun and the Moon are periodic by virtue of their motion and the orbital motion of the satellite: and so on. This periodicity of the perturbing forces is reflected in the expansion of their disturbing potentials as functions of a satellite's orbital elements and the parameters of the perturbing force. In such expansions, sine and cosine terms occur, having arguments of the form $\delta\,\psi$, where

$$\psi = \alpha\omega + \zeta M + \beta\Omega + u \tag{2.1}$$

δ, α, β and ζ are integers and u is a linear function of certain of the perturbing parameters. In the case of lunisolar perturbations, u is a function of the argument of perigee, ω_D, the mean anomaly, M_D, and the longitude of the ascending node, Ω_D, of their respective orbits relative to the ecliptic plane. Each ψ, because of the approximately linear time dependence of ω, Ω, M and u (see section 2.3), will cause the magnitude of the corresponding sine or cosine term in the disturbing function expansion to oscillate periodically; the period of the oscillation being dependent on the non-angular elements of a satellite's orbit (i.e. the semi-major axis, a, the eccentricity, e, and the inclination i) and certain parameters of the perturbing forces. Under certain circumstances, the semi-major axis, eccentricity and inclination of a satellite's orbit are such that they cause the periods of some of these terms to become nearly infinite. In such a case, the argument, $\delta\psi$, associated with one of these terms will be approximately a constant, i.e.

$$\dot{\psi} = \alpha\dot{\omega} + \zeta\dot{M} + \beta\dot{\Omega} + \dot{u} \approx 0 \qquad (2.2)$$

A relationship of the type (2.2) is known as a <u>commensurability condition</u>; a satellite whose orbital elements satisfy such a condition is said to be in the <u>commensurability $\dot{\psi} \approx 0$</u>. The orbits of satellites which are in a commensurability (2.2) are called <u>resonance orbits</u>. A satellite existing in the commensurability $\dot{\psi} \approx 0$ will have an orbit which is in resonance with those terms in the disturbing potential expansion having arguments of the form $\delta\psi$, where δ is an integer; which may be either positive or negative. Each of these terms has an amplitude factor dependent on the satellite's a, e and i and certain parameters of the perturbing force. In the case of lunisolar gravity perturbations, these perturbing parameters are the mass of the disturbing body, the semi-major axis, a_D, the eccentricity, e_D, and the inclination, i_D, of the disturbing body's orbit relative to the ecliptic plane. For solar radiation pressure perturbations, the disturbing parameters are a_D, e_D, i_D and a factor dependent on the solar constant and the physical properties of the satellite. Since a satellite in the commensurability $\dot{\psi} \approx 0$ is in resonance with a number of terms having amplitude factors of differing magnitudes, the resonant changes in the satellite's orbital elements is the sum of the effects of each of these terms. The importance of each term being determined by the size of its amplitude factor. Furthermore, because of the quasi-secular nature of a resonant term, the changes in the orbital elements of a satellite produced by a resonant term are in general larger than those produced by a non-resonant term having an amplitude factor of comparable magnitude.

The literature on lunisolar gravity and solar radiation pressure resonance orbits is small. Cook (1962), in his paper on lunisolar perturbations of artificial satellites, mentioned the possibility of resonances occurring with some of the leading terms in the lunisolar gravity and solar radiation pressure disturbing potential expansions. The discussion given was valid only for those resonant terms which produce changes in a satellite's orbital eccentricity (i.e. those whose arguments depend on ω). The method used in obtaining the commensurability conditions, that of expanding the disturbing potential term by term and truncating the resulting series expansion, is totally unsuitable for a general discussion of lunisolar resonance orbits. Using this method of approach, only the leading terms in the lunisolar gravity and solar radiation pressure disturbing potential expansions which produce changes in e were considered. Consequently, the leading terms dependent on Ω, but independent of ω, were excluded. In any general theory of lunisolar resonance orbits, it is necessary to include all types of commensurabilities, and not just particular cases as was done by Cook.

In this chapter, a general discussion of lunisolar gravity and solar radiation pressure resonance orbits will be given with particular emphasis on the following aspects.

(i) The classification of resonance orbits in terms of the general commensurability condition (2.2).

(ii) The form of the general commensurability condition when expressed as a function of the satellite's non-angular orbital elements and the parameters of the lunisolar perturbations.

(iii) The predominant resonance terms for each class of commensurability.

(iv) Examples of important resonance orbits for both lunisolar gravity and solar radiation pressure perturbations.

(v) Criteria which determine whether, or not, resonance orbits exist for a particular commensurability condition.

2.2 The General Commensurability Condition for Lunisolar Gravity and Direct Solar Radiation Pressure Resonance Orbits

The lunisolar gravity and direct solar radiation pressure disturbing potentials, R, when expanded as a function of a satellite's orbital elements a, e, i, ω, Ω and M and the orbital elements of the lunar or solar orbits relative to the celestial equator, are of the form (Allan, 1969; Hughes, 1977)

$$R = C \sum_{n=t}^{\infty} \frac{a^n}{(a^*)^{n+1}} \sum_{m=0}^{n} K_m \frac{(n-m)!}{(n+m)!} \sum_{p=0}^{n} \bar{F}_{n,m,p}(i) \sum_{h=0}^{n} \bar{F}_{n,m,h}(i^*)$$

$$\times \sum_{q=-\infty}^{+\infty} G_{n,p,q}(e) \sum_{j=-\infty}^{+\infty} H_{n,h,j}(e^*)$$

$$\times \cos\left[(n-2p)\omega + (n-2p+q)M - (n-2h)\,\omega^* - (n-2h+j)M^* + m(\Omega - \Omega^*)\right] \tag{2.3}$$

where $C = GM_D$ and $t = 2$ for lunisolar gravity perturbations, M_D being the mass of the disturbing body (i.e. the Sun or Moon): for direct solar radiation pressure perturbations

$$C = \frac{-S_o \bar{A}(2-\epsilon)a^{*2}\sigma}{cm_s}$$

and t = 1; $\sigma = 1$ when the satellite is in sunlight, and $\sigma = 0$ when the satellite is in the Earth's shadow. The quantity K_m is such that $K_o = 1$ and $K_m = 2$ for $m > 0$. The functions $G_{n,p,q}(e)$ and $H_{n,h,j}(e^*)$ are the Hansen coefficients $X^{n,(n-2p)}_{(n-2p+q)}(e)$ and $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*)$ (see Appendix 1); the quantities $\bar{F}_{n,m,p}(i)$ and $\bar{F}_{n,m,h}(i^*)$ are the modified Allan inclination functions (Allan, 1965; Hughes, 1977), defined by

$$\bar{F}_{n,m,p}(i) = \frac{(n+m)!}{2^n(n-p)!p!} \sum_{\kappa} (-1)^{\kappa} \binom{2n-2p}{\kappa}\binom{2p}{n-m-\kappa} (\cos\tfrac{1}{2}i)^{3n-m-2p-2\kappa} (\sin\tfrac{1}{2}i)^{m-n+2p+2\kappa}$$

with a similar expression for $\bar{F}_{n,m,h}(i^*)$. The starred quantities in (2.3) refer to the lunar or solar orbits relative to the celestial equator.

The variations in the argument of perigee, ω_D, and the longitude of the ascending node, Ω_D, of the lunar orbit relative to the <u>ecliptic</u> plane are largely caused by the perturbing effect of the Sun's gravity, and are approximately linear with periods of the order of 9 years and 18.6 years respectively (Brown, 1895). However, the motions of ω^* and Ω^* for the Moon relative to the celestial equator as a result of the Sun's gravity are somewhat different. Cook (1962) has shown that Ω^* varies between -13^o and $+13^o$ roughly every 18.6 years, the corresponding change in ω^* is also non-linear, ω^* taking about 9 years to a complete a cycle of 360^o. In addition, the non-linear variation in Ω^* also causes the inclination, i^*, of the lunar orbit relative to the celestial equator to oscillate between the limits 18.4^o and 28.6^o in a period of 18.6 years (Kozai, 1965). It will therefore be found convenient in the subsequent analysis to expand the lunisolar gravity and direct solar radiation pressure disturbing potentials as functions of a satellite's orbital elements relative to the <u>celestial equator</u> and elements of the lunar or solar orbits (given the subscript D) relative to the <u>ecliptic plane</u>. In such an expansion the lunisolar elements a_D, e_D and i_D are approximately constants, whilst the three remaining elements ω_D, Ω_D and M_D vary almost linearly with time. For the Sun, the rates of change of ω_D, Ω_D and M_D are approximately 4.71×10^{-5} deg/day, 0 deg/day and 0.99 deg/day, respectively. The corresponding values for the Moon are 0.16 deg/day, -0.05 deg/day and 13.07 deg/day. If the lunisolar gravity and direct solar radiation pressure disturbing functions are expanded in terms of a satellite's orbital elements relative to the celestial equator and the lunar or solar elements relative to the ecliptic plane, then, following the method of Giacaglia (1974), equation (2.3) becomes

$$R = C \sum_{n=t}^{\infty} \frac{a^n}{(a_D)^{n+1}} \sum_{m=0}^{n} \sum_{s=0}^{n} (-1)^m K_m K_s \frac{(n-s)!}{(n+m)!} \sum_{p=0}^{n} \bar{F}_{n,m,p}(i) \sum_{h=0}^{n} \bar{F}_{n,s,h}(i_D)$$

$$\times \sum_{q=-\infty}^{+\infty} G_{n,p,q}(e) \sum_{j=-\infty}^{+\infty} H_{n,h,j}(e_D)$$

$$\times \left[(-1)^{n+m-s} U_n^{m,-s} \cos\{\Phi^{(+)}(n,m,p,q,h,j,s)\} + U_n^{m,s} \cos\{\Phi^{(-)}(n,m,p,q,h,j,s)\}\right] \quad (2.4)$$

where

$$U_n^{m,\pm s} = \frac{(-1)^{m\mp s}}{(n\pm s)!} \left(\cos \frac{E_D}{2}\right)^{m\pm s} \left(\sin \frac{E_D}{2}\right)^{\pm s-m} \frac{d^{n\pm s}}{dZ^{n\pm s}} \left[Z^{n-m}(Z-1)^{n+m}\right] , \quad (2.5)$$

$$Z = \cos^2 (E_D/2) , \quad E_D = \text{obliquity of the ecliptic} \quad (2.6)$$

$$\Phi^{(+)}(n,m,p,q,h,j,s) = \left[(n-2p)\omega + (n-2p+q)M + (n-2h)\omega_D + (n-2h+j)M_D + m\Omega + s(\Omega_D + \pi/2)\right]$$

and

$$\Phi^{(-)}(n,m,p,q,h,j,s) = [(n-2p)\omega + (n-2p+q)M - (n-2h)\omega_D - (n-2h+j)M_D + m\Omega - s(\Omega_D + \pi/2)] \qquad (2.7)$$

The Kaula inclination functions (Kaula, 1962) as used by Giacaglia, have been replaced in (2.4) by the equivalent, but simpler, Allan inclination functions, $\overline{F}_{n,m,p}(i)$ and $\overline{F}_{n,s,h}(i_D)$.

A satellite in the lunisolar commensurability

$$\dot{\psi} = \alpha\dot{\omega} + \zeta\dot{M} + \eta\dot{\omega}_D + \gamma\dot{M}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0 \qquad (2.8)$$

where $\alpha, \beta, \zeta, \eta, \gamma$ and k are integer constants, will have an orbit which is resonant with those $\Phi^{(+)}$ terms in (2.4) that are characterised by the set(s) of integers n,m,p,q,h,j,s and δ satisfying the relations

$$\begin{aligned} (n-2p)^+ &= \alpha\delta \\ (n-2p+q)^+ &= \zeta\delta \\ (n-2h)^+ &= \eta\delta \\ (n-2h+j)^+ &= \gamma\delta \\ m^+ &= \beta\delta \\ s^+ &= k\delta \end{aligned} \qquad (2.9)$$

Similarly, a satellite in the lunisolar commensurability (2.8) will have an orbit which is in resonance with those $\Phi^{(-)}$ terms in (2.4) that are characterised by the set(s) of integers n,m,p,q,h,j,s and v satisfying the relations

$$\begin{aligned} (n-2p)^- &= \alpha v \\ (n-2p+q)^- &= \zeta v \\ (n-2h)^- &= -\eta v \\ (n-2h+j)^- &= -\gamma v \\ m^- &= \beta v \\ s^- &= -k v \end{aligned} \qquad (2.10)$$

The (+) and (-) superscripts in (2.9) and (2.10) are used to distinguish the $\Phi^{(+)}$ and $\Phi^{(-)}$ resonance terms. In order to make the definition of the commensurability (2.8) unique, it is necessary to place two stipulations on equation (2.8). First, $\dot{\psi} \approx 0$ must be a prime quantity. Secondly, β is defined as a positive integer; if $\beta = 0$ then the next non-zero integer constant immediately to its left in equation (2.8) is defined as always being positive, i.e. $\gamma > 0$ if $\beta = 0$; if $\gamma = 0$ then $\eta > 0$: and so on. From this definition and the relations (2.9) it follows that, in order for a satellite in a commensurability (2.8) to be in resonance with $\Phi^{(+)}$ terms, k and δ must satisfy the conditions

(i)	$\beta > 0$	$\delta > 0$ AND $k \geqslant 0$	
(ii)	$k > 0$, $\beta = 0$	$\delta > 0$	
			(2.11)
(iii)	$k < 0$, $\beta = 0$	$\delta < 0$	
(iv)	$k = 0$, $\beta = 0$	$-\infty < \delta < \infty, \quad \delta \neq 0$	

Hence, for $\beta > 0$, a satellite in the lunisolar commensurability (2.8) can <u>only</u> be in resonance with $\underline{\Phi^{(+)}}$ terms <u>if $k \geqslant 0$</u>. However, if $\beta = 0$, then resonance will occur with $\Phi^{(+)}$ terms <u>irrespective</u> of the value of k. Similarly, in order for a satellite in a commensurability (2.8) to be in resonance with $\Phi^{(-)}$ terms, k and v must satisfy the conditions

(i) $\beta > 0$ $v > 0$ AND $k \leqslant 0$

(ii) $k > 0$, $\beta = 0$ $v < 0$

(iii) $k < 0$, $\beta = 0$ $v > 0$ (2.12)

(iv) $k = 0$, $\beta = 0$ $-\infty < v < \infty,\ v \neq 0$

Hence, for $\beta > 0$, a satellite in the lunisolar commensurability (2.8) can <u>only</u> be in resonance with <u>$\Phi^{(-)}$</u> terms provided <u>$k \leqslant 0$</u>. However, if $\beta = 0$ then resonance will occur with <u>$\Phi^{(-)}$</u> terms <u>irrespective</u> of the value of k. Finally, it follows from conditions (2.11) and (2.12) that, if $k = 0$, then a satellite in such a lunisolar commensurability will be in resonance with both $\Phi^{(+)}$ and $\Phi^{(-)}$ terms.

2.3 The Classification of Lunisolar Gravity and Direct Solar Radiation Pressure Resonance Orbits

For a close Earth satellite, the J_2 harmonic in the geopotential produces the greatest change in a satellite's perigee and node. The rates of change of a satellite's argument of perigee, ω, and its nodal longitude, Ω, caused by the J_2 harmonic only are such that (King-Hele, 1958)

$$\dot{\omega} \simeq 4.98 \left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} (5 \cos^2 i - 1) \quad \text{deg/day} \qquad (2.13)$$

and

$$\dot{\Omega} \simeq -9.97 \left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} \cos i \quad \text{deg/day} \qquad (2.14)$$

where R_E is the mean equatorial radius of the Earth. The corresponding rate of change of a satellite's mean anomaly, M, with the effect of the J_2 harmonic included is

$$\dot{M} \simeq 6135.7\left(\frac{R_E}{a}\right)^{3/2} + 9.97\left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-3/2} (1-3/2 \sin^2 i) \qquad (2.15)$$

If equations (2.13), (2.14) and (2.15) are substituted into (2.8), then $\dot{\psi} \approx 0$ can be written as

$$\dot{\psi} = 4.98\,\alpha \left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} (5 \cos^2 i - 1) + 6135.7\zeta \left(\frac{R_E}{a}\right)^{1.5}$$

$$+ 9.97\zeta \left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-3/2} \left(1 - \frac{3}{2} \sin^2 i\right) + \eta \dot{\omega}_D$$

$$+ \gamma \dot{M}_D - 9.97\beta \left(\frac{R_E}{a}\right)^{3.5} (1-e^2)^{-2} \cos i$$

$$+ k\dot{\Omega}_D \simeq 0 \,. \qquad (2.16)$$

Since the semi-major axis, a, the eccentricity, e, and the inclination, i, of a satellite's orbit are not constants, the a, e and i contained in equation (2.16) have to be regarded as mean values, or as constants of integration.

It is seen from (2.16) that $\dot{\psi}$ is, to a good approximation, a function of a satellite's non-angular orbital elements and the parameters $\dot{M}_D$, $\dot{\omega}_D$ and $\dot{\Omega}_D$ associated with the lunar or solar orbits relative to the ecliptic plane. Clearly, the characteristics of a resonance orbit (i.e. its orbital elements) depend entirely on the type of commensurability in which the satellite exists (i.e. upon the values of α, β, ζ, γ, η and k). The subdivision of resonance orbits according to their commensurability condition is therefore the obvious method of classification. The general commensurability condition (2.8) can be divided into the following fifteen types for both lunisolar gravity and solar radiation pressure perturbations.

(1) $\dot{\psi}_1 = \dot{\Omega} \simeq 0$

(2) $\dot{\psi}_2 = \dot{\omega} \simeq 0$

(3) $\dot{\psi}_3 = \alpha\dot{\omega} + \beta\dot{\Omega} \simeq 0$ (α and $\beta \neq 0$)

(4) $\dot{\psi}_4 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) \simeq 0$ (α and $\gamma \neq 0$)

(5) $\dot{\psi}_5 = \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \simeq 0$ (γ and $\beta \neq 0$)

(6) $\dot{\psi}_6 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \simeq 0$ (α, β and $\gamma \neq 0$)

(7) $\dot{\psi}_7 = \beta\dot{\Omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \simeq 0$ ($\beta \neq 0$: η and/or $k \neq 0$)

(8) $\dot{\psi}_8 = \alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \simeq 0$ ($\alpha \neq 0$: η and/or $k \neq 0$)

(9) $\dot{\psi}_9 = \alpha\dot{\omega} + \eta\dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \simeq 0$ (α and $\beta \neq 0$: η and/or $k \neq 0$)

(10) $\dot{\psi}_{10} = \alpha\dot{\omega} + \eta\dot{\omega}_D + \gamma\dot{M}_D + k\dot{\Omega}_D \simeq 0$ (α and $\gamma \neq 0$: η and/or $k \neq 0$)

(11) $\dot{\psi}_{11} = \eta\dot{\omega}_D + \gamma\dot{M}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \simeq 0$ (β and $\gamma \neq 0$: η and/or $k \neq 0$)

(12) $\dot{\psi}_{12} = \alpha\dot{\omega} + \gamma\dot{M}_D + \eta\dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \simeq 0$ (α, β and $\gamma \neq 0$: η and/or $k \neq 0$)

(13) $\dot{\psi}_{13} = \eta\dot{\omega}_D + k\dot{\Omega}_D \simeq 0$ (η and/or $k \neq 0$)

(14) $\dot{\psi}_{14} = \gamma\dot{M}_D + \eta\dot{\omega}_D + k\dot{\Omega}_D \simeq 0$ ($\gamma \neq 0$: η and/or $k \neq 0$)

(15) $\dot{\psi}_{15} = \alpha\dot{\omega} + \gamma\dot{M}_D + \zeta\dot{M} + \eta\dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \simeq 0$ ($\zeta \neq 0$)

For a close Earth satellite, the mean anomaly, M, executes several complete revolutions per day, therefore a satellite in a commensurability involving $\dot{M}$, in which ω, M_D and Ω change by only a few degrees per day at most, must have an orbit with an extremely large semi-major axis. Since most satellites launched to date have orbits with perigee heights of less than 3200 km, very few satellites, if any, will be in commensurabilities involving M. Such a satellite would in any case be too far away from the Earth to be observed. Consequently, commensurabilities of this type, i.e. $\dot{\psi}_{15} \simeq 0$ are only of minor importance, and will therefore not be considered further in this chapter. The type fourteen commensurability $\dot{\psi}_{14} = \gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D \simeq 0$ can also be neglected, since $\dot{M}_D$ is very much greater than $\dot{\omega}_D$ or $\dot{\Omega}_D$. For both the lunar and solar cases, commensurabilities of this type can only occur for large values of η and k. In such cases, the resonant terms will have large n values and, hence, small amplitude factors of the order of $(a/a_D)^n$, $a/a_D << 1$, for close satellites.

Since $\dot{\omega}_D$ and $\dot{\Omega}_D$ are small quantities for the solar orbit, it follows that

$$
\begin{aligned}
[\dot{\psi}_1]_{SUN} &\equiv [\dot{\psi}_7]_{SUN} \\
[\dot{\psi}_2]_{SUN} &\equiv [\dot{\psi}_8]_{SUN} \\
[\dot{\psi}_3]_{SUN} &\equiv [\dot{\psi}_9]_{SUN} \\
[\dot{\psi}_4]_{SUN} &\equiv [\dot{\psi}_{10}]_{SUN} \\
[\dot{\psi}_5]_{SUN} &\equiv [\dot{\psi}_{11}]_{SUN} \\
[\dot{\psi}_6]_{SUN} &\equiv [\dot{\psi}_{12}]_{SUN}
\end{aligned}
\qquad (2.17)
$$

or in the short-hand notation

$$[\dot{\psi}_i]_{SUN} \equiv [\dot{\psi}_{i+6}]_{SUN} \qquad i = 1,2 \ldots 6 \qquad (2.18)$$

Hence, in the subsequent discussion only $[\dot{\psi}_i]$, $i = 1,2, \ldots 6$ will be considered in connection with solar gravity and solar radiation pressure perturbations. In the case of the Moon, $\dot{\omega}_D$ and $\dot{\Omega}_D$ are approximately

0.11 deg/day and -0.05 deg/day, respectively, it is therefore necessary to consider $[\dot{\psi}_i]_{MOON}$ and $[\dot{\psi}_{i+6}]_{MOON}$, i = 1,2 ... 6 as separate commensurability types. If, however, $\eta\dot{\omega}_D + k\dot{\Omega}_D \simeq 0$, then $[\dot{\psi}_i]_{MOON} \equiv [\dot{\psi}_{i+6}]_{MOON}$, i = 1,2 ... 6. The conditions for which this occurs will be discussed later. We will now consider the commensurability types $\dot{\psi}_1 \approx 0$ to $\dot{\psi}_{13} \approx 0$ in greater detail.

2.4(1) The Type (1) Commensurability $\dot{\psi}_1 = \dot{\Omega} \approx 0$

A satellite in a type (1) commensurability is in resonance with those $\Phi^{(+)}$ terms in the appropriate lunisolar disturbing function expansion for which

$$
\begin{aligned}
(n-2p)^+ &= 0 \\
q^+ &= 0 \\
(n-2h+j)^+ &= 0 \qquad\qquad (2.19)\\
j^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^+ + 2h^+ & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^+ &= 1,2 \ldots n^+ \\
s^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 1,2 \ldots n^+ & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
\delta &= 1,2 \ldots \infty
\end{aligned}
$$

The arguments of these terms are of the form $\delta(\psi_1^{\,+})_{MOON}$ for the Moon (where $(\psi_1^{\,+})_{MOON} = \Omega$) and of the form $\delta(\psi_1^{\,+})_{SUN}$ for the Sun, with $(\psi_1^{\,+})_{SUN} = \beta\Omega + \eta\,\omega_D + k\Omega_D$ $(k \geqslant 0)$. Similarly, the $\Phi^{(-)}$ resonance terms are given by

$$
\begin{aligned}
(n-2p)^- &= 0 \\
q^- &= 0 \\
(n-2h+j)^- &= 0 \qquad\qquad (2.20) \\
j &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^-+2h^- & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^- &= 1,2 \ldots n^- \\
s^- &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 1,2 \ldots n^- & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
v &= 1,2 \ldots \infty
\end{aligned}
$$

The arguments of these $\Phi^{(-)}$ terms are of the form $v(\psi_1^-)_{MOON}$ for the Moon, where $(\psi_1^-)_{MOON} = \Omega$, and of the form $v(\psi_1^-)_{SUN}$ for the Sun, with $(\psi_1^-)_{SUN} = \beta\Omega + \eta\omega_D + k\Omega_D$ $(k \leqslant 0)$.

Since $\dot{\Omega} \approx 0$ for a satellite in a type (1) commensurability, a satellite will exist in this commensurability if it has a stationary ascending node. From equation (2.14), this occurs when $i \approx 90^o$: therefore the only constraint on a satellite's orbital elements for it to be in the commensurability $\dot{\psi}_1 \simeq 0$ is that it should have a polar orbit. A resonance orbit of this type is known as an <u>inclination dependent resonance orbit</u>.

The amplitude factor of any resonant term contains the factor $(a/a_D)^n\, G_{n,p,q}(e) \times H_{n,h,j}(e_D)$. Since $G_{n,p,q}(e)$ and $H_{n,h,j}(e_D)$ are of the order $e^{|q|}$ and $e_D^{|j|}$, the factor $(a/a_D)^n\, G_{n,p,q}(e)$ $H_{n,h,j}(e_D)$ is proportional to $(a/a_D)^n e^{|q|}\, e_D^{|j|}$. For a close satellite, $(a/a_D)^n$ is approximately $(1/50)^n$ for the Moon and $(1/17{,}000)^n$ for the Sun. The eccentricities, e_D, of the lunar and solar orbits are approximately (1/20) and (1/60) respectively. Clearly, the most important resonance terms for <u>any</u> commensurability are those for which

$(a/a_D)^n e^{|q|} e^{|j|}$ is a minimum. A situation might arise in which other factors greatly affect the size of a resonant term's amplitude factor, thus causing the predominant resonant terms to have n, q and j values different from those which result in $(a/a_D)^n e^{|q|} e_D^{|j|}$ being a minimum. For example, the magnitudes of the inclination functions $\overline{F}_{n,m,p}(i)$ and $\overline{F}_{n,s,h}(i_D)$ may produce such a situation. However, in the majority of cases, the predominant resonant terms can be determined from the condition that $(a/a_D)^n e^{|q|} e_D^{|j|}$ should be a minimum. Such resonant terms, in addition to satisfying the conditions (2.9) and (2.10) for their particular α, β, γ, η, ζ and k, must also satisfy the restrictions on n, m, p, q, h, j and s implicit in equation (2.4), i.e.

$2 \leqslant n \leqslant \infty$ - lunisolar gravity perturbations <u>only</u>

$1 \leqslant n \leqslant \infty$ - solar radiation pressure perturbations <u>only</u>

$$
\left.
\begin{array}{rcl}
0 \leqslant & m & \leqslant n \\
0 \leqslant & p & \leqslant n \\
0 \leqslant & h & \leqslant n \\
0 \leqslant & s & \leqslant n \\
-\infty \leqslant & q & \leqslant \infty \\
-\infty \leqslant & j & \leqslant \infty
\end{array}
\right\}
\quad \text{All lunisolar perturbations}
$$

and the orbital restrictions, $e < 1$, $(a/a_D)_{MOON} < 1$ and $(a/a_D)_{SUN} \ll 1$, which are valid for all Earth satellites. The n, m, p, q, h, j, s, δ and v values for the predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms of type (1) are given in tables 2.1(a) and 2.1(b), respectively.

TABLE 2.1(a)

The n,m,p,q,h,j,s and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in the Lunisolar Commensurability $\dot{\psi}_1 \approx 0$

n^+	m^+	p^+	q^+	h^+	j^+	s^+ Lunar Gravity	s^+ Solar Perturbations	δ^+
2	1	1	0	1	0	0	0,1,2	1
2	2	1	0	1	0	0	0,1,2	2

TABLE 2.1(b)

The n,m,p,q,h,j,s and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in the Lunisolar Commensurability $\dot{\psi}_1 \approx 0$

n^-	m^-	p^-	q^-	h^-	j^-	s^- Lunar Gravity	s^- Solar Perturbations	v
2	1	1	0	1	0	0	0,1,2	1
2	2	1	0	1	0	0	0,1,2	2

2.4(2) <u>The Type (2) Commensurability $\dot{\psi}_2 = \dot{\omega} \approx 0$</u>

A satellite exists in this type of lunisolar commensurability if its orbital inclination is approximately equal to that of the 'critical inclination' of 63.4^{o}. The $\Phi^{(+)}$ resonant terms are those for which

$$
\begin{aligned}
(n-2p)^{+} &= \delta \\
q^{+} &= -\delta \\
(n-2h+j)^{+} &= 0 \qquad\qquad (2.21) \\
j^{+} &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^{+}+2h^{+} & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^{+} &= 0 \\
s^{+} &\quad \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 1,2 \ldots n^{+} & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
-\infty \leqslant &\ \delta \leqslant \infty,\ \delta \neq 0.
\end{aligned}
$$

The arguments of these terms are of the form $\delta(\psi_2^{\,+})_{MOON}$ for the Moon, where $(\psi_2^{\,+})_{MOON} = \omega$, and of the form $\delta(\psi_2^{\,+})_{SUN}$ for the Sun, with $(\psi_2^{\,+})_{SUN} = \alpha\omega + \eta\,\omega_D + k\Omega_D$ ($k \geqslant 0$ if $\delta > 0$ and $k \leqslant 0$ if $\delta < 0$). Similarly, the $\Phi^{(-)}$ resonant terms for a type (2) commensurability are those for which

$$
\begin{aligned}
(n-2p)^{-} &= v \\
q^{-} &= -v \\
(n-2h+j)^{-} &= 0 \qquad\qquad (2.22) \\
j^{-} &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^{-}+2h^{-} & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^{-} &= 0 \\
s^{-} &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 1,2 \ldots n^{-} & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
-\infty &\leqslant v \leqslant \infty \quad v \neq 0
\end{aligned}
$$

The arguments of these resonant terms are of the form $v(\psi_2^-)_{MOON}$ for the Moon, where $(\psi_2^-)_{MOON} = \omega$, and of the form $v(\psi_2^-)_{SUN}$, with $(\psi_2^-)_{SUN} = \alpha\omega + \eta\,\omega_D + k\Omega_D$ ($k \geqslant 0$ if $v < 0$ and $k \leqslant 0$ if $v > 0$).

The n,m,p,q,h,j, and s values for the predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms of type (2) are given in tables 2.2.

TABLE 2.2(a)

The n,m,p,q,h,j,s, v and δ values for the Predominant $\Phi^{(\pm)}$ Resonant Terms of a Satellite in the Lunar Gravity Commensurability $\dot{\psi}_2 \approx 0$

n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
2	0	0	-2	1	0	0	2
2	0	2	2	1	0	0	-2

TABLE 2.2(b)

The n,m,p,q,h,j,s, δ and v values for the Predominant $\Phi^{(\pm)}$ Resonant Terms of a Satellite in the Solar Gravity Commensurability $\dot{\psi}_2 \approx 0$

RESTRICTIONS ON (a/a_D), e AND e_D	n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right)<1$	2	0	0	−2	1	0	0,1,2	2
		0	2	2	1	0	0,1,2	−2
if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right)>1$	3	0	1	−1	2	1	0,1,2,3	1
					1	−1	0,1,2,3	1
		0	2	1	2	1	0,1,2,3	−1
					1	−1	0,1,2,3	−1

TABLE 2.2(c)

The n,m,p,q,h,j,s, δ and v values for the Predominant $\Phi^{(\pm)}$ Resonant Terms of a Satellite in the Solar Radiation Pressure Commensurability $\dot{\psi}_2 \approx 0$

n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
1	0	0	−1	1	1	0,1	1
1	0	0	−1	0	−1	0,1	1
1	0	1	1	1	1	0,1	−1
1	0	1	1	0	−1	0,1	−1

2.4(3) <u>The Type (3) Commensurability $\dot{\psi}_3 = \alpha\dot{\omega} + \beta\dot{\Omega} \approx 0$</u>

The third important type of lunisolar commensurability, $\dot{\psi}_3 \approx 0$, is the generalised form of the two previous types, $\dot{\psi}_1 \approx 0$ and $\dot{\psi}_2 \approx 0$: $\dot{\psi}_3 \approx 0$ reduces to $\dot{\psi}_1 \approx 0$ and $\dot{\psi}_2 \approx 0$, if $\alpha = 0$ and $\beta = 0$, respectively. From equation (2.16), a satellite exists in the lunisolar commensurability $\dot{\psi}_3 \approx 0$, if its orbital inclination satisfies the quadratic equation

$$5\alpha \cos^2 i - 2\beta \cos i - \alpha = 0 \tag{2.23}$$

On solving (2.23), cosi is found to be given by

$$\cos i = \frac{\beta \pm (\beta^2 + 5\alpha^2)^{\frac{1}{2}}}{5\alpha} \tag{2.24}$$

Two cases of equation (2.20) need to be considered; first when α is positive, and second when α is negative: β, by definition, is always a positive integer. If α is positive, then

$$\cos i = \frac{\beta \pm (\beta^2 + 5\alpha^2)^{\frac{1}{2}}}{5\alpha} \tag{2.25}$$

From equation (2.25), it is seen that there are two possible values of i - i_1 and i_2 - which satisfy (2.23). If both solutions are real, then one - i_1, say - lies between 0° and 90°, and the other, i_2, lies between 90° and 180°. For both i_1 and i_2 to be real

$$\cos i_1 = \frac{\beta + (\beta^2 + 5\alpha^2)^{\frac{1}{2}}}{5\alpha} \leqslant 1 \tag{2.26}$$

α +ve

and

$$\cos i_2 = \frac{\beta - (\beta^2 + 5\alpha^2)^{\frac{1}{2}}}{5\alpha} \geqslant -1 \tag{2.27}$$

On simplification, (2.26) and (2.27) become

$$2|\alpha| \geqslant \beta \tag{2.28}$$

and

$$2|\alpha| + \beta \geqslant 0 \tag{2.29}$$

respectively. Since $2|\alpha| + \beta$ is always positive, then, for α positive, there must always exist a solution, i_2, of (2.23), which lies between 90° and 180° for all values of $|\alpha|$ and β ($\alpha > 0$). In addition, when α is positive, i_1 can only exist if β is less than twice α.

When α is negative, equation (2.24) can be written as

$$\cos i = \frac{-\beta \pm (\beta^2 + 5|\alpha|^2)^{\frac{1}{2}}}{5|\alpha|} \tag{2.30}$$

By analogy with the positive α case, it is found that, for $\alpha < 0$, there must always exist a solution, i_1 of (2.23) which lies between 0° and 90°. A second real solution of (2.23) exists if $2|\alpha| \geqslant \beta$. The solution (s) of equation (2.23) for $4 \geqslant \alpha \geqslant -4$ and $4 \geqslant \beta \geqslant 0$ are given in tables 2.3(a) and 2.3(b).

A satellite in a lunisolar commensurability of the type $\dot{\psi}_3 \approx 0$ is in resonance with those $\Phi^{(+)}$ terms for which

$$\begin{aligned}
(n-2p)^+ &= \alpha\delta \\
q^+ &= -\alpha\delta \\
(n-2h+j)^+ &= 0 \\
j^+ &= \begin{cases} 0 & \text{- lunar gravity perturbation} \\ -n^+ + 2h^+ & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^+ &= \beta\delta \\
s^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 1,2 \ldots n^+ & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
\delta &> 0
\end{aligned} \tag{2.31}$$

The arguments of these resonant terms are of the form $\delta(\psi_3^+)_{MOON}$ for the Moon, where $(\psi_3^+)_{MOON} = \alpha\omega + \beta\Omega$, and of the form $\delta(\psi_3^+)_{SUN}$ for the Sun, with $(\psi_3^+)_{SUN} = \alpha\omega + \eta\omega_D + \beta\Omega + k\Omega_D$ ($k \geqslant 0$).

Similarly, the resonant $\Phi^{(-)}$ terms for a type (3) commensurability are

$$
\begin{aligned}
(n-2p)^- &= \alpha v \\
q^- &= -\alpha v \\
(n-2h+j)^- &= 0 \qquad\qquad (2.32) \\
j^- &= \begin{cases} 0 & \text{- lunar gravity perturbation} \\ -n^- + 2h^- & \text{- solar gravity perturbations or solar radiation pressure perturbations} \end{cases} \\
m^- &= \beta v \\
s^- &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1,2 \ldots n^- & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
v &> 0
\end{aligned}
$$

As with the $\Phi^{(+)}$ resonant terms, the $\Phi^{(-)}$ resonant terms have arguments of the form $v(\psi_3^-)_{MOON}$ and $v(\psi_3^-)_{SUN}$, with $(\psi_3^-)_{MOON} = \alpha\omega + \beta\Omega$ and $(\psi_3^-)_{SUN} = \alpha\omega + \eta\,\omega_D + \beta\Omega + k\Omega_D$ $(k \leqslant 0)$. The values of n,m,p,q,h,j,s, δ and v for the predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms of type (3) for a particular α and β are given in tables 2.4(a), 2.4(b) and 2.4(c).

The most important commensurabilities of a particular type are those for which the amplitude factor, $(a/a_D)^n e^{|q|} e_D^{|j|}$ of the predominant resonant terms is an absolute minimum. In the case of lunar gravity perturbations, the most important type (3) commensurabilities are

$$
\begin{aligned}
&(1) \quad \dot{\omega} + \dot{\Omega} \approx 0 \\
&(2) \quad -\dot{\omega} + \dot{\Omega} \approx 0 \\
&(3) \quad 2\dot{\omega} + \dot{\Omega} \approx 0 \qquad\qquad (2.33) \\
&(4) \quad -2\dot{\omega} + \dot{\Omega} \approx 0
\end{aligned}
$$

The amplitude factors of the predominant resonant terms for the set of commensurabilities (2.33) are of the order $(a/a_D)^2e^2$. The corresponding resonant orbital inclinations for satellites existing in these commensurabilities can be found from table 2.3(a). They are:

$$\begin{array}{ll} (1) & 46.4^\circ \text{ and } 106.9^\circ \\ (2) & 73.2^\circ \text{ and } 133.6^\circ \\ (3) & 56.1^\circ \text{ and } 111.0^\circ \\ (4) & 69.0^\circ \text{ and } 123.9^\circ \end{array} \tag{2.34}$$

For solar gravity perturbations, the most important commensurabilities of type (3) are also the set (2.33). However, the amplitude factors of the predominant resonant terms are, in this case, of the order $\left(\frac{a}{s_D}\right)^2 e^2$, if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) < 1$, and of the order $\left(\frac{a}{a_D}\right)^3 ee_D$, if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) > 1$. Since $\left(\frac{a}{a_D}\right)_{SUN} \ll 1$ and $e_D \approx 1/60$, the eccentricity, e, of a satellite existing in such a solar gravity commensurability would need to be very small ($\approx 10^{-6}$, if $\left(\frac{a}{a_D}\right) = 1/17000$) for the predominant resonant terms to have amplitude factors of order $\left(\frac{a}{a_D}\right)^3 ee_D$.

TABLE 2.3(a)

Solution of Equation 2.23 for Lunar Gravity Perturbations when

$4 \geqslant \alpha \geqslant -4$: $0 \leqslant \beta \leqslant 4$

α	β	i_1^{o}	i_2^{o}	Predominant $\left(\frac{a}{a_D}\right)^n e^{\lvert q \rvert} e_D^{\lvert j \rvert}$ value	Commensurability Condition
0	1	90.0	-	$\left(\frac{a}{a_D}\right)^2$	$\dot{\Omega} \approx 0$
1	0	63.4	116.6	$\left(\frac{a}{a_D}\right)^2 e^2$	$\dot{\omega} \approx 0$
1	1	46.4	106.9	$\left(\frac{a}{a_D}\right)^2 e^2$	$\dot{\omega} + \dot{\Omega} \approx 0$
1	2	0.0	101.5	$\left(\frac{a}{a_D}\right)^4 e^2$	$\dot{\omega} + 2\dot{\Omega} \approx 0$
1	3	-	98.5	$\left(\frac{a}{a_D}\right)^6 e^2$	$\dot{\omega} + 3\dot{\Omega} \approx 0$
1	4	-	96.7	$\left(\frac{a}{a_D}\right)^8 e^2$	$\dot{\omega} + 4\dot{\Omega} \approx 0$
2	1	56.1	111.0	$\left(\frac{a}{a_D}\right)^2 e^2$	$2\dot{\omega} + \dot{\Omega} \approx 0$
2	3	33.0	103.8	$\left(\frac{a}{a_D}\right)^4 e^2$	$2\dot{\omega} + 3\dot{\Omega} \approx 0$
3	1	58.8	112.7	$\left(\frac{a}{a_D}\right)^6 e^6$	$3\dot{\omega} + \dot{\Omega} \approx 0$
3	2	53.1	109.5	$\left(\frac{a}{a_D}\right)^6 e^6$	$3\dot{\omega} + 2\dot{\Omega} \approx 0$
3	4	38.1	104.7	$\left(\frac{a}{a_D}\right)^8 e^6$	$3\dot{\omega} + 4\dot{\Omega} \approx 0$
4	1	60.0	113.6	$\left(\frac{a}{a_D}\right)^4 e^4$	$4\dot{\omega} + \dot{\Omega} \approx 0$
4	3	51.6	108.8	$\left(\frac{a}{a_D}\right)^4 e^4$	$4\dot{\omega} + 3\dot{\Omega} \approx 0$

TABLE 2.3(a) CONTINUED

α	β	i_1^{o}	i_2^{o}	Predominant $\left(\frac{a}{a_D}\right)^{n} e^{\lvert q \rvert} e_D^{\lvert j \rvert}$ value	Commensurability Condition
-1	1	73.2	133.6	$\left(\frac{a}{a_D}\right)^{2} e^{2}$	$-\dot{\omega} + \dot{\Omega} \approx 0$
-1	2	78.5	180.0	$\left(\frac{a}{a_D}\right)^{4} e^{2}$	$-\dot{\omega} + 2\dot{\Omega} \approx 0$
-1	3	81.5	-	$\left(\frac{a}{a_D}\right)^{6} e^{2}$	$-\dot{\omega} + 3\dot{\Omega} \approx 0$
-1	4	83.3	-	$\left(\frac{a}{a_D}\right)^{8} e^{2}$	$-\dot{\omega} + 4\dot{\Omega} \approx 0$
-2	1	69.0	123.9	$\left(\frac{a}{a_D}\right)^{2} e^{2}$	$-2\dot{\omega} + \dot{\Omega} \approx 0$
-2	3	76.2	147.0	$\left(\frac{a}{a_D}\right)^{4} e^{2}$	$-2\dot{\omega} + 3\dot{\Omega} \approx 0$
-3	1	67.3	121.9	$\left(\frac{a}{a_D}\right)^{6} e^{6}$	$-3\dot{\omega} + \dot{\Omega} \approx 0$
-3	2	70.5	126.9	$\left(\frac{a}{a_D}\right)^{6} e^{6}$	$-3\dot{\omega} + 2\dot{\Omega} \approx 0$
-3	4	75.3	141.9	$\left(\frac{a}{a_D}\right)^{8} e^{6}$	$-3\dot{\omega} + 4\dot{\Omega} \approx 0$
-4	1	66.4	120.0	$\left(\frac{a}{a_D}\right)^{4} e^{4}$	$-4\dot{\omega} + \dot{\Omega} \approx 0$
-4	3	71.2	128.4	$\left(\frac{a}{a_D}\right)^{4} e^{4}$	$-4\dot{\omega} + 3\dot{\Omega} \approx 0$

TABLE 2.3(b)

Solution of Equation (2.23) for Solar Gravity and Solar Radiation Pressure Perturbations when $4 \geqslant \alpha \geqslant -4$ AND $0 \leqslant \beta \leqslant 4$

α	β	i_1^o	i_2^o	Predominant Value of $\left(\frac{a}{a_D}\right)^{n} e^{\lvert q\rvert} e_D^{\lvert j\rvert}$ Gravity	Radiation	Commensurability Condition
0	1	90.0	-	$\left(\frac{a}{a_D}\right)^2$	$\left(\frac{a}{a_D}\right)^2$	$\dot{\Omega} \approx 0$
1*	0	63.4	116.6	* $\left(\frac{a}{a_D}\right)^2 e^2$	$\left(\frac{a}{a_D}\right) ee_D$	$\dot{\omega} \approx 0$
1*	1	46.4	106.9	* $\left(\frac{a}{a_D}\right)^2 e^2$	$\left(\frac{a}{a_D}\right) ee_D$	$\dot{\omega} + \dot{\Omega} \approx 0$
1	2	0.0	101.5	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\dot{\omega} + 2\dot{\Omega} \approx 0$
1	3	-	98.5	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\dot{\omega} + 3\dot{\Omega} \approx 0$
1	4	-	96.7	$\left(\frac{a}{a_D}\right)^5 ee_D$	$\left(\frac{a}{a_D}\right)^5 ee_D$	$\dot{\omega} + 4\dot{\Omega} \approx 0$
2	1	56.1	111.0	$\left(\frac{a}{a_D}\right)^2 e^2$	$\left(\frac{a}{a_D}\right)^2 e^2$	$2\dot{\omega} + \dot{\Omega} \approx 0$
2	3	33.0	103.8	$\left(\frac{a}{a_D}\right)^4 e^2$	$\left(\frac{a}{a_D}\right)^4 e^2$	$2\dot{\omega} + 3\dot{\Omega} \approx 0$
3	1	58.8	112.7	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$3\dot{\omega} + \dot{\Omega} \approx 0$
3	2	53.1	109.5	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$3\dot{\omega} + 2\dot{\Omega} \approx 0$
3	4	38.1	104.7	$\left(\frac{a}{a_D}\right)^5 e^3 e_D$	$\left(\frac{a}{a_D}\right)^5 e^3 e_D$	$3\dot{\omega} + 4\dot{\Omega} \approx 0$
4	1	60.0	113.6	$\left(\frac{a}{a_D}\right)^4 e^4$	$\left(\frac{a}{a_D}\right)^4 e^4$	$4\dot{\omega} + \dot{\Omega} \approx 0$
4	3	51.6	108.8	$\left(\frac{a}{a_D}\right)^4 e^4$	$\left(\frac{a}{a_D}\right)^4 e^4$	$4\dot{\omega} + 3\dot{\Omega} \approx 0$

* if $\left(\frac{ae_D}{a_D e}\right) > 1$ Predominant $\left(\frac{a}{a_D}\right)^{n} e^{\lvert q\rvert} e^{\lvert j\rvert}$ value is $\left(\frac{a}{a_D}\right)^3 ee_D$

TABLE 2.3(b) CONTINUED

α	β	i_1^{o}	i_2^{o}	Predominant value of $\left(\frac{a}{a_D}\right)^n e^{\lvert q\rvert} e_D^{\lvert j\rvert}$ Gravity	Radiation	Commensurability Condition
-1^*	1^*	73.2	133.6	$^*\left(\frac{a}{a_D}\right)e^2$	$\left(\frac{a}{a_D}\right)ee_D$	$-\dot{\omega}+\dot{\Omega}\approx 0$
-1	2	78.5	180.0	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\left(\frac{a}{a_D}\right)^3 ee_D$	$-\dot{\omega}+2\dot{\Omega}\approx 0$
-1	3	81.5	-	$\left(\frac{a}{a_D}\right)^3 ee_D$	$\left(\frac{a}{a_D}\right)^3 ee_D$	$-\dot{\omega}+3\dot{\Omega}\approx 0$
-1	4	83.3	-	$\left(\frac{a}{a_D}\right)^5 ee_D$	$\left(\frac{a}{a_D}\right)^5 ee_D$	$-\dot{\omega}+4\dot{\Omega}\approx 0$
-2	1	69.0	123.9	$\left(\frac{a}{a_D}\right)^2 e^2$	$\left(\frac{a}{a_D}\right)^2 e^2$	$-2\dot{\omega}+\dot{\Omega}\approx 0$
-2	3	76.2	147.0	$\left(\frac{a}{a_D}\right)^4 e^2$	$\left(\frac{a}{a_D}\right)^4 e^2$	$-2\dot{\omega}+3\dot{\Omega}\approx 0$
-3	1	67.3	121.3	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$-3\dot{\omega}+\dot{\Omega}\approx 0$
-3	2	70.5	126.9	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$\left(\frac{a}{a_D}\right)^3 e^3 e_D$	$-3\dot{\omega}+2\dot{\Omega}\approx 0$
-3	4	75.3	141.9	$\left(\frac{a}{a_D}\right)^5 e^3 e_D$	$\left(\frac{a}{a_D}\right)^5 e^3 e_D$	$-3\dot{\omega}+4\dot{\Omega}\approx 0$
-4	1	66.4	120.0	$\left(\frac{a}{a_D}\right)^4 e^4$	$\left(\frac{a}{a_D}\right)^4 e^4$	$-4\dot{\omega}+\dot{\Omega}\approx 0$
-4	3	71.2	128.4	$\left(\frac{a}{a_D}\right)^4 e^4$	$\left(\frac{a}{a_D}\right)^4 e^4$	$-4\dot{\omega}+3\dot{\Omega}\approx 0$

* if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) > 1$ Predominant $\left(\frac{a}{a_D}\right)^n e^{\lvert q\rvert} e_D^{\lvert j\rvert}$ is $\left(\frac{a}{a_D}\right)^3 ee_D$

TABLE 2.4(a)

The n,m,p,q,h,j,s, δ and v values for the Predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_3 \approx 0$

RESTRICTIONS ON α and β	n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
α +ve $\alpha > \beta$ α even	α	β	0	$-\alpha$	$\alpha/2$	0	0	1
α +ve $\alpha \geqslant \beta$ α odd	2α	2β	0	-2α	α	0	0	2
α +ve, $\alpha < \beta$ α odd, β odd or α odd, β even	2β	2β	$\beta-\alpha$	-2α	β	0	0	2
α +ve $\alpha < \beta$ α even, β odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\frac{(\beta+1)}{2}$	0	0	1
α -ve $\|\alpha\| > \beta$ $\|\alpha\|$ even	$\|\alpha\|$	β	$\|\alpha\|$	$\|\alpha\|$	$\|\alpha\|/2$	0	0	1
α -ve $\|\alpha\| \geqslant \beta$ $\|\alpha\|$ odd	$2\|\alpha\|$	2β	$2\|\alpha\|$	$2\|\alpha\|$	$\|\alpha\|$	0	0	2
α -ve $\|\alpha\| < \beta$ $\|\alpha\|$ even, β odd	$\beta+1$	β	$(\beta+\|\alpha\|+1)/2$	$\|\alpha\|$	$\frac{(\beta+1)}{2}$	0	0	1
α -ve, $\|\alpha\| < \beta$ $\|\alpha\|$ odd, β odd or $\|\alpha\|$ odd, β even	2β	2β	$\beta+\|\alpha\|$	$2\|\alpha\|$	β	0	0	2

TABLE 2.4(b)

The n,m,p,q,h,j,s, δ and v values of the Predominant Resonant Terms ($\Phi^{(+)}$ and $\Phi^{(-)}$) for a Satellite in a Solar Gravity or Solar Radiation Pressure Commensurability, $\dot{\psi}_3 \approx 0$

RESTRICTIONS ON α and β	n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
α +ve, $\alpha > \beta$, α even	α	β	0	$-\alpha$	$\alpha/2$	0	$0,1..\alpha$	1
α +ve, $\alpha \geqslant \beta$, α odd	α^*	β	0	$-\alpha$	$(\alpha+1)/2$	1	$0,1...\alpha$	1
	α^*	β	0	$-\alpha$	$(\alpha-1)/2$	-1	$0,1...\alpha$	1
α +ve, $\alpha < \beta$, α even, β odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$(\beta+1)/2$	0	$0,1..\alpha$	1
α +ve, $\alpha < \beta$, α odd, β odd	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+1)/2$	1	$0,1..\alpha$	1
	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-1)/2$	-1	$0,1..\alpha$	1
α +ve, $\alpha < \beta$, α odd, β even	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$(\beta+2)/2$	1	$0,1..\alpha$	1
	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\beta/2$	-1	$0,1..\alpha$	1

* See table 2.4(c) for predominant solar gravity terms if $\alpha = 1$.

TABLE 2.4(b) CONTINUED

RESTRICTIONS ON α and β	n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
α -ve $\lvert\alpha\rvert > \beta$ $\lvert\alpha\rvert$ even	$\lvert\alpha\rvert$	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\lvert\alpha\rvert/2$	0	$0,1\ldots\alpha$	1
α -ve $\lvert\alpha\rvert \geqslant \beta$ $\lvert\alpha\rvert$ odd	$\lvert\alpha\rvert^*$	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert+1)}{2}$	1	$0,1\ldots\alpha$	1
	$\lvert\alpha\rvert^*$	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert-1)}{2}$	-1	$0,1\ldots\alpha$	1
α -ve $\lvert\alpha\rvert < \beta$ $\lvert\alpha\rvert$ even, β odd	$\beta+1$	β	$(\beta+1+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+1)/2$	0	$0,1\ldots\alpha$	1
α -ve $\lvert\alpha\rvert < \beta$ $\lvert\alpha\rvert$ odd, β odd	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+1)/2$	1	$0,1\ldots\alpha$	1
	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta-1)/2$	-1	$0,1\ldots\alpha$	1
α -ve $\lvert\alpha\rvert < \beta$ $\lvert\alpha\rvert$ odd, β even	$\beta+1$	β	$(\beta+1+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+2)/2$	1	$0,1\ldots\alpha$	1
	$\beta+1$	β	$(\beta+1+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$\beta/2$	-1	$0,1\ldots\alpha$	1

* if $\alpha = -1$ then see table 2.4(c) for predominant solar gravity terms.

TABLE 2.4(c)

The n,m,p,q,h,j,s, δ and v values for the Predominant Resonant Terms ($\Phi^{(+)}$ and $\Phi^{(-)}$) for a Satellite in a Solar Gravity Commensurability

$\pm\dot{\omega} + \dot{\Omega} \approx 0$

RESTRICTIONS ON α and β	n^+,n^-	m^+,m^-	p^+,p^-	q^+,q^-	h^+,h^-	j^+,j^-	s^+,s^-	δ,v
$\alpha = 1,\ \beta = 1$, $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) < 1$	2	2	0	-2	1	0	0,1,2	2
$\alpha = 1,\ \beta = 1$, $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) > 1$	3	1	1	-1	2	1	0,1,2,3	1
	3	1	1	-1	1	-1	0,1,2,3	1
$\alpha = -1,\ \beta = 1$, $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) < 1$	2	2	2	2	1	0	0,1,2	2
$\alpha = -1,\ \beta = 1$, $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) > 1$	3	1	2	1	2	1	0,1,2,3	1
	3	1	2	1	1	-1	0,1,2,3	1

2.4(4) <u>The Type (4) Commensurability $\dot{\psi}_4 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) \approx 0$</u>

A satellite in a commensurability of the type $\dot{\psi}_4 \approx 0$ is in resonance with those $\Phi^{(+)}$ terms in the lunisolar disturbing function expansions for which

$$
\begin{aligned}
(n - 2p)^+ &= \alpha\delta \\
(n - 2h + j)^+ &= \gamma\delta \\
q^+ &= -\alpha\delta \qquad\qquad (2.35) \\
j^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ \gamma\delta - n^+ + 2h^+ & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases} \\
s^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1\ldots n^+ & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^+ &= 0
\end{aligned}
$$

The arguments of the resonant terms are of the form $\delta(\psi_4^+)_{MOON}$ for the Moon, where $(\psi_4^+)_{MOON} = \alpha\omega + \gamma(\omega_D + M_D)$, and of the form $\delta(\psi_4^+)_{SUN}$ for the Sun, with $(\psi_4^+)_{SUN} = \alpha\omega + \eta\omega_D + \gamma M_D + k\Omega_D$. The n,m,p,q,h,j,s and δ values for the predominant $\Phi^{(+)}$ resonant terms of a lunar gravity commensurability of type (4) are given in tables 2.5(a) and 2.5(c); the corresponding values for a solar gravity, or solar radiation pressure, commensurability, $\dot{\psi}_4 \approx 0$, are given in tables 2.5(b) and 2.5(d).

Similarly, a satellite in a commensurability of the type $\dot{\psi}_4 \approx 0$ will be in resonance with those $\Phi^{(-)}$ terms in the lunisolar disturbing function expansions (2.4) for which

$$(n-2p)^{-} = \alpha v$$

$$q^{-} = -\alpha v$$

$$(n-2h+j)^{-} = -\gamma v \qquad (2.36)$$

$$j^{-} = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -\gamma v - n^{-} + 2h^{-} & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$s^{-} = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1\ldots n^{-} & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$m^{-} = 0$$

The arguments of the $\Phi^{(-)}$ resonance terms are of the form $v(\psi^{-}_{4})_{MOON}$ for the Moon, with $(\psi^{-}_{4})_{MOON} = \alpha\omega + \gamma(\omega_D + M_D)$, and of the form $v(\psi^{-}_{4})_{SUN}$ for the Sun, where $(\psi^{-}_{4})_{SUN} = \alpha\omega + \eta\,\omega_D + \gamma M_D + k\Omega_D$. The n,m,p,q,h,j,s and v values for the predominant $\Phi^{(-)}$ resonant terms of a satellite in the commensurability $\dot{\psi}_4 \approx 0$ are listed in tables 2.5(e), 2.5(f), 2.5(g) and 2.5(h).

If β is put to zero in equation (2.16) along with k and ζ, then the orbital elements of a satellite in a lunisolar commensurability $\dot{\psi}_4 \approx 0$ must satisfy

$$24.9 \quad \alpha \cos^2 i + \gamma n_D y^{3.5} - 4.98\,\alpha \approx 0 \qquad (2.37)$$

where

$$n_D = (\dot{\omega}_D + \dot{M}_D) \qquad (2.38)$$

and

$$y = \frac{a}{R_E} (1-e^2)^{4/7} \qquad (2.39)$$

The lunisolar commensurability $\dot{\psi}_4 \approx 0$ represents a departure from the three previous types in that it is not entirely inclination-dependent, but depends also on the semi-major axis, a, and the eccentricity, e, of a satellite's orbit. When γ/α is positive, the maximum value of y occurs at $i = 90^\circ$. The maximum value of y for γ/α positive is given by

$$y_{max} = \left[\frac{4.98\,\alpha}{n_D\gamma}\right]^{2/7} \qquad \gamma/\alpha > 0 \qquad (2.40)$$

Similarly, when $\gamma/\alpha < 0$, the maximum value of y occurs at $i = 0^\circ$ and 180°, and its value is given by

$$y_{max} = \left[\frac{19.92}{n_D}\left|\frac{\alpha}{\gamma}\right|\right]^{2/7} \qquad \gamma/\alpha < 0 \qquad (2.41)$$

Now for a satellite to exist in orbit, $a(1-e) > R_E$. If y is written in the form

$$y = \frac{a(1-e)(1+e)}{R_E(1-e^2)^{3/7}}$$

then it is easily seen that y must always be greater than unity, since $(1+e)/(1-e^2)^{3/7} \geqslant 1$ and $a(1-e) > R_E$. It therefore follows from equations (2.40) and (2.41) that close satellites can exist in lunisolar commensurabilities of the type $\dot{\psi}_4 \approx 0$ if

$$\underline{\underline{4.98\,\alpha > \gamma\, n_D}} \qquad (2.42)$$

for $\gamma/\alpha > 0$ and

$$\underline{\underline{19.92\,|\alpha| > |\gamma n_D|}} \qquad (2.43)$$

for $\gamma/\alpha < 0$.

If the appropriate condition (2.42) or (2.43) is not met for a given α and γ, then no close satellites can exist in the lunisolar commensurability $\dot{\psi}_4 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) \approx 0$. When the appropriate values of n_D for the Moon and the Sun are substituted into equations (2.42) and (2.43), the criteria for the existence of satellites in solar and lunar commensurabilities of the type $\dot{\psi}_4 \approx 0$ become

$$\begin{array}{lll} 5.05\,|\alpha| > |\gamma| & \gamma/\alpha > 0 & \\ & & \text{SUN} \\ 20.2\,|\alpha| > |\gamma| & \gamma/\alpha < 0 & \end{array}$$

and (2.44)

$$\begin{array}{lll} 0.38\,|\alpha| > |\gamma| & \gamma/\alpha > 0 & \\ & & \text{MOON} \\ 1.51\,|\alpha| > |\gamma| & \gamma/\alpha < 0 & \end{array}$$

For solar radiation pressure perturbations, two commensurabilities of the type $\dot{\psi}_4 \simeq 0$ are theoretically possible, in which the amplitude factors, $(a/a_D)^n e^{|q|} e_D^{|j|}$, of the predominant resonant terms have an absolute maximum of $(a/a_D)e$. They are

$$\begin{array}{ll} (5) & \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \approx 0 \\ (6) & -\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \approx 0 \end{array} \qquad (2.45)$$

Reference to conditions (2.44) shows that both of the solar commensurabilities of the set (2.45) occur for close satellites. In the case of lunisolar gravity perturbations, the most important commensurabilities of the type $\dot{\psi}_4 \approx 0$ are (5) and (6) of the set (2.45), if $(a/a_D e) < 1$. The amplitude factors of the predominant resonant terms are of the order $(a/a_D)^2 e^2$. However, if $(a/a_D e) > 1$, then the most important type (4) commensurabilities for lunisolar gravity perturbations are the set (2.45) plus

$$\begin{array}{ll} (7) & \dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) \approx 0 \\ (8) & -\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) \approx 0 \end{array} \qquad (2.46)$$

The predominant resonant terms in this instance (i.e. $(a/a_D e) > 1$) have amplitude factors of order $(a/a_D)^3 e$. From conditions (2.44), it is obvious that both of the commensurabilities in the set (2.46) are possible for solar gravity perturbations, but neither are possible for lunar gravity perturbations. Finally, only commensurability (6) of the set (2.45) is possible for lunar gravity perturbations.

The graphs of the function (2.37) for the solar commensurabilities (5) and (6) of the set (2.45) are given in figures (2.1) and (2.2), respectively. The graphs of solar commensurabilities (7) and (8) are given in figures (2.3) and (2.4) and the lunar commensurability (6) in figure (2.5).

TABLE 2.5(a)

The n,m,p,q,h,j,s and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability of the type $\dot{\psi}_4 \approx 0$

RESTRICTIONS ON α and β	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve γ +ve, $\alpha \geqslant \gamma$, $\alpha+\gamma$ even	α^*	0	0	$-\alpha$	$(\alpha-\gamma)/2$	0	0	1
	α^*	0	α	α	$(\alpha+\gamma)/2$	0	0	-1
α +ve γ +ve, $\alpha > \gamma$, $\alpha+\gamma$ odd	2α	0	0	-2α	$\alpha-\gamma$	0	0	2
	2α	0	2α	2α	$\alpha+\gamma$	0	0	-2
α +ve γ +ve, $\gamma > \alpha$, $\alpha+\gamma$ even	γ	0	$(\gamma-\alpha)/2$	$-\alpha$	0	0	0	1
	γ	0	$(\gamma+\alpha)/2$	$+\alpha$	γ	0	0	-1
α +ve γ +ve, $\gamma > \alpha$, $\alpha+\gamma$ odd	2γ	0	$\gamma-\alpha$	-2α	0	0	0	2
	2γ	0	$\gamma+\alpha$	2α	2γ	0	0	-2
α -ve γ +ve, $\lvert\alpha\rvert \geqslant \gamma$, $\lvert\alpha\rvert+\gamma$ even	$\lvert\alpha\rvert^*$	0	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert-\gamma)}{2}$	0	0	1
	$\lvert\alpha\rvert^*$	0	0	$-\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert+\gamma)}{2}$	0	0	-1
α -ve γ +ve, $\lvert\alpha\rvert > \gamma$, $\lvert\alpha\rvert+\gamma$ odd	$2\lvert\alpha\rvert$	0	$2\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\gamma)$	0	0	2
	$2\lvert\alpha\rvert$	0	0	$-2\lvert\alpha\rvert$	$\lvert\alpha\rvert+\gamma$	0	0	-2
α -ve α +ve, $\gamma > \lvert\alpha\rvert$, $\lvert\alpha\rvert+\gamma$ even	γ	0	$\frac{(\gamma+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	0	0	0	1
	γ	0	$\frac{(\gamma-\lvert\alpha\rvert)}{2}$	$-\lvert\alpha\rvert$	γ	0	0	-1
α -ve γ +ve, $\gamma > \lvert\alpha\rvert$, $\lvert\alpha\rvert+\gamma$ odd	2γ	0	$\gamma+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	0	0	0	2
	2γ	0	$\gamma-\lvert\alpha\rvert$	$-2\lvert\alpha\rvert$	2γ	0	0	-2

* See Table 2.5(c) for predominant lunar gravity terms if $\alpha = \pm 1$

TABLE 2.5(b)

The n,m,p,q,h,j,s and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Solar Gravity or Solar Radiation Commensurability of the type $\dot{\psi}_4 \approx 0$

RESTRICTIONS ON α and γ	n^+	m^+	p^+	q^+	j^+	h^+	s^+	δ
α +ve γ +ve, $\alpha \geqslant \gamma$, $\alpha + \gamma$ even	α*	0	0	$-\alpha$	0	$(\alpha-\gamma)/2$	$0,1\ldots\alpha$	1
			α	α	0	$(\alpha+\gamma)/2$	$0,1\ldots\alpha$	-1
α +ve γ +ve, $\alpha > \gamma$, $\alpha + \gamma$ odd	α	0	0	$-\alpha$	1	$(\alpha-\gamma+1)/2$	$0,1\ldots\alpha$	1
			0	$-\alpha$	-1	$(\alpha-\gamma-1)/2$	$0,1\ldots\alpha$	1
			α	α	1	$(\alpha+\gamma+1)/2$	$0,1\ldots\alpha$	-1
			α	α	-1	$(\alpha+\gamma-1)/2$	$0,1\ldots\alpha$	-1
α +ve γ +ve, $\gamma > \alpha$	α*	0	0	$-\alpha$	$\gamma-\alpha$	0	$0,1\ldots\alpha$	1
			α	α	$\alpha-\gamma$	α	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\|\alpha\| \geqslant \gamma$, $\|\alpha\| + \gamma$ even	$\|\alpha\|$*	0	$\|\alpha\|$	$\|\alpha\|$	0	$(\|\alpha\|-\gamma)/2$	$0,1\ldots\alpha$	1
			0	$-\|\alpha\|$	0	$(\|\alpha\|+\gamma)/2$	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\|\alpha\| > \gamma$, $\|\alpha\| + \gamma$ odd	$\|\alpha\|$	0	$\|\alpha\|$	$\|\alpha\|$	1	$\frac{(\|\alpha\|-\gamma+1)}{2}$	$0,1\ldots\alpha$	1
			$\|\alpha\|$	$\|\alpha\|$	-1	$\frac{(\|\alpha\|-\gamma-1)}{2}$	$0,1\ldots\alpha$	1
			0	$-\|\alpha\|$	1	$\frac{(\|\alpha\|+\gamma+1)}{2}$	$0,1\ldots\alpha$	-1
			0	$-\|\alpha\|$	-1	$\frac{(\|\alpha\|+\gamma-1)}{2}$	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\gamma > \|\alpha\|$	$\|\alpha\|$*	0	0	$-\|\alpha\|$	$-\|\alpha\|+\gamma$	0	$0,1\ldots\alpha$	1
			$\|\alpha\|$	$\|\alpha\|$	$-\gamma+\|\alpha\|$	$\|\alpha\|$	$0,1\ldots\alpha$	-1

* If $\alpha = \pm 1$ then the predominant solar gravity terms can be found in Table 2.5(d).

TABLE 2.5(c)

The n,m,p,q,h,j,s and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\pm\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \approx 0$

RESTRICTIONS ON α and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	0	0	-2	0	0	0	2
			2	2	2	0	0	-2
$\alpha = 1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	0	1	-1	1	0	0	1
			2	1	2	0	0	-1
$\alpha = -1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	0	2	2	0	0	0	2
			0	-2	2	0	0	-2
$\alpha = -1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	0	2	+1	1	0	0	1
			1	-1	2	0	0	-1

TABLE 2.5(d)

The n,m,p,q,h,j,s and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Solar Gravity Commensurability $\pm\dot{\omega} + \gamma(\dot{\omega}_D \pm \dot{M}_D) \approx 0$

RESTRICTIONS ON α and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1 \quad \gamma = 1$	2	0	0	-2	0	0	0,1,2	2
if $\left(\frac{a}{a_D e}\right) < 1$			2	2	2	0	0,1,2	-2
$\alpha = 1 \quad \gamma = 1$	3	0	1	-1	1	0	0,1,2,3	1
if $\left(\frac{a}{a_D e}\right) > 1$			2	1	2	0	0,1,2,3	-1
$\alpha = 1 \quad \gamma = 2$	2	0	0	-2	0	+2	0,1,2	2
if $\left(\frac{a}{a_D e e_D}\right) < 1$			2	2	2	-2	0,1,2	-2
$\alpha = 1 \quad \gamma = 2$	3	0	1	-1	1	1	0,1,2,3	1
			1	-1	0	-1	0,1,2,3	1
if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	3	1	0,1,2,3	-1
			2	1	2	-1	0,1,2,3	-1
$\alpha = 1 \quad \gamma \geqslant 3$	2	0	0	-2	0	$2\gamma-2$	0,1,2	2
if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) < 1$			2	2	2	$2-2\gamma$	0,1,2	-2
$\alpha = 1 \quad \gamma \geqslant 3$	3	0	1	-1	0	$\gamma-3$	0,1,2,3	1
if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) > 1$			2	1	3	$3-\gamma$	0,1,2,3	-1

/cont..

TABLE 2.5(d) cont.

RESTRICTIONS ON α and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = -1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	0	2	2	0	0	0,1,2	2
			0	-2	2	0	0,1,2	-2
$\alpha = -1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	0	2	1	1	0	0,1,2,3	1
			1	-1	2	0	0,1,2,3	-1
$\alpha = -1 \quad \gamma = 2$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	0	2	0,1,2	2
			0	-2	2	-2	0,1,2	-2
$\alpha = -1 \quad \gamma = 2$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	2	-1	0,1,2,3	-1
			1	-1	3	1	0,1,2,3	-1
			2	+1	0	-1	0,1,2,3	+1
			2	+1	1	1	0,1,2,3	+1
$\alpha = -1 \quad \gamma \geqslant 3$ if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) < 1$	2	0	2	2	0	$2\gamma-2$	0,1,2	2
			0	-2	2	$2-2\gamma$	0,1,2	-2
if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) > 1$	3	0	2	1	0	$\gamma-3$	0,1,2,3	1
			1	-1	3	$3-\gamma$	0,1,2,3	-1

TABLE 2.5(e)

The $\bar{n}$, $\bar{m}$, $\bar{p}$, $\bar{q}$, $\bar{h}$, $\bar{j}$, $\bar{s}$ and v values for Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability of the Type $\dot{\psi}_4 \approx 0$

RESTRICTIONS ON α and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α +ve γ +ve	α^*	0	0	$-\alpha$	$(\alpha+\gamma)/2$	0	0	1
$\alpha \geqslant \gamma$								
$\alpha+\gamma$ even	α^*	0	α	α	$(\alpha-\gamma)/2$	0	0	-1
α +ve γ +ve	2α	0	0	-2α	$\alpha+\gamma$	0	0	2
$\alpha > \gamma$								
$\alpha+\gamma$ odd	2α	0	2α	2α	$\alpha-\gamma$	0	0	-2
α +ve γ +ve	γ	0	$(\gamma-\alpha)/2$	$-\alpha$	γ	0	0	1
$\gamma > \alpha$								
$\alpha+\gamma$ even	γ	0	$(\gamma+\alpha)/2$	α	0	0	0	-1
α +ve γ +ve	2γ	0	$\gamma-\alpha$	-2α	2γ	0	0	2
$\gamma > \alpha$								
$\alpha+\gamma$ odd	2γ	0	$\gamma+\alpha$	2α	0	0	0	-2
α -ve γ +ve	$\|\alpha\|^*$	0	$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|+\gamma)/2$	0	0	1
$\|\alpha\| \geqslant \gamma$								
$\|\alpha\|+\gamma$ even	$\|\alpha\|^*$	0	0	$-\|\alpha\|$	$(\|\alpha\|-\gamma)/2$	0	0	-1
α -ve γ +ve	$2\|\alpha\|^*$	0	$2\|\alpha\|$	$2\|\alpha\|$	$(\|\alpha\|+\gamma)$	0	0	2
$\|\alpha\| > \gamma$								
$\|\alpha\|+\gamma$ odd	$2\|\alpha\|$	0	0	$-2\|\alpha\|$	$\|\alpha\|-\gamma$	0	0	-2
α -ve γ +ve	γ	0	$(\gamma+\|\alpha\|)/2$	$\|\alpha\|$	γ	0	0	1
$\|\gamma\| > \|\alpha\|$								
$\|\alpha\|+\gamma$ even	γ	0	$(\gamma-\|\alpha\|)/2$	$-\|\alpha\|$	0	0	0	-1
α -ve γ +ve	2γ	0	$\gamma+\|\alpha\|$	$2\|\alpha\|$	2γ	0	0	2
$\|\gamma\| > \|\alpha\|$								
$\|\alpha\|+\gamma$ odd	2γ	0	$\gamma-\|\alpha\|$	$-2\|\alpha\|$	0	0	0	-2

* See table 2.5(g) for predominant lunar gravity terms if $\alpha = \pm 1$

TABLE 2.5(f)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Solar Gravity or Solar Radiation Pressure Commensurability of the Type $\dot{\psi}_4 \approx 0$

RESTRICTIONS ON α and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α +ve γ +ve, $\alpha \geqslant \gamma$, $\alpha + \gamma$ even	α *	0	0	$-\alpha$	$(\alpha+\gamma)/2$	0	$0,1\ldots\alpha$	1
			α	α	$(\alpha-\gamma)/2$	0	$0,1\ldots\alpha$	-1
α +ve γ +ve, $\alpha > \gamma$, $\alpha + \gamma$ odd	α	0	0	$-\alpha$	$(\alpha+\gamma+1)/2$	1	$0,1\ldots\alpha$	1
			0	$-\alpha$	$(\alpha+\gamma-1)/2$	-1	$0,1\ldots\alpha$	1
			α	α	$(\alpha-\gamma+1)/2$	1	$0,1\ldots\alpha$	-1
			α	α	$(\alpha-\gamma-1)/2$	-1	$0,1\ldots\alpha$	-1
α +ve γ +ve, $\gamma > \alpha$	α *	0	0	$-\alpha$	α	$\alpha-\gamma$	$0,1\ldots\alpha$	1
			α	α	0	$\gamma-\alpha$	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\lvert\alpha\rvert \geqslant \gamma$, $\lvert\alpha\rvert + \gamma$ even	$\lvert\alpha\rvert$ *	0	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\gamma)/2$	0	$0,1\ldots\alpha$	1
			0	$-\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\gamma)/2$	0	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\lvert\alpha\rvert > \gamma$, $\lvert\alpha\rvert + \gamma$ odd	$\lvert\alpha\rvert$	0	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\gamma+1)/2$	1	$0,1\ldots\alpha$	1
			$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\gamma-1)/2$	-1	$0,1\ldots\alpha$	1
			0	$-\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\gamma+1)/2$	1	$0,1\ldots\alpha$	-1
			0	$-\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\gamma-1)/2$	-1	$0,1\ldots\alpha$	-1
α -ve γ +ve, $\lvert\gamma\rvert > \lvert\alpha\rvert$	$\lvert\alpha\rvert$ *	0	0	$-\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$-\gamma+\lvert\alpha\rvert$	$0,1\ldots\alpha$	1
			$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	0	$\gamma-\lvert\alpha\rvert$	$0,1\ldots\alpha$	-1

* If $\alpha = \pm 1$ then predominant solar gravity terms can be found in table 2.5(h).

TABLE 2.5(g)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability of the form

$$+\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \approx 0$$

RESTRICTIONS ON α and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = 1$ $\gamma = 1$	2	0	0	−2	2	0	0	2
if $\left(\frac{a}{a_D e}\right) < 1$			2	2	0	0	0	−2
$\alpha = 1$ $\gamma = 1$	3	0	1	−1	2	0	0	1
if $\left(\frac{a}{a_D e}\right) > 1$			2	1	1	0	0	−1
$\alpha = -1$ $\gamma = 1$	2	0	2	2	2	0	0	2
if $\left(\frac{a}{a_D e}\right) < 1$			0	−2	0	0	0	−2
$\alpha = -1$ $\gamma = 1$	3	0	2	1	2	0	0	1
if $\left(\frac{a}{a_D e}\right) > 1$			1	−1	1	0	0	−1

TABLE 2.5(h)

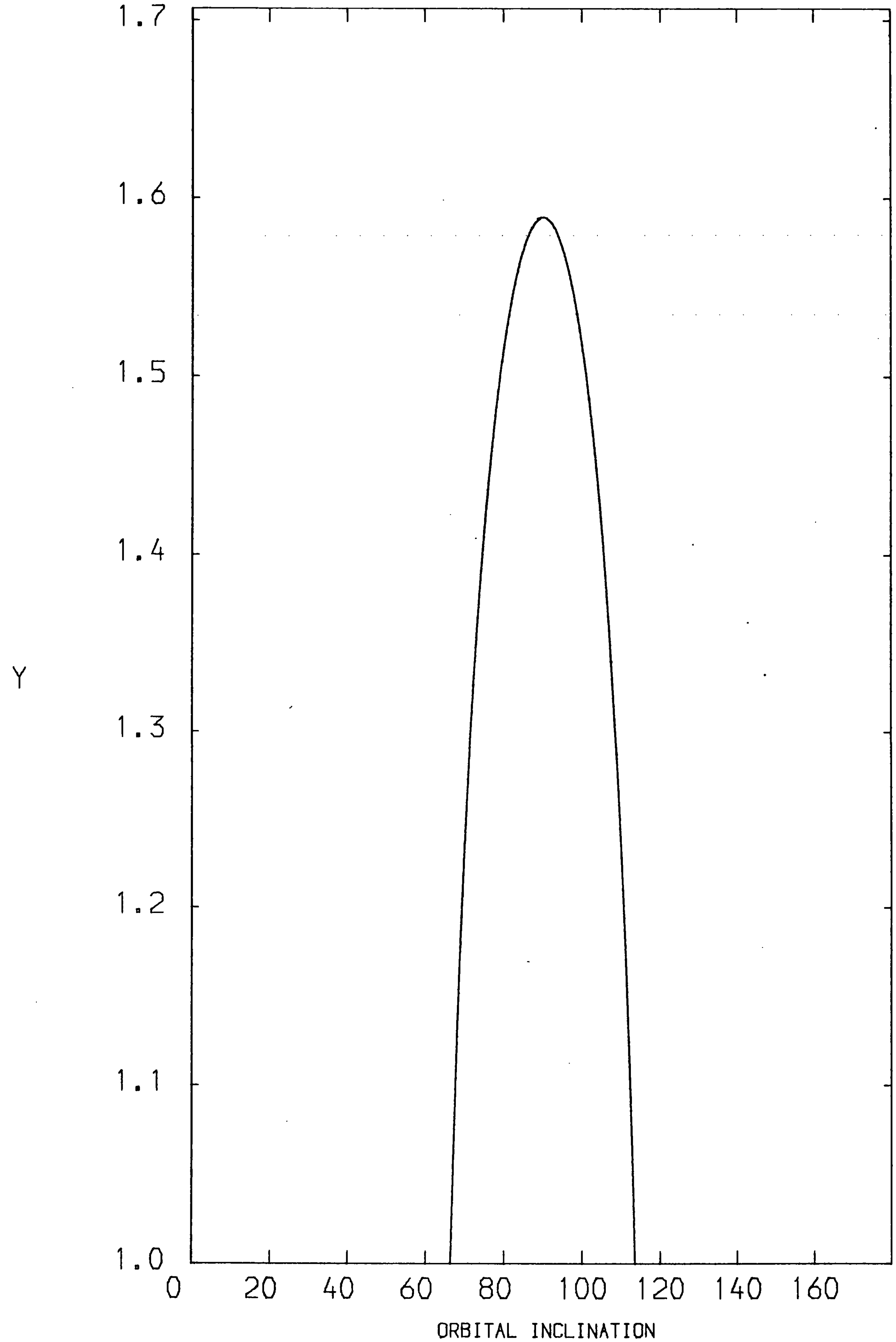
The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in a Solar Gravity Commensurability $\pm\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) \approx 0$

RESTRICTIONS ON α and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = 1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	0	0	-2	2	0	0,1,2	2
			2	2	0	0	0,1,2	-2
$\alpha = 1 \quad \gamma = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	0	1	-1	2	0	0,1,2,3	1
			2	1	1	0	0,1,2,3	-1
$\alpha = 1 \quad \gamma = 2$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	2	-2	0,1,2	2
			2	2	0	2	0,1,2	-2
$\alpha = 1 \quad \gamma = 2$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	3	1	0,1,2,3	1
			1	-1	2	-1	0,1,2,3	1
			2	1	1	1	0,1,2,3	-1
			2	1	0	-1	0,1,2,3	-1
$\alpha = 1 \quad \gamma \geqslant 3$ if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) < 1$	2	0	0	-2	2	$2-2\gamma$	0,1,2	2
			2	2	0	$2\gamma-2$	0,1,2	-2
$\alpha = 1 \quad \gamma \geqslant 3$ if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) > 1$	3	0	1	-1	3	$3-\gamma$	0,1,2,3	1
			2	1	0	$\gamma-3$	0,1,2,3	-1

/cont..

TABLE 2.5(h) cont.

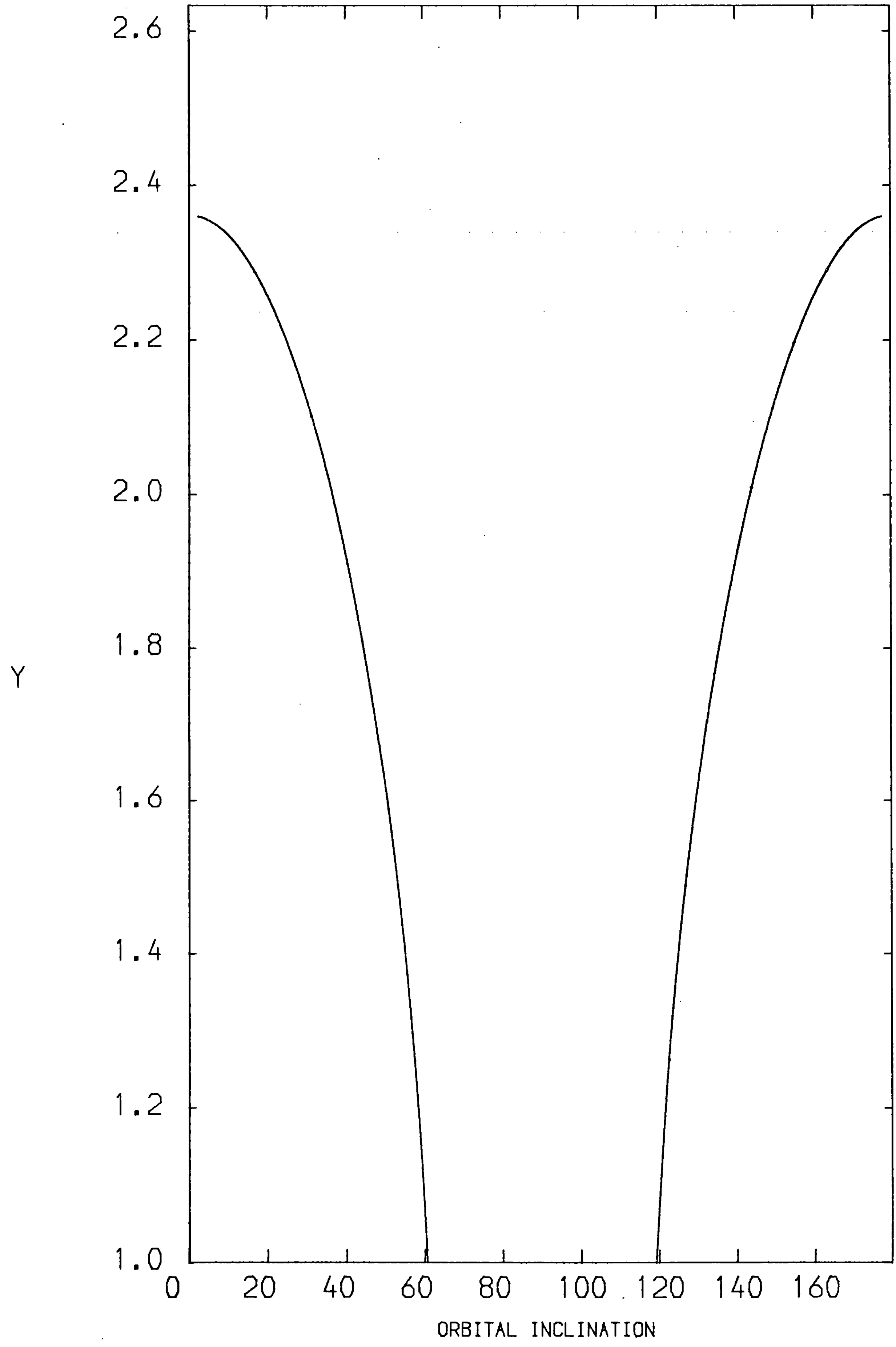
RESTRICTIONS ON α and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = -1$ $\gamma = 1$	2	0	2	2	2	0	0,1,2	2
if $\left(\frac{a}{a_D e}\right) < 1$			0	-2	0	0	0,1,2	-2
$\alpha = -1$ $\gamma = 1$	3	0	2	1	2	0	0,1,2,3	1
if $\left(\frac{a}{a_D e}\right) > 1$			1	-1	1	0	0,1,2,3	-1
$\alpha = -1$ $\gamma = 2$	2	0	2	2	2	-2	0,1,2	2
if $\left(\frac{a}{a_D e e_D}\right) < 1$			0	-2	0	2	0,1,2	-2
$\alpha = -1$ $\gamma = 2$	3	0	2	1	2	-1	0,1,2,3	1
			2	1	3	1	0,1,2,3	1
if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	0	-1	0,1,2,3	-1
			1	-1	1	1	0,1,2,3	-1
$\alpha = -1$ $\gamma \geqslant 3$	2	0	2	2	2	$2-2\gamma$	0,1,2	2
if $\left(\frac{a}{a_D e e_D{}^{(1+\gamma)}}\right) < 1$			0	-2	0	$2\gamma-2$	0,1,2	-2
$\alpha = -1$ $\gamma \geqslant 3$	3	0	2	1	3	$3-\gamma$	0,1,2,3	1
if $\left(\frac{a}{a_D e e_D{}^{(1+\gamma)}}\right) > 1$			1	-1	0	$\gamma-3$	0,1,2,3	-1



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$\dot{\omega}+\dot{\omega}_D+\dot{M}_D=0$

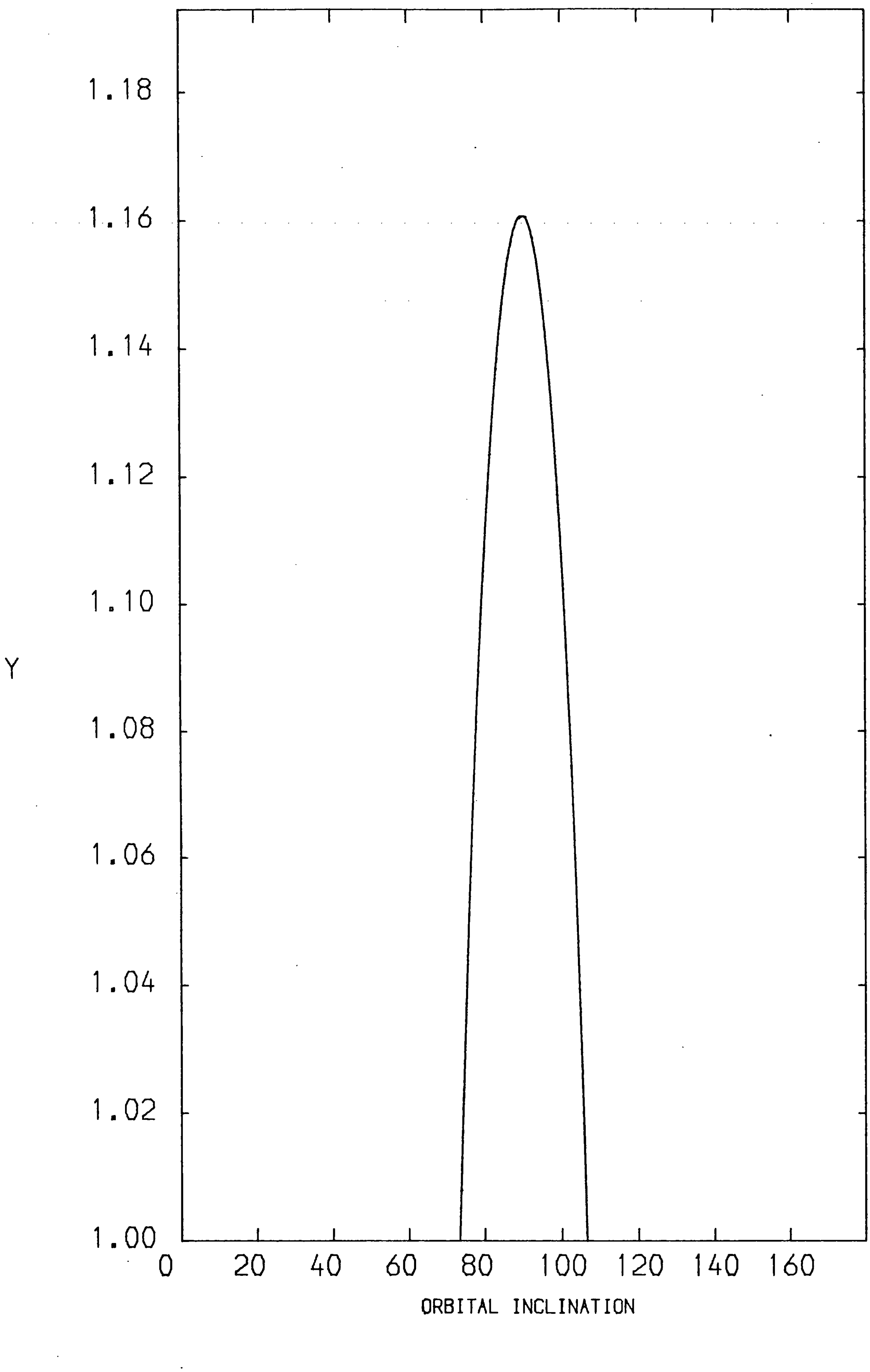
Figure 2.1



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D=0$

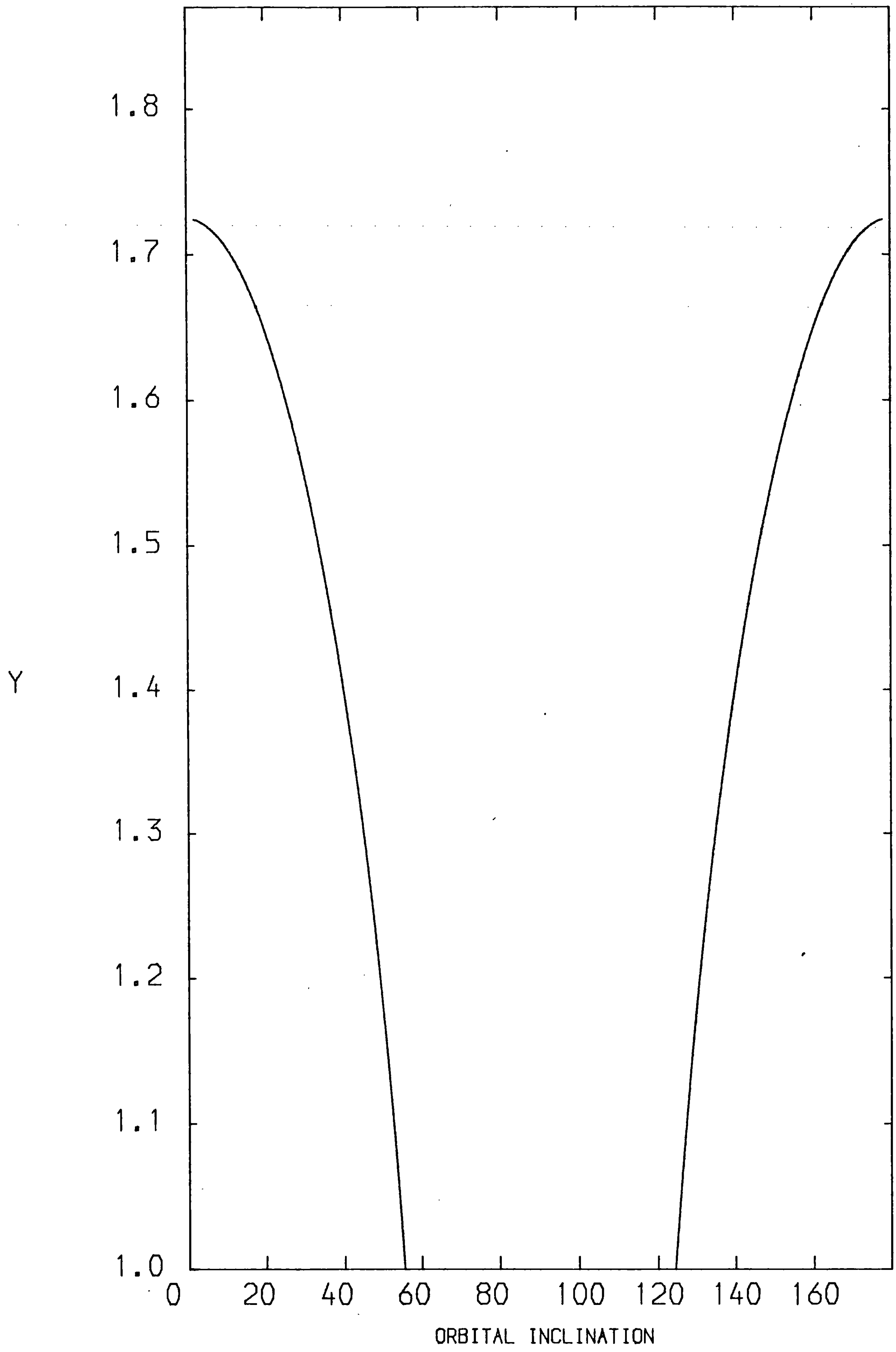
Figure 2.2



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$\dot{\omega} + 3\dot{\omega}_{D} + 3\dot{M}_{D} = 0$

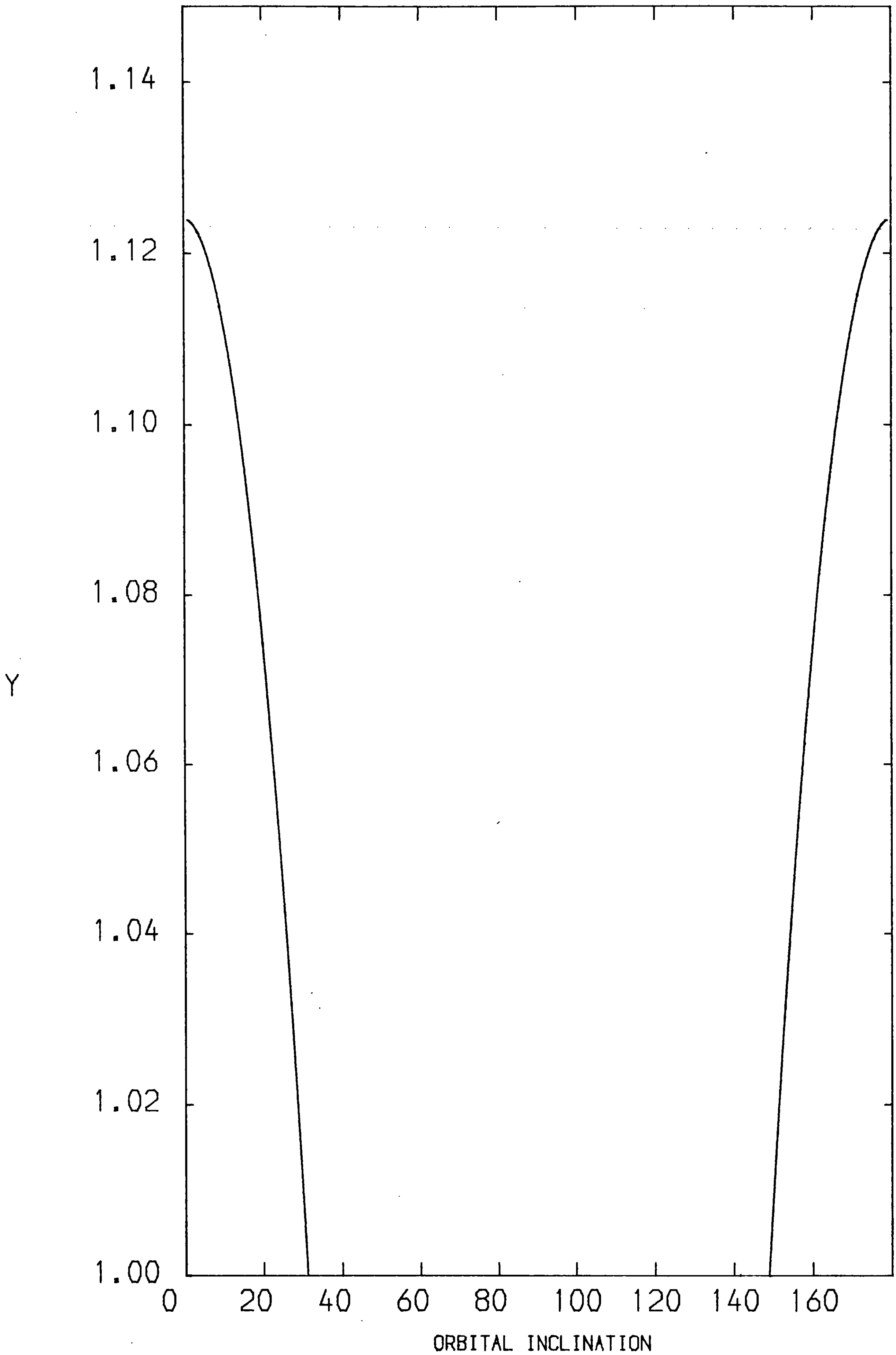
Figure 2.3



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-\dot{\omega} + 3\dot{\omega}_D + 3\dot{M}_D = 0$

Figure 2.4



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D=0$

Figure 2.5

2.4(5) The Type (5) Commensurability $\dot{\psi}_5 = \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \approx 0$

A satellite in a lunisolar commensurability of the type $\dot{\psi}_5 \approx 0$ is in resonance with $\Phi^{(+)}$ terms in the disturbing function expansions (2.4) for which

$$(n^+ - 2p^+) = 0$$

$$(n - 2p + q)^+ = 0$$

$$(n - 2h + j)^+ = \gamma\delta \qquad (2.47)$$

$j^+ =$ 0 - lunar gravity perturbations

$\gamma\delta - n^+ + 2h^+$ - Solar gravity or solar radiation pressure perturbations

$s^+ =$ 0 - lunar gravity perturbations

$0,1\ldots n^+$ - Solar gravity or solar radiation pressure perturbations

$m^+ = \beta\delta$

The arguments of the resonant terms are of the form $\delta(\psi_5^+)_{MOON}$ for the Moon, where $(\psi_4^+)_{MOON} = \gamma(\omega_D + M_D) + \beta\Omega$, and of the form $\delta(\psi_5^+)_{SUN}$ for the Sun, with $(\psi_5^+)_{SUN} = \eta\omega_D + \gamma M_D + \beta\Omega + k\Omega_D$. The n,m,p,q,h,j,s and δ values for the predominant $\Phi^{(+)}$ resonant terms of a lunar gravity commensurability of type (5) are given in table 2.6(a). The corresponding values for a solar gravity, or solar radiation pressure, commensurability $\dot{\psi}_5 \approx 0$ are given in table 2.6(b).

Similarly, a satellite in a commensurability of the type $\dot{\psi}_5 \approx 0$ will be in resonance with those $\Phi^{(-)}$ terms for which

$$(n - 2p)^- = 0$$

$$q^- = 0$$

$$(n - 2h + j)^- = -\gamma v \qquad (2.48)$$

$$j^- = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -\gamma v - n^- + 2h^- & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$s^- = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1\ldots n^- & \text{- Solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$m^- = \beta v$$

The arguments of the $\Phi^{(-)}$ resonance terms are of the form $v(\psi_5^-)_{MOON}$ for the Moon, where $(\psi_4^-)_{MOON} = \gamma(\omega_D + M_D) + \beta\Omega$, and of the form $v(\psi_5^-)_{SUN}$ for the Sun, with $(\psi_5^-)_{SUN} = \gamma M_D + \eta\omega_D + \beta\Omega + k\Omega_D$. The n,m,p,q,h,j,s and v values for the predominant $\Phi^{(-)}$ resonant terms of a lunar gravity commensurability of type (5) are given in table 2.6(c). The corresponding values for a solar gravity, or solar radiation pressure, commensurability $\dot{\psi}_5 \approx 0$ are given in table 2.6(d).

For a close satellite to exist in a lunisolar commensurability of type (5) its orbital elements must satisfy

$$\gamma(\dot{\omega}_D + \dot{M}_D)y^{3.5} - 9.97\ \beta\cos i \approx 0 \qquad (2.49)$$

The maximum value of y_{max} for a given γ and β is such that

$$y_{max} = \left(\frac{9.97\beta}{|\alpha|\ n_D}\right)^{2/7} \qquad (2.50)$$

Since y is always greater than unity, the lunisolar commensurability $\dot{\psi}_5 = \gamma/(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \approx 0$ will exist if

$$9.97\beta > |\gamma|\ n_D \qquad (2.51)$$

On substituting the appropriate values of n_D for the Sun and Moon into equation (2.51), it is found that a close satellite will exist in the solar commensurability $\dot{\psi}_5 \approx 0$ if

$$10.12\beta > |\alpha| \tag{2.52}$$

and in the lunar gravity commensurability $\dot{\psi}_5 \approx 0$ if

$$0.75\beta > |\gamma| \tag{2.53}$$

In the case of a type (5) commensurability, the theoretically possible commensurabilities for which the predominant resonant terms in (2.4) have n values of 2, q values of 0 and j values of zero are

$$\left.\begin{array}{ll} (9) & (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\ (10) & 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\ (11) & -(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\ (12) & -2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \end{array}\right\} \tag{2.54}$$

From equations (2.52) and (2.53), it is easily seen that <u>no</u> commensurabilities of the set (2.54) are possible for lunar gravity perturbations, whilst for solar perturbations all four of the set (2.54) are possible. The graphs of the orbits of satellites in these commensurabilities are given in figures (2.6) to (2.9).

TABLE 2.6(a)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability of Type $\dot{\psi}_5 \approx 0$

RESTRICTIONS ON β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
β +ve γ+ve $\gamma > \beta$ γ even	γ	β	$\gamma/2$	0	0	0	0	1
β +ve γ+ve $\gamma \geqslant \beta$ γ odd	2γ	2β	γ	0	0	0	0	2
β +ve γ+ve $\beta > \gamma$ β odd γ even	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta+1-\gamma)/2$	0	0	1
β +ve γ+ve $\beta>\gamma$ β odd γ odd or β even γ odd	2β	2β	β	0	$\beta-\gamma$	0	0	2
β +ve γ-ve $\|\gamma\| > \beta$ $\|\gamma\|$ even	$\|\gamma\|$	β	$\|\gamma\|/2$	0	$\|\gamma\|$	0	0	1
β +ve γ-ve $\|\gamma\| \geqslant \beta$ $\|\gamma\|$ odd	$2\|\gamma\|$	2β	$\|\gamma\|$	0	$2\|\gamma\|$	0	0	2
β +ve γ-ve $\beta > \|\gamma\|$ β odd $\|\gamma\|$ even	$\beta+1$	β	$(\beta+1)/2$	0	$\frac{(\beta+1+\|\gamma\|}{2}$	0	0	1
β+ve γ-ve $\beta>\|\gamma\|$: β odd $\|\gamma\|$ odd or β even $\|\gamma\|$ odd	2β	2β	β	0	$\beta+\|\gamma\|$	0	0	2

TABLE 2.6(b)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Solar Gravity or Solar Radiation Pressure Commensurability of the Type $\dot{\psi}_5 \approx 0$

RESTRICTIONS ON β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
β +ve γ+ve $\gamma > \beta$ γ odd β even	β	β	$\beta/2$	0	0	$\gamma-\beta$	$0,1..\beta$	1
β +ve γ +ve $\gamma \geqslant \beta+1$: β odd γ even or β odd γ odd	$\beta+1$	β	$(\beta+1)/2$	0	0	$\gamma-\beta-1$	$0,1..\beta+1$	1
β +ve γ +ve $\gamma = \beta = 1$	2	2	1	0	0	0	0,1,2	2
β +ve γ +ve $\beta > \gamma$ β even γ odd	β	β	$\beta/2$	0	$(\beta-\gamma+1)/2$ $(\beta-\gamma-1)/2$	1 -1	$0,1..\beta$ $0,+1..\beta$	1
β +ve γ +ve $\beta > \gamma$ β odd γ even	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta-\gamma+1)/2$	0	$0,1..\beta+1$	1
β +ve γ+ve $\beta > \gamma$ β odd γ odd	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta-\gamma+2)/2$ $(\beta-\gamma)/2$	1 -1	$0,1..\beta+1$ $0,1..\beta+1$	1

/cont..

TABLE 2.6(b) Cont.

RESTRICTIONS ON β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
β +ve γ-ve $\lvert\gamma\rvert > \beta$ $\lvert\gamma\rvert$ odd β even	β	β	$\beta/2$	0	β	$\beta-\lvert\gamma\rvert$	$0,1..\beta$	1
β +ve γ-ve $\lvert\gamma\rvert \geqslant \beta+1$ β odd $\lvert\gamma\rvert$ odd, β odd $\lvert\gamma\rvert$ even	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\beta+1$	$\beta+1-\lvert\gamma\rvert$	$0..1..\beta+1$	1
β +ve γ-ve $\beta = 1$ $\gamma = -1$	2	2	1	0	2	0	0,1,2	2
β +ve γ-ve $\beta > \lvert\gamma\rvert$ β even $\lvert\gamma\rvert$ odd	β	β	$\beta/2$	0	$\frac{(\beta+\lvert\gamma\rvert+1)}{2}$ $\frac{(\beta+\lvert\gamma\rvert-1)}{2}$	1 -1	$0,1..\beta$	1
β +ve γ-ve $\beta > \lvert\gamma\rvert$ β odd $\lvert\gamma\rvert$ even	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta+1+\lvert\gamma\rvert)}{2}$	0	$0..1.\beta+1$	1
β +ve γ-ve $\beta > \lvert\gamma\rvert$ β odd $\lvert\gamma\rvert$ odd	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta+\lvert\gamma\rvert+2)}{2}$ $\frac{(\beta+\lvert\gamma\rvert)}{2}$	1 -1	$0..1..\beta+1$	1

TABLE 2.6(c)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability of the Type

$$\dot{\psi}_5 \approx 0$$

RESTRICTIONS ON β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
β +ve γ+ve $\gamma > \beta$ γ even	γ	β	$\gamma/2$	0	γ	0	0	1
β +ve γ +ve $\gamma > \beta$ γ odd	2γ	2β	γ	0	2γ	0	0	2
β +ve γ +ve $\beta > \gamma$ β odd γ even	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta+1+\gamma)/2$	0	0	1
β+ve γ+ve $\beta>\gamma$ γ odd β odd or γ odd β even	2β	β	β	0	$\beta+\gamma$	0	0	2
β +ve γ -ve $\lvert\gamma\rvert>\beta$ $\lvert\gamma\rvert$ even	$\lvert\gamma\rvert$	β	$\lvert\gamma\rvert/2$	0	0	0	0	1
β +ve γ -ve $\lvert\gamma\rvert > \beta$ $\lvert\gamma\rvert$ odd	$2\lvert\gamma\rvert$	2β	$\lvert\gamma\rvert$	0	0	0	0	2
β +ve γ-ve $\beta > \lvert\gamma\rvert$ $\lvert\gamma\rvert$ even β odd	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta+1-\lvert\gamma\rvert)/2$	0	0	1
β+ve γ-ve $\beta>\lvert\gamma\rvert$ $\lvert\gamma\rvert$ odd β odd or $\lvert\gamma\rvert$ odd β even	2β	2β	β	0	$\beta-\lvert\gamma\rvert$	0	0	2

TABLE 2.6(d)

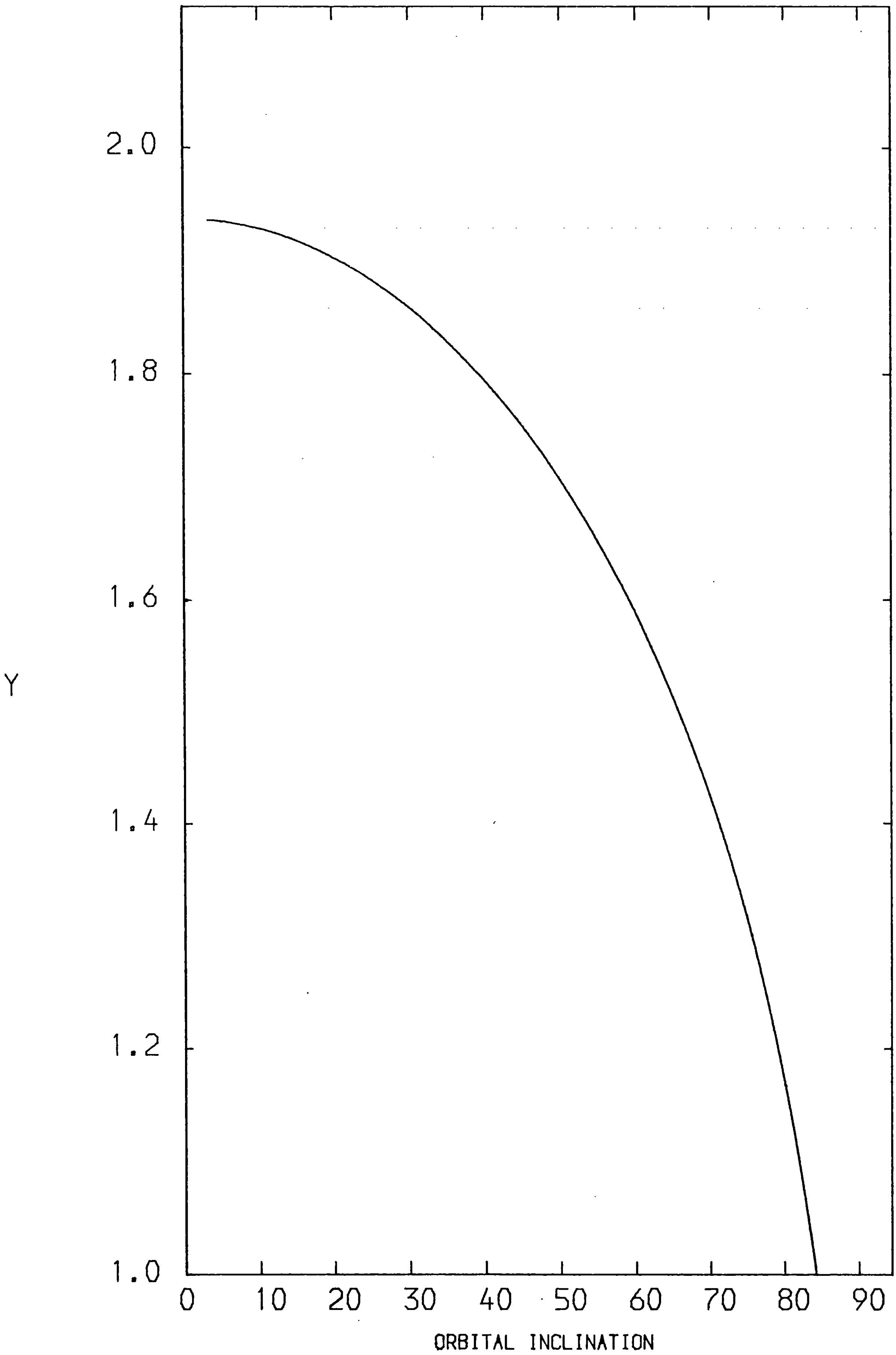
The $\bar{n},\bar{m},\bar{p},\bar{q},\bar{h},\bar{j},\bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Solar Gravity or Solar Radiation Pressure Commensurability of the Type $\dot{\psi}_5 \approx 0$

RESTRICTIONS ON β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
β +ve γ+ve $\gamma > \beta$ γ odd β even	β	β	$\beta/2$	0	β	$\beta-\gamma$	$0,1..\beta$	1
β +ve γ+ve $\gamma \geqslant \beta+1$ β odd γ even or β odd γ odd	$\beta+1$	β	$\frac{(1+\beta)}{2}$	0	$\beta+1$	$\beta+1-\gamma$	$0,1..\beta+1$	1
β +ve γ +ve $\beta = \gamma = 1$	2	2	1	0	2	0	0,1,2	2
β +ve γ +ve $\beta > \gamma$ β even γ odd	β	β	$\beta/2$	0	$\frac{(\beta+\gamma+1)}{2}$	1	$0,1..\beta$	1
					$\frac{(\beta+\gamma-1)}{2}$	-1	$0,1..\beta$	1
β +ve γ+ve $\beta > \gamma$ β odd γ even	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta+\gamma+1)}{2}$	0	$0..1..\beta+1$	1
β +ve γ+ve $\beta > \gamma$ β odd γ odd	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta+\gamma+2)}{2}$	1	$0..1..\beta+1$	1
					$(\beta+\gamma)/2$	-1	$0,1..\beta+1$	1

/cont..

TABLE 2.6(d) Cont.

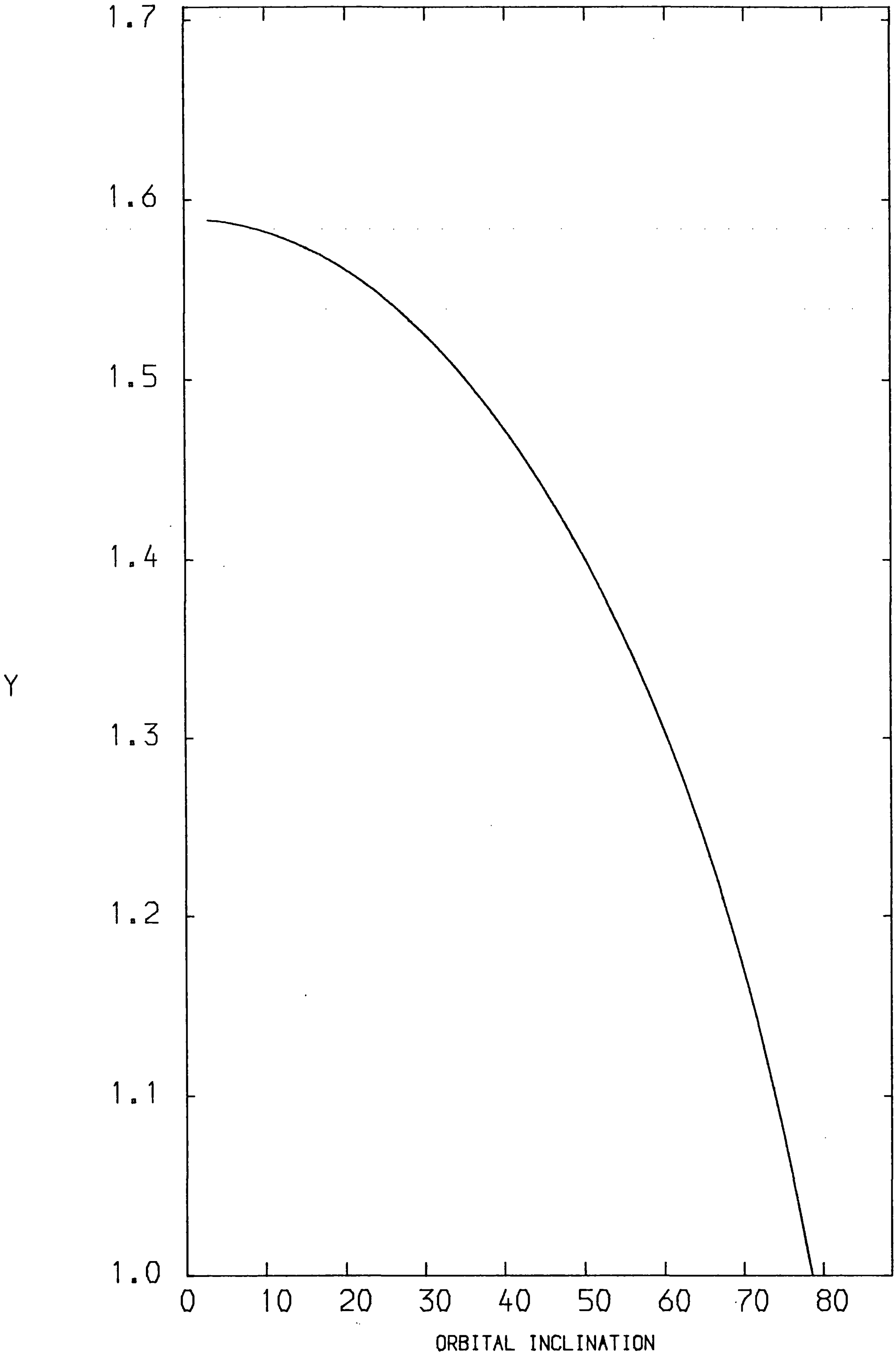
RESTRICTIONS ON β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
β +ve γ -ve $\lvert\gamma\rvert > \beta$ $\lvert\gamma\rvert$ odd β even	β	β	$\beta/2$	0	0	$\lvert\gamma\rvert-\beta$	$0,1..\beta$	1
β +ve γ -ve $\lvert\gamma\rvert \geqslant \beta+1$ β odd $\lvert\gamma\rvert$ odd, β odd γ even	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	0	$\lvert\gamma\rvert-\beta-1$	$0,1..\beta+1$	1
β +ve γ -ve $\lvert\gamma\rvert = \beta = 1$	2	2	1	0	0	0	0,1,2	2
β +ve γ -ve $\beta > \lvert\gamma\rvert$ β even $\lvert\gamma\rvert$ odd	β	β	$\beta/2$	0	$\frac{(\beta-\lvert\gamma\rvert+1)}{2}$	1	$0,1..\beta$	1
					$\frac{(\beta-\lvert\gamma\rvert-1)}{2}$	-1	$0,1..\beta$	1
β +ve γ -ve $\beta > \lvert\gamma\rvert$ β odd $\lvert\gamma\rvert$ even	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta+1-\lvert\gamma\rvert)}{2}$	0	$0,1..\beta+1$	1
β +ve γ-ve $\beta > \lvert\gamma\rvert$ β odd $\lvert\gamma\rvert$ odd	$\beta+1$	β	$\frac{(\beta+1)}{2}$	0	$\frac{(\beta-\lvert\gamma\rvert+2)}{2}$	1	$0,1..\beta+1$	1
					$(\beta-\lvert\gamma\rvert)/2$	-1	$0,1..\beta+1$	1



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$\dot{\omega}_D + \dot{M}_D + \dot{\Omega} = 0$

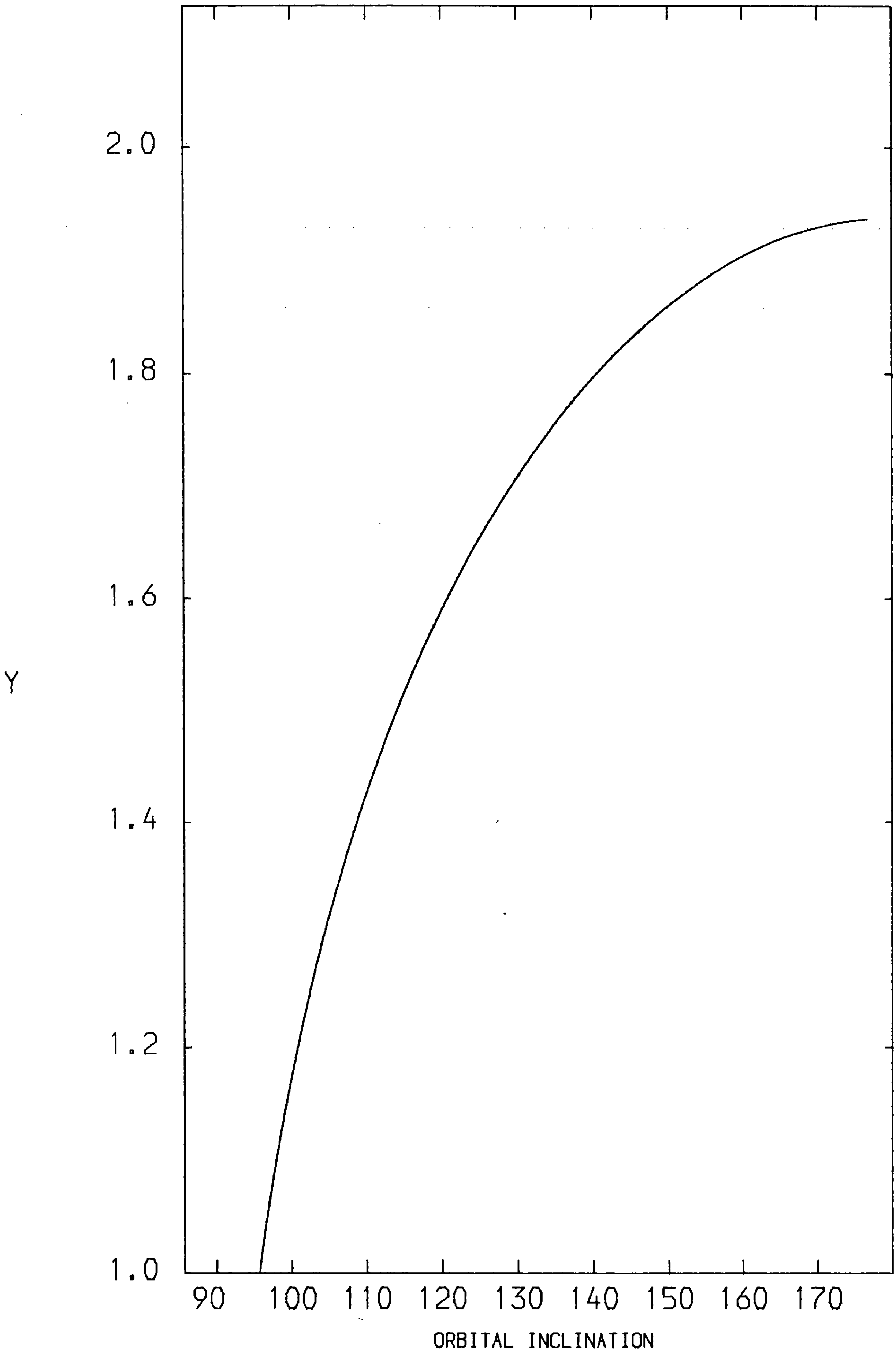
Figure 2.6



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$2\dot{\omega}_D + 2\dot{M}_D + \dot{\Omega} = 0$

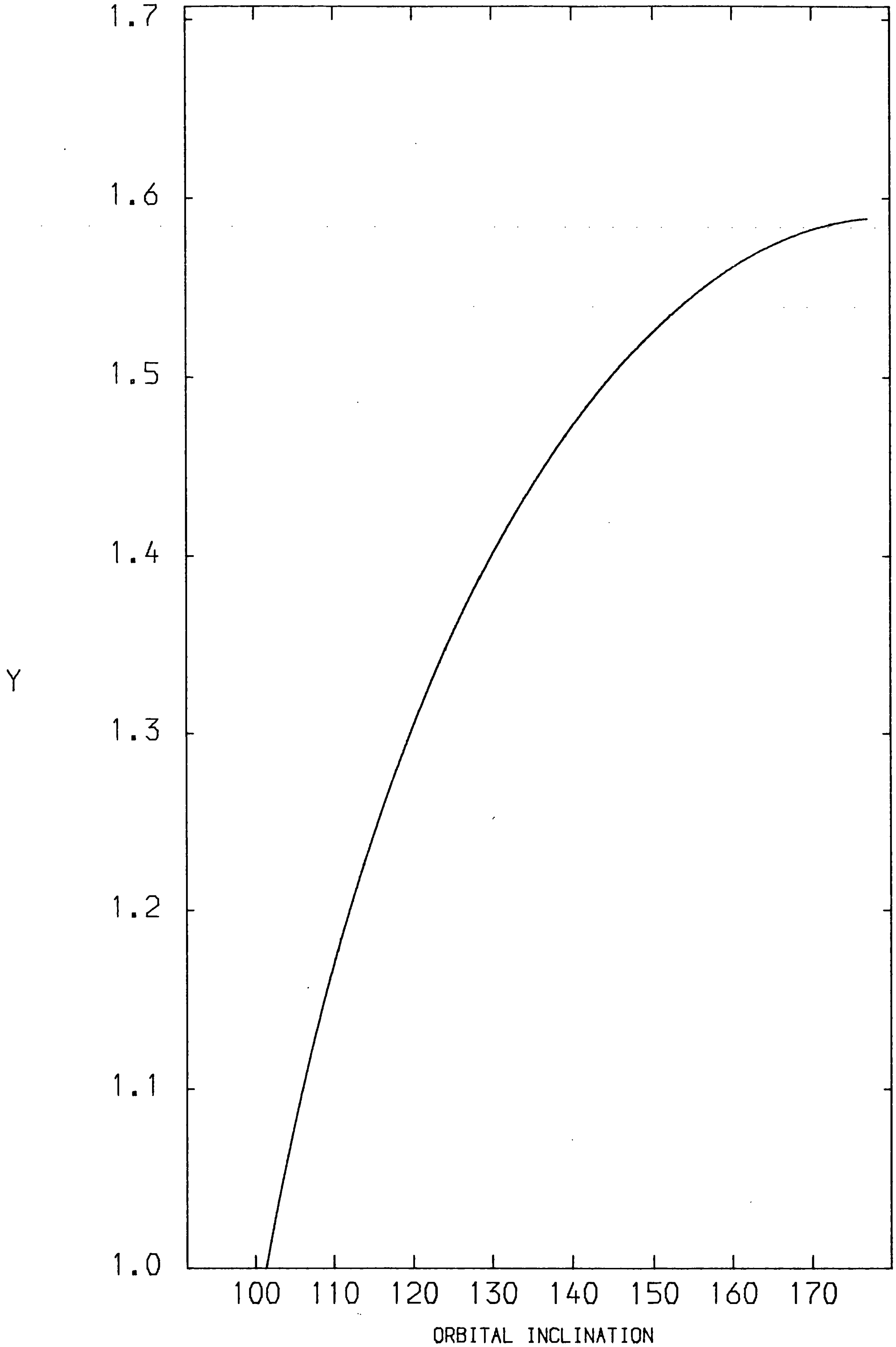
Figure 2.7



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-\dot{\omega}_D - \dot{M}_D + \dot{\Omega} = 0$

Figure 2.8



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-2\dot{\omega}_D - 2\dot{M}_D + \dot{\Omega} = 0$

Figure 2.9

2.4(6) The Type (6) Commensurability $\dot{\psi}_6 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \approx 0$

The sixth lunisolar commensurability condition is a combination of the five previous types: type (1) can be obtained if α and γ are zero; type (2) results if β and γ are put to zero; and so on. However, if α, β and γ are not zero, then the commensurability $\dot{\psi}_6 \approx 0$ has a set of properties different from those of type (1) to (5).

A satellite in a commensurability $\dot{\psi}_6 \approx 0$ is in resonance with those $\Phi^{(+)}$ terms in the lunisolar disturbing function expansions (2.4) for which

$$
\begin{aligned}
(n - 2p)^+ &= \alpha\delta \\
q^+ &= -\alpha\delta \\
(n - 2h + j)^+ &= \gamma\delta \\
j^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^+ + 2h^+ + \gamma\delta & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
s^+ &= \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1 \ldots n^+ & \text{- solar gravity or solar radiation pressure perturbations} \end{cases} \\
m^+ &= \beta\delta \\
\delta &> 0
\end{aligned}
\qquad (2.54)
$$

The arguments of the resonant terms are of the form $\delta(\psi_6^+)_{MOON}$ for lunar gravity perturbations, where $(\psi_6^+)_{MOON} = \alpha\omega + \gamma(\omega_D + M_D) + \beta\Omega$; and of the form $\delta(\psi_6^+)_{SUN}$ for solar perturbations, with $(\psi_6^+)_{SUN} = \alpha\omega + \gamma M_D + \eta\omega_D + \beta\Omega + k\Omega_D$. The $n^+, m^+, p^+, q^+, h^+, j^+$, s^+ and δ values for the predominant $\Phi^{(+)}$ resonant terms are given in tables 2.7(a) - 2.7(f). Similarly, a satellite in a lunisolar commensurability $\dot{\psi}_6 \approx 0$ is in resonance with those $\Phi^{(-)}$ terms for which

$$(n - 2p)^- = \alpha v$$

$$q^- = -\alpha v$$

$$(n - 2h + j)^- = -\gamma v$$

(2.55)

$$j^- = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ -n^- + 2h^- - \gamma v & \text{- solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$s^- = \begin{cases} 0 & \text{- lunar gravity perturbations} \\ 0,1\ldots n^- & \text{- solar gravity or solar radiation pressure perturbations} \end{cases}$$

$$m^- = \beta v$$

$$v > 0$$

The arguments of the $\Phi^{(-)}$ resonant terms are of the form $v(\psi_6^-)_{MOON}$ and $v(\psi_6^-)_{SUN}$, where $(\psi_6^-)_{MOON}$ and $(\psi_6^-)_{SUN}$ have the same form as in the $\Phi^{(+)}$ resonant case. The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values for the predominant $\Phi^{(-)}$ resonant terms are given in tables 2.7(g) - 2.7(l).

For a close satellite to exist in a given type (6) commensurability, the semi-major axis, a, the eccentricity, e, and the inclination, i, of its orbit must satisfy

$$24.9\ \alpha \cos^2 i - 9.97\ \beta \cos i - 4.98\ \alpha + \gamma\, n_D\, y^{3.5} \approx 0 \tag{2.56}$$

On solving for y, equation (2.56) can be written as

$$y = \left[(4.98\,\alpha - 24.9\,\alpha \cos^2 i + 9.97\beta \cos i)/\gamma\, n_D \right]^{2/7} \tag{2.57}$$

In order to obtain the criteria which determine whether resonance orbits exist for the commensurability

$$\dot{\psi}_6 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \approx 0$$

it is necessary to consider the four cases:

(a) α +ve γ +ve

(b) α +ve γ −ve

(c) α −ve γ +ve

(d) α −ve γ −ve

Case (a)

Define the function Z(i) by

$$Z(i) = (4.98\alpha - 24.9\alpha \cos^2 i + 9.97\beta \cos i)/\gamma n_D \tag{2.58}$$

The stationary values of Z(i) occur at $i = 0^\circ$, 180° and $\cos^{-1}\beta/4.99\alpha$. If $\beta > 4.99\alpha$, then $\cos^{-1}\beta/4.99\alpha$ is imaginary, and only $i = 0^\circ$ and 180° need be considered. If $\beta = 4.99\alpha$, then $\cos^{-1}\beta/4.99\alpha$ is 0°. In this case, the stationary values are also 0° and 180°. The second derivative of Z(i) with respect to i is found from equation (2.58) to be

$$\frac{d^2Z(i)}{di^2} = (49.8\alpha \cos 2i - 9.97\beta \cos i)/\gamma n_D \tag{2.59}$$

Since d^2Z/di^2 is positive when $i = 0^\circ$ and 180° for $4.99\alpha \geqslant \beta$, $Z(0^\circ)$ and $Z(180^\circ)$ are minima. When $i = \cos^{-1}(\beta/4.99\alpha)$, d^2Z/di^2 is negative for $4.99\alpha > \beta$, hence $Z(\cos^{-1}\beta/4.99\alpha)$ is a maxima such that

$$Z(\cos^{-1}\beta/4.99\alpha) = (4.98\alpha^2 + \beta^2)/\alpha\gamma n_D \tag{2.60}$$

For the commensurability $\dot{\psi}_6 = \alpha\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} \approx 0$ to exist,

y_{max} and, hence, Z_{max} must be greater than unity. The commensurability $\dot{\psi}_6 \approx 0$ therefore occurs for α +ve, γ +ve and $4.99\alpha \geqslant \beta$ if

$$\underline{4.98\,\alpha^2 + \beta^2 > \alpha\ \gamma n_D} \qquad \alpha\text{+ve}, \quad \gamma\text{+ve}, \ 4.99\,\alpha > \beta \tag{2.61}$$

When $\beta > 4.99\,\alpha$, d^2Z/di^2 is positive for $i = 180^o$, and negative for $i = 0^o$. Hence, if $\beta > 4.99\,\alpha$, $Z(0^o)$ is the only maxima, its value being given by

$$Z(0^o) = (9.97\beta - 19.92\,\alpha)/\gamma\, n_D \tag{2.62}$$

Therefore when $\beta > 4.99\,\alpha$, the commensurability $\dot{\psi}_6 \approx 0$ with α and γ positive can occur if

$$\underline{9.97\,\beta > 19.92\,\alpha + \gamma n_D} \qquad \alpha\text{+ve}, \quad \gamma\text{+ve}, \beta > 4.99\,\alpha \tag{2.63}$$

Proceeding as in case (a), it can be shown that, for cases (b), (c) and (d), the lunisolar commensurability $\dot{\psi}_6 \approx 0$ can occur if

<u>Case (b)</u> <u>α+ve, γ-ve</u>

$$\left.\begin{array}{ll} 19.92\,\alpha > |\gamma|\ n_D \pm 9.97\,\beta & \text{for } 4.99\,\alpha \geqslant \beta \\ 19.92\,\alpha > |\gamma|\ n_D - 9.97\,\beta & \text{for } \beta > 4.99\,\alpha \end{array}\right\} \tag{2.64}$$

<u>Case (c)</u> <u>α-ve, γ+ve</u>

$$\left.\begin{array}{ll} 19.92\,|\alpha| > \gamma n_D \pm 9.97\,\beta & \text{for } 4.99\,\alpha \geqslant \beta \\ 19.92\,|\alpha| > \gamma n_D - 9.97\,\beta & \text{for } \beta > 4.99\,\alpha \end{array}\right\} \tag{2.65}$$

Case (d) α -ve, γ-ve

$$4.98\ |\alpha|^2 + \beta^2 > |\alpha|\ |\gamma|\, n_D \quad \text{for } 4.99\,|\alpha| \geqslant \beta$$

$$9.97\beta > 19.92\,|\alpha| + |\gamma|\, n_D \quad \text{for } \beta > 4.99\,|\alpha| \tag{2.66}$$

For solar radiation pressure perturbations, four commensurabilities of type (6) are theoretically possible. For n=1 and j=0 resonant terms, they are:

$$\begin{aligned}
&(13) \quad \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(14) \quad \dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(15) \quad -\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(16) \quad -\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0
\end{aligned} \tag{2.67}$$

Similarly, for lunisolar gravity perturbations, eight commensurabilities of type (6) are theoretically possible which have n = 2 and j = 0 resonant terms: they are (13) - (16) of the set (2.67) plus the following

$$\begin{aligned}
&(17) \quad 2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(18) \quad 2\dot{\omega} - 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(19) \quad -2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
&(20) \quad -2\dot{\omega} - 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0
\end{aligned} \tag{2.68}$$

Consideration of equations (2.61) to (2.66) shows that close satellite orbits exist whose orbital elements satisfy the solar commensurability conditions (13) - (20) and the lunar commensurability conditions (14), (15), (18) and (19). The graphs of the function (2.58) for the solar commensurabilities (13) - (20) are given in figures (2.10) - (2.17), whilst the corresponding graphs for the lunar commensurabilities (14), (15), (18) and (19) are given in figures (2.18) - (2.21). The lunisolar gravity commensurabilities (13) - (20) are the most important for satellite orbits for which $\left(\frac{a}{a_D e}\right) < 1$. However, if $\left(\frac{a}{a_D e}\right) > 1$, then the most important lunisolar commensurabilities of type (6) are the set (2.67), and

$$
\begin{array}{llll}
(21) & \dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 & (31) & -\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
(22) & \dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 & (32) & -\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 \\
(23) & \dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 & (33) & -\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 \\
(24) & \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 & (34) & -\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 \\
(25) & \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 & (35) & -\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 \\
(26) & \dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 & (36) & -\dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \approx 0 \\
(27) & \dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 & (37) & -\dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 \\
(28) & \dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 & (38) & -\dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 \\
(29) & \dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 & (39) & -\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \approx 0 \\
(30) & \dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0 & (40) & -\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} \approx 0
\end{array}
\tag{2.69}
$$

The amplitude factors of the predominant resonant terms for these commensurabilities are of the order $\left(\frac{a}{a_D}\right)^3 e$. From equations (2.61) to (2.66), it is found that satellite orbits occur whose orbital elements satisfy the solar commensurability conditions (21) to (40) and the lunar commensurability conditions (25), (27), (28), (29), (30), (32), (33), (34), (35) and (40). Of the two types of commensurabilities (i.e. solar and lunar), the lunar gravity commensurabilities (2.68) and

(2.69) are likely to be the most important, since for the majority of Earth satellites the lunar value of $GM_D(a/a_D)^3 e/a_D$ is larger than the corresponding solar value. The lunar value of a/a_D is approximately 1/50 for a close satellite; therefore, when $e < 1/50$, the most important lunar gravity commensurabilities of type (6) are the ones with predominant resonant terms of order $(a/a_D)^3 e$. If $e > 1/50$, then the most important lunar commensurabilities for such a satellite are those which have predominant resonant terms of order $(a/a_D)^2 e^2$. The graphs of the function (2.58) for the lunar commensurabilities (25), (27), (28), (29), (30), (32), (33), (34), (35) and (40) are given in figures (2.22) to (2.31).

TABLE 2.7(a)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_6 \approx 0$ when $|\alpha|$ and/or $|\gamma| \geqslant \beta$

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α+ve γ+ve, $\alpha \geqslant \gamma$, $\alpha - \gamma$ even	α *	β	0	$-\alpha$	$(\alpha-\gamma)/2$	0	0	1
α+ve γ+ve, $\alpha > \gamma$, $\alpha - \gamma$ odd	2α	2β	0	-2α	$\alpha-\gamma$	0	0	2
α+ve γ+ve, $\gamma > \alpha$, $\gamma - \alpha$ even	γ	β	$(\gamma-\alpha)/2$	$-\alpha$	0	0	0	1
α+ve γ+ve, $\alpha > \gamma$, $\gamma - \alpha$ odd	2γ	β	$\gamma-\alpha$	-2α	0	0	0	2
α+ve γ-ve, $\alpha \geqslant \|\gamma\|$, $\alpha + \|\gamma\|$ even	α *	β	0	$-\alpha$	$\frac{(\alpha+\|\gamma\|)}{2}$	0	0	1
α+ve γ-ve, $\alpha > \|\gamma\|$, $\alpha + \|\gamma\|$ odd	2α	2β	0	-2α	$\alpha+\|\gamma\|$	0	0	2
α+ve γ-ve, $\|\gamma\| > \alpha$, $\|\gamma\| - \alpha$ even	$\|\gamma\|$	β	$\frac{(\|\gamma\|-\alpha)}{2}$	$-\alpha$	$\|\gamma\|$	0	0	1
α+ve γ-ve, $\|\gamma\| > \alpha$, $\|\gamma\| - \alpha$ odd	$2\|\gamma\|$	2β	$\|\gamma\|-\alpha$	-2α	$2\|\gamma\|$	0	0	2

* If $\alpha = 1$, $\gamma = \pm 1$. See table 2.7(c) for the predominant $\Phi^{(+)}$ resonant terms.

TABLE 2.7(a) cont.

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α-ve γ+ve $\lvert\alpha\rvert \geqslant \gamma$ $\lvert\alpha\rvert - \gamma$ even	$\lvert\alpha\rvert$ *	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert-\gamma)}{2}$	0	0	1
α-ve γ+ve $\lvert\alpha\rvert > \gamma$ $\lvert\alpha\rvert - \gamma$ odd	$2\lvert\alpha\rvert$	2β	$2\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\lvert\alpha\rvert-\gamma$	0	0	2
α-ve γ+ve $\gamma > \lvert\alpha\rvert$ $\lvert\alpha\rvert + \gamma$ even	γ	β	$\frac{(\lvert\alpha\rvert+\gamma)}{2}$	$\lvert\alpha\rvert$	0	0	0	1
α-ve γ+ve $\gamma > \lvert\alpha\rvert$ $\lvert\alpha\rvert + \gamma$ odd	2γ	2β	$\lvert\alpha\rvert+\gamma$	$2\lvert\alpha\rvert$	0	0	0	2
α-ve γ-ve $\lvert\alpha\rvert \geqslant \lvert\gamma\rvert$ $\lvert\alpha\rvert + \lvert\gamma\rvert$ even	$\lvert\alpha\rvert$ *	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert+\gamma)}{2}$	0	0	1
α-ve γ-ve $\lvert\alpha\rvert > \lvert\gamma\rvert$ $\lvert\alpha\rvert + \lvert\gamma\rvert$ odd	$2\lvert\alpha\rvert$	2β	$2\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\lvert\alpha\rvert+\lvert\gamma\rvert$	0	0	2
α-ve γ-ve $\lvert\gamma\rvert > \lvert\alpha\rvert$ $\lvert\alpha\rvert + \lvert\gamma\rvert$ even	$\lvert\gamma\rvert$	β	$(\lvert\gamma\rvert + \lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$\lvert\gamma\rvert$	0	0	1
α-ve γ-ve $\lvert\gamma\rvert > \lvert\alpha\rvert$ $\lvert\alpha\rvert + \lvert\gamma\rvert$ odd	$2\lvert\gamma\rvert$	2β	$\lvert\gamma\rvert+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$2\lvert\gamma\rvert$	0	0	2

* if $\alpha = -1$, $\gamma = \pm 1$ then see table 2.7(c) for the predominant $\Phi^{(+)}$ resonant terms.

Table 2.7(b)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Solar Commensurability of the Type $\dot{\psi}_6 \approx 0$ when $|\alpha| \geqslant \beta$

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α+ve γ+ve $\alpha \geqslant \gamma$ $\alpha - \gamma$ even	α^*	β	0	$-\alpha$	$(\alpha-\gamma)/2$	0	$0,1..\alpha$	1
α+ve γ+ve $\alpha > \gamma$	α^*	β	0	$-\alpha$	$(\alpha-\gamma+1)/2$	1	$0,1..\alpha$	1
$\alpha - \gamma$ odd	α^*	β	0	$-\alpha$	$(\alpha-\gamma-1)/2$	-1	$0,1..\alpha$	1
α+ve γ+ve $\gamma > \alpha$	α^*	β	0	$-\alpha$	0	$\gamma-\alpha$	$0,1..\alpha$	1
α+ve γ-ve $\alpha \geqslant \lvert\gamma\rvert$ $\alpha + \gamma$ even	α^*	β	0	$-\alpha$	$(\alpha+\lvert\gamma\rvert)/2$	0	$0,1..\alpha$	1
α+ve γ-ve $\alpha > \lvert\gamma\rvert$	α^*	β	0	$-\alpha$	$\frac{(\alpha+\lvert\gamma\rvert+1)}{2}$	1	$0,1..\alpha$	1
$\alpha + \gamma$ odd	α^*	β	0	$-\alpha$	$\frac{(\alpha+\lvert\gamma\rvert-1)}{2}$	-1	$0,1..\alpha$	1
α+ve γ-ve $\lvert\gamma\rvert > \alpha$	α^*	β	0	$-\alpha$	2α	$\alpha-\lvert\gamma\rvert$	$0,1..\alpha$	1

* if $\alpha = 1$ for a solar gravity commensurability then see table 2.7(d) for the predominant $\Phi^{(+)}$ resonant terms.

Table 2.7(b) continued

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α -ve γ +ve $\|\alpha\| \geqslant \gamma$ $\|\alpha\| - \gamma$ even	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|-\gamma)/2$	0	$0,1..\|\alpha\|$	1
α -ve γ +ve $\|\alpha\| > \gamma$ $\|\alpha\| - \gamma$ odd	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$\frac{(\|\alpha\|-\gamma+1)}{2}$	1	$0,1..\|\alpha\|$	1
α -ve γ +ve $\|\alpha\| - \gamma$ odd $\|\alpha\| > \gamma$	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$\frac{(\|\alpha\|-\gamma-1)}{2}$	-1	$0,1..\|\alpha\|$	1
α -ve γ +ve $\gamma > \|\alpha\|$	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	0	$\gamma-\|\alpha\|$	$0,1..\|\alpha\|$	1
α -ve γ -ve $\|\alpha\| \geqslant \|\gamma\|$ $\|\alpha\| + \|\gamma\|$ even	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$\frac{(\|\alpha\|+\|\gamma\|)}{2}$	0	$0,1..\|\alpha\|$	1
α -ve γ -ve $\|\alpha\| + \|\gamma\|$ odd $\|\alpha\| > \|\gamma\|$	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|+\|\gamma\|$ $+ 1)/2$	1	$0,1..\|\alpha\|$	1
α -ve γ -ve $\|\alpha\| + \|\gamma\|$ odd $\|\alpha\| > \|\gamma\|$	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|+\|\gamma\|$ $- 1)/2$	-1	$0,1..\|\alpha\|$	1
α -ve γ -ve $\|\gamma\| > \|\alpha\|$	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	$\|\alpha\|$	$\|\alpha\|-\|\gamma\|$	$0,1..\|\alpha\|$	1

* if $\alpha = -1$ then see table 2.7(d) for the predominant $\Phi^{(+)}$ solar gravity resonant terms.

Table 2.7(c)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability $\pm\dot{\omega} \pm (\dot{\omega}_D \pm \dot{M}_D) + \dot{\Omega} \approx 0$

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	0	0	0	2
$\alpha = 1, \beta = 1, \gamma = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	1	0	0	1
$\alpha = -1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	2	0	0	2
$\alpha = -1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	2	0	0	1
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	0	0	0	2
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	1	0	0	1
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	2	0	0	2
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	2	0	0	1

Table 2.7(d)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in a Solar Gravity Commensurability $\pm\dot{\omega} + \Upsilon(\dot{\omega}_D \pm \dot{M}_D) + \dot{\Omega} \approx 0$

RESTRICTIONS ON α, β and Υ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1, \Upsilon = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	0	0	0,1,2	2
$\alpha = 1, \Upsilon = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	1	0	0,1,2,3	1
$\alpha = 1, \Upsilon = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	0	-2	0	2	0,1,2	2
$\alpha = 1, \Upsilon = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	1	-1	1	1	0,1,2,3	1
					0	-1	0,1,2,3	1
$\alpha = 1, \Upsilon \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\Upsilon)}}\right) < 1$	2	2	0	-2	0	$2\Upsilon-2$	0,1,2	2
$\alpha = 1, \Upsilon \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\Upsilon)}}\right) > 1$	3	1	1	-1	0	$\Upsilon-3$	0,1,2,3	1
$\alpha = -1, \Upsilon = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	0	0	0,1,2	2
$\alpha = -1, \Upsilon = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	1	0	0,1,2,3	1
$\alpha = -1, \Upsilon = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	2	2	0	2	0,1,2	2

Table 2.7(d) continued

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha=-1, \gamma=2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	2	1	1	1	0,1,2,3	1
					0	-1	0,1,2,3	1
$\alpha=-1, \gamma \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D{}^{(1+\gamma)}}\right) < 1$	2	2	2	2	0	$2\gamma-2$	0,1,2	2
$\alpha=-1, \gamma \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D{}^{(1+\gamma)}}\right) > 1$	3	1	2	1	0	$\gamma-3$	0,1,2,3	1
$\alpha=-1, \gamma=-1, \beta=1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	2	0	0,1,2	2
$\alpha=-1, \gamma=-1, \beta=1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	2	0	0,1,2,3	1
$\alpha=-1, \gamma=-2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	2	2	2	-2	0,1,2	2
$\alpha=-1, \gamma=-2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	2	1	3	1	0,1,2,3	1
					2	-1	0,1,2,3	1
$\alpha=-1, \lvert\gamma\rvert \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D{}^{(\lvert\gamma\rvert+1)}}\right) < 1$	2	2	2	2	2	$2-2\lvert\gamma\rvert$	0,1,2	2
$\alpha=-1, \lvert\gamma\rvert \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D{}^{(\lvert\gamma\rvert+1)}}\right) > 1$	3	1	2	1	3	$3-\lvert\gamma\rvert$	0,1,2,3	1

Table 2.7(d) continued

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	2	0	0,1,2	2
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	2	0	0,1,2,3	1
$\alpha = 1, \gamma = -2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	0	-2	2	-2	0,1,2	2
$\alpha = 1, \gamma = -2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	1	-1	3	1	0,1,2,3	1
					2	-1	0,1,2,3	
$\alpha = 1, \lvert\gamma\rvert \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(\lvert\gamma\rvert+1)}}\right) < 1$	2	2	0	-2	2	$2-2\lvert\gamma\rvert$	0,1,2	2
$\alpha = 1, \lvert\gamma\rvert \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(\lvert\gamma\rvert+1)}}\right) > 1$	3	1	1	1	3	$3-\lvert\gamma\rvert$	0,1,2	1

Table 2.7(e)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability of the Type

$\dot{\psi}_6 \approx 0$ when $\beta > |\alpha|$ and $|\gamma|$

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve γ +ve $\beta-\alpha$ even $\beta-\gamma$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-\gamma)/2$	0	0	1
α +ve γ +ve $\beta-\alpha$ odd $\beta-\gamma$ odd	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+1-\gamma)/2$	0	0	1
α +ve γ +ve $\beta-\alpha$ odd, $\beta-\gamma$ even or $\beta-\alpha$ even, $\beta-\gamma$ odd	2β	2β	$\beta-\alpha$	-2α	$\beta-\gamma$	0	0	2
α +ve γ -ve $\beta-\alpha$ even $\beta+\|\gamma\|$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+\|\gamma\|)/2$	0	0	1
α +ve γ -ve $\beta-\alpha$ odd $\beta+\|\gamma\|$ odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\frac{(\beta+1+\|\gamma\|)}{2}$	0	0	1
α +ve γ -ve $\beta-\alpha$ even, $\beta+\|\gamma\|$ odd or $\beta-\alpha$ odd, $\beta+\|\gamma\|$ even	2β	2β	$\beta-\alpha$	-2α	$\beta+\|\gamma\|$	0	0	2
α -ve γ +ve $\beta+\|\alpha\|$ even $\beta-\gamma$ even	β	β	$(\beta+\|\alpha\|)/2$	$\|\alpha\|$	$(\beta-\gamma)/2$	0	0	1
α -ve γ +ve $\beta+\|\alpha\|$ odd $\beta-\gamma$ odd	$\beta+1$	β	$\frac{(\beta+1+\|\alpha\|)}{2}$	$\|\alpha\|$	$(\beta+1-\gamma)/2$	0	0	1
α -ve γ +ve $\beta+\|\alpha\|$ even, $\beta-\gamma$ odd or $\beta+\|\alpha\|$ odd, $\beta-\gamma$ even	2β	2β	$\beta+\|\alpha\|$	$2\|\alpha\|$	$\beta-\gamma$	0	0	2
α -ve γ -ve $\beta+\|\alpha\|$ even $\beta+\|\gamma\|$ even	β	β	$(\beta+\|\alpha\|)/2$	$\|\alpha\|$	$(\beta+\|\gamma\|)/2$	0	0	1

/cont....

Table 2.7(e) continued

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α -ve γ -ve $\beta + \lvert\alpha\rvert$ odd $\beta + \lvert\gamma\rvert$ odd	$\beta+1$	β	$\frac{(\beta+\lvert\alpha\rvert+1)}{2}$	$\lvert\alpha\rvert$	$\frac{(\beta+1+\lvert\gamma\rvert)}{2}$	0	0	1
α -ve γ -ve $\beta + \lvert\alpha\rvert$ even, $\beta + \lvert\gamma\rvert$ odd or $\beta + \lvert\alpha\rvert$ odd, $\beta + \lvert\gamma\rvert$ even	2β	2β	$\beta+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\beta+\lvert\gamma\rvert$	0	0	2

Table 2.7(f)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Solar Commensurability $\dot{\psi}_6 \approx 0$ when $\beta > |\alpha|$ and $|\gamma|$

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve γ +ve $\beta-\alpha$ even $\beta-\gamma$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-\gamma)/2$	0	$0,1..\beta$	1
α +ve γ +ve $\beta-\alpha$ odd $\beta-\gamma$ odd	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+1-\gamma)/2$	0	$0,1..\beta+1$	1
α +ve γ +ve $\beta-\alpha$ even $\beta-\gamma$ odd	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-\gamma+1)/2$ $(\beta-\gamma-1)/2$	1 -1	$0,1..\beta$ $0,1..\beta$	1 1
α +ve γ +ve $\beta-\alpha$ odd $\beta-\gamma$ even	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+2-\gamma)/2$ $(\beta-\gamma)/2$	1 -1	$0,1..\beta+1$ $0,1..\beta+1$	1 1
α +ve γ −ve $\beta-\alpha$ even $\beta+\|\gamma\|$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+\|\gamma\|)/2$	0	$0,1..\beta$	1
α +ve γ −ve $\beta-\alpha$ odd $\beta+\|\gamma\|$ odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\frac{(\beta+1+\|\gamma\|)}{2}$	0	$0,1..\beta+1$	1
α +ve γ −ve $\beta-\alpha$ even $\beta+\|\gamma\|$ odd	β	β	$(\beta-\alpha)/2$	$-\alpha$	$\frac{(\beta+\|\gamma\|+1)}{2}$ $\frac{(\beta+\|\gamma\|-1)}{2}$	1 -1	$0,1..\beta$ $0,1..\beta$	1 1
α +ve γ −ve $\beta-\alpha$ odd $\beta+\|\gamma\|$ even	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$\frac{(\beta+\|\gamma\|+2)}{2}$ $(\beta+\|\gamma\|)/2$	1 -1	$0,1..\beta+1$ $0,1..\beta+1$	1 1

Table 2.7(f) continued

RESTRICTIONS ON α, β and γ	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α -ve γ +ve; $\beta+\lvert\alpha\rvert$ even; $\beta-\gamma$ even	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta-\gamma)/2$	0	$0,1..\beta$	1
α -ve γ +ve; $\beta+\lvert\alpha\rvert$ odd; $\beta-\gamma$ odd	$\beta+1$	β	$\frac{(\beta+\lvert\alpha\rvert+1)}{2}$	$\lvert\alpha\rvert$	$(\beta+1-\gamma)/2$	0	$0,1..\beta+1$	1
α -ve γ +ve; $\beta+\lvert\alpha\rvert$ even; $\beta-\gamma$ odd	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta-\gamma+1)/2$	1	$0,1..\beta$	1
					$(\beta-\gamma-1)/2$	-1	$0,1..\beta$	1
α -ve γ +ve; $\beta+\lvert\alpha\rvert$ odd; $\beta-\gamma$ even	$\beta+1$	β	$\frac{(1+\beta+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$(\beta-\gamma+2)/2$	1	$0,1..\beta+1$	1
					$(\beta-\gamma)/2$	-1	$0,1..\beta+1$	1
α -ve γ -ve; $\beta+\lvert\alpha\rvert$ even; $\beta+\gamma$ even	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+\lvert\gamma\rvert)/2$	0	$0,1..\beta$	1
α -ve γ -ve; $\beta+\lvert\alpha\rvert$ odd; $\beta+\lvert\gamma\rvert$ odd	$\beta+1$	β	$\frac{(\beta+\lvert\alpha\rvert+1)}{2}$	$\lvert\alpha\rvert$	$\frac{(\beta+1+\lvert\gamma\rvert)}{2}$	0	$0,1..\beta+1$	1
α -ve γ -ve; $\beta+\lvert\alpha\rvert$ even; $\beta+\lvert\gamma\rvert$ odd	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$\frac{(\beta+1+\lvert\gamma\rvert)}{2}$	1	$0,1..\beta$	1
					$\frac{(\beta+\lvert\gamma\rvert-1)}{2}$	-1	$0,1..\beta$	1
α -ve γ -ve; $\beta+\lvert\alpha\rvert$ odd; $\beta+\lvert\gamma\rvert$ even	$\beta+1$	β	$\frac{(1+\beta+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$\frac{(\beta+\lvert\gamma\rvert+2)}{2}$	1	$0,1..\beta+1$	1
					$(\beta+\lvert\gamma\rvert)/2$	-1	$0,1..\beta+1$	1

Table 2.7(g)

The $\bar{n}$, $\bar{m}$, $\bar{p}$, $\bar{q}$, $\bar{h}$, $\bar{j}$, $\bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_6 \approx 0$ with $|\alpha|$ and/or $|\gamma| \geqslant \beta$

RESTRICTIONS ON α, β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{j}$	$\bar{h}$	$\bar{s}$	v
α +ve γ +ve $\alpha \geqslant \gamma$ $\alpha + \gamma$ even	α^*	β	0	$-\alpha$	0	$(\alpha+\gamma)/2$	0	1
α +ve γ +ve $\alpha > \gamma$ $\alpha + \gamma$ odd	2α	2β	0	-2α	0	$(\alpha+\gamma)$	0	2
α +ve γ +ve $\gamma > \alpha$ $\gamma - \alpha$ even	γ	β	$(\gamma-\alpha)/2$	$-\alpha$	0	γ	0	1
α +ve γ +ve $\gamma > \alpha$ $\gamma - \alpha$ odd	2γ	2β	$\gamma-\alpha$	-2α	0	2γ	0	2
α -ve γ -ve $\|\alpha\| \geqslant \|\gamma\|$ $\|\alpha\| - \|\gamma\|$ even	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	0	$\frac{(\|\alpha\|-\|\gamma\|)}{2}$	0	1
α -ve γ -ve $\|\alpha\| > \|\gamma\|$ $\|\alpha\| - \|\gamma\|$ odd	$2\|\alpha\|$	2β	$2\|\alpha\|$	$2\|\alpha\|$	0	$\|\alpha\|-\|\gamma\|$	0	2
α -ve γ -ve $\|\alpha\| > \|\gamma\|$ $\|\alpha\| + \|\gamma\|$ even	$\|\gamma\|$	β	$\frac{(\|\gamma\|+\|\alpha\|)}{2}$	$\|\alpha\|$	0	0	0	1
α -ve γ -ve $\|\gamma\| > \|\alpha\|$ $\|\gamma\| + \|\alpha\|$ odd	$2\|\gamma\|$	2β	$\|\gamma\|+\|\alpha\|$	$2\|\alpha\|$	0	0	0	2

* if $|\alpha| = +1$ see table 2.7(i) for the predominant $\Phi^{(-)}$ resonant terms

Table 2.7(g) continued

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	j^-	h^-	s^-	v
α -ve γ +ve $\|\alpha\| \geqslant \gamma$ $\|\alpha\| + \gamma$ even	$\|\alpha\|^*$	β	$\|\alpha\|$	$\|\alpha\|$	0	$(\|\alpha\|+\gamma)/2$	0	1
α -ve γ +ve $\|\alpha\| > \gamma$ $\|\alpha\| + \gamma$ odd	$2\|\alpha\|$	2β	$2\|\alpha\|$	$2\|\alpha\|$	0	$\|\alpha\| + \gamma$	0	2
α -ve γ +ve $\gamma > \|\alpha\|$ $\|\alpha\| + \gamma$ even	γ	β	$(\|\alpha\|+\gamma)/2$	$\|\alpha\|$	0	γ	0	1
α -ve γ +ve $\gamma > \|\alpha\|$ $\|\alpha\| + \gamma$ odd	2γ	2β	$\|\alpha\|+\gamma$	$2\|\alpha\|$	0	2γ	0	2
α +ve γ -ve $\alpha \geqslant \|\gamma\|$ $\alpha - \|\gamma\|$ even	$\|\alpha\|^*$	β	0	$-\alpha$	0	$(\alpha-\|\gamma\|)/2$	0	1
α +ve γ -ve $\alpha > \|\gamma\|$ $\alpha - \|\gamma\|$ odd	2α	2β	0	-2α	0	$\alpha-\|\gamma\|$	0	2
α +ve γ -ve $\|\gamma\| > \alpha$ $\|\gamma\| - \alpha$ even	$\|\gamma\|$	β	$(\|\gamma\|-\alpha)/2$	$-\alpha$	0	0	0	1
α +ve γ -ve $\|\gamma\| > \alpha$ $\|\gamma\| - \alpha$ odd	$2\|\gamma\|$	2β	$\|\gamma\|-\alpha$	-2α	0	0	0	2

Table 2.7(h)

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Solar Commensurability $\dot{\psi}_6 \approx 0$ with $|\alpha| \geqslant \beta$

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α +ve γ +ve $\alpha + \gamma$ even $\alpha \geqslant \gamma$	α^*	β	0	$-\alpha$	$(\alpha+\gamma)/2$	0	$0,1..\alpha$	1
α +ve γ +ve $\alpha > \gamma$	α^*	β	0	$-\alpha$	$(\alpha+\gamma+1)/2$	1	$0,1..\alpha$	1
$\alpha + \gamma$ odd	α^*	β	0	$-\alpha$	$(\alpha+\gamma-1)/2$	-1	$0,1..\alpha$	1
α +ve γ +ve $\gamma > \alpha$	α^*	β	0	$-\alpha$	α	$\alpha-\gamma$	$0,1..\alpha$	1
α +ve γ -ve $\alpha \geqslant \lvert\gamma\rvert$ $\alpha - \lvert\gamma\rvert$ even	α^*	β	0	$-\alpha$	$(\alpha-\lvert\gamma\rvert)/2$	0	$0,1..\alpha$	1
α +ve γ -ve $\alpha > \lvert\gamma\rvert$	α^*	β	0	$-\alpha$	$\frac{(\alpha-\lvert\gamma\rvert+1)}{2}$	1	$0,1..\alpha$	1
$\alpha - \lvert\gamma\rvert$ odd	α^*	β	0	$-\alpha$	$\frac{(\alpha-\lvert\gamma\rvert-1)}{2}$	-1	$0,1..\alpha$	1
α +ve γ -ve $\lvert\gamma\rvert > \alpha$	α^*	β	0	$-\alpha$	0	$\lvert\gamma\rvert-\alpha$	$0,1..\alpha$	1

* if $\alpha = 1$ then see table 2.7(j) for the predominant $\Phi^{(-)}$ solar gravity resonant terms

Table 2.7(h) continued

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α -ve γ +ve $\lvert\alpha\rvert \geqslant \gamma$ $\lvert\alpha\rvert + \gamma$ even	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\gamma)/2$	0	$0,1..\lvert\alpha\rvert$	1
α -ve γ +ve $\lvert\alpha\rvert > \gamma$ $\lvert\alpha\rvert + \gamma$ odd	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert+\gamma+1)}{2}$	1	$0,1..\lvert\alpha\rvert$	1
α -ve γ +ve $\lvert\alpha\rvert > \gamma$ $\lvert\alpha\rvert + \gamma$ odd	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert+\gamma-1)}{2}$	-1	$0,1..\lvert\alpha\rvert$	1
α -ve γ +ve $\gamma > \lvert\alpha\rvert$	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\lvert\alpha\rvert-\gamma$	$0,1..\lvert\alpha\rvert$	1
α -ve γ -ve $\lvert\alpha\rvert \geqslant \lvert\gamma\rvert$ $\lvert\alpha\rvert - \lvert\alpha\rvert$ even	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$\frac{(\lvert\alpha\rvert-\lvert\gamma\rvert)}{2}$	0	$0,1..\lvert\alpha\rvert$	1
α -ve γ -ve $\lvert\alpha\rvert > \lvert\gamma\rvert$ $\lvert\alpha\rvert - \lvert\gamma\rvert$ odd	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\lvert\gamma\rvert + 1)/2$	1	$0,1..\lvert\alpha\rvert$	1
α -ve γ -ve $\lvert\alpha\rvert > \lvert\gamma\rvert$ $\lvert\alpha\rvert - \lvert\gamma\rvert$ odd	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\lvert\gamma\rvert - 1)/2$	-1	$0,1..\lvert\alpha\rvert$	1
α -ve γ -ve $\lvert\gamma\rvert > \lvert\alpha\rvert$	$\lvert\alpha\rvert$*	β	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	0	$\lvert\gamma\rvert-\lvert\alpha\rvert$	$0,1..\lvert\alpha\rvert$	1

* if $\alpha = -1$ then see table 2.7(j) for predominant $\Phi^{(-)}$ solar gravity resonant terms

Table 2.7(i)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in a Lunar Gravity Commensurability $\pm\dot{\omega} \pm (\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega} \approx 0$

RESTRICTIONS ON α, β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = 1,\ \gamma = 1,\ \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	2	0	0	2
$\alpha = 1,\ \gamma = 1,\ \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	2	0	0	1
$\alpha = -1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	0	0	0	2
$\alpha = -1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	1	0	0	1
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	2	0	0	2
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	2	0	0	1
$\alpha = 1,\ \gamma = -1,\ \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	0	0	0	2
$\alpha = 1, \beta = 1, \gamma = -1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	1	0	0	1

Table 2.7(j)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values for the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in a Solar Gravity Commensurability

$\pm\dot{\omega} + \gamma(\dot{\omega}_D \pm \dot{M}_D) + \dot{\Omega} \approx 0$

RESTRICTIONS ON α, β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = 1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	2	0	0,1,2	2
$\alpha = 1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	2	0	0,1,2,3	1
$\alpha = 1, \gamma = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	0	-2	2	-2	0,1,2	2
$\alpha = 1, \gamma = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	1	-1	3	1	0,1,2,3	1
					2	-1	0,1,2,3	1
$\alpha = 1, \gamma \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) < 1$	2	2	0	-2	2	$2-2\gamma$	0,1,2	2
$\alpha = 1, \gamma \geqslant 3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\gamma)}}\right) > 1$	3	1	1	-1	3	$3-\gamma$	0,1,2,3	1
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	2	0	0,1,2	2
$\alpha = -1, \gamma = 1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	2	0	0,1,2,3	1
$\alpha = -1, \gamma = 2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	2	2	2	-2	0,1,2	2

Table 2.7(j) continued

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
$\alpha=-1, \gamma=2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	2	1	3	1	0,1,2,3	1
					2	-1	0,1,2,3	
$\alpha=-1, \gamma \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D}{}^{(1+\gamma)}\right) < 1$	2	2	2	2	2	$2-2\gamma$	0,1,2	1
$\alpha=-1, \gamma \geqslant 3, \beta=1$ if $\left(\frac{a}{a_D e e_D}{}^{(1+\gamma)}\right) > 1$	3	1	2	1	3	$3-\gamma$	0,1,2,3	1
$\alpha=-1, \gamma=-1, \beta=1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	2	2	0	0	0,1,2	2
$\alpha=-1, \gamma=-1, \beta=1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	2	1	1	0	0,1,2,3	1
$\alpha=-1, \gamma=-2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	2	2	0	2	0,1,2	2
$\alpha=-1, \gamma=-2, \beta=1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	2	1	1	1	0,1,2,3	1
					0	-1		
$\alpha=-1, \gamma \leqslant -3, \beta=1$ if $\left(\frac{a}{a_D e e_D}{}^{(1+\lvert\gamma\rvert)}\right) < 1$	2	2	2	2	0	$2\lvert\gamma\rvert-2$	0,1,2	2
$\alpha=-1, \gamma \leqslant -3, \beta=1$ if $\left(\frac{a}{a_D e e_D}{}^{(1+\lvert\gamma\rvert)}\right) > 1$	3	1	0	1	0	$\lvert\gamma\rvert-3$	0,1,2,3	1

Table 2.7(j) continued

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) < 1$	2	2	0	-2	0	0	0,1,2	2
$\alpha = 1, \gamma = -1, \beta = 1$ if $\left(\frac{a}{a_D e}\right) > 1$	3	1	1	-1	1	0	0,1,2,3	1
$\alpha = 1, \gamma = -2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	2	0	-2	0	2	0,1,2	2
$\alpha = 1, \gamma = -2, \beta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	1	1	-1	1 0	1 -1	0,1,2,3	1
$\alpha = 1, \gamma \leqslant -3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\lvert\gamma\rvert)}}\right) < 1$	2	2	0	-2	0	$2\lvert\gamma\rvert-2$	0,1,2	2
$\alpha = 1, \gamma \leqslant -3, \beta = 1$ if $\left(\frac{a}{a_D e e_D^{(1+\lvert\gamma\rvert)}}\right) > 1$	3	1	1	1	0	$\lvert\gamma\rvert-3$	0,1,2,3	1

Table 2.7(k)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability of the Type $\dot{\psi}_6 \approx 0$ when $\beta > |\alpha|$ and $|\gamma|$

RESTRICTIONS ON α, β and γ	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α +ve γ +ve $\beta - \alpha$ even $\beta + \gamma$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+\gamma)/2$	0	0	1
α +ve γ +ve $\beta - \alpha$ odd $\beta + \gamma$ odd	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+1+\gamma)/2$	0	0	1
α +ve γ +ve $\beta - \alpha$ odd, $\beta + \gamma$ even or $\beta - \alpha$ even, $\beta + \gamma$ odd	2β	2β	$\beta-\alpha$	-2α	$\beta+\gamma$	0	0	2
α +ve γ -ve $\beta - \alpha$ even $\beta - \|\gamma\|$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-\|\gamma\|)/2$	0	0	1
α +ve γ -ve $\beta - \alpha$ odd $\beta - \|\gamma\|$ odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\frac{(\beta+1-\|\gamma\|)}{2}$	0	0	1
α +ve γ -ve $\beta - \alpha$ even, $\beta - \|\gamma\|$ odd or $\beta - \alpha$ odd, $\beta - \|\gamma\|$ even	2β	2β	$\beta-\alpha$	-2α	$\beta-\|\gamma\|$	0	0	2
α -ve γ +ve $\beta + \|\alpha\|$ even $\beta + \gamma$ even	β	β	$(\beta+\|\alpha\|)/2$	$\|\alpha\|$	$(\beta+\gamma)/2$	0	0	1
α -ve γ +ve $\beta + \|\alpha\|$ odd $\beta + \gamma$ odd	$\beta+1$	β	$\frac{(\beta+1+\|\alpha\|)}{2}$	$\|\alpha\|$	$(\beta+1+\gamma)/2$	0	0	1
α -ve γ +ve $\beta + \|\alpha\|$ even, $\beta + \gamma$ odd $\beta + \|\alpha\|$ odd, $\beta+\gamma$ even	2β	2β	$\beta+\|\alpha\|$	$2\|\alpha\|$	$\beta+\gamma$	0	0	2

Table 2.7(k) continued

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α -ve γ -ve $\beta+\lvert\alpha\rvert$ even $\beta-\lvert\gamma\rvert$ even	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta-\lvert\gamma\rvert)/2$	0	0	1
α -ve γ -ve $\beta+\lvert\alpha\rvert$ odd $\beta-\lvert\gamma\rvert$ odd	$\beta+1$	β	$\dfrac{(\beta+\lvert\alpha\rvert+1)}{2}$	$\lvert\alpha\rvert$	$\dfrac{(\beta+1-\lvert\gamma\rvert)}{2}$	0	0	1
α -ve γ -ve $\beta+\lvert\alpha\rvert$ even, $\beta-\lvert\gamma\rvert$ odd <u>or</u> $\beta+\lvert\alpha\rvert$ odd, $\beta-\lvert\gamma\rvert$ even	2β	2β	$\beta+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\beta-\lvert\gamma\rvert$	0	0	2

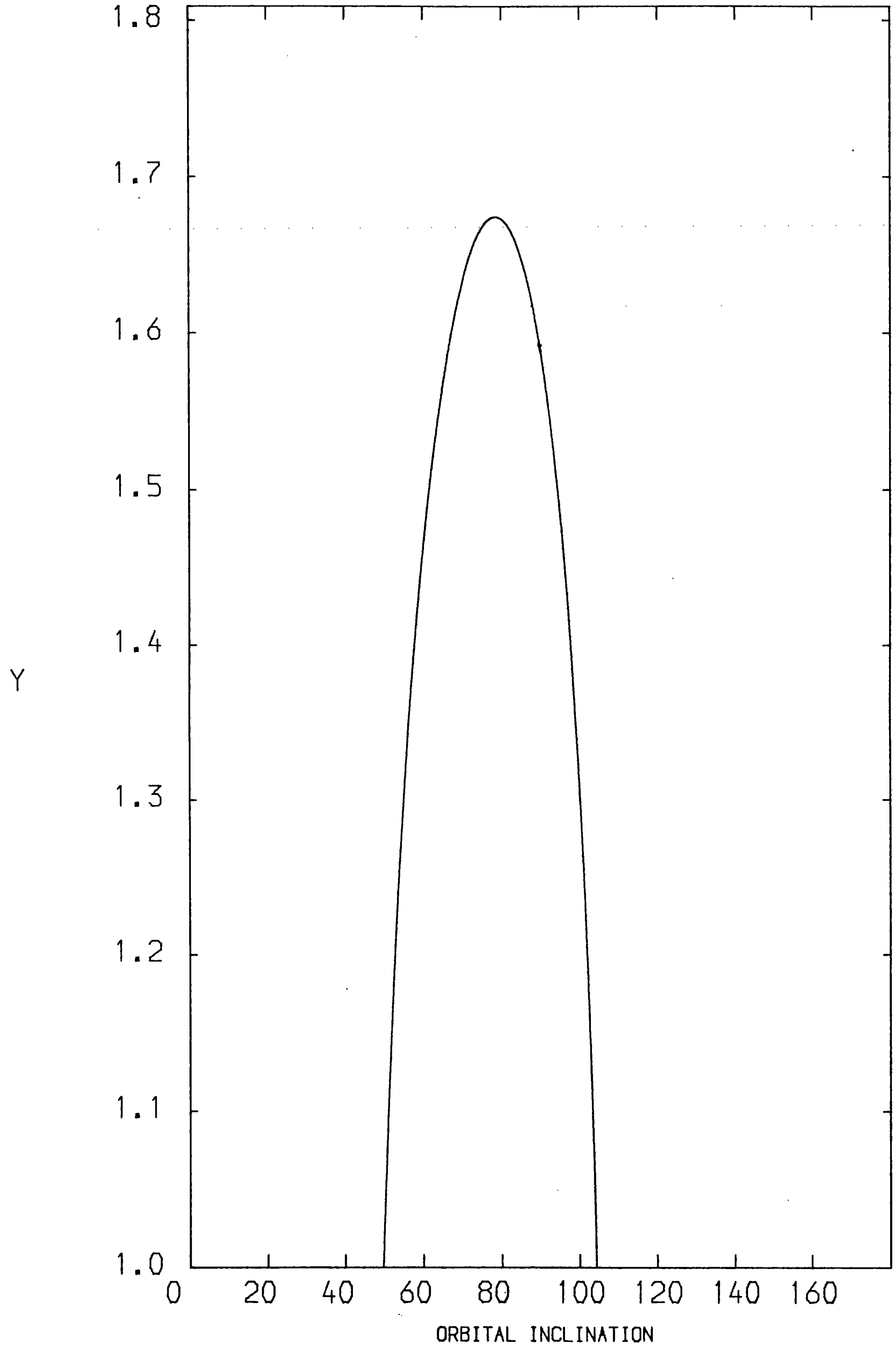
Table 2.7(1)

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values of the Predominant $\bar{\Phi}^{(-)}$ Resonant Terms for a Satellite in a Solar Commensurability $\dot{\psi}_6 \approx 0$ when $\beta > |\alpha|$ and $|\gamma|$

RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α +ve γ +ve $\beta-\alpha$ even $\beta+\gamma$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+\gamma)/2$	0	$0,1..\beta$	1
α +ve γ +ve $\beta-\alpha$ odd $\beta+\gamma$ odd	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+1+\gamma)/2$	0	$0,1..\beta+1$	1
α +ve γ +ve $\beta-\alpha$ even $\beta+\gamma$ odd	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta+\gamma+1)/2$ $(\beta+\gamma-1)/2$	1 -1	$0,1..\beta$	1
α +ve γ +ve $\beta-\alpha$ odd $\beta+\gamma$ even	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$(\beta+2+\gamma)/2$ $(\beta+\gamma)/2$	1 -1	$0,1..\beta+1$	1
α +ve γ -ve $\beta-\alpha$ even $\beta-\lvert\gamma\rvert$ even	β	β	$(\beta-\alpha)/2$	$-\alpha$	$(\beta-\lvert\gamma\rvert)/2$	0	$0,1..\beta$	1
α +ve γ -ve $\beta-\alpha$ odd $\beta-\lvert\gamma\rvert$ odd	$\beta+1$	β	$(\beta-\alpha+1)/2$	$-\alpha$	$\frac{(\beta+1-\lvert\gamma\rvert)}{2}$	0	$0,1..\beta+1$	1
α +ve γ -ve $\beta-\alpha$ even $\beta-\lvert\gamma\rvert$ odd	β	β	$\beta-\alpha$	$-\alpha$	$\frac{(\beta+1-\lvert\gamma\rvert)}{2}$ $\frac{(\beta-1-\lvert\gamma\rvert)}{2}$	1 -1	$0,1..\beta$	1
α +ve γ -ve $\beta-\alpha$ odd $\beta-\lvert\gamma\rvert$ even	$\beta+1$	β	$(\beta+1-\alpha)/2$	$-\alpha$	$\frac{(\beta+2-\lvert\gamma\rvert)}{2}$ $(\beta-\lvert\gamma\rvert)/2$	1 -1	$0,1..\beta+1$	1

Table 2.7(1) continued

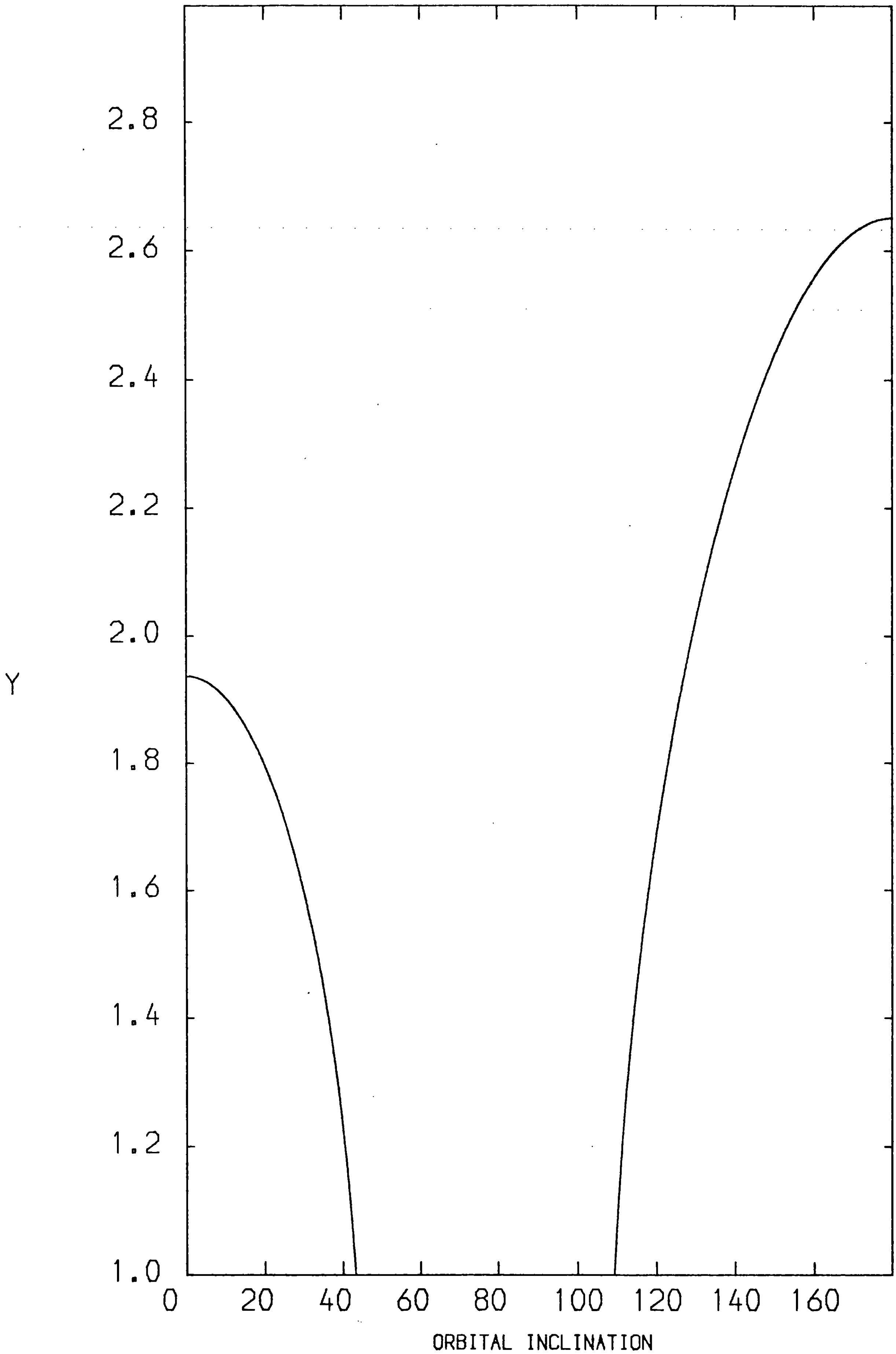
RESTRICTIONS ON α, β and γ	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α −ve γ +ve $\beta + \lvert\alpha\rvert$ even $\beta + \gamma$ even	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+\gamma)/2$	0	$0,1..\beta$	1
α −ve γ +ve $\beta + \lvert\alpha\rvert$ odd $\beta + \gamma$ odd	$\beta+1$	β	$\frac{(\beta+1+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$(\beta+1+\gamma)/2$	0	$0,1..\beta+1$	1
α −ve γ +ve $\beta + \lvert\alpha\rvert$ even $\beta + \lvert\gamma\rvert$ odd	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta+1+\gamma)/2$ $(\beta-1+\gamma)/2$	1 −1	$0,1..\beta$	1
α −ve γ +ve $\beta + \lvert\alpha\rvert$ odd $\beta + \gamma$ even	$\beta+1$	β	$\frac{(\beta+1+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$(\beta+\gamma+2)/2$ $(\beta+\gamma)/2$	1 −1	$0,1..\beta+1$	1
α −ve γ −ve $\beta + \lvert\alpha\rvert$ even $\beta - \lvert\gamma\rvert$ even	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$(\beta-\lvert\gamma\rvert)/2$	0	$0,1..\beta$	1
α −ve γ −ve $\beta + \lvert\alpha\rvert$ odd $\beta - \lvert\gamma\rvert$ odd	$\beta+1$	β	$\frac{(\beta+1+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$\frac{(\beta+1-\lvert\gamma\rvert)}{2}$	0	$0,1..\beta+1$	1
α −ve γ −ve $\beta + \lvert\alpha\rvert$ even $\beta - \lvert\gamma\rvert$ odd	β	β	$(\beta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	$\frac{(\beta+1-\lvert\gamma\rvert)}{2}$ $\frac{(\beta-1-\lvert\gamma\rvert)}{2}$	1 −1	$0,1..\beta$	1
α −ve γ −ve $\beta + \lvert\alpha\rvert$ odd $\beta - \lvert\gamma\rvert$ even	$\beta+1$	β	$\frac{(\beta+1+\lvert\alpha\rvert)}{2}$	$\lvert\alpha\rvert$	$\frac{(\beta+2-\lvert\gamma\rvert)}{2}$ $(\beta-\lvert\gamma\rvert)/2$	1 −1	$0,1..\beta+1$	1



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$\dot{\omega}+\dot{\omega}_D+\dot{M}_D+\dot{\Omega}=0$

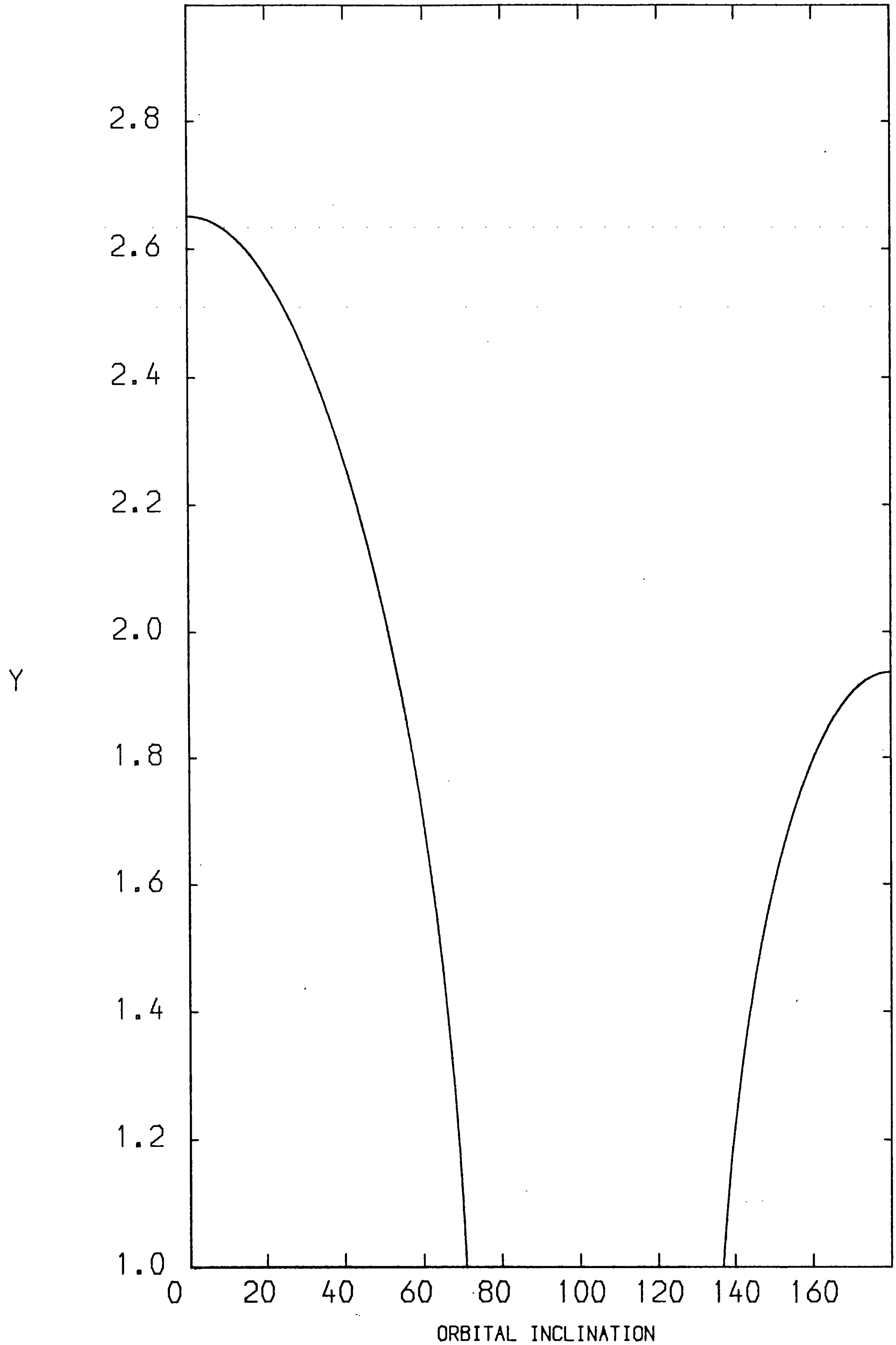
Figure 2.10



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$\dot{\omega}-\dot{\omega}_D-\dot{M}_D+\dot{\Omega}=0$

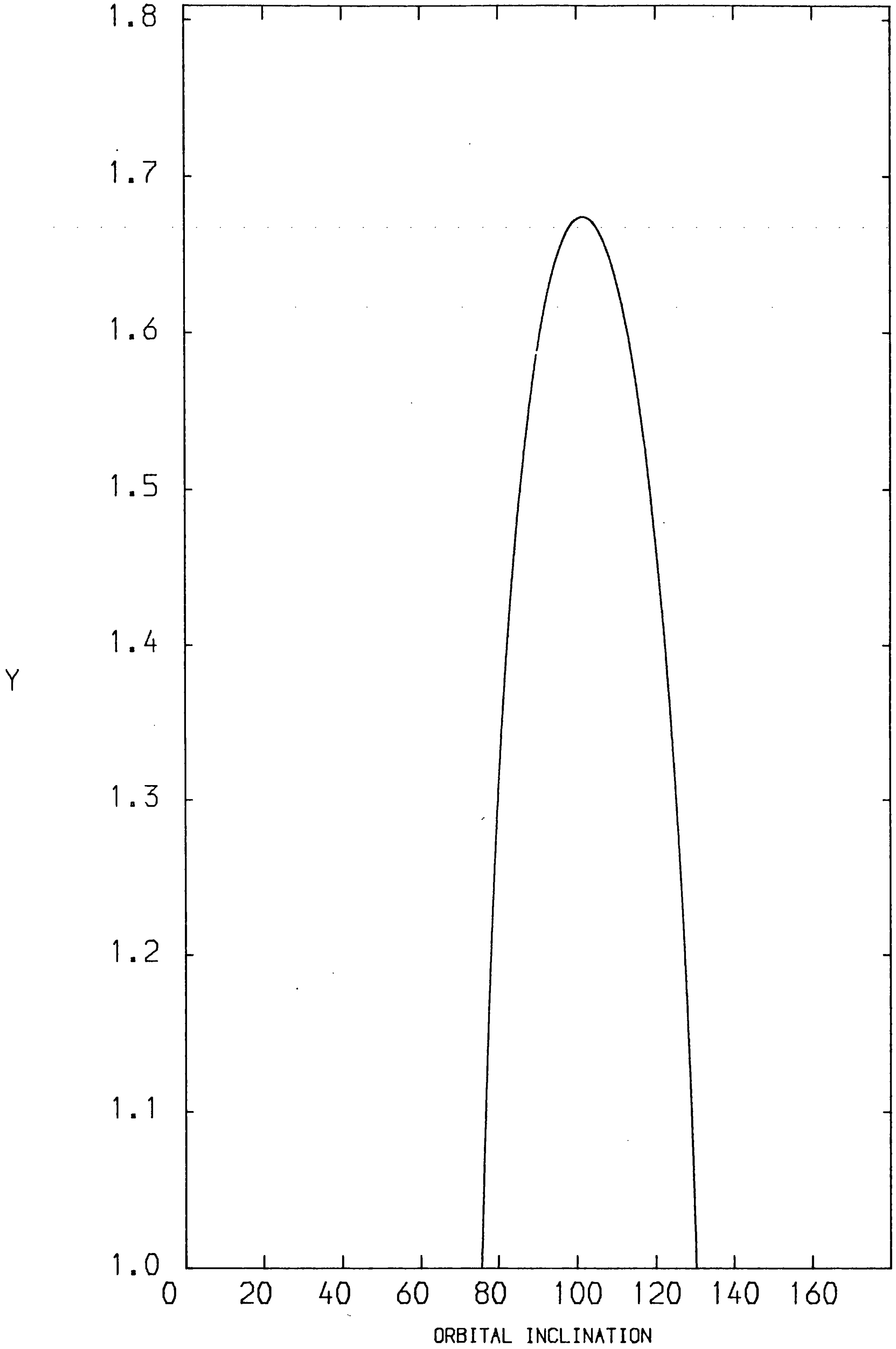
Figure 2.11



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D+\dot{\Omega}=0$$

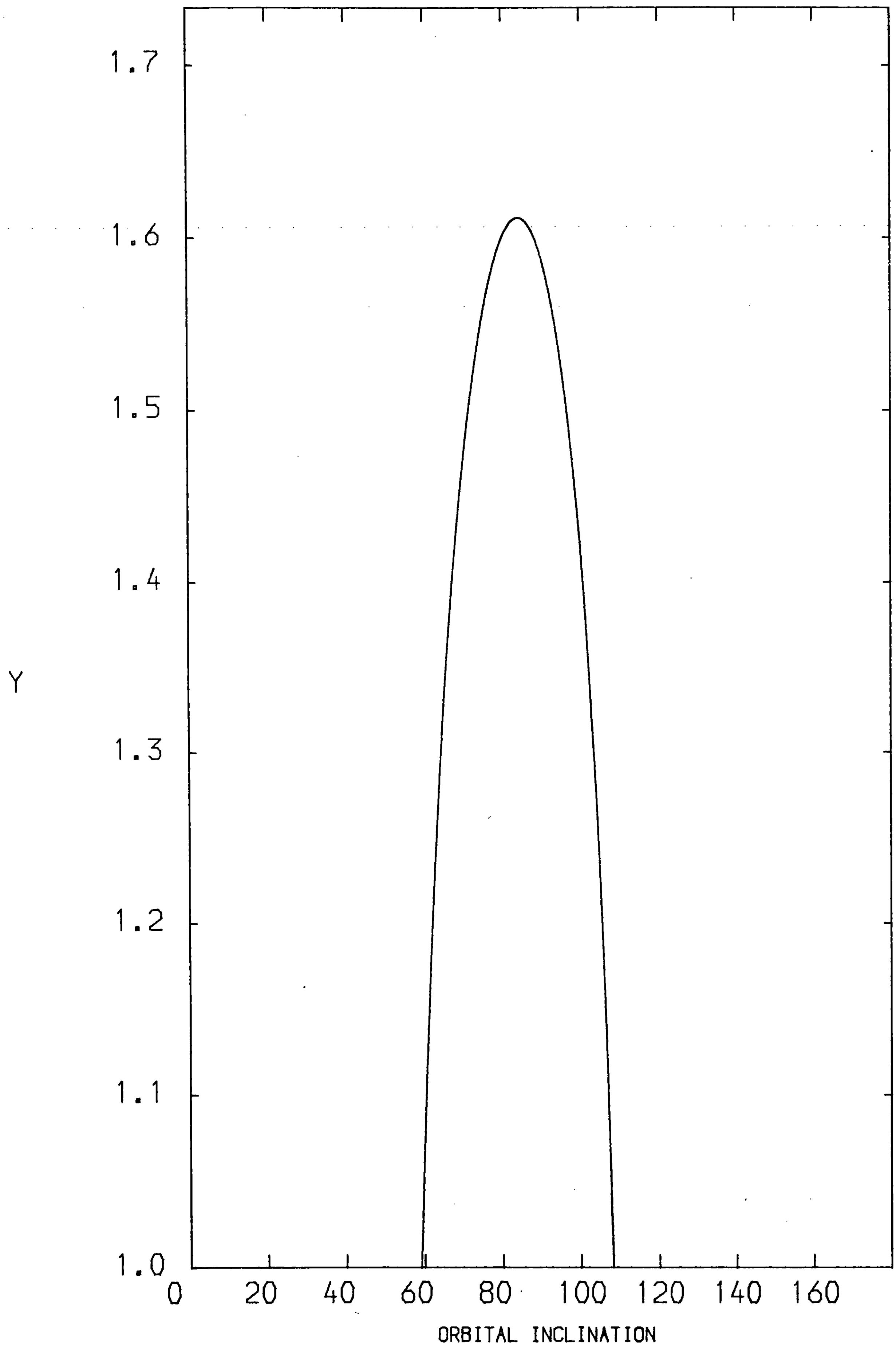
Figure 2.12



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-\dot{\omega}-\dot{\omega}_D-\dot{M}_D+\dot{\Omega}=0$

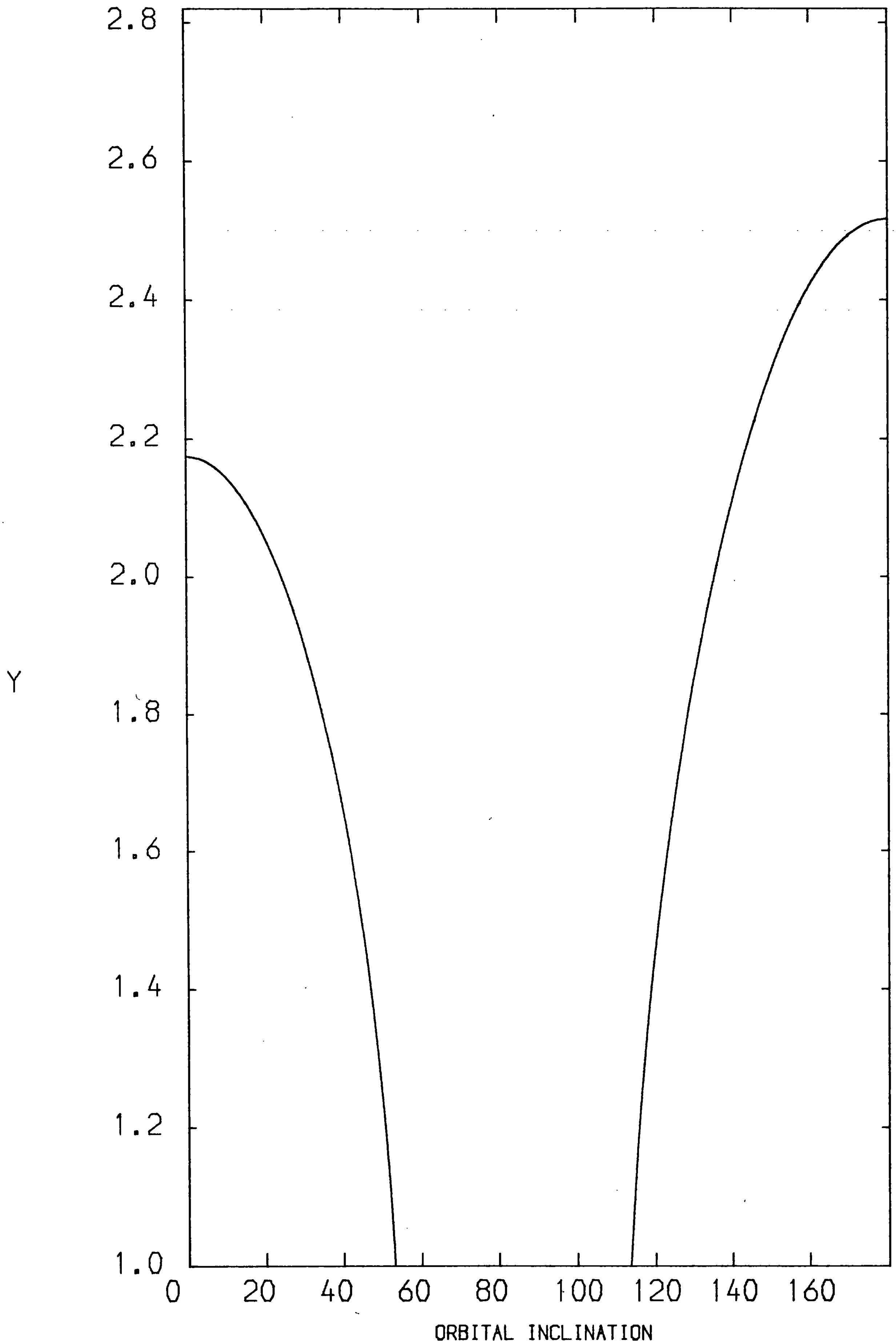
Figure 2.13



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$2\dot{\omega} + 2\dot{\omega}_D + 2\dot{M}_D + \dot{\Omega} = 0$

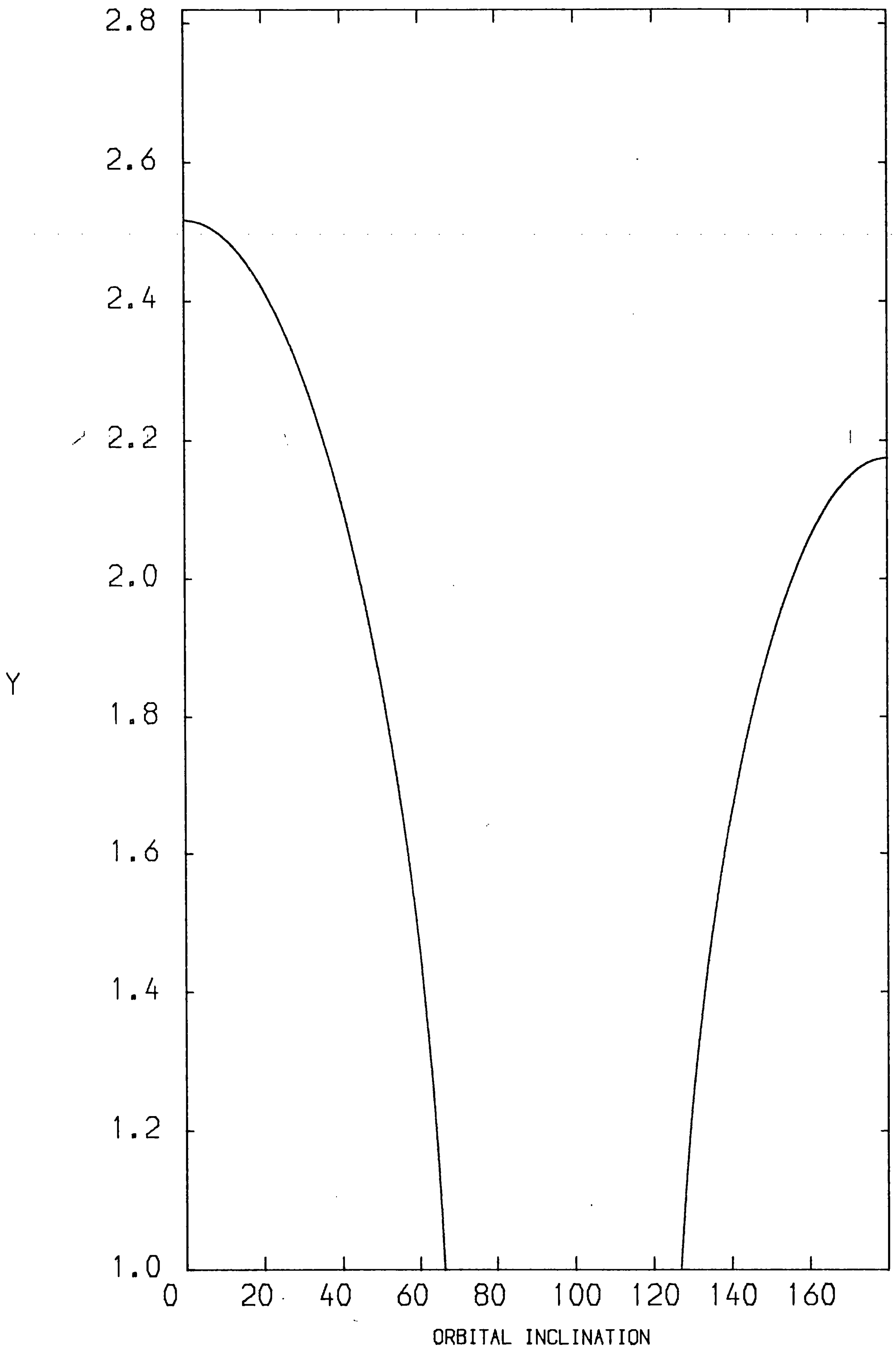
Figure 2.14



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$2\dot{\omega}-2\dot{\omega}_D-2\dot{M}_D+\dot{\Omega}=0$

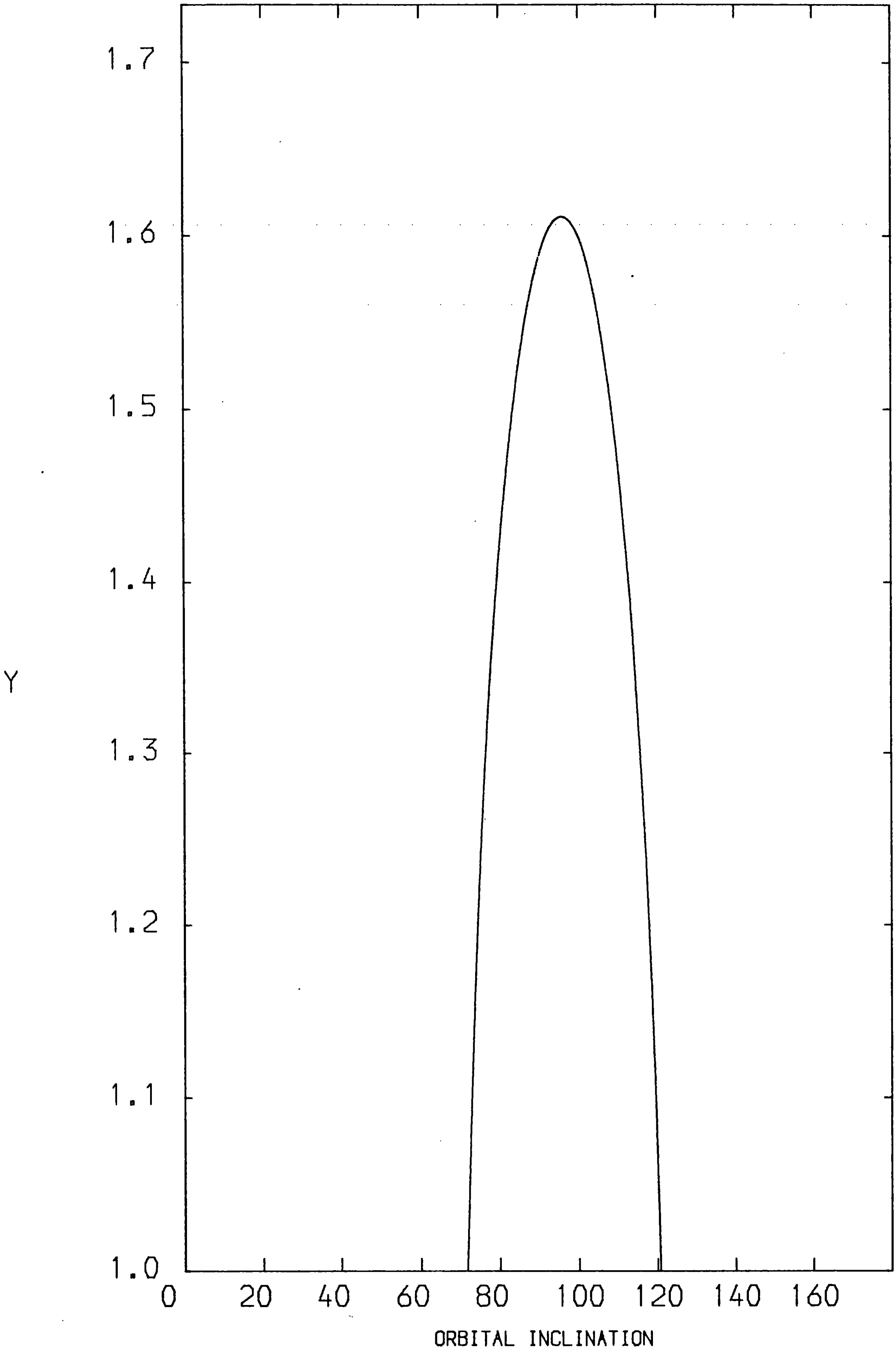
Figure 2.15



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$-2\dot{\omega}+ 2\dot{\omega}_D+ 2\dot{M}_D+\dot{\Omega}=0$

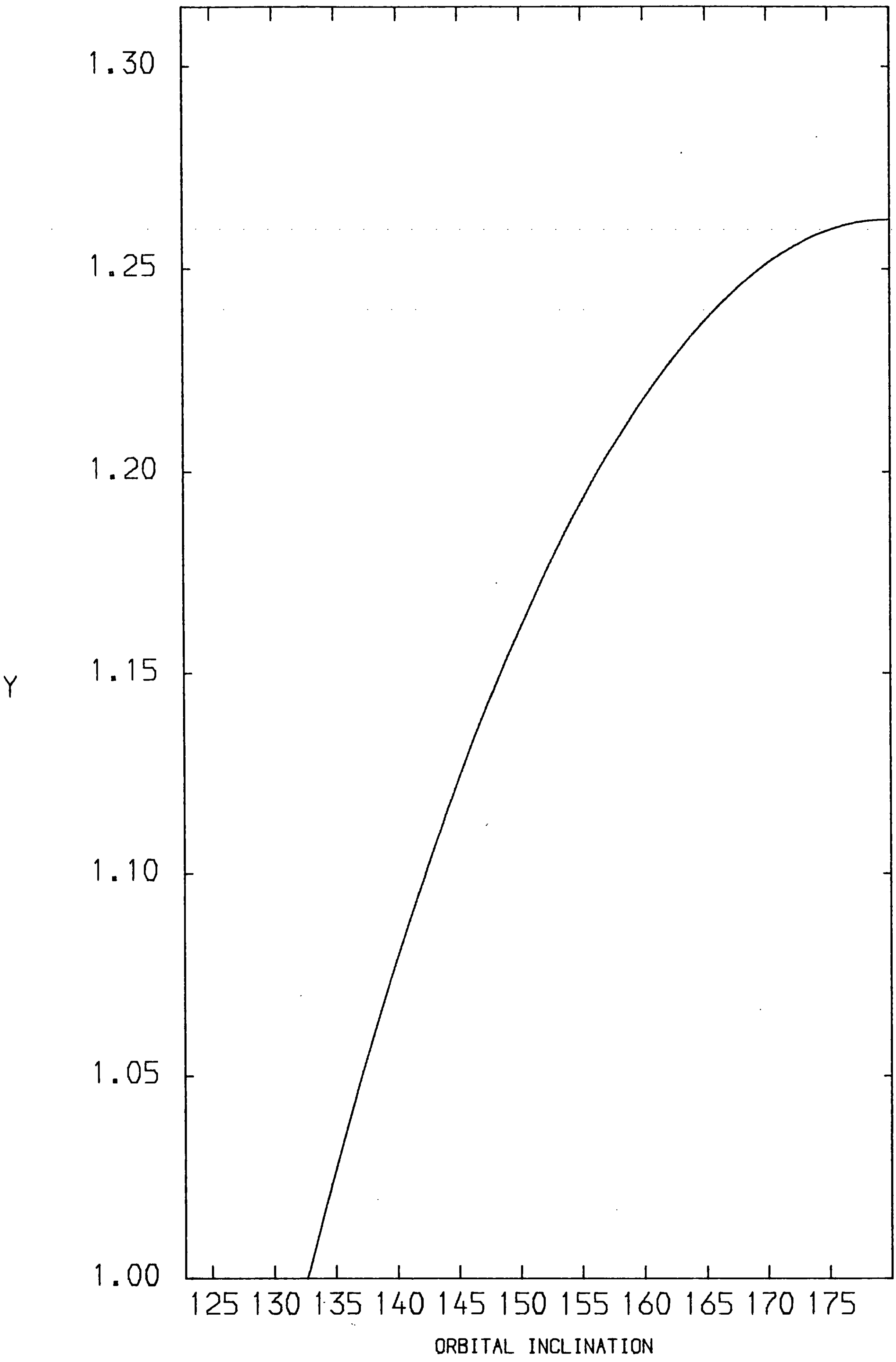
Figure 2.16



ORBITS FOR A SATELLITE IN THE SOLAR COMMENSURABILITY

$$-2\dot{\omega}-2\dot{\omega}_{\odot}-2\dot{M}_{\odot}+\dot{\Omega}=0$$

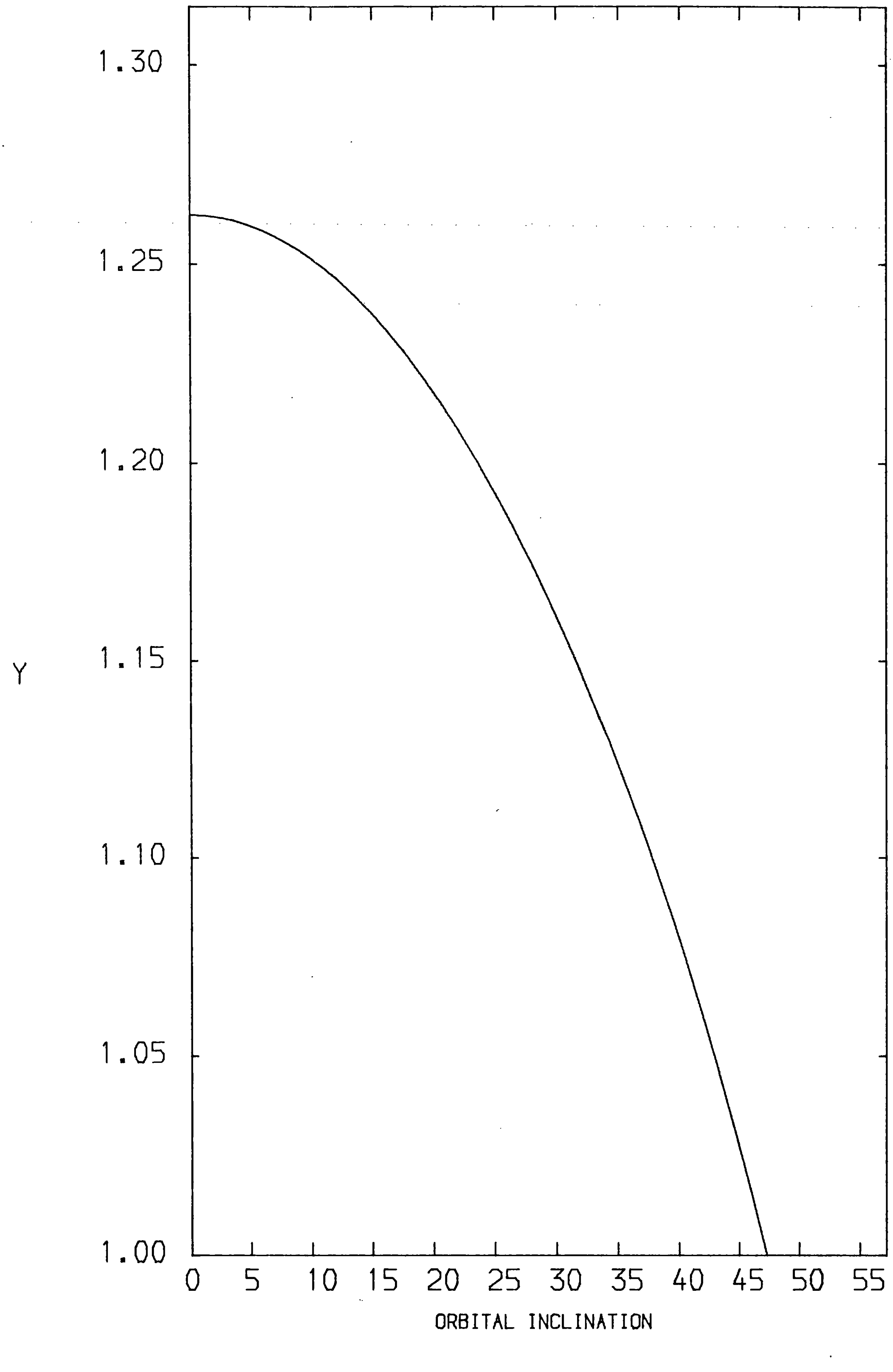
Figure 2.17



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$$\dot{\omega}-\dot{\omega}_D-\dot{M}_D+\dot{\Omega}=0$$

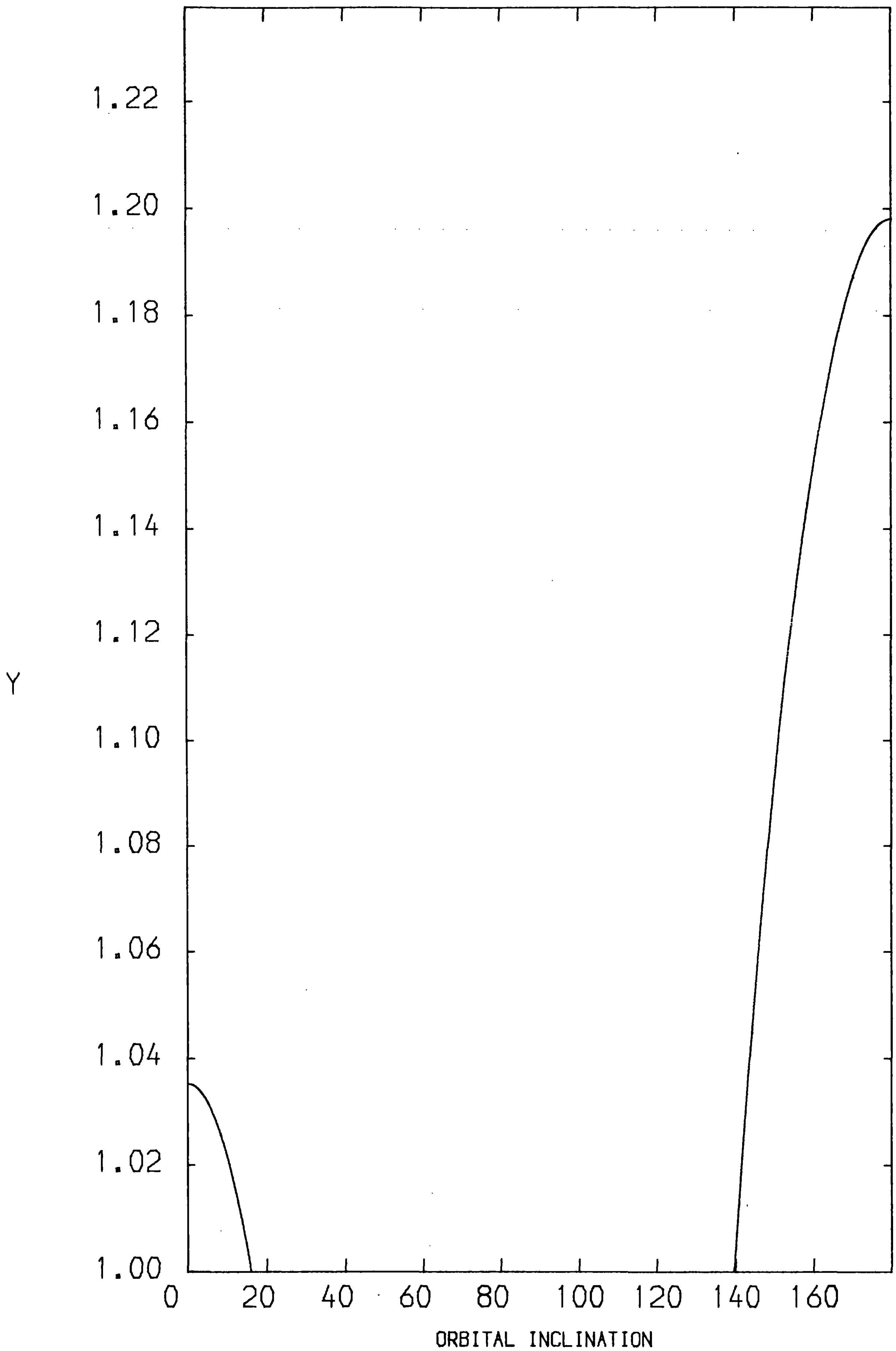
Figure 2.18



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D+\dot{\Omega}=0$

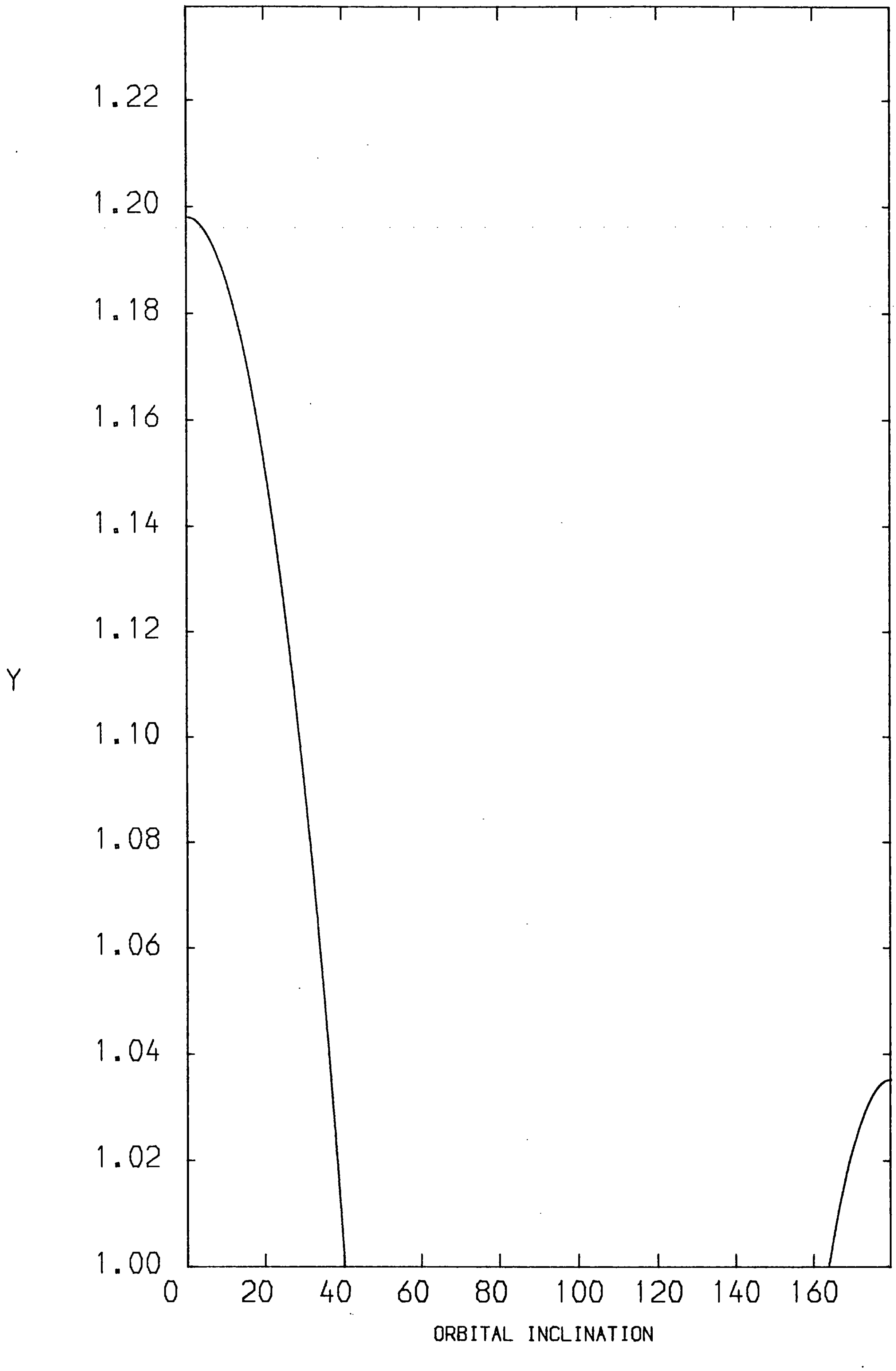
Figure 2.19



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$2\dot{\omega}-2\dot{\omega}_D-2\dot{M}_D+\dot{\Omega}=0$

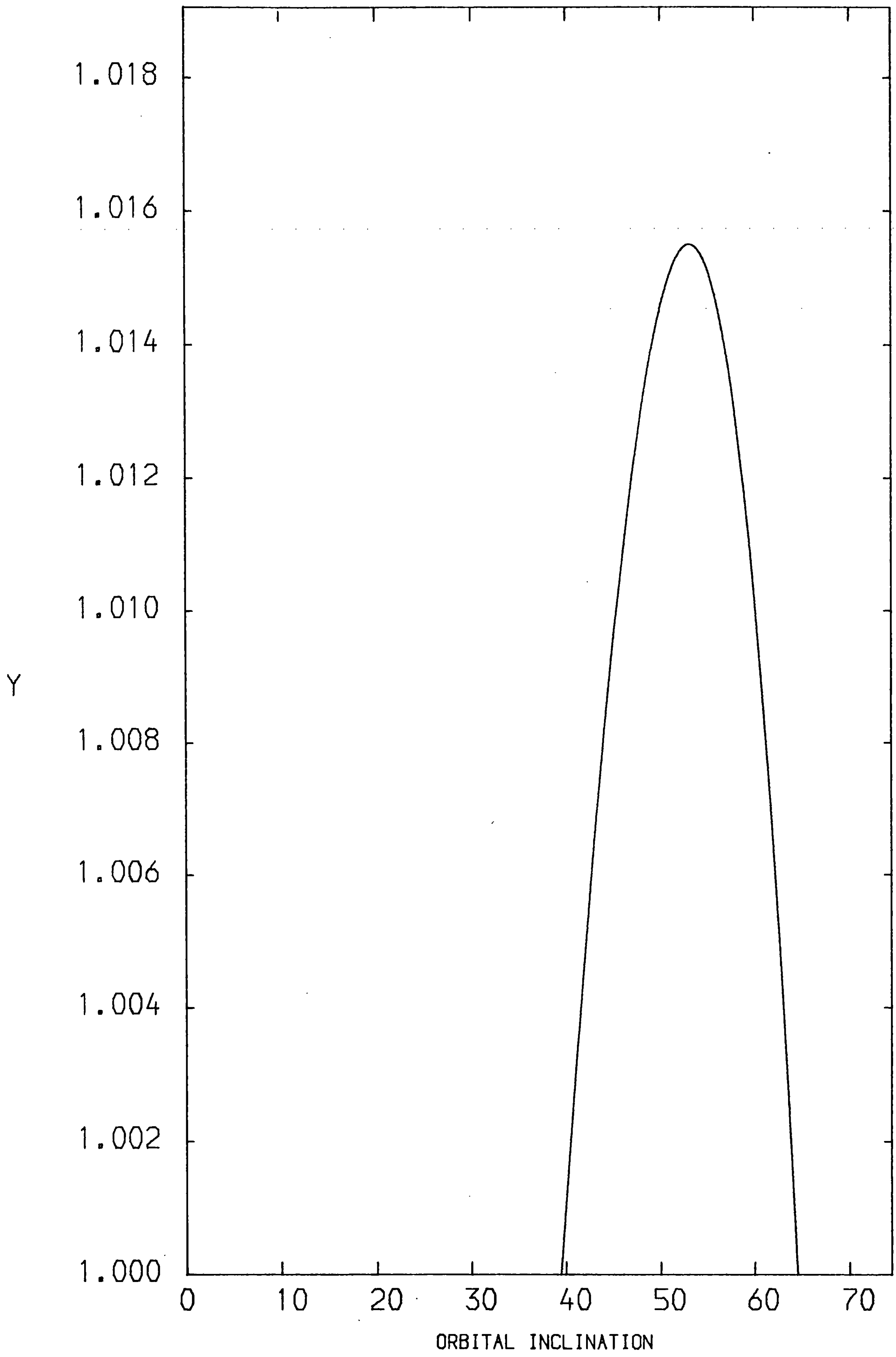
Figure 2.20



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-2\dot{\omega}+ 2\dot{\omega}_D+ 2\dot{M}_D+\dot{\Omega}=0$

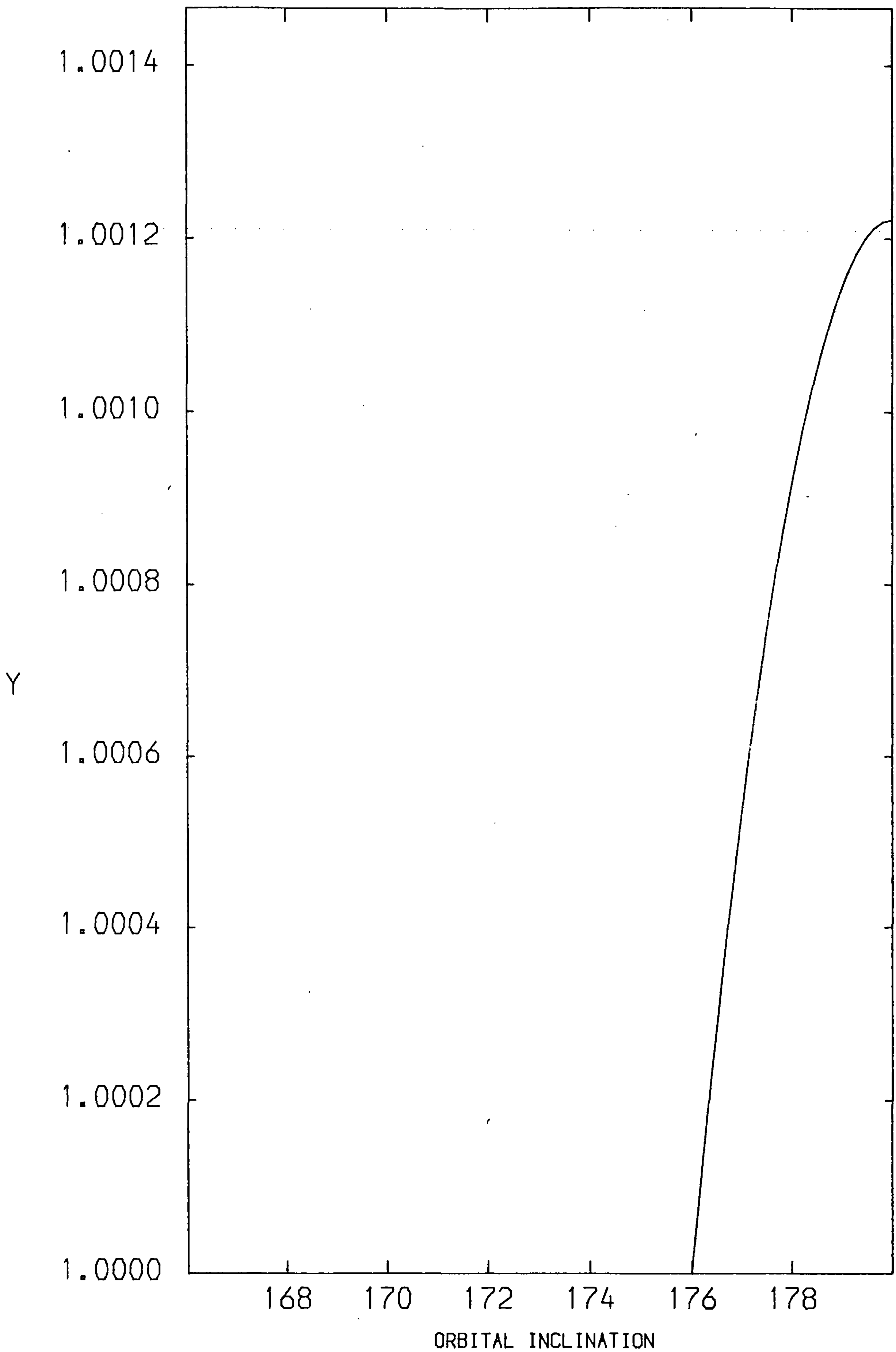
Figure 2.21



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}+\dot{\omega}_{D}+\dot{M}_{D}+ 3\dot{\Omega}=0$

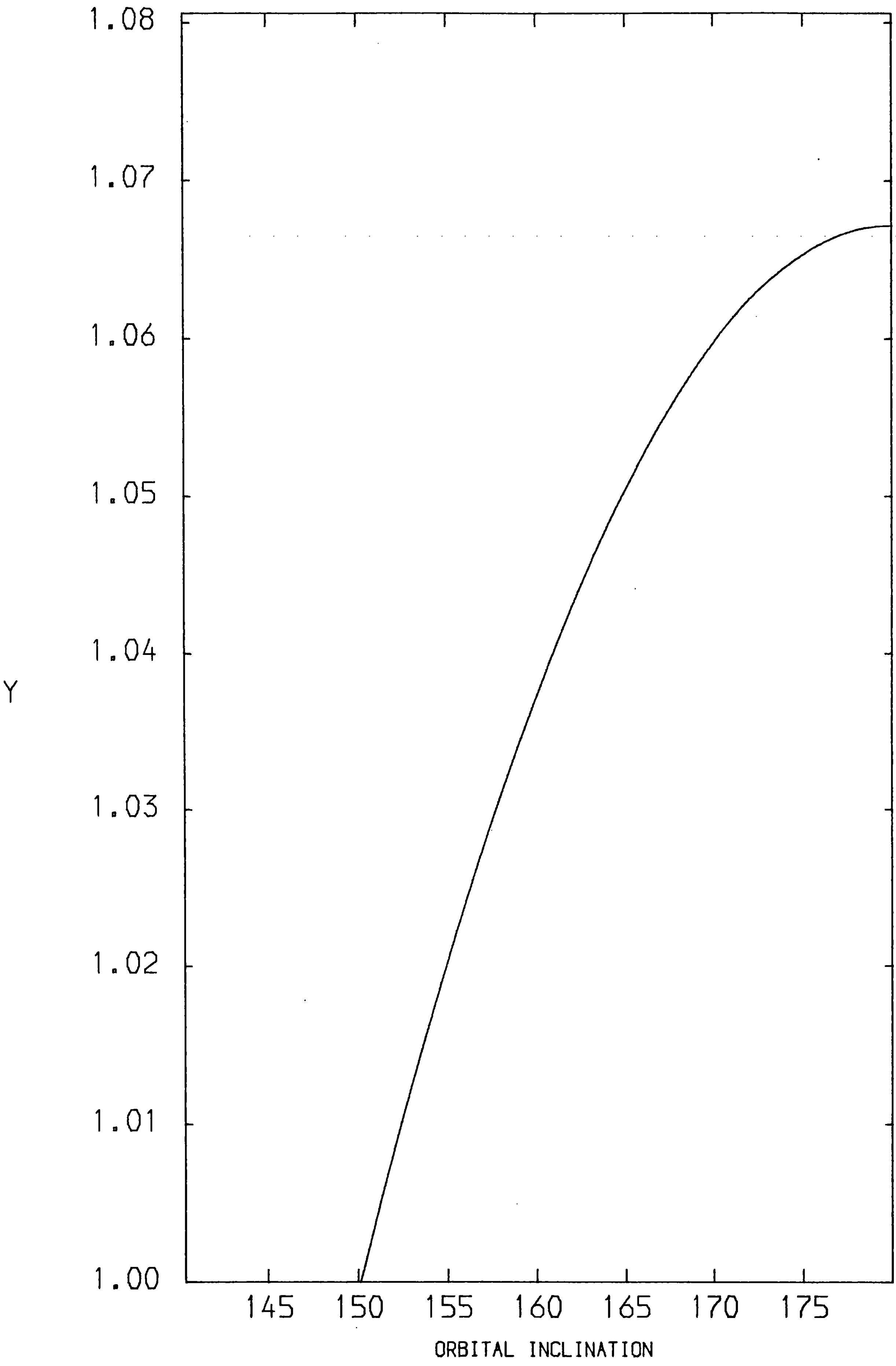
Figure 2.22



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-3\dot{\omega}_D-3\dot{M}_D+2\dot{\Omega}=0$

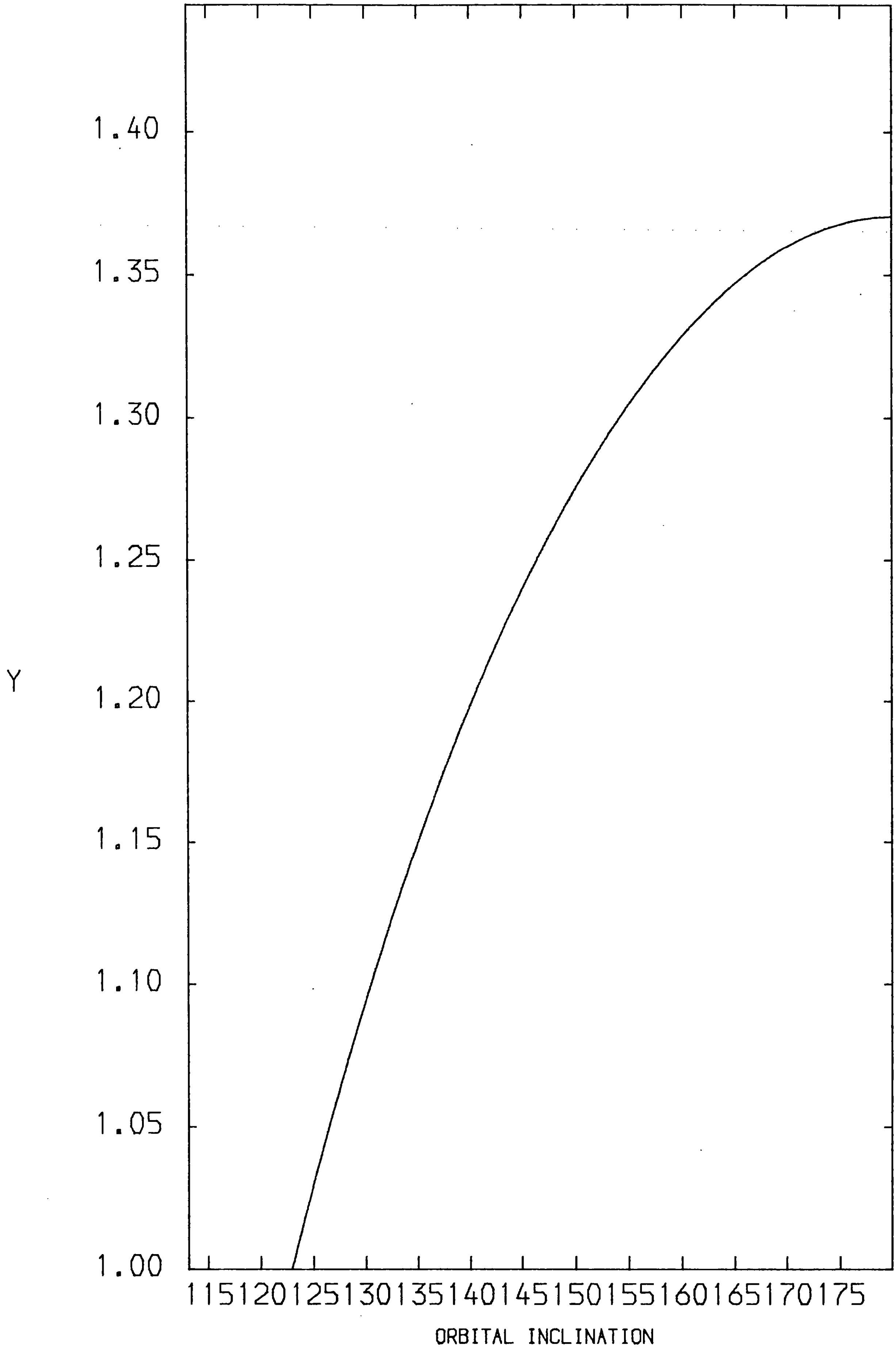
Figure 2.23



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-3\dot{\omega}_D-3\dot{M}_D+3\dot{\Omega}=0$

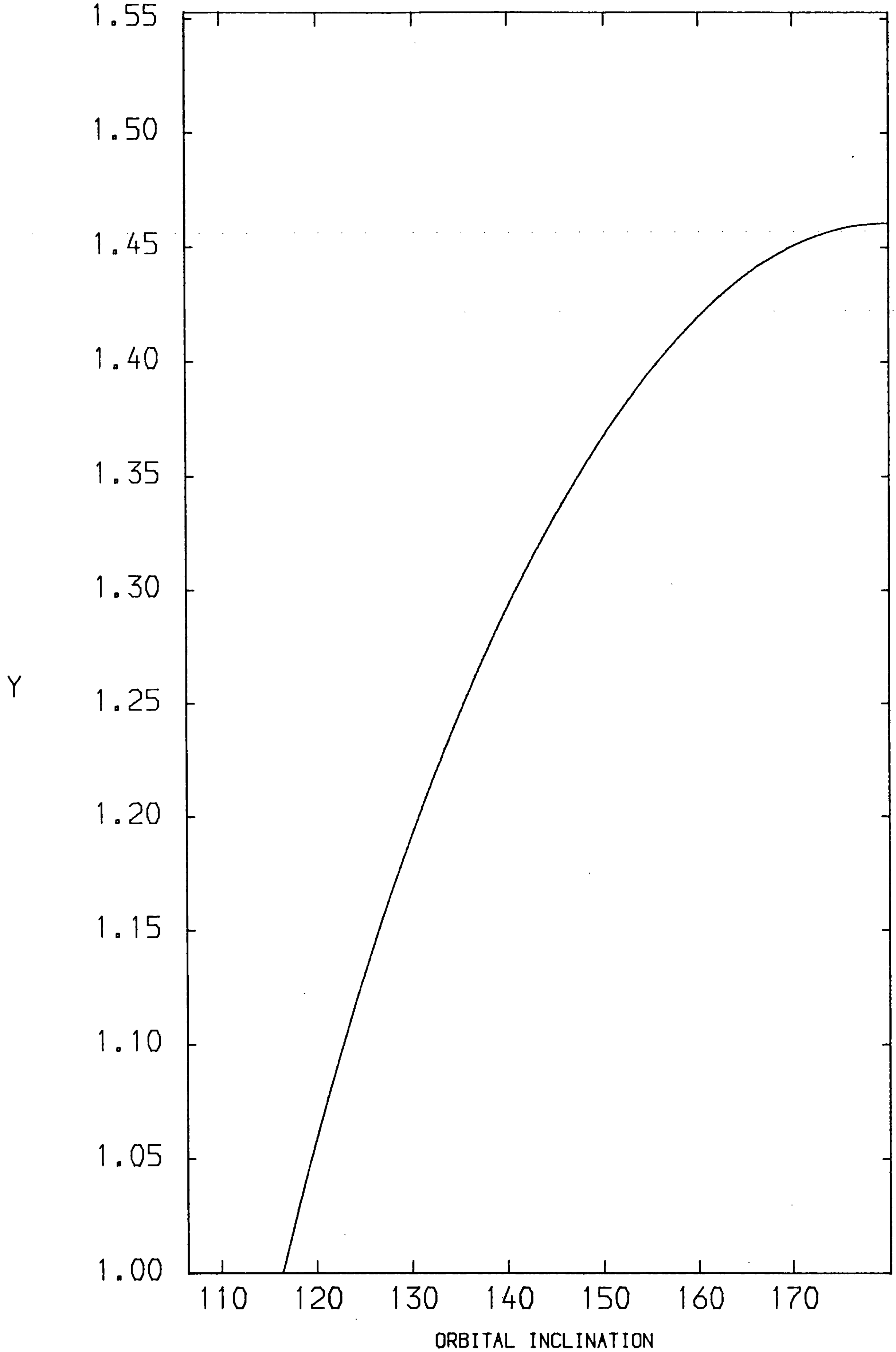
Figure 2.24



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-\dot{\omega}_D-\dot{M}_D+2\dot{\Omega}=0$

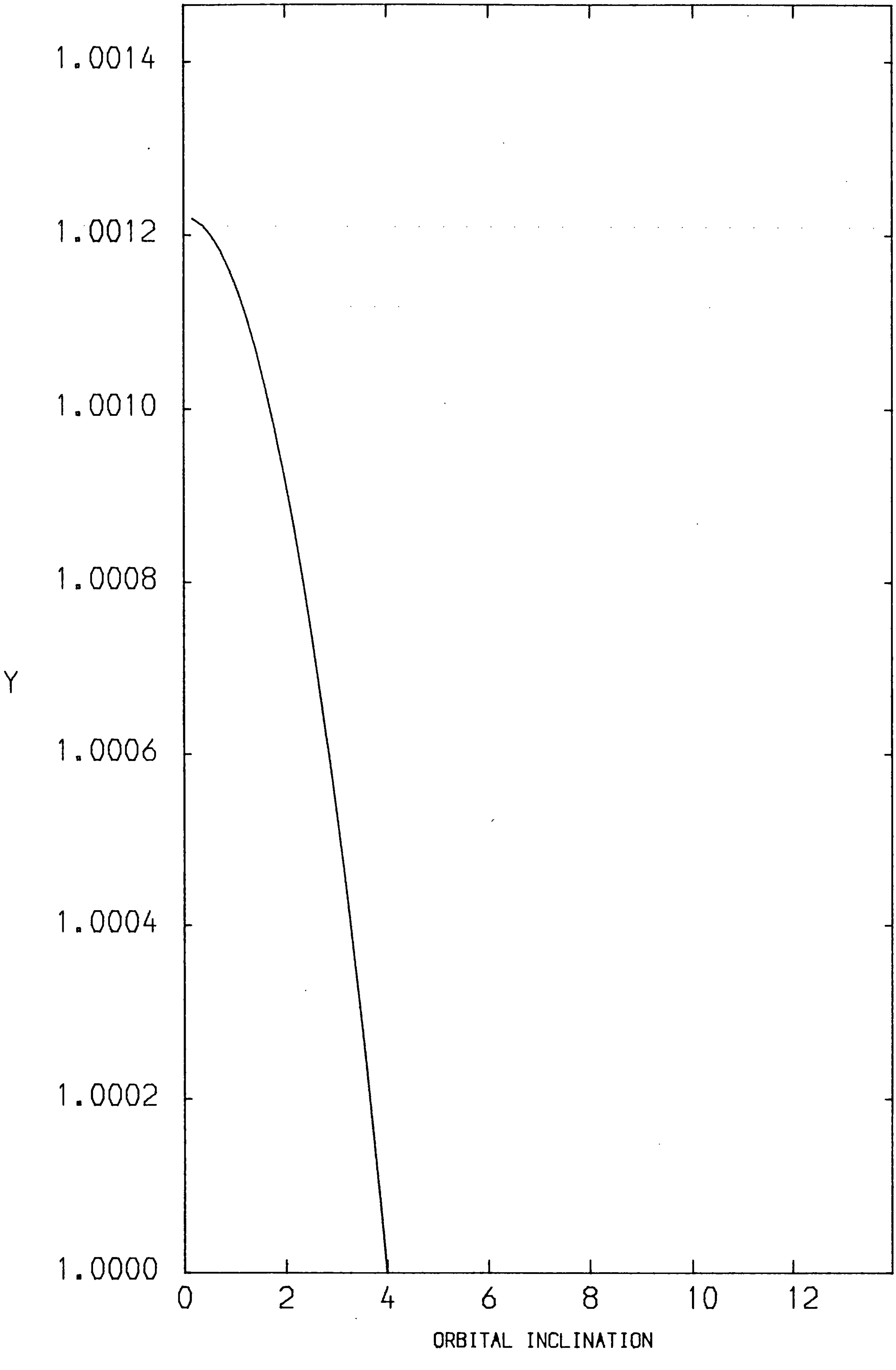
Figure 2.25



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-\dot{\omega}_D-\dot{M}_D+3\dot{\Omega}=0$

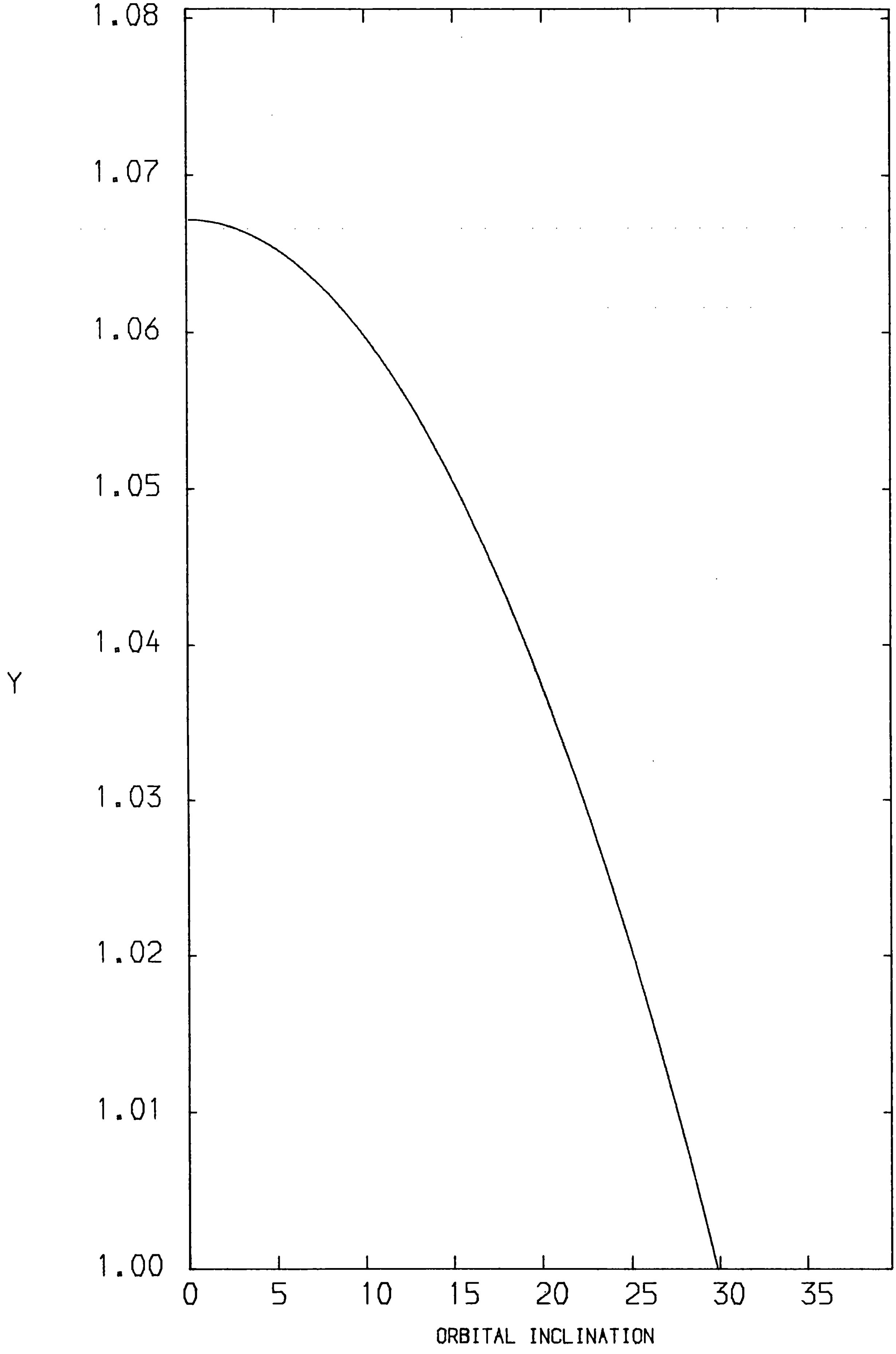
Figure 2.26



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$$-\dot{\omega} + 3\dot{\omega}_D + 3\dot{M}_D + 2\dot{\Omega} = 0$$

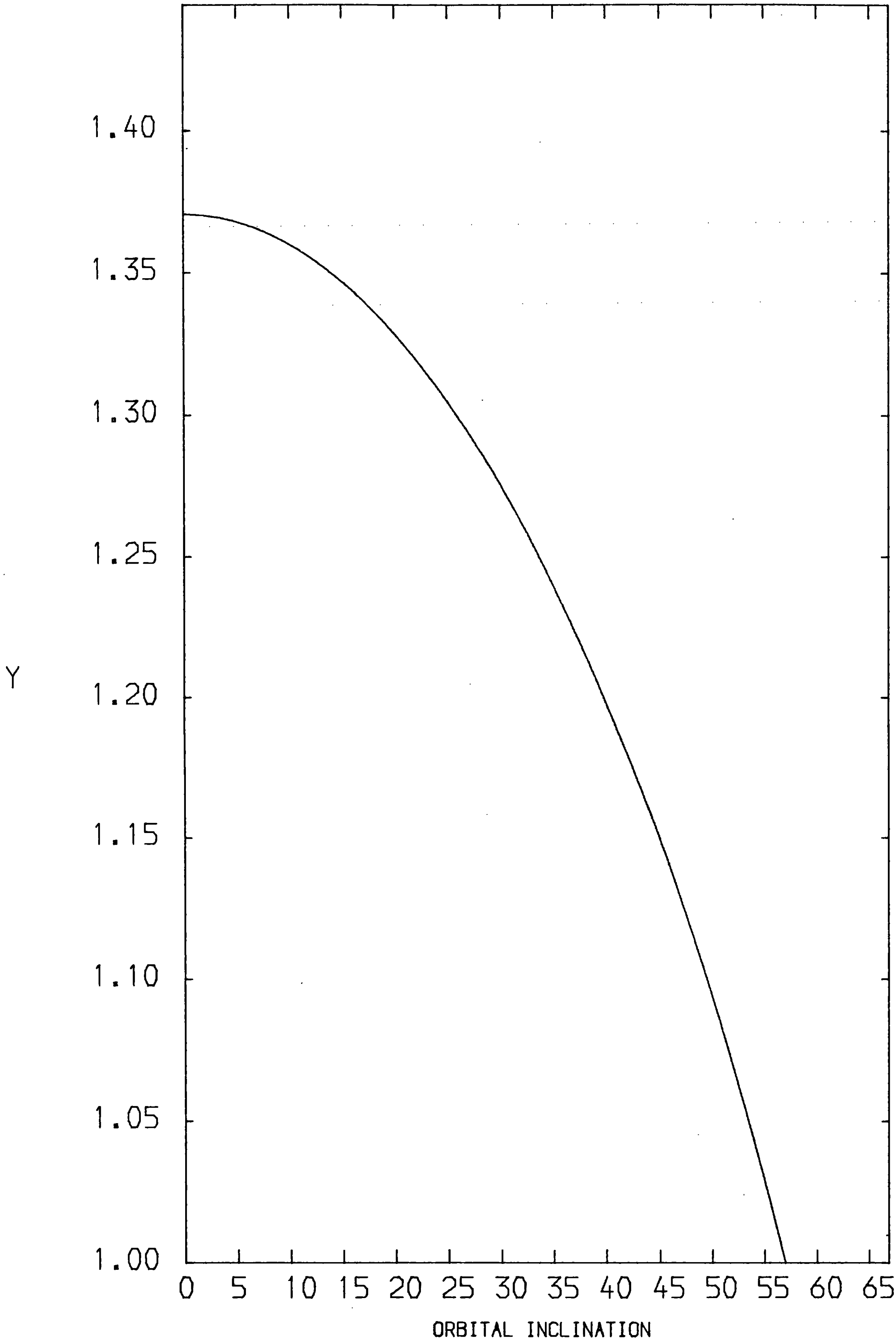
Figure 2.27



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega} + 3\dot{\omega}_D + 3\dot{M}_D + 3\dot{\Omega} = 0$

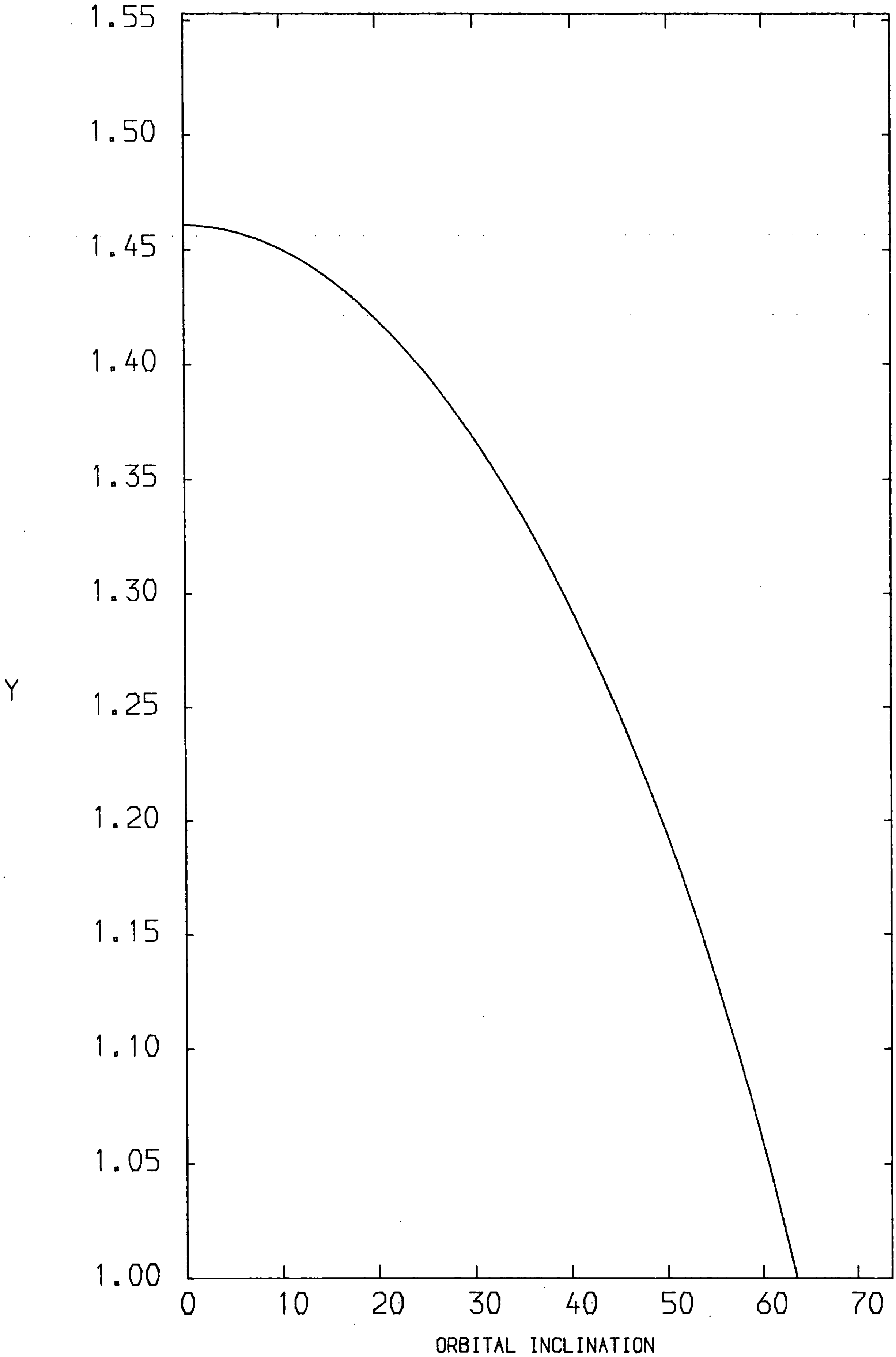
Figure 2.28



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D+ 2\dot{\Omega}=0$

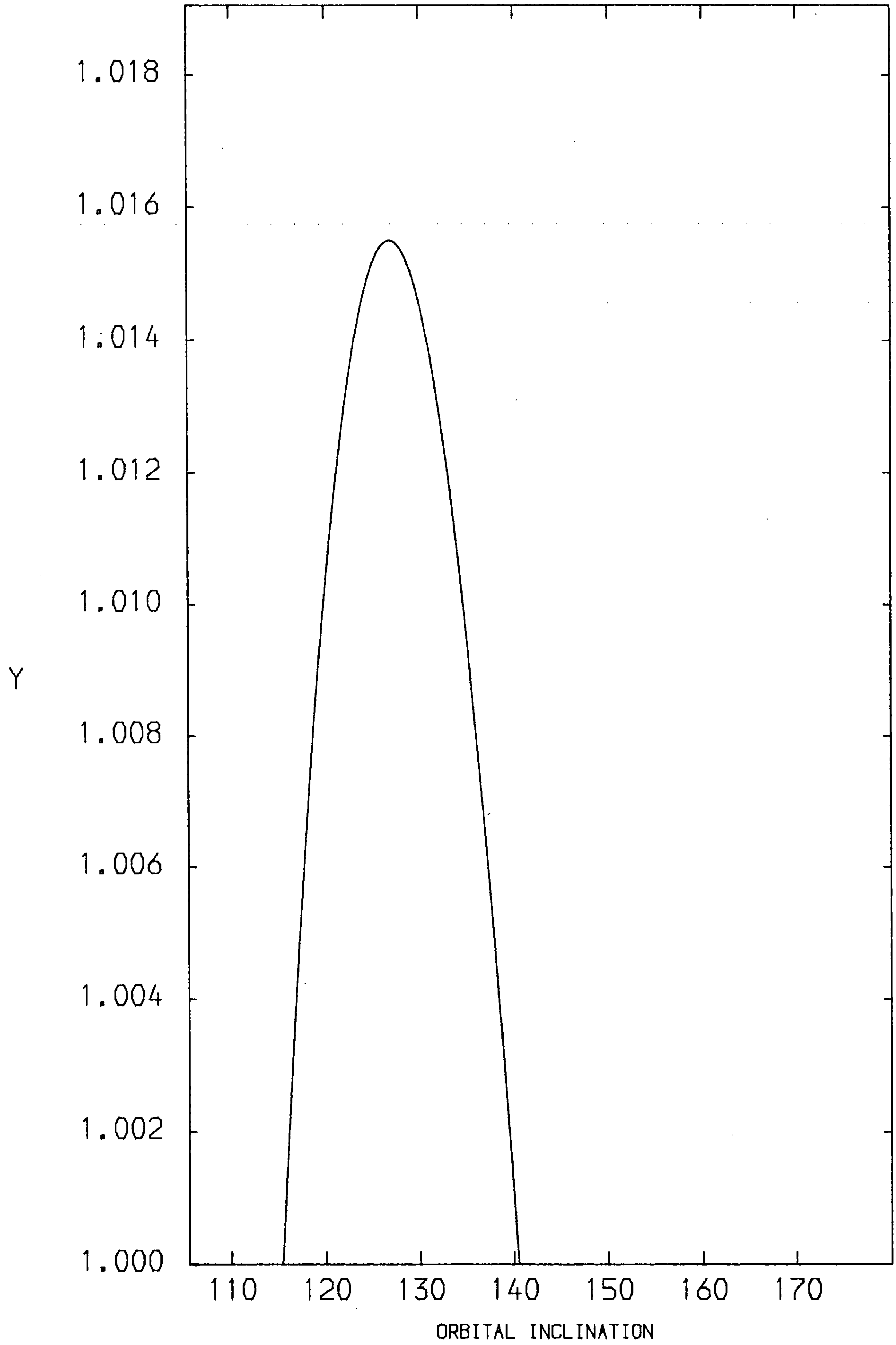
Figure 2.29



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D+3\dot{\Omega}=0$

Figure 2.30



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}-\dot{\omega}_{D}-\dot{M}_{D}+3\dot{\Omega}=0$

Figure 2.31

2.4(7) The Type (7) Lunar Gravity Commensurability

$\dot{\psi}_7 = \beta\dot{\Omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$

A satellite will exist in the lunar gravity commensurability $\dot{\psi}_7 \approx 0$ if its orbital elements satisfy the equation

$$9.97\beta \quad \cos i = (\eta\dot{\omega}_D + k\dot{\Omega}_D)\, y^{3.5} \tag{2.70}$$

The maximum value of y occurs at an inclination of $i = 0^\circ$, if $(\eta\dot{\omega}_D + k\dot{\Omega}_D) > 0$, and an inclination of 180°, if $(\eta\dot{\omega}_D + k\dot{\Omega}_D) < 0$. Such a resonance will be possible if $y_{max} > 1$, i.e. if

$$9.97\beta > |\eta\dot{\omega}_D + k\dot{\Omega}_D| \tag{2.71}$$

Since $\dot{\omega}_D$ and $\dot{\Omega}_D$ for the moon are approximately 0.1 deg/day and -0.05 deg/day, respectively, $|\eta\dot{\omega}_D + k\dot{\Omega}_D|$ will, in general, be small, provided η and k are small. However, large values of $|\eta|$ and $|k|$ imply large n values for the predominant resonance terms, and, hence, small amplitude factors. Consequently, all type (7) commensurabilities, except the very weakest, exist.

A satellite in a lunar gravity commensurability $\dot{\psi}_7 \approx 0$ is in resonance with those $\Phi^{(+)}$ terms in the lunar disturbing function expansion (2.4) for which

$$\begin{aligned}
(n-2p)^+ &= 0\\
q^+ &= 0\\
(n-2h+j)^+ &= 0\\
(n-2h^+) &= \eta\delta\\
m^+ &= \beta\delta\\
s^+ &= k\delta
\end{aligned} \tag{2.72}$$

$\delta > 0, \; k \geqslant 0$

The arguments of the resonant terms are of the form $\delta(\psi_7^{+})_{MOON}$, where $(\psi_7^{+})_{MOON} = \beta\Omega + k\Omega_D + \eta\omega_D$. The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the predominant $\Phi^{(+)}$ resonant terms of a $\dot{\psi}_7 \approx 0$ commensurability are given in tables 2.8(a) and 2.8(b).

Similarly, the $\Phi^{(-)}$ resonant terms are such that

$$
\begin{aligned}
(n-2p)^- &= 0 \\
q^- &= 0 \\
(n-2h+j)^- &= 0 \\
(n-2h)^- &= -\eta v \\
m^- &= \beta v \\
s^- &= -kv
\end{aligned}
\tag{2.73}
$$

$$v > 0,\ k \leqslant 0$$

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values for the predominant $\Phi^{(-)}$ resonant terms of type (7) are given in tables 2.8(c) and 2.8(d).

The most important commensurabilities of the type $\dot{\psi}_7 \approx 0$ are those for which the amplitude factor $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|}$ is an absolute minimum, i.e. $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|} = \left(\frac{a}{a_D}\right)^2$. The commensurabilities which satisfy such a requirement can be obtained from tables 2.8(a) - 2.8(d), and are found to be the following

$$
\begin{aligned}
\dot{\Omega} \pm \dot{\Omega}_D &\approx 0 \\
\dot{\Omega} \pm 2\dot{\Omega}_D &\approx 0 \\
2\dot{\Omega} \pm \dot{\Omega}_D &\approx 0
\end{aligned}
\tag{2.74}
$$

The orbital elements of the satellites which exist in these commensurabilities are given in figures (2.32) - (2.37).

Until now it has been assumed that $\eta\dot{\omega}_D + k\dot{\Omega}_D \neq 0$. However, if k and η are such that $2\eta = +k$, then $\eta\dot{\omega}_D + k\dot{\Omega}_D = 0$, since $\dot{\omega}_D \approx 0.16$ deg/day and $\dot{\Omega}_D \approx -0.05$ deg/day. The two lunar gravity commensurabilities $\dot{\psi}_1 \approx 0$ and $\dot{\psi}_7 \approx 0$ will therefore be equivalent, as will their corresponding resonance terms when $3\eta = +k$. The most important occurrences of such a situation arise when $\eta = -1$ and $k = -3$ and $\eta = 1$ and $k = +3$. The amplitude factors of the predominant resonant terms will be of the order $\left(\frac{a}{a_D}\right)^6 e_D^{\,2}$; which, for a close Earth satellite is about 10^{-9} times smaller than the amplitude factors of the most important type (1) and type (7) lunar gravity commensurabilities.

Table 2.8(a)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_7 = \beta\dot{\Omega}_D \pm \eta\dot{\omega}_D \pm k\dot{\Omega}_D \approx 0$ when β and/or $k \geqslant |\eta|$

RESTRICTIONS ON β, k and η	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\beta > k$ β even η +ve η even k +ve	β	β	$\beta/2$	0	$(\beta-\eta)/2$	$-\eta$	k	1
$\beta \geqslant k$ β odd η +ve η even k +ve	$\beta+1$	β	$(\beta+1)/2$	0	$\frac{(\beta+1-\eta)}{2}$	$-\eta$	k	1
$\beta \geqslant k$ η+ve k+ve β even η odd or β odd η odd	2β	2β	β	0	$\beta-\eta$	-2η	2k	2
$\beta > k$ β even η -ve $\|\eta\|$ even k +ve	β	β	$\beta/2$	0	$(\beta+\|\eta\|)/2$	$\|\eta\|$	k	1
$\beta \geqslant k$ β odd η -ve η even k +ve	$\beta+1$	β	$(\beta+1)/2$	0	$\frac{(\beta+1+\|\eta\|)}{2}$	$\|\eta\|$	k	1
$\beta \geqslant k$ η-ve k+ve β even $\|\eta\|$ odd or β odd, $\|\eta\|$ odd	2β	2β	β	0	$\beta+\|\eta\|$	$2\|\eta\|$	2k	2
k $> \beta$ k even η +ve η even k +ve	k	β	k/2	0	$(k-\eta)/2$	$-\eta$	k	1
k $> \beta$ k odd η +ve η even k +ve	k+1	β	(k+1)/2	0	$(k+1-\eta)/2$	$-\eta$	k	1
k $>\beta$ η+ve k+ve k odd η odd or k even η odd	2k	2β	k	0	$k-\eta$	-2η	2k	2

Table 2.8(a) continued

RESTRICTIONS ON β, k and η	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$k > \beta$ η-ve k+ve k even η even	k	β	k/2	0	$(k+\|\eta\|)/2$	$\|\eta\|$	k	1
$k > \beta$ η-ve k+ve k odd $\|\eta\|$ even	k+1	β	(k+1)/2	0	$\frac{(k+1+\|\eta\|)}{2}$	$\|\eta\|$	k	1
$k > \beta$ η-ve k+ve k odd $\|\eta\|$ odd or $\|\eta\|$ odd, k even	2k	2β	k	0	$k+\|\eta\|$	$2\|\eta\|$	2k	2

Table 2.8(b)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_7 = \beta\dot{\Omega} + k\dot{\Omega}_D + \eta\dot{\omega}_D \approx 0$ when $|\eta| > \beta$ and k

RESTRICTIONS ON β, η and k		n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
η +ve	η even k+ve	η	β	$\eta/2$	0	0	$-\eta$	k	1
η +ve	η odd k+ve	2η	2β	η	0	0	-2η	2k	2
η −ve	$\|\eta\|$ even k+ve	$\|\eta\|$	β	$\|\eta\|/2$	0	$\|\eta\|$	$\|\eta\|$	k	1
η −ve	$\|\eta\|$ odd k+ve	$2\|\eta\|$	2β	$\|\eta\|$	0	$\|\eta\|$	$2\|\eta\|$	2k	2

Table 2.8(c)

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_7 = \beta\dot{\Omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$ when β and/or $|k| \geqslant |\eta|$

RESTRICTIONS ON β, k and η	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
$\beta \geqslant \|k\|$ η +ve, k-ve β even η even	β	β	$\beta/2$	0	$(\beta+\eta)/2$	η	$\|k\|$	1
$\beta \geqslant \|k\|$ η +ve, k-ve β odd η even	$\beta+1$	β	$(\beta+1)/2$	0	$(\beta+1+\eta)/2$	η	$\|k\|$	1
$\beta \geqslant \|k\|$ η +ve, k-ve η odd β even, or η odd β odd	2β	2β	β	0	$\beta+\eta$	2η	$2\|k\|$	2
$\beta \geqslant \|k\|$ η -ve, k-ve β even $\|\eta\|$ even	β	β	$\beta/2$	0	$(\beta-\|\eta\|)/2$	$-\|\eta\|$	$\|k\|$	1
$\beta \geqslant \|k\|$ η -ve, k-ve β odd $\|\eta\|$ even	$\beta+1$	β	$(\beta+1)/2$	0	$\dfrac{(\beta+1-\|\eta\|)}{2}$	$-\|\eta\|$	$\|k\|$	1
$\beta \geqslant \|k\|$ η -ve, k-ve η odd β even, or η odd β odd	2β	2β	β	0	$\beta-\|\eta\|$	$-2\|\eta\|$	$2\|k\|$	2
$\|k\| > \beta$ η +ve, k-ve η even $\|k\|$ even	$\|k\|$	β	$\|k\|/2$	0	$(\|k\|+\eta)/2$	η	$\|k\|$	1
$\|k\| > \beta$ η +ve, k-ve $\|k\|$ odd, η even	$\|k\|+1$	β	$(\|k\|+1)/2$	0	$\dfrac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	1
$\|k\|>\beta$ η +ve, k-ve η odd $\|k\|$ even or η odd $\|k\|$ odd	$2\|k\|$	2β	$\|k\|$	0	$\|k\|+\eta$	2η	$2\|k\|$	2

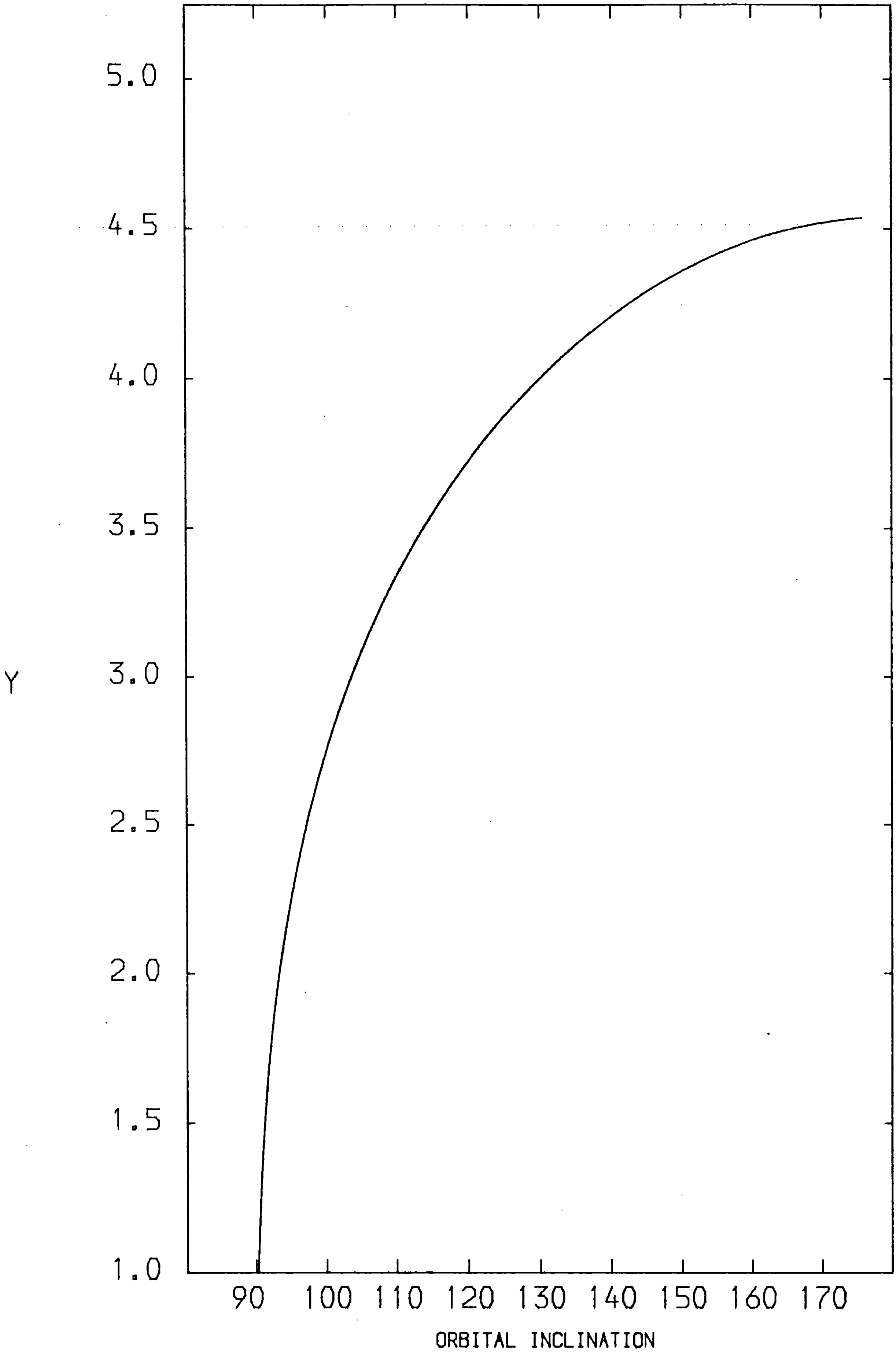
Table 2.8(c) continued

RESTRICTIONS ON β, k and η	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\lvert k\rvert > \beta$, η-ve, k-ve $\lvert k\rvert$ even η even	$\lvert k\rvert$	β	$\lvert k\rvert/2$	0	$\frac{(\lvert k\rvert-\lvert\eta\rvert)}{2}$	$-\lvert\eta\rvert$	$\lvert k\rvert$	1
$\lvert k\rvert > \beta$, η-ve, k-ve $\lvert k\rvert$ odd η even	$\lvert k\rvert+1$	β	$(\lvert k\rvert+1)/2$	0	$(\lvert k\rvert+1-\lvert\eta\rvert)/2$	$-\lvert\eta\rvert$	$\lvert k\rvert$	1
$\lvert k\rvert > \beta$, η-ve, k-ve $\lvert k\rvert$ odd $\lvert\eta\rvert$ odd or $\lvert k\rvert$ even, $\lvert\eta\rvert$ odd	$2\lvert k\rvert$	2β	$\lvert k\rvert$	0	$\lvert k\rvert-\lvert\eta\rvert$	$-2\lvert\eta\rvert$	$2\lvert k\rvert$	2

Table 2.8(d)

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_7 = \beta\dot{\Omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$ when $|\eta| > \beta$ and $|k|$

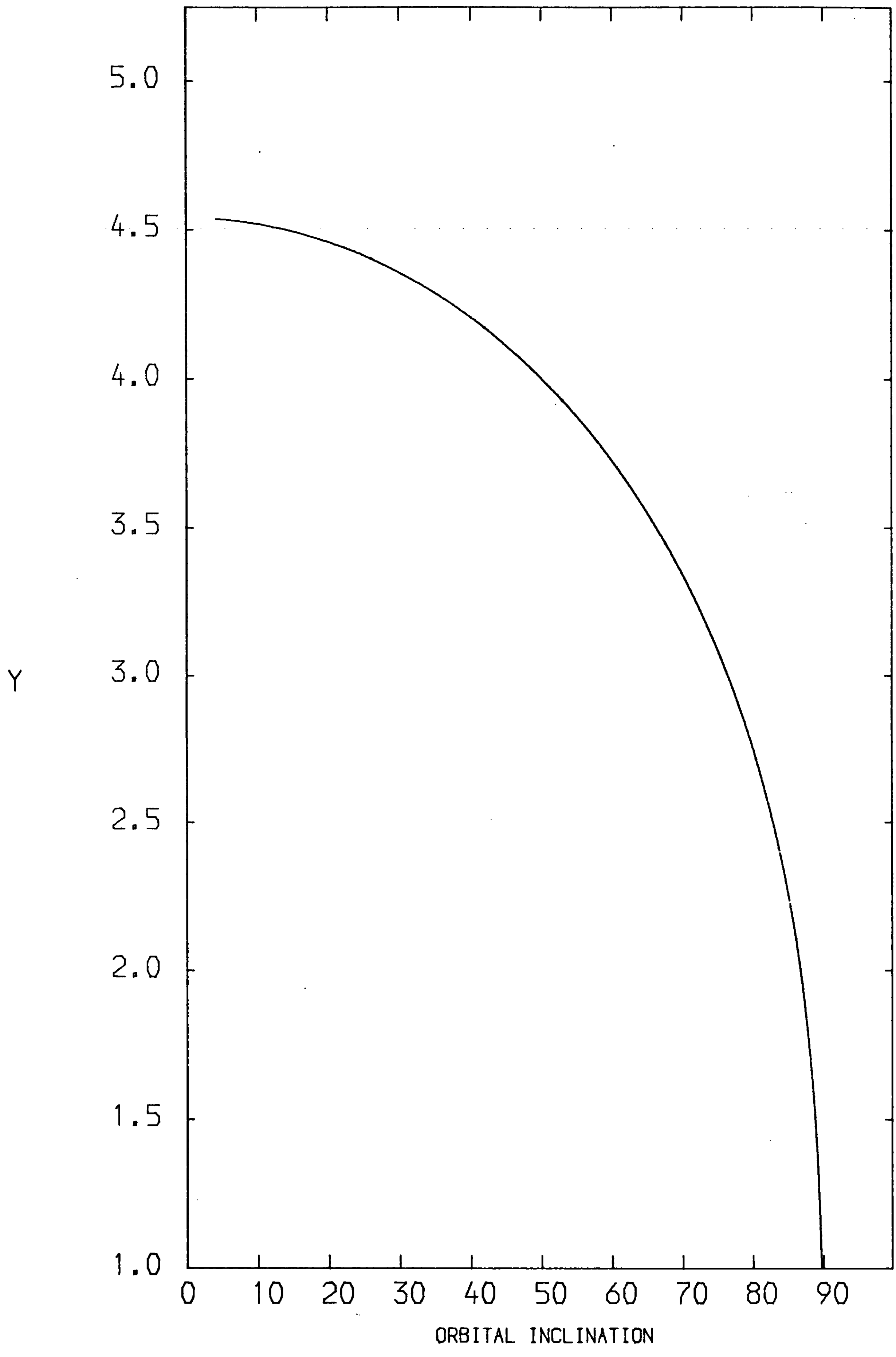
RESTRICTIONS ON β, η and k		n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
η +ve k-ve	η even	η	β	$\eta/2$	0	η	η	$\|k\|$	1
η +ve k-ve	η odd	2η	2β	η	0	2η	2η	$2\|k\|$	2
η -ve k-ve	$\|\eta\|$ even	$\|\eta\|$	β	$\|\eta\|/2$	0	0	$-\|\eta\|$	$\|k\|$	1
η -ve k-ve	$\|\eta\|$ odd	$2\|\eta\|$	2β	$\|\eta\|$	0	0	$-2\|\eta\|$	$2\|k\|$	2



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\Omega}+\dot{\Omega}_D=0$

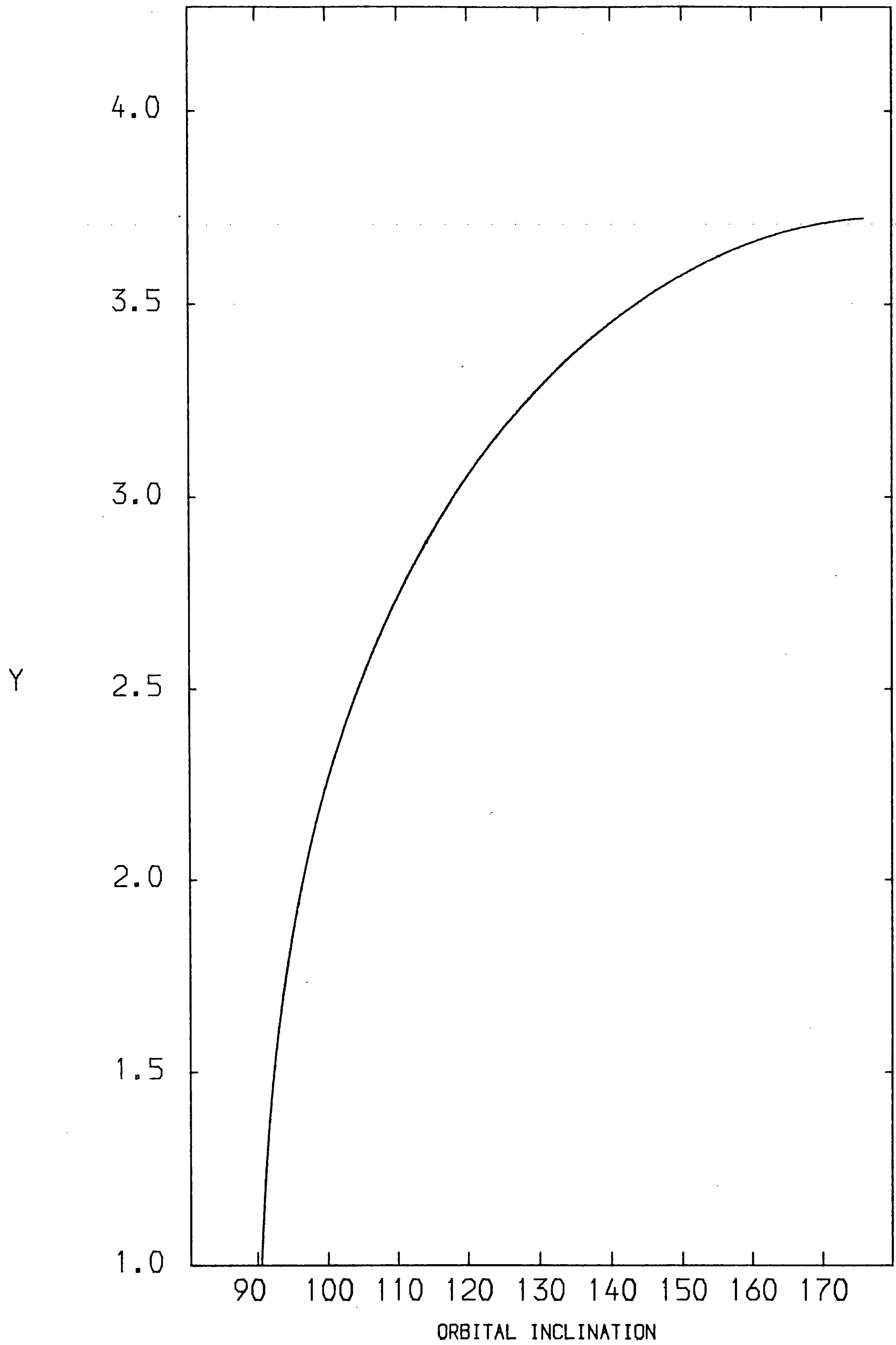
Figure 2.32



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\Omega}-\dot{\Omega}_D=0$

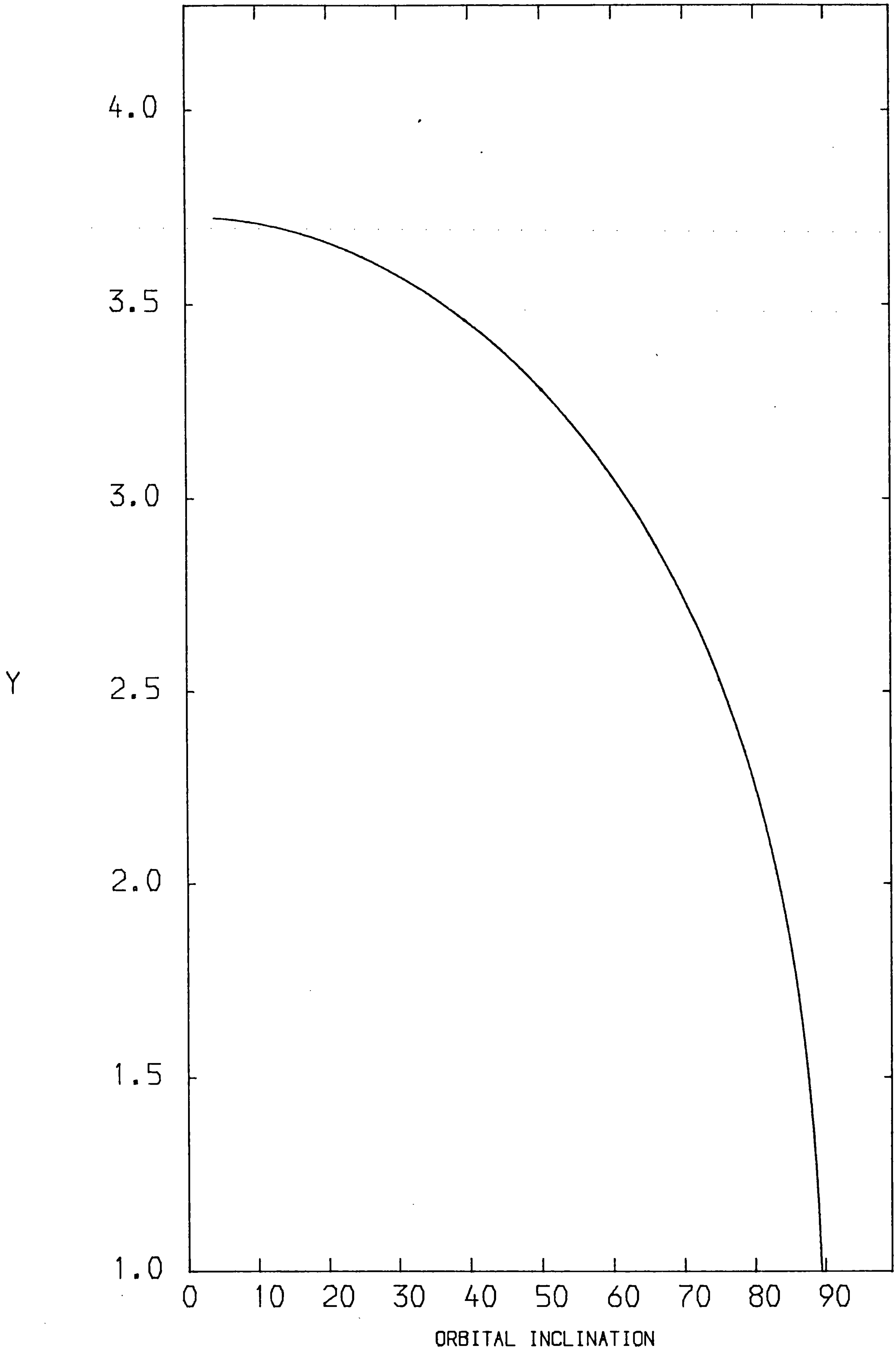
Figure 2.33



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\Omega} + 2\dot{\Omega}_D = 0$

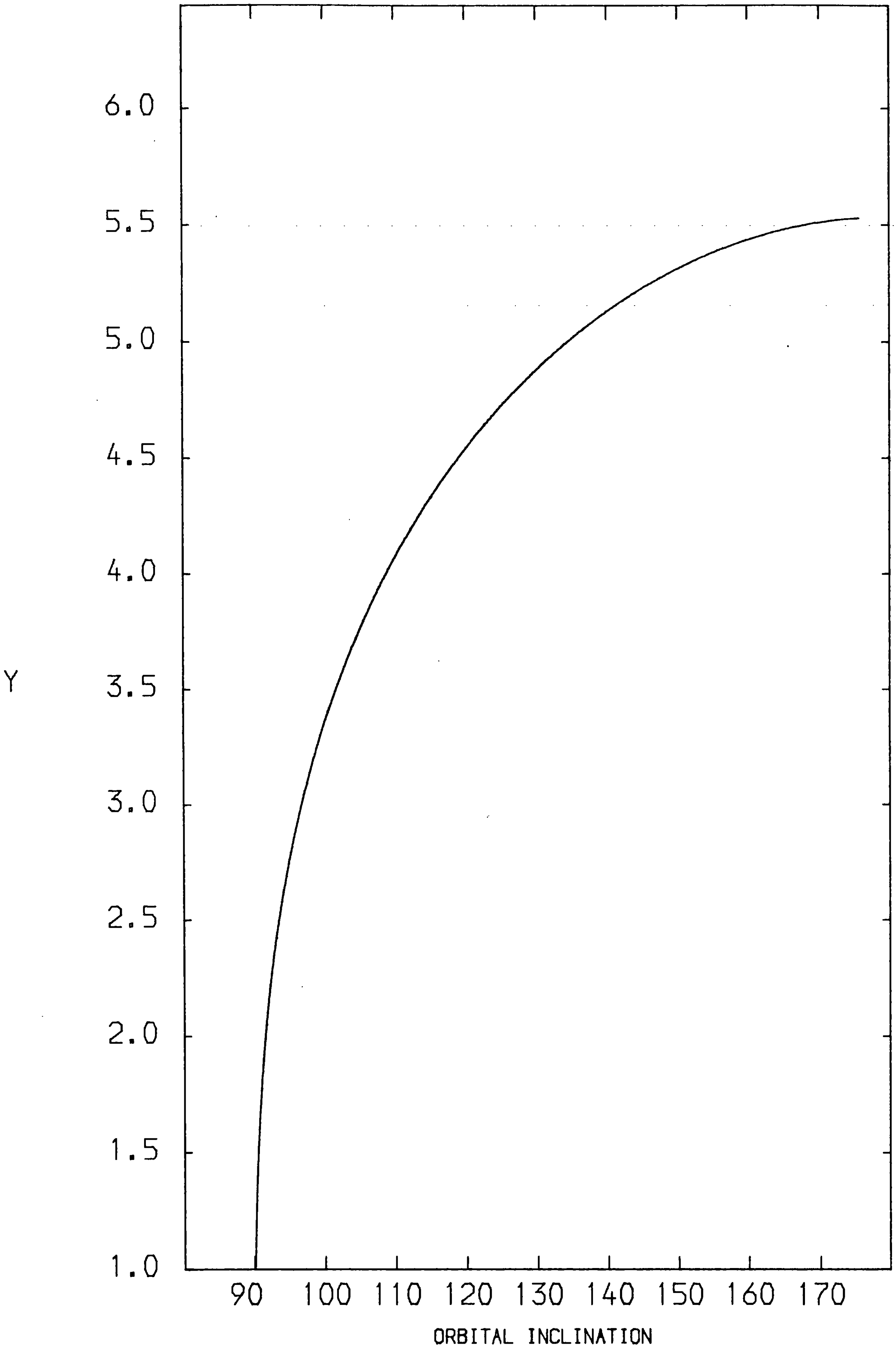
Figure 2.34



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\Omega}-2\dot{\Omega}_D=0$

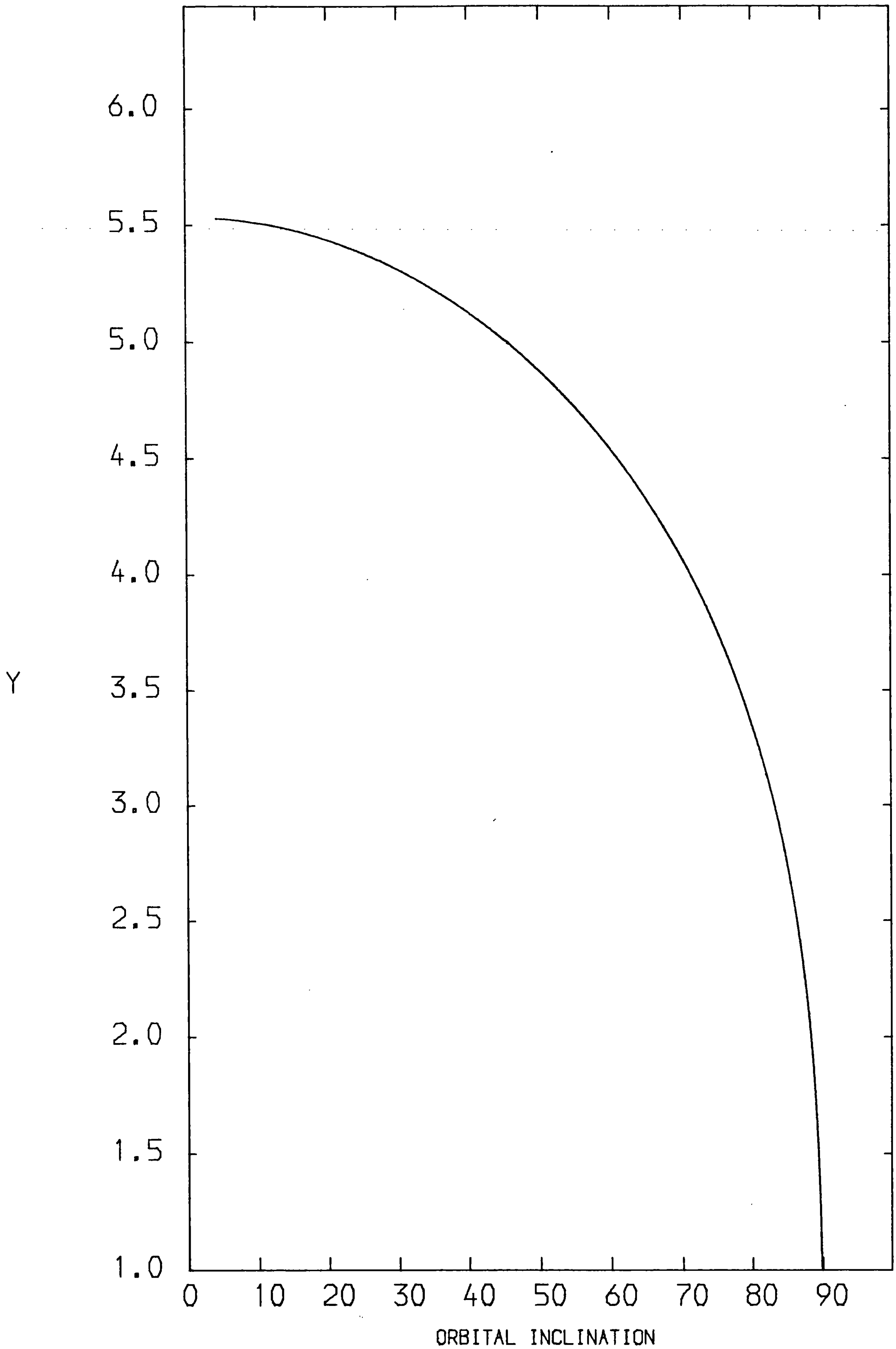
Figure 2.35



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$2\dot{\Omega}+\dot{\Omega}_D=0$

Figure 2.36



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$2\dot{\Omega}-\dot{\Omega}_D=0$

Figure 2.37

2.4(8) <u>The Lunar Gravity Commensurability $\dot{\psi}_8 = \alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$</u>

A satellite in the lunar gravity commensurability $\dot{\psi}_8 \approx 0$ is in resonance with those $\Phi^{(+)}$ terms in the lunar disturbing function expansion (2.4) for which

$$\begin{aligned} (n-2p)^+ &= \alpha\delta \\ q^+ &= -\alpha\delta \\ (n-2h)^+ &= \eta\delta \\ j^+ &= -\eta\delta \\ s^+ &= k\delta \\ m^+ &= 0 \end{aligned} \tag{2.75}$$

The arguments of the resonant terms are of the form $\delta(\psi_8^+)_{MOON}$, where $(\psi_8^+)_{MOON} = \alpha\omega + \eta\omega_D + k\Omega_D$. The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the predominant $\Phi^{(+)}$ resonant terms of a type (8) commensurability are given in tables 2.9(a) to 2.9(f).

Similarly, the $\Phi^{(-)}$ resonant terms are such that

$$\begin{aligned} (n-2p)^- &= \alpha v \\ q^- &= -\alpha v \\ (n-2h)^- &= -\eta v \\ j^- &= \eta v \\ s^- &= -kv \\ m^- &= 0 \end{aligned} \tag{2.76}$$

The corresponding values for the predominant resonant terms are given in tables 2.9(g) - 2.9(m).

In order for a satellite to exist in a type (8) lunar gravity commensurability, the orbital elements must satisfy the equation

$$4.98\,\alpha\ (5\cos^2 i - 1) + (\eta\dot{\omega}_D + k\dot{\Omega}_D)\, y^{3.5} \approx 0 \tag{2.77}$$

However, the commensurability $\dot{\psi}_8 \approx 0$ will only occur if the maximum value of y is greater than unity, i.e. if

$$y_{max} = -\frac{19.92\,\alpha}{(\eta\dot{\omega}_D + k\dot{\Omega}_D)} > 1 \qquad \alpha/(\eta\dot{\omega}_D + k\dot{\Omega}_D) < 0$$

(2.78

or $$y_{max} = \frac{4.98\,\alpha}{(\eta\dot{\omega}_D + k\dot{\Omega}_D)} > 1 \qquad \alpha/(\eta\dot{\omega}_D + k\dot{\Omega}_D) > 0$$

Since $(\eta\dot{\omega}_D + k\dot{\Omega}_D)$ is usually small (except for some very weak resonances), the inequalities (2.78) will normally be satisfied, and therefore all strong type (8) resonances will occur.

The most important type (8) commensurabilities are the ones for which

$$\dot{\omega} \pm \dot{\Omega}_D \approx 0$$

$$2\dot{\omega} \pm \dot{\Omega}_D \approx 0 \qquad (2.79)$$

the amplitude factors being of order $\left(\frac{a}{a_D}\right)^2 e^2$. However, if $\left(\frac{ae_D}{a_D e}\right) > 1$, then the most important lunar gravity commensurabilities of type (8) are those with amplitude factors of order $\left(\frac{a}{a_D}\right)^3 ee_D$, i.e. the set

$$\pm\, \dot{\omega} + \dot{\omega}_D \approx 0$$

$$\pm\, \dot{\omega} + \dot{\omega}_D \pm \dot{\Omega}_D \approx 0$$

$$\pm\, \dot{\omega} + \dot{\omega}_D \pm 2\dot{\Omega}_D \approx 0 \qquad (2.80)$$

$$\pm\, \dot{\omega} + \dot{\omega}_D \pm 3\dot{\Omega}_D \approx 0$$

For close Earth satellites a/a_D is approximately 1/50, therefore the set

(2.80) will only predominate over the set (2.79) for eccentricities, which are less than about 10^{-3}. The two commensurabilities $\pm \dot{\omega} + \dot{\omega}_D + 3\dot{\Omega}_D \approx 0$ are of the type discussed in the previous section, where η and k are such that $\eta \dot{\omega}_D + k\dot{\Omega}_D \approx 0$.

Table 2.9(a)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in the Lunar Gravity Commensurability $\dot{\psi}_8 = \alpha\dot{\omega} + \eta\dot{\omega}_D \approx 0$

RESTRICTIONS ON α and η	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve η +ve			0	$-\alpha$	$(\alpha-\eta)/2$	$-\eta$	0	1
$\alpha \geqslant \eta$	α^*	0						
$\alpha + \eta$ even			α	α	$(\alpha+\eta)/2$	η	0	-1
α +ve η +ve			0	-2α	$\alpha-\eta$	-2η	0	2
$\alpha > \eta$	2α	0						
$\alpha + \eta$ odd			2α	2α	$\alpha+\eta$	2η	0	-2
α +ve η +ve			$(\eta-\alpha)/2$	$-\alpha$	0	$-\eta$	0	1
$\eta > \alpha$	η	0						
$\alpha + \eta$ even			$(\eta+\alpha)/2$	α	η	η	0	-1
α +ve η +ve			$\eta-\alpha$	-2α	0	-2η	0	2
$\eta > \alpha$	2η	0						
$\alpha + \eta$ odd			$\eta+\alpha$	2α	2η	2η	0	-2
α -ve η +ve			$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|-\eta)/2$	$-\eta$	0	1
$\|\alpha\| \geqslant \eta$	$\|\alpha\|^*$	0						
$\|\alpha\| + \eta$ even			0	$-\|\alpha\|$	$(\|\alpha\|+\eta)/2$	η	0	-1
α -ve η +ve			$2\|\alpha\|$	$2\|\alpha\|$	$\|\alpha\|-\eta$	-2η	0	2
$\|\alpha\| > \eta$	$2\|\alpha\|$	0						
$\|\alpha\| + \eta$ odd			0	$-2\|\alpha\|$	$\|\alpha\|+\eta$	2η	0	-2

Table 2.9(a) continued

RESTRICTIONS ON α and η	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α -ve η +ve			$(\eta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	0	$-\eta$	0	1
$\eta > \lvert\alpha\rvert$	η	0						
$\lvert\alpha\rvert + \eta$ even			$(\eta-\lvert\alpha\rvert)/2$	$-\lvert\alpha\rvert$	η	η	0	-1
α -ve η +ve			$\eta+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	0	-2η	0	2
$\eta > \lvert\alpha\rvert$	2η	0						
$\lvert\alpha\rvert + \eta$ odd			$\eta-\lvert\alpha\rvert$	$-2\lvert\alpha\rvert$	2η	2η	0	-2

* if $\alpha = \pm 1$, then see table 2.9(c) for predominant $\Phi^{(+)}$ resonant terms.

Table 2.9(b)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_8 = \alpha\dot{\omega} + k\dot{\Omega}_D \approx 0$

RESTRICTIONS ON α and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve k+ve $\alpha > k$ α even	α	0	0	$-\alpha$	$\alpha/2$	0	k	1
α +ve k+ve $\alpha \geqslant k$ α odd	2α	0	0	-2α	α	0	2k	2
α +ve k+ve $k > \alpha$ α odd k odd or α odd k even	2k	0	$k-\alpha$	-2α	k	0	2k	2
α +ve k+ve α even $k > \alpha$ k odd	k+1	0	$(k+1-\alpha)/2$	$-\alpha$	$(k+1)/2$	0	k	1
α +ve k−ve $\alpha > \|k\|$ α even	α	0	α	α	$\alpha/2$	0	$\|k\|$	−1
α +ve k−ve $\alpha \geqslant \|k\|$ α odd	2α	0	2α	2α	α	0	$2\|k\|$	−2
α +ve k−ve $\|k\| > \alpha$ α even $\|k\|$ odd	$\|k\|+1$	0	$\frac{(\|k\|+1+\alpha)}{2}$	α	$(\|k\|+1)/2$	0	$\|k\|$	−1
α +ve k−ve $\|k\| > \alpha$ α odd $\|k\|$ odd or α odd $\|k\|$ even	$2\|k\|$	0	$\|k\|+\alpha$	2α	$\|k\|$	0	$\|k\|$	−2

Table 2.9(c)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values for the Predominant $\Phi^{(+)}$ Resonant Terms of a Satellite in the Lunar Gravity Commensurability $\pm\dot{\omega} + \dot{\omega}_D \approx 0$

RESTRICTIONS ON α and η	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1 \quad \eta = 1$ if $\left(\frac{a}{a_D e_D e}\right) < 1$	2	0	0	-2	0	-2	0	2
	2	0	2	2	2	2	0	-2
$\alpha = 1 \quad \eta = 1$ if $\left(\frac{a}{a_D e_D e}\right) > 1$	3	0	1	-1	1	-1	0	1
	3	0	2	1	2	1	0	-1
$\alpha = -1 \quad \eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	0	-2	0	2
	2	0	0	-2	2	2	0	-2
$\alpha = -1 \quad \eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	1	-1	0	1
	3	0	1	-1	2	1	0	-1

Table 2.9(d)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in the Lunar Gravity Commensurability $\alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$ $|\alpha|$ and/or $|\eta| \geqslant k$

RESTRICTIONS ON α, η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve η +ve k+ve $\alpha \geqslant \eta$ α even η even or α odd η odd	α^*	0	0	$-\alpha$	$(\alpha-\eta)/2$	$-\eta$	k	1
α +ve η +ve k+ve $\alpha > \eta$ α odd η even or α even η odd	2α	0	0	-2α	$\alpha-\eta$	-2η	$2k$	2
α +ve η +ve k−ve $\alpha \geqslant \eta$ α even η even or α odd η odd	α^*	0	α	α	$(\alpha+\eta)/2$	η	$\|k\|$	−1
α +ve η +ve k−ve $\alpha > \eta$ α even η odd or α odd η even	2α	0	2α	2α	$\alpha+\eta$	2η	$2\|k\|$	−2
α +ve η +ve k+ve $\eta > \alpha$ α even η even or α odd η odd	η	0	$(\eta-\alpha)/2$	$-\alpha$	0	$-\eta$	k	1
α +ve η +ve k+ve $\eta > \alpha$ α odd η even or α even η odd	2η	0	$\eta-\alpha$	-2α	0	-2η	$2k$	2
α +ve η +ve k−ve $\eta > \alpha$ α even η even or α odd η odd	η	0	$(\eta+\alpha)/2$	α	η	η	$\|k\|$	−1
α +ve η +ve k−ve $\eta > \alpha$ α odd η even or α even η odd	2η	0	$\eta+\alpha$	2α	2η	2η	$2\|k\|$	−2

Table 2.9(d) continued

RESTRICTIONS ON α, η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α-ve η +ve k+ve $\lvert\alpha\rvert \geqslant \eta$ $\lvert\alpha\rvert$ even η even or $\lvert\alpha\rvert$ odd η odd	$\lvert\alpha\rvert^*$	0	$\lvert\alpha\rvert$	$\lvert\alpha\rvert$	$(\lvert\alpha\rvert-\eta)/2$	$-\eta$	k	1
α-ve η+ve k+ve $\lvert\alpha\rvert > \eta$ $\lvert\alpha\rvert$ odd η even or $\lvert\alpha\rvert$ even η odd	$2\lvert\alpha\rvert$	0	$2\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\lvert\alpha\rvert-\eta$	-2η	2k	2
α-ve η+ve k-ve $\lvert\alpha\rvert \geqslant \eta$ $\lvert\alpha\rvert$ even η even or $\lvert\alpha\rvert$ odd η odd	$\lvert\alpha\rvert^*$	0	0	$-\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\eta)/2$	η	$\lvert k\rvert$	-1
α-ve η+ve k-ve $\lvert\alpha\rvert > \eta$ $\lvert\alpha\rvert$ odd η even or $\lvert\alpha\rvert$ even η odd	$2\lvert\alpha\rvert$	0	0	$-2\lvert\alpha\rvert$	$(\lvert\alpha\rvert+\eta)$	2η	$2\lvert k\rvert$	-2
α-ve η+ve k+ve $\eta > \lvert\alpha\rvert$ $\lvert\alpha\rvert$ even η even or $\lvert\alpha\rvert$ odd η odd	η	0	$(\eta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	0	$-\eta$	k	1
α-ve η+ve k+ve $\eta > \lvert\alpha\rvert$ $\lvert\alpha\rvert$ odd η even or $\lvert\alpha\rvert$ even η odd	2η	0	$\eta+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	0	-2η	2k	2
α-ve η+ve k-ve $\eta > \lvert\alpha\rvert$ $\lvert\alpha\rvert$ even η even or $\lvert\alpha\rvert$ odd η odd	η	0	$(\eta-\lvert\alpha\rvert)/2$	$-\lvert\alpha\rvert$	η	η	$\lvert k\rvert$	-1
α-ve η+ve k-ve $\eta > \lvert\alpha\rvert$ $\lvert\alpha\rvert$ even η odd or $\lvert\alpha\rvert$ odd η even	2η	0	$\eta-\lvert\alpha\rvert$	$-2\lvert\alpha\rvert$	2η	2η	$2\lvert k\rvert$	-2

* if $\lvert\alpha\rvert = 1$ then see table 2.9(e) for the predominant resonant terms.

Table 2.9(e)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\pm\dot{\omega} + \dot{\omega}_D \pm \dot{\Omega}_D \approx 0$

RESTRICTIONS ON α, η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
$\alpha = 1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	0	-2	2	2
$\alpha = 1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	1	-1	1	1
$\alpha = 1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	2	2	2	-2
$\alpha = 1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	2	1	1	-1
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	0	-2	2	2
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	1	-1	1	1
$\alpha = -1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	2	2	2	-2
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	2	1	1	-1

Table 2.9(f)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the Predominant $\bar{\Phi}^{(+)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$ $|k| > |\alpha|$ and $|\eta|$

RESTRICTIONS ON α, η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α +ve η+ve k+ve $k-\alpha$ even $k-\eta$ even	k	0	$(k-\alpha)/2$	$-\alpha$	$(k-\eta)/2$	$-\eta$	k	1
α +ve η +ve k+ve $k-\alpha$ odd $k-\eta$ odd	k+1	0	$(k+1-\alpha)/2$	$-\alpha$	$(k+1-\eta)/2$	$-\eta$	k	1
α +ve η +ve k+ve $k-\alpha$ even $k-\eta$ odd or $k-\alpha$ odd $k-\eta$ even	2k	0	$k-\alpha$	-2α	$k-\eta$	-2η	2k	2
α +ve η +ve k-ve $\|k\| + \alpha$ even $\|k\| + \eta$ even	$\|k\|$	0	$(\|k\|+\alpha)/2$	α	$(\|k\|+\eta)/2$	η	$\|k\|$	-1
α +ve η+ve k-ve $\|k\| + \alpha$ odd $\|k\| + \eta$ odd	$\|k\|+1$	0	$\frac{(\|k\|+1+\alpha)}{2}$	α	$\frac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	-1
α +ve η +ve k-ve $\|k\|+\alpha$ odd $\|k\|+\eta$ even or $\|k\|+\alpha$ even $\|k\|+\eta$ odd	$2\|k\|$	0	$\|k\|+\alpha$	2α	$\|k\|+\eta$	2η	$2\|k\|$	-2
α -ve η +ve k+ve $k+\|\alpha\|$ even $k-\eta$ even	k	0	$(k+\|\alpha\|)/2$	$\|\alpha\|$	$(k-\eta)/2$	$-\eta$	k	1
α -ve η +ve k+ve $k+\|\alpha\|$ odd $k-\eta$ odd	k+1	0	$\frac{(k+1+\|\alpha\|)}{2}$	$2\|\alpha\|$	$(k+1-\eta)/2$	$-\eta$	k	1

Table 2.9(f) continued

RESTRICTIONS ON α , η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
α -ve η +ve k+ve $k+\lvert\alpha\rvert$ odd $k-\eta$ even or $k+\lvert\alpha\rvert$ even $k-\eta$ odd	2k	0	$k+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$(k-\eta)$	-2η	2k	2
α -ve η +ve k-ve $\lvert k\rvert - \lvert\alpha\rvert$ even $\lvert k\rvert + \eta$ even	$\lvert k\rvert$	0	$\frac{(\lvert k\rvert-\lvert\alpha\rvert)}{2}$	$-\lvert\alpha\rvert$	$(\lvert k\rvert+\eta)/2$	η	$\lvert k\rvert$	-1
α -ve η +ve k-ve $\lvert k\rvert - \lvert\alpha\rvert$ odd $\lvert k\rvert + \eta$ odd	$\lvert k\rvert+1$	0	$(\lvert k\rvert+1-\lvert\alpha\rvert/2$	$-\lvert\alpha\rvert$	$\frac{(\lvert k\rvert+1+\eta)}{2}$	η	$\lvert k\rvert$	-1
α -ve η +ve k-ve $\lvert k\rvert-\lvert\alpha\rvert$ odd, $\lvert k\rvert+\eta$ even or $\lvert k\rvert-\lvert\alpha\rvert$ even, $\lvert k\rvert+\eta$ odd	$2\lvert k\rvert$	0	$\lvert k\rvert-\lvert\alpha\rvert$	$-2\lvert\alpha\rvert$	$\lvert k\rvert+\eta$	2η	$2\lvert k\rvert$	-2

Table 2.9(g)

The $n^-, m^-, p^-, q^-, h^-, j^-, s^-$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms of a Satellite in the Lunar Gravity Commensurability $\dot{\psi}_8 = \alpha\dot{\omega} + \eta\dot{\omega}_D \approx 0$

RESTRICTIONS ON α and η	n^-	m^-	p^-	q^-	h^-	j^-	s^-	
α +ve η +ve			0	$-\alpha$	$(\alpha+\eta)/2$	η	0	1
$\alpha \geqslant \eta$	α^*	0						
$\alpha + \eta$ even			α	$+\alpha$	$(\alpha-\eta)/2$	$-\eta$	0	-1
α +ve η +ve			0	-2α	$\alpha+\eta$	2η	0	2
$\alpha > \eta$	2α	0						
$\alpha + \eta$ odd			2α	2α	$\alpha-\eta$	-2η	0	-2
α +ve η +ve			$(\eta-\alpha)/2$	$-\alpha$	η	η	0	1
$\eta > \alpha$	η	0						
$\alpha + \eta$ even			$(\eta+\alpha)/2$	α	0	$-\eta$	0	-1
α +ve η +ve			$\eta-\alpha$	-2α	2η	2η	0	2
$\eta > \alpha$	2η	0						
$\alpha + \eta$ odd			$\eta+\alpha$	2α	0	-2η	0	-2
α -ve η +ve			$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|+\eta)/2$	η	0	1
$\|\alpha\| \geqslant \eta$	$\|\alpha\|^*$	0						
$\|\alpha\| + \eta$ even			0	$-\|\alpha\|$	$(\|\alpha\|-\eta)/2$	$-\eta$	0	-1
α -ve η +ve			$2\|\alpha\|$	$2\|\alpha\|$	$\|\alpha\|+\eta$	2η	0	2
$\|\alpha\| > \eta$	$2\|\alpha\|$	0						
$\|\alpha\| + \eta$ odd			0	$-2\|\alpha\|$	$\|\alpha\|-\eta$	-2η	0	-2

/continued

Table 2.9(g) continued

RESTRICTIONS ON α and η	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α -ve η +ve $\eta > \alpha$ $\lvert\alpha\rvert + \eta$ even	η	0	$(\eta+\lvert\alpha\rvert)/2$	$\lvert\alpha\rvert$	η	η	0	1
			$(\eta-\lvert\alpha\rvert)/2$	$-\lvert\alpha\rvert$	0	$-\eta$	0	-1
α -ve η +ve $\lvert\alpha\rvert + \eta$ odd	2η	0	$\eta+\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	2η	2η	0	2
			$\eta-\lvert\alpha\rvert$	$-2\lvert\alpha\rvert$	0	-2η	0	-2

* if $\alpha = \pm 1$ then see table 2.9(i) for predominant $\Phi^{(-)}$ resonant terms.

Table 2.9(h)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$, and v values of the Predominant $\bar{\Phi}^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\dot{\psi}_8$ = $\alpha\dot{\omega} + k\dot{\Omega}_D \approx 0$

RESTRICTIONS ON α and k	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α+ve k+ve $\alpha > k$ α even	α	0	α	α	$\alpha/2$	0	k	-1
α+ve k+ve $\alpha \geqslant k$ α odd	2α	0	2α	2α	α	0	2k	-2
α+ve k+ve $k > \alpha$ α odd k even or α odd k odd	2k	0	$k+\alpha$	2α	k	0	2k	-2
α+ve k+ve $k > \alpha$ α even k odd	k+1	0	$(k+1+\alpha)/2$	α	$(k+1)/2$	0	k	-1
α+ve k-ve $\alpha > \|k\|$ α even	α	0	0	$-\alpha$	$\alpha/2$	0	$\|k\|$	1
α+ve k-ve $\alpha \geqslant \|k\|$ α odd	2α	0	0	-2α	α	0	$2\|k\|$	2
α+ve k-ve $\|k\| > \alpha$ α even $\|k\|$ odd	$\|k\|+1$	0	0	$-\alpha$	$(\|k\|+1)/2$	0	$\|k\|$	1
α+ve k-ve $\|k\| > \alpha$ α odd $\|k\|$ odd or α odd $\|k\|$ even	$2\|k\|$	0	0	-2α	$\|k\|$	0	$2\|k\|$	2

Table 2.9(i)

The n^-, m^-, p^-, q^-, h^-, j^-, s^- and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in the Lunar Gravity Commensurabilities $\pm\dot{\omega} + \dot{\omega}_D \approx 0$

RESTRICTIONS ON α and η	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
$\alpha = 1$ $\eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	2	2	0	2
			2	2	0	-2	0	-2
$\alpha = 1$ $\eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	2	1	0	1
			2	1	1	-1	0	-1
$\alpha = -1$ $\eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	2	2	0	2
			0	-2	0	-2	0	-2
$\alpha = -1$ $\eta = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	2	1	0	1
			1	-1	1	-1	0	-1

Table 2.9(j)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability

$\alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0 \qquad |\alpha|$ and/or $|\eta| \geqslant |k|$

RESTRICTIONS ON α, η and k	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α +ve η +ve k+ve $\alpha \geqslant \eta$ α even η even or α odd η odd	α^*	0	α	α	$(\alpha-\eta)/2$	$-\eta$	k	-1
α +ve η +ve k+ve $\alpha > \eta$ α odd η even or α even η odd	2α	0	2α	2α	$\alpha-\eta$	-2η	2k	-2
α +ve η +ve k-ve $\alpha \geqslant \eta$ α even η even or α odd η odd	α^*	0	0	$-\alpha$	$(\alpha+\eta)/2$	η	$\|k\|$	1
α +ve η +ve k-ve $\alpha > \eta$ α even η odd or α odd η even	2α	0	0	-2α	$\alpha+\eta$	2η	$\|2k\|$	2
α +ve η +ve k+ve $\eta > \alpha$ α even η even or α odd η odd	η	0	$(\eta+\alpha)/2$	α	0	$-\eta$	k	-1
α +ve η +ve k+ve $\eta > \alpha$ α odd η even or α even η odd	2η	0	$\eta+\alpha$	2α	0	-2η	2k	-2
α +ve η +ve k-ve $\eta > \alpha$ α even η even or α odd η odd	η	0	$(\eta-\alpha)/2$	$-\alpha$	η	η	$\|k\|$	1
α +ve η +ve k-ve $\eta > \alpha$ α odd η even or α even η odd	2η	0	$\eta-\alpha$	-2α	2η	2η	$\|2k\|$	2

Table 2.9(j) continued

RESTRICTIONS ON α, η and k	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α -ve η +ve k+ve $\|\alpha\| \geqslant \eta$ $\|\alpha\|$ even η even **or** $\|\alpha\|$ odd η odd	$\|\alpha\|^*$	0	0	$-\|\alpha\|$	$(\|\alpha\|-\eta)/2$	$-\eta$	k	-1
α -ve η +ve k+ve $\|\alpha\| > \eta$ $\|\alpha\|$ odd η even **or** $\|\alpha\|$ even η odd	$2\|\alpha\|$	0	0	$-2\|\alpha\|$	$\|\alpha\|-\eta$	-2η	2k	-2
α -ve η +ve k-ve $\|\alpha\| \geqslant \eta$ $\|\alpha\|$ even η even **or** $\|\alpha\|$ odd η odd	$\|\alpha\|^*$	0	$\|\alpha\|$	$\|\alpha\|$	$(\|\alpha\|+\eta)/2$	η	$\|k\|$	1
α -ve η +ve k-ve $\|\alpha\| > \eta$ $\|\alpha\|$ odd η even **or** $\|\alpha\|$ even η odd	$2\|\alpha\|$	0	$2\|\alpha\|$	$2\|\alpha\|$	$\|\alpha\|+\eta$	2η	$2\|k\|$	2
α -ve η +ve k+ve $\eta > \|\alpha\|$ $\|\alpha\|$ even η even **or** $\|\alpha\|$ odd η odd	η	0	$(\eta-\|\alpha\|)/2$	$-\|\alpha\|$	0	$-\eta$	k	-1
α -ve η +ve k+ve $\eta > \|\alpha\|$ $\|\alpha\|$ odd η even **or** $\|\alpha\|$ even η odd	2η	0	$\eta-\|\alpha\|$	$-2\|\alpha\|$	0	-2η	2k	-2
α -ve η +ve k-ve $\eta > \|\alpha\|$ $\|\alpha\|$ even η even **or** $\|\alpha\|$ odd η odd	η	0	$(\eta+\|\alpha\|)/2$	$\|\alpha\|$	η	η	$\|k\|$	1
α -ve η +ve k-ve $\eta > \|\alpha\|$ $\|\alpha\|$ even η odd **or** $\|\alpha\|$ odd η even	2η	0	$\eta+\|\alpha\|$	$2\|\alpha\|$	2η	2η	$2\|k\|$	2

* if $|\alpha| = 1$ then see table 2.9(k) for the predominant resonant terms.

Table 2.9(k)

The $\bar{n}$, $\bar{m}$, $\bar{p}$, $\bar{q}$, $\bar{h}$, $\bar{j}$, $\bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability $\pm\dot{\omega} + \dot{\omega}_D \pm \dot{\Omega}_D \approx 0$

RESTRICTIONS ON α, η and k	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
$\alpha = 1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	0	-2	2	-2
$\alpha = 1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	1	-1	1	-1
$\alpha = 1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	2	2	2	2
$\alpha = 1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	2	1	1	1
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	0	-2	0	-2	2	-2
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	1	-1	1	-1	1	-1
$\alpha = -1$ $\eta = 1$ $k = -1$ if $\left(\frac{a}{a_D e e_D}\right) < 1$	2	0	2	2	2	2	2	2
$\alpha = -1$ $\eta = 1$ $k = 1$ if $\left(\frac{a}{a_D e e_D}\right) > 1$	3	0	2	1	2	1	1	1

Table 2.9(1)

The $\bar{n}, \bar{m}, \bar{p}, \bar{q}, \bar{h}, \bar{j}, \bar{s}$ and v values of the Predominant $\Phi^{(-)}$ Resonant Terms for a Satellite in a Lunar Gravity Commensurability

$\alpha\dot{\omega} + \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0 \quad |k| > |\beta|$ and $|\eta|$

RESTRICTIONS ON α, η and k	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
α +ve η +ve k+ve k + α even k − η even	k	0	$(k+\alpha)/2$	α	$(k-\eta)/2$	$-\eta$	k	−1
α +ve η +ve k+ve k + α odd k − η odd	k+1	0	$(k+1+\alpha)/2$	α	$(k+1-\eta)/2$	$-\eta$	k	−1
α +ve η +ve k+ve k + α even, k − η odd or k+ α odd, k − η even	2k	0	$k+\alpha$	2α	$k-\eta$	-2η	2k	−2
α +ve η +ve k−ve $\|k\|$ − α even $\|k\|$ + η even	$\|k\|$	0	$(\|k\|-\alpha)/2$	$-\alpha$	$(\|k\|+\eta)/2$	η	$\|k\|$	1
α +ve η +ve k−ve $\|k\|$ − α odd $\|k\|$ + η odd	$\|k\|+1$	0	$\frac{(\|k\|+1-\alpha)}{2}$	$-\alpha$	$\frac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	1
α +ve η +ve k−ve $\|k\|$ − α odd, $\|k\|$ + η even or $\|k\|$ − α even, $\|k\|$ + η odd	$2\|k\|$	0	$\|k\|-\alpha$	-2α	$\|k\|+\eta$	2η	$2\|k\|$	2
α −ve η +ve k+ve k − $\|\alpha\|$ even k − η even	k	0	$(k-\|\alpha\|)/2$	$-\|\alpha\|$	$(k-\eta)/2$	$-\eta$	k	−1
α −ve η +ve k+ve k − $\|\alpha\|$ odd k − η odd	k+1	0	$\frac{(k+1-\|\alpha\|)}{2}$	$-\|\alpha\|$	$(k+1-\eta)/2$	$-\eta$	k	−1
α −ve η +ve k+ve k− $\|\alpha\|$ even, k − η odd or k− $\|\alpha\|$ odd, k−η even	2k	0	$k-\|\alpha\|$	$-2\|\alpha\|$	$k-\eta$	-2η	2k	−2

Table 2.9(1) continued

RESTRICTIONS ON α, η and k	n^-	m^-	p^-	q^-	h^-	j^-	s^-	v
α -ve η +ve k-ve $\|k\| + \|\alpha\|$ even $\|k\| + \eta$ even	$\|k\|$	0	$\frac{(\|k\|+\|\alpha\|)}{2}$	$\|\alpha\|$	$(\|k\|+\eta)/2$	η	$\|k\|$	1
α -ve η +ve k-ve $\|k\| + \|\alpha\|$ odd $\|k\| + \eta$ odd	$\|k\|+1$	0	$(\|k\|+1+$ $\|\alpha\|/2$	$\|\alpha\|$	$\frac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	1
α -ve η +ve k-ve $\|k\|+\|\alpha\|$even, $\|k\|+\eta$odd or $\|k\|+\|\alpha\|$ odd, $\|k\|+\eta$even	$2\|k\|$	0	$\|k\|+\|\alpha\|$	$2\|\alpha\|$	$(\|k\|+\eta)$	2η	$2\|k\|$	2

2.4(9) The Type (9) Lunar Gravity Commensurability $\dot{\psi}_9 = \alpha\dot{\omega} + \eta\dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0$

The 9th type of commensurability $\dot{\psi}_9 = \alpha\dot{\omega} + \eta\dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0$ needs very little discussion, since most of its properties can be inferred from those already obtained for the commensurability type $\dot{\psi}_6 \approx 0$.

A satellite which exists in a lunar gravity commensurability of the type $\dot{\psi}_9 \approx 0$, will be in resonance with those $\Phi^{(+)}$ terms in (2.4) for which

$$\begin{aligned}
(n-2p)^+ &= \alpha\delta \\
q^+ &= -\alpha\delta \\
(n-2h)^+ &= \eta\delta \\
j^+ &= -\eta\delta \\
m^+ &= \beta\delta \\
s^+ &= k\delta
\end{aligned} \tag{2.81}$$

$$\delta > 0$$

and those $\Phi^{(-)}$ terms for which

$$\begin{aligned}
(n-2p)^- &= \alpha v \\
q^- &= -\alpha v \\
(n-2h)^- &= -\eta v \\
j^- &= \eta v \\
m^- &= \beta v \\
s^- &= -kv
\end{aligned} \tag{2.82}$$

$$v > 0$$

The predominant $\Phi^{(+)}$ resonant terms for a satellite in a type (9) lunar gravity commensurability, such that $|k| < \alpha, \beta$ and η, can easily be obtained from tables 2.7(a), 2.7(c) and 2.7(e), if γ is replaced by η, $j = 0$ by $j = -\eta\delta$, and $s^+ = 0$ by $s^+ = k\delta$. If $|k|$ is greater than α, β and η, then $|k|$ replaces β in the n^+ column, corresponding changes are made in the p^+ and h^+ columns. Similar remarks apply to the $\Phi^{(-)}$ resonant terms, which can be obtained from the tables 2.7(g), 2.7(i) and 2.7(k). However, in this instance, $j^- = 0$ is replaced by $j^- = \eta v$. The conditions for the existence of a particular type (9) commensurability are given by the equations 2.61 - 2.66, with Υn_D replaced by $\eta\dot{\omega}_D + k\dot{\Omega}_D$.

The most important type (9) commensurabilities are those for which $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|}$ is an absolute minimum, i.e. the commensurabilities

$$\begin{aligned}
&\pm\ \dot{\omega} + \dot{\Omega} \pm \dot{\Omega}_D \approx 0\\
&\pm\ 2\dot{\omega} + \dot{\Omega} \pm \dot{\Omega}_D \approx 0\\
&\pm\ 2\dot{\omega} + 2\dot{\Omega} \pm \dot{\Omega}_D \approx 0\\
&\pm\ 2\dot{\omega} + \dot{\Omega} \pm 2\dot{\Omega}_D \approx 0
\end{aligned} \tag{2.83}$$

$$\text{if } \left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) < 1$$

and the set

$$\pm\ \dot{\omega} \pm \dot{\omega}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0 \tag{2.84}$$

if $\left(\frac{a}{a_D}\right)\left(\frac{e_D}{e}\right) > 1$, where $\beta = 1,2,3$ and $k = -1,-2,-3,1,2,3$.

The predominant amplitude factors of the set (2.83) are of order $\left(\frac{a}{a_D}\right)^2 e^2$; whilst those in the set (2.84) are of order $\left(\frac{a}{a_D}\right)^3 ee_D$.

From equations (2.61) to (2.66), it is found that all 88 commensurabilities in the sets (2.83) and (2.84) can exist.

2.4(10) The Type (10) Lunar Gravity Commensurability

$$\dot{\psi}_{10} = \alpha\dot{\omega} + \eta\dot{\omega}_D + \gamma\dot{M}_D + k\dot{\Omega}_D \approx 0$$

As was the case with type (9), the commensurability type $\dot{\psi}_{10} \approx 0$ needs very little discussion, most of its properties being analogous to those of type (8). The $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for a satellite in a lunar gravity commensurability are given by

$$\begin{aligned}
(n-2p)^{+} &= \alpha\delta \\
q^{+} &= -\alpha\delta \\
(n-2h)^{+} &= \eta\delta \\
j^{+} &= \gamma\delta - \eta\delta \\
m^{+} &= 0 \\
s^{+} &= k\delta
\end{aligned} \qquad (2.85)$$

and

$$\begin{aligned}
(n-2p)^{-} &= \alpha v \\
q^{-} &= -\alpha v \\
(n-2h)^{-} &= -\eta v \\
j^{-} &= -\gamma v + \eta v \\
m^{-} &= 0 \\
s^{-} &= -kv
\end{aligned} \qquad (2.86)$$

respectively.

The predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms can be easily obtained from the tables 2.9(a) - 2.9(l) if η is considered to be always positive; $j^{+} = -\eta\delta$ is replaced by $j^{+} = \gamma\delta - \eta\delta$, and $j^{-} = \eta v$ is replaced by $j^{-} = \eta v - \gamma v$. Keeping η positive goes against the previously adopted sign convention, which would have made γ always positive. However, this change is convenient in that it readily gives the predominant resonant terms for a type (10) commensurability.

The conditions for the existence of a type (10) lunar gravity commensurability can be inferred from those already obtained for the commensurabilities $\dot{\psi}_4 \approx 0$, if $n_D \gamma / \alpha$ in equations (2.42) and (2.43) is replaced by $(\gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D)/\alpha$, viz

$$4.98\,\alpha > \gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D \tag{2.87}$$

for $\quad (\gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D)/\alpha > 0$

and

$$19.92\,\alpha > \gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D \tag{2.88}$$

when $\quad (\gamma \dot{M}_D + \eta \dot{\omega}_D + k\dot{\Omega}_D)/\alpha < 0$

The most important lunar gravity commensurabilities of type (10) when $\left(\frac{a}{a_D}\right)\left(\frac{1}{e}\right) < 1$ are those which have predominant amplitude factors proportional to $\left(\frac{a}{a_D}\right)^2 e^2$, i.e. the set

$$\begin{aligned} &\pm \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D \approx 0 \\ &\pm 2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D \approx 0 \end{aligned} \tag{2.89}$$

However, if $\left(\frac{a}{a_D}\right)\left(\frac{1}{e}\right) > 1$, then the most important type (10) commensurabilities are the set

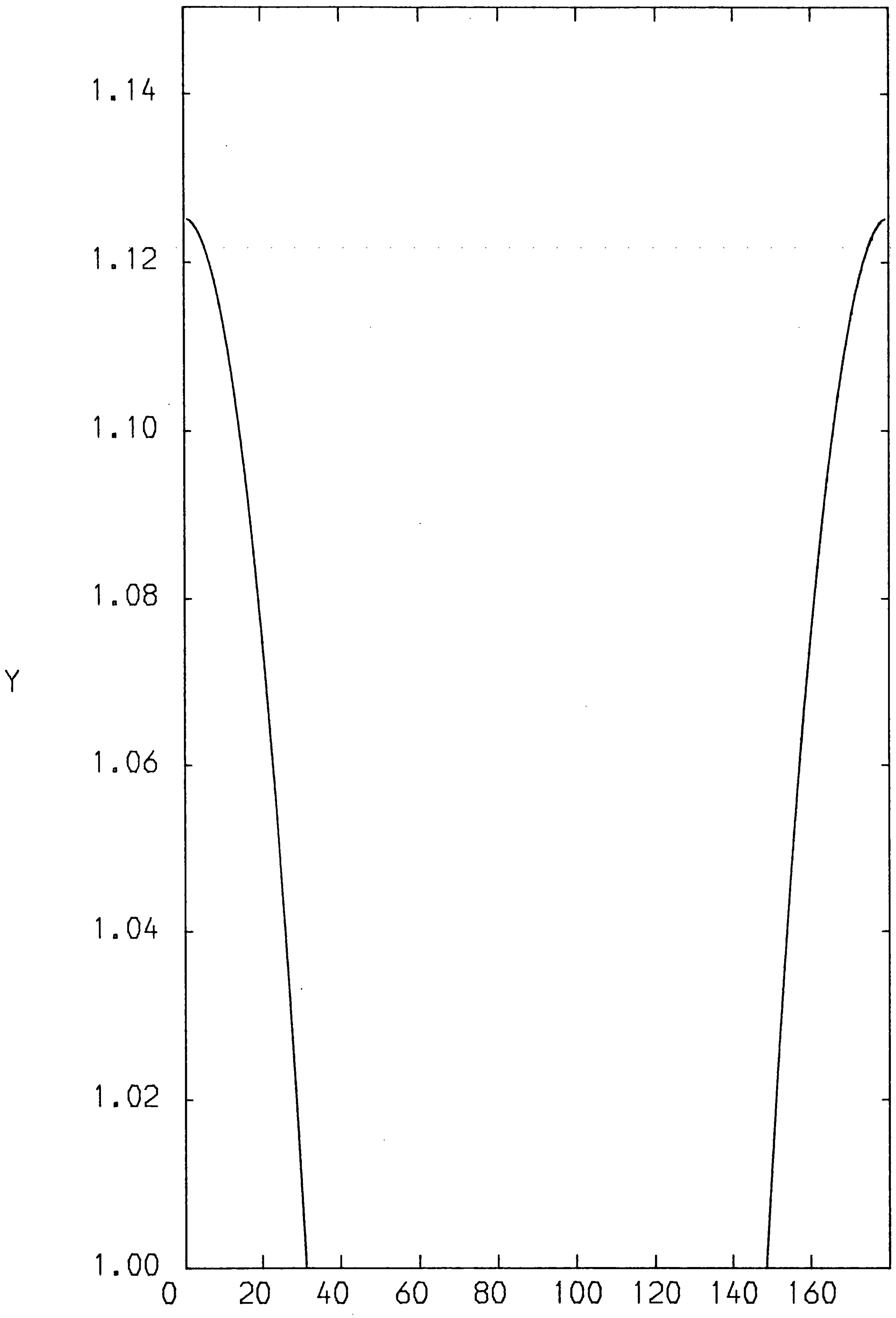
$$\begin{aligned} &\pm \dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D \approx 0 \\ &\pm \dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + k\dot{\Omega}_D \approx 0 \end{aligned} \tag{2.90}$$

where $k = -1,-2,-3,1,2,3$. The amplitude factors of the set (2.90) are proportional to $\left(\frac{a}{a_D}\right)^3 e$. From equations (2.87) and (2.88), it can easily be seen that the only commensurabilities in the sets (2.89) and

(2.90) which exist are

$$\left.\begin{array}{l} -\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D \approx 0 \\ \\ -2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D \approx 0 \end{array}\right. \qquad (2.91)$$

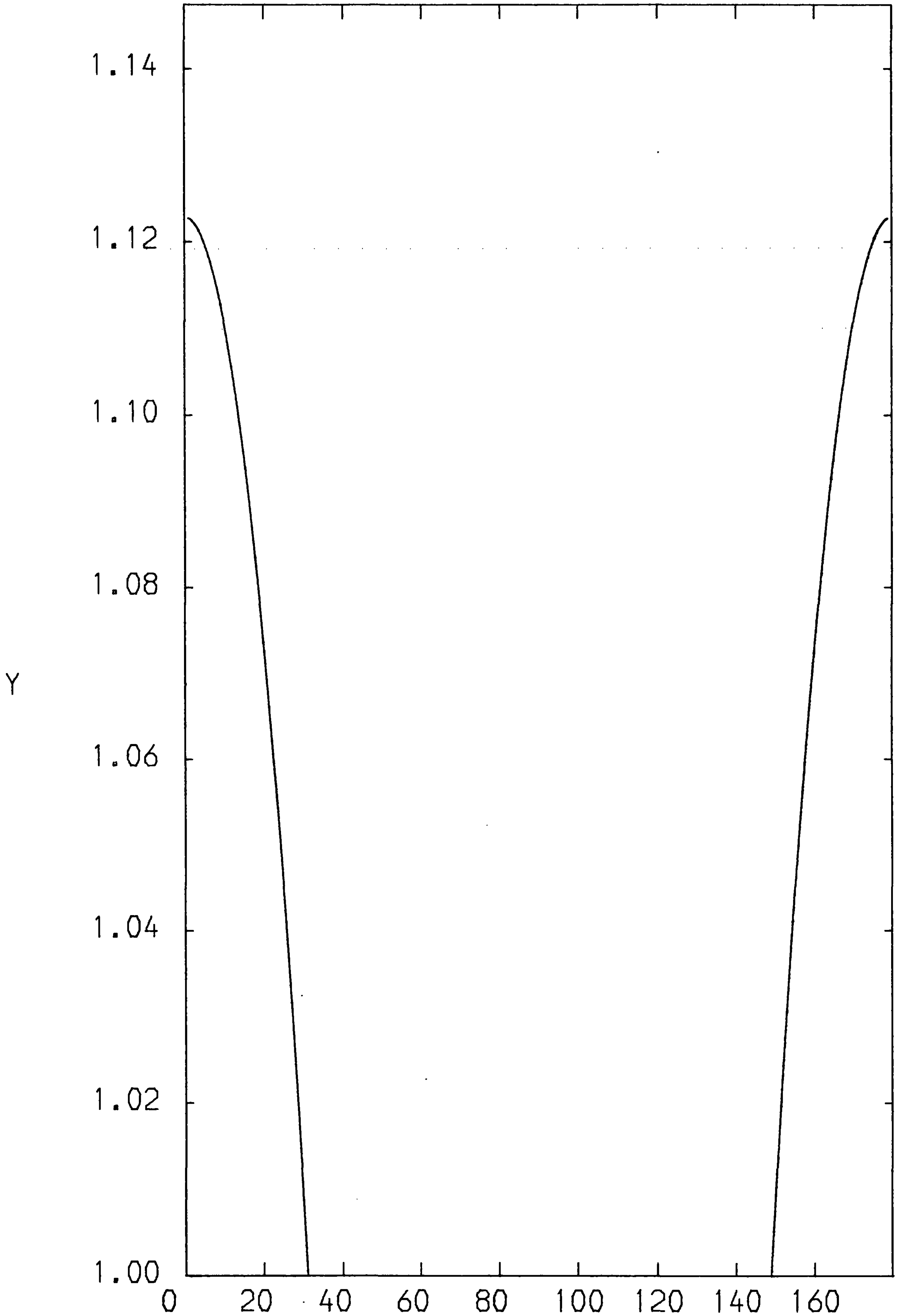
The satellite orbits which exist in the set of commensurabilities (2.91) are given in figures (2.38) - (2.41).



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D+\dot{\Omega}_D=0$

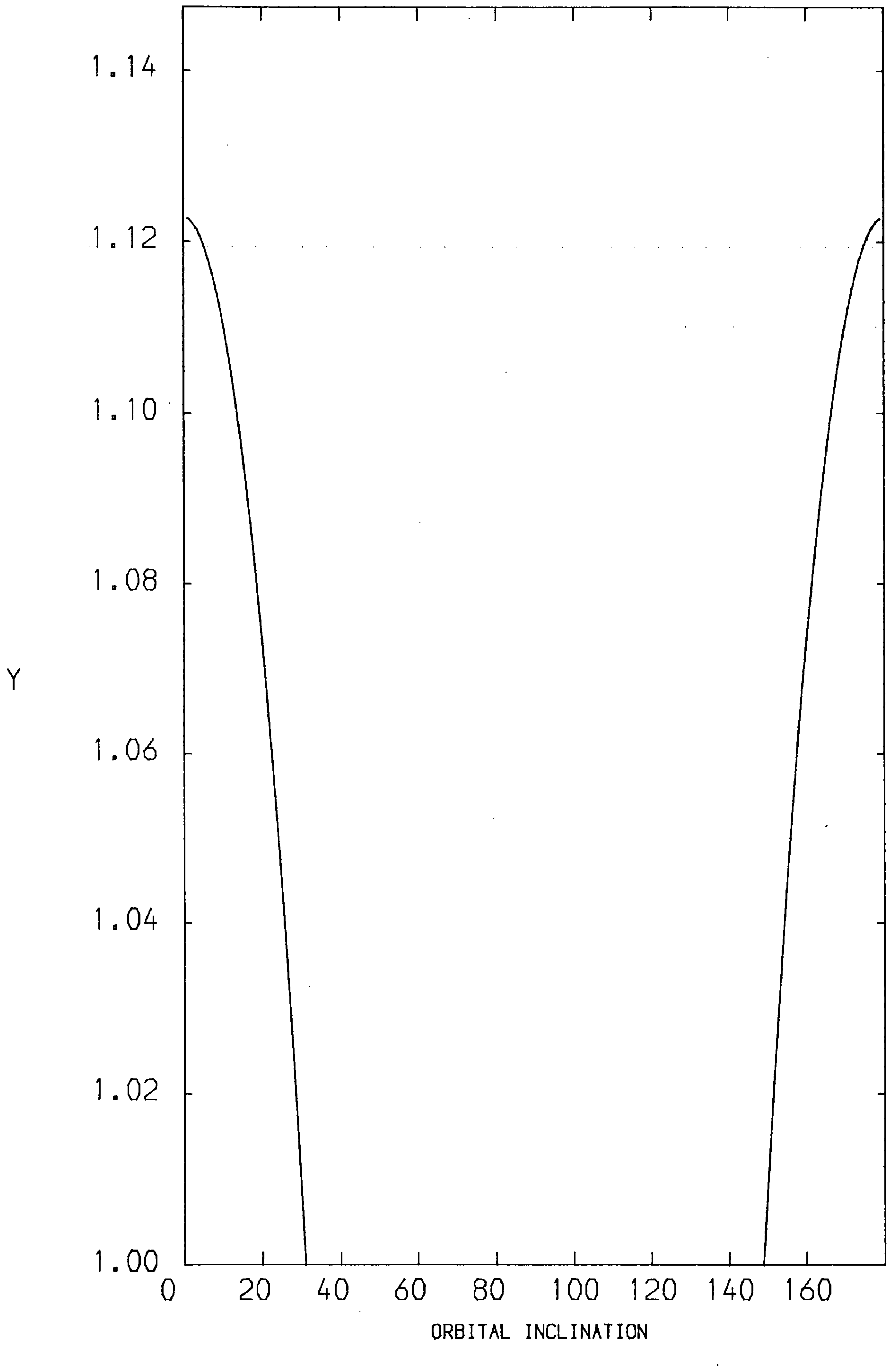
Figure 2.38



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$-\dot{\omega}+\dot{\omega}_D+\dot{M}_D-\dot{\Omega}_D=0$

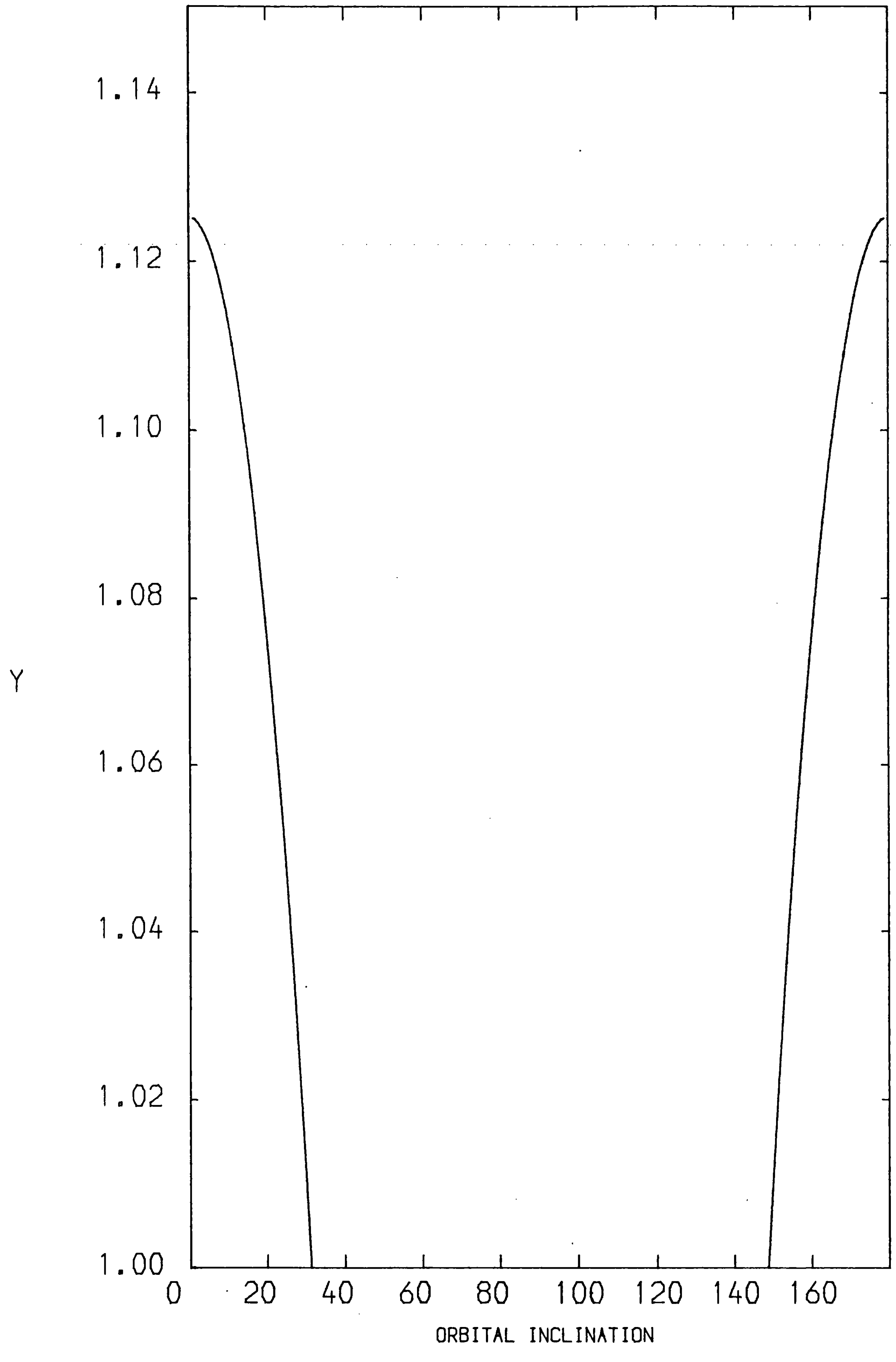
Figure 2.39



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-\dot{\omega}_D-\dot{M}_D+\dot{\Omega}_D=0$

Figure 2.40



ORBITS FOR A SATELLITE IN THE LUNAR COMMENSURABILITY

$\dot{\omega}-\dot{\omega}_D-\dot{M}_D-\dot{\Omega}_D=0$

Figure 2.41

2.4(11) The Type (11) Lunar Gravity Commensurability

$$\dot{\psi}_{11} = \eta\dot{\omega}_D + \gamma\dot{M}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

A satellite exists in a type (11) lunar gravity commensurability if its orbital elements satisfy the equation

$$9.97\beta \cos i = (\gamma\dot{M}_D + \eta\dot{\omega}_D + k\dot{\Omega}_D)\, y^{3.5} \qquad (2.92)$$

In order for a particular type (11) commensurability to occur, the maximum value of y in equation (2.92) must be greater than unity, i.e. if

$$9.97\beta > |\gamma\dot{M}_D + \eta\dot{\omega}_D + k\dot{\Omega}_D| \qquad (2.93)$$

On substituting in the appropriate values of $\dot{M}_D$, $\dot{\omega}_D$ and $\dot{\Omega}_D$ for the Moon, equation (2.93) becomes

$$9.97\beta > |13.07\gamma + 0.16\eta - 0.05\,k| \qquad (2.94)$$

The $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for a satellite in a commensurability $\dot{\psi}_{11} \approx 0$ are such that

$$\begin{aligned}
(n-2p)^+ &= 0 \\
q^+ &= 0 \\
(n-2h)^+ &= \eta\delta \\
(n-2h+j)^+ &= \gamma\delta \\
m^+ &= \beta\delta \\
s^+ &= k\delta
\end{aligned} \qquad (2.95)$$

$$\delta > 0$$

and

$$\begin{aligned}
(n-2p)^- &= 0 \\
q^- &= 0 \\
(n-2h)^- &= -\eta v \\
(n-2h+j)^- &= -\gamma v
\end{aligned} \qquad (2.96)$$

$$m^- = \beta v$$

$$s^- = -kv$$

$$v > 0$$

respectively. The predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms can be easily obtained from tables 2.8(a) - 2.8(d), if $j^+ = -\eta\delta$ is replaced by $j^+ = \gamma\delta - \eta\delta$, and $j^- = \eta v$ is replaced by $j^- = \eta v - \gamma v$.

The most important type (11) commensurabilities are those for which $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|}$ is an absolute minimum, i.e. the commensurabilities

$$
\begin{aligned}
&\pm (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&\pm 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&\pm 2(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&\pm 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm 2\dot{\Omega}_D \approx 0
\end{aligned}
\tag{2.97}
$$

The predominant amplitude factors of the set (2.97) are of order $\left(\frac{a}{a_D}\right)^2$.

However, consideration of equation (2.94) shows that none of the set (2.97) can exist. The next most important commensurabilities which can exist are those of order $\left(\frac{a}{a_D}\right)^4$, e.g.

$$+ (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} + \dot{\Omega}_D \approx 0$$

Such commensurabilities are, however, very weak, and, therefore, of no great importance: we will not consider them further.

2.4(12) The Type (12) Lunar Gravity Commensurability

$$\dot{\psi}_{12} = \alpha\dot{\omega} + \eta\dot{\omega}_D + \gamma\dot{M}_D + \beta\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

The lunar gravity commensurabilities of the type $\dot{\psi}_{12} \approx 0$ are the most complicated commensurabilities discussed so far in that they have the greatest number of terms. However, their properties are easily found from those already obtained for other commensurability types.

A satellite which exists in a type (12) lunar gravity commensurability is in resonance with those $\Phi^{(+)}$ and $\Phi^{(-)}$ terms in (2.4) for which

$$\begin{aligned}
(n-2p)^+ &= \alpha\delta \\
q^+ &= -\alpha\delta \\
(n-2h)^+ &= \eta\delta \\
(n-2h+j)^+ &= \gamma\delta \\
m^+ &= \beta\delta \\
s^+ &= k\delta
\end{aligned} \tag{2.98}$$

$$\delta > 0$$

and

$$\begin{aligned}
(n-2p)^- &= \alpha v \\
q^- &= -\alpha v \\
(n-2h)^- &= -\eta v \\
(n-2h+j)^- &= -\gamma v \\
m^- &= \beta v \\
s^- &= -kv
\end{aligned} \tag{2.99}$$

$$v > 0$$

The predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for a type (12) commensurability are obtained in the same manner as those of type (9), with the exception that $j^+ = 0$ is replaced by $j^+ = -\eta\delta + \gamma\delta$, and $j^- = 0$ by $\eta v - \gamma v$. The existence conditions for a particular type (12) commensurability can be obtained from equations (2.61) - (2.66) if γn_D is replaced by $\gamma\dot{M}_D + \eta\dot{\omega}_D + k\dot{\Omega}_D$. Finally, the most important type (12) commensurabilities will be those which have predominant resonant terms of order $\left(\frac{a}{a_D}\right)^2 e^2$ when $\left(\frac{a}{a_D}\right)\left(\frac{1}{e}\right) < 1$, i.e. the set

$$\begin{aligned}
&\pm\dot{\omega} \pm (\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D + \dot{\Omega} \approx 0 \\
&\pm 2\dot{\omega} \pm 2(\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D + \dot{\Omega} \approx 0 \\
&\pm 2\dot{\omega} \pm 2(\dot{\omega}_D + \dot{M}_D) \pm 2\dot{\Omega}_D + \dot{\Omega} \approx 0 \\
&\pm 2\dot{\omega} \pm 2(\dot{\omega}_D + \dot{M}_D) \pm \dot{\Omega}_D + 2\dot{\Omega} \approx 0
\end{aligned} \tag{2.100}$$

and those which have predominant resonant terms of order $\left(\frac{a}{a_D}\right)^3 e$ when $\left(\frac{a}{a_D}\right)\left(\frac{1}{e}\right) > 1$, i.e. the set

$$\pm\dot{\omega} + \gamma(\dot{\omega}_D + \dot{M}_D) + \beta\dot{\Omega} + k\dot{\Omega} \approx 0 \tag{2.101}$$

where $\gamma = -1,-3,1,3$; $\beta = 1,2,3$ and $k = -1,-2,-3,1,2,3$. However from equations (2.61) - (2.66) with γn_D replaced by $\gamma\dot{M}_D + \eta\dot{\omega}_D + k\dot{\Omega}_D$ it is easily seen that, of the set (2.100), only

$$\begin{aligned}
&\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&-\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&2\dot{\omega} - 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0 \\
&-2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm \dot{\Omega}_D \approx 0
\end{aligned} \tag{2.102}$$

$$2\dot{\omega} - 2(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \pm \dot{\Omega}_D \approx 0$$

$$-2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} \pm \dot{\Omega}_D \approx 0$$

$$2\dot{\omega} - 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm 2\dot{\Omega}_D \approx 0$$

$$-2\dot{\omega} + 2(\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} \pm 2\dot{\Omega}_D \approx 0$$

are possible. Similarly, of the set (2.101), only

$$\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$-\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + \dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$\dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} + k^1\dot{\Omega}_D \approx 0$$

$$\dot{\omega} - 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$-\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 2\dot{\Omega} + k^1\dot{\Omega}_D \approx 0$$

$$-\dot{\omega} + 3(\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$-\dot{\omega} + (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

$$-\dot{\omega} - (\dot{\omega}_D + \dot{M}_D) + 3\dot{\Omega} + k\dot{\Omega}_D \approx 0$$

are possible, where k^1 takes the values $-1, 1, 2, 3$.

2.4(13) The Type (13) Lunisolar Commensurability Condition

$$\dot{\psi}_{13} = \eta \dot{\omega}_D + k\dot{\Omega}_D \approx 0$$

The last commensurability condition to be discussed is somewhat unusual in that it is entirely independent of a satellite's orbital elements, depending solely on the nature of the lunar and solar orbits. All satellites therefore exist in every type (13) commensurability. The $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for such a commensurability are given by

$$\left.\begin{aligned}
(n-2p)^+ &= 0 \\
q^+ &= 0 \\
(n-2h)^+ &= \eta\delta \\
j^+ &= -\eta\delta \\
m^+ &= 0 \\
s^+ &= k\,\delta
\end{aligned}\right\} \qquad (2.104)$$

and

$$\left.\begin{aligned}
(n-2p)^- &= 0 \\
q^- &= 0 \\
(n-2h)^- &= -\eta\, v \\
j^- &= \eta\, v \\
m^- &= 0 \\
s^- &= -kv
\end{aligned}\right\} \qquad (2.105)$$

respectively.

For the Sun, $\eta \dot{\omega}_D + k\dot{\Omega}_D$ is approximately zero for all normal values of η and k. However, for the Moon, $\eta \dot{\omega}_D + k\dot{\Omega}_D$ is approximately zero when $\eta = 1$, $k = 3$. (It is only necessary to consider the solution $\eta = 1$ and $k = 3$, since all other solutions give equivalent commensurability conditions.) It will therefore be found convenient to discuss the lunar and solar cases separately - starting with the lunar commensurabilities of type (13).

The predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for the lunar case are such that

$$n^+ = 6,\ m^+ = 0,\ p^+ = 3,\ q^+ = 0,\ h^+ = 2,\ j^+ = -2,\ s^+ = 6,\ \delta = 2$$

and

$$n^- = 6,\ m^- = 0,\ p^- = 3,\ q^- = 0,\ h^- = 2,\ j^- = -2,\ s^- = 6,\ v = -2$$

The amplitude factors of these terms are proportional to $\left(\frac{a}{a_D}\right)^6 e_D^{\ 2}$.

Lunar gravity resonances of type (13) are therefore very weak, and will not greatly affect a satellite's motion.

The solar commensurabilities of type (13) are more complicated owing to the greater variation of allowed values for η and k (η being always positive). The most important type (13) solar commensurability occurs when $\eta = 0$ and $k = 1$, i.e. $\dot{\Omega}_D \approx 0$, the predominant amplitude factor being proportional to $\left(\frac{a}{a_D}\right)^2$. The predominant $\Phi^{(+)}$ and $\Phi^{(-)}$ resonant terms for the general type (13) solar commensurability condition $\eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$ are given in tables 2.10(a) and 2.10(b).

Table 2.10(a)

The $n^+, m^+, p^+, q^+, h^+, j^+, s^+$ and δ values of the predominant $\Phi^{(+)}$ Resonant Terms for a Satellite in a Type (13) Solar Commensurability $\dot{\psi}_{13} = \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$

RESTRICTIONS ON η and k	n^+	m^+	p^+	q^+	h^+	j^+	s^+	δ
η+ve k+ve $\eta > k$ η even	η	0	$\eta/2$	0	0	$-\eta$	k	1
η+ve k+ve $\eta \geqslant k$ η odd	2η	0	η	0	0	-2η	2k	2
η+ve k+ve $k > \eta$ k odd η even	k+1	0	(k+1)/2	0	$(k+1-\eta)/2$	$-\eta$	k	1
η+ve k+ve $k > \eta$ k odd η odd or k even η odd	2k	0	k	0	$k-\eta$	-2η	2k	2
η +ve k-ve $\eta > \|k\|$ η even	η	0	$\eta/2$	0	η	η	$\|k\|$	-1
η+ve k-ve $\eta > \|k\|$ η odd	2η	0	η	0	2η	2η	$2\|k\|$	-2
η+ve k-ve $\|k\| > \eta$ $\|k\|$ odd η even	$\|k\|+1$	0	$\frac{(\|k\|+1)}{2}$	0	$\frac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	-1
η+ve k-ve $\|k\| > \eta$ k odd η odd $\|k\|$ even η odd	$2\|k\|$	0	$\|k\|$	0	$\|k\| + \eta$	2η	$2\|k\|$	-2

Table 2.10(b)

The $\bar{n},\bar{m},\bar{p},\bar{q},\bar{h},\bar{j},\bar{s}$ and v values of the Predominant $\bar{\Phi}^{(-)}$ Resonant Terms for a Satellite in a Type (13) Solar Commensurability $\dot{\psi}_{13} = \eta\dot{\omega}_D + k\dot{\Omega}_D \approx 0$

RESTRICTIONS ON η and k	$\bar{n}$	$\bar{m}$	$\bar{p}$	$\bar{q}$	$\bar{h}$	$\bar{j}$	$\bar{s}$	v
η+ve k+ve $\eta > k$ η even	η	0	$\eta/2$	0	0	$-\eta$	k	−1
η+ve k+ve $\eta \geqslant k$ η odd	2η	0	η	0	0	-2η	2k	−2
η+ve k+ve $k > \eta$ k odd η even	k+1	0	(k+1)/2	0	$(k+1-\eta)/2$	$-\eta$	k	−1
η +ve k+ve $k > \eta$ k odd η odd or k even η odd	2k	0	k	0	$k-\eta$	-2η	2k	−2
η +ve k−ve $\eta > \|k\|$ η even	η	0	$\eta/2$	0	η	η	$\|k\|$	1
η +ve k−ve $\eta \geqslant \|k\|$ η odd	2η	0	η	0	2η	2η	$2\|k\|$	2
η +ve k−ve $\|k\| > \eta$ $\|k\|$ odd η even	$\|k\|+1$	0	$(\|k\|+1)/2$	0	$\frac{(\|k\|+1+\eta)}{2}$	η	$\|k\|$	1
η +ve k−ve $\|k\| > \eta$ $\|k\|$ even η odd or $\|k\|$ odd, η odd	$2\|k\|$	0	$\|k\|$	0	$\|k\|+\eta$	2η	$2\|k\|$	2

2.5 Discussion and Conclusion

In this thesis, the properties of a class of artificial satellite orbits known as lunisolar resonance orbits have been analysed. Special emphasis has been given to five particular aspects of such orbits - classification, orbital elements of resonant satellites, predominant resonant terms in lunisolar disturbing function(s), important examples and existence. Some general conclusions can now be drawn concerning the orbits.

Firstly, lunisolar resonance orbits can be divided into fifteen distinct types - a particular type being classified according to its commensurability condition. Further analysis shows that these fifteen types can be reduced to 3 basic classes - inclination dependent, y and i dependent, and orbit independent. Types (1), (2) and (3) belong to class (1); types (4) - (12) together with type (15) belong to class 2; the third basic class consists of types (13) and (14).

Secondly, the commensurability condition for the fifteen types of lunisolar resonance orbits can be expressed as a simple relationship between the three non-angular elements of a satellite orbit (i.e. a, e and i), if the satellite is sufficiently close to the Earth. A satellite will exist in a particular commensurability if its orbital elements satisfy such a relationship.

Thirdly, any satellite existing in a particular lunisolar commensurability is in resonance with an infinity of terms in the lunisolar disturbing function expansion(s), not all terms having equal weight owing to their differing amplitude factors. It has been shown that the most important resonant terms are likely to be those for which the amplitude factor $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|}$ is a minimum. This criterion has been used to determine the predominant resonant terms for the various commensurability types.

Fourthly, each type of resonance orbit has its most important commensurabilities, namely those for which $\left(\frac{a}{a_D}\right)^n e^{|q|} e_D^{|j|}$ is an absolute minimum. So far as inclination dependent resonances are concerned, the most important have been shown to occur at inclinations of 46.4°, 56.1°, 63.4°, 69.0°, 73.2°, 106.9°, 111.0°, 116.6°, 123.4°, 90·0° and 133.6°. In the case of y and i dependent resonances, the orbits which exist in 41 of the most important commensurabilities have been obtained. (The results are given in Figures 2.1 to 2.41.)

Lastly, criteria have been found which determine whether, or not, a particular lunisolar commensurability can exist, i.e. whether satellite orbits exist which satisfy the commensurability condition. It has been shown that <u>all</u> inclination dependent commensurabilities are possible, each giving at least one resonant inclination, with a maximum of two. y and i dependent resonances only exist if the maximum value of y obtained from the commensurabilities is greater than unity.

APPENDIX 1

Some Important Hansen Coefficients

A.1 Introduction

Hansen coefficients are series in the orbital eccentricity of a satellite, or disturbing body, resulting from the expansion of the perturbing potential (force) in the terms of the Keplerian elements of the satellite and the disturbing body. The coefficients $G_{n,p,q}(e)$ and $H_{n,h,j}(e_D)$ used in the text, are defined by

$$G_{n,p,q}(e) = X^{n,(n-2p)}_{(n-2p+q)}(e) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n \cos\{(n-2p)f - (n-2p+q)M\}\, dM \tag{1}$$

and

$$H_{n,h,j}(e_D) = X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a_D}{r_D}\right)^{n+1} \cos\{(n-2h)f_D - (n-2h+j)M_D\}\, dM_D \tag{2}$$

The subscript D in (2) refers to the orbit of the disturbing body.

Plummer (1918) has shown how to develop (1) as a series in e and the subsidiary variable β, given by

$$\beta = e/\{1 + (1 - e^2)^{\frac{1}{2}}\} \tag{3}$$

The result is

$$X^{n,(n-2p)}_{(n-2p+q)}(e) = (1+\beta^2)^{-(n+1)}\sum_{\theta=-\infty}^{+\infty} J_\theta\{(n-2p+q)e\}\; X^{n,(n-2p)}_{(n-2p+q),\theta}(e) \tag{4}$$

where

$$X^{n,(n-2p)}_{(n-2p+q),\theta}(e) = \begin{cases} (-\beta)^{q-\theta}\binom{2p+1}{q-\theta}F(-2p+q-\theta-1,-2n+2p-1,q-\theta+1;\beta^2) & \theta \leqslant q \\ (-\beta)^{\theta-q}\binom{2n-2p+1}{\theta-q}F(-2n+2p-q+\theta-1,-2p-1,-q+\theta+1;\beta^2) & \theta \geqslant q \end{cases} \tag{5}$$

$J_\Theta(ve)$ is the ordinary Bessel function of degree, Θ, i.e.

$$J_\Theta(ve) = \frac{\left(\frac{ve}{2}\right)^\Theta}{\Theta!}\left[1 - \frac{\left(\frac{ve}{2}\right)^2}{1.(\Theta+1)} + \frac{\left(\frac{ve}{2}\right)^4}{1.2(\Theta+1).(\Theta+2)} - \dots\right] \qquad \Theta \geqslant 0$$

$$(-1)^{|\Theta|} \; J_{|\Theta|}(ve) \qquad \Theta \leqslant 0 \tag{6}$$

and F is a hypergeometric function given by

$$F(a,b,c;\, x) = \left(1 + \frac{a.b.x}{1.c} + \frac{a.(a+1).b.(b+1)}{1.2.c.(c+1)}\, x^2 + \dots\right) \tag{7}$$

The leading term in the series for $X^{n,(n-2p)}_{(n-2p+q)}(e)$ is of order $e^{|q|} + 0(e^{|q|+2})$. Similarly, the result for $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D)$ in terms of e_D and β_D, where

$$\beta_D = e_D / \{1 + (1 - e_D{}^2)^{\frac{1}{2}}\} \tag{8}$$

is such that

$$X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D) = (1+\beta_D{}^2)^n \sum_{\Theta=-\infty}^{+\infty} J_\Theta\{(n-2h+j)e_D\}\, X^{-(n+1),(n-2h)}_{(n-2h+j),\Theta}(e_D) \tag{9}$$

$X^{-(n+1),(n-2h)}_{(n-2h+j),\Theta}(e_D)$ is given by

$$X^{-(n+1),(n-2h)}_{(n-2h+j),\Theta}(e_D) = \begin{cases} (-\beta_D)^{j-\Theta}\dbinom{-2n+2h-j}{j-\Theta} F(2n-2h+j-\Theta, 2h, j-\Theta+1;\beta_D{}^2) & \Theta \leqslant j \\ (-\beta_D)^{\Theta-j}\dbinom{-2h}{\Theta-j} F(2h-j+\Theta, 2n-2h, -j+1+\Theta;\, \beta_D{}^2) & \Theta \geqslant j \end{cases} \tag{10}$$

The leading terms in the series for $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D)$ is of order $e_D{}^{|j|} + 0(e_D{}^{|j|+2})$. Let us now consider the Hansen coefficients, $G_{n,p,q,}(e)$ and $H_{n,h,j}(e_D)$, in greater detail, starting with $G_{n,p,q}(e)$.

A.2 The Hansen Coefficients $G_{n,p,q}(e)$ $\{X_{(n-2p+q)}^{n,(n-2p)}(e)\}$

In Chapter 2, it was shown that, for close Earth satellites, the only important lunisolar commensurabilities are those for which $(n-2p+q) = 0$. Consequently, it is only necessary to consider here the Hansen coefficient $X_0^{n,(n-2p)}(e)$. If $(n-2p+q) = 0$, then the integral in (1) reduces to

$$X_0^{n,(n-2p)}(e) = \int_0^{2\pi} \left(\frac{r}{a}\right)^n \cos\{(n-2p)\ f\}\ dM \tag{11}$$

On changing the integration variable from the mean anomaly, M, to the eccentric anomaly, E, equation (11) becomes

$$X_0^{n,(n-2p)}(e) = \frac{1}{2\pi} \int_0^{2\pi} (1 - e\cos E)^{n+1} \cos mf\ dE \tag{12}$$

If we use the relation

$$\cos f = \frac{(\cos E - e)}{(1 - e\cos E)}$$

then (12) can be further transformed into

$$X_0^{n,\pm m}(e) = \frac{1}{2\pi}\int_0^{2\pi} (1-e\cos E)^{n+1-m} \left[\sum_{u=0}^{[m/2]} (-1)^u \binom{m}{2u} (\cos E-e)^{m-2u} \{(1-e\cos E)^2 - (\cos E-e)^2\}^u\right] dE \tag{13}$$

where $m = |n-2p|$, and $[m/2]$ signifies the integer part of m/2. Equation (13) when integrated is

$$X^{n,\pm m}_{0(e)} = \sum_{t=0}^{n+1-m} \sum_{u=0}^{[m/2]} \sum_{v=0}^{u} \sum_{\omega=0}^{u} \sum_{x=t+2v}^{m-2u+t+2v} \frac{(-1)^{x+u+\omega-v}}{2^y} \binom{n+1-m}{t}$$

$$\times \binom{m}{2u} \binom{u}{v} \binom{u}{\omega} \binom{m-2u}{x-t-2v} \binom{y}{y/2} e^x \qquad (14)$$

with $y = m+2t-2u+2v+2\omega-x$. It is to be noted that, in (14), y can only take even values.

A.3 The Hansen Coefficients $H_{n,h,j}(e_D)$ $\{X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D)\}$.

Since some important lunisolar commensurabilities exist for which $(n-2h+j) \neq 0$, it will be necessary to consider $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e_D)$ when $(n-2h+j) \neq 0$, but first let us discuss the Hansen coefficients $X^{-(n+1),(n-2h)}_{0(e_D)}$. If $(n-2h+j) = 0$, then the integral (2) reduces to

$$X^{-(n+1),(n-2h)}_{0(e_D)} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a_D}{r_D}\right)^{n+1} \cos\{(n-2h) f_D\} \, dM_D \tag{15}$$

On changing the integration variable from the mean anomaly, M_D, to the true anomaly, f_D, equation (15) becomes

$$X^{-(n+1),\pm m}_{0(e_D)} = \frac{1}{2\pi(1-e_D^2)^{(2n-1)/2}} \int_0^{2\pi} (1+e_D \cos f_D)^{n-1} \cos mf_D \, df_D \tag{16}$$

where $m = |n-2h|$. If (16) is integrated, we find that

$$X^{-(n+1),\pm m}_{0(e_D)} = \frac{1}{(1-e_D^2)^{(2n-1)/2}} \sum_{i=m}^{n-1} \frac{1}{2^i} \binom{n-1}{i} \binom{i}{(i-m)/2} e_D^i . \tag{17}$$

It is to be noted in (17) that $(i-m)$ can only take even values. Also if $m = n$, $X^{-(n+1),n}_{0(e_D)} = 0$.

If $(n-2h+j) \neq 0$, then equations (9) and (10) have to be used to calculate $H_{n,h,j}(e_D)$. The results for $j = 0, \pm 1$ and ± 2 up to order (e_D^2) are given by

$$H_{n,h,0}(e_D) = 1 + \frac{(n-3n^2 + 16hn - 16h^2)}{4} e_D^2 + 0(e_D^4) \tag{18}$$

$$H_{n,h,1}(e_D) = \frac{(3n-4h+1)}{2} e_D + 0(e_D{}^3) \tag{19}$$

$$H_{n,h,-1}(e_D) = \frac{(4h-n+1)}{2} e_D + 0(e_D{}^3) \tag{20}$$

$$H_{n,h,+2}(e_D) = \frac{(9n^2+16h^2-24hn-18h+14n+4)}{8} e_D{}^2 + 0(e_D{}^4) \tag{21}$$

and

$$H_{n,h,-2}(e_D) = \frac{(n^2+16h^2-8hn+18h-4n+4)}{8} e_D{}^2 + 0(e_D{}^4) \tag{22}$$

Table T.1

Hansen Coefficients $X_0^{n,m}(e)$: $1 \leqslant n \leqslant 5, 0 \leqslant m \leqslant n$

m \ n	1	2	3	4	5
0	$1+\frac{e^2}{2}$	$1+\frac{3e^2}{2}$	$1+3e^2+\frac{3e^4}{8}$	$1+5e^2+\frac{15e^4}{8}$	$1+\frac{15e^2}{2}+\frac{45e^4}{8}+\frac{5e^6}{16}$
1	$-\frac{3e}{2}$	$-2e-\frac{e^3}{2}$	$-\frac{5e}{2}-\frac{15e^3}{8}$	$-3e-\frac{9e^3}{2}-\frac{3e^5}{8}$	$-\frac{7e}{2}-\frac{35e^4}{4}-\frac{35e^6}{16}$
2	-	$\frac{5e^2}{2}$	$\frac{15e^2}{4}+\frac{15e^4}{8}$	$\frac{21e^2}{4}+\frac{21e^4}{8}$	$7e^2+7e^4+\frac{7e^6}{16}$
3	-	-	$-\frac{35e^3}{8}$	$-7e^3-\frac{7e^5}{8}$	$-\frac{21e^3}{2}-\frac{63e^5}{16}$
4	-	-	-	$\frac{63e^4}{8}$	$\frac{105e^4}{8}+\frac{21e^6}{16}$
5	-	-	-	-	$-\frac{231e^5}{16}$

Table T.2

Hansen Coefficients $X_0^{-(n+1),m}(e_D)$: $1 \leqslant n \leqslant 5,\ 0 \leqslant m \leqslant n-1$

m \ n	1	2	3	4	5
0	$\frac{1}{R}$	$\frac{1}{R^3}$	$\frac{1}{R^5} + \frac{e_D^2}{2R^5}$	$\frac{1}{R^7} + \frac{3e_D^2}{2R^7}$	$\frac{1}{R^9} + \frac{3e_D^2}{R^9} + \frac{3e_D^4}{8R^9}$
1	-	$\frac{e_D}{2R^3}$	$\frac{e_D}{R^5}$	$\frac{3e_D}{2R^7} + \frac{3e_D^3}{8R^7}$	$\frac{2e_D}{R^9} + \frac{3e_D^3}{2R^9}$
2	-	-	$\frac{e_D^2}{4R^5}$	$\frac{3e_D^2}{4R^7}$	$\frac{3e_D^2}{2R^9} + \frac{e_D^4}{4R^9}$
3	-	-	-	$\frac{e_D^3}{8R^7}$	$\frac{e_D^3}{2R^9}$
4	-	-	-	-	$\frac{e_D^4}{16R^9}$

where $R = (1 - e_D^2)^{\frac{1}{2}}$

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S.HUGHES Ph.D. THESIS 1978

A STUDY OF ARTIFICIAL SATELLITE RESONANCE ORBITS DUE TO LUNISOLAR PERTURBATIONS

S. HUGHES

Abstract

A study of artificial satellite resonance orbits due to lunisolar perturbations is given. Particular emphasis being placed on the following aspects -

1. The classification of resonance orbits according to their commensurability condition.
2. The form of the commensurability condition when expressed in terms of the orbital elements of a satellite.
3. The predominant resonant terms for each commensurability condition.
4. Criteria which determine the existence or non-existence of a particular commensurability condition.

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A study of near-circular satellite orbits: with an application to lunisolar perturbations

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A solution is obtained for the motion of a satellite in a near-circular orbit when acted upon by the zonal harmonics in the geopotential. It can be applied, in particular, to the evaluation of the indirect effects of the zonal harmonics on a satellite which is also acted upon by lunisolar perturbations. This approach avoids some of the limitations inherent in earlier solutions. Expressions are obtained for the time variations of all the elliptic elements a, e, I, ω, Ω and M, and the solution is valid to the first order of the zonal harmonics J_n ($n > 2$) and up to the second order in J_2. It can be applied to non-equatorial satellites with orbital eccentricities < 0.03 and inclinations not near the critical inclination of 63.4°.

1. INTRODUCTION

In any accurate theory for the motion of a close Earth satellite perturbed by lunisolar gravity or solar radiation pressure it is necessary to include the indirect effects of the zonal harmonics, since these produce changes in the amplitude and argument factors of the terms contained in the lunisolar disturbing functions. Before this can be done, a solution has to be obtained for the direct effect of the zonal harmonics on a close Earth satellite orbit. The solution obtained must be in a form which makes the calculation of the indirect effects of the zonal harmonics as simple as possible. Although a theory intended for the evaluation of indirect effects does not have to be as accurate as a theory designed for the direct effect of the zonal harmonics (see, for example, Aksnes 1970; Kinoshita 1976), it should account for the first-order effects of J_n ($n > 2$) and the effect of J_2 up to the second order (i.e. J_2^2). Since most satellites launched to date have small orbital eccentricities ($e < 0.03$), a solution suitable for near-circular orbits is generally applicable. For this reason, only near-circular orbits ($e < 0.03$) will be considered in the subsequent analysis.

A number of papers have discussed the motion of satellites having small-eccentricity orbits; but all have been subject to certain limitations. Kozai (1959) and Izsak (1963) have used the Lagrangian planetary equations (Smart 1953) in the variables ξ and η defined by

$$\left.\begin{aligned}\xi &= e\cos\omega,\\ \eta &= e\sin\omega,\end{aligned}\right\} \tag{1}$$

5-2

to obtain the first-order effects of the zonal harmonics J_2, J_3 and J_5 on the eccentricity, e, and argument of perigee, ω. The variation of the other orbital elements was not considered. Chebotarev (1963) has obtained expressions giving the time variations of all the orbital elements; but his analysis is restricted to the J_2 harmonic only. Cook (1966) has extended the analysis to include the first-order effect of J_2 and the general odd zonal harmonic, but, like Kozai and Izsak, he has limited his discussion to the elements e and ω only.

If the indirect effects of the zonal harmonics are to be studied for a satellite which is also acted upon by lunisolar perturbations, then in a Lagrangian treatment it will be found necessary to solve the six Lagrangian planetary equations by successive approximation, rather than by direct integration of the differential equations. A canonic method of approach is therefore more suitable when indirect effects have to be considered. In such a treatment, it is only necessary to perform one integration and six differentiations in order to obtain the time variations of the orbital elements.

In this paper, Delaunay–von Zeipel contact transformations will be used to obtain expressions for the time variations of all the orbital elements of a satellite when the eccentricity of the orbit is small (< 0.03). The first-order effects of the zonal harmonics J_n ($n > 2$), and the effects of J_2 up to the second order, will be included in the analysis.

2. The canonic equations when e is small

Brouwer (1959) and Kozai (1962) have used Delaunay–von Zeipel contact transformations (von Zeipel 1916) to study the motion of close Earth satellites when perturbed by the zonal harmonics. However, the solutions obtained for the variables ω and the anomaly, M, contain singularities when $e = 0$. Consequently, the results are only valid for large or moderate eccentricity orbits ($e > 0.03$). This situation arises because the perigee, and, hence, ω and M, becomes ill-defined as e tends to zero; with the result that the Delaunay variables L, G, H, l, g and h used by Brouwer and Kozai are inappropriate for the study of near-circular orbits. The Delaunay variables (Smart 1953) L, G, H, l, g and h are related to the osculating elliptic elements by the expressions,

$$\left.\begin{aligned} L &= (\mu a)^{\frac{1}{2}}, & l &= M, \\ G &= L(1-e^2)^{\frac{1}{2}}, & g &= \omega, \\ H &= G\cos I, & h &= \Omega, \end{aligned}\right\} \tag{2}$$

where μ is the gravitational constant for the Earth, a the semi-major axis of the orbit, I the inclination and Ω the longitude of the ascending node.

The Poincaré variables (Smart 1953) x_i and y_i ($i = 1, 2, 3$), defined by

$$\left.\begin{aligned} x_1 &= L, & y_1 &= l+g, \\ x_2 &= (x_1)^{\frac{1}{2}} e\sin g + O(e^3), & y_2 &= (x_1)^{\frac{1}{2}} e\cos g + O(e^3), \\ x_3 &= H, & y_3 &= h \end{aligned}\right\} \tag{3}$$

appear, however, to be more suitable for use when e is small. For small e the terms of $O(e^3)$ are negligible and will therefore not be considered further. The variables x_i and y_i $(i = 1, 2, 3)$ are the canonic equivalents of the Lagrangian variables a, ξ, η, I, Ω and $(\omega + M)$ usually adopted for the study of near-circular orbits (Chebotarev 1963).

The Cartesian vector equation of motion for a satellite perturbed by the zonal harmonics in the geopotential is

$$\ddot{\boldsymbol{r}} + \mu \boldsymbol{r}/r^3 = \boldsymbol{\nabla} R, \tag{4}$$

where R is the longitudinally independent part of the geopotential, such that

$$R = -\sum_{n=2}^{\infty} \frac{\mu}{R_{\mathrm{E}}} J_n \left(\frac{R_{\mathrm{E}}}{r}\right)^{n+1} P_n(\sin\theta), \tag{5}$$

where J_n is the zonal harmonic coefficient of degree n, R_{E} is the mean equatorial radius of the Earth, and P_n (sin θ) is the Legendre polynomial of degree n and argument θ (with θ the geocentric latitude of the satellite). If R is expressed as a function of the variables x_i and y_i $(i = 1, 2, 3)$, then the canonic equations in x_i and y_i are

$$\left.\begin{aligned} \dot{x}_i &= \partial \bar{R}/\partial y_i \\ \dot{y}_i &= -\partial \bar{R}/\partial x_j \end{aligned}\right\} \quad (i = 1, 2, 3) \tag{6}$$

with

$$\bar{R} = \mu^2/(2x_1^2) + R(x, y). \tag{7}$$

The expression for the expansion of R as a function of the elliptic elements a, e, I, ω, Ω and M (Cook 1966) is

$$\begin{aligned} R = &-(\mu/R_{\mathrm{E}}) \sum_{n=2}^{\infty} J_n (R_{\mathrm{E}}/a)^{n+1} P_n(\cos I)\, P_n(0) \sum_{\nu=-\infty}^{+\infty} X_\nu^{-(n+1),0}(e) \cos \nu M \\ &-(2\mu/R_{\mathrm{E}}) \sum_{n=2}^{\infty} \sum_{s=1}^{n} J_n (R_{\mathrm{E}}/a)^{n+1} ((n-s)\,!/(n+s)\,!)\, P_n^s(\cos I)\, P_n^s(0) \\ &\times \sum_{\nu=-\infty}^{+\infty} X_\nu^{-(n+1),s}(e) \cos\{s(\omega - \tfrac{1}{2}\pi) + \nu M\}, \end{aligned} \tag{8}$$

where P_n^s $(\cos I)$ is the Legendre associated function of degree n, order s and argument $\cos I$. The quantities

$$X_\nu^{-(n+1),0}(e) \quad \text{and} \quad X_\nu^{-(n+1),s}$$

are the Hansen coefficients expressed in terms of the eccentricity, e, and defined by

$$X_\nu^{-(n+1),s}(e) = 1/2\pi \int_0^{2\pi} (a/r)^{n+1} \cos(sf - \nu M)\, \mathrm{d}M, \tag{9}$$

where f is the true anomaly of the osculating orbit. For small e, $X_\nu^{-(n+1),s}(e)$ is of the order $e^{|(s-\nu)|}$. Since e is small $(e < 0.03)$, and the J_2 harmonic is approximately 10^3 times larger than subsequent J_n harmonics, only secular terms up to order

$J_2 e^2$ and J_n, long-period terms up to order $J_n e$, and short-period terms of order J_2, need be considered further. Equation (8) now becomes

$$\begin{aligned} R = \frac{\mu J_2}{4R_{\mathrm{E}}}\left(\frac{R_{\mathrm{E}}}{a}\right)^3 (3\cos^2 I - 1)(1+\tfrac{3}{2}e^2) - \frac{\mu}{R_{\mathrm{E}}}\sum_{n=4}^{\infty} J_n \left(\frac{R_{\mathrm{E}}}{a}\right)^{n+1} P_n(\cos I)\, P_n(0) \\ - \frac{\mu}{R_{\mathrm{E}}}\sum_{n=3}^{\infty} J_n \left(\frac{R_{\mathrm{E}}}{a}\right)^{n+1} \frac{(n-1)}{n(n+1)} P_n^1(\cos I) P_n^1(0)\, e \sin\omega \\ + \frac{\mu J_2}{4R_{\mathrm{E}}}\left(\frac{R_{\mathrm{E}}}{a}\right)^3 P_2^2(\cos I)\cos 2(\omega + M). \end{aligned} \tag{10}$$

In obtaining (10), use has been made of the following identities (see appendix):

$$X_0^{-(n+1),1}(e) = \tfrac{1}{2}(n-1)\, e + O(e^3) \tag{11}$$

$$X_0^{-3,0}(e) = 1 + \tfrac{3}{2}e^2 + O(e^4) \tag{12}$$

$$X_0^{-(n+1),0}(e) = 1 + O(e^2). \tag{13}$$

On using (10) and the relations (2) and (3), $\bar{R}$ can be expressed as a function of x_i and y_i. If terms of $O(J_2 e^4)$ and $O(J_n e^2)$ are neglected, equation (10) becomes

$$\begin{aligned} \bar{R} = \frac{\mu^2}{2x_1^2} + \frac{J_2\mu^4 R_{\mathrm{E}}^2}{4x_1^6}\left(\frac{3x_3^2}{x_1^2} - 1\right) + \frac{3J_2 R_{\mathrm{E}}^2\mu^4}{8x_1^7}\left(\frac{5x_3^2}{x_1^2} - 1\right) x_2^2 \\ + \frac{3J_2 R_{\mathrm{E}}^2\mu^4}{8x_1^7}\left(\frac{5x_3^2}{x_1^2} - 1\right) y_2^2 - \sum_{n=4}^{\infty}\frac{\mu}{R_{\mathrm{E}}} J_n \left(\frac{R_{\mathrm{E}}\mu}{x_1^2}\right)^{n+1} P_n(x_3/x_1)\, P_n(0) \\ - \sum_{n=3}^{\infty}\frac{\mu}{R_{\mathrm{E}}}\frac{J_n}{\sqrt{x_1}}\left(\frac{R_{\mathrm{E}}\mu}{x_1^2}\right)^{n+1}\frac{(n-1)}{n(n+1)} P_n^1(x_3/x_1) P_n^1(0)\, x_2 \\ + \frac{J_2\mu^4 R_{\mathrm{E}}^2}{4x_1^6} P_2^2(x_3/x_1)\cos 2y_1. \end{aligned} \tag{14}$$

In deriving equation (14), use has been made of the identity

$$\cos I = \frac{x_3}{x_1}\left(1 + \left\{\frac{x_2^2 + y_2^2}{2x_1}\right\}\right) + O(e^4).$$

which is easily obtained from relations (3).

3. Delaunay–von Zeipel contact transformations

Suppose the variables x_i and y_i $(i = 1, 2, 3)$ are changed to new variables x_i^* and y_i^* $(i = 1, 2, 3)$ by means of the general functional equations

$$\left.\begin{aligned} x_i &= x_i(x^*, y^*, t) \\ y_i &= y_i(x^*, y^*, t) \end{aligned}\right\} \quad (i = 1, 2, 3). \tag{15}$$

If the change of variables is such that x_i^* and y_i^* $(i = 1, 2, 3)$ are canonically conjugate with reference to a Hamiltonian $R^*(x^*, y^*, t)$, i.e.

$$\left.\begin{aligned} \dot{x}_i^* &= \partial R^*/\partial y_i^* \\ \dot{y}_i^* &= -\partial R^*/\partial x_i^* \end{aligned}\right\} \quad (i = 1, 2, 3) \tag{16}$$

then the transformation of variables is called a 'contact transformation'. The general theorem stating the conditions for a variable transformation to be canonic is as follows (Spiegel 1967).

If there exists a generating function $S(x^*, y)$ such that

$$\left.\begin{aligned} x_i &= \partial S(x^*, y)/\partial y_i, \\ y_i^* &= \partial S(x^*, y)/\partial x_i^*, \end{aligned}\right\} \tag{17}$$

then x_i^* and y_i^* $(i = 1, 2, 3)$ are canonically conjugate with reference to the Hamiltonian $R^*(x^*, y^*)$ given by

$$R^*(x^*, y^*) = \bar{R}(x, y). \tag{18}$$

$\bar{R}(x, y)$ can be expressed as a function of x_i and y_i once S is known. In the Delaunay–von Zeipel method, $S(x^*, y)$ is chosen so that, to the order of accuracy required, $\bar{R}(x, y)$ becomes a function of x_i^* $(i = 1, 2, 3)$ only. Since $\bar{R} = R^*$, the new Hamiltonian R^* is independent of y_i^*; consequently, x_i^* is a constant and y_i^* a linear function of the time. The equations (17) are then used to obtain the old canonic variables x_i and y_i as functions of the new canonic variables x_i^* and y_i^*. Finally, the relations (3) give expressions for the time variations of the satellite's osculating elements, a, e, I, ω, Ω and M.

Let S be chosen so that

$$S = \sum_{i=1}^{3} x_i^* y_i + S_1(x^*, y_2) + S_2(x^*, y_1), \tag{19}$$

where $S_1(x^*, y_2)$ and $S_2(x^*, y_1)$ are the perturbed parts of S, i.e. each have a small parameter dependent on the J_n coefficients as a factor. With the use of equation (19), the equations (17) become

$$x_i = x_i^* + \frac{\partial S_1}{\partial y_i} + \frac{\partial S_2}{\partial y_i}, \tag{20}$$

$$y_i = y_i^* - \frac{\partial S_1}{\partial x_i^*} - \frac{\partial S_2}{\partial x_i^*}. \tag{21}$$

On substituting equations (20) into the right hand side of (14), and expanding by Taylor's theorem, neglecting powers of

$$J_n\left(\frac{\partial S_2}{\partial y_1}\right) (n > 2), \left(\frac{\partial S_2}{\partial y_1}\right)^3, \quad J_2 x_2^2\left(\frac{\partial S_2}{\partial y_1}\right), \quad J_2\left(\frac{\partial S_2}{\partial y_1}\right) y_2^2, \quad J_n y_2\left(\frac{\partial S_2}{\partial y_1}\right)\left(\frac{\partial S_1}{\partial y_2}\right) \quad \text{and above,}$$

$\bar{R}$ becomes

$$\begin{aligned} \bar{R} = {} & \frac{\mu^2}{2x_1^{*2}} + \frac{J_2 R_E^2 \mu^4}{4x_1^{*6}}\left(\frac{3x_3^{*2}}{x_1^{*2}} - 1\right) + \frac{3J_2 R_E^2 \mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}} - 1\right) x_2^{*2} \\ & - \sum_{n=4}^{\infty} J_n \frac{\mu}{R_E}\left(\frac{R_E\, \mu}{x_1^{*2}}\right)^{n+1} P_n(x_3^*/x_1^*)\, P_n(0) + \frac{3J_2 R_E^2 \mu^4}{4x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}} - 1\right)\left(\frac{\partial S_1}{\partial y_2}\right) x_2^* \\ & + \frac{3J_2 R_E^2 \mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}} - 1\right)\left(\frac{\partial S_1}{\partial y_2}\right)^2 + \frac{3J_2 R_E^2 \mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}} - 1\right) y_2^2 \end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=3}^{\infty}\frac{J_n}{\sqrt{x_1^*}}\frac{\mu}{R_E}\left(\frac{R_E\mu}{x_1^{*2}}\right)^{n+1}\frac{(n-1)}{n(n+1)}P_n^1(x_3^*/x_1^*)\,P_n^1(0)\,x_2^* \\
&-\sum_{n=3}^{\infty}\frac{J_n}{\sqrt{x_1^*}}\frac{\mu}{R_E}\left(\frac{R_E\mu}{x_1^{*2}}\right)^{n+1}\frac{(n-1)}{n(n+1)}P_n^1(x_3^*/x_1^*)\,P_n^1(0)\frac{\partial S_1}{\partial y_2} \\
&+\frac{J_2\mu^4R_E^2}{4x_1^{*6}}P_2^2(x_3^*/x_1^*)\cos 2y_1 \\
&+\frac{J_2\mu^4R_E^2}{4x_1^{*6}}\frac{dP_2^2(x_3^*/x_1^*)}{dx_1^*}\cos 2y_1\frac{\partial S_2}{\partial y_1}-\frac{\mu^2}{x_1^{*3}}\left(\frac{\partial S_2}{\partial y_1}\right)+\frac{3\mu^2}{2x_1^{*4}}\left(\frac{\partial S_2}{\partial y_1}\right)^2 \\
&-\frac{3J_2}{2x_1^*}\frac{\mu}{R_E}\left(\frac{\mu R_E}{x_1^{*2}}\right)^3P_2^2(x_3^*/x_1^*)\cos 2y_1\frac{\partial S_2}{\partial y_1}.
\end{aligned}
\tag{22}
$$

If S_1 and S_2 are chosen such that

$$
\begin{aligned}
&\frac{3J_2R_E^2\mu^4}{4x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}}-1\right)x_2^*\frac{\partial S_1}{\partial y_2}+\frac{3J_2R_E^2\mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}}-1\right)\left(\frac{\partial S_1}{\partial y_2}\right)^2 \\
&\qquad-\sum_{n=3}^{\infty}\frac{J_n}{\sqrt{x_1^*}}\frac{\mu}{R_E}\left(\frac{R_E\mu}{x_1^{*2}}\right)^{n+1}\frac{(n-1)}{n(n+1)}P_n^1(x_3^*/x_1^*)\,P_n^1(0)\left(\frac{\partial S_1}{\partial y_2}\right) \\
&\qquad\qquad+\frac{3J_2R_E^2\mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}}-1\right)y_2^2=0,
\end{aligned}
\tag{23}
$$

$$
\frac{\partial S_2}{\partial y_1}=\frac{J_2\mu^2}{4x_1^{*3}}R_E^2\,P_2^2(x_3^*/x_1^*)\cos 2y_1
\tag{24}
$$

then $\bar{R}$ becomes, on neglecting short-period terms of order J_2^2,

$$
\begin{aligned}
\bar{R}=R^*=\;&\frac{\mu^2}{2x_1^{*2}}+\frac{J_2R_E^2\mu^4}{4x_1^{*6}}\left(\frac{3x_3^{*2}}{x_1^{*2}}-1\right)+\frac{3J_2R_E^2\mu^4}{8x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}}-1\right)x_2^{*2} \\
&-\sum_{n=4}^{\infty}\frac{J_n\mu}{R_E}\left(\frac{R_E\mu}{x_1^{*2}}\right)^{n+1}P_n(x_3^*/x_1^*)\,P_n(0) \\
&-\sum_{n=3}^{\infty}\frac{J_n\mu}{\sqrt{x_1^*}\,R_E}\left(\frac{R_E\mu}{x_1^{*2}}\right)^{n+1}\frac{(n-1)}{n(n+1)}P_n^1(x_3^*/x_1^*)\,P_n^1(0)\,x_2^* \\
&+\frac{J_2^2\mu^6R_E^4}{32x_1^{*9}}P_2^2(x_3^*/x_1^*)\frac{dP_2^2(x_3^*/x_1^*)}{dx_1^*}-\frac{9J_2^2\mu^6R_E^4}{64x_1^{*10}}[P_2^2(x_3^*/x_1^*)]^2.
\end{aligned}
\tag{25}
$$

$\bar{R}$ is now a function of x_i^* only.

4. The solution

The short-period terms depend on the anomaly, M, which has a period of less than 2 h for a close satellite. These terms are therefore of minor importance in a first-order theory, simply causing a rapid oscillation in the otherwise smooth time variations of the orbital elements. However, in a higher-order theory, short-period terms can combine to produce secular and long-period changes in the orbital elements of a satellite. It is for this reason that the short-period terms of order J_2 have been retained. Now that these second-order secular variations have been obtained, the short-period terms are no longer required, and will not be considered further.

Let A and P be defined by

$$A = \frac{3J_2 R_E^2 \mu^4}{4x_1^{*7}}\left(\frac{5x_3^{*2}}{x_1^{*2}} - 1\right) \tag{26}$$

and

$$P = \frac{\mu}{R_E}\sum_{n=3}^{\infty}\frac{J_n}{\sqrt{x_1^*}}\left(\frac{R_E \mu}{x_1^{*2}}\right)^{n+1}\frac{(n-1)}{n(n+1)}P_n^1(x_3^*/x_1^*)\,P_n^1(0). \tag{27}$$

Equation (23) then becomes

$$\tfrac{1}{2}\left(\frac{\partial S_1}{\partial y_2}\right)^2 + \left(x_2^* - \frac{P}{A}\right)\frac{\partial S_1}{\partial y_2} + \tfrac{1}{2}y_2^2 = 0. \tag{28}$$

The solution of (28) is

$$\partial S_1/\partial y_2 = (P/A - x_2^*) \pm [(x_2^* - P/A)^2 - y_2^2]^{\frac{1}{2}}, \tag{29}$$

which, when integrated, yields

$$S_1 = \left(\frac{P}{A} - x_2^*\right) y_2 \pm \tfrac{1}{2}(x_2^* - P/A)^2 \arcsin\left[\frac{y_2}{(x_2^* - P/A)}\right] \pm \tfrac{1}{2}y_2[(x_2^* - P/A)^2 - y_2^2]^{\frac{1}{2}}. \tag{30}$$

For the moment, both the positive and negative solutions for S_1 will be retained: a full discussion of the duality of S_1 will be given in § 5. Equations (20) and (21) can now be used to obtain relations connecting the old canonic variables x_i and y_i to the new canonic variables x_i^* and y_i^* in the following form

$$x_1 = x_1^*, \tag{31}$$

$$x_2 = P/A \pm (x_2^* - P/A)\cos\{y_2^*/(x_2^* - P/A)\}, \tag{32}$$

$$x_3 = x_3^*, \tag{33}$$

$$y_1 = y_1^* + \frac{1}{A}[-y_2^* \pm (x_2^* - P/A)\sin\{y_2^*/(x_2^* - P/A)\}]\left[\frac{P}{A}\frac{\partial A}{\partial x_1^*} - \frac{\partial P}{\partial x_1^*}\right], \tag{34}$$

$$y_2 = \pm (x_2^* - P/A)\sin\{y_2^*/(x_2^* - P/A)\}, \tag{35}$$

$$y_3 = y_3^* + \frac{1}{A}[-y_2^* \pm (x_2^* - P/A)\sin\{y_2^*/(x_2^* - P/A)\}]\left[\frac{P}{A}\frac{\partial A}{\partial x_3^*} - \frac{\partial P}{\partial x_3^*}\right]. \tag{36}$$

It only remains to determine the expressions for the time variations of x_i^* and y_i^*. Since $\bar{R} = R^*$ is a function of the variables x_i^* $(i = 1, 2, 3)$ only, the variables x_i^* are constants and the variables y_i^* are linear functions of the time.

The canonic equations for the variables x_i^* and y_i^* are

$$\left.\begin{aligned} \dot{x}_i^* &= \partial R^*/\partial y_i^* \quad (37)\\ \dot{y}_i^* &= -\partial R^*/\partial x_i^* \quad (38)\end{aligned}\right\} \quad (i = 1, 2, 3),$$

where R^* is given by equation 25).

The solution obtained here does not suffer from the difficulty usually encountered in von Zeipel's method: that of obtaining x_i and y_i as functions of x_i^* and y_i^* when

S is a function of the old canonic variables, y_i, and the new canonic variables, x_i^*. Usually, in order to express x_i and y_i as a function of x_i^* and y_i^* it is necessary to proceed by a method of successive approximation. However, in the present solution, x_i and y_i have been expressed exactly in terms of x_i^* and y_i^*.

Integration of (37) and (38) yields

$$x_1^* = \mathring{x}_1^*, \tag{39}$$

$$x_2^* = \mathring{x}_2^*, \tag{40}$$

$$x_3^* = \mathring{x}_3^*, \tag{41}$$

$$y_1^* = \mathring{y}_1^* - (\partial R^*/\partial x_1^*)\,t, \tag{42}$$

$$y_2^* = \mathring{y}_2^* - (\partial R^*/\partial x_2^*)\,t, \tag{43}$$

$$y_3^* = \mathring{y}_3^* - (\partial R^*/\partial x_3^*)\,t, \tag{44}$$

where $\mathring{x}_1^*$, $\mathring{x}_2^*$, $\mathring{x}_3^*$, $\mathring{y}_1^*$, $\mathring{y}_2^*$ and $\mathring{y}_3^*$ are the constants of integration determined from the initial conditions. Expressions for $\partial R^*/\partial x_1^*$, $\partial R^*/\partial x_2^*$ and $\partial R^*/\partial x_3^*$ can readily be obtained from (25).

5. Discussion

If the quantities B and β are defined as

$$B = (x_2^* - P/A) \tag{45}$$

and

$$\beta = \tfrac{1}{2}\pi - \mathring{y}_2^*/(x_2^* - P/A), \tag{46}$$

then equations (32) and (35) can be written in the form

$$x_2 = P/A \pm B\sin(At+\beta), \tag{47}$$

$$y_2 = \pm B\cos(At+\beta), \tag{48}$$

which are the canonic equivalents of Cook's equations (Cook 1966) for the time variations of the Lagrangian variables ξ and η, respectively, namely

$$\eta = C/K + B\sin(Kt+\beta),$$

$$\xi = B\cos(Kt+\beta),$$

where

$$C = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \sum_{n=3}^{\infty} J_n \left(\frac{R_\mathrm{E}}{a}\right)^n \frac{(n-1)}{n(n+1)} P_n^1(\cos I)\, P_n^1(0),$$

$$K = \tfrac{3}{4}\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} J_2 \left(\frac{R_\mathrm{E}}{a}\right)^2 (5\cos^2 I - 1).$$

If the positive solution of (30) is chosen, then equations (47) and (48), when multiplied by $1/\sqrt{x_1^*}$, are identical with Cook's equations. If the negative solution is chosen and B redefined as $-B$, then (47) and (48), when multiplied by $1/\sqrt{x_1^*}$, are again identical with Cook's equations. It therefore follows that the duality of sign occurring in equation (30) can be removed by a suitable choice of the arbitrary constant B. A full discussion of equations (47) and (48) has been given by Cook.

The time variation of the elliptic elements a, e, I, ω, Ω and M can be recovered from the relations

$$\left.\begin{aligned} a &= x_1^2/\mu, \quad \omega = \arctan(x_2/y_2), \\ e &= (1/\sqrt{x_1})(x_2^2+y_2^2)^{\frac{1}{2}}, \quad \Omega = y_3, \\ I &= \arccos\left[\frac{x_3}{x_1}\left(1-\frac{x_2^2-y_2^2}{x_1}\right)^{-\frac{1}{2}}\right], \quad M = y_1-\omega, \end{aligned}\right\} \tag{49}$$

The method of approach presented here has three major advantages over a Lagrangian treatment. First, the time variations for all the elliptic elements have been obtained. Secondly, in order to obtain the solution it is only necessary to perform one major integration; that of solving (29) for S_1, whereas in a Lagrangian approach a, e, i, ω, Ω and M have to be obtained from the solution of six simultaneous first-order differential equations. Finally, a canonic approach is more suitable for evaluating the indirect effects of the zonal harmonics on satellites affected by lunisolar gravity and solar radiation pressure perturbations. The inclusion of these indirect effects is directly obtained by the von Zeipel method when the lunisolar disturbing function is added to $\bar{R}$.

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Appendix. Explicit expressions for the Hansen coefficients $X_0^{-(n+1),m}(e)$

By definition, $X_0^{-(n+1),m}(e)$ is given by (Plummer 1918)

$$X_0^{-(n+1),m}(e) = \frac{1}{2\pi}\int_0^{2\pi} (a/r)^{n+1}\cos mf\,\mathrm{d}M. \tag{50}$$

By using the well-known elliptic relations

$$\mathrm{d}M = (r/a)^2(1-e^2)^{-\frac{1}{2}}\,\mathrm{d}f \tag{51}$$

and

$$r = \frac{a(1-e^2)}{(1+e\cos f)}, \tag{52}$$

equation (50) becomes

$$X_0^{-(n+1),m}(e) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1+e\cos f)^{n-1}}{(1-e^2)^{\frac{1}{2}(2n-1)}}\cos mf\,\mathrm{d}f \tag{53}$$

$$= \frac{1}{2\pi(1-e^2)^{\frac{1}{2}(2n-1)}}\int_0^{2\pi}\left(\sum_{q=0}^{n-1}\binom{n-1}{q}e^q\cos^q f\right)\cos mf\,\mathrm{d}f, \tag{54}$$

where $\binom{n-1}{q}$ is the binomial coefficient $\dfrac{(n-1)!}{q!(n-1-q)!}$. Clearly, equation (54) can be developed as a series of multiple cosines in f, together with a constant factor. On integration only the constant factor remains. The constant term results from the combination of the $\cos mf$ term inside the brackets in (54) with the $\cos mf$ term outside the bracket. The $\cos mf$ term inside the bracket is

$$\sum_{q=0}^{(n-1)} \frac{1}{2^{q-1}} \binom{q}{\tfrac{1}{2}(q-m)} \cos mf, \quad m \neq 0, \tag{55}$$

and, for $m = 0$,

$$\sum_{q=0}^{n-1} \frac{1}{2^q} \binom{q}{\tfrac{1}{2}q}. \tag{56}$$

Substituting (55) and (56) into (54), when $m \neq 0$ and when $m = 0$, respectively, and integrating yields

$$X_0^{-(n+1),0}(e) = \frac{1}{(1-e^2)^{\frac{1}{2}(2n-1)}} \sum_{q=0}^{n-1} \frac{1}{2^q} \binom{n-1}{q} \binom{q}{\tfrac{1}{2}q} e^q \tag{57}$$

and

$$X_0^{-(n+1),m}(e) = \frac{1}{(1-e^2)^{\frac{1}{2}(2n-1)}} \sum \frac{1}{2^q} \binom{n-1}{q} \binom{q}{\tfrac{1}{2}(q-m)} e^q. \tag{58}$$

The summations in (57) and (58) are limited to even $\frac{1}{2}(q-m)$, $q \geqslant m$ and $m \leqslant n-1$, therefore

$$X_0^{-(n+1),m}(e) = \frac{1}{(1-e^2)^{\frac{1}{2}(2n-1)}} \sum_{q=m}^{n-1} \frac{1}{2^q} \binom{n-1}{q} \binom{q}{\tfrac{1}{2}(q-m)} e^q. \tag{59}$$

If $m = n$, then $X_0^{-(n+1),n}(e) = 0$.

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SATELLITE ORBITS PERTURBED BY DIRECT SOLAR RADIATION PRESSURE: GENERAL EXPANSION OF THE DISTURBING FUNCTION

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Abstract—An expression is derived for the solar radiation pressure disturbing function on an Earth satellite orbit which takes into account the variation of the solar radiation flux with distance from the Sun's centre and the absorption of radiation by the satellite. This expression is then expanded in terms of the Keplerian elements of the satellite and solar orbits using Kaula's method. The Kaula inclination functions are replaced by an equivalent set of modified Allan inclination functions.

The resulting expression reduces to the form commonly used in solar radiation pressure perturbation studies (e.g. Aksnes, 1976), when certain terms are neglected. If, as happens quite often in practice, a satellite's orbit is in near-resonsnce with certain of these neglected terms, these near-resonant terms can cause changes in the satellite's orbital elements comparable to those produced by the largest term in Aksnes's expression. A new expression for the solar radiation pressure disturbing function expansion is suggested for use in future studies of satellite orbits perturbed by solar radiation pressure.

1. INTRODUCTION

In order to exploit the full accuracy of laser observations of artificial satellites for geophysical purposes it is necessary to have a good theory for the motion of an artificial satellite perturbed by solar radiation; such a theory must include an accurate expression for the solar radiation pressure disturbing function expanded in terms of the Keplerian elements of the satellite and solar orbits.

A number of authors have suggested models for the effect of direct solar radiation pressure on artificial satellites; all of these are, however, subject to certain limitations. Cook (1962) proposes a model in which the solar radiation flux is assumed constant and directed along the line of centres joining the Sun and the Earth. Kaula (1962) gives an expression for the solar radiation pressure disturbing potential expanded as a function of the satellite's orbital elements, but the expansion was derived using the same assumptions as Cook. This type of expansion has been used in a number of studies of solar radiation pressure perturbations, e.g. Brouwer (1963), Gooding (1966) and, more recently, Aksnes (1976).

It may be argued that an accurate expression for the solar radiation pressure disturbing function expansion is unnecessary because of the uncertainties in the area-to-mass ratio and reflection characteristics of a satellite, together with limitations in the theory for the "shadow" and "albedo" effects. But errors of the order of several per cent incurred in the removal of direct solar radiation pressure perturbation will not be negligible, when information about the higher-order harmonics in the geopotential is being sought from laser observations. An accurate expression is therefore necessary.

An obvious improvement to the existing model is to assume that the solar radiation flux varies inversely as the square of the distance from the Sun's centre and is directed towards the satellite along the line of centres joining the Sun and the satellite. A model for the solar radiation pressure disturbing potential and its expansion based on these assumptions will now be derived.

2. THE FORM OF THE SOLAR RADIATION PRESSURE DISTURBING POTENTIAL

Let S_0 be the solar radiation flux at a distance a^* from the Sun's centre equal to the semi-major axis of the Earth's orbit, then the solar radiation flux, S, incident on a sunlit satellite distant Δ from the Sun is given by

$$S=\frac{S_0 a^{*2}}{\Delta^2}. \tag{1}$$

If $\bar{A}$ is the average cross-sectional area of the satellite exposed to the Sun's radiation, ε is the fraction of incident radiation absorbed by the satellite, and c is the speed of light, then the magnitude of the radiation force, $F_{\rm rad}$, when the satellite is in sunlight, is

$$F_{\rm rad}=\frac{S_0 a^{*2}\bar{A}(2-\varepsilon)}{\Delta^2 c}. \tag{2}$$

In practice, the accurate theory developed here would be applicable primarily for satellites with constant $\bar{A}$ and ε, that is, for spheres of uniform surface textures, such as Lageos (1976-39A).

The vector form of (2) is

$$\mathbf{F}_{\text{rad}} = \frac{-S_0 a^{*2} \bar{A}(2-\varepsilon)}{cm_s \Delta^3} \sigma \underline{\Delta}, \tag{3}$$

where $\sigma = 1$ when the satellite is in sunlight and $\sigma = 0$ when in shadow, m_s is the mass of the satellite and Δ is the position vector of the Sun from the satellite. If (x, y, z) and (X, Y, Z) are the Cartesian co-ordinates of the satellite and the Sun, respectively, relative to a set of axes centred at the Earth and in directions given by the unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, then (3) can be written as:

$$\mathbf{F}_{\text{rad}} = \frac{-S_0 \bar{A} a^{*2}(2-\varepsilon)}{cm_s} \sigma \times \left[\frac{(X-x)\mathbf{i} + (Y-y)\mathbf{j} + (Z-z)\mathbf{k}}{\{(X-x)^2 + (Y-y)^2 + (Z-z)^2\}^{3/2}} \right]. \tag{4}$$

On defining the solar radiation pressure disturbing potential Φ_{rad} by

$$\mathbf{F}_{\text{rad}} = \nabla \Phi_{\text{rad}} = \left(\mathbf{i} \frac{\partial \Phi_{\text{rad}}}{\partial x} + \mathbf{j} \frac{\partial \Phi_{\text{rad}}}{\partial y} + \mathbf{k} \frac{\partial \Phi_{\text{rad}}}{\partial z} \right), \tag{5}$$

then on equating equation (4) and (5) and integrating, Φ_{rad} is found to be given by

$$\Phi_{\text{rad}} = \frac{-S_0 \bar{A} a^{*2}(2-\varepsilon)}{cm_s \Delta} \sigma \tag{6}$$

Now Δ can be written as

$$\Delta = R \left[1 + \left(\frac{r}{R} \right)^2 - \frac{2r}{R} \cos \delta \right]^{1/2.} \tag{7}$$

where r is the distance of the satellite from the Earth's centre, R is the Earth–Sun distance and δ is the angle subtended at the Earth's centre by the satellite and the Sun. Since Δ^{-1} is a generating function for the Legendre polynomials of argument $\cos \delta$, (6) becomes

$$\Phi_{\text{rad}} = \frac{-S_0 \bar{A}(2-\varepsilon)}{cm_s R} \sigma \sum_{n=1}^{\infty} P_n(\cos \delta) \frac{r^n}{R^n}. \tag{8}$$

The $n = 0$ term has been omitted from (8) because it is independent of the satellite's co-ordinates, and will therefore produce no changes in its orbital elements. Equation (8) is of a comparable form to the expression for the lunisolar gravity disturbing potential, viz.

$$\frac{GM_D^*}{R} \sum_{n=2}^{\infty} P_n(\cos \delta) \frac{r^n}{R^n}, \tag{9}$$

where M_D^* is the mass of the Sun or Moon, although the index of summation and the constant term are different for the solar radiation pressure case. Consequently, Kaula's method (Kaula, 1962) for expanding (9) in terms of the Keplerian elements of the satellite and solar orbits can be applied to (8). If Kaula's inclination functions are replaced by the simpler, but equivalent, Allan inclination functions (Allan, 1965), then (8) becomes

$$\begin{aligned}
\Phi_{\text{rad}} = {} & \frac{-S_0 \bar{A}(2-\varepsilon) a^{*2}}{cm_s} \sigma \\
& \times \sum_{n=1}^{\infty} \frac{a^n}{(a^*)^{n+1}} \sum_{m=0}^{n} (-1)^{n-m} K_m \frac{(n-m)!}{(n+m)!} \\
& \times \sum_{p=0}^{n} F_{n,m,p}(i) \sum_{h=0}^{n} F_{n,m,h}(i^*) \\
& \times \sum_{q=-\infty}^{+\infty} X_{(n-2p+q)}^{n,(n-2p)}(e) \sum_{j=-\infty}^{+\infty} X_{(n-2h+j)}^{-(n+1),(n-2h)}(e^*) \\
& \times \cos[(n-2p)\omega + (n-2p+q)M - (n-2h)\omega^* \\
& - (n-2h+j)M^* + m(\Omega - \Omega^*)],
\end{aligned} \tag{10}$$

where the asterisked quantities refer to the solar orbit relative to the celestial equator, and

$$K_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0. \end{cases} \tag{11}$$

The $F_{n,m,p}(i)$ and $F_{n,m,h}(i^*)$ are the Allan inclination functions defined by

$$\begin{aligned}
F_{n,mp}(i) = {} & \frac{(n+m)!(\sqrt{-1})^{n-m}}{2^n p!(n-p)!} \\
& \times \sum_K (-1)^k \binom{2n-2p}{k} \binom{2p}{n-m-k} \\
& \times (\cos \tfrac{1}{2} i)^{3n-m-2p-2k} (\sin \tfrac{1}{2} i)^{m-n+2p+2k}
\end{aligned} \tag{12}$$

with a similar expression for $F_{n,m,h}(i^*)$. The quantities $\binom{2n-2p}{k}$ and $\binom{2p}{n-m-k}$ in (12) are the binomial coefficients.

$$\frac{(2n-2p)!}{k!(2n-2p-k)!} \text{ and } \frac{2p!}{(n-m-k)!(2p-n+m+k)!},$$

respectively. The eccentricity functions, $X_{(n-2p+q)}^{n,(n-2p)}(e)$ and $X_{(n-2h+j)}^{(n+1),(n-2h)}(e^*)$, are the usual Hansen coefficients (Plummer, 1918). The a, e, i, ω, Ω and M, etc., are the usual symbols denoting the Keplerian elements of an orbit. Since

$$(-1)^{n-m}(\sqrt{-1})^{2(n-m)}=1$$

and $\Omega^*=0$, by definition, for the Sun, then on replacing $X^{n,(n-2p)}_{(n-2p+q)}(e)$ and $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*)$ with the shorthand notation $G_{n,p,q}(e)$ and $H_{n,h,j}(e^*)$, equation (10) becomes

$$\Phi_{\text{rad}}=\frac{-S_0\bar{A}(2-\varepsilon)}{cm_s}\sigma\sum_{n=1}^{\infty}\frac{a^n}{(a^*)^{n-1}}$$

$$\times\sum_{m=0}^{n}k_m\frac{(n-m)!}{(n+m)!}\sum_{p=0}^{n}\bar{F}_{n,m,p}(i)\sum_{h=0}^{n}\bar{F}_{n,m,h}(i^*)$$

$$\times\sum_{q=-\infty}^{+\infty}G_{n,p,q}(e)\sum_{j=-\infty}^{+\infty}H_{n,h,j}(e^*)$$

$$\times\cos[(n-2p)\omega+(n-2p+q)M$$

$$-(n-2h)\omega^*-(n-2h+j)M^*+m\Omega], \qquad (13)$$

where

$$\bar{F}_{n,m,p}(i)=\frac{1}{(\sqrt{-1})^{n-m}}\times F_{n,m,p}(i),$$

with a similar expression for $\bar{F}_{n,m,h}(i^*)$. The quantities $\bar{F}_{n,m,p}(i)$ and $\bar{F}_{n,m,h}(i^*)$ are now real quantities.

The usual expression for the solar radiation pressure disturbing function can be obtained if only the long-period terms having $n=1$ and $j=0$ are considered: such terms are characterized by the set of integers n, m, p, q, h and j given by

$$n=1\left[\begin{array}{l} m=0\left[\begin{array}{l} p=0\begin{bmatrix} h=0\\ h=1\end{bmatrix} q=-1\\ p=1\begin{bmatrix} h=0\\ h=1\end{bmatrix} q=+1\end{array}\right]\\ m=1\left[\begin{array}{l} p=0\begin{bmatrix} h=0\\ h=1\end{bmatrix} q=-1\\ p=1\begin{bmatrix} h=0\\ h=1\end{bmatrix} q=+1\end{array}\right]\end{array}\right] j=0.$$

From the definition of a Hansen coefficient (see Appendix 1), $G_{1,0,-1}(e)$ and $G_{1,1,1}(e)$ are given by

$$G_{1,0,-1}(e)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)\cos f\,\mathrm{d}M \qquad (14)$$

$$G_{1,1,1}(e)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)\cos(-f)\,\mathrm{d}M, \qquad (14)$$

where f is the true anomaly of the orbit. On evaluating the integrals,

$$G_{1,0,-1}(e)=G_{1,1,1}(e)=\frac{-3c}{2}. \qquad (15)$$

From equation (39) derived in Appendix 1, $H_{1,1,0}(e^*)$ and $H_{1,0,0}(e^*)$ are given by

$$H_{1,1,0}(e^*)=H_{1,0,0}(e^*)=1-\frac{e^{*2}}{2}+0(e^{*4}). \qquad (16)$$

If equation (12) is used to evaluate the Allan inclination functions for terms with $n=1$ and $j=0$, then, with the aid of (15) and (16), the expression for the corresponding disturbing potential is

$$\Phi^0_{\text{rad}}=\frac{3S_0\bar{A}(2-\varepsilon)ae(1-e^{*2}/2)}{2cm_s}\sigma$$

$$\times[2csc^*s^*\cos\{\omega-(\omega^*+M^*)\}$$

$$-2csc^*s^*\cos\{\omega+(\omega^*+M^*)\}$$

$$+c^2c^{*2}\cos\{\omega-(\omega^*+M^*)+\Omega\}$$

$$+c^2s^{*2}\cos\{\omega+(\omega^*+M^*)+\Omega\}$$

$$+s^2c^{*2}\cos\{-\omega-(\omega^*+M^*)+\Omega\}$$

$$+s^2s^{*2}\cos\{-\omega+(\omega^*+M^*)+\Omega\}], \qquad (17)$$

where $c=\cos\frac{1}{2}i$, $s=\sin\frac{1}{2}i$, $c^*=\cos\frac{1}{2}i^*$ and $s^*=\sin\frac{1}{2}i^*$. Equation (17) is Aksnes's expression for the solar radiation pressure disturbing function expansion, although the form of the constant factor is slightly different here, as a result of allowing for the variation of solar flux with distance.

The next most important terms are those for which $n=1$ and $j=\pm1$, i.e. terms of $0(ee^*)$. If equations (35) and (36) are used to evaluate the Hansen coefficients $H_{1,0,1}(e^*)$, $H_{1,1,1}(e^*)$, $H_{1,1,-1}(e^*)$ and $H_{1,0,-1}(e^*)$ to $0(e^*)$, then

$$\begin{aligned} H_{1,0,1}(e^*)&=H_{1,1,-1}(e^*)=2e^*+0(e^{*3})\\ H_{1,1,1}(e^*)&=H_{1,0,-1}(e^*)=0+0(e^{*3}). \end{aligned} \qquad (18)$$

Using equation (12) to evaluate the appropriate inclination functions and making use of equations (18), the expression for the long-period terms with

$n=1$ and $j=\pm 1$ is Φ^1_{rad}, where

$$\Phi^1_{\text{rad}} = \frac{3ee^*S_0\bar{A}(2-\varepsilon)a\sigma}{cm_s}$$
$$\times[2csc^*s^*\cos(\omega-\omega^*-2M^*)$$
$$-2csc^*s^*\cos(\omega+\omega^*+2M^*)$$
$$+c^2c^{*2}\cos(\omega-\omega^*-2M^*+\Omega)$$
$$+c^2s^{*2}\cos(\omega+\omega^*+\Omega)$$
$$+s^2c^{*2}\cos(-\omega-\omega^*-2M^*+\Omega)$$
$$+s^2s^{*2}\cos(-\omega+\omega^*+2M^*+\Omega)]. \quad (19)$$

Do any other n terms, apart from $n=1$, contribute significantly to the disturbing function expansion? Since for most close Earth satellites e is small ($e<0.03$) and $a/a^* \simeq 1/20{,}000$, it appears that only the $n=2$ terms of long-period and zero order in e and e^* need to be considered. These are the terms for which $n=2$, $p=1$, $q=0$ and $j=0$, that is

$$n=2\left\{\begin{bmatrix} m=0 \\ m=1 \\ m=2 \end{bmatrix}\right\} p=1\left\{\begin{bmatrix} h=0 \\ h=1 \\ h=2 \end{bmatrix}\right\} q=0\left\{\left[\, j=0.\right.\right. \quad (20)$$

If the solar Hansen coefficients for the terms in (20) which are of the order $1+0(e^{*2})$ are put equal to 1, then on using (12) to evaluate the appropriate inclination functions and the identity $X_0^{2,0}(e) = 1+3e^2/2$ (see Appendix 1), the disturbing potential Φ^2_{rad} is given by

$$\Phi^2_{\text{rad}} = \frac{-S_0\bar{A}(2-\varepsilon)a^2}{cm_sa^*}\left(1+\frac{3e^2}{2}\right)\sigma$$
$$\times[-\tfrac{3}{2}(\tfrac{3}{2}\sin^2 i-1)c^{*2}s^{*2}\cos 2(\omega^*+M^*)$$
$$+\tfrac{1}{4}(\tfrac{3}{2}\sin^2 i-1)(\tfrac{3}{2}\sin^2 i^*-1)$$
$$+3cs(c^2-s^2)c^*s^{*3}\cos\{2(\omega^*+M^*)+\Omega\}$$
$$-3cs(c^2-s^2)c^{*3}s^*\cos\{-2(\omega^*+M^*)+\Omega\}$$
$$+3cs(c^2-s^2)c^*s^*(c^{*2}-s^{*2})\cos\Omega$$
$$+\tfrac{3}{2}c^2s^2c^{*4}\cos\{-2(\omega^*+M^*)+2\Omega\}$$
$$+3c^2s^2c^{*2}s^{*2}\cos 2\Omega$$
$$+\tfrac{3}{2}c^2s^2s^{*4}\cos\{2(\omega^*+M^*)+2\Omega\}]. \quad (21)$$

It can be seen from equation (21) that the inclusion of the $n=2$ terms of zero order in e and e^* leads to the appearance of a secular term, albeit with small amplitude. Such a secular term produces secular changes in the argument of perigee, ω, the longitude of the ascending node, Ω, and the mean anomaly M, of the satellite's orbit when substituted into the Lagrangian planetary equations (Smart, 1953) for $\dot{\omega}$, $\dot{\Omega}$ and $\dot{M}$.

Similar expressions can be found for long-period terms of $0(ee^{*2})$ with $n=1$, and long-period terms of $0(e^*)$ with $n=2$. These are less likely to be important and are given in Appendix 2.

3. DISCUSSION

The ratio of the terms outside the square brackets in (19) and (17) is $2e^*=0.033$, so that Φ^1_{rad} is by no means negligible; and its effect will be enhanced if some of the cosine terms are near resonant, as quite often happens in practice. For example, if a satellite has the following set of elements: $a=6960$ km, $e=0.007$ and $i=56.06°$, then, on using the well-known expressions (King-Hele, 1958)

$$\dot{\omega} \simeq 5.0\left(\frac{R_E}{a}\right)^{3.5}(1-e^2)^{-2}(5\cos^2 i-1)\ \text{deg/day} \quad (22)$$

and

$$\dot{\Omega} \simeq -10.0\left(\frac{R_E}{a}\right)^{3.5}(1-e^2)^{-2}\cos i\ \text{deg/day} \quad (23)$$

(where R_E is the mean equatorial radius of the Earth) the argument of the first cosine term in (19) is found to vary at the rate of 0.08 deg/day, whilst the arguments of the other cosine terms are found to vary at rates exceeding 2 deg/day. Similarly, the arguments of the cosine terms in (17) vary at rates exceeding 1 deg/day.

Let Y_i ($i=1, 2, \ldots 6$) be the orbital elements of the satellite such that

$$\begin{array}{ll} Y_i = a & Y_4=\omega \\ Y_2=e & Y_5=\Omega \\ Y_3=i & Y_6=M. \end{array}$$

The Lagrangian planetary equations for Y_i when perturbed by one term in (13) can be written on integration as (Smart, 1953)

$$Y_i \approx Y_i^0 + \frac{Z_i(Y)}{\dot{\psi}}\left[\begin{matrix}\sin \\ \cos\end{matrix}(\psi) - \begin{matrix}\sin \\ \cos\end{matrix}(\psi^0)\right] \quad (i=1,2,3\ldots 6), \quad (24)$$

where $Z_i(Y)$ is a function of the satellite's orbital elements, ψ is the argument of the perturbing term and $\dot{\psi}$ its time derivative. The symbol in square brackets in (24) means either $\sin\psi$ or $\cos\psi$; $\sin\psi$

should be used for Y_4, Y_5 and Y_6, and $\cos\psi$ should be used for Y_1, Y_2 and Y_3; Y_i^0 and ψ^0 are the values of Y_i and ψ at time $t=0$. If Q_i is the ratio of the change in Y_i due to a near-resonant term in (13) to that produced by a non-resonant term, then

$$Q_i \approx \frac{{}_R Z_i(Y)\dot{\psi}}{Z_i(Y)\dot{\psi}_R}\left[\left(\begin{matrix}\sin\\ \cos\end{matrix}(\psi_R) - \begin{matrix}\sin\\ \cos\end{matrix}(\psi_R{}^0)\right)\Big/ \left(\begin{matrix}\sin\\ \cos\end{matrix}(\psi) - \begin{matrix}\sin\\ \cos\end{matrix}(\psi^0)\right)\right] \quad (i=1,2,3\dots 6). \tag{25}$$

The subscript R denotes the near-resonant part of (25). If terms of $0(e^2)$ are neglected, then the ratios $S_i = {}_R Z_i(Y)/Z_i(Y)$ $(i=1,2,3\dots 6)$ are given by

$$\begin{aligned}
S_1 &= \frac{(n-2p+q)_R A_R}{(n-2p+q)A}\\
S_2 &= \frac{q_R A_r}{qA}\\
S_3 &= \frac{(\cot i(n-2p) - m \operatorname{cosec} i)_R A_R}{(\cot i(n-2p) - m \operatorname{cosec} i)A}\\
S_4 &= \frac{\left(\frac{\partial A_R}{\partial e} - e\cot i\frac{\partial A_R}{\partial i}\right)_R}{\left(\frac{\partial A}{\partial e} - e\cot i\frac{\partial A}{\partial i}\right)}\\
S_5 &= \frac{\partial A_R}{\partial i}\Big/\frac{\partial A}{\partial i}\\
S_6 &= \frac{\left(\frac{\partial A_R}{\partial e} + 2ae\frac{\partial A_R}{\partial a}\right)}{\left(\frac{\partial A}{\partial e} + 2ae\frac{\partial A}{\partial a}\right)}.
\end{aligned} \tag{26}$$

Suppose the integration of the Lagrangian planetary equations is carried out from $t=0$ to $T/2$, where T is the period of the near-resonant term; equation (25) can then be written as

$$|Q_i| > \left|S_i \frac{\dot{\psi}^0}{\dot{\psi}_R{}^0}\begin{matrix}\sin\\ \cos\end{matrix}(\psi_R{}^0)\right|.$$

The maximum values of $|\sin\psi_R{}^0|$ and $|\cos\psi_R{}^0|$ are both unity. Consequently, if $|\sin\psi_R{}^0| = 1$, then $|\cos\psi_R{}^0| = 0$, and the $|Q_i|$ values for ω, Ω and M are such that

$$|Q_i| > \left|\frac{S_i\dot{\psi}^0}{\dot{\psi}_R{}^0}\right|, \tag{27}$$

whilst the $|Q_i|$ values for a, e and i are zero. Similarly, if $|\cos\psi_R{}^0| = 1$, then $|\sin\psi_R{}^0| = 0$, the $|Q_i|$ values for ω, Ω and M are now zero, whilst the $|Q_i|$ values for a, e and i are given by (27). In practice $|\cos\psi_R{}^0|$ and $|\sin\psi_R{}^0|$ will have values between zero and unity. For the sake of argument let us assume that they are both equal, i.e. $|\sin\psi_R{}^0| = |\cos\psi_R{}^0| = 1/\sqrt{2}$: equation (27) is now replaced by

$$|Q_i| = \frac{1}{\sqrt{2}}\left|S_i\frac{\dot{\psi}^0}{\dot{\psi}_R{}^0}\right| \quad i=1,2,\dots 6. \tag{28}$$

In the subsequent discussion it is assumed that equation (28) holds. Equation (28) is not exact since the effect of the Earth's shadow has been neglected but it should serve to give some indication of the relative magnitudes of near-resonant and non-resonant changes in a satellite's orbital elements, over a half cycle of the near-resonance. Furthermore in any accurate study involving near-circular orbits ($e<0.03$) the set of elements a, $e\cos\omega$, $e\sin\omega$, Ω, i and $\omega+M$ (Cook, 1966) should be used so as to avoid small divisors in e^{-1} occurring in the solution for the time variations of ω and M. However in the qualitative discussion given here the set of elements a, e, i, ω, Ω and M should suffice.

If a satellite has the orbital elements: $a = 6960$ km, $e=0.007$ and $i=56.06°$, then $|Q_2|$ and $|Q_3|$ values for the near-resonant term and the largest term in (17) are 0.20 and 0.24, respectively. It is therefore possible for a near-resonance with a Φ^1_{rad} term to produce changes in a satellite's orbital eccentricity and inclination which are 20 and 24%, respectively, of those produced by the main term in Φ^0_{rad}. The satellites 1965-53B, 1965-53C, 1965-53D, 1965-53E, 1965-53F, 1968-70A and 1968-70B have orbital elements approximately equal to those considered in the above example. It is easy to find orbits for which $\dot{\psi}_R{}^0 < 0.08$ deg/day: for example, changing i to 56.21° would reduce $\dot{\psi}_R{}^0$ to 0.04 deg/day. The changes in e and i would then be 40 and 48% of those produced by the main term in Φ^0_{rad}. It should be remembered, of course, that these changes occur over a half-cycle of the near-resonant term, that is, over about 6 yr if $\dot{\psi}_R = 0.08$ deg/day. The effects would therefore be of little importance in numerical day-to-day analysis of actual orbits; but it would be vital to include them in any accurate theoretical study covering several years.

Since the Φ^2_{rad} terms are non-zero when $\varepsilon = 0$, the effects of thses terms in comparison with the Φ^0_{rad} and Φ^1_{rad} terms, which are proportional to e, increases as e tends to zero. Consequently, for a circular orbit only the Φ^2_{rad} terms are present in the disturbing function, when $\Phi_{\text{rad}} = \Phi^0_{\text{rad}} + \Phi^1_{\text{rad}} + \Phi^2_{\text{rad}}$. If the eccentricity is small (which is true for most

Earth satellites launched to date) then the occurrence of a near-resonance with a Φ^2_{rad} term can produce changes in the satellite's orbital elements which are comparable to those produced by the main term in Φ_{rad}. For example, if a satellite has the following set of orbital elements: $a = 7700$ km, $e = 0.002$ and $i = 90.1°$, then $\dot{\Omega} \simeq 0.01$ deg/day and $\dot{\omega} \simeq -2.6$ deg/day. Such a satellite is in near-resonance with two terms in Φ^2_{rad}, but only the cos 2Ω term need be considered since the value of the inclination function of the cos Ω term is small when $i \simeq 90°$ and is zero for a satellite in an exact polar orbit. The largest $|Q_3|$ value for this near-resonant term and the term in (17) which produces the largest change in i is 0.12. So when the eccentricity is small, a satellite in a near-resonance with a Φ^2_{rad} term can undergo changes in its orbital inclination which are 12% of those produced by the main term in Φ^0_{rad}, though again only on a long time scale, about 25 yr.

Errors of the order of magnitude discussed in the above examples incurred in the removal of solar radiation pressure perturbations on artificial satellites cannot be ignored if accurate geophysical information is sought from long-term laser observations of artificial satellites. It is therefore essential in future studies of solar radiation pressure perturbations on such satellites to include the Φ^1_{rad} and Φ^2_{rad} terms in the expression for Φ_{rad}.

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APPENDIX 1

Important Hansen Coefficients

By definition, the Hansen coefficient $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*)$ is given by (Plummer, 1918)

$$X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^{n+1} \times \cos\{(n-2h)f-(n-2h+j)M\}\,dM, \quad (29)$$

where f is the true anomaly of the orbit. Plummer has shown how to develop (29) as a series in e^* and the subsidiary variable β defined by

$$\beta = e^*/\{1+(1-e^{*2})^{1/2}\}. \quad (30)$$

The result is

$$X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*) = (1+\beta^2)^n \sum_{\theta=-\infty}^{+\infty} J_\theta\{(n-2h+j)e^*\} \times X^{-(n+1),(n-2h)}_{(n-2h+j),\theta}(e^*), \quad (31)$$

where

$$X^{-(n+1),(n-2h)}_{(n-2h+j),\theta}(e^*) = \begin{cases} (-\beta)^{j-\theta}\begin{pmatrix}-2n+2h\\ j-\theta\end{pmatrix}F(2n-2h+j-\theta, 2h, j-\theta+1;\beta^2) & \text{for } (j-\theta)\geqslant 0\\ (-\beta)^{-j+\theta}\begin{pmatrix}-2h\\ -j+\theta\end{pmatrix}F(\theta+2h-j, 2n-2h, -j+\theta+1;\beta^2) & \text{for } -j+\theta\geqslant 0. \end{cases} \quad (32)$$

Here $J_\theta\{(n-2h+j)e^*\}$ is an ordinary Bessel function of degree θ and the symbol F denotes a hypergeometric function. It is clear from (31) and (32) that for small e^*, the Hansen coefficient $X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*)$ is of order $e^{*|j|}$. Since $\beta = \frac{1}{2}e^* + 0(e^{*3})$ and

$$J_\theta\{(n-2h+j)e^*\} = \frac{1}{\theta!}\left(\frac{(n-2h+j)e^*}{2}\right)^\theta + 0(e^{*\theta+2}) \quad \theta\geqslant 0$$

$$J_\theta\{(n-2h+j)e^*\} = (-1)^{|\theta|}J_{|\theta|}\{(n-2h+j)e^*\} \quad \theta\leqslant 0.$$

It follows that

$$X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*) = \sum_{\theta=0}^{|j|}\{-\tfrac{1}{2}e^*\}^{|j|}\frac{\{-(n-2h+i)\}^\theta}{\theta!} \times \begin{pmatrix}-2n+2h\\ j-\theta\end{pmatrix} + 0\{e^{*j+2}\} \quad \text{if } j\geqslant 0 \quad (33)$$

or

$$X^{-(n+1),(n-2h)}_{(n-2h+j)}(e^*) = \sum_{\theta=0}^{|j|}\{-\tfrac{1}{2}e^*\}^{|j|}\frac{(n-2h+j)^\theta}{\theta!} \times \begin{pmatrix}-2h\\ -j-\theta\end{pmatrix} + 0\{e^{*|j|+2}\} \quad \text{if } j\leqslant 0. \quad (34)$$

Hence $H_{n,h,1}(e^*)$ and $H_{n,h,-1}(e^*)$ are obtained to order e^* if $j=1$ and $j=-1$ are substituted into (33) and (34), respectively. On simplification, it is found that

$$H_{n,h,1}(e^*) = \frac{(3n-4h+1)e^*}{2} + 0(e^*3) \quad (35)$$

and

$$H_{n,h,-1}(e^*)=\frac{(4h-n+1)e^*}{2}+0(e^{*3}). \tag{36}$$

Similarly, if $j=\pm 2$, the expressions for $H_{n,h,+2}(e^*)$ and $H_{n,h,-2}(e^*)$ to $0(e^{*2})$ are

$$H_{n,h,+2}(e^*)=\frac{(9n^2+16h^2-24hn-18h+14n+4)e^{*2}}{8} \tag{37}$$

and

$$H_{n,h,-2}(e^*)=\frac{(n^2+16h^2-8hn+18h-4n+4)e^{*2}}{8}. \tag{38}$$

If the appropriate expressions for the Bessel and hypergeometric functions are used, then, after a considerable amount of algebra, $H_{n,h,0}(e^*)$ is found to be given up to $0(e^{*2})$ by

$$H_{n,h,0}(e^*)=1+\frac{(n-3n^2+16hn-16h^2)\,e^{*2}}{4}.$$

APPENDIX 2

Additional Terms in the Expression for $\Phi_{\rm rad}$

On making use of equations (12), (13), (37) and (38), the expression $\Phi^3_{\rm rad}$ for the long-period terms of $0(ee^{*2})$ and with $n=1$ is found to be

$$\begin{aligned}\Phi^3_{\rm rad}=&\frac{81S_0\bar{A}(2-\varepsilon)aee^{*2}}{16cm_s}\sigma\\
&\times\Big[2csc^*s^*\cos(\omega-\omega^*-3M^*)\\
&-2csc^*s^*\cos(\omega+\omega^*+3M^*)\\
&+c^2c^{*2}\cos(\omega-\omega^*-3M^*+\Omega)\\
&+c^2s^{*2}\cos(\omega+\omega^*-M^*+\Omega)\\
&+s^2c^{*2}\cos(-\omega-\omega^*-3M^*+\Omega)\\
&+s^2s^{*2}\cos(-\omega+\omega^*+3M^*+\Omega)\\
&+\tfrac{2}{27}csc^*s^*\cos(\omega-\omega^*+M^*)\\
&-\tfrac{2}{27}csc^*s^*\cos(\omega+\omega^*-M^*)\\
&+\frac{c^2c^{*2}}{27}\cos(\omega-\omega^*+M^*+\Omega)\\
&+\frac{c^2s^{*2}}{27}\cos(\omega+\omega^*+3M^*+\Omega)\\
&+\frac{s^2c^{*2}}{27}\cos(-\omega-\omega^*+M^*+\Omega)\\
&+\frac{s^2s^{*2}}{27}\cos(-\omega+\omega^*-M^*+\Omega)\Big].\end{aligned} \tag{40}$$

Similarly, the expression $\Phi^4_{\rm rad}$ for the long-period terms in $\Phi_{\rm rad}$ of $0(e^*)$ having n values of 2 is

$$\begin{aligned}\Phi^4_{\rm rad}=&\frac{-S_0\bar{A}(2-\varepsilon)a^2e^*}{2cm_sa^*}\sigma\left(1+\frac{3e^2}{2}\right)\\
&\times\Big[-\tfrac{21}{2}(\tfrac{3}{2}\sin^2 i-1)c^{*2}s^{*2}\cos(2\omega^*+3M^*)\\
&+\tfrac{3}{2}(\tfrac{3}{2}\sin^2 i-1)c^{*2}s^{*2}\cos(2\omega^*+M^*)\\
&+\tfrac{3}{2}(\tfrac{3}{2}\sin^2 i-1)(\tfrac{3}{2}\sin^2 i^*-1)\cos\omega^*\\
&-21cs(c^2-s^2)c^{*3}s^*\cos(-2\omega^*-3M^*+\Omega)\\
&+3cs(c^2-s^2)c^{*3}s^*\cos(-2\omega^*-M^*+\Omega)\\
&+9cs(c^2-s^2)c^*s^*(c^{*2}-s^{*2})\cos(-M^*+\Omega)\\
&+9cs(c^2-s^2)c^*s^*(c^{*2}-s^{*2})\cos(M^*+\Omega)\\
&-3cs(c^2-s^2)c^*s^{*3}\cos(2\omega^*+M^*+\Omega)\\
&+21cs(c^2-s^2)c^*s^{*3}\cos(2\omega^*+3M^*+\Omega)\\
&+\tfrac{21}{2}c^2s^2c^{*4}\cos(-2\omega^*-3M^*+2\Omega)\\
&-\frac{c^2s^2c^{*4}}{2}\cos(-2\omega^*-M^*+2\Omega)\\
&+9c^2s^2c^{*2}s^{*2}\cos(-M^*+2\Omega)\\
&+9c^2s^2c^{*2}s^{*2}\cos(M^*+2\Omega)\\
&-\tfrac{3}{2}c^2s^2s^{*4}\cos(2\omega^*+M^*+2\Omega)\\
&+\tfrac{21}{2}c^2s^2s^{*4}\cos(2\omega^*+3M^*+2\Omega)\Big].\end{aligned} \tag{41}$$