Controller Size Reduction in Advanced Control System Design

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by

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Statement of Originality

The accompanying thesis submitted for the degree of Doctor of Philosophy entitled Controller Size Reduction in Advanced Control System Design is based on work conducted by the author in the Department of Engineering of the University of Leicester mainly during the period between October 1990 and September 1993. All the work recorded in this thesis is original unless otherwise acknowledged in the text or by references. None of the work has been submitted for another degree in this or any other University.

Byung-Wook Choi

Byung-Wook Choi November 10, 1993

To

my parents

my late parents-in-law

my wife, Choon-Sil

and

my daughter, Sue-Jin and my son, Young-Jin.

Abstract

The work described in this thesis was undertaken to obtain a unified treatment to the controller size reduction problem in advanced robust control system design. A common feature in state-space solutions to advanced control system design, such as parametrizations of all stabilizing controllers and $\mathcal{H}_\infty$ suboptimal controllers, is that a free parameter matrix is contained in the parametrization to give the designer freedom in designing the required controllers. However, this free parameter can provide unnecessarily high order controllers.

This thesis presents a new methodology for controller size reduction. The methodology utilizes the parametrization of all stabilizing controllers and $\mathcal{H}_\infty$ suboptimal controllers, and then generates a set of low-order stabilizing controllers and a set of low-order $\mathcal{H}_\infty$ suboptimal controllers, respectively. The central idea is to achieve a low-order realization of a full-order controller, by deriving and solving two simultaneous matrix equations in order to eliminate unobservable states. Orthogonal canonical forms are employed to solve these simultaneous equations. A consequence of the algorithms employed is that the order of the controller is reduced from $n + n_q$ (or $n + n_\phi$) to n_q (or n_ϕ), where n is the order of the weighted plant and n_q (or n_ϕ) is the order of the free parameter.

In design applications, a possible solution to the problem of combining the objectives of robust stability and performance requirements is to use a loop shaping design procedure based on normalized coprime factor plant descriptions. The methodology obtained for low-order $\mathcal{H}_\infty$ suboptimal controllers is extended, with slight modifications, to one and two degree-of-freedom loop shaping design procedures.

The results are illustrated by numerical examples. Finally, a practical industrial problem of designing a low-order controller for a tetrahedral robot is considered by applying the methodology developed in the thesis.

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Chapter 1

Introduction

1.1 Background

Advanced control system design addresses multivariable plants, high performance requirements, significant model uncertainties, and disturbance signals. Each of these characteristics forces the control system to be complex in some sense. A good measure of the complexity of a linear state-space model is the dimension of its states, sometimes referred to as the system order. Examples of high-order systems are: nuclear power plant (> 500 states); chemical processes (> 100 states); compact disc players (> 100 states); flexible spacecraft (> 20 states), etc.

The complexity of a linear controller, like the plant under control, can also be measured by the dimension of its states. In present-day robust/optimal control system design with its emphasis on robust stability and robust performance of the closed-loop system, the complexity of the controller can easily exceed that of the plant. In particular, $\mathcal{H}_\infty$ optimal design typically leads to complex controllers, i.e., controllers of high state dimensions. For example, a standard $\mathcal{H}_\infty$ design procedure generates a controller of the same order as that of the generalized plant model (including actuators and weighting functions) to be controlled.

The high complexity of a controller adversely affects cost, commissioning, and maintainability. In contrast, a **lower-order controller** would be desirable, because it provides easier implementation (i.e., less on-line computational requirements), easier test and commissioning, easier maintenance, and easier training for a plant's operating personnel, etc. It is known, however, that there might be undesirable deterioration in the closed-loop performance when a low-order controller is being used in the place of a full-order controller.

It is therefore natural to wish to minimize control system complexity for high-order plant, subject to the achievement of satisfactory performance specifications in the face of uncertainty. That is, it may be reasonable for us to look for a reduced-order controller instead of the best full-order controller thereby trading-off complexity against reduced performance. The concept of 'simplicity' in designing controllers is analogous in spirit to the following philosophy:

Things should be as simple as possible, but no simpler.

It is desirable therefore to reduce controller complexity from both the design process and the final controller. The problem of reducing the dynamic order (i.e., McMillan degree) of the controller while retaining closed-loop stability and performance will be called the **Controller Size Reduction Problem**. It has become a research topic of great practical significance and has attracted much interest from researchers.

1.2 Motivation

While model reduction techniques have been well developed, the problem of controller size reduction is not easy and remains an open problem because of the following reasons:

- The resulting controller is required to have the lowest possible order, for 'simplicity', while capturing the important features of the control problem.
- Despite many previous efforts at the problem, a number of important issues - stability, robustness and performance - have not been properly addressed. Therefore, trade-offs between the controller size reduction and the allowable performance deterioration (due to controller size reduction) are to be made.
- The methodology for controller size reduction must be flexible enough to deal with controller structures, and should not lead to sophisticated procedures.

In advanced control system design, robust/optimal controllers may be designed for complex multivariable feedback systems to achieve stringent performance objectives in the presence of (unstructured) uncertainty. However, algorithms for synthesizing such controllers, e.g., $\mathcal{H}_\infty$ optimization, are not able to explicitly constrain the complexity of the control law. A common feature in the available state-space solutions for

- parametrizations of all stabilizing controllers [75]
- parametrizations of $\mathcal{H}_\infty$ suboptimal controllers [18],[27]
- parametrizations of robustly stabilizing controllers (using the normalized coprime factor plant descriptions) [28],[47]
- parametrizations of two degree-of-freedom (2-DOF) $\mathcal{H}_\infty$ controllers [43]

is that a free parameter matrix $Q(s)$ (or $\Phi(s)$) is contained in the parametrization to give the designer freedom in designing the required controllers. However, the freedom in this parameter usually leads to unnecessarily high order controllers. Although a good model reduction technique can be applied to the control law to reduce its complexity, an alternative methodology for reducing controller size is seen as an important and valuable research goal.

1.3 Related Literature

The literature includes much work on controller size reduction as well as on model reduction. In the account that follows, however, only the major results which provide a fundamental basis and motivation for the direction of this thesis are mentioned. More details of related literature can be found in the main text.

Fundamental to control system design is the requirement of internal stability. A celebrated solution to the stabilization problem is the parametrization of all stabilizing feedback controllers for a given plant, using a free parameter. This was initially developed by Youla *et al.* [75] and generalized by Desoer *et al.* [15]. Such a parametrization provides a basis to the $\mathcal{H}_\infty$ (sub)optimal control problem. The latter is an important problem in advanced robust control system design, where stabilizing controllers which satisfy an upper bound on the $\mathcal{H}_\infty$-norm of a certain closed-loop transfer function matrix are to be found. Glover and Doyle [27] and Doyle *et. al* [18] have recently provided an elegant state-space solution to this problem via two Riccati equations. In this, the set of all $\mathcal{H}_\infty$ suboptimal controllers is parametrized using plant data and a free parameter.

A simple, yet very useful design procedure based on $\mathcal{H}_\infty$ optimization and ideas from classical control was introduced by McFarlane and Glover [47]. It is called the Loop Shaping Design Procedure (LSDP), and is essentially a one degree of freedom (1-DOF) design scheme. First the plant is modified by pre- and post-compensating weights to shape the open-loop singular values so that they correspond to good closed-loop performance and robust stability properties as described by Doyle and Stein [19]. Then an $\mathcal{H}_\infty$ robust stabilization property is solved to maximize robustness with respect to perturbations in the normalized coprime factors of the shaped plant.

To introduce more performance objectives into the control problem, a 2-DOF scheme can be employed, e.g., Youla and Bongiorno [74]. An extended loop shaping design procedure in an $\mathcal{H}_\infty$ setting was recently proposed by Limebeer *et al.* [43] building on the 1-DOF LSDP of McFarlane and Glover. In this 2-DOF scheme the extra degree of freedom is used for model-matching the closed-loop transfer function to an ideal response.

1.4 Contribution and Organization of Thesis

In this thesis, we present a new and unified methodology to the problem of controller size reduction in advanced robust control system design. By utilizing and suitably choosing a free parameter, which is commonly involved in important parametrizations of controllers such as the Youla parametrization [75], Glover and Doyle's parametrization [27] and Glover and McFarlane's parametrization [28], an observable (or controllable) realization of a controller of low-order is obtained in each case. Consequently, the methodology improves the usefulness of such parametrizations in practice.

Design objectives for advanced robust control systems are more than just $\mathcal{H}_\infty$-norm bounds. For example, we might have time domain requirements on rise time, settling time, overshoot, and undershoot, and there might be requirements on the controller size. The work in this thesis contributes to optimal yet small size controller design. The major contributions of this thesis are covered in Chapters 4 to 8 and are considered to be:

- A low-order stabilization problem is solved in Chapter 4 using the Youla parametrization of all stabilizing controllers. A constructive algorithm for computing the low-order stabilizing controllers is developed in state-space form. The main elements of Chapter 4 have been published in [12],[30].

- It is shown in Chapter 5 that the order of $\mathcal{H}_\infty$ suboptimal controllers parametrized by Glover and Doyle may be reduced, in a constructive way, by suitably choosing a free parameter $\Phi(s)$. The bulk of Chapter 5 has been presented in [31].

- Constructive algorithms are similarly given for computing low-order controllers from the 1-DOF and 2-DOF LSDP in Chapters 6 and 7, respectively. The results of Chapter 6 have been presented in [13].

- Illustrative examples are given to verify the theory developed and to show the details of the steps involved. Further, in Chapter 8, the theory developed is applied to a practical industrial problem.

This thesis consists of 9 chapters, and we next give an outline of the main contents of the chapters that follow.

Chapter 2: Preliminaries

In this chapter, we review relevant results from linear systems theory. Included are orthogonal canonical forms, the Sylvester equation, the Lyapunov equation, coprime factorizations, norms of systems, relations between the Riccati equation and an $\mathcal{H}_\infty$-norm bound, and linear fractional transformations.

Chapter 3: Model Reduction and Controller Size Reduction

In this chapter, an overview of the model reduction and controller size reduction problem is given. Balanced realizations are introduced, followed by model reduction techniques such as modal residualization, balanced truncation, Hankel norm model reduction, and coprime factor model reduction. Some existing results on controller size reduction are also described.

Chapter 4: Low-Order Stabilizing Controller Design

In this chapter, we derive an algorithm to generate a set of low-order stabilizing controllers, using the well-known Youla parametrization [75] of all stabilizing feedback controllers. The algorithm requires a solution to two simultaneous matrix equations. Pole assignability via output feedback is shown utilizing a separation property. An explicit formula for low-order stabilizing controllers is derived as a special case. Relevant issues are discussed, and a constructive algorithm is given together with numerical examples.

Chapter 5: Low-Order $\mathcal{H}_\infty$ Sub-Optimal Controller Design

In this chapter, low-order $\mathcal{H}_\infty$ suboptimal controllers are derived by extending the concept developed in Chapter 4 to the state-space solution of Glover and Doyle [27] for the standard $\mathcal{H}_\infty$ suboptimal control problem. A constructive algorithm is given which requires a solution to two simultaneous matrix equations, as in Chapter 4, but subject to an $\mathcal{H}_\infty$-norm constraint. Two numerical examples are presented to illustrate the results.

Chapter 6: Low-Order Robust Sub-Optimal Controller Design

In this chapter, the so-called $\mathcal{H}_\infty$ Loop Shaping Design Procedure (1-DOF LSDP) of McFarlane and Glover [47] is considered. It is shown that the procedures discussed in Chapter 5 can be carried over to the 1-DOF LSDP, with only slight modifications, to derive low-order robust suboptimal controllers. An example is presented.

Chapter 7: An Extension to $\mathcal{H}_\infty$ 2-DOF Controller Design

In this chapter, the results of Chapters 5 and 6 are further extended to the two degree-of -freedom $\mathcal{H}_\infty$ controller design procedure (2-DOF LSDP) of Limebeer *et al.* [43].

Chapter 8: Application to the GEC-Alsthom Tetrahedral Robot

In this chapter, we consider a practical industrial problem of designing a low-order robust suboptimal controller for a tetrahedral robot, *Tetrabot*. The results of this thesis are applied to the problem to demonstrate their effectiveness and comparisons are made with a "central" $\mathcal{H}_\infty$ optimal controller which can be obtained using standard $\mathcal{H}_\infty$ algorithms.

Chapter 9: Conclusions and Future Research

This final chapter contains concluding remarks and suggestions for further research.

1.5 Notation

1.5.1 Symbols

All systems in this thesis are linear, multivariable, finite-dimensional and time-invariant, and possess real-rational transfer function matrices. The work is carried out in continuous time unless otherwise stated.

A (proper) transfer function matrix is represented in terms of state-space data by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

alternatively written as (A, B, C, D), where A, B, C and D are real matrices of appropriate dimensions and I is the identity matrix. If $D = 0$, the zero matrix, then the system is strictly proper and we shall write (A, B, C). The matrix A is *asymptotically stable* if and only if each of its eigenvalues has a strictly negative real part. In that case the system (A, B, C, D) is also called asymptotically stable.

Standard notation is used as far as possible as listed below.

ν_c	denotes the controllability index.
ν_o	denotes the observability index.
$\mathcal{N}$	denotes the order of a full-order controller.
$\mathcal{N}_{central}$	denotes the order of a central controller.
$\mathcal{N}_{low}$	denotes the order of a low-order controller.
s	Laplace variable. ($s = jw$ yields the frequency response.)
$\mathcal{R}e[x]$	Real part of x.
$\mathcal{R}^n$	n-dimensional real Euclidean space.
$\mathcal{C}^n$	n-dimensional complex Euclidean space.
$\mathcal{R}^{m \times l}$	Set of real $m \times l$ matrices.
$\mathcal{C}^{m \times l}$	Set of complex $m \times l$ matrices.
A_{ij}	The (i, j) element of A.
I_{m_i}	$m_i \times m_i$ identity matrix.
$0_{m \times l}$	$m \times l$ zero matrix.
A^T	Transpose of real matrix A.
A^H	Transpose of complex conjugate of matrix A.
$A^\dagger$	Pseudo-inverse of matrix A.
$A \geq 0$	Matrix A is positive semi-definite.
$A > 0$	Matrix A is positive definite.
$A^{1/2}$	For matrix $A \geq 0$, any square matrix B such that $A = B^H B$.
$\det(A)$	Determinant of matrix A.
$\mathrm{rank}(A)$	Rank of matrix A.
$\mathrm{tr}(A)$	Trace of matrix A.
$\lambda_i(A)$	The i-th eigenvalue of matrix A.
$\lambda_{max}(A)$	Largest eigenvalue of matrix A.
$\lambda_{min}(A)$	Smallest eigenvalue of matrix A.

$\rho(A)$	Spectral radius of A, i.e., $\max\{\lvert\lambda\rvert : \lambda \in \lambda(A)\}$.
$\sigma_i(A)$	The i-th singular value of matrix A.
$\sigma_{max}(A)$	Largest singular value of matrix A.
$\sigma_{min}(A)$	Smallest singular value of matrix A.
$G(s)$	denotes a transfer function matrix.
$G^*(s)$	$:= G^T(-s)$, i.e., the parahermitian conjugate of $G(s)$.
$G^H(s)$	$:= G^T(\overline{s})$, i.e., the complex-conjugate transpose of $G(s)$.
$\deg(G)$	Degree of $G(s)$, i.e., the number of states of $G(s)$.
$\sigma_{max}(G)$	Largest singular value of $G(s)$.
$\sigma_{min}(G)$	Smallest singular value of $G(s)$.
$\sigma_i^H(G)$	Hankel singular value of $G(s)$.
$\mathcal{RL}_\infty$	Lebesgue space of real rational matrices whose elements are proper and have no poles in the jw-axis.
$\mathcal{RH}_\infty$	Hardy space of real rational matrices whose elements are stable and proper.
$\mathcal{RL}_2$	Lebesgue space of real rational matrices whose elements are strictly proper and have no poles in the jw-axis.
$\mathcal{RH}_2$	Hardy space of real rational matrices whose elements are stable and strictly proper.
$\lVert x(t)\rVert_2$	$\mathcal{L}_2$-norm of a real vector valued signal $x(t)$.
$\lVert x(t)\rVert_{rms}$	RMS norm of a real vector valued signal $x(t)$.
$\lVert G(s)\rVert_2$	$\mathcal{H}_2$-norm of a transfer function matrix $G(s)$.
$\lVert G(s)\rVert_\infty$	$\mathcal{H}_\infty$-norm of $G(s)$.
$\lVert G(s)\rVert_{rms}$	RMS gain of $G(s)$, equal to its $\mathcal{H}_\infty$-norm.
$\lVert G(s)\rVert_H$	Hankel-norm of $G(s)$.
$\in$	'An element of'.
$\forall$	'For all'.
$\neq$	'Is not equal to'.
$X := Y$	'Y is defined as X'.
$X =: Y$	'X is defined as Y'.

Finally, the notation $X := \mathbf{Ric}\begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix}$ is used to denote a stabilizing solution matrix X to an algebraic Riccati equation:

$$E^T X + XE - XWX + Q = 0.$$

1.5.2 Abbreviations

ARE	Algebraic Riccati Equation
BIBO	Bounded-Input Bounded-Output
CAA	Controllability Argument Approach
CAD	Computer Aided Design
CLHP	Closed Left-Half Plane
CLTF	Closed-Loop Transfer Function
CRHP	Closed Right-Half Plane
DOF	Degree of Freedom
LCF	Left Coprime Factorization
LFT	Linear Fractional Transformation
LQG	Linear Quadratic Gaussian
LSDP	Loop Shaping Design Procedure
LTI	Linear Time-Invariant
MIMO	Multi-Input Multi-Output
OAA	Observability Argument Approach
PI	Proportional plus Integral
PID	Proportional plus Integral and Derivative
RCF	Right Coprime Factorization
RMS	Root Mean Square
SISO	Single-Input Single-Output
SVD	Singular Value Decomposition

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we introduce some well-known results on continuous time linear time-invariant systems, which will be particularly useful in the following chapters.

Section 2.2, on linear systems theory, includes the state-space form of a transfer function matrix, operations in linear systems, controllability and observability. Orthogonal canonical forms are introduced in Section 2.3, and algebraic equations such as the Sylvester and Lyapunov equations are reviewed in Section 2.4. In Section 2.5, definitions and an important theorem on coprime factorizations are given. Relevant norms of systems are introduced in Section 2.6, and a useful relationship between the Riccati equation and an $\mathcal{H}_\infty$-norm bound is discussed in Section 2.7. Finally, linear fractional transformations are reviewed in Section 2.8, together with an alternative chain scattering description.

2.2 Some Systems Theory

2.2.1 Transfer Functions

Consider a linear state-space model $G(s)$ described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0 \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

where $x(t) \in \mathcal{R}^n$ is the state vector of the system, $u(t) \in \mathcal{R}^m$ is the control vector and $y(t) \in \mathcal{R}^l$ is a vector of measurements, and $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{l\times n}$, and $D \in \mathcal{R}^{l\times m}$ are real matrices.

Taking Laplace transforms of (2.1)-(2.2), the resulting transfer function is

$$G(s) = C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.3}$$

Note that (2.3) says nothing more than (2.1)-(2.2); in other words, there is no implication that the realization is minimal or not.

Throughout this thesis, we will assume that the direct transmission matrix of the plant model is a zero matrix ($D = 0$), that is, the plant is strictly proper. This assumption of strictly properness is not essential and can be removed at the expense of more cumbersome formulas. Thus, this assumption may be justified not only by computational simplicity but also by the fact that most real systems are indeed strictly proper.

2.2.2 Operations on Linear Systems

Under a state similarity transformation, $\hat{x} = Tx$, system $G(s) = (A, B, C, D)$ becomes

$$G(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] \tag{2.4}$$

where T is invertible.

Given two systems defined by $G_1(s) = (A_1, B_1, C_1, D_1)$ and $G_2(s) = (A_2, B_2, C_2, D_2)$, a state-space representation of the *series-connected system* is given by

$$\begin{aligned} G_1(s) \times G_2(s) &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \times \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] \qquad (2.5) \\ &= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right] \qquad (2.6) \end{aligned}$$

There may be cancellations between the poles of one system and the transmission zeros of the other, in which case this realization will not be minimal even if $G_1(s)$ and $G_2(s)$ were minimal. A minimal realization can be obtained as described, for example, in Maciejowski [46, Section 8.3.5].

Given $G_1(s)$ and $G_2(s)$ as above, the state-space representation of the *parallel-connected system* is given by

$$\begin{aligned} G_1(s) + G_2(s) &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right] \qquad (2.7) \end{aligned}$$

Again, this realization may not be minimal even if $G_1(s)$ and $G_2(s)$ are minimal.

Given the system $G(s) = (A, B, C, D)$, we can obtain a state-space realization for the *inverse system* as

$$G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right] \tag{2.8}$$

provided the system is square (i.e., $l = m$) and D is nonsingular. If the system is not square, then a pseudo inverse $D^{\dagger}$ of D can be used in the place of D^{-1}.

Finally, a state-space realization for the *dual system* is given by

$$G^T(s) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right] \tag{2.9}$$

and the *parahermitian conjugate system* by

$$G^*(s) = G^T(-s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]. \tag{2.10}$$

2.2.3 Controllability and Observability

For the system (2.1)-(2.2), the pair (A, B) is *controllable* if, for each time $t_1 > 0$ and final state x_1, there exists a continuous input $u(t)$ such that the solution of (2.1) satisfies $x(t_1) = x_1$.

Lemma 2.1 *The following are equivalent:*

1. (A, B) is controllable.

2. The matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has independent rows.

3. The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has rank n for all eigenvalues of A in C.

4. $\lambda_i(A+BF)$ $(i = 1, \cdots, n)$ can be freely assigned subject to complex conjugate pairs by suitable choice of F.

The pair (A, B) is *stabilizable* when there exists an F such that $A + BF$ is stable.

Lemma 2.2 *The following are equivalent:*

1. (A, B) *is stabilizable.*

2. The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has rank* n *for all eigenvalues of* A *in CRHP.*

3. The uncontrollable modes of the system matrix A *are stable.*

We will now consider the dual notions of observability and detectability for the system (2.1)-(2.2). The pair (A, C) is *observable* if, for every $t_1 > 0$, the function $y(t)$, $t \in [0, t_1]$, uniquely determines the initial state x_0.

Lemma 2.3 *The following are equivalent:*

1. (A, C) *is observable.*

2. The matrix $\begin{bmatrix} C^T & A^TC^T & \cdots & (A^T)^{n-1}C^T \end{bmatrix}^T$ *has independent columns.*

3. The matrix $\begin{bmatrix} A^T - \lambda I & C^T \end{bmatrix}^T$ *has rank* n *for all eigenvalues of* A *in* $\mathcal{C}$*.*

4. $\lambda_i(A+HC)$ $(i = 1, \cdots, n)$ *can be freely assigned subject to complex conjugate pairs by suitable choice of* H*.*

5. (A^T, C^T) *is controllable.*

The pair (A, C) is *detectable* when there exists an H such that $A+HC$ is stable.

Lemma 2.4 *The following are equivalent:*

1. (A, C) *is detectable.*

2. The matrix $\begin{bmatrix} A^T - \lambda I & C^T \end{bmatrix}^T$ *has rank* n *for all eigenvalues of* A *in CRHP.*

3. The unobservable modes of the system matrix A *are stable.*

4. (A^T, C^T) *is stabilizable.*

The following definitions, first introduced by Luenberger [45], are of particular importance in this thesis.

Definition 2.5 The *controllability index*, ν_c, of the system (2.1)-(2.2) is the least positive integer for which the matrix

$$\begin{bmatrix} B & AB & \cdots & A^{\nu_c - 1}B \end{bmatrix}$$

has rank n. Similarly, the *observability index*, ν_o, of the system (2.1)-(2.2) is the least positive integer for which the matrix

$$\begin{bmatrix} C^T & A^T C^T & \cdots & (A^T)^{\nu_o - 1} C^T \end{bmatrix}^T$$

has rank n.

A realization of $G = (A, B, C, D)$ is *minimal* if and only if (A, B) is controllable and (A, C) is observable.

Consider the system (2.1)-(2.2) again and assume that C is of full rank. In this case, it can also be assumed that the matrix C takes the form

$$C = \begin{bmatrix} I_l & 0_{l \times (n-l)} \end{bmatrix}. \tag{2.11}$$

It is then convenient to partition x, A and B as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.12}$$

and accordingly write the system (2.1)-(2.2) in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.13}$$

$$y = \begin{bmatrix} I_l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.14}$$

The following Lemma 2.6 relates observability to that of particular partitioned block matrices and will be of use later.

Lemma 2.6 (Gopinath [29]) *If (A, C) is completely observable, then so is (A_{22}, A_{12}).*

Proof: Reference [29] gives a proof. We give an alternative proof below. Since (A, C) is completely observable, the matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ is of full column rank for all eigenvalues of A in $\mathcal{C}$. Using (2.11)-(2.12), we have

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & A_{22} - \lambda I \\ I & 0 \end{bmatrix} \tag{2.15}$$

and thus the right-hand side of (2.15) is of full rank. This is satisfied if and only if $\begin{bmatrix} A_{12} \\ A_{22} - \lambda I \end{bmatrix}$ is of full column rank. By Lemma 2.3 this is equivalent (A_{22}, A_{12}) is completely observable. ∎

2.3 Orthogonal Canonical Forms

Orthogonal transformations are useful for reducing a linear system (2.1)-(2.2) into a canonical form in a numerically stable way, e.g., Petkov *et al.* [55] and Van Dooren *et al.* [69]. Such orthogonal canonical forms play an important role in obtaining low-order controllers as described in this thesis.

Using an orthogonal matrix U, a controllable pair (A, B) can be reduced to the so-called *orthogonal canonical form* $(A_c, B_c) := (U^T A U, U^T B)$ with

$$A_c = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,\nu_c-1} & A_{1,\nu_c} \\ A_{21} & A_{22} & \cdots & A_{2,\nu_c-1} & A_{2,\nu_c} \\ 0 & A_{32} & \cdots & A_{3,\nu_c-1} & A_{3,\nu_c} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\nu_c,\nu_c-1} & A_{\nu_c,\nu_c} \end{bmatrix} \quad B_c = \begin{bmatrix} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.16}$$

where ν_c is the controllability index of (A, B), B_1 is $m_1 \times m$ and $A_{i,i-1}(i = 2, \cdots, \nu_c)$ are $m_i \times m_{i-1}$ matrices,

$$\begin{aligned} \text{rank}(B_1) &= m_1 \\ \text{rank}(A_{i,i-1}) &= m_i \qquad i = 2, \cdots, \nu_c \end{aligned}$$

and the numbers

$$m_1 \geq m_2 \geq \cdots \geq m_{\nu_c}$$
$$m_1 + m_2 + \cdots + m_{\nu_c} = n$$

are the *conjugate Kronecker indices* of the pair (A, B). The form (2.16) is also said to be the *block-Hessenberg form* (or staircase form) of the pair (A, B).

A dual realization is available, by working with the pair (A^T, C^T) instead of (A, B) and then transposing the result. That is, an observable pair (A, C) can be reduced into the orthogonal canonical form (A_o, C_o):

$$A_o = U^T A U = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & A_{\nu_o-1,\nu_o} \\ A_{\nu_o,1} & A_{\nu_o,2} & A_{\nu_o,3} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix} \tag{2.17}$$

$$C_o = CU = \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.18}$$

where ν_o is the observability index of (A, C), C_1 is $l \times l_1$ and $A_{i,i+1}(i = 1, \cdots, \nu_o - 1)$ are $l_i \times l_{i+1}$ matrices,

$$\text{rank}(C_1) = l_1$$
$$\text{rank}(A_{i,i+1}) = l_{i+1} \qquad i = 1, \cdots, \nu_o - 1$$

and the numbers

$$l_1 \geq l_2 \geq \cdots \geq l_{\nu_o}$$
$$l_1 + l_2 + \cdots + l_{\nu_o} = n$$

are the conjugate Kronecker indices of the pair (A, C).

Remark 2.7 The orthogonal canonical forms shown above provide a numerically reliable way to check for the controllability and observability of system (2.1)-(2.2). The matrix A_{ν_c,ν_c-1} (respectively A_{ν_o-1,ν_o}) is of full rank if the system is completely controllable (respectively observable), otherwise $A_{\nu_c,\nu_c-1} = 0$ (respectively $A_{\nu_o-1,\nu_o} = 0$).

Remark 2.8 Stable computational algorithms for finding these orthogonal canonical forms are available using the so-called staircase algorithm, e.g., Konstantinov *et al.* [40] and Boley [4].

Following Konstantinov *et al.* [39], the pair (A, B) can be further transformed into the orthogonal canonical form (A_c, B_c) as in (2.19)-(2.20) below, provided the pair (A, B) is completely controllable:

$$A_c = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,\nu_c-1} & A_{1,\nu_c} \\ \begin{bmatrix} I_{m_2} & 0 \end{bmatrix} & A_{22} & \cdots & A_{2,\nu_c-1} & A_{2,\nu_c} \\ 0 & \begin{bmatrix} I_{m_3} & 0 \end{bmatrix} & \cdots & A_{3,\nu_c-1} & A_{3,\nu_c} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} I_{m_{\nu_c}} & 0 \end{bmatrix} & A_{\nu_c,\nu_c} \end{bmatrix} \tag{2.19}$$

$$B_c = \begin{bmatrix} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.20}$$

We next give an algorithm for obtaining the form (A_c, B_c) as above, since the dual result of this form will be frequently used in this thesis.

1. Given $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$, find a controllability index ν_c.

2. Do a singular value decomposition (SVD) on B as $B = U_1\Sigma_1V_1^T$, and then build $B^{(1)}$ as

$$B^{(1)} := U_1^T B G_1 = \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}$$

by choosing a suitable matrix G_1, where $m_1 = \text{rank}(B)$. Let $P_1 := U_1^{-T}$ and $Q_1 := G_1$. Then

$$A^{(1)} := P_1^{-1}AP_1.$$

3. Partition $A^{(1)}$ as

$$A^{(1)} := \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix}$$

and let $m_2 := \operatorname{rank}(A_{21}^{(1)})$.

4. Do a SVD on $A_{21}^{(1)}$ as $A_{21}^{(1)} = U_2\Sigma_2 V_2^T$, and then build $\bar{A}_{21}^{(1)}$ as

$$\bar{A}_{21}^{(1)} := U_2^T A_{21}^{(1)} G_2 = \begin{bmatrix} I_{m_2} \\ 0 \end{bmatrix}$$

by choosing a suitable matrix G_2. Let

$$P_2 := \begin{bmatrix} G_2 & 0 \\ 0 & U_2^{-T} \end{bmatrix} \quad \text{and} \quad Q_2 := \begin{bmatrix} G_2 & 0 \\ 0 & I_{m_1} \end{bmatrix}$$

then

$$\begin{aligned} A^{(2)} &:= (P_1P_2)^{-1}AP_1P_2 \\ B^{(2)} &:= (P_1P_2)^{-1}BQ_1Q_2. \end{aligned}$$

5. Partition $A^{(2)}$ as

$$A^{(2)} := \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & A_{13}^{(2)} \\ I_{m_2} & A_{22}^{(2)} & A_{23}^{(2)} \\ 0 & A_{32}^{(2)} & A_{33}^{(2)} \end{bmatrix}$$

and let $m_3 := \operatorname{rank}(A_{32}^{(2)})$.

6. Do a SVD on $A_{32}^{(2)}$ as $A_{32}^{(2)} = U_3\Sigma_3 V_3^T$, and then build $\bar{A}_{32}^{(2)}$ as

$$\bar{A}_{32}^{(2)} := U_3^T A_{32}^{(2)} G_3 = \begin{bmatrix} I_{m_3} \\ 0 \end{bmatrix}$$

by choosing a suitable matrix G_3. Let

$$P_3 := \begin{bmatrix} G_3 & 0 \\ 0 & U_3^{-T} \end{bmatrix} \quad \text{and} \quad Q_3 := \begin{bmatrix} G_3 & 0 \\ 0 & I_{m_2} \end{bmatrix}$$

then

$$\begin{aligned} A^{(3)} &:= (P_1P_2P_3)^{-1}AP_1P_2P_3 \\ B^{(3)} &:= (P_1P_2P_3)^{-1}BQ_1Q_2Q_3. \end{aligned}$$

7. Repeat this procedure until $m_{\nu_c} := \mathrm{rank}(A^{(\nu_c-1)}_{\nu_c,\nu_c-1})$.

8. Finally, we obtain

$$\begin{aligned} A^{(\nu_c-1)} &:= (P_1P_2\cdots P_{\nu_c})^{-1}A(P_1P_2\cdots P_{\nu_c}) = A_c \\ B^{(\nu_c-1)} &:= (P_1P_2\cdots P_{\nu_c})^{-1}B(Q_1Q_2\cdots Q_{\nu_c}) = B_c. \end{aligned}$$

By a dual argument, the pair (A, C) can be further transformed into the orthogonal canonical form (A_o, C_o) as in (2.21)-(2.22) below, provided the pair (A, C) is completely observable:

$$\begin{aligned} A_o &= MAM^{-1} \\ &= \begin{bmatrix} A_{11} & \begin{bmatrix} I_{l_2} \\ 0 \end{bmatrix} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \begin{bmatrix} I_{l_3} \\ 0 \end{bmatrix} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & \begin{bmatrix} I_{l_{\nu_0}} \\ 0 \end{bmatrix} \\ A_{\nu_o,1} & A_{\nu_o,2} & A_{\nu_o,3} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix} \end{aligned} \tag{2.21}$$

$$\begin{aligned} C_o &= NCM^{-1} \\ &= \begin{bmatrix} \begin{bmatrix} I_{l_1} \\ 0 \end{bmatrix} & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{2.22}$$

where the matrix M is a product of an orthogonal matrix and diagonal matrices, and N is a nonsingular matrix.

2.4 Algebraic Equations

In this section, we give some properties of the Sylvester and Lyapunov equations. The Sylvester equation and a standard linear equation are to be solved simultaneously in our methodology for deriving low-order controllers to be presented later.

2.4.1 Sylvester Equation

The standard Sylvester equation is of the form

$$AX + XB = C \tag{2.23}$$

where $A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{m \times m}$ and $C \in \mathcal{R}^{n \times m}$ are given matrices. Necessary and sufficient conditions for the existence and uniqueness of a solution X to the standard Sylvester equation are as follows, e.g., Kučera [41]:

- A solution $X \in \mathcal{R}^{n \times m}$ to the standard equation *exists* if and only if the matrices $\begin{bmatrix} B & 0 \\ C & -A \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & -A \end{bmatrix}$ are similar.
- The standard equation has a *unique* solution X if and only if $\mathcal{R}e[\lambda_i(A)] + \mathcal{R}e[\lambda_j(B)] \neq 0 \quad \forall i = 1, \cdots, n$ and $\forall j = 1, \cdots, m$.

The equation (2.23) is also called the general Lyapunov equation. In particular, if $B = A^T$, (2.23) is reduced to the standard Lyapunov equation.

2.4.2 Lyapunov Equation

For the standard Lyapunov equation

$$A^T X + XA + Q = 0 \tag{2.24}$$

with given real matrices $A \in \mathcal{R}^{n \times n}$ and $Q \in \mathcal{R}^{n \times n}$, $Q = Q^T > 0$, the following facts are well known, e.g., Barnett [2]:

1) If A is stable, then the solution is $X = \int_0^\infty e^{A^T t} Q e^{At}\, dt$.
2) If A is stable, then X is unique and positive definite. ($X \geq 0$ if $Q \geq 0$).

Given the solution X to (2.24), we may conclude the following stability properties of the matrix A:
1) $\mathcal{R}e[\lambda_i(A)] \leq 0$ if $X > 0$ and $Q \geq 0$.
2) A is asymptotically stable if $X > 0$ and $Q > 0$.

2.5 Coprime Factorizations

A number of well-known results on *coprime factorizations* found in Vidyasagar [70] will be used in this thesis, and are summarized below.

Definition 2.9 Suppose $M, N \in \mathcal{RH}_\infty$ have the same number of columns. Then M and N are *right coprime* if and only if there exist $U, V \in \mathcal{RH}_\infty$ such that

$$UN + VM = I. \qquad (2.25)$$

The relation (2.25) is called the (right) Bezout identity. It is possible to represent a possibly unstable transfer function in terms of two stable, coprime factors using a *right coprime factorization* which is defined as follows.

Definition 2.10 The pair (N, M), where $M, N \in \mathcal{RH}_\infty$, is a *Right Coprime Factorization* (RCF) of $G(s)$ if and only if
(a) M is square and $\det(M) \neq 0$
(b) $G = NM^{-1}$, and
(c) N and M are right coprime.

Left coprimeness and a *Left Coprime Factorization* (LCF) can be defined in an analogous way. Thus if $(\tilde{M}, \tilde{N})$ is a LCF of $G(s)$, then $G = \tilde{M}^{-1}\tilde{N}$.

Lemma 2.11 *Let (N, M), $(\tilde{M}, \tilde{N})$ be any RCF and LCF of $G(s)$. Suppose $U, V \in \mathcal{RH}_\infty$ satisfy*

$$UN + VM = I. \tag{2.26}$$

Then there exist $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix} V & U \\ -\tilde{N} & -\tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{U} \\ N & -\tilde{V} \end{bmatrix} = I. \tag{2.27}$$

The ordered pair of matrices in (2.27) is called a *doubly coprime factorization* of $G(s)$. State space constructions of a doubly coprime factors will be described later in Chapter 4, Subsection 4.2.2.

For a given $G(s)$, there are infinitely many coprime pairs. A special pair is a *normalized coprime factorization* which satisfies

$$\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I \qquad \text{(for an LCF)} \tag{2.28}$$

$$M^*M + N^*N = I \qquad \text{(for an RCF)} \tag{2.29}$$

and will be treated later in Chapter 6, Section 6.2.

2.6 Norms of Systems

In this section we review methods of measuring the size of an LTI system with input u and transfer function matrix $G(s)$. Of interest are the $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm.

The $\mathcal{H}_2$*-norm* of the stable transfer function matrix $G(s)$ is defined as

$$||G(s)||_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G^*(jw)G(jw))\,dw\right)^{1/2} \tag{2.30}$$

and measures, for example, the RMS response of its output when the input is a white noise process, e.g., Boyd and Barratt [7, p.110]. The LQG theory is concerned with minimizing $||T||_2$ for a suitably specified $T(s)$.

The $\mathcal{H}_\infty$-*norm* of the stable transfer function matrix $G(s)$ is defined as

$$\begin{aligned} ||G(s)||_\infty &:= \sup_{w\in\mathcal{R}} \left(\lambda_{max}(G^*(jw)G(jw))\right)^{1/2} \\ &= \sup_{w\in\mathcal{R}} \left(\sigma_{max}(G(jw))\right) \end{aligned} \tag{2.31}$$

and is importantly interpreted as the $\mathcal{L}_2$ or RMS gain of the system $G(s)$. This is because the RMS gain of a transfer function matrix is defined as

$$||G(s)||_{rms} := \sup_{||u||_{rms}\neq 0} \frac{||Gu||_{rms}}{||u||_{rms}} \tag{2.32}$$

which coincides with its $\mathcal{L}_2$ gain

$$||G(s)||_{rms} = \sup_{||u||_2\neq 0} \frac{||Gu||_2}{||u||_2} \tag{2.33}$$

where

$$\begin{aligned} ||u(t)||_2 &:= \left(\int_0^\infty u^2(t)\,dt\right)^{1/2} \\ ||u(t)||_{rms} &:= \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T u^2(t)\,dt\right)^{1/2}. \end{aligned}$$

Remark 2.12 If the transfer function matrix is scalar, then

$$||g(s)||_\infty = \sup_{w\in\mathcal{R}} |\, g(jw) \,|\,. \tag{2.34}$$

That is, the ∞-norm is the highest value of the Bode magnitude plot. On the Nyquist diagram, the ∞-norm is the maximum modulus of the frequency response $G(jw)$ over all real frequency w, i.e., the maximum distance from the origin to the Nyquist diagram of $G(s)$.

In control theory, the $\mathcal{H}_\infty$-norm of the closed-loop transfer function matrix can be interpreted as the worst case energy gain (actually, $||G||_\infty^2$ is the worst case energy gain, since $||u||_2^2$ represents energy). Hence, minimizing the $\mathcal{H}_\infty$-norm of a transfer function matrix is equivalent to minimizing the energy in the output signal due to the energy in the input signal. A further property of the $\mathcal{H}_\infty$-norm of a closed-loop transfer function matrix will be used in Chapters 5, 6 and 7.

That is, the $\mathcal{H}_\infty$-norm of a closed-loop transfer function matrix is a particularly useful measure to minimize because it enables robust stability guarantees to be made.

2.7 Algebraic Riccati Equations and $\mathcal{H}_\infty$-Norm Bounds

The algebraic Riccati equation (ARE) plays a key role in optimal control theory. In this section, an important relationship between the solution to a certain ARE and an $\mathcal{H}_\infty$-norm bound is considered. It will be used in connection with an $\mathcal{H}_\infty$-norm constraint on some auxiliary dynamics $\Phi(s)$ - a free parameter matrix - to be discussed in Chapters 5, 6 and 7.

Consider the following ARE:

$$E^T X + XE - XWX + Q = 0 \tag{2.35}$$

where $E, W, Q \in \mathcal{R}^{n\times n}$, $W = W^T$ and $Q = Q^T$. For (2.35), we define a corresponding Hamiltonian matrix as

$$\mathcal{M} = \begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix} \tag{2.36}$$

and define a unique *stabilizing* solution to (2.35) by $X = X^T$ and $\mathcal{R}e[\lambda_i(E - WX)] < 0$, [16]. We may denote the stabilizing solution via its Hamiltonian matrix as

$$X := \mathbf{Ric}\,[\mathcal{M}]\,. \tag{2.37}$$

The existence of ARE solutions is closely related to satisfying $\mathcal{H}_\infty$-norm bounds as is now described. Let a transfer function matrix $G(s)$ of appropriate dimensions be

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

For the following ARE:

$$\begin{aligned} &(A - BR_\gamma^{-1}D^TC)^T X_\gamma + X_\gamma (A - BR_\gamma^{-1}D^TC) \\ &-\gamma X_\gamma B R_\gamma^{-1} B^T X_\gamma - \gamma C^T S_\gamma^{-1} C = 0 \end{aligned} \tag{2.38}$$

where

$$\begin{aligned} R_\gamma &= D^TD - \gamma^2 I \\ S_\gamma &= DD^T - \gamma^2 I \end{aligned}$$

the Lemmas 2.13 to 2.17 below show the relationships between the stabilizing solution of the ARE (2.38), its positive definiteness and an $\mathcal{H}_\infty$-norm bound on $G(s)$. A proof of Lemma 2.13 can be found in [16] and proofs of Lemmas 2.14 to 2.17 are given in Appendix A.

Lemma 2.13 *There exists a unique stabilizing solution of the ARE (2.38) if and only if the Hamiltonian matrix*

$$\mathcal{M}_\gamma := \begin{bmatrix} A - BR_\gamma^{-1}D^TC & -\gamma B R_\gamma^{-1} B^T \\ \gamma C^T S_\gamma^{-1} C & -(A - BR_\gamma^{-1}D^TC)^T \end{bmatrix} \tag{2.39}$$

has no $j\omega$-axis eigenvalues.

Lemma 2.14 *The ARE (2.38) has a unique stabilizing solution*

$$X_\gamma := \mathbf{Ric}\,[\mathcal{M}_\gamma] \tag{2.40}$$

if A is stable and $||G(s)||_\infty < \gamma$.

Lemma 2.15 *$||G(s)||_\infty < \gamma$ if the ARE (2.38) has a stabilizing solution X_γ and A is stable.*

Lemma 2.16 *If the ARE (2.38) has a solution X_γ and A is stable and $\sigma_{max}(D) < \gamma$, then the solution X_γ is positive definite.*

Lemma 2.17 *If the ARE (2.38) has a positive definite solution X_γ, then A is stable and $||G(s)||_\infty \leq \gamma$.*

2.8 Linear Fractional Transformations

In this section, we review the Linear Fractional Transformation (LFT) which provides a general framework for $\mathcal{H}_\infty$ (sub)optimal design in Chapter 5. The alternative Chain Scattering Description (CSD) used in Chapter 6 is also briefly described.

Consider the augmented (or generalized) plant $P(s)$:

$$\dot{x} = Ax + B_2 u + B_1 w \tag{2.41}$$

$$y = C_2 x + D_{22} u + D_{21} w \tag{2.42}$$

$$z = C_1 x + D_{12} u + D_{11} w \tag{2.43}$$

where $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the vector of control signals, $w(t) \in \mathcal{R}^k$ is the vector of all signals entering the system, $y(t) \in \mathcal{R}^l$ is the vector of measured outputs, and $z(t) \in \mathcal{R}^j$ are the controlled outputs. The vector w may include, for example, reference inputs, disturbances and sensor noise, while z is a vector of all the signals required to characterize the behaviour of the closed-loop system, which includes errors, process outputs and control inputs.

This system $P(s)$ is shown in Figure 2.1 with a linear controller $K(s)$. The combination is referred to as the Standard Feedback Control Configuration.

Suppose that $P(s)$ is partitioned as

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.44}$$

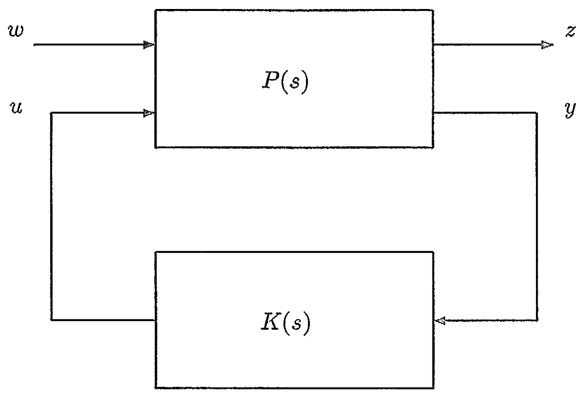


Figure 2.1: Standard Feedback Control Configuration.

$$:= \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \tag{2.45}$$

so that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{2.46}$$

with

$$P_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij} \qquad i, j = 1, 2 \tag{2.47}$$

where $A : n \times n$, $B_j : n \times m_j$, $C_i : p_i \times n$, $D_{ij} : p_i \times m_j$ $(i, j = 1, 2)$ are real matrices.

Then we obtain, using $u = K(s)y$,

$$z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w$$

and the closed-loop transfer function matrix mapping input w to output z, i.e.,

$$\mathcal{F}_l(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{2.48}$$

is called a *lower* Linear Fractional Transformation on $K(s)$ with the coefficient matrix $P(s)$. In $\mathcal{H}_\infty$ design the generalized plant $P(s)$ would include the nominal plant, weighting functions, and interconnections required to make the closed-loops, for example, from w to z in Figure 2.1. The LFT represents a means of standardizing a wide variety of feedback arrangements, and is frequently used in $\mathcal{H}_\infty$ control theory.

Let the representation of the controller be

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

then a state-space realization of the LFT, $\mathcal{F}_l(P, K)$, can be expressed, e.g., Postlethwaite *et al.* [57], by

$$\left[\begin{array}{cc|c} A + B_2 D_k \mathcal{D} C_2 & B_2 \tilde{\mathcal{D}} C_k & B_1 + B_2 D_k \mathcal{D} D_{21} \\ B_k \mathcal{D} C_2 & A_k + B_k D_{22} \tilde{\mathcal{D}} C_k & B_k \mathcal{D} D_{21} \\ \hline C_1 + D_{12} D_k \mathcal{D} C_2 & D_{12} \tilde{\mathcal{D}} C_k & D_{11} + D_{12} D_k \mathcal{D} D_{21} \end{array}\right] \tag{2.49}$$

where $\mathcal{D} := (I - D_{22} D_k)^{-1}$ and $\tilde{\mathcal{D}} := (I - D_k D_{22})^{-1}$. This realization is useful for the computation of the LFT. It should be noted that the realization (2.49) may not be a minimal realization.

If P_{21}^{-1} exists (which implies $p_2 = m_1$), an alternative expression for $\mathcal{F}_l(P, K)$ is given by a chain scattering description, namely:

$$\mathcal{F}_l(P, K) = CSD(G, K) := (G_{11} K + G_{12})(G_{21} K + G_{22})^{-1} \tag{2.50}$$

where $G(s)$ is a $(p_1 + m_1) \times (m_2 + p_2)$ matrix such that

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{2.51}$$

and can be expressed as

$$G = \begin{bmatrix} P_{12} - P_{11} P_{21}^{-1} P_{22} & P_{11} P_{21}^{-1} \\ -P_{21}^{-1} P_{22} & P_{21}^{-1} \end{bmatrix} =: \Gamma(P). \tag{2.52}$$

Note that the symbol Γ is used to denote the transformation from a linear fractional transformation matrix P to a chain scattering description matrix G.

Conversely, when G_{22}^{-1} exists, P can be expressed as

$$P = \begin{bmatrix} G_{11} & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{21} & G_{22} \\ I & 0 \end{bmatrix}^{-1} \tag{2.53}$$

$$= \begin{bmatrix} G_{12}G_{22}^{-1} & G_{11} - G_{12}G_{22}^{-1}G_{21} \\ G_{22}^{-1} & -G_{22}^{-1}G_{21} \end{bmatrix} =: \Gamma^{-1}(G). \tag{2.54}$$

Chapter 3

Model Reduction and Controller Size Reduction

3.1 Introduction

In modelling a dynamic system, the designer often tries to generate a reduced-order model which still gives a good representation of the true system. A compromise is usually to be made between the simplicity of the model and the accuracy of the results from control systems analysis. The reduced model is treated as if it represented the true system. It is hoped that the resulting control works when applied to the true system. The designer is generally satisfied if he/she can obtain a reasonably simplified model that is adequate for the problem under consideration.

Unlike model reduction procedures, a controller size reduction procedure should take into account the presence of the plant and thus the closed-loop considerations, although some model reduction techniques can be used for controller size reduction.

In this chapter, an overview of the methods for model reduction and controller size reduction is given. The chapter is organized as follows. Balanced realizations are first introduced in Section 3.2. These represent a convenient structure for model reduction and/or controller size reduction. In Section 3.3, some model reduction techniques such as balanced truncation, Hankel norm model reduction, and coprime factor model reduction are reviewed. In Section 3.4, some existing approaches to controller size reduction are briefly described with some examples. Concluding remarks are given in Section 3.5.

3.2 Balanced Realizations

A balanced realization of a transfer function matrix serve as a starting point either in model reduction or in conventional controller size reduction techniques. Hence a brief review is given here.

Let $G(s) = (A, B, C, D)$ be an asymptotically stable and minimal, $l \times m$ system having n states. The associated controllability gramian is defined as

$$L_c := \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt \tag{3.1}$$

and the observability gramian as

$$L_o := \int_0^\infty e^{A^Tt}C^TCe^{At}\,dt. \tag{3.2}$$

By integrating the corresponding matrix differential equations:

$$\frac{d}{dt}e^{At}BB^Te^{A^Tt} = Ae^{At}BB^Te^{A^Tt} + e^{At}BB^Te^{A^Tt}A^T$$

$$\frac{d}{dt}e^{A^Tt}C^TCe^{At} = A^Te^{A^Tt}C^TCe^{At} + e^{A^Tt}C^TCe^{At}A$$

from 0 to ∞, respectively, it can be shown that L_c and L_o satisfy the following Lyapunov equations

$$AL_c + L_cA^T + BB^T = 0 \tag{3.3}$$

$$A^T L_o + L_o A + C^T C = 0. \tag{3.4}$$

The controllability gramian L_c is symmetric, positive definite and may be solved for S such that

$$L_c = S^T S \tag{3.5}$$

using Cholesky factorization. Similarly, the observability gramian L_o may be factored as

$$L_o = R^T R, \qquad L_o > 0. \tag{3.6}$$

Hankel singular values of the system $G(s)$ are defined to be the positive square roots of the eigenvalues of $L_c L_o$ (or equivalently $L_o L_c$), i.e.,

$$\sigma_i^H := [\lambda_i(L_c L_o)]^{1/2} = [\lambda_i(L_o L_c)]^{1/2} \qquad i = 1, \cdots, n. \tag{3.7}$$

Define U and V to be the singular vectors of the singular value decomposition of the product SR^T. Then

$$SR^T = U \Sigma V^T \tag{3.8}$$

where

$$\Sigma = \text{ diag } (\sigma_i(SR^T)). \tag{3.9}$$

Suppose the state is transformed by a nonsingular matrix T_b to $\hat{x} = T_b x$ to yield the realization

$$G(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} T_b A T_b^{-1} & T_b B \\ \hline C T_b^{-1} & D \end{array}\right]. \tag{3.10}$$

Then the gramians L_c and L_o are transformed to

$$\hat{L}_c = T_b L_c T_b^T \tag{3.11}$$

$$\hat{L}_o = T_b^{-T} L_o T_b^{-1} \tag{3.12}$$

and thus are *not* invariant under coordinate transformations. However, the Hankel singular values are invariant since

$$\lambda_i(\hat{L}_c\hat{L}_o) = \lambda_i(T_b L_c L_o T_b^{-1}) = \lambda_i(L_c L_o).$$

If a nonsingular matrix T_b is chosen as

$$T_b = \Sigma^{1/2} U^T S^{-T} \tag{3.13}$$

then the gramians are equal and diagonal, i.e.,

$$\hat{L}_c = \hat{L}_o = \Sigma = \text{ diag } (\sigma_1^H, \sigma_2^H, \cdots, \sigma_n^H) \tag{3.14}$$

and by convention $\sigma_1^H \geq \sigma_2^H \geq \cdots \geq \sigma_n^H > 0$, where σ_i^H $(i = 1, \cdots, n)$ are the Hankel singular values of the system $G(s)$ since

$$\begin{aligned} \{\sigma_i^H\} &= \{[\lambda_i(L_c L_o)]^{1/2}\} \\ &= \{[\lambda_i(S^T S R^T R)]^{1/2}\} \\ &= \{[\lambda_i(R S^T S R^T)]^{1/2}\} \\ &= \{\sigma_i(S R^T)\} \end{aligned}$$

where $\{\sigma_i(SR^T)\}$ is the set of singular values of SR^T.

The state-space realization (3.10) is called a balanced realization, proposed by Moore [50], which implies that the observability and controllability gramians are both equal to the diagonal matrix of the Hankel singular values. The states of such a realization are *balanced* between controllability and observability. Thus they represent a convenient structure for model reduction since those states having *weak* controllability and observability can be neglected without causing any imbalance in controllability and observability properties of the remaining states. Hence, the Hankel singular values give a good indication of the 'minimal' dimension of a system.

3.3 Model Reduction Techniques

A model reduction problem is an approximation of the original system by a low-order system, but does not necessarily mean a 'minimal' realization (defined in Section 2.2). Model reduction is particularly concerned with the plant rather than controller, although its techniques can also be applied to the controller size reduction. The problem has received much attention and is reasonably well solved. In this section, we briefly review some important model reduction techniques.

3.3.1 Using Modal Residualization

The design of control schemes for linear SISO systems often hinges on knowledge of the transfer function of such systems. Suppose, in frequency domain terms, a stable transfer function matrix is expanded in partial fraction form as

$$G(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \cdots + \frac{R_n}{s-p_n} \tag{3.15}$$

with the (complex) poles p_i ordered so that $||\frac{R_i}{s-p_i}||$ are in descending order. Here the norm, $||\frac{R_i}{s-p_i}||$, can be either the $\mathcal{H}_2$-norm or the $\mathcal{H}_\infty$-norm and indicates the amount of contribution in a general transient response. This procedure is called modal residualization. Some term(s) having smaller contribution to the effects on the system response may be neglected, e.g., Franklin *et al.* [25, p.63]. So, model reduction using modal residualization is performed by truncating those of negligible norm, to give:

$$G_r(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \cdots + \frac{R_k}{s-p_k} \tag{3.16}$$

$$||G(s) - G_r(s)|| \leq ||\frac{R_{k+1}}{s-p_{k+1}}|| + \cdots + ||\frac{R_n}{s-p_n}|| \tag{3.17}$$

where $G_r(s)$ is a truncated transfer function, $k \leq n$. This method is a somewhat crude approach and thus might not be optimal, even though it can often be successful. A direct transmission matrix can be introduced in the state-space system of the truncated model to ensure that the reduced model has the same steady state response as the original model. The method can be extended to MIMO systems in that there will be a transfer function for each input and output pairing.

3.3.2 Balanced Truncated Model Reduction

Balanced truncation as initiated by Moore [50] is a powerful model reduction technique for LTI systems. It is based on the balanced realization described in the previous section.

Inspecting the Hankel singular values of a system will often reveal that some of them are quite small compared to others. The states corresponding to those small Hankel singular values are both difficult to control and observe. In other words, more energy is required to excite them and their effect on the output is also small. These *less significant* states may therefore be eliminated. This results in a lower-order model for the system.

If the realization $(\hat{A}, \hat{B}, \hat{C})$ of $G(s)$ is balanced and the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ in (3.10) and the balanced gramian Σ in (3.14) are partitioned conformally as

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \tag{3.18}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{3.19}$$

where the dimensions of $\hat{A}_{11}$ and Σ_1 are $k \times k$ for an integer $k < n$ such that $\sigma_k^H \gg \sigma_{k+1}^H$, then a balanced truncated model $G_r(s)$ of reduced order k is obtained as

$$G_r(s) = \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1 \tag{3.20}$$

by neglecting states associated with small Hankel singular values, Σ_2, in (3.19).

Pernebo and Silverman [54] showed that the reduced order model $G_r(s) = (\hat{A}_{11}, \hat{B}_1, \hat{C}_1)$ is balanced and that if $\sigma_k^H > \sigma_{k+1}^H$, then it is asymptotically stable and minimal.

This balanced truncated model reduction technique has the following frequency domain $\mathcal{L}_\infty$-norm error bound, Enns [21] and Glover [26]:

$$||G(s) - G_r(s)||_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i^H \tag{3.21}$$

where $G_r(s)$ is a truncated balanced realization and the σ_i^H's $(i = k+1, \cdots, n)$ are all considered to be small Hankel singular values of $G(s)$ that can be discarded.

A limitation of this balanced truncated model reduction technique is that it requires the original system to be *minimal* and *asymptotically stable.* When a balanced realization is computed, practical difficulties may arise for system models having uncontrollable and unobservable states because the balancing transformations are generally singular for such systems. These difficulties are overcome by Tombs and Postlethwaite [68] and by Safonov and Chiang [62]. For *unstable* plants, using the fact that the coprime factors of any minimal system are always asymptotically stable, Meyer [49] developed a coprime factor model reduction method by extending (unweighted) balanced truncated model reduction to the case where the system is not stable. This will be described later.

3.3.3 Hankel Norm Model Reduction

The *Hankel norm* of $G(s)$, denoted by $||G(s)||_H$, is the largest Hankel singular value of $G(s)$ and is defined as

$$||G(s)||_H := \{\lambda_{max}(\hat{L}_c\hat{L}_o)\}^{1/2} = \sigma_1^H \tag{3.22}$$

where $\hat{L}_c$ and $\hat{L}_o$ are as in (3.14). The Hankel norm is interpreted as the largest $\mathcal{L}_2$-gain from past inputs $u(t)$ to future outputs $y(t)$, in the following sense:

$$||G(s)||_H = \sup\frac{||y(t)||_2}{||u(t)||_2}, \qquad y(t)=0, \quad t<0 \qquad u(t)=0, \quad t>0.$$

It is noted that only the dynamic part (A, B, C) of $G(s)$ influences the Hankel norm.

The optimal Hankel norm model reduction problem is to choose a reduced order model $G_r(s)$ of McMillan degree $k < n$ such that the Hankel norm of the error system $G(s) - G_r(s)$, i.e., $||G(s) - G_r(s)||_H$ is minimized. This Hankel norm model reduction gives an $\mathcal{L}_\infty$-norm error bound, Glover [26], as

$$||G(s) - \tilde{G}_r(s)||_\infty \leq \sum_{i=k+1}^{n} \sigma_i^H \tag{3.23}$$

where $\tilde{G}_r(s)$ is the k-th order Hankel approximate with a particular choice of feed through term D.

3.3.4 Coprime Factor Model Reduction

Model reduction techniques such as balanced truncated model reduction and Hankel norm model reduction require that the model to be reduced is stable. The normalized coprime factor model reduction devised by Meyer [49] extends the balanced truncated model reduction to *unstable* plants, using the fact that coprime factors of any minimal system are always asymptotically stable.

Model reduction in a (left) coprime factor framework can be described as follows, McFarlane *et al.* [48]:

1. Write $G(s)$, the transfer function to be reduced (with degree n), as $G = \tilde{M}^{-1}\tilde{N}$ where $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$ are left coprime factors of $G(s)$.

2. Using an appropriate model reduction technique, either balanced truncated model reduction or Hankel norm model reduction, approximate $[\tilde{N}, \tilde{M}]$ of degree n by $[\tilde{N}_r, \tilde{M}_r]$ of degree k, $k < n$.

3. Form the reduced order transfer function $G_r(s)$ (with degree k) by $G_r = \tilde{M}_r^{-1}\tilde{N}_r$.

3.3.5 Model Reduction and Robustness

In doing model reduction for control purposes, singular value Bode plots of the reduced plant model and the modelling error are good indicators to check whether a given reduced model is sufficiently accurate to be used in the design of a control system with a prescribed bandwidth w_b. Using the $\mathcal{L}_\infty$ error bound, it is possible to associate a *robust frequency* w_r with a reduced model such that the model may be reliably used for controller design, Safonov *et al.* [63]. The robust frequency w_r is an upper bound on the bandwidth w_b of any multivariable control system to ensure robust stability. For example, in order to prevent a sufficient condition for stability from being violated at some frequency within the bandwidth, w_b should be less than w_r. For details, refer to [63].

3.4 Approaches to Controller Size Reduction

Some model reduction techniques have been used for controller size reduction. For example, Yousuff and Skelton [76] stated that, if the controller is stable, the balanced truncated model reduction technique can be used directly on it so that its uncontrollable or unobservable modes can be eliminated.

However, unlike model reduction procedures where only a dynamic system model is simplified, any controller size reduction procedure should take into account the presence of the plant and thus closed-loop considerations. That is, a controller size reduction procedure should preserve the closed-loop objectives such as closed-loop stability, closed-loop performance (without any serious deterioration), and robustness properties, etc. So, after controller size reduction, it is necessary to reanalyze the design to check that any degradation in performance is not too significant. In this sense, controller size reduction is fundamentally different from model reduction.

Various controller size reduction methods have been studied and classified in Anderson and Liu [1] into the following three categories:

(1) plant model reduction followed by controller design
(2) controller design followed by controller size reduction
(3) direct low-order controller design.

These different approaches to controller size reduction are illustrated in Figure 3.1. In this section, we summarize the procedure of each approach, its applications and some comments available on it.

3.4.1 Plant Model Reduction followed by Controller Design

In this approach, a model reduction technique is first applied to a high-order plant based on open-loop system considerations. Then a controller is designed to meet the control specifications, based on the reduced-order plant. A general comment on this approach is, as quoted from [1], that:

> reducing the order of the plant by approximation at an early step in the process may lead to the undesirable propagation of the effects of that approximation and make the ultimate effect unclear.

For example, the controller designed on the basis of the reduced order model applies controls to the true system and hence can inadvertently excite parts of the system that have been ignored (this is called control spillover). This category includes Choi *et al.* [14], McFarlane *et al.* [48], Postlethwaite and Feng [56] and Steinbuch [66].

3.4.2 Controller Design followed by Controller Size Reduction

In this approach, a standard controller of full-order is first designed to meet stability and/or performance requirements and then some model reduction method is applied to the full-order controller. Care should be taken during the controller size reduction step to ensure that the stability and/or performance achieved in the first step does not seriously degrade. Yousuff and Skelton [76] using LQG control, and McFarlane *et al.* [48], Bongers and Bosgra [5], Mustafa and Glover [51] using $\mathcal{H}_\infty$ control are examples which belong to this category.

A similar approach was introduced by Jonckheere and Silverman [35] but using a closed-loop methodology. That is, the open-loop system (which may be unstable) is first compensated with a standard LQG controller, where two algebraic Riccati equations are needed - one for filtering and one for control. Balancing the solutions to these two Riccati equations, so that they are equal and diagonal, exposes the difficulty of filtering and controlling each state. By truncating the states corresponding to small LQG-characteristic values (i.e., the diagonal elements of the solution to the LQG-balanced Riccati equations), a reduced-order plant or reduced-order LQG controller is obtained.

It is, however, pointed out in [76] that this scheme does not guarantee to eliminate any uncontrollable or unobservable modes in the controller which can arise in LQG design, since the LQG characteristic values are not measures of the observability/controllability of the controller states. The notion of LQG-characteristic values is extended in [51] to $\mathcal{H}_\infty$-characteristic values, which are then used as a basis for $\mathcal{H}_\infty$-balanced truncation.

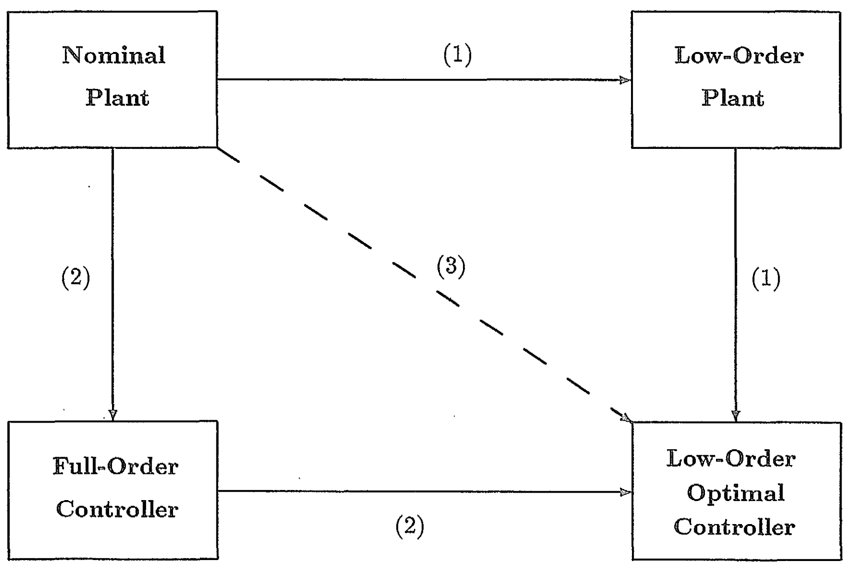
3.4.3 A Direct Approach

In this approach, the order of the controller is constrained (or fixed) prior to the controller design process and then the parameters defining a low-order controller are obtained, for example, either (i) by optimization of a performance index (or cost functional), Bernstein and Haddad [3] for $\mathcal{H}_\infty$ control, or (ii) by using the bounded real lemma, Hsu *et al.* [32], or (iii) by using a Lyapunov based approach, Iwasaki and Skelton [34].

Amongst others, we note that [32] obtains an observer-based controller of order $n - p_2$ which stabilizes the plant and also satisfies $||\mathcal{F}_l(P, K)||_\infty < \gamma$, where n and p_2 are the dimensions of the state and the measured output, respectively. This resultant order is the same as we obtain later in Chapter 5, although the methodologies are different.

3.5 Concluding Remarks

In this chapter, we briefly reviewed some important model reduction techniques and current controller size reduction approaches. In the following chapters, we will present a new methodology for controller size reduction in advanced robust control system design. The methodology developed in this thesis may be considered to be a direct approach, since it generates a low-order controller of a certain order by solving two matrix equations which are formed by plant data and the free parameter matrix.



(1) Plant Model Reduction followed by Controller Design

(2) Controller Design followed by Controller Size Reduction

(3) Direct Approach

Figure 3.1: Diagram for Controller Size Reduction.

Chapter 4

Low-Order Stabilizing Controller Design

4.1 Introduction

One of the most fundamental requirements in control system design is to make the closed-loop system internally stable. This is called a *stabilization problem*. The parametrization of all stabilizing feedback controllers for a given plant, initially developed by Youla *et al.* [75] and generalized by Desoer *et al.* [15], is a celebrated solution to the stabilization problem and provides a fundamental basis to the $\mathcal{H}_\infty$ optimal control problem. The order $\mathcal{N}$ of a stabilizing controller by such a parametrization, however, can be unnecessarily "high", since it can be shown that

$$\mathcal{N} \leq \deg(G) + \deg(Q)$$

where $Q(s)$ is a dynamic free parameter in the parametrization to give the designer freedom in designing a required controller, and $G(s)$ is the plant to be controlled.

In this chapter, we utilize the parametrization of all stabilizing controllers for the low-order stabilizing controller design problem, and present a constructive way to find a set of **low-order stabilizing controllers** of a certain order. The key idea is to achieve a low-order realization (4.22) of a "full-order" controller $K_{stab}(s)$ by eliminating unobservable states. We show that, if a low-order realization is possible, the order of the controller is reduced from a full-order of $n + n_q$ to n_q, where n is the order of $G(s)$ and n_q is the order of $Q(s)$. We further develop an algorithm to determine how low n_q might be using a sequence of matrix transformations as summarized in Theorem 4.8, Corollary 4.9, and Theorem 4.12. The algorithm checks the conditions in Theorem 4.12 and Corollary 4.9 successively until both are met. Then a low-order stabilizing controller $K^r_{stab}(s)$ is computed by the low-order realization (4.22). Our result is that a low-order stabilizing controller of order $n - l$ always exists, where l is the number of plant outputs. Moreover, we show that the order may be much less than $n - l$, depending upon the existence of a special form of a matrix $\hat{F}$ in (4.55). In summary, the order $\mathcal{N}_{low}$ of low-order stabilizing controllers as developed in this chapter is shown to satisfy

$$\mathcal{N}_{low} \leq \deg(G) - l.$$

The chapter is structured as follows. In Section 4.2, the notion of internal stability is reviewed, and then the parametrization of all stabilizing controllers is outlined together with the ability to assign poles. Section 4.3 is a central part of the chapter, where we show that a low-order realization (4.22) of all stabilizing controllers can be derived if two simultaneous matrix equations, (4.20)-(4.21), have a solution. We then solve the first equation (4.20) using an orthogonal canonical form, and the second one (4.21) by a standard linear equation solution on the assumption of the existence of a matrix $\hat{F}$ having a special form as in (4.55). In Section 4.4, we examine how to find such a special form of matrix $\hat{F}$ since its existence is the only constraint to deriving low-order stabilizing controllers. Then, in Section 4.5, a CAD algorithm for low-order stabilizing controller design is presented. In Section 4.6, an explicit formula for a set of low-order stabilizing controllers, as a special case, is derived using a special canonical form. We show in Section 4.7 that the low-order stabilizing controllers as characterized in the chapter provides a mechanism for closed-loop pole assignability via output feedback using a separation property. In Section 4.8, bounds on the order of low-order stabilizing controllers are discussed, and compared with some other existing results on the low-order stabilization problem. In Section 4.9, we confirm Kimura's results, [38], on pole assignment by gain output feedback using our approach and discuss an improvement on Kimura's. In Section 4.10, some illustrative examples are presented to validate the algorithm developed in the chapter and its relevant features. Concluding remarks are given in Section 4.11.

4.2 Observer-Based Stabilizing Controllers

The parametrization of the set of all stabilizing controllers in terms of a stable parameter matrix was first introduced by Youla *et al.* [75], based on fractional factorizations over the set of polynomial matrices. Youla's parametrization, however, may cause the stabilizing controller to be improper. Desoer *et al.* [15] removed this drawback by generalizing Youla's parametrization based on fractional factorizations over the set of proper stable rational matrices. As a result, they showed that the set of all proper stabilizing controllers can be characterized in terms of a proper stable parameter matrix. To use the proper stabilizing controller parametrization, a convenient state-space method for computing the fractional factorizations over the set of proper stable rational matrices was proposed by Nett *et al.* [52]. Also, Doyle [16] showed that the proper stabilizing controller parametrization can be realized as an observer-based controller with an added proper stable parameter matrix.

We begin this section by reviewing the notion of *internal stability*, and then outline the proper stabilizing controller parametrization. Finally, closed-loop pole assignability by stabilizing controllers is considered.

4.2.1 Internal Stability

Consider a positive-feedback configuration, as shown in Figure 4.1, where

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{4.1}$$

is assumed to be stabilizable and detectable.

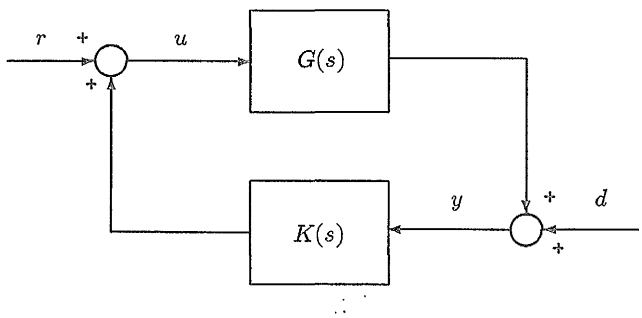


Figure 4.1: Diagram for Internal Stability.

From the feedback system of Figure 4.1, we have the following input-output relationship:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I + KSG & KS \\ SG & S \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} \tag{4.2}$$

where

$$S := (I - GK)^{-1} \tag{4.3}$$

is a sensitivity function matrix. The standard definition of internal stability is given below. This definition requires all the closed-loop transfer functions to be both stable and proper.

Definition 4.1 The feedback system of Figure 4.1 is *internally stable* if and only if
(a) S, KS, SG, $I + KSG \in \mathcal{RH}_\infty$
and
(b) $\det(I - GK)(\infty) \neq 0$.

It is important to realize from this definition, as clearly stated in [17, p.36], that it is not enough to look only at closed-loop transfer functions, such as from r to y. This transfer function could in fact be stable, so that y is bounded when r is bounded (BIBO stable), and yet an internal signal could be unbounded, probably causing internal damage to the physical system.

In the following we will use 'closed-loop stable' to mean 'internally stable' unless otherwise stated.

4.2.2 Stabilizing Controller Parametrization

Consider the feedback configuration of Figure 4.1 again, where $G(s) \in \mathcal{RL}_\infty^{l\times m}$ is a given plant of n states to be controlled, and $K(s)$ is a controller to be designed for internal stabilization. Without loss of generality, we assume $G(s)$ is minimal and, for simplicity, strictly proper (i.e., $D = 0$).

Let $G(s)$ have a doubly coprime factorization

$$\begin{aligned} G(s) &= N(s)M(s)^{-1} && (4.4)\\ &= \tilde{M}(s)^{-1}\tilde{N}(s) && (4.5) \end{aligned}$$

and also let $U(s), V(s), \tilde{U}(s)$ and $\tilde{V}(s)$ satisfy the Bezout identity, i.e.,

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.6}$$

where the transfer matrices $N, M, \tilde{N}, \tilde{M}, U, V, \tilde{U}, \tilde{V}$ all belong to $\mathcal{RH}_\infty$.

Then it is well known, for example, in [46, p.280] that the set of all stabilizing controllers for the given plant $G(s)$ is given by

$$\begin{aligned} K_{stab}(s) &= (U + MQ)(V + NQ)^{-1} && (4.7) \\ &= (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) && (4.8) \end{aligned}$$

for any $Q(s)$ which belongs to $\mathcal{RH}_\infty$.

The parametrization of $K_{stab}(s)$ in (4.7) and (4.8) is very powerful, because

- it provides the full set of the stabilizing controllers by means of fractional representations, once we know one stabilizing controller for the plant.
- the full set of the stabilizing controllers is simply characterized by a free parameter matrix $Q(s) \in \mathcal{RH}_\infty$.
- a closed-loop transfer function matrix related to performance can be written as a simple affine function of $Q(s)$, which is then useful for the solution of an $\mathcal{H}_\infty$ optimal control problem.

The stabilizing controller $K_{stab}(s)$ in Figure 4.1 can now be replaced by the block diagram of Figure 4.2, which is an observer-based stabilizing controller with added dynamics $Q(s)$.

The transfer matrices N, M, $\tilde{N}$, $\tilde{M}$, U, V, $\tilde{U}$, $\tilde{V}$ can each be expressed in state-space form as follows, after choosing real matrices F and H such that $A + BF$ and $A + HC$ are stable, [52]:

$$\begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C & 0 & I \end{array}\right] \tag{4.9}$$

$$\begin{bmatrix} \tilde{V}(s) & \tilde{U}(s) \\ \tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & B & H \\ \hline -F & I & 0 \\ C & 0 & I \end{array}\right]. \tag{4.10}$$

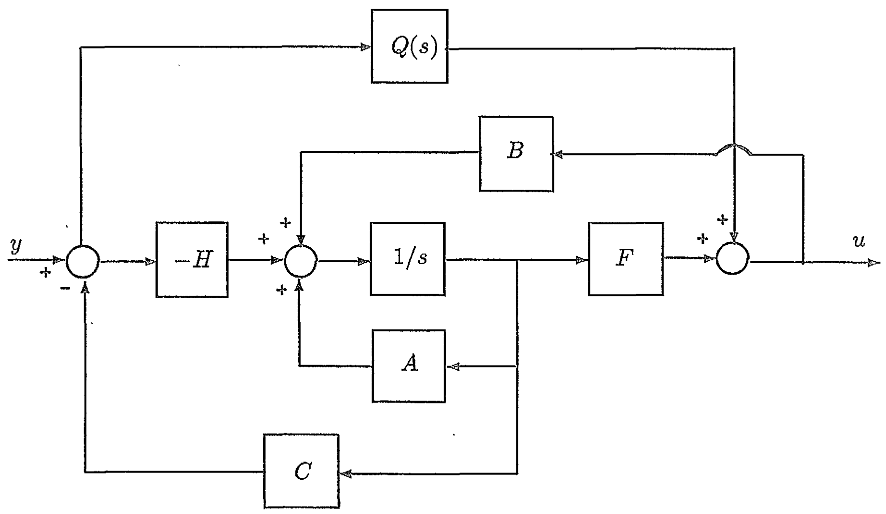


Figure 4.2: Observer-Based Stabilizing Controller with $Q(s)$.

Now suppose that $Q(s) \in \mathcal{RH}_\infty^{m\times l}$ in (4.7) and (4.8) has a state-space realization

$$Q(s) := \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right] \tag{4.11}$$

where the dimensions of matrices A_q, B_q, C_q and D_q are $n_q \times n_q$, $n_q \times l$, $m \times n_q$ and $m \times l$, respectively. Then, from (4.7), or alternatively from (4.8), we have a state-space realization of all stabilizing controllers $K_{stab}(s)$ given by

$$K_{stab}(s) = \left[\begin{array}{cc|c} A+BF+HC-BD_qC & BC_q & -H+BD_q \\ -B_qC & A_q & B_q \\ \hline F-D_qC & C_q & D_q \end{array}\right] \tag{4.12}$$

$$:= \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]. \tag{4.13}$$

It is noted that $K_{stab}(s)$ in (4.12) has a "formal" order (or "full-order"):

$$\deg(K_{stab}) = n + n_q$$

which is a state dimension equal to the sum of the states of $G(s)$ and $Q(s)$ together. By the term "formal" order we mean the order before any pole-zero cancellations which might occur have been removed. It is also noted that $K_{stab}(s)$ is strictly proper if and only if $Q(s)$ is strictly proper.

By taking $Q(s) = 0$ either in (4.7) or in (4.8), a stabilizing controller $K_{stab}(s)$ of order n is obtained as

$$K_{stab}(s) = \left[\begin{array}{c|c} A + BF + HC & -H \\ \hline F & 0 \end{array}\right]. \tag{4.14}$$

4.2.3 Pole Assignability of Stabilizing Controllers

In this section, we examine the poles of the closed-loop system formed as in Figure 4.1 by the plant $G(s)$ in (4.1) and the controller $K_{stab}(s)$ in (4.12). We will consider a state-space realization of $(I - GK_{stab})^{-1}$.

The A-matrix, A_{cl}, of $(I - GK_{stab})^{-1}$ can be expressed in state-space form as

$$A_{cl} := \begin{bmatrix} A + BF + HC - BD_qC & BC_q & -HC + BD_qC \\ -B_qC & A_q & BC_q \\ BF - BD_qC & BC_q & A + BD_qC \end{bmatrix}. \tag{4.15}$$

By applying a state similarity transformation to (4.15) using a nonsingular matrix T given by

$$T := \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{4.16}$$

we obtain

$$T^{-1}A_{cl}T = \begin{bmatrix} A + HC & 0 & 0 \\ -B_qC & A_q & 0 \\ BF - BD_qC & BC_q & A + BF \end{bmatrix} \tag{4.17}$$

which is *similar* to A_{cl}. So, it is obvious that, when the controller $K_{stab}(s)$ in (4.12) is applied to the plant $G(s)$, the resulting closed-loop poles are the union of

- the observer poles (the eigenvalues of $A + HC$),
- the state feedback controller poles (the eigenvalues of $A + BF$), and
- the poles of the augmented dynamics (the eigenvalues of A_q).

This fact shows that the familiar separation property of observer-based controllers still remains when a proper stable parameter matrix $Q(s)$ is added to as shown in Figure 4.2. In reality, as stated in Vidyasagar [71], not all of the poles will necessarily appear since the realizations constructed at the various stages need not be minimal. But it is certain that no new poles will appear other than the ones mentioned above.

4.3 Low-Order Stabilizing Controllers

The realization of the controller in (4.12) may not be minimal. Chang and Yousuff [9] showed that, if the realization (4.12) is not minimal, the uncontrollable or unobservable modes of the controller are some subset of the eigenvalues of $A+BF$ and $A+HC$. However, they did not address the problem of deriving such a subset. The primary purpose of this chapter is to find a *subset* for which a certain number of modes are unobservable or uncontrollable. As a consequence, we can find a set for which the order of the controllers does not exceed n_q, i.e., the number of states of the free parameter matrix $Q(s)$. Furthermore, we will show that n_q can be less than or equal to $n - l$.

In Subsection 4.3.1, a low-order realization of all stabilizing controllers is characterized via two simultaneous matrix equations, (4.20)-(4.21). The size of the solution matrix $X \in \mathcal{R}^{n_q \times n}$ to these two equations, if X exists, will determine the order of the low-order stabilizing controllers. We will consider two cases: (1) $n_q = n - l$ in Subsection 4.3.2 and (2) $n_q < n - l$ in Subsection 4.3.3.

4.3.1 Derivation of a Low-Order Realization

One way to derive a set of low-order stabilizing controllers is to apply a change of state coordinates on (4.12). We begin by applying a state similarity transformation T_x to $K_{stab}(s)$ in (4.12):

$$K_{stab}(s) = \left[\begin{array}{cc|c} A + BF + HC - BD_qC & BC_q & -H + BD_q \\ -B_qC & A_q & B_q \\ \hline F - D_qC & C_q & D_q \end{array}\right]$$

and

$$T_x := \begin{bmatrix} I_n & 0 \\ X & I_{n_q} \end{bmatrix} \tag{4.18}$$

results in an alternative realization given by

$$K_{stab}(s) = \left[\begin{array}{c|c} T_xA_kT_x^{-1} & T_xB_k \\ \hline C_kT_x^{-1} & D_k \end{array}\right] =: \left[\begin{array}{cc|c} K_{a11} & K_{a12} & K_{b1} \\ K_{a21} & K_{a22} & K_{b2} \\ \hline K_{c1} & K_{c2} & K_d \end{array}\right] \tag{4.19}$$

where

$$\begin{aligned}
K_{a11} &= A + BF + HC + BD_qC - BC_qX \\
K_{a12} &= BC_q \\
K_{a21} &= XA + XBF + XHC + XBD_qC + B_qC - XBC_q - A_qX \\
K_{a22} &= XBC_q + A_q \\
K_{b1} &= H + BD_q \\
K_{b2} &= XH + XBD_q + B_q \\
K_{c1} &= F + D_qC - C_qX \\
K_{c2} &= C_q \\
K_d &= D_q.
\end{aligned}$$

Putting K_{a21} and K_{c1} equal to zero in (4.19), we have the following **two matrix equations:**

$$A_qX - X(A + HC) = B_qC \tag{4.20}$$

$$C_qX - D_qC = F. \tag{4.21}$$

So, if there exists a solution matrix $X \in \mathcal{R}^{n_q \times n}$ to (4.20)-(4.21), $K_{stab}(s)$ in (4.12) can be reduced to the following *lower-order realization*:

$$K^r_{stab}(s) = \left[\begin{array}{c|c} A_q + XBC_q & B_q + XH + XBD_q \\ \hline C_q & D_q \end{array}\right]. \tag{4.22}$$

We refer to the controllers defined by (4.22) as **low-order stabilizing controllers**. The order of low-order stabilizing controllers is in fact:

$$\deg(K^r_{stab}) = n_q$$

which is the same as that of $Q(s)$, and thus it is obvious that $K^r_{stab}(s)$ in (4.22) is of *lower-order* than $K_{stab}(s)$ in (4.12).

We obtained the low-order realization $K^r_{stab}(s)$ of (4.22) by deleting the *unobservable states* contained in $K_{stab}(s)$, and thus the realization of $K^r_{stab}(s)$ may be completely observable as shown in the following Lemma.

Lemma 4.2 *Suppose there exists a solution matrix X to (4.20)-(4.21). Then the realization of $K^r_{stab}(s)$ in (4.22) is completely observable if and only if the free parameter matrix $Q(s)$ is chosen to be completely observable.*

Proof: (*Sufficiency*) Since the pair (A_q, C_q) is completely observable, there exists an H_q such that eigenvalues of $A_q + H_qC_q$ can be arbitrarily assigned by suitable choice of H_q. Now define $\hat{H}_q := H_q - XB$. Then, since

$$(A_q + XBC_q) + \hat{H}_qC_q = A_q + H_qC_q \tag{4.23}$$

the eigenvalues of $(A_q + XBC_q) + \hat{H}_qC_q$ can also be arbitrarily assigned by suitable choice of H_q. Hence, the pair $(A_q + XBC_q, C_q)$ is completely observable, i.e., $K^{r}_{stab}(s)$ is completely observable.

(*Necessity*) Since the pair $(A_q + XBC_q, C_q)$ is completely observable, there exists an $\hat{H}_q$ such that eigenvalues of $(A_q + XBC_q) + \hat{H}_q C_q$ can be arbitrarily assigned. Hence, by (4.23), the eigenvalues of $A_q + H_q C_q$ can also be arbitrarily assigned by suitable choice of H_q. This implies that the pair (A_q, C_q) is completely observable, i.e., $Q(s)$ is completely observable. ∎

Realization (4.22) is in a convenient form for computing a set of low-order stabilizing controllers, when X exists and is determined by equations (4.20)-(4.21). Figure 4.3 shows the closed-loop system formed by the plant $G(s)$ in (4.1) and the low-order output feedback stabilizing controller $K^{r}_{stab}(s)$ in (4.22).

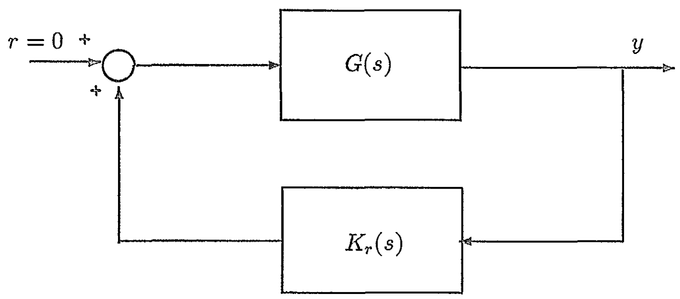


Figure 4.3: Output Feedback System with a Low-Order Dynamic Controller.

Equation (4.20), which is a Sylvester equation (or a general Lyapunov equation), and linear equation (4.21) are of crucial importance in determining a low-order controller. Both equations are solved in the following subsections.

Remark 4.3 Alternatively we can apply a state similarity transformation T_y to $K_{stab}(s)$ in (4.12) using a nonsingular matrix

$$T_y := \begin{bmatrix} I_n & Y \\ 0 & I_{n_q} \end{bmatrix}. \tag{4.24}$$

We then can obtain a lower-order realization of order n_q as

$$K^r_{stab}(s) = \left[\begin{array}{c|c} A_q + B_qCY & B_q \\ \hline C_q + D_qCY - FY & D_q \end{array}\right] \tag{4.25}$$

if there exists a matrix $Y \in \mathcal{R}^{n \times n_q}$ which satisfies the following two matrix equations:

$$\begin{aligned} (A+BF)Y - YA_q &= BC_q \qquad (4.26) \\ YB_q + BD_q &= H. \qquad (4.27) \end{aligned}$$

Corollary 4.4 *Suppose there exists a solution matrix Y to (4.26)-(4.27). Then the realization of $K^r_{stab}(s)$ in (4.25) is completely controllable if and only if $Q(s)$ is chosen to be completely controllable.*

The procedure to be presented for solving two equations (4.20)-(4.21) is based on the observability of the pair $(A+HC, C)$, whereas the procedure for solving (4.26)-(4.27) is based on the controllability of the pair $(A+BF, B)$. We shall call the former the *Observability Argument Approach* (OAA), and the latter the *Controllability Argument Approach* (CAA). In the following, we will develop and state our results by the OAA and give the results by the CAA if necessary.

4.3.2 Stabilizing Controllers of Order $n - l$

It is well known in observer theory, [45], that if the rank of the matrix C is l, then a state observer of order $n - l$ can be constructed to generate all the state variables. A state feedback matrix can then be used to stabilize the plant, given that the pair (A, B) is controllable. This implies the existence of stabilizing controllers of order $n - l$. We therefore consider the special case of order $n_q = n - l$ in this subsection.

We first assume without loss of generality that C is full row rank and that by a change of coordinates C takes the form

$$C = \begin{bmatrix} I_l & 0_{l\times(n-l)} \end{bmatrix} \tag{4.28}$$

i.e., C is partitioned into an $l \times l$ identity matrix and an $l \times (n-l)$ zero matrix. Let A be partitioned conformally as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{4.29}$$

where the matrices $A_{11}, A_{12}, A_{21}, A_{22}$ have dimensions of $l \times l$, $l \times (n-l)$, $(n-l) \times l$ and $(n-l) \times (n-l)$, respectively.

Suppose that X and H take the forms

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \tag{4.30}$$

and

$$H = \begin{bmatrix} H_1^T & H_2^T \end{bmatrix}^T \tag{4.31}$$

where the dimensions of the matrices X_1, X_2, H_1, H_2 are $(n-l) \times l$, $(n-l) \times (n-l)$, $l \times l$ and $(n-l) \times l$, respectively.

From equation (4.20), we have the following two equations:

$$A_q X_1 - \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} A_{11} + H_1 \\ A_{21} + H_2 \end{bmatrix} = B_q \tag{4.32}$$

$$A_q X_2 - \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = 0. \tag{4.33}$$

The next Lemma shows the existence of the solution matrices X_1 and X_2 to (4.33).

Lemma 4.5 *For any stable A_q, there always exists a matrix X_1 and a non-singular matrix X_2 satisfying equation (4.33).*

Proof: Equation (4.33) can be rewritten as

$$X_2A_{22} + X_1A_{12} = A_qX_2. \tag{4.34}$$

Recall from Lemma 2.6 that the pair (A_{22}, A_{12}) is completely observable if the pair (A, C) is completely observable. It is therefore clear that, using Lemma 2.3, the eigenvalues of $A_{22} + \Psi A_{12}$ can be freely assigned by a suitable choice of Ψ, since the pair (A, C) is completely observable. In other words, there always exist a matrix Ψ and a nonsingular matrix W such that

$$A_{22} + \Psi A_{12} = W^{-1}A_qW$$

i.e.,

$$WA_{22} + W\Psi A_{12} = A_qW$$

for any stable A_q. Comparing this with (4.34), it is then obvious that we may take $X_1 = W\Psi$ and $X_2 = W$ as solutions to (4.34), and in addition, that X_2 is nonsingular. ■

Notice from Lemma 4.5 that the nonsingular X_2 may *not* be unique since the similarity transformation matrix W is not unique. However, if we are only interested in the eigenvalues of A_q and not its exact form, then it follows from the proof of Lemma 4.5 that there exists a Ψ such that $A_{22} + \Psi A_{12}$ has the same eigenvalues as those of A_q. In this case, therefore, we may simply let $A_q = A_{22} + \Psi A_{12}$, and consequently $X_2 = I$ and $X_1 = \Psi$.

Having found the solution matrices X_1 and X_2 from (4.33), B_q can then be obtained from (4.32). Meanwhile, since the second equation (4.21) can be rewritten as

$$\begin{bmatrix} C_q & -D_q \end{bmatrix} \begin{bmatrix} X \\ C \end{bmatrix} = F \tag{4.35}$$

where

$$\begin{bmatrix} X \\ C \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ I & 0 \end{bmatrix}$$

is square and full rank, C_q and D_q can be computed by

$$\begin{aligned} C_q &= F_2 {X_2}^{-1} && (4.36) \\ D_q &= -F_1 + F_2 {X_2}^{-1} X_1 && (4.37) \end{aligned}$$

for any $F := \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ partitioned conformally. So, we now have the following Theorem.

Theorem 4.6 *Let C be full row rank. Then the system $G(s) = (A, B, C)$ always has low-order stabilizing controllers of order:*

$$\deg(K_{stab}^r) = n - l.$$

To summarize, we have shown in this subsection that under the assumption of full rank C there always exist stabilizing controllers of order $n - l$, and that in the formulae for such controllers the only constraints on the choice of F and H are that $A + BF$ and $A + HC$ are stable.

4.3.3 Stabilizing Controllers of Order Less Than $n - l$

Let us reconsider equation (4.20):

$$A_q X - X(A + HC) = B_q C.$$

In order to find stabilizing controllers of order less than $n - l$ we now look for a full rank solution $X \in \mathcal{R}^{n_q \times n}$ with order $n_q < n - l$.

Equation (4.20) will have a unique solution if A_q is chosen such that $\mathcal{R}e[\lambda_i(A_q)] \neq \mathcal{R}e[\lambda_j(A + HC)]$. The standard approach to solve a Sylvester equation (4.20) may be useful when we are interested in the element matrices - A_q, B_q, C_q, D_q - of the free parameter $Q(s)$, since A_q and B_q can be *a priori* chosen arbitrarily at the designer's discretion. Here we are interested in the solution X having the smallest possible n_q.

Unlike the standard Sylvester equation considered in Subsection 2.4.1, equation (4.20) has a great deal of freedom in the coefficient matrices, specifically in A_q, B_q and H. We therefore adopt, in this subsection, a different approach to solve (4.20) making use of the freedom available. In our approach, A_q and H will be first chosen arbitrarily but with the smallest possible n_q, subject to the stability of A_q and $A+HC$. Then the solution X of full rank n_q is found and B_q is finally decided in due course.

Assumption 4.7 We assume that the plant $G(s) = (A, B, C)$ is minimal and C is full rank. (The full rankness of C can be relaxed.)

As we saw in Chapter 2, Section 2.3, the pair (A, C) can be transformed into the orthogonal canonical form (A_o, C_o):

$$A_o = MAM^{-1} = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & A_{\nu_o-1,\nu_o} \\ A_{\nu_o,1} & A_{\nu_o,2} & A_{\nu_o,3} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix} \tag{4.38}$$

$$C_o = NCM^{-1} = \begin{bmatrix} I_{l_1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.39}$$

where ν_o is the *observability index* of (A, C), and $A_{i,i} (i = 1, \cdots, \nu_o)$ are $l_i \times l_i$ matrices, and the numbers

$$l = l_1 \geq l_2 \geq \cdots \geq l_{\nu_o} \qquad l_1 + l_2 + \cdots + l_{\nu_o} = n$$

are the conjugate Kronecker indices of the pair (A, C).

When the observable canonical form (A_o, C_o) is derived, the transformed B-matrix is denoted as $B_o := MB$. Using the form (A_o, C_o) of (4.38)-(4.39), the two equations (4.20)-(4.21) can therefore be transformed into:

$$A_q\bar{X} - \bar{X}(A_o + \bar{H}C_o) = \bar{B}_q C_o \tag{4.40}$$

$$C_q\bar{X} - \bar{D}_q C_o = \bar{F} \tag{4.41}$$

where

$$\begin{aligned}
\bar{X} &= XM^{-1} && (4.42)\\
\bar{H} &= MHN^{-1} && (4.43)\\
\bar{B}_q &= B_qN^{-1} && (4.44)\\
\bar{D}_q &= D_qN^{-1} && (4.45)\\
\bar{F} &= FM^{-1}. && (4.46)
\end{aligned}$$

The following Theorem gives the (possibly small) dimensions of a solution $\bar{X}$ to equation (4.40).

Theorem 4.8 *Equation (4.40) has full row rank solutions $\bar{X}$ of dimensions $l_{\nu_o} \times n$.*

Proof: Partition $\bar{X} \in \mathcal{R}^{l_{\nu_o} \times n}$ and $\bar{H} \in \mathcal{R}^{n \times l}$ as

$$\bar{X} = \left[\begin{array}{cccc} \bar{X}_1 & \bar{X}_2 & \cdots & \bar{X}_{\nu_o} \end{array} \right] \qquad \bar{H} = \left[\begin{array}{c} \bar{H}_1 \\ \bar{H}_2 \\ \vdots \\ \bar{H}_{\nu_o} \end{array} \right]$$

where $\bar{X}_i \in \mathcal{R}^{l_{\nu_o} \times l_i}$ and $\bar{H}_i \in \mathcal{R}^{l_i \times l}$.

Then from equation (4.40) we have the two equations:

$$A_q\bar{X}_1 - \left[\begin{array}{ccc} \bar{X}_1 & \cdots & \bar{X}_{\nu_o} \end{array} \right] \left[\begin{array}{c} A_{11} + \bar{H}_1 \\ A_{21} + \bar{H}_2 \\ \vdots \\ A_{\nu_o,1} + \bar{H}_{\nu_o} \end{array} \right] = \bar{B}_q \qquad (4.47)$$

$$A_q \left[\begin{array}{ccc} \bar{X}_2 & \cdots & \bar{X}_{\nu_o} \end{array} \right] - \left[\begin{array}{ccc} \bar{X}_2 & \cdots & \bar{X}_{\nu_o} \end{array} \right] \tilde{A}_1 = \bar{X}_1\tilde{C}_1 \qquad (4.48)$$

where

$$\tilde{C}_1 := \left[\begin{array}{ccccc} A_{12} & 0 & 0 & \cdots & 0 \end{array} \right]$$

$$\tilde{A}_1 := \begin{bmatrix} A_{22} & A_{23} & 0 & \cdots & 0 \\ A_{32} & A_{33} & A_{34} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,2} & A_{\nu_o-1,3} & A_{\nu_o-1,4} & \cdots & A_{\nu_o-1,\nu_o} \\ A_{\nu_o,2} & A_{\nu_o,3} & A_{\nu_o,4} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix}.$$

Note that the pair $(\tilde{A}_1, \tilde{C}_1)$ is completely observable since (A, C) is completely observable. This can be easily proved by constructing the observability matrix of $(\tilde{A}_1, \tilde{C}_1)$ and then checking its full rankness.

For a given A_q (stable), $\bar{B}_q$ can be obtained directly from (4.47) with known $\bar{X}_1$ and $[\bar{X}_2 \quad \cdots \quad \bar{X}_{\nu_o}]$.

Equation (4.48) is similar in form to equation (4.40), and can thus be solved in the same fashion. Repeat this procedure until we get

$$A_q \bar{X}_{\nu_o} - \bar{X}_{\nu_o} A_{\nu_o,\nu_o} = \bar{X}_{\nu_o-1} A_{\nu_o-1,\nu_o}$$

i.e.,

$$\bar{X}_{\nu_o}(A_{\nu_o,\nu_o} + \bar{X}_{\nu_o}^{-1} \bar{X}_{\nu_o-1} A_{\nu_o-1,\nu_o}) \bar{X}_{\nu_o}^{-1} = A_q.$$

Since $(A_{\nu_o,\nu_o}, A_{\nu_o-1,\nu_o})$ is observable, we can always find an $\bar{X}_{\nu_o-1}$ and a nonsingular $\bar{X}_{\nu_o}$ for any stable A_q. Calculating backwards, we can find $\bar{X}_{\nu_o-2}, \bar{X}_{\nu_o-3}, \cdots, \bar{X}_1$ and hence $\bar{B}_q$. The solution $\bar{X}$ is full row rank, since $\bar{X}_{\nu_o}$ is invertible. This completes the proof. ∎

Note that the solution $\bar{X}$ as above is independent of $\bar{H}$, but depends on A and A_q only. A candidate for $\bar{X}_{\nu_o}$ is simply the identity matrix (i.e., $\bar{X}_{\nu_o} = I_{l_{\nu_o}}$), since we are more interested in the eigenvalues of A_q than the structure of A_q.

Theorem 4.8 applies directly to equation (4.20), and thus (4.20) has full row rank solutions $X \in \mathcal{R}^{l_{\nu_o} \times n}$. And as expected from Theorem 4.8, we have the following Corollary.

Corollary 4.9 *For $n_q = l_{\nu_o}$, $n_q = l_{\nu_o} + l_{\nu_o-1}$, $\cdots$, $n_q = l_{\nu_o} + \cdots + l_2 = n - l$, respectively, there always exist full row rank solutions to equation (4.40) and thus (4.20).*

Corollary 4.9 indicates that the next step in finding a solution to (4.40) is to increase the value of n_q from l_{ν_o} to $l_{\nu_o} + l_{\nu_o-1}$, and thus the next order of controllers will be $l_{\nu_o} + l_{\nu_o-1}$.

By a dual controllability argument approach (CAA), using the controllable canonical form (A_c, B_c), we have the following two Corollaries.

Corollary 4.10 *Equation (4.26) has full column rank solutions Y of dimensions $n \times m_{\nu_c}$.*

Corollary 4.11 *For $n_q = m_{\nu_c}$, $n_q = m_{\nu_c} + m_{\nu_c-1}$, $\cdots$, $n_q = m_{\nu_c} + \cdots + m_2 = n - m$, respectively, there always exist full column rank solutions to equation (4.26).*

Let us now turn to the problem of solving the second equation (4.41):

$$C_q\bar{X} - \bar{D}_q C_o = \bar{F}$$

where $\bar{F}$ can be arbitrarily chosen by the designer subject to the stability of $A_o + B_o\bar{F}$. We rewrite equation (4.41) as

$$\begin{bmatrix} C_q & -\bar{D}_q \end{bmatrix} \begin{bmatrix} \bar{X} \\ C_o \end{bmatrix} = \bar{F} \tag{4.49}$$

and suppose without loss of generality that the solution $\bar{X} \in \mathcal{R}^{l_{\nu_o} \times n}$ to equation (4.41) is

$$\bar{X} = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \cdots & \bar{X}_{\nu_o-1} & \bar{X}_{\nu_o} \end{bmatrix} \tag{4.50}$$

with $\bar{X}_{\nu_o} = I_{l_{\nu_o}}$. Further, define a nonsingular matrix T_2

$$T_2 := \begin{bmatrix} I_{l_1} & 0 & 0 & \cdots & 0 \\ 0 & I_{l_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{l_{\nu_o-1}} & 0 \\ -\bar{X}_1 & -\bar{X}_2 & \cdots & -\bar{X}_{\nu_o-1} & I_{l_{\nu_o}} \end{bmatrix} \tag{4.51}$$

such that

$$\hat{X} := \bar{X}T_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{l_{\nu_o}} \end{bmatrix}. \tag{4.52}$$

Then, we have

$$\hat{A} := T_2^{-1}A_oT_2 \quad \hat{B} := T_2^{-1}B_o \quad \hat{C} := C_oT_2 = C_o \tag{4.53}$$

and equation (4.49) becomes

$$\begin{bmatrix} C_q & -\bar{D}_q \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{l_{\nu_o}} \\ I_l & 0 & \cdots & 0 & 0 \end{bmatrix} = \bar{F}T_2 =: \hat{F}. \tag{4.54}$$

Thus, if there exists $\hat{F}$ in the special form

$$\hat{F} = \begin{bmatrix} \hat{F}_1 & 0 & \cdots & 0 & \hat{F}_{\nu_o} \end{bmatrix} \tag{4.55}$$

then we can find C_q and D_q as

$$C_q = \hat{F}_{\nu_o} \tag{4.56}$$

$$D_q = -\hat{F}_1N \tag{4.57}$$

using (4.54)-(4.55) and (4.45).

So far, we have shown how to find the element matrices which are required for computing a realization of $K_{stab}^{r}(s)$ in (4.22) of order l_{ν_o}. In addition, it was shown in Lemma 4.2 that $K_{stab}^{r}(s)$ is completely observable if and only if the pair (A_q, C_q) is completely observable. Indeed, the system (A_q, B_q, C_q, D_q) can always be made observable in a reduced form, say $(\hat{A}_q, \hat{B}_q, \hat{C}_q, \hat{D}_q)$, via a similarity transformation. It is easy to show that these new element matrices still satisfy the two matrix equations with a solution of *smaller* size and the matrix F will not be affected. It is therefore possible to reduce the order of controller further. Consequently, we have a stabilizing controller of order not exceeding l_{ν_o} as stated formally in the next Theorem.

Theorem 4.12 *The system (A, B, C) has stabilizing controllers of order less than or equal to l_{ν_o}, i.e.,*

$$\mathcal{N}_{low} \leq l_{\nu_o}$$

if there exists an $\hat{F}$ as in (4.55) which makes $\hat{A} + \hat{B}\hat{F}$ stable.

The stability requirement of $\hat{A} + \hat{B}\hat{F}$ may prohibit a selection of $\hat{F}$ which also satisfies (4.55). Suppose that the second equation (4.41) does not have a solution $\bar{X} \in \mathcal{R}^{l_{\nu_o} \times n}$. This implies that there does not exist an $\hat{F}$ as in (4.55), and consequently that the algorithm fails to find a low-order controller of order l_{ν_o}. In such a case, we should increase the value of n_q from l_{ν_o} to the next value, i.e., $l_{\nu_o} + l_{\nu_o - 1}$ (as per Corollary 4.9) and try to find a state feedback matrix $\hat{F}$ of similar structure to (4.55) but now with more degrees of freedom. Thus, if it is found, the next order of controllers will be $l_{\nu_o} + l_{\nu_o - 1}$. This procedure can be repeated up to $n_q = n - l$ until a suitable $\hat{F}$ is found.

The determination of $\hat{F}$ is considered, in the next section, in some detail. It will be shown later in Section 4.9 that a suitable $\hat{F}$ always exists if n_q is chosen such that the inequality $n_q \geq n - m - l + 1$ holds.

Remark 4.13 The system can be stabilized by a *static* output feedback matrix if there exists an $\hat{F}$ as in (4.55) with $\hat{F}_{\nu_o} = 0$. Similarly, the existence of an $\hat{F}$ satisfying (4.55) with $\hat{F}_1 = 0$ leads to *strictly proper* low-order stabilizing controllers.

4.4 Determination of $\hat{F}$

As discussed in Section 4.3, $\hat{F}$ shall be of the form:

$$\hat{F} = \begin{bmatrix} \hat{F}_1 & 0 & \cdots & 0 & \hat{F}_{\nu_o} \end{bmatrix}$$

as in (4.55) for the existence of C_q and D_q. This introduces a constraint in the algorithm developed in this chapter for generating low-order stabilizing controllers. The exception is the case of $n_q = n - l$, in which there is no restriction on $\hat{F}$, as already discussed in Subsection 4.3.2. This raises the following question: *Under what conditions can we ensure the existence of a state feedback matrix $\hat{F}$ such that $\hat{A} + \hat{B}\hat{F}$ be stable and $\hat{F}$ is of the form in (4.55)?* In the following, some different methods are suggested to answer this question.

4.4.1 Method I: via a Search

Suppose that $\bar{F}$ is partitioned as

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 & \cdots & \bar{F}_{\nu_o-1} & \bar{F}_{\nu_o} \end{bmatrix} \tag{4.58}$$

where $\bar{F}_i \quad (i = 1, ..., \nu_o)$ is an $m \times l_i$ matrix. Then, using (4.51), $\hat{F}$ becomes

$$\hat{F} = \bar{F}T_2 = \begin{bmatrix} \hat{F}_1 & \hat{F}_2 & \cdots & \hat{F}_{\nu_o-1} & \hat{F}_{\nu_o} \end{bmatrix} \tag{4.59}$$

where

$$\hat{F}_i = \bar{F}_i - \bar{F}_{\nu_o}\bar{X}_i \qquad (i = 1, \cdots, \nu_o - 1) \tag{4.60}$$

$$\hat{F}_{\nu_o} = \bar{F}_{\nu_o}. \tag{4.61}$$

In order for $\hat{F}$ in (4.59) to have the special form (4.55), $\bar{F}$ in (4.58) ought to be of the following form:

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_{\nu_o}\bar{X}_2 & \cdots & \bar{F}_{\nu_o}\bar{X}_{\nu_o-1} & \bar{F}_{\nu_o} \end{bmatrix} \tag{4.62}$$

where the $\bar{X}_i \quad (i = 2, \cdots, \nu_o - 1)$ are already known, and $\bar{F}_1$ and $\bar{F}_{\nu_o}$ can each be arbitrary $m \times l_1$ and $m \times l_{\nu_o}$ matrices, respectively. Then, $\bar{F}$ can be determined relatively easily because only two parameters, $\bar{F}_1$ and $\bar{F}_{\nu_o}$, are to be tuned to obtain $\hat{F}$ of the special form in (4.55).

A fundamental restriction on $\bar{F}$ is, however, that $A_o + B_o\bar{F}$ is stable. That is, we must have

$$\mathcal{R}e[\lambda_i(A_o + B_o\bar{F})] < 0 \qquad \forall\, i. \tag{4.63}$$

Note that

$$\mathcal{R}e[\lambda_i(A + BF)] = \mathcal{R}e[\lambda_i(A_o + B_o\bar{F})] = \mathcal{R}e[\lambda_i(\hat{A} + \hat{B}\hat{F})].$$

The eigenvalues of $A_o + B_o\bar{F}$ can be easily obtained using readily available algorithms in, for example, Matlab. So, we may determine a required $\hat{F}$ as follows:

step 1: Given A_o, B_o and $\bar{X}$, choose $\bar{F}$ in the form of (4.62), by selecting $\bar{F}_1$ and $\bar{F}_{\nu_o}$.

step 2: Check if the stability constraint (4.63) is satisfied.
If *yes*, go to step 3.
If *no*, go to step 1 to choose an alternative $\bar{F}$.

step 3: Compute $\hat{F} = \bar{F}T_2$.

Example 1 in Section 4.10 demonstrates this approach.

4.4.2 Method II: via an Algebraic Riccati Equation

Using equation (4.54):

$$\hat{F} = \begin{bmatrix} C_q & -\bar{D}_q \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{n_q} \\ I_l & 0 & \cdots & 0 & 0 \end{bmatrix}$$

the problem of making $\hat{A}+\hat{B}\hat{F}$ stable can be shown to be equivalent to a stabilization problem via static output feedback, because of the following equality:

$$\hat{A} + \hat{B}\hat{F} = \hat{A} + \hat{B}W\check{C} \tag{4.64}$$

where

$$W := \begin{bmatrix} C_q & -\bar{D}_q \end{bmatrix} : m \times (l + n_q) \tag{4.65}$$

$$\check{C} := \begin{bmatrix} \hat{X} \\ \hat{C} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & I_{n_q} \\ I_l & 0 & \cdots & 0 \end{bmatrix} : (l + n_q) \times n. \tag{4.66}$$

Namely, a system $\hat{G}(s) := (\hat{A}, \hat{B}, \check{C})$ which has n states, m inputs and $(l + n_q)$ outputs can be stabilized via a static output feedback matrix $W \in \mathcal{R}^{m \times (l+n_q)}$, where n, m and l are all *fixed* but n_q can be *varied.* Hence, if we can find a smaller sized W, we can make $\hat{A} + \hat{B}\hat{F}$ stable with a smaller sized $\hat{X}$.

To cope with the problem of making $\hat{A} + \hat{B}W\check{C}$ stable, we consider a feedback system $\hat{H}(s)$ which comprises an open-loop plant $\hat{G}(s)$ and a static output feedback $W \in \mathcal{R}^{m\times(l+n_q)}$. Then, the closed-loop system $\hat{H}(s)$ can be expressed as

$$\hat{H}(s) := \left[\begin{array}{c|c} \hat{A} + \hat{B}W\check{C} & \hat{B} \\ \hline \check{C} & 0 \end{array} \right] \tag{4.67}$$

and the stability of $\hat{A} + \hat{B}W\check{C}$ is guaranteed if the feedback system $\hat{H}(s)$ of (4.67) belongs to $\mathcal{RH}_\infty$.

Using Lemma 2.17 in Chapter 2, we can derive the following Lemma to check whether there exists a W having smaller dimension such that the system $\hat{H}(s)$ is stable, and to find such a W if it exists.

Lemma 4.14 *Let $\hat{H}(s)$ be as in (4.67) and $\gamma > 0$. Then the system $\hat{H}(s)$ is stable with an $\mathcal{H}_\infty$-norm bound, $||\hat{H}(s)||_\infty \leq \gamma$, if there exists a unique symmetric positive definite solution P to the following ARE:*

$$(\hat{A} + \hat{B}W\breve{C})^T P + P(\hat{A} + \hat{B}W\breve{C}) + \frac{1}{\gamma}P\hat{B}\hat{B}^T P + \frac{1}{\gamma}\breve{C}^T\breve{C} = 0 \tag{4.68}$$

for the designer selected W.

Given $\hat{A}, \hat{B}$ and $\breve{C}$, the existence of a positive definite solution P to ARE (4.68) depends on the choice of the matrix W and also on γ. The scalar γ can be *any* size, since it has nothing to do with the poles of the closed-loop system formed by the plant $G(s)$ and the low-order stabilizing controller $K_{stab}^r(s)$, but a large γ may be preferable for ensuring the existence of a positive definite solution P.

Therefore, $\hat{F}$ can be determined by the following procedure:

step 1 Given $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{X}$, construct $\breve{C}$ as in (4.66).

step 2 Set γ to be a large value, e.g., $\gamma = 10^3$.

step 3 Select an arbitrary output feedback matrix W.

step 4 Solve ARE (4.68) for P.
If $P > 0$, go to step 5.
If $P \leq 0$, go to step 3 to choose an alternative W.

step 5 Compute $\hat{F} = W\breve{C}$.

Example 1a in Section 4.10 follows this approach.

4.4.3 An Optimization Method

An optimization technique is considered here to supplement the methods described in previous subsections which depend on trial and error techniques.

Firstly, the following procedure is suggested to strengthen the method in Subsection 4.4.1. Let

$$\Xi := \hat{A} + \hat{B}\hat{F} \in \mathcal{R}^{n \times n}$$

and let f be the following vector containing all the entries of $\hat{F}$

$$f := [f_{1*}, f_{2*}, \cdots, f_{m*}]^T \in \mathcal{R}^{mn}$$

where f_{i*} denotes i-th row of $\hat{F}$.

step 1 Select an initial guess $f^{(0)}$. (iteration index $j = 0$.)

<u>(at j-th iteration)</u>

step 2 Calculate eigenvalues of Ξ and check if $\mathcal{R}e[\lambda_i(\Xi)] < 0 \quad \forall i$.

If *yes*, stop.

If *no*, go to step 3.

step 3 Transform Ξ into its Jordan form J by a similarity transformation using a nonsingular matrix S such that

$$S^{-1}\Xi S = J.$$

Note that J will be in diagonal form if Ξ is not *defective* (i.e., Ξ has n linearly independent eigenvectors).

step 4 If there are any eigenvalues with multiplicity greater than 1, modify $f^{(j)}$ and go to step 3 to ensure that the eigenvalues are distinct.

step 5 Use a gradient method to obtain a new parameter $f^{(j+1)}$, on observing the following equality:

$$\frac{\partial \lambda_i(\Xi)}{\partial f_k} = s_{*i}^T \frac{\partial \Xi}{\partial f_k} s_{*i}$$

for $i = 1, \cdots, n$ and $k = 1, \cdots, mn$, where s_{*i} denotes i-th column of S, i.e.,

$$S = [s_{*1}, s_{*2}, \cdots, s_{*n}] \in \mathcal{R}^{n \times n}.$$

step 6 Set $j = j + 1$ and then go to step 2.

Secondly, as an alternative to the method in Subsection 4.4.2, a readily available Matlab command such as *attgoal.m* (in the Optimization Toolbox) may be used to find a desired output feedback matrix W.

4.5 A Low-Order Stabilizing Controller Design Algorithm

The aim of this section is to present a CAD algorithm for low-order stabilizing controller design, summarizing the procedures described in the chapter so far.

step 1: Given a minimal realization of the plant $G(s) = (A, B, C) \in \mathcal{RL}_\infty^{l \times m}$, choose an observer gain matrix $H \in \mathcal{R}^{n \times l}$ subject to the stability of $A + HC$.

step 2: Transform the pair (A, C) into the required canonical form (A_o, C_o) as in (4.38)-(4.39), and then find the observability index ν_o and the Kronecker indices, $l_1, l_2, \cdots, l_{\nu_o}$.

step 3: Set $n_q = l_{\nu_o}$.

step 4: Choose any stable $A_q \in \mathcal{R}^{n_q \times n_q}$.

step 5: Find the solution matrix $\bar{X}$ to (4.40), setting $\bar{X}_{\nu_o} = I_{l_{\nu_o}}$.

step 6: Compute $B_q \in \mathcal{R}^{n_q \times l}$ using (4.44) and (4.47).

step 7: Define T_2 as in (4.51).

step 8: Determine an $\hat{F}$ in the special form (4.55) which also satisfies the stability of $\hat{A} + \hat{B}\hat{F}$, using either method I or method II or an optimization method suggested in Section 4.4.
If $\hat{F}$ is found, go to step 9.
If $\hat{F}$ is not found, go to step 3 to increase n_q to the next level, e.g., from l_{ν_o} to $l_{\nu_o} + l_{\nu_o - 1}$.

step 9: Compute C_q and D_q as per (4.56) and (4.57), respectively.

step 10: Compute a low-order stabilizing controller as per (4.22).

4.6 Explicit Formulae of Low-Order Stabilizing Controllers

In this section we derive explicit formulae of low-order stabilizing controllers, using a certain canonical transformation.

Suppose a pair (A, C) is transformed to a more special canonical form (A_s, C_s), as is found in Yokoyama and Kinnen [73], as

$$A_s = \begin{bmatrix} A_{11} & \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & \cdots & 0 \\ A_{21} & 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & 0 & 0 & \cdots & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ A_{\nu_o,1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.69}$$

$$C_s = \begin{bmatrix} I_l & 0_{l\times(n-l)} \end{bmatrix}. \tag{4.70}$$

Then, low-order stabilizing controllers may be explicitly expressed in state-space form in terms of the plant data and A_q only. Two different cases, $n_q = l_{\nu_o}$ and $n_q = n - l$ are considered.

4.6.1 Case 1 : $n_q = l_{\nu_o}$

For this situation, we can obtain the solution $\bar{X}$ to equation (4.40) as

$$\bar{X} = \begin{bmatrix} \begin{bmatrix} A_q{}^{\nu_o-1} & 0 \end{bmatrix} & \begin{bmatrix} A_q{}^{\nu_o-2} & 0 \end{bmatrix} & \cdots & \begin{bmatrix} A_q & 0 \end{bmatrix} & I_{l_{\nu_o}} \end{bmatrix} \tag{4.71}$$

by direct calculation with $\bar{X}_{\nu_o} = I_{l_{\nu_o}}$.

Define T_2 as in (4.51) and follow the same procedure as in Subsection 4.3.3. Then, we have $\hat{X}$, A_q, B_q, C_q and D_q as below:

$$\begin{aligned}
\hat{X} &= \begin{bmatrix} 0 & \cdots & 0 & I_{l_{\nu_o}} \end{bmatrix} \\
A_q &= \text{(any given stable matrix with distinct eigenvalues)} \\
B_q &= (\begin{bmatrix} A_q{}^{\nu_o} & 0 \end{bmatrix} - \begin{bmatrix} A_q{}^{\nu_o-1} & 0 \end{bmatrix} A_{11} - \begin{bmatrix} A_q{}^{\nu_o-2} & 0 \end{bmatrix} A_{21} \\
&\quad - \cdots - \begin{bmatrix} A_q & 0 \end{bmatrix} A_{\nu_o-1,1} - A_{\nu_o,1} - \hat{H}_{\nu_o})N \\
C_q &= \hat{F}_{\nu_o} \\
D_q &= -\hat{F}_1 N
\end{aligned}$$

where the dimensions of the matrices are $\hat{X} : l_{\nu_o} \times n$, $A_q : l_{\nu_o} \times l_{\nu_o}$, $B_q : l_{\nu_o} \times l$, $C_q : m \times l_{\nu_o}$ and $D_q : m \times l$.

These give a set of stable transfer functions $Q(s)$, of order l_{ν_o}, in terms of all stable A_q and some $\hat{H}$ and $\hat{F}$. On substituting this characterization for $Q(s)$ into the formulae for $K^r_{stab}(s)$ in (4.22), we obtain a set of stabilizing controllers with order not bigger than l_{ν_o}. This is summarized in the following Theorem.

Theorem 4.15 *If the assumption in Theorem 4.12 is satisfied, then a set of stabilizing controllers of order less than or equal to l_{ν_o} is given by*

$$K^r_{stab}(s) := \left[\begin{array}{c|c} A_{kr} & B_{kr} \\ \hline C_{kr} & D_{kr} \end{array}\right] \tag{4.72}$$

where

$$\begin{aligned}
A_{kr} &= A_q + \hat{B}_{\nu_o}\hat{F}_{\nu_o} && (4.73)\\
B_{kr} &= (\begin{bmatrix} A_q{}^{\nu_o} & 0 \end{bmatrix} - \begin{bmatrix} A_q{}^{\nu_o-1} & 0 \end{bmatrix} A_{11} - \begin{bmatrix} A_q{}^{\nu_o-2} & 0 \end{bmatrix} A_{21} \\
&\quad - \cdots - \begin{bmatrix} A_q & 0 \end{bmatrix} A_{\nu_o-1,1} - A_{\nu_o,1} - \hat{B}_{\nu_o}\hat{F}_1)N && (4.74)\\
C_{kr} &= \hat{F}_{\nu_o} && (4.75)\\
D_{kr} &= -\hat{F}_1 N. && (4.76)
\end{aligned}$$

Proof: By direct manipulation. ∎

Similar formulae for order larger than l_{ν_o} (e.g., $l_{\nu_o}+l_{\nu_o-1}, \cdots$) can be obtained in the same manner. Note that the matrix H plays no direct role in the formulae for the set of low-order stabilizing controllers $K^r_{stab}(s)$, but is present in the formulae for the free stable parameter matrix $Q(s)$.

4.6.2 Case 2 : $n_q = n - l$

In this case we assume that C_s of (4.70) is partitioned as

$$C_s = \begin{bmatrix} I_l & 0_{l\times(n-l)} \end{bmatrix}$$

and A_s of (4.69) is partitioned conformally as

$$A_s = \begin{bmatrix} A_{s11} & A_{s12} \\ A_{s21} & A_{s22} \end{bmatrix}$$

where

$$\begin{aligned} A_{s11} &= A_{11} : l \times l \\ A_{s12} &= \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & 0 & \cdots & 0 \end{bmatrix} : l \times (n-l) \\ A_{s22} &= \begin{bmatrix} 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} : (n-l) \times (n-l). \end{aligned}$$

By assuming without loss of generality that A_q has the form complying with that of A_{s22}, i.e.,

$$A_q := \begin{bmatrix} A_{q,2} & \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & \cdots & 0 \\ A_{q,3} & 0 & \begin{bmatrix} I \\ 0 \end{bmatrix} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{q,\nu_o-1} & 0 & 0 & \cdots & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ A_{q,\nu_o} & 0 & 0 & \cdots & 0 \end{bmatrix} : (n-l) \times (n-l) \tag{4.77}$$

we can obtain the solution $\bar{X}$ to equation (4.40) as

$$\bar{X} = \begin{bmatrix} \bar{X}_1 & I_{n-l} \end{bmatrix} : (n-l) \times n \tag{4.78}$$

where

$$\bar{X}_1 = \begin{bmatrix} A_q \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 \end{bmatrix} : (n-l) \times l.$$

Define T_2 as in (4.51) and proceed in the same manner as in Subsection 4.3.3. Then, we have $\hat{X}$, B_q, C_q, and D_q as follows:

$$\begin{aligned} \hat{X} &= \begin{bmatrix} 0 & I \end{bmatrix} \\ B_q &= ({A_q}^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - A_q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} A_{s11} - A_{s21} - \hat{H}_2)N \\ C_q &= \hat{F}_2 \\ D_q &= -\hat{F}_1 N \end{aligned}$$

where $\hat{F}_1$, $\hat{F}_2$ and $\hat{H}_2$ are partitions of $\hat{F}$ and $\hat{H}$ defined by $\hat{F} = \begin{bmatrix} \hat{F}_1 & \hat{F}_2 \end{bmatrix}$ and $\hat{H} = \begin{bmatrix} \hat{H}_1^T & \hat{H}_2^T \end{bmatrix}^T$.

Consequently, the stabilizing controllers of order $n-l$ are given by

$$K_{stab}^r(s) := \left[\begin{array}{c|c} A_{kr} & B_{kr} \\ \hline C_{kr} & D_{kr} \end{array}\right]$$

where

$$
\begin{aligned}
A_{kr} &= A_q + \hat{B}_2\hat{F}_2 && (4.79)\\
B_{kr} &= (A_q{}^2\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix} - A_q\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}A_{s11} - A_{s21} - \hat{B}_2\hat{F}_1)N && (4.80)\\
C_{kr} &= \hat{F}_2 && (4.81)\\
D_{kr} &= -\hat{F}_1 N && (4.82)
\end{aligned}
$$

and $\hat{B}_2$ is defined by $\hat{B} = \begin{bmatrix} \hat{B}_1^T & \hat{B}_2^T \end{bmatrix}^T$.

Remark 4.16 For $n_q = n - l$, no restriction on $\hat{F}$ is imposed, i.e., any F which makes $A + BF$ stable would be a suitable choice.

4.7 Pole Assignability of Low-Order Stabilizing Controllers

In this section, we examine the poles of the closed-loop system in Figure 4.3 formed by the plant $G(s)$ in (4.1) and the low-order controller $K^r_{stab}(s)$ in (4.22). We consider the A-matrix of the state-space realization of $(I - GK^r_{stab})^{-1}$.

A state-space realization of $(I - GK^r_{stab})^{-1}$ can be expressed as :

$$
\left[\begin{array}{cc|c}
A - BD_qC & BC_q & -BD_q\\
-B_qC - XHC - XBD_qC & A_q + XBC_q & -B_q - XH - XBD_q\\
\hline
C & 0 & I
\end{array}\right]. \quad (4.83)
$$

On applying a state coordinate change to (4.83) using the transformation matrix T_x as in (4.18), we have the following representation:

$$
(I - GK^r_{stab})^{-1} = \left[\begin{array}{cc|c}
A - BD_qC + BC_qX & BC_q & -BD_q\\
-B_qC - X(A + HC) + A_qX & A_q & -B_q - XH\\
\hline
C & 0 & I
\end{array}\right]
$$

$$= \left[\begin{array}{cc|c} A+BF & BC_q & -BD_q \\ 0 & A_q & -B_q - XH \\ \hline C & 0 & I \end{array}\right] \tag{4.84}$$

making use of equations (4.20)-(4.21). Representation (4.84) is *similar* to (4.83).

It follows from (4.84) that, given the existence of a solution X to (4.20)-(4.21), the poles of the closed-loop system are the union of the eigenvalues of $A + BF$ and the eigenvalues of A_q.

By definition, both $A + BF$ and A_q are stable. This means that the low-order controller $K^r_{stab}(s)$ is guaranteed to always satisfy the closed-loop stability constraint. Note that the separation property of all stabilizing controllers, as described in Subsection 4.2.3, is still present when the full-order stabilizing controller $K_{stab}(s)$ is replaced by low-order stabilizing controller $K^r_{stab}(s)$. Hence all the closed-loop poles are assignable. Also note that the observer poles (i.e., eigenvalues of $A + HC$) are not closed-loop poles when a low-order controller $K^r_{stab}(s)$ is used. Numerical examples in Section 4.10 illustrate this separation property precisely.

The separation property of the closed-loop poles can also be found, for example, in an observer-based controller [45] and in an LQG compensator, e.g., [46, p.227].

Remark 4.17 We have the total freedom in selecting the poles of $Q(s)$, namely, the eigenvalues of A_q, whereas the restriction on $\hat{F}$ to have the form in (4.55) in turn restricts the eigenvalues of $\hat{A}+\hat{B}\hat{F}$. This restriction was discussed in some detail in Section 4.4.

Remark 4.18 When a CAA is adopted, the separation property will arise between the eigenvalues of $A + HC$ and the poles of $Q(s)$, since it can then be shown that

$$(I - GK_{stab}^r)^{-1} = \left[\begin{array}{cc|c} A + HC & 0 & BD_q + YB_q \\ -B_qC & A_q & B_q \\ \hline -C & CY & I \end{array}\right].$$

4.8 Lower-Bounds on the Controller Order

In this section, we briefly summarize some existing results on stabilization by reduced-order controllers, and compare them with the new results developed in this thesis.

The following notation is used.
$\mathcal{N}_{low}$: a lower bound on the order of a dynamic stabilizing controller.
n : number of states of the plant.
l : number of outputs of the plant.
m : number of inputs of the plant.

4.8.1 Existing Results

The low-order stabilization problem has received much attention by a number of researchers. Despite this effort, the low-order stabilization problem is still an open problem in the sense that most of the existing results provide only *sufficient* conditions for the existence of stabilizing controllers of a certain order. We summarize in this subsection some of the existing results on the lower-bounds of the dynamic order of stabilizing controllers.

(1) Luenberger [45]: Corresponding to an nth-order system having l linearly independent outputs, a reduced-order observer of order

- $\mathcal{N}_{low} = n - l$

can be constructed having arbitrary eigenvalues.

(2) Brasch and Pearson [8]: For *arbitrary* closed-loop pole placement, a dynamic compensator should have at least an order

- $\mathcal{N}_{low} = \min(\nu_c - 1, \nu_o - 1)$

where ν_c is the controllability index and ν_o is the observability index of the plant.

(3) Kimura [38]: *Almost arbitrary* closed-loop pole assignability is possible by constant gain output feedback if $n \leq m + l - 1$. However, a dynamic controller is required to achieve closed-loop stability if $n > m + l - 1$. In this case, the minimum order of the dynamic controller is

- $\mathcal{N}_{low} = n - m - l + 1.$

(4) Linnemann [44]: SISO systems can be stabilized by a controller having the minimum order

- $\mathcal{N}_{low} = n - 1 - k$

where k is the order of the first Hurwitz polynomial in the sequence of remainders occurring in the Euclidean algorithm on its application to the numerator and denominator polynomials of an nth-order system transfer function.

(5) Smith and Sondergeld [65]: The arbitrary closed-loop pole placement procedure of Brasch and Pearson [8] may generate an *unstable* controller for closed-loop stabilization. In certain cases, the instability of a controller appears to result in poor overall system sensitivity to variations in controller parameters, [64]. This leads to a consideration of the strong stabilization problem, which

means the stabilization of a closed-loop feedback system by an asymptotically stable controller. For strong stabilization of a single-loop plant, Smith and Sondergeld showed the following results:

- $\mathcal{N}_{low} = d - 1$ (if z=0)
- $\mathcal{N}_{low} = n + d - 2$ (if z=1 and the plant is strongly stabilizable)
- $\mathcal{N}_{low} \to \infty$ (if $z \geq 2$)

where d is the relative degree of the plant and z is the number of zeros of the plant in the CRHP.

4.8.2 New Results

It was shown in Section 4.3 that the order of stabilizing controller, $\mathcal{N}_{low}(= n_q)$, can be varied from l_{ν_o} (at its lowest) to $n - l$ (at its highest). It is obvious that, when the observability index ν_o is equal to 2, the order of controller is fixed at $\mathcal{N}_{low} = n - l$, and that if $\nu_o \geq 3$, $\mathcal{N}_{low} = l_{\nu_o}$, where l_{ν_o} is the dimension of block matrix A_{ν_o,ν_o} in (4.38).

Similarly, using the controllability argument approach, the order of the controller can be fixed at $\mathcal{N}_{low} = n - m$ when $\nu_c = 2$, or have a lower bound of $\mathcal{N}_{low} = m_{\nu_c}$ when $\nu_c \geq 3$.

Combining the above two results, we have the following lower bounds on the order of controller :

$$\begin{aligned}
\mathcal{N}_{low} &= \min(n - l, n - m) \quad \text{if } \nu_c = \nu_o = 2 && (4.85)\\
&= \min(n - m, l_{\nu_o}) \quad \text{if } \nu_c = 2, \nu_o \geq 3 && (4.86)\\
&= \min(m_{\nu_c}, n - l) \quad \text{if } \nu_c \geq 3, \nu_o = 2, && (4.87)\\
&= \min(m_{\nu_c}, l_{\nu_o}) \quad \text{if } \nu_c, \nu_o \geq 3. && (4.88)
\end{aligned}$$

This observation indicates that, when either $n - l$ or $n - m$ is large, there is considerable scope for reducing the order of the dynamic stabilizing controller.

4.8.3 Comparison and Comments

Luenberger's result (1) on low-order observers is standard. Results (2) and (3) in Subsection in 4.8.1 are, in general, considered by Keel *et al.* [37] to be too conservative due to the essential requirement of *arbitrary* pole placement, when stabilization is the only requirement. Indeed, the order of stabilizing controllers can be further reduced by, for example, the new results in this thesis. Numerical examples in Section 4.10 verify this claim. Two results (4) and (5) in Subsection 4.8.1 are applicable only to the SISO case, whereas our new results can be applied to the MIMO case as well. Among others, Kimura's results (3) will be revisited in Section 4.9 to investigate more of the implications of the new results developed in this chapter.

4.9 Comparison with Kimura's Results

In [38], Kimura showed the following results on pole assignment by output feedback:

1. If a system having n states, m inputs and l outputs is minimal, and $m + l - 1 \geq n$ (i.e., $m + l > n$), then an almost arbitrary set of distinct closed-loop poles is assignable by constant gain output feedback.

2. The minimum order of the dynamic compensator required for almost arbitrary closed-loop pole assignments is not greater than $n - m - l + 1$.

In this section, using the approach described in Section 4.3, we show that Kimura's results above can be confirmed, and that it is possible to improve upon the second result.

4.9.1 Confirmation

Let the order of low-order stabilizing controllers, n_q, satisfy

$$n_q = l_{\nu_o} + l_{\nu_o - 1} + \cdots + l_k \geq n - m - l + 1 \tag{4.89}$$

and let the matrix $\hat{F}_{k\nu_o}$ have the form

$$\hat{F}_{k\nu_o} = \begin{bmatrix} \hat{F}_k & \hat{F}_{k+1} & \cdots & \hat{F}_{\nu_o - 1} & \hat{F}_{\nu_o} \end{bmatrix} : m \times n_q. \tag{4.90}$$

Then, from equations (4.53) and (4.55), we have the following:

$$\hat{A} + \hat{B}\hat{F} = \hat{A} + \hat{B} \begin{bmatrix} \hat{F}_1 & 0 & \cdots & 0 & \hat{F}_{k\nu_o} \end{bmatrix}. \tag{4.91}$$

By a similarity transformation using a nonsingular $n \times n$ matrix T_3 given by

$$T_3 = \begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_{n-l-n_q} \\ 0 & I_{n_q} & 0 \end{bmatrix} \tag{4.92}$$

we have

$$\tilde{A} = T_3{}^{-1}\hat{A}T_3 \qquad \tilde{B} = T_3{}^{-1}\hat{B} \tag{4.93}$$

and

$$\tilde{F} = \hat{F}T_3 = \begin{bmatrix} \hat{F}_1 & \hat{F}_{k\nu_o} & 0 \end{bmatrix} \tag{4.94}$$

using (4.91)-(4.92).

Therefore for the existence of low-order stabilizing controllers of order n_q satisfying (4.89), it is now required that

$$\begin{aligned} \tilde{A} + \tilde{B}\tilde{F} &= \tilde{A} + \tilde{B} \begin{bmatrix} \hat{F}_1 & \hat{F}_{k\nu_o} & 0 \end{bmatrix} && (4.95) \\ &= \tilde{A} + \tilde{B}\tilde{K}\tilde{C} && (4.96) \end{aligned}$$

is stable, where

$$\tilde{K} := \left[\begin{array}{cc} \hat{F}_1 & \hat{F}_{k\nu_o} \end{array} \right] : m \times (l + n_q) \tag{4.97}$$

and

$$\tilde{C} := \left[\begin{array}{cc} I & 0 \end{array} \right] : (l + n_q) \times n. \tag{4.98}$$

Note that $\tilde{C} \neq \hat{C}T_3$.

From equation (4.96), the problem of finding an $\tilde{F}$ such that $\tilde{A} + \tilde{B}\tilde{F}$ is stable is equivalent to a static output feedback stabilization problem for a system comprising $(\tilde{A}, \tilde{B}, \tilde{C})$. Therefore, low-order stabilizing controllers of order n_q $(\geq n - m - l + 1)$ will exist if there exists a gain output feedback matrix $\tilde{K}$ for the system $(\tilde{A}, \tilde{B}, \tilde{C})$ having n states, m inputs and $(l + n_q)$ outputs.

Indeed, by Kimura's first result, there always exists such a gain matrix $\tilde{K}$ for almost arbitrary pole assignability if $n_q \geq n-m-l+1$. For the system $(\tilde{A}, \tilde{B}, \tilde{C})$, this can be easily proved. That is, using condition (4.89), we have

$$m + (l + n_q) - 1 \geq m + (l + n - m - l + 1) - 1$$

which is identical to the following inequality:

$$m + (l + n_q) - 1 \geq n. \tag{4.99}$$

Inequality (4.99) meets the condition of Kimura's first result, namely, the condition for the existence of gain output feedback, and therefore $\tilde{K}$ exists.

Thus, in the approach for deriving low-order stabilizing controllers (as described in Section 4.3), if we increase n_q until it exceeds $n - m - l + 1$, then we can always find $\hat{F}$ (or F) which not only satisfies the special form as in (4.55) but also makes $\hat{A} + \hat{B}\hat{F}$ (or $A + BF$) stable. Consequently, we can have dynamic output feedback controllers of order $n - m - l + 1$ which stabilize the plant. This is a confirmation of Kimura's second result.

Remark 4.19 If $n_q = n - m - l + 1$, we have a stabilization problem by a static output feedback: i.e., find $\tilde{K}$ such that $\tilde{A} + \tilde{B}\tilde{K}\tilde{C}$ is stable where $\tilde{K}$ is $m \times (n - m + 1)$ and $\tilde{C}$ is $(n - m + 1) \times n$.

4.9.2 Improvement

Equations (4.41) and (4.54) in Section 4.3 are here reviewed in line with Kimura's result. In Subsection 4.4.2, we have shown that the problem of making $\hat{A} + \hat{B}\hat{F}$ stable is equivalent to an output feedback stabilization problem of making $\hat{A} + \hat{B}W\breve{C}$ stable, with

$$W = \begin{bmatrix} C_q & -\bar{D}_q \end{bmatrix} \qquad \text{and} \qquad \breve{C} = \begin{bmatrix} \hat{X} \\ \hat{C} \end{bmatrix}$$

as in (4.65)-(4.66).

As shown in the previous subsection, if the dimension of $\hat{X} \in \mathcal{R}^{n_q \times n}$ reaches or exceeds $n - m - l + 1$ (i.e., $n_q \geq n - m - l + 1$), it is guaranteed that a static output feedback matrix W exists such that the eigenvalues of $\hat{A} + \hat{B}W\breve{C}$ can almost arbitrarily be assigned. This is due to Kimura's first result, and is summarized in the following Theorem.

Theorem 4.20 *The system (A, B, C) always has low-order stabilizing controllers of order $n_q \geq n - m - l + 1$.*

Moreover, in some cases, a suitable W will exist for an $\hat{X}$ of smaller dimension (i.e., $n_q < n - m - l + 1$). In Subsection 4.4.2, we considered how to find such a W. This implies that the approach developed in this chapter could find controllers of smaller order than predicted by Kimura's result. Example 2 in Section 4.10 illustrates this. Other developments on the existence problem of a static output feedback matrix can be found in Oh *et al.* [53] and the references therein, where the problem is solved via an optimization technique.

4.10 Illustrative Examples

In this section, we present two numerical examples to illustrate the new algorithm for low-order stabilizing controller design. All calculations were performed using Matlab on a SUN work station. The first example is a MIMO system and we use the observability approach, whereas the second is SISO and we use the controllability approach.

4.10.1 Example 1

The state matrices for this example are taken from Kimura [38], and describe a system with 3 inputs, 2 outputs and 5 states. The plant is given by $G(s) = (A, B, C, D)$, where

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding transfer function matrix is

$$G(s) = \frac{1}{s^5 - 1} \begin{bmatrix} s^3 & s^2 & s \\ s^4 & s^3 & s^2 \end{bmatrix}$$

and the open-loop poles are at

$$-0.8090 \pm j0.5878, \quad 0.3090 \pm j0.9511, \quad 1.0000.$$

This example is already in *observable canonical form* as in (4.38)-(4.39), with $l_1 = 2, \quad l_2 = 1, \quad l_3 = 1$ and $l_{\nu_o} = 1$, where $\nu_o = 4$ is the observability index.

So, we have

$$A_o = A, \quad B_o = B, \quad C_o = C, \quad M = I_5, \quad N = I_2$$

and choose an observer gain matrix H such that

$$\lambda(A + HC) = -1, -2, -3, -4, -5.$$

Since $l_{\nu_o} = 1$, we set $n_q = 1$ and choose the arbitrary stable A_q as $A_q = -2$. Then X and B_q are determined to be

$$\begin{aligned} X &= \begin{bmatrix} 0 & -8 & 4 & -2 & 1 \end{bmatrix} \\ B_q &= \begin{bmatrix} -0.7320 & -0.0065 \end{bmatrix}. \end{aligned}$$

A nonsingular matrix T_2 is then constructed by (4.51). If we choose the state feedback matrix $\bar{F}$ as

$$\bar{F} = \begin{bmatrix} -12 & -71 & -8 & 4 & -2 \\ -1 & 20 & -24 & 12 & -6 \\ 1 & -80 & -40 & 20 & -10 \end{bmatrix}$$

then the eigenvalues of $A_o + B_o\bar{F}$ are

$$-63.3498, \quad -5.7614 \pm j4.8267, \quad -0.1153, \quad -0.0121$$

and

$$\hat{F} = \bar{F}T_2 = \begin{bmatrix} -12 & -87 & 0 & 0 & -2 \\ -1 & -28 & 0 & 0 & -6 \\ 1 & -160 & 0 & 0 & -10 \end{bmatrix}$$

which is of the special form in (4.55). C_q and D_q follow simply using (4.56) and (4.57). Finally, using (4.22), a low-order stabilizing controller $K_{stab}^r(s)$ is computed as

$$K_{stab}^r(s) = \left[\begin{array}{c|c} A_{kr} & B_{kr} \\ \hline C_{kr} & D_{kr} \end{array}\right] = \left[\begin{array}{r|rr} 10 & -91 & -888 \\ \hline -2 & 12 & 87 \\ -6 & 1 & 28 \\ -10 & -1 & 160 \end{array}\right]$$

which is a 1st-order (unstable) controller. The resulting closed-loop poles are given by

$$-63.3498, \quad -5.7614 \pm j4.8267, \quad -0.1153, \quad -0.0121, \quad -2.0000$$

which are the union of the eigenvalues of $A_o + B_o\bar{F}$ and A_q.

Example 1a To demonstrate method II described in Subsection 4.4.2 for determining W and thus $\hat{F}$, we reconsider the plant model in Example 1. Again, we aim to find a 1st-order controller (i.e., $n_q = 1$). So, we set $A_q = -2$ and find X as before. Then $\breve{C}$ matrix is built as

$$\breve{C} = \begin{bmatrix} \hat{X} \\ \hat{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

If we choose $\gamma = 10^3$ and an output feedback matrix $W \in \mathcal{R}^{3\times 3}$ as

$$W = \begin{bmatrix} -2 & -12 & -55 \\ -6 & -1 & 55 \\ -10 & 1 & -16 \end{bmatrix}$$

then a positive definite solution P to ARE (4.68) exists. Having chosen such a W and found a positive definite solution P, the state feedback matrix $\hat{F} = W\breve{C}$, from equation (4.64), is computed as

$$\hat{F} = \begin{bmatrix} -12 & -55 & 0 & 0 & -2 \\ -1 & 55 & 0 & 0 & -6 \\ 1 & -16 & 0 & 0 & -10 \end{bmatrix}$$

from which C_q and D_q are easily obtained. The eigenvalues of $\hat{A} + \hat{B}\hat{F}$ are at

$$-16.9831 \pm j8.7452, \quad -8.8934, \quad -0.1285, \quad -0.0120.$$

So, we may construct another low-order stabilizing controller of order 1 given by

$$K_{stab}^{r}(s) = \frac{1}{s-10}\begin{bmatrix} 12s+62 & 55s+802 \\ s+536 & -55s+4606 \\ -s+920 & 16s+6600 \end{bmatrix}.$$

This controller generates the closed-loop poles at

$$-16.9831 \pm j8.7452, \quad -8.8934, \quad -0.1285, \quad -0.0120, \quad -2.0000$$

which also illustrates the separation property.

4.10.2 Example 2

To demonstrate the controllability argument approach (CAA), we consider the following SISO unstable plant $G(s) = (A, B, C)$:

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right].$$

Its transfer function matrix is

$$G(s) = \frac{2s+3}{s^3 - 8s - 6}$$

and the open-loop poles are at

$$3.1474, \quad -0.8186, \quad -2.3289.$$

The plant matrices (A, B, C) can be transformed into the *controllable canonical form* (A_c, B_c, C_c) :

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0.5000 & 6.2501 & 1.2353 & 1 \\ 1.0000 & 0.3824 & 1.1626 & 0 \\ 0.0000 & 1.0000 & -0.8824 & 0 \\ \hline 0.0000 & 1.1781 & 0.7276 & 0 \end{array}\right]$$

having $m_1 = 1, \quad m_2 = 1$ and $m_{\nu_c} = 1$, where $\nu_c = 3$ is controllability index.

Since $m_{\nu_c} = 1$, we may start with $n_q = 1$. We set $A_q = -4$ and then find Y as

$$Y = \left[\begin{array}{ccc} 12.4998 & -3.1175 & 1.0000 \end{array}\right]^T.$$

If we choose an observer gain matrix H as

$$H = \begin{bmatrix} -50.000 & -6.235 & 2.000 \end{bmatrix}^T$$

the eigenvalues of $A + HC$ are at

$$-2.0986 \pm j6.7293, \quad -1.6930.$$

Then, a 1st-order (stable) controller $K^r_{stab}(s)$ is obtained using (4.25) as

$$K^r_{stab}(s) = \frac{74.9996s + 224.0003}{s + 9.8902}$$

which results in closed-loop poles at

$$-2.0986 \pm j6.7293, \quad -1.6930, \quad -4.0000.$$

This example demonstrates pole assignability via the separation property between $A + HC$ and A_q.

4.10.3 Analysis and Comments

The examples shown illustrate the following attractive features :

- The computational algorithm for computing low-order stabilizing controllers is valid and easily implemented.
- A separation property holds for the closed-loop poles:
  The closed-loop poles are the union of the eigenvalues of $A + BF$ and the poles of $Q(s)$, in the observability argument approach as in Example 1. In the controllability arguments approach used in Example 2, the separation property is between $A + HC$ and $Q(s)$.
- The stability of the closed-loop is preserved when the full-order controller is replaced by the low-order controller.

In Examples 1 and 2, using the same $Q(s)$, we compute the $(n + n_q)$th-order controller $K_{stab}(s)$ as per (4.12), and then examine the *normalized* Hankel singular values of $K_{stab}(s)$. The results are given in Table 4.1, and clearly show that the low-order controllers obtained in the previous subsections are minimal realizations of the "formal" order controllers. That is, controllers having the same orders may be obtained using existing model reduction techniques. Indeed, using Matlab files *balmr.m* (for the balanced truncated model reduction) and *ohklmr.m* (for the optimal Hankel norm model reduction), we obtained the low-order controllers of the same order as those we derived here.

Example 1 $(n = 5, n_q = 1)$	Example 1a $(n = 5, n_q = 1)$	Example 2 $(n = 3, n_q = 1)$
1.0E-00	1.0E-00	1.0E-00
0	0	1.7E-13
0	0	4.0E-14
0	0	0
0	0	
0	0	

Table 4.1. Normalized Hankel Singular Values of $K_{stab}(s)$

Frequency responses can be used as a performance measure to evaluate the low-order controllers. For comparison purposes, the frequency responses (singular value plots in general) of the formal order controllers and the low-order controllers are shown in Figure 4.4 for Example 1 and in Figure 4.5 for Example 2.

Table 4.2 below compares the orders of stabilizing controllers which we obtained in Examples 1 and 2 by means of the algorithm presented in this chapter, with those predicted by others [8],[38],[44],[45].

	Example 1	Example 2
number of states (n)	5	3
number of inputs (m)	3	1
number of outputs (l)	2	1
controllability index (ν_c)	3	3
observability index (ν_o)	4	3
Brasch/Pearson [8]	2	2
Kimura [38]	1	2
Linnemann [44]	N/A	1
Luenberger [45]	3	2
New results	1	1

Table 4.2. Orders of Low-Order Stabilizing Controllers.

4.11 Concluding Remarks

A methodology for determining a set of stabilizing controllers of smallest possible order was presented in this chapter. Working from the celebrated parametrization of all stabilizing controllers in terms of a free stable parameter matrix $Q(s)$, we derived a low-order realization $K^r_{stab}(s)$ of (4.22) on the assumption of the existence of a solution matrix X to two simultaneous matrix equations, (4.20)-(4.21). The derivation was based on eliminating any unobservable states in the formal order controllers $K_{stab}(s)$ given by (4.12). Two equations were solved to find the smallest possible size of stabilizing controllers, using an orthogonal canonical transformation. As a result, we have shown that the order of stabilizing controllers may be less than or equal to l_{ν_o}.

The algorithm presented in this chapter for deriving low-order stabilizing controllers can be used either as a form of minimal realization or of model reduction of all stabilizing controllers $K_{stab}(s)$ in (4.12), depending on the choice of the special form of $\hat{F}$ as in (4.55).

Some related issues, such as the determination of $\hat{F}$ in the form (4.55), pole assignability of closed-loop poles by low-order stabilizing controllers, and confirmation of Kimura's results, were also considered. Finally, two numerical examples were given to illustrate the application of the constructive algorithm and the results on controller size.

Although stabilization is of fundamental importance in control system design, it is not enough on its own to guarantee good performance. In the following chapters, the main idea and the solution method described in this chapter will be extended to $\mathcal{H}_\infty$ design where robust stability and some performance objectives can be addressed together.

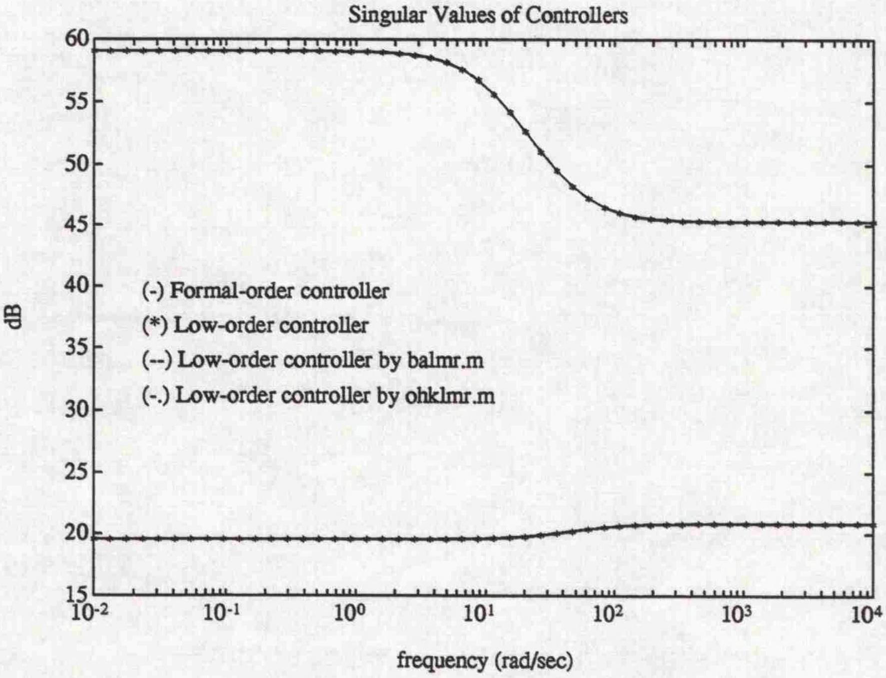


Figure 4.4: Singular Values of Controllers for Example 1.

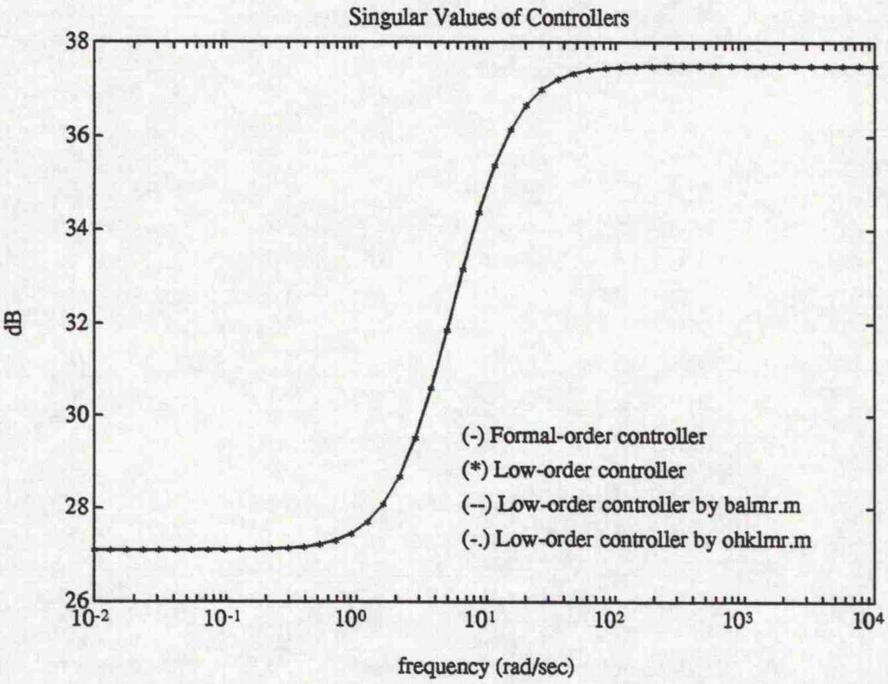


Figure 4.5: Singular Values of Controllers for Example 2.

Chapter 5

Low-Order $\mathcal{H}_\infty$ Sub-Optimal Controller Design

5.1 Introduction

An important problem in advanced control system design is the $\mathcal{H}_\infty$ *suboptimal control problem*, where stabilizing controllers which satisfy an upper bound on the $\mathcal{H}_\infty$-norm of a certain closed-loop transfer function matrix are to be found. The problem has recently seen the elegant state-space solutions, obtained by Glover and Doyle [27] and Doyle *et. al* [18], via two Riccati equations. There, the set of all $\mathcal{H}_\infty$ suboptimal controllers is parametrized using linear fractional transformations and the so-called $\mathcal{Q}$*-parametrization.*

Even though there is the freedom in the $\mathcal{Q}$-parametrization to meet certain control design objectives, the $\mathcal{H}_\infty$ suboptimal controller may have a "high" order, i.e.,

$$\mathcal{N} \leq \deg(G) + \deg(W) + \deg(\Phi)$$

where $G(s)$ is the nominal plant to be controlled, $W(s)$ is the frequency weighting functions selected by the designer and $\Phi(s)$ is a free stable transfer function matrix.

There are some approaches currently available for finding low-order $\mathcal{H}_\infty$ controllers. For example, an early result by Limebeer and Hung [42] showed that for the $\mathcal{H}_\infty$ suboptimal control problem, a controller with degree no greater than that of the generalized plant (i.e., the nominal plant plus the weighting functions) exists; Mustafa and Glover [51] developed an $\mathcal{H}_\infty$-balancing method in which the $\mathcal{H}_\infty$ characteristic values are computed and its small values are then truncated; and more recently, Hsu *et al.* [32] and Iwasaki and Skelton [34] gave some interesting results.

In this chapter, we present a new approach to low-order $\mathcal{H}_\infty$ suboptimal controller design, which is similar in spirit to that developed in Chapter 4 of this thesis. We have shown in Chapter 4 that low-order stabilizing controllers can be derived by suitably choosing the free parameter matrix $Q(s)$ in the Youla parametrization of all stabilizing controllers. So, given the results of Chapter 4, it is natural to ask whether a size reduction on a class of $\mathcal{H}_\infty$ suboptimal controllers is possible in the spirit of low-order stabilizing controllers. It is the purpose of this chapter to show that this is the case. Indeed, we will extend the methodology developed in Chapter 4 to characterize **low-order $\mathcal{H}_\infty$ suboptimal controllers**, $K_\infty^r(s)$, of order less than the order of the generalized plant, while keeping the $\mathcal{H}_\infty$-norm of a closed-loop transfer function matrix within the prescribed value. So, the approach to be presented only requires the solution to two simultaneous matrix equations, (5.34)-(5.35), and the satisfaction of an $\mathcal{H}_\infty$-norm bound, (5.38). Consequently, it is shown that the order of $\mathcal{H}_\infty$ suboptimal controllers may be reduced to

$$\mathcal{N}_{low} = \deg(G) + \deg(W) - p_2$$

or less for some plants, where p_2 is the number of plant outputs.

The chapter is organized as follows. In Section 5.2 the now standard state-space solution to the general $\mathcal{H}_\infty$ suboptimal problem is briefly outlined. In Section

5.3, we derive a low-order realization, (5.33), of $\mathcal{H}_\infty$ suboptimal controllers and show that controllers of order $n - p_2$ (or less) exist providing an $\mathcal{H}_\infty$-norm constraint is satisfied. The problem of the $\mathcal{H}_\infty$-norm constraint is considered in Section 5.4. A CAD algorithm for low-order $\mathcal{H}_\infty$ suboptimal controller design is presented in Section 5.5. A related problem of controller size reduction in $\mathcal{H}_2$(LQG) controllers is considered in Section 5.6. In Section 5.7 some numerical examples are given to illustrate the results of the chapter. Conclusions are given in Section 5.8.

5.2 State-Space Formulae for the $\mathcal{H}_\infty$ Sub-Optimal Controller

Consider a generalized plant, $P(s)$ described by

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{5.1}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{5.2}$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \tag{5.3}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $w(t) \in \mathcal{R}^{m_1}$ is the exogenous input vector, $u(t) \in \mathcal{R}^{m_2}$ is the control input vector, $z(t) \in \mathcal{R}^{p_1}$ is the error vector, and $y(t) \in \mathcal{R}^{p_2}$ is the observation vector. The generalized plant $P(s)$ is given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{5.4}$$

The $\mathcal{H}_\infty$ optimization problem is to find

$$\inf_{K stabilizing} ||\mathcal{F}_l(P, K)||_\infty = \gamma_{min} \tag{5.5}$$

where the controller $K(s)$ to be designed is chosen over all controllers which internally stabilize the generalized plant $P(s)$. To represent both robust stability and performance objectives in the same $\mathcal{H}_\infty$ minimization framework, an $\mathcal{H}_\infty$ optimization problem specification typically combines a number of frequency weighted closed-loop transfer functions and minimizes the $\mathcal{H}_\infty$-norm of the composite transfer function matrix. In general, it is not possible to solve for γ_{min} exactly and hence the so-called $\mathcal{H}_\infty$ suboptimal control problem was introduced. The reduction in stability margin incurred as a result of using a slightly suboptimal cost can be compensated by improved performance.

The $\mathcal{H}_\infty$ suboptimal problem of finding a stabilizing controller $K_\infty(s)$ such that

$$||\mathcal{F}_l(P, K_\infty)||_\infty < \gamma \tag{5.6}$$

for some prespecified value of $\gamma(> \gamma_{min})$ has been efficiently solved by Glover and Doyle [27] and Doyle *et al.* [18], using two Riccati equations.

This class of $\mathcal{H}_\infty$ suboptimal control problem is considered in this chapter to develop low-order $\mathcal{H}_\infty$ suboptimal controllers.

The questions we consider in this chapter are: *Under what assumption does there exist controllers $K^r_\infty(s)$ of size less than n which also meet the $\mathcal{H}_\infty$-norm constraint:*

$$||\mathcal{F}_l(P, K^r_\infty)||_\infty < \gamma \tag{5.7}$$

and how can we find such reduced-order controllers?

Glover and Doyle [27] have stated necessary and sufficient conditions for the existence of a stabilizing controller solving (5.6), and parametrize all such controllers. A brief summary of their work is given next.

The following assumptions are made on $P(s)$:

A1. (A, B_2, C_2, D_{22}) is stabilizable and detectable.

A2. $\text{Rank}(D_{12}) = m_2$ and $\text{rank}(D_{21}) = p_2$.

A3. D_{12} and D_{21} are transformed into

$$D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \quad \text{and} \quad D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$$

by a scaling of u and y, together with a unitary transformation of w and z. And D_{11} is partitioned as

$$D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$$

where D_{1122} has m_2 rows and p_2 columns.

A4. The physical plant is strictly proper, consequently $D_{22} = 0$.

A5. Rank $\begin{bmatrix} A - jwI & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \quad \forall w \in \mathcal{R}.$

A6. Rank $\begin{bmatrix} A - jwI & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \quad \forall w \in \mathcal{R}.$

These assumptions are required for the following reasons:
A1 is to ensure the existence of a stabilizing controller $K_\infty(s)$; A2 is to ensure the properness of $K(s)$; A3 is for dimensional compatibility with D_{12} and D_{21}; A4 is for simplicity only and thus can be relaxed; finally, both A5 and A6 are to avoid pole-zero cancellations on the imaginary axis and to prevent $P(s)$ from having transmission zeros on the jw-axis.

Define

$$R := D_{1*}^T D_{1*} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.8}$$

and

$$\tilde{R} := D_{*1} D_{*1}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.9}$$

where

$$D_{1*} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \quad \text{and} \quad D_{*1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}. \tag{5.10}$$

Let $X_\infty \geq 0$ and $Y_\infty \geq 0$ be the stabilizing solutions of the following Riccati equations:

$$X_\infty := \mathbf{Ric} \begin{bmatrix} A - BR^{-1}D_{1*}^T C_1 & -BR^{-1}B^T \\ -C_1^T(I - D_{1*}R^{-1}D_{1*}^T)C_1 & -(A - BR^{-1}D_{1*}^T C_1)^T \end{bmatrix} \tag{5.11}$$

and

$$Y_\infty := \mathbf{Ric}\begin{bmatrix} A - B_1 D_{*1}^T \tilde{R}^{-1} C & -C^T \tilde{R}^{-1} C \\ -B_1(I - D_{*1}^T \tilde{R}^{-1} D_{*1}) B_1^T & -(A - B_1 D_{*1}^T \tilde{R}^{-1} C)^T \end{bmatrix}. \quad (5.12)$$

Now define a state feedback matrix F as

$$F \; := \; -R^{-1}(D_{1*}^T C_1 + B^T X_\infty) = \begin{bmatrix} F_{11} \\ F_{12} \\ F_2 \end{bmatrix} \quad (5.13)$$

where F_{11}, F_{12} and F_2 have $m_1 - p_2, p_2$ and m_2 rows, respectively, and define an observer gain matrix H as

$$H := -(B_1 D_{*1}^T + Y_\infty C^T)\tilde{R}^{-1} = \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} \quad (5.14)$$

where H_{11}, H_{12} and H_2 have $p_1 - m_2, m_2$ and p_2 columns, respectively.

The central results for the algorithm are stated in the following Theorem.

Theorem 5.1 (Glover and Doyle [27])
(1) A stabilizing controller exists, such that $||\mathcal{F}_l(P, K_\infty)||_\infty < \gamma$, if and only if
(i)

$$\gamma > \max(\sigma_{max}[D_{1111}, D_{1112}], \sigma_{max}[D_{1111}^T, D_{1121}^T]) \quad (5.15)$$

and
(ii) there exist solutions $X_\infty \geq 0$ and $Y_\infty \geq 0$ of (5.11) and (5.12), respectively, such that

$$\rho(X_\infty Y_\infty) < \gamma^2. \quad (5.16)$$

(2) If (i) and (ii) above are satisfied, then all (rational) stabilizing controllers $K_\infty(s)$, for which $||\mathcal{F}_l(P, K_\infty)||_\infty < \gamma$, are given by

$$K_\infty = \mathcal{F}_l(K_a, \Phi) \quad (5.17)$$

for any rational $\Phi(s) \in \mathcal{RH}_\infty^{m_2 \times p_2}$ *such that* $||\Phi(s)||_\infty < \gamma$, *where* K_a *has the realization*

$$K_a(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right] \tag{5.18}$$

and

$$\hat{D}_{11} = -D_{1121}D_{1111}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122} \tag{5.19}$$

$\hat{D}_{12} \in \mathcal{R}^{m_2 \times m_2}$ *and* $\hat{D}_{21} \in \mathcal{R}^{p_2 \times p_2}$ *are any matrices satisfying*

$$\begin{aligned} \hat{D}_{12}\hat{D}_{12}^T &= I - D_{1121}(\gamma^2 I - D_{1111}^T D_{1111})^{-1}D_{1121}^T & (5.20) \\ \hat{D}_{21}^T\hat{D}_{21} &= I - D_{1112}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} & (5.21) \end{aligned}$$

and

$$\begin{aligned} \hat{B}_2 &= (B_2 + H_{12})\hat{D}_{12} & (5.22) \\ \hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12})Z_\infty & (5.23) \\ \hat{B}_1 &= -H_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} & (5.24) \\ \hat{C}_1 &= F_2 Z_\infty + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & (5.25) \\ \hat{A} &= A + HC + \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & (5.26) \end{aligned}$$

where

$$Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}. \tag{5.27}$$

Figure 5.1 shows a diagram of an $\mathcal{H}_\infty$ suboptimal controller comprising of $K_a(s)$ and $\Phi(s)$, as in (5.17).

The all-solution controllers $K_\infty(s)$ such that $||\mathcal{F}_l(P, K_\infty)||_\infty < \gamma$ are parameterized by the free stable $\Phi(s)$ constrained by $||\Phi(s)||_\infty < \gamma$. In the next section, we will use the parameterization of $K_\infty(s)$ given by (5.17) to reduce the order of the $\mathcal{H}_\infty$ suboptimal controllers.

Note that if $D_{11} = 0$ then the formulae in Theorem 5.1 are considerably simplified, Doyle *et al.* [18].

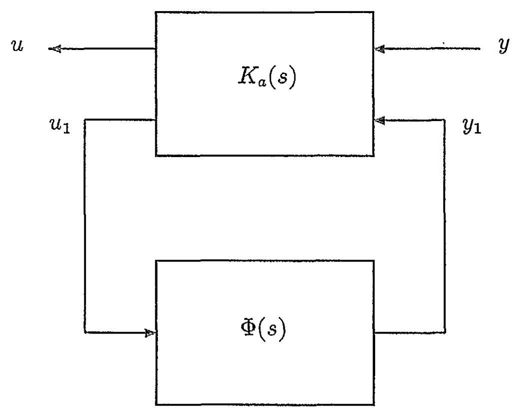


Figure 5.1: Diagram for an $\mathcal{H}_\infty$ Sub-optimal Controller, $K_\infty = \mathcal{F}_l(K_a, \Phi)$.

5.3 Low-Order $\mathcal{H}_\infty$ Sub-Optimal Controllers

The formulae cited in the previous section represent an important result in optimal control theory. It is clear that we are always able to obtain suboptimal controllers of size equal to, or less than, n by simply choosing $\Phi(s)$ as a constant matrix with largest singular value less than γ, provided the feasibility conditions (5.15) and (5.16) are met.

For convenience, we rewrite all (rational) stabilizing $\mathcal{H}_\infty$ controllers in (5.17) as

$$K_\infty(s) = \mathcal{F}_l(K_a, \Phi) \tag{5.28}$$

for any rational $\Phi(s) \in \mathcal{RH}_\infty^{m_2 \times p_2}$ such that $||\Phi(s)||_\infty < \gamma$.

Let $\Phi(s)$ in (5.28) have a state-space realization

$$\Phi(s) := \left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right] \in \mathcal{RH}_\infty \tag{5.29}$$

where A_ϕ is stable, and $A_\phi : n_\phi \times n_\phi$, $B_\phi : n_\phi \times p_2$, $C_\phi : m_2 \times n_\phi$ and $D_\phi : m_2 \times p_2$, respectively.

Then, using (5.18) and (5.29), $K_\infty(s)$ of (5.28) can be expressed as

$$K_\infty(s) = \left[\begin{array}{cc|c} \hat{A} + \hat{B}_2 D_\phi \hat{C}_2 & \hat{B}_2 C_\phi & \hat{B}_1 + \hat{B}_2 D_\phi \hat{D}_{21} \\ B_\phi \hat{C}_2 & A_\phi & B_\phi \hat{D}_{21} \\ \hline \hat{C}_1 + \hat{D}_{12} D_\phi \hat{C}_2 & \hat{D}_{12} C_\phi & \hat{D}_{11} + \hat{D}_{12} D_\phi \hat{D}_{21} \end{array}\right] \tag{5.30}$$

making use of a state-space realization of an LFT given in [57]. From (5.30), we see that all controllers $K_\infty(s)$ have a state dimension of $(n + n_\phi)$ if there are no pole-zero cancellations between $K_a(s)$ and $\Phi(s)$, since

$$\begin{aligned} \deg(K_\infty) &= \deg(K_a) + \deg(\Phi) - \alpha \\ &= n + n_\phi - \alpha \end{aligned}$$

where α is the number of cancellations between $K_a(s)$ and $\Phi(s)$.

Remark 5.2 The "central" (or maximum entropy) controller, which is obtained from (5.28) by taking $\Phi(s) = 0$, has the realization

$$K_\infty(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right].$$

This "central" controller has an order:

$$\begin{aligned} \mathcal{N}_{central} &= \deg(G) + \deg(W) \\ &= n \end{aligned}$$

and is widely used in $\mathcal{H}_\infty$ control. The freedom in the parametrization cannot easily be used to yield desirable closed-loop properties and hence usually is ignored.

5.3.1 Derivation of Reduced-Order Controllers

We begin by applying a state similarity transformation to $K_\infty(s)$ of (5.30) using a nonsingular matrix

$$T_1 := \begin{bmatrix} I_n & 0 \\ X & I_{n_\phi} \end{bmatrix} \tag{5.31}$$

to find a new realization given by

$$K_\infty(s) := \left[\begin{array}{cc|c} K_{a11} & K_{a12} & K_{b1} \\ K_{a21} & K_{a22} & K_{b2} \\ \hline K_{c1} & K_{c2} & K_d \end{array}\right] \tag{5.32}$$

where

$$\begin{aligned}
K_{a11} &= \hat{A} + \hat{B}_2 D_\phi \hat{C}_2 - \hat{B}_2 C_\phi X \\
K_{a12} &= \hat{B}_2 C_\phi \\
K_{a21} &= X\hat{A} + X\hat{B}_2 D_\phi \hat{C}_2 + B_\phi \hat{C}_2 - X\hat{B}_2 C_\phi X - A_\phi X \\
K_{a22} &= X\hat{B}_2 C_\phi + A_\phi \\
K_{b1} &= \hat{B}_1 + \hat{B}_2 D_\phi \hat{D}_{21} \\
K_{b2} &= X\hat{B}_1 + X\hat{B}_2 D_\phi \hat{D}_{21} + B_\phi \hat{D}_{21} \\
K_{c1} &= \hat{C}_1 + \hat{D}_{12} D_\phi \hat{C}_2 - \hat{D}_{12} C_\phi X \\
K_{c2} &= \hat{D}_{12} C_\phi \\
K_d &= \hat{D}_{11} + \hat{D}_{12} D_\phi \hat{D}_{21}.
\end{aligned}$$

From the realization of $K_\infty(s)$ in (5.32), we may obtain a *reduced-order realization*, which we refer to as the **low-order $\mathcal{H}_\infty$ suboptimal controller**:

$$K_\infty^r(s) = \left[\begin{array}{c|c} X\hat{B}_2 C_\phi + A_\phi & X\hat{B}_1 + X\hat{B}_2 D_\phi \hat{D}_{21} + B_\phi \hat{D}_{21} \\ \hline \hat{D}_{12} C_\phi & \hat{D}_{11} + \hat{D}_{12} D_\phi \hat{D}_{21} \end{array}\right] \tag{5.33}$$

if there exists a matrix $X \in \mathcal{R}^{n_\phi \times n}$ satisfying $K_{a21} = 0$ and $K_{c1} = 0$, i.e., the following two matrix equations are satisfied:

$$A_\phi X - X\tilde{A} = B_\phi \hat{C}_2 \tag{5.34}$$

$$C_\phi X - D_\phi \hat{C}_2 = \tilde{F} \tag{5.35}$$

where

$$\tilde{A} = \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 \tag{5.36}$$

$$\tilde{F} = \hat{D}_{12}^{-1} \hat{C}_1. \tag{5.37}$$

It is emphasized that $K_\infty^r(s)$ in (5.33) has an order:

$$\deg(K_\infty^r) = n_\phi$$

at most, which is obviously less than $n + n_\phi$, the "formal" order of $K_\infty(s)$ of (5.30), occurring in the case of $\alpha = 0$. The realization (5.33) is in a convenient form for computing a set of low-order $\mathcal{H}_\infty$ suboptimal controllers.

It is also interesting to see that the two equations (5.34)-(5.35) are similar to those required for finding low-order stabilizing controllers as described in Chapter 4. Thus, the methodology proposed in Chapter 4 for solving those equations might be applicable here. However, in the present chapter, we have a new constraint, i.e.,

$$||\Phi(s)||_\infty = ||C_\phi(sI - A_\phi)^{-1}B_\phi + D_\phi||_\infty < \gamma \tag{5.38}$$

and, in addition, the freedom in the choice of $\tilde{F}$ is considerably limited. These two constraints indicate that the problem of finding low-order $\mathcal{H}_\infty$ suboptimal controllers is more difficult than that of finding low-order stabilizing controllers.

Remark 5.3 The closed-loop transfer function (CLTF) formed by the generalized plant $P(s)$ and the low-order controller $K_\infty^r(s)$ can be computed from the following linear fractional transformation:

$$\text{CLTF} := \mathcal{F}_l(P, K_\infty^r)$$

using $P(s)$ in (5.2)-(5.4) and $K_\infty^r(s)$ in (5.33).

5.3.2 $\mathcal{H}_\infty$ Sub-Optimal Controllers of Order $n - p_2$

The existence of low-order $\mathcal{H}_\infty$ suboptimal controllers depends on the solution of the two simultaneous matrix equations, (5.34)-(5.35), subject to an $\mathcal{H}_\infty$-norm constraint, (5.38). In what follows we show how to find a solution matrix X and a suitable free parameter $\Phi(s)$ to satisfy the two matrix equations.

The first equation (5.34):

$$A_\phi X - X\tilde{A} \;=\; B_\phi \hat{C}_2$$

can be rather easily solved for $X \in \mathcal{R}^{n_\phi \times n}$, whereas the solvability of the second equation (5.35):

$$C_\phi X - D_\phi \hat{C}_2 \;=\; \tilde{F}$$

may be limited due to the lack of freedom on $\tilde{F}$. In this subsection, we consider the special case of $n_\phi = n - p_2$ in which C_ϕ and D_ϕ can always be found regardless of the structure of $\tilde{F}$.

Assumption 5.4 Assume that $\hat{C}_2 \in \mathcal{R}^{p_2 \times n}$ is full row rank.

To solve the problem for the case of $n_\phi = n - p_2$, suppose that $\hat{C}_2$ takes the form

$$\hat{C}_2 = \begin{bmatrix} I_{p_2} & 0_{p_2 \times (n-p_2)} \end{bmatrix}. \tag{5.39}$$

Then, partition $\tilde{A} \in \mathcal{R}^{n \times n}$ conformally as

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \tag{5.40}$$

where $\tilde{A}_{11} : p_2 \times p_2$, $\tilde{A}_{12} : p_2 \times (n-p_2)$, $\tilde{A}_{21} : (n-p_2) \times p_2$, and $\tilde{A}_{22} : (n-p_2) \times (n-p_2)$.

Note from Lemma 2.6 that the pair $(\tilde{A}_{22}, \tilde{A}_{12})$ is completely observable if the pair $(\tilde{A}, \hat{C}_2)$ is completely observable.

Recall that we set $n_\phi = n - p_2$, which will be the order of the low-order $\mathcal{H}_\infty$ suboptimal controller. Now suppose that $X \in \mathcal{R}^{(n-p_2) \times n}$ and $\tilde{F} \in \mathcal{R}^{m_2 \times n}$ are partitioned as

$$X \;=\; \begin{bmatrix} X_1 & X_2 \end{bmatrix} \tag{5.41}$$

$$\tilde{F} \;=\; \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix} \tag{5.42}$$

where $X_1 : (n-p_2)\times p_2$, $X_2 : (n-p_2)\times(n-p_2)$, $\tilde{F}_1 : m_2\times p_2$, and $\tilde{F}_2 : m_2\times(n-p_2)$.

Then, from (5.34), we have the following two equations:

$$\begin{aligned} A_\phi X_1 - X_1\tilde{A}_{11} - X_2\tilde{A}_{21} &= B_\phi && (5.43) \\ A_\phi X_2 - X_1\tilde{A}_{12} - X_2\tilde{A}_{22} &= 0. && (5.44) \end{aligned}$$

Equation (5.44) is equivalent to:

$$X_2\tilde{A}_{22} + X_1\tilde{A}_{12} = A_\phi X_2 \quad (5.45)$$

As shown in Lemma 4.5, for any stable A_ϕ, there always exists a matrix X_1 and a nonsingular X_2 which satisfy equation (5.45), provided the pair $(\tilde{A}, \hat{C}_2)$ is completely observable.

Remark 5.5 The assumption of observability of the pair $(\tilde{A}, \hat{C}_2)$ - and hence observability of the pair $(\tilde{A}_{22}, \tilde{A}_{12})$ - can be relaxed when finding the solution matrix X, since X can also be found even when the pair $(\tilde{A}, \hat{C}_2)$ is *not* completely observable. For details, refer to Remark 5.9 later.

Having found X_1 and X_2, B_ϕ can be obtained from (5.43), and C_ϕ and D_ϕ can be computed from (5.35) as

$$\begin{aligned} C_\phi &= \tilde{F}_2 X_2^{-1} && (5.46) \\ D_\phi &= \tilde{F}_2 X_2^{-1} X_1 - \tilde{F}_1 && (5.47) \end{aligned}$$

making use of (5.41)-(5.42).

Hence, we now have X, A_ϕ, B_ϕ, C_ϕ and D_ϕ required for the realization (5.33), and can therefore compute an $\mathcal{H}_\infty$ suboptimal controller of order $n-p_2$, provided the $\mathcal{H}_\infty$-norm constraint, $||\Phi(s)||_\infty < \gamma$, is satisfied.

Without loss of generality, the identity matrix can be chosen as a candidate for X_2 (i.e., $X_2 = I_{n-p_2}$). In this instance, all the state-space matrices of the free parameter matrix $\Phi(s)$ can be found, in terms of an arbitrary matrix

$X_1 \in \mathcal{R}^{(n-p_2)\times p_2}$, by

$$\begin{aligned}
A_\phi &= \tilde{A}_{22} + X_1\tilde{A}_{12} &(5.48)\\
B_\phi &= A_\phi X_1 - X_1\tilde{A}_{11} - \tilde{A}_{21} &(5.49)\\
C_\phi &= \tilde{F}_2 &(5.50)\\
D_\phi &= \tilde{F}_2 X_1 - \tilde{F}_1 &(5.51)
\end{aligned}$$

subject to the stability of A_ϕ. Hence, a solution matrix X and a suitable free parameter $\Phi(s)$ to the two simultaneous matrix equations (5.34)-(5.35) are both characterized in terms of just X_1, which can be chosen arbitrarily subject to the stability of A_ϕ. In addition, such a characterization of the parameters A_ϕ, B_ϕ, C_ϕ and D_ϕ in terms of X_1 may simplify the solvability of the $\mathcal{H}_\infty$-norm constraint on $\Phi(s)$ which we will discuss later in Section 5.4. Consequently, we have low-order $\mathcal{H}_\infty$ suboptimal controllers of order $n - p_2$ as stated in the Theorem below.

Theorem 5.6: *The generalized plant $P(s)$ described by (5.4) has low-order $\mathcal{H}_\infty$ suboptimal controllers of order:*

$$\mathcal{N}_{low} = n - p_2$$

if $\hat{C}_2$ in (5.23) is full row rank and if the $\mathcal{H}_\infty$-norm constraint on $\Phi(s)$ of (5.38) is satisfied.

Remark 5.7: In solving (5.44), X_2 need not be the identity matrix, as shown in Lemma 4.5. In general, we need to choose A_ϕ as an arbitrary but stable matrix. Then, X_1 and X_2 can be calculated from (5.45) and consequently B_ϕ, C_ϕ and D_ϕ all depend only on A_ϕ. A_ϕ is a suitable candidate if the resulting $\Phi(s)$ satisfies the $\mathcal{H}_\infty$-norm constraint of (5.38).

An alternative solution to the two equations (5.34)-(5.35) is given in Appendix B, using an orthogonal canonical form as in (2.21)-(2.22).

5.3.3 $\mathcal{H}_\infty$ Sub-Optimal Controllers of Order Less Than $n - p_2$

Recall that the order of $K^r_\infty(s)$ is n_ϕ, which is the number of rows of the solution matrix X. In this subsection, we consider the possibility of lowering the order of $K^r_\infty(s)$, that is, finding a lower n_ϕ. The approach adopted in Subsection 4.3.3 is largely used here.

Suppose the pair $(\tilde{A}, \hat{C}_2)$ is completely observable. Then the pair $(\tilde{A}, \hat{C}_2)$ is reduced to the orthogonal canonical form (A_o, C_o):

$$A_o = M\tilde{A}M^{-1} = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & A_{\nu_o-1,\nu_o} \\ A_{\nu_o,1} & A_{\nu_o,2} & A_{\nu_o,3} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix} \tag{5.52}$$

$$C_o = N\hat{C}_2M^{-1} = \begin{bmatrix} I_{p_2} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{5.53}$$

as in Chapter 4.

Using the form (A_o, C_o), the two equations (5.34)-(5.35) can be transformed into:

$$A_\phi\bar{X} - \bar{X}A_o = \bar{B}_\phi C_o \tag{5.54}$$

$$C_\phi\bar{X} - \bar{D}_\phi C_o = \bar{F} \tag{5.55}$$

where

$$\bar{X} = XM^{-1} \tag{5.56}$$

$$\bar{B}_\phi = B_\phi N^{-1} \tag{5.57}$$

$$\bar{D}_\phi = D_\phi N^{-1} \tag{5.58}$$

$$\bar{F} = \tilde{F}M^{-1}. \tag{5.59}$$

Assuming that the pair (A_o, C_o) is completely observable, we have the following Theorem which shows the possibility of lowering the order of $K_\infty^r(s)$, since n_ϕ may be reduced down to $l_{\nu_o}(\leq p_2)$. (This assumption is not restrictive. The case in which this assumption does not hold is discussed in Remark 5.9.)

Theorem 5.8 *Equation (5.54) has full row rank solutions $\bar{X} \in R^{l_{\nu_o} \times n}$.*

Proof: The proof is similar to that of Theorem 4.8. ∎

From the above discussion it can be seen that if we set n_ϕ to be l_{ν_o}, $l_{\nu_o} + l_{\nu_o - 1}$, $\cdots$, n, then the corresponding solutions $\bar{X}$ to equation (5.54) may be found.

For an arbitrarily chosen A_ϕ, $\bar{B}_\phi$ can be obtained to satisfy (5.54), and the corresponding C_ϕ and $\bar{D}_\phi$ can be found from (5.55). Using a similarity transformation given by a nonsingular matrix T_2, as in (4.51), such that

$$\begin{aligned} \bar{X}T_2 &= \begin{bmatrix} 0 & 0 & \cdots & 0 & I \end{bmatrix} \\ C_oT_2 &= \begin{bmatrix} I & 0 & \cdots & 0 & 0 \end{bmatrix} \end{aligned}$$

we can rewrite equation (5.55) as

$$\begin{bmatrix} C_\phi & -\bar{D}_\phi \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & I \\ I & 0 & \cdots & 0 & 0 \end{bmatrix} = \bar{F}T_2. \tag{5.60}$$

So, if there exists an $\bar{F}$ such that $\bar{F}T_2$ has the form

$$\bar{F}T_2 = \begin{bmatrix} * & 0 & \cdots & 0 & * \end{bmatrix} \tag{5.61}$$

then we can find C_ϕ and $\bar{D}_\phi$ to satisfy (5.55), where $*$ denotes a nonzero non-specified block matrix. However, unlike the low-order stabilizing controller case in Chapter 4, there is little freedom on $\bar{F}$ since $\bar{F} = \hat{D}_{12}^{-1}\hat{C}_1 M^{-1}$ is almost completely determined by plant data. Although, the particular form of $\bar{F}$ will exclude some choices of n_ϕ, we may still expect low-order $\mathcal{H}_\infty$ suboptimal controllers of order less than $n - p_2$.

Remark 5.9 In solving equation (5.54), we assumed the pair (A_o, C_o) is completely observable. The case when this assumption is violated is now considered. In this case, A_{ν_o-1,ν_o} in (5.52) will be a zero block, as mentioned earlier in Remark 2.7 of Chapter 2. Then, it can easily be verified by taking $\bar{X}_{\nu_o} = 0$ that

$$\bar{X} = \begin{bmatrix} \bar{X}_1 & \cdots & \bar{X}_{\nu_o-1} & 0_{n_\phi \times l_{\nu_o}} \end{bmatrix} \tag{5.62}$$

will be a solution to (5.54), where $\bar{X}_1, \cdots, \bar{X}_{\nu_o-1}$ are obtained for a reduced-size pair $(\hat{A}_o, \hat{C}_o)$ by the same procedure as before, where

$$\hat{A}_o = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-2,1} & A_{\nu_o-2,2} & A_{\nu_o-2,3} & \cdots & A_{\nu_o-2,\nu_o-1} \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & A_{\nu_o-1,\nu_o-1} \end{bmatrix} \tag{5.63}$$

$$\hat{C}_o = \begin{bmatrix} I_{p_2} & 0 & 0 & \cdots & 0 \end{bmatrix} : p_2 \times (n - l_{\nu_o}). \tag{5.64}$$

5.4 $\mathcal{H}_\infty$-Norm Constraint on $\Phi(s)$

A sufficient condition for the existence of low-order $\mathcal{H}_\infty$ suboptimal controllers is the $\mathcal{H}_\infty$-norm constraint on $\Phi(s)$ given in (5.38). That is, having found all the element matrices required for $K_\infty^r(s)$ in (5.33), the $\mathcal{H}_\infty$-norm constraint on $\Phi(s)$ has to be checked. As we saw earlier in Chapter 2, Section 2.8, the following Lemma shows a connection between the $\mathcal{H}_\infty$-norm bound of a transfer function matrix and the existence of a positive definite solution X_ϕ to a certain ARE.

Lemma 5.10 *If the following ARE*

$$\begin{aligned} &(A_\phi - B_\phi R_\phi^{-1} D_\phi^T C_\phi)^T X_\phi + X_\phi (A_\phi - B_\phi R_\phi^{-1} D_\phi^T C_\phi) \\ &-\gamma X_\phi B_\phi R_\phi^{-1} B_\phi^T X_\phi - \gamma C_\phi^T S_\phi^{-1} C_\phi = 0 \end{aligned} \tag{5.65}$$

has a positive definite solution X_ϕ, then A_ϕ is stable and $||\Phi(s)||_\infty \leq \gamma$, where

$$\begin{aligned} R_\phi &= D_\phi^T D_\phi - \gamma^2 I_{p2} \\ S_\phi &= D_\phi D_\phi^T - \gamma^2 I_{m_2}. \end{aligned}$$

In what follows, we propose some methods for finding a positive definite solution X_ϕ to ARE (5.65).

5.4.1 Search Method I

Since A_ϕ, B_ϕ, C_ϕ and D_ϕ are characterized in terms of X_1 as in (5.48)-(5.51), the existence of a positive definite solution matrix X_ϕ to ARE (5.65) depends on the choice of X_1. Note that, from Lemma 5.10, the positive definiteness of X_ϕ guarantees the stability of A_ϕ. This implies that X_1 can be chosen, without considering the stability of A_ϕ, such that the solution X_ϕ to ARE (5.65) is positive definite. So, by a search over X_1, we may achieve the constraint of $||\Phi(s)||_\infty$ in (5.38).

To select an effective candidate X_1 as an initial point, we consider a result on the $\mathcal{H}_\infty$-norm bound. It is known [6] that a lower bound γ_{lb} and an upper bound γ_{ub} on $||G(s)||_\infty$, for $G(s) = (A, B, C, D)$, are given by

$$\begin{aligned} \gamma_{lb} &= \max\{\sigma_{max}(D), \sigma_1^H\} \\ \gamma_{ub} &= \sigma_{max}(D) + 2\sum_{i=1}^{n} \sigma_i^H \end{aligned}$$

where σ_i^H are the Hankel singular values of $G(s)$. An interpretation from this is that the D-matrix can play an important role on the $\mathcal{H}_\infty$-norm bounds. We therefore attempt to make D_ϕ as small as possible and, using (5.51), select an initial X_1 as

$$X_1 = \tilde{F}_2^\dagger \tilde{F}_1. \tag{5.66}$$

The following procedure is therefore proposed to find a positive definite solution X_ϕ to ARE (5.65).

step 1: Set $X_2 = I_{n_\phi}$.

step 2: Select an initial $X_1 \in \mathcal{R}^{n_\phi \times p_2}$ as per (5.66).

step 3: Compute A_ϕ, B_ϕ, C_ϕ and D_ϕ as per (5.48)-(5.51).

step 4: Solve ARE (5.65) for X_ϕ.
If $X_\phi > 0$, stop.
If $X_\phi \leq 0$, go to step 2 to choose an alternative X_1.

5.4.2 Search Method II

In Appendix B, the matrices A_ϕ, B_ϕ, C_ϕ and D_ϕ are characterized in terms of $\bar{X}_1$. In particular, when $\bar{X}_2 = I_{n_\phi}$, A_ϕ is given in (B.16) by

$$A_\phi = \left[\begin{array}{cc} A_{o22} + \bar{X}_1 & A_{o23} \end{array} \right]$$

with A_{o22} and A_{o23} fixed, and B_ϕ, C_ϕ and D_ϕ by (B.18)-(B.20), respectively. Hence, if A_ϕ is first chosen with some freedom in the left partition block (i.e., $A_{o22} + \bar{X}_1$), then $\bar{X}_1$ can be decided correspondingly from the above equation. Bearing this fact and the stability requirement of A_ϕ in mind, a relatively crude search is proposed, in this subsection, as follows.

step 1: Set $\bar{X}_2 = I_{n_\phi}$.

step 2: Select an arbitrary A_ϕ of the form:

$$A_\phi = \left[\begin{array}{cc} -\alpha I_{(n-p_2)\times l_2} & A_{o23} \end{array} \right]$$

with a small number of α, say $\alpha = 10^{-3}$.

step 3: Compute X, B_ϕ, C_ϕ and D_ϕ as per (B.17)-(B.20).

step 4: Solve ARE (5.65) for X_ϕ.
If $X_\phi > 0$, stop.
If $X_\phi \leq 0$, go to step 2 to increase α. (A reasonable upper limit on α would be a suboptimal γ, i.e., $\alpha \leq \gamma$.)

5.4.3 An Optimization Method

Alternatively, an optimization technique may be adopted here. That is, to meet the requirements that $\Phi(s) \in \mathcal{RH}_\infty$ and $||\Phi(s)||_\infty < \gamma$, we define γ_1 as

$$\gamma_1 := \gamma - \epsilon \tag{5.67}$$

where ϵ is a small positive number. We may then consider a set of constraints (5.68)-(5.71) as below. Using the fact that a symmetric matrix can be transformed to a diagonal one by an orthogonal transformation, we can try to find an $\bar{X}_\phi$ and an orthogonal matrix U such that $\bar{X}_\phi(:= U^T X_\phi U)$ solves the following ARE:

$$\begin{aligned} U^T(A_\phi - B_\phi R_\phi^{-1} D_\phi^T C_\phi)^T U \bar{X}_\phi + \bar{X}_\phi U^T(A_\phi - B_\phi R_\phi^{-1} D_\phi^T C_\phi) U \\ -\gamma_1 \bar{X}_\phi U^T B_\phi R_\phi^{-1} B_\phi^T U \bar{X}_\phi - \gamma_1 U^T C_\phi^T S_\phi^{-1} C_\phi U = 0 \end{aligned} \tag{5.68}$$

and satisfies the following constraints:

$$\bar{X}_\phi = \mathrm{diag}(\bar{x}_{\phi,1}, \cdots, \bar{x}_{\phi,n_\phi}) \tag{5.69}$$

$$\bar{x}_{\phi,i} \geq \bar{x}_{\phi,i+1} \qquad i = 1, \cdots, n_\phi - 1 \tag{5.70}$$

$$\bar{x}_{\phi,n_\phi} > 0. \tag{5.71}$$

This is a standard constrained optimization problem, for which various algorithms may be applied.

5.5 A Low-Order $\mathcal{H}_\infty$ Sub-Optimal Controller Design Algorithm

In this subsection, we present a CAD algorithm for low-order $\mathcal{H}_\infty$ suboptimal controller design, in which low-order controllers of order $n - p_2$ can be designed using the procedures described earlier in the chapter.

step 1: Build a generalized plant $P(s)$ as per (5.4), including weighting functions where necessary.

step 2: Find γ_{min} such that

$$\gamma_{min} = \inf_{K stabilizing} ||\mathcal{F}_l(P, K)||_\infty$$

using reliable algorithms in, for example, Matlab and then select a suboptimal γ to be $\gamma > \gamma_{min}$.

step 3: Compute all element matrices of $K_a(s)$ in (5.18), as per (5.19)-(5.26). This can be easily implemented using reliable algorithms in, for example, Matlab.

step 4: Compute $\tilde{A}$ and $\tilde{F}$ as per (5.36)-(5.37), and partition $\tilde{A}$, $\hat{C}_2$ and $\tilde{F}$ as per (5.39)-(5.40) and (5.42).

step 5: Set $X_2 = I_{n-p_2}$.
† Note that, as stated in Remark 5.7, X_2 need not necessarily be the identity matrix.

step 6: Select an arbitrary X_1, and then compute A_ϕ, B_ϕ, C_ϕ and D_ϕ as per (5.48)-(5.51).
†† Alternatively, as stated in Appendix B, first choose A_ϕ as per (B.16) and then compute X, B_ϕ, C_ϕ and D_ϕ as per (B.17)-(B.20).

step 7: Solve ARE (5.65) for X_ϕ.
If $X_\phi > 0$, go to step 8.
If $X_\phi \leq 0$, go to step 6 to repeat.

step 8: Compute a low-order $\mathcal{H}_\infty$ suboptimal controller, $K_\infty^r(s)$, as per (5.33).

The procedure described above is based on the given $P(s)$ and the suboptimal γ, and thus modifications to the procedure may be required, if necessary, in connection with the choices of weighting functions and a suboptimal γ.

5.6 A Related Problem: Low-Order $\mathcal{H}_2$ Sub-Optimal Controller Design

The aim of this section is to show that the approach developed so far can be carried over to the design of low-order $\mathcal{H}_2$ suboptimal controllers.

Again consider $P(s)$ in (5.4) assuming $D_{11} = 0$ and $D_{22} = 0$, i.e.,

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{5.72}$$

The standard $\mathcal{H}_2$(LQG) design problem is to find a controller that minimizes the $\mathcal{H}_2$-norm of the transfer function from w to z, $T_{zw}(s)$, where w represents a vector of zero mean white noise signals and z is used to define performance objectives.

We review in this section the characterization of all stabilizing controllers $K_2(s)$ of the $\mathcal{H}_2$ *suboptimal problem* which satisfy $||T_{zw}(s)||_2 < \gamma$ for some prespecified γ, Doyle *et al.* [18], and derive a reduced-order realization $K_2^r(s)$.

The following assumptions are made on $P(s)$:

A1. (A, B_1, C_1) is stabilizable and detectable.

A2. (A, B_2, C_2) is stabilizable and detectable.

A3. $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

A4. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Assumption A3 means that there is no cross weighting between the state and control input, and that the control weighting matrix is the identity. Assumption A4 is dual to A3 and concerns how the exogenous signal w enters $P(s)$: w includes both plant disturbances and sensor noise, which are orthogonal, and the sensor noise weighting is normalized and nonsingular.

Now define $X_2 \geq 0$ and $Y_2 \geq 0$ to be the stabilizing solutions of the following Riccati equations:

$$X_2 := \mathbf{Ric} \begin{bmatrix} A & -B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} \tag{5.73}$$

$$Y_2 := \mathbf{Ric} \begin{bmatrix} A^T & -C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}. \tag{5.74}$$

Having obtained X_2 and Y_2, define

$$F_2 := -B_2^T X_2 \tag{5.75}$$

$$H_2 := -Y_2 C_2^T \tag{5.76}$$

and

$$G_c(s) := \left[\begin{array}{c|c} A + B_2 F_2 & I \\ \hline C_1 + D_{12} F_2 & 0 \end{array}\right] \tag{5.77}$$

$$G_f(s) := \left[\begin{array}{c|c} A + H_2 C_2 & B_1 + H_2 D_{21} \\ \hline I & 0 \end{array}\right]. \tag{5.78}$$

The following Theorem describes all $\mathcal{H}_2$ suboptimal controllers.

Theorem 5.11 (Doyle *et al.* [18]) *The family of all proper, real-rational stabilizing controllers $K_2(s)$ such that $||T_{zw}||_2 < \gamma$ are given by*

$$K_2(s) = \mathcal{F}_l(M_2, \Theta) \tag{5.79}$$

for any $\Theta(s) \in \mathcal{RH}_2$ such that $||\Theta(s)||_2^2 < \gamma^2 - (||G_c B_1||_2^2 + ||F_2 G_f||_2^2)$, where

$$M_2(s) \quad = \quad \left[\begin{array}{c|cc} A + B_2F_2 + H_2C_2 & -H_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]. \tag{5.80}$$

Let $\Theta(s) \in \mathcal{RH}_2$ in (5.79) have a state-space realization

$$\Theta(s) := \left[\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array}\right] \tag{5.81}$$

where $A_\theta \in \mathcal{R}^{n_\theta \times n_\theta}$. Then, in a similar manner to the $\mathcal{H}_\infty$ suboptimal problem case, as described in Section 5.3, we may obtain a *reduced-order realization* $K_2^r(s)$ as

$$K_2^r(s) = \left[\begin{array}{c|c} A_\theta + XB_2C_\theta & B_\theta + XB_2D_\theta - XH_2 \\ \hline C_\theta & D_\theta \end{array}\right] \tag{5.82}$$

if there exists a matrix $X \in \mathcal{R}^{n_\theta \times n}$ satisfying the following two matrix equations:

$$A_\theta X - X(A + H_2C_2) \quad = \quad -B_\theta C_2 \tag{5.83}$$

$$C_\theta X + D_\theta C_2 \quad = \quad F_2. \tag{5.84}$$

So, it is obvious that the design of low-order $\mathcal{H}_2$ suboptimal controllers can be achieved in a similar manner to that developed above.

Remark 5.12 The same argument can be applied to the unconstrained $\mathcal{H}_2$ optimal state-feedback controller, which minimizes $||T_{zw}||_2$ when the plant state is available for feedback, as defined in Rotea and Khargonekar [60, Theorem 2.6].

5.7 Illustrative Examples

In this section, we present two examples to illustrate the results obtained in this chapter. In each case, we consider a weighted mixed sensitivity $\mathcal{H}_\infty$ design problem of finding a low-order stabilizing controller $K_\infty^r(s)$ such that

$$\left\| \begin{bmatrix} \gamma W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty < 1 \tag{5.85}$$

where $S := (I + GK_\infty^r)^{-1}$ is the sensitivity function, $T := GK_\infty^r(I + GK_\infty^r)^{-1}$ is the complementary sensitivity function, and $W_i \quad (i = 1, 2)$ are weighting functions used to tailor the solution to meet design specifications. It is known, for example in [61], that the requirements for disturbance attenuation and robust stability can be readily handled by this formulation. In the standard configuration of Figure 2.1, the problem has a generalized plant

$$\left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|c} \gamma W_1 & -\gamma W_1 G \\ 0 & W_2 G \\ \hline I & -G \end{array}\right]. \tag{5.86}$$

This represents a 2-block $\mathcal{H}_\infty$ problem.

5.7.1 Example 1: SISO Hydraulic Actuator Design

We will consider a SISO digital hydraulic actuator design example, taken from Chiang and Safonov [11]. The continuous time hydraulic actuator model $G(s)$ is given by

$$G(s) = \frac{9000}{s^3 + 30s^2 + 700s + 1000}.$$

To design a digital control system, the following approach is adopted here:

1. Convert the continuous-time plant $G(s)$ to the discrete-time plant $G(z)$, by augmenting the plant with a zero-order-hold (ZOH) and including a sampler as shown in Figure 5.2.

2. Convert $G(z)$ into the w-plane, see for example Franklin and Powell [24, Section 5.5].

3. Proceed with the design method described in this chapter, as if it were in the s-plane, since control systems in the w-domain can be analyzed and designed using continuous techniques and then transformed back for discretization, [24].

4. After the design is done, convert the controller back into the z-plane via the inverse w-transform.

We omit step 4 in this example, and therefore the data shown below are in the w-domain except when otherwise stated.

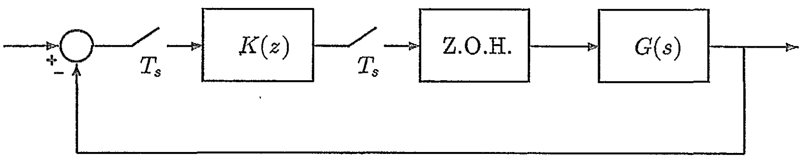


Figure 5.2: Digital controller by Z.O.H.

The model in the w-domain, $G(\mathrm{w})$, will now be used in problem (5.85) with the weights as in [11], i.e.,

$$W_1(s) = \frac{(\frac{s}{30}+1)^2}{0.01(s+1)^2}$$

$$W_2(s) = \frac{\frac{s}{10}+1}{3.16(\frac{s}{300}+1)}$$

and the same γ given as 1.5.

For this problem, we have $n = 6$ and $p_2 = 1$ and thus $n_\phi = n - p_2 = 5$. Therefore we generally expect to find a 6th-order controller as a "central" solution. In the following we derive a 5th-order controller, using the solution method described in Appendix B.

After finding the two equations corresponding to (5.34)-(5.35), we set $n_\phi = n - p_2 = 5$, $X_2 = I_5$ and choose $A_\phi = \begin{bmatrix} * & A_{o23} \end{bmatrix}$ as per (B.16):

$$A_\phi = \begin{bmatrix} -0.20 & 1 & 0 & 0 & 0 \\ 0.06 & -1.98E+01 & 1 & 0 & 0 \\ 0.09 & 2.33E+02 & -1.53E+02 & 1 & 0 \\ 0.04 & -5.40E+04 & 2.12E+04 & -1.53E+02 & 1 \\ -0.10 & 2.78E+01 & -1.57E+00 & -9.20E-03 & -1.52 \end{bmatrix}.$$

We then find X, B_ϕ, C_ϕ and D_ϕ as per (B.17)-(B.20). This results in $||\Phi(s)||_\infty = 1.0565 \quad (< \gamma)$. Hence, using (5.33) we can compute a 5th-order $\mathcal{H}_\infty$ suboptimal controller $K_\infty^r(\mathrm{w})$ which is found to be stable. The controller $K_\infty^r(\mathrm{w}) = (A_{kr}, B_{kr}, C_{kr}, D_{kr})$ is given by

$$\begin{aligned} A_{kr} &= \begin{bmatrix} -4.07E+02 & -1.78E+00 & 2.63E-01 & 2.52E-04 & 2.83E-02 \\ -1.26E+05 & -8.82E+02 & 9.15E+00 & 7.79E-02 & 8.76E+00 \\ 1.15E+06 & 8.11E+03 & -2.32E+02 & 2.89E-01 & -7.99E+01 \\ 1.39E+08 & 8.96E+05 & 1.21E+04 & -2.34E+02 & -9.66E+03 \\ -1.06E+07 & -7.24E+04 & 6.83E+02 & 6.54E+00 & 7.34E+02 \end{bmatrix} \\ B_{kr} &= \begin{bmatrix} 3.22E-01 & 1.01E+02 & -9.29E+02 & -1.11E+05 & 8.49E+03 \end{bmatrix}^T \\ C_{kr} &= \begin{bmatrix} -3.69E+05 & -2.51E+03 & 2.37E+01 & 2.27E-01 & 2.55E+01 \end{bmatrix} \\ D_{kr} &= 294.58. \end{aligned}$$

The poles of the closed-loop are at

$$\{-364.24, -332.69, -202.08, -300.00, -14.38 \pm j19.86, \\ -14.10 \pm j7.40, -1.00 \pm j0.0000005, -1.53\}$$

and the $\mathcal{H}_\infty$-norm of the cost function is

$$||\mathcal{F}_l(P, K_\infty^r)||_\infty = 0.9546$$

which is obviously less than the prescribed value of $\gamma = 1.5$.

5.7.2 Example 2: MIMO Fighter Design

In this subsection we consider a MIMO fighter design example, taken from Safonov and Chiang [61]. The plant is the 2-input 2-output HIMAT experimental aircraft and is unstable. The longitudinal dynamics of the aircraft (trimmed at an altitude of 25000 ft and a speed of 0.9 Mach) are modelled by the state-space description $G(s) = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right] =$

$$\left[\begin{array}{rrrrrr|rr}
-2.2567E-02 & -36.6170 & -18.8970 & -3.2090E+01 & 3.2509 & -.7626 & 0 & 0 \\
9.2572E-05 & -1.8997 & .9831 & -7.2562E-04 & -.1710 & -.0005 & 0 & 0 \\
1.2338E-02 & 11.7200 & -2.6316 & 8.7582E-04 & -31.6040 & 22.3960 & 0 & 0 \\
0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -30.0000 & 0 & 30 & 0 \\
0 & 0 & 0 & 0 & 0 & -30.0000 & 0 & 30 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right].$$

The singular value design specifications for this example are as follows:
(1) Robustness specification: -40 db/decade roll-off and at least -20 db/decade at 100 rad/sec.
(2) Performance specification: Minimize the sensitivity function as much as possible.

The weighting functions $W_1(s)$ and $W_2(s)$ are taken as

$$\begin{aligned}
W_1(s) &= \frac{0.01s+1}{s+0.01} \times I_2 \\
W_2(s) &= \frac{s^2}{1000}\begin{bmatrix} 1 & 0 \\ 0 & 0.0005s+1 \end{bmatrix}
\end{aligned}$$

from [61] with $\gamma = 16.8$.

With these weightings, the order of the generalized plant is eight. Thus, an 8th-order controller arises from a "central" solution, while we can generate a

6th-order controller since $n_\phi = n - p_2 = 8 - 2 = 6$. As in Example 1, after deriving the two equations corresponding to (5.34) and (5.35), we set $n_\phi = 6$, $X_2 = I_6$ and choose A_ϕ as

$$A_\phi = \begin{bmatrix} -.0050 & 0 & 1 & 0 & 0 & 0 \\ 0 & -.0050 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1.05E+01 & 6.84E-03 & 1 & 0 \\ 0 & 0 & 7.40E+03 & -6.71E+00 & 0 & 1 \\ 0 & 0 & -3.86E-01 & 3.51E-05 & -.363 & -.0247 \\ 0 & 0 & 5.91E-01 & -1.69E-04 & 1.210 & -.1620 \end{bmatrix}$$

we then find X, B_ϕ, C_ϕ and D_ϕ, for which we have $||\Phi(s)||_\infty = 0.9790 \quad (< \gamma)$. Hence, we can compute a 6th-order $\mathcal{H}_\infty$ suboptimal controller $K_\infty^r(s)$ which is found to be stable. The controller $K_\infty^r(s) = (A_{kr}, B_{kr}, C_{kr}, D_{kr})$ is given by

$$A_{kr} = \begin{bmatrix} -2.659E+3 & 1.576 & -1.109E+1 & -8.224E-3 & -1.561E+1 & 2.479E+0 \\ -1.064E+6 & 595.8 & -2.518E+3 & -6.850E-0 & -3.773E+3 & 5.646E+2 \\ -7.690E+4 & 46.44 & -4.173E+2 & -1.190E-1 & -5.114E+2 & 8.222E+1 \\ -1.310E+7 & 6.344 & -6.816E+4 & -1.585E+1 & -9.396E+4 & 1.516E+4 \\ 4.899E+4 & -29.58 & 2.585E+2 & 8.070E-2 & 3.257E+2 & -5.235E+1 \\ -3.630E+4 & 21.90 & -1.902E+2 & -6.194E-2 & -2.393E+2 & 3.842E+1 \end{bmatrix}$$

$$B_{kr} = \begin{bmatrix} 1.843E+2 & 2.363E+3 \\ 1.015E+4 & 1.048E+6 \\ 7.006E+3 & 6.656E+4 \\ 1.605E+6 & 1.073E+7 \\ -4.426E+3 & -4.238E+4 \\ 3.199E+3 & 3.151E+4 \end{bmatrix}$$

$$C_{kr} = \begin{bmatrix} 2.972E+2 & -0.205 & 3.270E+1 & -2.851E-3 & 3.791E+1 & -6.306E-1 \\ -2.724E+3 & 1.608 & -1.194E+1 & -9.314E-3 & -1.551E+2 & 2.457E+0 \end{bmatrix}$$

$$D_{kr} = \begin{bmatrix} -1.648E+2 & -8.498E+1 \\ 4.656E+1 & 2.605E+3 \end{bmatrix}.$$

The closed-loop poles are positioned at

$$\{-2001.30, -26.28, -28.93, -6.51, -20.53 \pm j19.93, -21.86 \pm j18.45,$$
$$-0.69 \pm j0.25, -0.259, -0.021, -0.010, -0.010\}$$

and the $\mathcal{H}_\infty$-norm of the cost function is found to be

$$||\mathcal{F}_l(P, K_\infty^r)||_\infty = 0.9775 \quad (< \gamma).$$

5.7.3 Analysis and Comparison

Both examples show that the stability property of the closed-loop system is preserved when the $\mathcal{H}_\infty$ suboptimal controllers are replaced by the low-order controllers derived using the methods of this chapter. However, unlike the low-order stabilizing controller case in Chapter 4, the coupling in the Riccati equations (5.11)-(5.12) breaks the separation principle for the low-order $\mathcal{H}_\infty$ suboptimal controller case.

In both examples, for the same $P(s)$ and $\Phi(s)$, we compute the $(n + n_\phi)$th-order controller $K_\infty(s)$ as per (5.32) and then examine the *normalized* Hankel singular values of $K_\infty(s)$. These are given in Table 5.1. The results exhibit clearly that the low-order controllers obtained in the previous subsections are minimal realizations of the "formal" order controllers, in the sense of balanced truncated model reduction on the controllers.

Example 1	$(n = 6, n_\phi = 5)$	Example 2	$(n = 8, n_\phi = 6)$
1.0000E-00	1.4498E-01	1.0000E-00	6.6079E-02
9.0051E-02	3.3346E-02	2.9207E-02	1.1058E-02
2.5656E-04	5.3015E-12	2.3364E-03	2.6847E-04
1.7450E-14	1.0963E-15	9.5744E-06	2.9367E-09
7.2240E-16	3.7094E-17	9.2848E-10	7.9077E-10
4.2444E-20		4.8605E-10	7.6090E-11
		7.3000E-13	0

Table 5.1. Normalized Hankel Singular Values of $K_\infty(s)$

With the same $P(s)$ but with $\Phi(s) = 0$ we have computed "central" controllers of order n, i.e., 6th-order for Example 1 and 8th-order for Example 2, using Matlab files *hinf.m* (in the Robust Control Toolbox) and *hinfsyn.m* (in the μ-Analysis & Synthesis Toolbox). The frequency responses of these "central" controllers and the controllers given by formulae (5.32)-(5.33) are shown in

Figure 5.3 for Example 1 and in Figure 5.7 for Example 2. As seen in both figures and as expected theoretically, both controllers generated by formulae (5.32)-(5.33), although having different sizes, have exactly the same frequency responses.

Figure 5.4 for Example 1 and Figure 5.8 for Example 2 show that, although the "central" controllers cause a certain roll-off at high frequency, both controllers by (5.32)-(5.33) make the cost flat over frequency. On the other hand, both figures show that, since the cost generated by (5.33) is the same as that by (5.32), the closed-loop performance as well as the robustness of the closed-loop stability has not been degraded by the use of the low-order controller (5.33), instead of the "formal" order controller (5.32), in the feedback system.

Figures 5.5 and 5.9 show the singular values of the sensitivity functions, and Figures 5.6 and 5.10 the singular values of the complementary sensitivity functions.

5.8 Concluding Remarks

This chapter has considered the problem of reducing the order of $\mathcal{H}_\infty$ suboptimal controllers, i.e., stabilizing controllers which satisfy an $\mathcal{H}_\infty$-norm constraint on a prescribed closed-loop transfer function. Starting from a parametrization (5.17) of all solutions to the general $\mathcal{H}_\infty$ suboptimal control problem, we first derived a low-order realization (5.33) on the assumption of the existence of a solution matrix X to two simultaneous matrix equations, (5.34)-(5.35), which are similar in structure to those in Chapter 4 but are to be solved subject to an $\mathcal{H}_\infty$-norm constraint, (5.38).

The aim was to eliminate any unobservable modes in the "formal" order of controllers given by (5.17). We then showed how to solve the two matrix equations using an orthogonal canonical transformation. We further showed that the $\mathcal{H}_\infty$-norm constraint can be tackled by checking the positive definiteness of a solution matrix X_ϕ to a certain ARE (5.65). We showed, as a result, that the order of the low-order $\mathcal{H}_\infty$ suboptimal controllers may be equal to $n - p_2$ (or less). The algorithm developed in the chapter was summarized and demonstrated by two numerical examples. The examples showed that the low-order $\mathcal{H}_\infty$ suboptimal controllers preserved the closed-loop performance as well as closed-loop stability, without any degradation.

It should be noted that the existence of a positive definite matrix X_ϕ to ARE (5.65) depends heavily on the choice of the matrix A_ϕ which can be arbitrarily chosen as a stable matrix. It is also noted that the dual approach of eliminating the uncontrollable states to find a set of low-order $\mathcal{H}_\infty$ suboptimal controllers can be carried out in a similar manner.

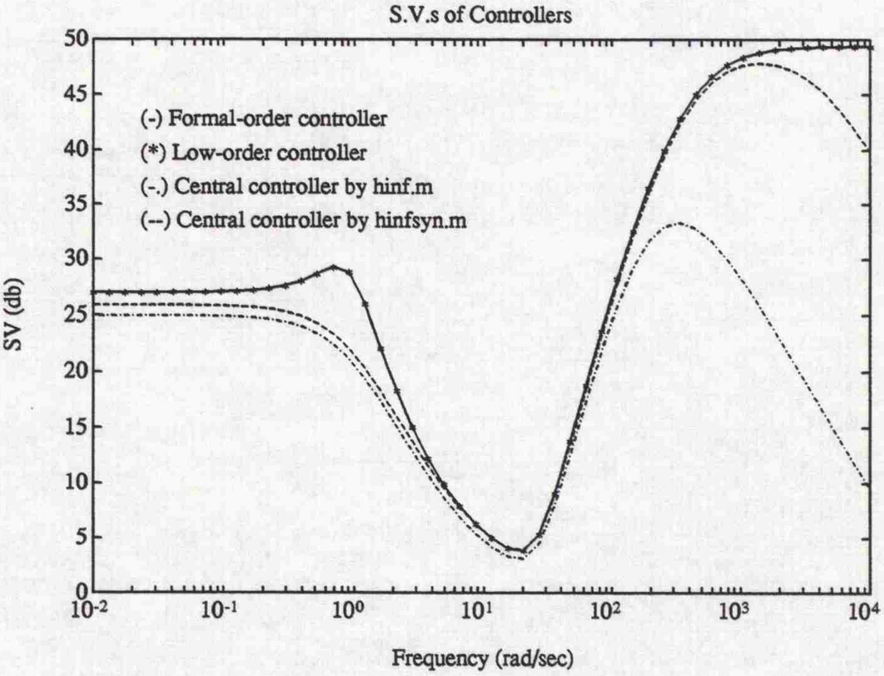


Figure 5.3: Singular Values of Controllers for Example 1.

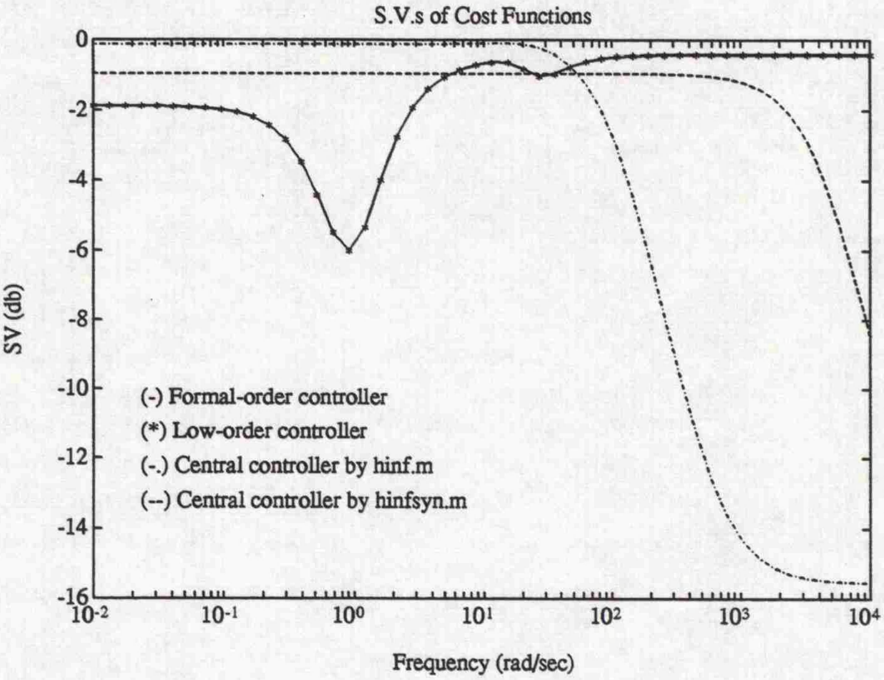


Figure 5.4: Singular Values of Cost Functions for Example 1.

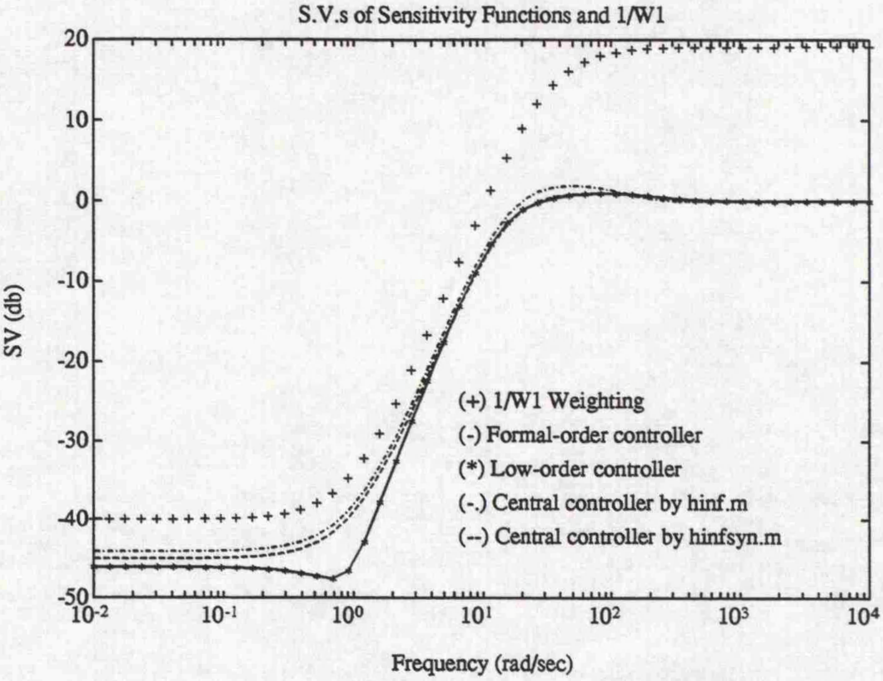


Figure 5.5: Sensitivity and Weighting W_1^{-1} for Example 1.

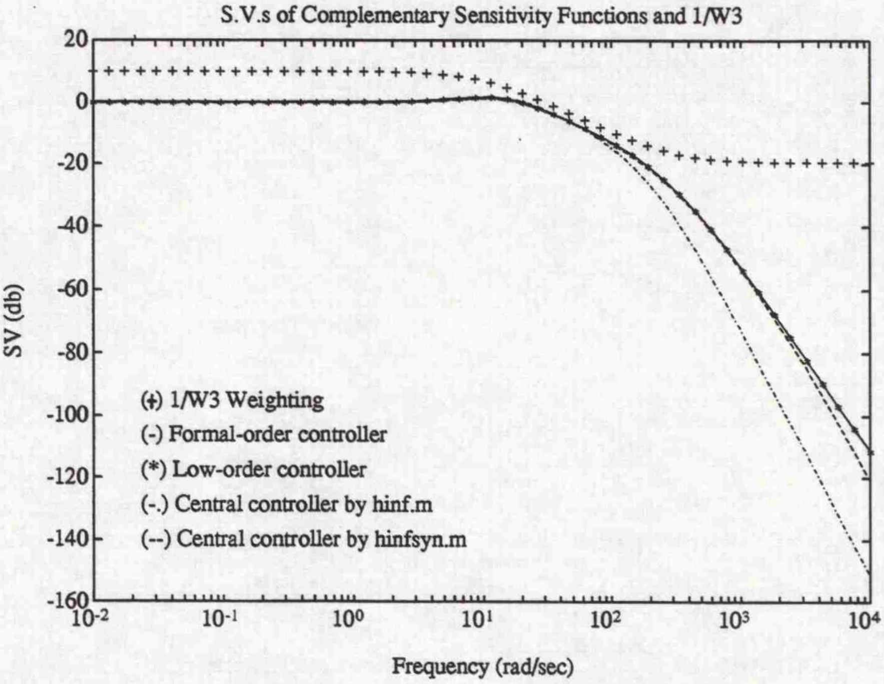


Figure 5.6: Complementary Sensitivity and Weighting W_3^{-1} for Example 1.

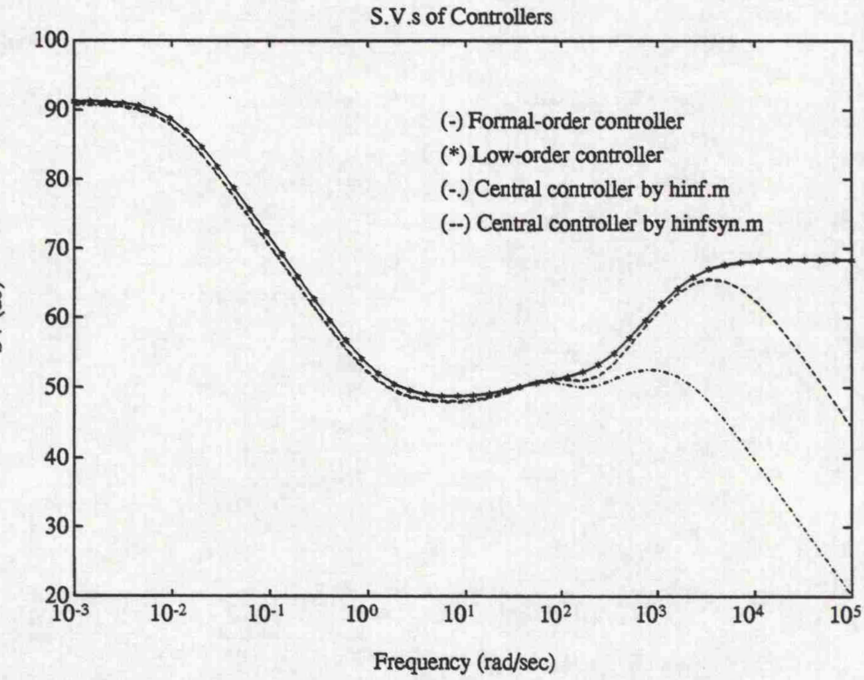


Figure 5.7: Singular Values of Controllers for Example 2.

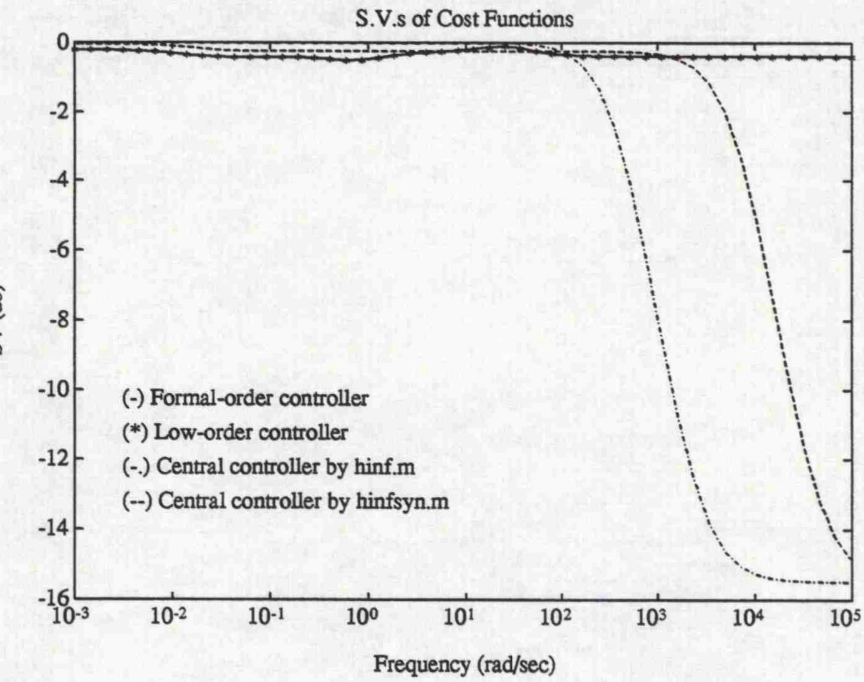


Figure 5.8: Singular Values of Cost Functions for Example 2.

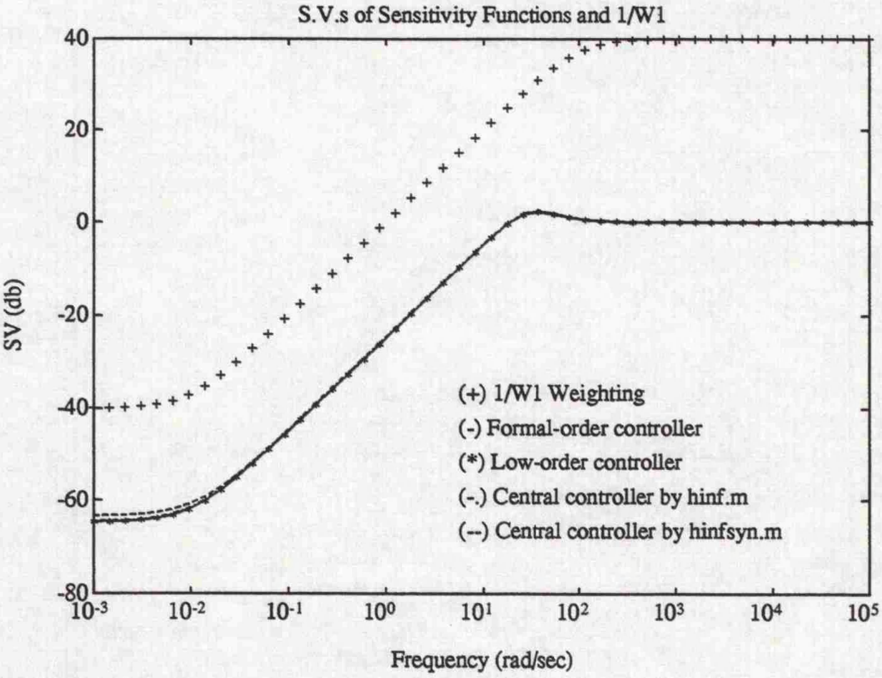


Figure 5.9: Sensitivity and Weighting W_1^{-1} for Example 2.

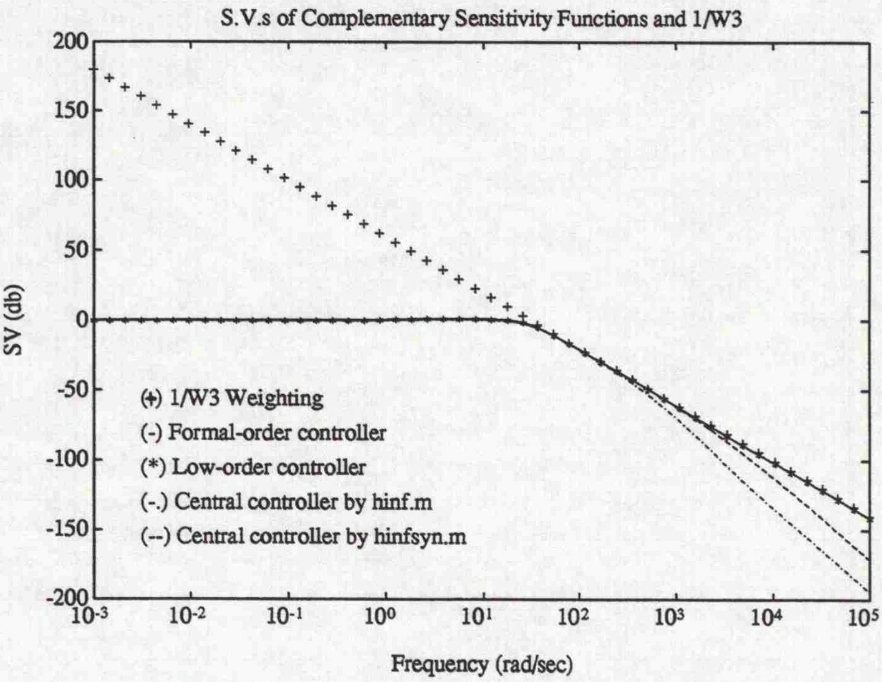


Figure 5.10: Complementary Sensitivity and Weighting W_3^{-1} for Example 2.

Chapter 6

Low-Order Robust Sub-Optimal Controller Design

6.1 Introduction

An important development in robust control system design was the *robust stabilization problem* of Glover and Mcfarlane [28] in which uncertainty is modelled by norm bounded perturbations on the factors in a normalized coprime factorization of the plant. This method was enhanced by McFarlane and Glover [47] to meet specifications on performance, by combining the robust stabilization problem with classical loop shaping techniques. The method was termed the *Loop Shaping Design Procedure* (LSDP) and has been used to great effect on the design of controllers for a number of real problems, e.g., Hyde and Glover [33], McFarlane and Glover [47], Walker *et al.* [72].

However, the order of the resultant controller is likely to be "high", i.e.,

$$\mathcal{N} \leq \deg(G) + 2\deg(W) + \deg(\Phi)$$

where $G(s)$ is the nominal plant, $W(s)$ is the weighting function and $\Phi(s)$ is a stable free parameter. For designing low-order controllers, model reduction techniques can be applied to the stable coprime factors of either the full-order plant or the full-order controller. For examples, see McFarlane *et al.* [48] and Bongers and Bosgra [5].

In this chapter, we present a different approach to the design of low-order robust $\mathcal{H}_\infty$ controllers using LSDP. It is similar in spirit to the methodology developed in Chapter 4 for stabilizing controllers and in Chapter 5 for $\mathcal{H}_\infty$ suboptimal controllers. We will show that we can derive low-order robust controllers of order $\mathcal{N}_{low}$, where

$$\mathcal{N}_{low} = \deg(G) + 2\deg(W) - p$$

by removing unobservable (or uncontrollable) states via a suitable choice of $\Phi(s)$, where p is the number of plant outputs.

The chapter is organized as follows. In Section 6.2, the normalized LCF robust stabilization problem and its optimal/suboptimal solutions are reviewed. The results are then used in Section 6.3 to construct a closed-form expression, (6.23), of low-order robust suboptimal controllers, which can be easily implemented provided a solution to two equations, (6.24)-(6.25), exists and an $\mathcal{H}_\infty$-norm constraint, $||\Phi(s)||_\infty \leq 1$, is satisfied. The methodology used in Chapter 5 is used here with some slight modifications. In Section 6.4, the LSDP is described in some detail, to show how low-order robust suboptimal controllers can be designed in a straightforward manner. Section 6.5 presents a CAD algorithm for the approach. An illustrative numerical example is given in Section 6.6. Conclusions are given in Section 6.7.

6.2 The Normalized LCF Robust Stabilization Problem

In this section, the normalized left coprime factorization (LCF) robust stabilization problem and its optimal/suboptimal solutions are summarized.

As we saw in Chapter 2, Section 2.6, the plant model $G(s)$ can be factored as

$$G(s) = \tilde{M}(s)^{-1}\tilde{N}(s) \tag{6.1}$$

and the factorization (6.1) is said to be a normalized LCF of $G(s)$ if $\tilde{M}(s), \tilde{N}(s) \in \mathcal{RH}_\infty$ and

$$\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I. \tag{6.2}$$

Let $G(s) = (A, B, C) \in \mathcal{RL}_\infty^{p\times m}$ be a *strictly proper* system having n_g states. Then, a state space construction of a normalized LCF of the *strictly proper* system $G(s)$ is given by

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} := \left[\begin{array}{c|cc} A+HC & B & H \\ \hline C & 0 & I \end{array}\right] \tag{6.3}$$

where

$$H = -ZC^T \tag{6.4}$$

and the matrix $Z \geq 0$ is the unique stabilizing solution to the algebraic Riccati equation

$$AZ + ZA^T - ZC^TCZ + BB^T = 0. \tag{6.5}$$

Consider now the configuration of Figure 6.1 in which a *perturbed model* $G_\Delta(s)$ is defined as

$$G_\Delta(s) = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N) \tag{6.6}$$

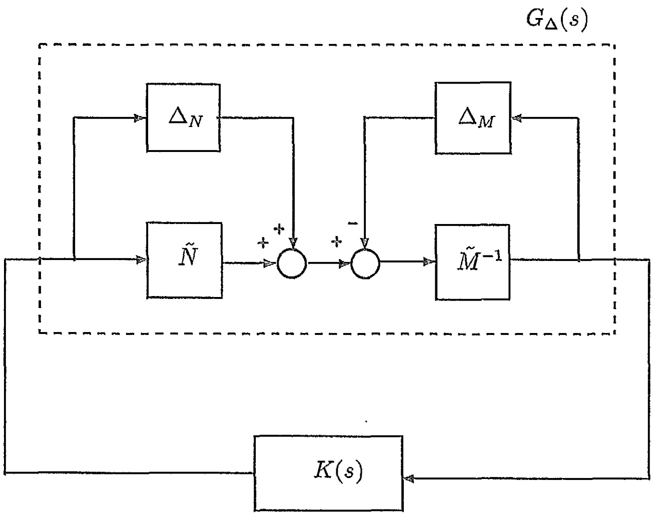


Figure 6.1: Coprime Factor Robust Stabilization Problem.

where $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$.

To maximize the class of perturbed models defined by (6.6) such that the configuration of Figure 6.1 is stable, we need to find the controller $K(s)$ which stabilizes the nominal closed-loop system and which minimizes γ where

$$\gamma = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\tilde{M}^{-1} \right\|_\infty. \tag{6.7}$$

This is the problem of robust stabilization of normalized coprime factor plant descriptions as introduced in Glover and McFarlane [28]. From the small gain theorem, the closed-loop system will remain stable if

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < \gamma^{-1}. \tag{6.8}$$

6.2.1 Optimal Solutions

The minimum value of γ for all stabilizing controllers $K(s)$ is

$$\gamma_o = \inf_{K stabilizing} || \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\tilde{M}^{-1}||_\infty \tag{6.9}$$

and is given in [28] by

$$\gamma_o = (1 - || \begin{bmatrix} \tilde{N}, & \tilde{M} \end{bmatrix} ||_H^2)^{-1/2}. \tag{6.10}$$

From [28]

$$|| \begin{bmatrix} \tilde{N}, & \tilde{M} \end{bmatrix} ||_H^2 = \lambda_{max}(ZY(I + ZY)^{-1}) \tag{6.11}$$

where the matrix $Y \geq 0$ is the unique stabilizing solution to the ARE

$$A^TY + YA - YBB^TY + C^TC = 0. \tag{6.12}$$

Hence from (6.11), it can be shown that the optimal value γ_o can easily be computed by

$$\gamma_o = (1 + \lambda_{max}(ZY))^{1/2} \tag{6.13}$$

without any iteration. In the following subsection, we will consider the associated suboptimal control problem and outline a characterization of all suboptimally robust controllers together with a "central" suboptimal controller.

6.2.2 Robust Sub-Optimal Controllers

A related problem to the optimal $\mathcal{H}_\infty$ problem posed in Subsection 6.2.1 is the *suboptimal problem* of obtaining the set of stabilizing controllers $K(s)$ such that

$$|| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\tilde{M}^{-1}||_\infty \leq \gamma \tag{6.14}$$

where $\gamma(>\gamma_o)$ is some prespecified tolerance level for the allowable uncertainty.

A state-space characterization of all suboptimally robust controllers for the normalized LCF robust stabilization problem is given in [28]. That is, all suboptimally robust controllers $K_{nlcf}(s)$ such that (6.14) is satisfied for $\gamma > \gamma_o$ are given by a chain scattering description:

$$K_{nlcf}(s) = (L_{11}\Phi + L_{12})(L_{21}\Phi + L_{22})^{-1} \tag{6.15}$$

for any $\Phi(s) \in \mathcal{RH}_\infty^{m \times p}$ with $||\Phi(s)||_\infty \leq 1$, where

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & -\gamma^2 Q^{-T}B & \gamma^2\beta^{-1}Q^{-T}ZC^T \\ \hline F & I_m & 0 \\ C & 0 & -\beta^{-1}I_p \end{array}\right] \tag{6.16}$$

where

$$\begin{aligned} \beta &= (\gamma^2-1)^{1/2} && (6.17) \\ F &= -B^TY && (6.18) \\ Q &= (1-\gamma^2)I_n + YZ. && (6.19) \end{aligned}$$

Remark 6.1 The "central" suboptimal controller which corresponds to $\Phi(s) = 0$ is given by

$$K_{nlcf}^c(s) = \left[\begin{array}{c|c} A+BF+\gamma^2Q^{-T}ZC^TC & \gamma^2Q^{-T}ZC^T \\ \hline B^TY & 0 \end{array}\right]. \tag{6.20}$$

Remark 6.1 above shows that a "central" suboptimal controller is synthesized by the solution of two ARE's, (6.5) and (6.12), without an iterative search on γ which is normally required to solve $\mathcal{H}_\infty$ problems.

6.3 Low-Order Robust Sub-Optimal Controller Design

From our point of view in controller size reduction, it is noted that the characterization, (6.15), of all suboptimally robust controllers involves a free parameter matrix $\Phi(s)$ in a chain scattering description, which will play a key role in deriving low-order robust controllers in this chapter.

Let $\Phi(s) \in \mathcal{RH}_\infty^{m\times p}$ in (6.15) have a state-space realization

$$\Phi(s) := \left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right] \tag{6.21}$$

where $A_\phi \in \mathcal{R}^{n_\phi \times n_\phi}$ is stable. Then, substitution of (6.16) and (6.21) into (6.15) results in the full-order controller of the form:

$$K_{nlcf}(s) = \left[\begin{array}{cc|c} K_{a11} & K_{a12} & K_{b1} \\ K_{a21} & K_{a22} & K_{b2} \\ \hline K_{c1} & K_{c2} & K_d \end{array}\right] \tag{6.22}$$

where

$$\begin{aligned}
K_{a11} &= A + BF + \gamma^2 Q^{-T} Z C^T C - \gamma^2 \beta Q^{-T} B D_\phi C \\
K_{a12} &= -\gamma^2 Q^{-T} B C_\phi \\
K_{a21} &= \beta B_\phi C \\
K_{a22} &= A_\phi \\
K_{b1} &= \gamma^2 \beta Q^{-T} B D_\phi - \gamma^2 Q^{-T} Z C^T \\
K_{b2} &= -\beta B_\phi \\
K_{c1} &= F + \beta D_\phi C \\
K_{c2} &= C_\phi \\
K_d &= -\beta D_\phi .
\end{aligned}$$

Note from (6.22) that the feedback controller $K_{nlcf}(s)$ will have the "formal" order:

$$\deg(K_{nlcf}) = n_g + n_\phi.$$

As in Chapters 4 and 5, the realization of (6.22) can be reduced via a state similarity transformation into the following *low-order form*, which we refer to as a **low-order robust suboptimal controller**:

$$K^r_{nlcf}(s) = \left[\begin{array}{c|c} -\gamma^2 XQ^{-T}BC_\phi + A_\phi & \gamma^2\beta XQ^{-T}BD_\phi - \gamma^2 XQ^{-T}ZC^T - \beta B_\phi \\ \hline C_\phi & -\beta D_\phi \end{array}\right] \tag{6.23}$$

if there exists a matrix $X \in \mathcal{R}^{n_\phi \times n}$ satisfying the following two matrix equations:

$$A_\phi X - X\tilde{A} = B_\phi \tilde{C} \tag{6.24}$$
$$C_\phi X - D_\phi \tilde{C} = F \tag{6.25}$$

where

$$\tilde{A} = A + BF + \gamma^2 Q^{-T}BF + \gamma^2 Q^{-T}ZC^TC \tag{6.26}$$
$$\tilde{C} = \beta C. \tag{6.27}$$

Again, $K^r_{nlcf}(s)$ in (6.23) has an order of at most

$$\deg(K^r_{nlcf}) = n_\phi$$

which is obviously less than the "formal" order of $K_{nlcf}(s)$ of (6.22).

6.3.1 Two Matrix Equations

It is not surprising to see that the two equations (6.24)-(6.25) are similar in structure to (5.34)-(5.35), and thus can be solved in a similar manner. We therefore leave out the details and state the following Theorem.

Theorem 6.2 *The plant $G(s)$ described by (6.1) has low-order robust suboptimal controllers of order:*

$$deg(K^r_{nlcf}) = n_g - p \tag{6.28}$$

if C in (6.27) is full row rank and if the $\mathcal{H}_\infty$-norm constraint of $||\Phi(s)||_\infty \leq 1$ is satisfied.

Note that, since F in (6.25) has no freedom at all, the possibility of further reduction in controller size may not be expected from our algorithm developed.

6.3.2 $\mathcal{H}_\infty$-Norm Constraint on $||\Phi(s)||_\infty$

The $\mathcal{H}_\infty$-norm constraint of $||\Phi(s)||_\infty \leq 1$ required in the characterization (6.15) can also be tackled in the same manner as in Section 5.4, with $\gamma = 1$.

6.4 The Loop Shaping Design Procedure

Before describing a CAD algorithm for low-order robust controller design, we will first see, in this section, how the attractive robust stabilization problem can be enhanced to give a reliable multivariable loop shaping design procedure. Our method for designing low-order robust stabilizing controllers can be applied after loop shaping and therefore can be incorporated in the LSDP.

The classical loop shaping approach to control system design aims to achieve certain specifications on the closed-loop system by selecting a controller which appropriately *shapes* the magnitude of the open-loop transfer function. It has been applied to industrial systems over several decades and, for SISO systems and loosely coupled systems, the approach has worked well. But for truly multivariable systems a reliable generalization of the approach has only recently emerged. Based on the idea that a satisfactory definition of gain (range of gain) for a transfer function matrix is given by the singular values of the transfer function matrix, Doyle and Stein [19] in the early 1980's showed how the classical loop shaping ideas of feedback design could be generalized to multivariable systems. The term *multivariable loop shaping* is now widely accepted to mean the shaping of singular values of appropriately specified transfer function matrix, Postlethwaite and Skogestad [58].

Multivariable loop shaping is in general non-trivial. A satisfactory loop shaping design procedure in conjunction with $\mathcal{H}_\infty$ control methods was recently developed by McFarlane and Glover [47]. In this the normalized LCF robust stabilization problem described in Section 6.2 is extended to include performance requirements. The resulting design procedure will be called the one degree-of-freedom loop shaping design procedure (1-DOF LSDP), and is outlined as follows:-

1. The nominal plant $G(s)$ is modified using a pre-compensator $W_1(s)$ and/or post-compensator $W_2(s)$, so that the shaped plant

$$G_s = W_2 G W_1 \tag{6.29}$$

has desired open-loop singular values. (See Figure 6.2-a)). At low frequencies, the open-loop gains should be made large enough for good disturbance rejection and command following, while at high frequencies the loop gains should be made small enough to provide robustness and to reduce the effects of sensor noise. Further, the open-loop singular values can be given a particular cross-over frequency keeping in mind the desired closed-loop bandwidth and time response requirements. Now, define

$$\begin{aligned} n_s &:= \deg(G_s) \\ n_g &:= \deg(G) \\ n_{w_1} &:= \deg(W_1) \\ n_{w_2} &:= \deg(W_2). \end{aligned}$$

Therefore

$$n_s = n_g + n_{w_1} + n_{w_2} \tag{6.30}$$

represents the order of the shaped plant.

2. A feedback controller, $K_{s,nlcf}(s)$, for the shaped plant $G_s(s)$ is then synthesized which robustly stabilizes the normalized LCF of $G_s(s)$, as shown in Figure 6.2-b):

$$G_s = \tilde{M}_s^{-1}\tilde{N}_s. \tag{6.31}$$

The formulae given in (6.15) for an optimal or suboptimal controller (given γ) can be applied to $G_s(s)$ to get the "formal" order controller $K_{s,nlcf}(s)$. From $K_{s,nlcf}(s)$, a low-order robust suboptimal controller, $K^r_{s,nlcf}(s)$, can be obtained. Note, from (6.22) and (6.30), that $K_{s,nlcf}(s)$ will have a "formal" order:

$$\deg(K_{s,nlcf}) = n_s + n_\phi \tag{6.32}$$

while the low-order robust suboptimal controller $K^r_{s,nlcf}(s)$ will have an order:

$$\deg(K^r_{s,nlcf}) = n_s - p \tag{6.33}$$

as predicted by Theorem 6.2.

3. The *final* feedback controller, $K_{f,nlcf}(s)$, for the plant $G(s)$ is then constructed by simply combining the full-order feedback controller $K_{s,nlcf}(s)$ for the shaped plant $G_s(s)$ with the weights to give

$$K_{f,nlcf} = W_1 K_{s,nlcf} W_2 \tag{6.34}$$

as shown in Figure 6.2-c). Similarly, the final low-order feedback controller, $K^r_{f,nlcf}(s)$, for the plant $G(s)$ is given by

$$K^r_{f,nlcf} = W_1 K^r_{s,nlcf} W_2 \tag{6.35}$$

with order:

$$\begin{aligned}\deg(K^{r}_{f,nlcf}) &= (n_s - p) + n_{w_1} + n_{w_2} \\ &= n_g + 2(n_{w_1} + n_{w_2}) - p \\ &=: \mathcal{N}_{low}.\end{aligned} \tag{6.36}$$

Essentially, with the 1-DOF LSDP, the weights $W_1(s)$ and $W_2(s)$ are the design parameters which are chosen both to shape the open-loop singular values and to ensure that the optimal value γ_o is not too large (usually less than 4). A large value of γ_o indicates that the specified singular value shapes are incompatible with robust stability requirements. The choice of suboptimal γ is another design parameter for the low-order robust suboptimal controller design and is closely related to the $\mathcal{H}_\infty$-norm constraint, $||\Phi(s)||_\infty \leq 1$.

6.5 A Low-Order Robust Controller Design Algorithm

In this section, we present a CAD algorithm for low-order robust suboptimal controller design, summarizing the procedure described above.

step 1: Choose weightings, $W_1(s)$ and $W_2(s)$, to compensate the open-loop plant $G(s)$ so that the shaped plant $G_s(s)$ has singular values of desired shape.

step 2: Compute γ_o as per (6.13).
If $\gamma_o \leq 4$, go to step 3.
If $\gamma_o > 4$, repeat step 1 to choose suitable weights.

step 3: Select a suboptimal γ such that $\gamma > \gamma_o$.

step 4: Compute β, F and Q as per (6.17)-(6.19).

step 5: Compute $\tilde{A}$ and $\tilde{C}$ as per (6.26)-(6.27), and partition the pair $(\tilde{A}, \tilde{C})$ as in (5.39)-(5.40) of Chapter 5.

step 6: Set $X_2 = I_{n-p}$.
† Note that, as stated in Remark 5.7, X_2 need not necessarily be the identity matrix.

step 7: Select an arbitrary X_1, and then compute A_ϕ, B_ϕ, C_ϕ and D_ϕ as per (5.48)-(5.51).
†† Alternatively, as stated in Appendix B, choose first A_ϕ as per (B.16) and then compute X, B_ϕ, C_ϕ and D_ϕ as per (B.17)-(B.20).

step 8: Solve ARE (5.65) for X_ϕ, with $\gamma = 1$.
If $X_\phi > 0$, go to step 9.
If $X_\phi \leq 0$, go to step 7 and repeat.

step 9: Compute a low-order robust suboptimal controller, $K^r_{s,nlcf}(s)$, as per (6.23).

step 10: To obtain the final low-order feedback controller, $K^r_{f,nlcf}(s)$, pre/post multiply $K^r_{s,nlcf}(s)$ by the weighing functions as in (6.35).

The procedure described above is based on the given weighting functions and the suboptimal γ, and thus modifications to the procedure may be required. For example, if one desires to improve the design, then he/she can modify the weighting functions in step 1 and repeat the design procedure; if one fails to find a positive definite solution to ARE (5.65) during steps 7 and 8, the selected suboptimal γ in step 3 should be increased.

6.6 Illustrative Example

In this section a numerical example for a SISO flexible spacecraft design is described to illustrate the results of the chapter.

The system is a satellite with two highly flexible solar arrays attached, as considered in [47]. The plant model is described in state-space form by

$$\begin{aligned} \dot{x} &= Ax + Bu + Bv \\ y &= Cx \end{aligned}$$

where u is the control torque (Nm), v is a constant disturbance torque (Nm), y is the roll angle measurement (rad) of the satellite, and

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.7319 \times 10^{-5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -w^2 & -2\zeta w & 3.7859 \times 10^{-4} \\ \hline 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

where $w = 1.539$ rad/sec is the frequency of the flexure mode, and $\zeta = 0.003$ is the flexural damping ratio.

For the loop shaping procedure, weighting functions are chosen to be

$$W_2(s) = 10000 \times \frac{s + 0.4}{s}$$

and $W_1(s) = I$, as in [47]. Then, we have $n_s = 5$ and $p = 1$, and thus $n_\phi = 4$. The optimal value of γ is then obtained as $\gamma_o = 2.34$ using (6.13) and the "central" suboptimal controller of order 5 is obtained using (6.20).

We select a suboptimal $\gamma = 3 \quad (> \gamma_o)$, and find all the element matrices required for the low-order robust suboptimal controller, by choosing an A_ϕ as

$$A_\phi = \begin{bmatrix} -0.0510 & 1.0000 & 0.0000 & -0.0000 \\ 0 & -0.5222 & 1.0000 & 0.0000 \\ 0 & -1.6359 & 0.0481 & 1.0000 \\ 0 & -0.0018 & -0.0181 & -0.3912 \end{bmatrix}.$$

Then, the $\mathcal{H}_\infty$-norm constraint is satisfied with $||\Phi(s)||_\infty = 0.9763$. So, using the realization (6.23), a low-order robust suboptimal controller $K^r_{s,nlcf}(s)$ of order 4 is computed as

$$K^r_{s,nlcf}(s) = \left[\begin{array}{cccc|c} -3.6961 & 0.5004 & -0.0170 & 0.2405 & -0.0057 \\ 2.4283 & -0.1894 & 1.0113 & -0.1602 & 0.0028 \\ 5.7072 & -0.8537 & 0.0747 & 0.6235 & 0.0086 \\ -0.0427 & -0.0076 & -0.0183 & -0.3884 & 0.0001 \\ \hline -1347.9 & -184.80 & -6.3000 & 88.900 & -1.8373 \end{array}\right].$$

The normalized Hankel singular values of $K_{s,nlcf}(s)$ of (6.22) are given by Table 6.1 below:

1.0000	0.9990	0.0125	0.0021	0.0000
0.0000	0.0000	0.0000	0.0000	

Table 6.1. Normalized Hankel Singular Values of $K_{s,nlcf}(s)$

and justify the controller size reduction in the sense of balanced truncation.

Figures 6.3 to 6.5 show step responses for the two controllers, while Figures 6.6 to 6.8 show frequency responses. All the figures indicate that the low-order suboptimal controller can replace the "central" suboptimal controller, without any serious deterioration in performance. In particular, Figure 6.7 compares sensitivity measures and indicates that the closed-loop performance can be well achieved by a low-order robust suboptimal controller. In Figure 6.8, singular values for the shaped and achieved open-loop systems are shown to be closely compatible even when a low-order robust suboptimal controller is used.

6.7 Concluding Remarks

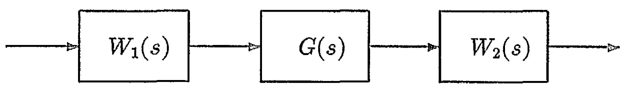
In this chapter, we addressed controller size reduction in the 1-DOF LSDP. The controllers for the 1-DOF LSDP are characterized in terms of the shaped plant $G_s(s)$ and a free parameter $\Phi(s)$. However, the order of a controller using the 1-DOF LSDP is likely to be "high". To reduce this order, by removing unobservable states via a suitable choice of $\Phi(s)$, we have shown how to derive low-order robust suboptimal controllers of order:

$$\mathcal{N}_{low} = \deg(G) + 2\deg(W) - p.$$

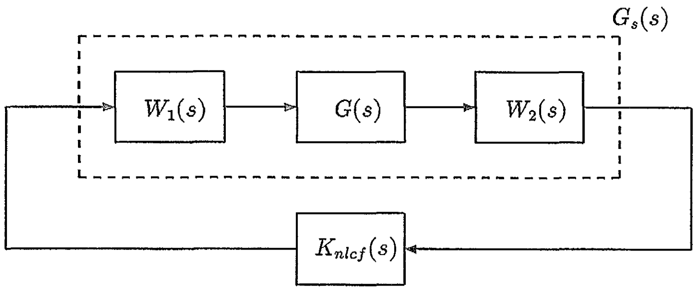
A numerical example was given to illustrate how a low-order controller could replace the "central" controller of order:

$$\mathcal{N}_{central} = \deg(G) + 2\deg(W)$$

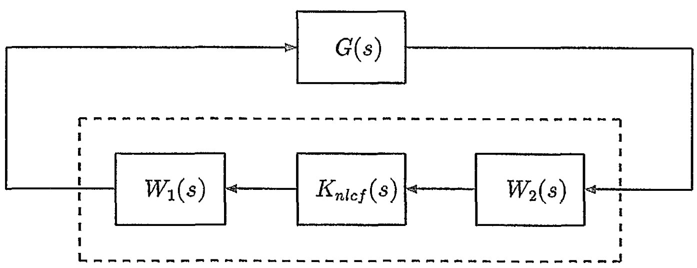
without any serious deterioration in performance.



a) The Shaped Plant



b) $\mathcal{H}_\infty$ Compensation



c) Final Controller

Figure 6.2: The 1-DOF Loop Shaping Design Procedure.

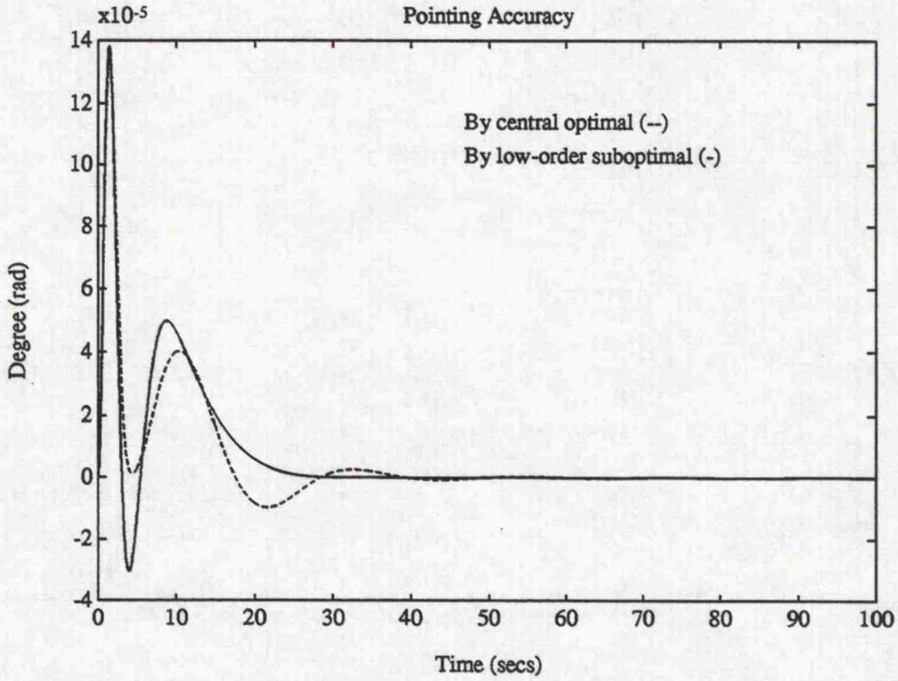


Figure 6.3: Output Response to Disturbance Input Step.

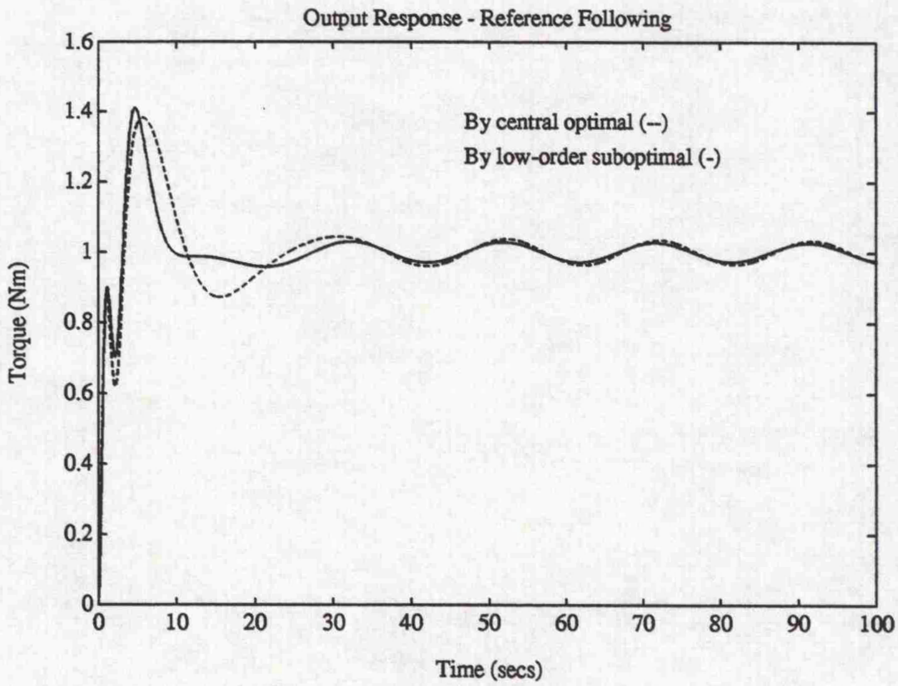


Figure 6.4: Output Response - Reference Following.

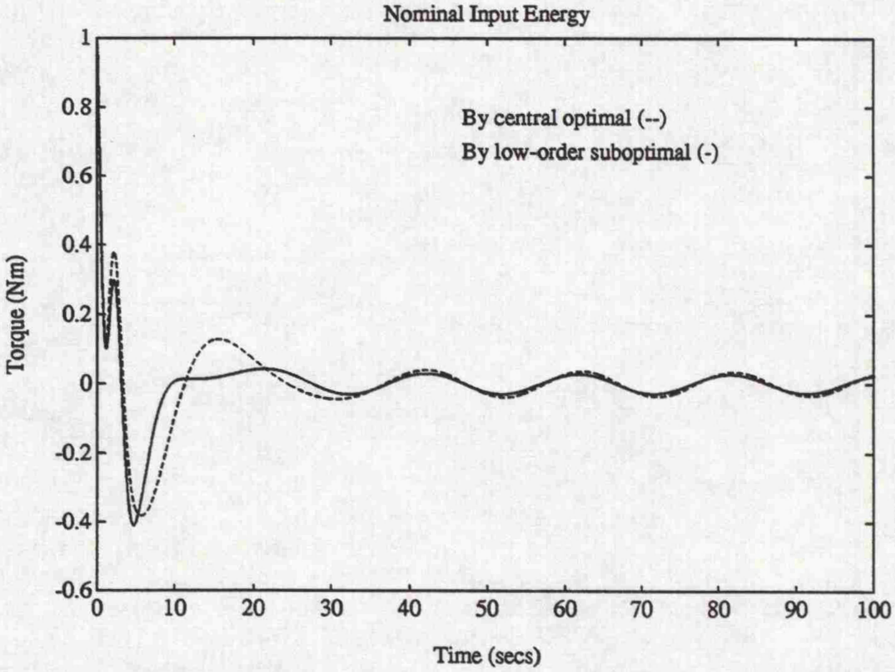


Figure 6.5: Output Response - Nominal Input Energy.

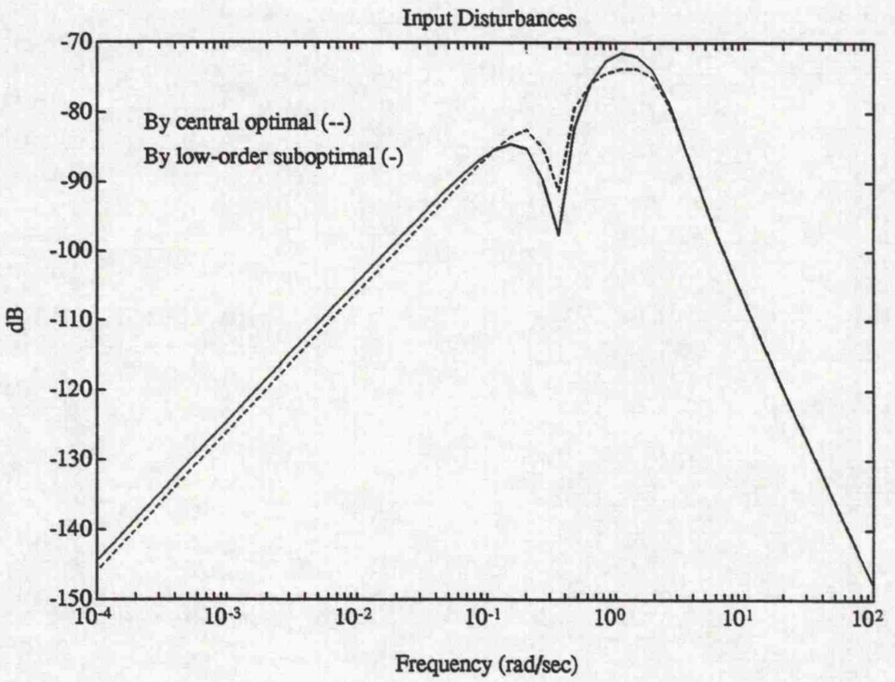


Figure 6.6: Singular Values for Input Disturbances.

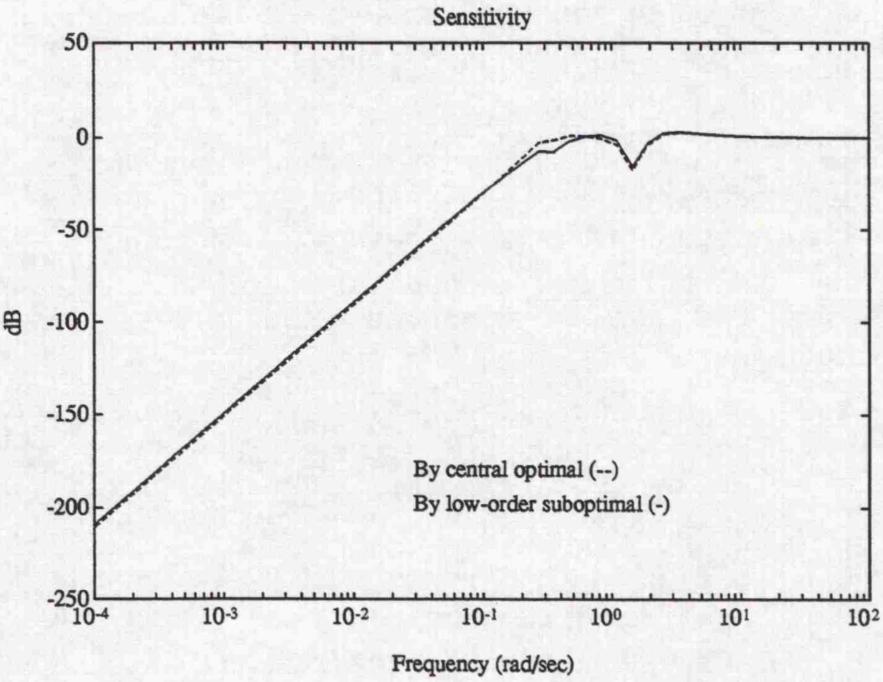


Figure 6.7: Singular Values for Sensitivity Measure.

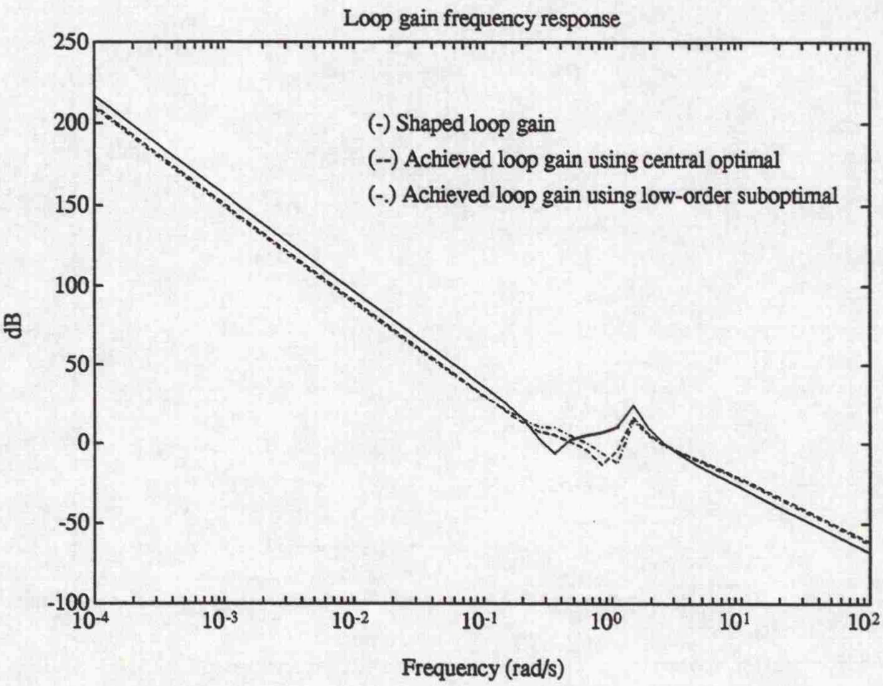


Figure 6.8: Singular Values for Shaped and Achieved Systems.

Chapter 7

An Extension to 2-DOF $\mathcal{H}_\infty$ Controller Design

7.1 Introduction

To introduce performance objectives into the control problem, a two degree of freedom (2-DOF) scheme is often employed, e.g., Youla and Bongiorno [74]. Limebeer *et al.* [43] have recently enhanced the model matching properties of the 1-DOF LSDP by extending the design procedure to a 2-DOF scheme. The *feedback* part of the controller is designed to meet robust stability and disturbance rejection requirements in a manner identical to the 1-DOF LSDP. An additional *prefilter* part of the controller is then introduced to force the response of the closed-loop system to follow that of a specified model $M_o(s)$. However, the order of the resultant controller is again likely to be "high", i.e.,

$$\mathcal{N} \leq \deg(G) + \deg(M_o) + 2\deg(W) + \deg(\Phi)$$

where $G(s)$ is the nominal plant, $M_o(s)$ is a reference model, $W(s)$ represents the weighting functions and $\Phi(s)$ is a free stable parameter.

In this chapter, we review the 2-DOF $\mathcal{H}_\infty$ design procedure of [43] and show that the methodology presented in previous chapters can be extended to this 2-DOF setting to reduce the controller order down to:

$$\mathcal{N}_{low} = \deg(G) + \deg(M_o) + 2\deg(W) - p_g - m_m$$

where p_g is the number of outputs of $G(s)$ and m_m is the number of inputs of $M_o(s)$.

The chapter is organized as follows. In Section 7.2, the 2-DOF $\mathcal{H}_\infty$ design is summarized. A CAD algorithm for deriving low-order 2-DOF $\mathcal{H}_\infty$ suboptimal controllers is described in Section 7.3.

7.2 A 2-DOF $\mathcal{H}_\infty$ Design

The control scheme for the 2-DOF $\mathcal{H}_\infty$ design is shown in Figure 7.1. The reference model $M_o(s)$ defines the desired response of the output. Although shown in Figure 7.1 it is only part of the problem formulation and not the controller implementation.

If the 2-DOF controller $K(s)$ is partitioned as $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ the control signal is given by

$$u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \beta \\ y \end{bmatrix} \tag{7.1}$$

in which K_1 is the prefilter, K_2 is the feedback controller, y is the measured output of the system and β is related to the reference input r as described below. The prefilter is included in the system to ensure that

$$||T_{y\beta} - M_o||_\infty \leq \gamma\rho^{-2} \tag{7.2}$$

where $T_{y\beta}$ is the closed-loop transfer function mapping β to y. Examination of

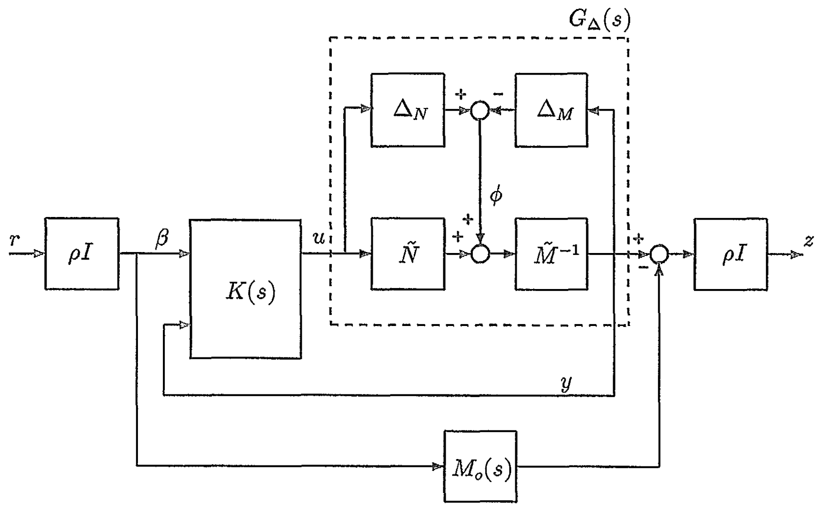


Figure 7.1: The 2-DOF $\mathcal{H}_\infty$ Controller Design Scheme.

Figure 7.1 shows that

$$\begin{bmatrix} u \\ y \\ z \end{bmatrix} = \left[\begin{array}{c|c} \rho(I-K_2G)^{-1}K_1 & K_2(I-GK_2)^{-1}\tilde{M}^{-1} \\ \rho(I-GK_2)^{-1}GK_1 & (I-GK_2)^{-1}\tilde{M}^{-1} \\ \hline \rho^2((I-GK_2)^{-1}GK_1 - M_o) & \rho(I-GK_2)^{-1}\tilde{M}^{-1} \end{array}\right] \begin{bmatrix} r \\ \phi \end{bmatrix} \tag{7.3}$$

in which the scaling factor ρ is used to weight the relative importance of robust stability compared with robust model matching.

Equation (7.3) has several important properties. The (1,2) partition is equal to equation (6.7) in Chapter 6, and thus is associated with robust stability optimization. Indeed, if ρ is set to zero, the 2-DOF $\mathcal{H}_\infty$ problem reduces to the standard robust stability problem described in Chapter 6, Section 6.2. The (2,1) partition is used for matching the closed-loop response to the ideal response, and the (1,1) block can be interpreted as limiting actuator use when following references. The scaling factor $\rho \geq 1$ is introduced into the problem to emphasize the (2,1) partition and to de-emphasize (relatively speaking) the (1,1) and (2,2) partitions.

Setting the problem up in a generalized regulator framework for $\mathcal{H}_\infty$ optimization, the standard control plant for this type of presentation is given by

$$\left[\begin{array}{c} u \\ y \\ z \\ \hline \beta \\ y \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} r \\ \phi \\ \hline u \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & I \\ 0 & \tilde{M}^{-1} & G \\ -\rho^2 M_o & \rho\tilde{M}^{-1} & \rho G \\ \hline \rho I & 0 & 0 \\ 0 & \tilde{M}^{-1} & G \end{array}\right] \left[\begin{array}{c} r \\ \phi \\ \hline u \end{array}\right] \tag{7.4}$$

from Figure 7.1.

Suppose the plant $G(s) \in \mathcal{RL}_\infty^{p_g \times m_g}$ has n_g states and is strictly proper. Suppose also that the reference model $M_o(s) \in \mathcal{RL}_\infty^{p_m \times m_m}$ has n_m states. Then, defining

$$G(s) := \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & 0 \end{array}\right] \quad \text{and} \quad M_o(s) := \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

and substituting these into equation (7.4) gives the generalized plant $P(s)$ as

$$P(s) = \left[\begin{array}{c||c|c} A & B_1 & B_2 \\ \hline\hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] := \left[\begin{array}{cc||cc|c} A_g & 0 & 0 & ZC_g^T & B_g \\ 0 & A_o & -B_o & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & I_{m_g} \\ C_g & 0 & 0 & I_{p_g} & 0 \\ \rho C & \rho^2 C_o & -\rho^2 D_o & \rho I_{p_g} & 0 \\ \hline 0 & 0 & \rho I_{m_m} & 0 & 0 \\ C_g & 0 & 0 & I_{p_g} & 0 \end{array}\right] \tag{7.5}$$

where the matrix $Z \geq 0$ is the unique stabilizing solution to the algebraic Riccati equation

$$A_g Z + Z A_g^T - Z C_g^T C_g Z + B_g B_g^T = 0. \tag{7.6}$$

Now, by submitting this generalized plant $P(s)$ of (7.5) to the standard $\mathcal{H}_\infty$ suboptimal problem described in Chapter 5, we can produce 2-DOF $\mathcal{H}_\infty$ controllers and also corresponding low-order controllers. As expected from the results in Chapter 5, the order of low-order 2-DOF $\mathcal{H}_\infty$ controllers will be equal to

$$\begin{aligned} \mathcal{N}_{low} &= (n_s + n_m) - (m_m + p_g) \\ &= (n_s - p_g) + (n_m - m_m) \end{aligned} \tag{7.7}$$

or less, where

$$\begin{aligned} n_s &= \text{number of states of the (shaped) plant} \\ p_g &= \text{number of outputs of the plant} \\ n_m &= \text{number of states of the reference model} \\ m_m &= \text{number of inputs of the reference model.} \end{aligned}$$

A constructive algorithm for low-order 2-DOF $\mathcal{H}_\infty$ controller design is discussed in the next section.

7.3 A Low-Order 2-DOF $\mathcal{H}_\infty$ Controller Design Algorithm

We now present a CAD algorithm for *Low-Order 2-DOF $\mathcal{H}_\infty$ Controller Design*, by combining the theory of Chapter 6 with the low-order $\mathcal{H}_\infty$ suboptimal controller design described in Chapter 5.

step 1: Select a simple step response model for the closed-loop system; that is, select $M_o(s)$ in Figure 7.1. This is usually a diagonal matrix of first or second-order lags. The speed of response of the ideal model must be realistic so as to avoid poor robust stability properties and excessive control signals.

step 2: Select loop shaping weights for the open-loop plant. This is used to meet the closed-loop performance specifications.

step 3: Find the minimal achievable value of γ (i.e., γ_o) which may be calculated using (6.13).
If $\gamma_o \leq 4$, go to step 4.
If $\gamma_o > 4$, go to step 2 to select alternative weights.

step 4: Select the scaling factor ρ for the 2-DOF problem to be $1 \leq \rho \leq 3$. The scaling of ρ is a compromise between robust stability and model matching. The smaller value of ρ generates the smaller value of γ_{min} in step 6 below, Postlethwaite and Skogestad [58].

step 5: Build a generalized plant $P(s)$ as in (7.5).

step 6: Find γ_{min} such that

$$\gamma_{min} = \inf_{K stabilizing} ||\mathcal{F}_l(P, K)||_\infty$$

using reliable algorithms in, for example, Matlab. This will always be higher than γ_o. In [43], a range of $1.2\gamma_o \leq \gamma_{min} \leq 3\gamma_o$ was suggested to give a good compromise between the robust stability and robust performance objectives. The loop shapes for the feedback loop will not be altered significantly provided a low value for γ_{min} is achieved. The reciprocal of γ_{min} is roughly proportional to the multivariable stability margin, [58].

step 7: Select a suboptimal γ such that $\gamma > \gamma_{min}$.

step 8: Compute all element matrices of $K_a(s)$ in (5.18), as per (5.19)-(5.26), for the generalized plant $P(s)$.

step 9: Compute $\tilde{A}$ and $\tilde{F}$ as per (5.36)-(5.37), and partition $\tilde{A}$, $\hat{C}_2$ and $\tilde{F}$ as per (5.39)-(5.40) and (5.42).

step 10: Set $X_2 = I_{n_s-p_g+n_m-m_m}$.
† Note that, as stated in Remark 5.7, X_2 need not necessarily be the identity matrix.

step 11: Select an arbitrary X_1, and then compute A_ϕ, B_ϕ, C_ϕ and D_ϕ as per (5.48)-(5.51).
†† Alternatively, as stated in Appendix B, first choose A_ϕ as per (B.16) and then compute X, B_ϕ, C_ϕ and D_ϕ as per (B.17)-(B.20).

step 12: Solve ARE (5.65) for X_ϕ.
If $X_\phi > 0$, go to step 13.
If $X_\phi \leq 0$, go to step 11 and repeat.

step 13: Compute a low-order 2-DOF $\mathcal{H}_\infty$ controller by substituting all the element matrices into (5.33).

step 14: Pre/post-multiply a low-order 2-DOF $\mathcal{H}_\infty$ controller obtained by the loop shaping weights to compute the *final* low-order 2-DOF $\mathcal{H}_\infty$ controller and rescale the prefilter to achieve perfect steady state model matching.

The procedure described above is based on the given $M_o(s)$, weighting functions and the suboptimal γ, and thus modifications to the procedure may be required in a similar manner described in Chapter 6, Section 6.5.

7.4 Concluding Remarks

In this chapter, controller size reduction in the 2-DOF LSDP was considered, by combining the low-order 1-DOF LSDP controller design of Chapter 6 and the low-order $\mathcal{H}_\infty$ suboptimal controller design of Chapter 5. As a consequence, it was shown that the order of low-order 2-DOF $\mathcal{H}_\infty$ controllers can be reduced down to

$$\mathcal{N}_{low} = (n_s - p_g) + (n_m - m_m)$$

or less. A CAD algorithm for the design was also presented.

Chapter 8

Application to the GEC-Alsthom Tetrahedral Robot

8.1 Introduction

In this chapter we apply the theory developed in this thesis to a practical system, the *Tetrabot* which is a tetrahedral robot designed by GEC-Alsthom as a fast and accurate assembly robot. The Tetrabot is a novel device with a parallel-serial structure designed to overcome many of the inherent disadvantages of conventional serial robots. In developing a controller for the Tetrabot, the designers of the robot adopted a conventional approach, by ignoring both the nonlinearities and dynamic coupling between robot joints, Dwolatzky and Thornton [20]. An evaluation of their design indicates a need for more sophisticated controllers capable of coping with multivariable nonlinear systems. Postlethwaite and Feng [56] first designed an $\mathcal{H}_\infty$ optimal controller with relatively good performance.

The aim of this chapter is to demonstrate the effectiveness, for the class of robots represented by the Tetrabot, of a low-order robust suboptimal $\mathcal{H}_\infty$ controller as devised in this thesis.

The chapter is organized as follows. In Section 8.2, a linear model to be used in the controller design is introduced, together with control objectives. Then, a low-order controller is designed in Section 8.3, to demonstrate the effectiveness of the methodology developed in this thesis. Concluding remarks are given in Section 8.4.

8.2 Model and Control Objectives

Unlike the conventional robots of a serial configuration, the Tetrabot is a serial-parallel configuration robot. The Tetrabot shown in Figure 8.1 consists of a parallel structure (of three linear actuator rods in 3 degrees-of-freedom) combined with a serial structure (of three wrist links in another 3 degrees-of-freedom). With this configuration, the Tetrabot has the advantages of being potentially stiffer and more accurate than a conventional serial structure. A conceptual diagram of the mechanical structure is shown in Figure 8.2.

The original control design for the Tetrabot was based on the assumption that all six joints are decoupled, and a PI controller was used to position each loop, Dwolatzky and Thornton [20]. They observed, however, that large displacements and high speeds caused the Tetrabot to display considerable overshoot and recommended that more advanced control strategies should be investigated.

8.2.1 Dynamic Modelling

For advanced control strategies, Feng *et al.* [22] developed a reasonably comprehensive nonlinear model of the Tetrabot dynamics. From this model, linear models about various operating points can be obtained for controller design studies. One such linearized model for which we will design a low-order controller is given by:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{8.1}$$

where

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & 0 & -2.7258 & 1.4443 & 1.4441 \\ 0 & 0 & 0 & 1.4446 & -2.7252 & 1.4444 \\ 0 & 0 & 0 & 1.4446 & 1.4448 & -2.7256 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \end{bmatrix} \\
B &= \begin{bmatrix} -1.8295 & -0.5688 & -0.5688 \\ -0.5688 & -1.8295 & -0.5688 \\ -0.5688 & -0.5688 & -1.8295 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
C &= \begin{bmatrix} 0 & 0 & 0 & 0.9994 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9994 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9994 \end{bmatrix} \\
D &= 0_{3\text{x}3}.
\end{aligned}$$

The model represents the parallel structure only which can realistically be assumed to be decoupled from the 3-axis wrist. The open-loop poles of the model are at

$$\pm 0.4042, \quad 0.0 \pm j2.0420, \quad 0.0 \pm j2.0421.$$

The frequency responses of the open-loop singular values are shown in Figure 8.3.

The input variables are:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{m1} \\ T_{m2} \\ T_{m3} \end{bmatrix} = \begin{bmatrix} \text{Drive torque of actuator drive rod 1} \\ \text{Drive torque of actuator drive rod 2} \\ \text{Drive torque of actuator drive rod 3} \end{bmatrix}$$

the output variables are:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} \text{Length of rod 1} \\ \text{Length of rod 2} \\ \text{Length of rod 3} \end{bmatrix}$$

and the state variables are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \\ l_1 \\ l_2 \\ l_3 \end{bmatrix}.$$

8.2.2 Performance Specification

Control objectives are listed as below:

(1) good steady state behaviour

(2) small overshoot at high speeds for large displacements

(3) robustness with respect to nonlinearities, dynamic coupling and variable payloads.

8.3 Low-Order Controller Design

Postlethwaite and Feng [56] designed an $\mathcal{H}_\infty$ controller for the position control of the Tetrabot and showed reasonably good performance of the $\mathcal{H}_\infty$ controller with respect to robust stability and disturbance rejection, in simulation on a full nonlinear model. The resultant $\mathcal{H}_\infty$ controller had a dynamic order of 15, including the necessary weight. We next design a low-order robust suboptimal controller for the Tetrabot.

8.3.1 The 1-DOF LSDP Scheme

We will consider the low-order robust suboptimal controller design method discussed in Chapter 6. A diagram for the controller design is given in Figure 8.4. For the loop shaping procedure, a first-order weighting function is chosen as

$$W(s) = \frac{100(s+1.5)}{(s+0.00005)} \times I_3$$

then we have $n_s = 9$, $p = 3$ and thus $n_\phi = 6$ for this case.

In the following we obtain, for comparison purpose, a low-order robust suboptimal controller of order 6 and also the "central" robust optimal controller of order 9. The optimal value γ is found to be $\gamma_o = 2.76$ and we select a suboptimal value of $\gamma = 15 \quad (> \gamma_o)$. Then, all tuning matrices required for low-order robust suboptimal controller are obtained, by choosing an A_ϕ as

$$A_\phi = \begin{bmatrix} -0.1010 & 0 & 0 & 1.0000 & -0.0000 & 0.0000 \\ 0 & -0.1010 & 0 & -0.0000 & 1.0000 & 0.0000 \\ 0 & 0 & -0.1010 & 0.0000 & -0.0000 & 1.0000 \\ 0 & 0 & 0 & -147.8358 & -0.0000 & 0.0001 \\ 0 & 0 & 0 & 0.0006 & -43.6246 & -0.0000 \\ 0 & 0 & 0 & 0.0007 & 0.0000 & -43.6247 \end{bmatrix}.$$

Then, the $\mathcal{H}_\infty$-norm constraint is satisfied with $||\Phi(s)||_\infty = 0.9850$. So, a low-order robust suboptimal controller $K^r_{s,nlcf}(s)$ of order 6 is computed by (6.24).

The controller $K^{r}_{s,nlcf}(s) = (A_{kr}, B_{kr}, C_{kr}, D_{kr})$ is given by

$$
\begin{aligned}
A_{kr} &= 1.0E+04 \times \\
&\begin{bmatrix} -0.0149 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0055 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0055 & -0.0000 & -0.0000 & 0.0000 \\ 2.1769 & -0.0000 & -0.0000 & -0.0022 & -0.0000 & -0.0000 \\ -0.0000 & 0.2300 & 0.0000 & 0.0000 & -0.0005 & -0.0000 \\ -0.0000 & -0.0000 & 0.2300 & 0.0000 & 0.0000 & -0.0005 \end{bmatrix} \\
B_{kr} &= 1.0E+03 \times \\
&\begin{bmatrix} -0.0339 & -0.0339 & -0.0339 \\ -0.0082 & -0.0030 & 0.0112 \\ -0.0082 & 0.0112 & -0.0030 \\ 5.0022 & 5.0023 & 5.0022 \\ 0.3537 & 0.1294 & -0.4832 \\ 0.3537 & -0.4832 & 0.1294 \end{bmatrix} \\
C_{kr} &= \begin{bmatrix} 137.4239 & 48.6494 & 48.6520 & 0.7931 & 0.8078 & 0.8078 \\ 137.4252 & 17.8092 & -66.4580 & 0.7931 & 0.2957 & -1.1035 \\ 137.4251 & -66.4588 & 17.8054 & 0.7931 & -1.1035 & 0.2956 \end{bmatrix} \\
D_{kr} &= \begin{bmatrix} 8.5659 & 2.2130 & 2.2130 \\ 2.2130 & 8.5660 & 2.2130 \\ 2.2130 & 2.2130 & 8.5660 \end{bmatrix}.
\end{aligned}
$$

Figure 8.5 shows the frequency responses of the two controllers, and Figure 8.6 shows the frequency responses of the shaped and stabilized open-loop systems. Figures 8.7 and 8.8 show the plant output response to the output disturbance (from d_o to y) and the control efforts to an input disturbance (from d_i to u), respectively. The disturbance is quickly rejected, but with a relatively large control energy in a transient period. Figure 8.9 shows reference following (from r to y), and indicates good decoupling but there is still a large overshoot as mentioned in [20]. The (output) sensitivity function is shown in Figure 8.10.

It should be noted that the $\mathcal{H}_\infty$-norm constraint of $||\Phi(s)||_\infty \leq 1$ resulted in the suboptimal γ being chosen relatively large compared with the optimal γ_o. This consequently leads to a degradation in performance.

8.4 Concluding Remarks

We have demonstrated in this chapter how a low-order suboptimal controller can work in the place of a corresponding "central" optimal controller. This was done by applying the design methodology to the GEC-Alsthom Tetrabot and by showing that the low-order controller as designed here, in general, achieves similar performance to the corresponding "central" controller.

The choice of weighting functions used to shape the nominal plant model is crucially important for the success of the loop shaping design procedure for both the "central" optimal and low-order suboptimal controller designs, but this choice may not be at all related to the controller size reduction techniques developed in this thesis. In addition, the choice of the design parameter γ required for the low-order robust suboptimal controller design is dependent on the $\mathcal{H}_\infty$-norm constraint. This was seen in the example where the choice of suboptimal γ needed to be large in order for the constraint $||\Phi(s)||_\infty \leq 1$ to be met. Consequently, some loss of performance arose since the loop shaping methodology requires γ to be small.

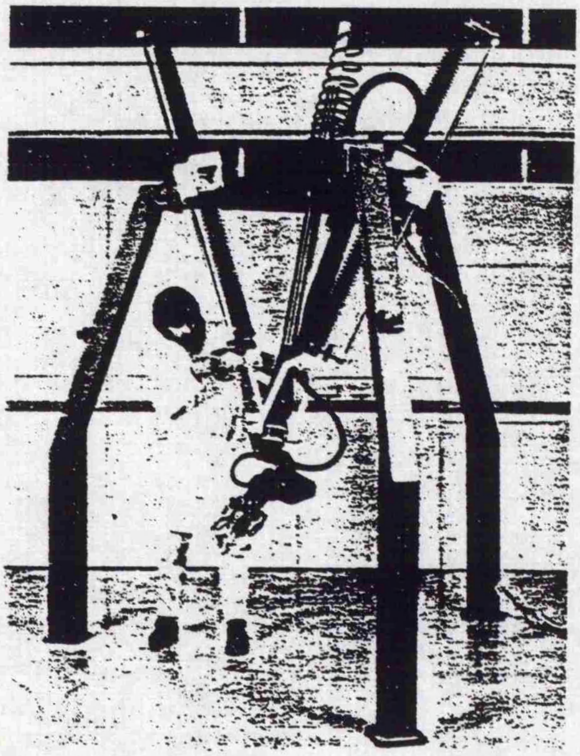


Figure 8.1: The GEC-Alsthom Tetrabot.

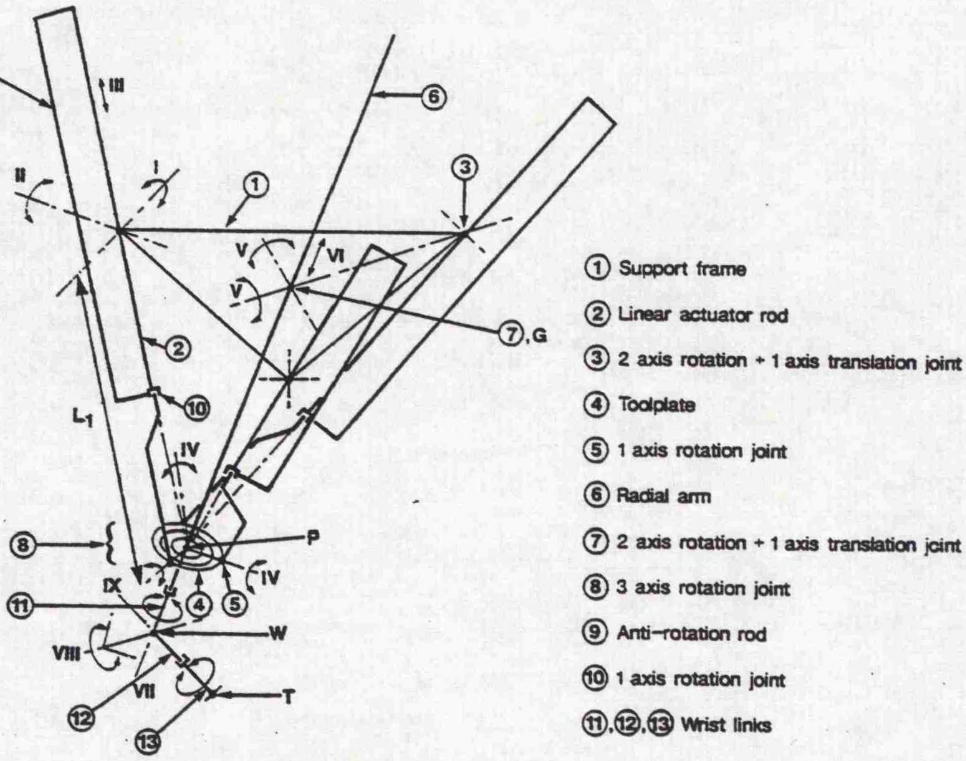


Figure 8.2: The Mechanical Structure of the Tetrabot.

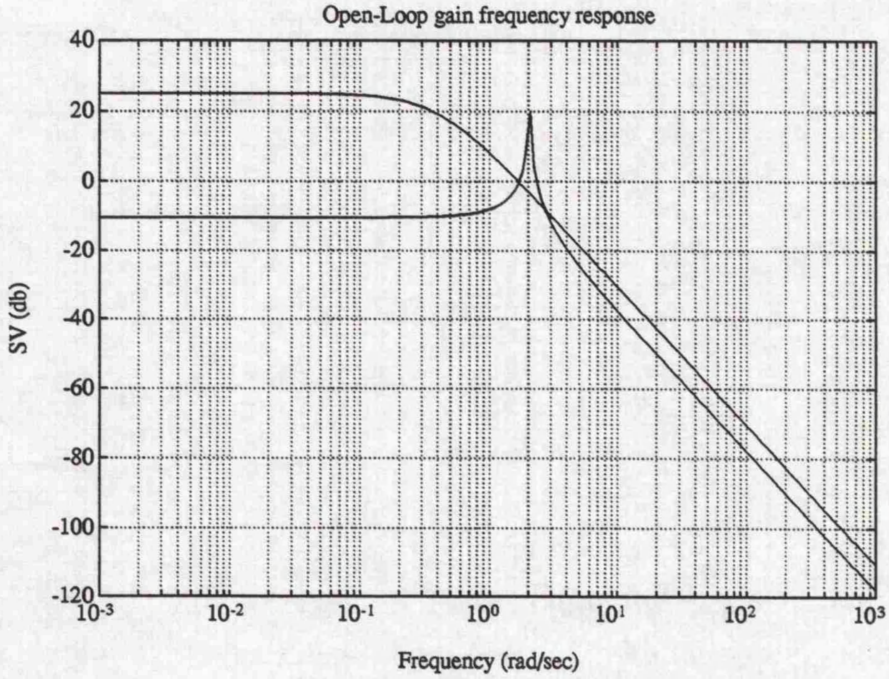


Figure 8.3: Open-Loop Gain of the Nominal Tetrabot Model.

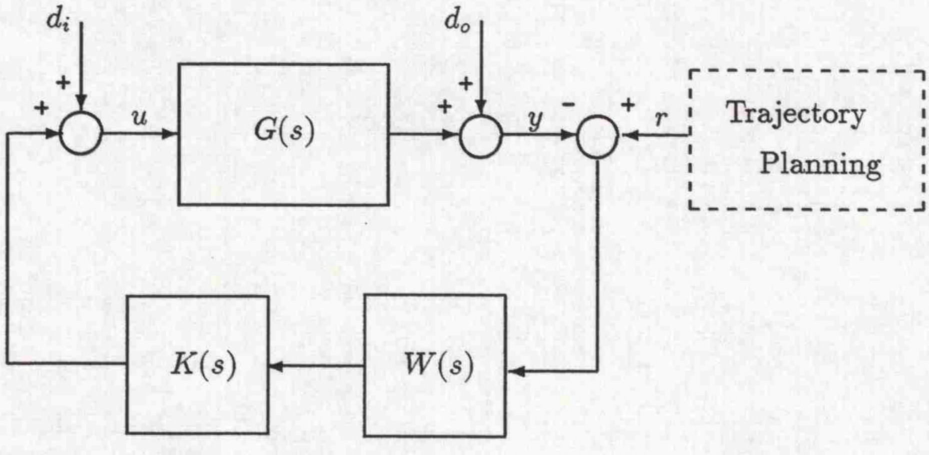


Figure 8.4: Diagram for Controller Design (1-DOF LSDP).

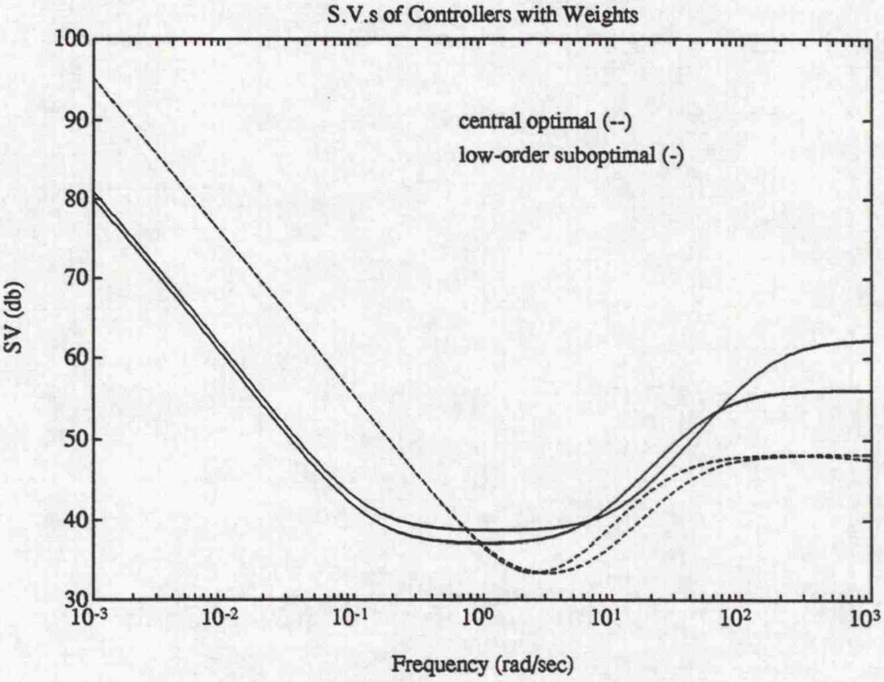


Figure 8.5: Singular Values of Controllers.

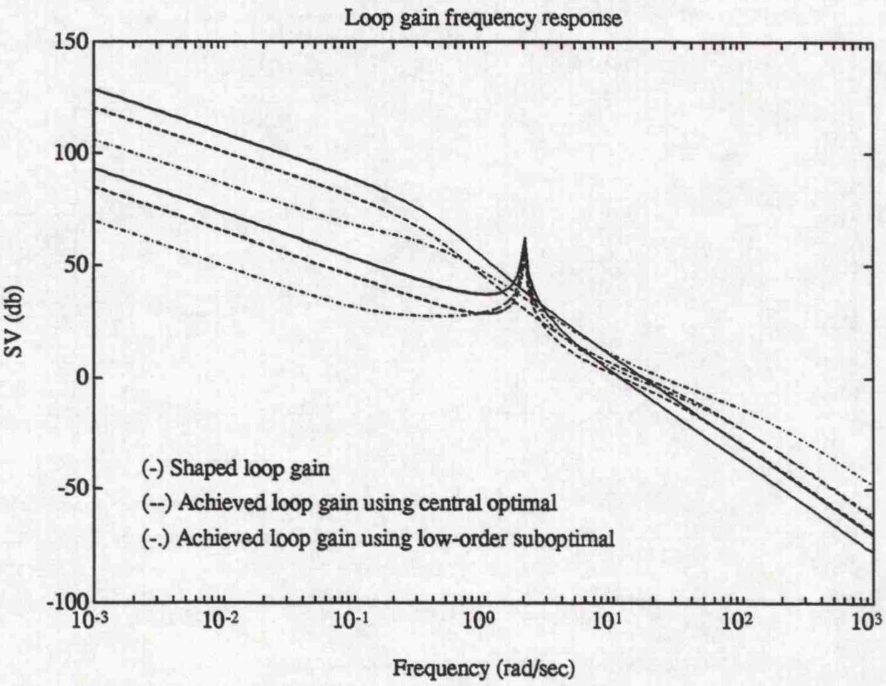


Figure 8.6: Singular Values for Shaped and Achieved Systems.

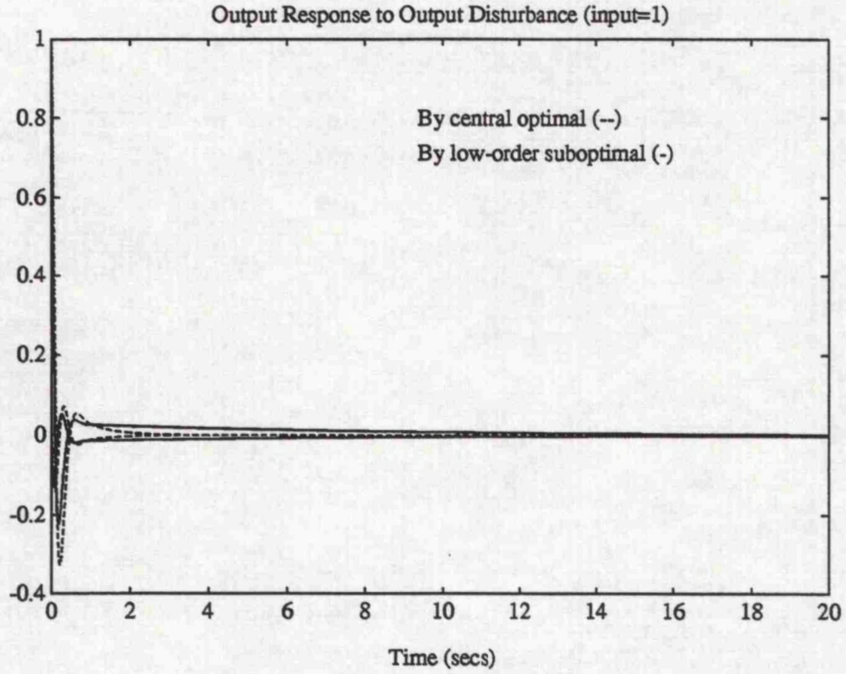


Figure 8.7: Output Response to Output Disturbance Step.

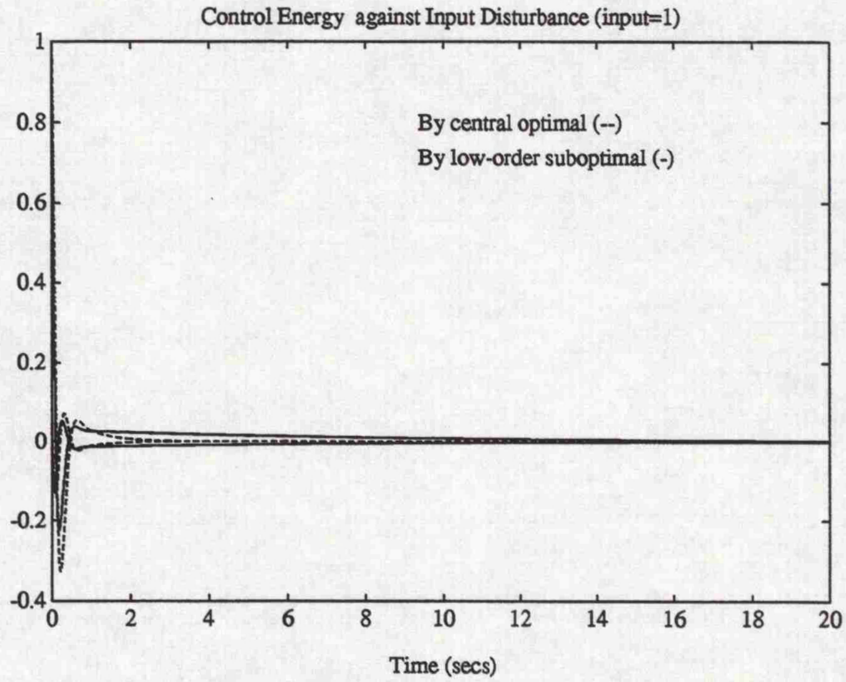


Figure 8.8: Control Efforts to Input Disturbance Step.

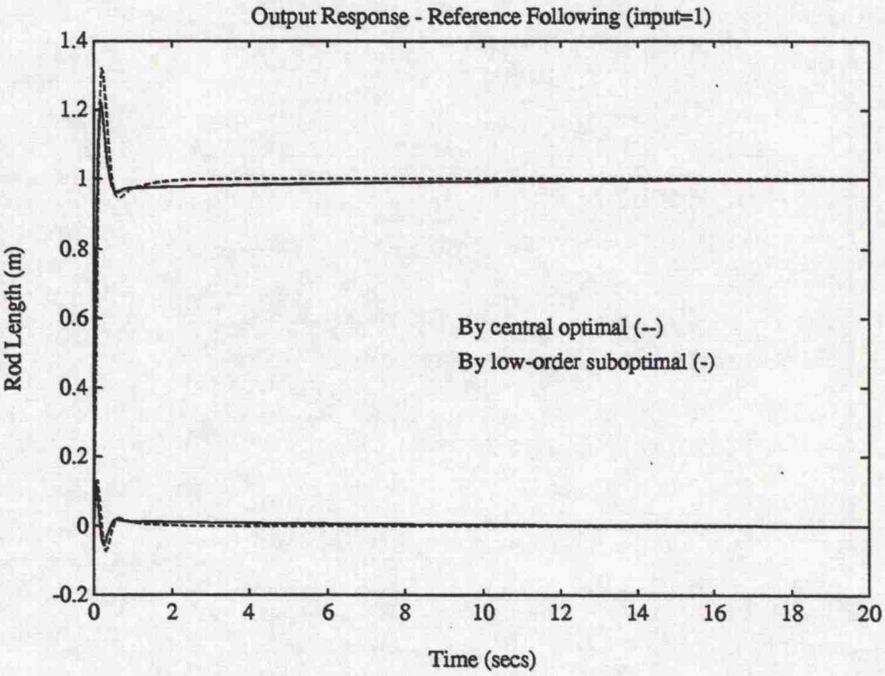


Figure 8.9: Output Response - Reference Following.

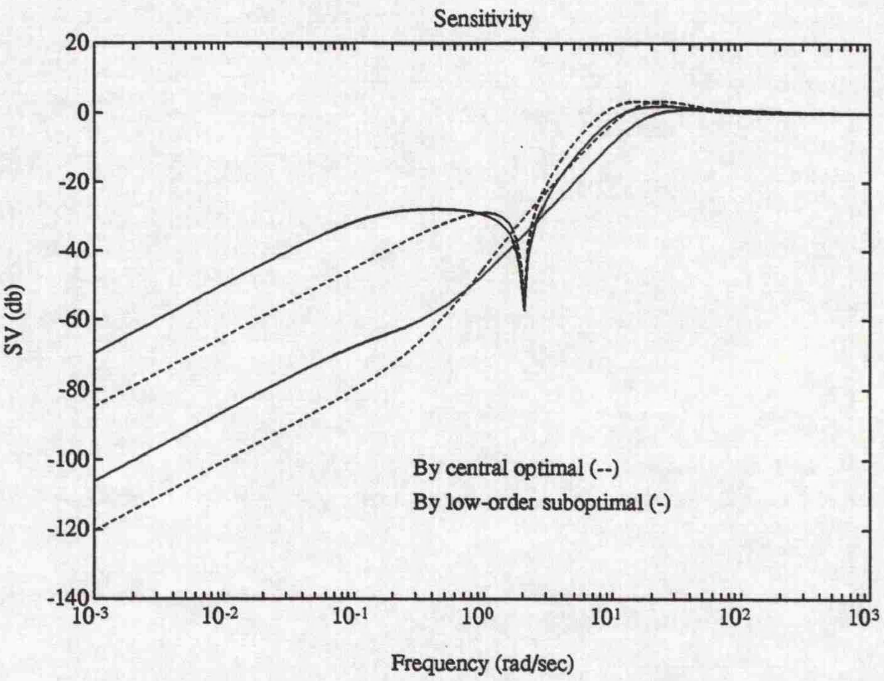


Figure 8.10: Singular Values of the Sensitivity Function.

Chapter 9

Conclusions and Future Research

9.1 Conclusions

This thesis has addressed the problem of controller size reduction in advanced robust control system design. Methods have been given:

- to reduce the order of stabilizing controllers
- to reduce the order of $\mathcal{H}_\infty$ suboptimal controllers
- to reduce the order of robust suboptimal controllers (for the 1-DOF LSDP)
- to reduce the order of 2-DOF $\mathcal{H}_\infty$ controllers.

A common component in the synthesis of advanced robust controllers is a parametrization of controllers via a *stable free parameter*, $Q(s)$ or $\Phi(s)$, which can lead to controllers of "high" order. For this reason, model reduction techniques are frequently used. In this thesis, we have given an alternative methodology for obtaining low-order controllers. The central idea in all the design methods considered was to take the parametrization of controllers and to show that the order of such controllers could be reduced, by eliminating any unobservable (or uncontrollable) states, if the corresponding free parameter, $Q(s)$ or $\Phi(s)$, solved two simultaneous matrix equations (in all cases) and an $\mathcal{H}_\infty$-norm constraint (in the $\mathcal{H}_\infty$ cases). Orthogonal canonical forms were employed to solve the two matrix equations, and a certain Riccati equation was used to tackle the $\mathcal{H}_\infty$-norm constraint.

It was shown that in each design method the low-order realizations could be expressed in state-space form following relatively easy computations, based on constructive algorithms. As expected, the low-order controllers performed in exactly the same way as in the "formal" order counterparts. The following results on the size of controllers were obtained:

1. The order of all stabilizing controllers may be less than or equal to the number of plant outputs.
2. The order of all $\mathcal{H}_\infty$ suboptimal controllers may be equal to the order of the generalized plant minus the number of plant outputs.
3. The order of the 1-DOF robust suboptimal controllers may be equal to the order of the shaped plant minus the number of plant outputs.
4. The order of the 2-DOF $\mathcal{H}_\infty$ suboptimal controllers may be equal to the order of the generalized plant (including reference model) minus the sum of the number of plant outputs and the number of reference model inputs.

Numerical examples were given to illustrate the algorithms developed and to confirm the results on controller orders. Examples showed that the low-order controller could work in place of the "central" one but with some deterioration in performance.

9.2 Recommendations for Future Research

The methodology and the associated algorithms for controller size reduction as presented here raise a number of questions which require further research:

1. In the design method for low-order stabilizing controllers in Chapter 4, there is considerable freedom in choosing the parameter matrices H and A_q, while a special form of $\hat{F}$ as in (4.55) is required to obtain controllers of the smallest possible order.
   How do we search over the space of H and A_q to obtain such a special form of $\hat{F}$, when it exists, without missing it?
   How do we check if the space of $\hat{F}$ of a certain order is empty?

2. In low-order $\mathcal{H}_\infty$ suboptimal controller design (Chapter 5), low-order robust suboptimal controller design (Chapter 6), and low-order 2-DOF $\mathcal{H}_\infty$ controller design (Chapter 7), the difficult part of each of the algorithms is the selection of a free parameter matrix $\Phi(s)$ which meets an $\mathcal{H}_\infty$-norm constraint. Some approaches to tackle this constraint were given, but there is a scope for more systematic and/or effective methods, particularly in view of the freedom in A_ϕ as previously mentioned.
   How can we tackle this problem in a more systematic way?
   How can we check if the space of $\Phi(s)$ of a certain order is empty?

3. In low-order $\mathcal{H}_\infty$ suboptimal controller design (Chapter 5), the possibility of lowering the order to less than $n - p_2$ is limited by the structure of $\tilde{F} = \hat{D}_{12}^{-1}\hat{C}_1$, where $\hat{C}_1$ depends on $\hat{D}_{21}$.
   How should we choose $\hat{D}_{12}$ and $\hat{D}_{21}$ from (5.20)-(5.21) for this purpose?

4. For the 1-DOF LSDP (Chapter 6) and the 2-DOF LSDP (Chapter 7), a reasonable value of a suboptimal γ is required to ensure that the loop shapes can be well approximated together with robust stability. In low-order controller design, this requirement may, in some cases (as seen in Chapter 8), be violated due to the $\mathcal{H}_\infty$-norm constraint on $\Phi(s)$.

Can we find a systematic way to relate a suitable selection of $\Phi(s)$ with a reasonable value of γ?

5. A general question which applies to all the design methods considered is: *How can we find an "optimal" controller amongst the set of low-order controllers?*

   Here, by "optimal", we mean the controller which best meets the performance requirements amongst the set of low-order controllers.

Appendices

Appendix A: Proofs of Lemmas 2.14 - 2.17

1. Proof of Lemma 2.14

Suppose A is stable and $||G||_\infty < \gamma$. Then $G^*G - \gamma^2 I < 0$. So, $G^*G - \gamma^2 I$ is invertible for all frequencies and $(G^*G - \gamma^2 I)^{-1}$ has no jw-axis poles, where the state-space realization of $(G^*G - \gamma^2 I)^{-1}$ is given by

$$
\begin{aligned}
(G^*G - \gamma^2 I)^{-1} &= \left[\begin{array}{cc|c} A - BR_\gamma^{-1}D^T C & BR_\gamma^{-1}B^T & BR_\gamma^{-1} \\ -\gamma^2 C^T S_\gamma^{-1} C & -(A - BR_\gamma^{-1}D^T C)^T & C^T D R_\gamma^{-1} \\ \hline -R_\gamma^{-1}D^T C & R_\gamma^{-1}B^T & R_\gamma^{-1} \end{array}\right] \\
&=: \left[\begin{array}{c|c} H_\gamma & * \\ \hline * & * \end{array}\right].
\end{aligned}
$$

Thus, the A-matrix of $(G^*G - \gamma^2 I)^{-1}$, H_γ, has no jw-axis eigenvalues. Further, since we have the following equality:

$$
\begin{bmatrix} I & 0 \\ 0 & -\gamma I \end{bmatrix}^{-1} H_\gamma \begin{bmatrix} I & 0 \\ 0 & -\gamma I \end{bmatrix} = \begin{bmatrix} A - BR_\gamma^{-1}D^T C & -\gamma BR_\gamma^{-1}B^T \\ \gamma C^T S_\gamma^{-1} C & -(A - BR_\gamma^{-1}D^T C)^T \end{bmatrix}
$$

the Hamiltonian matrix $\mathcal{M}_\gamma$ in (2.39) has no jw-axis eigenvalues. Hence, using Lemma 2.13, we conclude that the ARE (2.38) has a unique stabilizing solution X_γ. ∎

2. Proof of Lemma 2.15

From the assumption that X_γ is a solution to the ARE (2.38), we define $M(s)$ as

$$M(s) := \left[\begin{array}{c|c} A & B \\ \hline -(-R_\gamma)^{-1/2}(\gamma B^T X_\gamma + D^T C) & (-R_\gamma)^{1/2} \end{array}\right]. \tag{A.1}$$

Then it can be shown, by direct manipulations, that the following equality holds:

$$\gamma^2 I - G^* G = M^* M. \tag{A.2}$$

Thus, $G^*G - \gamma^2 I \leq 0$. Since X_γ is the stabilizing solution to the ARE (2.38), the Hamiltonian matrix $\mathcal{M}_\gamma$ in (2.39) has no jw-axis eigenvalues (from Lemma 2.13) and thus $(G^*G - \gamma^2 I)^{-1}$ has no jw-axis poles, as seen in the proof of Lemma 2.14 above. This implies that $G^*G - \gamma^2 I < 0$. Then, $||G||_\infty < \gamma$ since A is stable. ∎

3. Proof of Lemma 2.16

Rewrite the ARE (2.38) as

$$A^T X_\gamma + X_\gamma A + \Omega = 0 \tag{A.3}$$

where

$$\Omega := -\gamma(B^T X_\gamma + \gamma^{-1} D^T C)^T R_\gamma^{-1} (B^T X_\gamma + \gamma^{-1} D^T C) + \gamma^{-1} C^T C. \tag{A.4}$$

Ω in (A.4) is positive definite since $\sigma_{max}(D) < \gamma$. So, from the Lyapunov equation (A.3) it is easy to see that, if A is stable, the solution X_γ is positive definite. ∎

4. Proof of Lemma 2.17

Since X_γ is the solution to the ARE (2.38), the equality $\gamma^2 I - G^* G = M^* M$ in (A.2) holds. So, we have $||G||_\infty \leq \gamma$. In addition, from the Lyapunov equation (A.3), it is easily verified that A is stable since the solution X_γ is positive definite. This completes the proof. ■

Appendix B: Alternative Solution to Two Equations (5.34)-(5.35)

Let the pair $(\tilde{A}, \hat{C}_2)$ be transformed into the orthogonal canonical form (A_o, C_o) $= (M\tilde{A}M^{-1}, N\hat{C}_2M^{-1})$, as in (2.21)-(2.22):

$$
\begin{aligned}
A_o &= \begin{bmatrix} A_{11} & \begin{bmatrix} I_{l_2} \\ 0 \end{bmatrix} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \begin{bmatrix} I_{l_3} \\ 0 \end{bmatrix} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_{\nu_o-1,1} & A_{\nu_o-1,2} & A_{\nu_o-1,3} & \cdots & \begin{bmatrix} I_{l_{\nu_o}} \\ 0 \end{bmatrix} \\ A_{\nu_o,1} & A_{\nu_o,2} & A_{\nu_o,3} & \cdots & A_{\nu_o,\nu_o} \end{bmatrix} \\
C_o &= \begin{bmatrix} I_{l_1} & 0 & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}
$$

where ν_o is the observability index of $(\tilde{A}, \hat{C}_2)$, and $A_{i,i}(i = 1, \cdots, \nu_o)$ are $l_i \times l_i$ matrices, and the numbers

$$
p_2 = l_1 \geq l_2 \geq \cdots \geq l_{\nu_o} \qquad l_1 + l_2 + \cdots + l_{\nu_o} = n
$$

are the conjugate Kronecker indices of the pair $(\tilde{A}, \hat{C}_2)$.

Using the form (A_o, C_o), the two equations (5.34)-(5.35) can therefore be transformed into:

$$
\begin{aligned}
A_\phi \bar{X} - \bar{X} A_o &= \bar{B}_\phi C_o && \text{(B.1)} \\
C_\phi \bar{X} - \bar{D}_\phi C_o &= \bar{F} && \text{(B.2)}
\end{aligned}
$$

where

$$
\bar{X} = XM^{-1} \qquad \text{(B.3)}
$$

$$\begin{aligned}\bar{B}_\phi &= B_\phi N^{-1} &\text{(B.4)}\\ \bar{D}_\phi &= D_\phi N^{-1} &\text{(B.5)}\\ \bar{F} &= \tilde{F} M^{-1}. &\text{(B.6)}\end{aligned}$$

To solve the problem for the case of $n_\phi = n - p_2$, we partition the form (A_o, C_o) as shown below:

$$A_o = \left[\begin{array}{c|c}A_{o11} & \left[\begin{array}{c|c}\begin{bmatrix} I_{l_2} \\ 0 \end{bmatrix} & 0_{p_2\times(n-p_2-l_2)}\end{array}\right] \\ \hline A_{o21} & \begin{array}{cc}A_{o22} & A_{o23}\end{array}\end{array}\right] \tag{B.7}$$

$$C_o = \left[\begin{array}{c|cc} I_{p_2} & 0_{p_2\times l_2} & 0_{p_2\times(n-p_2-l_2)}\end{array}\right] \tag{B.8}$$

where $A_{o11} := A_{11} : p_2 \times p_2$, $A_{o21} : (n-p_2)\times p_2$, $A_{o22} : (n-p_2)\times l_2$, and $A_{o23} : (n-p_2)\times(n-p_2-l_2)$. The following Fact B1 is useful in solving (B.1)-(B.2).

Fact B1 *The pair* $(\begin{bmatrix} A_{o22} & A_{o23}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix} I_{l_2} \\ 0\end{bmatrix} & 0\end{bmatrix})$ *is completely observable if the pair* $(\tilde{A}, \hat{C}_2)$ *is completely observable.*

In the following, we assume for the sake of simplicity that $l_2 = p_2$, and suppose that $\bar{X}$ is partitioned as

$$\bar{X} = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 \end{bmatrix} : \quad (n-p_2)\times n$$

with $\bar{X}_1$ and $\bar{X}_2$ having p_2 and $n-p_2$ columns, respectively. Then, from (B.1), we have the following two equations:

$$\begin{aligned} A_\phi \bar{X}_1 - \bar{X}_1 A_{o11} - \bar{X}_2 A_{o21} &= \bar{B}_\phi &\text{(B.9)}\\ A_\phi \bar{X}_2 - \bar{X}_1 \begin{bmatrix} I & 0 \end{bmatrix} - \bar{X}_2 \begin{bmatrix} A_{o22} & A_{o23}\end{bmatrix} &= 0. &\text{(B.10)}\end{aligned}$$

Equation (B.10) is equivalent to

$$\bar{X}_2(\begin{bmatrix} A_{o22} & A_{o23}\end{bmatrix} + \bar{X}_2^{-1}\bar{X}_1 \begin{bmatrix} I & 0 \end{bmatrix})\bar{X}_2^{-1} = A_\phi \tag{B.11}$$

from which we can find a matrix $\bar{X}_1$ and an invertible matrix $\bar{X}_2$ for any *stable* A_ϕ. Obviously, $\bar{X}$ obtained is of full rank. And, in turn, $\bar{B}_\phi$ can be computed from (B.9).

In order to solve (B.2), partition $\bar{F}$ as

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix} : \quad m_2 \times n$$

where $\bar{F}_1$ and $\bar{F}_2$ have p_2 and $n - p_2$ columns, respectively. Equation (B.2) can then be rewritten as

$$C_\phi \begin{bmatrix} \bar{X}_1 & \bar{X}_2 \end{bmatrix} - \bar{D}_\phi \begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix} \tag{B.12}$$

and, from (B.12), C_ϕ and $\bar{D}_\phi$ are computed as follows:

$$C_\phi = \bar{F}_2 \bar{X}_2^{-1} \tag{B.13}$$

$$\bar{D}_\phi = C_\phi \bar{X}_1 - \bar{F}_1. \tag{B.14}$$

Hence, we have A_ϕ, $\bar{X}$, $\bar{B}_\phi$, C_ϕ and $\bar{D}_\phi$, and can therefore compute an $\mathcal{H}_\infty$ sub-optimal controller of $(n - p_2)$th order by making use of the realization (5.33), provided A_ϕ is chosen such that $||\Phi(s)||_\infty < \gamma$. Note that X, B_ϕ and D_ϕ can be computed from (B.3)-(B.5), respectively.

Without loss of generality, the identity matrix can be chosen as a candidate for $\bar{X}_2$ in (B.11), i.e., $\bar{X}_2 = I$. In this instance, (B.11) becomes

$$\begin{bmatrix} A_{o22} & A_{o23} \end{bmatrix} + \bar{X}_1 \begin{bmatrix} I & 0 \end{bmatrix} = A_\phi. \tag{B.15}$$

Thus, $\bar{X}_1$ can be chosen arbitrarily subject to the stability of A_ϕ, where A_ϕ is computed from (B.15) by

$$A_\phi = \begin{bmatrix} A_{o22} + \bar{X}_1 & A_{o23} \end{bmatrix} \tag{B.16}$$

with A_{o22} and A_{o23} fixed. Consequently, the matrices X, B_ϕ, C_ϕ and D_ϕ are computed by

$$X = \begin{bmatrix} \bar{X}_1 & I \end{bmatrix} M \tag{B.17}$$

$$
\begin{aligned}
B_\phi &= (A_\phi \bar{X}_1 - \bar{X}_1 A_{o11} - A_{o21})N && \text{(B.18)}\\
C_\phi &= \bar{F}_2 && \text{(B.19)}\\
D_\phi &= (\bar{F}_2 \bar{X}_1 - \bar{F}_1)N. && \text{(B.20)}
\end{aligned}
$$

Hence, a solution matrix X and a suitable free parameter $\Phi(s)$ to the two simultaneous matrix equations (5.34)-(5.35) are all characterized in terms of $\bar{X}_1$ only, which can be chosen arbitrarily subject to A_ϕ in (B.16) being stable.

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