Sliding Mode Schemes Using Output Information with Application to Heating Plant Problems

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Abstract

This thesis considers the problem of developing sliding mode output tracking controllers for uncertain systems when output information alone is available. Two different approaches to controller design are proposed. The first approach is an observer based scheme which utilizes integral action. A new framework is proposed for the design of a class of sliding mode observers. The attainment of a canonical form, which is central to the framework, is a necessary and sufficient condition for the existence of a class of observers insensitive to matched uncertainty. The results supersede previous work in this area which necessitated checking the validity of a structural constraint between the state space matrices of the system. A formal analysis is undertaken of the combined plant/observer dynamics obtained when using a sliding mode control law incorporating integral action. It is demonstrated that the combined system is quadratically stable, despite the presence of a class of bounded matched uncertainty. Furthermore, the control law and the observer system can be designed independently; in other words, the 'separation principle' for linear systems also holds for this class of uncertain system and controller/observer pair. The second approach considers an output feedback stabilization problem where the class of hyperplanes and control laws considered is restricted to those which require only output information. The analysis is performed in essentially the same framework as that developed for the sliding mode observer. It enables the class of systems considered by other workers in this field to be extended and provides a practical realizable controller which requires no additional assumptions. These results are used in a model-reference framework to obtain a tracking controller.

Successful attempts to implement these new theoretical ideas on an experimental furnace at the Gas Research Centre at Loughborough are documented. A single-input single-output sliding mode scheme to control temperature is described. Details are given of a more ambitious, multivariable scheme to regulate both temperature and excess oxygen by the manipulation of the fuel and air flows.

To my parents

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Chapter 1

Introduction

1.1 An Overview

Much of the research in the area of control systems theory during the seventies and eighties focussed on the issue of *robustness* – i.e. designing controllers with the ability to maintain performance/stability in the presence of discrepancies between the plant and model. One nonlinear approach to robust controller design which emerged during this period is the Variable Structure Control Systems methodology. Although the approach is applicable to systems of nonlinear differential equations, much of the research has been directed towards linear systems.

Variable Structure Control Systems comprise a collection of different – usually quite simple – feedback control laws and a decision rule. Depending on the 'state' of the system, the decision rule, often termed the *switching function*, determines which of the control laws is 'on-line' at any one time. Unlike for example a gain-scheduling methodology, the decision rule is designed to force the system states to reach, and subsequently remain on, a pre-defined surface within the state-space. The dynamical behaviour of the system when confined to the surface is described as an *ideal sliding motion*. The advantages of obtaining such a motion are twofold: firstly there is a reduction in order (and for nonlinear systems, by an appropriate choice of surface, the reduced order motion may be chosen to be linear); secondly the sliding motion is insensitive to parameter variations implicit in the input channels. The latter property of invariance towards so-called *matched uncertainty* makes the methodology an attractive one for designing robust controllers for uncertain systems. The design approach comprises two components: the design of a surface/manifold in the state space so that the reduced order sliding motion satisfies the specifications imposed on the designer; and the synthesis of a control law, discontinuous about the sliding surface, such that the trajectories of the closed loop motion are directed towards the surface. An alternative interpretation of the latter property is that the discontinuous control action renders the sliding surface invariant and (at least locally) attractive.

The closed-loop dynamical behaviour obtained from using a Variable Structure Control law comprises two distinct types of motion. The initial phase, occurring whilst the states are being driven towards the surface, is in general affected by any matched disturbances present. Only when the states reach the surface and the sliding motion takes place does the system become insensitive to all matched uncertainty. Ideally the control system should be designed so that the initial (disturbance affected) phase is as short as possible.

The motivation for considering differential equations with discontinuous right hand sides is that they occur naturally in physical systems; for example, the resistance force due to dry Colulomb friction takes opposite signed values depending on the direction of motion [80]. Also in the case of certain electric motors and power converters the control action is naturally discontinuous [64]. In other systems, especially mechanical ones, the introduction of discontinuous control action will not induce an ideal sliding motion. Imperfections in the process such as delays and hysteresis will conspire to induce a high frequency motion known as *chattering.* This is characterised by the states repeatedly crossing rather than remaining on the surface. Such a motion is highly undesirable in practice and will result in unnecessary wear and tear on the actuator components. It is usual in such situations to modify the discontinuous control action so that rather than forcing the states to lie on the sliding surface they are forced to remain within an (arbitrarily) small boundary layer about the surface. In the literature this is often referred to as a *pseudo sliding motion.* The total invariance properties associated with ideal sliding will be lost. However, an arbitrarily close approximation to ideal sliding can usually be obtained. Formally these results can be explored in terms of the notion of *practical stability* [60].

This thesis considers *uncertain linear systems* of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(x,u,t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

where A, B and C are known matrices and the function $f(\cdot)$ is unknown but bounded. From the previous discussions, the sliding mode approach is very much embedded within the state-space methodology. Indeed, most of the published work relies on all the internal states being available to the controller. Under this assumption, provided the uncertain linear system described above is completely controllable and the uncertainty is matched and bounded, regulatory controllers can be designed which guarantee asymptotic convergence to zero of the states (for example [60]).

In reality, the idealized situation described above is quite rare. Usually certain states will be either impossible or prohibitively expensive to measure. Alternatively the linear model may represent an approximation of a distributed parameter system or else have been obtained via system identification. As a result the internal states will have no physical meaning. In either situation much of the sliding mode literature is no longer directly applicable.

The underlying assumption of this thesis is that only output information is available to the control law. The objective of the early chapters is to isolate the class of systems for which a stable sliding motion may be achieved using only output information. In this case it is sensible to consider only the regulation of the outputs or output tracking problems. In Chapter 4 output tracking is achieved using an integral action methodology and in Chapter 5 a model reference framework is adopted. In the first approach an observer is used to estimate the internal states which enables an established state feedback controller to be used. In the model reference case a control law is synthesised directly which requires only output information. Throughout, the emphasis has been placed on:

- isolating the class of systems for which the results apply in a way that is easily recognizable – preferably in terms of system invariants.
- providing realizable controllers possessing the required robustness characteristics

which can be readily synthesised within Computer Aided Control System Design packages such as MATLAB.

The early chapters of the thesis consider these problems and formulate new robust output tracking sliding mode schemes. The remainder of the thesis documents the application of these new results to control problems associated with gas fired furnaces.

All the work pertains to the control of a single burner furnace of a design commonly used in the process industry for the firing of ceramics. This design has been chosen as the starting point for the development of controllers because British Gas Research and Technology has an experimental furnace of this type which was made available for controller evaluation purposes. The control inputs to this particular furnace configuration are the fuel and air flow rates to the burner – specifically gas valve position and fuel/air ratio trim. The outputs are the furnace temperature as measured by a thermocouple embedded in one of the walls; and the percentage of oxygen remaining in the combustion products as measured by a probe in the flue. Under typical furnace operating conditions the temperature is required to follow a pre-specified temperature/time profile which imparts to the furnace load an appropriate temperature history. At the same time the burner is required to operate in a safe and energy efficient way. For a given fuel flow rate an appropriate (theoretical) air flow rate is required to completely oxidise the hydrocarbons in the fuels. Operating the furnace in such a fashion – called *stoichiometric combustion* – is the most energy efficient and results in no excess oxygen in the flue products. An inadequate flow of oxygen results in the potentially dangerous situation of running the furnace with unburnt fuel appearing in the flue products. Conversely, surplus oxygen increases the flow rate in the flue, which in turn results in increased energy losses. The concentration of oxygen in the flue products therefore represents a measure of the efficiency of combustion and needs to be tightly controlled.

Currently, control of temperature and excess oxygen is affected separately. A controller to manipulate the fuel flow rate so that the temperature tracks the required profile is designed based on the assumption of a fixed fuel/air ratio. Independently another controller is synthesised to manipulate the air flow so that an appropriate concentration of oxygen is maintained in the flue products. At present sophisticated Proportional/Integral/Derivative (PID) controllers are commercially available to control the fuel/temperature loop. These have appropriate in-built logic to prevent undesirable behaviour such as integral wind-up. A device called an Electronic Ratio Controller (ERC) is also marketed which mains an appropriate fuel/air ratio by manipulating the air flow. Further details of the ERC are given in [32].

Within the context of this thesis, the problem initially addressed is that of designing a sliding mode controller to adjust the fuel flow in order to control temperature, under the assumption of a fixed fuel/air ratio. After successfully implementing a robust output tracking controller/observer pair to control temperature on the experimental furnace at the Gas Research Centre, a more ambitious multivariable problem was tackled. The ERC has a 'trim signal' which allows the set fuel/air ratio to be varied. This signal was considered as another input to the system – the objective being to control the excess oxygen level in flue by manipulating the fuel/air ratio set point. Details of the design process, from system identification through to implementation, are given in the final chapters.

1.2 Thesis Organisation

In essence the early chapters develop new theoretical results whilst the latter pertain to the design of furnace controllers based on these ideas. Details of the mathematical notation used, together with preliminary results pertaining to matrix determinants and symmetric positive definite matrices are given in Appendix A.

Chapter 2 presents an introduction to Variable Structure Control Systems with a sliding mode. Mention is made of the mathematical difficulties which are encountered when attempting to describe solutions of differential equations with discontinuities in the right hand side. Such considerations lead to the important concept of the *equivalent control* action, which enables the reduced order motion to be calculated. The invariance of the reduced order sliding motion to disturbances occurring in the range of the input distribution matrix is also demonstrated. A theorem is cited from the literature which provides sufficient conditions for a sliding motion to occur using Lyapunov-like stability notions. This theorem will be used in later chapters to demonstrate that the proposed control laws do indeed guarantee that a sliding motion will be attained. A canonical form will be described which is often termed as *regular form.* This form provides a convenient platform from which to design the sliding surface. Different strategies for designing hyperplanes for multivariable systems are discussed which attempt to reduce the sensitivity of the sliding motion to the effects of unmatched disturbances. Finally, some continuous approximations to Variable Structure Controllers are presented. These are commonly used at the implementation stage to alleviate the problem of chattering. The chapter emphasizes the state-space nature of the approach and (in general) the need for full state availability.

Chapter 3 considers the dual problem of synthesising sliding mode observers. Here the objective is to induce a sliding motion on the hyperplane in the state estimation error space defined by the kernel of the output distribution matrix. The intention is that, in the case when only plant output information is available, such an observer could be used to generate estimates of the internal states. The controller designs of Chapter 2 can then be realized using the state estimates. A review of recent work in the area of sliding mode observers is presented and the limitations of these methods are discussed. The problem posed is, under what circumstances can a sliding mode observer be designed which provides quadratic stability of the state error system despite the presence of matched uncertainty? A new framework for designing such observers is proposed which can be viewed as a special case of the regular form. Necessary and sufficient conditions for the attainment of the canonical form are established which rely only on the computation of certain system invariants. The relationship between this approach and previous work in this area is then considered. It is emphasised that the constraints imposed by previous workers are encompassed within the new framework. It is stressed that it is more straightforward to establish the validity of the system invariants which are required for the framework to be attained than to establish the validity of the structural constraints imposed by the previous workers. At the end of the chapter non-trivial numerical examples are given to demonstrate the efficacy of the design method. The chapter is based on the material in the paper by Edwards & Spurgeon [22].

Chapter 4 considers the problem of formulating a robust output tracking controller which requires only plant output information. It was stated at the outset that the underlying assumption of the thesis is that only output information is available. Therefore it is pertinent to consider the regulation only of the outputs rather than the internal states. The control law considered is a modification of the one proposed in [11] which incorporates output demand following requirement through the use of integral action. In order to realize this controller all internal states must be available to the control law. To circumvent this difficulty, a sliding mode observer of the type considered in Chapter 3 will be used to generate estimates of the unavailable states for use in the control law. The key issue is therefore to establish the stability of the combined plant and observer system. In the literature very little work has been published in this area. Usually, in such a situation, sliding mode controllers have been implemented via linear high gain asymptotic observers [80]. The chapter undertakes a rigorous closed loop analysis of the sliding mode observer and sliding mode controller pair. Using Lyapunov techniques it is demonstrated that provided the system requirements of Chapter 3 are satisfied, quadratic stability of the overall dynamics is obtained. In this situation no loss in performance results from the restriction that only plant output information is available. In addition it is demonstrated that the observer and controller pair can be designed independently.

Chapter 5 examines from a different perspective the problem of output tracking using only plant output information . The approach in this chapter is to synthesise sliding surfaces and corresponding sliding mode controllers which require only output information. This obviates the necessity to design an observer and results in a much less computationally intense 'static' control system. At face value, the problem considered for most of the chapter is that of sliding mode output feedback stabilization. At the end of the chapter it is demonstrated that by use of a model-reference framework the output feedback stabilization results are pertinent to the design of an output tracking controller. The two issues associated with the design of sliding mode schemes, namely the hyperplane design and the realization of a controller, are addressed. It is demonstrated that with minor modifications the framework introduced in Chapter 3 can be used to great effect in this context also. It enables the class of systems considered by other workers in this field to be extended and provides an explicitly realizable controller which requires no additional assumptions. Several non-trivial examples are given which demonstrate the numerical tractability of the approach. The theoretical results obtained in this chapter have been accepted for publication in the International Journal of Control.

Chapter 6 describes the temperature modelling work carried out to simulate a single burner furnace operating with a fixed fuel/air ratio. Heat transfer at high temperature within an enclosure is dominated by radiation effects. The chapter describes the *Zone Method* approach to modelling radiation exchange attributed to Hottel & Cohen [35]. Since its inception this approach has remained almost unchallenged with regard to accuracy of solution. A description of the radiation concepts required is given – particularly the different types of exchange areas, leading up to the definition of *directed flux areas.* These quantities are the crucial components of the Zone method. A general algorithm for computing the solution to unsteady radiation exchange in enclosures is outlined. This forms the basis of the furnace simulation. The remainder of the chapter discusses in detail the modifications made to tailor the simulation to specifically represent the experimental furnace at the Gas Research Centre. In particular, details of the actuator and valve characteristic are given, together with a description of the circulatory cooling water arrangement. The end result is a simulation which gives reasonably good agreement with the available furnace data and provides an ideal test-bed for evaluating the performance of different control laws.

Chapter 7 documents the work undertaken to design sliding mode temperature controllers. The nonlinear simulation described in the previous chapter was used to obtain a linear model representing the dynamics relating valve position to temperature. As a result of the method used to obtain the linear model, the internal states do not have any physical meaning and consequently only output information is available to the control law; the theoretical results of Chapters 4 and 5 are consequently applicable for the purpose of control law design. The design procedure is described step by step for the case of the robust controller/observer pair and for the input/output model reference approach. The results obtained from using the two controllers on the simulation are compared.

Finally, and more importantly, the results of trials of the controller/observer pair implemented on the experimental furnace at the Gas Research Center are presented. For comparison, the results from a commercially available auto-tuned PID controller are given. The sliding mode controller is demonstrated to be comparable to the PID in terms of performance in the nominal case, and shown to possess greater robustness when the furnace setup is deliberately and significantly perturbed.

Chapter 8 describes a multivariable scheme for controlling both temperature and excess oxygen in a single burner furnace. The inputs to the system are considered to be the fuel valve position and the oxygen trim signal for the ERC – the control objective being to manipulate these values so that both the temperature and excess oxygen follow a specified profile. The furnace simulation described in Chapter 6 is not appropriate for controller synthesis because it assumes a fixed fuel/air ratio. Instead a nominal linear model obtained directly from the furnace using a system identification approach is employed. The chapter discusses the design procedure and presents the results of successful trials implementing the control law on the experimental rig.

Chapter 9 summarises the contributions of the thesis together with recommendations for future work.

Chapter 2

Variable Structure Control Systems with Sliding Modes

2.1 Introduction

Variable Structure Control Systems evolved from the pioneering work in Russia of Emel'yanov and Barbashin in the early 1960's. The ideas did not appear outside of Russia until the mid 1970's when a book by Itkis [40] and a survey paper by Utkin [78] were published in English. Variable Structure Systems concepts have subsequently been utilized in the design of robust regulators [60], model-reference systems [93, 72], adaptive schemes [38], tracking systems [11, 67] and state observers [80, 83]. The ideas have successfully been applied to problems as diverse as automatic flight control, control of electrical motors, chemical processes, helicopter stability augmentation, space systems and robotics.

Variable Structure Control Systems, as the name suggests, are a class of systems whereby the 'control law' is deliberately changed during the control process according to some defined 'rules' which depend on the state of the system. For the purpose of illustration consider the double integrator given by

$$\ddot{y}(t) = u(t) \tag{2.1}$$

Using the feedback control law

$$u(t) = -ky(t) \qquad\qquad k > 0 \tag{2.2}$$

results in simple harmonic motion characterised by an elliptical phase portrait which may be considered to be marginally stable.

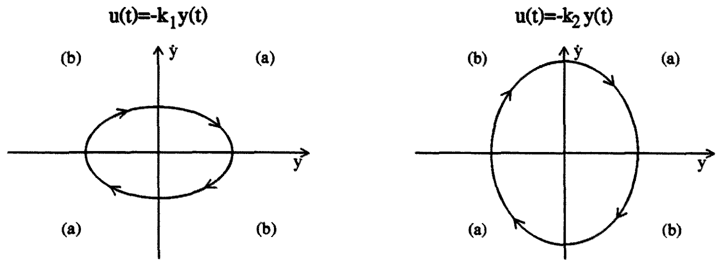


Figure 2.1: Phase Portraits of Simple Harmonic Motion

Consider instead the control law

$$u^*(t) = \begin{cases} -k_2 y(t) & \text{if } y\dot{y} > 0 \\ -k_1 y(t) & \text{otherwise} \end{cases} \tag{2.3}$$

where $k_2 > 1 > k_1 > 0$. The phase plane $(y, \dot{y})$ is partitioned by the switching rule into four quadrants separated by the axes as shown in Figure 2.1. This system clearly fits the description of a Variable Structure Control System given earlier. The control law $u = -k_2 y$ will be in effect in the quadrants of the phase plane labelled (a). In this region the distance from the origin of the points in the phase portrait decrease along the system trajectory. Likewise, in region (b) when the control law $u = -k_1 y$ is in operation, the distance from the origin of the points in the phase portrait also decreases. The phase portrait for the closed loop system under the variable structure control law u^* is obtained by splicing together the appropriate regions from the two phase portraits in Figure 2.1. An asymptotically stable motion results as shown in Figure 2.2. By introducing a rule for switching between two control structures, which independently do not provide stability, an asymptotically stable system has been obtained.

A more significant example results from using the control law

$$u^*(t) = \begin{cases} -1 & \text{if } s(y, \dot{y}) > 0 \\ 1 & \text{if } s(y, \dot{y}) < 0 \end{cases} \tag{2.4}$$

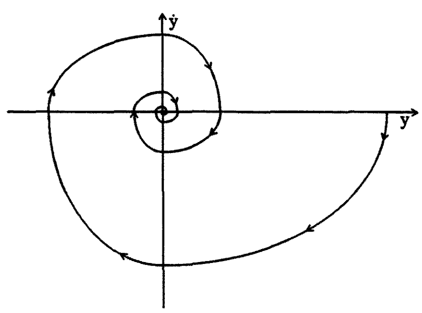


Figure 2.2: Phase Portrait of the System under VSC

where the *switching function* is defined by

$$s(y, \dot{y}) = my + \dot{y} \qquad m > 0 \tag{2.5}$$

The reason for the use of the term switching function is clear, since $s(\cdot)$ is used for deciding which control structure is in use at point $(y, \dot{y})$ in the phase plane. For large values of $\dot{y}$ the phase portrait, obtained from joining the parabolic components of the constituent laws, is shown in Figure 2.3.

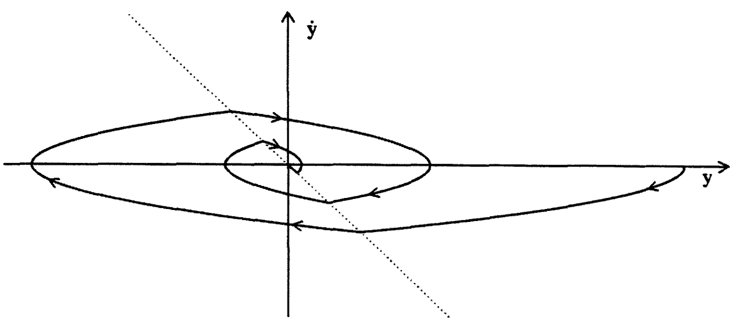


Figure 2.3: Phase Portrait of the System for large $\dot{y}$

However for values of $\dot{y}$ satisfying the inequality $m|\dot{y}| < 1$ then

$$s\dot{s} = s\,(m\dot{y} + \ddot{y}) = s\,(m\dot{y} - \text{sgn}(s)) < -|s|\,(1 - m|\dot{y}|) < 0$$

or equivalently

$$\lim_{s \to 0^+} \dot{s} < 0 \qquad \text{and} \qquad \lim_{s \to 0^-} \dot{s} > 0$$

Consequently when $m|\dot{y}| < 1$ the system trajectories on either side of the line

$$\mathcal{L}_s = \{(y, \dot{y}) \ : \ s(y, \dot{y}) = 0\} \tag{2.6}$$

point towards the line. Intuitively high frequency switching between the two control structures will take place as the system trajectories repeatedly cross the line $\mathcal{L}_s$. This high frequency motion is described as *chattering*. If infinite frequency switching were possible, intuitively at least, the motion would be trapped or constrained to remain on the line $\mathcal{L}_s$. The motion when confined to the line $\mathcal{L}_s$ satisfies the first order differential equation obtained from rearranging $s(y, \dot{y}) = 0$, namely

$$\dot{y}(t) = -my(t) \tag{2.7}$$

This represents a first order decay and the trajectories will 'slide' along the line $\mathcal{L}_s$ to the origin (Figure 2.4). Such dynamical behaviour is described as an *ideal sliding mode* and the line $\mathcal{L}_s$ is termed the *sliding surface*. During the sliding motion the system behaves as a reduced order free motion with all the control effort expended in ensuring that $s(y, \dot{y}) = 0$. The reduced order dynamics can be seen to depend only on the choice of the gradient of the line $\mathcal{L}_s$. It should be noted that the control action required to bring about such a motion is discontinuous, and on the sliding surface is not even defined.

If a new variable

$$x \stackrel{s}{=} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

is introduced then equation (2.1) can be written in *state-space* form as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.8}$$

The switching function can also be conveniently expressed in matrix terms as

$$s(y, \dot{y}) = \begin{bmatrix} m & 1 \end{bmatrix} x(t) \tag{2.9}$$

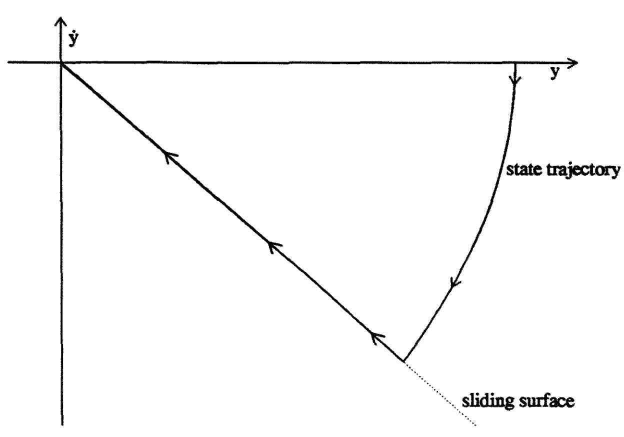


Figure 2.4: Phase Portrait of a Sliding Motion

This suggests that differential equations written in state-space form constitute a natural framework in which to explore the properties of Variable Structure Control Systems for multi-input linear systems. Without any apparent increase in complexity consider the nth order linear time invariant system with m inputs given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.10}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $1 \leq m \leq n$. Without loss of generality it can be assumed that the input distribution matrix B has full rank. Define a switching function $s : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be the linear map $s(x) = Sx$ where $S \in \mathbb{R}^{m \times n}$ is of full rank and let $\mathcal{S}$ be the hyperplane defined by

$$\mathcal{S} = \{x \in \mathbb{R}^n \ \ : \ \ s(x) = 0\} \tag{2.11}$$

Suppose $u^*(s(x), x)$ represents a Variable Structure Control law where the changes in control strategy depend on the value of the switching function. The control law considered as a map $u^* : x \mapsto u^*(x)$ is therefore discontinuous. It is natural to explore the possibility of choosing the control action and selecting the switching strategy so that an ***ideal sliding motion*** takes place on the hyperplane, i.e. there exists a time t_s such that

$$Sx(t) = 0 \qquad \text{for all } t > t_s \tag{2.12}$$

For the purpose of this thesis, the definition of a sliding motion given above will suffice. A more general notion and a more rigorous definition of sliding is given in DeCarlo [16].

Before proceeding any further, a fundamental question must be answered regarding the dynamical behaviour described as sliding. It was noted that the control action viewed as a function $x \mapsto u^*(x)$ is discontinuous which implies the differential equation describing the closed loop system has a discontinuous 'right hand side'. From a rigorous mathematical view point the classical theory of differential equations is not applicable since Lipschitz conditions are usually invoked to guarantee the existence of a unique solution. From a physical point of view the existence of a unique well defined solution is of fundamental importance. In practice an ideal sliding motion is not attainable - imperfections such as delays, hysteresis and unmodelled dynamics will result in a chattering motion in a neighbourhood of the sliding surface. Such a system will usually fall within the scope of classical differential equation theory. The ideal sliding motion can therefore be thought of as the limiting solution obtained as the imperfections diminish. A formal discussion along these lines appears in [80]. The solution concept proposed by Filippov [28] for differential equations with discontinuous right hand sides is also described in [80]. A rigorous approach for a certain class of Variable Structure Controllers is given by Leitmann in [47]. A more recent approach using the theory of differential inclusions is taken by Ryan [59]. Probably the most intuitively appealing approach is the method of 'equivalent control' proposed by Utkin [78] which is described below.

2.2 The Method of Equivalent Control

This section describes a method of establishing the control action necessary to maintain a sliding motion on $\mathcal{S}$, and the equations representing the dynamical behaviour of the states when constrained to the surface. At this point it should be stressed that the method about to be described is not confined to linear systems and hyperplanes, but since these are representative of the systems and surfaces that will be considered in the subsequent chapters, it is convenient to restrict the exposition to these objects. More general nonlinear systems and sliding surfaces are considered in the tutorial paper by DeCarlo *et al.*[16] and by Utkin in [80].

Suppose at time $t = t_s$ the systems states lie on the surface $\mathcal{S}$ and an ideal sliding motion takes place. This can be expressed mathematically as $Sx(t) = 0$ and $\dot{s}(x) = S\dot{x}(t) = 0$ for all $t \geq t_s$. Substituting for $\dot{x}(t)$ from (2.10) gives

$$S\dot{x}(t) = SAx(t) + SBu(t) = 0 \qquad \text{for all } t \geq t_s \tag{2.13}$$

Suppose the matrix S is designed so that the square matrix SB is nonsingular[1]. The *equivalent control*, written as u_{eq}, is defined to be the unique solution to the algebraic equation (2.13), namely

$$u_{eq}(t) = -(SB)^{-1}SAx(t) \tag{2.14}$$

The ideal sliding motion is then given by substituting the expression for the equivalent control into equation (2.10) which results in a free motion

$$\dot{x}(t) = \left(I_n - B(SB)^{-1}S\right)Ax(t) \qquad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \tag{2.15}$$

The linear operator $P_s \stackrel{s}{=} (I_n - B(SB)^{-1}S)$ is a *projection operator* (see Zinober [91]) and in particular satisfies

$$SP_s = 0 \qquad \text{and} \qquad P_sB = 0 \tag{2.16}$$

The system matrix governing the sliding motion P_sA therefore belongs to $\mathrm{N}(S)$ and consequently the sliding motion is of reduced order. Alternatively the ideal sliding motion can be written in the form

$$\dot{x}(t) = (A - BK)x(t) \qquad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \tag{2.17}$$

where $K \stackrel{s}{=} (SB)^{-1}SA$. The equivalent control can be considered to be the linear state feedback component necessary to maintain the reduced order motion.

From equation (2.15) it can be seen that the sliding motion depends on the choice of sliding surface, although the precise effect is not readily apparent. A convenient way to shed light on the problem is to first transform the system into a suitable canonical form. In this form the system is decomposed into two connected subsystems, one acting in $\mathrm{R}(B)$ and the other in $\mathrm{N}(S)$. Since by assumption the $\mathrm{rank}(B) = m$ there exists an orthogonal matrix $T \in \mathbb{R}^{n \times n}$ such that

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2.18}$$

[1]The singular case when $\det(SB) = 0$ is considered in Utkin [80].

where $B_2 \in \mathbb{R}^{m\times m}$ and is nonsingular[2]. Let $z = Tx$ and partition the new coordinates so that

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{2.19}$$

where $z_1 \in \mathbb{R}^{n-m}$ and $z_2 \in \mathbb{R}^m$. The nominal linear system (2.10) can then be written as

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) \tag{2.20}$$

$$\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + B_2u(t) \tag{2.21}$$

which is referred to as *regular form*. Equation (2.20) is referred to as describing the *null-space dynamics* and equation (2.21) as describing the *range-space dynamics*. Supposing the matrix defining the hyperplane (in the new coordinate system) is compatibly partitioned as

$$\tilde{S} = ST^{\mathrm{T}} = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \qquad \text{where } S_1 \in \mathbb{R}^{m\times(n-m)} \text{ and } S_2 \in \mathbb{R}^{m\times m}$$

then $\det(SB) = \det(S_2)\det(B_2)$. Therefore necessary and sufficient conditions for the matrix SB to be nonsingular is that $\det(S_2) \neq 0$. By design assume this to be the case. The hyperplane matrix can therefore be written as

$$\tilde{S} = S_2 \begin{bmatrix} M & I_m \end{bmatrix} \tag{2.22}$$

where $M \stackrel{s}{=} S_2^{-1}S_1$. During an ideal sliding motion

$$S_1z_1(t) + S_2z_2(t) = 0 \qquad \text{for all } t > t_s \tag{2.23}$$

and therefore formally expressing $z_2(t)$ in terms of $z_1(t)$ and substituting for $z_2(t)$ in equation (2.20) gives

$$\dot{z}_1(t) = (A_{11} - A_{12}M)\, z_1(t) \tag{2.24}$$

It can be seen that S_2 has no direct effect on the dynamics of the sliding motion and acts only as a scaling factor for the switching function.

In the context of designing a regulator the matrix $\tilde{A}_{11} \stackrel{s}{=} A_{11} - A_{12}M$ must have stable eigenvalues. The hyperplane design problem can therefore be considered to be one of

[2]In practice such a matrix can be found by 'QR' decomposition (see [92])

choosing a state feedback matrix M to stabilize the reduced order system (A_{11}, A_{12}). Because of the special structure of the regular form, and using the fact that $\det(B_2) \neq 0$, it follows that

$$\begin{aligned} \text{rank}\begin{bmatrix} sI-A & B \end{bmatrix} &= \text{rank}\begin{bmatrix} sI-A_{11} & -A_{12} & 0 \\ -A_{21} & sI-A_{22} & B_2 \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} sI-A_{11} & A_{12} \end{bmatrix} + m \qquad \text{for all } s \in \mathbb{C} \end{aligned}$$

This implies

$$\text{rank}\begin{bmatrix} sI-A & B \end{bmatrix} = n \Leftrightarrow \text{rank}\begin{bmatrix} sI-A_{11} & A_{12} \end{bmatrix} = n-m$$

and from the PBH rank test[3] it follows that (A, B) is controllable if and only if the pair (A_{11}, A_{12}) is controllable. Therefore provided the original pair (A, B) is controllable then any robust linear state feedback method can be applied to designing M. Several approaches have been proposed including quadratic minimisation [81], eigenvalue placement in a region [86] and eigenstructure assignment methods [92].

For the system given in regular form, it can easily be verified that

$$A_{eq} \stackrel{s}{=} P_s A = \begin{bmatrix} A_{11} & A_{12} \\ -MA_{11} & -MA_{12} \end{bmatrix} \tag{2.25}$$

and furthermore

$$\begin{bmatrix} A_{11} & A_{12} \\ -MA_{11} & -MA_{12} \end{bmatrix} \equiv \begin{bmatrix} I & 0 \\ -M & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -M & I \end{bmatrix}^{-1}$$

which implies $\lambda(A_{eq}) = \lambda(\tilde{A}_{11}) \cup \{0\}^m$. Let λ_i be a nonzero eigenvalue of A_{eq} with a corresponding right eigenvector v_i. Then from equation (2.16) it follows that

$$SA_{eq} = 0 \Rightarrow SA_{eq}v_i = 0 \Rightarrow \lambda_i S v_i = 0 \Rightarrow Sv_i = 0$$

and the eigenvectors of the eigenvalues describing the sliding motion belong to $\mathrm{N}(S)$. This fact is used in the design of hyperplanes using eigenvector methods.

Up to this point the concept of sliding modes and sliding surfaces has been introduced in a rigorous way for multi-input linear systems. The relationship between the choice

[3]For details of the PBH rank test for controllability see A.3 in Appendix A.

of sliding surface and the resulting sliding motion has been examined and an explicit relationship has been identified which is convenient from the design viewpoint. Two issues of importance however have not yet been addressed:

- no details have been given relating to possible controller design which guarantee the existence of a sliding motion;
- the effect that uncertainties or disturbances may have on a system when using a variable structure control law.

The next section considers the latter issue and outlines the robustness properties obtained as a result of the additional complexity introduced by the switching strategy.

2.3 Invariance Conditions

Consider the *uncertain linear system* given by

$$\dot{x}(t) = Ax(t) + Bu(t) + D\xi(x,t) \tag{2.26}$$

where the matrix $D \in \mathbb{R}^{n \times l}$ and the function $\xi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^l$ can be thought of as representing uncertainty in the known matrices A and B, or alternatively, as an exogenous disturbance acting on the system. Suppose a controller exists which induces a sliding motion on the surface $\mathcal{S}$ despite the presence of $\xi(\cdot)$. If at time t_s the states lie on $\mathcal{S}$ and subsequently remain there, then arguing as in §2.2, the control action necessary to maintain such a motion is given by

$$u_{eq}(t) = -(SB)^{-1}\left(SAx(t) + SD\xi(t,x)\right) \qquad \text{for } t \geq t_s \tag{2.27}$$

It should be noted that this equivalent control action is dependent on the exogenous signal. Since this control law is never realized in practice however, this dependence does not present any difficulty. Substituting for (2.27) in the uncertain system (2.26) it follows that the sliding motion satisfies

$$\dot{x}(t) = P_s Ax(t) + P_s D\xi(x,t) \qquad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \tag{2.28}$$

Suppose $\mathrm{R}(D) \subset \mathrm{R}(B)$ then there exists a matrix $R \in \mathbb{R}^{m \times l}$ such that $D = BR$. As a result it follows that $P_s D = P_s BR = 0$ since $P_s B = 0$ by the projection property described in equation (2.16). Therefore

$$\dot{x}(t) = P_s A x(t) \qquad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \tag{2.29}$$

and the signal $\xi(\cdot)$ does not appear in the equation representing the sliding motion. Hence the reduced order motion is insensitive to disturbances occurring in $\mathrm{R}(B)$. This condition was originally formulated by Draženović [21]. In the literature any disturbances or uncertainty which acts in the $\mathrm{R}(B)$ is referred to as *matched uncertainty.* This invariance property with respect to matched uncertainty makes Variable Structure Systems a powerful tool for controlling uncertain systems and is the motivation for the continuing research interest in the area.

Any residual uncertainty is described as *unmatched* and appears in the null space dynamic equation (2.24). Clearly the unmatched uncertainty will impact directly on the sliding motion and its effect will depend solely on the inherent disturbance rejection properties of the closed loop matrix $\tilde{A}_{11}$.

The effect of unmatched uncertainty will be explored in due course but first sufficient conditions for the existence of a sliding mode will be derived. This is vital to the design of the Variable Structure Control System since the robustness property outlined above will exist only if a sliding motion can be maintained.

2.4 Conditions for the Existence of Sliding Mode

From the definition of ideal sliding given in §2.1 it is clear that in the neighbourhood of S the system trajectories must be directed towards the surface, i.e. the surface must be locally stable. As a result if $s_i(x)$ is the ith component of the switching function then the problem can be viewed as one of stability of the 'states' $(s_1, s_2, .., s_m)$. One approach to the problem of determining the stability of (nonlinear) systems is that of Lyapunov[4]. The method obviates the need to obtain an analytical solution to the differential equation describing the system when assessing the stability properties. Using

[4]A brief review of Lyapunov stability ideas is given in Appendix B.

a Lyapunov type approach, a sufficient condition for the existence of a sliding domain is given by :

Theorem 2.1 *For the domain $\mathcal{D} \subset \mathcal{S}$ to be the domain of a sliding mode it is sufficient that in some region $\Omega \subset \mathbb{R}^n$ where $\mathcal{D} \subset \Omega$ there exists a continuously differentiable scalar function $V : \mathbb{R}_+ \times \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfying*

1) $V(t,x,s)$ is positive definite with respect to s i.e. $V(t,x,s) > 0$ if $s \neq 0$ for all $x \in \Omega$, and on the spheres $\|s\| = r$ for all $x \in \Omega$

- $\inf_{\|s\|=r} V(t,x,s) = h_r$ *and* $h_r > 0$ *for* $r \neq 0$
- $\sup_{\|s\|=r} V(t,x,s) = H_r > 0$

where h_r and H_r depend on r.

2) The total time derivative of $V(t,x,s)$ has a negative supremum for every $x \in \Omega$ except at points on the switching surface where the control action may be undefined and hence the derivative of $V(t,x,s)$ does not exist.

Proof (see Utkin [78]) ∎

The simplest Lyapunov function is the quadratic form

$$V(s) = \sum_{i=1}^{m} s_i^2(x) \tag{2.30}$$

This is clearly positive definite with respect to $\mathcal{S}$ and satisfies the supremum and infimum conditions. The total time derivative is given by

$$\dot{V} = 2\sum_{i=1}^{m} s_i \dot{s}_i$$

and therefore a sufficient condition for the derivative to be strictly decreasing is that

$$s_i \dot{s}_i < 0 \qquad \text{for } i = 1, 2, ..., m \tag{2.31}$$

This condition is the one most often cited in the literature when demonstrating the existence of a sliding mode. Indeed, certain control law design methods are based on selecting the control structure so that the condition (2.31) is satisfied. These and other control design methods are discussed in the next section.

2.5 Variable Structure Control Law Design

Early controller designs for single input systems were built around the relay structure given in the first section with the discontinuous element being multiplied by either a fixed scalar, or a scalar function of the states, so that the condition (2.31) is satisfied. For example consider the (uncertain) double integrator system

$$\ddot{y}(t) = u(t) + \xi(t, y, \dot{y}) \tag{2.32}$$

where the exogenous signal $\xi(\cdot)$ is unknown but bounded by a known constant k. Define a switching function $s(y, \dot{y}) = my + \dot{y}$ where $m > 0$ and let the Variable Structure Control law be given by

$$u^*(t) = -\rho \ \mathrm{sgn}(s) \tag{2.33}$$

where $\rho = k + r$ for some positive scalar r. Then

$$s\dot{s} = s(m\dot{y} + \ddot{y}) = s\,(m\dot{y} + \xi - \rho \ \mathrm{sgn}(s)) < -|s|\,(\rho - k - |m\dot{y}|) = -|s|(r - m|\dot{y}|)$$

and therefore in the domain $\Omega = \{(y, \dot{y}) \ : \ |m\dot{y}| < r\}$ the condition for the existence of a sliding mode (2.31) is satisfied. A sliding motion will take place governed by a first order decay which is insensitive to the disturbance $\xi(\cdot)$. The region in which sliding occurs, which is defined by the scalar r, is sometimes referred to as the *sliding patch* [65]. More generally for single input systems in phase canonic form, necessary and sufficient conditions on the magnitude of the gains required to ensure the existence of a sliding motion, are given in [78].

The first control laws for multivariable systems were extensions of single input ideas. If u_i represents the ith component of the control vector then

$$u_i = g_i(x) \ \mathrm{sgn}(s_i) \tag{2.34}$$

where $s_i(\cdot)$ represents the ith component of the switching function. The scalar functions $g_i(\cdot)$ are selected to ensure the sliding condition given in (2.31) is satisfied. The design process is greatly simplified if the hyperplane is chosen so that the matrix SB is diagonal and so there is no 'interaction' between the control components. Earlier it was shown that $SB = S_2 B_2$ and that the matrix $S_2 \in \mathbb{R}^{m \times m}$ had no effect on the sliding motion dynamics. Therefore if $S_2 \stackrel{s}{=} \Lambda B_2^{-1}$ for some diagonal design matrix $\Lambda \in \mathbb{R}^{m \times m}$ it follows that $SB = \Lambda$. This 'diagonalization' method and the so called 'hierarchical' design procedure are described in [16].

A more recent and more convenient control structure for multivariable systems is that proposed by Ryan & Corless [60]. They consider an uncertain system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t,x) + g(t,x,u) \tag{2.35}$$

where the functions $f : \mathbb{R} \times \mathbb{R}^n \to (\mathcal{R}(B))^\perp$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathcal{R}(B)$ are <u>unknown</u> but bounded and satisfy

$$\|f(t,x)\| \leq k_1\|x\| + k_2 \tag{2.36}$$

$$\|g(t,x,u)\| \leq k_3\|u\| + \alpha(t,x) \tag{2.37}$$

where $k_1, k_2, k_3 \geq 0$ are known constants with $k_3 < 1/\|B_2^{-1}\|$ and $\alpha(\cdot)$ is a known strongly Carathéodory function. These assumptions are required to rigorously guarantee the existence of a solution. For details and definitions see [60]. Here by definition, the functions $f(\cdot)$ and $g(\cdot)$ represent the unmatched and matched uncertainty components respectively. After changing coordinates with respect to T defined in (2.18) it follows that

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + \tilde{f}(t,x) \tag{2.38}$$

$$\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + B_2u(t) + \tilde{g}(t,x,u) \tag{2.39}$$

and because T is orthogonal

$$\|\tilde{f}(t,z)\| \leq k_1\|z\| + k_2 \tag{2.40}$$

$$\|\tilde{g}(t,z,u)\| \leq k_3\|u\| + \tilde{\alpha}(t,z) \tag{2.41}$$

Suppose the matrix M defining the hyperplane in equation (2.22) has been chosen by some appropriate design procedure. Define a nonsingular linear change of coordinates by

$$T_\phi \stackrel{s}{=} \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \tag{2.42}$$

If new coordinates are defined as

$$\begin{bmatrix} z_1 \\ \phi \end{bmatrix} = T_\phi \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{2.43}$$

where z_1 and z_2 are defined as in equation (2.19) then

$$\dot{z}_1(t) = \tilde{A}_{11}z_1(t) + A_{12}\phi(t) + \tilde{f}(t,z) \tag{2.44}$$

$$\dot{\phi}(t) = \tilde{A}_{21}z_1(t) + \tilde{A}_{22}\phi(t) + B_2u(t) + \tilde{g}(t,z,u) + M\tilde{f}(t,z) \tag{2.45}$$

where $\tilde{A}_{11} = A_{11} - A_{12}M$, $\tilde{A}_{21} = M\tilde{A}_{11} + A_{21} - A_{22}M$ and $\tilde{A}_{22} = MA_{12} + A_{22}$. The control law proposed by Ryan & Corless comprises two components; a linear component to stabilize the nominal linear system; and a discontinuous component. Specifically

$$u(t) = u_l(t) + u_n(t) \tag{2.46}$$

where the linear component is given by

$$u_l(t) \stackrel{s}{=} B_2^{-1}\left(-\tilde{A}_{21}z_1(t) - (\tilde{A}_{22} - A_{22}^s)\phi(t)\right) \tag{2.47}$$

where $A_{22}^s \in \mathbb{R}^{m\times m}$ is any stable design matrix. The nonlinear component is defined to be

$$u_n(t) \stackrel{s}{=} -\rho(t,z)B_2^{-1}\frac{P_2\phi(t)}{\|P_2\phi(t)\|} \qquad \text{for } \phi(t) \neq 0 \tag{2.48}$$

where $P_2 \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix satisfying the Lyapunov equation

$$P_2A_{22}^s + (A_{22}^s)^{\mathrm{T}}P_2 = -I \tag{2.49}$$

and the scalar function $\rho(t,z)$, which depends only on the magnitude of the uncertainty, is any function satisfying

$$\rho(t,z) \geq (\|M\|(k_1\|z\| + k_2) + k_3\|u_l\| + \tilde{\alpha}(t,z) + \gamma_2)/(1 - k_3\|B_2^{-1}\|) \tag{2.50}$$

where $\gamma_2 > 0$ is a design parameter. It can easily be shown that any function satisfying equation (2.50) also satisfies

$$\rho(t,z) \geq \|M\tilde{f}(t,z)\| + \|\tilde{g}(t,z,u)\| + \gamma_2 \tag{2.51}$$

and is therefore greater in magnitude than the unmatched uncertainty occurring in equation (2.45). Also, it follows that the quantity $\phi(t) = 0 \Leftrightarrow Sx(t) = 0$ and therefore

$$\mathcal{S} = \{(z_1, \phi) \in \mathbb{R}^n \;\; : \;\; \phi = 0\} \tag{2.52}$$

As a consequence the unit vector control law (2.48) is defined everywhere except on the hyperplane $\mathcal{S}$.

Remarks

- In the single input case the *unit vector* component $\frac{P_2\phi}{\|P_2\phi\|} = \text{sgn}(\phi)$ when $\phi \neq 0$ and the control structure (2.48) becomes the scaled relay structure (2.34) described earlier.
- In the absence of uncertainty, and using only the linear component of the control action, the closed-loop poles of the linear system are given by $\lambda(\tilde{A}_{11}) \cup \lambda(A_{22}^s)$.

The next subsection demonstrates that the control law (2.46) induces a sliding motion on $\mathcal{S}$ in finite time despite the presence of uncertainty.

2.5.1 Existence of a Sliding mode

Substituting the control law (2.46) into (2.45) and simplifying, results in the system

$$\begin{aligned} \dot{z}_1(t) &= \tilde{A}_{11}z_1(t) + A_{12}\phi(t) + \tilde{f}(t,x) && (2.53)\\ \dot{\phi}(t) &= A_{22}^s\phi(t) - \rho(t,x)\tfrac{P_2\phi(t)}{\|P_2\phi(t)\|} + \tilde{g}(t,z,u) + M\tilde{f}(t,z) \quad \text{for } \phi(t) \neq 0 && (2.54)\end{aligned}$$

Choosing $V(\phi) = \phi^{\mathrm{T}}P_2\phi$ and differentiating it can be verified that the total time derivative satisfies

$$\begin{aligned} \dot{V}(\phi) &\leq -\|\phi\|^2 - 2\rho(t,z)\|P_2\phi\| + 2\phi^{\mathrm{T}}P_2\left(\tilde{g}(t,z,u) + M\tilde{f}(t,z)\right)\\ &\leq -\|\phi\|^2 - 2\|P_2\phi\|\left(\rho(t,z) - \|g(t,z,u)\| - \|M\tilde{f}(t,z)\|\right)\\ &\leq -\|\phi\|^2 - 2\gamma_2\|P_2\phi\| && (2.55)\end{aligned}$$

except on $\mathcal{S}$. Therefore from Theorem 2.1 the surface $\mathcal{S}$ is the domain of a sliding mode. Furthermore, a sliding motion is attained in finite time because

$$\|P_2\phi\|^2 = (P_2^{\frac{1}{2}}\phi)^{\mathrm{T}}P_2(P_2^{\frac{1}{2}}\phi) \;\geq\; \lambda_{min}(P_2)\|P_2^{\frac{1}{2}}\phi\|^2 = \lambda_{min}(P_2)V(\phi) \qquad (2.56)$$

and so from inequalities (2.55) and (2.56)

$$\dot{V} \leq -2\gamma_2\sqrt{\lambda_{min}(P_2)}\sqrt{V} \qquad (2.57)$$

Integrating equation (2.57) implies that the time taken to reach the sliding surface $\mathcal{S}$ denoted by t_s satisfies

$$t_s \leq \gamma_2^{-1}\sqrt{V(\phi_0)/\lambda_{min}(P_2)} \qquad (2.58)$$

where ϕ_0 represents the initial value of $\phi(t)$ at $t = 0$. All that remains is to investigate the dynamical behaviour when confined to the hyperplane $\mathcal{S}$.

2.5.2 Description of the Sliding Motion

The equation representing the motion when confined to the sliding surface is obtained by substituting $\phi = 0$ into equation (2.53) giving

$$\dot{z}_1(t) = \tilde{A}_{11} z_1(t) + \tilde{f}(t, z) \tag{2.59}$$

For single input systems the eigenvalues of $\tilde{A}_{11} = A_{11} - A_{12}M$ uniquely define the row vector M but for multivariable systems the additional degrees of freedom exist. These degrees of freedom can be used to specify the eigenstructure to make the eigenvectors of $\tilde{A}_{11}$ as insensitive to disturbances as possible[5]. More generally, Lyapunov ideas can be used to investigate the robustness of this subsystem with respect to the disturbance. Let the matrix $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ be the unique symmetric positive definite solution to the Lyapunov equation

$$P_1 \tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}} P_1 = -Q_1 \tag{2.60}$$

where $Q_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ is a symmetric positive definite design matrix. Let the function $V(z_1) = z_1^{\mathrm{T}} P_1 z_1$ be a candidate quadratic Lyapunov function for (2.59). Taking the total time derivative along the system trajectories gives

$$\begin{aligned} \dot{V}(z_1) &= -z_1^{\mathrm{T}} Q_1 z_1 + 2 z_1^{\mathrm{T}} P_1 \tilde{f}(t, z) \\ &\leq -z_1^{\mathrm{T}} Q_1 z_1 + 2\|P_1 z_1\| \|\tilde{f}\| \\ &\leq -\lambda_{min}(Q_1)\|z_1\|^2 + 2\lambda_{max}(P_1)\|z_1\|\|\tilde{f}\| \\ &= -\|z_1\|\lambda_{max}(P_1)\left(\mu\|z_1\| - 2\|\tilde{f}\|\right) \end{aligned}$$

where $\mu \stackrel{s}{=} \lambda_{min}(Q_1)/\lambda_{max}(P_1)$. Therefore if

$$\|\tilde{f}(t, z)\| < \tfrac{1}{2}\mu\|z_1\| \tag{2.61}$$

then $\dot{V}(z_1) < 0$ and the system (2.59) is quadratically stable. Ideally the symmetric positive definite matrix Q_1 should be chosen to maximize the value of the scalar μ.

[5]This was one of the hyperplane design procedures identified in §2.2; for details see [20].

Patel & Toda [57] show that the maximum value is given by $\hat{\mu} = 1/\lambda_{max}(P_1)$ when the design matrix $Q_1 = I$. Furthermore

$$\hat{\mu} \leq -2 \max \left[\text{Re } \lambda(\tilde{A}_{11})\right] \tag{2.62}$$

with equality if the matrix $\tilde{A}_{11}$ is normal[6]. This agrees with the eigenstructure hyperplane design methodology which seeks to synthesise a matrix $\tilde{A}_{11}$ whose eigenvectors are as close to orthogonal as possible.

When dealing with uncertain systems, it may not be possible to guarantee asymptotic stability. A more practical concept is that of *ultimate boundedness* with respect to some bounded set $\mathcal{E} \subset \mathbb{R}^n$ (see Appendix B). Essentially, ultimate boundness ensures that in finite time the solution enters the set $\mathcal{E}$ and remains there for all subsequent time. Usually the set $\mathcal{E}$ is an acceptably small neighbourhood of the origin. This concept is often termed *practical stability* [60].

During the sliding motion $z_2 = -M_1 z_1$ and therefore $\|z\| \leq (\sqrt{1 + \|M\|^2})\|z_1\|$. Consequently the bound on the unmatched uncertainty (2.40) can be written as

$$\|\tilde{f}(t,z)\| < \tilde{k}_1\|z_1\| + k_2 \tag{2.63}$$

where $\tilde{k}_1 = k_1\sqrt{1 + \|M\|^2}$. Then arguing as before, and assuming $2\tilde{k}_1 < \hat{\mu}$, it follows that

$$\begin{aligned} \dot{V}(z_1) &\leq -\|z_1\|\lambda_{max}(P_1)\left(\hat{\mu}\|z_1\| - 2\|\tilde{f}\|\right) \\ &\leq -\|z_1\|\lambda_{max}(P_1)\left(\hat{\mu}\|z_1\| - 2\tilde{k}_1\|z_1\| - 2k_2\right) \end{aligned}$$

and consequently $\dot{V}(z_1) < 0$ if $z_1 \notin \mathcal{E}_1$ where

$$\mathcal{E}_1 = \{z_1 \in \mathbb{R}^{(n-m)} \ : \ \|z_1\| < 2k_2/(\hat{\mu} - 2\tilde{k}_1) + \epsilon\} \tag{2.64}$$

for some small scalar $\epsilon > 0$. Therefore the states $z_1(\cdot)$ are ultimately bounded with respect to the ellipsoid $\mathcal{E}_1$.

An alternative approach for analysing stability is considered by Spurgeon & Davies [70]. A different uncertainty structure is assumed to that of (2.40) – (2.41) namely

$$\begin{aligned} \tilde{f}(t,z) &= \tilde{f}_1(t,z)z_1 + \tilde{f}_2(t,z) && (2.65) \\ \tilde{g}(t,z,u) &= \tilde{g}_1(t,z,u)u + \tilde{g}_2(t,z) && (2.66) \end{aligned}$$

[6]A matrix $N \in \mathbb{R}^{n \times n}$ is said to be *normal* if it has n orthonormal eigenvectors $\Leftrightarrow$ $NN^* = N^*N$.

where the unknown vector functions $\tilde{f}_2(\cdot)$ and $\tilde{g}_2(\cdot)$ are bounded and the square matrices $\tilde{f}_1(\cdot)$ and $\tilde{g}_1(\cdot)$ are uncertain and time varying. In particular $\tilde{f}_1(\cdot)$ satisfies the *quadratic stability constraint*

$$P_1\tilde{f}_1(t,z) + \tilde{f}_1(t,z)^{\mathrm{T}}P_1 < (1-\nu)I_{n-m} \tag{2.67}$$

for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$ for some scalar $\nu > 0$. Then arguing as above

$$\dot{V}(z_1) \leq -\nu\|z_1\|^2 + 2\lambda_{max}(P_1)\|\tilde{f}_2\|\|z_1\|$$

and ultimate boundedness can be established for the states z_1.

The guaranteed attainment of a sliding mode in finite time requires that in the control law (2.50) the parameter $\gamma_2 > 0$. As a consequence the control action is discontinuous across $\mathcal{S}$. Unfortunately in practice the implementation of such a control law would produce a chattering motion in a boundary of the surface $\mathcal{S}$ rather than an ideal sliding motion. In certain situations such high frequency switching could excite unmodelled dynamics and therefore the control law would not be considered acceptable. Several modifications have been proposed to overcome this difficultly which are described in the next section.

2.6 Continuous Approximations

The most obvious approach is to 'soften' the discontinuity in the control law. Slotine & Sastry [67] and many other workers utilize a *boundary layer* approach whereby the discontinuous component is replaced by the *continuous* nonlinear approximation

$$u_n^\delta(t) = \begin{cases} -\rho(t,x)B_2^{-1}\frac{P_2\phi(t)}{\|P_2\phi(t)\|} & \text{if } \|P_2\phi\| \geq \delta \\ -\delta^{-1}\rho(x,t)B_2^{-1}P_2\phi(t) & \text{otherwise} \end{cases} \tag{2.68}$$

The positive scalar δ defines the size of the boundary layer in which a high gain linear feedback control law operates. An ideal sliding motion will no longer take place and the system will not be totally invariant to matched uncertainty. However the range-space state $\phi(\cdot)$ will be uniformly bounded with respect to the ellipsoid

$$\mathcal{E}_b = \{\phi \in \mathbb{R}^m \;\; : \;\; \phi^{\mathrm{T}}P_2^2\phi < \delta^2\} \tag{2.69}$$

Although no ideal sliding takes place, the states $z_1(\cdot)$ remain ultimately bounded with respect to a neighborhood of the origin (albeit a larger one than before).

Ryan & Corless [60] use a variant on this *parallel boundary layer* approach and advocate the power law interpolation structure

$$u_n^\delta(t) = \begin{cases} -\rho(t,x)B_2^{-1}\|\eta(t,x)\|^{-1}\eta(t,x) & \|\eta(t,x)\| > \delta \\ -\rho(t,x)\delta^{q-1}B_2^{-1}\|\eta(t,x)\|^{-q}\eta(t,x) & 0 < \|\eta(t,x)\| \le \delta \\ 0 & \eta(t,x) = 0 \end{cases} \tag{2.70}$$

where $\eta(t,x) \stackrel{s}{=} \rho(t,x)P_2\phi(t)$, the design scalar $q \in [0\ 1)$ and as before δ is a small positive scalar. An alternative differentiable approximation is given by Burton & Zinober [6] and Spurgeon & Davies [70] who consider

$$u_n^\delta(t) = -\rho(t,x)B_2^{-1}\frac{P_2\phi(t)}{\|P_2\phi(t)\|+\delta} \tag{2.71}$$

where again δ is a small positive constant. As shown in Figure 2.5 the scalar δ does not

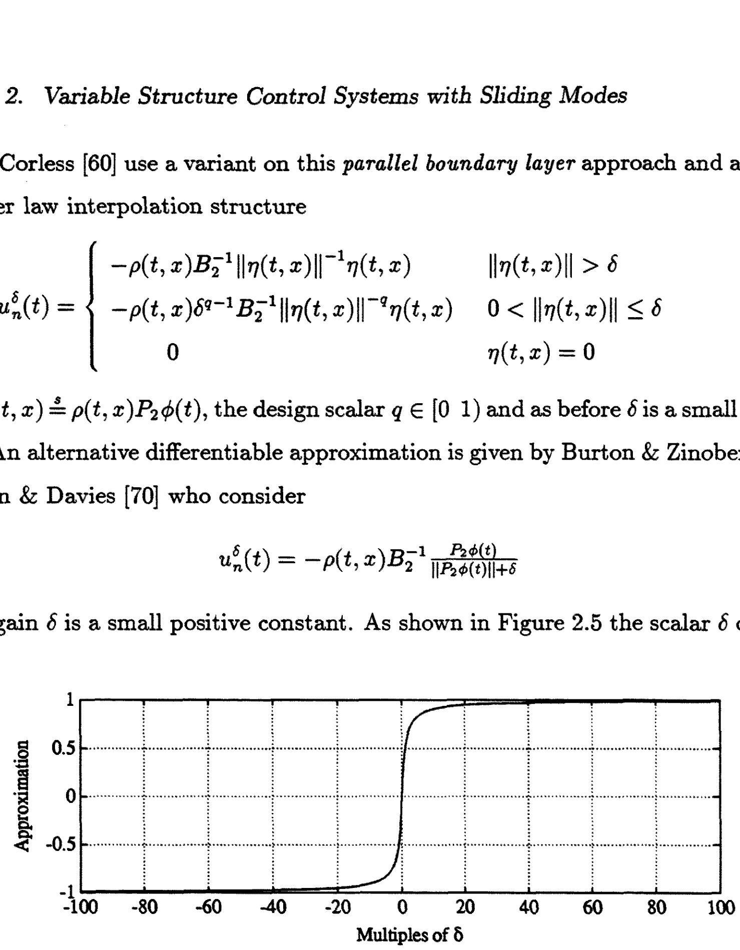


Figure 2.5: A differentiable approximation of the *signum* function

define the boundary layer (at $\|s\| = \delta$ the nonlinear control action is half of its maximum value), but as for the other parallel boundary techniques ultimate boundedness results can be demonstrated. The motion resulting from using the fractional approximation (2.71) is studied in detail by Burton & Zinober in [6] where, in particular, a formal examination of its chatter elimination properties is made. Other more complex approximations are proposed in [90] and [54].

A comparison of these different approaches to eliminate chattering (technically for the case of a tracking controller) is made by de Jager [15] . The steady state errors resulting from using the continuous approximations in the presence of a constant disturbance are compared. It is reported that no significant difference arises between the various methods to suggest using one method in favour of another. The effect of the introduction of integral action to remove steady state errors, which often occur as a result of the introduction of a boundary layer, is also examined.

2.7 Summary

A brief review of the key ideas associated with Variable Structure Control Systems has been presented. The notions of a sliding surface and a sliding mode have been introduced, together with sufficient conditions for their existence. These ideas are fundamental to the developments proposed in the chapters which follow. The exposition has been restricted to uncertain linear systems. Also only hyperplanes have been considered as potential sliding surfaces. Implicitly it has been assumed that all the states are available to the controller. This restriction is often cited as a limitation of the technique, since in practice, usually only certain 'outputs' are directly measurable. In the following chapters Variable Structure Control ideas will be examined for uncertain systems for which only outputs are available. In this situation, either the class of hyperplanes and controllers considered must be restricted to those requiring only the system output information, or alternatively estimates of the unavailable internal states must be generated for use by the control law. Both these possibilities are investigated.

Chapter 3

Sliding Mode Observers

3.1 Introduction

Many of the proponents of the theoretical developments in the area of sliding mode control systems have found it convenient to assume that the system state vector is available for use by the control scheme. In practice, it is not always possible or practical to measure the state vector and so, in order to exploit these control strategies, a suitable estimate of the state vector must be constructed for use in the original control law. The idea of using a dynamical system to generate estimates of the system – an *observer* – can be traced to Luenberger [50] who proposed a method for linear certain systems which now bears his name. Despite fruitful research and development activity in the area of variable structure control theory, relatively few authors have considered the application of the underlying principles to the problem of observer design in the case of uncertain/nonlinear systems. In the work of Utkin [79, 80] a discontinuous observer strategy is described whereby the error between the estimated and measured outputs is forced to exhibit a sliding mode. Dorling & Zinober [19] explore the practical application of this observer to an uncertain system and report difficulties in the selection of an appropriate switched gain which will cause the system to exhibit a sliding mode without excessive chattering. It should be noted that this study did not consider the application of a continuous approximation to the discontinuous observer action since the paper pre-dates most of the rigorous attempts at 'softening' the discontinuity described in § 2.6. Walcott & Żak [83, 84] use a Lyapunov based approach to formulate an observer design which under appropriate assumptions, exhibits asymptotic state error decay in the presence of bounded nonlinearities/uncertainties. In particular, this method seeks to render the observer error system totally insensitive to matched uncertainty. This approach is shown by Walcott *et al.*[82] to compare favourably with direct approaches to nonlinear observer designs when the nonlinearities present in the systems are assumed to be perfectly known. The strategy of Walcott & Żak, although intuitively appealing, necessitates the use of algebraic manipulation tools to effectively solve an associated constrained Lyapunov problem for systems of reasonable order.

This chapter seeks to build upon the existing contributions outlined above. A methodology is presented for determining the magnitude of the discontinuous gain required by the Utkin observer to ensure the existence of a sliding mode despite the presence of a class of bounded plant uncertainty. The proposed framework is then developed further to incorporate the effects of bounded matched or indeed other structured system uncertainty. This effectively considers the problem first posed by Walcott & Żak. However here the emphasis is placed upon the numerical tractability of the solution method. A different nonlinear structure is used which guarantees sliding on the manifold in the error space described by the null space of the input distribution matrix. Also the precise class of systems to which these results apply is explicitly identified. A number of tutorial design examples including robust observer designs for selected aerospace systems are presented.

3.2 Current Conceptions of Discontinuous Observers

Consider the uncertain dynamical system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(x,u,t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $p \geq m$; in addition the matrices B and C are assumed to be of full rank. The unknown function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ represents the system uncertainty. Since the approaches of different authors will be compared, different restrictions will be imposed on $f(\cdot)$, but loosely speaking the uncertainty is assumed to be bounded by known constants. The problem to be considered is that of reconstructing the state variables using only measured output information.

3.2.1 Utkin Observer

Consider initially the system described above under the added assumptions that the pair (A, C) is observable and $f(x, u, t) \equiv 0$. As the outputs are to be considered, it is logical to effect a change of coordinates so that the outputs appear as components of the states. Without loss of generality it can be assumed that the output distribution matrix can be written as

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{3.2}$$

where $C_1 \in \mathbb{R}^{p \times (n-p)}$, $C_2 \in \mathbb{R}^{p \times p}$ and $\det(C_2) \neq 0$. Consequently the transformation

$$T_c = \begin{bmatrix} I_{n-p} & 0 \\ C_1 & C_2 \end{bmatrix} \tag{3.3}$$

is nonsingular and with respect to this new coordinate system it can be seen that the new output distribution matrix $CT_c^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}$. If the other system matrices are written as

$$T_c A T_c^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \text{and} \qquad T_c B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

then the nominal system can be written as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}y(t) + B_1u(t) \\ \dot{y}(t) &= A_{21}x_1(t) + A_{22}y(t) + B_2u(t) \end{aligned} \tag{3.4}$$

where

$$\begin{bmatrix} x_1 \\ y \end{bmatrix} = T_c x$$

and $x_1 \in \mathbb{R}^{n-p}$. The observer proposed by Utkin [79] has the form

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_{11}\hat{x}_1(t) + A_{12}\hat{y}(t) + B_1u(t) + L\nu \\ \dot{\hat{y}}(t) &= A_{21}\hat{x}_1(t) + A_{22}\hat{y}(t) + B_2u(t) - \nu \end{aligned} \tag{3.5}$$

where $(\hat{x}_1, \hat{y})$ are the state estimates for (x_1, y), $L \in \mathbb{R}^{(n-p)\times p}$ is a constant feedback gain matrix and the discontinuous vector ν is defined component-wise by

$$\nu_i = M \ \mathrm{sgn}(\hat{y}_i - y_i) \qquad\qquad \text{for } M \in \mathbb{R}_+ \tag{3.6}$$

If the errors between the estimates and the true states are written as $e_1 = \hat{x}_1 - x_1$ and $e_y = \hat{y} - y$ then from equations (3.4) and (3.5) the following error system is obtained

$$\begin{aligned} \dot{e}_1(t) &= A_{11}e_1(t) + A_{12}e_y(t) + L\nu \\ \dot{e}_y(t) &= A_{21}e_1(t) + A_{22}e_y(t) - \nu \end{aligned} \tag{3.7}$$

Since the pair (A, C) is observable the pair (A_{11}, A_{21}) is also observable[1]. As a consequence L can be chosen to make the spectrum of $A_{11} + LA_{21}$ lie in $\mathbb{C}_-$. Define a further change of coordinates by

$$T_s = \begin{bmatrix} I_{n-p} & L \\ 0 & I_p \end{bmatrix} \qquad \text{and let} \qquad \begin{bmatrix} x'_1 \\ y \end{bmatrix} = T_s \begin{bmatrix} x_1 \\ y \end{bmatrix}$$

The error system with respect to these new coordinates can be written as

$$\dot{e'}_1(t) = A'_{11}e'_1(t) + A'_{12}e_y(t) \tag{3.8}$$

$$\dot{e}_y(t) = A_{21}e'_1(t) + A'_{22}e_y(t) - \nu \tag{3.9}$$

where $A'_{11} = A_{11} + LA_{21}$, $A'_{12} = A_{12} + LA_{22} - A'_{11}L$ and $A'_{22} = A_{22} - A_{21}L$. It can be shown using singular perturbation theory, that for large enough M, a sliding motion can be induced on the output error state in equation (3.9). It follows that after some finite time t_s, for all subsequent time, $e_y = 0$ and $\dot{e}_y = 0$. Equation (3.8) then reduces to

$$\dot{e'}_1(t) = A'_{11}e'_1(t) \tag{3.10}$$

which by choice of L represents a stable system and so $e'_1 \to 0$ as $t \to \infty$. Consequently $\hat{x}_1 \to x_1$ and the remaining states can be constructed in the original coordinate system as

$$\hat{x}_2 = C_2^{-1}(y - C_1\hat{x}_1) \tag{3.11}$$

The major practical difficulty in this approach is the selection of an appropriate scalar gain M to induce a sliding motion in finite time. Dorling & Zinober [19] report the need

[1] This is the dual result to the controllability problem discussed in §2.2.

to modify the gain parameter M during the time interval in order to reduce excessive chattering; it should be noted however that their study does not attempt to investigate the effect of a continuous approximation to ν. In addition, the above analysis does not formally address robustness issues.

3.2.2 A Modification to the Utkin Observer

Consider the effect of adding a negative output error feedback term to each equation of the Utkin Observer (3.5). This results in a new error system governed by

$$\begin{aligned} \dot{e'}_1(t) &= A'_{11}e'_1(t) + A'_{12}e_y(t) - G_1 e_y(t) \\ \dot{e}_y(t) &= A_{21}e'_1(t) + A'_{22}e_y(t) - G_2 e_y(t) - \nu \end{aligned} \tag{3.12}$$

By selecting $G_1 = A'_{12}$ and $G_2 = A'_{22} - A^s_{22}$, where A^s_{22} is any stable design matrix of appropriate dimension, then

$$\begin{aligned} \dot{e'}_1(t) &= A'_{11}e'_1(t) \\ \dot{e}_y(t) &= A_{21}e'_1(t) + A^s_{22}e_y(t) - \nu \end{aligned} \tag{3.13}$$

In this form the (nominal) error system is asymptotically stable for $\nu \equiv 0$ because the poles of the combined system are given by $\lambda(A'_{11}) \cup \lambda(A^s_{22})$ and so lie in the open left half complex plane. In the original Utkin observer, the switching action ν was potentially required to make the error system stable. The addition of a Luenberger type gain matrix, feeding back the output error, yields the potential to provide robustness against certain classes of uncertainty using the discontinuous component ν. This approach is conceptually similar to that proposed by Slotine *et al.*[65] in that the output errors are fed back in both a linear and discontinuous way. However the problem formulation is quite different since Slotine *et al.* propose an observer for the system $\dot{x} = f(x,t)$ and seek only to extend the region in which sliding takes place – the so-called 'sliding patch' – by incorporating a linear term. In this chapter an approach more akin to that of Walcott & Żak [83, 84] will be adopted whereby global error convergence will be sought for a class of uncertain systems.

3.2.3 Walcott & Żak Observer

The problem considered by Walcott & Żak [83] involves estimating the states of a system such as that described in equation (3.1) so that the error system is quadratically stable despite the presence of matched uncertainty. Walcott & Żak assume that

A1) the unknown function in (3.1) is of the form

$$f(x,u,t) = B\xi(x,t) \tag{3.14}$$

where the function $\xi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ is unknown, but bounded, so that

$$\|\xi(x,t)\| \leq \rho \qquad \forall x \in \mathbb{R}^n, t \geq 0$$

A2) there exists a $G \in \mathbb{R}^{n\times p}$ such that $A_0 = A - GC$ has stable eigenvalues and there exists a Lyapunov pair (P,Q) for A_0 such that the structural constraint

$$C^{\mathrm{T}}F^{\mathrm{T}} = PB \tag{3.15}$$

is satisfied for some $F \in \mathbb{R}^{m\times p}$.

Utilizing these assumptions they propose an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - G(C\hat{x}(t) - y(t)) + \nu \tag{3.16}$$

where $e = \hat{x} - x$ and

$$\nu = \begin{cases} -\rho\frac{P^{-1}C^{\mathrm{T}}F^{\mathrm{T}}FCe}{\|FCe\|} & \text{if } FCe \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

By using $V(e) = e^{\mathrm{T}}Pe$ as a candidate Lyapunov function, Walcott & Żak [83] show that $\dot{V}(e) < 0$ for $e \neq 0$, thus $e(t) \to 0$ exponentially and a sliding motion is induced on the surface in the state error space given by

$$\mathcal{S}_{wz} = \{\ e \in \mathbb{R}^n \ : \ FCe = 0\} \tag{3.18}$$

The crucial problem is therefore to compute the gain matrix G such that, for the resulting closed loop matrix A_0, there exists a matrix pair (F,P) satisfying :

- P is a Lyapunov matrix for A_0
- F satisfies the structural constraint $C^{\mathrm{T}}F^{\mathrm{T}} = PB$.

Steinberg & Corless [73] demonstrate that a sufficient condition for the existence of a Lyapunov matrix P which satisfies the constraint $C^{\mathrm{T}}F^{\mathrm{T}} = PB$ is that the modified transfer function $G_F(s) = FC(A_0 - sI)^{-1}B$ is *strictly positive real.*[2] The paper by Steinberg & Corless does not however offer a solution to the problem of finding a suitable G and F. An indication of the class of systems (A, B, C) for which this problem has a solution may be obtained from the observation that strictly positive real systems are both minimum phase and relative degree one (see for example [66]). These facts are <u>not</u> noted by Walcott & Żak who instead propose an algorithm for the design of P which can be summarised as follows :

1) Choose the spectrum of A_0, and compute G accordingly.
2) Solve the structural constraint *symbolically* to obtain an expression for P_F in terms of the entries of F, ensuring that P_F is symmetric.
3) Compute $Q(P_F)$ symbolically in terms of the entries of P_F using the expression $Q(P_F) = -(P_F A_0 + A_0^{\mathrm{T}} P_F)$.
4) Choose the elements of F and P_F to ensure $Q(P_F)$ is symmetric positive definite by checking that all the principal minors are positive.

From a computational point of view, the algebra associated with the manipulation and solution of the inequalities resulting from step 4 is impractical without the use of a symbolic manipulation package for systems of order 4 and above. Also no indication is given of the class of systems for which the algorithm will produce a successful design. The remainder of this chapter explores the potential for synthesising a design procedure for an observer of the type given in (3.16) which exploits the properties of strictly positive systems and obviates the need for the use of a symbolic manipulation package.

[2] A transfer function $G(s)$ is defined to be *strictly positive real* if it is stable and the hermitian matrix defined by $G(i\omega) + G^{\mathrm{T}}(-i\omega) > 0$ for all real ω.

3.3 Synthesis of a Discontinuous Observer

Consider the dynamical system given in (3.1) and suppose that following a linear change of coordinates with respect to a nonsingular matrix T the system can be written as

$$\begin{aligned} \dot{x}_1(t) &= \mathcal{A}_{11}x_1(t) + \mathcal{A}_{12}y(t) \\ \dot{y}(t) &= \mathcal{A}_{21}x_1(t) + \mathcal{A}_{22}y(t) + B_2u(t) + \xi \end{aligned} \tag{3.19}$$

where $x_1 \in \mathbb{R}^{(n-p)}$, $y \in \mathbb{R}^p$ and the matrix $\mathcal{A}_{11}$ has stable eigenvalues. The uncertain function $\xi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$ is unknown, but assumed to be bounded so that

$$\|\xi(x,u,t)\| \leq r_1\|u\| + \alpha(t,y) \qquad \forall y \in \mathbb{R}^p,\ u \in \mathbb{R}^m,\ t \geq 0$$

where r_1 is a known scalar and $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}_+$ is a known function. Consider an observer of the form

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \mathcal{A}_{11}\hat{x}_1(t) + \mathcal{A}_{12}\hat{y}(t) - \mathcal{A}_{12}e_y(t) \\ \dot{\hat{y}}(t) &= \mathcal{A}_{21}\hat{x}_1(t) + \mathcal{A}_{22}\hat{y}(t) + B_2u(t) - (\mathcal{A}_{22} - \mathcal{A}_{22}^s)e_y(t) + \nu \end{aligned} \tag{3.20}$$

The discontinuous vector ν is defined by

$$\nu = \begin{cases} -\rho(u,y)\frac{P_2e_y}{\|P_2e_y\|} & \text{if } e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

where the scalar function $\rho : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_+$ satisfies

$$\rho(u,y) > r_1\|u\| + \alpha(t,y) + \gamma_o \tag{3.22}$$

$P_2 \in \mathbb{R}^{p\times p}$ is symmetric positive definite and γ_o is a positive scalar. If the state estimation errors are defined as $e_1 = \hat{x}_1 - x_1$ and $e_y = \hat{y} - y$ then it is straightforward to show

$$\dot{e}_1(t) = \mathcal{A}_{11}e_1(t) \tag{3.23}$$

$$\dot{e}_y(t) = \mathcal{A}_{21}e_1(t) + \mathcal{A}_{22}^s e_y(t) + \nu - \xi \tag{3.24}$$

The lower block triangular structure has been shown to occur quite naturally as a result of the state-space representation chosen and the output error feedback gains employed.

Proposition 3.1 *There exist a family of symmetric positive definite matrices P_2 such that the uncertain dynamical error system above is quadratically stable.*

Proof

Let $Q_1 \in \mathbb{R}^{(n-p)\times(n-p)}$ and $Q_2 \in \mathbb{R}^{p\times p}$ be symmetric positive definite design matrices and define $P_2 \in \mathbb{R}^{p\times p}$ to be the unique symmetric positive definite solution to the Lyapunov equation

$$P_2\mathcal{A}_{22}^s + (\mathcal{A}_{22}^s)^{\mathrm{T}}P_2 \;=\; -Q_2$$

Using the computed value of P_2 define

$$\hat{Q} \;=\; \mathcal{A}_{21}^{\mathrm{T}}P_2Q_2^{-1}P_2\mathcal{A}_{21} + Q_1$$

and notice that $\hat{Q} = \hat{Q}^{\mathrm{T}} > 0$. Let $P_1 \in \mathbb{R}^{(n-p)\times(n-p)}$ be the unique symmetric positive definite solution to the Lyapunov equation

$$P_1\mathcal{A}_{11} + \mathcal{A}_{11}^{\mathrm{T}}P_1 \;=\; -\hat{Q}$$

Consider the quadratic form given by

$$V(e_1, e_y) = e_1^{\mathrm{T}}P_1e_1 + e_y^{\mathrm{T}}P_2e_y \tag{3.25}$$

as a candidate Lyapunov function. The derivative along the system trajectory can be shown to be

$$\dot{V} = -e_1^{\mathrm{T}}\hat{Q}e_1 + e_1^{\mathrm{T}}\mathcal{A}_{21}^{\mathrm{T}}P_2e_y + e_y^{\mathrm{T}}P_2\mathcal{A}_{21}e_1 - e_y^{\mathrm{T}}Q_2e_y + 2e_y^{\mathrm{T}}P_2\nu - 2e_y^{\mathrm{T}}P_2\xi \tag{3.26}$$

It is easy to verify that

$$\begin{aligned}&(e_y - Q_2^{-1}P_2\mathcal{A}_{21}e_1)^{\mathrm{T}}Q_2(e_y - Q_2^{-1}P_2\mathcal{A}_{21}e_1) \equiv \\ &\quad e_y^{\mathrm{T}}Q_2e_y - e_1^{\mathrm{T}}\mathcal{A}_{21}^{\mathrm{T}}P_2e_y - e_y^{\mathrm{T}}P_2\mathcal{A}_{21}e_1 + e_1^{\mathrm{T}}\mathcal{A}_{21}^{\mathrm{T}}P_2Q_2^{-1}P_2\mathcal{A}_{21}e_1\end{aligned} \tag{3.27}$$

Substituting the identity (3.27) into equation (3.26) and writing for notational convenience $(e_y - Q_2^{-1}P_2\mathcal{A}_{21}e_1)$ as $\tilde{e}_y$ then

$$\begin{aligned}\dot{V} &= -e_1^{\mathrm{T}}\hat{Q}e_1 + e_1^{\mathrm{T}}\mathcal{A}_{21}^{\mathrm{T}}P_2Q_2^{-1}P_2\mathcal{A}_{21}e_1 - \tilde{e}_y^{\mathrm{T}}Q_2\tilde{e}_y + 2e_y^{\mathrm{T}}P_2\nu - 2e_y^{\mathrm{T}}P_2\xi \\ &= -e_1^{\mathrm{T}}Q_1e_1 - \tilde{e}_y^{\mathrm{T}}Q_2\tilde{e}_y + 2e_y^{\mathrm{T}}P_2\nu - 2e_y^{\mathrm{T}}P_2\xi\end{aligned}$$

$$
\begin{aligned}
&= -e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}_y^{\mathrm{T}} Q_2 \tilde{e}_y - 2\rho(u,y)\|P_2 e_y\| - 2e_y^{\mathrm{T}} P_2 \xi \\
&\leq -e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}_y^{\mathrm{T}} Q_2 \tilde{e}_y - 2\rho(u,y)\|P_2 e_y\| + 2(r_1\|u\| + \alpha(y))\|P_2 e_y\| \\
&\leq -e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}_y^{\mathrm{T}} Q_2 \tilde{e}_y - 2\gamma_o\|P_2 e_y\| \\
&< 0 \qquad\qquad \text{for } (e_1, e_y) \neq 0
\end{aligned}
$$

and hence the error system is quadratically stable. ∎

Consider the hyperplane in the error space given by

$$\mathcal{S}_o = \{e \in \mathbb{R}^n \ : \ Ce = 0\} \tag{3.28}$$

then from the previous proposition

Corollary 3.1 *A sliding motion takes place on the surface $\mathcal{S}_o$ defined above.*

Proof

Consider the quadratic form

$$V_s(e_y) = e_y^{\mathrm{T}} P_2 e_y$$

as a potential Lyapunov function. From equation (3.24) and arguing as in Proposition 3.1 it follows that

$$
\begin{aligned}
\dot{V}_s &= -e_y^{\mathrm{T}} Q_2 e_y + 2e_y^{\mathrm{T}} P_2 \mathcal{A}_{21} e_1 + 2e_y^{\mathrm{T}} P_2(\nu - \xi) \\
&\leq 2\|P_2 e_y\|\|\mathcal{A}_{21} e_1\| - 2\gamma_o\|P_2 e_y\|
\end{aligned}
$$

In the domain $\Omega = \{(e_1, e_y) \ : \ \|\mathcal{A}_{21} e_1\| < \gamma_o\}$ it follows that $\dot{V}_s < 0$. From Theorem 2.1 a sliding motion takes place in the domain Ω, which, from Proposition 3.1, is entered in finite time. ∎

If $\hat{x}$ represents the state estimate for x and $e = \hat{x} - x$ then the robust observer can conveniently be written as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - G_l(Ce(t)) + G_n\nu \tag{3.29}$$

where

$$G_l = T^{-1}\begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{22} - \mathcal{A}_{22}^s \end{bmatrix} \qquad\qquad G_n = T^{-1}\begin{bmatrix} 0 \\ I_p \end{bmatrix} \tag{3.30}$$

and

$$\nu = \begin{cases} -\rho(u,y)\frac{P_2 Ce}{\|P_2 Ce\|} & \text{if } Ce \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

A methodology for determining the magnitude of the discontinuous observer gain has been provided. This can be implemented using any available numerical package for control systems design and ensures asymptotic stability of the error states for the proposed uncertainty class. In the discontinuous component given in equation (3.31) the unit vector structure has been pre-multiplied by a scalar function. In principle a more complicated diagonal matrix of gain functions could be employed to individually scale each of the channels.

The observer formulation (3.29)–(3.31) is different to that of Walcott & Żak [83] for the case when $p > m$ since their results guarantee sliding will take place on the surface in the error space given by $\{e \in \mathbb{R}^n \ : \ FCe = 0\}$. This does not imply $Ce = 0$ since the null space of F is nonempty and therefore the observer does not necessarily track the system outputs perfectly. In the above formulation this is guaranteed. The usefulness of Proposition 3.1 will depend on being able to identify the class of systems which can be placed in the canonical form of equation (3.19). This will be addressed in the following sections.

3.4 The Walcott & Żak Observer revisited

Consider the uncertain linear system (3.1) together with assumptions A1 and A2 given earlier in §3.2.3 . It will be shown that if a Walcott & Żak observer exists for a given system and, in particular if the Lyapunov matrix (3.15) is known, then this observer can be analysed within the framework developed in Proposition 3.1. It is necessary to introduce first a lemma which considers the effect of a change of basis on the structural constraint.

Lemma 3.1 *Let the system (A_0, B, C) be given with A_0 stable, and let $(\tilde{A}_0, \tilde{B}, \tilde{C})$ be related to (A_0, B, C) by a nonsingular similarity transformation T. If P is a Lyapunov matrix for A_0 which satisfies the structural constraint $C^{\mathrm{T}}F^{\mathrm{T}} = PB$ then the matrix*

$\tilde{P} \stackrel{s}{=} (T^{-1})^{\mathrm{T}} P T^{-1}$ *is a Lyapunov matrix for* $\tilde{A}_0$ *which satisfies the structural constraint* $\tilde{C}^{\mathrm{T}} F^{\mathrm{T}} = \tilde{P}\tilde{B}$

Proof By direct validation ∎

The main result of this section will now be proved :

Proposition 3.2 *Let* (A, B, C) *be a system for which there exists a Walcott & Żak observer characterized by the matrix pair* (P, F)*. Then there exists a nonsingular similarity transformation so that the triple with respect to the new coordinates* $(\bar{A}, \bar{B}, \bar{C})$ *and the observer pair* $(\bar{P}, F)$ *exhibits the following properties :*

1) $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$ *where* $\bar{A}_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ *and is stable*

2) $\bar{B} = \begin{bmatrix} 0 \\ \bar{P}_{22} F^{\mathrm{T}} \end{bmatrix}$ *where* $\bar{P}_{22} \in \mathbb{R}^{p\times p}$ *and is symmetric positive definite*

3) $\bar{C} = \begin{bmatrix} 0 & I_p \end{bmatrix}$

4) $\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}$ *where the matrices* $\bar{P}_1 \in \mathbb{R}^{(n-p)\times(n-p)}$ *and* $\bar{P}_2 \in \mathbb{R}^{p\times p}$*.*

Proof

Let (P, F) represent the nonlinear components of the observer and let G be an appropriate (Luenberger) gain matrix so that $A_0 = A - GC$ is stable. If the output distribution matrix is not in the form $C = \begin{bmatrix} 0 & I_p \end{bmatrix}$ apply transformation (3.3) in §3.2.1 and use Lemma 3.1 to calculate the observer pair in these new coordinates. For convenience assume this is already the case. Partition the Lyapunov matrix as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}$$

where $P_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $P_{12} \in \mathbb{R}^{(n-p)\times p}$ and $P_{22} \in \mathbb{R}^{p\times p}$. Because P is symmetric positive definite the sub-block P_{11} is also symmetric positive definite and in particular is nonsingular. Change coordinates with respect to the transformation

$$T = \begin{bmatrix} I_{n-p} & P_{11}^{-1} P_{12} \\ 0 & I_p \end{bmatrix}$$

In the new coordinate system

$$\bar{C} = CT^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}$$

and so the third property is satisfied.

Let

$$P^{-1} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$$

for appropriately defined sub-blocks. By assumption $B = P^{-1}C^{\mathrm{T}}F^{\mathrm{T}}$ and therefore

$$\begin{aligned} \bar{B} = TB &= \begin{bmatrix} I & P_{11}^{-1}P_{12} \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} \begin{bmatrix} 0 \\ I_p \end{bmatrix} F^{\mathrm{T}} \\ &= \begin{bmatrix} 0 \\ \tilde{P}_{22}F^{\mathrm{T}} \end{bmatrix} \qquad \text{since } P_{11}\tilde{P}_{12} + P_{12}\tilde{P}_{22} = 0 \end{aligned}$$

and so the second property is proved.

From Lemma 3.1 the matrix $\bar{P} = (T^{-1})^{\mathrm{T}}PT^{-1}$ is a Lyapunov matrix for $\bar{A}_0$ and by direct computation

$$\begin{aligned} \bar{P} &= \begin{bmatrix} I_{n-p} & 0 \\ -P_{12}^{\mathrm{T}}P_{11}^{-1} & I_p \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix} \begin{bmatrix} I_{n-p} & -P_{11}^{-1}P_{12} \\ 0 & I_p \end{bmatrix} \\ &= \begin{bmatrix} P_{11} & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \end{aligned}$$

where $\bar{P}_2 = P_{22} - P_{12}^{\mathrm{T}}P_{11}^{-1}P_{12}$ and so $\bar{P}$ has the block diagonal structure of property 4.

By definition $\bar{P}$ is a Lyapunov matrix for $\bar{A}_0$ and therefore

$$\bar{P}\bar{A}_0 + \bar{A}_0^{\mathrm{T}}\bar{P} = -\bar{Q} \tag{3.32}$$

for some $\bar{Q}$ which is symmetric positive definite. If $(\bar{A}_0)_{11}$ and $\bar{Q}_{11}$ are the top left sub-blocks of $\bar{A}_0$ and $\bar{Q}$ respectively then from equation (3.32) it follows that

$$P_{11}(\bar{A}_0)_{11} + (\bar{A}_0)_{11}^{\mathrm{T}}P_{11} = -\bar{Q}_{11} \tag{3.33}$$

where both P_{11} and $\bar{Q}_{11}$ are symmetric positive definite. Equation (3.33) represents a Lyapunov equation for $(\bar{A}_0)_{11}$ which is therefore stable. However, because of the structure of $\bar{C}$, if $\bar{A}$ is partitioned as in the proposition statement, it follows immediately that $\bar{A}_{11} = (\bar{A}_0)_{11}$ and the first property has been demonstrated. ∎

The above proposition has shown that given a system (A, B, C) for which there exists a Walcott & Żak Observer then there exists a nonsingular similarity transformation so that in these coordinates the system triple $(\bar{A}, \bar{B}, \bar{C})$ can be written as

$$\begin{aligned} \dot{x}_1(t) &= \mathcal{A}_{11}x_1(t) + \mathcal{A}_{12}y(t) \\ \dot{y}(t) &= \mathcal{A}_{21}x_1(t) + \mathcal{A}_{22}y(t) + \bar{B}_2u(t) + \bar{B}_2\xi \end{aligned} \tag{3.34}$$

where the matrix $\mathcal{A}_{11}$ is stable. This is identical to the uncertain system given in equation (3.19) and is thus appropriate for the design and analysis method presented in Proposition 3.1. The transformation used to obtain the canonical form was, however, given in terms of the Lyapunov matrix. Proposition 3.2 shows the feasibility of using the canonical form (3.19) but is not practical from the design point of view. The remainder of this chapter will determine the class of systems which fall within the design framework of Proposition 3.1.

3.5 Conditions for the existence of sliding mode observers

Let (A, B, C) represent the linear part of the uncertain system given in (3.1) where the matrices B and C of full rank and $p > m$. Consider the problem of constructing an observer for the uncertain system of the form

$$\dot{z}(t) = Az(t) + Bu(t) - G_lCe(t) + G_n\nu \tag{3.35}$$

where $e = z - x$, ν is discontinuous about the hyperplane $\mathcal{S}_o = \{e \in \mathbb{R}^n \ : \ Ce = 0\}$ and $G_l, G_n \in \mathbb{R}^{n \times p}$ are appropriate gain matrices. The purpose of this section is to determine the class of systems for which the observer (3.35) provides quadratic stability of the error system despite the presence of bounded matched uncertainty. Two lemmas will first be introduced – the first provides a canonical form for observer design and the second helps isolate the class of linear systems to which the results apply.

Lemma 3.2 *Let (A, B, C) be a linear system with $p > m$ and rank $(CB) = m$. Then a change of coordinates exists so that the triple with respect to the new coordinates $(\bar{A}, \bar{B}, \bar{C})$ has the following structure :*

a) *the system matrix can be written as*

$$\bar{A} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{211} & \\ & A_{22} \\ A_{212} & \end{array}\right] \tag{3.36}$$

where $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $A_{211} \in \mathbb{R}^{(p-m)\times(n-p)}$ and when partitioned have the structure

$$A_{11} = \begin{bmatrix} A_{11}^{o} & A_{12}^{o} \\ 0 & A_{22}^{o} \end{bmatrix} \quad \text{and} \quad A_{211} = \begin{bmatrix} 0 & A_{21}^{o} \end{bmatrix} \tag{3.37}$$

where $A_{11}^{o} \in \mathbb{R}^{r\times r}$ and $A_{21}^{o} \in \mathbb{R}^{(p-m)\times(n-p-r)}$ for some $r \geq 0$ and the pair (A_{22}^{o}, A_{21}^{o}) is completely observable.

b) *the input distribution matrix has the form*

$$\bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{3.38}$$

where $B_2 \in \mathbb{R}^{m\times m}$ is nonsingular.

c) *the output distribution matrix has the form*

$$\bar{C} = \begin{bmatrix} 0 & T_o \end{bmatrix} \quad \text{where } T_o \in \mathbb{R}^{p\times p} \text{ and is orthogonal} \tag{3.39}$$

Proof

A succession of linear transformations will be demonstrated which bring about the required canonical form. Define

$$T_c = \begin{bmatrix} N_c \\ C \end{bmatrix} \tag{3.40}$$

where $N_c \in \mathbb{R}^{(n-p)\times n}$ is any matrix whose rows span the null space of C. By construction T_c is nonsingular. If (A_c, B_c, C_c) are the state space matrices resulting from changing coordinates with respect to T_c then

$$C_c \stackrel{s}{=} CT_c^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}$$

Partition $B_c \stackrel{s}{=} T_c B$ in a compatible way so that

$$B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \qquad \text{where } B_{c1} \in \mathbb{R}^{(n-p)\times m} \text{ and } B_{c2} \in \mathbb{R}^{p\times m}$$

Then $CB = C_c B_c = B_{c2}$ and so by assumption $\text{rank}(B_{c2}) = m$. Hence in particular the square matrix $B_{c2}^{\mathrm{T}} B_{c2} \in \mathbb{R}^{m\times m}$ is nonsingular. Let $T_{22} \in \mathbb{R}^{p\times p}$ be any orthogonal matrix such that

$$T_{22} B_{c2} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad \text{where } B_2 \in \mathbb{R}^{m\times m} \text{ and is nonsingular} \tag{3.41}$$

Consequently the transformation

$$T_b = \begin{bmatrix} I_{n-p} & T_{12} \\ 0 & T_{22} \end{bmatrix} \qquad \text{where } T_{12} \stackrel{s}{=} -B_{c1}(B_{c2}^{\mathrm{T}} B_{c2})^{-1} B_{c2}^{\mathrm{T}} \tag{3.42}$$

is nonsingular. Let (A_b, B_b, C_b) be the state space matrices obtained following a change of coordinates with respect to T_b then

$$B_b = T_b B_c = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

and

$$C_c = C_c T_b^{-1} = \begin{bmatrix} 0 & T_{22}^{\mathrm{T}} \end{bmatrix}$$

Partition the matrix $A_b = T_b A_c T_b^{-1}$ as

$$A_b = \left[\begin{array}{c|c} A_{11}^b & A_{12}^b \\ \hline A_{211}^b & \\ & A_{22}^b \\ A_{212}^b & \end{array}\right]$$

where $A_{11}^b \in \mathbb{R}^{(n-p)\times(n-p)}$, $A_{211}^b \in \mathbb{R}^{(p-m)\times(n-p)}$ and $A_{22}^b \in \mathbb{R}^{p\times p}$. Examine the observability of the pair (A_{11}^b, A_{211}^b). Let $T_{obs} \in \mathbb{R}^{(n-p)\times(n-p)}$ be any matrix which puts (A_{11}^b, A_{211}^b) into the following observability canonical form

$$T_{obs} A_{11}^b T_{obs}^{-1} = \begin{bmatrix} A_{11}^{\circ} & A_{12}^{\circ} \\ 0 & A_{22}^{\circ} \end{bmatrix} \quad \text{and} \quad A_{211}^b T_{obs}^{-1} = \begin{bmatrix} 0 & A_{21}^{\circ} \end{bmatrix}$$

where $A_{11}^{o} \in \mathbb{R}^{r\times r}$, $A_{21}^{o} \in \mathbb{R}^{(p-m)\times(n-p-r)}$, the pair (A_{22}^{o}, A_{21}^{o}) is observable and $r \geq 0$ represents the number of unobservable states of $(A_{11}^{b}, A_{211}^{b})$. Finally let

$$T_a = \begin{bmatrix} T_{obs} & 0 \\ 0 & I_p \end{bmatrix} \tag{3.43}$$

and change coordinates to generate the matrices $(\bar{A}, \bar{B}, \bar{C})$. By construction the matrix $\bar{A} \stackrel{s}{=} T_a A_b T_a^{-1}$ has the required structure and

$$\bar{B} \stackrel{s}{=} T_a B_b = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \text{and} \quad \bar{C} \stackrel{s}{=} C_b T_a^{-1} = \begin{bmatrix} 0 & T_o \end{bmatrix}$$

where $T_o \stackrel{s}{=} T_{22}^{\mathrm{T}}$, which is the required canonical form. ∎

The proof of this lemma has been included since it is constructive in nature and will form the theoretical basis of an algorithm that will be developed to design sliding mode observers. In order to ensure compatibility in the partition of the state space matrices in the statement of Lemma 3.2 let

$$A_{21} = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} \tag{3.44}$$

where $\bar{B}_2 \in \mathbb{R}^{p\times m}$ is defined as

$$\bar{B}_2 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{3.45}$$

The next lemma identifies the relationship between the canonical form of Lemma 3.2 and the *invariant zeros* of the linear system.

Lemma 3.3 *Let (A, B, C) be a given system in the canonical form of Lemma 3.2 then the invariant zeros of the system are the r eigenvalues of A_{11}^{o}.*

Proof

Let $P(s)$ denote Rosenbrock's System Matrix

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

The invariant zeros of (A, B, C) are defined[3] to be

$$\{s \in \mathbb{C} \ : \ P(s) \text{ loses normal rank}\}$$

By assumption (A, B, C) is in the canonical form of Lemma 3.2 and therefore

$$\begin{aligned} P(s) \text{ loses rank } &\Leftrightarrow \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ -A_{21} & sI - A_{22} & \bar{B}_2 \\ 0 & -T_o & 0 \end{bmatrix} \text{ loses rank} \\ &\Leftrightarrow \begin{bmatrix} sI - A_{11} & 0 \\ -A_{21} & \bar{B}_2 \end{bmatrix} \text{ loses rank} \end{aligned}$$

since T_o is nonsingular. Substituting for A_{11} from (3.37) and for A_{21} from (3.44) and repartitioning gives

$$\begin{bmatrix} sI - A_{11} & 0 \\ -A_{21} & \bar{B}_2 \end{bmatrix} \equiv \left[\begin{array}{cc|c} sI - A_{11}^o & -A_{12}^o & \\ 0 & sI - A_{22}^o & 0 \\ 0 & -A_{21}^o & \\ \hline * & * & B_2 \end{array}\right]$$

where $*$ represent matrix sub-blocks which play no part in the analysis. Because B_2 is nonsingular, the expression on the right, and hence $P(s)$, loses rank if and only if

$$\begin{bmatrix} sI - A_{11}^o & -A_{12}^o \\ 0 & sI - A_{22}^o \\ 0 & -A_{21}^o \end{bmatrix} \text{ loses rank}$$

By construction the pair (A_{22}^o, A_{21}^o) is completely observable and therefore from the PBH observability rank test

$$\text{rank} \begin{bmatrix} sI - A_{22}^o \\ -A_{21}^o \end{bmatrix} = n - p - r \text{ for all } s \in \mathbb{C}$$

Therefore

$$P(s) \text{ loses rank } \Leftrightarrow \det(sI - A_{11}^o) = 0$$

and so the zeros of (A, B, C) are the eigenvalues of A_{11}^o. ∎

[3]For a definition of *invariant zeros* see MacFarlane & Karcanias [51].

The main result of this section (and the chapter) will now be proved

Proposition 3.3 *An observer of the form (3.35) providing asymptotic error decay in the presence of bounded matched uncertainty exists if and only if the nominal linear system satisfies*

- *rank*$(CB) = m$
- *any invariant zeros of* (A, B, C) *must be lie in* $\mathbb{C}_-$.

Proof

(proof of necessity)

Let G_l and G_n be appropriate gain matrices so that $A_0 = A - G_l C$ is stable, and a sliding mode insensitive to matched uncertainty exists on the hyperplane in the error space given by $\mathcal{S}_o = \{e \in \mathbb{R}^n \ : \ Ce = 0\}$. The error system satisfies

$$\dot{e}(t) = A_0 e(t) - B\xi(x,t) + G_n \nu \tag{3.46}$$

For a unique 'equivalent control' to exist $\det(CG_n) \neq 0$ and arguing as in §2.2 the sliding motion satisfies

$$\dot{e}(t) = \left(I - G_n(CG_n)^{-1}C\right) A_0 e(t) + \left(I - G_n(CG_n)^{-1}C\right) B\xi(x,t) \tag{3.47}$$

To be insensitive to the uncertainty it follows that

$$\left(I - G_n(CG_n)^{-1}C\right) B = 0$$

or equivalently

$$B = G_n(CG_n)^{-1}(CB) \tag{3.48}$$

It follows immediately from (3.48) that $\operatorname{rank}(CB) = m$ and therefore it can be assumed without loss of generality that the system (A, B, C) is in the canonical form given in Lemma 3.2. If the nonlinear gain matrix is partitioned so that

$$G_n = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad \text{where } G_1 \in \mathbb{R}^{(n-p)\times p} \text{ and } G_2 \in \mathbb{R}^{p\times p} \tag{3.49}$$

then $CG_n = T_o G_2$ and so $\det(G_2) \neq 0$. From equation (3.47) and using arguments similar to those presented in §2.2 it follows that the poles of the (linear) reduced order motion are given by

$$\lambda\left((A_0)_{11} - G_1 G_2^{-1} (A_0)_{21}\right) \tag{3.50}$$

where $(A_0)_{11}$ and $(A_0)_{21}$ represent the top-left and bottom-left sub-blocks of the closed loop matrix A_0 partitioned in a compatible way to the canonical form. By definition the matrix $A_0 = A - G_l C$ and so

$$(A_0)_{11} = A_{11} - (G_l C)_{11}$$

where $(G_l C)_{11}$ represents the top-left sub-block of the square matrix $G_l C$. However it is easy to check that $(G_l C)_{11} = 0$ for all $G_l \in \mathbb{R}^{n \times p}$ and so $(A_0)_{11} = A_{11}$. Similarly it can be shown that $(A_0)_{21} = A_{21}$ and consequently

$$\lambda\left((A_0)_{11} - G_1 G_2^{-1} (A_0)_{21}\right) = \lambda\left(A_{11} - G_1 G_2^{-1} A_{21}\right) \tag{3.51}$$

From equation (3.48) it follows that

$$G_1 G_2^{-1} \bar{B}_2 = 0$$

which after considering the structure of $\bar{B}_2$ implies

$$G_1 G_2^{-1} = \begin{bmatrix} \bar{G} & 0 \end{bmatrix} \quad \text{where } \bar{G} \in \mathbb{R}^{(n-p)\times(p-m)}$$

and therefore from the definition of A_{21} it follows that

$$A_{11} - G_1 G_2^{-1} A_{21} = A_{11} - \bar{G} A_{211}$$

By construction the pair (A_{11}, A_{211}) is such that

$$\{\text{zeros of } (A, B, C)\} = \lambda\left(A_{11}^{o}\right) \subset \lambda\left(A_{11} - \bar{G} A_{211}\right) \quad \text{for all } \bar{G} \in \mathbb{R}^{(n-p)\times(p-m)}$$

and therefore for a stable sliding motion any invariant zeros must lie in $\mathbb{C}_-$.

(proof of sufficiency)

Conversely let (A, B, C) represent the system and suppose rank $(CB) = m$ and any invariant zeros lie in $\mathbb{C}_-$. Without loss of generality it can be assumed that the system is already in the canonical form of Lemma 3.2 where the matrix A_{11}^{o} is stable. As a consequence there exists a matrix $L \in \mathbb{R}^{(n-p)\times(p-m)}$ such that $\bar{A}_{11} + L\bar{A}_{211}$ is stable. Define a nonsingular transformation as

$$\bar{T} = \begin{bmatrix} I_{n-p} & \bar{L} \\ 0 & T_o \end{bmatrix} \qquad \text{where } \bar{L} = \begin{bmatrix} L & 0_{(n-p)\times m} \end{bmatrix}$$

After changing coordinates with respect to $\bar{T}$, the new output distribution matrix becomes

$$\tilde{C} = C\bar{T}^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}$$

From the definition of $\bar{L}$ and $\bar{B}_2$

$$\bar{L}\bar{B}_2 = \begin{bmatrix} L & 0 \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = 0$$

and so the input distribution matrix is given by

$$\tilde{B} = \bar{T}B = \begin{bmatrix} \bar{L}\bar{B}_2 \\ T_o\bar{B}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ T_o\bar{B}_2 \end{bmatrix}$$

Finally it can be shown by direct evaluation that

$$(\tilde{A})_{11} = (\bar{T}\bar{A}\bar{T}^{-1})_{11} = \bar{A}_{11} + LA_{211}$$

which is stable by choice of L. The system triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is now in the canonical form (3.19) and from Proposition 3.1 a robust observer exists. ∎

3.6 Remarks

- In the special case where $m = p$ an observer of the form (3.35) which is insensitive to bounded matched uncertainty exists if and only if

  1) $\det(CB) \neq 0$

  2) the invariant zeros of (A, B, C) lie in $\mathbb{C}_-$.

  i.e. the triple (A, B, C) is minimum phase and relative degree one. In this case the restriction that $\det(CB) \neq 0$ guarantees the existence of exactly $n - p$ invariant zeros and therefore the reduced order sliding motion is totally determined by these zeros. This is in agreement with the results of El-Ghezawi *et al.*[23].

- The minimum phase relative degree one condition arises totally from the uncertainty class considered.

- For an observer of the form (3.35) to exist the nominal linear system need not be completely observable. For example, consider the second order system

$$A = \begin{bmatrix} -1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

  where $b_2 \neq 0$ and any uncertainty is bounded as in §3.3. This system is not completely observable but is minimum phase and relative degree one and so a robust sliding mode observer can be designed.

- Lemma 3.2 in conjunction with Proposition 3.3, yields an algorithm for the construction of a robust discontinuous observer which is suitable for implementation on any standard numerical matrix computation package such as MATLAB and does not require the use of symbolic manipulation.

- The canonical of Lemma 3.2 is a suitable form for designing a Walcott & Żak style observer because if $F^{\mathrm{T}} \stackrel{s}{=} P_2 B_2$ then it can be verified that the block diagonal Lyapunov matrix associated with the quadratic form in equation (3.25) satisfies the structural constraint (3.15). This will be discussed further in §4.4 where an analytic solution to the problem posed by Walcott & Żak will be given.

- It must be noted however, that at no time was it necessary to appeal to the fact that B represented the system input distribution matrix. Consequently the design procedure can be used to synthesise the construction of a robust observer for a system with structured uncertainty which can be expressed as

$$f(x, u, t) \quad = \quad \mathcal{B}\xi(x, u, t)$$

  where $\mathcal{B}$ is any matrix in $\mathbb{R}^{n \times l}$ $(l \leq p)$ and $\xi(\cdot)$ is bounded, by considering the 'system' $(A, \mathcal{B}, C)$. It is therefore possible to design an observer for a system which is non-minimum phase provided any invariant zeros of $(A, \mathcal{B}, C)$ lie in $\mathbb{C}_{-}$.

3.7 Selected Design Examples

3.7.1 Example 1 : Pendulum [Walcott & Żak]

The equations of motion in state space form for a pendulum, taken from Walcott & Żak [83], can be written as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1)\end{aligned} \tag{3.52}$$

Therefore

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the matched 'uncertain' bounded function $\xi(x_1, x_2, t) = -\sin(x_1)$. By design the output distribution matrix $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Consider the problem of finding a robust observer. Change coordinates with respect to

$$T_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

so that the output is a state variable. The system triple becomes

$$\tilde{C} = CT_c^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \tilde{B} = T_c B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \tilde{A} = T_c A T_c^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

which is in the canonical form (3.19). A robust observer exists for this system because $A_{11} = -1$ which is stable. Letting the design matrix $A_{22}^s = -1$, the feedback gain matrix

$$\tilde{G}_l = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and $\lambda(A_0) = \{-1, -1\}$. Defining $Q_2 = 2$ and solving the Lyapunov equation for A_{22}^s and Q_2 gives $P_2 = 1$. In the original coordinates the gain matrices become

$$\begin{aligned}G_l &= T_c^{-1}\tilde{G}_l = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ G_n &= T_c^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{aligned}$$

which is identical to the example given in [83].

3.7.2 Pendulum Simulation

Figure 3.1 demonstrates the nonlinear observer system tracking the output from the pendulum when the initial conditions of the true states and observer states are deliberately set to different values.

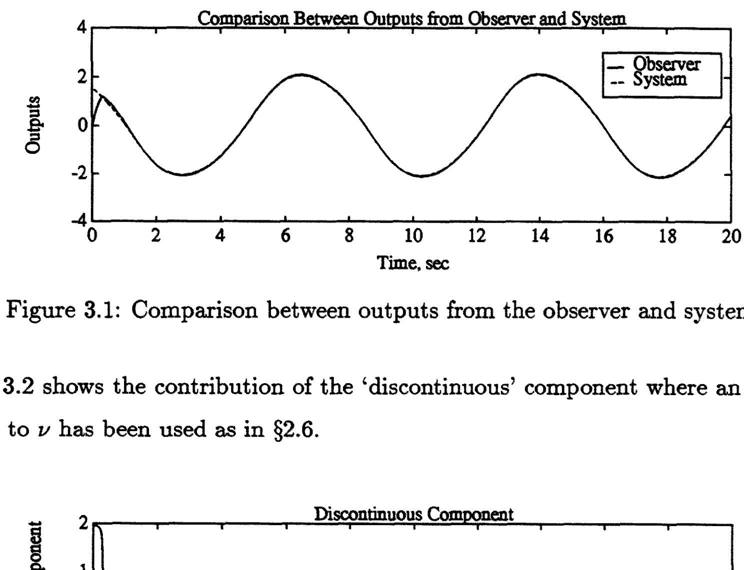


Figure 3.1: Comparison between outputs from the observer and system

Figure 3.2 shows the contribution of the 'discontinuous' component where an approximation to ν has been used as in §2.6.

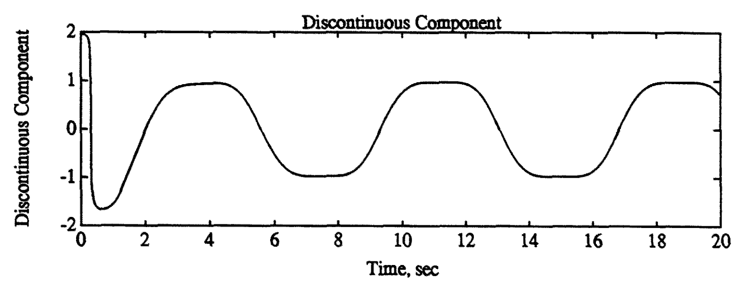


Figure 3.2: Discontinuous unit vector component

It can be seen from Figure 3.1 that after 0.5 seconds a 'sliding motion' takes place – i.e. the difference between the output of the pendulum and the output of the observer is zero. In this situation the true states are available for comparative purposes. A comparison of the true and estimated states is given in Figure 3.3. After approximately 4 seconds visually perfect replication of the states is taking place.

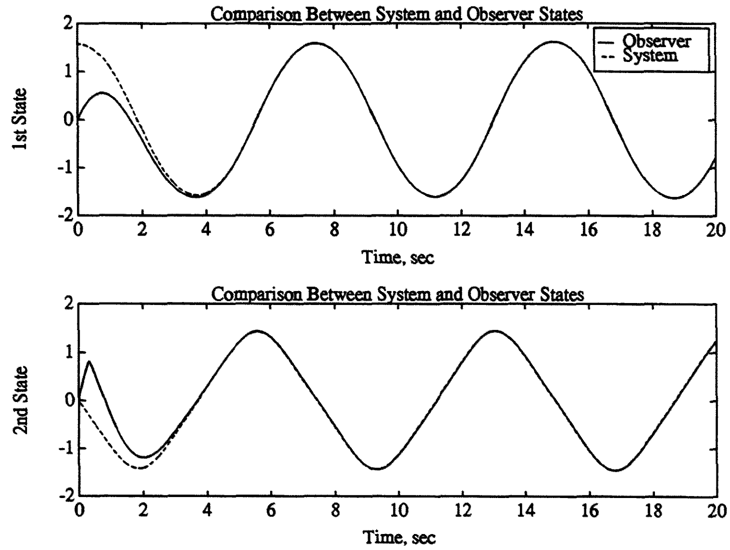


Figure 3.3: Comparison between system and observer states

If the nonlinear component is removed by setting ρ to zero, the resulting Luenberger Observer behaves as shown in Figure 3.4.

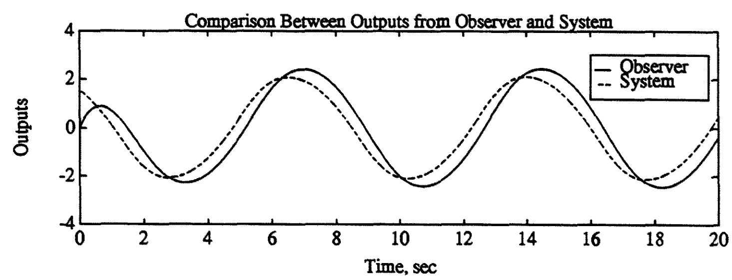


Figure 3.4: Comparison between outputs from the observer and system

There appears to be a distinct phase discrepancy between the outputs of the system and the outputs of the observer which is due to the presence of the nonlinear 'sine' term. In practice such an observer would probably be regarded as unacceptable. As expected from the poor output tracking exhibited in Figure 3.4, imprecise replication of the internal states takes place (Figure 3.5).

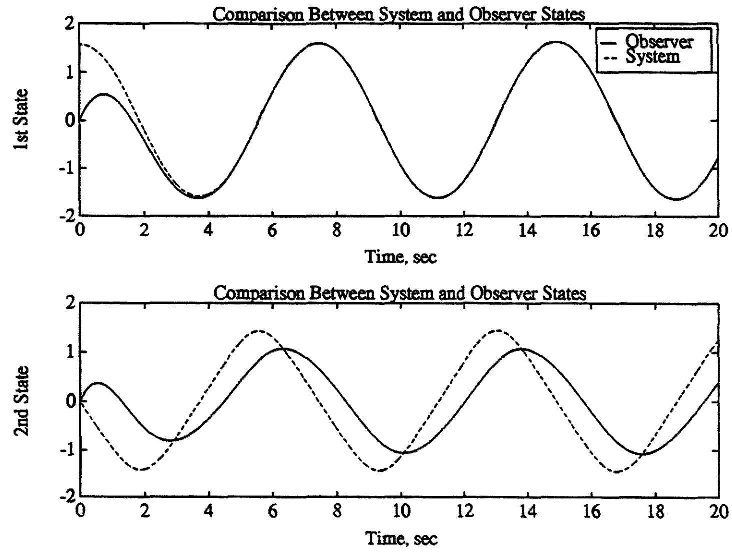


Figure 3.5: Comparison between system and observer states

3.7.3 Example 2 : L-1011 Fixed-Wing Aircraft [Sobel & Shapiro]

Consider the lateral axis model of a L-1011 at cruise flight conditions which appears in Sobel & Shapiro [69], with the actuator dynamics removed. The state vector is represented by

$$x = \begin{bmatrix} \phi \\ r \\ p \\ \beta \\ x_5 \end{bmatrix} \quad \begin{array}{l} \text{bank angle (rad)} \\ \text{yaw rate (rad/s)} \\ \text{roll rate (rad/s)} \\ \text{sideslip angle (rad)} \\ \text{washed out filter state} \end{array}$$

The system triple (A, B, C) is given by

$$A = \begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0 & -0.1540 & -0.0042 & 1.5400 & 0 \\ 0 & 0.2490 & -1.0000 & -5.2000 & 0 \\ 0.0386 & -0.9960 & -0.0003 & -0.1170 & 0 \\ 0 & 0.5000 & 0 & 0 & -0.5000 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ -0.7440 & -0.0320 \\ 0.3370 & -1.1200 \\ 0.0200 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The inputs are

$$u = \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \quad \begin{array}{l} \text{rudder deflection (rad)} \\ \text{aileron deflection (rad)} \end{array}$$

and the outputs

$$y = \begin{bmatrix} r_{wo} \\ p \\ \beta \\ \phi \end{bmatrix} \quad \begin{array}{l} \text{washed out yaw rate} \\ \text{roll rate (rad/s)} \\ \text{sideslip angle (rad)} \\ \text{bank angle (rad)} \end{array}$$

It can easily be checked that 'CB' is of full rank and a realization of (A, B, C) in the canonical form is given by

$$A = \left[\begin{array}{r|rrrr} -0.0133 & 0.0007 & 0.0172 & -0.2838 & -0.6266 \\ \hline 0 & 0 & -0.0008 & -0.9110 & 0.4125 \\ -0.7071 & 0.0386 & -0.0858 & 0.4167 & 0.9210 \\ -0.1227 & 0.0004 & 4.1018 & -0.8028 & 0.4471 \\ 0.1545 & 0.0009 & -3.5336 & 0.0580 & -0.8691 \end{array}\right]$$

from which the sub-matrices

$$A_{11} = \begin{bmatrix} -0.0133 \end{bmatrix} \quad \text{and} \quad A_{211} = \begin{bmatrix} 0 \\ -0.7071 \end{bmatrix}$$

can be identified. The pair (A_{11}, A_{211}) is trivially observable and so a robust observer exists.

Remark

Since the pair (A_{11}, A_{211}) is observable (or equivalently the system (A, B, C) has no invariant zeros), the poles of $A_{11} + LA_{211}$ can be placed arbitrarily, and there is no loss of design freedom in the placement of the poles of A_0.

3.8 Summary

The potential for robust control provided by discontinuous control action, particularly the total parameter insensitivity to and disturbance rejection of a particular class of uncertainty is widely reported. The possibilities presented by the underlying theory to the problem of observer design is, however, much less well explored. This chapter has outlined the major contributions to date in the area of discontinuous observer design and has further developed the existing ideas. An algorithm for the design of a robust discontinuous observer has been presented which only requires tools available in any standard commercially available numerical computer-aided control system design package. This algorithm contrasts with previous developments which required the use of an algebraic manipulation package to solve a constrained Lyapunov problem for systems of reasonable order. Also, the class of systems to which these results apply has been explicitly identified. Numerical examples which include a realistic fixed wing aircraft study demonstrates the practicality of the results. In practice such an observer would form part of a control strategy where the estimated states would be used in a state feedback control law. The next chapter examines the overall closed loop performance of the observer when used in conjunction with a robust output tracking state feedback control law.

Chapter 4

Output Tracking with a Sliding Mode Controller and Observer Pair

4.1 Introduction

In this chapter, a nonlinear controller/observer pair will be presented, based on sliding mode ideas, which provides robust output tracking of a reference signal using only measured output information. The proposed control law is essentially that of Davies & Spurgeon [11] which incorporates a demand following requirement through the use of integral action. The fundamental assumption in this existing contribution is that all the internal system states are available, which limits the practical applicability, and is contrary to the underlying thrust of this thesis. The approach adopted in this chapter, is to use the observer described previously to generate estimates of the unavailable internal states. This, of course, immediately raises the question of whether the resulting combined plant/observer dynamical system is robustly stable. The problem when sliding mode controllers are used in conjunction with sliding mode observers has not been studied extensively in the literature. Work analysing the closed loop stability when using a sliding mode controller and an asymptotic observer appears in the work of Utkin [80], Young & Kwatny [88] and in the paper Breinl & Leitmann [3]. In the latter, closed loop asymptotic stability is demonstrated when using the 'insensitive observer' described in Breinl & Muller [4]. In particular, the well known 'separation principle' for linear systems[1] is shown to be valid. Walcott & Żak [84] consider a sliding mode

[1]For example see Chapter 13 in *Modern Control Theory* by Brogan [5].

controller/observer pair and claim that the results of Breinl & Leitmann [3] imply that the separation principle is valid in this situation also. They do not, however, attempt any formal analysis. In this chapter, a rigorous closed loop stability analysis will be performed using quadratic stability concepts. The chapter begins by outlining the assumptions required and the class of uncertainty considered.

4.2 System Description

Consider an uncertain dynamical square system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t, x, u) \\ y(t) &= Cx(t) \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m = p < n$. Assume that the nominal linear system (A, B, C) is known and that the input and output matrices B and C are both of full rank. The unknown function $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ represents any nonlinearities plus any model uncertainties in the system which are assumed to be matched and bounded, i.e.

$$f(t, x, u) = B\left(g_1(t, x, u)u + g_2(t, x)\right) \tag{4.2}$$

where

$$\|g_1(t, x, u)\| \leq k_{g_1} \qquad \text{and} \qquad \|g_2(t, x)\| \leq \alpha(t, y) \tag{4.3}$$

for some known scalar k_{g_1} and known function $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}_+$. In addition, it is assumed that the nominal linear system (A, B, C) satisfies

A1) the pair (A, B) is controllable

A2) $\det(CB) \neq 0$

A3) the invariant zeros of (A, B, C) are in $\mathbb{C}_-$

The assumption that the system is square is required since the observer formulation of the previous chapter requires there to be at least as many outputs as inputs. Conversely, the control scheme of Davies & Spurgeon requires at least as many inputs as outputs – a square system is therefore required. The assumptions A2 and A3 can be recognised from §3.5 as the necessary and sufficient requirement for the existence of a sliding mode observer which is insensitive to matched uncertainty.

It can be assumed without loss of generality that the system is already in regular form, i.e.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{4.4}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ and $C_2 \in \mathbb{R}^{p\times p}$. The square matrix B_2 is nonsingular because the input distribution matrix is assumed to be of full rank. The square matrix C_2 is nonsingular because $C_2B_2 = CB$ is nonsingular by assumption and B_2 is nonsingular by construction.

4.3 Controller Formulation

Consider initially the development of a tracking control law for the nominal linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.5}$$

where the matrix pair (A, B) is assumed to be in regular form as in (4.4). The control law described here is based on that described by Davies & Spurgeon [11] and utilizes an integral action methodology. Consider the introduction of additional states $x_r \in \mathbb{R}^p$ satisfying

$$\dot{x}_r(t) = r(t) - y(t) \tag{4.6}$$

where the differentiable signal $r(t)$ satisfies

$$\dot{r}(t) = \Gamma\left(r(t) - R\right) \tag{4.7}$$

with $\Gamma \in \mathbb{R}^{p\times p}$ a stable design matrix and R a constant demand vector. Augment the states with the integral action states and define

$$\tilde{x} \stackrel{s}{=} \begin{bmatrix} x_r \\ x \end{bmatrix} \tag{4.8}$$

and partition the augmented states as

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.9}$$

where $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. The (augmented) nominal system can then be conveniently written in the form

$$\begin{aligned} \dot{x}_1(t) &= \tilde{A}_{11}x_1(t) + \tilde{A}_{12}x_2(t) + \tilde{T}_i r(t) \qquad (4.10) \\ \dot{x}_2(t) &= \tilde{A}_{21}x_1(t) + A_{22}x_2(t) + B_2 u(t) \qquad (4.11) \end{aligned}$$

where

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & A_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|c} 0 & -C_1 & -C_2 \\ 0 & A_{11} & A_{12} \\ \hline 0 & A_{21} & A_{22} \end{array}\right] \tag{4.12}$$

and the gain on the 'demand' signal $r(t)$ is given by

$$\tilde{T}_i = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \tag{4.13}$$

The proposed controller seeks to induce a sliding motion on the surface

$$\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^{n+p} \quad : \quad S_1 x_1 + S_2 x_2 = S_r r\} \tag{4.14}$$

where $S_1 \in \mathbb{R}^{m\times n}$, $S_2 \in \mathbb{R}^{m\times m}$ and $S_r \in \mathbb{R}^{p\times p}$ are design parameters which govern the reduced order motion. Let $S_2 = \Lambda B_2^{-1}$ where Λ is a nonsingular diagonal design matrix which satisfies

$$k_{g_1}\kappa(\Lambda) < 1 \tag{4.15}$$

If a controller exists which induces an ideal sliding motion on $\mathcal{S}$ then the ideal sliding motion is given by

$$\dot{x}_1(t) = (\tilde{A}_{11} - \tilde{A}_{12}M)x_1(t) + \left(\tilde{A}_{12}S_2^{-1}S_r + \tilde{T}_i\right) r(t) \tag{4.16}$$

where $M \stackrel{s}{=} S_2^{-1}S_1$. In order for the hyperplane design methods of Chapter 2 to be valid, it is necessary for the matrix pair $(\tilde{A}_{11}, \tilde{A}_{12})$ to be completely controllable. Necessary conditions on the original system are given by the following result.

Lemma 4.1 *If (A, B, C) is completely controllable and has no invariant zeros at the origin then the matrix pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is completely controllable.*

Proof

Denote Rosenbrock's System Matrix by

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

The invariant zeros of the triple (A, B, C) are given by

$$\{s \in \mathbb{C} \;:\; \det P(s) = 0\}$$

Therefore the system has zeros at the origin, if and only if, $\det P(0) = 0$. Because (A, B, C) is already in 'regular form' and B_2 is nonsingular

$$\begin{aligned} \det P(0) = 0 \;&\Leftrightarrow\; \det \begin{bmatrix} -C & 0 \\ -A & B \end{bmatrix} = 0 \\ &\Leftrightarrow\; \det \begin{bmatrix} -C_1 & -C_2 & 0 \\ -A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} = 0 \\ &\Leftrightarrow\; \det \begin{bmatrix} C_1 & C_2 \\ A_{11} & A_{12} \end{bmatrix} = 0 \end{aligned}$$

Utilising the PBH rank test the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} sI_p & C_1 & -C_2 \\ 0 & sI - A_{11} & A_{12} \end{bmatrix} = n \quad \text{for all s} \tag{4.17}$$

If $s = 0$ then

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sI_p & C_1 & -C_2 \\ 0 & sI - A_{11} & A_{12} \end{bmatrix} = n \;&\Leftrightarrow\; \det \begin{bmatrix} C_1 & -C_2 \\ -A_{11} & A_{12} \end{bmatrix} \neq 0 \\ &\Leftrightarrow\; \det \begin{bmatrix} C_1 & C_2 \\ A_{11} & A_{12} \end{bmatrix} \neq 0 \\ &\Leftrightarrow\; (A, B, C) \text{ has no zeros at the origin} \end{aligned}$$

Otherwise $s \neq 0$ and

$$\operatorname{rank}\begin{bmatrix} sI_p & C_1 & C_2 \\ 0 & sI - A_{11} & A_{12} \end{bmatrix} = n \quad \Leftrightarrow \quad \operatorname{rank}\begin{bmatrix} sI - A_{11} & A_{12} \end{bmatrix} = n - p$$

However (A, B) is controllable if and only if (A_{11}, A_{12}) is controllable (see §2.2) and therefore by assumption

$$\operatorname{rank}\begin{bmatrix} sI - A_{11} & A_{12} \end{bmatrix} = n - p \quad \text{for all } s$$

from the PBH rank test applied to (A_{11}, A_{12}). Therefore assertion (4.17) is true and $(\tilde{A}_{11}, \tilde{A}_{12})$ is controllable. ■

The assumptions of §4.2, together with Lemma 4.1, imply that the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is completely controllable and consequently an appropriate matrix M may be chosen by any robust linear design technique described in §2.2. For the development that follows it is convenient to introduce a new coordinate system as in [60]. Specifically define

$$T_\phi = \begin{bmatrix} I_n & 0 \\ S_1 & S_2 \end{bmatrix} \tag{4.18}$$

which is nonsingular because by construction S_2 is nonsingular. Let

$$\begin{bmatrix} x_1 \\ \phi \end{bmatrix} \stackrel{s}{=} T_\phi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then with respect to these new coordinates the nominal system can be written as

$$\dot{x}_1(t) = \bar{A}_{11}x_1(t) + \bar{A}_{12}\phi(t) + \tilde{T}_i r(t) \tag{4.19}$$

$$\dot{\phi}(t) = S_2\bar{A}_{21}x_1(t) + S_2\bar{A}_{22}S_2^{-1}\phi(t) + \Lambda u(t) + S_1\tilde{T}_i r(t) \tag{4.20}$$

where $\bar{A}_{11} = \tilde{A}_{11} - \tilde{A}_{12}M$, $\bar{A}_{21} = M\bar{A}_{11} + \tilde{A}_{21} - \tilde{A}_{22}M$, $\bar{A}_{22} = M\tilde{A}_{12} + A_{22}$ and for convenience $\bar{A}_{12} = \tilde{A}_{12}S_2^{-1}$. Define a linear operator $u_L(\cdot)$ as

$$u_L(x_1, \phi, r) = \Lambda^{-1}\left(-S_2\bar{A}_{21}x_1 + (\Phi - S_2\bar{A}_{22}S_2^{-1})\phi - \left(\Phi S_r + S_1\tilde{T}_i\right)r + S_r\dot{r}\right) \tag{4.21}$$

where Φ is any stable design matrix. The control law proposed by Davies & Spurgeon [11] is then given by

$$u = u_L(x_1, \phi, r) + \nu_c \tag{4.22}$$

where ν_c is the discontinuous vector given by

$$\nu_c = \begin{cases} -\rho_c(u_L, y)\Lambda^{-1}\frac{\bar{P}_2(\phi - S_r r)}{\|\bar{P}_2(\phi - S_r r)\|} & \text{if } \phi \neq S_r r \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

where $\bar{P}_2$ is a symmetric positive definite matrix satisfying

$$\bar{P}_2^{-1}\Phi^{\mathrm{T}} + \Phi\bar{P}_2^{-1} = -\hat{Q}_2 \tag{4.24}$$

for some positive definite design matrix $\hat{Q}_2$. The reason for the use of this modified Lyapunov equation will be made clear in §4.5. The positive scalar function which multiplies the unit vector component of the controller is given by

$$\rho_c(u_L, y) = \|\Lambda\|\rho_o(u_L, y) + \gamma_c \tag{4.25}$$

where γ_c is a positive design scalar and $\rho_o(\cdot)$ is a positive scalar function depending on the uncertainty structure which will be determined in the next section. Substituting for $\Lambda^{-1} = B_2^{-1}S_2^{-1}$ and $S_1 = S_2 M$ in equation (4.21) it follows that, in terms of the original coordinates

$$u_L(\tilde{x}, r) = L\tilde{x} + L_r r + L_{\dot{r}}\dot{r} \tag{4.26}$$

where the gains are defined as

$$\begin{aligned} L &= -B_2^{-1}\Big[\big(\ \bar{A}_{21} + \bar{A}_{22}M - \Phi' M\big) \ \Big|\ \big(\bar{A}_{22} - \Phi'\big)\Big] && (4.27) \\ L_r &= -B_2^{-1}\left(\Phi' S_2^{-1}S_r + M\tilde{T}_i\right) && (4.28) \\ L_{\dot{r}} &= B_2^{-1}S_2^{-1}S_r && (4.29) \end{aligned}$$

where $\Phi' \stackrel{s}{=} S_2^{-1}\Phi S_2$. This recovers the notation of Davies & Spurgeon [11]. Generally speaking, this control law relies on all the internal states being available. The approach adopted here is to use the observer defined in the previous chapter to provide estimates of the internal states.

4.4 Nonlinear Observer Formulation in Regular Form

In the previous chapter the observer design was performed in the canonical form of Lemma 3.2. For reasons of convenience and numerical accuracy, it may be preferable to design the observer in the coordinates of the 'regular form'. Consider the observer structure from §3.4 given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - GCe(t) + B\nu_o \tag{4.30}$$

where $\hat{x}$ is an estimate for the true states x, and $e \stackrel{s}{=} \hat{x} - x$ is the state estimation error. The output error feedback gain matrix G is chosen so that the closed loop matrix $A_0 \stackrel{s}{=} A - GC$ is stable and has a Lyapunov matrix P satisfying

$$PA_0 + A_0^{\mathrm{T}}P = -Q \tag{4.31}$$

for some positive definite Q and the structural constraint

$$PB = C^{\mathrm{T}}F^{\mathrm{T}} \tag{4.32}$$

for some nonsingular matrix $F \in \mathbb{R}^{m \times m}$. The discontinuous vector ν_o is given by

$$\nu_o = \begin{cases} -\rho_o(u_L, y)\frac{FCe}{\|FCe\|} & \text{if } Ce \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.33}$$

where $\rho_o(u_L, y)$ is the scalar function

$$\rho_o(u_L, y) = \left(k_{g_1}\|u_L\| + \alpha(t, y) + k_{g_1}\gamma_c\|\Lambda^{-1}\| + \gamma_o\right) / \left(1 - k_{g_1}\kappa(\Lambda)\right) \tag{4.34}$$

where γ_o is a strictly positive scalar. The formulation given in (4.30) – (4.34) is essentially that of Walcott & Żak [83]. Such an observer exists and exhibits a sliding mode on the surface

$$\mathcal{S}_o = \{ \, e \in \mathbb{R}^n \;\; : \;\; FCe = 0 \, \} \tag{4.35}$$

because of the results of Chapter 3 and assumptions A2 and A3. Since the system is square, the matrix F is nonsingular. Therefore sliding on the surface $\mathcal{S}_o$ implies the output of the observer is identical to that of the plant. This formulation is therefore equivalent to the one proposed in §3.5. The decision to use the Walcott & Żak formulation will result in a simplification of the analysis in the subsequent chapters. For compatibility with the state space partition used throughout the chapter let

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{4.36}$$

where $G_2 \in \mathbb{R}^{p \times p}$. The following subsection uses the results of the previous chapter to synthesise directly the observer given in equation (4.30).

Assume the system is in the form of (4.4). The sub-block C_2 is nonsingular and hence the linear transformation

$$T = \begin{bmatrix} I_{n-p} & 0 \\ C_1 & C_2 \end{bmatrix} \tag{4.37}$$

is also nonsingular. The transformation $x \mapsto Tx$ induces a set of coordinates in which the system triple is in the canonical form of Lemma 3.2. It can easily be verified that in the notation of §3.3

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12}C_2^{-1} \\ C_1A_{11} - C_2A_{22}C_2^{-1}C_1 + C_2A_{21} & C_1A_{12}C_2^{-1} + C_2A_{22}C_2^{-1} \end{bmatrix} \tag{4.38}$$

Let A_{22}^s be a stable design matrix as in §3.3 which assigns suitable linear dynamics to the output error system. Using the synthesis procedure of §3.3, the linear output error feedback gain matrix G, in the coordinates of the regular form, is given explicitly by

$$G = \begin{bmatrix} A_{12}C_2^{-1} \\ A_{22}C_2^{-1} - C_2^{-1}A_{22}^s \end{bmatrix} \tag{4.39}$$

If P_2 is a symmetric positive definite Lyapunov matrix for A_{22}^s then define

$$F \stackrel{s}{=} (P_2C_2B_2)^{\mathrm{T}} \tag{4.40}$$

From the results of Chapter 3, if the symmetric positive definite matrix P_1 is an appropriate Lyapunov matrix for $\mathcal{A}_{11}$ then

$$\begin{aligned} P &= \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix} \\ &= \begin{bmatrix} P_1 + C_1^{\mathrm{T}}P_1C_1 & C_1^{\mathrm{T}}P_2C_2 \\ C_2^{\mathrm{T}}P_2C_1 & C_2^{\mathrm{T}}P_2C_2 \end{bmatrix} \end{aligned} \tag{4.41}$$

is a Lyapunov matrix for $A_0 = A - GC$ which satisfies $PB = C^{\mathrm{T}}F^{\mathrm{T}}$. This provides a method of designing a Walcott & Żak observer, without the use of symbolic computation which was hinted at in §3.6.

4.5 Closed Loop Analysis

In this section the effect of using the state estimates in the control law will be explored. In particular, stability of the combined closed loop will be demonstrated. From equation (4.30) the system representing the uncertain and observer dynamics is given by

$$\dot{e}(t) = A_0 e(t) + B\nu_o - f(t,x,u) \tag{4.42}$$

$$\dot{\hat{x}}_r(t) = r(t) - \hat{y}(t) + e_y(t) \tag{4.43}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) - Ge_y(t) + B(\hat{u}(t) + \nu_o) \tag{4.44}$$

where $\hat{y} = C\hat{x}, e_y = Ce$ and $\hat{u}$ is the control action obtained from using the state estimates. It is convenient to repartition equations (4.43) and (4.44) to obtain coordinates

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} x_r \\ \hat{x} \end{bmatrix} \tag{4.45}$$

where $\hat{x}_1 \in \mathbb{R}^n$ and $\hat{x}_2 \in \mathbb{R}^m$ and then change coordinates using the transformation (4.18) to generate equations in terms of $(\hat{x}_1, \hat{\phi})$, which represent estimates of (x_1, ϕ). It can be verified that

$$\dot{\hat{x}}_1(t) = \bar{A}_{11}\hat{x}_1(t) + \bar{A}_{12}\left(\hat{\phi}(t) - S_r r(t)\right) + \left(\tilde{T}_i + \bar{A}_{12}S_r\right) r(t) - \bar{G}_1 e_y(t) \tag{4.46}$$

$$\dot{\hat{\phi}}(t) = \Phi\left(\hat{\phi}(t) - S_r r(t)\right) + \Lambda\hat{\nu}_c + S_r\dot{r}(t) - (S_1\bar{G}_1 + S_2G_2)e_y(t) + \Lambda\nu_o \tag{4.47}$$

where the output error gain matrix

$$\bar{G}_1 \stackrel{s}{=} \begin{bmatrix} -I_p \\ G_1 \end{bmatrix}$$

Here the linear component of the control action has been included and the equations simplified. For convenience define

$$e_r(t) \stackrel{s}{=} r(t) - R \tag{4.48}$$

From equation (4.7) it follows that the evolution of $e_r(t)$ satisfies

$$\dot{e}_r(t) = \Gamma e_r(t) \tag{4.49}$$

If the affine change of coordinates $(\hat{x}_1, \hat{\phi}) \mapsto \zeta$ where

$$\zeta \stackrel{s}{=} \begin{bmatrix} \hat{x}_1 - (\bar{A}_{11})^{-1}(\bar{A}_{12}S_r + \tilde{T}_i)R \\ \hat{\phi} - S_r r \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

is introduced then

$$\dot{\zeta}_1(t) = \bar{A}_{11}\zeta_1(t) + \bar{A}_{12}\zeta_2(t) + \left(\tilde{T}_i + \bar{A}_{12}S_r\right) e_r(t) - \bar{G}_1 e_y(t) \tag{4.50}$$

$$\dot{\zeta}_2(t) = \Phi\zeta_2(t) + \Lambda\hat{\nu}_c - \bar{G}_2 e_y(t) + \Lambda\nu_o \tag{4.51}$$

where

$$\bar{G}_2 = S_1\bar{G}_1 + S_2 G_2 \tag{4.52}$$

Equations (4.50) and (4.51) may then be written more conveniently as

$$\dot{\zeta}(t) = A_c\zeta(t) + \bar{G}_r e_r(t) - \bar{G}Ce(t) + \bar{\Lambda}\left(\nu_o + \hat{\nu}_c\right) \tag{4.53}$$

for appropriate choices of gain matrices $\bar{G}$, $\bar{G}_r$ and $\bar{\Lambda}$. The linear closed loop dynamics are determined by the matrix

$$A_c = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \Phi \end{bmatrix}$$

which is stable, with eigenvalues given by $\lambda(\bar{A}_{11}) \cup \lambda(\Phi)$. For the closed loop analysis which follows, a block diagonal Lyapunov matrix for A_c will be sought of the form

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \tag{4.54}$$

where the lower right matrix sub-block $\bar{P}_2$ is the 'Lyapunov matrix' for Φ used in the unit vector component of the control law. Using the Lyapunov matrix P for A_0 given in (4.31) define

$$P_G \stackrel{s}{=} \begin{bmatrix} P & 0 \\ 0 & \bar{P} \end{bmatrix} \tag{4.55}$$

where the matrix $\bar{P}$ will be chosen so that P_G is a Lyapunov matrix for the overall closed loop system matrix

$$A_G \stackrel{s}{=} \begin{bmatrix} A_0 & 0 \\ -\bar{G}C & A_c \end{bmatrix} \tag{4.56}$$

By direct calculation

$$P_G A_G + A_G^{\mathrm{T}} P_G = \begin{bmatrix} -Q & C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}^{\mathrm{T}} \\ \bar{P}\bar{G}C & \bar{P}A_c + A_c^{\mathrm{T}}\bar{P} \end{bmatrix} \tag{4.57}$$

where Q is defined in the Lyapunov equation (4.31). The expression on the right hand side of equation (4.57) is negative definite if and only if

$$\bar{P}A_c + A_c^{\mathrm{T}}\bar{P} + \bar{P}\bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G}\bar{P} < 0 \tag{4.58}$$

Consider the problem of finding a block diagonal symmetric positive definite matrix $\bar{P}$ as in equation (4.54) so that

$$\bar{P}A_c + A_c^{\mathrm{T}}\bar{P} + \bar{P}\bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G}\bar{P} + \bar{P}\bar{Q}\bar{P} = 0 \tag{4.59}$$

for some symmetric positive definite design matrix $\bar{Q}$.

Lemma 4.2 *A block diagonal solution to equation (4.59) exists for a family of positive definite matrices $\bar{Q}$.*

Proof

Solving equation (4.59) is equivalent to finding a block diagonal solution to the Lyapunov equation

$$A_c\bar{P}^{-1} + \bar{P}^{-1}A_c^{\mathrm{T}} = -\hat{Q} \tag{4.60}$$

where

$$\hat{Q} = \bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G} + \bar{Q} \tag{4.61}$$

because the matrix $\bar{P}$ is required to be positive definite and therefore invertible. For convenience let

$$\bar{P}^{-1} = \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & \hat{P}_2 \end{bmatrix}$$

then from the definition of A_c it follows that

$$A_c\bar{P}^{-1} + \bar{P}^{-1}A_c^{\mathrm{T}} = \begin{bmatrix} \hat{P}_1\bar{A}_{11}^{\mathrm{T}} + \bar{A}_{11}\hat{P}_1 & \bar{A}_{12}\hat{P}_2 \\ \hat{P}_2\bar{A}_{12}^{\mathrm{T}} & \hat{P}_2\Phi^{\mathrm{T}} + \Phi\hat{P}_2 \end{bmatrix}$$

and therefore a parameterization of $\hat{Q}$ is given by

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 + \bar{A}_{12}\hat{P}_2\hat{Q}_2^{-1}\hat{P}_2\bar{A}_{12}^{\mathrm{T}} & -\bar{A}_{12}\hat{P}_2 \\ -\hat{P}_2\bar{A}_{12}^{\mathrm{T}} & \hat{Q}_2 \end{bmatrix}$$

where the matrices $\hat{P}_1$ and $\hat{P}_2$ solve the pair of Lyapunov equations given by

$$\hat{P}_2\Phi^{\mathrm{T}} + \Phi\hat{P}_2 = -\hat{Q}_2 \tag{4.62}$$

$$\hat{P}_1\bar{A}_{11}^{\mathrm{T}} + \bar{A}_{11}\hat{P}_1 = -\bar{A}_{12}\hat{P}_2\hat{Q}_2^{-1}\hat{P}_2\bar{A}_{12}^{\mathrm{T}} - \hat{Q}_1 \tag{4.63}$$

where $\hat{Q}_1$ and $\hat{Q}_2$ are arbitrary symmetric positive definite matrices of appropriate dimension. If

$$Q'_{22} \stackrel{s}{=} CQ^{-1}C^{\mathrm{T}} \tag{4.64}$$

then it follows that

$$\bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}} = \begin{bmatrix} \bar{G}_1Q'_{22}\bar{G}_1^{\mathrm{T}} & \bar{G}_1Q'_{22}\bar{G}_2^{\mathrm{T}} \\ \bar{G}_2Q'_{22}\bar{G}_1^{\mathrm{T}} & \bar{G}_2Q'_{22}\bar{G}_2^{\mathrm{T}} \end{bmatrix}$$

Rearranging equation (4.61) it follows that

$$\bar{Q} = \begin{bmatrix} \hat{Q}_1 + \bar{A}_{12}\hat{P}_2\hat{Q}_2^{-1}\hat{P}_2\bar{A}_{12}^{\mathrm{T}} - \bar{G}_1Q'_{22}\bar{G}_1^{\mathrm{T}} & -\bar{A}_{12}\hat{P}_2 - \bar{G}_1Q'_{22}\bar{G}_2^{\mathrm{T}} \\ -\hat{P}_2\bar{A}_{12}^{\mathrm{T}} - \bar{G}_2Q'_{22}\bar{G}_1^{\mathrm{T}} & \hat{Q}_2 - \bar{G}_2Q'_{22}\bar{G}_2^{\mathrm{T}} \end{bmatrix} \tag{4.65}$$

which by careful choice of $\hat{Q}_1$ and $\hat{Q}_2$ can be made positive definite. A necessary condition for $\bar{Q}$ to be positive definite is that

$$\hat{Q}_2 > \bar{G}_2Q'_{22}\bar{G}_2^{\mathrm{T}} \tag{4.66}$$

Let $\hat{Q}_2$ be any positive definite matrix satisfying the matrix inequality above. Solving the Lyapunov equation (4.62) provides the matrix $\hat{P}_2$. Consequently each element in the matrix $\bar{Q}$ given in equation (4.65) is specified except for $\hat{Q}_1$. A necessary and sufficient condition from (4.65) for $\bar{Q}$ to be positive definite is that

$$\begin{aligned} \hat{Q}_1 > {} & \left(\bar{A}_{12}\hat{P}_2 + \bar{G}_1Q'_{22}\bar{G}_2^{\mathrm{T}}\right)\left(\hat{Q}_2 - \bar{G}_2Q'_{22}\bar{G}_2^{\mathrm{T}}\right)^{-1}\left(\hat{P}_2\bar{A}_{12}^{\mathrm{T}} + \bar{G}_2Q'_{22}\bar{G}_1^{\mathrm{T}}\right) \\ & + \bar{G}_1Q'_{22}\bar{G}_1^{\mathrm{T}} - \bar{A}_{12}\hat{P}_2\hat{Q}_2^{-1}\hat{P}_2\bar{A}_{12}^{\mathrm{T}} \end{aligned} \tag{4.67}$$

The block diagonal solution to (4.59) is given by

$$\bar{P} = \begin{bmatrix} \hat{P}_1^{-1} & 0 \\ 0 & \hat{P}_2^{-1} \end{bmatrix}$$

and the proof is complete. ∎

Let P_r be a Lyapunov matrix for Γ satisfying

$$P_r\Gamma + \Gamma^{\mathrm{T}}P_r = -\tilde{G}_r^{\mathrm{T}}\bar{Q}^{-1}\tilde{G}_r - Q_r \tag{4.68}$$

for some symmetric positive definite matrix Q_r. Because it has been assumed that all the uncertainty is matched, it follows that the system given by (4.42)-(4.44) can then be written as

$$\dot{e}(t) = A_0e(t) + B\left(\nu_o - g_1(t,x,u)u - g_2(t,x)\right) \tag{4.69}$$

$$\dot{\zeta}(t) = A_c\zeta(t) + \bar{G}_r e_r(t) - \bar{G}Ce(t) + \bar{\Lambda}(\nu_o + \hat{\nu}_c) \tag{4.70}$$

The main result of the chapter will now be proved :

Proposition 4.1 *The quadratic form* $V(e,\zeta,e_r) \stackrel{s}{=} e^{\mathrm{T}}Pe + \zeta^{\mathrm{T}}\bar{P}\zeta + e_r^{\mathrm{T}}P_r e_r$ *is a Lyapunov function for the system given above*

Proof

Taking derivatives along the trajectory gives

$$\begin{aligned}\dot{V} = & -e^{\mathrm{T}}Qe - 2e^{\mathrm{T}}PB\left(g_1u + g_2\right) + 2e^{\mathrm{T}}PB\nu_o - \zeta^{\mathrm{T}}\bar{P}\bar{Q}\bar{P}\zeta \\ & -\zeta^{\mathrm{T}}\bar{P}\bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta - 2\zeta^{\mathrm{T}}\bar{P}\bar{G}Ce + 2\zeta^{\mathrm{T}}\bar{P}\bar{\Lambda}\nu_o + 2\zeta^{\mathrm{T}}\bar{P}\bar{\Lambda}\hat{\nu}_c \\ & +2\zeta^{\mathrm{T}}\bar{P}\bar{G}_r e_r - e_r^{\mathrm{T}}Q_r e_r - e_r^{\mathrm{T}}\bar{G}_r^{\mathrm{T}}\bar{Q}^{-1}\bar{G}_r e_r\end{aligned} \tag{4.71}$$

Utilising the structural constraint (4.32) and the uncertainty structure given in equations (4.2) and (4.3) it follows that

$$\begin{aligned}2e^{\mathrm{T}}PB\left(\nu_o - g_1u - g_2\right) &= 2e^{\mathrm{T}}C^{\mathrm{T}}F^{\mathrm{T}}\nu_o - 2e^{\mathrm{T}}C^{\mathrm{T}}F^{\mathrm{T}}\left(g_1u + g_2\right) \\ &\le 2\|FCe\|\left(\|g_1\|\|\hat{u}\| + \|g_2\| - \rho_o(\hat{u}_L,y)\right) \\ &\le 2\|FCe\|\left(k_{g_1}\|\hat{u}\| + \alpha(t,y) - \rho_o(\hat{u}_L,y)\right)\end{aligned} \tag{4.72}$$

Recalling the definition of $\rho_o(\hat{u}_L,y)$ from (4.34)

$$\rho_o(\hat{u}_L,y) = \left(k_{g_1}\|\hat{u}_L\| + \alpha(t,y) + k_{g_1}\gamma_c\|\Lambda^{-1}\| + \gamma_o\right) / \left(1 - k_{g_1}\kappa(\Lambda)\right)$$

Rearrangement yields

$$\rho_o(\hat{u}_L,y) = k_{g_1}\left(\|\hat{u}_L\| + \|\Lambda^{-1}\|(\rho_o(\hat{u}_L,y)\|\Lambda\| + \gamma_c)\right) + \alpha(t,y) + \gamma_o$$

$$
\begin{aligned}
&= k_{g_1}\left(\|\hat{u}_L\| + \|\Lambda^{-1}\|\rho_c(u_L, y)\right) + \alpha(t,y) + \gamma_o \\
&> k_{g_1}\left(\|\hat{u}_L\| + \|\hat{\nu}_c\|\right) + \alpha(t,y) + \gamma_o \\
&\geq k_{g_1}\|\hat{u}_L + \hat{\nu}_c\| + \alpha(t,y) + \gamma_o \\
&= k_{g_1}\|\hat{u}\| + \alpha(t,y) + \gamma_o
\end{aligned} \tag{4.73}
$$

and therefore from inequalities (4.72) and (4.73) it follows that

$$
2e^{\mathrm{T}}C^{\mathrm{T}}F^{\mathrm{T}}\nu_o - 2e^{\mathrm{T}}PB\,(g_1 u + g_2) \;\leq\; -2\gamma_o\|FCe\| \tag{4.74}
$$

Also

$$
\begin{aligned}
2\zeta^{\mathrm{T}}\bar{P}\bar{\Lambda}(\nu_o + \hat{\nu}_c) &\leq 2\zeta_2^{\mathrm{T}}\bar{P}_2\Lambda\nu_o + 2\zeta_2^{\mathrm{T}}\bar{P}_2\Lambda\nu_c \\
&= 2\zeta_2^{\mathrm{T}}\bar{P}_2\Lambda\nu_o - 2\rho_c(\hat{u}_L, y)\|\bar{P}_2\zeta_2\| \\
&\leq 2\|\bar{P}_2\zeta_2\|\left(\rho_o(\hat{u}_L, y)\|\Lambda\| - \rho_c(\hat{u}_L, y)\right) \\
&= -2\gamma_c\|\bar{P}_2\zeta_2\|
\end{aligned} \tag{4.75}
$$

Using inequalities (4.74) and (4.75) in equation (4.71) it follows that

$$
\begin{aligned}
\dot{V} &\leq -e^{\mathrm{T}}Qe - 2\gamma_o\|FCe\| - \zeta^{\mathrm{T}}\bar{P}\bar{Q}\bar{P}\zeta - \zeta^{\mathrm{T}}\bar{P}\bar{G}CQ^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta - 2\zeta^{\mathrm{T}}\bar{P}\bar{G}Ce \\
&\qquad -2\gamma_c\|\bar{P}_2\zeta_2\| + 2\zeta^{\mathrm{T}}\bar{P}\bar{G}_r e_r - e_r^{\mathrm{T}}Q_r e_r - 2e_r^{\mathrm{T}}\bar{G}_r^{\mathrm{T}}\bar{Q}^{-1}\bar{G}_r e_r \\
&\equiv -(e + Q^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta)^{\mathrm{T}}Q(e + Q^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta) - \zeta^{\mathrm{T}}\bar{P}\bar{Q}\bar{P}\zeta - 2\gamma_o\|FCe\| \\
&\qquad -2\gamma_c\|\bar{P}_2\zeta_2\| + 2\zeta^{\mathrm{T}}\bar{P}\bar{G}_r e_r - e_r^{\mathrm{T}}Q_r e_r - 2e_r^{\mathrm{T}}\bar{G}_r^{\mathrm{T}}\bar{Q}^{-1}\bar{G}_r e_r \\
&\equiv -(e + Q^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta)^{\mathrm{T}}Q(e + Q^{-1}C^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{P}\zeta) - e_r^{\mathrm{T}}Q_r e_r - 2\gamma_o\|FCe\| \\
&\qquad -2\gamma_c\|\bar{P}_2\zeta_2\| - (\zeta - \bar{P}^{-1}\bar{Q}^{-1}\bar{G}_r e_r)^{\mathrm{T}}\bar{P}\bar{Q}\bar{P}(\zeta - \bar{P}^{-1}\bar{Q}^{-1}\bar{G}_r e_r) \\
&< 0 \qquad \text{if } (e, \zeta, e_r) \neq 0
\end{aligned}
$$

■

Corollary 4.1 *The output of the uncertain system $y(t)$ tracks the reference signal $r(t)$ asymptotically*

Proof

From Proposition 4.1, the system (4.70) is quadratically stable and therefore $\dot{\zeta} \to 0$ which implies $\dot{\hat{x}}_1 \to 0$. From the definition of $\hat{x}_1$, the first p states are the integral action states. Consequently $\dot{x}_r \to 0$ and it follows that $y \to r$ asymptotically. ■

Corollary 4.2 *Sliding motions are induced on the surfaces $\mathcal{S}$ and $\mathcal{S}_o$ defined in (4.14) and (4.35) respectively.*

Proof

It is sufficient to demonstrate that a sliding motion occurs on the surface in the combined state space given by

$$\mathcal{S}_c = \{(e, \zeta_1, \zeta_2) \in \mathbb{R}^{2n+p} \quad : \quad Ce = 0 \text{ and } \zeta_2 = 0\}$$

which contains the surfaces $\mathcal{S}$ and $\mathcal{S}_o$. Consider the quadratic form

$$V_c = e^{\mathrm{T}} C^{\mathrm{T}} P_2 Ce + \zeta_2^{\mathrm{T}} \bar{P}_2 \zeta_2$$

as a candidate Lyapunov function. From (4.69) it follows that

$$C\dot{e}(t) = CA_0 e(t) + C_2 B_2 (\nu_o - g_1(t, x, u)u(t) - g_2(t, x))$$

and from the definition of F in equation (4.40) it follows that

$$P_2 = F^{\mathrm{T}} (CB)^{-1}$$

Therefore

$$P_2 C\dot{e}(t) = F^{\mathrm{T}} (CB)^{-1} CA_0 e(t) + F^{\mathrm{T}} (\nu_o - g_1(t, x, u)u(t) - g_2(t, x))$$

and using inequality (4.74) it follows that

$$e^{\mathrm{T}} C^{\mathrm{T}} P_2 C\dot{e} \leq \|FCe\| \|(CB)^{-1} CA_0 e\| - \gamma_o \|FCe\|$$

Taking derivatives along the trajectories and using equation (4.51) in conjunction with inequality (4.75) it follows that

$$\begin{aligned}
\dot{V}_c &= 2e^{\mathrm{T}} C^{\mathrm{T}} P_2 C\dot{e} + \zeta_2^{\mathrm{T}} \bar{P}_2 \dot{\zeta}_2 + \dot{\zeta}_2^{\mathrm{T}} \bar{P}_2 \zeta_2 \\
&= 2e^{\mathrm{T}} C^{\mathrm{T}} P_2 C\dot{e} - \zeta_2^{\mathrm{T}} \bar{P}_2 \hat{Q}_2 \bar{P}_2 \zeta_2 + 2\zeta_2^{\mathrm{T}} \bar{P}_2 \Lambda (\hat{\nu}_c + \nu_o) - 2\zeta_2^{\mathrm{T}} \bar{P}_2 \bar{G}_2 Ce \\
&\leq 2e^{\mathrm{T}} C^{\mathrm{T}} P_2 C\dot{e} - 2\zeta_2^{\mathrm{T}} \bar{P}_2 \bar{G}_2 Ce - 2\gamma_c \|\bar{P}_2 \zeta_2\| \\
&\leq -2\|FCe\| \left(\gamma_o - \|(CB)^{-1} CA_0 e\|\right) - 2\|\bar{P}_2 \zeta_2\| \left(\gamma_c - \|\bar{G}_2 Ce\|\right)
\end{aligned}$$

In the domain $\Omega = \{(e, \zeta_1, \zeta_2) \in \mathbb{R}^{2n+p} \ : \ \|(CB)^{-1} CA_0 e\| < \gamma_o \text{ and } \|\bar{G}_2 Ce\| < \gamma_c\}$ the inequality $\dot{V}_c < 0$ is satisfied, and from Theorem 2.1 a sliding motion takes place in the domain Ω. From Proposition 4.1 the error states $e(t) \to 0$ as $t \to \infty$ and hence the trajectories enter the domain Ω in finite time and sliding takes place.

∎

Remark

An intuitive argument supporting the analysis of this section runs as follows:

Assuming a sliding motion is attained on $\mathcal{S}_o$, then from equation (4.70) it follows that

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t) - B\nu_{eq} \tag{4.76}$$

where ν_{eq} is the 'equivalent control' necessary to maintain the motion on $\mathcal{S}_o$ and is a function of the uncertainty. The control law $\hat{u}$ can be thought of as 'controlling' the dynamical system (4.76) given above. The term '$B\nu_{eq}$' can be considered as matched uncertainty and therefore by an appropriate choice of sliding mode control law can be rejected. The output $\hat{y} = C\hat{x}$ therefore behaves in an ideal way. However since sliding occurs on the hyperplane $\mathcal{S}_o$ it follows that the output of the uncertain systems $y \equiv \hat{y}$. Therefore the output of the uncertain system behaves in an ideal nominal way.

4.6 Implementation Issues

The observer described in §4.4 has only the stable matrix A_{22}^s and its associated Lyapunov matrix P_2 in the way of design freedom. As a result, this section concentrates on the design of the nonlinear control law. The following subsection considers the design of the hyperplane.

4.6.1 Hyperplane Design

The hyperplane defined in §4.3 is given by

$$\mathcal{S} = \{\tilde{x} \in \mathbb{R}^{n+p} \quad : \quad S\tilde{x} = S_r r\} \tag{4.77}$$

where r represents the reference signal, $S_r \in \mathbb{R}^{m \times m}$ is a design parameter and

$$S = S_2 \begin{bmatrix} M & I_p \end{bmatrix}$$

The square design matrix S_2 has no effect on the dynamics of the reduced order motion and serves only as a scaling of the hyperplane. From equation (4.16) the design matrix $M \in \mathbb{R}^{m \times n}$ can be viewed as a state feedback matrix for the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ and hence must be chosen so that

$$\lambda\left(\tilde{A}_{11} - \tilde{A}_{12}M\right) \subset \mathbb{C}_-$$

Many different approaches to the design of this matrix have been detailed in the literature. These were summarised in §2.2 and will not be discussed here. This section concentrates instead on the choice of the matrix S_r. In the controller framework under discussion, the choice of S_r may be viewed as affecting the values of the integral action components since from equation (4.16) at steady state

$$(\tilde{A}_{11} - \tilde{A}_{12}M)x_1 + \left(\tilde{A}_{12}S_2^{-1}S_r + \tilde{T}_i\right) r = 0 \tag{4.78}$$

where $\tilde{T}_i$ is defined in equation (4.13). One possibility is to choose a value if S_r so that, for the nominal system at steady state, the integral action states are zero. Equation (4.78) can be re-written as

$$x_1 = -\bar{A}_{11}^{-1}\left(\tilde{A}_{12}S_2^{-1}S_r + \tilde{T}_i\right) r \tag{4.79}$$

where $\bar{A}_{11} \stackrel{s}{=} \tilde{A}_{11} - \tilde{A}_{12}M$, which is stable by design and therefore invertible. The steady state values of the integral action states are therefore given by

$$x_r = -\tilde{T}_i^{\mathrm{T}}\bar{A}_{11}^{-1}\left(\tilde{A}_{12}S_2^{-1}S_r + \tilde{T}_i\right) r \tag{4.80}$$

Define K_s to be the static gain of the square system $(\bar{A}_{11}, \tilde{A}_{12}, \tilde{T}_i^{\mathrm{T}})$. Provided K_s is nonsingular, choosing

$$S_r = S_2K_s^{-1}\tilde{T}_i^{\mathrm{T}}\bar{A}_{11}^{-1}\tilde{T}_i \tag{4.81}$$

implies $x_r = 0$. The following lemma demonstrates that the assumptions in §4.2 guarantee that K_s is nonsingular.

Lemma 4.3 *The static gain of the square system $(\bar{A}_{11}, \tilde{A}_{12}, \tilde{T}_i^{\mathrm{T}})$ is nonsingular*

Proof

Let K_s represent the static gain of the system $(\bar{A}_{11}, \tilde{A}_{12}, \tilde{T}_i^{\mathrm{T}})$ then by direct evaluation it follows that

$$\begin{bmatrix} I & 0 \\ \tilde{T}_i^{\mathrm{T}}\bar{A}_{11}^{-1} & K_s \end{bmatrix} \equiv \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{T}_i^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -M & I \end{bmatrix} \begin{bmatrix} \bar{A}_{11}^{-1} & -\bar{A}_{11}^{-1}\tilde{A}_{12} \\ 0 & I \end{bmatrix}$$

Therefore K_s is of full rank if and only if

$$\det \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{T}_i^{\mathrm{T}} & 0 \end{bmatrix} \neq 0$$

Substituting for $\tilde{A}_{11}$ and $\tilde{A}_{12}$ from equation (4.12) and for $\tilde{T}_i$ from equation (4.13) it follows that

$$\det\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{T}_i^{\mathrm{T}} & 0 \end{bmatrix} \neq 0 \;\Leftrightarrow\; \det\begin{bmatrix} 0 & C_1 & C_2 \\ 0 & A_{11} & A_{12} \\ I & 0 & 0 \end{bmatrix} \neq 0$$

$$\Leftrightarrow\; \det\begin{bmatrix} C_1 & C_2 \\ A_{11} & A_{12} \end{bmatrix} \neq 0$$

In the proof of Lemma 4.1 it was shown that

$$\det\begin{bmatrix} C_1 & C_2 \\ A_{11} & A_{12} \end{bmatrix} = 0 \;\Leftrightarrow\; (A, B, C) \text{ has zeros at the origin}$$

This is guaranteed by assumption A2 in §4.2 and so gain K_s is nonsingular. ∎

The next section considers the design of the nonlinear component of the control law.

4.6.2 Design of the Unit Vector Controller

From the perspective of design, apart from the scalar functions which pre-multiply the unit vector components, the only dependence between the controller and observer is through the choice of the controller Lyapunov matrix $\bar{P}_2$. This matrix defines the unit vector component of the control law

$$\nu_c = \begin{cases} -\rho_c(u_L, y)\Lambda^{-1}\frac{\bar{P}_2(\phi - S_r r)}{\|\bar{P}_2(\phi - S_r r)\|} & \text{if } \phi \neq S_r r \\ 0 & \text{otherwise} \end{cases} \tag{4.82}$$

and satisfies

$$\bar{P}_2^{-1}\Phi^{\mathrm{T}} + \Phi\bar{P}_2^{-1} = -\hat{Q}_2 \tag{4.83}$$

where Φ is the stable design matrix defining the linear range space dynamics. The matrix $\hat{Q}_2$ is a symmetric positive definite design matrix which must satisfy

$$\hat{Q}_2 > \bar{G}_2^{\mathrm{T}} Q'_{22} \bar{G}_2 \tag{4.84}$$

where $\bar{G}_2$ is defined as in (4.52) and Q'_{22} in equation (4.64). For a given observer design the right hand side of (4.84) is completely determined and as a consequence this appears to restrict the choice of $\bar{P}_2$. After a moments thought, this turns out not to be the case. Suppose an observer has been designed and hence $\bar{G}_2^{\mathrm{T}} Q'_{22} \bar{G}_2$ is determined. Let $\hat{Q}_2$ be any symmetric positive definite matrix and let $\bar{P}_2$ be the unique solution to the Lyapunov equation (4.83). If λ is any positive scalar then the matrix pair $(\bar{P}_2^{\lambda}, \hat{Q}_2^{\lambda}) \stackrel{s}{=} (\frac{1}{\lambda}\bar{P}_2, \lambda \hat{Q}_2)$ satisfies

$$(\bar{P}_2^{\lambda})^{-1}\Phi^{\mathrm{T}} + \Phi(\bar{P}_2^{\lambda})^{-1} = -\hat{Q}_2^{\lambda} \tag{4.85}$$

It is easy to verify that

$$\hat{Q}_2^{\lambda} > \bar{G}_2^{\mathrm{T}} Q'_{22} \bar{G}_2 \quad \Leftrightarrow \quad \lambda > \lambda_{max}\left(\hat{Q}_2^{-\frac{1}{2}} \bar{G}_2^{\mathrm{T}} Q'_{22} \bar{G}_2 \hat{Q}_2^{-\frac{1}{2}}\right)$$

Consequently for any given $\hat{Q}_2$, for a large enough value of λ, the matrix $\bar{P}_2^{\lambda}$ is an appropriate choice for the unit vector control component ν_c. From (4.82) it can be seen that ν_c is independent of the value of λ. As a result the controller and observer can be designed independently, i.e. the separation principle holds.

4.7 Summary

A practical nonlinear control strategy has been presented which provides robust output tracking despite the presence of uncertainty. The strategy uses only measured output information and constructs estimates of the internal states, required for the control law, using a nonlinear observer. It has been demonstrated for the class of controllers and observers considered that the separation principle holds. A rigorous analysis of the closed loop performance of the controller/observer pair has been undertaken which indicates that for a class of matched uncertainty, asymptotic tracking of a constant reference signal will be achieved.

The assumptions require that the nominal linear system is minimum phase and relative degree one. This implies that, in the state space, the hyperplane defined by the null space of C provides a stable reduced order ideal sliding motion. This suggests that it may be possible to design a sliding mode controller using only output information thus obviating the need to employ an observer. This approach is investigated in the next chapter.

Chapter 5

Sliding Mode Controllers Using Output Information

5.1 Introduction

The problem considered here involves designing the sliding surface and the variable structure control law in such a way that only output information is required and no observer need be used. For linear systems with no uncertainty this problem has been investigated by White [85] and more recently by El-Khazali & DeCarlo [24, 25]. For uncertain linear systems an approach has recently been reported by Hui & Żak [39] and Żak & Hui [89] who propose an algorithm for output dependent hyperplane design, which is based upon eigenvector methods. The majority of this chapter is devoted to developing a framework for designing a regulator for uncertain systems. At the end of the chapter, it will be shown that these ideas can be incorporated into a model reference structure to generate an output tracking controller. The control law that will be developed is similar to the unit vector structure employed in [60] except only output information is required.

5.2 Problem Formulation

Consider an uncertain dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t, x, u) \\ y(t) &= Cx(t) \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \leq p < n$. Assume that the nominal linear system (A, B, C) is known and that the input and output matrices B and C are both of full rank. In addition, it is assumed that :

A1) rank $(CB) = m$

A2) the unknown function $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ which represents the system nonlinearities plus any model uncertainties in the system satisfies the matching condition

$$f(t, x, u) = B\xi(t, x, u) \tag{5.2}$$

where the bounded function $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies

$$\|\xi(t, x, u)\| < k_1\|u\| + \alpha(t, y) \tag{5.3}$$

for some known function $\alpha : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ and positive constant $k_1 < 1$.

The intention is to develop a control law which induces a sliding motion on the surface

$$\mathcal{S} \;=\; \{x \in \mathbb{R}^n \;: FCx = 0\} \tag{5.4}$$

for some selected matrix $F \in \mathbb{R}^{m \times p}$. A control law of the form

$$u(t) \;=\; G_l y(t) - \nu_y \tag{5.5}$$

will be sought where G_l is a fixed gain matrix and the discontinuous vector

$$\nu_y = \begin{cases} \rho(u, y)\frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

and $\rho(u, y)$ is some positive scalar function of the inputs and outputs.

The reason for imposing the restriction A1 at the outset, is that for a unique equivalent control to exist, the matrix $FCB \in \mathbb{R}^{m \times m}$ must have full rank[1]. It is well known that

$$\text{rank}(FCB) \leq \min\{\text{rank}(F),\ \text{rank}(CB)\}$$

[1]This differs from the approach of White [85] who considers the case when FCB is rank deficient. In this situation an ideal sliding motion as defined in §2.2 cannot be guaranteed; for details see Utkin [80].

and so in order for FBC to have full rank both F and CB must have rank m. The matrix F is a design parameter and therefore by choice can be chosen to be of full rank. A necessary condition therefore for the matrix FBC to be full rank is that $\text{rank}(CB) = m$. The first problem which must be considered is how to choose F so that the associated sliding motion is stable. Before describing the new results a review of recent developments which have appeared in the literature will be made.

5.3 Recent Developments

This section discusses the approaches of El-Khazali & DeCarlo [24, 25] and Żak & Hui [89]. The assumptions of the previous section are common to both the above; any additional ones peculiar to one approach will be indicated at the appropriate time.

5.3.1 The Approach of El-Khazali & DeCarlo

El-Khazali & DeCarlo [24] consider only nominal linear systems and do not consider the effect of uncertainty. In addition to A1 they assume that the nominal linear system (A, B, C) is both observable and controllable and already in regular form with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{5.7}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ and the remaining sub-blocks are partitioned accordingly. Their work requires that the pair $(\hat{A}_{11}, \hat{C}_1)$ is completely observable where

$$\hat{A}_{11} \stackrel{s}{=} A_{11} - A_{12}C_2^{-L}C_1 \tag{5.8}$$

$$\hat{C}_1 \stackrel{s}{=} M^{\mathrm{T}}C_1 \tag{5.9}$$

with $M \in \mathbb{R}^{p\times(p-m)}$ a full rank left annihilator of C_2, and $C_2^{-L} \in \mathbb{R}^{m\times p}$ a left pseudo-inverse of C_2. If the sub-block A_{12} has either full row or column rank, and the 'Kimura-Davison' conditions [13, 14, 44, 45], written as

$$n < m + p - 1 \qquad \text{and} \qquad p > m \geq 2$$

are satisfied, then there exists a matrix $\Gamma \in \mathbb{R}^{m\times(p-m)}$ such that $\lambda(\hat{A}_{11} - A_{12}\Gamma\hat{C}_1)$ can be assigned arbitrarily close to any given self conjugate set of $n - m$ complex numbers.

El-Khazali & DeCarlo then show a choice for the sliding surface matrix is

$$F = C_2^{-L} + \Gamma M^{\mathrm{T}} \tag{5.10}$$

In [25], an alternative hyperplane design algorithm is proposed based on eigenstructure ideas similar to those presented in §2.2. Essentially, the authors show that the eigenvectors associated with the eigenvalues of the reduced order motion must lie in $\mathrm{N}(C)$. An algorithm to design the hyperplane which guarantees this requirement is provided. For the algorithm to be valid, all the assumptions outlined above need to be met.

5.3.2 The Approach of Żak & Hui

The conceptual approach of Żak & Hui [89] is similar to that of El-Khazali & DeCarlo in that it relies on establishing an appropriate eigenstructure for the reduced order sliding motion. Assumptions A1 and A2 are required together with controllability and observability of the nominal linear system. Żak and Hui prove that the problem of designing a suitable hyperplane matrix is equivalent to finding a matrix $W \in \mathbb{R}^{n \times (n-m)}$ of full rank such that

1) $\mathrm{R}(W) \cap \mathrm{R}(B) = \{0\}$

2) $AW - W\Lambda \subset \mathrm{R}(B)$

3) $\operatorname{rank}(CW) = p - m$

where Λ is the diagonal matrix formed from a self conjugate set of complex numbers $\{\lambda_1 \ldots \lambda_{n-m}\}$ which represent the proposed eigenvalues of the sliding motion. For a given matrix W satisfying the above conditions, the hyperplane matrix can be shown to be any matrix $F \in \mathbb{R}^{m \times p}$ such that $FCW = 0$. Their approach introduced an interesting new perspective on the problem of hyperplane design. However, the problem of computing the matrix W, which is in itself a far from trivial problem, is not addressed. It is noted by Żak & Hui that any invariant zeros of the nominal system must be included in the set of proposed sliding mode eigenvalues. A necessary condition for the existence of a hyperplane is therefore that the nominal systems is minimum phase. In addition the structural constraint

$$FCA = MC \tag{5.11}$$

where $M \in \mathbb{R}^{m \times p}$ must be satisfied for the control law proposed by Żak and Hui to be realised.

In the following sections, a new framework for the design of output feedback sliding mode controllers is proposed which is similar to the one used for observer design in §3.3. The approach is quite different to that of Hui & Żak and is applicable to a wider class of systems than those considered by El-Khazali & DeCarlo.

5.4 A Framework for Hyperplane Design

The general case when there are more outputs than inputs will be considered first. This situation turns out to exhibit a richer mathematical structure than the case of equal numbers of inputs and outputs. Necessary and sufficient conditions will be developed for the existence of a control law of the form given in (5.5), which quadratically stabilizes the uncertain system given in (5.1). In addition, it will be shown that a sliding mode is exhibited. The following lemma provides a canonical form for the system triple (A, B, C) which will be used throughout the analysis. It is essentially identical to that given in Lemma 3.2 for the design of sliding mode observers, except the sub-blocks that compose the system matrix have been partitioned in a different way.

Lemma 5.1 *Let (A, B, C) be a linear system with $p > m$ and rank $(CB) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure*

a) *the system matrix can be written as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \textit{where } A_{11} \in \mathbb{R}^{(n-m)\times(n-m)} \tag{5.12}$$

and the sub-block A_{11} when partitioned has the structure

$$A_{11} = \left[\begin{array}{cc|c} A_{11}^{o} & A_{12}^{o} & \\ 0 & A_{22}^{o} & A_{12}^{m} \\ \hline 0 & A_{21}^{o} & A_{22}^{m} \end{array}\right] \tag{5.13}$$

where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{22}^o \in \mathbb{R}^{(n-p-r)\times(n-p-r)}$ *and* $A_{21}^o \in \mathbb{R}^{(p-m)\times(n-p-r)}$ *for some* $r \geq 0$ *and the pair* (A_{22}^o, A_{21}^o) *is completely observable. In addition the invariant zeros of* (A, B, C) *are the eigenvalues of* A_{11}^o.

b) *the input distribution matrix has the form*

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \text{where } B_2 \in \mathbb{R}^{m\times m} \text{ and is nonsingular} \tag{5.14}$$

c) *the output distribution matrix has the form*

$$C = \begin{bmatrix} 0 & T \end{bmatrix} \quad \text{where } T \in \mathbb{R}^{p\times p} \text{ and is orthogonal} \tag{5.15}$$

Proof (identical to Lemma 3.2 and Lemma 3.3) ∎

As in the case of the observer design, the canonical form of the above lemma will be instrumental in the proposed framework for output feedback controller design. Suppose a controller exists which induces a stable sliding motion on the surface $\mathcal{S}$ defined in equation (5.4). As argued earlier in §5.2, for a unique equivalent control to exist, $\text{rank}(CB) = m$. Therefore, in all the analysis which follows, it can be assumed without loss of generality that the system is already in the canonical form of Lemma 5.1. It will be useful to partition the matrix FC in a compatible way to the canonical form. To this end let

$$FT = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$$

where $F_1 \in \mathbb{R}^{m\times(p-m)}$ and $F_2 \in \mathbb{R}^{m\times m}$ and T is the matrix from equation (5.15). As a result

$$FC = \begin{bmatrix} F_1C_1 & F_2 \end{bmatrix} \tag{5.16}$$

where $C_1 \in \mathbb{R}^{(p-m)\times(n-m)}$ is defined to be

$$C_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)} \end{bmatrix} \tag{5.17}$$

Therefore $FCB = F_2B_2$ and in particular the square matrix F_2 is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form of Lemma 5.1 can be viewed as a special case of the 'regular form' normally used in sliding mode controller design, the reduced order sliding motion is governed by a free motion with system matrix

$$A_{11}^{s} \stackrel{s}{=} A_{11} - A_{12}F_2^{-1}F_1C_1 \tag{5.18}$$

which must be therefore be stable. If $K \in \mathbb{R}^{m\times(p-m)}$ is defined as $K = F_2^{-1}F_1$ then

$$A_{11}^{s} = A_{11} - A_{12}KC_1 \tag{5.19}$$

and the problem of designing a suitable hyperplane is therefore equivalent to an output feedback problem for the system (A_{11}, A_{12}, C_1). It is well established that invariant zeros play an important part in the sliding motion – in particular Hui & Żak identify that any invariant zeros of (A, B, C) must appear in the sliding mode dynamics when only output information is used. Intuitively if the original system has invariant zeros, then from Lemma 5.1 the eigenvalues of A_{11}^{o} must appear in the spectrum of A_{11}^{s} and so the poles of $A_{11} - A_{12}KC_1$ cannot be assigned arbitrarily by choice of K. The intention is to construct a new sub-system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ which is both controllable and observable with the property that $\lambda(A_{11}^{s}) = \lambda(A_{11}^{o}) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1K\tilde{C}_1)$. To this end, partition the matrices A_{12} and A_{12}^{m} as

$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^{m} = \begin{bmatrix} A_{121}^{m} \\ A_{122}^{m} \end{bmatrix} \tag{5.20}$$

where $A_{122} \in \mathbb{R}^{(n-m-r)\times m}$ and $A_{122}^{m} \in \mathbb{R}^{(n-m-r)\times(p-m)}$ and form a new sub-system represented by the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ where

$$\tilde{A}_{11} \stackrel{s}{=} \begin{bmatrix} A_{22}^{o} & A_{122}^{m} \\ A_{21}^{o} & A_{22}^{m} \end{bmatrix} \quad \text{and} \quad \tilde{C}_1 \stackrel{s}{=} \begin{bmatrix} 0_{(p-m)\times(n-p-r)} & I_{(p-m)} \end{bmatrix} \tag{5.21}$$

Lemma 5.2 *The spectrum of A_{11}^{s} contains the invariant zeros of (A, B, C) and in particular decomposes as*

$$\lambda(A_{11} - A_{12}KC_1) = \lambda(A_{11}^{o}) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1)$$

Proof

Using the partition in equation (5.13) by definition

$$
\begin{aligned}
A_{11} - A_{12}KC_1 &= \left[\begin{array}{cc|c} A^o_{11} & A^o_{12} & \multirow{2}{*}{A^m_{12}} \\ 0 & A^o_{22} & \\ \hline 0 & A^o_{21} & A^m_{22} \end{array}\right] - \left[\begin{array}{cc} 0_{(n-m)\times(n-p)} & A_{12}K \end{array}\right] \\
&= \left[\begin{array}{cc} A^o_{11} & [A^o_{12} \;\; A^m_{121}] \\ 0 & \tilde{A}_{11} \end{array}\right] - \left[\begin{array}{cc} 0 & A_{121}K \\ 0 & A_{122}K \end{array}\right] \\
&= \left[\begin{array}{cc} A^o_{11} & \left[\begin{array}{c|c} A^o_{12} & A^m_{121} - A_{121}K \end{array}\right] \\ 0 & \tilde{A}_{11} - A_{122}K\tilde{C}_1 \end{array}\right]
\end{aligned}
$$

Therefore $\lambda(A^s_{11}) = \lambda(A_{11} - A_{12}KC_1) = \lambda(A^o_{11}) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1)$ as claimed. From Lemma 5.1 the spectrum of A^o_{11} represents the invariant zeros of (A, B, C). ■

It follows directly that for a stable sliding motion, the invariant zeros of the system (A, B, C) must lie in the open left half plane and the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ must be stabilizable with respect to output feedback[2]. The matrix A_{122} is not necessarily full rank. Suppose $\text{rank}(A_{122}) = m'$ then it is possible to construct a matrix of elementary column operations $T_{m'} \in \mathbb{R}^{m \times m}$ such that

$$
A_{122}T_{m'} = \left[\begin{array}{cc} \tilde{B}_1 & 0 \end{array}\right] \tag{5.22}
$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r)\times m'}$ and is of full rank. If $K_{m'} = T^{-1}_{m'}K$ and $K_{m'}$ is partitioned compatibly as

$$
K_{m'} = \left[\begin{array}{c} K_1 \\ K_2 \end{array}\right] \quad \text{where } K_1 \in \mathbb{R}^{m'\times(p-m)} \text{ and } K_2 \in \mathbb{R}^{(m-m')\times(p-m)}
$$

then

$$
\tilde{A}_{11} - A_{122}K\tilde{C}_1 = \tilde{A}_{11} - \left[\begin{array}{cc} \tilde{B}_1 & 0 \end{array}\right] K_{m'}\tilde{C}_1 = \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1
$$

and $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ is stabilizable by output feedback if and only if $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilizable by output feedback. (The reason for the somewhat tortuous argument to establish the sub-system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ will be made clear later in the chapter when the

[2]The phrase *'stabilizable with respect to output feedback'* will indicate that for the linear system (A, B, C) there exists a fixed gain K such that the matrix $A - BKC$ is stable.

results of Davison *et al.*[13, 14] and Kimura [44, 45] will be utilised.) As a result of the preceding argument, the 'only if' direction of the following proposition has been established.

Proposition 5.1 *There exists a matrix F defining a surface $\mathcal{S}$ which provides a stable sliding motion with a unique equivalent control if and only if*

- *the invariant zeros of (A, B, C) lie in $\mathbb{C}_-$*
- *the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilizable with respect to output feedback.*

In the next section, a proof of the 'if' direction is given – specifically a control law will be synthesised explicitly which induces a stable sliding motion on $\mathcal{S}$ and quadratically stabilizes the uncertain system (5.1).

5.5 Controller Formulation

Let (A, B, C) be the nominal linear part of the uncertain system given in (5.1). Suppose that $p > m$, $\text{rank}(CB) = m$ and any invariant zeros of the nominal linear system lie in the open left half plane. Without loss of generality, the triple (A, B, C) can be assumed to be in the canonical form of Lemma 5.1 and from equations (5.20), (5.21) and (5.22) the sub-system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ can be determined. Assume this subsystem is output feedback stabilizable so there exists a $K_1 \in \mathbb{R}^{m' \times (p-m)}$ such that $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$ is stable. Let

$$K = T_{m'} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \quad \text{where } K_2 \in \mathbb{R}^{(m-m') \times (p-m)} \text{ and is arbitrary} \tag{5.23}$$

and matrix $T_{m'} \in \mathbb{R}^{m \times m}$ is defined in equation (5.22). Then it follows that

$$\lambda(A_{11} - A_{12} K C_1) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1)$$

and so the matrix $A_{11} - A_{12} K C_1$ is stable. Choose

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^{\mathrm{T}}$$

where $F_2 \in \mathbb{R}^{m \times m}$ is nonsingular and will be defined later. Introduce a nonsingular state transformation $x \mapsto \bar{T}x = z$ where

$$\bar{T} = \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \tag{5.24}$$

and C_1 is defined in (5.17). In this new coordinate system the system triple $(\bar{A}, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad F\bar{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \tag{5.25}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$. The square system $(\bar{A}, \bar{B}, F\bar{C})$ is relatively degree one and minimum phase because $\bar{A}_{11}$ is stable. It is well known that such systems are output feedback stabilizable [31]. Let P be the symmetric positive definite matrix partitioned as in (5.25)

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{5.26}$$

where the sub-block P_2 is a design matrix and P_1 satisfies the Lyapunov equation

$$P_1\bar{A}_{11} + \bar{A}_{11}^{\mathrm{T}}P_1 = -Q_1 \tag{5.27}$$

for some symmetric positive definite matrix Q_1. If the design matrix $F_2 \stackrel{s}{=} B_2^{\mathrm{T}}P_2$ then the matrix P satisfies the structural constraint

$$P\bar{B} = \bar{C}^{\mathrm{T}}F^{\mathrm{T}} \tag{5.28}$$

For notational convenience let

$$Q_2 \stackrel{s}{=} P_1\bar{A}_{12} + \bar{A}_{21}^{\mathrm{T}}P_2 \tag{5.29}$$

$$Q_3 \stackrel{s}{=} P_2\bar{A}_{22} + \bar{A}_{22}^{\mathrm{T}}P_2 \tag{5.30}$$

and define

$$\gamma_0 \stackrel{s}{=} \tfrac{1}{2}\,\lambda_{max}\left((F_2^{-1})^{\mathrm{T}}(Q_3 + Q_2^{\mathrm{T}}Q_1^{-1}Q_2)F_2^{-1}\right) \tag{5.31}$$

This scalar is well defined since the matrix on the right is symmetric and therefore has no complex eigenvalues.

Lemma 5.3 *The symmetric matrix* $L(\gamma) \stackrel{s}{=} PA_0 + A_0^{\mathrm{T}}P$ *where* $A_0 = \bar{A} - \gamma\bar{B}F\bar{C}$ *is negative definite if and only if* $\gamma > \gamma_0$.

Proof

Using the definition of A_0 and the structural constraint (5.28) it follows that

$$\begin{aligned} L(\gamma) \stackrel{s}{=} PA_0 + A_0^{\mathrm{T}}P &= P\bar{A} + \bar{A}^{\mathrm{T}}P - \gamma P\bar{B}F\bar{C} - \gamma\bar{C}^{\mathrm{T}}F^{\mathrm{T}}\bar{B}^{\mathrm{T}}P \\ &= P\bar{A} + \bar{A}^{\mathrm{T}}P - 2\gamma(F\bar{C})^{\mathrm{T}}F\bar{C} \end{aligned}$$

Using the partitions of $\bar{A}, F\bar{C}$ and P from equations (5.25) and (5.26) respectively and the definitions of Q_2 and Q_3 it follows that

$$L(\gamma) = \begin{bmatrix} -Q_1 & Q_2 \\ Q_2^{\mathrm{T}} & Q_3 - 2\gamma F_2^{\mathrm{T}}F_2 \end{bmatrix}$$

Using the properties of symmetric matrices in Appendix A, and the fact that Q_1 is positive definite, it follows that

$$\begin{aligned} L(\gamma) < 0 &\Leftrightarrow Q_3 - 2\gamma F_2^{\mathrm{T}}F_2 + Q_2^{\mathrm{T}}Q_1^{-1}Q_2 < 0 \\ &\Leftrightarrow 2\gamma F_2^{\mathrm{T}}F_2 > Q_3 + Q_2^{\mathrm{T}}Q_1^{-1}Q_2 \\ &\Leftrightarrow 2\gamma I_m > (F_2^{-1})^{\mathrm{T}}(Q_3 + Q_2^{\mathrm{T}}Q_1^{-1}Q_2)F_2^{-1} \\ &\Leftrightarrow 2\gamma > \lambda_{max}\left((F_2^{-1})^{\mathrm{T}}(Q_3 + Q_2^{\mathrm{T}}Q_1^{-1}Q_2)F_2^{-1}\right) \end{aligned}$$

and the proof is complete. ∎

Define a variable structure control law, depending only on outputs, by

$$u(t) = -\gamma Fy(t) - \nu_y \tag{5.32}$$

where $\gamma > \gamma_0$ and ν_y is the discontinuous vector given by

$$\nu_y = \begin{cases} \rho(u, y)\frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

and $\rho(u, y)$ is the positive scalar function

$$\rho(u, y) = (k_1\gamma\|Fy\| + \alpha(t, y) + \gamma_2) \,/\, (1 - k_1) \tag{5.34}$$

where γ_2 is a positive design scalar which will be shown to define the region in which sliding takes place. In the new coordinate system, the uncertain system (5.1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}(u(t) + \xi(t, x, u)) \tag{5.35}$$

Proposition 5.2 *The variable structure control law above quadratically stabilizes the uncertain system given in equation (5.35).*

Proof

Consider as a candidate Lyapunov function the positive definite expression

$$V(z) \stackrel{s}{=} z^{\mathrm{T}}Pz \tag{5.36}$$

Taking derivatives along the system trajectory, and using the structural constraint from equation (5.28) gives

$$\begin{aligned}
\dot{V} &= z^{\mathrm{T}}\left(\bar{A}^{\mathrm{T}}P + P\bar{A} - 2\gamma(F\bar{C})^{\mathrm{T}}F\bar{C}\right)z + 2z^{\mathrm{T}}P\bar{B}(\xi - \nu_y) \\
&= z^{\mathrm{T}}L(\gamma)z + 2y^{\mathrm{T}}F^{\mathrm{T}}(\xi - \nu_y) \\
&\leq z^{\mathrm{T}}L(\gamma)z - 2y^{\mathrm{T}}F^{\mathrm{T}}\nu_y + 2\|Fy\|\|\xi\| \\
&= z^{\mathrm{T}}L(\gamma)z - 2\rho(u,y)\|Fy\| + 2\|Fy\|\|\xi\| \\
&< z^{\mathrm{T}}L(\gamma)z - 2\|Fy\|\left(\rho(u,y) - k_1\|u\| - \alpha(t,y)\right)
\end{aligned}$$

But by definition

$$\rho(u,y) = \left(k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2\right)/(1 - k_1)$$

and so by rearranging

$$\begin{aligned}
\rho(u,y) &= k_1\rho(u,y) + k_1\gamma\|Fy\| + \alpha(t,y) + \gamma_2 \\
&\geq k_1(\|\nu_y\| + \gamma\|Fy\|) + \alpha(t,y) + \gamma_2 \\
&\geq k_1\|u\| + \alpha(t,y) + \gamma_2
\end{aligned} \tag{5.37}$$

Using (5.37) in the inequality for the Lyapunov derivative

$$\dot{V} < z^{\mathrm{T}}L(\gamma)z - 2\gamma_2\|Fy\| < 0 \quad \text{if } z \neq 0 \text{ and } \gamma > \gamma_0$$

and therefore the system is quadratically stable. ∎

Corollary 5.1 *A sliding motion takes place on the surface $\mathcal{S}$ defined in equation (5.4) in the domain $\Omega = \{z \in \mathbb{R}^n \ : \ \|B_2^{-1}A_0^L z\| < \gamma_2\}$ where the matrix sub-block A_0^L represents the last p rows of the closed loop matrix A_0.*

Proof

Let the 'switching function' $s(z) = F\bar{C}z$ then from equation (5.35) it follows that

$$\dot{s} = F\bar{C}A_0 z + F_2 B_2 (\xi - \nu_y)$$

Let the 'Lyapunov function' $V_c : \mathbb{R}^m \rightarrow \mathbb{R}$ be defined by

$$V_c(s) = 2s^{\mathrm{T}}(F_2^{-1})^{\mathrm{T}} P_2 F_2^{-1} s$$

Using the fact that $F_2^{\mathrm{T}} = P_2 B_2$ it follows that $(F_2^{-1})^{\mathrm{T}} P_2 F_2^{-1} F\bar{C}A_0 = B_2^{-1}A_0^L$ and arguing as in Proposition 5.2 it can be verified that

$$\dot{V}_c \;=\; 2s^{\mathrm{T}}B_2^{-1}A_0^L z + 2s^{\mathrm{T}}(\xi - \nu) \;\leq\; 2\|s\|\|B_2^{-1}A_0^L z\| - 2\gamma_2\|s\|$$

Therefore $\dot{V}_c < 0$ if $z \in \Omega$ and from Theorem 2.1 in §2.4 a sliding motion takes place in the domain Ω. From Proposition 5.2 the states $z(t)$ are quadratically stable and so in finite time $z(t) \in \Omega$ and a sliding motion will be attained. ∎

The 'if' direction of Proposition 5.1 has been proved and hence in conjunction with the results of §5.4 the proof of Proposition 5.1 is complete.

Remark

From the statement of Corollary 5.1 it is clear that the parameter γ_2 defines the region Ω in which sliding takes place. As a result, outside of this region the system trajectories may pierce the sliding surface without sliding taking place and the control law will behave in a similar fashion to a relay. Because quadratic stability has been demonstrated, eventually the trajectories will enter Ω and sliding will take place.

5.6 Design Considerations

This short section addresses the practical problem of designing the sliding surface hyperplane matrix F and determining when the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is output feedback stabilizable.

Lemma 5.4 *If the pair (A, B) is completely controllable then the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is completely controllable and $(\tilde{A}_{11}, \tilde{C}_1)$ is completely observable.*

Proof

Because the pair (A, B) is in the canonical form of Lemma 5.1 which is a special case of the 'regular form' used for sliding mode design, it is well known that pair (A, B) is completely controllable if and only if the pair (A_{11}, A_{12}) is completely controllable. Therefore from the PBH rank test

$$\text{rank}\begin{bmatrix} sI - A_{11} & A_{12} \end{bmatrix} = n - m \quad \text{for all } s \in \mathbb{C}$$

Substituting for A_{11} from (5.13) and A_{12} from (5.20) gives

$$\text{rank}\begin{bmatrix} sI - A_{11}^o & [A_{12}^o \;\; A_{121}^m] & A_{121} \\ 0 & sI - \tilde{A}_{11} & A_{122} \end{bmatrix} = n - m \quad \text{for all } s \in \mathbb{C}$$

This implies

$$\text{rank}\begin{bmatrix} sI - \tilde{A}_{11} & A_{122} \end{bmatrix} = n - m - r \quad \text{for all } s \in \mathbb{C}$$

and therefore $(\tilde{A}_{11}, A_{122})$ is completely controllable by the PBH test. By construction $(\tilde{A}_{11}, A_{122})$ is controllable if and only if $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable and so the first part of the lemma is established. Applying the PBH observability rank test to the pair $(\tilde{A}_{11}, \tilde{C}_1)$

$$\begin{aligned}
\text{rank}\begin{bmatrix} sI - \tilde{A}_{11} \\ \tilde{C}_1 \end{bmatrix} &= \text{rank}\begin{bmatrix} sI - A_{22}^o & -A_{122}^m \\ -A_{21}^o & sI - A_{22}^m \\ 0 & I_{p-m} \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} sI - A_{22}^o \\ -A_{21}^o \end{bmatrix} + (p - m) \quad \text{for all } s \in \mathbb{C}
\end{aligned}$$

Now (A_{22}^o, A_{21}^o) is observable and so by the PBH observability test

$$\text{rank}\begin{bmatrix} sI - A_{22}^o \\ -A_{21}^o \end{bmatrix} = n - p - r \quad \text{for all } s \in \mathbb{C}$$

and therefore

$$\text{rank}\begin{bmatrix} sI - \tilde{A}_1 \\ \tilde{C}_1 \end{bmatrix} = n - m - r \quad \text{for all } s \in \mathbb{C}$$

and hence $(\tilde{A}_{11}, \tilde{C}_1)$ is completely observable ■

Suppose (A, B) is controllable and the scalar inequality

$$n \leq p + m' + r - 1 \tag{5.38}$$

holds. This inequality can be rewritten as

$$(n - m - r) \leq (p - m) + m' - 1$$

and therefore the 'Kimura-Davison' conditions are satisfied for the system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ with m' inputs and $p - m$ outputs which from Lemma 5.4 is controllable, observable and has input and output distribution matrices of full rank. Under these circumstances the poles of $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$ can be assigned arbitrarily by output feedback for 'almost all pairs $(\tilde{B}_1, \tilde{C}_1)$'. For details see Davsion [13], Davison & Wang [14], Kimura [44, 45] and more recently Misra & Patel [52].

5.7 A Design Framework for Square Systems

Up to this point it has been assumed that the uncertain system has more outputs than inputs. In this section, square systems are considered. The analysis is similar to that presented for the case of more outputs than inputs and so only brief details will be given. Suppose a control law exists to induce a stable sliding motion with unique equivalent control on $\mathcal{S}$ given in equation (5.4). As before, FCB must be of full rank which implies the $\det(CB) \neq 0$. In this case it can be shown that the canonical form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & I_m \end{bmatrix} \tag{5.39}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $B_2 \in \mathbb{R}^{m\times m}$ is nonsingular can always be attained. It can also be shown that the invariant zeros of the nominal linear system are the eigenvalues of A_{11}. Arguing as in §2.2 the reduced order sliding motion is governed by the matrix sub-block A_{11} which consequently must be stable. Necessary conditions for the existence of a controller are therefore :

- $\det(CB) \neq 0$
- the invariant zeros of (A, B, C) must lie in $\mathbb{C}_-$

Conversely, given any (A, B, C) which satisfies the above conditions, then without loss of generality, it can be assumed that the system is in the form of (5.39) with A_{11} having stable eigenvalues. The matrices A, B and FC can be seen to have the same structure as the matrices $\bar{A}, \bar{B}$ and $F\bar{C}$ comprising the control law design canonical form given in (5.25). The control law defined in equations (5.32)–(5.34) after replacing F_2 with F can be shown to quadratically stabilize the uncertain system. The conditions given above are therefore necessary and sufficient.

5.8 Remarks

- The approach presented in §5.4 and §5.5 is applicable to a wider class of systems than those considered by El-Khazali & DeCarlo [25, 26]. Firstly, the sub-block A_{12} given in equation (5.7) is not required to fulfil any rank conditions; and secondly, the results of El-Khazali & DeCarlo do not apply to systems which have invariant zeros. The latter comment can be justified as follows:

  Recalling the notation of §5.3.1, El-Khazali & DeCarlo require the pair $(\hat{A}_{11}, \hat{C}_1)$ to be completely observable where

$$\hat{A}_{11} \;=\; A_{11} - A_{12}C_2^{-L}C_1 \quad \text{and} \quad \hat{C}_1 \;=\; M^{\mathrm{T}}C_1$$

  with $M \in \mathbb{R}^{p\times(p-m)}$ a full rank left annihilator of C_2 and $C_2^{-L} \in \mathbb{R}^{m\times p}$ a left pseudo-inverse of C_2. By assumption, $\text{rank}(CB) = m$ and therefore the results of Lemma 5.1 are applicable. Consequently, without loss of generality,

$$A_{11} \;=\; \begin{bmatrix} A_{11}^{o} & [A_{12}^{o}\ A_{121}^{m}] \\ 0 & \tilde{A}_{11} \end{bmatrix}, \qquad \begin{bmatrix} C_1 & C_2 \end{bmatrix} \;=\; \left[\begin{array}{cc|c} 0 & T_1 & T_2 \end{array}\right] \tag{5.40}$$

  where $A_{11}^{o} \in \mathbb{R}^{r\times r}$ for $r \geq 0$ and $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ is an orthogonal matrix where $T_1 \in \mathbb{R}^{p\times(p-m)}$ and $T_2 \in \mathbb{R}^{m\times p}$. As a result of the orthogonality conditions

$$T_1^{\mathrm{T}}C_2 \stackrel{s}{=} T_1^{\mathrm{T}}T_2 = 0 \qquad \text{and} \qquad \text{rank}(T_1) = p - m$$

  so $M = T_1$ is a choice for the full rank left annihilator of C_2. Also the orthogonality conditions imply

$$T_2^{\mathrm{T}}C_2 \stackrel{s}{=} T_2^{\mathrm{T}}T_2 = I_m$$

and so $C_2^{-L} = T_2$ is a left pseudo-inverse of C_2. Therefore

$$\hat{A}_{11} \;=\; A_{11} - A_{12}C_2^{-L}C_1 \;=\; A_{11} - A_{12}T_2^{\mathrm{T}}\begin{bmatrix} 0 & T_1 \end{bmatrix} \;=\; A_{11} \qquad (5.41)$$

and

$$\hat{C}_1 \;=\; M^{\mathrm{T}}C_1 \;=\; T_1^{\mathrm{T}}\begin{bmatrix} 0 & T_1 \end{bmatrix} \;=\; \begin{bmatrix} 0 & I_{(p-m)} \end{bmatrix} \qquad (5.42)$$

If (A, B, C) has invariant zeros then from Lemma 5.1 the zeros are the eigenvalues of A_{11}° and $r > 0$. Examining the structure of A_{11} from (5.40) and $\hat{C}_1$ from (5.42) it can be seen that

$$\lambda(A_{11}^{\circ}) \subset \lambda(\hat{A}_{11} - G\hat{C}_1)$$

for all $G \in \mathbb{R}^{(n-m)\times p}$ and so $(\hat{A}_{11}, \hat{C}_1)$ is not completely observable and the approach of Khazali & DeCarlo is not applicable.

- The controller synthesis presented in §5.5 is more straightforward than that proposed by El-Khazali & DeCarlo [25] and does not require the structural constraint (5.11) imposed by Żak & Hui.

5.9 Numerical Design Examples

The following non-trivial examples substantiate the practicality of the proposed design procedure.

5.9.1 Example 1 [Hui & Żak]

Consider the system from Hui & Żak [39]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{1}{3} & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & \frac{8}{3} & 1 \\ 4 & \frac{2}{3} & -2 \end{bmatrix}$$

In the canonical form of Lemma 5.1 the system becomes

$$A_c = \begin{bmatrix} -1.5816 & 0.0192 & 0.1457 \\ 1.4071 & 0.3845 & -1.7080 \\ 0.2953 & 0.3400 & 0.1971 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ -3.9016 \end{bmatrix}$$

and

$$C_c = \begin{bmatrix} 0 & 0.3417 & -0.9398 \\ 0 & 0.9398 & 0.3417 \end{bmatrix}$$

from which $B_2 = -3.9016$ and the orthogonal matrix

$$T = \begin{bmatrix} 0.3417 & -0.9398 \\ 0.9398 & 0.3417 \end{bmatrix}$$

can be identified. Also it can be verified that the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_{11})$ is given by

$$\tilde{A}_{11} = \begin{bmatrix} -1.5816 & 0.0192 \\ 1.4071 & 0.3845 \end{bmatrix} \qquad \tilde{B}_1 = \begin{bmatrix} 0.1457 \\ -1.7080 \end{bmatrix} \qquad \tilde{C}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Here $r = 0$ and so the original system does not possess any invariant zeros. Arbitrary placement of the poles of $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$ where $K_1 \in \mathbb{R}$ is clearly not possible. However if $K_1 = -1.0556$ then $\lambda(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1) = \{-1, -2\}$ from which

$$\begin{aligned} F &= F_2 \begin{bmatrix} K & 1 \end{bmatrix} T^{\mathrm{T}} \\ &= F_2 \begin{bmatrix} -1.3005 & -0.6503 \end{bmatrix} \end{aligned} \tag{5.43}$$

where F_2 is a non-zero scalar which will be computed later. Transforming the system into the control law design canonical form using $\bar{T}$ defined in (5.24) generates the sub-block

$$\bar{A}_{11} = \begin{bmatrix} -1.5816 & 0.1729 \\ 1.4071 & -1.4184 \end{bmatrix}$$

where $\lambda(\bar{A}_{11}) = \{-1, -2\}$ by construction. It can be verified that

$$P_1 = \begin{bmatrix} 0.3368 & 0.1891 \\ 0.1891 & 0.5401 \end{bmatrix}$$

is a Lyapunov matrix for $\bar{A}_{11}$ and that if the design parameter $P_2 = 1$ then $F_2 = -3.9016$. It can be checked that $\gamma_0 = 0.2452$ and substituting for F_2 in (5.43) gives

$$F = \begin{bmatrix} 5.0741 & 2.5370 \end{bmatrix}$$

Note that if $F_2 = -1.5378$ then $F = \begin{bmatrix} 2 & 1 \end{bmatrix}$ which is the solution obtained in [39].

5.9.2 Example 2 [Helicopter]

The following eighth order linear system, models the small-perturbation rigid body motion of a helicopter about the hover condition. The states are given by

$$x = \begin{bmatrix} \theta \\ \phi \\ p \\ q \\ r \\ u \\ v \\ w \end{bmatrix} \quad \begin{array}{l} \text{Pitch attitude (rad)} \\ \text{Roll attitude (rad)} \\ \text{Body roll rate (rad/s)} \\ \text{Body pitch rate (rad/s)} \\ \text{Body yaw rate (rad/s)} \\ \text{Forward velocity (ft/s)} \\ \text{Lateral velocity (ft/s)} \\ \text{Normal velocity (ft/s)} \end{array}$$

The system matrices are

$$A = \begin{bmatrix} 0 & 0 & 0 & 0.9986 & 0.0534 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & -0.0032 & 0.0595 & 0 & 0 & 0 \\ 0 & 0 & -11.5705 & -2.5446 & -0.0636 & 0.1068 & -0.0949 & 0.0071 \\ 0 & 0 & 0.4394 & -1.9982 & 0 & 0.0167 & 0.0185 & -0.0012 \\ 0 & 0 & -2.0409 & -0.4590 & -0.7350 & 0.0193 & -0.0046 & 0.0021 \\ -32.1036 & 0 & -0.5034 & 2.2979 & 0 & -0.0212 & -0.0212 & 0.0158 \\ 0.1022 & 32.0578 & -2.3472 & -0.5036 & 0.8349 & 0.0212 & -0.0379 & 0.0004 \\ -1.9110 & 1.7138 & -0.0040 & -0.0574 & 0 & 0.0140 & -0.0009 & -0.2905 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1243 & 0.0828 & -2.7525 & -0.0179 \\ -0.0364 & 0.4751 & 0.0143 & 0 \\ 0.3045 & 0.0150 & -0.4965 & -0.2067 \\ 0.2877 & -0.5445 & -0.0164 & 0 \\ -0.0191 & 0.0164 & -0.5445 & 0.2348 \\ -4.8206 & -0.0004 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0595 & 0.0533 & -0.9968 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0535 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with inputs

$$u = \begin{bmatrix} \theta_{0d} \\ \theta_{ls} \\ \theta_{lc} \\ \theta_{0t} \end{bmatrix} \quad \begin{array}{l} \text{Main rotor collective (deg)} \\ \text{Longitudinal Cyclic (deg)} \\ \text{Lateral Cyclic (deg)} \\ \text{Tail rotor collective (deg)} \end{array}$$

and outputs

$$y = \begin{bmatrix} \dot{h} \\ \theta \\ \phi \\ \dot{\psi} \\ q \\ p \end{bmatrix} \quad \begin{array}{l} \text{Heave velocity (ft/s)} \\ \text{Pitch attitude (rad)} \\ \text{Roll attitude (rad)} \\ \text{Heading rate (ft/s)} \\ \text{Body pitch rate (rad/s)} \\ \text{Body roll rate (rad/s)} \end{array}$$

In the canonical form of Lemma 5.1

$$A_c = \left[\begin{array}{cccc|cccc} -0.0015 & -0.0003 & 31.3425 & 7.2071 & -0.0007 & -0.0013 & -0.0060 & -0.0011 \\ -0.0003 & -0.0054 & -7.2003 & 31.2859 & -0.0062 & -0.0051 & 0.0046 & -0.0172 \\ 0 & 0 & 0 & 0 & 0.0792 & -0.0127 & 1.0001 & 0.0010 \\ 0 & 0 & 0 & 0 & 0.1336 & 0.9919 & -0.0006 & -0.0298 \\ \hline -0.0004 & -0.0114 & 0 & 0 & -0.7244 & 0.0760 & -0.0199 & -0.0279 \\ 0.0991 & -0.1014 & 0 & 0 & -1.8671 & -10.9450 & 4.3139 & 0.2854 \\ -0.0346 & 0 & 0 & 0 & 0.1526 & 1.0816 & -2.6711 & -0.0206 \\ -0.0076 & -0.0098 & 0.0001 & 0 & 0.1074 & 0.5664 & -0.0174 & -0.3060 \end{array}\right]$$

from which $r = p - m$ and so

$$A_{11}^{\circ} = \begin{bmatrix} -0.0015 & -0.0003 \\ -0.0003 & -0.0054 \end{bmatrix} \quad \text{and} \quad A_{22}^{m} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It can be checked that $\lambda(A_{11}^{\circ}) = \{-0.0014, -0.0054\}$ and so the method is applicable. Also from equation (5.21)

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A_{122} = \begin{bmatrix} 0.0792 & -0.0127 & 1.0001 & 0.0010 \\ 0.1336 & 0.9919 & -0.0006 & -0.0298 \end{bmatrix}$$

A possible choice for $\tilde{B}_1$ is

$$\tilde{B}_1 = \begin{bmatrix} -1.0034 & 0 \\ 0.0026 & -1.0013 \end{bmatrix}$$

where

$$T_{m'} = \begin{bmatrix} -0.0790 & -0.1337 & -0.9879 & 0.0012 \\ 0.0127 & -0.9906 & 0.1330 & 0.0298 \\ -0.9968 & -0.0021 & 0.0800 & -0.0007 \\ -0.0010 & 0.0297 & -0.0027 & 0.9996 \end{bmatrix}$$

Because $\tilde{C}_1$ is the identity and $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable the poles of $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$ where $K_1 \in \mathbb{R}^{2\times2}$ can be assigned arbitrarily.

Remark

It should be noted that this system does not satisfy the assumptions of El-Khazali & DeCarlo [25] because of the presence of the invariant zeros at $\{-0.0014, -0.0054\}$.

5.10 Sliding Mode Model Reference Systems

In the previous chapter, output tracking was achieved by 'shaping' the plant to include integral action. An alternative approach is to employ a *model following* strategy whereby the plant is required to follow the dynamic behaviour of a specified plant model. Most work to incorporate sliding mode ideas within a model reference framework, assumes that all plant states are available – for example, the work of Ambrosino *et al.*[1] and Spurgeon & Patton [71]. Work using only input/output information has appeared within the context of adaptive control where the variable structure ideas are utilised in the parameter adaptation mechanism. A survey of these methods appears in [37]. In this section, a model following problem is considered. In keeping with the overall thrust of the thesis, it will be assumed that only output information is available. The intention is to formulate a control law using the model states and the output information from the plant which forces the discrepancy between the states of the plant and the model to zero. This section will demonstrate a link between such a problem and the output regulation problem that has occupied the majority of the chapter.

Consider the uncertain system described in §5.2 and define a new dynamical system – the ideal model – according to

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_m r(t) \\ y_m(t) &= C x_m(t) \end{aligned} \tag{5.44}$$

where $x_m \in \mathbb{R}^n$ are the model states and $r(t)$ are the reference inputs. Here it will be assumed that

$$A_m = A + BL_x \quad \text{and} \quad B_m = BL_r \tag{5.45}$$

for some appropriate fixed gains L_x and L_r and that the model state space matrix A_m is stable. This, together with the 'matching' assumption A2 from §5.1, guarantees that the conditions for perfect model following of Chan [7] and Erzberger [27] are satisfied. Formally it is desired to synthesise a control law so that the error

$$e(t) \stackrel{s}{=} x_m(t) - x(t) \tag{5.46}$$

is quadratically stable.

If all the states of the uncertain system are available, the approach of Spurgeon *et al.*[72] can be used to formulate a control law, which induces a sliding motion in the error space, making the system insensitive to the matched uncertainty. Here, as for the regulator problem discussed previously, it will be assumed that only plant output information is available. Using equations (5.1) and (5.44) the error system satisfies

$$\dot{e}(t) = Ae(t) + B\left(L_x x_m(t) + L_r r(t) - \xi - u\right) \tag{5.47}$$

If the proposed control law has the form

$$u(t) \;=\; L_x x_m(t) + L_r r(t) - u_e(t) \tag{5.48}$$

where $u_e(t)$ is a control component law depending only on $e_y = Ce$ then

$$\dot{e}(t) \;=\; Ae(t) + B\left(u_e(t) - \xi(t,x,u)\right) \tag{5.49}$$

The problem is therefore to construct a control law $u_e(t)$ using only output information to stabilize the uncertain error system (5.49). This, of course, is the regulator problem considered in the rest of this chapter and hence all the results obtained there are applicable. This framework will subsequently be used to develop temperature controllers for gas fired furnaces.

5.11 Summary

A new design procedure has been presented to synthesise robust output feedback controllers for uncertain systems based on sliding mode concepts. The class of systems to which the results apply has been identified, and includes the requirement that the nominal linear system is minimum phase. Emphasis has been placed on the tractability of the associated design procedure. The design of the sliding surface utilizes established output feedback eigenvalue assignment results. It has been demonstrated that all the assumptions imposed on the system pertain to the design of the sliding surface. The proposed controller which guarantees attainment of a sliding mode despite the presence of uncertainty, requires no additional assumptions. The practicality of the procedure has been substantiated by two non-trivial design examples. The last section demonstrated how the previous output feedback results can be viewed within the context of a model reference approach.

The remainder of the thesis is devoted to the application of the ideas of Chapters 3,4 and 5 to control problems associated with gas fired furnaces. The next chapter develops a nonlinear model of high temperature heat transfer within a single burner gas fired furnace. This model will be used as a test-bed for comparing the effectiveness of the different sliding mode control schemes.

Chapter 6

Modelling of Temperature in Gas Fired Furnaces

6.1 Introduction

The remainder of this thesis will consider the application of the theoretical results in the earlier chapters, to control problems associated with gas fired furnaces. The furnace configuration considered can be represented schematically as shown below.

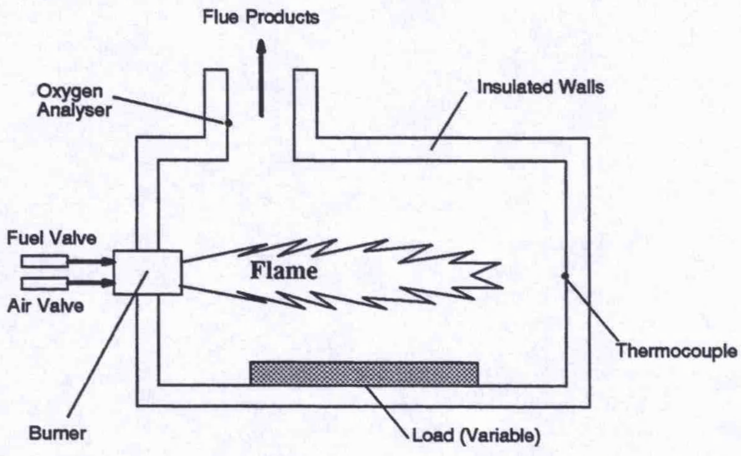


Figure 6.1: Schematic of the furnace considered

The reason for studying this particular design, is that it is representative of an experimental furnace at the Gas Research Centre which is used for examining the efficacy of new control schemes. In order to demonstrate the viability of the proposed approach, a detailed mathematical model of the experimental furnace was first developed using established furnace modelling techniques. The sliding mode control schemes were first demonstrated to perform well on the simulation before attempting to implement them on the actual plant. This chapter documents the modelling and validation work undertaken in this regard. The furnace in Figure 6.1 can be thought of as a gas filled enclosure bounded by insulating surfaces and containing a heat sink. The heat input is achieved by a burner located in one of the end walls, and the combustion products are evacuated via a flue in the roof. This is perhaps the simplest design possible – a single burner and a single flue – but such a plant could legitimately represent an industrial kiln for the firing of pottery or ceramics.

From a control systems perspective, the inputs to the system are the fuel and air flow rates, and the outputs are the furnace temperature (as measured by the thermocouple) and the percentage of oxygen present in the combustion products. In an industrial situation, the internal furnace temperature would be required to exhibit a specific time/temperature profile comprising, say, a period of low fire, a ramp to a higher temperature, a period of soak and finally a return to ambient temperature. During normal furnace operation efficient fuel combustion must also be maintained. For a given mass of fuel, a theoretical mass of oxygen is required to completely oxidize the hydrocarbons – so called *stoichiometric combustion*. An inadequate air supply will result in incomplete combustion with a corresponding loss in thermal energy release. Conversely, excess air whilst guaranteeing complete combustion, will give rise to unnecessary enthalpy losses through the flue, due to the increased flue flow rate. Efficient combustion is ensured by controlling the amount of oxygen in the combustion products as measured by the oxygen analyser positioned in the flue.

The dominant mechanism of heat exchange within furnaces operating at high temperature is radiation. Consequently, the key component in any mathematical model is the technique used to solve the radiation exchange problem within the enclosure. Different approaches to this problem will be discussed in the next section.

6.2 Temperature Modelling in Furnaces

The established method of calculating the radiation exchange within an enclosure is the *Zone Method* [35]. In this approach the surfaces and interior volumes are divided into sub-surfaces and sub-volumes small enough to be considered isothermal. The integro-differential equations governing radiation exchange are reduced to algebraic and finite-difference equations, which can be solved numerically. The method is very computationally intense, and as a result, simplifications have appeared in the literature – most recently, the *Imaginary Planes Method* [8] and the variation described in [33].

The *Monte Carlo* simulation method described in [74] uses statistical techniques to simulate radiation exchange. A large number of 'bundles' of radiant energy are emitted from random points, in random directions from probability density functions which agree with the physical laws governing radiation.The radiation exchange is then determined from the average behaviour of these bundles. This technique is also very computationally demanding, but does have the advantage that irregular enclosure geometries can be dealt with.

More approximate solutions can be obtained by using *Flux Methods* [63, 17]. These rely on converting the laws of conservation of radiation intensity into a coupled set of differential equations which can be solved using finite difference techniques [68]. Although these provide less accurate solutions than the previously mentioned techniques, they are less computationally demanding. The 'Zone' and 'Monte Carlo' methods require the volumetric combustion distribution and gas velocity/mass flow to be specified, and find the unknown temperature distributions. Flux methods are usually used as part of the complete solution process, usually with computational fluid dynamic schemes, to predict both flow and temperature distributions.

Before discussing the zone method in detail some definitions and elementary results in radiation theory will be presented. The work which follows may be found in standard references for example Hottel & Sarofim [36] or Love [48]. The treatment given here is not intended to be exhaustive – only details necessary for the exposition which follows are included.

6.3 Surface Radiation Concepts

Infra-red emission from a *perfect emitter* or *black body* obeys Planck's Law. Integrating over all possible wavelengths leads to the well known Stefan–Boltzman fourth power law

$$E_b = \sigma T^4 \tag{6.1}$$

where E_b is known as the *black body emissive power*. This represents the rate of energy release from a unit incremental area on a black surface at a temperature T into the enclosing hemisphere. The constant σ is known as the Stefan-Boltzman constant.

Another concept, which is often used as a starting point in treatises on radiation, is that of *radiative intensity* which is usually denoted as I_b. This is defined to be the rate of energy release from a unit differential area on a black surface, through a unit differential solid angle normal to the surface. If the intensity $I_{b,\eta}$ at an angle η to the normal satisfies the cosine law

$$I_{b,\eta} = I_b \cos \eta \tag{6.2}$$

then the surface is said to be a *Lambert Surface* . Under this assumption it can be shown that the black body emissive power and intensity are related by the equation

$$E_b = \pi I_b \tag{6.3}$$

Most engineering surfaces are not perfect emitters and the concept of *emissivity* is introduced to make allowance for this. A surface's emissivity, ϵ_λ, is the ratio of energy emitted from the surface at wavelength λ compared to that emitted from a black surface at the same temperature. Analogously, most engineering surfaces do not absorb all the radiation incident upon them, and the *absorptivity*, written as α_λ, is defined as the fraction of the energy absorbed by the surface compared to that absorbed by a black surface exposed to the same incident radiation. Kirchhoff's Law states that at thermal equilibrium the absorptivity and emissivity are equal. If this is assumed to hold under non-equilibrium conditions, and the emissivity and absorptivity are independent of wavelength, then the surface is known as a *grey surface*. Consequently for grey surfaces

$$\alpha = \epsilon = 1 - \rho \tag{6.4}$$

where the quantity ρ is known as the *reflectivity* and the subscript λ, indicating dependence on wavelength, has been dropped. A *grey Lambert surface* is therefore one for which equations (6.2) and (6.4) are true. For the furnace under consideration, the enclosure in which the radiation exchange takes place contains gases which take part in the heat transfer process. The notions of absorptivity and emissivity must be extended to encompass this commonly encountered situation.

6.3.1 Attenuation Laws and Gas Absorptivity/Emissivity

Consider a beam of radiation propagating in an absorbing medium. Assuming no scattering occurs, the Brouger-Lambert law of attenuation states that the fractional decrease in intensity is proportional to the incremental distance traversed

$$-\frac{dI}{I} = kdx \tag{6.5}$$

where the quantity k known as the *absorption coefficient*. Integrating the above expression and assuming k to be constant, gives

$$I = I_0 e^{-kl} \tag{6.6}$$

where I_0 is the initial value of the intensity and l represents the length traversed. In equation (6.6) the exponential factor by which the intensity is reduced is called the *transmittance*. The quantity of energy absorbed by the medium – the *absorptivity* and written as α_g is given by

$$\alpha_g = 1 - e^{-kl} \tag{6.7}$$

Usually the absorption coefficient for gases is expressed in terms of the partial pressure and so

$$\alpha_g = 1 - e^{-k_p pl} \tag{6.8}$$

where p is the partial pressure of the absorbing gas. If Kirchhoff's Law is assumed to hold in conditions of non-thermal equilibrium and k_p is independent of wavelength, then the medium is described as *grey* and the emissivity ϵ_g satisfies

$$\epsilon_g = \alpha_g = 1 - e^{-k_p pl} \tag{6.9}$$

In radiation modelling, 'real gases' are often approximated as the weighted sums of a finite number of grey gases so that

$$\epsilon_g = \sum_n a_{g,n}(T_g)(1 - e^{-k_{p,n} pl}) \tag{6.10}$$

The scalar $k_{p,n}$ is the absorption coefficient of the n^{th} grey gas and the $a_{g,n}$ are weighting coefficients which depend on the gas temperature T_g. The weights are required to satisfy

$$a_{g,n}(\cdot) > 0 \tag{6.11}$$

$$\sum_n a_{g,n}(\cdot) = 1 \tag{6.12}$$

so that the properties of emissivity are maintained. Essentially this can be thought of as a curve fitting procedure; for further details see Hottel & Sarofim [36]. Similarly for absorptivity

$$\alpha_g = \sum_n a_{s,n}(T_s)(1 - e^{-k_{p,n}pl}) \tag{6.13}$$

where T_s is the temperature of the emitting surface and the weighting functions satisfy similar constraints to (6.11) and (6.12). For practical purposes, the $a_{s,n}$ and $a_{g,n}$ are usually low order polynomials, and can be thought of as coefficients that 'fit' the exponential curves to actual gas data. In [36] it is argued that for engineering purposes only three grey gases are required for reasonable accuracy with one of the $k_{p,n}$ being identically zero.

Finally an expression comparable to the one for emissive power for a surface can be established for a gas volume. It can be shown that the rate of energy loss from a gas volume V, at temperature T_g, with an absorption coefficient k, into the surrounding 4π steradians is given by

$$\dot{Q} = 4kV\sigma T_g^4 \tag{6.14}$$

The next section introduces the concept of *exchange areas* which will play a fundamental role in the evaluation of radiation exchange in an enclosure.

6.3.2 Introduction to Exchange Areas

Consider two isothermal grey Lambert surfaces A_1 and A_2 at temperatures T_1 and T_2 respectively, separated by a grey absorbing medium as shown in Figure 6.2. Let dA_1 and dA_2 be incremental areas on the respective surfaces and consider the rate of energy exchange between them.

The radiative intensity from dA_1 at an angle η_1 to the surface normal is given by

$$\cos\eta_1 dA_1 E_1/\pi$$

where the grey body emissive power $E_1 = \epsilon\sigma T_1^4$. The rate of energy transfer, through the solid angle $d\omega$ subtended by dA_2, which reaches dA_2 is

$$e^{-kr}\cos\eta_1 d\omega dA_1 E_1/\pi$$

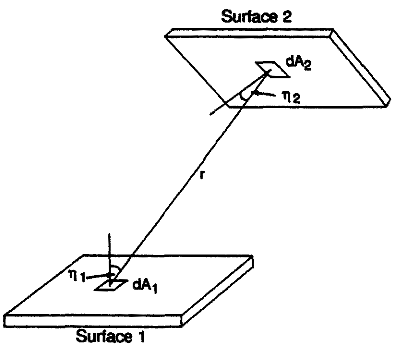


Figure 6.2: Direct exchange area between surfaces

The solid angle

$$d\omega = dA_2 \cos\eta_2 / r^2 \tag{6.15}$$

and so the rate of energy transfer from dA_1 to dA_2 is

$$\frac{\cos\eta_1 \cos\eta_2 e^{-kr}}{\pi r^2} dA_1 dA_2 E_1$$

Therefore the rate of energy transfer from surface A_1 to A_2 is given by

$$\dot{Q}_{12} = (\overline{s_1 s_2}) E_1 \tag{6.16}$$

where

$$(\overline{s_1 s_2}) \stackrel{s}{=} \int_{A_2} \int_{A_1} \frac{\cos\eta_1 \cos\eta_2 e^{-kr}}{\pi r^2} dA_1 dA_2 \tag{6.17}$$

and the quantity $(\overline{s_1 s_2})$ is the *direct exchange area* between surface A_1 and A_2. These ideas can be extended to radiation exchange between surfaces and gas volumes, and also gas volume to gas volume exchange. The direct exchange area between a volume i and a surface j, written $(\overline{g_i s_j})$, is defined to be

$$(\overline{g_i s_j}) \stackrel{s}{=} \int_{A_j} \int_{V_i} \frac{k cos\eta_j e^{-kr}}{\pi r^2} dV_i dA_j \tag{6.18}$$

Similarly for volume-to-volume exchanges

$$(\overline{g_i g_j}) \stackrel{s}{=} \int_{V_j} \int_{V_i} \frac{k^2 e^{-kr}}{\pi r^2} dV_i dV_j \tag{6.19}$$

Details of the derivation of these integral equations appear in Rhine & Tucker [58]. The direct exchange areas can be shown to satisfy

- Reciprocity Conditions –

$$(\overline{s_i s_j}) = (\overline{s_j s_i}) \qquad (\overline{s_i g_j}) = (\overline{g_j s_i}) \qquad (\overline{g_i g_j}) = (\overline{g_j g_i}) \tag{6.20}$$

- Summation Constraints –

$$\sum_j (\overline{s_i s_j}) + \sum_j (\overline{s_i g_j}) = A_i \tag{6.21}$$

$$\sum_j (\overline{g_i s_j}) + \sum_j (\overline{g_i g_j}) = 4kV_i \tag{6.22}$$

The expressions for direct exchange areas given above are, in general, difficult to evaluate. In the case where the intervening medium is non-absorbing, the integrals can be evaluated analytically for simple geometrical configurations, see for example [36]. However when $k \neq 0$ no analytic solutions exist and a numerical approach must be adopted. In the paper by Hottel & Cohen [35] the exchange areas between squares and cubes of common side b are given in charts as functions of the so-called *optical length* kb. These charts can be used to calculate exchange areas in rectangular geometries where the surfaces and volumes can be built up from squares and cubes by using so-called *view-factor algebra* (for example see Hottel & Sarofim [36]). Similar charts, which extend the maximum range of optical lengths, are presented in [76]. Exponential correlations in a form suitable for inclusion into the source code of furnace models are also included. Direct numerical integration techniques are discussed in [61] where in particular the area integrals are reduced to line integrals by the use of Stoke's Theorem, and the efficiency of different numerical schemes are considered. 'Monte Carlo' methods are also discussed in the literature – a sketch of the technique appears in [58] and in the paper by Tucker & Ward [77]. The Monte Carlo method has the advantage that it is applicable to irregular enclosure geometries and configurations where 'shading' or partial obscuring takes place.

The methods outlined above are only approximate and the resulting direct exchange areas are unlikely to satisfy the summation constraints (6.21) – (6.22). This deficiency is usually overcome by 'smoothing' [46]. Here the (approximate) exchange area coefficients are perturbed to satisfy the constraints in a way that the sum of the squares of the perturbations is minimised. A critique of the method appears in [53].

It was noted earlier that a grey surface reflects a fraction ρ of the radiation incident upon it. This presents an added complication when calculating the radiation exchanges in an enclosed environment, since $(\overline{s_i s_j})E_i$ does not represent the total fraction of energy leaving A_i that impinges on A_j, because rays may reach A_j after being reflected ad infinitum from all other surfaces within the enclosure. In the next section a modification will be introduced to circumvent this difficulty.

6.3.3 Total Exchange Areas and Directed Flux Areas

To take account of multiple reflections a quantity called the *total exchange area* is introduced. The total exchange area $(\overline{S_i S_j})$ between two surfaces i and j (and distinguished from direct exchange areas by the use of upper case letters) is defined to be the fraction of the total energy leaving A_i, that reaches A_j, making allowance for all multiple reflections from all other surfaces. Analogous definitions can be made for total exchange areas for volume-surface and volume-volume exchange. Total exchange areas are calculated from their direct counterparts by algebraic methods. The most easily understood exposition is given in the paper by Noble [55]. If the exchange areas are viewed as elements of appropriate matrices so that, for example $(\overline{s_i s_j})$ is the ijth element of $(\overline{ss}) \in \mathbb{R}^{n \times n}$, then for a collection of surfaces $\{A_1, \ldots A_n\}$ and gas volumes $\{V_1, \ldots V_m\}$, the total surface to surface exchange areas are given by

$$(\overline{SS}) = \{\epsilon \delta_{ij} A_i\} \epsilon R^{-1} (\overline{ss}) \tag{6.23}$$

where δ_{ij} is the Kroneker Delta and the matrix $R \in \mathbb{R}^{n \times n}$ is defined element-wise as

$$R_{ij} \stackrel{s}{=} \{\delta_{ij} A_i\} - \rho(\overline{s_i s_j}) \tag{6.24}$$

Also the total exchange areas for gas volumes are given in matrix terms by

$$\begin{aligned} (\overline{GS}) &= \{\delta_{ij} A_i\} \epsilon R^{-1} (\overline{sg}) && (6.25) \\ (\overline{GG}) &= (\overline{gs}) \rho R^{-1} (\overline{sg}) + (\overline{gg}) && (6.26) \end{aligned}$$

Proofs of these relationships are given in Rhine & Tucker [58]. The above discussion has assumed that both the surfaces and gas are grey. In the discussion on emissivity/absorptivity it was explained that 'real gases' are often approximated by a weighted sum of different grey gases. This additional complexity can be carried through by the use of *directed flux area.* The directed exchange area between surface i and surface j, and written $(\overrightarrow{S_iS_j})$, is defined to be

$$(\overrightarrow{S_iS_j}) = \sum_n a_{s,n}(T_i)\ (\overline{S_iS_j})_n \tag{6.27}$$

where the $a_{s,n}$'s are the temperature dependent weighting factors defined earlier, T_i is the temperature of surface i, and $(\overline{S_iS_j})_n$ is the total exchange area obtained using the absorption coefficient k_n. This makes $(\overrightarrow{S_iS_j})$ temperature dependent and destroys the reciprocity constraint. Analogously

$$(\overrightarrow{S_iG_j}) = \sum_n a_{s,n}(T_i)\ (\overline{S_iG_j})_n \tag{6.28}$$

$$(\overrightarrow{G_iS_j}) = \sum_n a_{g,n}(T_{g,i})\ (\overline{G_iS_j})_n \tag{6.29}$$

$$(\overrightarrow{G_iG_j}) = \sum_n a_{g,n}(T_{g,i})\ (\overline{G_iG_j})_n \tag{6.30}$$

where $T_{g,i}$ is the temperature of the ith gas volume.

In addition to the radiative exchange taking place within an enclosure, energy will be lost from the system to the environment as a result of *conduction* through the bounding walls. The differential equations governing unsteady heat conduction will be described in the next section

6.4 Unsteady Heat Conduction in Solids

Consider a slab of uniform composition in which there exists a temperature gradient in the through thickness. Fourier's Law for unsteady heat conduction states that the heat flux at any point, in a given direction, is proportional to the temperature gradient in that direction. Therefore the heat flux in the through thickness direction of the slab is given by

$$\dot{q} = -\lambda \frac{\partial \phi}{\partial x} \tag{6.31}$$

where $\phi(x,t)$ is the temperature field and the proportionality constant λ is referred to as the *conductivity*. By considering an energy balance on an incremental layer of depth dx, it can be shown that within the solid, the temperature distribution must satisfy

$$\rho c \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}(\lambda \frac{\partial \phi}{\partial x}) \tag{6.32}$$

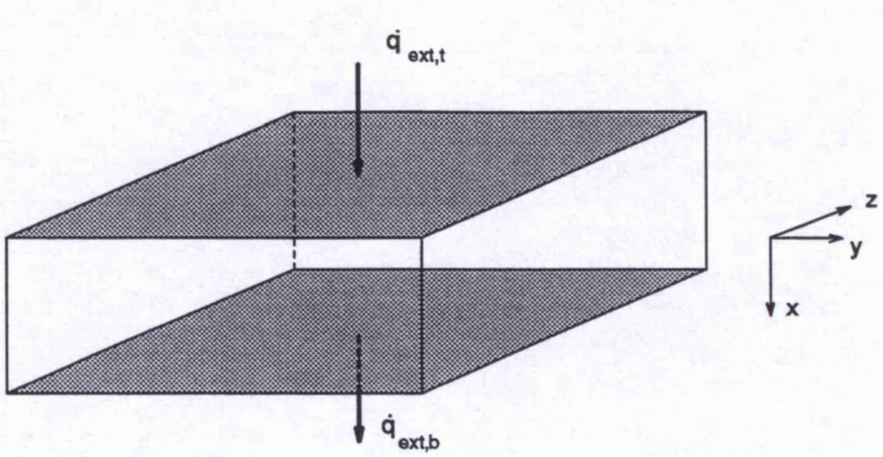


Figure 6.3: Heat conduction in a uniform slab

where ρ and c represent the material's *density* and *specific heat* respectively. In order to solve equation (6.32) boundary equations need to be specified at the top and bottom surfaces. These surfaces will be subjected to some form of energy exchange with the surrounding environment – for example radiative exchange with other surfaces or convective heat exchange with respect to a fluid passing over it. The boundary equations are obtained by equating the net flux resulting form the external energy exchange to the conductive heat flux at the surface i.e.

$$-\lambda \frac{\partial \phi_i}{\partial x}\bigg|_{x=surf} = \dot{q}_{ext} \tag{6.33}$$

If it is assumed that the thermal properties λ, ρ and c are constants, then for simple boundary conditions it is possible to solve the conduction equation analytically – for example using Fourier transforms – otherwise numerical techniques must be employed. Numerical techniques for this problem are well established and allow temperature dependent thermal properties and realistic boundary conditions to be considered [68]. For simple regular geometries such as spheres, cylinders and slabs, finite difference methods are usually employed. Suppose a solution to (6.32) is required over the domain $\{(x,t) : x \in [0\ d],\ t \in [0\ t_1]\}$ then the finite difference approach is to cover the domain with a mesh and evaluate the solution only at the nodes. The partial differential equations are converted to difference equations which relate the value at a node to the values of neighbouring ones. Let δt and δx be strictly positive scalars which represent the inter-node distances in the time and spatial directions respectively. Define $\phi_i^j = \phi(i\delta x, j\delta t)$

then the partial derivatives can be approximated as

$$\frac{\partial \phi}{\partial x} \simeq (\phi^j_{i+1} - \phi^j_i)/\delta x \quad \text{and} \quad \frac{\partial \phi}{\partial t} \simeq (\phi^{j+1}_i - \phi^j_i)/\delta t$$

and consequently an expression for the conduction equation in finite difference form is

$$(\rho c)_i(\phi^{j+1}_i - \phi^j_i)/\delta t = \left(\lambda_i(\phi^j_{i+1} - \phi^j_i)/\delta x - \lambda_{i-1}(\phi^j_i - \phi^j_{i-1})/\delta x\right)/\delta x \tag{6.34}$$

where λ_i and $(\rho c)_i$ are the thermal properties evaluated at the temperature $(\phi^j_{i+1}+\phi^j_i)/2$. Equation (6.34) is an explicit method since it can easily be re-arranged to give ϕ^{j+1}_i in terms of ϕ^j_{i+1}, ϕ^j_i, ϕ^j_{i-1}. In this way, the solution can be found by 'marching forward' time-step by time-step. Usually, for reasons of numerical stability, implicit schemes are chosen. The so-called Crank-Nicholson [10] representation is

$$\begin{aligned}(\rho c)_i(\phi^{j+1}_i - \phi^j_i)/\delta t = &\left(\lambda_i \left((\phi^{j+1}_{i+1} - \phi^{j+1}_i)/\delta x + (\phi^j_{i+1} - \phi^j_i)/\delta x\right)\right. \\ &\left. -\lambda_{i-1} \left((\phi^{j+1}_i - \phi^{j+1}_{i-1})/\delta x + (\phi^j_i - \phi^j_{i-1})/\delta x\right)\right)/2\delta x \end{aligned} \tag{6.35}$$

where the right-hand-side may be thought to represent the average flux between the times $j\delta t$ and $j\delta t + \delta t$. The Crank-Nicholson finite difference equation can be rewritten as

$$a_{i,i-1}\phi^{j+1}_{i-1} + a_{i,i}\phi^{j+1}_i + a_{i,i+1}\phi^{j+1}_{i+1} = b_{i,i-1}\phi^j_{i-1} + b_{i,i}\phi^j_i + b_{i,i+1}\phi^j_{i+1} \tag{6.36}$$

for $i = 2, 3, ...n-1$ where n is the number of spatial nodes and $a_{i,j}$ and $b_{i,j}$ are appropriately defined scalar coefficients. Finite difference approximations are also required for the boundary equations. The most common approach is to consider an energy balance on the 'half element' at the surface (see Figure 6.4). If the temperature varies linearly between the top two nodes then the temperature in the middle of the half element, ϕ^j_M, is given by

$$\phi^j_M = (3\phi^j_1 + \phi^j_2)/4$$

By considering the flux entering the half element and performing an energy balance, it can be verified that

$$(\rho c)_1(\phi^{j+1}_M - \phi^j_M)/\delta x = \tfrac{1}{2}\left(2\dot{q}_{ext,t} - \lambda_1\left((\phi^{j+1}_2 - \phi^{j+1}_1)/\delta x + (\phi^j_2 - \phi^j_1)/\delta x\right)\right)\delta t \tag{6.37}$$

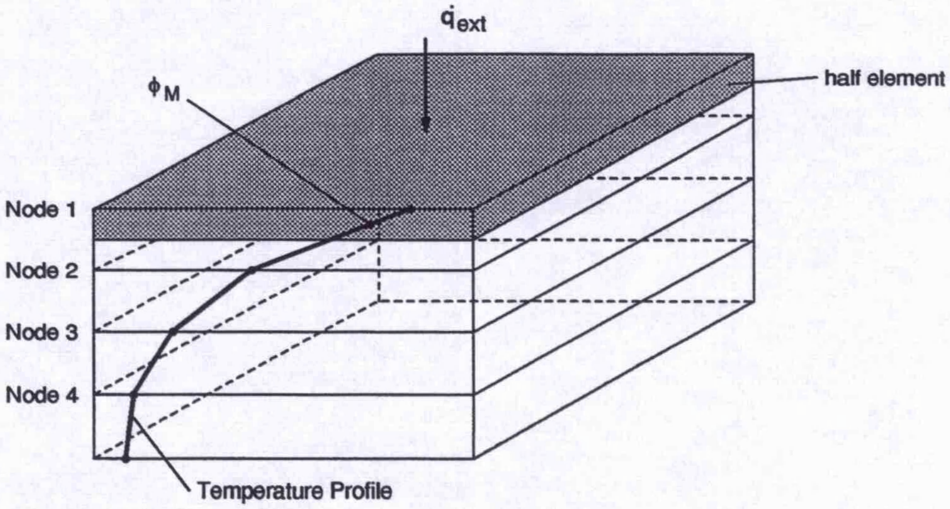


Figure 6.4: Half element at the surface

After substituting for ϕ_M^j and ϕ_M^{j+1} the following implicit relationship can be derived

$$a_{1,1}\phi_1^{j+1} + a_{1,2}\phi_2^{j+1} = b_{1,1}\phi_1^j + b_{1,2}\phi_2^j + \delta t \dot{q}_{ext,t} \tag{6.38}$$

A similar expression can be derived for the boundary condition on the lower surface. If a vector of node temperatures is introduced

$$\underline{\phi}^j \stackrel{s}{=} \begin{bmatrix} \phi_1^j \\ \phi_2^j \\ \vdots \\ \phi_n^j \end{bmatrix} \tag{6.39}$$

then the finite difference equations (6.36) and (6.38) can be conveniently written in matrix form as

$$A\underline{\phi}^{j+1} = \underline{d} \tag{6.40}$$

where $\underline{d}$ is a vector function of the node temperatures at the previous time step $\underline{\phi}^j$ and the energy fluxes at the surfaces $\dot{q}_{ext,t}$ and $\dot{q}_{ext,b}$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \dots & \dots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix} \tag{6.41}$$

Because of the special tridiagonal structure of the matrix A, an explicit 'back substitution' solution to the matrix equation (6.40) can be found [68]. As in the case of the explicit representation, by marching forward time-step by time-step the solution can be obtained. The finite difference approximations just described and the concept of directed flux areas described in the previous section, are the fundamental components of the 'zone method' approach to transient temperature modelling which will be described in the next section.

6.5 Description of the 'Zone Method'

The '*Zone Method*' [35] is the most well established technique for modelling radiation exchange in an enclosure. The method involves the breaking up of all relevant surfaces and volumes into a patch-work of sub-surfaces and sub-volumes, termed *zones*, which are small enough to be considered isothermal.

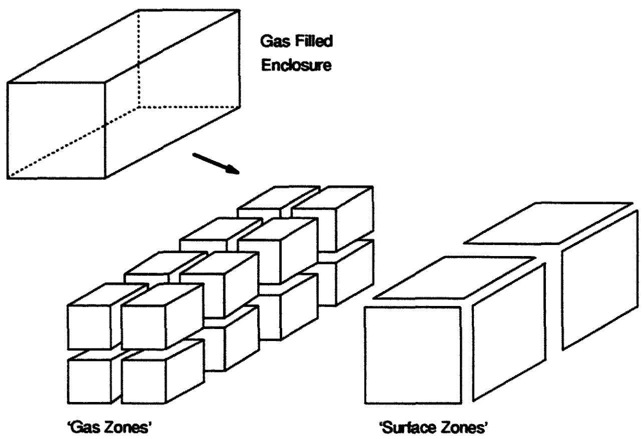


Figure 6.5: Schematic of the Zone Method

Directed flux areas are then calculated for every possible pair of zones using the procedures described earlier. Heat balances are then used to derive equations connecting the surface and gas volume temperatures. These are solved in conjunction with finite difference equations which represent the heat loss by conduction through the enclosure walls. The heat balance equations contain terms that require knowledge of the gas mass flow rates and also the volumetric energy release distribution. This requirement is perhaps the most serious draw-back, since, because of the computational burden inherent in the method, these quantities need to be evaluated in isolation - either by direct measurement, a CFD approach, or by empirical correlation.

6.5.1 Model Assumptions

The assumptions inherent in the 'Zone Method' modelling technique will now be described:

A1) the interior surfaces of the enclosure are divided into a total of N surface zones of area $\{A_1, ..., A_N\}$ respectively. Each surface is assumed to be a grey–Lambert surface and the ith surface zone is assumed to be isothermal and at temperature T_i.

A2) the enclosure is assumed to be filled with a gas which can be modelled as a finite sum of grey gases. The interior volume is divided up into M isothermal gas zones of volume $\{V_1, ..., V_M\}$ respectively with the ith volume being at temperature $T_{g,i}$. The volumetric energy release resulting from combustion in the ith zone, and denoted by $\dot{Q}_{comb,i}$, is assumed to be known.

A3) Heat loss through the enclosure walls is modelled as a 1-dimensional transient conduction problem. Each surface zone is considered (independently) to be one surface of a rectangular slab, in which there exists a temperature gradient only in the direction perpendicular to the surface. If $\dot{q}_{net,i}$ represents the net energy flux at the ith surface in the direction perpendicular to and towards the surface, and ϕ_i represents the temperature distribution in the slab, then

$$-\lambda \frac{\partial \phi_i}{\partial x}\bigg|_{x=surf,int} = \dot{q}_{net,i} \tag{6.42}$$

On the opposite (outside) surface, it is assumed that radiation to the surroundings, and natural convection occurs. Therefore the boundary condition is

$$-\lambda \frac{\partial \phi_i}{\partial x}\bigg|_{x=surf,ext} = h_i(\phi_{i,ext} - T_{Amb}) = \dot{q}_{ext,i} \tag{6.43}$$

where $\phi_{i,ext}$ is the external temperature of the slab, T_{Amb} is the ambient temperature of the surroundings, and h_i is the heat transfer coefficient.

6.5.2 Surface and Gas Volume Heat Balances

Although the dominant mechanism of heat exchange in high temperature enclosures is radiation, the subordinate effect of convection and bulk enthalpy movement, as a result of the flow of gas inside, are taken into account in the heat balances. Let $E_{b,i} = \sigma T_i^4$ and $E_{g,i} = \sigma T_{g,i}^4$ then an energy balance on the ith surface gives

$$A_i \; \dot{q}_{net,i} = \sum_j (\overrightarrow{S_j S_i}) E_{b,j} + \sum_j (\overrightarrow{G_j S_i}) E_{g,j} - A_i \epsilon_i E_{b,i} + \dot{Q}_{conv,i} \tag{6.44}$$

where $\dot{Q}_{conv,i}$ represents convection gains by the surface as a result of the flow of hot gas across the surface. For each gas zone the following quasi steady-state condition is imposed

$$\sum_j (\overrightarrow{G_j G_i}) E_{g,j} + \sum_j (\overrightarrow{S_j G_i}) E_{b,j} - \sum_n 4 a_{g,n} k_n V_i E_{g,i} + \dot{Q}_{enth,i} - \dot{Q}'_{conv,i} + \dot{Q}_{comb,i} = 0 \tag{6.45}$$

where $\dot{Q}'_{conv,i}$ is the total heat loss from gas volume i to all contiguous surfaces as a result of convection and $\dot{Q}_{enth,i}$ is the net enthalpy gain taking into account the flow of combustion products, enthalpy loss through a flue, or gains from any fuel/air inputs to the gas volume.

6.5.3 Algorithm for Zone Method

In the previous section $N+M$ nonlinear equations were derived in terms of the unknown temperatures $\{T_i\}_{i=1,N}$ and $\{T_{g,j}\}_{j=1,M}$ and also the surface fluxes $\{\dot{q}_{net,i}\}_{i=1,N}$ which relate the energy balances to the temperature distributions in the enclosure walls (via the boundary conditions (6.42)). The following algorithm solves these coupled algebraic and partial differential equations using a finite difference approach to update the surface and gas temperatures over a finite time interval.

1) Set the gas zone temperatures $\{T_{g,j}\}_{j=1,M}$ and the through thickness wall temperatures $\{\underline{\phi}_i^t\}_{i=1,N}$ to the 'initial condition' values. (NB: the first element of $\underline{\phi}_i^t$ is by definition the surface temperature T_i)

2) Using the current values of $\{T_{g,j}\}_{j=1,M}$, $\{\underline{\phi}_i^t\}_{i=1,N}$ and the current flow values, calculate the enthalpy and convection terms $\{\dot{Q}'_{enth,j}\}_{j=1,M}$ and $\{\dot{Q}_{conv,j}\}_{j=1,M}$ that appear in the gas balance equations.

3) Using the current values of $\{\underline{\phi}_i^t\}_{i=1,N}$, the enthalpy and convection terms from step 2, and the combustion distribution $\{\dot{Q}_{comb,j}\}_{j=1,M}$, solve the M nonlinear gas volume heat balance equations (6.45), using some appropriate numerical scheme such as Newton-Raphson[1] to obtain 'new' gas temperatures $\{T^*_{g,j}\}_{i=j,M}$. (For a well-defined system the nonlinear equations representing the heat balances are sufficiently 'well behaved' and in practice no numerical difficulties have been encountered.)

4) Calculate the net heat flux at the surface zones $\{\dot{q}_{net,i}\}_{i=1,N}$ from the surface energy balance equation (6.44)) using the old values of temperature $\{\underline{\phi}_i^t\}_{i=1,N}$ and the new gas temperatures $\{T^*_{g,j}\}_{j=1,M}$. Also evaluate the external boundary condition $\{\dot{q}_{ext,i}\}_{i=1,N}$ using equation (6.43).

5) Use $\{\dot{q}_{ext,i}\}_{i=1,N}$ and $\{\dot{q}_{net,i}\}_{i=1,N}$ and the Crank-Nicholson finite difference scheme given in equation (6.35) to compute the new internal node temperatures after a time increment $\{\underline{\phi}_i^{t+\delta t}\}_{i=1,N}$. (Therefore in steps 2–5 the gas and surface temperature values have both been updated over the interval δt.)

6) If a 'control scheme' is included then use the new temperature values to compute the 'control action' and consequently update the combustion distribution and flow patterns.

7) Increment the variable calculating elapsed simulation time by δt.

8) Unless the simulation is complete, return to step 2.

Up to this point the radiation modelling techniques including the 'zone method' have been discussed in general terms. The next section describes in detail the features of a particular mathematical model of an experimental gas fired furnace which has been developed. In particular certain non-generic features have been added to make this suitable as a tool for evaluating the performance of controllers.

[1] See for example [34].

6.6 The Zone Method Applied to an Experimental Furnace

The Gas Research Centre at Loughborough has an experimental single-burner gas fired furnace which is used to test the performance of new temperature control schemes. The furnace is virtually identical to the generic heating plant given earlier except for the presence of a water cooling system, which effectively represents a load, as shown in Figure 6.6.

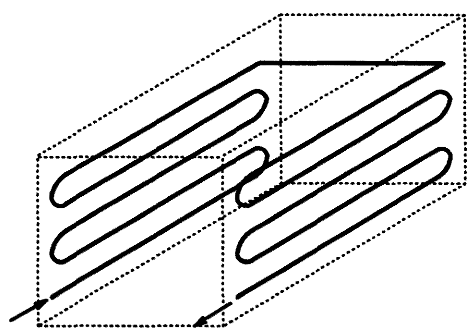


Figure 6.6: Schematic of the pipe arrangement in the water cooling system

A mathematical model of this furnace has been developed to evaluate the performance of the sliding mode controllers before implementation on the experimental rig. The Zone Method approach has been employed for the modelling process for two reasons; firstly it is the method against which all subsequent methods are judged in the literature; and secondly because validated Fortran code written by British Gas plc [56] already existed which drastically reduced the development time. The code was written to simulate general furnace geometries and has been modified to include the effects of the coolant water and the valve arrangement which governs the fuel flow to the burner. A detailed description of the model follows:

- A 'single-gas-zone' approach has been taken – i.e. the interior volume of the furnace is considered as a single gas volume and the surrounding six surfaces as the surface zones. The side, top and back walls are assumed to be made from low density ceramic fibre. The base and front walls are assumed to consist of fire brick which is denser but more conductive. The thermal properties of the insulation are taken from Junot *et al.*[41]. In addition both side walls are assumed to have identical temperature distributions.

- The furnace is assumed to be filled with the combustion products plus any excess air – i.e. air greater than that required for stoichiometric combustion. The excess air is assumed to play no part in the radiation exchange process with only the CO_2 and H_2O present in the combustion products emitting and absorbing radiation. The radiative properties of these gases have been modelled using a three term mixed grey gas approach so that

$$\alpha_g = \epsilon_g = \sum_{i=1}^{3} a_{g,i}(T_g) \left(1 - e^{-k_i(p_c+p_h)x}\right) \tag{6.46}$$

  where p_c and p_h represent the partial pressures of CO_2 and H_2O in the flue products respectively. The amount of excess air present in the furnace affects the value of p_c and p_h and consequently the values of the direct exchange areas which have been calculated using the Monte Carlo method [77]. Since these are computed 'off-line' – i.e. outside the simulation loop – the model must assume a fixed fuel/air ratio. The weighting polynomials $a_{g,i}$ are assumed to be affine functions of the gas temperature so that

$$a_{g,i}(T_g) = p_{1,i} + p_{2,i}T_g \tag{6.47}$$

  The values of the absorption and polynomial coefficients used in the simulation are given in Chapter 10 of Rhine & Tucker [58].

- As a result of the single-gas-zone approach, the term $\dot{Q}_{comb}$ representing the energy release as a result of combustion, can be obtained by multiplying the fuel mass flow rate by the calorific value of natural gas. The air entering the burner is preheated by the combustion products as an energy saving device. In the simulation the temperature of the air entering the furnace $T_{air,in}$ is assumed to satisfy the relationship

$$T_{air,in} = T_{amb} + \eta_{eff}(T_g - T_{amb}) \tag{6.48}$$

  where T_{amb} is the ambient temperature of the surroundings and η_{eff} is the thermal efficiency of the recovery device [58]. The bulk enthalpy term $\dot{Q}_{enth}$ comprises the rate of energy input into the furnace as a result of the fuel and air in-flow, minus the rate of energy loss through the flue. Data relating to the enthalpy of the flue products and the enthalpy of the air are given in Appendix 1 of [58].

- The effect of the water cooling arrangement has been modelled by incorporating an extra term in the heat balance equation of the surface zone which represents the side wall. Specifically

$$A_{i_w}\ \dot{q}_{net,i_w} = \sum_j (\overrightarrow{S_j S_{i_w}}) E_{b,j} + \sum_j (\overrightarrow{G_j S_{i_w}}) E_{g,j} - A_{i_w} \epsilon E_{b,i_w} + \dot{Q}_{conv,i_w} - A_{i_w} \dot{q}_w \quad (6.49)$$

where $\dot{q}_w$ represents the heat flux to the water as a result of forced convection and is assumed to have the structure

$$\dot{q}_w = h_1 (T_{i_w} - T_w)^{h_2} \quad (6.50)$$

where T_w is the in-coming water temperature and h_1 and h_2 are scalars. More detailed modelling would be difficult in view of the complicated nature of the water circulation with respect to the zone geometry. The evaluation of suitable coefficients for equation (6.50) will be discussed in the next subsection.

6.6.1 The Quantitative Effects of the Water flow

The coefficients h_1 and h_2 defining the heat flux to the water have been determined experimentally. The furnace at the Gas Research Centre is fitted internally with five thermocouples – two are located on the inside of each side wall and one on the front wall directly facing the burner. In addition, thermocouples in the water flow measure the in-going and out-going temperature, and a meter in the water supply pipe enables the water volume flow rate to be calculated. From this information the rate of energy loss to the water can be calculated as

(the increase in water temperature) $\times$ (mass flow rate) $\times$ (specific heat)

Figure 6.7 shows the heat flux to the coolant water plotted against the difference between the wall temperature and incoming water temperature. This experimental data was obtained whilst the furnace was subjected to a series of typical heating and cooling cycles. The values of h_1 and h_2 in equation (6.50) were estimated from a least squares curve fit. The resulting curve is plotted in Figure 6.7 together with the original data.

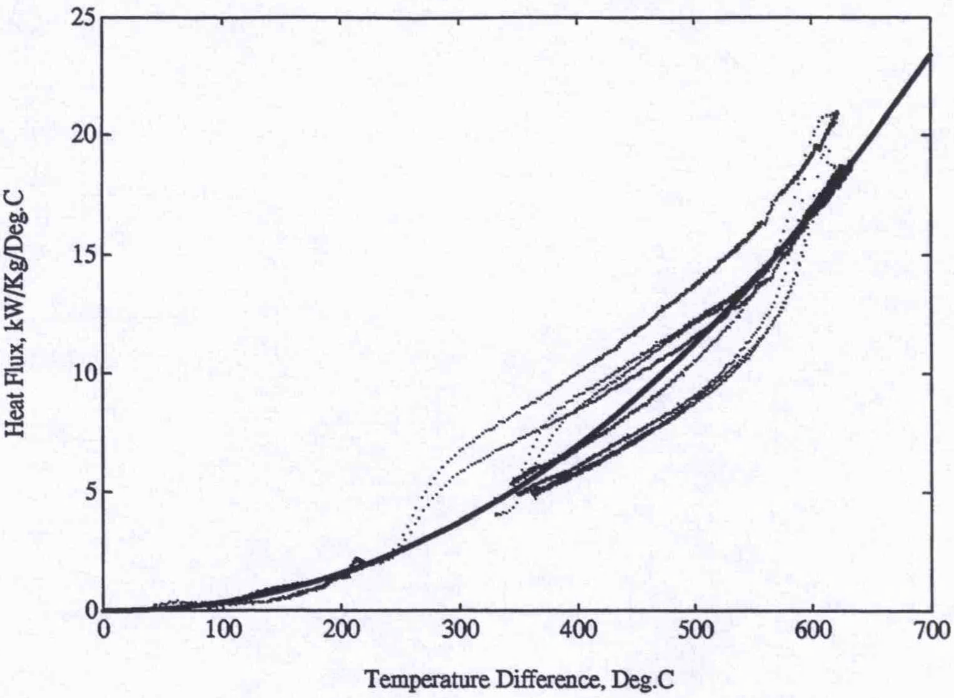


Figure 6.7: Measured heat Flux compared to the empirical relationship

6.6.2 Calibration of Fuel Flow Valve

In order to simulate the experimental furnace in a form suitable for evaluating the effect of different control schemes, it has been necessary to derive a relationship between the voltage applied to the valve positioner system and the resulting fuel flow rate. An experiment was conducted at the Gas Research Centre whereby the excitation signal shown in Figure 6.8 was applied to the valve positioner system. The steps in the 'staircase' function were long enough to allow steady state flow to be achieved in the fuel line.

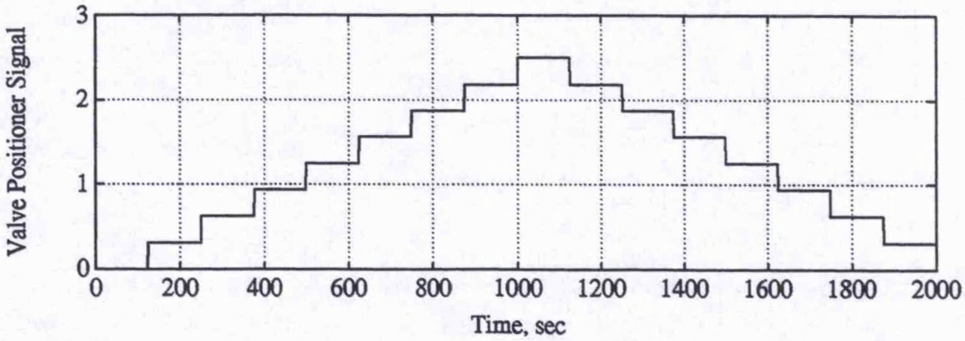


Figure 6.8: Valve positioner excitation signal

Unfortunately the volume flow meter in the fuel line was not able to provide an analogue signal that could be logged continuously. However it was possible to monitor a voltage representing the differential pressure across an orifice plate in the fuel flow. It is well known that for an incompressible fluid passing through an orifice plate the velocity is proportional to the square root of the differential pressure[2]. In this way the fuel volume flow could be indirectly monitored and the onset of steady state identified. Once the differential pressure signal attained steady state a manual reading of the fuel flow was made from the digital display on the flow meter. The resulting set of discrete data points, relating steady state volume flow to the valve positioner signal, is plotted in Figure 6.9. A curve of the form

$$f(v) = c_0 + c_1 \tanh(c_2 v + c_3) \tag{6.51}$$

was fitted through the observation data points which is also shown in Figure 6.9.

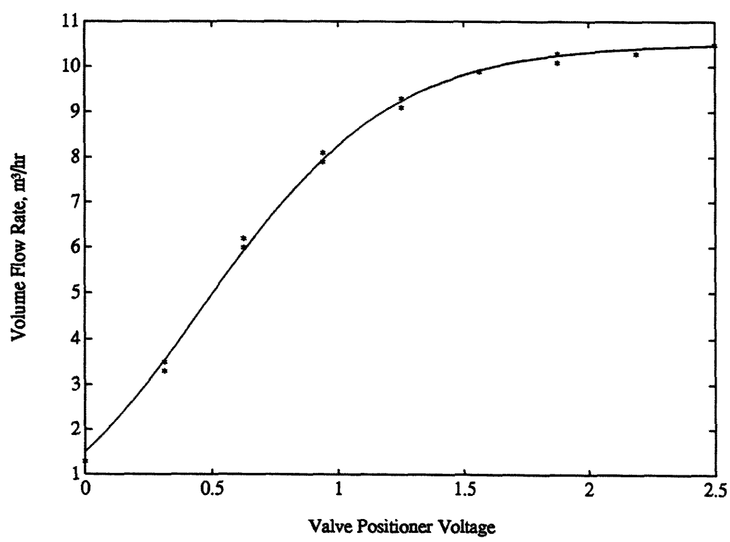


Figure 6.9: Steady state volume flow rate against valve position

In this way $f(v)$ represents the steady state volume flow rate obtained from applying a voltage v to the valve positioner. An expression relating the valve positioner voltage to

[2]Such a device is usually referred to as a Venturi flow meter – see for example [75].

the volumetric energy release in the furnace can be obtained as

$$\dot{Q}_{comb}(v) = CV_{net}\ \rho_f\ f(v) \tag{6.52}$$

where ρ_f is the density of the fuel and CV_{net} is the calorific value.

6.6.3 Model Validation

The modelling work just described is not intended to produce the most accurate furnace model possible. However, it is important that the simulation is realistic. In order to validate the model, a control signal from the experimental furnace was used as an open-loop input to the simulation, and the respective outputs compared. This represents quite a severe test of the quality of the simulation. The results are given graphically in Figure 6.10, where the solid line represents the output from the model.

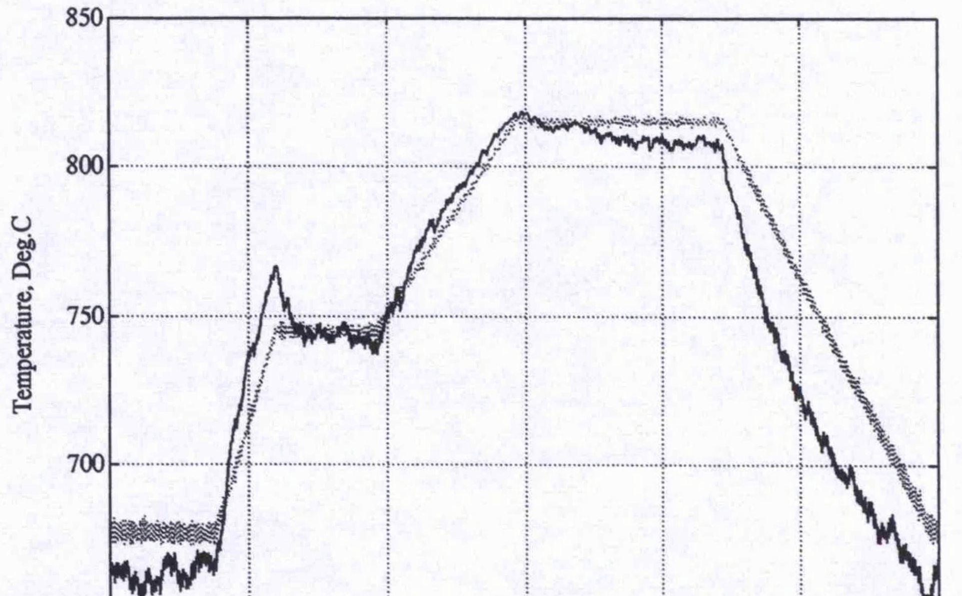


Figure 6.10: Comparison between the plant and model outputs

It can be seen that the simulation is representative of the real furnace, and thus is suitable as an initial testbed for the sliding mode control schemes that will be developed in the next chapter.

6.7 Summary

This chapter has described the salient features of a mathematical model developed to estimate the temperature distribution in an experimental single burner gas fired furnace. The model utilizes the most reliable method of computing the radiation exchange in an enclosure filled with a participating medium – namely the Zone Method – and transient conduction analysis in the walls which make up the enclosure boundaries. The model has been validated against open loop data and has been found to give good agreement. The intention is to use this nonlinear model to develop and test sliding mode controllers based on the theoretical ideas proposed in the earlier chapters. With this in mind, the effect of the valve controlling the fuel flow has been incorporated into the model, so that the dynamic temperature response resulting from manipulating the fuel flow can be calculated. Inherently the model assumes a fixed fuel/air ratio and hence is only suitable as a dynamical tool for evaluating the effect of fuel flow on temperature. The next chapter outlines the development and analysis of sliding mode temperature controllers for single burner configurations.

Chapter 7

Sliding Mode Temperature Control Schemes

7.1 Introduction

In this chapter, the theoretical results described earlier in this thesis will be used to develop temperature controllers for furnaces of the type discussed in the previous chapter. The control problem considered here involves the manipulation of the fuel flow rate so that the measured temperature at some point in the furnace follows some specified temperature/time profile. It is assumed that an independent controller already exists which regulates the air flow so that for a given fuel flow rate an appropriate air flow rate is maintained to ensure combustion efficiency and an appropriate concentration of unburnt oxygen in the flue products. Currently, temperature control in single burner furnaces is usually achieved through the use of traditional Proportional/Integral/Derivative (PID) controllers. Such controllers are commercially available with appropriate protective logic to prevent undesirable effects such as integral wind-up. More advanced 'modern control' approaches have been studied including a bilinear self-tuning strategy [30]. Indeed work is in progress to develop the later approach into a commercially available 'off the shelf' product [29]. More recently the use of various neural network strategies have been explored as in [49, 12]. This chapter considers a sliding mode approach.

The temperature control problem described above is consistent with the simulation described in the previous chapter which relied on assumption that the composition of the products of combustion remained constant. The simulation can therefore be used as a test-bed for evaluating the performance of the temperature controllers that will subsequently be designed. The nonlinear finite difference and algebraic equations that comprise the model do not lend themselves easily to the problem of control law design. Instead a system identification approach will be used to obtain a nominal linear model of the system. The inherent nonlinearities present will be treated as bounded uncertainty and will be incorporated into the (true) uncertainties associated with the valve dynamics and external disturbances. As a result, measurement of internal states of the linear model will not be possible – indeed because of the identification approach adopted, the internal states will not necessarily have any physical meaning. The approaches of Chapters 4 and 5 are therefore applicable.

Later in the chapter a quantitative measure of control system performance, which is typically used for closed-loop analysis within the industry, is described. The performance of the proposed sliding mode schemes can then be formally explored – initially using the simulation described in Chapter 6 as test-bed. The results of trials conducted on the experimental furnace at the Gas Research Centre will also be presented. For comparative purposes, the typical performance of a sophisticated commercial PID controller will also be shown. Using a quantitative measure of closed loop performance – which considers the tracking accuracy, the control effort, and the degree to which frequency components are present in the control action – the effectiveness of the sliding mode scheme will be assessed.

7.2 Identification of a Nominal Linear Model

The sliding mode control schemes of Chapter 4 and 5 require knowledge of the nominal linear system triple (A, B, C). In order to generate a linear model relating the valve position to the thermocouple temperature, a system identification approach was adopted. The identification was made about a steady state operating temperature of approximately 675° using a Pseudo Random Binary Sequence (PRBS) as the excitation signal. Discrete time transfer functions of the form

$$G(z) = \frac{z^{-q}\,(b_m z^{-m} + \cdots + b_1 z^{-1} + b_0)}{a_n z^{-n} + \cdots + a_1 z^{-1} + a_0}$$

for a selection values of $q, m, n \in \mathbb{N}$ were fitted to the input/output data using the ARX model in the MATLAB Identification toolbox. The validity of the linear models was examined by comparing the predicted output with new input/output data obtained from using a different excitation signal.

Model Parameters			Integral of	Model Parameters			Integral of
n	m	q	Absolute Error	n	m	q	Absolute Error
2	1	1	38.93	4	3	1	39.29
2	2	1	43.00	4	4	1	39.26
3	1	1	36.67	5	1	1	36.63
3	2	1	37.83	5	2	1	37.85
3	3	1	38.54	5	3	1	39.92
4	1	1	36.67	5	4	1	38.55
4	2	1	37.72	5	5	1	39.02

Table 7.1: Goodness of fit for different model orders

Table 7.1 gives the integral of absolute error for different values of n and m. It was found that fitting higher order delays i.e. $q > 1$ resulted in unstable models. These were rejected on physical grounds and are not included in the table. A pragmatic choice of linear model is given by $m = 2$ and $n = 3$ which represents the relative degree one system with the lowest prediction error.

After first transforming the discrete time transfer function to continuous time, a minimal order realization, relating thermocouple temperature voltage to the valve positioner signal was obtained. In state space form this may be represented as

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.0001 & -0.0082 & -0.1029 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0.0001 & 0.0022 & 0.0053 \end{bmatrix}$$

Conveniently this realization is already in regular form. The invariant zeros of this linear system are given by $\{-0.3749, -0.0358\}$ and therefore all the preceding theory is valid. The remainder of the chapter is devoted to synthesising and analysing sliding mode temperature controllers based on this nominal linear model. Before describing the proposed designs, it is convenient to describe the quantitative measures of performance which will be used to analyse the closed loop response.

7.3 A Quantitative Measure of Controller Performance

To compare the performance of the nonlinear controller/observer pair with other control schemes, a quantitative measure of performance is proposed, which is similar to that proposed by Goodhart *et al.*[30]. It takes account of how accurately the output tracks the demand signal, the amount of control action used, and the degree to which high frequency components appear in the control action. If $\{y_i\}$,$\{u_i\}$ and $\{r_i\}$ are finite sequences which represent the output, input and demand signals respectively which are sampled at a fixed rate, then formally the index comprises :

1) **Mean Absolute Error** - representing the accuracy of the tracking performance, is defined by :

$$\bar{e} = \sum_{i=1}^{N_s} |y_i - r_i|/N_s \qquad \text{where } N_s \text{ is the number of samples}$$

2) **Mean Control Action** - reflecting the amount of control action used, is defined in the obvious way as

$$\bar{u} = \sum_{i=1}^{N_s} u_i/N_s$$

3) **Measure of Excitation** - which assesses the degree to which high frequency components appear in the control signal, is defined as

$$u_{ex} = \sum_{i=1}^{N_s} |u_i - (u_f)_i|/N_s \qquad \text{where } \{u_f\} = f(\{u\})$$

and f is a linear low pass filter designed to remove the high frequency components from the control action (which are unrepresentative in terms of the movement of the valve).

From a sliding mode perspective the control signal u_f may be viewed as the equivalent control action necessary to maintain a sliding motion. The choice of filter is therefore somewhat arbitrary. Here a discrete linear filter with no phase distortion $f : \{x_i\} \to \{y_i\}$ is proposed. This is similar to the one, used in the preparation of the data prior to identification, suggested by Backx [2]. Formally the filter comprises:

1) A linear first order filter defined by

$$z_i = \alpha z_{i-1} + (1-\alpha)x_i$$

where $\alpha = 0.2$ which generates an intermediate sequence $\{z_i\}_{i=0}^N$.

2) Since the filtering takes place off-line it is possible to define a new sequence $\{z_i'\}_{i=0}^N$ according to $z_i' = z_{N-i}$.

3) Applying the same first order filter to $\{z_i'\}_{i=0}^N$ results in another intermediate sequence $\{y_i'\}_{i=0}^N$ satisfying

$$y_i' = \alpha y_{i-1}' + (1-\alpha)z_i'$$

4) Finally define $\{y_i\}_{i=0}^N$ according to $y_i = y_{N-i}'$

The overall effect $\{x_i\} \rightarrow \{y_i\}$ is a linear (low pass) filter with no phase distortion.

7.4 Design of the Controller Observer Pair

This section discusses the design of an observer/controller pair for the furnace system based on the theory of Chapter 4. From the results described there the controller and observer can be designed independently. The observer design is discussed first.

7.4.1 Observer Design

Before discussing the design procedure adopted it should be noted that because of the structure of the realization, any variation in the elements in the last row of the system matrix (which are the coefficients of the characteristic polynomial) occurs in $\mathrm{R}(B)$, and so can be considered as matched uncertainty. The observer is therefore insensitive to changes in the poles of the system. Utilising the notation §4.4 it follows that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -0.0001 & -0.0082 & -0.1029 \end{array}\right]$$

and

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{cc|c} 0.0001 & 0.0022 & 0.0053 \end{array}\right]$$

If the stable design matrix $A_{22}^s = -0.2$ it follows immediately from equation (4.39) that

$$G = \begin{bmatrix} A_{12}C_2^{-1} \\ A_{22}C_2^{-1} - C_2^{-1}A_{22}^s \end{bmatrix} = \begin{bmatrix} 0 \\ 188.8498 \\ 18.3328 \end{bmatrix}$$

Because the system is single-input single-output there is no need to compute the sliding surface matrix F. All that remains is to compute the nonlinear scalar gain function given in (4.34). In this case, the approach that has been adopted is to let

$$\rho_o(u_L, y) = r_1|y| + r_2|u_L(\cdot)| + \gamma_o \tag{7.1}$$

where $u_L(\cdot)$ represents the linear component of the control action and the positive scalars r_1, r_2 and γ_o are to be chosen empirically. This will be discussed in more detail in a later subsection.

7.4.2 Controller Design

Because the system is already in regular form, the augmented system from equation (4.12) can easily be identified to be

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & -0.0001 & -0.0022 & -0.0053 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & -0.0001 & -0.0082 & -0.1029 \end{array}\right]$$

As predicted from the theory, the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is completely controllable. The poles of the ideal sliding motion dynamics have been chosen to be $\{-0.025,\ -0.03 \pm 0.025i\}$, which represent dynamics marginally faster than the dominant pole of the open-loop plant. The unique M such that $\sigma(\tilde{A}_{11} - \tilde{A}_{12}M) = \{-0.025,\ -0.03 \pm 0.025i\}$ is given by

$$M = \begin{bmatrix} -0.5372 & 0.0019 & 0.0822 \end{bmatrix}$$

For single-input single-output systems no additional design freedom is provided by the scalar Λ. For simplicity it has been chosen so that $S_2 = 1$. The transformation T_ϕ

from equation (4.18) is now completely determined, and in the new coordinates the (augmented) system matrix can be shown to be

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} -0.0028 & -0.0001 & -0.0017 & -0.0053 \\ 0 & 0 & 1 & 0 \\ 0.5372 & -0.0019 & -0.0822 & 1 \\ \hline -0.0096 & 0.0000 & -0.0037 & -0.0179 \end{array}\right]$$

The remaining pole, associated with the range space dynamics, has been assigned the value -0.1 by selecting the design matrix $\Phi = -0.1$. Again because of the single-input single-output nature of the system there is no need to calculate the Lyapunov matrix $\bar{P}_2$ associated with the unit vector controller. The nonlinear component of the controller is given by

$$\nu_c = -\rho_c(u_L, y)\ \mathrm{sgn}(S\tilde{x} - S_r r) \tag{7.2}$$

where the gain function $\rho_c(\cdot) = \rho_o(\cdot) + \gamma_c$ for some positive scalar γ_c; and the hyperplane matrix is given by

$$S = \begin{bmatrix} -0.5372 & 0.0019 & 0.0822 & 1.0000 \end{bmatrix}$$

As argued in §4.6, because of the minimum phase relative degree one assumption, it is always possible to choose S_r so that, for the nominal system, at steady state $x_r = 0$. Using equation (4.81), for the system under consideration $S_r = 26.1642$. From equations (4.27) - (4.29) the gains are given by

$$\begin{aligned} L &= \begin{bmatrix} 0.0537 & -0.0002 & -0.0030 & -0.0821 \end{bmatrix} \\ L_r &= 3.1536 \\ L_{\dot{r}} &= 26.1642 \end{aligned}$$

The control law is given by

$$u(t) = L\tilde{x}(t) + L_r r(t) + L_{\dot{r}}\dot{r}(t) + \nu_c \tag{7.3}$$

where ν_c is defined as in (7.2). The stable matrix Γ from equation (4.7) affects the closed loop performance. Here it has been chosen to tailor the step response of the nominal closed loop. Figure 7.1 shows the step response for different values of Γ in the interval $[-0.04, -0.015]$.

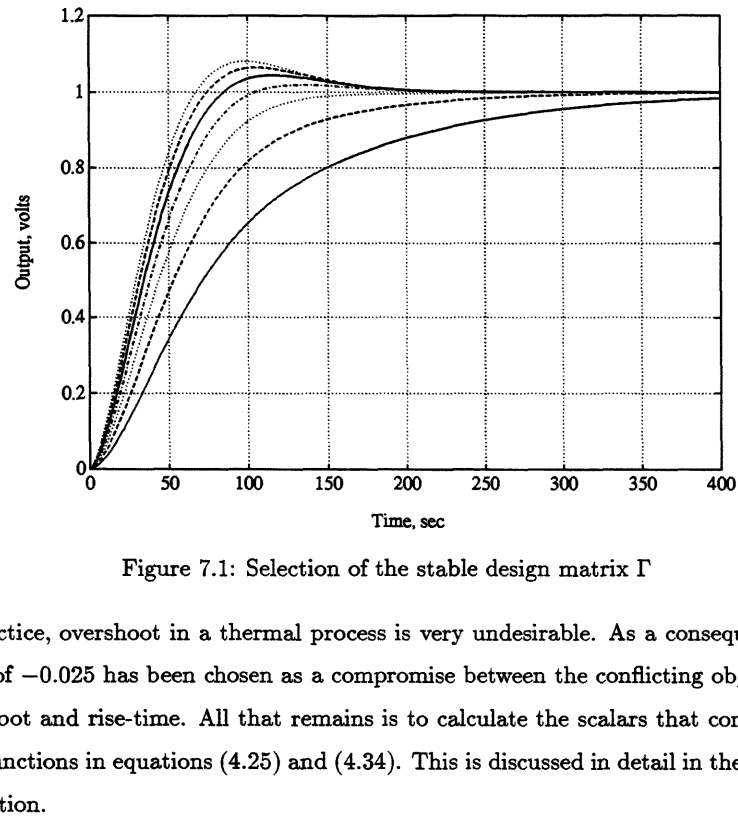


Figure 7.1: Selection of the stable design matrix Γ

In practice, overshoot in a thermal process is very undesirable. As a consequence, the value of -0.025 has been chosen as a compromise between the conflicting objectives of overshoot and rise-time. All that remains is to calculate the scalars that comprise the gain functions in equations (4.25) and (4.34). This is discussed in detail in the following subsection.

7.4.3 Design of the Nonlinear Gain Function

Formally, the nonlinear gains are related to the magnitudes of the uncertainty bounds, which in this situation are not available. An estimate of the design constants in the nonlinear gain function can be obtained by considering the range of allowable inputs. For the system under consideration, the input is restricted to the interval $[0, 4]$ Volts. Consequently it is reasonable to require that the inequality $r_1|y| < 4$ to be satisfied. In practice, as a result of the chosen temperature profile to be tracked, it was found that y is of the order 1. In this way, a sensible upper bound on r_1 can be obtained.

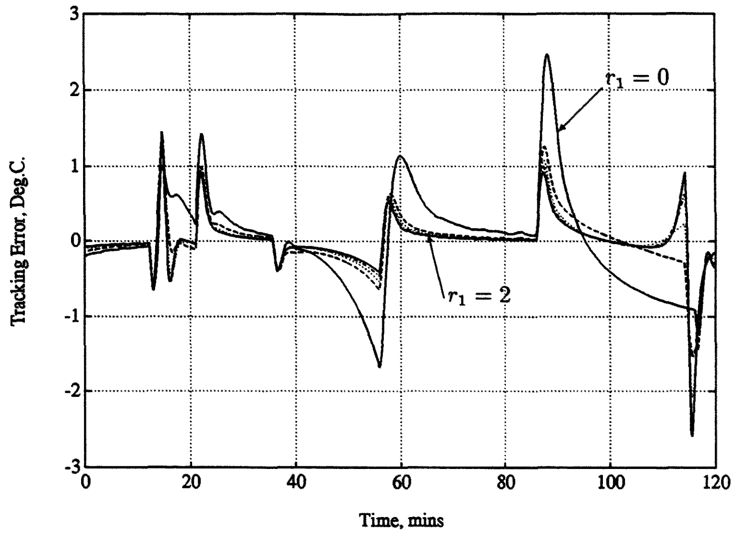


Figure 7.2: Simulations With Different Nonlinear Components

Figure (7.2) represents nonlinear simulation tests, using the observer designed in the previous subsection, with the nonlinear gain

$$\rho_o(u_L, y) = r_1|y| \qquad \text{for } r_1 \in \{0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2\}$$

This demonstrates the increase in performance obtained as a result of increasing r_1 and hence the nonlinear component of the control action. The final value of r_1 was chosen as a trade-off between the tracking error and the control effort required.

7.4.4 Furnace Simulations

Under typical operating conditions, the controller is required to take the furnace from one operating temperature to another, along a specified trajectory. A typical temperature demand signal, which will subsequently be used for the simulations and plant trials, is given in Figure 7.3.

Consider the differential equation

$$\dot{r}(t) = \Gamma\left(r(t) - R(t - \tfrac{1}{\Gamma})\right) \tag{7.4}$$

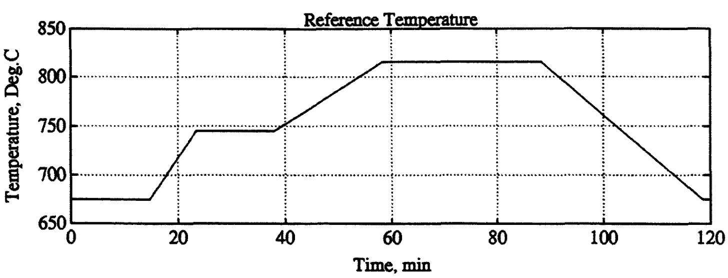


Figure 7.3: Typical temperature reference signal

which effectively replaces (4.7) in §4.3. It is assumed that the intervals which define the piece-wise linear components of the reference, are large enough compared to the time constant of Γ so that 'steady state' occurs. It can be seen that on each interval $\ddot{R}(t) = 0$ and therefore $\dot{R}(t) = \alpha$ for some scalar. If $e_r \stackrel{s}{=} r(t) - R(t)$ then from equation (7.4) it follows immediately that

$$\begin{aligned} \dot{e}_r(t) &= \Gamma r(t) - \Gamma R(t - \tfrac{1}{\Gamma}) - \alpha \\ &= \Gamma e_r(t) + \Gamma R(t) - \Gamma R(t - \tfrac{1}{\Gamma}) - \alpha \\ &= \Gamma e_r(t) \end{aligned}$$

Therefore $e_r(t) \to 0$ and the solution $r(t)$ to equation (7.4) follows the profile in Figure 7.3 asymptotically.

Figure 7.4 represents the response of the furnace simulation under nonlinear control with the tracking error given in Figure 7.5. This performance is very good even on the

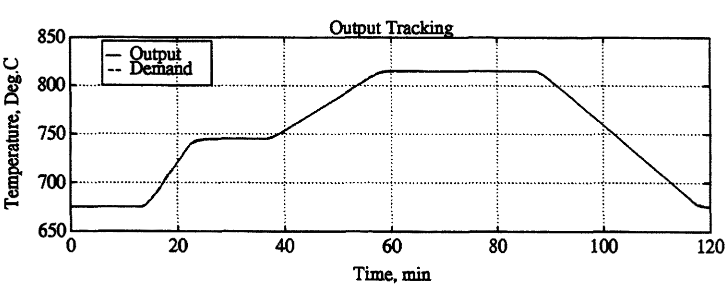
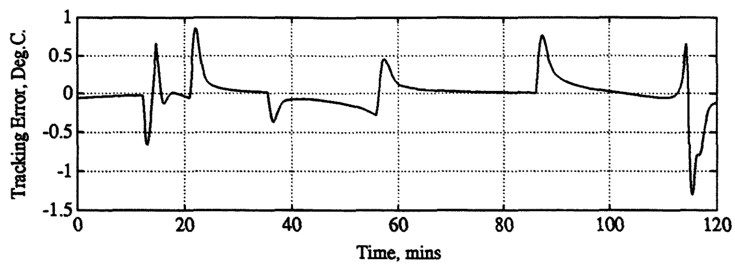


Figure 7.4: Nonlinear simulation under sliding mode control

parts of the demand profile that comprise transients between steady state operating



Absolute Error	Mean Control	Excitation	Overall
0.179	1.073	0.001	1.253

Figure 7.5: Tracking error using the sliding mode controller

points, for which asymptotic tracking is not guaranteed theoretically. The next section considers the design of an output feedback model reference controller.

7.5 Model Reference Controller

This section considers the design of a model reference based controller using output information. Because the linear model is minimum phase and of relative degree one, the theory in Chapter 5 is applicable.

7.5.1 Design of the Reference Model

Initially the design of the ideal reference model is considered. The theory in §5.10 assumes that the ideal reference model is given by

$$\begin{aligned} \dot{x}_m(t) &= (A + BL_x)\, x_m(t) + BL_r R(t) \\ y_m(t) &= C x_m(t) \end{aligned} \tag{7.5}$$

where (A, B, C) represents the nominal linear system and L_x and L_r represent some appropriate gain matrices. In this case study, the feedback matrix L_x was chosen so that the poles of the ideal model are $\{-0.020, -0.053 \pm 0.050i\}$. It can be verified that

$$L_x = \begin{bmatrix} -0.00004 & 0.00080 & -0.02308 \end{bmatrix} \tag{7.6}$$

is the appropriate model gain. After selecting L_x, the scalar gain L_r was chosen so that the transfer function from the reference signal to the (model) output given by

$$G(s) = C\left(sI - A - BL_x\right)^{-1} BL_r \tag{7.7}$$

has unity D.C. gain. Therefore, at steady state, the output from the ideal model tracks the reference signal perfectly. In this example

$$L_r = 1.4961 \tag{7.8}$$

Unfortunately, a high percentage of the reference profile is made up of 'ramps' between different steady state operating points. During these periods the limiting value of the tracking error between the reference and output will be finite but nonzero. A practical way of overcoming this, is to replace the state space differential equation above with

$$\dot{x}_m(t) = (A + BL_x)\, x_m(t) + BL_r R(t + t_o) \tag{7.9}$$

for some non-zero scalar t_o. If the transfer function from the reference to the output given in equation (7.7) is written as

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} \tag{7.10}$$

then, because the D.C. gain is unity, it follows that $a_0 = b_0$. The invariant zeros of $G(s)$ are identical to those of (A, B, C) and therefore by assumption $G(s)$ has no zeros at the origin. As a consequence $a_0 = b_0 \neq 0$. Using the fact that $\ddot{R}(t) = 0$ it follows that $\dot{R}(t) = \alpha$ for some fixed scalar. If the tracking error is given by $e(t) = R(t) - y(t)$, then from (7.10) and using the fact that $R^{(i)}(t) = 0$ for $i \geq 2$, it follows that

$$\begin{aligned} e^{(n)}(t) + a_{n-1}e^{(n-1)}(t) + \cdots + a_1\dot{e}(t) + a_0 e(t) &= a_1\dot{R}(t) + a_0 R(t) - b_1\dot{R}(t+t_o) - b_0 R(t+t_o) \\ &= (a_1 - b_1)\alpha + a_0 R(t) - a_0 R(t+t_o) \\ &= (a_1 - b_1 - a_0 t_o)\alpha \end{aligned}$$

If the time shift is chosen as

$$t_o = \frac{1}{a_0}(a_1 - b_1) \tag{7.11}$$

which is possible since $a_0 \neq 0$, it follows that $e(t) \to 0$ for large values of t. To complete the control scheme proposed in Chapter 5, all that remains is to synthesise an output feedback regulator. This is described in the next subsection.

7.5.2 Sliding Mode Output Feedback Design

Using the transformation from Lemma 5.1 in §5.4 it can be verified that the nominal linear system representing the temperature dynamics can be written as

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -0.0134 & -0.4106 & 188.8498 \\ \hline -0.0000 & -0.0006 & 0.3077 \end{array}\right] \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ 0.0053 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

This is the canonical form of (5.39) given in §5.7, which implies $\hat{B}_2 = 0.0053$. In this case study, the block diagonal matrix P_2 has been chosen so that $F = 1$. Three design parameters remain, namely, the elements that comprise the symmetric positive definite matrix Q_1 which forms the right hand side of equation (5.27). Once this matrix is defined, the Lyapunov matrix P_1 for the stable matrix $\hat{A}_{11}$ is uniquely defined, and so is the scalar γ_0 from equation (5.31). The output feedback gain γ in equation (5.32) must be greater than γ_0 and so from the point of view of implementation, a low value for γ_0 is preferable[1]. In this particular design

$$Q_1 = \begin{bmatrix} 0.0010 & -0.0005 \\ -0.0005 & 0.0010 \end{bmatrix}$$

gives $\gamma_0 = 65.5813$. Choosing $\gamma = 70$ results in the error system having poles at $\{-0.0347, -0.2195 \pm 0.3116i\}$. The control law is then given by

$$u(t) = L_x x_m(t) + L_r R(t + t_o) + \gamma F e_y(t) + \rho(u_L, y)\ \mathrm{sgn}\,(e_y(t)) \tag{7.12}$$

where $e_y(t)$ represents the difference between the output of the ideal model and the plant. The scalar function

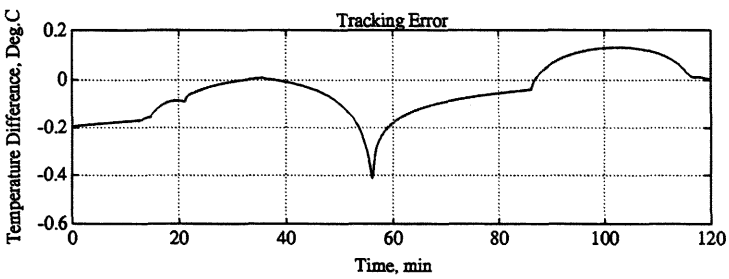
$$\rho(u_L, y) = r_1 |u_L(\cdot)| + r_2 |y| + r_3 \tag{7.13}$$

where $u_L(\cdot)$ is the linear component of the control action in (7.12) and r_1, r_2 and r_3 are suitably chosen positive scalars. As for the controller/observer pair these have been chosen by inspection.

[1]In this case it is possible to show that the infimum for $\gamma_0 = 58.11$. For single input systems, in order for the closed loop matrix $A_0 = \hat{A} - \gamma\hat{B}F\hat{C}$ to have a block diagonal Lyapunov Matrix, the bottom right element of A_0 must be negative which implies $\hat{A}_{22} - \gamma\hat{B}_2 < 0$. This generates the bound given above.

7.5.3 Furnace Simulation

This subsection describes the results of using the output feedback model reference controller on the furnace simulation. Again, for implementation purposes, the discontinuous component of the control law has been smoothed as in §2.6. Figure 7.6 represents the output tracking error from the simulation (and is a scaled version of the switching function).



Absolute Error	Mean Control	Excitation	Overall
0.095	1.076	0.001	1.172

Figure 7.6: Tracking error response under model-reference control

The results can been seen to be very good, as confirmed by the performance index. As a result of the excellent performance provided by the sliding mode strategies on the simulation, sufficient confidence was established in the approach to warrant implementation on the real furnace. The results of these trials are described in the next section

7.6 Plant Trials

The experimental furnace at the Gas Research Centre is fitted with five thermocouples – one on the end wall opposite the burner and a pair on each side wall. This section initially considers the performance of the sliding mode observer/controller pair with regard to the end-wall thermocouple; this represents the 'nominal' situation. To examine the effectiveness of the proposed scheme it is sensible to compare its performance with that of a 'well tuned' PID, the controller settings being found via an auto tuning algorithm.

7.6.1 End Wall Thermocouple Trials

Figure 7.7 represents a typical furnace response under PID control. The corresponding control action given is in Figure 7.8.

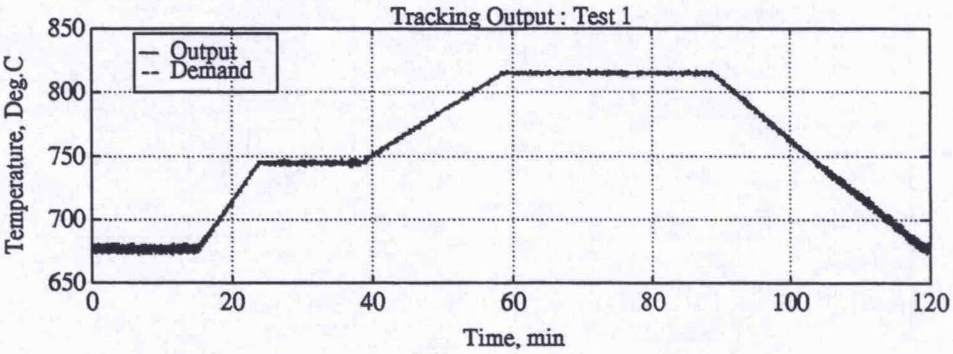


Figure 7.7: Experimental furnace under PID Control

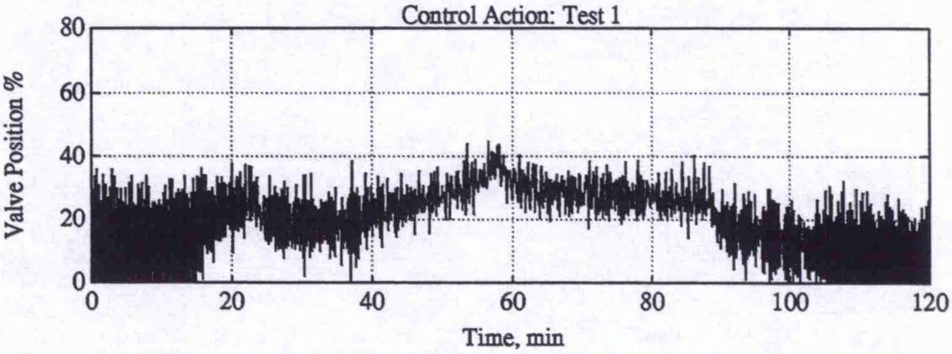


Figure 7.8: PID Control Action

The controller/observer pair was implemented for trial purposes using a portable PC. An input/output card provided the interface between the PC and the analogue thermocouple and valve positioner signals. The controller/observer algorithm was coded in Turbo Pascal, compiled and run as an executable file. The temperature signal was sampled, the output voltage updated and transmitted 10 times a second. This left sufficient CPU time to 'sample' the observer at a rate of 100 times as second. The controller sampling rate is comparable with that of the PID which has a sampling rate of 8 Hz. Figure 7.9 gives the response of the furnace, under nonlinear control, to the reference signal used earlier. Here the average tracking error is less than 1° which represents a level of accuracy probably greater than the measurement precision. The control excitation is higher than that of the simulation, but is considerably less than that of the PID.

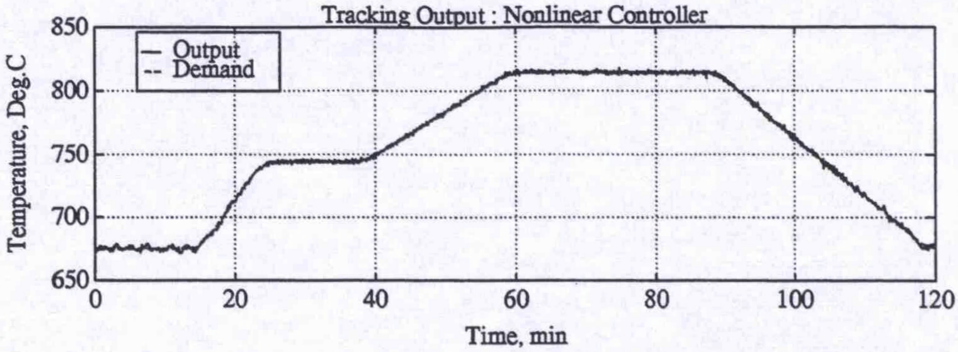


Figure 7.9: Response of End-Wall Thermocouple under Nonlinear Control

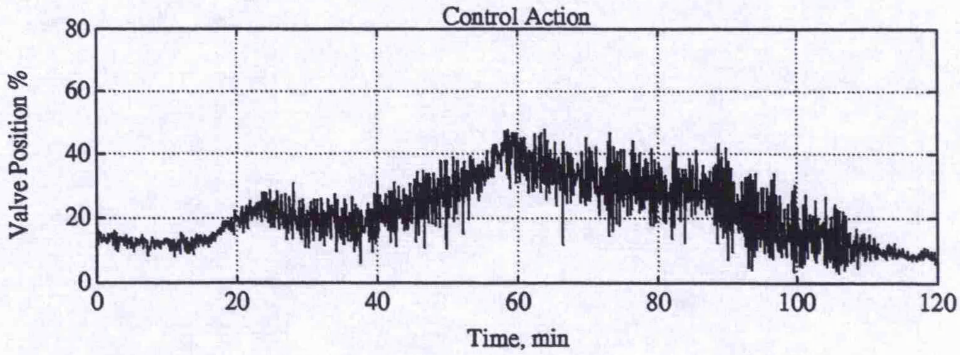


Figure 7.10: Nonlinear Control Action

Table 7.2 given below presents the performance indices for trials of the nonlinear controller whilst Table 7.3 shows the performance indices of PID controller trials conducted on the same furnace during early 1993.

Performance Indices for Nonlinear Controller				
Test	Absolute Error	Mean Control	Excitation	Overall
1	0.893	1.085	0.169	2.148
2	0.901	1.048	0.205	2.155

Table 7.2: Performance Indices for sliding mode scheme

The absolute error measure is consistently lower for the nonlinear controller as is the valve excitation. The PID performs better with regard to the mean control signal index. However this measure is heavily dependent on the operating history of the furnace immediately prior to any trial. This is demonstrated by the PID tests which represent

Performance Indices for PID Trials				
Test	Absolute Error	Mean Control	Excitation	Overall
1	1.333	1.022	0.287	2.643
2	1.433	1.001	0.313	2.757
3	1.440	0.971	0.301	2.712

Table 7.3: Performance Indices for the PID Controllers

three back-to-back trials; the mean control measure decreases as less fuel is used, because of the retention of heat by the furnace. The overall measure indicates that the nonlinear controller/observer pair is performing at least as well as a commercial PID controller for the nominal test. Robustness issues will now be addressed.

7.6.2 Robustness - Side Wall Thermocouple Trials

In order to explore the robustness of the nonlinear controller, trials were undertaken where the temperature signal was supplied from one of the side-wall thermocouples. This represents a seriously perturbed system from the point of view of control since it is known that under normal operating conditions while the end wall thermocouple operates in the region of 650-820°, the side-wall temperatures are in the range 450-650°. It is therefore not possible to use the original demand signal since the side walls cannot attain such elevated temperatures. Figure 7.11 demonstrates the results of a typical side-wall test using the same nonlinear controller/observer pair but a modified demand.

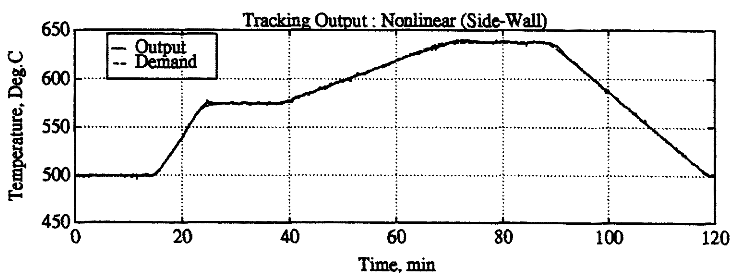


Figure 7.11: Response of Side-Wall Thermocouple under Nonlinear Control

Results of a comparable PID test are presented in Figure 7.12; as in the case of the nonlinear controller the original parameters have been left unaltered.

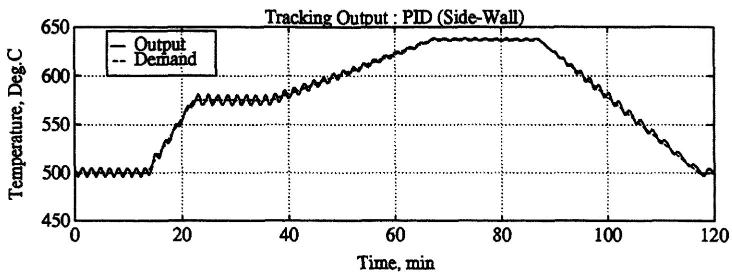


Figure 7.12: Response of Side-Wall Thermocouple under PID Control

These test results indicate that the nonlinear controller performs better from the point of view of robustness. In an industrial setting, retuning of the PID controller would be necessary. Here, this is circumvented by the use of a single robust nonlinear controller.

Performance Indices for Robustness Trials				
Controller	Absolute Error	Mean Control	Excitation	Overall
VSC	0.703	0.985	0.105	1.793
PID	2.138	1.060	0.218	3.414

Table 7.4: Performance Indices for Robustness Trials

7.7 Summary

In this chapter, the theoretical results of Chapters 4 and 5 have been used to design temperature control strategies for a single burner furnace. The furnace simulation developed in the previous chapter was used to obtain a nominal linear model representing the effect of changes in valve position on the temperature. Based on this linear model, a sliding mode controller/observer pair and a model reference output feedback design have been synthesised. The practical choice of parameters for both methodologies has been described. Encouraged by the excellent performance obtained from using the schemes on the furnace simulation, the controller/observer was implemented on the experimental furnace at the Gas Research Centre. No great improvement in performance was obtained in the nominal case. However the sliding mode strategy was demonstrably more robust to changes in the plant configuration. In order to exploit the true potential of the sliding mode strategies, a multivariable problem needs to be considered. This is undertaken in the next chapter where the control of both temperature and excess oxygen content will be considered.

Chapter 8

Multivariable Control of Temperature and Excess Oxygen

8.1 Introduction

The previous chapter described the successful work to incorporate a sliding mode strategy within the current control framework for the experimental furnace. This framework is usually described as 'gas led' since the air flow is modulated as a result of changes in the fuel flow to maintain the appropriate fuel/air ratio necessary for efficient combustion. An Electronic Ratio Controller (ERC) is an established commercially available device which maintains a set fuel/air ratio [32]. The fuel and air flows are measured using heated thermistors placed in by-pass lines around orifice plates in the flow paths of the air and gas. The relative flows are compared and fed-back to the ERC which makes appropriate adjustments to the air valve position. An additional input to the ERC exists referred to as the trim signal. This allows the fuel/air ratio set-point to be adjusted and may be considered as another control input.

An effective measure of combustion efficiency is the percentage of oxygen present in the flue products. An absence of oxygen in the flue gases implies that insufficient oxygen is being supplied. This may lead to a situation where excess fuel is present in the flue gases which is both inefficient and potentially dangerous. Conversely, unnecessarily high levels of oxygen imply unnecessarily high air flow rates, which increase the flue mass flows, leading to additional energy losses. Therefore an additional control problem to be considered is that of regulating the excess oxygen level. Zirconia based devices, which continuously measure the concentration of oxygen present in a sample of gas, are commercially available. Such a product, adapted from automotive technology to monitor exhaust gas oxygen levels in engines with catalytic converters, are low cost and rugged enough for the environment in which they have to operate [32]. The intention is to develop a control scheme, to provide reference tracking for both temperature and excess oxygen, by modulating the fuel flow and oxygen trim signal. The proposed scheme is outlined in Figure 8.1.

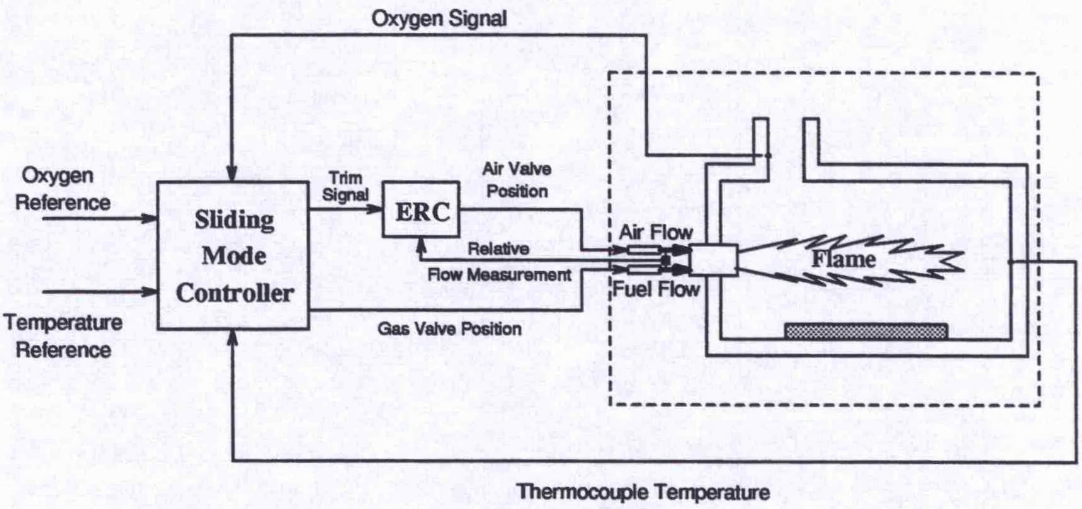


Figure 8.1: Schematic of proposed multivariable control scheme

This chapter charts the development of such a control system, from the identification of a nominal linear model, through to successful implementation. As in the case of temperature control, the controller/observer pair proposed in Chapter 4 has been used. A discussion of the selection of the design parameters is included, together with the results of successful trials to implement the proposed scheme on the experimental furnace at the Gas Research Centre. In the following section the identification procedure is described.

8.2 Identification of a Linear Model

The furnace simulation developed in Chapter 6 is not appropriate for the problem under consideration since it assumes a fixed fuel/air ratio. The composition of the combustion products affects the absorptivity properties and hence the exchange areas. Technically, the simulation could be altered to allow for this added complexity by modifying the code so that all the exchange areas are recomputed at each time step. It was felt that this modification would result in hopelessly long and infeasible run times. As a result, in order to obtain a linear model around which to base the controller designs, identification of the real plant was chosen as the most viable option.

A PRBS excitation signal, taking the discrete values 0 and 1.5 volts, was applied to the valve positioner signal in the fuel line as in §7.2. Unfortunately such a signal could not be used to excite the oxygen trim channel. For reasons of safety, the burner is only permitted to operate 'off ratio' for a short period of time, after which a relay 'alarm' shuts down the flow of fuel to the burner. Using a PRBS signal in the oxygen trim channel with its large sudden amplitude changes would undoubtedly provoke such a shut down and hence is not suitable for identification purposes. Instead, an excitation signal comprising the sum of several sinusoids of different frequency and amplitude was used. Such a signal is of course not ideal for identification purposes but represented the best that could be done under the circumstances.

Since the intention is to use the sliding mode schemes proposed in the early chapters, only relative degree one model structures were considered. The ARX command in the MATLAB system identification toolbox was used to fit discrete time transfer functions of different orders and structures to the furnace input-output data. As in §7.2, the validity of the resulting models was compared by examing the prediction error resulting from the use of different input-output data. A (continuous time) low order, relative degree one minimum phase model, which provides good agreement with the input-output data, is given by

$$
A = \begin{bmatrix} -0.4738 & 0.0279 & 0.0644 & 0.0573 \\ 0.1319 & -0.1965 & -0.0482 & 0.0692 \\ -0.0473 & 0.0217 & -0.0783 & -0.0670 \\ -0.0144 & -0.0001 & 0.0076 & -0.0260 \end{bmatrix} \quad B = \begin{bmatrix} -0.4726 & -0.0201 \\ 0.1528 & 0.0789 \\ -0.0144 & -0.0677 \\ 0.0038 & -0.0074 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 0 & 0 & -0.1197 & 0.6694 \\ 0 & 0 & -0.2811 & -0.2850 \end{bmatrix} \tag{8.1}
$$

This system is stable and has invariant zeros at $\{-0.1633, \ -1.2059\}$. The first input channel represents the fuel valve position signal and the second the oxygen trim signal.

The first output is the furnace temperature as measured by the thermocouple; and the second output is the signal from the zirconia probe which monitors the percentage of oxygen in the flue. It should be remembered that because of the limitations imposed on the identification signal, this model may exhibit significant mismatches with the true behaviour of the system.

The realization above is, of course, not in regular form. A convenient way to achieve regular form for general systems originally suggested in [92], is by the use of 'QR Reduction'. Given any matrix $B \in \mathbb{R}^{n \times m}$, QR Reduction generates an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ so that $B = QR$ where

$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

and $R_1 \in \mathbb{R}^{m \times m}$ is upper triangular. Numerical algorithms to effect such a decomposition are an intrinsic feature of most matrix computation packages. Consequently the orthogonal matrix $T \in \mathbb{R}^{n \times n}$ defined as

$$T = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & 1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} Q^{\mathrm{T}}$$

is an appropriate linear transformation to place a general system in regular form. It can be verified that

$$\begin{aligned} A &= \begin{bmatrix} -0.0186 & -0.0065 & 0.0190 & 0.0129 \\ 0.0026 & -0.1354 & 0.0310 & 0.0040 \\ -0.0972 & 0.0695 & -0.1273 & 0.0530 \\ -0.0193 & -0.0155 & -0.1121 & -0.4934 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -0.0960 \\ 0.4969 & 0.0453 \end{bmatrix} \\ C &= \begin{bmatrix} 0.6707 & -0.1085 & -0.0286 & 0.0086 \\ -0.2750 & -0.1933 & -0.2175 & 0.0060 \end{bmatrix} \end{aligned} \tag{8.2}$$

is a realization of (8.1) in regular form. The following section describes the design procedures adopted to synthesise a controller/observer pair. The design of an observer is discussed first.

8.3 Observer Design

Since the system is square, the reduced order dynamics of the estimation error are completely determined by the invariant zeros of the system. By choosing a priori the stable matrix A_{22}^s to be diagonal, any diagonal positive definite matrix P_2 will always be a Lyapunov matrix for A_{22}^s. The diagonal matrix P_2 can then be considered independently as a design matrix. Here it has been chosen in an effort to ensure the diagonal entries of the matrix $F = (P_2 C_2 B_2)^{\mathrm{T}}$ are of compatible order. It was reasoned that the diagonal elements of F act as weighting parameters which govern the distribution of the nonlinear control action between the individual input channels[1]. By choosing

$$P_2 = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$$

it follows that

$$F = \begin{bmatrix} 0.0426 & 0.0060 \\ 0.0313 & 0.0423 \end{bmatrix}$$

The diagonal elements of A_{22}^s were chosen to make the condition number of the observer closed loop matrix $A_0 = A - GC$ small. By inspection, it was found that choosing

$$A_{22}^s = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$$

fulfils this requirement, giving a condition number for A_0 of approximately 38.26. From equation (4.39) the linear gain matrix is given by

$$G_l = \begin{bmatrix} 1.7201 & -0.3132 \\ 0.6269 & -0.2251 \\ 8.1321 & -2.7820 \\ 0.4512 & 0.4562 \end{bmatrix}$$

As in the single-input single-output design in the previous chapter, the nonlinear gain function will need to be determined empirically.

[1]For diagonal matrices this is certainly the case.

8.4 Design of the Controller

Forming the augmented plant from (4.12) it follows that in the notation of §4.3

$$\left[\begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 0 & -0.6707 & 0.1085 & 0.0286 & -0.0086 \\ 0 & 0 & 0.2750 & 0.1933 & 0.2175 & -0.0060 \\ 0 & 0 & -0.0186 & -0.0065 & 0.0190 & 0.0129 \\ 0 & 0 & 0.0026 & -0.1354 & 0.0310 & 0.0040 \\ \hline 0 & 0 & -0.0972 & 0.0695 & -0.1273 & 0.0530 \\ 0 & 0 & -0.0193 & -0.0155 & -0.1121 & -0.4934 \end{array}\right]$$

Early attempts to select the poles of the reduced order sliding motion by inspection, resulted in controllers which exhibited high levels of control activity. To circumvent this, a preliminary linear LQR design was made based on the augmented system, with the cost functional biased towards penalising the use of control effort. The four slowest poles of the resulting closed loop matrix were used as initial values for the poles of the reduced order motion. These were subsequently manually adjusted in an effort to improve the conditioning of the closed loop matrix. Ultimately the sliding mode eigenvalues were taken to be $\{-0.0849, -0.0118, -0.0357 \pm 0.0220i\}$. The matrix M from equation (4.16) which acts as a feedback matrix for the pair $(\tilde{A}_{11}, \tilde{A}_{12})$, was obtained from using the MATLAB command 'place', which uses the robust pole placement algorithm of Kautsky *et al.*[43]. The resulting matrix which defines the hyperplane is given by

$$M = \left[\begin{array}{cccc} 0.0376 & 0.0379 & -0.1258 & -2.7493 \\ -0.2317 & -0.0930 & 5.7274 & 4.0709 \end{array}\right]$$

The design parameter Λ was chosen as

$$\Lambda = \left[\begin{array}{cc} 0.1 & 0 \\ 0 & 0.1 \end{array}\right]$$

which completes the hyperplane design. If the stable design matrix Φ, which assigns the poles of the range space dynamics, is block diagonal, a diagonal solution for $\bar{P}_2$ can be attained from the modified Lyapunov equation (4.62) by a suitable choice of $\hat{Q}_2$. A diagonal structure for $\bar{P}_2$ is preferable since this 'decouples' the nonlinear components of the control action. The eigenvalues for the range space dynamics were take to be the unused poles from the initial LQR design, namely $\{-0.1800 - 0.5009\}$. The diagonal elements of Φ were arranged so that the slowest pole was associated with the oxygen trim channel. Consciously, every effort was made not to apply an aggressive control signal into this channel to guard against violating the fuel/air ratio error shut down alarm. The linear components of the control law from equations (4.27) – (4.29) can be shown to be

$$
\begin{aligned}
L &= \begin{bmatrix} 0.1080 & 0.0379 & -2.7802 & -0.5288 & -0.2363 & 0.2641 \\ 0.0706 & 0.0711 & -1.4509 & -0.4258 & -0.2679 & 0.4139 \end{bmatrix} \\
L_r &= \begin{bmatrix} 5.6968 & -0.3288 \\ 2.4412 & 3.3682 \end{bmatrix} \\
L_{\dot{r}} &= \begin{bmatrix} 20.9894 & -1.9125 \\ 11.3840 & 16.5172 \end{bmatrix}
\end{aligned}
$$

As in the previous chapter, the sliding surface design parameter S_r was chosen so that, in the nominal case, the steady state values of the integrator states are zero. In this particular case study

$$
\begin{aligned}
S &= \begin{bmatrix} -0.0431 & -0.0151 & 1.1407 & 0.5580 & 0.0950 & 0.2012 \\ -0.0392 & -0.0395 & 0.1310 & 2.8634 & -1.0415 & 0.0000 \end{bmatrix} \\
S_r &= \begin{bmatrix} 2.0989 & -0.1912 \\ 1.1384 & 1.6517 \end{bmatrix}
\end{aligned}
$$

The design matrix Γ from equation (4.7) in §4.3 has been chosen to tailor the step response of the closed-loop system in the nominal case. The stable design matrix has been chosen to be diagonal, since it makes no sense to introduce coupling between the reference signals. The multivariable equation (4.7) can be represented as the pair of scalar equations

$$\dot{r}_i(t) = \Gamma_i \left(r_i(t) - R_i(t - \tfrac{1}{\Gamma_i}) \right) \qquad \text{for } i = 1, 2$$

where Γ_1 and Γ_2 are the diagonal elements of Γ. If $\ddot{R}_i(t) = 0$ on some interval then $r_i(t) \rightarrow R_i(t)$ asymptotically. In this particular design $\Gamma_1 = -0.02$ and $\Gamma_2 = -0.05$. This reflects the different speeds of response required in the temperature and oxygen channels respectively. Again because of the identification procedure adopted, no information was available to compute the nonlinear gain functions as in (4.34). Furthermore, no linear model was available to test the design prior to implementation. The scalars were chosen experimentally during the trials at the Gas Research Centre.

8.5 Trial Results

As in Chapter 7, the control scheme was implemented on a portable PC containing appropriate interface cards to perform the required analogue $\leftrightarrow$ digital conversions. As before, the plant output signals were sampled and the control outputs updated ten times a second. In order to compare the multivariable temperature results with the single-input single-output ones from the previous chapter, a similar temperature demand profile was used. It is observed by Goodhart in [29] that the excess oxygen reference signal cannot be chosen without regard to the temperature set points. Care has therefore been taken to provide a realistic and attainable excess oxygen profile. The scalars comprising the nonlinear gain functions were chosen initially very conservatively under the restriction that the gain $\rho_c(\cdot)$ be bounded by unity. The rationale for this conservatism was that, generally speaking, increasing the nonlinear control component increases the 'aggressiveness' of the control signal, which was considered to be undesirable in the oxygen trim channel.

The results of an early design, with a low degree of nonlinear control action, are given in Figures 8.2 and 8.3.

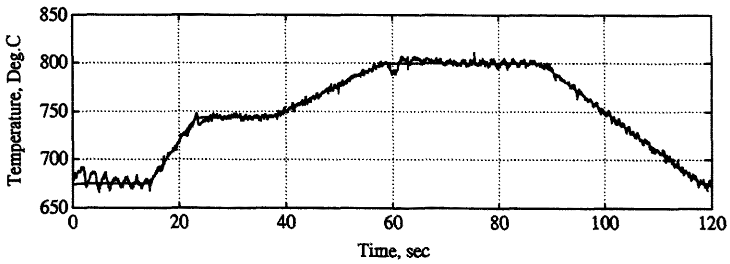


Figure 8.2: Furnace temperature compared to the reference (1st design)

The temperature performance measures in Table 8.1 demonstrate that a loss in tracking performance, compared to the single-input single-output results in the previous chapter,

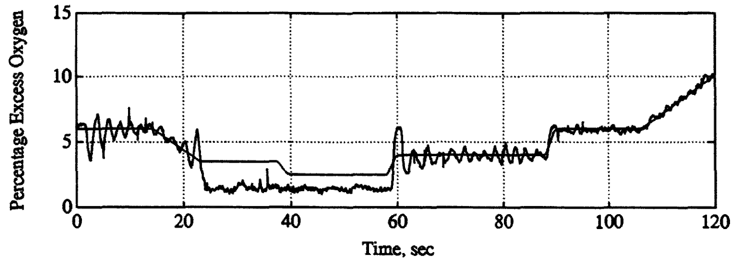


Figure 8.3: Excess oxygen in the flue compared to the reference (1st design)

has occurred. The tracking performance with regard to the excess oxygen demand is even worse. This is not entirely unexpected in view of the limited input-output data that was associated with this channel and hence the low degree of confidence in the nominal linear model. Importantly, no difficulties were encountered with respect to the burner shut-down alarm. As a result, sufficient confidence was established in the scheme to warrant increasing the nonlinear gain. Figures 8.4 and 8.5 present results obtained from using the scalar gains

$$\rho_o(u_L, y) = 0.2\ \|u_L(\cdot)\| + 0.2\ \|y\| + 0.25$$

and

$$\rho_c(u_L, y) = 0.1\ \rho_o(u_L, y) + 0.05$$

A significant improvement in the tracking performance is obtained in both channels.

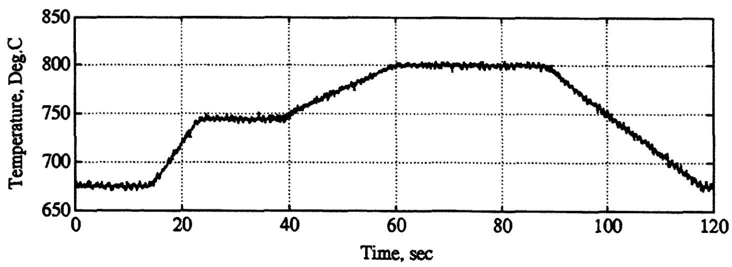


Figure 8.4: Furnace temperature compared to the reference (2nd design)

The relevant performance measure in the temperature channel is given in Table 8.1.

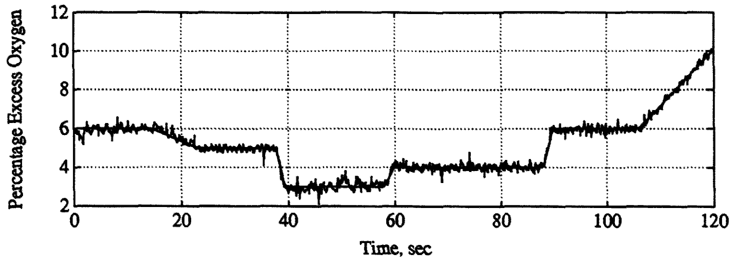


Figure 8.5: Excess oxygen in the flue compared to the reference (2nd design)

These responses are manifestly better than the previous design but still not as good as the single-input single-output results from §7.6. Due to time limitations this response could not be (directly) improved up on.

It was felt that the thermocouple signal was subject to noise disturbances. It was observed that peaks of up to 10° occasionally occurred between consecutive, essentially 'steady state', temperature readings. At no point in the theoretical chapters of this thesis has the effect of output noise been considered. As a short term solution, a simple linear first order filter was applied to the temperature signal, prior to use in the observer. The resulting closed loop performance is shown in Figures 8.6 and 8.7.

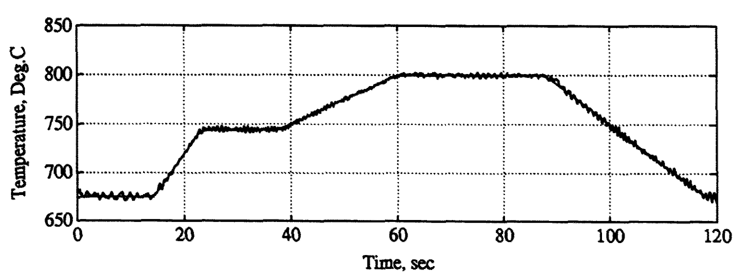


Figure 8.6: Furnace temperature compared to the reference (3rd design)

The results shown in Table 8.1 demonstrate that the insertion of a filter does indeed improve upon the earlier designs, although the results are still not comparable to those in §7.6. It should be noted however that between the trials presented above, and the ones given in Chapter 7, the furnace had been totally dismantled and reassembled at a different site. This may in itself account for the difference in performance.

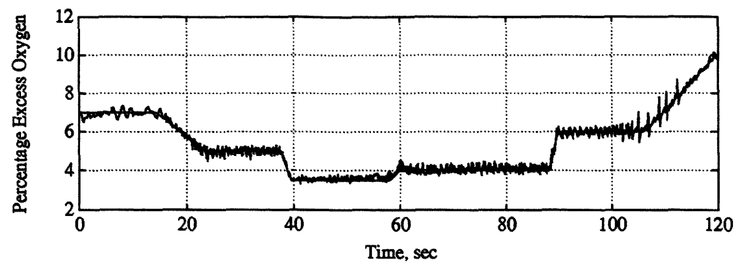


Figure 8.7: Excess oxygen in the flue compared to the reference (3rd design)

Temperature Performance Indices for Multivariable Trials				
Figure	Absolute Error	Mean Control	Excitation	Overall
8.2	2.433	1.267	0.096	3.806
8.4	1.627	1.093	0.095	2.815
8.6	1.394	1.143	0.116	2.654

Table 8.1: Performance Indices for Multivariable Trials

8.6 Summary

This chapter has considered the problem of controlling both the temperature, and oxygen content in the combustion products. The oxygen trim signal to the ERC, which alters the fuel/air ratio set-point, has been used as an additional input. The simulation of Chapter 6 is not appropriate in this situation because it assumes a fixed fuel/air ratio. Therefore the nominal linear model around which the control system was designed was obtained by means of system identification of the real plant. The theoretical results of Chapter 4 were once again successfully used to design a controller/observer pair. Because a model was not available to test the controller prior to implementation, the design scalars that define the nonlinear gain functions were selected *in situ* during the trials. Overall, good tracking performance was obtained for both reference signals, although the temperature tracking was not quite as precise as that obtained in the single-input single-output trials of the previous chapter.

Chapter 9

Conclusions and Future Work

9.1 Concluding Remarks

This thesis has considered the problem of designing sliding mode output tracking controllers for uncertain linear systems when only output information is available to the control law. The theoretical ideas proposed in the early chapters have been successfully implemented on an experimental gas fired furnace.

Two distinct approaches to controller design have been proposed: firstly an observer based scheme which utilizes integral action; and secondly, an output feedback stabilization approach within a model reference framework. The perceived contribution of this work is outlined below:

- The first method uses previously developed state feedback sliding mode theory. The novelty of the approach is in the use of a sliding mode observer to estimate the unknown internal states. A new canonical form has been developed for the design of a class of sliding mode observers. The attainment of this canonical form was demonstrated to be a necessary and sufficient condition for the existence of a class of observers insensitive to matched uncertainty. It has also been proved that the class of systems which can be transformed into the canonical form are precisely those systems which are minimum phase and relative degree one. Therefore necessary and sufficient conditions in terms of the nominal linear system have been established for the existence of a sliding mode observer insensitive to matched uncertainty. This supersedes previous work in this area which relied on checking the validity of a structural constraint between the state space matrices of the system. Establishing whether a system is minimum phase and relatively degree one can readily be accomplished. In addition, the canonical form provides a framework for designing sliding modes observers, which circumvents the need to solve explicitly a structurally constrained Lyapunov equation.

- The use of a controller/observer pair immediately raises the question of whether the combined closed-loop system is stable. This query has been answered in Chapter 4 using a quadratic stability argument. It is demonstrated that for uncertain systems, whose nominal linear representation is minimum phase and relative degree one, asymptotic tracking of a constant reference signal can be achieved, despite the presence of bounded matched uncertainty. Furthermore it is demonstrated that the control law and the observer system can be designed independently (apart from the scalar functions which multiply the discontinuous components of the controller and observer and depend on the magnitude of the uncertainty). In other words, the well known separation principle for linear systems also holds for this class of uncertain system and controller/observer pair.

- An alternative control law has also been considered based on a model reference structure. Sliding mode approaches within model reference frameworks have been well documented in the literature. The novelty of this contribution is that only the reference signal, the model states and plant output are used to formulate the control law. By considering the error system between the model states and (unknown) plant states it has been shown that the problem can be re-formulated as one of output feedback stabilization. The majority of Chapter 5 is devoted to addressing this problem and proposes a new control scheme. The analysis is performed in essentially the same framework as that developed for the sliding mode observer. This framework is demonstrated to encompass a wider class of systems than those considered by other investigators currently working in this field. The proposed design procedure utilizes well established linear output feedback pole placement results, rather than eigenstructure arguments which is a feature of current approaches to the problem. The proposed control law is more straight forward in terms of realizability than those suggested by other workers.

- Control schemes based on these new theoretical ideas have been successfully implemented on an experimental furnace at the Gas Research Centre at Loughborough. Initially, a single-input single-output sliding mode controller/observer pair was used to replace a PID in the temperature control loop. The results were compared and demonstrate that the sliding mode scheme performed at least as well in this nominal situation. It is not perhaps totally surprising that in a nominal, single-input single-output configuration, the performance of a PID is difficult to improve upon. The sliding mode strategy did however exhibit greater robustness with respect to severe changes in the plant configuration, when a differently situated thermocouple was used to supply the temperature control signal. The results suggest that, to fully exploit the potential of the new sliding mode schemes, a multivariable problem needs to be considered.

- A suitable problem, associated with the same single burner furnace, is the regulation of both temperature and excess oxygen by the manipulation of the fuel and air flows. The linear model around which the control system was designed was obtained by identification of input-output data obtained from the furnace. It should be stressed that the input-output data was somewhat limited – especially in the excess oxygen channel. The inherent furnace safety procedures associated with the ERC severely curtailed the degree of excitation which could be introduced into this channel. It is therefore expected that potentially there is a large degree of mismatch between the real system and the model. The fact that comparable tracking performance can still be obtained is a testament to the robustness of the sliding mode controller/observer strategy.

These successful implementations should convince those sceptical of the merits of introducing 'discontinuous' control action into a real system of the robustness and practicality of sliding mode schemes.

9.2 Recommendations for Future Work

From a theoretical standpoint, the approaches of Chapters 3, 4 and 5 utilize a discontinuous component and induce ideal sliding motions. For implementation (and even simulation), the discontinuous component has been replaced by a continuous approximation. By reformulating the propositions, especially those in Chapter 4, the effect of this modification could be rigorously explored using a 'practical stability' argument as used in [60, 70]. In this way formal performance bounds could be obtained.

As a result of the work undertaken to implement the controllers, several practical issues have been raised :

- In this thesis no attempt has been made to examine the effect of output noise on the performance of the sliding mode control strategies. Relevant work in this area, i.e. relating to sliding mode observers, appears in Slotine *et al.*[65] where 'intriguing results' are reported 'requiring further investigation'. Also, in Dorling & Zinober [19], the effect of noise on sliding mode observers has been explored through simulation. In the case of the multivariable control law implemented in Chapter 8, the effect of noise on the thermocouple signal was noted and was thought to degrade the performance. It was more noticeable than in the previous single-input single-output trials which were conducted prior to the furnace being dismantled and reassembled at a different site. As described in Chapter 8 a simple first order filter was used to suppress the noise. No theoretical justification or consideration of the consequences has been examined. This represents an obvious area requiring attention in the future. An initial starting point may be the work of Young & Drakunov [87].

- In a similar vein, work is currently in progress to formulate an appropriate strategy to design the control law hyperplane, in an optimal way, to minimise the effect of noise. It can be argued that noise in the system prevents a sliding motion taking place on the hyperplane in the observer error space. The noise will enter the control system through the linear observer gain G from equation (4.30). The initial research direction is to consider LQR ideas. It is tentatively suggested that the optimal hyperplane is one such that the invariant zeros of the system appear in the reduced order dynamics. This represents an area of on-going research.

In Chapter 5 a sufficient condition for the design of the sliding mode output feedback hyperplane was given in terms of 'Kimura-Davison' constraints for a reduced order sub-system. A well established approach to circumvent such constraints, is to design a compensator to provide additional degrees of freedom. Such an approach for sliding mode output feedback problems is considered by Diong & Medanic [18] and El-Khazali & DeCarlo in [26]. A similar approach, within the context of the framework developed in Chapter 5, is worth considering. It is believed that this may broaden the class of systems which could be considered from the point of view of design.

A more divergent and speculative area for future research lies in the investigation of similar problems to those considered in this thesis, within a discrete time framework. From the point of view of implementation, it was found that relative to the sampling rates usually used for thermal processes, the sampling period required for the sliding mode approach was very small. A discrete time formulation would consequently be beneficial. More importantly though, it can be argued that the constraints on the systems considered in this thesis stem from the (sufficient) condition that the transfer function $G_F(s) = FC(sI - A)^{-1}B$ from §3.2.3 must be strictly positive real. A recent result for single-input single-output discrete time systems indicates the corresponding requirement for positive realness does not have to be satisfied [62]. This suggests that for discrete time systems, a wider class of systems can be considered. This represents an important avenue for further work.

All the practical work documented in this thesis has been directed towards a single burner furnace – essentially because of its availability. The potential of such furnaces to provide challenging control problems has more or less been exhausted. Future work should clearly be directed towards more complicated multi-burner geometries. An industrially wide spread, multi-burner facility, is the continuous 'pusher' furnace used for reheating ferrous metals. In these configurations, the load enters at one end, and moves continuously through the furnace on a bogey, emerging at the other end having been subjected to an appropriate temperature/time history. This represents a fundamentally different mode of operation to the furnace described in this thesis, since the temperature history of the load results from its movement through the furnace. The control strategy is to maintain different parts of the furnace at different fixed set points. It can be envisaged that setting up of a bank of PID controllers to maintain a longitudinal temperature gradient, bearing in mind that interaction will take place between the different zones within the furnace, is difficult, and represents a truly multivariable control problem. It is believed that in such a situation, the multivariable potential of the sliding mode schemes may be exploited. To this end, it is proposed that the furnace simulation in Chapter 6 be modified to represent such a heating plant. This is a straight forward modification, and indeed, is more in keeping with the original intention of the model codes. From such a nonlinear model, a nominal linear system can be identified. Again such an approach will result in a control problem in which only outputs are available and for which the strategies of Chapters 4 and 5 are appropriate.

Appendix A

Mathematical Notation and Preliminaries

A.1 Mathematical Notation

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathrm{Re}[z]$	the real part of the complex number z
$\mathrm{Im}[z]$	the imaginary part of the complex number z
$\mathbb{R}_+$	the set of strictly positive real numbers
$\{a_i\}$	a sequence (possibly infinite) of real numbers
$\mathbb{C}_-$	the open left half of the complex plane
$\mathbb{R}^{n\times m}$	the set of real matrices with n rows and m columns
$\lvert a\rvert$	the absolute values of the real number a
$\mathrm{sgn}(\cdot)$	the signum function
A^{T}	the transpose of the matrix A
$\det(A)$	the determinant of the square matrix A
A^{-1}	the inverse of the square matrix A
$A^{\dagger}$	the (left) pseudo-inverse of the matrix A
$\mathrm{rank}(A)$	the rank of the matrix A
$\lambda(A)$	the spectrum of the square matrix A i.e. the set of eigenvalues
$\lambda_{max}(A)$	the largest eigenvalue of the square matrix A
$\lambda_{min}(A)$	the smallest eigenvalue of the square matrix A
$\kappa(A)$	the spectral condition number

$\mathrm{R}(A)$	the range space of the matrix A (viewed as a linear operator)
$\mathrm{N}(A)$	the null space of the matrix A (viewed as a linear operator)
I_n	the $n \times n$ identity matrix
$A > 0$	implies the square matrix A is symmetric positive definite
$A > B$	implies the square matrix $A - B$ is symmetric positive definite
$\|\cdot\|$	the Euclidean norm for vectors and the spectral norm for matrices
$\equiv$	equivalent to
$\times$	Cartesian product
$\perp$	orthogonal complement
$\stackrel{s}{=}$	equal to by definition

A.2 Mathematical Preliminaries

Fact 1 (Matrix determinants)
Let A be a real symmetric matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where the top left and bottom right sub-blocks are square then it is well known that

- $\det(A) = \det(A_{11})\ \det(A_{22})$
- $\lambda(A) = \lambda(A_{11}) \cup \lambda(A_{22})$

Fact 2 (Symmetric positive definiteness)
Let P be a real symmetric matrix partitioned so that

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}$$

where the matrix sub-blocks P_{11} and P_{22} are square then it can be shown that

$$\begin{aligned} P > 0 \ &\Leftrightarrow\ P_{11} > 0 \quad \text{and} \quad P_{22} > P_{12}^{\mathrm{T}} P_{11}^{-1} P_{12} \\ &\Leftrightarrow\ P_{22} > 0 \quad \text{and} \quad P_{11} > P_{12} P_{22}^{-1} P_{12}^{\mathrm{T}} \end{aligned}$$

Another useful property of symmetric positive definite matrices is that

$$P > 0 \ \Rightarrow\ P_{11} > 0 \quad \text{and} \quad P_{22} > 0$$

A.3 Controllability and Observability

Consider the linear time invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$.

Definition A.1 *The system is said to be completely controllable if given any initial condition $x(t_0)$ there exists an input function on the finite interval $[t_0,\ t_1]$ such that $x(t_1) = 0$.*

Definition A.2 *The linear system is said to be completely observable if the output function $y(t)$ over some time interval $[t_0,\ t_1]$ uniquely determines the initial condition $x(t_0)$.*

Theorem A.1 *Given any pair (A, B) the following conditions are all equivalent :*

- *(A, B) is completely observable*
- *the controllability matrix $\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$*
- *the matrix $\begin{bmatrix} sI - A & B \end{bmatrix}$ has full rank for all $s \in \mathbb{C}$*
- *the spectrum of $(A + BF)$ can be assigned arbitrarily by choice of $F \in \mathbb{R}^{m \times n}$.*

■

The third condition, which is often the most convenient method of establishing controllability, is often referred to as the PHB rank test. (For details see Kailath [42].)

Theorem A.2 *The pair (A, C) is completely observable if and only if the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is completely controllable*

■

From the duality theorem above, the results of Theorem A.1 can be modified to provide a list of equivalent statements for observability.

Appendix B

Lyapunov Stability

Consider the nonlinear system given by

$$\dot{x}(t) = F(x, t) \tag{B.1}$$

where $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ with $F(0, t) = 0$ i.e. the system has an equilibrium point at the origin. Loosely speaking if a generalised energy function can be found which is nonzero except at an equilibrium point and whose total time derivative decreases along the system trajectories then the equilibrium point is stable (for rigorous details see [66]). The key point is that this approach obviates the need to obtain a (analytical) solution to the nonlinear differential equation when assessing its stability properties. Unfortunately no systematic way exists to synthesis Lyapunov functions for nonlinear systems. Define a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ to be the *quadratic form*

$$V(x) = x^{\mathrm{T}} P x \tag{B.2}$$

where $P \in \mathbb{R}^{n \times n}$ is some symmetric positive definite matrix. By construction the function is nonzero except at the origin.

Definition B.1 *The (origin) of the system (B.1) is said to be quadratically stable if there exists a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$ such that the total time derivative satisfies*

$$\dot{V}(x) = 2x^{\mathrm{T}} P f(x, t) \leq -x^{\mathrm{T}} Q x \tag{B.3}$$

This implies $\|x(t)\| < e^{-\alpha t}$ where $\alpha = \lambda_{min}(P^{-1}Q)$ and hence the origin is asymptotically stable. If $F(x, t) = Ax(t)$ then it is well known that A has stable eigenvalues if and only if given any symmetric positive definite matrix Q there exists an unique symmetric positive definite matrix P satisfying the *Lyapunov Equation*

$$PA + A^{\mathrm{T}}P = -Q \tag{B.4}$$

Consequently any stable linear system is quadratically stable. A symmetric positive definite matrix P satisfying (B.4) will be referred to as a *Lyapunov Matrix* for the matrix A. Details of quadratic stability and its relationship with other stability theory concepts in particular H_∞ ideas are given in [9].

When dealing with uncertain systems, it may not be possible to guarantee asymptotic stability. Consider the nonlinear system (B.1) and suppose it is subject to an imprecisely known exogenous signal $\xi(\cdot)$ so that

$$\dot{x}(t) = F(x, t, \xi) \tag{B.5}$$

Let $\mathcal{E} \subset \mathbb{R}^n$ be a bounded set then :

Definition B.2 *The solution $x(\cdot)$ to the uncertain system (B.5) is said to ultimately bounded with respect to the set $\mathcal{E}$ if*

- *on any finite interval the solution remains bounded i.e. if $\|x(t_0)\| < \delta$ then $\|x(t)\| < d(\delta)$ for any $t \in [t_0,\ t_1)$*
- *in finite time the solution enters $\mathcal{E}$ and remains there for all subsequent time.*

The set $\mathcal{E}$ is usually an acceptably small neighbourhood of the origin and the concept is often termed *practical stability*.

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