SOME ASPECTS OF DYNAMICAL SYSTEMS

by

D.K.Arrowsmith, B.Sc.

A thesis presented for the degree of Doctor of Philosophy

at the University of Leicester

September, 1972

FOREWORD

I should like to thank my supervisors, Professor T.V.Davies of the University of Leicester and Professor E.C.Zeeman of the University of Warwick, for their advice and encouragement.

The work contained here was completed during the tenure of an award by the Science Research Council. A substantial part of Chapter 2 has since been accepted for publication [36] and is to appear shortly.

Mathematics Department,
University of Southampton.

September, 1972.

CONTENTS

CHAPTER I

INTRODUCTION

In this thesis we consider the properties of suspension flows of diffeomorphisms, particularly products of these flows. Also considered are the cohomology groups of the insets of certain diffeomorphisms where the non-wandering sets contain Cantor sets.

Differentiable dynamical systems to which the title refers had its origins with the work of Poincaré when the emphasis on the subject of differential equations was changed from the quantitative to the qualitative approach using topological (geometric) methods. Interest in this qualitative approach declined in the early twentieth century mainly because the period was taken up consolidating the foundations of algebraic topology which had its beginnings with Poincaré's homology theory.

During the 1950's and 1960's interest gradually returned to the subject, combining topology and differential equations in the spirit of Poincaré. In [26], the formulation of the fundamental question was stated as a precise mathematical problem; it was the problem of classification where one tries to give a generic system of differential equations. A differential equation on a smooth compact manifold M is defined by a vector field X on M such that the equation is given by $\dot{x} = X(x)$ for $x \in M$. By the existence theorem X determines a unique flow (or dynamical system). The solution curves of X which are tangent to X at every point of M are the orbits of the flow. (For precise definitions of flow and orbit see below.) The topology put on $\mathfrak{X}(M)$ the set of all vector fields on M, is the C^r-topology $(r \geqslant 0)$. In other words X, X' are close if their vectors and all their partial derivatives up to order r are close.

With this notation the problem is one of finding an open dense subspace of $\mathcal{X}(M)$ which can be described by some "simple" conditions. Several of the ideas and constructions used in [26] in an attempt to achieve this classification are investigated here.

We now define the fundamental definitions of dynamical systems which are constantly referred to in the following chapters.

1.1 Definition

A diffeomorphism f of a manifold M is a homeomorphism $f: M \to M$ such that f and f^{-1} are differentiable maps.

1.2 Definition

A flow ϕ of a manifold M is a differentiable map $\phi : M \times R \to M$, such that $\forall$ $t \in R$, $\phi_t : M \to M$ is a diffeomorphism, where $\phi_t(x) = \phi(x,t)$, and such that $\phi_s \phi_t = \phi_{s+t}$. In other words ϕ induces a 1-parameter subgroup $R \to \mathrm{Diff}(M)$.

Remark The vector field X is related to ϕ by

$$\left. \frac{d\phi_t(x)}{dt} \right|_{t=o} = X(x) .$$

1.3 Definition

Given flows ϕ, ψ on M,N respectively, then the product flow $\phi \times \psi$ on $M \times N$ is defined as

$$(\phi \times \psi)_t = \phi_t \times \psi_t : M \times N \to M \times N.$$

1.4 Definition

The orbit of $x \in M$ with respect to the diffeomorphism f [flow ϕ] is the set $\{f^n(x) | n \in Z$, the integers$\}$ $[\{\phi_t(x) | t \in R$, the reals$\}]$.

1.5 Definition

If $f \in \mathrm{Diff}(M)$, $x \in M$ is called a wandering point when there is a neighbourhood U of x such that $\bigcup_{|m|>0} f^m(U) \cap U = \Phi$, the empty set. A point will be called a non-wandering point if it is not a wandering point. Let the collection of all non-wandering points be denoted by $\Omega = \Omega(f)$.

1.6 Definition

Let ϕ be a flow on M; then we have x is a wandering point of ϕ if there is some neighbourhood U of x with $\bigcup_{|t|>|t_0|} \phi_t(U) \cap U = \Phi$, for some $t_0 > 0$. Let the set of non-wandering points be $\Omega = \Omega(\phi)$.

All the diffeomorphisms considered will satisfy Axiom A, defined as follows [26].

Axiom A. For $f \in \mathrm{Diff}(M)$, (a) the non-wandering set is hyperbolic, (b) the periodic points of f are dense in Ω.

Hyperbolicity is defined in detail in [26].

SPECTRAL DECOMPOSITION OF DIFFEOMORPHISMS. Suppose $f : M \to M$ is an Axiom A diffeomorphism. Them there is a unique way of writing Ω as the finite union of disjoint, closed, invariant indecomposable subsets (or "pieces") on each of which f is topologically transitive:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \dots \cup \Omega_k \ .$$

Corollary If $f : M \to M$ is as above one can write M canonically as a finite disjoint union of invariant subsets $M = \bigcup_{i=1}^{k} I(\Omega_i)$ where $I(\Omega_i) = \{x \in M \mid f^m(x) \to \Omega_i, m \to \infty\}$. $I(\Omega_i)$ will be called the inset of Ω_i.

It was the attempts of many mathematicians during the 1950's and 1960's to understand the global geometric picture of the phase portrait that resulted in [26] giving various conjectures and many of the important features in dynamical systems in higher dimensions. There are two definite parts in Smale's paper; they correspond to sections on diffeomorphisms and flows. The discussion centres around the non-wandering set.

A large part of the subsequent chapters is taken up investigating a functorially defined "operator" Σ which given the pair (M,f), where f is a diffeomorphism of M, gives $\Sigma(M,f)$ denoting a flow reflecting the qualitative features of the diffeomorphism f. The "suspension" Σ is defined as follows. Given a manifold M with diffeomorphism f then we have a diffeomorphism $\alpha : M \times R \to M \times R$, R the real line, defined by $\alpha(x,u) = (f(x),u-1)$. Identification of orbital points of α gives us a manifold M_f. Denote the projection $\pi_f\colon M \times R \to M_f$, then we define a flow ϕ on M_f by $\phi_v(\pi_f(x,t)) = \pi_f(x,t+v)$, $v \in R$. Let $\phi = \Sigma(M,f)$.

In chapter 2 we investigate the structure of products of suspended flows. We shall see later in the chapter that such structures occur naturally in applied mathematics. In chapter 3 we construct a more general operator Σ_j for any positive integer j which gives a flow derived from j-commuting diffeomorphisms of M. We consider j-suspension flows and also the product of j and k suspension flows. The results of chapter 3 follow a similar pattern to those of chapter 2. Chapter 4 is concerned with some algebraic properties of Σ and Σ_j. We show that Σ and Σ_j have functorial representations with respect to suitably defined categories. Also other operations on flows such as boundary flows and the relations they have with Σ and Σ_j are investigated.

There exist diffeomorphisms with non-wandering sets which have pieces (or subsets) homeomorphic to a Cantor sets, [25], [26], [33]. We look at particular diffeomorphisms, one of which is the Smale "Horse-Shoe" (for definitions see chapter 5), which exhibit these properties and consider the structure of the insets of the Cantor sets. We use compact Čech cohomology [8], [31], [34], to find the cohomology groups of these insets.

In Appendix 1, we consider an application of chapter 2 , §5 to the theory of Lie Group Bundles. Appendices 2 to 5 are devoted to definitions and a lemma.

In the rest of this introductory chapter we will give examples to illustrate the various aspects of dynamical systems which will be investigated.

1.7 <u>The Simple Harmonic Oscillator; $\ddot{x} + x = 0$.</u> ($\cdot$ denotes differentiation with respect to t).

We reduce this equation to a first order in two variables by taking $\dot{x} = y$. This gives

$$\dot{x} = y, \quad \dot{y} = -x \quad .$$

The orbits in (x,y) space (phase space) are concentric circles with respect to the origin as in Fig 1.1.

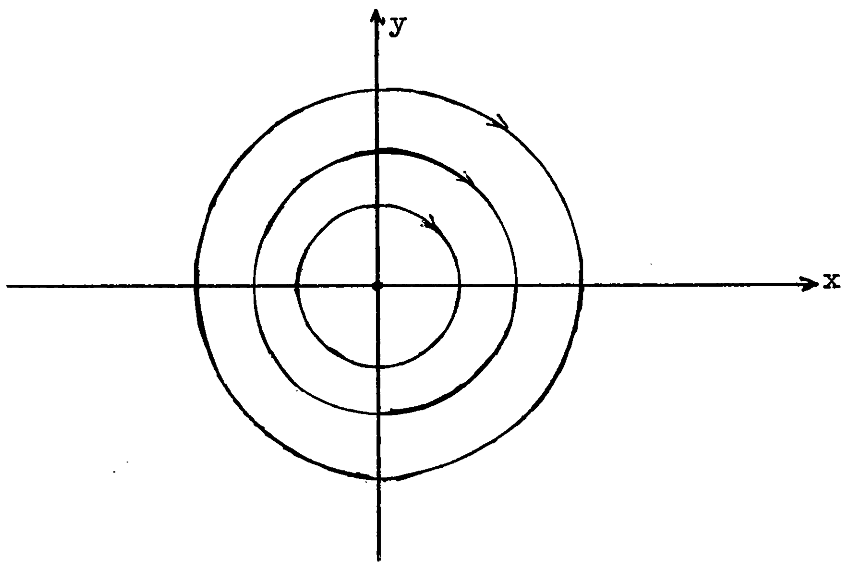


Fig 1.1

By taking the first integration of $\ddot{x} + x = 0$, we have $\frac{\dot{x}^2}{2} + \frac{x^2}{2} = E$ (a constant) and this is obviously the energy equation where energy levels correspond to the phase plane orbits of Fig 1.1.

If we restrict the energy E of the system such that $E \in [E_1, E_2]$ with $E_1 > 0$ then Fig 1.1 is restricted to an annulus of first integrals which one can easily see is a trivial suspended flow with respect to the identity diffeomorphism on the interval $\{(x,0) \mid \sqrt{2E_1} \leqslant x \leqslant \sqrt{2E_2}\}$.

1.8 The Biharmonic Oscillator; $\ddot{x} + \lambda_1 x = 0$, $\ddot{y} + \lambda_2 y = 0$.

Let the new variables u,v be defined as follows:

$$\dot{x} = u, \quad \dot{u} = -\lambda_1 x$$
$$\text{and} \quad \dot{y} = v, \quad \dot{v} = -\lambda_2 y \quad .$$

This gives a first order autonomous differential equation in the variables $(x,u,y,v) \in R^4$. The total energy of the system E is given by $\lambda_1 x^2 + u^2 + \lambda_2 y^2 + v^2 = 2E$. So we have the system represented in R^4 with energy surfaces on 3-spheres S^3. The energy levels are foliated by invariant 2-tori T^2. These tori are obtained as we vary the energy partition between the oscillators, say E_1, E_2. Then $\lambda_1 x^2 + u^2 = 2E_1$ and $\lambda_2 y^2 + v^2 = 2E_2$. We can take angular coordinates α, β, say, such that $x = (\sqrt{2E_1/\lambda_1})\cos\alpha$, $u = \sqrt{2E_1}\sin\alpha$, $y = (\sqrt{2E_2/\lambda_2})\cos\beta$ and $v = \sqrt{2E_2}\sin\beta$. Given the partition (E_1, E_2) then the state can be realised by (α, β), i.e. the flow is restricted to a torus and periodic solutions will exist if $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ are rationally related and almost periodic solutions will fill the torus when they are irrationally related. The phase plane solutions can be represented as a product flow where each flow of the product is a flow as in Fig 1.1 on R^2.

1.9 Weakly Coupled Oscillators; $\ddot{x} + \lambda_1 x + \epsilon y = 0$, $\ddot{y} + \lambda_2 y + \epsilon x = 0$.

We use the principle of normal modes to obtain a solution. Let the normal modes be (ξ, η) such that

$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$, for some non-singular 2×2 matrix A. Then we obtain

$$\begin{pmatrix}\ddot\xi\\ \ddot\eta\end{pmatrix} = -A\begin{pmatrix}\lambda_1 & \epsilon\\ \epsilon & \lambda_2\end{pmatrix}A^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$ and by choosing the matrix

$A\begin{pmatrix}\lambda_1 & \epsilon\\ \epsilon & \lambda_2\end{pmatrix}A^{-1}$ to be diagonal assuming $\begin{vmatrix}\lambda_1 & \epsilon\\ \epsilon & \lambda_2\end{vmatrix} \neq 0$ we have the equations

$$\ddot\xi + \lambda_1'\,\xi = 0$$
$$\ddot\eta + \lambda_2'\,\eta = 0$$

By considering the normal mode coordinates we have been able to reduce the weakly coupled oscillator to an equivalent system to the biharmonic oscillator. So we have been able to move from a system where we could not express the system as a product flow to the 'normal' system which is a product flow.

1.10 <u>The Van der Pol Oscillator; $\ddot x + \mu(x^2-1)\dot x + x = 0$, μ small.</u>

Let $\dot x = y$; then the phase portrait equations are

$$\dot x = y$$
$$\dot y = -\mu(x^2-1)y - x,$$ with solution curves givenin Fig 1.2.

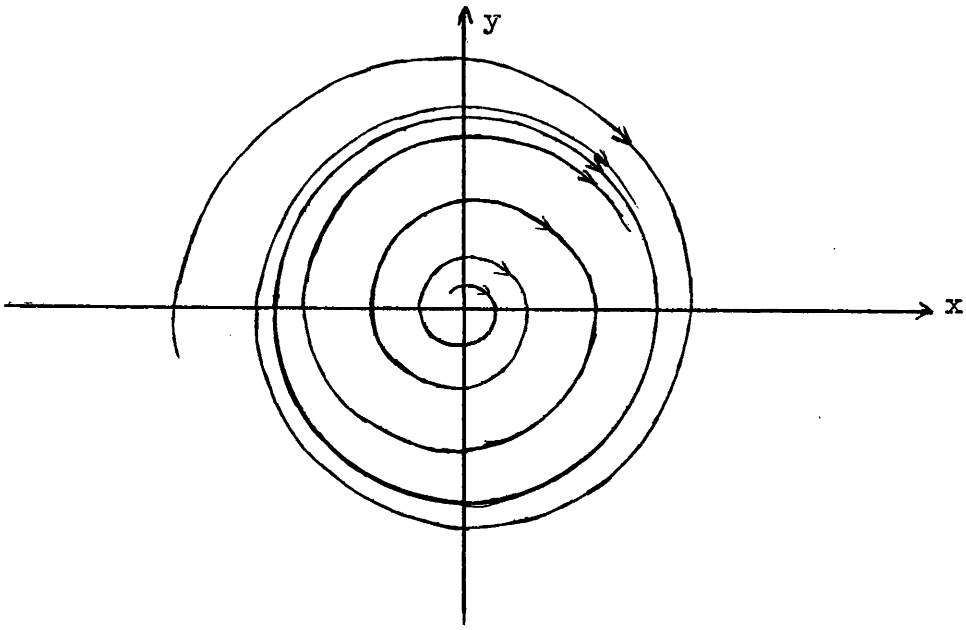


Fig 1.2

Note that for the choice of a small value for the parameter μ the differential equation 1.10 is only slightly perturbed from that of 1.7 which corresponds to $\mu = 0$. However the phase portrait even though only a small perturbation of 1.7, is topologically different to it, i.e. there is no homeomorphism between the phase portraits throwing orbits onto orbits. This phenomenon is described by saying the simple harmonic oscillator is not structurally stable. However the Van der Pol is structurally stable under sufficiently small perturbations. An extension of this equation to that of the forced Van der Pol equation will give us scope to discuss the phase portrait in the way the results of chapters 2 and 3 might be used in applications.

The phenomenon we will discuss is the "locking-on" of the forced Van der Pol oscillator. Often referred to as synchronization it was known to Huyghens [16] who observed that two clocks slightly out of synchronism when suspended on a wall became synchronized when fixed on a thin wooden board. It was not until 1922 [30] that the present theory of this effect was formulated by Van der Pol. The equation representing the forced motion is given canonically by

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = \cos\omega_0 t \quad .$$

Suppose that the natural frequency of the oscillator is ω. From the equation we see that the forcing oscillation has frequency ω_0. Whenever this system is observed beats are detected. This is because under observation the values ω and ω_0 are seen as rationally related. We are observing a real system and the rational or irrational relationship of ω and ω_0 is not transient but locally constant on the rationals. It is expected that as we allow $\omega_0 \to \omega$ the period of the beats will increase indefinitely. This is true except that the beats increase up to a certain limit of the difference $|\omega - \omega_0|$, at which the beats disappear suddenly and there remains only one frequency ω_0. It appears that the natural frequency of the oscillator ω suddenly locks onto the applied oscillating frequency ω_0 when ω_0 moves into the "zone of synchronization". The reason for this phenomenon is that prediction of the increasing period of the beats is made on the basis of linear theory and the synchronization occurs from the presence of non-linearity in the system. The theory has been studied in great detail and it is used for instance in keeping fluctuations in frequency of an electric current to a minimum by locking the frequency onto some constant frequency oscillator (such as a quartz oscillator). The language of the qualitative theory used to describe this theory is as follows.

In the phase plane of the forced Van der Pol we have a 1st order autonomous system in R^3 defined by

$$\dot{x} = y$$
$$\dot{y} = -\mu(x^2-1)y - x + \cos\omega_0\tau$$
$$\dot{\tau} = 1$$

where τ is the dummy variable $\tau = t$. It is better to identify points of $\tau \bmod(2\pi/\omega_0)$ and realise the flow on $S^1 \times R^2$ as a result of this identification. Also by identifications of τ we have an induced transformation of $R^2 \rightarrow R^2$ obtained from the intersections of orbits of the dynamical system with $\tau = 0$ and $\tau = (2\pi/\omega_0)$.

Locally a flow will be formed on a torus by considering the flow restricted to the product of the natural oscillation with the forced oscillation of the system. The resultant flow will be observed as rational flows according to the rationality between ω and ω_0. As the rational flows get closer to the diagonal flow, i.e. as $\omega_0 \rightarrow \omega$, we get a perturbatio to a diagonal flow when ω_0 is sufficiently close to ω. However, the diagonal flow is not stable and by the very phenomenon of synchronization in the real situation we must have stability, i.e. perturb the natural oscillator a little from ω_0 and it will "lock on" again. So we must have a stable perturbation of the diagonal flow which gives a diagonal attractor. By Peixoto's Theorem [19] the generic perturbation will be a flow with a finite number of attractors and repellors with orbits spiralling between them.

It can be seen from these examples how product flows naturally occur. In general, whenever we have two dynamical systems each with a stable attractor brought into contact, then this is represented by taking the product (or if one system is driving the other perhaps by a fibre bundle). However, the product of two stable attractors is no longer stable in general and therefore it is important to understand the product structure of such flows in order to understand what possible attractors can arise as a result of perturbations of the product flow.

The suspension of a diffeomorphism has been of great importance in converting qualitative statements on the nature of the non-wandering sets of diffeomorphisms to similar statements on the non-wandering sets of flows. For instance non-density theorems on diffeomorphisms have been instantly transferable to non-density theorems on flows [26].

An important step in analysing the qualitative behaviour of diffeomorphisms came with the discovery of Anosov Diffeomorphisms [3]. The original examples given by Thom are the induced diffeomorphisms $f : T^n \to T^n$ (n-torus) given by a linear automorphism $\hat{f}$* of R^n representable by a non-singular matrix. These diffeomorphisms exhibited the property that they were structurally stable, but that the non-wandering set was the whole manifold. These diffeomorphisms disproved the conjecture of 1960 that Morse-Smale diffeomorphisms [22] were a generic system on all smooth manifolds.

1.11 Definitions of Anosov Diffeomorphisms and Flows

A diffeomorphism f of M is said to be Anosov if $T(M)$, the tangent bundle of M with a Riemannian norm, has an invariant splitting

---

* the eigenvalues λ_i satisfy $|\lambda_i| \neq 1$

under df, the differential of f. The splitting is a Whitney sum $T(M) = T_c(M) \oplus T_e(M)$ such that there exist constants $0 < c_1 < 1 < c_2$ with $|df^n(v)| \leqslant c_1^n |v|$, $v \in T_c(M)$ and $|df^n(v)| \geqslant c_2^n |v|$, $v \in T_e(M)$. A flow ϕ is said to be Anosov on M if there exists an invariant Whitney Sum of $T(M) = T'(M) \oplus T_c(M) \oplus T_e(M)$ such that the components are invariant with respect to $d\phi_t$ where $T'(M)$ is the 1-dimensional bundle defined by differentiating ϕ_t with respect to t, $T_c(M)$ is a contracting bundle with constants $c, \lambda (> 0)$ such that $|d\phi_t(v)| \leqslant ce^{-\lambda t}|v|$, $v \in T_c(M)$, and $T_e(M)$ is an expanding bundle with constants $c_1, \mu\ (> 0)$ such that $|d\phi_t(v)| \geqslant c_1 e^{\mu t}|v|, v \in T_e(M)$.

At present there are very few ways of constructing Anosov flows. They are as follows:

1) Geodesic flows on the unit tangent bundle of Riemannian manifolds with negative curvature;

2) Suspensions of Anosov diffeomorphisms.

Here we have an important class of flows where one of the main methods of construction is by suspending Anosov diffeomorphisms.

In chapter 5 we consider the cohomology [34] of the insets of various diffeomorphisms. They are related to the "horse-shoe" diffeomorphism which is defined in chapter 5. It has part of its non-wandering set homeomorphic to a Cantor set. The diffeomorphisms are Ω stable and were constructed to show that non-wandering sets are not generically manifolds. These diffeomorphisms, particularly the "horse-shoe", came from a paper by Levinson [14] and the ideas presented there were developed by Smale. Levinson's paper was concerned with the Van der Pol equation with forcing term. It was consideration of the types of transformations of the plane onto itself, which could occur in the way indicated earlier, that led to these diffeomorphisms with complicated non-wandering sets.

Unfortunately chapter 5 is restricted in the sense that we have not yet given an answer to a question such as "Given any diffeomorphism $f : M \to M$ such that there exists an $\Omega_i \subset \Omega(f)$ which is homeomorphic to a Cantor set, then what is the cohomology group of its inset?"

An answer to such a question would be of great importance particularly in view of the following theorem.

SHUB (1972) [35] A Smale diffeomorphism has $\Omega = \{$finite number of fixed points, finite number of closed orbits, finite number of n-horse-shoes$\}$, and the usual tranversality conditions between insets and outsets. Then

1) Smale diffeomorphisms are structurally stable.

2) Smale diffeomorphisms are dense in the C^0-topology.

[n-horse-shoes are defined in chapter 5].

If we had an answer to the question above then we would be able to supersede the Morse-Smale Inequalities for Morse-Smale Diffeomorphisms by Smale Inequalities for the above system. What we can deduce from chapter 5 is that Čech cohomology seems a promising tool for the analysis of non-wandering sets and their insets in general. Čech cohomology was used successfully in the relatively simple case of Morse-Smale systems [22] where the non-wandering sets were only fixed points and closed orbits.

There also exists a corresponding theorem on Smale flows defined analogously to Smale diffeomorphisms. An interesting problem concerned with suspension and these two theorems is: "Is there a way of extending the idea of a suspension to obtain a way of converting the theorem on diffeomorphisms to the one on flows?". It has been suggested that this might take the form of a finite number of discs placed in the manifold so that every orbit intersects one of these discs at least once. We would then have mappings of these discs by following the flow between intersections. The mappings could then be investigated in the light of the theorem of SHUB and by using the density of Smale diffeomorphisms possibly obtain the density of Smale flows.

In the Appendix 1 we consider Lie Groups as a possible application of chapter 2, §5 .

CHAPTER 2

SUSPENDED PRODUCT AND BUNDLE DIFFEOMORPHISMS

We have seen in chapter 1 the importance of products of flows in investigating interrelated systems and here we consider products of suspended flows. The aim is to find some relationships between suspensions of various diffeomorphisms. For instance, suppose we take a product diffeomorphism $f \times g : M \times N \to M \times N$ then we also obtain the diffeomorphisms $f : M \to M$ and $g : N \to N$. We now have three pairs of manifolds and diffeomorphisms (M,f), (N,g) and $(M \times N, f \times g)$. Hence we can obtain three pairs of manifolds and suspension flows $(M_f, \Sigma(M,f))$, $(N_g, \Sigma(N,g))$ and $((M \times N)_{f \times g}, \Sigma(M \times N, f \times g))$. The relationship between the flows appears as a theorem in §1 together with a corollary, and the proof is given in §2. A similar theorem is given for bundle diffeomorphisms in §4 with a possible generalization in §5.

§1 STATEMENT OF PRODUCT THEOREM

We require the definitions of chapter 1 supplemented by the following:

2.1.1 Definition

Two flows ϕ and ψ on the manifolds M,N are said to be differentiably equivalent or more briefly, equivalent, if there exists a diffeomorphism h such that the diagram commutes

$$\begin{tikzcd} M \times R \arrow[r, "h \times 1"] \arrow[d, "\phi"'] & N \times R \arrow[d, "\psi"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

Remark. This is a very strong equivalence relation; much stronger than the usual topological equivalence because it preserves time and is a diffeomorphism of orbits onto orbits.

2.1.2 Definition. A flow ϕ on the bundle

$$V \longrightarrow B \xrightarrow{\pi} X$$

is a fibre flow if all the fibres are invariant under the flow.

2.1.3 Definition. If further to 2.1.2 the flows on all fibres are equivalent we call ϕ a uniform fibre flow (see 2.1.6 for a more precise definition).

2.1.4 PRODUCT THEOREM. Given diffeomorphisms f,g of the manifolds M,N respectively then the product flow $\Sigma(M,f) \times \Sigma(N,g)$ is a uniform fibre flow on the bundle

$$(M \times N)_{f\times g} \longrightarrow M_f \times N_g \xrightarrow{\pi} S^1.$$

Moreover the flow on the fibre is $\Sigma(M \times N,\ f \times g)$.

2.1.5 Corollary. Products of suspension flows can never be structurally stable.

Proof. By a theorem of Thom [20] a flow is not structurally stable if there exists a first integral for which the flow is invariant. The projection π will give the first integral. Hence we have non-stability.

Remark. Suppose f and g are Anosov diffeomorphisms, then $f \times g$ is also an Anosov diffeomorphism and so $\Sigma(M,f)$, $\Sigma(N,g)$ and $\Sigma(M \times N,\ f \times g)$ are stable Anosov flows. By the theorem $\Sigma(M \times N,\ f \times g)$ is the fibre flow which is stable. However the product flow $\Sigma(M,f) \times \Sigma(N,g)$ on $M_f \times N_g$ is unstable. If we perturb the fibre flow on $M_f \times N_g$ by adding a small gradient flow on S^1 with one sink and one source we obtain a structurally stable flow on $M_f \times N_g$ with one attracting fibre and one repelling fibre. Both of these fibres have restricted flows which are Anosov.

2.1.6 Definition. Given a bundle

$$V \longrightarrow B \xrightarrow{\pi} X \text{ and a flow } \phi \text{ on } B, \text{ then}$$

ϕ is a uniform fibre flow with respect to the bundle if $\exists$ a flow ψ on V and $\exists$ an atlas of bundle charts (U,h) where $U \subset X$ and $h : U \times V \to \pi^{-1}(U)$ is a diffeomorphism such that the diagram commutes.

$$\begin{tikzcd}
U \times V \times R \arrow[r, "h \times 1"] \arrow[d, "1\times\psi"'] & \pi^{-1}(U) \times R \arrow[d, "\phi"] \\
U \times V \arrow[r, "h"] \arrow[d, "\rho"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\
U \arrow[r, "1"] & U
\end{tikzcd}$$

[ρ is the natural projection of the first factor of the product.]

Notation. Given the manifold M and the diffeomorphism $f: M \to M$ then let Π_f denote the projection map

$$\Pi_f : M \times R \longrightarrow M_f .$$

2.1.7 Example. Here we take the simplest possible case of M and N being point manifolds. It was this example which motivated the theorem for general products. Let $f = g = 1$ be the identity maps on M and N respectively. The theorem is proved by the following series of lemmas. The first of these lemmas is of general use in chapters 2 and 3 and is constantly referred to.

2.1.8 LEMMA. [29] Let G be a Lie Group acting on a space X. Consider the set of orbits X/G and the canonical map $\Pi_X : X \to X/G$. Then defining the orbit space X/G to be the set of orbits with the quotient topology,

1) $\Pi_X : X \to X/G$ is an open map

2) the topology on X/G is characterized as being the unique topology making the map continuous and open.

Proof. The orbits of X by the action G are $0_x = \{\phi_g(x) | g \in G\}$, where ϕ is the action $\phi : X \times G \to X$ such that $\phi_1 = 1_X$ and $\phi_{g_1} \cdot \phi_{g_2} = \phi_{g_1 \cdot g_2}$.

Let $U \subset X$ be open; then $\phi_g(U)$ is open and therefore $\Pi(U) = \Pi_X^{-1}(\Pi_X(U))$ is open being the union of all sets $\phi_g(U)$, $g \in G$, ($\Pi(U)$ is just the orbit of U under the induced G operation).

By definition of the quotient topology, $\Pi_X(U)$ is open. To prove 2), consider more generally a map $\tau : X \to Y$ from a set X to a set Y. Two topologies on Y making τ continuous and open necessarily coincide. Because if 0 is an open set of Y in one topology, $\tau^{-1}(0)$ is open in X and $\tau(\tau^{-1}(0)) = 0$ is also open in the other topology.

The proofs concerning the continuity of maps between quotient manifolds are usually approached first of all by checking that the maps are well defined. The continuity is proved by using the openness properties of the orbit projection maps Π_f, etc.

2.1.9 LEMMA. $\Sigma(M,f)$ and $\Sigma(N,g)$ are both flows on circles C_1, C_2 say. Let $\phi = \Sigma(M,f)$ and $\psi = \Sigma(N,g)$ then ϕ_1 and ψ_1 are identity diffeomorphisms.

Proof. Let $\alpha : M \times R \to M \times R$ be defined by $\alpha(P,t) = (f(P),t-1) = (P,t)$ where P represents the point manifold M. Then $M_f = M \times R/\sim$, where

$\sim$ denotes identification of orbital points of α.

We have the map $\Pi_f : M \times R \to M_f$ and so M_f is given by

$$M_f = \{\Pi_f(P,t) | \Pi_f(P,t) = \Pi_f(P,t-1)\}.$$ Hence M_f is a circle,

say C_1. The flow ϕ is defined by $\phi_v \Pi_f(P,t) = \Pi_f(P,t+v)$, $v \in R$. Putting $v = 1$ and using the recurrence relations we have ϕ_1 is the identity map of M_f. In a completely analogous way we obtain that N_g is a circle C_2 say and the flow ψ is such that ψ_1 is the identity diffeomorphism of C_2.

2.1.10 LEMMA. The product flow $\phi \times \psi$ on $C_1 \times C_2$ is equivalent to the diagonal flow λ on the torus T^2.

Proof. Define $T^2 = R^2/Z^2$, the quotient space of the real plane by the lattice of integers. Let $\sigma : R^2 \to T^2$ be the natural projection map.

The diagonal flow λ on T^2 is defined as:

$$\lambda_t \sigma(u,v) = \sigma(u+t, v+t) \quad .$$

Choose fixed points $C_1^o, C_2^o \in C_1, C_2$ respectively. Then all points of C_1 and C_2 are representable as $(\phi_{t_1}(C_1^o), \psi_{t_2}(C_2^o))$ for some $t_1, t_2 \in R$. Define $h : C_1 \times C_2 \to T^2$ by $h(\phi_{t_1}(C_1^o), \psi_{t_2}(C_2^o)) = \sigma(t_1, t_2)$.

Using 2.1.9 we get

1) h is well defined.

$$\begin{array}{ccc} (\phi_{t_1}(C_1^o), \psi_{t_2}(C_2^o)) & = & (\phi_{t_1+\ell_1}(C_1^o), \psi_{t_2+\ell_2}(C_2^o)), \; \ell_1, \ell_2 \in Z. \\ \Big\downarrow h & & \Big\downarrow h \\ \sigma(t_1, t_2) & = & \sigma(t_1+\ell_1, t_2+\ell_2) \end{array}$$

2) h is 1-1.

$$h^{-1}(\sigma(t_1,t_2)) = h^{-1}(\sigma(t_1+\ell_1,t_2+\ell_2)) = \{(\phi_{t_1+\ell_1}(C_1^o)),\psi_{t_2+\ell_2}(C_2^o))\mid \ell_1,\ell_2 \in Z\}$$

$$= (\phi_{t_1}(C_1^o),\ \psi_{t_2}(C_2^o))$$

3) h is a homeomorphism.

Consider the diagram

$$\begin{tikzcd} M \times N \times R \times R \arrow[r, "h'"] \arrow[d, "\Pi_f \times \Pi_g"'] & R \times R \arrow[d, "\sigma"] \\ C_1 \times C_2 \arrow[r, "h"] & T^2 \end{tikzcd}$$

Define $h' : M \times N \times R \times R \to R \times R$ by $h'(P,Q,t_1,t_2) = (t_1,t_2)$ then h' is obviously a diffeomorphism. The diagram commutes. Given an open set $O \subset T^2$ we have from the commutativity,

$$h^{-1}(O) = (\Pi_f \times \Pi_g)\, h'^{-1}\, \sigma^{-1}(O)\ .$$

But h' and σ are continuous and from 2.1.10 $\Pi_f \times \Pi_g$ is open and so h is continuous. Also, given an open set $O' \subset C_1 \times C_2$ we have

$(h^{-1})^{-1}(O') = h(O') = \sigma h'(\Pi_f \times \Pi_g)^{-1}(O')$. The openness of σ and h' and the continuity of $\Pi_f \times \Pi_g$ ensure h^{-1} is continuous. Hence h is a diffeomorphism, because all maps used are differentiable of class C^r $(r \geqslant 1)$.

2.1.11 <u>LEMMA</u>. There is a diffeomorphism k of $T^2 \to T^2$ which takes the natural generators S^1_1, S^1_2 onto the generators S^1_1, D where D is a diagonal of the torus such that λ is a uniform fibre flow on T^2 with the bundle representation

$$D \longrightarrow S^1_1 \times D(= T^2) \xrightarrow{\pi'} S^1_1\ .$$

<u>Proof</u>. The diffeomorphism k is induced by the linear automorphism $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of R^2. The fibre flow on D is the restriction of λ to D.

2.1.12 LEMMA. The flow on D say $\mu = \lambda | D : D \times R \to D$ is equivalent to $\Sigma(M \times N, f \times g)$.

Proof. The manifold $M \times N$ is a point (P,Q); the map $f \times g : M \times N \to M \times N$ is the identity. Then as before $\Sigma(M \times N, f \times g)$ is a flow on a circle. A time-preserving diffeomorphism of D onto $(M \times N)_{f \times g}$ will give the equivalence.

2.1.13 LEMMA. The product flow $\Sigma(M,f) \times \Sigma(N,g)$ is a uniform fibre flow on the product

$$(M \times N)_{f \times g} \longrightarrow M_f \times N_g \ (= S^1 \times (M \times N)_{f \times g}) \xrightarrow{\pi} S^1$$

The flow on the fibre is $\Sigma(M \times N, f \times g)$.

§2 PROOF OF PRODUCT THEOREM

2.2.1 Definition. Let $\bar{f} : (M \times N)_{f \times g} \longrightarrow (M \times N)_{f \times g}$ be the diffeomorphism defined by $\bar{f}\,\Pi_{f \times g}(x,y,t) = \Pi_{f \times g}(f(x),y,t)$ where $x \in M$, $y \in N$ and $t \in R$.

2.2.2 THEOREM. The flow λ on $((M \times N)_{f \times g})_{\bar{f}}$ defined by $\lambda_v(\Pi_{\bar{f}}(\Pi_{f \times g}(x,y,t),u)) = \Pi_{\bar{f}}(\Pi_{f \times g}(x,y,t+v),u)$ is equivalent to the product flow $\phi \times \psi$ on $M_f \times N_g$ where $\phi = \Sigma(M,f)$ and $\psi = \Sigma(N,g)$.

This is proved by the following lemmas.

2.2.3 LEMMA. There are maps q_1, q_2 such that

$$q_1 : ((M \times N)_{f \times g})_{\bar{f}} \longrightarrow M_f$$

$$q_2 : ((M \times N)_{f \times g})_{\bar{f}} \longrightarrow N_g$$

These maps are defined by

$$q_1 : \Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) \longrightarrow \Pi_f(x,t+u)$$

$$q_2 : \Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) \longrightarrow \Pi_g(y,t)$$

<u>Proof</u>. 1) q_1 is continuous.

Consider the following diagram.

$$\begin{tikzcd}
M \times N \times R \times R \arrow[rr, "q'_1"] \arrow[d, "\Pi_{f\times g}\times 1"'] & & M \times R \arrow[dd, "\Pi_f"] \\
(M \times N)_{f\times g} \times R \arrow[d, "\Pi_{\bar f}"'] & & \\
((M \times N)_{f\times g})_{\bar f} \arrow[rr, "q_1"] & & M_f
\end{tikzcd}$$

The maps $\Pi_{f\times g}$ and 1 are open and so $\Pi_{f\times g} \times 1$ is open. Also the maps $\Pi_{\bar f}$ and Π_f are open. Define the map q'_1 by $q'_1(x,y,t,u) = (x,t+u)$. Then q'_1 is continuous.

The map q_1 is well defined because we have the following diagram.

$$\begin{tikzcd}
\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) \arrow[d, "q_1"'] & = & \Pi_{\bar f}(\Pi_{f\times g}(f^{n_1+n_2}(x),g^{n_1}(y),t-n_1),u-n_2), \; n_1, n_2 \in Z \arrow[d, "q_1"] \\
\Pi_f(x,t+u) & = & \Pi_f(f^{n_1+n_2}(x),t+u-(n_1+n_2))
\end{tikzcd}$$

The diagram commutes

$$\Pi_f q'_1(x,y,t,u) = \Pi_f(x,t+u)$$

$$q_1\Pi_{\bar f}(\Pi_{f\times g}\times 1)(x,y,t,u) = q_1\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) = \Pi_f(x,t+u) \ .$$

Consider an open set $O_1 \subset M_f$, then

$$(\Pi_f q'_1)^{-1}(O_1) = (q_1 \Pi_{\bar f}(\Pi_{f\times g} \times 1))^{-1}(O_1)$$

$$\Rightarrow \quad (q'_1)^{-1}\, \Pi_f^{-1}(O_1) = (\Pi_{f\times g} \times 1)^{-1}\, \Pi_{\bar f}^{-1}\, q_1^{-1}(O_1)$$

$$\Rightarrow \quad q_1^{-1}(O_1) = \Pi_{\bar f}(\Pi_{f\times g} \times 1)(q'_1)^{-1}\Pi_f^{-1}(O_1).$$ Hence $q_1^{-1}(O_1)$ is open

Thus q_1 is a continuous map.

2) q_2 is continuous.

Consider the following diagram.

$$\begin{tikzcd}
M \times N \times R \times R \arrow[r, "q'_2"] \arrow[d, "\Pi_{f\times g} \times 1"'] & N \times R \arrow[dd, "\Pi_g"] \\
(M \times N)_{f\times g} \times R \arrow[d, "\Pi_{\bar f}"'] & \\
((M \times N)_{f\times g})_{\bar f} \arrow[r, "q_2"] & N_g
\end{tikzcd}$$

The maps $\Pi_g, \Pi_{f\times g} \times 1$ and $\Pi_{\bar f}$ are open as before. Define the map

q'_2 by $q'_2(x,y,t,u) = (y,t)$. The map q'_2 is continuous.

The map q_2 is well defined because we have the following diagram.

$$\begin{tikzcd}
\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) \arrow[d, "q_2"'] & = & \Pi_{\bar f}(\Pi_{f\times g}(f^{n_1+n_2}(x), g^{n_1}(y), t-n_1), u-n_2) \arrow[d, "q_2"] \\
\Pi_g(y,t) & = & \Pi_g(g^{n_1}(y), t-n_1)
\end{tikzcd}$$

$n_1, n_2 \in Z$.

The diagram commutes

$$\Pi_g q'_2(x,y,t,u) = \Pi_g(y,t)$$

$$q_2 \Pi_{\bar f}(\Pi_{f\times g} \times 1)(x,y,t,u) = q_2(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u)) = \Pi_g(y,t) .$$

Consider an open set $O_2 \subset N_g$, then

$$(\Pi_g q'_2)^{-1}(O_2) = (q_2 \Pi_{\bar{f}}(\Pi_{f\times g} \times 1))^{-1}(O_2)$$

$$\Rightarrow \quad (q'_2)^{-1}\Pi_g^{-1}(O_2) = (\Pi_{f\times g} \times 1)^{-1}\Pi_{\bar{f}}^{-1} q_2^{-1}(O_2)$$

$$\Rightarrow \quad q_2^{-1}(O_2) = \Pi_{\bar{f}}(\Pi_{f\times g} \times 1)(q'_2)^{-1}\Pi_g^{-1}(O_2).$$ Hence $q_2^{-1}(O_2)$ is open.

Thus q_2 is a continuous map.

2.2.4 LEMMA. Let $(x_0,s_0) \in M \times R$. There exists a homeomorphism r_1 of $q_1^{-1}(\Pi_f(x_0,s_0))$ with N_g, where r_1 is the restriction of q_2 to $q_1^{-1}(\Pi_f(x_0,s_0))$.

Proof. Given $(x_0,s_0) \in M \times R$, $\Pi_f(x_0,s_0) = \Pi_f(f^n(x_0),s_0-n)$, $n \in Z$ and so we have

$$q_1^{-1}(\Pi_f(x_0,s_0)) = \{\Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u) \mid x = f^n(x_0), t+u = s_0-n, y \in N\}$$

$$= \{\Pi_{\bar{f}}(\Pi_{f\times g}(x_0,y,t),u) \mid t+u = s_0, y \in N\}.$$

Consider the restriction $q_2 | q_1^{-1}(\Pi_f(x_0,s_0)) = r_1$; then

$$r_1(\Pi_{\bar{f}}(\Pi_{f\times g}(x_0,y,t),s_0-t)) = \Pi_g(y,t)$$

1) r_1 is well defined

$$\begin{array}{ccc}
\Pi_{\bar{f}}(\Pi_{f\times g}(x_0,y,t),s_0-t) & = & \Pi_{\bar{f}}(\Pi_{f\times g}(x_0,g^{n_1}(y),t-n_1,s_0-(t-n_1))) \\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_1} \\
\Pi_g(y,t) & = & \Pi_g(g^{n_1}(y), t-n_1)
\end{array}$$

2) r_1 is 1-1.

From the definition of r_1, $r_1^{-1} = q_1^{-1}(\Pi_f(x_0,s_0)) \cap q_2^{-1}$.

Hence $r_1^{-1}(\Pi_g(y_0,t_0)) = q_1^{-1}(\Pi_f(x_0,s_0)) \cap \{\Pi_{\bar f}(\Pi_{f\times g}(x,y_0,t_0),u) \mid x \in M,\ u \in R\}$

$$= \Pi_{\bar f}(\Pi_{f\times g}(x_0,y_0,t_0),s_0-t_0).$$

3) r_1 is a homeomorphism.

Since r_1 is 1-1 and the restriction of a continuous map we need only to prove the continuity of r_1^{-1}. Consider the following diagram where $S_1 = \{(x_0,y,t,s_0-t) \mid y \in N,\ t \in R\}$ and $S_2 = \{(\Pi_{f\times g}(x_0,y,t),s_0-t) \mid y \in N,\ t \in R\}$

$$\begin{tikzcd} N \times R \arrow[r, "r'_1"] \arrow[dd, "\Pi_g"'] & S_1 \arrow[d, "\Pi_{f\times g}\times 1|S_1"] \\ & S_2 \arrow[d, "\Pi_{\bar f}|S_2"] \\ N_g \arrow[r, "r_1^{-1}"] & q_1^{-1}(\Pi_f(x_0,s_0)) \end{tikzcd}$$

Define r'_1 by $r'_1(y,t) = (x_0,y,t,s_0-t)$ which is obviously a continuous map. The diagram commutes:

$$(\Pi_{\bar f}|S_2)(\Pi_{f\times g}\times 1|S_1)r'_1(y,t) = \Pi_{\bar f}(\Pi_{f\times g}(x_0,y,t),s_0-t).$$

$$r_1^{-1}\,\Pi_g(y,t) = \Pi_{\bar f}(\Pi_{f\times g}(x_0,y,t),\ s_0-t)\ .$$

Consider an open set $O_3 \subset q_1^{-1}(\Pi_f(x_0,s_0))$. Then we have

$$(r_1^{-1}\Pi_g)^{-1}(O_3) = ((\Pi_{\bar f}|S_2)(\Pi_{f\times g}\times 1|S_1)r'_1)^{-1}(O_3)$$

$$\Rightarrow \quad \Pi_g^{-1}(r_1^{-1})^{-1}(O_3) = (r'_1)^{-1}(\Pi_{f\times g}\times 1|S_1)^{-1}(\Pi_{\bar f}|S_2)^{-1}(O_3)$$

$$\Rightarrow \quad r_1(O_3) = \Pi_g(r'_1)^{-1}(\Pi_{f\times g}\times 1|S_1)^{-1}(\Pi_{\bar f}|S_2)^{-1}(O_3) \quad \text{and so}$$

$r_1(O_3)$ is open, i.e. r_1^{-1} is a continuous map.

2.2.5 LEMMA. Let $(y_0,t_0) \in N\times R$. There is a homeomorphism r_2 of $q_2^{-1}(\Pi_g(y_0,t_0))$ with M_f, where r_2 is the restriction of q_1 to $q_2^{-1}(\Pi_g(y_0,t_0))$.

<u>Proof</u>. Given $(y_0,t_0) \in N\times R, \Pi_g(y_0,t_0) = \Pi_g(g^n(y_0),t_0-n)$, $n \in Z$ and so we have

$$q_2^{-1}(\Pi_g(y_0,t_0)) = \{\Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u) \mid y = g^n(y_0),\ t = t_0-n,\ x\in M,\ u\in R\}$$

$$= \{\Pi_{\bar{f}}(\Pi_{f\times g}(x,y_0,t_0),u) \mid x\in M,\ u\in R\}.$$

Consider the restriction $q_1 | q_2^{-1}(\Pi_g(y_0,t_0)) = r_2$, then

$$r_2(\Pi_{\bar{f}}(\Pi_{f\times g}(x,y_0,t_0),u)) = \Pi_f(x,t_0+u).$$

1) r_2 is well-defined.

$$\begin{array}{ccc}
\Pi_{\bar{f}}(\Pi_{f\times g}(x,y_0,t_0),u) & = & \Pi_{\bar{f}}(\Pi_{f\times g}(f^{n_1}(x),y_0,t_0),u-n_1) \\
\Big\downarrow r_2 & & \Big\downarrow r_2 \\
\Pi_f(x,t_0+u) & = & \Pi_f(f^{n_1}(x),t_0+u-n_1)
\end{array}$$

2) r_2 is 1-1.

From the definition of $r_2 : r_2^{-1} = q_2^{-1}(\Pi_g(y_0,t_0)) \cap q_1^{-1}$. Hence

$$r_2^{-1}(\Pi_f(x_0,s_0)) = q_2^{-1}(\Pi_g(y_0,t_0)) \cap \{\Pi_{\bar{f}}(\Pi_{f\times g}(x_0,y,t),u) \mid t+u = s_0, y \in N\}$$

$$= \Pi_{\bar{f}}(\Pi_{f\times g}(x_0,y_0,t_0),s_0-t_0).$$

3) r_2 is a homeomorphism.

Since r_2 is a 1-1 and the restriction of a continuous map we need only to prove the continuity of r_2^{-1}. Consider the following diagram where $T_1 = \{(x,y_0,t_0,s-t_0) \mid x\in M,\ s\in R\}$ and $T_2 = \{(\Pi_{f\times g}(x,y_0,t_0),s-t_0) \mid x\in M,\ s\in R\}$.

$$\begin{tikzcd}
M \times R \arrow[r, "r'_2"] \arrow[dd, "\Pi_f"'] & T_1 \arrow[d, "\Pi_{f\times g} \times 1 | T_1"] \\
& T_2 \arrow[d, "\Pi_{\bar{f}} | T_2"] \\
M_f \arrow[r, "r_2^{-1}"] & q_2^{-1}(\Pi_g(y_0,t_0))
\end{tikzcd}$$

Define r'_2 by $r'_2(x,s) = (x,y_0,t_0,s-t_0)$ which again is obviously a continuous map. The diagram commutes:

$$(\Pi_{\bar{f}}| T_2)(\Pi_{f\times g} \times 1 | T_1) r'_2(x,s) = \Pi_{\bar{f}}(\Pi_{f\times g}(x,y_0,t_0,s-t_0)$$

$$r_2^{-1}\,\Pi_f(x,s) = \Pi_{\bar{f}}(\Pi_{f\times g}(x,y_0,t_0),s-t_0) \,.$$

Consider an open set $O_4 \subset q_2^{-1}(\Pi_g(y_0,t_0))$. Then we have

$$(r_2^{-1}\Pi_f)^{-1}(O_4) = ((\Pi_{\bar{f}}| T_2)(\Pi_{f\times g} \times 1 | T_1) r'_2)^{-1}(O_4)$$

$$\Rightarrow \Pi_f^{-1}(r_2^{-1})^{-1}(O_4) = r'^{-1}_2(\Pi_{f\times g} \times 1 | T_1)^{-1}(\Pi_{\bar{f}}| T_2)^{-1}(O_4)$$

$\Rightarrow \quad r_2(O_4) = \Pi_f r'^{-1}_2(\Pi_{f\times g} \times 1 | T_1)^{-1}(\Pi_{\bar{f}}| T_2)^{-1}(O_4)$. The openness of $r_2(O_4)$ follows because Π_f is open, r'_2 is continuous and $\Pi_{f\times g} \times 1 | T_1$ and $\Pi_{\bar{f}}| T_2$ are restrictions of continuous maps.

2.2.6 <u>LEMMA</u>. Denote an element of $((M\times N)_{f\times g})_{\bar{f}}$ by z then the mapping

$$\kappa : ((M\times N)_{f\times g})_{\bar{f}} \to M_f \times N_g$$

$$z \mapsto (q_1(z), q_2(z))$$

is a diffeomorphism.

<u>Proof</u>. 1) κ is 1-1.

Let $(a,b) \in M_f \times N_g$. Then $\kappa^{-1}(a,b) = \{q_1^{-1}(a) \cap q_2^{-1}(b)\}$.

Suppose we have elements $z_1, z_2 \in \kappa^{-1}(a,b)$ then $z_1 \in q_1^{-1}(a)$, $q_2^{-1}(b)$

and $z_2 \in q_1^{-1}(a)$, $q_2^{-1}(b)$. We have $q_2(z_1) = q_2(z_2)$. However $z_1, z_2 \in q_1^{-1}(a)$; hence $(q_2 | q_1^{-1}(a))(z_1) = (q_2 | q_1^{-1}(a))(z_2)$. But we have proved that $q_2 | q_1^{-1}(a)$ is a homeomorphism and so $z_1 = z_2$. Hence κ is 1-1.

2) κ is continuous.

The continuity of κ follows from the fact that q_1, q_2 are continuous

We now use the compactness of the manifolds M, N to obtain compact manifolds $((M\times N)_{f\times g})_{\bar{f}}$ and $M_f \times N_g$. Hence we have κ is a 1-1 continuous map between compact spaces and so is a homeomorphism, [11].

<u>Remark</u>. It follows that κ is a diffeomorphism since it is constructed from differentiable maps.

2.2.7 <u>LEMMA</u>. The flow λ on $((M\times N)_{f\times g})_{\bar{f}}$ defined in 2.2.2 is well defined.

<u>Proof</u>.

$$\begin{tikzcd}
\Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u) \arrow[d, "\lambda_v"'] & = & \Pi_{\bar{f}}(\Pi_{f\times g}(f^{n_1+n_2}(x), g^{n_1}(y), t-n_1), u-n_2) \arrow[d, "\lambda_v"] \\
\Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t+v),u) & = & \Pi_{\bar{f}}(\Pi_{f\times g}(f^{n_1+n_2}(x), g^{n_1}(y), t-n_1+v), u-n_2).
\end{tikzcd}$$

2.2.8 <u>LEMMA</u>. The following diagram commutes where $\phi = \Sigma(M,f)$.

$$\begin{tikzcd}
((M\times N)_{f\times g})_{\bar{f}} \times R \arrow[r, "q_1\times 1"] \arrow[d, "\lambda"'] & M_f \times R \arrow[d, "\phi"] \\
((M\times N)_{f\times g})_{\bar{f}} \arrow[r, "q_1"] & M_f
\end{tikzcd}$$

Proof.

$$\phi_v q_1(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u)) = \phi_v \Pi_f(x,t+u) = \Pi_f(x,t+u+v)$$

$$q_1\lambda_v(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u)) = q_1(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t+v),u) = \Pi_f(x,t+u+v).$$

2.2.9 LEMMA. The following diagram commutes where $\psi = \Sigma(N,g)$.

$$\begin{tikzcd}
((M\times N)_{f\times g})_{\bar f}\times R \arrow[r, "q_2\times 1"] \arrow[d, "\lambda"'] & N_g\times R \arrow[d, "\psi"] \\
((M\times N)_{f\times g})_{\bar f} \arrow[r, "q_2"] & N_g
\end{tikzcd}$$

Proof.

$$\psi_v q_2(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u)) = \psi_v \Pi_f(y,t) = \Pi_g(y,t+v)$$

$$q_2\lambda_v(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u)) = q_2(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t+v),u)) = \Pi_g(y,t+v).$$

Lemmas 2.2.6, 2.2.7, 2.2.8 and 2.2.9 give the equivalence of Theorem 2.2.2.

2.2.10 LEMMA. The flow λ on $((M\times N)_{f\times g})_{\bar f}$ is a uniform fibre flow on the bundle

$$(M\times N)_{f\times g} \longrightarrow ((M\times N)_{f\times g})_{\bar f} \xrightarrow{\pi'} S^1$$

Moreover the flow on the fibre is $\Sigma(M\times N, f\times g)$.

Proof. The projection π' is defined by

$\pi'(\Pi_{\bar f}(\Pi_{f\times g}(x,y,t),u) \longrightarrow p(u)$ where p is the natural projection $p : R \longrightarrow R/Z$.

Let $\{U_1, U_2\}$ be an open covering of R/Z defined as follows.

$$U_1 = \{p(u) | 0 < u < 1\}$$

$$U_2 = \{p(u) | 3/4 < u < 5/4\}.$$

Define $h_1 : U_1 \times (M \times N)_{f\times g} \to \pi'^{-1}(U_1)$ by $h_1(p(u), \Pi_{f\times g}(x,y,t))$ $= \Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u)$ where $0 < u < 1$ is the value we take for u, and $h_2 : U_2 \times (M\times N)_{f\times g} \to \pi'^{-1}(U_2)$ by $h_2(p(u), \Pi_{f\times g}(x,y,t))$ $= \Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u)$ where $3/4 < u < 5/4$ is the value taken by u.

Using $\{(U_1,h_1),(U_2,h_2)\}$ we have a fibre flow, with $\mu = \Sigma(M\times N, f\times g)$.

$$\begin{tikzcd}
(p(u), \Pi_{f\times g}(x,y,t), v) \arrow[r, "h_1\times 1"] \arrow[d, "1\times\mu"'] & (\Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t),u),v) \arrow[d, "\lambda"] \\
(p(u), \Pi_{f\times g}(x,y,t+v)) \arrow[r, "h_1"] \arrow[d, "\rho"'] & \Pi_{\bar{f}}(\Pi_{f\times g}(x,y,t+v,u) \arrow[d, "\pi'"] \\
p(u) \arrow[r, "1"] & p(u)
\end{tikzcd}$$

$[u \in (0,1)]$. A similar result holds for (U_2, h_2).

2.2.11 <u>LEMMA</u> With the results of 2.2.2, 2.2.6 and 2.2.10, we have the uniform fibre flow in 2.1.4 by taking the projection $\pi = \pi'\kappa^{-1}$: $M_f \times N_g \to S^1$ and the covering $\{U_1, U_2\}$ of S^1 with coordinate functions $h'_1 = \kappa h_1$ and $h'_2 = \kappa h_2$.

<u>Proof</u>. The following diagrams commute for $i = (1,2)$.

$$\begin{tikzcd}
U_i \times (M\times N)_{f\times g} \times R \arrow[r, "h_i \times 1"] \arrow[d, "1\times\mu"'] & \pi'^{-1}(U_i)\times R \arrow[r, "\kappa\times 1"] \arrow[d, "\lambda"] & \kappa\pi'^{-1}(U_i)\times R \arrow[d, "\phi\times\psi"] \\
U_i\times (M\times N)_{f\times g} \arrow[r, "h_i"] \arrow[d, "\rho"'] & \pi'^{-1}(U_i) \arrow[r, "\kappa"] \arrow[d, "\pi'"] & \kappa\pi'^{-1}(U_i) \arrow[d, "\pi'\kappa^{-1}"] \\
U_i \arrow[r, "1"] & U_i \arrow[r, "1"] & U_i
\end{tikzcd}$$

The commutativity of the left-hand side of the diagram follows from Lemma 2.2.10. The commutativity $\kappa\lambda = (\phi\times\psi)(\kappa\times 1)$ follows from Theorem 2.2.2. The commutativity of $\pi'\kappa^{-1}\kappa = 1\cdot\pi'$ follows immediately. Using this information we have

$$(\phi\times\psi)(\kappa h_i\times 1) = (\phi\times\psi)(\kappa\times 1)(h_i\times 1) = \kappa\lambda(h_i\times 1)$$
$$= \kappa h_i(1\times\mu) = (\kappa h_i)(1\times\mu)$$
$$(\pi'\kappa^{-1})(\kappa h_i) = \pi' h_i = 1\rho \ .$$

It follows that the diagram commutes for $i = 1,2$.

$$\begin{tikzcd}
U_i \times (M\times N)_{f\times g} \times R \arrow[r, "(\kappa h_i)\times 1"] \arrow[d, "1\times\mu"'] & (\pi'\kappa^{-1})^{-1}(U_i)\times R \arrow[d, "\phi\times\psi"] \\
U_i\times (M\times N)_{f\times g} \arrow[r, "(\kappa h_i)"] \arrow[d, "\rho"'] & (\pi'\kappa^{-1})^{-1}(U_i) \arrow[d, "\pi'\kappa^{-1}"] \\
U_i \arrow[r, "1"] & U_i
\end{tikzcd}$$

<u>Remark</u>. Given any open covering $\{U_i\}$ of S^1 by careful choice of the diffeomorphiems h_i then fibre flows can be constructed by a similar method to the above.

§3. REMARKS ON THE PRODUCT THEOREM

Here we will discuss the choice of various mappings and particularly $\bar{f}$. First of all we will show that the bundle notation in the theorem is necessary.

2.3.1 LEMMA. The result is not trivial, i.e. the bundle in 2.2.2 is not a trivial product.

$$(M\times N)_{f\times g} \longrightarrow M_f \times N_g \xrightarrow{\pi} S^1$$

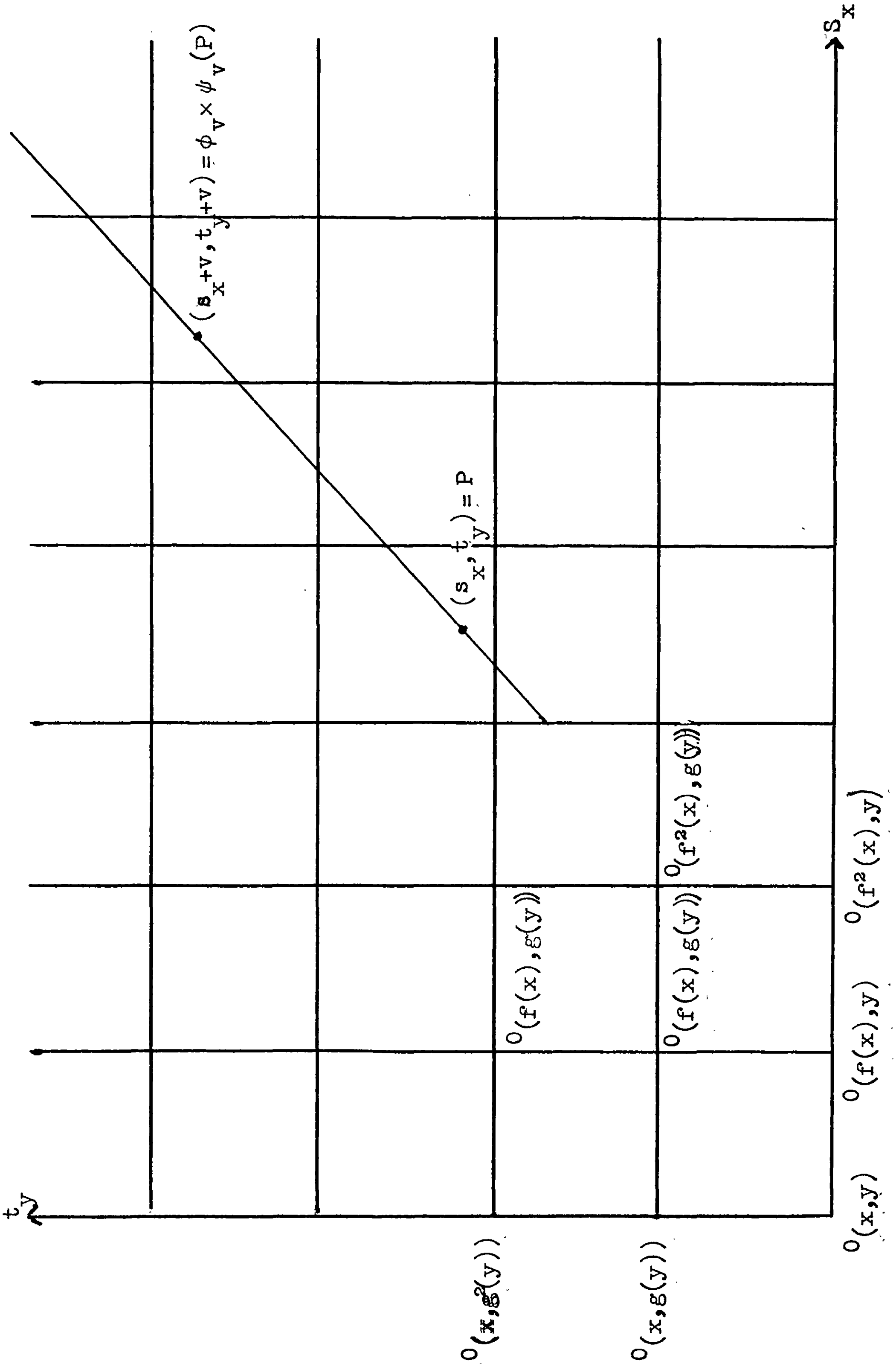
Proof. This follows from the fact that $\bar{f} : (M\times N)_{f\times g}\circlearrowleft$ is not always diffeotopic to 1. However a simple counter-example will illustrate the lemma more clearly. Define

$S_1^1 = \{u \bmod 1 | u \in R\} = M$ and $f : M \to M$ is defined by $f(u \bmod 1) = (-u \bmod 1)$; and

$S_2^1 = \{v \bmod 1 | v \in R\} = N$ and $g : N \to N$ is defined by $g(v \bmod 1) = (-v \bmod 1)$. The diffeomorphisms f,g are reflections of the circles S_1^1, S_2^1 respectively. Hence $\Sigma(M,f)$ and $\Sigma(N,g)$ are flows on the Klien bottles κ_1, κ_2 say. The diffeomorphism $f\times g$ is a map of the torus $T^2 = S_1^1 \times S_2^1$ which reflects in both S_1^1 and S_2^1. The mapping $(f\times g)^* : R^2 \to R^2$ which induces the diffeomorphism $f\times g : T^2 \to T^2$ is given by $(f\times g)^* = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : R^2 \to R^2$. The group generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is C_2 the cyclic group of order 2. Then $T^2_{f\times g}$ is not the product bundle $S^1 \times T^2$. Now $\bar{f} : T^2_{f\times g} \to T^2_{f\times g}$ is defined by $\bar{f}\,\Pi_{f\times g}(u \bmod 1,\ v \bmod 1,\ t) = \Pi_{f\times g}(-u \bmod 1,\ v \bmod 1,\ t)$ and so $(T^2_{f\times g})_{\bar{f}}$ is a bundle with structure group $<\bar{f}>$ generated by the diffeomorphism $\bar{f}$ and so is of order 2. Hence $(T^2_{f\times g})_{\bar{f}} \neq S^1 \times T^2_{f\times g}$ so that $(T^2_{f\times g})_{\bar{f}}$ is not a trivial bundle and the lemma follows.

It should be pointed out that the diffeomorphism $\bar{f}: (M\times N)_{f\times g} \to (M\times N)_{f\times g}$ defined in 2.2.1 is not the only choice that could be made to give a similar theorem. We can best discuss the possibilities of other diffeomorphisms $h: (M\times N)_{f\times g} \to (M\times N)_{f\times g}$ that can be used by consideration of Fig.2.1

Fig. 2.1



Let P represent $(\Pi_f(x,s),\Pi_g(y,t))$ and $\phi_v \times \psi_v(P) = (\Pi_f(x,s+v),\Pi_g(y,t+v))$.

This diagram and its interpretation gives the intuitive insight into the theory of this chapter.

Assume we take a (s,t) coordinate system. At the origin of the (s,t) system we associate the point $(x,y) \in M \times N$. We wish to relate this diagram to $M_f \times N_g$, then it is a necessity that all points of the integer lattice have to be associated with some points $(x',y') \in M \times N$. For instance let $(n_1,n_2) \in Z^2 \subset R^2$ then with respect to the origin $0_{(x,y)}$ this point represents $(\Pi_f(x,n_1), \Pi_g(y,n_2)) \in M_f \times N_g$. But the point also has a representation $\Pi_f(f^{n_1}(x),0), \Pi_g(g^{n_2}(y),0)$. This means that we should also associate to (n_1,n_2) the origin of a coordinate system with respect to $(f^{n_1}(x),g^{n_2}(y)) \in M \times N$.

Since the product flow $\phi \times \psi$ on $M_f \times N_g$ has the effect of increasing both s and t by the same amount the orbits are represented by diagonal lines of slope $+1$. Let the orbit that passes through $0_{(x,y)}$ be given by $D_{(x,y)}$ and let $U = \{u | u \in R\}$. We redefine points of the plane as points of $D_{(x,y)} \times U$ as in Fig. 2.2.

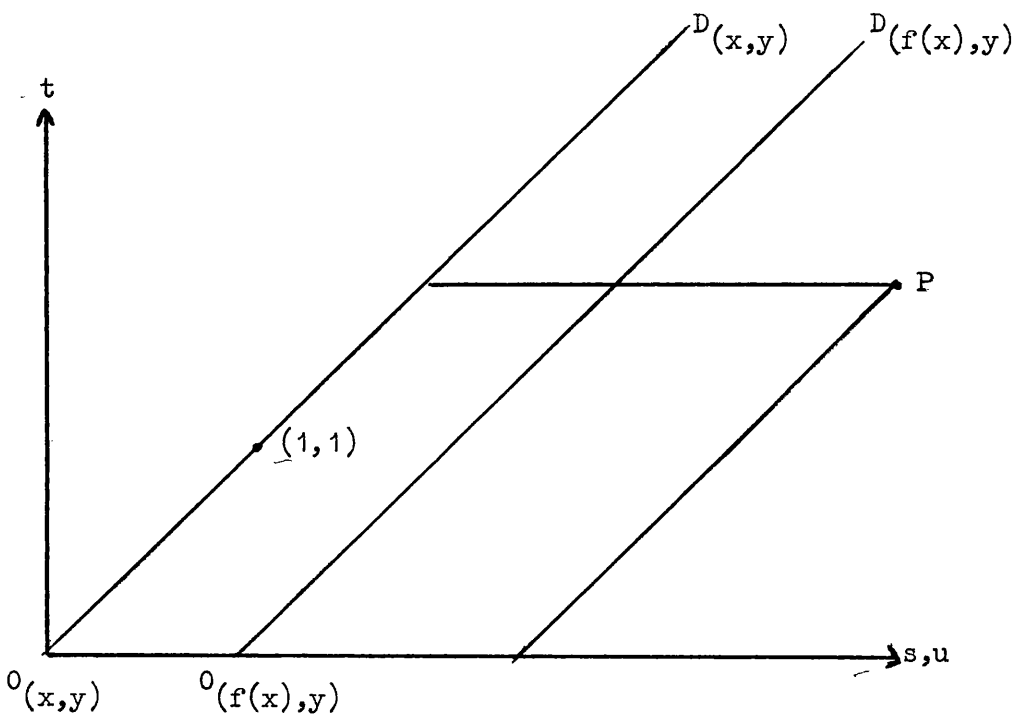


Fig. 2.2

The "U" real line is also along the s-axis; a point P of the (s,t) plane is given in $(D_{(x,y)},U)$ coordinates by projections parallel to U and $D_{(x,y)}$. Then the point $P = (\Pi_f(x,s), \Pi_g(y,t))$ becomes $(\Pi_f(x,t), \Pi_g(y,t)) \in D(x,y)$ and $s-t \in U$. It is by noting the representations of the points in the plane with respect to other origins that we obtain the bundle structure of 2.2.2. The crucial fact to notice is that $\{D_{(x,y)} | x \in M,\ y \in N\}$ aan be identified with orbits of $\Sigma(M\times N, f\times g)$.

The map $\overline{f}$ was deduced from noting in Fig. 2.2 that $P = (t,u)_{(x,y)} \in D_{(x,y)} \times U$ and also $P = (t,u-1)_{(f(x),y)} \in D_{(f(x),y)} \times U$. The coordinate U was measured in the horizontal direction as an apparently arbitrary direction. This is in fact the case and we can take U in any direction. Another possibility would be to take U vertically as in Fig. 2.3.

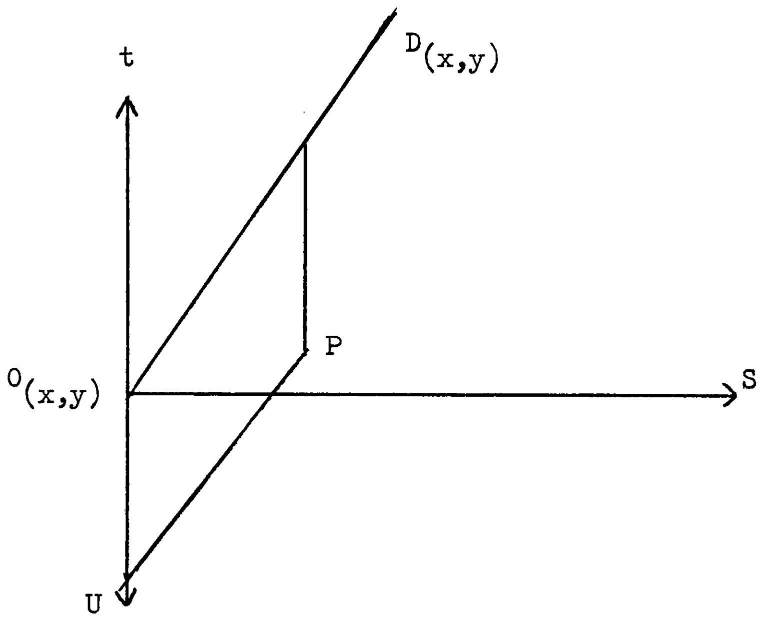


Fig. 2.3

In Fig. 2.3 relative to $(D_{(x,y)},U)$ coordinates we have the following relation:

$$P = (t,u)_{(x,y)} = (t,u-1)_{(x,g^{-1}(y))}.$$

2.3.2 LEMMA. Given two diffeomorphisms f_1, f_2 of M; then M_{f_1} and M_{f_2} are fibre diffeomorphic if f_1 and f_2 are diffeotopic.

Proof. We have the bundles

$$M \longrightarrow M_{f_1} \xrightarrow{\pi_1} S^1$$

$$M \longrightarrow M_{f_2} \xrightarrow{\pi_2} S^1$$

Let Π_{f_1} and Π_{f_2} be the usual projection maps. Suppose that the diffeotopy between f_1 and f_2 is $\{d_u | 0 \leqslant u \leqslant 1\}$ where $d_0 = f_1$ and $d_1 = f_2$. Then the fibre diffeomorphism h is given by

$$h(\Pi_{f_1}(x,t)) = \Pi_{f_2}(f_1^{-1} d_t(x), t)$$

2.3.3 LEMMA. Any two manifolds $(M\times N_{f\times g})_{h_1}$ and $((M\times N)_{f\times g})_{h_2}$ used in 2.2.2 obtained by taking U in the directions $s = (\tan\theta_1)t$ and $s = (\tan\theta_2)t$ respectively are fibre diffeomorphic. [θ is measured anticlockwise from $t=0$ and $-3\pi/4 < \theta_1, \theta_2 < \pi/4$.]

Proof. They are fibre diffeomorphic by 2.3.2. The diffeomorphisms h_1 and h_2 are diffeotopic. The diffeotopy is constructed by taking the interval of diffeomorphisms corresponding to $\theta \in [\theta_1, \theta_2]$ assuming $\theta_1 \leqslant \theta_2$.

§4. SUSPENSIONS OF BUNDLE DIFFEOMORPHISMS

We wish to prove a result for bundle diffeomorphisms similar to the one proved for product diffeomorphisms in §1. We use the definitions already given in earlier sections. In this section we commence with a bundle of manifolds

$$N \longrightarrow B \xrightarrow{\pi} M$$

and let (h,f) be the bundle diffeomorphism in the commutative diagram.

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

<u>Notation</u>. Let N_x denote the fibre over $x \in M$, i.e. $\pi^{-1}(x) = N_x$.

2.4.1 <u>Definition</u>. Let $\bar{h} : B_h \to B_h$ be the diffeomorphism defined by $\bar{h}\,\Pi_h(b,t) = \Pi_h(h(b), t)$ $b \in B$, $t \in R$.

2.4.2 <u>Definition</u>. The flow λ on the bundle

$$V \longrightarrow W \xrightarrow{\sigma} X$$

is called a <u>local product flow</u> if $\exists$ flows ϕ on X and ψ on V and an atlas of local bundle charts (U,α) such that for $x \in U \subset X$, $\exists$ $R[x]$, an open set of R with $0 \in R[x]$ such that the diagram commutes.

$$\begin{tikzcd} \{x\} \times V \times R[x] \arrow[r, "\alpha|\{x\} \times 1"] \arrow[d, "\phi \times \psi"'] & \sigma^{-1}(x) \times R[x] \arrow[d, "\lambda"] \\ U \times V \arrow[r, "\alpha"] \arrow[d, "\text{projection}"'] & \sigma^{-1}(U) \arrow[d, "\sigma"] \\ U \arrow[r, "1"] & U \end{tikzcd}$$

2.4.3 <u>BUNDLE THEOREM</u>. Given the bundle of manifolds

$N \longrightarrow B \xrightarrow{\pi} M$, with bundle diffeomorphism (h,f)

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

then $\exists$ a uniform fibre flow λ on the bundle

$$B_h \longrightarrow (B_h)_{\bar{h}} \xrightarrow{\pi'} S^1, \quad \text{where the flow on the fibre}$$

is $\Sigma(B,h)$, such that λ is also a local product flow on the bundle

$$N_1 \longrightarrow (B_h)_{\bar{h}} \xrightarrow{\sigma} M_f \ .$$

Moreover the flow on M_f is $\Sigma(M,f)$ and the flow on N_1 is $\Sigma(N,1)$.

Corollary. A local product flow of the type above is not structurally stable.

Proof. The projection π' provides a first integral for which the flow is invariant, and so by the theorem of Thom [20] the flow is not structurally stable.

The Bundle Theorem is derived from the following observations.

2.4.4 LEMMA. There exists a map $q: (B_h)_{\bar{h}} \to M_f$ defined by

$$q(\Pi_{\bar{h}}(\Pi_h(b,t),u)) = \Pi_f(\pi(b),t+u).$$

Proof. 1) q is well-defined.

Let $n_1, n_2 \in Z$. Then

$$\begin{array}{ccc}
\Pi_{\bar{h}}(\Pi_h(b,t),u) & = & \Pi_{\bar{h}}(\Pi_h(h^{n_1}h^{n_2}(b),t-n_1),u-n_2) \\
\Big\downarrow q & & \Big\downarrow q \\
\Pi_f(\pi(b),t+u) & = & \Pi_f(\pi(h^{n_1}h^{n_2}(b)),t+u-(n_1+n_2))
\end{array}$$

2) q is continuous.

Consider the following diagram.

$$\begin{tikzcd}
B \times R \times R \arrow[r, "q'"] \arrow[d, "\Pi_h \times 1"'] & M \times R \arrow[dd, "\Pi_f"] \\
B_h \times R \arrow[d, "\Pi_{\bar{h}}"'] & \\
(B_h)_{\bar{h}} \arrow[r, "q"] & M_f
\end{tikzcd}$$

Define q' by $q'(b,t,u) = (\pi(b),t+u)$; then q' is obviously continuous. The diagram commutes:

$$\Pi_f q'(b,t,u) = \Pi_f(\pi(b),t+u)$$

$$q\,\Pi_{\bar{h}}(\Pi_h \times 1)(b,t,u) = q\,\Pi_{\bar{h}}(\Pi_h(b,t),u) = \Pi_f(\pi(b),t+u).$$

Take an open set $0 \subseteq M_f$; then

$$(q\,\Pi_{\bar{h}}(\Pi_h \times 1))^{-1}(0) = (\Pi_f q')^{-1}(0)$$

$$\Rightarrow \quad q^{-1}(0) = \Pi_{\bar{h}}(\Pi_h \times 1)q'^{-1}\Pi_f^{-1}(0).$$

Because $\Pi_{\bar{h}}$ and $\Pi_h \times 1$ are open maps and q' and Π_f are continuous it follows that $q^{-1}(0)$ is open and so q is continuous.

2.4.5 <u>LEMMA</u>. Given $(x_0,s_0) \in M \times R$; then

$$q^{-1}(\Pi_f(x_0,s_0)) = \{\Pi_{\bar{h}}(\Pi_h(b,t),u) \mid t+u = s_0,\ \pi(b) = x_0\}$$

<u>Proof</u>. Given $(x_0,s_0) \in M \times R$; $q^{-1}(\Pi_f(x_0,s_0)) = q^{-1}(\Pi_f(f^n(x_0),s_0-n))$, $\forall n \in Z$.

We therefore have

$$q^{-1}(\Pi_f(x_0,s_0)) = \{\Pi_{\bar{h}}(\Pi_h(b,t),u) \mid \pi(b) = f^n(x_0);\ \ t+u = s_0-n\}$$

$$= \{\Pi_{\bar{h}}(\Pi_h(b,t),u) \mid \pi(b) = x_0, t+u = s_0\}$$

using the relations given. This follows because $\Pi_{\bar{h}}(\Pi_h(b,t),u) =$

$= \Pi_{\bar{h}}(\Pi_h(h^{-n}(b),t+n),u)$ where $\pi(h^{-n}(b)) = x_0$ and $t+n+u = s_0$.

2.4.6 LEMMA. Given $(x_0,s_0) \in M \times R$, then there exists a homeomorphism

$$k_{(x_0,s_0)} : (N_{x_0})_1 \longrightarrow q^{-1}(\Pi_f(x_0,s_0)).$$

Proof. The definition of $k_{(x_0,s_0)}$ uses the lemma 2.4.5. Let us represent points of $q^{-1}(\Pi_f(x_0,s_0))$ as in 2.4.5. ; then we define $k_{(x_0,s_0)}$ by $k_{(x_0,s_0)}(\Pi_1(b,t)) = \Pi_{\bar{h}}(\Pi_h(b,t),s_0-t)$, $\pi(b) = x_0$.

1) $k_{(x_0,s_0)}$ is well-defined.

$$\begin{tikzcd} \Pi_1(b,t) \arrow[d, "k_{(x_0,s_0)}"'] \arrow[r, phantom, "="] & \Pi_1(b,t-n), \ n \in Z \arrow[d, "k_{(x_0,s_0)}"] \\ \Pi_{\bar{h}}(\Pi_h(b,t),s_0-t) \arrow[r, phantom, "="] & \Pi_{\bar{h}}(\Pi_h(b,t-n),s_0-t+n) \end{tikzcd}$$

$$\left[\Pi_{\bar{h}}(\Pi_h(b,t-n),s_0-t+n) = \Pi_{\bar{h}}(\Pi_h(h^{-n}h^n(b),t),s_0-t)\right]$$

2) $k_{(x_0,s_0)}$ is continuous .

Consider the following diagram where $B_1 = \{(b,t,s_0-t) | \pi(b) = x_0\}$ and $B_2 = \{(\Pi_h(b,t),s_0-t) | \pi(b) = x_0\}$.

$$\begin{tikzcd} N_{x_0} \times R \arrow[r, "k'_{(x_0,s_0)}"] \arrow[dd, "\Pi_1"'] & B_1 \arrow[d, "\Pi_h \times 1 \mid B_1"] \\ & B_2 \arrow[d, "\Pi_{\bar{h}} \mid B_2"] \\ (N_{x_0})_1 \arrow[r, "k_{(x_0,s_0)}"] & q^{-1}(\Pi_f(x_0,s_0)) \end{tikzcd}$$

Define $k'_{(x_0,s_0)}$ by $k'_{(x_0,s_0)}(b,t) = (b,t,s_0-t)$, $\pi(b) = x_0$.

The diagram commutes.

$$k_{(x_0,s_0)}\Pi_1(b,t) = \Pi_{\bar{h}}(\Pi_h(b,t),s_0-t), \quad \pi(b) = x_0.$$

$$(\Pi_{\bar{h}}|B_2)(\Pi_h\times 1|B_1)k'_{(x_0,s_0)}(b,t) = \Pi_{\bar{h}}(\Pi_h(b,t),s_0-t), \quad \pi(b)=x_0 .$$

It follows that

$$k^{-1}_{(x_0,s_0)}(0) = \Pi_1(k'_{(x_0,s_0)})^{-1}(\Pi_h\times 1|B_1)^{-1}(\Pi_{\bar{h}}|B_2)^{-1}(0),$$

for an open set $0 \subseteq q^{-1}(\Pi_f(x_0,s_0))$. Hence we have $k_{(x_0,s_0)}$ is continuous since Π_1 is open, $k'_{(x_0,s_0)}$ is continuous and $\Pi_h\times 1|B_1$ and $\Pi_{\bar{h}}|B_2$ are restrictions of continuous maps and therefore continuous.

3) $k_{(x_0,s_0)}$ is a homeomorphism.

We have $k_{(x_0,s_0)}$ is a $1-1$ map. Also $(N_{x_0})_1$ is a compact space because N is compact. The space $q^{-1}(\Pi_f(x_0,s_0))$ is closed in $(B_h)_{\bar{h}}$ since q is continuous and $\Pi_f(x_0,s_0)$ is closed in M_f. Because $(B_h)_{\bar{h}}$ is compact under the assumption that B is compact, we have a $1-1$ continuous map between compact spaces and so $k_{(x_0,s_0)}$ is a homeomorphism.

2.4.7 LEMMA There exists an equivalence between the flow $\Sigma(N_{x_0},1)$ on $(N_{x_0})_1$ and the flow $\Sigma(N,1)$ on N_1.

Proof. A diffeomorphism between N_{x_0} and N is used to construct the equivalence. The diffeomorphism is provided by a restriction of a local chart map from 2.4.9.

2.4.8 LEMMA. The uniform fibre flow λ of the bundle theorem is defined by

$$\lambda_v(\Pi_{\bar{h}}(\Pi_h(b,t),u)) = \Pi_{\bar{h}}(\Pi_h(b,t+v),u) .$$

Proof. The atlas of local bundle charts which gives the lemma is defined as follows.

Let the covering $\{U_1, U_2\}$ of S^1 be defined by $U_1 = \{p(u) | 0 < u < 1\}$ where p is projection $p: R \to R/Z = S^1$ and $U_2 = \{p(u) | 3/4 < u < 5/4\}$.

Define $\alpha_1 : U_1 \times B_h \to \pi'^{-1}(U_1)$ by $\alpha_1(p(u), \Pi_h(b,t))$

$= \Pi_{\bar{h}}(\Pi_h(b,t),u)$, where u is taken such that $0 < u < 1$, and

$\alpha_2 : U_2 \times B_h \to \pi'^{-1}(U_2)$ by $\alpha_2(p(u), \Pi_h(b,t))$

$= \Pi_{\bar{h}}(\Pi_h(b,t),u)$ where u is taken such that $3/4 < u < 5/4$.

[For a proof that these constructions give diffeomorphisms see appendix 2.] The proof now follows taking the flow on the fibre B_h to be $\mu = \Sigma(B,h)$ and the atlas $\{(U_1,\alpha_1),(U_2,\alpha_2)\}$. We have commutativity

$$\begin{tikzcd}
(p(u), \Pi_h(n,t), v) \arrow[r, "\alpha_1 \times 1"] \arrow[d, "1 \times \mu"'] & (\Pi_{\bar{h}}(\Pi_h(b,t),u), v) \arrow[d, "\lambda"] \\
(p(u), \Pi_h(b,t+v)) \arrow[r, "\alpha_1"] \arrow[d, "\rho"'] & \Pi_{\bar{h}}(\Pi_h(b,t+v),u) \arrow[d, "\pi'"] \\
p(u) \arrow[r, "1"] & p(u)
\end{tikzcd}$$

where $u \in (0,1)$. A similar result holds when we take (U_2,α_2).

2.4.9 Definition. Let (U', β) be an atlas of local bundle charts for the bundle

$$N \longrightarrow B \xrightarrow{\pi} M$$

2.4.10 Definition. Take an open covering $\{U\}$ of M_f to be defined as follows:

$$\left\{\{\Pi_f(x,s)|x \in U' ; s \in (0,1)\},\ \{\Pi_f(x,s)| x \in U',\ s \in (3/4,5/4)\}\right\}.$$

2.4.11 <u>LEMMA</u>. Given the open covering $\{U\}$ of M_f then $\exists$ an atlas of local bundle charts (U,γ) for the bundle

$$N_1 \rightarrowtail (B_h)_{\bar{h}} \xrightarrow{q} M_f$$

<u>Proof</u>. Let us take $U = \Pi_f(U' \times U_1)$ say. Then define a function

$$\gamma : U \times N_1 \to q^{-1}(u) \subset (B_h)_{\bar{h}} \text{ by } \gamma(\Pi_f(x,s), \Pi_1(y,t))$$

$= \Pi_{\bar{h}}(\Pi_h(\beta(x,y),t),s-t)$. In the definition we stipulate that at all times s must take a value such that $0 < s < 1$.

1) γ is well-defined.

Under the conditions on the definition of γ the point $(\Pi_f(x,s), \Pi_1(y,t))$ is defined uniquely up to $(\Pi_f(x,s), \Pi_1(y,t)) = (\Pi_f(x,s), \Pi_1(y,t-n))$, $n \in Z$; but

$$\begin{array}{ccc} (\Pi_f(x,s), \Pi_1(y,t)) & = & (\Pi_f(x,s), \Pi_1(y,t-n)) \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\gamma} \\ (\Pi_{\bar{h}}(\Pi_h(\beta(x,y),t),s-t) & = & \Pi_{\bar{h}}(\Pi_h(\beta(x,y),t-n),s-(t-n)) \end{array}.$$

Hence γ is well-defined.

2) γ preserves the bundle structure, i.e. $q\gamma : U \times N_1 \to U$ is the projection from the first factor.

$$\begin{aligned} q\gamma(\Pi_f(x,s), \Pi_1(y,t)) &= q(\Pi_{\bar{h}}(\Pi_h(\beta(x,y),t),s-t)) \\ &= \Pi_f(\Pi_1(\beta(x,y)),s) \\ &= \Pi_f(x,s), \end{aligned}$$ [β is a local chart map and so $\pi\beta$=projection onto first factor for local charts in bundle B].

3) γ is a homeomorphism.

Consider the following diagram.

$$\begin{tikzcd}
U' \times U_1 \times N \times R \arrow[r, "\gamma'"] \arrow[dd, "\Pi_f|(U' \times U_1) \times \Pi_1"'] & B \times R \times R \arrow[d, "\Pi_h \times 1"] \\
& B_h \times R \arrow[d, "\Pi_{\bar h}"] \\
\Pi_f(U' \times U_1) \times N_1 \arrow[r, "\gamma"] & (B_h)_{\bar h}
\end{tikzcd}$$

Define γ' by $\gamma'(x,s,y,t) = (\beta(x,y), y.s-t)$, $0 < s < 1$. Hence γ' is a diffeomorphism because β is a diffeomorphism. The diagram commutes. Let $(x,s,y,t) \in U' \times U_1 \times N \times R$; then

$$\gamma(\Pi_f \times \Pi_1)(x,s,y,t) = \gamma(\Pi_f(x,s), \Pi_1(y,t))$$

$$= \Pi_{\bar h}(\Pi_h(\beta(x,y),t),s-t)$$

$$\Pi_{\bar h}(\Pi_h \times 1)\gamma'(x,s,y,t) = \Pi_{\bar h}(\Pi_h \times 1)(\beta(x,y),t,s-t)$$

$$= \Pi_{\bar h}(\Pi_h(\beta(x,y),t),s-t) .$$

Now γ is onto $q^{-1}(U) = q^{-1}(\Pi_f(U' \times U_1))$ which is open in $(B_h)_{\bar h}$. Suppose 0 is an open set of $q^{-1}(U)$ then 0 is an open set of $(B_h)_{\bar h}$. Hence by commutativity of the diagram we get

$$\gamma^{-1}(0) = (\Pi_f|(U' \times U_1))\gamma'^{-1}(\Pi_h \times 1)^{-1}(\Pi_{\bar h})^{-1}(0).$$

Since Π_f is restricted to an open subspace of $M \times R$ the restricted map of Π_f is open. Hence $\gamma^{-1}(0)$ is open and so γ is continuous. To check that γ is 1-1 we have

$$\begin{array}{ccc}
\Pi_{\bar h}(\Pi_h\,\beta(x,y),t),s-t) & = & \Pi_{\bar h}(\Pi_h(\beta(x,y),t-n),s-(t-n)),\ n \in Z. \\
\Big\downarrow \gamma^{-1} & & \Big\downarrow \gamma^{-1} \\
(\Pi_f(x,s),\ \Pi_1(y,t)) & = & (\Pi_f(x,s),\ \Pi_1(y,t-n))
\end{array}$$

Hence γ is 1-1. To prove γ^{-1} is continuous we require a restriction of the diagram which gave the continuity of γ. It is the following diagram. Let $\gamma'(U' \times U_1 \times N \times R) = A$

$$\begin{tikzcd}
U' \times U_1 \times N \times R \arrow[dd, "\Pi_f | (U' \times U_1) \times \Pi_1"'] & A \subset B \times R \times R \arrow[l, "\gamma'^{-1}"'] \arrow[d, "\Pi_h \times 1 | A"] \\
 & \Pi_h \times 1(A) \subset B_h \times R \arrow[d, "\Pi_{\bar h} | \Pi_h \times 1(A)"] \\
\Pi_f(U' \times U_1) \times N_1 & \Pi_{\bar h}(\Pi_h \times 1)(A) \subset (B_h)_{\bar h} \arrow[l, "\gamma^{-1}"']
\end{tikzcd}$$

The fact that the previous diagram commutes and γ and γ' are 1-1 give commutativity of the above diagram. The set A is open in $B \times R \times R$ and so the restriction map of $\Pi_h \times 1$ is open. It follows that $\Pi_h \times 1(\gamma'(U' \times U_1 \times N \times R))$ is an open subspace of $B_h \times R$. Hence the restriction of $\Pi_{\bar h}$ is open. Using the openness of these maps we obtain the continuity of γ^{-1}. Hence γ is a homeomorphism. In all the defining diagrams of γ and γ^{-1} differentiable maps are used and we therefore obtain γ is a diffeomorphism.

If we had considered the open set $\Pi_f(U' \times U_2)$ instead of $\Pi_f(U' \times U_1)$, then completely analogous theory with the restriction of s to be such that $3/4 < s < 5/4$ gives us the other local bundle chart diffeomorphisms.

Hence we have an atlas of bundle charts.

2.4.12 LEMMA. The flow λ in the Bundle Theorem is the local product flow as indicated using the atlas of local bundle charts (U,γ).

Proof. Suppose $\Pi_f(x,s) \in U = \Pi_f(U' \times U_1)$. Then let $\{v \mid v \in R[\Pi_f(x,s)]\}$ be such that $\Pi_f(x,s+v) \in U$. We have commutativity in the following diagram where $\phi = \Sigma(M,f)$ and $\psi = \Sigma(N,1)$.

$$\begin{tikzcd}
\{\Pi_f(x,s)\} \times N_1 \times R[\Pi_f(x,s)] \arrow[r, "\gamma \times 1"] \arrow[d, "\phi \times \psi"'] & q^{-1}(\Pi_f(x,s)) \times R[\Pi_f(x,s)] \arrow[d, "\lambda"] \\
U \times N_1 \arrow[r, "\gamma"] \arrow[d, "\text{projection}"'] & q^{-1}(U) \arrow[d, "q"] \\
U \arrow[r, "1"] & U
\end{tikzcd}$$

§5. GENERALIZATION

It can be seen that the Bundle Theorem is not a direct generalization of the Product Theorem in the sense that a restriction of the former to product spaces would not give us the latter.

There are various difficulties in producing such a theorem and it is only possible to go so far in the development of the theory. Here we will give the starting definitions that would be required. The two lemmas which appear here will not be proved as they have their analogous statements in §4 and are proved by the same techniques. However an application to the structure of Lie Group Bundles exists and is given in Appendix 1, but it is not applicable to dynamical systems and only gives the manifold structure and mentions nothing of flows.

2.5.1 Definition. Given the bundle of manifolds

$N \longrightarrow B \xrightarrow{\pi} M$ then let (h,f) and (k,f) be two commutative bundle diffeomorphisms

$$\begin{array}{ccc} B & \xrightarrow{h,k} & B \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

Remark. Note $hk^{-1} : \pi^{-1}(x) \to \pi^{-1}(x)$, $\forall x \in M$, because h and k both have the same induced base map.

2.5.2 Definition. Given h,k as above define $\bar{k} : B_h \to B_h$ by $\bar{k}(\Pi_h(b,t)) = \Pi_h(k(b),t)$.

2.5.3 LEMMA. Given the bundle

$$B_h \longrightarrow (B_h)_{\bar{k}} \xrightarrow{\pi'} S^1 \quad , \text{ then there exists}$$

a map $q : (B_h)_{\bar{k}} \longrightarrow M_f$ which is defined by

$$q(\Pi_{\bar{k}}(\Pi_h(b,t),u)) = \Pi_f(\pi(b),t+u).$$

2.5.4 LEMMA. Given $(x_0,s_0) \in M \times R$ then

$$q^{-1}(\Pi_f(x_0,s_0)) \text{ is homeomorphic to } (N_{x_0})_{hk^{-1}}$$

At this point we do not have a lemma analogous to 2.4.7. and even if we had then there are difficulties in finding the atlas of local chart bundles for the new bundle structure obtained from $(B_h)_{\bar{k}}$.

CHAPTER 3

j-SUSPENSIONS

In this chapter we extend the basic definition of suspension as defined in chapter 1. The operator Σ produces a flow $\Sigma(M,f)$ on a manifold M_f such that $\dim(M_f) = \dim(M) + 1$. Here we consider j-commuting diffeomorphisms of a manifold M and construct a flow on a manifold of j dimensions higher than the dimension of M. We prove results analogous to those of chapter 2 with some additional ones.

§1. DEFINITIONS AND NOTATION

We use the definitions of chapters 1 and 2 together with:

3.1.1 Definition. Given commuting diffeomorphisms $f_1,\dots,f_j$ of a compact manifold M, then there exists a flow ϕ on a manifold $M_{\hat{f}}$ where $\hat{f}$ denotes the j-tuple $(f_1,\dots,f_j)$ of j dimensions higher than that of M called the j-suspension of $(f_1,\dots,f_j)$ denoted by $\Sigma_j(M,f_1,\dots,f_j)$. The construction is as follows.
Let $\alpha_i : M \times R^j \to M \times R^j$, $i \in \{1,\dots,j\}$ be defined by
$\alpha_i(x,t_1,\dots,t_j) = (f_i(x),t_1,\dots,t_i - 1,\dots,t_j)$. Then
$\{\alpha_1^{n_1} \cdot \dots \cdot \alpha_j^{n_j}\} = Z \oplus Z \oplus \dots \oplus Z = G$ (j-terms) acts freely on $M \times R^j$ and the resultant orbit space under G is the manifold $M_{\hat{f}}$. Furthermore the flow ψ on $M \times R^j$ defined by $\psi_v(x,t_1,\dots,t_j) = (x,t_1 + v,\dots,t_j + v)$ induces a flow ϕ on $M_{\hat{f}}$.

§2. j-SUSPENSIONS OF PRODUCT DIFFEOMORPHISMS

We consider a product manifold $M \times N$ and commuting product diffeomorphisms $f_1 \times g_1,\dots,f_j \times g_j : M \times N \to M \times N$ composed of the diffeomorphisms $f_i : M \to M$ and $g_i : N \to N$, $i \in \{1,\dots,j\}$.

We have the following theorem.

3.2.1 THE j-PRODUCT THEOREM. Given the j-tuples of diffeomorphisms $f_1,\dots,f_j$ and $g_1,\dots,g_j$ of M and N respectively, then the product flow $\Sigma_j(M,\hat{f}) \times \Sigma_j(N,\hat{g})$ is a uniform fibre flow on the bundle

$$(M \times N)_{\hat{f} \times \hat{g}} \longrightarrow M_{\hat{f}} \times N_{\hat{g}} \xrightarrow{\pi} T^j$$

Moreover the flow on the fibre is $\Sigma_j(M \times N, \hat{f} \times \hat{g})$.
[T^j is the j-dimensional torus].

Corollary. The product of two j-suspension flows is not structurally stable.

Proof. We use the Integral Invariant Theorem of Thom [20]. We can obtain a first integral by combining the projections $\pi : M_{\hat{f}} \times N_{\hat{g}} \to T^j$ and $p_1 : T^j \to S^1$, the projection onto the first generator of the torus T^j. The projection $p_1\pi : M_{\hat{f}} \times N_{\hat{g}} \to S^1$ gives a first integral for which the flow is invariant.

The theorem is proved by the following observations.

3.2.2 Definition. Let $\hat{h}$ denote the j-tuple of diffeomorphisms $h_1,\dots,h_j$ where $h_i : (M \times N)_{\hat{f} \times \hat{g}} \to (M \times N)_{\hat{f} \times \hat{g}}$ is defined by $h_i\Pi_{\hat{f} \times \hat{g}}(x,y,\underline{t}) = \Pi_{\hat{f} \times \hat{g}}(f_i(x),y,\underline{t})$, [$\underline{t} \in R^j$ denotes the j-tuple $(t_1,\dots,t_j)$].

3.2.3 THEOREM. The flow λ on $((M \times N)_{\hat{f} \times \hat{g}})_{\hat{h}}$ defined by $\lambda_v(\Pi_{\hat{h}}(\Pi_{\hat{f} \times \hat{g}}(x,y,\underline{t}),\underline{u})) = \Pi_h(\Pi_{\hat{f} \times \hat{g}}(x,y,\underline{t}+\underline{v}),\underline{u})$ where $x \in M$, $y \in N$ and $\underline{t},\underline{u},\underline{v} = (v,\dots,v) \in R^j$ is equivalent to the product flow $\phi \times \psi$ on $M_{\hat{f}} \times N_{\hat{g}}$ where $\phi = \Sigma_j(M,\hat{f})$ and $\psi = \Sigma_j(N,\hat{g})$.

This theorem is crucial to 3.2.1. and is proved by the following lemmas.

3.2.4 LEMMA. There are maps q_1, q_2 defined as follows:

$$q_1 : ((M \times N)^{\wedge}_{f\times g})^{\wedge}_{h} \to M^{\wedge}_{f}$$

$$q_2 : ((M \times N)^{\wedge}_{f\times g})^{\wedge}_{h} \to N^{\wedge}_{g}$$

The maps q_1, q_2 are defined by:

$$q_1(\Pi^{\wedge}_{h}(\Pi^{\wedge}_{f\times g}(x,y,t_1,\ldots,t_j),u_1,\ldots,u_j)) = \Pi^{\wedge}_{f}(x,t_1+u_1,\ldots,t_j+u_j)$$

$$q_2(\Pi^{\wedge}_{h}(\Pi^{\wedge}_{f\times g}(x,y,t_1,\ldots,t_j),u_1,\ldots,u_j)) = \Pi^{\wedge}_{g}(y,t_1,\ldots,t_j)$$

Proof. 1) q_1 is continuous.

From the definition of q_1 we have q_1 is well-defined.

$$\begin{array}{ccc}
\Pi^{\wedge}_{h}(\Pi^{\wedge}_{f\times g}(x,y,\underline{t}),\underline{u}) & = & \Pi^{\wedge}_{h}(\Pi^{\wedge}_{f\times g}(f^{n_1+m_1}_1\cdots f^{m_j+n_j}_j(x),g^{n_1}_1\cdots g^{n_j}_j(y),t_1-n_1,\ldots,t_j-n_j),\; u_1-m_1,\ldots,u_j-m_j) \\
\Big\downarrow q_1 & & \Big\downarrow q_1 \\
\Pi^{\wedge}_{f}(x,t_1+u_1,\ldots,t_j+u_j) & = & \Pi^{\wedge}_{f}(f^{n_1+m_1}_1\cdots f^{n_j+m_j}_j(x),t_1+u_1-(n_1+m_1),\ldots,t_j+u_j-(n_j+m_j))
\end{array}$$

Consider the following diagram.

$$\begin{tikzcd}
M \times N \times R^j \times R^j \arrow[r, "q_1'"] \arrow[d, "\Pi^{\wedge}_{f\times g} \times 1"'] & M \times R^j \arrow[dd, "\Pi^{\wedge}_{f}"] \\
(M \times N)^{\wedge}_{f\times g} \times R^j \arrow[d, "\Pi^{\wedge}_{h}"'] & \\
((M \times N)^{\wedge}_{f\times g})^{\wedge}_{h} \arrow[r, "q_1"] & M^{\wedge}_{f}
\end{tikzcd}$$

From 2.1.8 we have $\Pi_{\hat{f}\times\hat{g}}\times 1$, $\Pi_{\hat{h}}$ and $\Pi_{\hat{f}}$ are open maps. Let q_1' be defined by $q_1'(x,y,t_1,\ldots,t_j,u_1,\ldots,u_j) = (x,t_1+u_1,\ldots,t_j+u_j)$. Then q_1' is obviously continuous. The diagram commutes:

$$\Pi_{\hat{f}}q_1' = q_1\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}\times 1) : M\times N\times R^J\times R^J \longrightarrow M_{\hat{f}}$$

Let $O_1 \subset M_{\hat{f}}$ be an open set; then

$$(\Pi_{\hat{f}}q_1')^{-1}(O_1) = (q_1\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}\times 1))^{-1}(O_1)$$

$$\Rightarrow (q_1')^{-1}\Pi_{\hat{f}}^{-1}(O_1) = (\Pi_{\hat{f}\times\hat{g}}\times 1)^{-1}\Pi_{\hat{h}}^{-1}q_1^{-1}(O_1)$$

$$\Rightarrow q_1^{-1}(O_1) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}\times 1)(q_1')^{-1}\Pi_{\hat{f}}^{-1}(O_1).$$ Hence $q_1^{-1}(O_1)$ is open.

Thus q_1 is continuous.

2) q_2 is continuous.

From the definition of q_2 we have q_2 is well-defined.

$$\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),\underline{u}) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(f_1^{n_1+m_1}\cdots f_j^{n_j+m_j}(x), g_1^{n_1}\cdots g_j^{n_j}(y), t_1-n_1,\ldots,t_j-n_j), u_1-m_1,\ldots,u_j-m_j)$$

$$\Big\downarrow q_2 \qquad\qquad\qquad \Big\downarrow q_2$$

$$\Pi_{\hat{g}}(y,t_1,\ldots,t_j) = \Pi_{\hat{g}}(g_1^{n_1}\cdots g_j^{n_j}(y), t_1-n_1,\ldots,t_j-n_j).$$

Consider the following diagram.

$$\begin{tikzcd}
M\times N\times R^J\times R^J \arrow[r, "q_2'"] \arrow[d, "\Pi_{\hat{f}\times\hat{g}}\times 1"'] & N\times R^J \arrow[dd, "\Pi_{\hat{g}}"] \\
(M\times N)_{\hat{f}\times\hat{g}}\times R^J \arrow[d, "\Pi_{\hat{h}}"'] & \\
((M\times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \arrow[r, "q_2"] & N_{\hat{g}}
\end{tikzcd}$$

Define q_2' by $q_2'(x,y,t_1,\ldots,t_j, u_1,\ldots,u_j) = (y,t_1,\ldots,t_j)$. Then q_2' is obviously continuous. The diagram commutes:

$$\Pi_{\hat{g}}q_2' = q_2\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}} \times 1) : M \times N \times R^J \times R^J \to N_{\hat{g}}$$

Let $O_2 \subset N_{\hat{g}}$ be an open set; then

$(\Pi_{\hat{g}}q_2')^{-1}(O_2) = (q_2\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}} \times 1))^{-1}(O_2)$

$\Rightarrow (q_2')^{-1}\Pi_{\hat{g}}^{-1}(O_2) = (\Pi_{\hat{f}\times\hat{g}} \times 1)^{-1}\Pi_{\hat{h}}^{-1}q_2^{-1}(O_2)$

$\Rightarrow q_2^{-1}(O_2) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}} \times 1)(q_2')^{-1}\Pi_{\hat{g}}^{-1}(O_2)$. Hence $q_2^{-1}(O_2)$ is open.

It then follows that q_2 is continuous.

3.2.5 LEMMA. Given $(x_0,s_1,\ldots,s_J) \in M \times R^J$, then there is a homeomorphism r_1 of $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$ with $N_{\hat{g}}$ where r_1 is the restriction of q_2 to $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$.

Proof. The set $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J)) = \{\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y,t_1,\ldots,t_J),u_1,\ldots,u_J) \mid s_i = t_i + u_i,\ i = 1,2,\ldots,j\}$.

The map r_1 is defined by

$$r_1\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y,t_1,\ldots,t_J),s_1-t_1,\ldots,s_J-t_J) = \Pi_{\hat{g}}(y,t_1,\ldots,t_J)$$

Since r_1 is the restriction of a continuous map, it is continuous.

Note that $r_1^{-1} = q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J)) \cap q_2^{-1}$; using this we get

$$r_1^{-1}(\Pi_{\hat{g}}(y_0,t_1,\ldots,t_J)) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y_0,t_1,\ldots,t_J),s_1-t_1,\ldots,s_J-t_J).$$

So we have r_1 is 1-1 and continuous. But r_1 is a map between compact spaces. This follows because $\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J)$ is closed in $M_{\hat{f}}$ and so $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$ is closed in $((M\times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ and therefore compact. Hence r_1 is a 1-1 continuous map of compact spaces and so r_1 is a homeomorphism.

3.2.6 LEMMA. Given $(y_0,t_1,\dots,t_j) \in N \times R^j$, then there is a homeomorphism r_2 of $q_2^{-1}(\Pi_{\hat{g}}(y_0,t_1,\dots,t_j))$ with $M_{\hat{f}}$, where r_2 is the restriction of q_1 to $q_2^{-1}(\Pi_{\hat{g}}(y_0,t_1,\dots,t_j))$.

Proof. By reasoning as before, we get

$$q_2^{-1}(\Pi_{\hat{g}}(y_0,t_1,\dots,t_j)) = \{\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y_0,t_1,\dots,t_j),u_1,\dots,u_j)\}.$$

The map r_2 is defined by

$$r_2(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y_0,t_1,\dots,t_j),u_1,\dots,u_j)) = \Pi_{\hat{f}}(x,t_1+u_1,\dots,t_j+u_j)$$

We get r_2 is continuous because it is the restriction of a continuous mapping q_1. Also r_2 is 1-1. Note that $r_2^{-1} = q_2^{-1}(\Pi_{\hat{g}}(y_0,t_1,\dots,t_j)) \cap q_1^{-1}$ and so it follows that $r_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\dots,s_j)) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y_0,t_1,\dots,t_j)s_1-t_1,\dots,s_j-t_j)$. Hence r_2 is both 1-1 and continuous. Again as for r_1 in 3.2.5, r_2 is a map between compact spaces, and so r_2 is a homeomorphism.

3.2.7 LEMMA. Denote an element of $((M\times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ by z, then the mapping

$$\kappa : z \longrightarrow (q_1(z),\ q_2(z)), \text{ is a homeomorphism.}$$

$$\kappa : ((M\times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \longrightarrow M_{\hat{f}}\times N_{\hat{g}}\ .$$

Proof. As for 2.2.6.

Remark. In fact κ is a diffeomorphism since all maps used in the construction are differentiable of class C^r $(r \geqslant 1)$.

3.2.8 LEMMA. The flow λ on $((M\times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ defined in 3.2.3 is well-defined.

Proof. As for 2.2.7.

3.2.9 LEMMA. The following diagram commutes where $\phi = \Sigma_J(M,\hat{f})$

$$\begin{tikzcd} ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \times R \arrow[r, "q_1 \times 1"] \arrow[d, "\lambda"'] & M_{\hat{f}} \times R \arrow[d, "\phi"] \\ ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \arrow[r, "q_1"] & M_{\hat{f}} \end{tikzcd}$$

Proof. Follows directly from the definitions.

3.2.10. LEMMA. The following diagram commutes where $\psi = \Sigma_J(N,\hat{g})$.

$$\begin{tikzcd} ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \times R \arrow[r, "q_2 \times 1"] \arrow[d, "\lambda"'] & N_{\hat{g}} \times R \arrow[d, "\psi"] \\ ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \arrow[r, "q_2"] & N_{\hat{g}} \end{tikzcd}$$

Proof. Follows directly from the definitions.

3.2.11 LEMMA. The flow λ on $((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ is a uniform fibre flow on the bundle

$$(M \times N)_{\hat{f}\times\hat{g}} \longrightarrow ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \xrightarrow{\pi'} T^J$$

Moreover the flow on the fibre is $\mu = \Sigma_J(M \times N, f \times g)$.

Proof. The projection π' is defined by

$\pi'(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),\underline{u}) = p(\underline{u})$ where p is the natural projection $p : R^J \to R^J/Z^J = T^J$.

Let $\{V_\iota \mid i \in I\}$ be a finite class of open subsets of R^J such that the projection $p : R^J \to R^J/Z^J$ is a homeomorphism on restriction to each V_ι and $\{U_\iota \mid U_\iota = p(V_\iota)\}$ is an open covering of $T^J = R^J/Z^J$. It can be easily shown that such a cover exists.

Define $\alpha_\iota : U_\iota \times (M \times N)_{\hat{f}\times\hat{g}} \to \pi^{\prime -1}(U_\iota)$ by $\alpha_\iota(p(\underline{u}), \Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}))$
$= \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),\underline{u})$ where $\underline{u} \in V_\iota$.

Using the atlas $\{(U_\iota,\alpha_\iota)\}$ we have a fibre flow with $\mu = \Sigma_J(M \times N,\hat{f}\times \hat{g})$

$$\begin{tikzcd}
(p(\underline{u}),\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),v) \arrow[r, "\alpha_i \times 1"] \arrow[d, "1\times\mu"'] & (\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),u),v) \arrow[d, "\lambda"] \\
(p(\underline{u}),\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}+\underline{v})) \arrow[r, "\alpha_i"] \arrow[d, "\rho"'] & \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}+\underline{v}),\underline{u}) \arrow[d, "\pi'"] \\
p(\underline{u}) \arrow[r, "1"] & p(\underline{u})
\end{tikzcd}$$

where $\underline{u} \in V_\iota$ and $\underline{v} = (v,v,\dots,v) \in R^J$.

3.2.12 LEMMA Using the results of 3.2.3, 3.2.7 and 3.2.11 we have the uniform fibre flow of 3.2.1 by taking the projection $\pi = \pi'\kappa^{-1}$: $M_f \times N_g \to T^J$ and the covering $\{U_\iota\}$ with coordinate functions $\beta_\iota = \kappa\alpha_\iota$

Proof. The following diagram commutes for $i \in I$.

$$\begin{tikzcd}
U_i \times (M \times N)_{\hat{f}\times\hat{g}} \times R \arrow[r, "\alpha_i \times 1"] \arrow[d, "1\times\mu"] & \pi'^{-1}(U_i) \times R \arrow[r, "\kappa \times 1"] \arrow[d, "\lambda"] & \kappa\pi'^{-1}(U_i) \times R \arrow[d, "\phi\times\psi"] \\
U_i \times (M \times N)_{\hat{f}\times\hat{g}} \arrow[r, "\alpha_i"] \arrow[d, "\rho"] & \pi'^{-1}(U_i) \arrow[r, "\kappa"] \arrow[d, "\pi'"] & \kappa\pi'^{-1}(U_i) \arrow[d, "\pi'\kappa^{-1}"] \\
U_i \arrow[r, "1"] & U_i \arrow[r, "1"] & U_i
\end{tikzcd}$$

The commutativity of the diagram follows from lemmas 3.2.7, 3.2.9, 3.2.10 and 3.2.11.

It follows that the diagram below commutes for $i \in I$.

$$\begin{tikzcd}
U_i \times (M \times N)_{\hat{f}\times\hat{g}} \times R \arrow[r, "(\kappa\alpha_i) \times 1"] \arrow[d, "1\times\mu"] & (\pi'\kappa^{-1})^{-1}(U_i) \arrow[d, "\phi\times\psi"] \\
U_i \times (M \times N)_{\hat{f}\times\hat{g}} \arrow[r, "\kappa\alpha_i"] \arrow[d, "\rho"] & (\pi'\kappa^{-1})^{-1}(U_i) \arrow[d, "\pi'\kappa^{-1}"] \\
U_i \arrow[r, "1"] & U_i
\end{tikzcd}$$

§3. (j,k)-SUSPENSIONS

3.3.1 Definition. We define a (j,k) suspension of a manifold M and its diffeomorphisms $f_1,\ldots,f_j$ and a manifold N and its diffeomorphisms $g_1,\ldots,g_k$ to be the product flow $\Sigma_j(M,\hat{f}) \times \Sigma_k(N,\hat{g})$ on $M_{\hat{f}} \times N_{\hat{g}}$.

3.3.2 Definition. Let $\hat{f} \times \hat{g}$ denote the k-tuple of diffeomorphisms $f_1 \times g_1, f_2 \times g_2, \ldots, f_j \times g_j, 1 \times g_{j+1}, \ldots, 1 \times g_k : M \times N \to M \times N$. Suppose $j < k$.

3.3.3 Definition. Let h_i, $i = 1, \ldots, j$ be the diffeomorphisms defined by

$$h_i(\Pi_{\hat{f}\times\hat{g}}(x,y,t_1,\ldots,t_k)) = \Pi_{\hat{f}\times\hat{g}}(f_i(x),y,t_1,\ldots,t_k).$$

Remark. In §2 we considered the special case of $j=k$. Obviously we lose no generality in this section by taking $j < k$ as in 3.3.2.

3.3.4 THE (j,k) PRODUCT THEOREM. Given the commuting j,k tuples of diffeomorphisms $f_1, \ldots, f_j$, $g_1, \ldots, g_k$ of the manifolds M,N respectively then the (j,k) suspension of these diffeomorphisms is a uniform fibre flow on

$$(M \times N)_{\hat{f}\times\hat{g}} \longrightarrow M_{\hat{f}} \times N_{\hat{g}} \xrightarrow{\pi} T^j$$

Moreover the flow on the fibre is $\Sigma_k(M \times N, \hat{f} \times \hat{g})$.

Corollary. A (j,k) suspension is not structurally stable.

Remark. The lemmas required for this theorem follows a similar pattern to those of chapter 2, §4. The maps are not completely analogous because of the non-symmetrical condition $j < k$. However, some of the proofs will be abbreviated because of the similarity.

3.3.5 THEOREM. Using 3.3.3, the flow λ on $((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ defined by

$$\lambda_v(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,t_1,\ldots,t_k),u_1,\ldots,u_j)) = \lambda_v(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,t_1+v,t_2+v,\ldots,t_k+v), u_1,\ldots,u_j))$$

is equivalent to the product flow $\phi \times \psi$ on $M_{\hat{f}} \times N_{\hat{g}}$ where $\phi = \Sigma_j(M,\hat{f})$ and $\psi = \Sigma_k(N,\hat{g})$.

To prove this, from which 3.3.4 follows, we require the following lemmas.

3.3.6 LEMMA. There are maps q_1, q_2 such that

$$q_1 : ((M \times N)_{\hat{f} \times \hat{g}})_{\hat{h}} \longrightarrow M_{\hat{f}}$$

$$q_2 : ((M \times N)_{\hat{f} \times \hat{g}})_{\hat{h}} \longrightarrow N_{\hat{g}}$$

These maps are defined by :

$$q_1(\Pi_{\hat{h}}(\Pi_{\hat{f} \times \hat{g}}(x,y,t_1,\ldots,t_k),u_1,\ldots,u_j)) = \Pi_{\hat{f}}(x,t_1+u_1,\ldots,t_j+u_j)$$

$$q_2(\Pi_{\hat{h}}(\Pi_{\hat{f} \times \hat{g}}(x,y,t_1,\ldots,t_k),u_1,\ldots,u_j)) = \Pi_{\hat{g}}(y,t_1,\ldots,t_k)$$

Proof. 1) q_1 is well-defined and continuous. The continuity of q_1 is proved by consideration of the diagram.

$$\begin{tikzcd}
M \times N \times R^k \times R^j \arrow[r, "q_1'"] \arrow[d, "\Pi_{\hat{f} \times \hat{g}} \times 1"'] & M \times R^j \arrow[dd, "\Pi_{\hat{f}}"] \\
(M \times N)_{\hat{f} \times \hat{g}} \times R^j \arrow[d, "\Pi_{\hat{h}}"'] & \\
((M \times N)_{\hat{f} \times \hat{g}})_{\hat{h}} \arrow[r, "q_1"] & M_{\hat{f}}
\end{tikzcd}$$

Define q_1' by $q_1'(x,y,t_1,\ldots,t_k,u_1,\ldots,u_j) = (x,t_1+u_1,\ldots,t_j+u_j)$. Then the diagram commutes. By the techniques as used before we obtain the continuity of q_1.

2) q_2 is well-defined and continuous. The continuity of q_2 is proved by consideration of the diagram.

$$\begin{tikzcd}
M \times N \times R^k \times R^J \arrow[r, "q_2'"] \arrow[d, "\Pi_{\hat{f}\times\hat{g}} \times 1"'] & N \times R^J \arrow[dd, "\Pi_{\hat{g}}"] \\
(M \times N)_{\hat{f}\times\hat{g}} \times R^J \arrow[d, "\Pi_{\hat{h}}"'] & \\
((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \arrow[r, "q_2"] & N_{\hat{g}}
\end{tikzcd}$$

Define q_2' by $q_2'(x,y,t_1,\ldots,t_K,u_1,\ldots,u_J) = (y,t_1,\ldots,t_K)$. Then the diagram commutes. Again using the techniques of 3.2.4 we have that q_2 is a continuous map.

3.3.7 LEMMA. Given $(x_0,s_1,\ldots,s_J) \in M \times R^J$, there is a homeomorphism r_1 of $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$ with $N_{\hat{g}}$, where r_1 is the restriction of q_2 to $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$.

Proof. From the definition of q_1 it follows that $q_1^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$

$$= \{\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y,t_1,\ldots,t_K),u_1,\ldots,u_J) \mid y \in N, t_1+u_1 = s_1,\ldots,t_J+u_J = s_J\}.$$

From the definition of r_1 we have:

$$r_1(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y,t_1,\ldots,t_K),u_1,\ldots,u_J)) = \Pi_{\hat{g}}(y,t_1,\ldots,t_K).$$

It follows that r_1 is 1-1 because

$$r_1^{-1}(\Pi_{\hat{g}}(y,t_1,\ldots,t_K)) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x_0,y,t_1,\ldots,t_K),s_1-t_1,\ldots,s_J-t_J) .$$

Hence r_1 is a 1-1 continuous map being the restriction of $q_2 : ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \to N_{\hat{g}}$. The assumption of the compactness of M and N give us that r_1 is a 1-1 continuous map between compact spaces and so r_1 is a homeomorphism.

3.3.8 LEMMA. Given $(y_0, t_1', \ldots, t_k') \in N \times R^k$, there is a homeomorphism r_2 of $q_2^{-1}(\Pi_{\hat g}(y_0, t_1', \ldots, t_k'))$ with $M_{\hat f}$, where r_2 is the restriction of q_1 to $q_2^{-1}(\Pi_{\hat g}(y_0, t_1', \ldots, t_k'))$.

Proof. The set $q_2^{-1}(\Pi_{\hat g}(y_0, t_1', \ldots, t_k')) = \{\Pi_{\hat h}(\Pi_{\hat f \times \hat g}(x, y_0, t_1', \ldots, t_k'), u_1, \ldots, u_j) \mid x \in M, (u_1, \ldots, u_j) \in R^j\}$.

From the definition of r_2 we have:

$$r_2(\Pi_{\hat h}(\Pi_{\hat f \times \hat g}(x, y_0, t_1', \ldots, t_k'), u_1, \ldots, u_j)) = \Pi_{\hat f}(x, t_1' + u_1, \ldots, t_j' + u_j).$$

It follows that r_2 is 1-1 because

$$r_2^{-1}(\Pi_{\hat f}(x_0, s_1, \ldots, s_j)) = \Pi_{\hat h}(\Pi_{\hat f \times \hat g}(x_0, y_0, t_1', \ldots, t_k'), s_1 - t_1', \ldots, s_j - t_j')\} .$$

Hence r_2 is a continuous 1-1 map between compact spaces for reasons which we have previously discussed and so r_2 is a homeomorphism.

3.3.9 LEMMA. Denote an element of $((M \times N)_{\hat f \times \hat g})_{\hat h}$ by z, then the mapping $\kappa : z \longrightarrow (q_1(z), q_2(z))$ is a diffeomorphism.

$$\kappa : ((M \times N)_{\hat f \times \hat g})_{\hat h} \longrightarrow M_{\hat f} \times N_{\hat g}$$

Proof. As for 2.2.6.

3.3.10 LEMMA. The flow λ on $((M \times N)_{\hat f \times \hat g})_{\hat h}$ defined by 3.3.5 is well-defined.

Proof. As for 2.2.7.

3.3.11 LEMMA. The following diagram commutes where $\phi = \Sigma_J(M, \hat f)$.

$$\begin{tikzcd}
((M \times N)_{\hat f \times \hat g})_{\hat h} \times R \arrow[r, "q_1 \times 1"] \arrow[d, "\lambda"'] & M_{\hat f} \times R \arrow[d, "\phi"] \\
((M \times N)_{\hat f \times \hat g})_{\hat h} \arrow[r, "q_1"] & M_{\hat f}
\end{tikzcd}$$

Proof. Follows directly from the definitions.

3.3.12 LEMMA. The following diagram commutes where $\psi = \Sigma_K(N,\hat{g})$

$$\begin{array}{ccc} ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \times R & \xrightarrow{q_2 \times 1} & N_{\hat{g}} \times R \\ \big\downarrow \lambda & & \big\downarrow \psi \\ ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} & \xrightarrow{q_2} & N_{\hat{g}} \end{array}$$

Proof. Follows directly from the definitions.

Lemmas 3.3.9, 3.3.11 and 3.3.12 give the equivalence of theorem 3.3.5.

3.3.13 LEMMA. The flow λ on $((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}}$ is a uniform fibre flow on the bundle

$$(M \times N)_{\hat{f}\times\hat{g}} \longrightarrow ((M \times N)_{\hat{f}\times\hat{g}})_{\hat{h}} \xrightarrow{\pi'} T^J$$

and the flow on the fibre is $\Sigma_K(M \times N, \hat{f}\times\hat{g})$.

Remark. The proof is completely analogous to that of 3.2.11; however we need the definition of the atlas of local charts and the projection π' for the next lemma. With the usual notation:

$U_i = \{p(\underline{u}) \mid \underline{u} \in V_i\}$. Define π' by $\pi'(\Pi_{\hat{h}}(\Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,t_1,\ldots,t_K),u_1,\ldots,u_j)) = p(u_1,\ldots,u_j)$ and the atlas of local charts is given by $\{(U_i,\alpha_i)\}$ where $\alpha_i : U_i \times (M \times N)_{\hat{f}\times\hat{g}} \to \pi'(U_i)$ is defined by $\alpha_i(p(\underline{u}), \Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t})) = \Pi_{\hat{h}}(\Pi_{\hat{f}\times\hat{g}}(x,y,\underline{t}),\underline{u})$ where $\underline{u} \in V_i$, $\forall\ i \in I$

3.3.14 LEMMA. The projection $\pi = \pi'\kappa^{-1} : M_{\hat{f}} \times N_{\hat{g}} \to T^j$ and the atlas of local charts $\{(U_i, \kappa\alpha_i) \mid i \in I\}$ give the structure for the uniform fibre flow of 3.3.4.

§4. j-SUSPENSIONS OF BUNDLE DIFFEOMORPHISMS

Consider the bundle of manifolds

$$N \longrightarrow B \xrightarrow{\pi} M$$

and the j-tuple of commutative bundle diffeomorphisms $(h_1, f_1), \dots (h_j, f_j)$ such that the diagrams commute for $i = 1, 2, \dots, j$.

$$\begin{tikzcd} B \arrow[r, "h_i"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ M \arrow[r, "f_i"] & M \end{tikzcd}$$

3.4.1 Definition. Let the diffeomorphisms $\hat{h}' = (h_1', \dots, h_j')$ be defined by $h_i' : B_{\hat{h}} \to B_{\hat{h}}$ where $\hat{h} = (h_1, \dots, h_j)$ and $h_i'(\Pi_{\hat{h}}(b, t_1, \dots, t_j)) = \Pi_{\hat{h}}(h_i(b), t_1, \dots, t_j)$, $i = 1, \dots, j$.

Notation Let $N_x = \pi^{-1}(x)$, the fibre over x in B.

Notation Let $\hat{1}$ denote the j-tuple of identity diffeomorphisms $1 : N_x \to N_x$.

3.4.2 THE j-BUNDLE THEOREM. Given the bundle of manifolds

$$N \longrightarrow B \xrightarrow{\pi} M$$

and j-commuting bundle diffeomorphisms $(h_1, f_1), \dots, (h_j, f_j)$ with j derived diffeomorphisms $h_i' : B_{\hat{h}} \to B_{\hat{h}}$, then $\exists$ a uniform fibre flow λ on the bundle

$$B_{\hat{h}} \longrightarrow (B_{\hat{h}})_{\hat{h}'} \xrightarrow{\pi'} T^j,$$ where the flow on the fibre is

$\Sigma_j(B, h)$, such that λ is also a local product flow on the bundle

$$N_{\hat{1}} \longrightarrow (B_{\hat{h}})_{\hat{h}'} \xrightarrow{\sigma} M_{\hat{f}}$$

The flow on $M_{\hat{f}}$ is $\Sigma_J(M,\hat{f})$ and the flow on $N_{\hat{1}}$ is $\Sigma_J(N,\hat{1})$.

Remark. Most of the lemmas required for 3.4.2 follow from similar statements to those in the proof of previous theorems of this chapter. However for completeness we will state the lemmas without proof except for those concerned with construction of atlases of local charts which we shall give.

3.4.3 LEMMA. There exists a continuous map $q: (B_{\hat{h}})_{\hat{h}'} \to M_{\hat{f}}$ defined by

$$q(\Pi_{\hat{h}'}(\Pi_{\hat{h}}(b,\underline{t}),\underline{u}) = \Pi_{\hat{f}}(\pi(b),t_1+u_1,\ldots,t_J+u_J)$$

3.4.4 LEMMA. Given $(x_0,s_1,\ldots,s_J) \in M \times R^J$; then

$$q^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J)) = \{\Pi_{\hat{h}'}(\Pi_{\hat{h}}(b,t_1,\ldots,t_J),s_1-t_1,\ldots,s_J-t_J) | \pi(b)=x_0\}.$$

3.4.5 LEMMA. Given $(x_0,s_1,\ldots,s_J) \in M \times R^J$; then there exists a homeomorphism

$$\kappa_{(x_0,s_0)} : (N_{x_0})_{\hat{1}} \longrightarrow (q^{-1}(\Pi_{\hat{f}}(x_0,s_1,\ldots,s_J))$$

3.4.6 LEMMA. There exists an equivalence between the flow $\Sigma_J(N_{x_0},\hat{1})$ on $(N_{x_0})_{\hat{1}}$ and the flow $\Sigma_J(N,\hat{1})$ on $N_{\hat{1}}$.

3.4.7 LEMMA. The fibre flow λ of the theorem is defined by

$$\lambda_v(\Pi_{\hat{h}'}(\Pi_{\hat{h}}(b,\underline{t}),\underline{u}) = \Pi_{\hat{h}'}(\Pi_{\hat{h}}(b,t_1+v,\ldots,t_J+v),u_1,\ldots,u_J).$$

3.4.8 Definition. Let (U',β) be an atlas of local bundle charts for the bundle

$$N \longrightarrow B \xrightarrow{\pi} M$$

3.4.9 Definition. Take an open covering $\{U\}$ of $M_{\hat{f}}$ to be defined with typical element

$$U = \{\Pi_{f}(x,\underline{u}) \mid x \in U', \underline{u} \in V_{\iota}\}$$

3.4.10 LEMMA. Given the open covering $\{U\}$ of $M_{\hat{f}}$ then $\exists$ an atlas of local bundle charts (U,γ) for the bundle

$$N_{\hat{1}} \longrightarrow (B_{\hat{h}})_{\hat{h}'} \xrightarrow{q} M_{\hat{f}}$$

Construction of atlas. Let us take $U = \Pi_{\hat{f}}(U' \times V_{\iota})$ say. Then $\gamma : U \times N_{\hat{1}} \to q^{-1}(U)$ is defined by $\gamma(\Pi_{\hat{f}}(x,s_1,\ldots,s_j), \Pi_{\hat{1}}(y,t_1,\ldots,t_j))$ $= \Pi_{\hat{h}'}(\Pi_{\hat{h}}(\beta(x,y),t_1,\ldots,t_j),s_1 - t_1,\ldots,s_j - t_j)$. Again in this definition we stipulate that $\underline{s} = (s_1,\ldots,s_j)$ must take values such that $\underline{s} \in V_{\iota}$.

3.4.11 LEMMA. The flow λ in the j-Bundle Theorem is the local product flow as indicated in 3.4.2 using the atlas of local bundle charts, (U,γ).

§5. THE RELATIONSHIP BETWEEN j and j-1-SUSPENSIONS

3.5.1 THEOREM. Given the manifold M and j-commuting diffeomorphisms $f_1,\ldots,f_j$ and denoting $f_1,\ldots,f_j$ by $\hat{f}$ and $f_1,\ldots,f_{j-1}$ by $\hat{h}$ then $\Sigma_j(M,\hat{f})$ is a local product flow on the bundle

$$M_{\hat{h}} \longrightarrow M_{\hat{f}} \xrightarrow{\pi} S^1$$

The flow on the fibre is $\Sigma_{j-1}(M,\hat{h})$ and the flow on the base S^1 is the unit flow.

Remark. This gives a relationship between Σ_j and Σ_{j-1}. An example of Σ_j for $j=2$ is the locking on phenomenon of the Van der Pol Oscillator discussed in Chapter 1. This is the simplest example of the point manifolds with identity diffeomorphisms giving the diagonal flow on the torus, a perturbation of which is the local flow when the oscillator runs synchronously.

3.5.2 Definition. Let $\bar{f}_j : M_{\hat{h}} \to M_{\hat{h}}$ be the diffeomorphism defined by $\bar{f}_j\Pi_{\hat{h}}(x,t_1,\ldots,t_{j-1}) = \Pi_{\hat{h}}(f_j(x),t_1,\ldots,t_{j-1})$.

3.5.3 LEMMA. There exists a diffeomorphism κ between $M_{\hat{f}}$ and $(M_{\hat{h}})_{\bar{f}_j}$.

Proof. Define $\kappa : M_{\hat{f}} \to (M_{\hat{h}})_{\bar{f}_j}$ by $\kappa\Pi_{\hat{f}}(x,t_1,\ldots,t_j) = \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1,\ldots,t_{j-1}),t_j)$. Then κ is well-defined because

$$\begin{array}{ccc} \Pi_{\hat{f}}(x,t_1,\ldots,t_j) & = & \Pi_{\hat{f}}(f_1^{n_1}\ldots f_j^{n_j}(x),t_1-n_1,\ldots,t_j-n_j) \\ \Big\downarrow{\scriptstyle\kappa} & & \Big\downarrow{\scriptstyle\kappa} \\ \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1,\ldots,t_{j-1}),t_j) & = & \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(f_1^{n_1}\ldots f_j^{n_j}(x),t_1-n_1,\ldots,t_{j-1}-n_{j-1}),t_j-n_j) \end{array}$$

It is easily checked that κ is a 1-1 map. We have that κ is a homeomorphism by consideration of the following diagram.

$$\begin{tikzcd}
M \times R^J \arrow[rr, shift left, "\kappa_1"] \arrow[dd, "\Pi_{\hat{f}}"'] & & (M \times R^{J-1}) \times R \arrow[ll, shift left, "\kappa_1^{-1}"] \arrow[d, "\Pi_{\hat{h}} \times 1"] \\
 & & M_{\hat{h}} \times R \arrow[d, "\Pi_{\bar{f}_J}"] \\
M_{\hat{f}} \arrow[rr, shift left, "\kappa"] & & (M_{\hat{h}})_{\bar{f}_J} \arrow[ll, shift left, "\kappa^{-1}"]
\end{tikzcd}$$

Define the diffeomorphism κ_1 by $\kappa_1(x,t_1,\dots,t_J) = ((x,t_1,\dots, t_{J-1}),t_J)$. We then have

$$\kappa\Pi_{\hat{f}} = \Pi_{\bar{f}_J}(\Pi_{\hat{h}} \times 1)\kappa_1 \ : M \times R^J \to (M_{\hat{h}})_{\bar{f}_J}$$

$$\Rightarrow \quad (\kappa\Pi_{\hat{f}})^{-1}(O_1) = (\Pi_{\bar{f}_J}(\Pi_{\hat{h}}\times 1)\kappa_1)^{-1}(O_1), \text{ for an open set } O_1 \subseteq (M_{\hat{h}})_{\bar{f}_J}$$

$$\Rightarrow \quad \kappa^{-1}(O_1) = \Pi_{\hat{f}}\kappa_1^{-1}(\Pi_{\hat{h}}\times 1)^{-1}(\Pi_{\bar{f}_J})^{-1}(O_1) \text{ and so } \kappa^{-1}(O_1) \text{ is open.}$$

Therefore κ is continuous.

Also we have

$$\Pi_{\hat{f}}\kappa_1^{-1} = \kappa^{-1}\Pi_{\bar{f}_J}(\Pi_{\hat{h}}\times 1) \ : M\times R^{J-1} \times R \to M_{\hat{f}}$$

Using this commutativity and the fact that κ_1 is a diffeomorphism and all the projection maps are open we have that κ^{-1} is continuous. Hence κ is a homeomorphism. Again since differentiable maps are used to define κ and κ^{-1} we have that κ is a diffeomorphism.

3.54 LEMMA. The flow λ on the bundle

$$M_{\hat{h}} \longrightarrow (M_{\hat{h}})_{\bar{f}_J} \xrightarrow{\pi'} S^1$$

where λ is defined by $\lambda_v(\Pi_{\bar{f}_J}(\Pi_{\hat{h}}(x,t_1,\dots,t_{J-1}),t_J)) = \Pi_{\bar{f}_J}(\Pi_{\hat{h}}(x,t_1+ v,\dots, t_{J-1}+ v),t_J+v)$, $v \in R$, is a local product flow where the flow on the fibre is $\Sigma(M,\hat{h})$ and the flow on the base is the unit flow, ϕ.

__Proof.__ Define the atlas of charts (U,α) as follows. Let $\{U_1,U_2\}$ be a covering of S^1 where $U_1 = \{p(u) \mid 0 < u < 1\}$ and $U_2 = \{p(u) \mid 3/4 < u < 5/4\}$ where p is the natural projection $p : R \to R/Z = S^1$.

Define $\alpha_1 : U_1 \times M_{\hat{h}} \to \pi'^{-1}(U_1)$ by $\alpha_1(p(u), \Pi_{\hat{h}}(x,t_1,\dots,t_j)) = \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1,\dots,t_{j-1}),u)$ such that in the definition u is made to take a value $0 < u < 1$. Similarly we define $\alpha_2 : U_2 \times M_{\hat{h}} \to \pi'^{-1}(U_2)$ by $\alpha_2(\bar{p}(u), \Pi_{\hat{h}}(x,t_1,\dots,t_{j-1})) = \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1,\dots,t_{j-1}),u)$ where $3/4 < u < 5/4$. For reasons as to why α_1, α_2 are diffeomorphisms see appendix 2.

This atlas of local bundle charts is the one which we use to give us the local product flow. Choose u_0 such that $0 < u_0 < 1$. Let $R[p(u_0)] = \{p(u) \mid -u_0 < v < 1 - u_0\}$.

Then we have commutativity in the following diagram.

$$\begin{tikzcd}
(p(u_0), \Pi_{\hat{h}}(x,t_1,\dots,t_{j-1}), v) \arrow[r, "\alpha_1 \times 1"] \arrow[d, "\phi \times \Sigma_{j-1}(M,\hat{h})"'] & (\Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1,\dots,t_{j-1}),u_0),v) \arrow[d, "\lambda"] \\
(p(u_0+v), \Pi_{\hat{h}}(x,t_1+v,\dots,t_{j-1}+v)) \arrow[r, "\alpha_1"] \arrow[d, "\rho"'] & \Pi_{\bar{f}_j}(\Pi_{\hat{h}}(x,t_1+v,\dots,t_{j-1}+v),u_0+v) \arrow[d, "\pi'"] \\
p(u_0+v) \arrow[r, "1"] & p(u_0+v)
\end{tikzcd}$$

A similar commutativity is obtained for α_2.

3.55 __LEMMA.__ The diffeomorphism κ of 3.5.3 gives the equivalence of the flow λ on $(M_{\hat{h}})_{\bar{f}_j}$ and $\Sigma_j(M,\hat{f})$ on $M_{\hat{f}}$.

__Proof.__ The following diagram commutes.

$$\begin{CD} M_{\hat{f}} \times R @>{\kappa \times 1}>> (M_{\hat{h}})_{\bar{f}_J} \times R \\ @V{\Sigma_J(M,\hat{f})}VV @VV{\lambda}V \\ M_{\hat{f}} @>{\kappa}>> (M_{\hat{h}})_{\bar{f}_J} \end{CD}$$

3.5.6 LEMMA. The atlas $\{(U_1, \kappa^{-1}\alpha_1), (U_2, \kappa^{-1}\alpha_2)\}$ gives the local product flow $\phi \times \Sigma_{J-1}(M,\hat{h})$ on the bundle

$$M_{\hat{h}} \longrightarrow M_{\hat{f}} \xrightarrow{\pi} S^1$$

Proof. This follows from 3.5.3 and 3.5.4.

CHAPTER 4

ALGEBRAIC PROPERTIES OF Σ

§ 0. INTRODUCTION

In the previous two chapters we have loosely referred to Σ as an 'operator' purely as a descriptive term. In this chapter we will set up categories and show that Σ can be represented algebraically as a covariant functor with respect to these categories. Also considered is the inter-relation between Σ and other operations on manifolds such as the boundary of a manifold and the interior of a manifold. Similar results are stated for Σ_j, $(j \in Z_+)$.

§ 1. NOTATION and DEFINITIONS

We again consider all manifolds to be compact and of class C^r $(r \geqslant 1)$ and all diffeomorphisms to be of class C^r, $(r \geqslant 1)$. We define the various operations

4.1.1 Notation. Let M be a manifold with boundary and let $\partial(M)$ denote the boundary of M, which is a submanifold of M.

4.1.2 Notation. Let M be as in 4.1.1 and take $I(M)$ to denote the interior of M, which is a submanifold of M.

4.1.3 Definition. Let $D(M)$ denote the double of the manifold M. It is defined as follows. Let M be a manifold with non-empty boundary. Take $M_0 = M \times 0$ and $M_1 = M \times 1$ as two copies of M. Then $D(M)$ is $M_0 \cup M_1$, with points $(x,0)$ and $(x,1)$ identified for $x \in \partial(M)$. A definition of the differentiable structure appears in Appendix 4.

The definitions of category and functor also appear in Appendix 3.

§ 2. FUNCTORIAL REPRESENTATION OF Σ

4.2.1 Definition. Let the category $\mathcal{M}$ have as objects the pairs (M,f) where M is a manifold and f is a diffeomorphism $f: M \to M$. Let the set of morphisms between (M,f) and (N,g) denoted by $[(M,f),(N,g)]$ be defined as the set of triples $\{(k,f,g) | k: M \to N$ and $kf = gk: M \to N\}$.

4.2.2 Definition. Let the category $\mathcal{F}_s$ have as objects the flows $\Sigma(M,f)$ which are suspensions of diffeomorphisms. Let the set of morphisms between $\Sigma(M,f)$ and $\Sigma(N,g)$, denoted by $[\Sigma(M,f), \Sigma(N,g)]$, be defined as follows. The triple $(k,f,g)^* \in [\Sigma(M,f), \Sigma(N,g)]$ if $k: M \to N$ and $kf = gk: M \to N$. Let Π_f and Π_g be the usual projection maps. Then this means there is a well-defined map $k(f,g): M_f \to N_g$ given by $k(f,g)\Pi_f(x,s) = \Pi_g(k(x),s)$.

4.2.3 LEMMA Σ is a functor from the category $\mathcal{M}$ to the category $\mathcal{F}_s$

$$\Sigma : \mathcal{M} \to \mathcal{F}_s.$$

Proof. 1) $\mathcal{M}$ is a category.

Referring to the definition in Appendix 2. We have already satisfied conditions C1 and C2. Consider the triple $((E,e),(F,f),(G,g))$ of objects of $\mathcal{M}$. Then a map is defined

$$[(E,e),(F,f)] \times [(F,f),(G,g)] \to [(E,e),(G,g)]$$

$$((p,e,f),(q,f,g)) \to (qp,e,g).$$

This follows because

$$pe = fp : E \to F \text{ and } qf = gq : F \to G$$

$$\Rightarrow \quad q = gqf^{-1} : F \to G$$

$$\Rightarrow \quad qpe = gqf^{-1}fp : E \to G$$

$\Rightarrow$ $qpe = gqp : E \to G$

$\Rightarrow$ $(qp,e,g) \in [(E,e),(G,g)]$, hence C3 is satisfied.

To satisfy C4 we take the identity morphism $1_{(E,e)} \in [(E,e),(E,e)]$ as the triple $(1_E,e,e)$. We have C1 – C4 satisfied.

AC1. Let $(p.e.f) \in [(E,e),F,f)],(q,f,g) \in [(F,f),(G,g)]$ and $(r,g,h) \in [(G,g),(H,h)]$. Then we have associativity:

$$((r,g,h)(q,f,g))(p,e,f) = (rq,f,h)(p,e,f) = (rqp,e,h)$$

$$(r,g,h)((q,f,g)(p,e,f)) = (r,g,h)(qp,e,g) = (rqp,e,h)$$

AC2. Let $(p,e,f) \in [(E,e),(F,f)]$. Then

$$(p,e,f).1_{(E,e)} = (p,e,f).(1_E,e,e) = (p1_E,e,f) = (p,e,f)$$

$$1_{(F,f)}(p,e,f) = (1_F,f,f)(p,e,f) = (1_Fp,e,f) = (p,e,f)$$

2) $\mathcal{J}_s$ is a category.

We have defined the objects to satisfy C1. We must check that the morphisms of $[\Sigma(E,e), \Sigma(F,f)]$ are well defined. Let $(p,e,f)^*$ be such a morphism then $(p,e,f)^*\Pi_e(x,s) = \Pi_f(p(x),s)$. However $\Pi_e(x,s) = \Pi_e(e^n(x),s-n)$, $n \in Z$ and so $(p,e,f)^*\Pi_e(e^n(x),s-n) = \Pi_f(pe^n(x),s-n) = \Pi_f(f^np(x),s-n) = \Pi_f(p(x),s)$. This ensures C2 is satisfied. To check C3 let us consider the morphisms $(p,e,f)^* : \Sigma(E,e) \to \Sigma(F,f)$ and $(q,f,g)^* : \Sigma(F,f) \to \Sigma(G,g)$. Then we have the map $((p,e,f)^*(q,f,g)^* \to (qp,e,g)^*$. The morphism $(qp,e,g)^* : \Sigma(E,e) \to \Sigma(G,g)$ is well defined because $qpe = gpq : E \to G$. The condition C4 is satisfied by taking $1_{\Sigma(M,f)}$ as the morphism $(i_M,f,f)^*$. The two axioms AC1 and AC2 can now be easily checked as above for the category .

3) <u>Σ is a covariant functor</u>.

We define the functor as follows $\Sigma : \mathcal{M} \to \mathcal{J}_s$.

i) Given $(E,e) \in \mathcal{M}$, then Σ gives $\Sigma(E,e) \in \mathcal{J}_s$

ii) Given $(p,e,f) \in [(E,e),F,f)]$ then $\Sigma(p,e,f) = (p,e,f)^*$.

The conditions F1, F2 hold.

F1. $\Sigma(1_E,e,e) = (1_E,e,e)^* = 1_{\Sigma(E,e)}$

F2. $\Sigma((q,f,g)(p,e,f)) = \Sigma(qp,e,g) = (qp,e,g)^*$

$\Sigma(q,f,g)\Sigma(p,e,f) = (q,f,g)^*(p,e,f)^* = (qp,e,g)^*$

§3. FLOWS ON BOUNDARIES OF MANIFOLDS

4.3.1 THEOREM Given the manifold M and diffeomorphism $f: M \to M$, let $\phi = \Sigma(M,f)$ be the suspension flow on M_f. Let $\phi : M_f \times R \to M_f$ be the induced group action on R. Then $\phi|\partial(M_f) \times R : \partial(M_f) \times R \to \partial(M_f)$ is well-defined and $\phi|\partial(M_f) \times R$ is equivalent to $\Sigma(\partial M, f_\partial)$ where $f_\partial = f|\partial M : \partial M \to \partial M$. This is proved by the following lemmas.

4.3.2 LEMMA. Given a manifold M with boundary and a diffeomorphism $f : M \to M$ then $f_\partial = f|\partial M$ is a diffeomorphism of ∂M onto ∂M.

Proof. Let the differentiable structure on M be the collection of coordinate neighbourhoods (U,h). Take a point $x \in \partial M$, then we have $f(x) \in \partial M$. To prove this, suppose not, then $f(x) \in V$ where V is some coordinate neighbourhood of $f(x)$ and by assumption $k(f(x)) \in k(V)$ where $k(V)$ is open in R^m. Let U be a coordinate neighbourhood of x, then $h(U) \subset H^m$ with $h(x) \in R^{m-1} = \partial H^m$, ($H^m$ = half m-space). We then have that the diffeomorphism kfh^{-1} takes an open neighbourhood of $h(x)$ in H^m into an open neighbourhood of $kf(x)$ in R^m with $h(x) \in \partial H^m$. This contradicts Brouwer's theorem on invariance of domain [31]. So $f(x) \in \partial M$ and we have $f|\partial M : \partial M \to f(\partial M) \subset \partial M$. In fact $f(\partial M) = \partial M$. This follows because f^{-1} is also a diffeomorphism $f^{-1} : M \to M$ and so $f^{-1}(\partial M) \subset \partial M \Rightarrow$ $\partial M = f\ (f^{-1}(\partial M)) \subset f(\partial M) \subset \partial M \Rightarrow f(\partial M) = \partial M$.

4.3.3 LEMMA. The manifolds $(\partial M)_{f_\partial}$ and $\partial(M_f)$ are diffeomorphic.

Proof. With the usual notation we have that $\partial(M_f) = (\Pi_f|\partial M \times R)(\partial M \times R)$. Consider the following diagram where $d : (\partial M)_{f_\partial} \to \partial(M_f)$ is defined by $d(\Pi_{f_\partial}(x,t)) = \Pi_f(x,t)$, $x \in \partial M$, $t \in R$.

$$\begin{array}{ccc} \partial M \times R & \xrightarrow{\ 1\ } & \partial M \times R \\ {\scriptstyle \Pi_{f_\partial}}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_f|\partial M \times R} \\ (\partial M)_{f_\partial} & \xrightarrow{\ d\ } & (M_f) \end{array}$$

It can be easily checked that the diagram commutes

1) d is well-defined. This follows because $f_\partial(x) = f(x)$, for $x \in \partial M$.

2) d is 1-1. We have
$$\begin{aligned} d^{-1}(\Pi_f(x,s)) &= d^{-1}(\Pi_f(f^n(x),s-n)) \\ &= \{\Pi_{f_\partial}(f^n(x),s-n) \mid x \in \partial M,\ n \in Z\} \\ &= \{\Pi_{f_\partial}(f_\partial^{\,n}(x),s-n) \mid x \in \partial M,\ n \in Z\} \\ &= \Pi_{f_\partial}(x,s). \end{aligned}$$

3) d is continuous.

This follows from the commutativity of the diagram in the usual way noting that Π_{f_∂} is open and $\Pi_f|\partial M \times R$ is the restriction of a continuous map, Π_f. We have that d is 1-1 and continuous. Our assumption of M being compact gives us that ∂M is compact and so M_f, ∂M_{f_∂} are compact. Hence both ∂M_f and ∂M_{f_∂} are compact. The map d is between compact spaces and so is a homeomorphism.

In fact since d is defined in terms of differentiable maps we have that d is a diffeomorphism.

4.3.4 LEMMA. The diffeomorphism d gives the equivalence of theorem 4.3.1.

Proof. The diagram commutes.

$$\begin{tikzcd}
(\partial M)_{f_\partial} \times R \arrow[r, "d \times 1"] \arrow[d, "\lambda"'] & \partial(M_f)\times R \arrow[d, "\phi|\partial(M_f)\times R"] \\
(\partial M)_{f_\partial} \arrow[r, "d"] & \partial(M_f)
\end{tikzcd}$$

where $\lambda = \Sigma(\partial M, f_\partial)$.

Remark. By taking the inclusion $i_{\partial M} : \partial M \to M$ we obtain a morphism $(i_{\partial M}, f_\partial, f) \in [(\partial(M), f_\partial), (M,f)]$. The functor Σ gives $\Sigma(\partial(M), f_\partial)$ and $\Sigma(M,f)$ and the morphism $(i_{\partial M}, f_\partial, f)^* \in [\Sigma(\partial M, f_\partial), \Sigma(M,f)]$

§4. FLOWS ON INTERIORS OF MANIFOLDS

4.4.1 THEOREM. Given the manifold M and diffeomorphism $f : M \to M$, let $\phi = \Sigma(M,f)$ be the suspension flow on M_f. Then $\phi | I(M_f) \times R : I(M_f)\times R \to I(M_f)$ is equivalent to $\Sigma(I(M), f_I)$ where $f_I = f|I(M)$.

Remark. It follows from the fact that f_∂ is a diffeomorphism of ∂M onto itself that we have

$$f_I : I(M) \to I(M) \quad \text{with} \quad f_I = f|I(M)$$

We need the following lemmas for theorem 4.4.1.

4.4.2 LEMMA. The manifolds $(I(M))_{f_I}$ and $I(M_f)$ are diffeomorphic.

Proof. We use the fact that $I(M_f) = \{\Pi_f(x,t) | x \in I(M), t \in R\}$.

Define $d_1 : I(M)_{f_I} \to I(M_f)$ by $d_1(\Pi_{f_I}(x,t)) = \Pi_f(x,t), x \in I(M), t \in R$.

The diagram commutes

$$\begin{tikzcd} I(M)\times R \arrow[r, "1"] \arrow[d, "\Pi_{f_I}"'] & I(M)\times R \arrow[d, "\Pi_f \mid I(M)\times R"] \\ I(M)_{f_I} \arrow[r, "d_1"] & I(M_f) \end{tikzcd}$$

As for d we can easily check that d_1 is well defined and 1-1. The continuity of d_1 follows in the usual way from use of the commutativity of the diagram. To prove the continuity of d_1^{-1} consider the following diagram.

$$\begin{tikzcd} I(M)\times R \arrow[r, "i\times 1"] \arrow[d, "\Pi_{f_I}"'] & M\times R \arrow[d, "\Pi_f"] \\ I(M)_{f_I} \arrow[r, "d_1"] & M_f \end{tikzcd}$$

where $i : I(M) \to M$ is the natural inclusion map (a diffeomorphism onto its image). Because $I(M)$ is open in M the inclusion map $i : I(M) \to M$ is opeh. Let 0 be an open set of $I(M)_{f_I}$. Then

$$d_1(0) = d_1\Pi_{f_I}(\Pi_{f_I}^{-1}(0)) = \Pi_f(i \times 1)\Pi_{f_I}^{-1}(0).$$

However $d_1(0) \subset I(M_f) \subset M_f$ and since $I(M_f) \subset M_f$ is an open subset it follows that $d_1(0)$ is an open set of $I(M)$.

Hence d_1 is a homeomorphism. Because d_1 is constructed with differentiable maps d_1 is in fact a diffeomorphism.

4.4.3 <u>IEMMA</u>. The diffeomorphism d_1 gives the equivalence in theorem 4.4.1.

Proof. The diagram commutes

$$\begin{array}{ccc} I(M)_{f_I} \times R & \xrightarrow{d_1 \times 1} & I(M_f) \times R \\ \downarrow \mu & & \downarrow \phi | I(M_f) \times R \\ I(M)_{f_I} & \xrightarrow{d_1} & I(M_f) \end{array}$$

where $\mu = \Sigma(I(M), f_I)$.

Remark. By taking the inclusion $i_M : I(M) \to M$ we obtain a morphism $(i_M, f_I, f) \in [(I(M), f_I), (M, f)]$. The functor Σ gives the objects $\Sigma(I(M), f_I)$ and $\Sigma(M, f)$ and the morphism $(i_M, f_I, f)^*$.

4.4.4 LEMMA. We have the relation

$$\Sigma(i_M, f_I, f) = (i_M, f_I, f)^* \Sigma \; : \; \mathcal{M} \to \mathcal{F}_s .$$

§5. FLOWS ON DOUBLE MANIFOLDS

In 4.1.3, the definition of the double of a manifold, no mention was made of its differentiable structure. This is supplied in Appendix 4 with the notation which we use here.

4.5.1 Definition. Given a diffeomorphism $f : M \to M$ we define a homeomorphism $f_D : D(M) \to D(M)$ as follows

$$f_D \, i_0(x,0) = i_0(f(x),0), \; (x,0) \in M_0$$

$$f_D \, i_1(x,1) = i_1(f(x),1), \; (x,1) \in M_1$$

Remark. From consideration of the differentiable structure on $D(M)$ we cannot deduce in general that $f_D : D(M) \to D(M)$ is a diffeomorphism. To by-pass this problem we must restrict our attention to the sub-category $\mathcal{N} \subseteq \mathcal{M}$ where $\mathcal{N} = \{(M,f) | (D(M), f_D) \in \mathcal{M}\}$.

4.4.2 THEOREM. The flow $\Sigma(D(M),f_D)$ on $D(M)_{f_D}$ is topologically equivalent to the natural flow on $D(M_f)$ induced by the flow $\Sigma(M,f)$ on M_f if the flows are defined.

Remark. The natural flow ψ on $D(M_f)$ induced by $\Sigma(M,f)$ on M_f is obtained by using the natural C^r embeddings $j_0 : (M_f)_0 \to D(M_f)$ and $j_1 : (M_f)_1 \to D(M_f)$. Let ψ be the flow on $D(M_f)$ such that the diagram commutes

$$\begin{tikzcd}
(M_f)_i \times R \arrow[r, "j_i \times 1"] \arrow[d, "\phi"'] & j_i((M_f)_i) \times R \arrow[d, "\psi"] \\
(M_f)_i \arrow[r, "j_i"] & j_i((M_f)_i)
\end{tikzcd}$$

for $i = 0,1$.. The flow $\phi = \Sigma(M,f)$. The supposition of 4.4.2 is that the flow ψ defined in this way is differentiable.

4.4.3 LEMMA. There is a homeomorphism $h : D(M_f) \to D(M)_{f_D}$.

Proof. Let π, σ be the usual projections $\pi : M \times R \to M_f$ and $\sigma : D(M) \times R \to D(M)_{f_D}$.

Define the maps h_0, h_1 as follows

$$h_0 : j_0(M_f)_0 \to D(M)_{f_D}$$

$$h_1 : j_1(M_f)_1 \to D(M)_{f_D}$$

by $h_0(j_0(\pi(x,s),0)) = \sigma(i_0(x,0),s)$ and $h_1(j_1(\pi(x,s),1) = \sigma(i_1(x,1),s)$. In fact h_0 and h_1 are homeomorphisms onto their images and this is proved now.

The maps h_0, h_1 are well defined. This is easily checked as follows. For h_0 we have

$$h_0(j_0(\pi(x,s),0)) = h_0(j_0(\pi(f^n(x),s-n),0) = \sigma(i_0(f^n(x),0),s-n)$$
$$= \sigma(f_D^n i_0(x,0),s-n) = \sigma(i_0(x,0),s), \quad \forall\, n \in Z.$$

We will prove that h_0 is a homeomorphism (h_1 follows by direct analogy).

Consider the following diagram

$$\begin{tikzcd}
M \times R \times \{0\} \arrow[r, "h_0'"] \arrow[dd, "j_0\pi"'] & M \times \{0\} \times R \arrow[d, "i_0 \times 1"] \\
 & i_0(M_0) \times R \arrow[d, "\sigma_0"] \\
j_0(M_f)_0 \arrow[r, "h_0"] & i_0(M_0)_{f_{D'}}
\end{tikzcd}$$

where $f_{D'} = f_D | i_0(M_0)$ and $\sigma_0 = \sigma | i_0(M_0)$. The diffeomorphism h_0' is the map $h_0'(x,t,0) = (x,0,t)$. The bicontinuity of h_0 follows in the usual way from the commutative diagram. All the maps in the diagram are differentiable and so we get h_0 is a diffeomorphism. The fact of h_1 being a diffeomorphism follows in a similar way.

Define the map $h : D(M_f) \to D(M)_{f_D}$ by

1) $h | j_0(M_f)_0 = h_0$

2) $h | j_1(M_f)_1 = h_1$

From the definitions $h_0 | (j_0(M_f)_0 \cap j_1(M_f)_1) = h_1 | (j_0(M_f)_0 \cap j_1(M_f)_1)$. Using the lemma of Appendix 4 h is a homeomorphism.

4.4.4 LEMMA. The homeomorphism h gives the topological equivalence of the two flows of 4.4.2.

Proof. The diagrams commute.

$$\begin{tikzcd}
D(M_f) \times R \arrow[r, "h \times 1"] \arrow[d, "\psi"'] & D(M)_{f_D} \times R \arrow[d, "\Sigma(D(M), f_D)"] \\
D(M_f) \arrow[r, "h"] & D(M)_{f_D}
\end{tikzcd}$$

where ψ is the induced flow on $D(M_f)$ by $\Sigma(M,f)$ on M_f.

§6. ALGEBRAIC PROPERTIES OF Σ_j, $j \in Z$

In a similar manner to those definitions of §2 we have a functorial representation of Σ_j between categories.

4.5.1 Definition. Let $\mathcal{M}_j$ be the category consisting of objects $(M,f_1,\ldots,f_j)$ where $f_1,\ldots,f_j$ are commutative diffeomorphisms of M. The morphisms $[(M,\hat{f}),(N,\hat{g})]$ where $\hat{f}$ represents $(f_1,\ldots,f_j)$ and $\hat{g}$ represents $(g_1,\ldots,g_j)$ are triples, $(k,\hat{f},\hat{g})$ such that k is a continuous map $k : M \to N$ such that $kf_i = g_i k \quad \forall\, i \in \{1,\ldots,j\}$.

4.5.2 Definition. Let $\mathcal{F}_{s_j}$ be the category consisting of objects $\Sigma_j(M,f)$ and morphisms $(k,\hat{f},\hat{g})^* \in [\Sigma(M,\hat{f}), \Sigma(N,\hat{g})]$ such that $k : M \to N$ is a continuous map where $kf_i = g_i k : M \to N$.

Remark. This condition defines a map $k(\hat{f},\hat{g}) : M_{\hat{f}} \to N_{\hat{g}}$ by $k\pi_{\hat{f}}(x,s) = \pi_{\hat{g}}(k(x),s)$.

4.5.3 LEMMA. Σ_j is a functor $\Sigma_j : \mathcal{M}_j \to \mathcal{F}_{s_j}$.

Construction. The functorial relations are

1) $(M,\hat{f}) \xrightarrow{\Sigma_j} \Sigma_j(M,\hat{f})$

2) $(k,\hat{f},\hat{g}) \xrightarrow{\Sigma_j} (k,\hat{f},\hat{g})^*$

CHAPTER 5

CANTOR-TYPE DIFFEOMORPHISMS

§0. INTRODUCTION

In this chapter we discuss the cohomological analysis of some diffeomorphisms which have non-wandering sets containing Cantor sets. The main example which is described in detail is the diffeomorphism originally derived from the study of recurrence in the forced Van der Pol equation. Also described briefly is the 'n horse-shoe' referred to in Chapter 1, which is crucial in the analysis of generic systems on n-manifolds.

The cohomologies of the insets of the diffeomorphisms given are investigated using Čech Cohomology theory. We do this by giving a cell-decomposition of insets of an Axiom A diffeomorphism based on [21] which is useful for the application of this cohomology theory.

§1. THE 'HORSE-SHOE DIFFEOMORPHISM" [26]

The 'horse-shoe' diffeomorphism is given by the map $f : Q \to R^2$ where $Q = \{(x,y), |x| \leqslant 1, |y| \leqslant 1\}$ and R^2 is the two dimensional real plane. It is usually extended to a diffeomorphism of S^2, the two sphere.

The diffeomorphism can be described geometrically as follows.

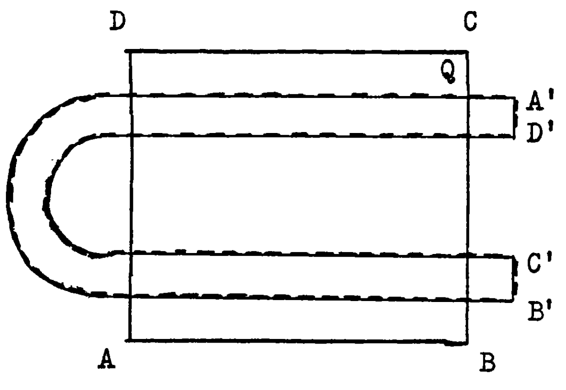


Fig. 5.1

Let f take Q onto the dotted image with $A \to A'$, $B \to B'$, $C \to C'$ and $D \to D'$. Also we require that f is linear on $f^{-1}(f(Q) \cap Q) = P_0 \cup P_1$ where P_0, P_1 are shown in Fig. 5.2; it follows that $f(P_0) = Q_0$ and $f(P_1) = Q_1$ where $Q_0 \cup Q_1 = f(Q) \cap Q$.

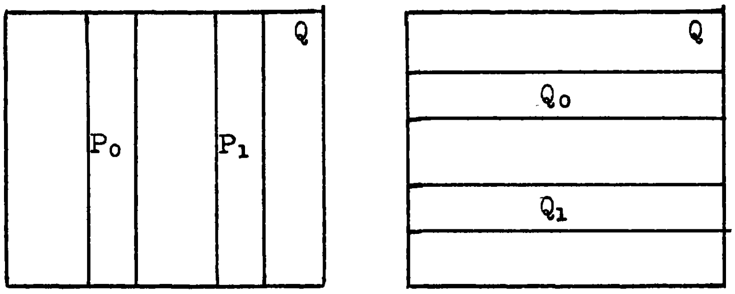


Fig. 5.2.

We now define a system of subsets of Q.

5.1.1 Definition. Let $Q^{(1)} = f(Q) \cap Q$ $(= Q_0 \cup Q_1)$. From this define inductively $Q^{(n)} = f(Q^{(n-1)}) \cap Q$. Let $Q^{(0)} = Q$ and $Q^{(-1)} = f^{-1}(Q) \cap Q$ $(= P_0 \cup P_1)$ and inductively $Q^{(-n)} = f^{-1}(Q^{-(n-1)}) \cap Q$.

Define $\Lambda = \bigcap_{i \in Z} Q^{(i)}$, then Λ is the non-wandering set of the diffeomorphism $f : Q \to R^2$ restricted to Q.

5.1.2 LEMMA. Λ is homeomorphic to a Cantor set.

Proof. From the linearity conditions on f we have $Q^{(0)} \supset Q^{(1)} \supset \dots \supset Q^{(n)} \supset \dots$, and $Q^{(n)}$ consists of 2^n horizontal segments. By taking the intersections of all these sets this gives the classic construction of $I \times C_2$ where $I \times C_2 = \{(x,y) \,|\, |x| \leqslant 1,\ y \in C_2\}$ where C_2 is the usually constructed Cantor set. Similarly we have $Q^{(0)} \supset Q^{(1)} \supset \dots \supset Q^{(-n)} \supset \dots$, and this gives us on taking intersections, $C_1 \times I = \{(x,y) \,|\, x \in C_1, |y| \leqslant 1\}$. So we have $\Lambda = (I \times C_2) \cap (C_1 \times I) = C_1 \times C_2$, a product of Cantor sets which is itself a Cantor set.

5.1.3 Definition. Let $I_Q(\Lambda)$ be the inset of Λ on Q of the diffeomorphism f and let $O_Q(\Lambda)$ be the outset of Λ on Q of f. ($O_Q(\Lambda)$ being the inset of Λ on Q of f^{-1}).

5.1.4 LEMMA. $I_Q(\Lambda) = \{(x,y) \in Q \mid x \in C_1\}$ and $O_Q(\Lambda) = \{(x,y) \in Q \mid y \in C_2\}$.

Proof. First of all we prove that $\{(x,y) \in Q \mid x \in C_1\} \subset I_Q(\Lambda)$.
Consider $(x,y) \in (C_1 \times I) \cap Q = \bigcap_{n \geqslant 0} Q^{(-n)}$. Then $(x,y) \in Q^{(-n)}$, $\forall\, n \geqslant 0$. But we have $Q^{(-n)} \subset Q^{-(n-1)} \subset \ldots \subset Q^{(0)}$ so that $f(x,y) \in f(Q^{(-n)}) \subset \ldots \subset f(Q^{(0)})$. Also $(x,y) \in Q^{(-1)}$ and so $f(x,y) \in f(Q^{(-1)}) = Q^{(1)}$; similarly it can be shown $f^2(x,y) \in Q^{(2)}$, $f^3(x,y) \in Q^{(3)}$ and so on. Since $\bigcap_{n \geqslant 0} Q^{(n)} = I \times C_2$ is closed we have $\lim_{m \to \infty} f^m(x,y) \subseteq \bigcap_{n \geqslant 0} Q^{(n)}$,

($\lim_{m \to \infty} f^m(x,y)$ is not necessarily a single point)

Because $(x,y) \in Q^{(-n)}$, $\forall\, n \geqslant 0$ it follows that $f(x,y) \in f(Q^{(-n)}) = f(f^{-1}(Q^{-(n-1)}) \cap Q) \subset f(f^{-1}(Q^{-(n-1)}) \cap f(Q) \subset Q^{-(n-1)} \cap f(Q) \subset Q^{-(n-1)}$.
From this it follows $f^m(x,y) \in \bigcap_{n \geqslant 0} Q^{(-n)}$ if $(x,y) \in \bigcap_{n \geqslant 0} Q^{(-n)}$ for $m \geqslant 0$.
The set $\bigcap_{n \geqslant 0} Q^{(-n)} (= I \times C_2)$ is closed and hence $\lim_{m \to \infty} f^m(x,y) \in \bigcap_{n \geqslant 0} Q^{(-n)}$.
Combining the results we have $\lim_{m \to \infty} f^m(x,y) \in \left[\bigcap_{n \geqslant 0} Q^{(-n)}\right] \cap \left[\bigcap_{n \geqslant 0} Q^{(n)}\right] = \Lambda$.
Hence $\{(x,y) \in Q \mid x \in C_1\} \subset I_Q(\Lambda)$.

Now consider the set $T = \{(x,y) \in Q \mid x \notin C_1\}$. Let $(x,y) \in T$. Then $(x,y) \notin Q^{(-n_0)} \Rightarrow f^{n_0}(x,y) \notin Q^{(0)}$. This means that all points of T are eventually expelled from Q under the diffeomorphism f. Once a point is expelled from Q in the global extension of f it never returns to Q and so

$$I_Q(\Lambda) = \{(x,y) \in Q \mid x \in C_1\}$$

$$O_Q(\Lambda) = \{(x,y) \in Q \mid y \in C_2\}.$$

5.1.5 <u>Definition</u>. Let S be a finite set with discrete topology and define X_S to be the set of functions from Z to S provided with the compact-open topology, (Z has the discrete topology also). If $a \in X_S$ the value of a at $m \in Z$ will be denoted by a_m. Thus a may be thought of as a doubly infinite sequence of elements of S, i.e. $a = (\ldots a_{-1}a_0a_1\ldots)$. Define $\alpha : X_S \to X_S$ by $(\alpha(a))_m = a_{m-1}$. The map α is a homeomorphism called the shift automorphism of X_S. The pair (X_S,α) is called a <u>shift of finite type</u>.

5.1.6 <u>LEMMA</u>.[26]. On Λ f is topologically conjugate to a shift automorphism.

<u>Proof</u>. We need to produce a homeomorphism $h : X_S \to \Lambda$ such that the diagram commutes

$$\begin{tikzcd} X_S \arrow[r, "\alpha"] \arrow[d, "h"'] & X_S \arrow[d, "h"] \\ \Lambda \arrow[r, "f"] & \Lambda \end{tikzcd}$$

Given a point $z \in \Lambda$ then it is captured by two infinite sets of inclusions. Let the components of $Q^{(1)}$ be Q_{11}, Q_{12} and $Q^{(2)}$ be $Q_{21}, Q_{22}, \ldots, Q_{24}$, and in general $Q^{(n)}$ be $Q_{n1}, \ldots, Q_{n2^n}$. Similarly let $Q^{(-1)}$ be $P_{11}, P_{12}, \ldots,$ and $Q^{(-n)}$ be $P_{n1}, \ldots, P_{n2^n}$. Then z is contained in the intersection of a bi-infinite sequence $\ldots, Q_{n i_n}, \ldots, Q_{1 i_1}, P_{1 j_1}, \ldots, P_{n j_n}, \ldots$. We then note that $Q_{n i_n}$ is equal to $f^n(Q_0) \cap Q$ or $f^n(Q_1) \cap Q$ and $P_{n j_n}$ is equal to $f^{-n}(Q_0) \cap Q$ or $f^{-n}(Q_1) \cap Q$. Define $h : X_S \to \Lambda$ where $S = \{0,1\}$

$$h(a) = \bigcap_{n=-\infty}^{+\infty} f^n(Q_{a_n}).$$

Then $h(a)$ certainly represents a point of Λ. From the very construction and uniqueness of the bi-infinite sequence of inclusions we have h is 1-1. Basic open sets of X_S containing $a \in X_S = \{b \mid a_i = b_i \mid i| \leqslant n_0\}$ for any n_0.

$$h\{b \mid a_i = b_i \mid i| \leqslant n_0\} = \left\{\bigcap_{n=-\infty}^{+\infty} f^n(Q_{b_n}) \mid a_i = b_i \mid i| \leqslant n_0\right\}$$

$$= \Lambda \cap f^{-n_0}(Q_{a_{-n_0}}) \cap f^{n_0}(Q_{a_{n_0}})$$

$$= \text{open set of } \Lambda$$

It follows that h is an open map.

The basic open sets of Λ are of the form $\Lambda \cap f^{-n_0}(Q_{i_1}) \cap f^{n_0}(Q_{i_2})$ for $i_1, i_2 \in \{0,1\}$. Given $\Lambda \cap f^{-n_0}(Q_{i_1}) \cap f^{n_0}(Q_{i_2})$.

$$h^{-1}(\Lambda \cap f^{-n_0}(Q_{i_1}) \cap f^{n_0}(Q_{i_2}))$$

$$= \{b \mid f^{-n_0}(Q_{i_1}) \cap f^{n_0}(Q_{i_2}) \cap f^m(Q_{b_m}) \neq \Phi,\ -n_0 \leqslant m \leqslant n_0\}$$

$$= \{b \mid b_i = a_i, -n_0 \leqslant i \leqslant n_0,\ f^i(Q_{a_i}) \cap f^{-n_0}(Q_{i_1}) \cap f^{n_0}(Q_{i_2}) \neq \Phi\}.$$

Hence h is a homeomorphism.

We have, given $a \in X_S$

$$h\alpha(a) = h\alpha(a) = \bigcap_{n=-\infty}^{+\infty} f^n(Q_{(\alpha(a))_n})$$

$$= \bigcap_{n=-\infty}^{+\infty} f^n(Q_{a_{n-1}})$$

$$= f \bigcap_{n=-\infty}^{+\infty} f^{n-1}(Q_{a_{n-1}})$$

$$= fh(a).$$

This proves the conjugacy.

We now extend the diffeomorphism $f : Q \to R^2$ to a global diffeomorphism of the two-sphere S^2[26]. This is done by extending f to a map $f_0 : D^2 \to D^2$ by mapping G diffeomorphically onto G' and F diffeomorphically onto F' as in Fig. 5.3. The map is defined so that it contracts F onto F'

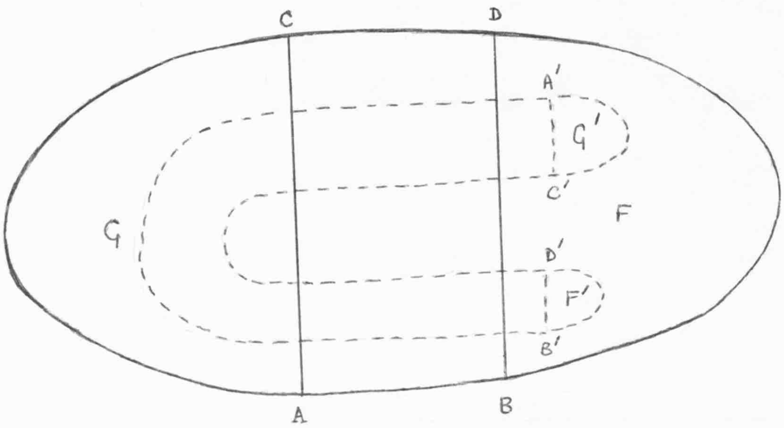


Fig. 5.3

about some fixed point p_0 in F'. Thus f_0 will be a diffeomorphism of D^2 onto a subset of D^2 so that the non-wandering set is the disjoint union of Λ and p_0. Finally f_0 is extended to $g : S^2 \to S^2$ so that $\Omega(g) = \Lambda \cup p_0 \cup q_0$ where q_0 is an expanding point outside of D^2. Other diffeomorphisms can be constructed with similar properties to the diffeomorphism already discussed. We take images of Q under similar linearity conditions to obtain the following diagrams.

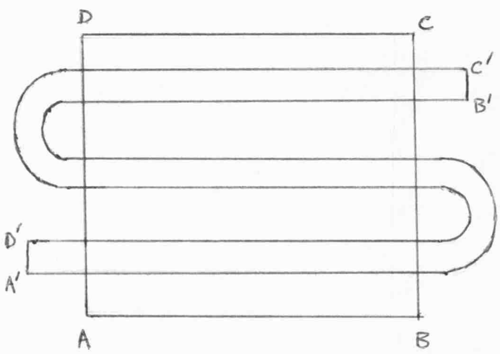


Fig. 5.4

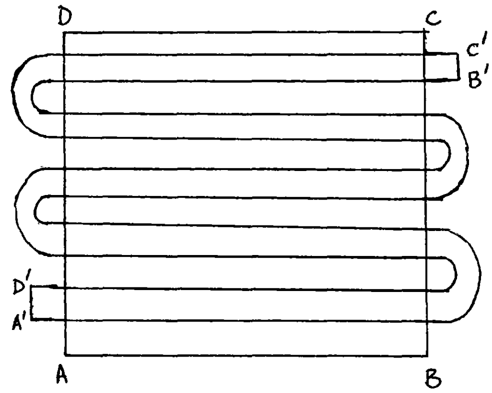


Fig. 5.5

Locally the diffeomorphisms are conjugate to shift automorphisms on 3 and 5 symbols respectively.

§2. A CELL-DECOMPOSITION OF A SPACE

Let X be a compact space and suppose there exists a finite class of subspaces of X, $\{X_m | m \in (1,\dots,n_0)\}$ such that $X = \bigcup_{m=1}^{n_0} X_m$. We can define a sequence of subspaces as follows.

Let $Y_0 = \cup\{X_m | \partial X_m \subseteq X_m\}$. Suppose $Y_0,\dots,Y_n$ are already defined, then define inductively

$$Y_{n+1} = \cup\{X_m | \partial X_m \subset Y_n\}.$$

If such a decomposition of X exists i.e. $Y_n = X$ for some n then we have the following lemma.

5.2.1 LEMMA. Y_n is closed in X and Y_n is closed in Y_{n+1}.

Proof. Let $X_m \in Y_0$; then $X_m \subseteq C\ell(X_m) = X_m \cup \partial X_m = X_m$. Hence $X_m \in Y_0 \Rightarrow X_m$ is closed: So Y_0 is a finite union of closed sets $\Rightarrow Y_0$ is closed. We now proceed by induction. Suppose Y_n is closed; then $Y_{n+1} = \cup\{X_m | \partial X_m \subset Y_n\}$ and so $\partial Y_{n+1} \subset \cup\{\partial X_m | \partial X_m \subset Y_n\} \subset Y_n \subset Y_{n+1}$. Hence Y_{n+1} is closed. The subspace Y_n is closed in Y_{n+1} follows immediately from Y_n being closed in X.

The type of cohomology used will be Compact Čech Theory [34]. By using the exactness axiom we obtain

5.2.2 <u>LEMMA</u>. The following cohomology triangles are exact where H^* is the cohomology functor.

$$\begin{tikzcd} & H^*X \arrow[dl, "j^*"'] & \\ H^*Y_n \arrow[rr, "\delta"] & & H^*(X-Y_n) \arrow[ul, "\tau"'] \end{tikzcd}$$

$$\begin{tikzcd} & H^*Y_{n+1} \arrow[dl, "j_1^*"'] & \\ H^*Y_n \arrow[rr, "\delta_1"] & & H^*(Y_{n+1}-Y_n) \arrow[ul, "\tau_1"'] \end{tikzcd}$$

<u>Proof</u>. Follows immediately from applying 5.2.1 to the axioms of H^*

<u>Remark</u>. The homomorphism j^* is induced from the inclusion map and δ is the cohoundary homomorphism.

We now wish to find a decomposition of a manifold into insets of diffeomorphisms so that we can make use of 5.2.2. Suppose we have an Axiom A diffeomorphism f on a compact manifold M. Then we can use the Spectral Decomposition Theorem of Smale (see Chapter 1).

5.2.3 LEMMA. Let $f \in \mathrm{Diff}(M)$ satisfy Axiom A and suppose

(1) The decomposition of $\Omega(f) = \{\Omega_1, \dots, \Omega_k\}$;

(2) f satisfies the no cycle property

i.e. $\nexists$ a sequence $\Omega_{i_1}, \dots, \Omega_{i_n}$ such that

$$I(\Omega_{i_j}) \cap O(\Omega_{i_{j+1}}) \neq \Phi \quad 1 \leqslant j \leqslant n-1 \quad \text{and} \quad i_1 = i_n \ ;$$

(3) If $\partial I(\Omega_\alpha) \cap I(\Omega_\beta) \neq \Phi, \dots,$ then $\exists$ $\alpha = i_1, \dots, i_m = \beta$ such that $O(\Omega_\alpha) \cap I(\Omega_{i_2}) \neq \Phi, \dots, O(\Omega_{i_j}) \cap I(\Omega_{i_{j+1}}) \neq \Phi, \dots,$ and

$$O(\Omega_{i_{m-1}}) \cap I(\Omega_\beta) \neq \Phi \ .$$

Let $Y_0 = \bigcup_\iota \{I(\Omega_\iota) = \Omega_\iota\}$ and inductively

$$Y_{j+1} = \bigcup_\iota \{I(\Omega_\iota) | \partial I(\Omega_\iota) \subset Y_j\}.$$

Then $Y_0 \subset Y_1 \subset \dots \subset Y_n = M$ for some n.

Proof. Let us show that $Y_0 \neq \Phi$. Let $\alpha(p)$ and $w(p)$ denote the past and future limit sets of the orbit of p by the diffeomorphism f.

Consider first of all Ω_1 ; If $I(\Omega_1) \neq \Omega_1$ then $\exists$ $p_1 \in M - \Omega_1$ such that $w(p_1) \subset \Omega_1$. Let $\alpha(p_1) \subset \Omega_j$; $j \neq 1$ by condition (2). So assume $j = 2$. Then by exactly similar arguments either $I(\Omega_2) = \Omega_2$ or $\exists$ $p_2 \in M - \Omega_2$ such that $w(p_2) \subset \Omega_2$, Let us suppose the latter, then by condition 2 again $\alpha(p_2) \subset \Omega_j$ where $j \neq 1$ or 2. Assume $\alpha(p_1) \subset \Omega_3$. Continuing in this way we must arrive at Ω_ρ such that $\Omega_\rho = I(\Omega_\rho)$, otherwise we will violate the no cycle property. So $Y_0 \neq \Phi$

We also need to show that there exists $Y_n = M$ for some n. Since there are only a finite number of Ω_ι's we need to show that if $Y_\iota \neq M$, then $Y_{\iota+1} - Y_\iota \neq \Phi$.

Let $\Omega_{i_1}, \Omega_{i_2}, \dots, \Omega_{i_\rho}$ be the Ω_ι's in $M - Y_\iota$. We can assume every orbit entering Ω_{i_1} originates in Y_ι, otherwise by an argument similar to the one above we could violate the no-cycle property in $\{\Omega_{i_1}, \dots, \Omega_{i_\rho}\}$.

It follows $\partial I(\Omega_{i_1}) \subset Y_\ell$. Suppose not, then $\partial I(\Omega_{i_1}) \cap \Omega_j \neq \Phi$ for some $\Omega_j \in \{\Omega_{i_1},\ldots,\Omega_{i_p}\}$. But by condition (3) $\exists$ a sequence of orbits from Ω_{i_1} to Ω_j. So there is (using no cycle property) some m such that $\Omega_m \subset M - Y_\ell$ such that $\exists x \in Y_\ell$ with $\alpha(x) \in \Omega_m$. But if $x \in Y_\ell$ then $x \in I(\Omega_{i_q})$ say where $I(\Omega_{i_q}) \subset Y_\ell$. Y_ℓ is invariant under f and Y_ℓ is closed so $\partial I(\Omega_{i_q}) \subset Y_\ell$ but $\partial I(\Omega_{i_q}) \cap \Omega_m \neq \Phi$ since $\alpha(x) \subset \Omega_m$ and $w(x) \subset \Omega_{i_q}$ so $\Omega_m \cap Y_\ell \neq \Phi \Rightarrow \Omega_m \subset Y_\ell$ which gives the contradiction. So no orbit enters Y_ℓ and therefore $\partial I(\Omega_{i_1}) \subset Y_\ell$ and $\Omega_{i_1} \not\subset Y_\ell$.

§3. THE COHOMOLOGY GROUPS OF THE INSETS OF THE 'HORSE-SHOE" DIFFEOMORPHISM

The horse-shoe diffeomorphism $g : S^2 \to S^2$ has the non-wandering set $\Omega(g) = \Lambda \cap p_0 \cup q_0$ where Λ = Cantor set, p_0 = sink and q_0 = source. Because q_0 is a source $I(q_0) = q_0$. The inset of $p_0, I(p_0)$ will be an open disc because locally we have an open disc as inset.

Using the cell decomposition we have

$$Y_0 = \{I(q_0)\},\ Y_1 = \{I(q_0) \cup I(\Lambda)\},\ Y_2 = \{I(q_0) \cup I(\Lambda) \cup I(p_0)\} = S^2.$$

5.3.1 LEMMA. $H^*(I(\Lambda))$ is cohomologically trivial.

This results from the following considerations. We take the compact pair (S^2, Y_1) noting $S^2 - Y_1 = I(p_0)$, an open disc. We also note that Y_1 is 1-dimensional from 5.1.4 and so $H^*(Y_1)$ will be $\{A,B,0,0,\ldots\}$ for some groups A,B. $H^*(S^2) = \{Z,0,Z,0,0,\ldots\}$ and $H^*(S^2 - Y_1) = \{0,0,Z,0,0,\ldots\}$ if we take the coefficient group to be the integers, Z. Using 5.2.2 we have the following exact sequence.

$$0 \to 0 \to A \to Z \to 0 \to 0 \to B \to Z \to Z \to 0 \to \text{, etc.}$$

By consideration of exactness we have $A = Z$ and $B = 0$. We now use the exactness of the pair (Y_1, Y_0) with $H^*Y_0 = \{Z,0,0,\ldots\}$. If we take $H^*(I(\Lambda)) = H^*(Y_1 - Y_0) = \{A',B',0,\ldots\}$ we obtain the exact sequence

$$0 \to A' \to Z \xrightarrow{j^*} Z \to B' \to 0 \to 0 \to \text{, etc.}$$

Two possible solutions are

$$H^*(I(\Lambda)) = \{Z,Z,0,\ldots\} \quad \text{or} \quad H^*(I(\Lambda)) = \underline{0} \; .$$

The following lemma gives 5.3.1.

5.3.2 LEMMA. The inclusion map $j : Y_0 \to Y_1$ is such that $j^* : H^*(Y_1) \to H^*(Y_0)$ is onto.

Proof. Let $p : Y_1 \to Y_0$ be the projection map. Then $pj = 1 : Y_0 \to Y_0$. Since the cohomology is functorial $(pj)^* = 1 \Rightarrow j^*p^* = 1$ and so j^* is onto.

The requirement j^* is onto gives:

$$H^*(I(\Lambda)) = \underline{0} \; .$$

§4. THE COHOMOLOGY GROUPS OF THE INSETS OF DIFFEOMORPHISMS RELATED TO THE 'HORSE-SHOE'

Consider the diffeomorphism $h : Q \to R^2$ represented by Fig. 5.4 with linearity conditions on $h^{-1}(h(Q) \cap Q)$. Then arguments similar to those of §3 give us an extension to $h_0 : D^2 \to D^2$ as in Fig. 5.6.

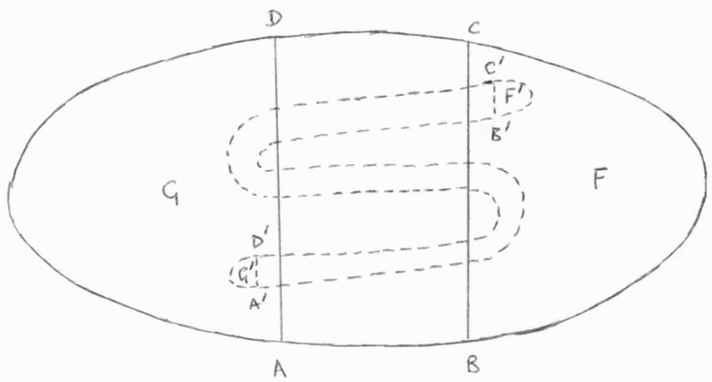


Fig. 5.6

The diffeomorphism h_0 takes G onto G' and F onto F'. The maps are defined to be contractions of F and G onto F' and G' respectively. The contractions are defined about fixed points $p_0 \in F$ and $p_1 \in G$. The diffeomorphism h_0 is extended to a diffeomorphism $k : S^2 \to S^2$ by adding an expanding fixed point outside of D^2. The non-wandering set

$\Omega(k) = \Lambda' \cup p_0 \cup p_1 \cup q_0$ where Λ' is a Cantor set, p_0 and p_1 are point sinks and q_0 is a point source.

The Cantor set Λ' is such that $k : \Lambda' \to \Lambda'$ is topologically conjugate to a shift automorphism $\alpha : X_S \to X_S$ where S is a finite set of 3 symbols. This is proved by a similar analysis to that of §1 .

5.4.1 LEMMA. $H^*(I(\Lambda')) = \{0,Z,0,0,\ldots\}$.

Proof. The cell decomposition is as follows:

$$Y_0 = \{I(q_0) = q_0\};\quad Y_1 = \{I(q_0) \cup I(\Lambda')\};\quad Y_2 = S^2.$$

Now $Y_2 = I(q_0) \cup I(\Lambda') \cup I(p_0) \cup I(p_1)$ and so if we take the pair (S^2,Y_1) then $S^2 - Y_1 = I(p_0) \cup I(p_1)$. Hence $H^*(S^2 - Y_1) = \{0,0,Z \oplus Z,0,\ldots\}$. Also $H^*(S^2) = \{Z,0,Z,0,0,\ldots\}$.

We use the exactness of the cohomology sequence for the pair (S^2,Y_1) to get $H^*(Y_1) = \{Z,Z,0,0,\ldots\}$. Considering the exactness of the pair (Y_0,Y_1) and using the fact that $j^* : H^*(Y_1) \to H^*(Y_0)$ is onto we get $H^*(I(\Lambda'))$ as required.

If we extend the diffeomorphism represented in Fig.5.5 to a diffeomorphism of the sphere in a way analogous to that for h above, then we shall obtain a non-wandering set $\Lambda'' \cup p_0 \cup p_1 \cup q_0$ of a Cantor set, two sinks and a source.

5.4.2 LEMMA. $H^*(I(\Lambda'')) = \{0,Z,0,0,\ldots\}$.

An immediate extension of the horse-shoe diffeomorphism in 2-dimensions was given in [25] for n-dimensions. This is embeddable in a sphere S^n. However the methods required to analyse the cohomology groups of the insets are completely analogous to those of §3 and nothing new is required in the analysis.

§5. THE SPINNING DIFFEOMORPHISM

In [26] a general contruction is given for a diffeomorphism whose non-wandering set is locally the product of a Cantor Set and a manifold conjectured as generically the most complicated non-wandering set that exists. The aim of this section is to show that we can still use the techniques of Čech Theory to give the cohomology groups of the insets of this non-wandering set.

We will consider the particular construction obtained as an illustration of the general construction of [26].

Let the manifold under consideration be S^1 and let $f : S^1 \to S^1$ be the expanding endomorphism given complex analytically by $z \longrightarrow z^2$. Embed S^1 in $D^2 \times S^1$ as $(0) \times S^1$. Let λ be such that $0 < \lambda < 1$ and $g_\lambda : D^2 \times S^1 \to D^2 \times S^1$ be defined as $g_\lambda(x,y) = (\lambda x, y)$. Next let $\phi : (0) \times S^1 \to D^2 \times S^1$ be a C^1 approximation of the map $0 \times S^1 \to D^2 \times S^1$, $(0,y) \longrightarrow (0,f(y))$ such that ϕ is an embedding. Let T be a tubular neighbourhood of $\phi(S^1)$ with fibres being the various components of $T \cap (D^2 \times y)$; $y \in S^1$. Now extend ϕ to $\psi : D^2 \times S^1 \to T$ in a fibre preserving way so that ψ is a diffeomorphism. Consider $h = \psi g_\lambda$: $D^2 \times S^1 \to D^2 \times S^1$. If λ is chosen sufficiently small so that there are no self intersections of T, then

$$\Lambda = \bigcap_{m > 0} h^m(D^2 \times S^1) \text{ has hyperbolic structure.}$$

Locally Λ is the product of a Cantor set and an open interval. The Figure 5.7 shows the solid torus $T = S^1 \times D^2$ and its image (which is dotted) under h.

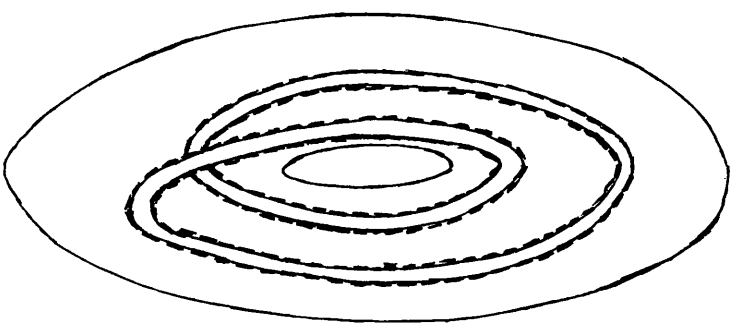


Fig.5.7

The solid torus T goes into itself as another torus T' winding twice around the original torus. This statement can be made more precise using cohomology theory. Before we analyse the cohomology of the non-wandering set, let us extend this to a diffeomorphism f of S^3.

The construction is given in [33]. Let S^3 be the join of two smooth 1-spheres S_A and S_B. Then S_A and S_B have tubular neighbourhoods A and B which are chosen so that $A \cup B = S^3$ and $A \cap B = T^2$, a smooth torus. Let S_1 be a smooth unknotted 1-sphere passing twice around B_1 in its interior and let S_{-1} be situated in A, just as S_1 is in B.

Let $T_1 \subset \mathrm{Int}(B)$ be the boundary of a tubular neighbourhood (S_{-1}) of S_{-1}. Then the linking numbers are $\ell(S_{-1},S_B) = 2 = \ell(S_A,S_1)$. Hence there is a diffeomorphism $f : S^3 \to S^3$ taking S_{-1} to S_A and S_B to S_1. We take f to send $\nu^{\ell}(S_{-1})$ to A and B to $\nu^{\ell}(S_1)$.

We then have the non-wandering set:

$$\Omega = \Omega(f) = \Lambda_+ \cup \Lambda_- \text{ where } \Lambda_+ = \left\{\lim_{n\to+\infty} f^n(x),\ x \in S^3 - \Omega\right\}$$

$$\text{and } \Lambda_- = \left\{\lim_{n\to-\infty} f^n(x),\ x \in S^3 - \Omega\right\}$$

Since under this diffeomorphism $\{x | x \in S^3 - \Omega\}$ are eventually swept into either B by f or A by f^{-1} we have:

$$\Lambda_+ = \bigcap_{n \geqslant 0} \{f^n(x) | x \in B\} \quad \text{and} \quad \Lambda_- = \bigcap_{n\geqslant 0} \{f^{-n}(x) | x \in A\} .$$

From the construction since $f|B$ and $f^{-1}|A$ are essentially the same, i.e. both map the solid tori, B,A into their interiors winding them twice around the original tori in the same way we have that Λ_+ and Λ_- are homeomorphic to each other.

Let us now return to the diffeomorphism $h : D^2 \times S^1 \to D^2 \times S^1$. Again denote $D^2 \times S^1$ by T. Our aim is to find the cohomology groups of $\bigcap_{n=0}^{\infty} h^n(T)$ and the following lemmas use the extensive notation of directed sets and their associated limiting systems. The main elements of this theory [8] which are used here are given in Appendix 5.

5.5.1 LEMMA. [8] If (X,π) is an inverse system over the directed set M and for each relation $\alpha < \beta$ in M we have π_α^β is a 1-1 map of X_β into (onto) X_α, then for each $\alpha \in M$, π_α is a 1-1 map of X_∞ into (onto) X_α.

5.5.2 LEMMA. [8] If (X,π) is an inverse system of compact spaces over M, then

$$\pi_\alpha(X_\infty) = \bigcap_{\alpha<\beta} \pi_\alpha^\beta(X_\beta)$$

5.5.3 LEMMA. [8] The inverse limit of the following inverse system is homeomorphic to $\bigcap_{n=0}^{\infty} h^n(T)$ ($=\Lambda$, the non-wandering set of h).

$$T_0 \xleftarrow{h} T_1 \xleftarrow{h} T_2 \xleftarrow{h} \cdots \xleftarrow{h} T_n \xleftarrow{h} T_{n+1} \xleftarrow{h} \cdots$$

where $T_i = T \;\forall i \in Z_+$.

Proof. From 5.5.1 and 5.5.2 we have

$\pi_0(T_\infty) = \bigcap_{n=0}^{\infty} h^n(T)$, where T_∞ is the inverse limit of the inverse system. The map π_0 is 1-1 and into by 5.5.1. because $\pi_\alpha^\beta = h^{\beta-\alpha}: T \to T$ which is 1-1 and into.

To show π_0 is a homeomorphism we need to show π_0^{-1} is continuous where $\pi_0^{-1} : \bigcap_{n=0}^{\infty} h^n(T) \to T_\infty$. Now sets of the following type $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } T_\alpha\}$ are a base for the topology in T_∞. Given a basic set $\pi_\alpha^{-1}(U_\alpha)$ for some U_α open in T_α, then $((\pi_0)^{-1})^{-1}\pi_\alpha^{-1}(U_\alpha) = \pi_0(\pi_\alpha^{-1}(U_\alpha)) = \pi_0\pi_\alpha^{-1}(U_\alpha)$. The relation $\pi_0 = \pi_0^\alpha \pi_\alpha : T_\infty \to T_0$ gives us $\pi_0(U) = \pi_0^\alpha \pi_\alpha(U)$ where U is a basic open set of T_∞. But such an open set U is of the form $\pi_\alpha^{-1}(U_\alpha)$ for some open set U_α in T_α. Therefore $\pi_0(U) = \pi_0(\pi_\alpha^{-1}(U_\alpha)) = \pi_0^\alpha \pi_\alpha(\pi_\alpha^{-1}(U_\alpha)) = \pi_0^\alpha(U_\alpha) = h^\alpha(U_\alpha)$ which is open because h is a diffeomorphism. So π_0^{-1} is continuous and therefore π_0 is a homeomorphism.

5.5.4 LEMMA. $H^*(\Lambda) = \{Z, G, 0, 0, \ldots\}$ where the group $G = \{m/2^n | m,n \in Z\}$.

Proof. For the diffeomorphism $h : T \to T$ we have induced cohomology map $h^* : H^*T \to H^*T$. We obtain the direct system

$$H^*T_0 \xrightarrow{h^*} H^*T_1 \xrightarrow{h^*} H^*T_2 \longrightarrow \ldots \text{ with direct limit } H^*(T_\infty)$$

Now $T_\infty = \bigcap_{n=0} h^n(T) = \Lambda$ and so $H^*(T_\infty) = H^*(\Lambda)$. We have $H^0(\Lambda) = \mathbb{Z}$ because Λ is closed and connected. Restricting our attention to $H^1(\Lambda)$ we have the direct system:

$$Z \xrightarrow{(h^*)_1} Z \xrightarrow{(h^*)_1} \ldots \qquad \ldots \longrightarrow H^1(\Lambda) .$$

where $(h^*)_1 : H^1(T) \to H^1(T)$ is given by $g \longrightarrow 2g$ where $g \in Z$. Let $(m,n) \in H^1(\Lambda)$ be the element represented by $m \in H^1(T_n)$. Then using the definition of $(h^*)_1$ we have the relation $(m,n) = (2m,n+1)$. The addition law is $(m,n) + (m',n) = (m+m',n)$. Let $G = \{m/2^n | m,n \in Z\}$ under the usual addition; then

$\{(m,n) | (m,n) = (2m,n+1),\ m,n \in Z\}$ is isomorphic to G by ϕ say where $\phi(m,n) = m/2^n$. Since Λ is 1-dimensional we have determined the cohomology.

Remark. Had we taken the situation where the torus T winds inside itself p-times instead of twice, then the theory would have been analogous with $G = \{m/p^n | m,n \in Z\}$ with the usual addition.

5.5.4 LEMMA. The inset $I(\Lambda_+)$ of the diffeomorphism $f : S^3 \to S^3$ has cohomology $H^*(I(\Lambda_+)) = \{0,0,G,Z,0,0,\ldots\}$.

Proof. The exactness of the cohomology triangle for the compact pair (S^3,Λ_-) gives $H^*(S^3 - \Lambda_-) = H^*(I(\Lambda_+)) = \{0,0,G,Z,0,\ldots\}$ or $\{Z,Z,G,Z,0,\ldots\}$. The openness of $I(\Lambda_+)$ gives $H^*(I(\Lambda_+)) = \{0,0,G,Z,0,\ldots\}$.

§6. n-HORSE-SHOE FLOW

In this section we wish to give the generalization of the horse-shoe diffeomorphism and its counterpart in flows and suggest the programme to be followed from the elementary investigations of this chapter of which a brief mention was made in the introduction. First of all we require:

5.6.1 Definition of Subshift of Finite Type

In 5.1.5 we defined the shift (X_S,α) of finite type. Let $M = (m_{jk})$ be a $n \times n$ matrix of 0's and 1's where $n = |S|$.

Define Y_S be a subset of X_S to be the set of all sequences a such that if $a_i = s_j$ and $a_{i+1} = s_k$ for any $i \in Z$, then $m_{jk} = 1$.

Then $(Y_S,f|Y_S)$ is a subshift of finite type.

5.6.2 Definition of n-horse-shoe flow

Let $f : R^{n-1} \to R^{n-1}$ be a diffeomorphism and suppose the restriction to a Cantor Set $C \subset R^{n-1}$ is a subshift of finite type i.e. a (n-1)-horse-shoe then the suspension of the restricted diffeomorphism $f|C : C \to C$, $\Sigma\ (C,f|C)$ is called a n-horse-shoe flow.

Example. The "Horse-Shoe" Diffeomorphism

If we consider the restriction of $f : Q \to R^2$ to $f^{-1}(Q \cap f(Q))$ then by the definition of f the restriction is a linear map. The following picture is obtained for the restriction mapping where the dotted lines are the images under f.

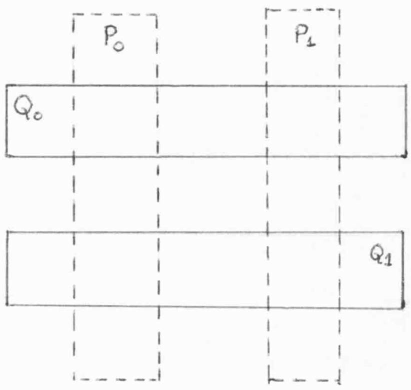


Fig. 5.8

We have $f(P_0) = Q_0$ and $f(P_1) = Q_1$. Note from §1, $f|C \times C$ is conjugate to a finite shift automorphism and so there is no restriction on the sequences and therefore the shift matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In fact the matrix M can be derived by noticing the intersections of P_0, P_1 with $f(P_0), f(P_1)$. Since all possible intersections occur the matrix is M as above.

<u>Example</u>. <u>A local diffeomorphism conjugate to a subshift</u>

Suppose we have the following 4 blocks each homeomorphic to $D^{\ell} \times D^{n-\ell}$ in R^n and a diffeomorphism $f : R^n \to R^n$ such that the restriction of f to the blocks is a linear map. Let the blocks be denoted by symbols S_1, S_2, S_3, S_4. Let the dotted lines denote the images under f where $S_i' = f(S_i)$, $i = 1,2,3$ and 4.

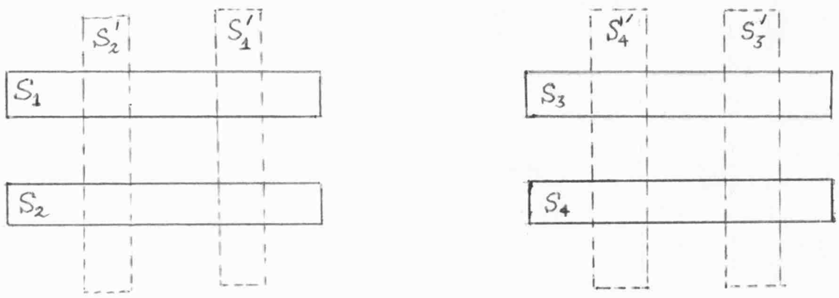


Fig. 5.9

The matrix representing the subshift automorphism is

$$M' = \begin{pmatrix} I_2 & O_2 \\ O_2 & I_2 \end{pmatrix} \text{ where } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Since we are not specifying that f should force all possible intersections we have this diffeomorphism gives a subshift.

The cohomological types of the insets of the diffeomorphisms considered here have not yet in general been investigated. Connected with this is the following problem briefly mentioned at the end of Chapter 1, the answer to which would give us generalizations of the Morse-Smale Inequalities for Smale Diffeomorphisms.

Problem. Given a Smale Diffeomorphism (p.12) then does the local behaviour of the diffeomorphism, which when restricted to a Cantor set is conjugate to a subshift of finite type, determine the cohomological type of the associated inset.

APPENDIX 1

LIE GROUP BUNDLES

Here we give some theory of Lie Groups to illustrate how the theory of Chapter 2, §5 could be used.

Consider a Lie Group B with a closed normal subgroup G. Then the group B has the structure of a bundle space B with base B/G and fibre G where the projection $\pi : B \to B/G$ is the natural homomorphism. Bundle mappings of B are obtained by letting B act on itself by left translations. Let $\bar{b}_0$ be the bundle mapping $\bar{b}_0 : B \to B$ induced by left translations of $b_0 \in B$. Therefore the map $\bar{b}_0$ is defined by $\bar{b}_0(b) = b_0 b$, $b \in B$. Let $\underline{b}_0$ be the induced map of the base such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "\bar{b}_0"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ B/G \arrow[r, "\underline{b}_0"] & B/G \end{tikzcd}$$

The map $\underline{b}_0$ is defined by $\underline{b}_0(bG) = b_0 bG$. Let $b_0, b_1 \in B$ and suppose we require $\underline{b}_0 = \underline{b}_1$; then $b_0 bG = b_1 bG$, $\forall\, b \in G \Rightarrow b_0 = b_1 g_0$ for some $g_0 \in G$. Suppose G is non-trivial and choose in the first instance $g_0 \neq 1$; then $\bar{b}_0 \neq \bar{b}_1$ but $\underline{b}_0 = \underline{b}_1$.

Another condition of Chapter 2, §5 is that the bundle mappings commute, i.e. $\bar{b}_0 \bar{b}_1 = \bar{b}_1 \bar{b}_0 \Rightarrow b_0 b_1 = b_1 b_0 \Rightarrow b_1 g_0 b_1 = b_1 b_1 g_0 \Rightarrow g_0 b_1 = b_1 g_0$. Hence to satisfy the condition we require $g_0 \in Z_G(b_1)$, the centralizer group. If $Z_G(b_1)$ is non-trivial we have $b_0, b_1 : B \to B$ such that $\bar{b}_0, \bar{b}_1$ are not equal but induce the same base map.

(A.1).1 Definition. Let $B_{\bar{b}_0}$ be the suspended manifold of B by the map $\bar{b}_0$. Define $\bar{b}_1' : B_{\bar{b}_0} \to B_{\bar{b}_0}$ by $\bar{b}_1' \pi_{\bar{b}_0}(b,t) = \pi_{\bar{b}_0}(b_1 b,t)$.

(A.1).2 LEMMA. Given the Lie Group bundle

$$G \longrightarrow B \xrightarrow{\pi} B/G, \text{ where } G \text{ is a closed normal subgroup}$$

and suppose $Z_G(B)$, the centralizer of B in G is non-trivial, then $\exists$ two different diffeomorphisms $\bar{b}_0, \bar{b}_1$ of B which induce the same base diffeomorphism and the bundle

$$B_{\bar{b}_0} \longrightarrow (B_{\bar{b}_0})_{\bar{b}_1'} \xrightarrow{\pi'} S^1$$

also has the structure of the bundle

$$G_{\bar{g}_0} \longrightarrow (B_{\bar{b}_0})_{\bar{b}_1'} \xrightarrow{q} (B/G)_{\underline{b}_0}$$

Remark. The assumption of the non-triviality of $Z_G(B)$ gives us a structure in the bundle case which is the logical extension of the product. However in the general bundle case no such extension exists.

Proof. From 2.5.3 we have that there exists a map

$$q : (B_{\bar{b}_0})_{\bar{b}_1'} \longrightarrow (B/G)_{\underline{b}_0}$$

where q is defined $q(\pi_{\bar{b}_1'}(\pi_{\bar{b}_0}(b,t),u)) = \pi_{\underline{b}_0}(\pi(b),t+u)$ such that $q^{-1}(\pi_{\underline{b}_0}(x,s)) = (G_x)_{\bar{b}_0(\bar{b}_1)^{-1}}$ for $x \in B/G$.

Given any $x \in B/G$ $\exists$ a diffeomorphism taking G_x onto the fibre G_1. The diffeomorphism is given as follows. Let $x = bG$, say, then $\exists$ an induced diffeomorphism

$$\bar{b}^{-1} : G_x \to G_1 \text{ , then it follows that the}$$

diffeomorphism $\bar{b}_0(\bar{b}_1)^{-1} : G_x \to G_x$ induces the diffeomorphism

$$(\bar{b}^{-1})\bar{b}_0(\bar{b}_1)^{-1}(\bar{b}^{-1})^{-1} : G_1 \to G_1$$

i.e. given $g \in G_1$ then under the induced diffeomorphism we have $g \longmapsto b^{-1}b_0b_1^{-1}bg$. Now $b^{-1}b_0b_1^{-1}b = b^{-1}b_1g_0b_1^{-1}b = b^{-1}g_0b$ because $g_0 \in Z_G(b_1)$. If under the supposition of the theorem we choose g_0 not only in the centralizer of b_1 but in the centralizer of B, then the induced map is $g \longrightarrow g_0g$ which is independent of $x \in B/G$.
It then follows that $(G_x)_{\bar{b}_0(\bar{b}_1)^{-1}}$ is diffeomorphic to $G_{\bar{g}_0}$ $\forall$ $x \in B/G$ and so the lemma is proved.

APPENDIX 2

ATLAS OF LOCAL BUNDLE CHARTS

At several points in Chapters 2 and 3 an atlas of local bundle charts has been given for the bundle

$$M \longrightarrow M_f \xrightarrow{\pi} S^1$$

with the usual notation where M and the diffeomorphism $f : M \longrightarrow M$ have been defined in various ways. It is the intention here to indicate why the bundle chart maps are in fact diffeomorphisms.

Construction of local bundle charts

Let p denote the natural projection $p : R \to R/Z = S^1$. Let the open covering $\{U_1, U_2\}$ of S^1 be defined

$$U_1 = \{p(u) \mid 0 < u < 1\}, \quad U_2 = \{p(u) \mid 3/4 < u < 5/4\} .$$

Then we define the atlas $\{(U_1,h_1), (U_2,h_2)\}$ by

$$h_1 : U_1 \times M \to \pi^{-1}(U_1)$$
$$(p(u),x) \longrightarrow \pi_f(x,u) , \quad 0 < u < 1$$
$$h_2 : U_2 \times M \to \pi^{-1}(U_2)$$
$$(p(u),x) \longrightarrow \pi_f(x,u) , \quad 3/4 < u < 5/4.$$

To show that h_1 is a diffeomorphism consider the following diagram.

$$\begin{tikzcd} M \times (0,1) \arrow[rr, "d_1"] \arrow[dr, "\pi_f"'] & & U_1 \times M \arrow[dl, "h_1"] \\ & \pi^{-1}(U_1) & \end{tikzcd}$$

The diagram commutes where d_1 is defined as the natural diffeomorphism $d_1(x,s) = (p(s),x)$. By definition π_f is a differentiable map and from the commutativity gives us

$$h_1 = \pi_f d_1^{-1} : U_1 \times M \to \pi^{-1}(U_1).$$

It follows that h_1 is differentiable. It can be checked that h_1 is a homeomorphism. Hence h_1 is a diffeomorphism. We can similarly show that h_2 is a diffeomorphism.

APPENDIX 3

CATEGORIES AND FUNCTORS

(A.3).1 Definition of Category

A category C consists of

(C1) a class of objects A,B,C,...;

(C2) for each pair (A,B) of objects a set [A,B], where the elements are called morphisms from A to B with domain A and range B (we write $\alpha : A \to B$ or $A \xrightarrow{\alpha} B$ for $\alpha \in [A,B]$), these sets being pairwise disjoint, i.e. $(A,B) \neq (A',B')$ implies $[A,B] \cap [A',B'] = \Phi$;

(C3) for each triple (A,B,C) of objects a map

$$[A,B] \times [B,C] \longrightarrow [A,C]$$
$$(\alpha,\beta) \longrightarrow \beta\alpha$$

called composition of morphisms;

(C4) for each object A an element $1_A \in [A,A]$ called identity morphisms;

these data being subject to the two axioms:

(AC1) If $\alpha \in [A,B]$, $\beta \in [B,C]$, $\gamma \in [C,D]$ then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$

(AC2) If $\alpha \in [A,B]$, then $\alpha 1_A = \alpha$, $1_B\alpha = \alpha$.

(A.3.)2 Definition of Functor.

Let $\mathcal{C}$ and $\mathfrak{D}$ be categories. A covariant functor $F : \mathcal{C} \to \mathfrak{D}$ is the assignment

(F1) of an object FA of $\mathfrak{D}$ to each object A of $\mathcal{C}$;

(F2) of a morphism $F\alpha : FA \to FB$ of $\mathfrak{D}$ to each morphism $\alpha : A \to B$ of $\mathcal{C}$;

subject to the axioms

(AF1) $F(1_A) = 1_{FA}$

(AF2) $F(\beta\alpha) = F(\beta) . F(\alpha)$.

APPENDIX 4

THE DOUBLE OF A MANIFOLD

(A.4).1 Definition [18]. Let M be a C^r-manifold with non-empty boundary. Then $D(M)$ is the union of $M_0 = M \times 0$ and $M_1 = M \times 1$, with $(x,0)$ and $(x,1)$ identified whenever $x \in \partial(M)$. Define the differentiable structure on $D(M)$ as follows: Let $p_0 : U_0 \to \partial(M_0) \times [0,1)$ and $p_1 : U_1 \to \partial(M_1) \times (-1,0]$ be product neighbourhoods of the boundary of M_0 and M_1, and let $p : U \to \partial M \times (-1,1)$ be the homeomorphism induced by p_0 and p_1. A C^r differentiable structure on $D(M)$ is well-defined if we require (1) P to be a C^r diffeomorphism and (2) the inclusions of M_0 and M_1 in $D(M)$ to be C^r embeddings.

(A.4).2 LEMMA. Let X and Y be compact spaces and A and B closed subspaces of X such that $A \cup B = X$. Let f_1, f_2 be homeomorphisms $f_1 : A \to Y$ and $f_2 : B \to Y$ such that $f_1(A) \cup f_2(B) = Y$ and $f_1 | A \cap B = f_2 | A \cap B$, then $f : X \to Y$ defined by $f|A = f_1$ and $f|B = f_2$ is a homeomorphism.

Proof. Because of the overlap condition f is well-defined and 1-1. Given V, an open set of Y, then $f^{-1}(V) = f^{-1}((V \cap f_1(A)) \cup (V \cap f_2(B))) = f^{-1}(V \cap f_1(A)) \cup f^{-1}(V \cap f_2(B)) = f_1^{-1}(V \cap f_1(A)) \cup f_2^{-1}(V \cap f_2(B))$. We have that $V \cap f_1(A)$ and $V \cap f_2(B)$ are both closed in Y. Since f_1 and f_2 are homeomorphisms it follows that $f^{-1}(V)$ is the union of two closed sets and hence closed. Similarly given U closed in X then $f(U) = f((U \cap A) \cup (U \cap B)) = f(U \cap A) \cup f(U \cap B) = f_1(U \cap A) \cup f_2(U \cap B)$. Because U is closed, $U \cap A$ and $U \cap B$ are closed in A and B respectively and so $f_1(U \cap A)$, $f_2(U \cap B)$ are closed in $f_1(A)$ and $f_2(B)$ respectively since f_1 and f_2 are homeomorphisms. Hence since $f_1(A)$, $f_2(B)$ are closed in Y, then $f_1(U \cap A) \cup f_2(U \cap B)$ is closed in Y. Therefore f is closed and so f is a homeomorphism.

APPENDIX 5

DIRECT AND INVERSE SYSTEMS [8]

(A.5).1 Definition. A directed set M is a quasi-ordered set such that for each pair $\alpha, \beta \in M$, there exists a γ such that $\alpha < \gamma$ and $\beta < \gamma$.

(A.5).2 Definition. A direct system of sets $\{X, \pi\}$ over a directed set M is a function which attaches to $\alpha \in M$ a set X_α and to each pair α, β such that $\alpha < \beta$ in M, a map

$$\pi_\alpha^\beta : X_\alpha \to X_\beta$$

such that for each $\alpha \in M$

$$\pi_\alpha^\alpha = 1$$

and for $\alpha < \beta < \gamma$

$$\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$$

(A.5).3 Definition. Let $\{G, \pi\}$ be a direct system over the directed set M where each G_α is an abelian group and each π_α^β is a homomorphism. Let ΣG denote the direct sum of the groups $\{G_\alpha\}$. For each $\alpha < \beta$ if $g_\beta = \pi_\alpha^\beta g_\alpha$ then g_α and g_β are said to be related. Let Q be the subgroup of ΣG generated by all related elements. Then the direct limit of $\{G, \pi\}$ is the factor group

$$G^\infty = (\Sigma G)/Q .$$

(A.5).4 An inverse systems of sets $\{X, \pi\}$ over a directed set M is a function which attaches to each $\alpha \in M$ a set X_α, and to each pair (α, β) such that $\alpha < \beta$ in M, a map

$$\pi_\alpha^\beta : X_\beta \to X_\alpha$$

such that

$$\pi_\alpha^\alpha = 1, \qquad \alpha \in M$$

$$\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma, \quad \alpha < \beta < \gamma \text{ in } M.$$

(A.5).5 The <u>inverse limit</u> X_∞ of the inverse system of (A.5).4 is the subset of the product ΠX_α consisting of those functions $x = \{x_\alpha\}$ such that for each relation $\alpha < \beta$ in M

$$\pi_\alpha^\beta(x_\beta) = x_\alpha \ .$$

Define the projection $\pi_\beta : X_\infty \to X_\beta$ by $\pi_\beta(x) = x_\beta$.

(A.5).6 <u>LEMMA</u>. Given the inverse systems of sets $\{X,\pi\}$ with inverse limit X_∞ then the direct system obtained by action of the Cech cohomology functor H^* on $\{X,\pi\}$ has direct limit H^*X_∞ if the coefficient group is abelian. [This is the receiving category we have used in Chapter 5].

BIBLIOGRAPHY

[1] R.Abraham and J.Marsden, Foundations of mechanics, Benjamin, New York, 1967.

[2] R.Abraham and J.Robbin, Transversal mappings and flows, Benjamin, New York, 1967.

[3] D.V.Anosov, Geodesic flows on compact Riemannian manifolds of negative curvature, Trudy Mat.Inst.Steklov.90 (1967) = Proc.Steklov Math.Inst.

[4] V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics, Benjamin, New York, 1968.

[5] L.Auslander and R.E.Mackenzie, Differentiable Manifolds, McGraw Hill, 1963.

[6] M.L.Cartwright, Forced Oscillations in non-linear systems, contributions to the theory of non-linear oscillations, Ann.of Math. Studies (20).

[7] E.Coddington and N.Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.

[8] S.Eilenberg and N.Steenrod, Foundations of Algebraic Topology, Princeton (1952).

[9] P.Hilton and S.Wylie, Homology Theory, An Introduction to Algebraic Topology, C.U.P., 1960.

[10] D.Husemoller, Fibre Bundles, McGraw Hill, 1966.

[11] S.T.Hu, Elements of General Topology, Holden Day, 1964.

[12] S.Lang, Introduction to differentiable manifolds, Wiley, New York, 1962.

[13] S.Lefchetz, Differential Equations; Geometric Theory, Interscience, New York, 1957.

[14] N.Levinson, A second order Differential Equation with singular solutions, Ann. of Math.(2) 50 (1949), 127-153.

[15] L.Markus, Differentiable dynamical systems, Lecture Notes (Warwick University, 1969).

[16] N.Minorsky, Introduction to non-linear Mechanics, J.W.Edwards, Ann Arbor (1946).

[17] M.Morse and S.S.Cairns, Critical point theory in Global Analysis and Differentiable Topology, Academic Press, 1969.

[18] J.R.Munkres, Elementary Differential Topology, Ann.of Maths. 54.

[19] M.M.Peixoto, Structural stability on two-dimensional manifolds, Topology 1 (1962), 101-120.

[20] M.M.Peixoto, Qualitative Theory of Differential Equations and Structural Stability, Symposium on differential equations and dynamical systems (Puerto Rico) Academic Press, New York, 1967.

[21] H.Rosenberg, A generalization of Morse-Smale inequalities, Bull.Amer.Math. Soc.70 (1964), 422-427.

[22] S.Smale, Morse inequalities for a dynamical system, Bull.Amer.Math.Soc. 66 (1960), 43-49.

[23] " , Stable manifolds for differential equations and diffeomorphisms, Ann.Scuola Norm.Sup. Pisa (3) (1963), 97-116.

[24] " , A structurally stable differentiable homeomorphism with an infinite number of periodic points, Proc.Internat.Sympos, Non-linear Vibrations, Vol II, 1961, Izdat, Akad, Nauk,Ukrain SSR, Kiev, 1963.

[25] " , Diffeomorphisms with many periodic points, Differential and Combinatorial Topology, Princeton Univ. Press, 1965, 747-817.

[26] " , Differentiable dynamical systems, B.A.M.S., 73 (1967).

[27] N.Steenrod, Topology of Fibre Bundles, Princeton, 1951.

[28] P.Tondeur, Introduction to Lie Groups and Transformation Groups, Lect.Note. Math. (Springer-Verlag), 7.

[29] R.Thom, Sur une partition en cellules associes a une function sur une variete, C.R.Acad.Sci.Paris 228 (1949), 973-975.

[30] B. Van der Pol, "On a type of oscillation-Hysteresis in a simple triode generator", Phil.Mag., 43, 177 (Jan, 1922).

[31] H.Wallman and W.Hurewicz, Dimension Theory, Princeton Univ.Press, 1948.

[32] R.Williams, One dimensional non-wandering sets, Topology 6 (1967), 473-487.

[33] R.Williams, Non-Compact Lie Group Actions, Proc.Con.Transformation Groups, New Orleans, 1967.

[34] E.C.Zeeman, Compact Čech Cohomology Theory (Lecture notes, Warwick University, 1965).

[35] E.C.Zeeman, An Essay on Dynamical Systems (Warwick University, 1972).

[36] D.K.Arrowsmith, Products of Flows, Proc.Camb.Phil.Soc. (to appear).

ABSTRACT

Flows which are suspensions of auto-diffeomorphisms of manifolds are studied in this thesis. The structure of the product of two such suspended flows is investigated and its relation to product diffeomorphisms, together with some simple statements concerning Anosov flows are given. A generalization of suspension to deal with any finite number of commuting auto-diffeomorphisms is considered and analogous results to those obtained above are proved together with some additional ones.

A functorial representation is given for suspended flows. Other flow invariant operations on manifolds are considered for this class of flows.

Also considered are diffeomorphisms with non-wandering sets which have parts homeomorphic to Cantor Sets. The cohomologies of their insets are computed using Čech cohomology theory. This is a first step in the problem of using Morse Theory to obtain Morse-inequalities for Smale diffeomorphisms as defined in the introduction.

ERRATA

Page 10, line 23: These diffeomorphisms disprove the conjecture [22] that the set of Morse-Smale Diffeomorphisms on a smooth compact manifold is generic

Page 108, line 34: [34] M.A. Armstrong, Compact Čech Cohomology, (Lecture notes by E.C. Zeeman, University of Warwick, 1965.).