IDENTIFICATION AND SELF-ADAPTIVE CONTROL

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CONTENTS

NOTATION

$A(z^{-1})$	:	polynomial corresponding to output
a	:	n-vector of output parameters of pulse transfer function
$B(z^{-1})$	:	polynomial corresponding to input
b	:	n-vector of input parameters of pulse transfer function
c_t	:	perturbation input
E	:	expectation operator
$F(z^{-1})$	:	filter polynomial
$G(z^{-1})$	:	auto-regressive polynomial of the disturbance model
$H(z^{-1})$	:	polynomial representing the predictor
$H^*(z^{-1})$	:	polynomial representing the optimum predictor
$\bar{H}(z^{-1})$	:	polynomial representing the modified predictor
L	:	orthonormal matrix of input
k, ℓ	:	process and model time delay respectively
$M(z^{-1})$	:	model impulse response polynomial
N	:	number of data samples
$P(z^{-1})$	:	process impulse response polynomial
$Q(z^{-1})$	:	auto-regressive polynomial of the disturbance model
$R(z^{-1})$	:	auto-regressive polynomial of the estimated disturbance model
$S(z^{-1})$	:	moving-average polynomial of the disturbance model
$[\cdot]_T$	:	front-end truncation operator
t	:	discrete time
U	:	matrix containing shifted vectors of the input sequence u
u_t	:	input
$V(z^{-1})$	:	polynomial representing the process, process-model discrepancy
v	:	r-vector of non-satisfied assumptions
X	:	matrix of input output data
X_t	:	estimated disturbance

$x_t, \hat{x}_{t+k}$	:	disturbance and its k step ahead prediction
Y	:	Nxn matrix of shifted vectors of the output sequence y
y_t	:	output
y_t^d	:	desired set-point
z	:	forward shift operator
γ, ρ	:	covariance and correlation coefficient, respectively
ε_t	:	disturbance
ζ_t	:	zero mean independent random variable, i.e. white noise
η_t	:	total effect of disturbance on the output
θ	:	vector of parameters
μ	:	number of additional parameters of the modified predictor
σ_ζ^2	:	variance of ζ_t
$\Phi(z^{-1})$	:	polynomial representing the modified part of the predictor
Ψ	:	rxr symmetric matrix of the sum of squares of estimation errors
$\Omega(z^{-1})$	:	moving average filter of the correlation analysis

CHAPTER I

SURVEY

1.1. Introduction

This chapter presents a survey of recent literature in the area of system parameter estimation techniques applicable to single input, single output, linear discrete dynamic systems. Its main objective is to examine the structure of the assumptions made about the unknown disturbances, since the crucial factor in every identification experiment is whether these assumptions are satisfied by the unknown disturbances.

1.2. System description

A causal deterministic system can be described as [1]

$$y_t = f(y_{t-1}, y_{t-2}, \cdots, y_o, u_{t-1}, u_{t-2}, \ldots, u_o) \qquad (1.2.1)$$

where u_t and y_t represent input and output respectively. It is assumed that all initial conditions necessary to compute y_1 are known.

For stochastic systems where it is not possible to determine y_t exactly by previous inputs and outputs as in (1.2.1), it is appropriate to consider the probabilistic description of y_t. If the uncertainty of a stochastic system is represented by a disturbance term ε_t then a stochastic system can be described as

$$y_t = E(y_t | y_{t-1}, y_{t-2}, \ldots, y_o, u_{t-1}, u_{t-2}, \ldots, u_o) + \varepsilon_t \qquad (1.2.2)$$

where $E(y_t|.)$ is the conditional expectation given all past inputs and outputs. It is assumed that the information is available at time t = 1 about the previous behaviour of the system.

For practical reasons (1.2.2) is often expressed in a recursive way such as linear difference or state space equations. In this thesis the former representation will be used throughout, since it is more suitable for the single input, single output systems. Hence (1.2.2) will be written as

$$y_t = A(z^{-1})y_t + B(z^{-1})u_t + \varepsilon_t \qquad (1.2.3)$$

where $A(z^{-1}) = \sum_{i=1}^{n} a_i z^{-i}$, $B(z^{-1}) = \sum_{i=1}^{n} b_i z^{-i}$, z^{-1} being a delay operator, such that $z^{-1}y_t = y_{t-1}$.

The advantage of this representation is that only a few past values of y_t and u_t are needed at each step to determine y_t and also most of the control design techniques require such a description.

1.3. Problem of identification and parameter estimation

Identification is defined by Zadeh [2] as 'the determination on the basis of input and output, of a system within a specified class of systems, to which the system under test is equivalent.' Using this formulation, it is assumed that 'the system under test' is described by the linear difference equation (1.2.3). Naturally, in practice this assumption is rather restrictive, since it implies that the system can be modelled exactly. Nevertheless in a majority of engineering problems a certain amount of a priori knowledge is available which may be used to represent the system in the form (1.2.3) [3].

The above assumption obviously reduces 'a specified class of systems' to a following model

$$y_t = \hat{A}(z^{-1})y_t + \hat{B}(z^{-1})u_t + \hat{\varepsilon}_t \qquad (1.3.1)$$

where $\hat{A}(z^{-1}) = \sum_{i=1}^{\hat{n}} \hat{a}_i z^{-i}$, $\hat{B}(z^{-1}) = \sum_{i=1}^{\hat{n}} \hat{b}_i z^{-i}$, $\hat{\varepsilon}_t$ as an estimated value of the disturbance ε_t. Since there is now no uncertainty about the structure, the identification problem is reduced to that of estimation of unknown parameters $(\hat{a}_i, \hat{b}_i, \hat{n})$ on the basis of input output measurements such that the model and the system are equivalent.

Several different ways have been developed to define the equivalence [4],[5]. For purposes of discussion these will be broadly classified as

(i) minimization of prediction error,

(ii) minimization of output error,

(iii) minimization of estimation error.

The prediction of output y_t is an essential part of the modern control theory. In, e.g., the minimum variance control theory, the optimum input u_t must be designed so that the output variance $E(y_{t+1}^2)$ is minimum [6]. Therefore it is natural to estimate the model parameters which give the best prediction of output. In class (i) techniques this is achieved by minimizing some function of the prediction error

$$e_t = y_t - \hat{y}_{t|t-1} \tag{1.3.3}$$

where $\hat{y}_{t|t-1}$ denotes a first step ahead prediction of y_t given past values of inputs and outputs up to and including time t - 1. It is known that [6] the minimum mean square prediction $\hat{y}_{t|t-1}$ of y_t is given by

$$\hat{y}_{t|t-1} = E(y_t|y_{t-1}, \ldots, y_o, u_{t-1}, \ldots, u_o) \tag{1.3.4}$$

if ε_t is a zero-mean uncorrelated process then from (1.2.2) and (1.2.3) it follows that

$$\hat{y}_{t|t-1} = A(z^{-1})y_t + B(z^{-1})u_t \tag{1.3.5}$$

On substitution of (1.3.5) into (1.3.3), the prediction error takes the form

$$e_t = y_t - A(z^{-1})y_t - B(z^{-1})u_t \tag{1.3.6}$$

and the parameters (a_i, b_i, n) are estimated by minimizing a weighted sum of squares of e_t.

It is well known that the prediction error criterion is intimately connected with the maximum likelihood estimates if ε_t is also assumed to be normally distributed [1].

The main idea of class (ii) techniques is to estimate a model so that its output approximates the output of the unknown system as closely as possible. Writing the system (1.2.3) in the form

$$y_t = B(z^{-1})\,[1-A(z^{-1})]^{-1}\,u_t + n_t \tag{1.3.7}$$

where n_t is assumed as the total effect of all disturbances acting on the system output then comparing the system output y_t and the corresponding model output

$$\hat{y}_t = \hat{B}(z^{-1})[1-\hat{A}(z^{-1})]^{-1}\,u_t \tag{1.3.8}$$

an output error is defined as

$$e_t = y_t - \hat{y}_t = y_t - \hat{B}(z^{-1})[1-\hat{A}(z^{-1})]^{-1} u_t \qquad (1.3.9)$$

then the model parameters are estimated by minimizing e_t in a weighted mean-square sense. The use of non-linear model as (1.3.8) requires a degree of sophistication in the minimization technique employed. If however, the model is chosen as linear in the parameters such as an impulse response model

$$\hat{y}_t = \hat{P}(z^{-1})u_t \qquad (1.3.10)$$

where $\hat{P}(z^{-1}) = \sum_{i=0}^{\infty} \hat{P}_i z^{-i}$, then the minimization can be done by a simple least squares algorithm.

The philosophy of class (iii) techniques can be explained as follows. Denote all the parameters to be estimated as

$$\theta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)^T \qquad (1.3.11)$$

and the corresponding estimates as $\hat{\theta}$. Assuming that the order n is known, then $\hat{\theta}$ is determined such that the following scalar loss function

$$V = E[f(\theta-\hat{\theta})] \qquad (1.3.12)$$

is minimum. Such a formulation makes it possible to use the tools of basic estimation theory. In particular, if it is assumed that θ is a random variable and the joint probability distribution of $(\theta, y_t, \ldots, y_o, u_t \ldots, u_o)$ is available then the minimization of (1.3.12) for a wide range of reasonable loss functions yields the minimum variance estimate $\hat{\theta}$ of θ [7]. In cases where θ is unknown but not random or the probability distribution of θ is not available then the minimization of (1.3.12) leads to the maximum likelihood principle where $\hat{\theta}$ is obtained such that

$$p(y_t, y_{t-1}, \ldots, y_o | \hat{\theta})$$

is maximum.

The question of how to define the equivalence between the system and its corresponding model has not been completely answered yet. Often the choice is a compromise between the purpose of identification and the ease of computation of estimates. If, for example, the purpose is to design minimum variance

control strategies then the prediction error criterion may be convenient to employ. If on the other hand, all probabilistic properties of the disturbance are known, in particular, if the distribution is Gaussian then the use of criterion (iii) makes it possible to assign accuracies to parameter estimates and to test various statistical hypotheses [7]. For the criterion given in terms of the output error (1.3.9), estimation is simply a deterministic optimisation problem. However, under some suitable assumptions on the disturbance it may be possible to find a probabilistic interpretation and hence the accuracy of estimation can be estimated.

Parameter estimation methods can also be classified according to the experimental conditions, i.e.

a) data obtained from the open loop system with a pre-selected input signal,

b) input, output measurements from the closed loop system with or without an additional input perturbation.

It is one of the main assumptions of estimation that the disturbance must be independent of the input to the system. This assumption is in general violated if the system is operating under feedback control. This therefore requires that estimation be carried out on an uncontrolled system with a pre-selected input signal u_t - a condition that may well prove unrealistic for real life problems, nevertheless class (a) methods have the following advantages due to the pre-designed input signal.

- Significant simplifications in computation can be achieved by choosing an input signal of a special type such as a pseudo random binary sequence.

- Input signal can be designed persistently exciting [8] to satisfy identifiability conditions.

- In some cases it is also possible to reduce the estimation errors by a proper choice of input signal [9],[10],[11].

In practice, it is generally desirable and sometimes even necessary to estimate the system parameters while the system is operating under feedback control. Reasons for this can be summarized as follows [12]:

- The disturbance may contain some form of drift which usually causes large output fluctuations if the system is operated in open loop.

- Often a linear model of the process, valid around certain operating conditions, is estimated therefore a suitable controller must keep the system near these conditions.

Class(b) methods claim to use of input, output measurements from closed loop systems with or without an additional input perturbation. A detailed survey of these methods will be given in Chapter 3 and 4. It will only be noted here that estimation of such systems causes extra difficulties. In cases where an additional extra input perturbation is allowed, it is in general possible first to identify the closed loop system and hence infer the dynamics of the system if the controller is known. If it is not possible to use an additional perturbation signal then the identifiability cannot always be guaranteed [12].

In the following sections of this chapter a survey of recent parameter estimation techniques applicable to single input output systems operating in open loop will be given. The fundamental point of interest will be the assumed a priori knowledge of disturbances, since most estimation methods rely on the assumptions made about unknown disturbances which in turn completely determine the quality of solution. The non-satisfied assumptions always lead to incorrect estimates. As it was stated by Astrom and Eykhoff [13], this is probably the main reason why 'Most methods seem to work well on simulated data but not always that well on actual industrial data.'

In order to make a comparison of parallel estimation techniques first a brief summary of regression analysis will be made.

1.4. Regression analysis

Let y be a scalar continuous variable as a function of known continuous variables $x_1, x_2, \ldots, x_m$ and various unknown factors whose effect on y are represented by one continuous variable ε. In the absence of evidence to the contrary, the relationship between these variables can be assumed a linear one as

$$y=\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_m x_m + \varepsilon \qquad (1.4.1)$$

If a series of N experiments are performed, t^{th} experiment can be described by

$$y_t = \theta_1 x_{1,t} + \theta_2 x_{2,t} + \ldots + \theta_m x_{m,t} + \varepsilon_t \qquad (1.4.2)$$

using a vector-matrix formulation, the total set of N equations can be conveniently written as

$$y = X\theta + \varepsilon \qquad (1.4.3)$$

where $y = (y_1, y_2, \ldots, y_N)^T$, $X = \{x_{t,i}; i = 1, \ldots, m; t = 1, \ldots, N\}$ is an Nxm matrix and $\theta = (\theta_1, \theta_2, \ldots, \theta_m)^T$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)^T$ and $N > m$.

The estimation problem is now to ascertain θ from the given data X,y. In principle this problem is insoluble, since the number of unknowns in θ and ε is greater than that of equations, N. Consequently it follows that θ and ε can only be estimated if additional m equations can be found to complement (1.4.3) to (N+m) equations. It will be assumed that these m equations are linear in the form

$$E(M^T\varepsilon) = \gamma \qquad (1.4.4)$$

where M is an Nxm matrix and γ is an m vector. The assumptions (1.4.4) will be taken as m exact constraints on the estimate $\hat{\varepsilon}$ as

$$M^T\hat{\varepsilon} = \gamma \qquad (1.4.5)$$

Denoting $\hat{\theta}$ as an estimate of θ in (1.4.3) and collecting (1.4.3) with (1.4.5) it is found that estimates are obtained as the solution of following (N+m) equations

$$\begin{bmatrix} y \\ \gamma \end{bmatrix} = \begin{bmatrix} I_N & X \\ M^T & (0) \end{bmatrix} \begin{bmatrix} \hat{\varepsilon} \\ \hat{\theta} \end{bmatrix} \qquad (1.4.6)$$

where I_N is a unity matrix of N^{th} order. Assuming that the mxm matrix (M^TX) is non-singular then $\hat{\theta}$ and $\hat{\varepsilon}$ are obtained as

$$\hat{\theta} = (M^TX)^{-1} M^Ty = (M^TX)^{-1}\gamma \qquad (1.4.7)$$

$$\hat{\varepsilon} = [I_N - X(M^TX)^{-1} M^T]y + X(M^TX)^{-1}\gamma \qquad (1.4.8)$$

In regression analysis the assumptions (1.4.4) take the following form

$$E(X^T\varepsilon) = X^TE\varepsilon = \underline{0} \qquad (1.4.9)$$

The justification of this choice is that the elements of X matrix are completely known then it can be assumed that the effects of unknown elements, ε, are physically independent of those of the known, X. This physical independence can also be assumed to imply probabilistic independence and hence the first equality in (1.4.9) follows. It is also found convenient to assume that ε is a zero mean vector. However, if there is a known constant level this can easily be incorporated by considering $(\varepsilon - E\varepsilon)$ instead of $(E\varepsilon)$ in (1.4.9).

In terms of $\hat{\varepsilon}$ (1.4.9) takes the form

$$X^T\hat{\varepsilon} = \underline{0} \tag{1.4.10}$$

thus substituting $M = X$, $\gamma = \underline{0}$ into (1.4.7) and (1.4.8), the well-known estimation equations are obtained as

$$\hat{\theta} = (X^TX)^{-1} X^Ty \tag{1.4.11}$$

$$\hat{\varepsilon} = [I_N - X(X^TX)^{-1} X^T]y \tag{1.4.12}$$

For the existence of estimates the requirement is that (X^TX) must be non-singular, that is, X must be of rank m.

The above derivation of estimates $\hat{\theta},\hat{\varepsilon}$ has an intuitive appeal as far as the assumptions are concerned but it is not possible to say in what sense the estimates are optimal. It can however be shown that (1.4.11) and (1.4.12) are identical to the least squares estimates where θ is estimated such that the following error criterion

$$V = \hat{\varepsilon}^T\hat{\varepsilon} \tag{1.4.13}$$

is minimum. The elements of $\hat{\varepsilon}$ in least squares estimation are referred to as the residuals.

Comparison of the two different derivations of $\hat{\theta},\hat{\varepsilon}$ shows that the least squares criterion implicitly dictates what assumptions are made about the unknown disturbance ε. It would however be more desirable to use a criterion which minimizes the inevitable assumption errors. The reason for this is that the least squares criterion gives rise to models which best fit the experimental data, whereas in general the function of such models is used to make predictions about the future behaviour of the system. It is precisely in such situations that the models obtained by the consideration of assumption errors may be more

realistic. This point is particularly important for the dynamic systems if the final goal of identification is to design minimum variance control strategies.

1.5. Parameter estimation in linear dynamic systems

The system is described by an n^{th} order linear difference equation as

$$y_t = A(z^{-1})y_t + B(z^{-1})u_t + \varepsilon_t \tag{1.5.1}$$

where the polynomials $A(z^{-1})$ and $B(z^{-1})$ are defined as before. There is no loss of generality to assume that $A(z^{-1})$, $B(z^{-1})$ have the same order since if this is not the case the later coefficients can be replaced by zeros in either polynomial. It is also assumed that the system is stable, i.e.

$$1 - A(z^{-1}) = 0$$

has all roots inside the unit circle of the z-plane.

To evaluate $\{y_t;\ t = 1,2, \ldots,N\}$ it is required that all initial conditions $(y_o,\ldots,y_{1-n},u_{-1},\ldots,u_{1-n})$ be known. These values can be regarded as unknowns or simply assigned an arbitrary value, say zero. In the former case the unknown initial conditions are included in the estimated parameters [14]. But often these values cannot be estimated consistently [1]. For the latter case, if the data length is sufficiently long then as far as the large sample properties are concerned the assumption of zero initial conditions has little effect on the estimates. As it will also be seen in the following that by ignoring initial conditions it is possible to simplify notations considerably.

Now assuming that $u_t = 0$ for $t < 0$ and $y_t = 0$ for $t \leq 0$, the total set of N equations of (1.5.1) can be written as

$$y = Ya + Ub + \varepsilon \tag{1.5.2}$$

where $a = (a_1,a_2,\ldots,a_n)^T$, $b = (b_1,b_2,\ldots,b_n)^T$, $\varepsilon = (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N)^T$, $y = (y_1,y_2,\ldots,y_N)^T$, U and Y are Nxn matrices whose elements are the past values of input and output respectively. Since all initial values are

assumed null then U and Y matrices can be represented in a form using the NxN shift matrix [9]

$$S = \left[\begin{array}{c|c} \underline{0}^T & 0 \\ \hline I_{N-1} & \underline{0} \end{array}\right] \tag{1.5.3}$$

whose effect is a vector equivalent of the delay operator z^{-1}, for example, $Sy = (0,y_1,y_2,\ldots,y_{N-1})$. In terms of S, U and Y can be written as $U = (u,Su,\ldots,S^{n-1}u)$ and $Y = (Sy,S^2y,\ldots,S^ny)$ where $u = (u_o,u_1,\ldots,u_{N-1})^T$.

In the following it is also found convenient to write (1.5.2) as

$$y = X\theta + \varepsilon \tag{1.5.4}$$

where $X = (Y\,|\,U)$ is an Nx2n matrix of data and $\theta = \left(\begin{array}{c} a \\ \hline b \end{array}\right)$ is an 2n vector of unknown parameters. Correspondingly the model will be written as

$$y = X\hat{\theta} + \hat{\varepsilon} \tag{1.5.5}$$

and it will be assumed that the order n is known. The problem is now to estimate θ and ε given data X and y. First the least squares solution will be discussed.

1.6. Least squares estimation

The error criterion to be minimised is

$$V = \hat{\varepsilon}^T\hat{\varepsilon} = (y-X\hat{\theta})^T(y-X\hat{\theta}) \tag{1.6.1}$$

and the minimum value is obtained for

$$\hat{\theta} = (X^TX)^{-1}\,X^T_y \tag{1.6.2}$$

As it has been discussed in section 1.4, the least squares estimate is based on the assumptions

$$E(X^T\varepsilon) = E\left[\begin{array}{c} Y^T\varepsilon \\ \hline U^T\varepsilon \end{array}\right] = \underline{0} \tag{1.6.3}$$

which can be rewritten as

$$E\,\varepsilon^TS^iu = 0 \qquad i = 0,1,\ldots,\ n-1 \tag{1.6.4}$$

$$E\,\varepsilon^TS^jy = 0 \qquad j = 1,2,\ldots,\ n \tag{1.6.5}$$

The assumptions (1.6.4) can be justified on the grounds that the input signal u can be designed a priori and is completely known then it can be argued that the effect of unknown factors, ε, is independent of those of the known, u, and if it is also assumed that $E(\varepsilon) = \underline{0}$ then (1.6.4) follows.

To discuss the assumptions (1.6.5) it is found convenient to write the vector (Ya) of (1.5.2) as

$$Ya = \left(\sum_{i=1}^{n} a_i S^i\right) y = Ay \qquad (1.6.6)$$

where $A = \sum_{i=1}^{n} a_i S^i$ is an NxN lower triangular matrix with zero elements on the major diagonal. Now writing (1.5.2) as

$$y = Ay + Ub + \varepsilon \quad \text{or} \qquad (1.6.7)$$

$$y = (I-A)^{-1}(Ub + \varepsilon) \qquad (1.6.8)$$

then assumptions (1.6.5) take the following form

$$E\varepsilon^T S^i y = E\varepsilon^T S^i (I-A)^{-1}\varepsilon + E\varepsilon^T S^i (I-A)^{-1} Ub = 0$$

$$i = 1,2,\ldots,n \qquad (1.6.9)$$

Consider now the second term of (1.6.9) which can be written as

$$E\varepsilon^T S^i (I-A)^{-1} Ub = E\varepsilon^T S^i (I-A)^{-1} Bu = E\varepsilon^T S^i Pu = E\varepsilon^T S^i (U,\underline{U})p$$

$$i = 1,2,\ldots,n \qquad (1.6.10)$$

where $B = \sum_{i=1}^{n} b_i S^{i-1}$ and P is the impulse response matrix defined by $P = (I-A)^{-1}B = \sum_{j=0}^{r-1} p_j S^j$ where p_j is the j^{th} coefficient of the impulse response assumed of r sample intervals duration and $p = (p_o, p_1, \ldots, p_{r-1})^T$ and $\underline{U} = (S^n u, \ldots, S^{r-1} u)^T$. Previously in (1.6.4) it has been assumed that $E\varepsilon^T S^i u = 0$; i=0,1,..., n-1 , following the argument about the justification of this assumption, it may also be asserted that this condition will hold for i=n, n+1,..., n+r-1 and thus it follows from (1.6.10) that

$$E\varepsilon^T S^i (I-A)^{-1} Ub = 0 \quad i=1,2,\ldots,n \qquad (1.6.11)$$

hence the assumptions (1.6.5) now become

$$E\varepsilon^T S^i y = E\varepsilon^T S^i (I-A)^{-1} \varepsilon = 0 \qquad i=1,2,\ldots,n \tag{1.6.12}$$

The definition of impulse response matrix P states that $(I-A)^{-1}$, when expanded by the binomial theorem, is a convergent polynomial in S and whose coefficients after the first (r-n) are considered negligible. Let

$(I-A)^{-1} = \sum_{k=1}^{r-n} d_k S^k$ then (1.6.12) can be written

$$E\varepsilon^T S^i y = E \sum_{k=1}^{r-n} d_k \varepsilon^T S^{i+k} \varepsilon = 0 \qquad i = 1,2,\ldots,n \tag{1.6.13}$$

Thus a set of sufficient conditions for $E(X^T\varepsilon) = \underline{0}$ are

$$E\ \varepsilon^T S^i u = 0 \qquad i=0,1,\ldots, n+r-1 \tag{1.1.14}$$

$$E\ \varepsilon^T S^j \varepsilon = 0 \qquad j=1,2,\ldots, r \tag{1.6.15}$$

The above equations state the well-known assumptions that the disturbance should be uncorrelated with itself and with the input. They are also more specific in the sense that they give some idea of the range of shifts for which lack of correlation is required. It is well known that if the above assumptions are made stronger as

a) $\{\varepsilon_t\}$ is a stationary zero mean sequence with bounded fourth order moments, such that ε_t and ε_s are independent for $t \neq s$,

b) The input sequence $\{u_t\}$ is independent of $\{\varepsilon_t\}$,

c) The input sequence is persistently exciting of order n,

then $\hat{\theta}$ converges to θ with probability one as $N \to \infty$ and the covariance matrix of $\hat{\theta}$ is given by

$$\text{Cov}\ \hat{\theta} = E(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T = \sigma_\varepsilon^2 (X^T X)^{-1} \tag{1.6.16}$$

where $\sigma_\varepsilon^2 = E\varepsilon_t^2$ [15],[16].

The assumption that the disturbance is an uncorrelated sequence is so unrealistic in real life problems that the straightforward application of the method can lead to wrong conclusions. The inapplicability of least squares estimation in the case of correlated disturbances has given

rise to several other techniques which assume that appropriate pre-filtering of the data with some specified filter can convert the problem into a least squares framework.

1.7. Box and Jenkins technique

The system representation chosen by Box and Jenkins [17] is

$$y = Ay + Bu + \varepsilon \tag{1.7.1}$$

where A and B matrices are as defined before and ε is assumed to be an auto-regressive moving-average process as

$$\varepsilon = (I-Q)^{-1}(I+C)\zeta \tag{1.7.2}$$

where $C = \sum_{i=1}^{m} c_i S^i$, $Q = \sum_{i=1}^{m} q_i S^i$ and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)^T$ is an N vector of zero mean independent random variables ζ_t. For stationarity it is assumed that the polynomial $[1 - Q(z^{-1})]$ has all roots inside the unit circle of the z-plane. Since ε_t is now correlated the least squares estimates are not consistent. However, pre-filtering of the data u,y by the filter $(I-Q)(I+C)^{-1}$ will leave a transformed disturbance ζ which is uncorrelated. The difficulty is of course that the filter $(I-Q)(I+C)^{-1}$ is unknown and an iterative technique is required to generate successively improved estimates of Q and C. Details of the Box and Jenkins iterative procedure are as follows.

(1) Obtain least squares estimates $\hat{A}$, $\hat{B}$ for some given $\hat{Q}_i$ and $\hat{C}_i$ from the model

$$(I-\hat{A})\bar{y}_i = \hat{B}\bar{u}_i + \hat{\varepsilon} \tag{1.7.3}$$

where $\bar{u}_i = (I-\hat{Q}_i)(I+\hat{C}_i)^{-1}u$ and $\bar{y}_i = (I-\hat{Q}_i)(I+\hat{C}_i)^{-1}y$ are the filtered input and output data of the i^{th} iteration. Initially $\hat{Q}_i$, $\hat{C}_i$ can be taken null.

(2) Using residuals $\hat{\varepsilon}$ attempt to generate a new noise model as

$$\hat{\varepsilon} = (I-\hat{Q}_{i+1})^{-1}(I+\hat{C}_{i+1})\hat{\zeta} \tag{1.7.4}$$

(3) Revert to step (1) with re-application of the least squares routine.

The procedure terminates when $\hat{\varepsilon}$ is insignificantly different from ζ.

1.8. Generalized least squares technique

The contribution of Clarke with the generalized least squares [18] is that by representing the noise ε as a pure auto-regressive process

$$\varepsilon = (I-Q)^{-1}\zeta \tag{1.8.1}$$

the iterative procedure of the Box and Jenkins technique is considerably simplified. This follows from the fact that the estimation of the auto-regressive parameters in step (2) can be easily done by a simple least squares procedure. Three different versions of the generalized least squares method are available. In Clarke's original proposal the filtered data of the i^{th} iteration are obtained as

$$\bar{u}_i = (I-\hat{Q}_i)\,\bar{u}_{i-1} \qquad \bar{y} = (I-\hat{Q}_i)\bar{y}_{i-1}$$

where $\bar{u}_o = u$, $\bar{y}_o = y$, whereas the second version [19] obtains its filtered data from the original data as

$$\bar{u}_i = (I-\hat{Q}_i)u \qquad \bar{y}_i = (I-\hat{Q}_i)y$$

They correspond to different ways of estimating the correlation of the disturbance. The third version of the generalized least squares [20] is based on the following representation

$$y = (I-A)^{-1}\,Bu + \eta \tag{1.8.2}$$

where now η denotes the total effect of all disturbances acting on the system output y and is represented by an auto-regressive moving-average process

$$\eta = (I-Q)^{-1}\,(I+C)\zeta \tag{1.8.3}$$

Writing (1.8.2) as

$$(I-A)y = Bu + (I-A)(I-Q)^{-1}(I+C)\zeta \tag{1.8.4}$$

It follows that if a filter (I-F) can be found as

$$I-F = (I-A)^{-1}(I-Q)(I+C)^{-1} \tag{1.8.5}$$

then filtering of the data y and u by (I-F) in (1.8.4) leaves an uncorrelated disturbance term ζ. To find this filter, the procedure uses output error computed from the current estimates $\hat{A}_i$ and $\hat{B}_i$ as

$$\hat{\eta}_i = y - (I-\hat{A}_i)^{-1}\hat{B}_i u \tag{1.8.6}$$

and fits an auto-regressive model to $\hat{\eta}_i$ at the i^{th} iteration as

$$\hat{\eta}_i = (I-\hat{G}_i)^{-1}\hat{\zeta} \tag{1.8.7}$$

Comparison of (1.8.3) with (1.8.7) shows that $(I-\hat{G})$ is an approximation to $(I-Q)(I+C)^{-1}$, then the required filter at the $(i+1)^{th}$ iteration is obtained as

$$(I-\hat{F}_{i+1}) = (I-\hat{A}_i)^{-1}(I-\hat{G}_i) \tag{1.8.8}$$

The advantage of this version in comparison with the previous ones is that if the polynomial $1-A(z^{-1})$ has a root near unity, a high order auto-regressive model may be needed if the filter (I-F) is estimated by fitting an auto-regressive model to the residual

$$\hat{\varepsilon}_i = (I-\hat{A}_i)y - \hat{B}_i u \tag{1.8.9}$$

A similar technique to the third version was previously proposed by Steiglitz and McBride [21], where it was assumed that $\eta = \zeta$, i.e. white noise disturbance at the system output. An extension to the correlated output noise was also reported in [22]. However, it was assumed that the noise characteristics are known.

The convergence of generalised least squares technique was first discussed by Phillipson [23] and it was shown that the first version is convergent in the sense that the sum of squares of residuals $(\hat{\varepsilon}^T\hat{\varepsilon})$ decreases at each step. Later in [19], asymptotic properties of the estimates obtained from the second version were analyzed and it was

proved that the method may give wrong estimates if the signal to noise ratio is low enough.

1.9. Extended matrix method

This scheme [24] estimates the system and the disturbance parameters in one single procedure instead of separate algorithms as applied in the generalized least squares method. The disturbance is described as an auto-regressive moving-average process

$$\varepsilon = (I-Q)^{-1}\,(I+C)\zeta \tag{1.9.1}$$

combining (1.9.1) with the system representation

$$y = Ya + Ub + \varepsilon \tag{1.9.2}$$

(1.9.2) can be written as

$$\begin{aligned} y &= Ya + Ub + Q\varepsilon + C\zeta + \zeta \\ y &= Ya + Ub + \Omega q + \Gamma c + \zeta \\ y &= (Y \,\vdots\, U \,\vdots\, \Omega \,\vdots\, \Gamma) \begin{bmatrix} a \\ \hdashline b \\ \hdashline q \\ \hdashline c \end{bmatrix} + \zeta \end{aligned} \tag{1.9.3}$$

where $\Omega = (S\varepsilon,\ldots,S^m\varepsilon)$, $q = (q_1,\ldots,q_m)^T$, $\Gamma = (S\zeta,\ldots,S^m\zeta)$, $c = (c_1,c_2,\ldots,c_m)^T$, since ζ is a vector of independent zero mean sequence ζ_t then

$$E(Y \,\vdots\, U \,\vdots\, \Omega \,\vdots\, \Gamma)^T\zeta = \underline{0} \tag{1.9.4}$$

and hence the application of the least squares technique to (1.9.3) gives consistent estimates of (a,b,q,c). As the elements of Ω and Γ are unknown, these are replaced by their estimates $\hat{\Omega}$ and $\hat{\Gamma}$ whose elements are obtained from estimates $(\hat{a},\hat{b},\hat{q},\hat{c})$ of the i^{th} iteration as

$$\hat{\varepsilon}_t = y_t - \hat{A}_i(z^{-1})y_t - \hat{B}_i(z^{-1})u_t \tag{1.9.5}$$

$$\hat{\zeta}_t = \hat{\varepsilon}_t - \hat{Q}_i(z^{-1})\hat{\varepsilon}_t - \hat{C}_i(z^{-1})\hat{\zeta}_t \tag{1.9.6}$$

This technique is first discussed by Young [25] for the moving-average disturbance case.

One common feature of the estimation techniques discussed so far is that either by a suitable filtering of the data or by representing the system conveniently, the estimation problem is formulated such that the transformed disturbances are uncorrelated and hence satisfy the assumptions of the least squares technique. There are a number of other techniques whose basic philosophy is not to cast the problem into a least squares framework but to exploit the lack of correlation between input and disturbance to obtain consistent estimates. Three techniques which fall into this category will now be discussed.

1.10. Tally principle

The tally principle was first formulated by Peterka and Šmuk [26]. Following Gustavson [27], the technique can be described as follows. Let the system be represented as

$$y = Ya + Ub + \varepsilon = X\theta + \varepsilon \tag{1.10.1}$$

where it is only assumed that ε_t and u_s are uncorrelated for all t,s. If an Nxr matrix U_r is defined as $U_r = (u, Su, \ldots, S^{r-1}u)$ then this assumption can be written as

$$E(\varepsilon | U_r) = E\,\varepsilon + U_r\,\phi \tag{1.10.2}$$

subject to the condition that all r elements ϕ_i of the vector $\phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$ are identical to zero. For convenience here, it will be assumed that $E\,\varepsilon = \underline{0}$. Let $\hat{a}, \hat{b}$ denote arbitrary parameter vectors spanning the same space with the vectors a and b, since in general $\hat{a} \neq a$, $\hat{b} \neq b$ then

$$E(\hat{\varepsilon} | U_r) = U_r\,\hat{\phi} \neq \underline{0} \tag{1.10.3}$$

where $\hat{\varepsilon}$ is the residual vector corresponding to $\hat{a}$ and $\hat{b}$, i.e.

$$\hat{\varepsilon} = y - Y\hat{a} - U\hat{b} \tag{1.10.4}$$

Using (1.10.3) $\hat{\varepsilon}$ can be written as

$$\hat{\varepsilon} = U_r\,\hat{\phi} + \hat{e} \tag{1.10.5}$$

where the N vector $\hat{e}$ satisfies $E(\hat{e}|U_r) = \underline{0}$. From (1.10.4) and (1.10.5)

$$\hat{e} = y - Y\hat{a} - U\hat{b} - U_r\hat{\phi} = y - X\hat{\theta} - U_r\hat{\phi} \qquad (1.10.6)$$

The estimate $\hat{\theta}$ is obtained in the following two steps:

a) obtain $\hat{\phi}$ given $\hat{\theta}$ by minimising $\hat{e}^T\hat{e}$,

b) find $\hat{\theta}$ which minimizes $\hat{\phi}^T\hat{\phi}$.

The resulting estimate $\hat{\theta}$ can be written as

$$\hat{\theta} = [X^TU_r(U_r^TU_r)^{-2}U_r^TX]^{-1}\ X^TU_r(U_r^TU_r)^{-2}U_r^Ty \qquad (1.10.7)$$

The choice of number of vectors r in U_r has been discussed and it has been shown that r should be greater than the impulse response duration. The technique has also been used to estimate the system and the disturbance parameters in the Astrom's representation [28]

$$y = Ya + Ub + (I+C)\zeta \qquad (1.10.8)$$

1.11. Two stage least squares

The two stage least squares technique, proposed by Theil [29] for the solution of linear simultaneous equations model in econometrics, has later been applied for the estimation of system and disturbance parameters in the Astrom's model by Pandya and Pagurek [30]. The technique consists of the following stages.

Stage 1: Write (1.10.8) in the impulse response representation as

$$y = (I-A)^{-1}Bu + \eta = Pu + \eta = U_rp + \eta \qquad (1.11.1)$$

where $P = (I-A)^{-1}B$, $\eta = (I-A)^{-1}(I+C)\zeta$, $U_r = (u, Su, \ldots, S^{r-1}u)$, $p = (p_0, p_1, \ldots, p_{r-1})$ and r is the duration of impulse response. Since $E(U_r^T\eta) = \underline{0}$ that is, input and disturbance are uncorrelated then the least squares estimate $\hat{p}$ of p obtained as

$$\hat{p} = (U_r^TU_r)^{-1}U_r^Ty \qquad (1.11.2)$$

is consistent.

Stage 2: Using $\hat{p}$ in (1.11.2), obtain the estimated or fitted y as

$$\hat{y} = U_r\hat{p} = U_r(U_r^TU_r)^{-1}U_r^Ty = P(U_r)y \qquad (1.11.3)$$

where $P(U_r) = U_r(U_r^TU_r)^{-1}\ U_r^T$ is a projection matrix [31]. Since $Y = (Sy, S^2y, \ldots, S^ny)$ then correspondingly $\hat{Y}$ is

$$\hat{Y} = P(U_r)Y \qquad (1.11.4)$$

Using $\hat{Y}$, (1.11.1) can be written as

$$\begin{aligned} y &= \hat{Y}a + Ub + (Y-\hat{Y})a + (I+C)\zeta \\ &= P(U_r)Ya + P(U_r)Ub + (Y-\hat{Y})a + (I+C)\zeta \\ &= P(U_r)(Y\,|\,U)\begin{pmatrix}a\\ \hline b\end{pmatrix} + (Y-\hat{Y})a + (I+C)\zeta \\ &= P(U_r)\ X\theta + (Y-\hat{Y})a + (I+C)\zeta \end{aligned} \qquad (1.11.5)$$

where in the second equality the use is made of $P(U_r)U = U$ which follows from the fact that U lies in the range space of U_r. If θ is estimated from (1.11.5) by minimising $\hat{\underline{\varepsilon}}^T\hat{\underline{\varepsilon}}$ where

$$\hat{\underline{\varepsilon}} = y - P(U_r)\ X\hat{\theta} \qquad (1.11.6)$$

then the resulting least squares estimates $\hat{\theta}$ can be written as

$$\begin{aligned} \hat{\theta} &= [X^TP(U_r)X]^{-1}\ X^T\ P(U_r)y \quad \text{or} \\ &= (X^T\ U_r(U_r^TU_r)^{-1}\ U_r^TX]^{-1}\ X^TU_r(U_r^TU_r)^{-1}\ U_r^Ty \end{aligned} \qquad (1.11.7)$$

where the term $\{(Y-\hat{Y})\hat{a}\}$ has no effect on the solution, since from (1.11.4) and $[P(U_r)\ .\ P(U_r)] = P(U_r)$ it follows that $X^TP(U_r)(Y-\hat{Y})\hat{a} = \underline{0}$. The disturbance model $(I+\hat{C})$ is estimated by re-applying the technique to the residuals $\hat{\varepsilon} = (I+\hat{C})\zeta$.

Before proceeding further, it is to be noted here that since both the tally principle and the two-stage least squares technique exploit the same assumptions, namely the lack of correlation between input and disturbance, their resulting estimates (1.10.7) and (1.11.7) are quite similar. In fact for an uncorrelated input sequence, i.e. for

$$\frac{1}{N}\ (U_r^TU_r) \simeq I_r$$

they are equivalent.

Another technique in the same vein is the instrumental variable method.

1.12. Instrumental variable method

For the system represented by

$$y = Ya + Ub + \varepsilon = X\theta + \varepsilon \tag{1.12.1}$$

an instrumental variable estimate of θ has the following general form

$$\hat{\theta} = (Z^T X)^{-1} Z^T y \tag{1.12.2}$$

where Z is an (N x 2n) matrix of instrumental variables such that the following two limits exist.

$$\lim_{N \to \infty} \frac{1}{N} Z^T \varepsilon = \underline{0} \quad , \quad \lim_{N \to \infty} \frac{1}{N} Z^T X \text{ is non-singular} \tag{1.12.3}$$

By an application of the Slutzky theorem [32], it immediately follows that if the above limits exist then $\hat{\theta}$ converges to θ as $N \to \infty$ either in probability or with probability one depending on the specifications of stochastic limits in (1.12.3).

The idea of instrumental variables is very old in econometrics [33], in control engineering it was first introduced by Wong and Polak [34] and later used by Young [35] where the instrumental matrix Z is formed as $Z = (W \vdots U)$ where $W = (w, Sw, \ldots, S^{n-1}w)$ and w is generated from input only by use of a model

$$w = W\bar{a} + U\bar{b} \tag{1.12.4}$$

where $\bar{a}$ and $\bar{b}$ are the parameter vectors of the assumed model. In [34], [35] these are chosen as estimates of the true parameter vectors a,b and these estimates are obtained from the recursive computation of (1.12.2). Later it has been shown by Finigan and Rowe [36] and by Soderstrom [37] that if u_t is white noise and under some mild conditions on the system (1.12.1) and its noise free model (1.12.4), then for all choices of $\bar{a},\bar{b}$ the resulting Z is an instrumental matrix, that is, the limits in (1.12.3) are satisfied.

Writing (1.12.4) in the form

$$w = (I-\bar{A})^{-1}\,\bar{B}u = \bar{P}u = U_r\bar{p} \qquad (1.12.5)$$

where $\bar{A} = \sum_{i=1}^{n} \bar{a}_i S^i$ and $\bar{B} = \sum_{i=1}^{n} \bar{b}_i S^{i-1}$, the first stochastic limit of (1.12.3) can be written as

$$\lim_{N\to\infty} \frac{1}{N}\,\varepsilon^T S^i u = 0 \qquad i = 0,1,\ldots,\ r+n-2 \qquad (1.12.6)$$

which implies

$$E\varepsilon^T S^i u = 0 \qquad i = 0,1,\ldots,\ r+n-2 \qquad (1.12.7)$$

hence the choice of assumptions here is again the lack of correlation between input and disturbance.

The identification methods discussed so far in this chapter have one feature in common - in all methods no assumptions are made about the distribution of the disturbances. If there are grounds for specifying this distribution, it is in general possible to obtain more efficient estimates than those of the previous techniques. For example, the maximum likelihood method of Astrom and Bohlin [8] assume that the elements of ζ of the following model

$$y = Ya + Ub + (I+C)\zeta \qquad (1.12.8)$$

are normally distributed and the parameters are estimated by maximizing the joint probability density function of observations over the parameter space $(\hat{a},\hat{b},\hat{c}_1,\ldots,\hat{c}_m)$. Astrom and Bohlin have used a version of the generalized Newton-Raphson procedure for this maximization. There are also other available procedures, for example, the second version of the generalized least squares technique can be interpreted as the maximization of the density function by a relaxation method [19]. Various computational problems still remain to be solved; e.g. due to the multi-peak likelihood function, a chosen procedure may only converge to a local maximum point [38],[39]. It is well known that [8] if the algorithm converges to a global maximum then the resulting maximum likelihood estimates are consistent and efficient in the sense that the covariance of estimates

equals to the Cramer-Rao lower bound [40].

1.13. Conclusions

Will all these identification techniques work in real life problems? It is almost impossible to say, since the crucial factor in every identification experiment is whether the assumptions are met by the unknown disturbances. There have been successful industrial applications of these techniques, but it does not follow that disturbances will always fit the assumed pattern.

For an open loop identification experiment, the assumption that input and disturbance are independent is a natural one and can be justified from the physical considerations, but also assuming that the disturbance is an independent process is so unrealistic in practice that the least squares method must be discounted as a serious technique. The generalization of the disturbance model to an auto-regressive moving-average type in which the generating process is independent has given rise to several generalized least squares techniques where the implicit assumption is that the disturbance has a rational spectra [6] which may not be true in connection with real life processes [41]. The maximum likelihood method where the disturbance is assumed normally distributed has nice asymptotic properties, however the resulting estimates are, in general, not robust for a mild deviation of disturbance from the assumed pattern [41],[42].

So far the main concern has been with the validity of assumptions, but an equally important point is the sensitivity of estimates to the inevitable assumption errors. In the next chapter a new estimation method will be proposed which takes into account the violation of assumptions when deriving its estimates.

CHAPTER II

THE MODIFIED LEAST SQUARES ALGORITHM

2.1. Introduction

This chapter presents a new approach to system parameter estimation. It is assumed that the only reliable feature of disturbance is its independence of input. This yields a set of assuptions in excess of the minimal requirements and an endeavour has been made to exploit this excess to minimize the parameter estimation errors. The resulting estimates are equivalent to those of the two stage least squares method. However it is believed that the way of thinking about the estimation problem is original.

2.2. Modified least squares algorithm

As it has been surveyed in Chapter I, there are a number of techniques which derive their estimates by exploiting the assumptions that input and disturbance are statistically independent and hence uncorrelated. These assumptions will be represented by the following set of r equations

$$U_r^T \varepsilon \simeq \underline{0} \qquad (2.2.1)$$

where $U_r = (u, Su, \ldots, S^{r-1}u)$ is an Nxr matrix. The assumptions (2.2.1) effectively require that the component of ε lying in the range space of U_r is small. There also exists a matrix L which consists of normalized column vectors such that $L^TL = I_r$ and whose range space coincides with that of U_r. The equations (2.2.1) are effectively unchanged by substituting L for U_r, i.e.

$$L^T\varepsilon \simeq \underline{0} \qquad (2.2.2)$$

Although the calculation of L matrix introduces the problem of computing an ortonormal set of vectors spanning the space $(u, Su, \ldots, S^{r-1}u)$, this

can be done efficiently by Grahm-Schmidt orthonormalization procedure [43].

Following the vector-matrix representation of previous chapter, the N equations of the system

$$y_t = A(z^{-1})y_t + B(z^{-1})u_t + \varepsilon_t \qquad t = 1,2,\ldots,N \tag{2.2.3}$$

will be represented as

$$y = Ya + Ub + \varepsilon = X\theta + \varepsilon \tag{2.2.4}$$

where it is assumed that the order n of the polynomials $A(z^{-1}) = \sum_{i=1}^{n} a_i z^{-i}$ and $B(z^{-1}) = \sum_{i=1}^{n} b_i z^{-i}$ is known or at least an upper bound can be given. As it has been discussed in section 1.4, the estimation of θ and ε requires 2n additional assumptions to complement (2.2.4) to (N + 2n) equations. It is then necessary to reduce the r assumptions (2.2.2) to the 2n required for estimation. Denote the latter by

$$M^T\varepsilon \simeq \underline{0} \tag{2.2.5}$$

where M is an Nx2n matrix. It will be assumed that each of these 2n assumptions (2.2.5) will consist of some arbitrarily weighted sum of the original r assumptions and this will be expressed in terms of an arbitrary rx2n matrix F such that

$$M = LF \tag{2.2.6}$$

thus using the estimation equation (1.4.7) for $\gamma = \underline{0}$, the estimates are given by

$$\hat{\theta} = (F^TL^TX)^{-1} F^TL^Ty \tag{2.2.7}$$

For the existence of estimates it is required that the matrices F and L are such that $(F^TL^TX)^{-1}$ exists. This is the identifiability condition with respect to 2n assumptions. On substitution of y from (2.2.4), (2.2.7) can be written as

$$\hat{\theta} - \theta = (F^TL^TX)^{-1} F^TL^T\varepsilon \tag{2.2.8}$$

Since in general the assumptions (2.2.2) will not be completely met, some allowance must be given to the inevitable assumption errors. To do this ε can be decomposed into a component $\bar{\varepsilon}$ which exactly satisfies the assumptions, i.e. $L^T\bar{\varepsilon} = \underline{0}$ and a component Lv which violates them, i.e. $L^T\varepsilon = L^TLv \neq \underline{0}$ where the unknown r vector v is defined by $v = (L^TL)^{-1} L^T\varepsilon$. Under this decomposition the estimation errors are given by

$$\hat{\theta} - \theta = (F^TL^TX)^{-1} F^TL^TLv = (F^TL^TX)^{-1} F^Tv \tag{2.2.9}$$

and the sum of squares of estimation errors by

$$||\hat{\theta} - \theta||^2 = v^TF(X^TLF)^{-1}(F^TL^TX)^{-1}F^Tv = v^T\Psi v \tag{2.2.10}$$

where $\Psi = F(X^TLF)^{-1}(F^TL^TX)^{-1}F^T$ is an rxr symmetrix matrix. The objective is to select F such that the estimation error $||\hat{\theta} - \theta||^2$ is in some sense minimum taking into consideration that v is unknown. There is still an ambiguity in F since the effect of post-multiplying F by any non-singular matrix leaves the corresponding Ψ unchanged. This ambiguity is to be eliminated without loss of generality by imposing the constraints

$$X^TLF = I_{2n} \tag{2.2.11}$$

which ensures the required non-singularity of (X^TLF). The constraints (2.2.11) are satisfied by

$$F = L^TX(X^TLL^TX)^{-1} + K \tag{2.2.12}$$

where K is an arbitrary rx2n matrix subject to $X^TLK = (0)$. Substitution from (2.2.11) into (2.2.10) the sum of squares of estimation errors becomes

$$||\hat{\theta} - \theta||^2 = v^TFF^Tv \tag{2.2.13}$$

which is a positive semi-definite quadratic form of which the vector v is unknown. One criterion appropriate for minimization of (2.2.13) is

$$\text{tr } \Psi = \text{tr } FF^T = \text{tr } F^TF = \text{tr}(X^TLL^TX)^{-1} + \text{tr } K^TK \geq \text{tr}(X^TLL^TX)^{-1} \tag{2.2.14}$$

where the equality holds if and only if K is null. On this criterion therefore the best choice is

$$F = L^T X(X^T LL^T X)^{-1} \tag{2.2.15}$$

On substitution from (2.2.15) into (2.2.7) the estimation equations are obtained as

$$\hat{\theta} = (X^T LL^T X)^{-1} X^T LL^T y \tag{2.2.16}$$

The minimization of (tr Ψ) is a valid criterion since it is related to the magnitude of the quadratic form which is the sum of squares of estimation errors. It is also to be noted however that one reason for its choice is the fact that it gives rise to a comparatively simple estimation algorithm. Another possible choice of criterion is the minimization of the largest eigenvalue of Ψ. This choice follows from the relation that

$$v^T \Psi v \leq \lambda_m v^T v \tag{2.2.17}$$

where λ_m denotes the largest eignevalue of Ψ. It would then be appropriate to choose F such that λ_m is minimized. This criterion is highly favourable from an engineering point of view since it seeks to alleviate estimation errors in the worst possible situation, namely when v lies entirely in the range space of F, however for the less extreme cases it may increase the estimation errors. Furthermore choice of F to minimize λ_m is a problem which has not an easy answer. However it will be shown below that the choice of F which minimizes (trΨ) is one solution to this problem, i.e. it also minimizes λ_m.

From $\Psi = FF^T$, it can be seen from an induction proof that the following identity holds

$$\Psi^i \equiv F(F^T F)^{i-1} F^T \tag{2.2.18}$$

Let the largest eigenvalue λ_m occur with multiplicity p then it follows that

$$\lim_{i \to \infty} \operatorname{tr} \left(\frac{\Psi}{\lambda_m}\right)^i = p = \lim_{i \to \infty} \operatorname{tr} \frac{F(F^TF)^{i-1}F^T}{\lambda_m^i}$$

$$= \lim_{i \to \infty} \operatorname{tr} \left(\frac{F^TF}{\lambda_m}\right)^i \qquad (2.2.19)$$

the latter limit is p if and only if F^TF has a largest eigenvalue λ_m with multiplicity p, then

$$v^TF^TFv \leq \lambda_m v^Tv \qquad (2.2.20)$$

but using (2.2.12)

$$v^TF^TFv = v^T[(X^TLL^TX)^{-1} + K^TK]v \qquad (2.2.21)$$

thus from (2.2.20),(2.2.21) it follows that

$$\lambda_m \geq (v^Tv)^{-1} [v^T(X^TLL^TX)^{-1} v + v^TK^TKv] \qquad (2.2.22)$$

hence λ_m is minimized with respect to K by $Kv = \underline{0}$ which includes K = (0), that is, the choice of F with K = (0) is one solution to the minimization of the largest eigenvalue λ_m.

Comments

1) The modified least squares algorithm (2.2.16) has an interesting interpretation from the generalized least squares point of view. It can be shown that the estimation equation is based on minimisation of the criterion

$$V = \hat{\varepsilon}^TR\hat{\varepsilon} \qquad (2.2.23)$$

where the weighting matrix $R = LL^T = L(L^TL)^{-1}L^T$ is a projection matrix of rank r into the range space of L. Hence what is sought is that $\hat{\varepsilon}$ which has the smallest amplitude component in the r dimensional sub-space which is the range space of L. It will also be noted that in the extreme case where the number of assumptions r equals to the number of observations N, the modified least squares estimates are identical to that of the least squares, since if L is full rank then L^{-1} exists and hence $R = I_N$.

2) The modified least squares algorithm is identical to that of two stage least squares (1.9.16). This follows from the fact that since the column vectors of L matrix are linear combinations of the column vectors of U_r [43] then it is possible to write $L = U_r\Phi$ where Φ is an rxr non-singular matrix. On substitution of $L = U_r\Phi$ into R gives $R = U_r(U_r^TU_r)^{-1}\ U_r^T$ hence the modified least squares algorithm can also be written as

$$\hat{\theta} = [X^TU_r(U_r^TU_r)^{-1}\ U_r^TX]^{-1}\ X^TU_r(U_r^TU_r)^{-1}\ U_r^Ty \qquad (2.2.24)$$

which is identical to the two stage least squares estimates (1.11.7).

3) An important point in connection with identification is the choice of input. Because of the desirable properties, it has been common practice to use uncorrelated sequences such as pseudo-random binary sequences. There is a strong incentive to use such sequences for the modified least squares algorithm, since then the column vectors of U_r will be orthogonal and consequently there will be no need to employ time consuming orthogonalization procedure to generate L. More explicitly the required conditions are

$$\frac{1}{N}\ U_r^TU_r = \kappa\ I_r \qquad (2.2.25)$$

where κ is a constant depending on the signal amplitude. In this event the estimates are given by

$$\hat{\theta} = (X^TU_rU_r^TX)^{-1}\ X^TU_rU_r^Ty \qquad (2.2.26)$$

One of the unresolved difficulties associated with the modified least squares algorithm is the choice of r, the number of assumptions. In the next section one possible way to estimate r will be presented.

2.3. The choice of number of assumptions

This problem will only be discussed for an uncorrelated input sequence and hence the algorithm (2.2.26) will be used throughout the

discussion. It will be shown that if r is chosen greater than the impulse response duration plus the order of system then approximately no significant change will occur in the resulting estimates, that is, if the impulse response duration is denoted by J then

$$\hat{\theta}_r \simeq \hat{\theta}_{r-1} \quad \text{for} \quad r > J + n \tag{2.3.1}$$

where the suffix r of $\hat{\theta}$ indicates that the estimates are based on r assumptions.

By partitioning U_r as $U_r = (U_{r-1} \,\vdots\, S^{r-1}u)$, $(X^T U_r U_r^T X)^{-1}$ can be written as

$$(X^T U_r U_r^T X)^{-1} = [(X^T U_{r-1} U_{r-1}^T X) + X^T S^{r-1} u \, (S^{r-1}u)^T X]^{-1} \tag{2.3.2}$$

by applying the well-known matrix inversion lemma [44], (2.3.2) takes the form

$$(X^T U_r U_r^T X)^{-1} = \left[I_{2n} - \frac{(X^T U_{r-1} U_{r-1}^T X)^{-1} X^T S^{r-1} u \, (S^{r-1}u)^T X}{1 + (S^{r-1}u)^T X \, (X^T U_{r-1} U_{r-1}^T X)^{-1} X^T S^{r-1} u} \right] (X^T U_{r-1} U^T{}_{r-1} X)^{-1}$$

$$= \Gamma_{r-1} \, (X^T U_{r-1} \, U_{r-1}^T X)^{-1} \tag{2.3.3}$$

where Γ_{r-1} is the matrix given in square brackets. Using the same partitioning again

$$X^T U_r U_r^T y = X^T U_{r-1} \, U_{r-1}^T y + X^T S^{r-1} u \, (S^{r-1}u)^T y \tag{2.3.4}$$

substituting from (2.3.3) and (2.3.4) into (2.2.26), $\hat{\theta}_r$ can be written as

$$\hat{\theta}_r = \Gamma_{r-1} \, \hat{\theta}_{r-1} + \Gamma_{r-1} \, (X^T U_{r-1} \, U_{r-1}^T X)^{-1} X^T S^{r-1} u (S^{r-1}u)^T y \tag{2.3.5}$$

Now it will be shown that for $r > J + n$, $\Gamma_{r-1} \simeq I_{2n}$ and the second term of (2.3.5) is approximately zero.

From the system representation (2.2.4) it is possible to write

$$y = (I-A)^{-1} Bu + (I-A)^{-1}\varepsilon = Pu + (I-A)^{-1}\varepsilon = U_J \, p + (I-A)^{-1}\varepsilon \tag{2.3.6}$$

where the matrices A,B,P are as defined in Chapter I and $U_J = (u, Su, \ldots, S^{J-1}u)$. Now consider the term $(S^{r-1}u)^T y$ of (2.3.5) which can be written as

$$(S^{r-1}u)^T y = (S^{r-1}u)^T \, U_J p + (S^{r-1}u)^T \, (I-A)^{-1}\varepsilon \tag{2.3.7}$$

Assuming that the disturbance ε fits the required assumptions (2.2.1) for all r then the second term of (2.3.7) is approximately zero and hence

$$(S^{r-1}u)^T y \simeq (S^{r-1}u)^T U_J p = [(S^{r-1}u)^T u, \ldots, (S^{r-1}u)^T S^{J-1}u]p \tag{2.3.8}$$

it then follows that for an uncorrelated input sequence

$$(S^{r-1}u)^T y \simeq 0 \quad \text{for} \quad r > J \tag{2.3.9}$$

and hence the second term of (2.3.5) is approximately zero. Consider now the term $(S^{r-1}u)^T X$ in Γ_{r-1} which can be written by using the partitioning $X = (Y \vdots U)$ as

$$(S^{r-1}u)^T X = [(S^{r-1}u)^T Sy, \ldots, (S^{r-1}u)^T S^n y, (S^{r-1}u)^T u, \ldots, (S^{r-1}u)^T S^{n-1}u \tag{2.3.10}$$

the first n elements of this row vector are $(S^{r-1}u)^T S^i y$; $i = 1, \ldots, n$. Using y from (2.3.6) these can be written as

$$\begin{aligned}(S^{r-1}u)^T S^i y &= (S^{r-1}u)^T S^i U_J p + (S^{r-1}u)^T S^i (I-A)^{-1}\varepsilon \\ &= \sum_{j=0}^{J-1} p_j (S^{r-1}u)^T S^{j+i}u + (S^{r-1}u)^T S^i (I-A)^{-1}\varepsilon \; ; \; i=1,2,\ldots,n\end{aligned} \tag{2.3.11}$$

Following the same argument above, it can be seen that

$$(S^{r-1}u)^T S^i y \simeq 0 \quad i=1,\ldots,n \quad \text{for } r > J + n.$$

From orthogonality of input it also follows that for $r > J + n$ the later n elements of $(S^{r-1}u)^T X$ are also zero and hence

$$(S^{r-1}u)^T X \simeq 0 \quad \text{for } r > J + n \tag{2.3.12}$$

which implies approximately

$$\Gamma_{r-1} \simeq I_{2n} \quad \text{for } r > J + n \tag{2.3.13}$$

Incorporating (2.3.9) and (2.3.13) with (2.3.5) it follows that (2.3.1) holds, that is, the choice of r greater than the impulse response duration plus the order of system leaves the modified least squares estimates approximately unchanged. This conclusion is in fact in agreement with the choice of number of assumption in parallel estimation techniques such as tally principle [26] and two stage least squares technique [30] where in

the latter r is explicitly chosen as the duration of impulse response.

In cases where the impulse response duration is not known a priori a different approach is required to determine r. This can be based on the following reasoning. An increase in r permits a greater degree of arbitrariness in matrix F which can be exploited to reduce the ($tr\Psi$) criterion; but at the same time this is counteracted by the additional elements added to vector v. The inequality (2.2.17)with the choice of largest eigenvalue λ_m illustrates this dilemma. In this situation, a reasonable way to proceed is to assert that the similarity of assumptions suggest that they will be violated to the same degree and $v^T v$ will be, very roughly, proportional to r. This suggests that the choice of r can be based on a criterion ($r.tr\Psi$).

The minimum possible value of r can be determined by considering the non-singularity of $(X^T U_r U_r^T X)$ in (2.2.26). If $r < 2n$ then U_r is of rank less than 2n and hence $(X^T U_r U_r^T X)$ is of rank less than 2n and consequently is singular. Therefore r should be chosen at least greater than twice the system order.

The next section will consider the asymptotic properties of the modified least squares estimates.

2.4. Asymptotic properties of estimates

It will be shown in this section that under some mild conditions on the disturbance, the modified least squares estimates converge to their true values when $N \to \infty$. In order to prove this property following lemmas will be needed.

Lemma 1: If the covariance function of a zero-mean stochastic process x_t satisfies

$$|Ex_t x_s| \leq \gamma \frac{t^\alpha + s^\alpha}{1 + |t-s|^\beta} \qquad (2.4.1)$$

with $0 \leq 2\alpha < \beta < 1$ and $\gamma > 0$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} x_t = 0 \qquad (2.4.2)$$

with probability one and in mean square. Proof is given in [45].

Lemma 2: If

i) $\{u_t\}$ is a bounded sequence, $|u_t| \leq K_1$ say,

ii) ε_t is a zero-mean stationary stochastic process independent of u_t and whose covariance function satisfies

$$|E\varepsilon_{t+\tau}\, \varepsilon_t| \leq K_2\, \tau^{-\delta} \quad \text{for } \tau \geq 1, \quad 0 < \delta < 1 \qquad (2.4.3)$$

then for all finite $i,j << N$

$$\lim_{N \to \infty} \frac{1}{N} (S^i\varepsilon)^T S^j u = 0 \qquad (2.4.4)$$

with probability one and in mean square.

Proof:

$$\frac{1}{N} (S^i\varepsilon)^T S^j u = \frac{1}{N} \sum_{t=\max(i,j)}^{N-1} \varepsilon_{t-i+1}\, u_{t-j} \qquad (2.4.5)$$

Let us define $x_t = \varepsilon_{t-i+1}\, u_{t-j}$ then x_t is a zero-mean stochastic process whose covariance function satisfies, from assumptions (i) and (ii)

$$|Ex_t x_s| = |E\, \varepsilon_{t-i+1}\, \varepsilon_{s-i+1}\, u_{t-j}\, u_{s-j}| \leq K_1^2\, |E\varepsilon_{t-i+1}\, \varepsilon_{s-i+1}|$$

$$\leq K_1^2\, K_2 |t-s|^{-\delta} \qquad (2.4.6)$$

It will now be shown that for $\alpha = 0$, $\beta = \delta$ and $\gamma = K_1^2 .\ \max\left(\frac{E\varepsilon_t^2}{2}, K_2\right)$, (2.4.6) implies (2.4.1) and hence (2.4.4) holds with probability one and in mean square. Taking $\alpha = 0$, $\beta = \delta$ and $t = s$ in (2.4.1)

$$\gamma \geq \frac{1}{2} Ex_t^2 \qquad (2.4.7)$$

substituting $x_t = \varepsilon_{t-i+1}\, u_{t-j}$ into (2.4.7) and using $E\varepsilon_{t-i+1}^2 = E\varepsilon_t^2$, $Eu_{t-j}^2 \leq K_1^2$, it follows that

$$\gamma \geq K_1^2 \frac{E\varepsilon_t^2}{2} \qquad (2.4.8)$$

Once again using (2.4.1) for $\alpha = 0$, $\beta = \delta$, $t \neq s$ and $\gamma = K_1^2 K_2$

$$|Ex_t x_{t-\tau}| \leq K_1^2 K_2 \frac{2}{1+\tau^\delta} \tag{2.4.9}$$

where $\tau = t - s$. But from (2.4.6) for $s = t - \tau$

$$|Ex_t\ x_{t-\tau}| \leq K_1^2 K_2 \frac{1}{\tau^\delta} \tag{2.4.10}$$

comparison of (2.4.9) with (2.4.10) and noting that for $\tau \geq 1$ and $0 < \delta < 1$

$$\frac{1}{\tau^\delta} \leq \frac{2}{1+\tau^\delta} \tag{2.4.11}$$

hence (2.4.10) implies (2.4.9) then it follows that for $\alpha = 0$, $\beta = \delta$ and $\gamma = K_1^2 \max\left(\frac{E\varepsilon_t^2}{2}, K_2\right)$; (2.4.6) implies (2.4.1) and consequently (2.4.4) holds with probability one and in mean square.

From a practical viewpoint, convergence with probability one is very attractive, since it states that (2.4.4) holds for almost all sequences of the disturbance ε_t. However, it also requires that a rather restrictive assumption (2.4.3) must be satisfied by the covariance function of ε_t. By relaxing (2.4.3), the convergence in probability will be given in the next lemma.

Lemma 3: If

i) $|u_t| \leq K_1$ and

ii) ε_t is a zero-mean stationary stochastic process independent of u_t and whose covariance function tends to zero at a rate faster than $\frac{1}{N}$ as $N \to \infty$, i.e.

$$\lim_{N \to \infty} N\ |E\varepsilon_{t+N}\ \varepsilon_t| = 0 \tag{2.4.12}$$

then

$$\lim_{N \to \infty} \frac{1}{N} (S^i\varepsilon)^T S^j u = 0$$

in probability. Proof is given by Wong and Polak [34].

The consistency of the modified least squares estimates will now be proved in the following theorem.

Theorem 1: Consider the stable system

$$y_t = A(z^{-1})y_t + B(z^{-1})u_t + \varepsilon_t \tag{2.4.13}$$

and assume that

i) the polynomials $[1-A(z^{-1})]$ and $B(z^{-1})$ have no common factors,

ii) ε_t is a zero mean stationary stochastic process, independent of u_t and its covariance function satisfies (2.4.3),

iii) the input sequence $u = (u_o, u_1, \ldots, u_{N-1})$ is chosen such that

$$\lim_{N \to \infty} \frac{1}{N} (S^i u)^T S^j u = 0; \quad i \neq j$$

and without loss of generality that

$$\lim_{N \to \infty} \frac{1}{N} (S^i u)^T S^j u = 1 ; \quad i = j$$

then the modified least squares estimates $\hat{\theta} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n)^T$ converge to their true value θ with probability one and also in mean square.

Proof: From (2.2.4) and (2.2.26), the estimation errors can be written as

$$\hat{\theta} - \theta = \left(\frac{1}{N^2} X^T U_r U_r^T X\right)^{-1} \frac{1}{N^2} X^T U_r U_r^T \varepsilon \qquad (2.4.14)$$

Now, if it can be shown that

(I) $$\lim_{N \to \infty} \frac{1}{N^2} X^T U_r U_r^T \varepsilon = \underline{0}$$

and

(II) $$\lim_{N \to \infty} \frac{1}{N^2} (X^T U_r U_r^T X)$$ is positive-definite and hence its inverse is non-singular, then

$$\lim_{N \to \infty} \hat{\theta} = \theta \qquad (2.4.15)$$

with probability one and in mean square.

Proof of (I): By partitioning X as $X = (Y \vdots U_n)$, where the subscript n of U_n denotes the number of column vectors of U, i.e. $U_n = (u, Su, \ldots, S^{n-1}u)$, it is then possible to write

$$\frac{1}{N^2} X^T U_r U_r^T \varepsilon = \begin{bmatrix} \frac{1}{N^2} Y^T U_r U_r^T \varepsilon \\ \hline \frac{1}{N^2} U_n^T U_r U_r^T \varepsilon \end{bmatrix} \tag{2.4.16}$$

writing U_n as $U_n = U_r \begin{bmatrix} I_n \\ \hline (0) \end{bmatrix}$, the last n rows of (2.4.16) become

$$\frac{1}{N^2} U_n^T U_r U_r^T \varepsilon = [I_n \,|\, (0)] \left(\frac{1}{N} U_r^T U_r\right) \frac{1}{N} U_r^T \varepsilon$$

and

$$\lim_{N \to \infty} \frac{1}{N^2} U_n^T U_r U_r^T \varepsilon = [I_n \,|\, (0)] \, I_r \lim_{N \to \infty} \frac{1}{N} U_r^T \varepsilon$$

since from (2.4.4) of lemma 2 for i = 0 and j = 0,1,...,r-1 it follows that $\lim_{N \to \infty} \frac{1}{N} U_r^T \varepsilon = \underline{0}$, then

$$\lim_{N \to \infty} \frac{1}{N^2} U_n^T U_r U_r^T \varepsilon = \underline{0} \tag{2.4.17}$$

that is, the last n rows of (2.4.16) become zero. Now consider the first n rows of (2.4.16). Writing a set of N equations of (2.4.13) as in (2.3.6)

$$\begin{aligned} y = Ya + Ub + \varepsilon &= (I-A)^{-1} Bu + (I-A)^{-1}\varepsilon \\ &= Pu + (I-A)^{-1}\varepsilon = U_J p + \eta \end{aligned} \tag{2.4.18}$$

then the matrix Y can be written as

$$\begin{aligned} Y = (Sy, S^2y, \ldots, S^n y) &= (SU_J p, \ldots, S^n U_J p) + (S\eta, \ldots, S^n \eta) \\ &= D + W \quad \text{say} \end{aligned} \tag{2.4.19}$$

thus

$$\frac{1}{N^2} Y^T U_r U_r^T \varepsilon = \frac{1}{N^2} D^T U_r U_r^T \varepsilon + \frac{1}{N^2} W^T U_r U_r^T \varepsilon \tag{2.4.20}$$

Consider the first term of (2.4.20), and assume for convenience that r is chosen as the duration of impulse response, i.e. r = J, then defining an N vector $p_N = \begin{bmatrix} p \\ \hline \underline{0} \end{bmatrix}$, the column vectors of D matrix can be written as

$$D = (SU_r p, \ldots, S^n U_r p) = (SU_N p_N, \ldots, S^n U_N p_N) \tag{2.4.21}$$

Using the commutation properties of shift matrices, it is now possible to write the first term of (2.4.20) as

$$\begin{aligned}
\frac{1}{N^2} D^T U_r U_r^T \varepsilon &= \frac{1}{N^2} (Sp_N, \ldots, S^n p_N)^T U_N^T U_r U_r^T \varepsilon \\
&= \frac{1}{N^2} H_1^T \begin{bmatrix} U_r^T \\ \hline \bar{U}_{N-r}^T \end{bmatrix} U_r U_r^T \varepsilon \\
&= H_1^T \begin{bmatrix} \left(\frac{1}{N} U_r^T U_r\right)\left(\frac{1}{N} U_r^T \varepsilon\right) \\ \hline \left(\frac{1}{N} \bar{U}_{N-r}^T U_r\right)\left(\frac{1}{N} U_r^T \varepsilon\right) \end{bmatrix}
\end{aligned} \tag{2.4.22}$$

where $H_1 = (Sp_N, \ldots, S^n p_N)$ and $\bar{U}_{N-r} = (S^r u, \ldots, S^{N-1} u)$, then from $\lim_{N \to \infty} \frac{1}{N} U_r^T \varepsilon = \underline{0}$ and the finiteness of matrices $\lim_{N \to \infty} \frac{1}{N} U_r^T U_r$ and $\lim_{N \to \infty} \frac{1}{N} \bar{U}_{N-r}^T U_r$, it follows that

$$\lim_{N \to \infty} \frac{1}{N^2} D^T U_r U_r^T \varepsilon = \underline{0} \tag{2.4.23}$$

Consider now the second term of (2.4.20) which is

$$\frac{1}{N^2} W^T U_r U_r^T \varepsilon = \frac{1}{N^2} \begin{bmatrix} (U_r^T S\eta)^T U_r^T \varepsilon \\ \vdots \\ (U_r^T S^n \eta)^T U_r^T \varepsilon \end{bmatrix} \tag{2.4.24}$$

by expanding $(I-A)^{-1}$ as a polynomial in S, η will be written as

$$\eta = (I-A)^{-1} \varepsilon = \sum_{k=1}^{r-n} d_k S^k \varepsilon \tag{2.4.25}$$

substituting (2.4.25) into (2.4.24)

$$\frac{1}{N^2} W^T U_r U_r^T \varepsilon = \begin{bmatrix} \left(\frac{1}{N} \sum_{k=1}^{r-n} d_k U_r^T S^{k+1} \varepsilon\right)^T \left(\frac{1}{N} U_r^T \varepsilon\right) \\ \vdots \\ \left(\frac{1}{N} \sum_{k=1}^{r-n} d_k U_r^T S^{k+n} \varepsilon\right)^T \left(\frac{1}{N} U_r^T \varepsilon\right) \end{bmatrix} \tag{2.4.26}$$

then by the application of the Slutzky theorem [32] and from lemma 2

$$\lim_{N \to \infty} \frac{1}{N^2} W^T U_r U_r^T \varepsilon = \underline{0} \tag{2.4.27}$$

Combining (2.4.17), (2.4.23), (2.4.27) in (2.4.16) it then follows that

$$\lim_{N \to \infty} \frac{1}{N^2} X^T U_r U_r^T \varepsilon = \underline{0} \tag{2.4.28}$$

with probability one and also in mean square. QED(I).

Proof of (II): Partitioning X as before

$$\frac{1}{N^2} X^T U_r U_r^T X = \frac{1}{N^2} \left[\begin{array}{c|c} Y^T U_r U_r^T Y & Y^T U_r U_r^T U_n \\ \hline U_n^T U_r U_r^T Y & U_n^T U_r U_r^T U_n \end{array}\right] \tag{2.4.29}$$

and considering the limit values of matrices in (2.4.29), the following results are obtained in a similar way as in the previous proof

$$\lim_{N \to \infty} \frac{1}{N^2} U_n^T U_r U_r^T U_n = \lim_{N \to \infty} [I_n \,|\, (0)] \frac{1}{N} U_r^T U_r \frac{1}{N} U_r^T U_r \left[\begin{array}{c} I_n \\ \hline (0) \end{array}\right]$$

$$= I_n \tag{2.4.30}$$

$$\lim_{N \to \infty} \frac{1}{N^2} Y^T U_r U_r^T U_n = \lim_{N \to \infty} \frac{1}{N} (D + W)^T U_n$$

$$= \lim_{N \to \infty} \frac{1}{N} H_1^T U_N^T U_n$$

$$= \lim_{N \to \infty} H_1^T \left[\begin{array}{c} \frac{1}{N} U_n^T U_n \\ \hline \frac{1}{N} \bar{U}_{N-n}^T U_n \end{array}\right]$$

$$= \lim_{N \to \infty} H_{11}^T \tag{2.4.31}$$

where H_{11} is an nxn partitioned matrix of H_1, i.e.

$$H_1 = \begin{bmatrix} H_{11} \\ \hline H_{12} \end{bmatrix} \tag{2.4.32}$$

and

$$\lim_{N \to \infty} \frac{1}{N^2} Y^T U_r U_r^T Y = \lim_{N \to \infty} \frac{1}{N^2} (D^T U_r U_r^T D + D^T U_r U_r^T W + W^T U_r U_r^T D + W^T U_r U_r^T W) \tag{2.4.33}$$

starting from the first term of (2.4.33) and using the commutation properties of shift matrices

$$\lim_{N \to \infty} \frac{1}{N^2} D^T U_r U_r^T D = H_1^T \begin{bmatrix} I_r \\ \hline (0) \end{bmatrix} [I_r \,|\, (0)] H_1 \tag{2.4.34}$$

on partitioning H_1 as

$$H_1 = \begin{bmatrix} H_{13} \\ \hline H_{14} \end{bmatrix} \tag{2.4.35}$$

(2.4.34) takes the form

$$\lim_{N \to \infty} \frac{1}{N^2} D^T U_r U_r^T D = H_{13}^T H_{13} \tag{2.4.36}$$

where H_{13} and H_{14} are rxn and (N-r)xn matrices respectively. It can be shown by the application of lemma 2 that all the other terms of (2.4.33) tend to zero with probability one. Now (2.4.30), (2.4.31), (2.4.36) reduce (2.4.29) to

$$\lim_{N \to \infty} \frac{1}{N^2} X^T U_r U_r^T X = \begin{bmatrix} H_{13}^T H_{13} & H_{11}^T \\ \hline H_{11} & I_n \end{bmatrix} \tag{2.4.37}$$

Following the results given in [46], (2.4.37) is positive-definite if and only if

$$(H_{13}^T H_{13} - H_{11}^T H_{11}) \text{ is positive definite.} \tag{2.4.38}$$

It will now be shown that if the polynomials $[1-A(z^{-1})]$ and $B(z^{-1})$ contain no common factors then (2.4.38) holds. To show this, consider the matrix H_{13} in the partitioned form as

$$H_{13} = \left[\begin{array}{ccc} 0 & 0 \ldots & 0 \\ p_o & 0 & 0 \\ p_1 & p_o & 0 \\ \vdots & \vdots & \vdots \\ p_{n-2} & p_{n-3} & 0 \\ \hline p_{n-1} & p_{n-2} & p_o \\ \vdots & \vdots & \vdots \\ p_{r-2} & p_{r-3} & p_{r-n-1} \end{array}\right] = \left[\begin{array}{c} H_{11} \\ \hline H_{15} \end{array}\right] \qquad (2.4.39)$$

where H_{15} is an $(r-n)xn$ matrix. It then follows from this partitioning that

$$H_{13}^T H_{13} - H_{11}^T H_{11} = H_{15}^T H_{15} \qquad (2.4.40)$$

To prove the positive-definiteness of $H_{15}^T H_{15}$ it is sufficient to show that the column vectors of H_{15} are linearly independent. From the expansion of the pulse transfer function in partial fractions and assuming that poles are distinct, $P(z^{-1})$ can be written as

$$P(z^{-1}) = \frac{B(z^{-1})}{1-A(z^{-1})} = \sum_{i=1}^{n} g_i (1-\gamma_i z^{-1})^{-1} \qquad (2.4.41)$$

where $(1-\gamma_i z^{-1})$; $i=1,2,\ldots,n$ are the n factors of $[1-A(z^{-1})]$ and g_i's are a set of unique constants all differing from zero if $B(z^{-1})$ has no factors $(1-\gamma_i z^{-1})$. In terms of shift matrices S, (2.4.41) takes the form

$$P = (I-A)^{-1} B = \sum_{i=1}^{n} g_i (I-\gamma_i S)^{-1}$$

$$= \sum_{i=1}^{n} g_i (I+\gamma_i S+\gamma_i^2 S^2+\ldots+\gamma_i^{N-1} S^{N-1}) \qquad (2.4.42)$$

Since $P = p_o+p_1 S+\ldots+p_{J-1} S^{J-1}$ then equating the coefficients of corresponding powers of S in (2.4.42), p_i's can be written as

$$p_i = \sum_{\ell=1}^{n} g_\ell \gamma_\ell^i \qquad i=0,1,\ldots, J-1 \tag{2.4.43}$$

Defining the following n-vector $\underline{\gamma}_\ell$ as,

$$\underline{\gamma}_\ell = (1, \gamma_\ell, \gamma_\ell^2, \ldots, \gamma_\ell^{n-1})^T$$

it can be seen from (2.4.39) and (2.4.43) that the first n rows of H_{15} can be written as

$$\begin{bmatrix} p_{n-1} & p_{n-2} \cdots & p_o \\ p_n & p_{n-1} \cdots & p_1 \\ \vdots & \vdots & \vdots \\ p_{2n-2} & p_{2n-1} \cdots & p_{n-1} \end{bmatrix} = \left[\sum_{\ell=1}^{n} g_\ell \gamma_\ell^{n-1} \underline{\gamma}_\ell, \sum_{\ell=1}^{n} g_\ell \gamma_\ell^{n-2} \underline{\gamma}_\ell, \ldots, \sum_{\ell=1}^{n} g_\ell \underline{\gamma}_\ell \right] \tag{2.4.44}$$

But

$$\sum_{\ell=1}^{n} g_\ell \gamma_\ell^i \underline{\gamma}_\ell = \begin{bmatrix} 1 & 1 \ldots & 1 \\ \gamma_1 & \gamma_2 & \gamma_n \\ \vdots & \vdots & \vdots \\ \gamma_1^{n-1} & \gamma_2^{n-1} & \gamma_n^{n-1} \end{bmatrix} \begin{bmatrix} g_1\gamma_1^i \\ g_2\gamma_2^i \\ \vdots \\ g_n\gamma_n^i \end{bmatrix} = V \begin{bmatrix} g_1\gamma_1^i \\ g_2\gamma_2^i \\ \vdots \\ g_n\gamma_n^i \end{bmatrix} \tag{2.4.45}$$

where V is the Vandermonde matrix [47] then from substitution of (2.4.45) into (2.4.44), the determinant of (2.4.44) becomes

$$\begin{vmatrix} p_{n-1} & p_{n-2} \cdots & p_o \\ p_n & p_{n-1} \cdots & p_1 \\ \vdots & & \\ p_{2n-2} & p_{2n-1} \cdots & p_{n-1} \end{vmatrix} = |V| \begin{vmatrix} g_1\gamma_1^{n-1} & g_1\gamma_1^{n-2} & \cdots & g_1 \\ g_2\gamma_2^{n-1} & g_2\gamma_2^{n-2} & \cdots & g_2 \\ & & & \\ g_n\gamma_n^{n-1} & g_2\gamma_2^{n-2} & \cdots & g_n \end{vmatrix}$$

$$= \left[\prod_{1\le j<i\le n} (\gamma_i - \gamma_j) \right] \left[\prod_{1\le j<i\le n} (\gamma_i - \gamma_j) \prod_{\ell=1}^{n} g_\ell \right] \tag{2.4.46}$$

where the first term on the right hand side of the second equality is the Vandermonde determinant which is different from zero when $\gamma_i \neq \gamma_j$. The second term is the determinant of the matrix

$$\begin{bmatrix} g_1\gamma_1^{n-1} & \cdots & g_1 \\ \vdots & & \vdots \\ g_n\gamma_n^{n-1} & \cdots & g_n \end{bmatrix}$$

which can be easily obtained by noting that the rows of this matrix are in fact the columns of the Vandermonde matrix multiplied by one of the g_i's, hence the determinant is given by $|V|$ multiplied by $\prod_{\ell=1}^{n} g_\ell$. It then can be concluded that if

$$\gamma_i \neq \gamma_j \quad i,j = 1,2,\ldots,n \tag{2.4.47}$$

then the matrix $H_{15}^T H_{15}$ is of rank n and consequently $(H_{13}^T H_{13} - H_{11}^T H_{11})$ or equivalently

$$\lim_{N \to \infty} \frac{1}{N^2} X^T U_r U_r^T X \text{ is positive-definite. QED(II).}$$

Corollary: By the application of lemma 3 instead of lemma 2 in the proof of theorem 1, it can be shown in a completely similar manner that the convergence of the modified least squares estimates holds only in probability if the covariance function of the disturbance satisfies (2.4.12).

One practical difficulty associated with the modified least squares algorithm is that although its estimates are strongly consistent, the algorithm, in parallel with other techniques in its class, does not provide the accuracy of estimates, nor the possibility to use a statistical hypothesis test to estimate the model order. Writing the covariance matrix of the modified algorithm estimates from (2.2.4) and (2.2.26) as

$$\text{Cov}(\hat{\theta}) = E(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T = E[(X^T U_r U_r^T X)^{-1} X^T U_r U_r^T \varepsilon\varepsilon^T U_r U_r^T X (X^T U_r U_r^T X)^{-1}] \tag{2.4.48}$$

it can be seen that unless the covariance matrix of the disturbance is known there is no possibility to estimate the accuracy of estimated parameters. One conclusion that might be conjectured from the covariance matrix is that if the disturbances are independent with common variance

σ_ε^2, i.e. $E\varepsilon\varepsilon^T = \sigma_\varepsilon^2 I_N$ then (2.4.48) can be reduced to

$$\text{Cov}\ (\hat{\theta}) = \sigma_\varepsilon^2 [X^T U_r (U_r^T U_r)^{-1}\ U_r^T X]^{-1} \tag{2.4.49}$$

or equivalently in terms of orthonormal matrix L to

$$\text{Cov}\ (\hat{\theta}) = \sigma_\varepsilon^2 (X^T L L^T X)^{-1} \tag{2.4.50}$$

If this result is compared with the covariance matrix of the least squares estimates

$$\text{Cov}\ (\hat{\theta}) = \sigma_\varepsilon^2 (X^T X)^{-1} \tag{2.4.51}$$

since $\text{tr}(X^T L L^T X)^{-1} \geq \text{tr}(X^T X)^{-1}$, it can be concluded that in a situation where the data is ideally suitable for the application of the least squares, the modified algorithm in general produces estimates with higher variance than that of the least squares technique.

2.5. Examples

This section presents a series of results to demonstrate the properties of the modified least squares estimates. Mainly two aspects of the proposed algorithm are illustrated.

I) How much improvement can be obtained by the application of the modified least squares algorithm instead of the least squares if the disturbances are correlated?

II) How efficient is the modified least squares algorithm in a situation where the disturbances are uncorrelated and hence the data is ideally suitable for the application of the least squares technique?

To make these comparisons following error criteria were used.

i) Mean-squared parameter estimation errors of the modified least squares (MLS) algorithm related to that of the least squares (LS)

$$\delta_1 = \frac{||\hat{\theta}_{MLS} - \theta||}{||\hat{\theta}_{LS} - \theta||}$$

ii) Mean-squared impulse response estimation errors related to the mean-squared true impulse response

$$\delta_2 = \frac{||\hat{p}-p||}{||p||}$$

The criterion δ_1 is an overall measure of the performance of the (MLS) algorithm with respect to the (LS) technique. The impulse response error δ_2 is particularly relevant if the final goal of identification is to design minimum variance control strategies.

As test cases for comparison three simulated linear time invariant processes were chosen.

P1: Second order process

$1-A(z^{-1}) = 1-1.5z^{-1}+0.7z^{-2}$ $B(z^{-1}) = z^{-1}+0.5z^{-2}$

P2: Second order non-minimum phase process

$1-A(z^{-1}) = 1-1.425z^{-1}+0.496z^{-2}$ $B(z^{-1}) = -0.102z^{-1} + 0.173z^{-2}$

P3: Third order process

$1-A(z^{-1}) = 1-1.5z^{-1} + 0.705z^{-2} - 0.1z^{-3}$ $B(z^{-1}) = 0.065z^{-2} + 0.048z^{-3}-0.008z^{-4}$

Processes P2 and P3 have been proposed by Isermann [48] as test cases for a comparison of different identification methods. They have been derived from continuous processes with a zero-order hold. Process P1 seems to be a common example used by different authors, e.g. [16],[19],[27],[28], [49],[50].

The disturbance models used in the examples were in the following forms

$$\varepsilon_t = \zeta_t \quad , \quad \varepsilon_t = \frac{1}{1-0.9z^{-1}}\ \zeta_t \ , \ \varepsilon_t = (1+0.9z^{-1})$$

where the uncorrelated sequence $\{\zeta_t\}$ was generated by using a standard random number generator from the Harwell subroutine library of ELLIOT 4130. This generator first creates equally distributed random numbers and then, based on these, gives Gaussian distributed numbers with zero mean and unity variance. The sequence $\{\zeta_t\}$ was tested for randomness by calculating the auto-correlation function; it was observed from the sample mean values of the auto-correlation coefficients for 50 different samples of $\{\zeta_t\}$ that the sequence could be considered to be uncorrelated with 95% confidence

level for samples greater than about 200.

The input sequence $\{u_t\}$ was chosen to be a pseudo-random binary sequence (PRBS) of magnitude ±1. The sequence was generated by a suitable combination of 8-register (PRBS) generator such that the period was 255. This choice had the advantage that the column vectors of $U_r = (u, Su, \ldots, S^{r-1}u)$ were approximately orthogonal and hence there was no need to employ time consuming orthogonalization procedures to generate L matrix of (2.2.16). Therefore the modified least squares algorithm was in the form

$$\hat{\theta} = (X^T U_r U_r^T X)^{-1} X^T U_r U_r^T y$$

where the number of assumptions, r, was chosen as the impulse response duration of the process considered in each example.

To obtain better comparison, 50 runs with different noise data sets of length N = 255 were made. Then the sample mean values $\bar{\delta}_1$, $\bar{\delta}_2$ of errors δ_1 and δ_2 were calculated as, for example

$$\bar{\delta}_1 = \frac{1}{50} \sum_{i=1}^{50} \delta_{1,i}$$

Tables (2.5.1), (2.5.2) and (2.5.3) compare the modified least squares estimates with that of the least squares in the case of correlated disturbances for three different signal to noise ratios as

$$\lambda = \frac{\sigma_u^2}{\sigma_\zeta^2} = 0.1,\ 1.0,\ 10.0$$

where σ_u^2 and σ_ζ^2 denote the variance of input u_t and uncorrelated noise ζ_t respectively. It can be seen that as far as the parameter estimation error $\bar{\delta}_1$ is concerned, in general, the modified least squares method produced more reliable results except in the case of low signal to noise ratio $\lambda = 0.1$ where there was no considerable improvement. To assess this comparison quantitatively six different values of $\bar{\delta}_1$ for each λ, obtained from three examples are averaged and tabulated overleaf.

λ	Average value of $\bar{\delta}_1$
0.1	0.99
1.0	0.80
10.0	0.67

It is apparent from the averaged values of $\bar{\delta}_1$ that the modified least squares performed on the average better than the least squares particularly in the case of high signal to noise ratios.

The comparison based on the impulse response error measure $\bar{\delta}_2$ showed no clear improvement in favour of the modified least squares algorithm. In a total of 18 different cases considered in three examples, in only 9 cases the impulse response error of the modified least squares estimates were smaller than that of the least squares. It was also found that in most of these 9 cases the difference was not large enough to suggest any significant improvement.

To test the modified least squares algorithm under conditions ideal for the least squares technique, the same processes were simulated by 50 different uncorrelated sequences $\{\varepsilon_t=\zeta_t\}$. Tables (2.5.4), (2.5.5) and (2.5.6) represent the error measures together with sample variances of the estimated parameters for $\lambda = 1.0$. Comparison of $\bar{\delta}_1$ and $\bar{\delta}_2$ showed that in all cases the least squares technique produced more reliable results. Of particular interest in the examples was the comparison of sample variances of the estimated parameters. The modified least squares algorithm gave estimates with higher variances than that of the least squares. Especially for processes P2 and P3 the variances of $\{\hat{a}_i\}$ estimates were greater by a factor of 10 or so in the modified least squares algorithm.

In all examples represented above it was assumed that the process impulse response duration was known and consequently the number of assumptions, r, was chosen as the impulse response duration of the corresponding process. In cases where this information is lacking, r can be chosen by minimizing an approximate criterion $r\,\mathrm{tr}\Psi = r\,\mathrm{tr}(X^T U_r U_r^T X)^{-1}$ as described in section 2.3. Simulation studies showed that the algorithm with r determined from the

above criterion produced equally good estimates of the parameters in comparison with the choice of r based on the impulse response duration. A representative example of the results is given in table (2.5.7) where the disturbance was of an auto-regressive type and $\lambda = 1.0$. The error measures $\bar{\delta}_1$ and $\bar{\delta}_2$ were again based on 50 different samples. It was found that the value of r minimizing $\left[r\ tr(X^T U_r U^T X)^{-1}\right]$ was about 8 in comparison with the impulse response duration of about 33 sample intervals. This and other examples suggest that the choice of r is not too critical as long as r is greater than twice the actual process order.

In summary, simulation studies have shown that the modified least squares algorithm can produce better estimates than that of the least squares when the disturbances are correlated. Under conditions ideal for the least squares technique the modified algorithm does not perform well in the sense that the sum of squares of estimation errors and variances of estimates are larger than that of the least squares technique.

2.6. Conclusions

This chapter has presented an approach for the linear dynamic system parameters estimation. Although the derived algorithm is identical to that of the two stage least squares technique nevertheless it is believed that an attempt has been made to discover an appropriate way of thinking about the estimation problem. The essence of the solution is that, the assumptions must be made about the unknown disturbances and here reliance has been placed on its most plausible feature, namely its independence of the input. This yields a set of assumptions in excess of the minimal requirements and an endeavour has been made to exploit this excess to reduce the sum of squares of estimation errors.

The choice of the number of assumptions, r, has been discussed and it has been shown that if the process impulse response duration is known then the choice of r greater than the impulse response duration plus the system order do not produce any significant improvement

in the resulting estimates.

Under some mild assumptions for the disturbance and for an orthogonal input sequence it has been proved that the modified least squares estimates are strongly consistent.

Finally, the algorithm has been subjected to test by computer simulation and it has been shown that the technique can produce better results than that of the least squares when the disturbances are correlated.

Nature of Disturbance	Signal to Noise Ratio, λ	Parameter Estimation Error, $\bar{\delta}_1$	Impulse Response Error	
			$(\bar{\delta}_2)_{MLS}$	$(\bar{\delta}_2)_{LS}$
$\varepsilon_t = \frac{\zeta_t}{1-0.9z^{-1}}$	0.1	1.003	2.806	2.530
	1.0	0.504	0.495	1.248
	10.0	0.770	0.039	0.077
$\varepsilon_t=(1+0.9z^{-1})\zeta_t$	0.1	1.038	1.179	0.976
	1.0	0.657	0.227	0.318
	10.0	0.958	0.022	0.023

Table 2.5.1. Estimation errors for P1: $\frac{z^{-1}+0.5z^{-2}}{1-1.5z^{-1}+0.7z^{-2}}$

Nature of Disturbance	Signal to Noise Ratio, λ	Parameter Estimation Error, $\bar{\delta}_1$	Impulse Response Error	
			$(\bar{\delta}_2)_{MLS}$	$(\bar{\delta}_2)_{LS}$
$\varepsilon_t = \frac{\zeta_t}{1-0.9z^{-1}}$	0.1	1.001	57.451	65.114
	1.0	0.943	8.201	9.057
	10.0	0.461	2.665	5.114
$\varepsilon_t=(1+0.9z^{-1})\zeta_t$	0.1	0.998	18.695	18.322
	1.0	1.011	1.861	1.683
	10.0	0.479	0.472	0.579

Table 2.5.2. Estimation errors for P2: $\frac{-0.102z^{-1} + 0.173\ z^{-2}}{1-1.425\ z^{-1} + 0.496\ z^{-2}}$

Nature of Disturbance	Signal to Noise Ratio, λ	Parameter Estimation Error, δ_1	Impulse Response Error	
			$(\bar{\delta}_2)_{MLS}$	$(\bar{\delta}_2)_{LS}$
$\varepsilon_t = \frac{\zeta_t}{1-0.9z^{-1}}$	0.1	0.924	71.544	56.524
	1.0	0.821	6.893	5.254
	10.0	0.820	0.940	0.545
$\varepsilon_t=(1+0.9z^{-1})\zeta_t$	0.1	0.992	21.016	18.621
	1.0	0.875	1.877	1.525
	10.0	0.643	0.328	0.328

Table 2.5.3. Estimation errors for P3: $\frac{0.065z^{-2}+0.048z^{-3}-0.008z^{-4}}{1-1.5z^{-1}+0.705z^{-2}-0.100z^{-3}}$

Parameter Estimation Error $\bar{\delta}_1 = 1.128$	Impulse Response Error $\bar{\delta}_2$	Sample Variance of Estimates $\hat{a}_1$	$\hat{a}_2$	$\hat{b}_1$	$\hat{b}_2$
LS	0.104	0.001	0.001	0.004	0.005
MLS	0.125	0.002	0.002	0.004	0.005

Table 2.5.4. Comparison in the case of uncorrelated disturbances for

$$P1: \frac{z^{-1} + 0.5\ z^{-2}}{1-1.5\ z^{-1} + 0.7\ z^{-2}}$$

Parameter Estimation Error $\bar{\delta}_1 = 2.244$	Impulse Response Error $\bar{\delta}_2$	Sample Variance of Estimates $\hat{a}_1$	$\hat{a}_2$	$\hat{b}_1$	$\hat{b}_2$
LS	0.700	0.003	0.003	0.004	0.005
MLS	0.924	0.021	0.021	0.004	0.005

Table 2.5.5. Comparison in the case of uncorrelated disturbances for

$$P2: \frac{-0.102z^{-1} + 0.173\ z^{-2}}{1-1.425\ z^{-1} + 0.496\ z^{-2}}$$

Parameter Estimation Error $\bar{\delta}_1 = 2.168$	Impulse Response Error $\bar{\delta}_2$	Sample Variance of Estimates $\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
LS	0.819	0.004	0.013	0.006	0.005	0.004	0.003
MLS	1.142	0.042	0.132	0.059	0.005	0.005	0.003

Table 2.5.6. Comparison in the case of uncorrelated disturbances for

$$P3: \frac{0.065\ z^{-2} + 0.048\ z^{-3} - 0.008\ z^{-4}}{1-1.5\ z^{-1} + 0.705\ z^{-2} - 0.100\ z^{-3}}$$

	Parameter Estimation Error $\bar{\delta}_1$	Impulse Response Error $(\bar{\delta}_2)_{MLS}$
r obtained by minimizing r.trΨ	0.496	0.593
r = Impulse response duration	0.563	0.521

Table 2.5.7. Estimation errors for two different choices of number assumptions for

P1: $$\frac{z^{-1} + 0.5\ z^{-2}}{1-1.5\ z^{-1} + 0.7\ z^{-2}}$$

CHAPTER III

IDENTIFICATION AND SELF-ADAPTIVE CONTROL WITH THE BOX AND JENKINS CONTROL SYSTEM

3.1. Introduction

The parameter estimation techniques discussed in the previous chapters have one feature in common - the collection of a set of data from the process operated in open loop. However, in practice it is often desirable and even necessary to perform the experiments in closed loop for the following reasons [12]:

i) The process may be unstable without any control.

ii) Risks of loss of production or reduced efficiency may prevent open loop experiments.

iii) Most often the chosen model is linear and valid around certain operating conditions only, hence the process must be kept around these operating conditions by a suitable controller.

If the objective of the identification is to obtain a model that can be used to design control laws, it is sufficient to estimate the transfer functions describing the process and the noise characteristics. These transfer functions can be consistently estimated if either of the following conditions hold.

a) Two different linear controllers are used during the identification experiment [12], [52].

b) The controller is non-linear [12].

c) An additional persistently exciting signal can be applied to the process [12], [38].

d) There is an added disturbance to the feedback path [12], [51].

For the most common situation of time invariant linear controller, identification in closed loop is not always possible. It has been shown in [12] that the conditions which guarantee identifiability can only be

given in terms of process, model and controller structures.

Identifiability problems for closed loop processes also exist in many adaptive control situations where the controller parameters are determined by the current estimates of the process and the disturbance parameters by enforcing the certainty equivalence principle [52],[53], [54],[55]. Since the structure of the closed loop system is very complex in these cases, it is in general difficult to establish identifiability except in the case of independent disturbance [55].

In this chapter the identifiability problem is investigated for a feedback control system of Box and Jenkins type. Previous results have shown that the impulse response of the process can be consistently estimated if the process time delay is greater than the number of parameters of the assumed auto-regressive disturbance model [52],[56]. This chapter will present a new result that if the above condition is not satisfied then the identifiability is governed by the part of the disturbance model which cannot be estimated and hence the identifiability cannot always be guaranteed. However, further analysis has shown that the estimation procedure may be self-adaptive in the sense that if repeated applications of the procedure converge to a stationary solution then the resulting control system is optimum as far as the disturbances are concerned. Convergence is controlled by the unknown part of the disturbance which effectively limits the application of the method to a certain class of disturbance, namely those that are describable by auto-regressive models with a restricted number of coefficients.

3.2. The Box and Jenkins Control System

This section discusses the design of a control system for a linear time invariant sampled data process represented by its impulse response function $z^{-k}P(z^{-1})$ as

$$y_t = z^{-k}P(z^{-1})u_t + x_t \qquad (3.2.1)$$

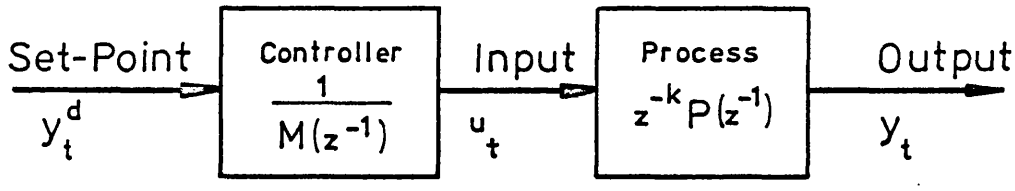

Fig.3.2.1. CONTROL TO THE SET-POINT

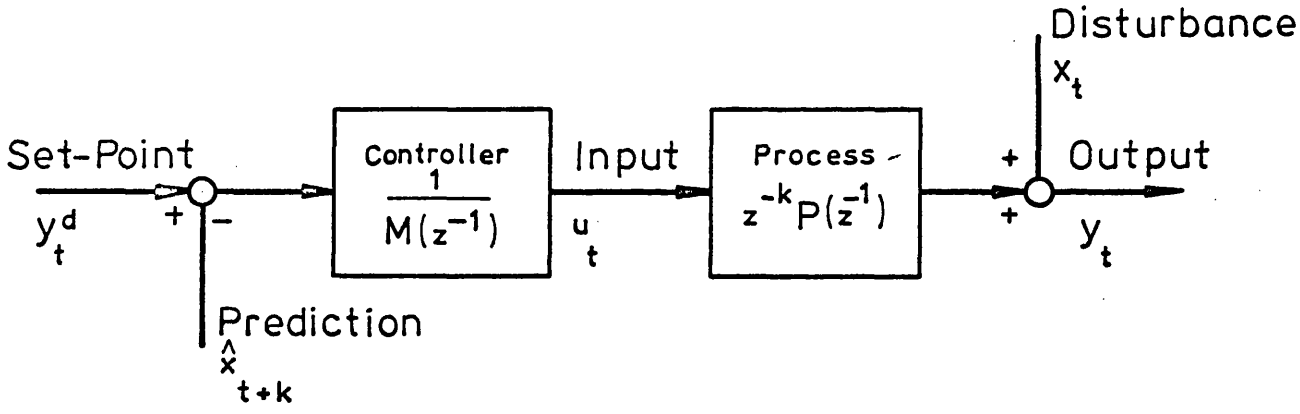

Fig.3.2.2. CONTROL TO THE SET-POINT AND TO THE EFFECT OF THE DISTURBANCES

where $P(z^{-1}) = \sum_{i=0}^{\infty} p_i z^{-i}$, k is an integer denoting the time delay of k sample intervals and x_t represents the disturbances added to the process output. The approach considered in this section is applicable to minimum phase, stable processes. The objective of the control system is to maintain the output of the process y_t at some desired set-point y_t^d which is assumed to change infrequently from one constant value to another constant value. Furthermore, the control system must minimize the effects of the unknown disturbances on the process output. Following the work of Phillipson [57] and assuming that the process dynamics are known, the control system can be designed by considering the above aspects of the control objective separately as

i) control to the desired set-point,

ii) control to the effects of the disturbances.

In the absence of disturbances, a feedforward controller of impulse response function $M^{-1}(z^{-1})$ can maintain the process output at the desired set-point value if the controller is such that $M(z^{-1}) = P(z^{-1})$. In this event it can be seen from Fig. (3.2.1) that $y_t = y_{t-k}^d$, i.e. y_t follows y_t^d exactly with a delay of k sample intervals and hence the first part of the control objective is realized.

The effects of the unknown disturbances can be minimized by using a k step ahead prediction $\hat{x}_{t+k}$ of x_{t+k} at time t as shown in Fig. (3.2.2). In this case the process output becomes

$$y_t = y_{t-k}^d + (x_t - z^{-k} \hat{x}_{t+k}) \qquad (3.2.2)$$

hence for constant desired set-point, the effects of the disturbances can be eliminated by use of a best possible k step ahead prediction $\hat{x}_{t+k}$ of x_{t+k} at time t. The prediction $\hat{x}_{t+k}$ can be generated

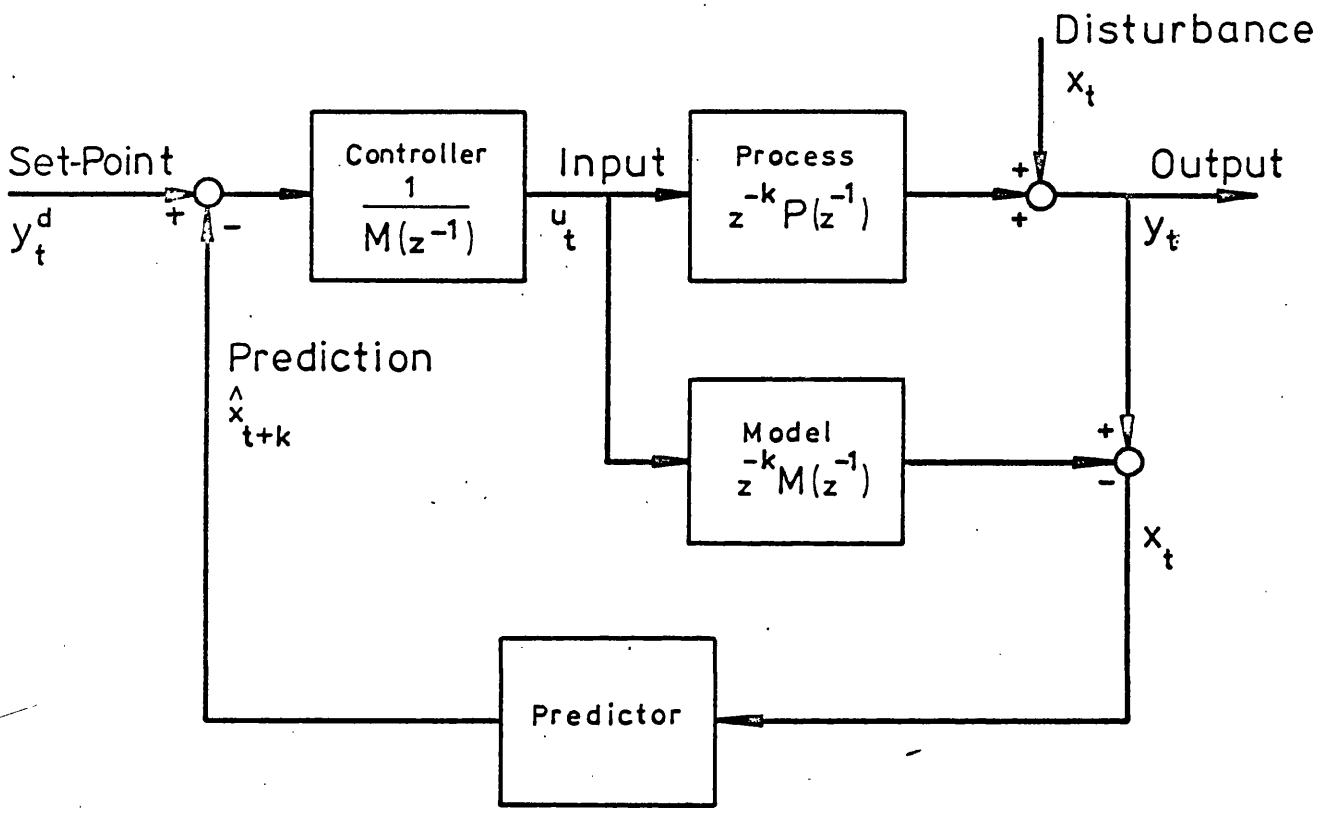

Fig. 3.2.3. THE OVERALL CONTROL SYSTEM

by using a Box and Jenkins type control system as shown in Fig. (3.2.3) which contains a model (equal to process) in parallel with the process to generate x_t and a predictor operating on the past and current values of x_t to generate $\hat{x}_{t+k}$. The process output from Fig. (3.2.3) is given by eqn. (3.2.2) and hence the control system meets the two aspects of the control objective.

The above derivation of the control system has an intuitive appeal as far as the control objective is concerned. However, it does not say in what sense the control system is optimum, since as it can be seen from (3.2.2) , the control system errors are entirely due to the prediction errors and hence the optimality can only be defined in terms of the type of predictor used. It will now be shown that if the disturbance x_t is a zero mean stationary process and the chosen predictor is linear minimum variance predictor then the control system of Fig. (3.2.3) is optimum in the sense that its control signal

$$u_t = \frac{1}{M(z^{-1})} \left(y_t^d - \hat{x}_{t+k}\right) \tag{3.2.3}$$

minimizes the mean squared output error

$$L = E(y_{t+k} - y_t^d)^2 \tag{3.2.4}$$

To prove this, it will be assumed that the disturbance x_t is represented by an infinite order moving average process as

$$x_t = [1 + S(z^{-1})]\, \zeta_t \tag{3.2.5}$$

where $S(z^{-1}) = \sum_{i=0}^{\infty} s_i z^{-i}$ and ζ_t is an independent, zero mean random variable such that $E(\zeta_t) = 0$, $E(\zeta_t \zeta_s) = 0 \quad t \neq s$, and $E\zeta_t = \sigma_\zeta^2$. It should however be noted here that the assumption of moving average disturbance model does not introduce any loss of generality here, since other forms of disturbance models can be converted to an

infinite order moving average form [14].

From the fundamental lemma of stochastic control theory [6], it is known that the minimization of (3.2.4) with respect to u_t is equivalent to that of

$$L = E[(y_{t+k}-y_t^d)^2 \mid y_t,y_{t-1},\ldots,u_t,u_{t-1},\ldots,y_t^d,y_{t-1}^d,\ldots] \tag{3.2.6}$$

i.e. the conditional output error variance given all past information up to and including time t. It will be assumed that y_t^d is known for all values of t. Substituting from (3.2.1) and (3.2.5) into (3.2.6) and denoting

$$E[(y_{t+k}-y_t^d)^2 \mid y_t,\ldots,u_t,\ldots,y_t^d,\ldots] = \langle (y_{t+k}-y_t^d)^2 \rangle \tag{3.2.7}$$

(3.2.6) can be written as

$$L = \langle (y_{t+k}-y_t^d)^2 \rangle = \langle \{P(z^{-1})u_t+[1+S(z^{-1})]\zeta_{t+k}-y_t^d\}^2 \rangle \tag{3.2.8}$$

The term $[1+S(z^{-1})]\zeta_{t+k}$ of (3.2.8) is to be partitioned as

$$[1+S(z^{-1})]\zeta_{t+k} = [1+S(z^{-1}) - S_T(z^{-1})]\zeta_{t+k} + S_T(z^{-1})\zeta_{t+k} \tag{3.2.9}$$

where $S_T(z^{-1}) = \sum_{i=k}^{\infty} s_i z^{-i}$, i.e. the subscript T denotes k^{th} order front-end truncation. From the properties of ζ_t

$$\langle [1+S(z^{-1}) - S_T(z^{-1})]\zeta_{t+k} \rangle = 0 \tag{3.2.10}$$

and since the values of $\{\zeta_t, \zeta_{t-1},\ldots\}$ are available at time t then from (3.2.9) and (3.2.10)

$$\langle [1+S(z^{-1})]\zeta_{t+k} \rangle = S_T(z^{-1})\zeta_{t+k} = z^k S_T(z^{-1})\zeta_t \tag{3.2.11}$$

and

$$\langle\{[1+S(z^{-1})]\zeta_{t+k}\}^2\rangle = (1+s_1^2+\ldots+s_{k-1}^2)\sigma_\zeta^2 + [z^k S_T(z^{-1})\zeta_t]^2 \tag{3.2.12}$$

thus completing the square in (3.2.8) and using (3.2.10),(3.2.11) and (3.2.12) the criterion (3.2.8) can be written as

$$L = (1+s_1^2+\ldots+s_{k-1}^2)\sigma_\zeta^2 + [z^k S_T(z^{-1})\zeta_t + P(z^{-1})u_t - y_t^d]^2 \tag{3.2.13}$$

and the minimum value of L is obtained for

$$u_t = \frac{1}{P(z^{-1})}[y_t^d - z^k S_T(z^{-1})\zeta_t] \tag{3.2.14}$$

Defining

$$H^*(z^{-1}) = S_T(z^{-1})[1+S(z^{-1})]^{-1} \tag{3.2.15}$$

and using (3.2.5), the optimum control law (3.2.14) can be rewritten as

$$u_t = \frac{1}{P(z^{-1})}[y_t^d - z^k H^*(z^{-1})x_t] \tag{3.2.16}$$

It is shown in section 3.5 that $z^k H^*(z^{-1})x_t$ is the minimum variance k step ahead linear prediction of x_{t+k} at time t and hence comparison of (3.2.3) with (3.2.16) shows that the control law (3.2.3) which is obtained by an heuristic argument is optimum in the sense that it minimizes the mean squared output error if $\hat{x}_{t+k}$ is the minimum variance prediction of x_{t+k}. Finally, from (3.2.1), (3.2.15) and (3.2.16), the optimum control error is

$$y_{t+k} - y_t^d = [1-H^*(z^{-1})]x_{t+k} = [1+S(z^{-1}) - S_T(z^{-1})]\zeta_{t+k} \tag{3.2.17}$$

from which the minimum control error variance becomes

$$L_{min} = (1+s_1^2+\ldots+s_{k-1}^2)\sigma_\zeta^2 \tag{3.2.18}$$

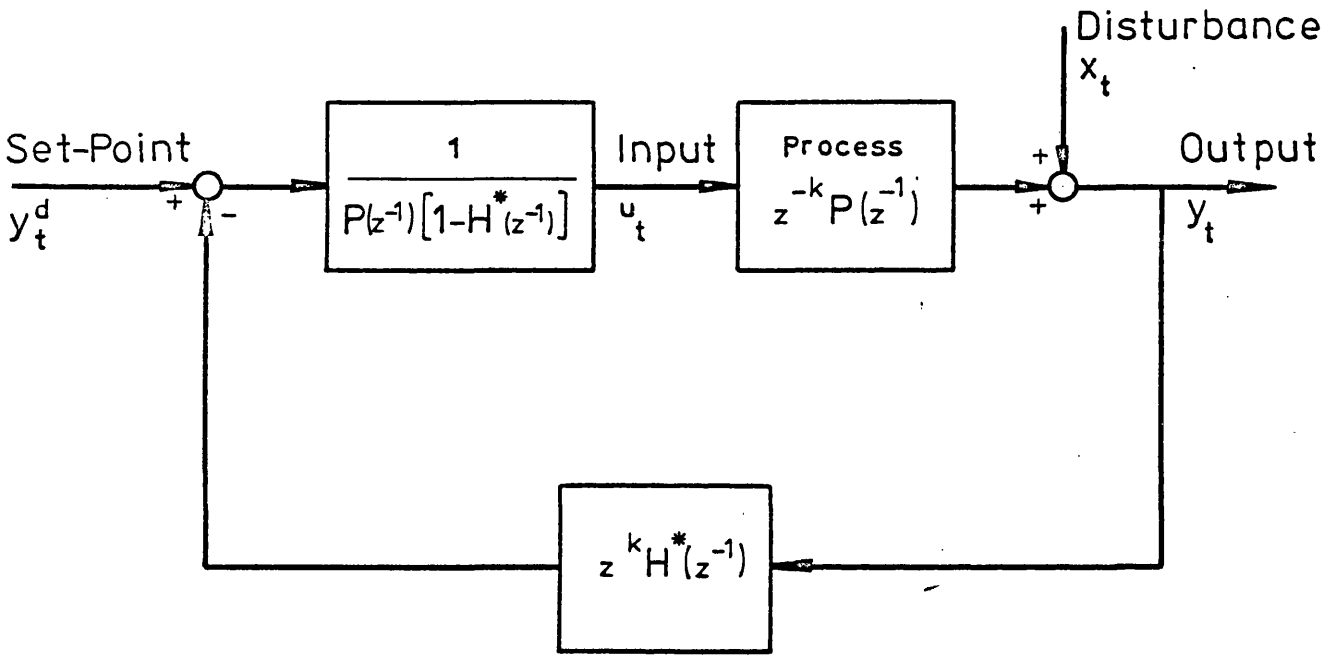

Fig. 3.2.4. OPTIMUM CONTROL SYSTEM IN CONVENTIONAL SERVO-MECHANISM REPRESENTATION

Comments

1) From (3.2.16) and (3.2.17) it follows that an optimum control requires known process dynamics and an optimum predictor and given these the control system errors are entirely due to the prediction errors for the disturbance.

2) It is known that [58] the optimum control law (3.2.16) also minimizes the criterion

$$\frac{1}{N} E \sum_{t=1}^{N} (y_{t+k} - y_t^d)^2$$

for any arbitrary N.

3) The block diagram of Fig. (3.2.3) is not the only way of implementation of the control system. Writing the control law (3.2.16) by use of (3.2.1), in the form

$$u_t = \frac{1}{P(z^{-1})[1-H^*(z^{-1})]} [y_t^d - z^k H^*(z^{-1}) y_t] \qquad (3.2.19)$$

a conventional servo-mechanism of Fig. (3.2.4) with forward path compensator $\frac{1}{P(z^{-1})[1-H^*(z^{-1})]}$ and a feedback path compensator $z^k H^*(z^{-1})$ can be obtained.

4) The use of an inverse process model in the forward path compensator has first been suggested in the Guillemin's cancellation procedure [59] where the process is preceded by a compensation network cancelling the undesirable dynamic modes of the process. The same feature also appears in the Smith controller where a good approximation to the inverse process model has been used to control the processes with time delays [60]. And finally, the control system of Fig. (3.2.3) is an interpretation of the control system proposed by Box and Jenkins [61], [62].

Adverse Comments

1) In the case of non-minimum phase processes the minimum variance control law (3.2.19) is extremely sensitive to variations in the model parameters [6], since the forward path compensator of Fig. (3.2.4) attempts to cancel directly the poles and zeros of the system transfer function corresponding to $P(z^{-1})$ by introducing equivalent poles and zeros. This results in an unstable mode if the cancellation is not exact.

2) The inverse process model as a controller will in general be a high pass filter so that the control will change rapidly and could be unacceptably large in amplitude as far as the saturation problem is concerned.

3) The minimum variance control law has no facility to adjust the controller if the closed loop performance becomes unsatisfactory after the implementation of the control law.

Following the work of Clarke and Hasting-James [63], these difficulties can be overcome at the expense of some deterioration in the control quality by suitable modifications in the criterion (3.2.4) as

$$L_1 = E[(y_{t+k} - y_t^d)^2 + \lambda u_t^2] \tag{3.2.20}$$

$$L_2 = E[(y_{t+k} - y_t^d)^2 + \lambda(u_t - u_{t-1})^2] \tag{3.2.21}$$

where λ is an arbitrary positive constant. It has been shown in [63] that the sub-optimum control law based on the criterion L_1 avoids instability problems associated with the non-minimum phase processes by an appropriate choice of λ. The criterion L_2 has the advantage that the variance of changes in control signal is weighted by an adjustable parameter λ which can be used to avoid saturation problems. Therefore use of modified criteria can give some degree of adjustment if the control system performance becomes unsatisfactory.

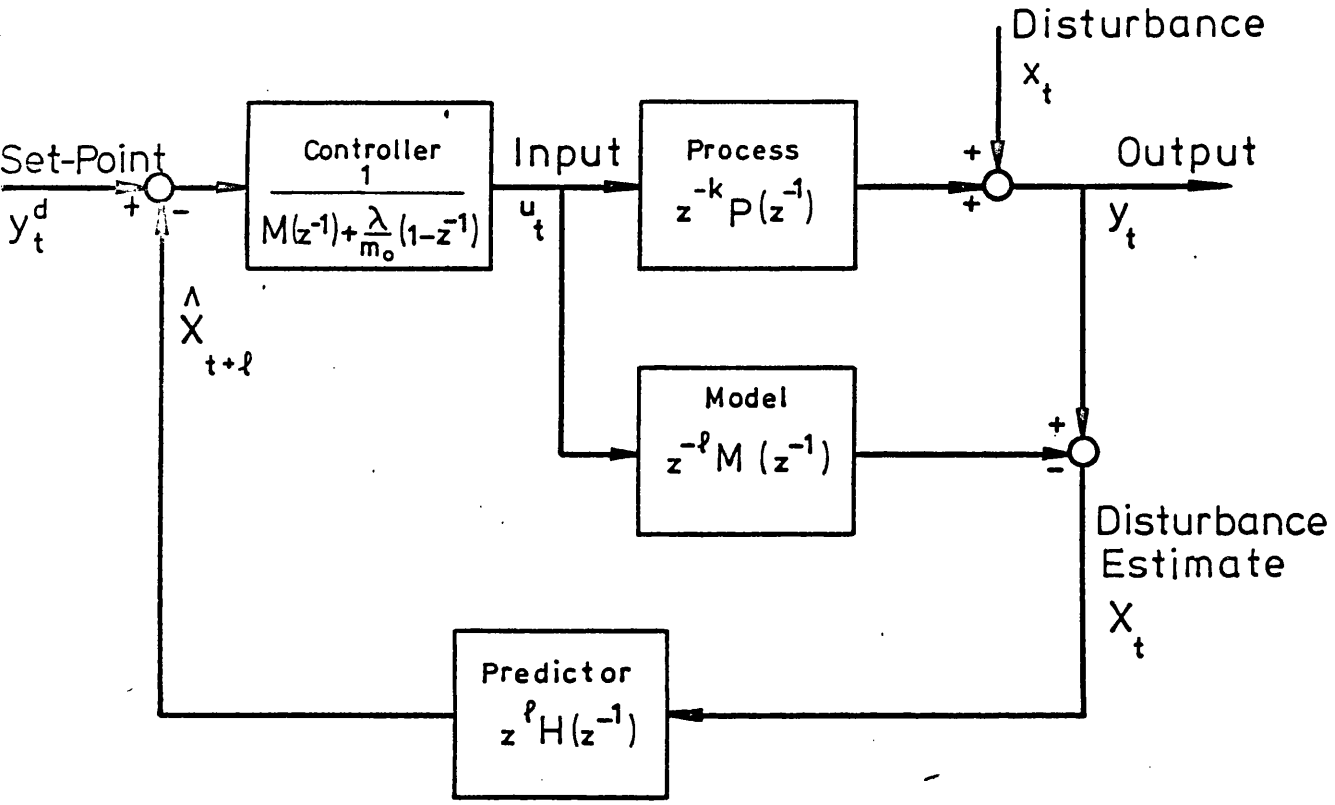

Fig. 3.2.5. THE BOX AND JENKINS CONTROL SYSTEM

$\lambda = 0$ IMPLEMENTS THE OPTIMUM CONTROL LAW 3.2.16

$\lambda \neq 0$ IMPLEMENTS THE SUB-OPTIMUM CONTROL LAW 3.2.22

Analogous calculations show that the criterion L_2 is minimized for the following sub-optimum control law

$$u_t = \frac{1}{P(z^{-1}) + \frac{\lambda}{P_o}(1-z^{-1})} [y_t^d - z^k H^*(z^{-1})x_t] \qquad (3.2.22)$$

and the corresponding result for the criterion L_1 is obtained by replacing $(1-z^{-1})$ in (3.2.22) by unity.

So far it has been assumed that the dynamics of the process and the disturbance are accurately known. In practice however, the exact knowledge of $P(z^{-1})$, $H^*(z^{-1})$ and k is seldom known. Let $M(z^{-1})$, $H(z^{-1})$ and ℓ be estimates of $P(z^{-1})$, $H^*(z^{-1})$ and k respectively. Then the practical implementation of the Box and Jenkins control system of Fig. (3.2.3) will be as in Fig. (3.2.5), where the signal

$$X_t = y_t - z^{-\ell} M(z^{-1}) u_t \qquad (3.2.23)$$

now represents an estimate of the disturbance x_t, since if $M(z^{-1}) = P(z^{-1})$ and $\ell = k$ then $X_t = x_t$. Substituting estimated parameters into (3.2.16), the practical implementation of the optimum control law is

$$u_t = \frac{1}{M(z^{-1})} [y_t^d - z^{\ell} H(z^{-1}) X_t] \qquad (3.2.24)$$

3.3. Stability of the Box and Jenkins control system

Since it is not realistic in practice to assume a perfect knowledge of the process impulse response function or the disturbance model, it is therefore of practical importance to analyse the stability of the control system where allowance is made for estimation errors.

An empirical solution to the stability problem has been given by Phillipson [57]. It has been shown that if an exponential smoothing

filter is used as a predictor, i.e.

$$z^{\ell}H(z^{-1}) = (1-\beta) \sum_{i=o}^{\infty} \beta^{i} z^{-i} \qquad (3.3.1)$$

where β as the variable parameter to be set on-line between values one and zero, then provided that the process and its model are stable and minimum phase and the steady state gains of process and model are of the same sign, it is always possible to stabilize the closed loop system by reducing the bandwidth of the exponential smoother used as a predictor.

It will now be shown that an alternative way to stabilize the Box and Jenkins control system is to use the adjustment parameter λ of the sub-optimum control law given by (3.2.22). It will be assumed that the process, model and predictor are stable and minimum phase and the predictor has a unity steady state gain, i.e.

$$H(z^{-1}) \Big|_{z=1} = 1 \qquad (3.3.2)$$

Denoting $M(z^{-1})$, $H(z^{-1})$ and ℓ as the estimates of $P(z^{-1})$, $H^{*}(z^{-1})$ and k, the sub-optimum control law (3.2.22) can be written in terms of estimates as

$$u_t = \frac{1}{M(z^{-1}) + \frac{\lambda}{m_o}(1-z^{-1})} [y_t^d - z^{\ell} H(z^{-1}) X_t] \qquad (3.3.3)$$

The closed loop response of the Box and Jenkins control system of Fig. (3.2.5) corresponding to the sub-optimum control law is given by

$$y_t = \frac{\{M(z^{-1})[1-H(z^{-1})] + \frac{\lambda}{m_o}(1-z^{-1})\}X_t + z^{-k}P(z^{-1})y_t^d}{M(z^{-1})[1-H(z^{-1})]+z^{\ell-k}P(z^{-1})H(z^{-1}) + \frac{\lambda}{m_o}(1-z^{-1})} \qquad (3.3.4)$$

and therefore the characteristic equation of the closed loop system is

$$M(z^{-1})[1-H(z^{-1})] + z^{\ell-k}P(z^{-1})\, H(z^{-1}) + \frac{\lambda}{m_o}(1-z^{-1}) = 0 \qquad (3.3.5)$$

It will now be assumed that $P(z^{-1})$, $M(z^{-1})$ and $H(z^{-1})$ can be written as conventional pulse transfer functions, i.e. as a ratio of two polynomials in z of equal degree ν with a multiplicative constant as

$$M(z^{-1}) = m_o \frac{M_1(z)}{M_2(z)}, \quad P(z^{-1}) = p_o \frac{P_1(z)}{P_2(z)}, \quad H(z^{-1}) = h_o z^{-\ell} \frac{H_1(z)}{H_2(z)} \tag{3.3.6}$$

then using (3.3.6) and clearing denominator terms, the characteristic equation becomes

$$m_o^2 z P_2(z) M_1(z)[z^{\ell}H_2(z) - h_oH_1(z)] + p_om_oh_oz^{\ell-k+1} P_1(z)H_1(z)M_2(z)$$

$$+ \lambda z^{\ell}(z-1) P_2(z) M_2(z) H_2(z) = 0 \tag{3.3.7}$$

The stability of the system requires that all roots of (3.3.7) must lie in the unit circle of the z-plane. Consider the root loci of (3.3.7) for λ over the interval $(0,\infty)$. The poles are given by

$$F_1(z) = m_o^2 z P_2(z) M_1(z)[z^{\ell}H_2(z) - h_oH_1(z)] + p_oh_om_oz^{\ell-k+1}P_1(z)H_1(z)M_2(z) = 0 \tag{3.3.8}$$

Inspection of the degrees of polynomials in (3.3.8) shows that the number of poles is $(3\nu + \ell + 1)$. There is an equal number of zeros given by their defining equation

$$F_2(z) = z^{\ell}(z-1) P_2(z) M_2(z) H_2(z) = 0 \tag{3.3.9}$$

If all zeros lay in the interior of the unit circle of the z-plane then there would exist a sufficiently large value of λ such that portions of the loci and consequently the closed loop poles would lie entirely within the unit circle. Since it has been assumed that process, model and predictor are all stable, this situation occurs with but one exception, namely the zero at (1,0). The locus terminating on this zero must lie on the ~~unit circle~~ real axis, since all coefficients of

the characteristic equation are real and hence the root locus diagram is symmetrical about the real axis. If this branch approaches (1,0) from the left then the closed loop system can be made stable for sufficiently large values of λ .

If this branch approaches (1,0) from the right then there will be a closed loop pole exterior to the unit circle and the closed loop system will be unstable for all λ . But this situation arises if there is an odd number of real poles to the right of (1,0) , since there is no zero at $(\infty,0)$. Thus a test has to be carried out on (3.3.8) to find whether there exist an even number of real poles in the range $(1,\infty)$. From the properties of algebraic equations [64], this test can be reduced to the requirement that

$$\text{Sign}[F_1(1)] = \text{Sign}[\lim_{z \to \infty} F_1(z)] \tag{3.3.10}$$

Since the degree of the first term of $F_1(z)$ is always higher than that of the second for $k > 0$, it follows that $\lim_{z \to \infty} F_1(z) = \infty$, and this requires that

$$F_1(1) = m_o^2 P_2(1) M_1(1)[H_2(1) - h_o H_1(1)] + p_o m_o h_o P_1(1) H_1(1) M_2(1) \tag{3.3.11}$$

should be positive. But from (3.3.2)

$$H(1) = h_o \frac{H_1(1)}{H_2(1)} = 1 \qquad \text{or,}$$

$$H_2(1) - h_o H_1(1) = 0 \tag{3.3.12}$$

then using (3.3.12), $F_1(1)$ becomes

$$F_1(1) = p_o m_o h_o P_1(1) H_1(1) M_2(1) \tag{3.3.13}$$

Since all roots of $P_1(z)$, $P_2(z)$, $M_1(z)$, $M_2(z)$, $H_1(z)$, $H_2(z)$ are required to be within the unit circle of the z-plane , it follows from

the Descartes rule of signs that [64] all the above polynomials are positive for $z=1$ and hence

$$\text{Sign}[F_1(1)] = \text{Sign}(p_o m_o h_o) \tag{3.3.14}$$

Then $F_1(1) > 0$ implies

$$p_o m_o h_o > 0 \tag{3.3.15}$$

Following [57], (3.3.15) can be equivalently written in terms of the steady-state gains of process and model as

$$m_o^2 h_o \frac{p_o P_1(1)/P_2(1)}{m_o M_1(1)/M_2(1)} > 0 \tag{3.3.16}$$

Therefore the conclusion is that if the steady state gains of process and model are of the same sign then the Box and Jenkins control system can always be made stable by increasing λ provided that the first coefficient of the unity steady gain predictor is positive.

Comments

1) The predictor with a steady state gain of unity is often desirable in practice. Since, as it will be shown in Section 3.6, if the disturbance has an unknown constant component, then unbiased prediction of the constant component requires that the predictor must have a unity steady state gain, i.e. $H(z^{-1})\,|_{z=1} = 1$.

2) If the unity steady state gain predictor is chosen as an exponential smoothing filter as in [57], i.e.

$$z^{\ell} H(z^{-1}) = \frac{1-\beta}{1-\beta z^{-1}} \qquad 0 \leq \beta < 1 \tag{3.3.17}$$

then since $h_o = 1-\beta > 0$ and hence (3.3.16) now states that the Box and Jenkins control system can always be stabilized by increasing λ provided that the steady state gains of process and model are of the same sign.

In conclusion, it has been shown in this section that it is always possible to set up a stable Box and Jenkins control system if the steady state gains of process and model are of the same sign. The following sections will represent some methods of estimation of disturbance models and the design of linear minimum variance predictors which will be used in the identification and self-adaptive control techniques developed in the later sections of this chapter.

3.4. Estimation of auto-regressive model parameters

Consider an auto-regressive, zero mean stationary random process X_t as

$$X_t = \sum_{i=1}^{n} r_i X_{t-i} + \zeta_t \qquad (3.4.1)$$

where ζ_t is an independent zero mean process, such that $E(\zeta_t) = 0$, $E(\zeta_t \zeta_s) = 0 \quad t \neq s$, $E(\zeta_t \zeta_s) = \sigma_\zeta^2 \quad t = s$. The problem is to estimate unknown parameters $\{r_i, n\}$ given a set of observations $\{X_t;\ t=1,2,\ldots,N\}$. Mann and Wald [15] have shown that the least squares estimates of the unknown parameters $\{r_i\}$ are consistent and asymptotically Gaussian distributed if the order n is known. Using the vector-matrix representation of previous chapters, the least squares estimates $\{\hat{r}_i\}$ can be written as

$$\hat{r} = (Z^T Z)^{-1} Z^T X \qquad (3.4.2)$$

where $\hat{r} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n)^T$, $Z = (SX, S^2X, \ldots, S^nX)$ and $X = (X_1, X_2, \ldots, X_N)^T$. The asymptotic properties of estimates can now be stated as

$$\text{i)} \quad \lim_{N \to \infty} \hat{r} = r \quad \text{with probability one} \qquad (3.4.3)$$

ii) $\sqrt{N}\ (\hat{r}_i - r_i)$; $i = 1,2,\ldots,n$ have a limiting Gaussian distribution with means zero and covariances given by the elements of the matrix $\sigma_\zeta^2 W^{-1}$ where W is defined as

$$W = E \begin{bmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-n+1} \end{bmatrix} [X_t, X_{t-1}, \ldots, X_{t-n+1}] = \begin{bmatrix} \gamma_o & \gamma_1 & \cdots & \gamma_{n-1} \\ \vdots & & & \\ \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_o \end{bmatrix}$$

and where γ_i; $i=0,1,\ldots,n-1$ are the auto-covariances of X_t, i.e. $\gamma_i = EX_tX_{t-i}$.

In a practical situation of the auto-regressive model fitting, the order of the model is not generally known. In fact as it will be the case in Section 3.8, the order may not be finite and in this case the chosen model order will only be an artificial variable giving rise to models which only approximate the true model.

One possible way to estimate the correct order n is to use the property (ii) in testing the following null hypothesis

$$H : r_n = 0 \quad \text{against alternative} \quad r_n \neq 0 .$$

Since $(\hat{r}_n - r_n)$ has a limiting normal distribution with mean zero and variance $\sigma_\zeta^2 (NW)^{-1}_{n,n}$, where the subscript n denotes the $(n,n)^{th}$ element of $(NW)^{-1}$, then if the hypothesis is true, the test statistic α_n defined below

$$\alpha_n = \frac{\hat{r}_n}{\sigma_\zeta \sqrt{(NW)^{-1}_{n,n}}} \tag{3.4.4}$$

has a normal distribution with zero mean and unity variance. Furthermore under the hypothesis it is also known that $\sigma_\zeta^2 (W)^{-1}_{n,n} = 1$, [65], and hence the test statistic α_n is simplified as

$$\alpha_n = \sqrt{N}\ \hat{r}_n \tag{3.4.5}$$

Then employing the two-sided ε-significance points $t(\varepsilon)$ of the normal distribution, the null hypothesis is rejected with $(1-\varepsilon)\%$ confidence if $|\alpha_n| > t(\varepsilon)$. Based on this result, a procedure to estimate the model order can now be given in the following steps.

1) Fit auto-regressive models of successive orders to $\{X_t\}$ by the least squares method. Let $\hat{r}_{\hat{n},i}$; $i=1,2,\ldots,\hat{n}$ denote an estimate of the i^{th} parameter of a model of order $\hat{n}$.

2) Test the hypothesis $\hat{r}_{\hat{n},\hat{n}} = 0$ as described above.

3) Repeat steps (1) and (2) until the hypothesis is accepted.

The computational burden of step (1) can be avoided if the least squares estimates are approximated by those that given by the Yule-Walker equations. The Yule-Walker estimates are obtained by approximating the elements of the matrices Z^TZ and Z^TX with the estimates of auto-covariance coefficients as

$$Z^TZ \simeq N \begin{bmatrix} \hat{\gamma}_o & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{\hat{n}-1} \\ \vdots & & & \\ \hat{\gamma}_{\hat{n}-1} & \hat{\gamma}_{\hat{n}-2} & \cdots & \hat{\gamma}_o \end{bmatrix}, \quad Z^TX \simeq N \begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_{\hat{n}} \end{bmatrix} \tag{3.4.6}$$

Defining auto-correlation coefficients as $\hat{\rho}_i = \dfrac{\hat{\gamma}_i}{\hat{\gamma}_o}$; $i=0,1,\ldots,\hat{n}$ and using (3.4.6) in (3.4.2), the Yule-Walker estimates are given as the solution of

$$\begin{bmatrix} 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{\hat{n}-1} \\ \vdots & 1 & & \vdots \\ \hat{\rho}_{\hat{n}-1} & \hat{\rho}_{\hat{n}-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \hat{r}_{\hat{n},1} \\ \vdots \\ \hat{r}_{\hat{n},\hat{n}} \end{bmatrix} = \begin{bmatrix} \hat{\rho}_1 \\ \vdots \\ \hat{\rho}_{\hat{n}} \end{bmatrix} \tag{3.4.7}$$

Durbin [66] has shown that eqn. (3.4.7) can be solved by the following recursive relations

$$\hat{r}_{\hat{n},\hat{n}} = \frac{\hat{\rho}_{\hat{n}} - \sum_{i=1}^{\hat{n}-1} \hat{r}_{\hat{n}-1,i}\, \hat{\rho}_{\hat{n}-i}}{1 - \sum_{i=1}^{\hat{n}-1} \hat{r}_{\hat{n}-1,i}\, \hat{\rho}_i} \tag{3.4.8}$$

$$\hat{r}_{\hat{n},j} = \hat{r}_{\hat{n}-1,j} - \hat{r}_{\hat{n}-1,\,\hat{n}-j}\, \hat{r}_{\hat{n},\hat{n}} \;;\; j=1,2,\ldots,n-1 \tag{3.4.9}$$

However the resulting estimates may become sensitive if $1-R(z^{-1}) = 0$ has a root near unity [14].

The procedure explained above assumes that there is a true order n and tests a null hypothesis about n. The idea that the order n is only an artificial variable for the purpose of approximating an infinite order auto-regressive process and as such is a parameter to be estimated was suggested by Akaike [67]. His procedure chooses the model order by minimizing 'the final prediction error' which is the mean squared one step ahead prediction error when the set of fitted coefficients $\{\hat{r}_i\}$ is applied to another independent realization of the process X_t. It was shown that the final prediction error (FPE) can be estimated as

$$\text{FPE}\ (\hat{n}) = \frac{N+\hat{n}}{N-\hat{n}}\, \hat{\sigma}_\zeta^2 \tag{3.4.10}$$

By scanning $\hat{n}$ successively from zero to some upper limit, the model order is estimated as the value of $\hat{n}$ which minimizes $FPE(\hat{n})$. The advantage of this approach is that no subjective element exists in the procedure as compared with the test of statistical hypothesis which requires the specification of the significance level. However, for a finite order auto-regressive model, the final prediction error test is not consistent without further modification in (3.4.10), [67].

3.5. The design of optimum linear predictor

In the derivation of the Box and Jenkins control system it has been shown that the prediction of the disturbance x_t plays a vital role as far as the disturbance rejection problem is concerned. This section will, in consequence, be concerned with the design of optimum linear predictor for a zero mean stationary disturbance x_t, represented by an auto-regressive process

$$x_t = \frac{\zeta_t}{1-Q(z^{-1})} \qquad (3.5.1)$$

where $Q(z^{-1}) = \sum_{i=1}^{m} q_i z^{-i}$ and $E(\zeta_t^2) = \sigma_\zeta^2$ are assumed to be known. The optimality is in the sense that the predictor minimizes the mean square prediction error.

Now consider first the one step ahead prediction. Let $\hat{x}_{t+1}$ denote the prediction of x_{t+1} given all past information up to and including time t, i.e. given $\underline{x}_t = (x_t, x_{t-1}, \ldots)$. From definition, $\hat{x}_{t+1}$ minimizes $E(x_{t+1}-\hat{x}_{t+1})^2$. Writing x_{t+1} from (3.5.1) as

$$x_{t+1} = \sum_{i=1}^{m} q_i\, x_{t+1-i} + \zeta_{t+1} = E(x_{t+1}|\underline{x}_t) + \zeta_{t+1}$$
$$= \langle x_{t+1} \rangle + \zeta_{t+1} \tag{3.5.2}$$

where $\langle . \rangle$ denotes $E(\cdot|\underline{x}_t)$, since $E(\langle x_{t+1} \rangle\, \zeta_{t+1}) = 0$ then it follows that

$$E(x_{t+1} - \hat{x}_{t+1})^2 = E(\langle x_{t+1} \rangle - \hat{x}_{t+1})^2 + \sigma_\zeta^2 \tag{3.5.3}$$

and the minimum value is obtained for

$$\hat{x}_{t+1} = \langle x_{t+1} \rangle = E(x_{t+1}|\underline{x}_t) \tag{3.5.4}$$

the minimum mean square prediction error is then

$$\min E(x_{t+1} - \hat{x}_{t+1})^2 = \sigma_\zeta^2 \tag{3.5.5}$$

For the two step ahead prediction, first writing from (3.5.1)

$$\langle x_{t+2} \rangle = E(x_{t+2}|\underline{x}_t) = q_1 \langle x_{t+1} \rangle + \sum_{i=2}^{m} q_i\, x_{t-i+2} \tag{3.5.6}$$

then it can be shown as in (3.5.2) that

$$x_{t+2} = q_1(x_{t+1} - \langle x_{t+1} \rangle) + \langle x_{t+2} \rangle + \zeta_{t+2}$$
$$= q_1\, \zeta_{t+1} + \langle x_{t+2} \rangle + \zeta_{t+2} \tag{3.5.7}$$

$\hat{x}_{t+2}$ is now obtained by minimizing

$$E(x_{t+2} - \hat{x}_{t+2})^2 = E(\langle x_{t+2} \rangle - \hat{x}_{t+2} + q_1\, \zeta_{t+1} + \zeta_{t+2})^2$$
$$= E(\langle x_{t+2} \rangle - \hat{x}_{t+2})^2 + \sigma_\zeta^2\,(1 + q_1^2) \tag{3.5.8}$$

for which the minimum is obtained for

$$\hat{x}_{t+2} = \langle x_{t+2} \rangle = E(x_{t+2}|\underline{x}_t) \tag{3.5.9}$$

with the corresponding mean square error

$$\min E(x_{t+2} - \hat{x}_{t+2})^2 = \sigma_\zeta^2 (1 + q_1^2) \qquad (3.5.10)$$

From (3.5.6), $\hat{x}_{t+2}$ can also be written in terms of $\hat{x}_{t+1}$ as

$$\hat{x}_{t+2} = q_1 \hat{x}_{t+1} + \sum_{i=2}^{m} q_i x_{t-i+2} \qquad (3.5.11)$$

Proceeding this way the well known result of Kalman [68] on prediction is obtained as

$$\hat{x}_{t+k} = \langle x_{t+k} \rangle = E(x_{t+k} | \underline{x}_t) \qquad (3.5.12)$$

and it can be shown that for the auto-regressive process (3.5.1), k^{th} step ahead prediction can be expressed in a recursive way as [14]

$$\left.\begin{aligned} \hat{x}_{t+k} &= \sum_{i=1}^{k-1} q_i \hat{x}_{t+k-i} + \sum_{i=k}^{m} q_i x_{t+k-i} \quad \text{for } k \leq m \\ \hat{x}_{t+k} &= \sum_{i=1}^{m} q_i \hat{x}_{t+k-i} \quad \text{for } k > m \end{aligned}\right\} \qquad (3.5.13)$$

In the context of the Box and Jenkins control system it is convenient to express $\hat{x}_{t+k}$ only in terms of the past values of x_t, i.e. $\underline{x}_t$. This can be achieved by first representing the auto-regressive process (3.5.1) as an infinite order moving average process as

$$x_t = \frac{\zeta_t}{1 - Q(z^{-1})} = [1 + S(z^{-1})]\, \zeta_t \qquad (3.5.14)$$

where $S(z^{-1}) = \sum_{i=1}^{\infty} s_i z^{-i}$. Then from (3.5.12)

$$\begin{aligned} \hat{x}_{t+k} = E(x_{t+k} | \underline{x}_t) &= s_k \zeta_t + s_{k+1} \zeta_{t-1} + \cdots \\ &= (s_k z^{-k} + s_{k+1} z^{-k-1} + \ldots)\, \zeta_{t+k} \\ &= S_T(z^{-1})\, \zeta_{t+k} = z^k S_T(z^{-1})\, \zeta_t \end{aligned} \qquad (3.5.15)$$

On substitution of ζ_t from (3.5.14), $\hat{x}_{t+k}$ can now be written as

$$\hat{x}_{t+k} = z^k S_T(z^{-1}) \left[1 + S(z^{-1})\right]^{-1} x_t$$

$$= z^k H^*(z^{-1}) x_t \qquad (3.5.16)$$

where $H^*(z^{-1})$ is defined as

$$H^*(z^{-1}) = S_T(z^{-1}) \left[1 + S(z^{-1})\right]^{-1} \qquad (3.5.17)$$

or equivalently in terms of the auto-regressive model, as

$$H^*(z^{-1}) = \{[1 - Q(z^{-1})]^{-1}\}_T [1 - Q(z^{-1})] \qquad (3.5.18)$$

Using (3.5.14) and (3.5.16) the optimum prediction error is obtained as a moving average process of order $k - 1$, i.e.

$$x_{t+k} - \hat{x}_{t+k} = [1 - H^*(z^{-1})] x_{t+k} = [1 + S(z^{-1}) - S_T(z^{-1})] \zeta_{t+k} \qquad (3.5.19)$$

and hence the minimum prediction error variance is

$$\min E(x_{t+k} - \hat{x}_{t+k})^2 = \sigma_\zeta^2 (1 + \sum_{i=1}^{k-1} s_i^2) \qquad (3.5.20)$$

Optimum predictors designed from auto-regressive models have the following useful practical properties.

1) The predictor $z^k H^*(z^{-1})$ for a stationary auto-regressive process is a moving average filter and therefore always stable.

2) For an auto-regressive process of order m, $H^*(z^{-1})$ is a finite memory filter with only m non-zero coefficients [52].

3) Let $z^i H_i^*(z^{-1})$ denote i^{th} step ahead prediction of x_t, that is $\hat{x}_{t+i} = z^i H_i^*(z^{-1}) x_t$. By substituting of $\hat{x}_{t+i}$ into (3.5.13) it can be shown that a k^{th} step ahead predictor can be calculated recursively as

$$\left.\begin{aligned} H_k^*(z^{-1}) &= \sum_{i=1}^{k-1} q_i\, H_{k-i}^*(z^{-1})z^{-i} + \sum_{i=k}^{m} q_i\, z^{-i} \quad \text{for } k \leq m \\ H_k^*(z^{-1}) &= \sum_{i=1}^{m} q_i\, H_{k-i}^*(z^{-1})z^{-i} \quad \text{for } k > m \end{aligned}\right\} \qquad (3.5.21)$$

where $H_1^*(z^{-1}) = Q(z^{-1})$.

3.6. Predictors with unity steady state gain

In the previous section it was assumed that x_t is a zero mean random disturbance. In the case of constant but unknown mean level, $z^k H^*(z^{-1})$ is no longer an optimum predictor. To see this, let the stationary non-zero mean disturbance be represented as

$$x'_t = x_t + \bar{x} \qquad (3.6.1)$$

where $\bar{x} = Ex'_t$ is unknown. If $\hat{x}'_{t+k}$ is determined from (3.5.16) as $\hat{x}'_{t+k} = H^*(z^{-1})x'_{t+k}$ then

$$E(x'_{t+k} - \hat{x}'_{t+k}) = (1 - \sum_{i=0}^{\infty} h_i^*)\bar{x} \qquad (3.6.2)$$

$$E(x'_{t+k} - \hat{x}'_{t+k})^2 = E(x_{t+k} - \sum_{i=0}^{\infty} h_i^* x_{t-i})^2 + (1 - \sum_{i=0}^{\infty} h_i^*)^2 \bar{x}^2 \qquad (3.6.3)$$

Therefore it follows that the prediction is biased and the mean square prediction error is increased by an amount which is proportional to $(1- \sum_{i=0}^{\infty} h_i^*)\bar{x}$. Hence the only way to eliminate the bias and the increased variance is to modify the predictor so that it has a unity steady state gain, i.e.

$$H^*(z^{-1})\big|_{z=1} = \sum_{i=0}^{\infty} h_i^* = 1 \qquad (3.6.4)$$

Consider now a situation where the auto-regressive polynomial $[1-Q(z^{-1})]$ can be factored as

$$1-Q(z^{-1}) = (1-\frac{1}{d}\sum_{i=1}^{d} z^{-i})^{f}\,[1-Q'(z^{-1})] \tag{3.6.5}$$

where d and f are positive integers. Then writing $[1-H^*(z^{-1})]$ from (3.5.17) and (3.6.5) as

$$1-H^*(z^{-1})=[1+S(z^{-1})-S_T(z^{-1})](1-\frac{1}{d}\sum_{i=1}^{d} z^{-i})^{f}\,[1-Q'(z^{-1})] \tag{3.6.6}$$

it can be seen that $H^*(z^{-1})\big|_{z=1} = 1$, i.e. the optimum predictor based on the auto-regressive model (3.6.5) has a unity steady state gain. The form (3.6.5) for $d=1$ has been used extensively by Box and Jenkins [14]. This corresponds to a non-stationary disturbance of accumulated type, since $[1-Q(z^{-1})]$ has roots on the unit circle. The case $f=0$ in (3.6.5) corresponds to the modification suggested by Turtle [52] where d was suitably chosen such that the mean squared error of the modified predictor is close to that of the optimum predictor. However, since $z=1$ is a factor of (3.6.5), this form also corresponds to a non-stationary disturbance model. Therefore the conclusion is that neither of the above form can be used if the disturbance is stationary.

One way to modify the predictor in the stationary case is to assume a linear prediction in the form [65]

$$\hat{x}_{t+k} = \sum_{i=0}^{w} h_i\, x_{t-i} \tag{3.6.7}$$

and to determine $\{h_i\}$ parameters so that the mean square prediction error

$$E(x_{t+k}-\sum_{i=0}^{w} h_i x_{t-i})^2=\gamma_o-2\sum_{i=0}^{w} h_i\gamma_{k+i}+\sum_{i=0}^{w}\sum_{j=0}^{w} h_i\, h_j\, \gamma_{|i-j|} \tag{3.6.8}$$

is minimized subject to the constraint $\sum_{i=0}^{w} h_i = 1$,

where $\gamma_i = Ex_t x_{t-i}$. Introducing a Lagrange multiplier ν, the problem becomes the minimization of

$$L = E(x_{t+k} - \sum_{i=0}^{w} h_i x_{t-i})^2 + \nu(\sum_{i=0}^{w} h_i - 1) \qquad (3.6.9)$$

Straightforward calculations show that (3.6.9) is minimized for

$$h = \begin{bmatrix} h_o \\ h_1 \\ \vdots \\ h_w \end{bmatrix} = W^{-1}\gamma + (J^T W^{-1} J)^{-1}(1 - J^T W^{-1}\gamma)W^{-1}J \qquad (3.6.10)$$

where $W = \begin{bmatrix} \gamma_o & \cdots & \gamma_w \\ \vdots & & \vdots \\ \gamma_w & \cdots & \gamma_o \end{bmatrix}$ is an $(w+1)x(w+1)$ matrix and non-singular for all w [65],

$\gamma = (\gamma_k, \gamma_{k+1}, \ldots, \gamma_{k+w})^T$ and J is an $(w+1)$ column vector with unity elements.

The choice of the number of predictor parameters can be based on the following reasoning. Since the disturbance is assumed to be stationary then, in practice, its infinite order moving average representation can be truncated after a certain number of terms, say $\bar{w}$, i.e.

$$1 + S(z^{-1}) \simeq 1 + \sum_{i=1}^{\bar{w}} s_i z^{-i} \qquad (3.6.11)$$

then corresponding to this model, auto-covariance coefficients γ_i can be written as

$$\left.\begin{aligned} \gamma_i &= \sigma_\zeta^2 \sum_{j=0}^{\bar{w}-i} s_{i+j}\, s_j \qquad i=0,1,\ldots,\bar{w} \\ \gamma_i &= 0 \qquad i > \bar{w} \end{aligned}\right\} \tag{3.6.12}$$

where $s_o=1$. It then follows that the choice of w as $k+w>\bar{w}$ adds only zero elements to γ . In consequence, a reasonable choice for the number of predictor parameters is that

$$w > \bar{w} - k \tag{3.6.13}$$

The main disadvantage of this technique is that the number of parameters in the moving average representation (3.6.11) is in general large and hence the computation of h_i's may prove to be time consuming.

The next section will represent a new technique for the design of unity steady state gain predictors.

3.7. The modified predictor

Consider the prediction error (3.5.19) corresponding to the optimum predictor, i.e.

$$x_{t+k} - \hat{x}_{t+k} = (1 + \sum_{i=1}^{k-1} s_i z^{-i})\zeta_{t+k} \tag{3.7.1}$$

and suppose that the modified predictor $z^k\bar{H}(z^{-1})$ is designed such that the resulting prediction error is

$$\begin{aligned} x_{t+k} - \bar{x}_{t+k} &= [1 - \bar{H}(z^{-1})]x_{t+k} \\ &= (1 + \sum_{i=1}^{k-1} s_i z^{-i})\zeta_{t+k} + (\sum_{i=k}^{k+\mu-1} \phi_i z^{-i})\zeta_{t+k} \\ &= (1 + \sum_{i=1}^{k-1} s_i z^{-i} + \sum_{i=k}^{k+\mu-1} \phi_i z^{-i})[1-Q(z^{-1})]x_{t+k} \end{aligned} \tag{3.7.2}$$

where $\bar{x}_{t+k}$ is the modified k step ahead prediction and the arbitrary coefficients $\{\phi_k, \phi_{k+1}, \ldots, \phi_{k+\mu-1}\}$ are chosen such that the prediction error variance is minimized subject to the constraint

$$1-\bar{H}(z^{-1})\big|_{z=1} = (1 + \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{k+\mu-1} \phi_i)(1 - \sum_{i=1}^{m} q_i) = 0 \tag{3.7.3}$$

Since the disturbance is stationary then $\sum_{i=1}^{m} q_i \neq 1$ and hence (3.7.3) is satisfied if and only if

$$1 + \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{k+\mu-1} \phi_i = 0 \tag{3.7.4}$$

Introducing a Lagrange multiplier ν, ϕ_i's are determined such that

$$L = E(x_{t+k} - \bar{x}_{t+k})^2 + \nu(1 + \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{k+\mu-1} \phi_i) \tag{3.7.5}$$

is minimum. Using (3.7.2) and taking expectation, (3.7.5) becomes

$$L = (1 + \sum_{i=1}^{k-1} s_i^2 + \sum_{i=k}^{k+\mu-1} \phi_i^2)\sigma_\zeta^2 + \nu(1 + \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{k+\mu-1} \phi_i) \tag{3.7.6}$$

which is minimized for

$$\phi_i = -\frac{1 + \sum_{j=1}^{k-1} s_j}{\mu} = \phi \quad \text{say} \quad ; \quad i = k, k+1, \ldots, k+\mu-1 \tag{3.7.7}$$

and the minimum value is

$$L_{min} = \sigma_\zeta^2 (1 + \sum_{i=1}^{k-1} s_i^2) + \frac{\sigma_\zeta^2}{\mu} (1 + \sum_{i=1}^{k-1} s_i)^2 \tag{3.7.8}$$

Since the first term of (3.7.8) is the optimum prediction error then it follows that the introduction of additional $\{\phi_i\}$ parameters increases the optimum prediction error by

$$\frac{\sigma_\zeta^2}{\mu}\left(1+\sum_{i=1}^{k-1} s_i\right)^2 \qquad (3.7.9)$$

On substitution of ϕ_i from (3.7.7) into (3.7.2) and defining $\Phi(z^{-1}) = \phi \sum_{i=k}^{k+\mu-1} z^{-i}$, the modified prediction $\bar{x}_{t+k}$ can be written as

$$\begin{aligned}\bar{x}_{t+k} &= z^k[-\Phi(z^{-1})+S_T(z^{-1})][1+S(z^{-1})]^{-1}x_t \\ &= z^k[-\Phi(z^{-1})+S_T(z^{-1})][1-Q(z^{-1})]x_t \\ &= z^k \bar{H}(z^{-1})x_t \qquad (3.7.10)\end{aligned}$$

where

$$\bar{H}(z^{-1}) = [-\Phi(z^{-1})+S_T(z^{-1})][1-Q(z^{-1})] \qquad (3.7.11)$$

Comparison of $\bar{H}(z^{-1})$ with $H^*(z^{-1})$ given by (3.5.17) shows that the modified predictor is similar in structure to the optimum predictor $z^k H^*(z^{-1})$. The difference is that in (3.7.11) the first μ parameters of $S_T(z^{-1})$ are modified to satisfy the constraint (3.7.3).

One main problem associated with the modified predictor is the choice of μ, for which at the moment a theoretical answer is lacking. However since the disturbance characteristics are assumed to be known then a reasonable choice of μ is such that the increase in the optimum prediction error variance (3.7.9) is as small as possible.

Comments

1) the modified predictor $z^k\bar{H}(z^{-1})$ is a finite memory filter with only $(m+\mu)$ parameters. This can be shown by writing $\bar{H}(z^{-1})$ as

$$\bar{H}(z^{-1}) = [\sum_{i=k}^{k+\mu-1}(s_i-\phi)z^{-i}][1-Q(z^{-1})] + (\sum_{i=k+\mu}^{\infty} s_i z^{-i})[1-Q(z^{-1})] \qquad (3.7.12)$$

where it was proved in [52] that the second term has only m non-zero coefficients which correspond to the powers of

$$\{z^{-(k+i)};\ i=\mu,\ \mu+1,\ \ldots,\ \mu+m-1\} \tag{3.7.13}$$

whereas the first term has $(\mu+m)$ parameters corresponding to the powers of

$$\{z^{-(k+j)};\ j=0,\ 1,\ \ldots,\ \mu-1,\ \mu,\ \mu+1,\ \ldots,\ \mu+m-1\} \tag{3.7.14}$$

then noting that the set given by (3.7.14) contains that of (3.7.13), it follows that only $(m+\mu)$ parameters of the modified predictor are different from zero.

2) The modified predictor has been obtained by considering only the unconditional expectation of the criterion (3.7.5). In consequence, the derived predictor does not take into account the fact that at time $(t+k)$, past information up to and including time t is available. Consider now the conditional version of the criterion (3.7.5), i.e.

$$L_c = E[(x_{t+k}-\bar{x}_{t+k})^2 | x_t, x_{t-1}, \ldots] + \nu\left(1 + \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{k+\mu-1} \phi_i\right) \tag{3.7.15}$$

Using (3.7.2) and defining

$$\underline{\phi} = (\phi_k, \phi_{k+1}, \ldots, \phi_{k+\mu-1})^T \quad \text{and} \quad \underline{W} = (\zeta_t, \zeta_{t-1}, \ldots, \zeta_{t-\mu+1})^T (\zeta_t, \zeta_{t-1}, \ldots, \zeta_{t-\mu+1})$$

it can be shown that L_c can be written as

$$L_c = \sigma_\zeta^2\left(1 + \sum_{i=1}^{k-1} s_i^2\right) + \underline{\phi}^T \underline{W}\, \underline{\phi} + \nu\left(1 + \sum_{i=1}^{k-1} s_i + \underline{\phi}^T J\right) \tag{3.7.16}$$

where J is an μ dimensional column vector with unity elements. Straightforward calculations give that L_c is minimized for

$$\underline{\phi} = -\left(J^T \underline{W}^{-1} J\right)^{-1} \left(1 + \sum_{i=1}^{k-1} s_i\right) \underline{W}^{-1} J \tag{3.7.17}$$

But since $|\underline{W}| = 0$ for all $\mu > 1$ then the modified predictor based on the criterion (3.7.15) is ill-defined. However, one particular feature of this approach is that if $\underline{\phi}$ is chosen as

$$\underline{\phi} = \kappa \underline{w} \tag{3.7.18}$$

where κ is an arbitrary constant and $\underline{w}$ is an eigenvector of $\underline{W}$ corresponding to the zero eigenvalue, then L_c is reduced to

$$L_c = \sigma_\zeta^2 (1 + \sum_{i=1}^{k-1} s_i^2) + \nu(1 + \sum_{i=1}^{k-1} s_i + \kappa \underline{w}^T J) \tag{3.7.19}$$

hence the choice of constant κ is such that

$$1 + \sum_{i=1}^{k-1} s_i + \kappa \underline{w}^T J = 0 \tag{3.7.20}$$

or equivalently

$$\kappa = - \frac{\sum_{i=1}^{\mu} w_i}{1 + \sum_{i=1}^{k-1} s_i} \tag{3.7.21}$$

reduces L_c to the mean squared prediction error corresponding to the optimum predictor, i.e.

$$L_c = \sigma_\zeta^2 (1 + \sum_{i=1}^{k-1} s_i^2) \tag{3.7.22}$$

It should however be admitted that since the eigenvector $\underline{w}$ is time variable then $\underline{\phi}$ given by (3.7.18) is not a practical solution for the design of unity steady state gain predictors.

So far some results have been presented from linear estimation and prediction theory. In the following sections these will be used to identify the unknown process and disturbance dynamics while the process is operating under the Box and Jenkins control system.

3.8. Updating the Box and Jenkins control system

It has been shown by Turtle and Phillipson [56] that under certain conditions on the auto-regressive disturbance model, it is possible to obtain the consistent estimates of the process and disturbance dynamics while the process is operating under feedback control of Box and Jenkins type. Since the main result of this chapter is based on the estimation procedure given in [56], a brief summary here is called for as an easy reference.

Consider the estimated disturbance signal $\{X_t\}$ of the Box and Jenkins control system of Fig. (3.2.5) and assume that the control system is stable and is designed to implement the optimum control law, i.e. $\lambda = 0$. Then using (3.2.1), (3.2.23) and (3.2.24) it can be shown that

$$X_t = \frac{1}{1+V(z^{-1})H(z^{-1})}\, x_t + \frac{V(z^{-1})}{1+V(z^{-1})H(z^{-1})}\, y^d_{t-\ell} \qquad (3.8.1)$$

where the polynomial $V(z^{-1})$ represents the differences between the process and its model and is defined as

$$V(z^{-1}) = \frac{z^{-k}\, P(z^{-1})}{z^{-\ell}\, M(z^{-1})} - 1 \qquad (3.8.2)$$

It is assumed that

i) The disturbance is represented by an auto-regressive process of order m , i.e.

$$x_t = \frac{1}{1-Q(z^{-1})}\, \zeta_t \quad , \quad Q(z^{-1}) = \sum_{i=1}^{m} q_i\, z^{-i} \qquad (3.8.3)$$

ii) The set point y^d_t is constant,

$$y^d_t = y^d \quad \text{say} \qquad (3.8.4)$$

then (3.8.1) can be written as

$$X_t = \frac{1}{[1-Q(z^{-1})][1+V(z^{-1})H(z^{-1})]}\zeta_t + \frac{V(1)}{1+V(1)\ H(1)}\ y^d \tag{3.8.5}$$

Since the second term of (3.8.5) is constant, there is no loss of generality in the following if this is considered to be zero. Then defining

$$1-R(z^{-1}) = [1-Q(z^{-1})][1+V(z^{-1})H(z^{-1})] = 1 - \sum_{i=1}^{\infty} r_i z^{-i} \tag{3.8.6}$$

X_t can be written as an infinite order auto-regressive process as

$$X_t = \frac{1}{1-R(z^{-1})}\zeta_t \tag{3.8.7}$$

It was shown in [56] that if $P(z^{-1})$, $M^{-1}(z^{-1})$ and $H(z^{-1})$ are all input output stable filters, i.e. if for instance $\sum_{i=0}^{\infty} |p_i| < \infty$ then $R(z^{-1})$ can be truncated after a certain number of terms, say n, i.e.

$$R(z^{-1}) \simeq \sum_{i=1}^{n} r_i\ z^{-i} \tag{3.8.8}$$

Now if the process is operated under the Box and Jenkins feedback control system over the interval $1 \leq t \leq N$, a set of data $\{X_t\ ;\ t=1,2,\ldots,N\}$ can be recorded. Then if an auto-regressive model of order $\hat{n}$ is fitted to $\{X_t\}$ by the least squares technique as described in section 3.4, the resulting estimated polynomial $\hat{R}(z^{-1}) = \sum_{i=1}^{\hat{n}} \hat{r}_i z^{-i}$ satisfies

$$X_t = \frac{1}{1-\hat{R}(z^{-1})}\hat{\zeta}_t \tag{3.8.9}$$

where $\{\hat{\zeta}_t\}$ denotes the residual sequence. Assuming that $\hat{n}$ is estimated such that $\hat{n} \geq n$ then $\hat{R}(z^{-1})$ is a consistent estimate of $R(z^{-1})$ and hence for a moderately large N, approximately

$$1-\hat{R}(z^{-1}) \simeq 1-R(z^{-1}) = 1-Q(z^{-1}) + [1-Q(z^{-1})]\, V(z^{-1})H(z^{-1}) \qquad (3.8.10)$$

Let $\hat{Q}(z^{-1})$, $\hat{V}(z^{-1}) = \dfrac{z^{-\hat{k}}\,\hat{P}(z^{-1})}{z^{-\ell}\,M(z^{-1})}$ denote estimates of $Q(z^{-1})$ and $V(z^{-1})$ respectively so that they satisfy

$$1-\hat{R}(z^{-1}) = [1-\hat{Q}(z^{-1})] + [1-\hat{Q}(z^{-1})]\, \hat{V}(z^{-1})\, H(z^{-1}) \qquad (3.8.11)$$

Now in eq.(3.8.11) $\hat{R}(z^{-1})$, $H(z^{-1})$ are known and $\hat{Q}(z^{-1})$, $\hat{V}(z^{-1})$ are unknown and to be determined. It is in general not possible to solve (3.8.11) for both $\hat{Q}(z^{-1})$ and $\hat{V}(z^{-1})$. If however $Q(z^{-1})$ was known or its estimate was obtained previously, then it would be possible to determine $\hat{V}(z^{-1})$ from (3.8.11). This effectively corresponds to the technique given by Florentin et al [69].

In the case of unknown $\hat{Q}(z^{-1})$, it is possible to solve for $\hat{Q}(z^{-1})$ and $\hat{V}(z^{-1})$ if the lowest power of z^{-1} in the second term of (3.8.10) is greater than the number of parameters of the auto-regressive disturbance model, i.e. if

$$\bar{m} \equiv \min(k,\ell) > m \qquad (3.8.12)$$

In this event the two polynomials on the right hand side of (3.8.10) have no common powers of z then correspondingly (3.8.11) can be separated into two equations as

$$1 - \sum_{i=1}^{\bar{m}-1} \hat{r}_i z^{-i} = 1 - \sum_{i=1}^{\bar{m}-1} \hat{q}_i z^{-i} \qquad (3.8.13)$$

$$-\hat{R}_T(z^{-1}) = -\sum_{i=\bar{m}}^{\hat{n}} \hat{r}_i z^{-i} = [1-\hat{Q}(z^{-1})]\, \hat{V}(z^{-1})\, H(z^{-1}) \qquad (3.8.14)$$

where $[\cdot]_T$ is now the $\bar{m}^{th}$ order truncation operator. The disturbance model parameters can be estimated from (3.8.13) as

$$\hat{q}_i = \hat{r}_i \qquad i= 1,2,\ldots,\bar{m}-1$$

or equivalently

$$\hat{Q}(z^{-1}) = \hat{R}(z^{-1}) - \hat{R}_T(z^{-1}) \tag{3.8.15}$$

Substituting of $\hat{Q}(z^{-1})$ into (3.8.14) and solving for $z^{-\hat{k}}\,\hat{P}(z^{-1})$, the estimate of the process dynamics are given as

$$z^{-\hat{k}}\,\hat{P}(z^{-1}) = z^{-\ell}\,M(z^{-1})\left[1 - \frac{\hat{R}_T(z^{-1})}{[1-\hat{R}(z^{-1})+\hat{R}_T(z^{-1})]H(z^{-1})}\right] \tag{3.8.16}$$

The Box and Jenkins control system can now be updated by calculating a new controller from the inverse of the process model $\hat{P}(z^{-1})$ and a new predictor from the estimated disturbance model $[1-\hat{Q}(z^{-1})]$ as described previously. In [56] this procedure was referred to as 'updating the Box and Jenkins control system by the separation technique analysis (STA)'.

The process and the disturbance parameters resulting from the (STA) are consistent provided that $\hat{R}(z^{-1})$ is consistently estimated. The estimates $\{\hat{q}_i\}$ are unbiased and their variances can be obtained from the main diagonal terms of the covariance matrix for the estimates $\{\hat{r}_i\}$. As far as the process parameters estimates are concerned, it is in general difficult to assess their statistical properties, since the estimation error equation derived by using (3.8.16), viz

$$z^{-k}P(z^{-1})-z^{-\hat{k}}\hat{P}(z^{-1}) = \frac{z^{-\ell}M(z^{-1})}{H(z^{-1})}\left[\frac{\hat{R}_T(z^{-1})}{1-\hat{Q}(z^{-1})} - \frac{R_T(z^{-1})}{1-Q(z^{-1})}\right] \tag{3.8.17}$$

is non-linear in $\hat{q}_i = \hat{r}_i$; $i=1,2,\ldots,\bar{m}-1$.

* It should be noted that since the right hand side of (3.8.16) is a rational function then the new process model $\hat{P}(z^{-1})$ is effectively defined in a transfer function form. The implication of this is that since the initial process model $M(z^{-1})$ is given in its impulse response form then there is is a change in nature of the model. It is assumed that this change can be circumvented by expansion of the right hand side and truncation of the series to give a $\hat{P}(z^{-1})$ in an impulse response form.

From its derivation it is clear that the only identifiability condition for the (STA) is the inequality given by (3.8.12). This effectively limits the application of the procedure to a certain class of disturbance, namely those that are describable by an auto-regressive model with a certain number of parameters. It is therefore of interest to analyse the properties of the procedure in the case where its identifiability condition is not satisfied.

* See footnote

3.9. Errors in separation technique analysis

As in the (STA), suppose that for the N observed values of $\{X_t\}$, an auto-regressive model

$$X_t = \frac{1}{1-\hat{R}(z^{-1})}\,\hat{\zeta}_t \tag{3.9.1}$$

is fitted by the least squares technique. From the system representation (3.8.1), the observed sequence $\{X_t\ ;\ t=1,2,\ldots,N\}$ is related to the disturbance occuring specifically over the same data gathering interval $1 \leqslant t \leqslant N$ as

$$X_t = \frac{1}{1+V(z^{-1})H(z^{-1})}\,x_t \tag{3.9.2}$$

thus (3.9.2) defines the disturbance x_t over the interval $1 \leqslant t \leqslant N$ as

$$x_t \equiv [1+V(z^{-1})\ H(z^{-1})]X_t \tag{3.9.3}$$

Since X_t is given by (3.9.1) then defining

$$1-G(z^{-1}) = [1+V(z^{-1})H(z^{-1})]^{-1}[1-\hat{R}(z^{-1})]$$

$$= 1 - \sum_{i=1}^{m'} g_i z^{-i} \tag{3.9.4}$$

x_t can now be written as

$$x_t = \frac{1}{1-G(z^{-1})}\,\hat{\zeta}_t \qquad (3.9.5)$$

Comparing (3.9.5) with the disturbance representation of the (STA), i.e.

$$x_t = \frac{1}{1-Q(z^{-1})}\,\zeta_t \qquad (3.9.6)$$

it can be stated that the difference between these two representations is that in (3.9.5) the polynomial $[1-G(z^{-1})]$ is a function of the data $\{X_t\}$ occuring specifically over the measurement interval and not some hypothetical model of a stationary random process as in (3.9.6).

Rewriting (3.9.4) as

$$1-\hat{R}(z^{-1}) = [1-G(z^{-1})] + [1-G(z^{-1})]\,V(z^{-1})\,H(z^{-1}) \qquad (3.9.7)$$

with the corresponding 'updating equation' (3.8.11) of [56], i.e.

$$1-\hat{R}(z^{-1}) = [1-\hat{Q}(z^{-1})] + [1-\hat{Q}(z^{-1})]\,\hat{V}(z^{-1})\,H(z^{-1}) \qquad (3.9.8)$$

it can be seen that new 'updating equation' (3.9.7) replaces an estimate of the hypothetical auto-regressive model $[1-Q(z^{-1})]$ by the polynomial $[1-G(z^{-1})]$ which is a function of the data obtained over the interval $1 \leq t \leq N$. Now assuming that the process time delay k is known and the model time delay ℓ is chosen equal to k, then an application of $\bar{m} = k^{th}$ order truncation operator to (3.9.7) gives

$$-\hat{R}_T(z^{-1}) = \sum_{i=\bar{m}}^{\hat{n}} \hat{r}_i z^{-i} = -G_T(z^{-1}) + [1-G(z^{-1})]\,V(z^{-1})H(z^{-1}) \qquad (3.9.9)$$

where $G_T(z^{-1}) = \sum_{i=\bar{m}}^{m'} g_i z^{-i}$. Substracting (3.9.9) from (3.9.7), viz

$$\hat{R}(z^{-1}) - \hat{R}_T(z^{-1}) = G(z^{-1}) - G_T(z^{-1}) \qquad (3.9.10)$$

and noting from (3.8.15) that the left hand side of (3.9.10) is $\hat{Q}(z^{-1})$. then

$$G(z^{-1}) - G_T(z^{-1}) = \hat{Q}(z^{-1}) \qquad (3.9.11)$$

This equation in connection with (3.9.10) states that if the process time delay is known then only $\hat{Q}(z^{-1})$ part of the actual disturbance model $G(z^{-1})$ can be estimated by fitting an auto-regressive model to $\{X_t\}$.

On substitution of $G(z^{-1})$ from (3.9.11), (3.9.9) can be written in terms of $\hat{Q}(z^{-1})$ as

$$- \hat{R}_T(z^{-1}) = - G_T(z^{-1}) + [1-\hat{Q}(z^{-1})]\ V(z^{-1})\ H(z^{-1}) - G_T(z^{-1})\ V(z^{-1})\ H(z^{-1}) \qquad (3.9.12)$$

rewriting the corresponding equation (3.8.14) of the (STA)

$$- \hat{R}_T(z^{-1}) = [1-\hat{Q}(z^{-1})]\ \hat{V}(z^{-1})\ H(z^{-1}) \qquad (3.9.13)$$

and equating the right hand sides of (3.9.12) and (3.9.13), $\hat{V}(z^{-1})$ can be found in terms of $V(z^{-1})$ as

$$\hat{V}(z^{-1}) = -H^{-1}(z^{-1}) G_T(z^{-1}) [1-\hat{Q}(z^{-1})]^{-1} + V(z^{-1}) - G_T(z^{-1}) [1-\hat{Q}(z^{-1})]^{-1} V(z^{-1}) \qquad (3.9.14)$$

Substituting $V(z^{-1}) = \dfrac{P(z^{-1})}{M(z^{-1})} - 1$ and $\hat{V}(z^{-1}) = \dfrac{\hat{P}(z^{-1})}{M(z^{-1})} - 1$

into (3.9.14) and rearranging the terms, (3.9.14) can be written as

$$P(z^{-1}) - \hat{P}(z^{-1}) = [P(z^{-1}) - M(z^{-1})]\ G_T(z^{-1}) [1-\hat{Q}(z^{-1})]^{-1} + M(z^{-1})\ H^{-1}(z^{-1})\ G_T(z^{-1})\ [1-\hat{Q}(z^{-1})]^{-1} \qquad (3.9.15)$$

Since $[P(z^{-1}) - M(z^{-1})]$ is the process-model error of the initial Box and Jenkins control system whereas $[P(z^{-1}) - \hat{P}(z^{-1})]$ is the estimation error corresponding to the updated control system then (3.9.15) describes the estimation error as a function of previous (initial) estimation error and a term dependent on the previous process model and the

predictor inverse. Reduction, or otherwise, of the estimation error is entirely dependent on the disturbance parameters in $G_T(z^{-1})$ and $\hat{Q}(z^{-1})$. Since $G_T(z^{-1})$ is unknown and cannot be estimated then it is impossible to assert the estimation error of the (STA) except in the case

$$G_T(z^{-1}) = 0 \tag{3.9.16}$$

which gives one step convergence to the true solution and effectively corresponds to the (STA) proposed by Turtle and Phillipson [56].

One feature of this new error equation (3.9.15) is that it was derived by the consideration of only one data sequence $\{X_t;\ t=1,2,\ldots,N\}$ and hence the statements made above are somewhat deterministic in the sense that they are true for only that particular realization of the data $\{X_t\}$. However, by analyzing the large sample properties of the error equation (3.9.15), it is possible to obtain parallel results in a probabilistic framework. In particular it is shown in Appendix I that if (3.9.16) is true, i.e. if the disturbance arises from an auto-regressive process $G(z^{-1}) = Q(z^{-1})$ and if $\hat{Q}(z^{-1})$ is a consistent estimate of $Q(z^{-1})$ then

$$\lim_{N \to \infty} E[\hat{P}(z^{-1}) | \hat{Q}(z^{-1})] = P(z^{-1}) \tag{3.9.17}$$

i.e. given $\hat{Q}(z^{-1})$, the process estimate $\hat{P}(z^{-1})$ is asymptotically unbiased.

3.10. Dynamics of error equation

Suppose now that the (STA) is repeatedly applied and during these applications the parameters of $\hat{Q}(z^{-1})$ and $G_T(z^{-1})$ were time invariant. This restriction which is necessary for the elementary study of dynamic behaviour can be lifted for large N. Now, replacing $M(z^{-1})$ by $\hat{P}_{i-1}(z^{-1})$, and $\hat{P}(z^{-1})$ by $\hat{P}_i(z^{-1})$ and noting that

$$H(z^{-1}) = \{[1-\hat{Q}(z^{-1})]^{-1}\}_T\,[1-\hat{Q}(z^{-1})] \qquad (3.10.1)$$

also remains constant then the error equation corresponding to the i^{th} iteration can be written as

$$P(z^{-1}) - \hat{P}_i(z^{-1}) = [P(z^{-1}) - \hat{P}_{i-1}(z^{-1})]\, G_T(z^{-1})[1-\hat{Q}(z^{-1})]^{-1} -$$

$$+ G_T(z^{-1})[1-\hat{Q}(z^{-1})]^{-1} H^{-1}(z^{-1})\, \hat{P}_{i-1}(z^{-1}) \qquad (3.10.2)$$

rearranging the terms and defining

$$\Delta\hat{P}_i(z^{-1}) = P(z^{-1}) - \hat{P}_i(z^{-1})$$

$$\alpha(z^{-1}) = G_T(z^{-1})[1-\hat{Q}(z^{-1})]^{-1}[1-H^{-1}(z^{-1})]$$

$$\beta(z^{-1}) = G_T(z^{-1})[1-\hat{Q}(z^{-1})]^{-1} H^{-1}(z^{-1})\, P(z^{-1})$$

(3.10.2) can be conveniently written as

$$\Delta\hat{P}_i(z^{-1}) = \alpha(z^{-1})\, \Delta\hat{P}_{i-1}(z^{-1}) + \beta(z^{-1}) \qquad (3.10.3)$$

If it is assumed that the repeated applications of the (STA) converges to a stationary solution $\hat{P}_s(z^{-1})$, then from (3.10.3)

$$\Delta\hat{P}_s(z^{-1}) = P(z^{-1}) - \hat{P}_s(z^{-1}) = \Delta\hat{P}_{i-1}(z^{-1}) = \Delta\hat{P}_i(z^{-1}) = \frac{\beta(z^{-1})}{1-\alpha(z^{-1})} \qquad (3.10.4)$$

combining (3.10.4) with (3.10.3), the error equation can now be expressed in terms of the stationary solution as

$$\Delta\hat{P}_i(z^{-1}) - \Delta\hat{P}_s(z^{-1}) = \alpha(z^{-1})\, [\Delta\hat{P}_{i-1}(z^{-1}) - \Delta\hat{P}_s(z^{-1})] \qquad (3.10.5)$$

which shows that the convergence occurs if

$$\lim_{j\to\infty} \alpha^j(z^{-1}) = \lim_{j\to\infty} G_T^j(z^{-1})[1-\hat{Q}(z^{-1})]^{-j}[1-H^{-1}(z^{-1})]^j = 0 \qquad (3.10.6)$$

This equation states the convergence condition as a function of the unknown disturbance dynamics $G_T(z^{-1})$ and the terms which are assumed to be known. Due to its unknown nature, little or nothing can

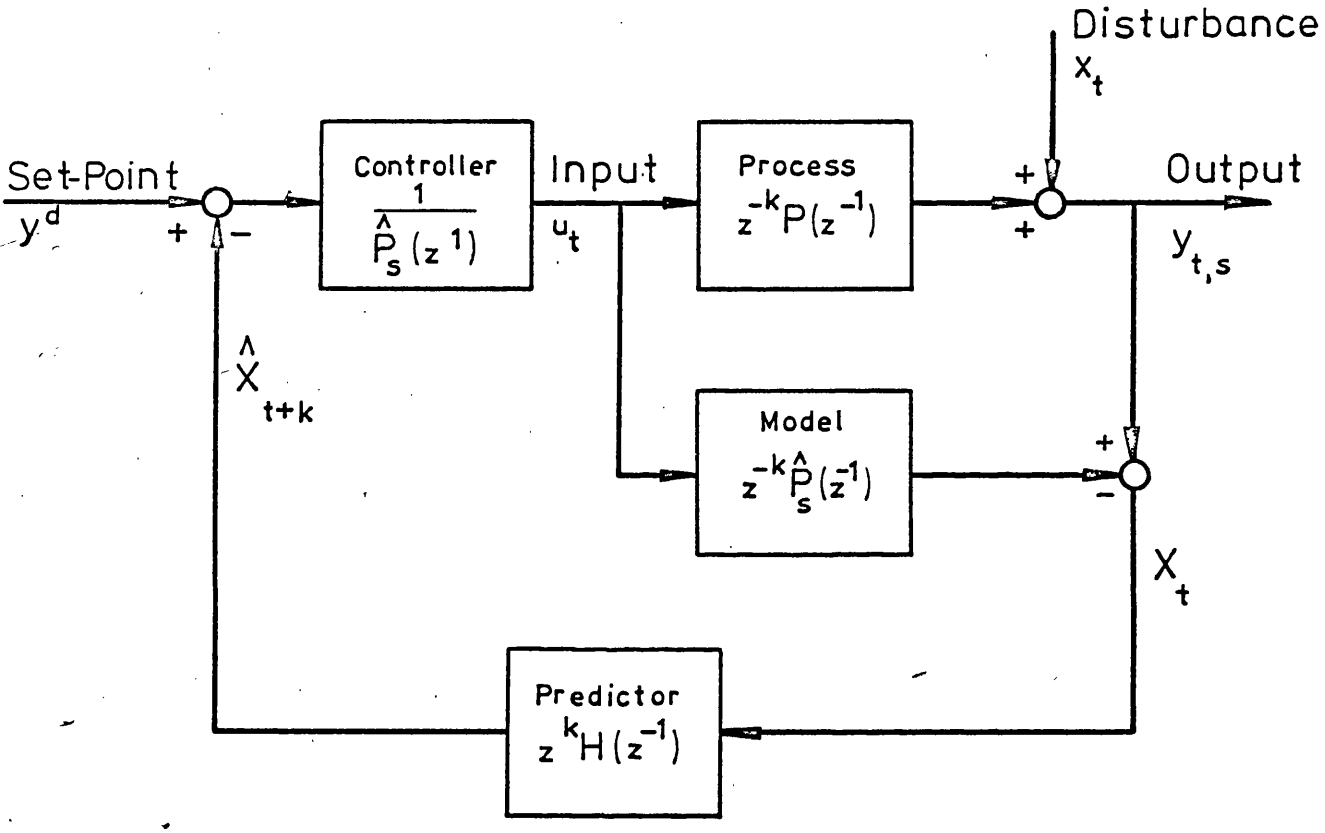

Fig. 3.11.1. THE STATIONARY SOLUTION

in general be said about convergence under repeated applications of separation technique analysis. Neither could this lead to a resulting state of zero estimation errors as inspection of the stationary solution

$$\hat{P}_s = \frac{1}{1+G_T(z^{-1})[1-G(z^{-1})]H^{-1}(z^{-1})} P(z^{-1}) \qquad (3.10.7)$$

will show.

3.11. Properties of the stationary solution

The stationary solution yields an incorrect estimate of the process impulse response; nevertheless it will be shown that the resulting control system is optimum as far as the disturbances are concerned.

Let $\hat{V}_s(z^{-1})$ denote the process-process model discrepancies corresponding to the stationary solution, i.e.

$$\hat{V}_s(z^{-1}) = \frac{P(z^{-1})}{\hat{P}_s(z^{-1})} - 1 \qquad (3.11.1)$$

Substituting of $\hat{P}_s(z^{-1})$ from (3.10.7), $\hat{V}_s(z^{-1})$ becomes

$$\hat{V}_s(z^{-1}) = G_T(z^{-1})\,[1-G(z^{-1})]^{-1}\,H^{-1}(z^{-1}) \qquad (3.11.2)$$

Consider now the Box and Jenkins control system under stationary solution conditions as shown in Fig. (3.11.1). The output component due to the disturbance x_t can be written as

$$y_{t,s} = \frac{1-H(z^{-1})}{1+H(z^{-1})\hat{V}_s(z^{-1})} x_t = \frac{1-H(z^{-1})}{[1+H(z^{-1})\hat{V}_s(z^{-1})][1-G(z^{-1})]} \hat{\zeta}_t$$

or by the use of (3.11.2)

$$y_{t,s} = \frac{1-H(z^{-1})}{1-\hat{Q}(z^{-1})} \hat{\zeta}_t = [1-H(z^{-1})][1+\hat{S}(z^{-1})]\hat{\zeta}_t \qquad (3.11.3)$$

where $1 + \hat{S}(z^{-1}) = [1-\hat{Q}(z^{-1})]^{-1}$ and for which the corresponding optimum predictor is given by (3.5.17), i.e.

$$H(z^{-1}) = \hat{S}_T(z^{-1}) \; [1+\hat{S}(z^{-1})]^{-1} \tag{3.11.4}$$

Hence substitution from (3.11.4), the output corresponding to the stationary solution becomes

$$y_{t,s} = [1+\hat{S}(z^{-1}) - \hat{S}_T(z^{-1})] \; \hat{\zeta}_t \tag{3.11.5}$$

On the other hand, it has been shown in Section 3.2 that the optimum component of output due to the disturbance x_t is

$$y_t = [1+S(z^{-1}) - S_T(z^{-1})] \; \hat{\zeta}_t \tag{3.11.6}$$

where

$$1 + S(z^{-1}) = \frac{1}{1-G(z^{-1})}$$

But

$$\begin{aligned} 1+S(z^{-1}) &= [1-G(z^{-1})]^{-1} = [1-\hat{Q}(z^{-1})-G_T(z^{-1})]^{-1} \\ &= [1-\hat{Q}(z^{-1})]^{-1} \{1-G_T(z^{-1})[1-\hat{Q}(z^{-1})]^{-1}\}^{-1} \\ &= [1+\hat{S}(z^{-1})] \; [1-\delta_T(z^{-1})]^{-1} \quad \text{say .} \\ &= [1+\hat{S}(z^{-1})] + \delta_T[1+\hat{S}(z^{-1})][1-\delta_T(z^{-1})]^{-1} \end{aligned} \tag{3.11.7}$$

and application $\bar{m} = k^{th}$ order front-end truncation gives

$$S_T(z^{-1}) = \hat{S}_T(z^{-1}) + \delta_T \; [1+\hat{S}(z^{-1})][1-\delta_T(z^{-1})]^{-1} \tag{3.11.8}$$

Then from (3.11.7) and (3.11.8)

$$1 + S(z^{-1}) - S_T(z^{-1}) = 1 + \hat{S}(z^{-1}) - \hat{S}_T(z^{-1}) \tag{3.11.9}$$

which implies that (3.11.5) and (3.11.6) are identical equations. This demonstrates the optimum nature of the stationary solution.

** The stationary solution (3.10.7) and its optimum property (Section 3.11) are obtained on the assumption that no expansion and truncation are carried out in calculations of updated process models $\hat{P}(z^{-1})$. This assumption effectively means that $\hat{P}(z^{-1})$, used in the error equation (3.9.15), is obtained from (3.8.16) without expansion and truncation.

In conclusion, it has been shown that if the repeated applications of the separation technique analysis converge to a stationary solution then the resulting Box and Jenkins control system output is equivalent to that of the optimum control system which could be obtained if the minimum variance control law was used in the case of known process dynamics.

The case where repeated applications use the modified predictor of Section 3.7 is discussed in Appendix II. It is shown that the Box and Jenkins control system corresponding to the stationary solution is also optimum in the sense that its output is equivalent to that of the control system operating with the modified predictor in the case of known process dynamics.

** See footnote

3.12. Convergence of the procedure

The main question for the successful application of such a self-adaptive procedure is of course: Will the repeated applications converge to the stationary solution? - A formal answer to this question can be given by summarizing the assumptions of the above analysis that if

i) the control system remains stable and

ii) the disturbance model $[1-G(z^{-1})]$ remains constant and

iii) (3.10.6) is satisfied, i.e.

$$\lim_{j \to \infty} \alpha^j(z^{-1}) = \lim_{j \to \infty} G_T^j(z^{-1})[1-\hat{Q}(z^{-1})]^{-j}\ [1-H^{-1}(z^{-1})]^j = 0$$

then the repeated applications of the separation technique analysis converge to its stationary solution. Unfortunately the convergence, or otherwise, is governed by the completely unknown part of the disturbance model $G_T(z^{-1})$ and in consequence a satisfactory practical answer to the convergence problem is lacking at the moment.

In parallel with the other self-adjusting techniques [54],[55] an important problem associated with the proposed procedure is the closed loop stability under repeated applications. A number of simulation studies have shown that if the coefficients of $G_T(z^{-1})$ are comparatively small in magnitude then the control system is insensitive to estimation errors as long as the process is stable and minimum phase. However for a significant $G_T(z^{-1})$, the resulting large errors in the estimated process parameters can cause instability, in particular, it was found in most cases that estimated process models were non-minimum phase and gave rise to unstable controllers.

The assumption that the disturbance model $[1-G(z^{-1})]$ remains constant can only be justified if the number of data tends to infinity. For a moderate amount of data, $[1-G(z^{-1})]$ is bound to change from one iteration to the other due to the sampling variabilities of estimated parameters in $\hat{R}(z^{-1})$.

Perhaps the most important limiting factor to the application of this procedure arises from the complex condition given in assumption (iii). Due to its unknown nature, it is not trivial to analyze the properties of (3.10.6). However it is of interest to note that if the coefficients of $G_T(z^{-1})$ are negligible, whether by accident or design, then the procedure may converge to the stationary solution. However, it is not practically possible to analyze what negligible exactly means.

These doubts concerning convergence added to the rather restrictive assumption that the process time delay k is known, do not recommend the procedure as a reliable method of system optimization.

TABLE 3.13.1. SEPARATION TECHNIQUE ANALYSIS I

Disturbance: $x_t = (1-0.8z^{-1}+0.06z^{-2}-0.07z^{-3}-0.11z^{-4}-0.09z^{-5}+0.25z^{-6})^{-1}\zeta_t$

$\sigma_\zeta^2 = 1.19$, $\bar{m}=k=\ell=2$, $m=6$, $N=300$

Process	Model	Predictor
$z^{-k}P(z^{-1})$	$z^{-k}M(z^{-1})$	$z^{k}H(z^{-1})$
$\dfrac{z^{-2}}{(1-0.5z^{-1})(1-0.3z^{-1})}$	$\dfrac{z^{-2}}{1-0.7z^{-1}}$	$\dfrac{0.8}{1-0.2z^{-1}}$

Estimated process models $\hat{P}(z^{-1})$

	$\hat{P}_o$	$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
$\hat{P}_1(z^{-1})$	0.96	0.68	0.17	-0.17	0.12	0.18	0.20
$\hat{P}_2(z^{-1})$	0.95	0.61	0.01	-0.42	0.03	0.34	0.43
$\hat{P}_3(z^{-1})$	0.94	0.75	0.36	0.01	0.42	0.61	0.28
$\hat{P}_4(z^{-1})$	1.25	0.99	0.69	0.47	0.44	0.39	0.33
$\hat{P}_5(z^{-1})$	0.96	0.64	0.08	-0.37	0.04	0.30	0.38
$\hat{P}_6(z^{-1})$	0.99	0.68	0.11	-0.35	0.05	0.31	0.40
$\hat{P}_7(z^{-1})$	0.98	0.67	0.10	-0.36	0.04	0.30	0.40
$\hat{P}_8(z^{-1})$	0.98	0.67	0.10	-0.36	0.04	0.30	0.40

Computed values of Var $\{y_t\}$

Iteration	1	2	3	4	5	6	7	8
Var $\{y_t\}$	2.63	2.00	3.19	25.4	2.42	1.98	1.97	1.97

Optimum output variance : 1.95

3.13. Examples

This section presents two successful examples of the repeated application of the separation technique analysis. It should however be noted that these are not typical examples of this self-adaptive procedure. In fact, it was observed in many simulation studies that the control system did not always converge to its optimum, and in most cases where the coefficients of $G_T(z^{-1})$ were significant, the procedure gave rise to unstable control systems. Therefore the purpose of this section is to demonstrate only the optimum nature rather than the effectiveness of the procedure.

As the first example consider the case shown in Table (3.13.1). The simulated process $z^{-k} P(z^{-1})$ was taken to be a second order pulse transfer function with the process model $z^{-k} M(z^{-1})$, a first order approximation; in particular, k was a delay of two sample intervals. The predictor $z^{k} H(z^{-1})$ was chosen to be a first order low-pass filter with a unity steady state gain and the controller was taken to be the exact inverse of the process model.

The simulated process was operated by the initial control system for 300 sample intervals with N = 300 numbers from the auto-regressive process $\{x_t\}$ shown at the top of Table (3.13.1). This yielded a record of $\{X_t;\ t=1,2,\ldots,300\}$ and following Section 3.4, an auto-regressive model $[1-\hat{R}(z^{-1})]$ was fitted by the least squares algorithm of eqn.(3.4.2). The order of the fitted model was estimated by minimizing the final prediction error criterion of eqn.(3.4.10). To illustrate the calculations involved, the results of the first iteration are given in Table (3.13.2). Now, since $k=\ell=2$ and hence $\bar{m} = \min(k,\ell) = 2$, then from updating equations (3.8.15), (3.8.16) of the separation technique analysis and using the parameters

TABLE 3.13.2 DETAILS OF THE FIRST ITERATION

True auto-regressive model for $\{X_t\}$: $R(z^{-1})$		Coefficients of $\hat{R}(z^{-1})$ with 95% confidence intervals		Final prediction error criterion (FPE)	
				$\hat{n}$	FPE($\hat{n}$)
r_1	0.80	$\hat{r}_1$	0.78± 0.11	1	1.48
r_2	-0.06	$\hat{r}_2$	0.03± 0.14	2	1.49
r_3	-0.01	$\hat{r}_3$	-0.03± 0.13	3	1.49
r_4	0.21	$\hat{r}_4$	0.24± 0.13	4	1.49
r_5	0.12	$\hat{r}_5$	0.10± 0.14	5	1.44
r_6	-0.24	$\hat{r}_6$	-0.36± 0.10	6	1.28
r_7	0.00			7	1.29
r_8	0.00			8	1.30
r_9	-0.02			9	1.30
r_{10}	-0.02			10	1.31

Estimated part of the disturbance model : $\hat{Q}(z^{-1})=0.78z^{-1}$

Predictor based on $\hat{Q}(z^{-1})$: $z^2H(z^{-1})=0.62$

Process $P(z^{-1})$		Updated process model $\hat{P}(z^{-1})$	
p_o	1.00	$\hat{p}_o$	0.96
p_1	0.80	$\hat{p}_1$	0.68
p_2	0.49	$\hat{p}_2$	0.18
p_3	0.27	$\hat{p}_3$	-0.17
p_4	0.14	$\hat{p}_4$	0.12
p_5	0.07	$\hat{p}_5$	0.18
p_6	0.04	$\hat{p}_6$	0.20

of the initial control system, the disturbance and process parameters estimates are obtained as

$$\hat{q}_1 = \hat{r}_1$$

$$z^{-2}\hat{P}(z^{-1}) = \frac{z^{-2}}{1-0.7z^{-1}} \left[1 - \frac{\left(\sum_{i=2}^{6} \hat{r}_i z^{-i} \right)(1-0.2z^{-1})}{(1-\hat{r}_1 z^{-1})\ 0.8\ z^{-2}} \right]$$

The estimates $\hat{Q}(z^{-1})$, and $\hat{P}(z^{-1})$ obtained in the first iteration are also shown in Table (3.13.2). As a comparison of $\hat{P}(z^{-1})$ with $P(z^{-1})$ will show, the process estimate $\hat{P}(z^{-1})$ is biased since the identifiability condition (3.8.12) of the separation technique analysis is not satisfied.

Based on the estimates of disturbance and process parameters, the initial control system was updated by calculating an optimum predictor from $\hat{Q}(z^{-1})$ as in eqn. (3.5.18), and a controller from the inverse of the process estimate $\hat{P}(z^{-1})$. Each time a record of 300 samples of $\{X_t\}$ became available, this updating procedure was repeated. To guarantee that the disturbance model $[1-G(z^{-1})]$ remained constant, the same disturbance sequence $\{x_t\}$ was used in all iterations. Estimated process parameters and sample variance of output $\{y_t\}$ resulted from each iteration are shown in Table (3.13.1). It can be seen, from the values of Var $\{y_t\}$, that the control system converged towards the optimum and in particular, that convergence was established by the 7^{th} iteration. Therefore a total of about 2100 data was needed to optimize the control system.

The second simulation study shown in Table (3.13.3) illustrates the behaviour of the procedure when the disturbance set was changed from one iteration to the other by choosing the set corresponding to the i^{th} iteration as $\{x_t\ ;\ t=(i-1)N+1,\ldots,iN\}$. An inspection of Var $\{y_t\}$, plotted in Fig. (3.13.4), shows that the control system remained around optimum after about 10 iterations. However since N=600 appeared to be

TABLE 3.13.3 SEPARATION TECHNIQUE ANALYSIS II

Disturbance: $x_t = (1-0.8z^{-1}-0.2z^{-2}+0.2z^{-3}-0.2z^{-4}+0.2z^{-5})^{-1}\ \zeta_t$

$\sigma_\zeta^2 = 1.0$, $\bar{m}=k=\ell=2$, $m=5$, $N=600$

Process	Model	Predictor
$z^{-k}P(z^{-1})$	$z^{-\ell}M(z^{-1})$	$z^{\ell}H(z^{-1})$
$\dfrac{z^{-2}}{1-0.5z^{-1}}$	$z^{-2}(0.5+0.4z^{-1})$	$0.85+0.10z^{-1}+0.05z^{-2}$

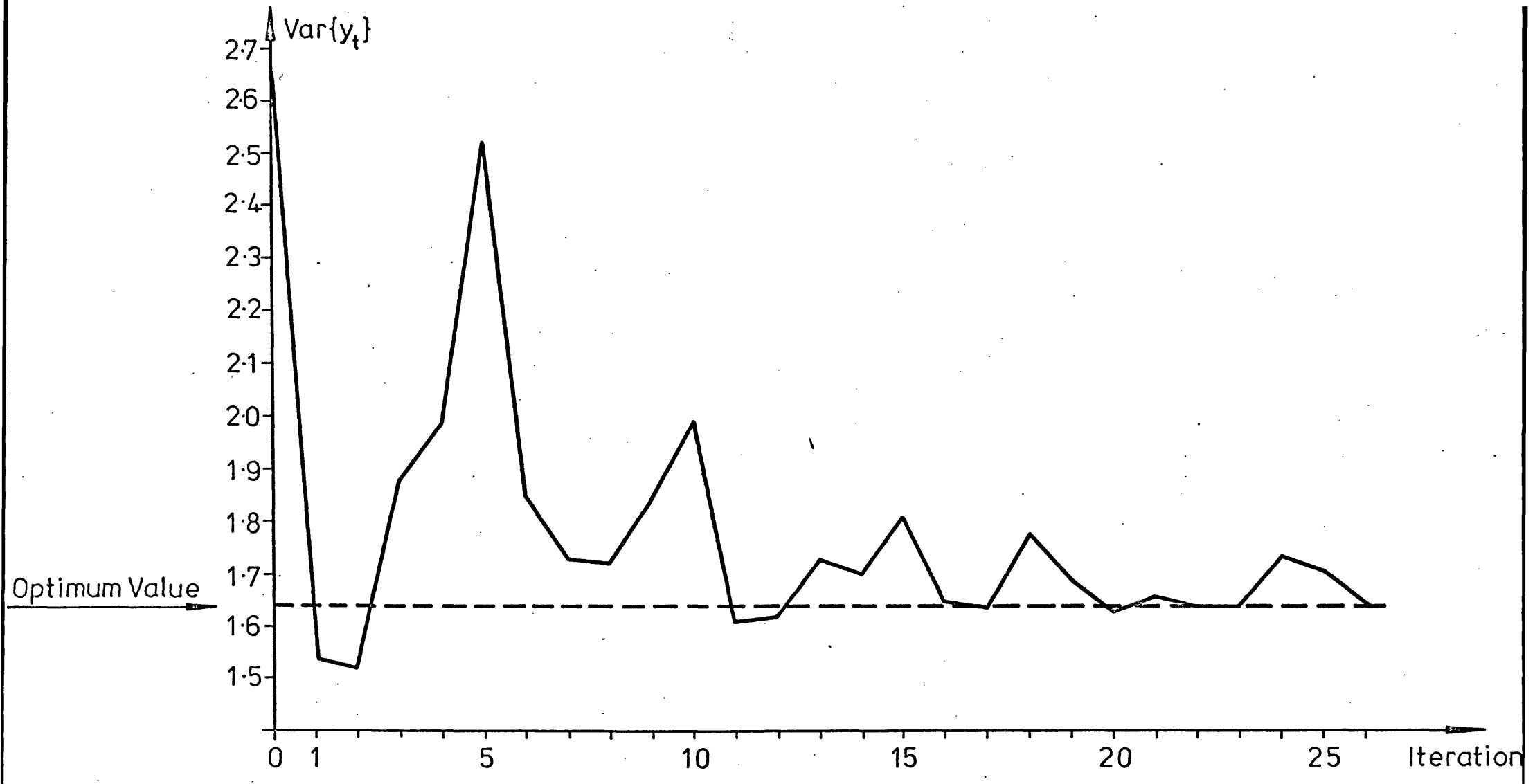

Fig. 3.13.4. EXAMPLE II: SAMPLE VARIANCE OF THE OUTPUT

$Var\{y_t\}_o$ corresponds to the initial control system

*** It will be noted that the examples differ from the theoretical procedure because of the expansion and truncation carried out in calculations of $\hat{P}(z^{-1})$. In both examples, the updated process models are obtained by expanding the right hand side of (3.8.16) and truncating the series after the impulse response durations of the true process models. For example, $P(z^{-1})$ of the first example has 7 significant coefficients and consequently it is assumed that the updated process models have impulse response durations of 7 sample intervals. It should however be noted that the truncation based on the true impulse response duration may introduce a second type of error if $G_T(z^{-1})$ is not negligible, since in this case $\hat{P}(z^{-1})$ may differ considerably from $P(z^{-1})$ and hence the above truncation procedure may not represent the true impulse response duration of $\hat{P}(z^{-1})$. A better representation of the theoretical algorithm would be obtained if the estimated impulse responses were extended up to their last significant coefficients.

a minimum data length, a total of about 6000 data was needed to obtain an approximate optimum control system. Also as is apparent from Fig. (3.13.4), the convergence was not monotonic and Var $\{y_t\}$ showed considerable fluctuations above the optimum variance. Furthermore the behaviour of the estimated process parameters, shown in Fig. (3.13.5), did not suggest clearly that the convergence occurred after about 10 iterations.

In conclusion, it has been shown in these examples that if the underlying assumptions of the proposed procedure are satisfied, whether by accident or design, then the control system converges towards the optimum. However due to the limitations imposed by the assumptions, to the relatively long time required for optimization and to the undesirable feature of the convergence as in the second example, it appears that the procedure cannot be recommended as a practical method of optimization.

*** See footnote

3.14. Conclusions

This chapter has presented some new results on the identification and control of linear time invariant processes operating under the Box and Jenkins feedback control system.

Firstly it has been shown that it is always possible to design a stable control system by using a sub-optimum control law, provided that the steady state gains of process and model are of the same sign and the first coefficient of the unity steady state gain predictor is positive. The design of unity gain predictors for stationary auto-regressive processes has been discussed and a new predictor has been designed by modifying the optimum prediction error such that the resulting predictor has a unity steady state gain.

Secondly and more importantly, the identifiability problem of the separation technique analysis has been discussed in detail. A new error analysis has shown explicitly that if the unknown part of

the disturbance model, $G_T(z^{-1})$, is negligible, whether by accident or design, then the errors of the estimated process parameters are negligible. Since $G_T(z^{-1})$ reflects the unknown property of the disturbance, the identifiability cannot always be guaranteed. However further investigations showed that if the repeated applications of the separation technique analysis converge to a stationary solution then the resulting control system is optimum as far as the disturbances are concerned.

Convergence is governed by the unknown part of the disturbance model $G_T(z^{-1})$ and at the moment a theoretical analysis to express the convergence conditions in terms of some known or estimated parameters is lacking. In consequence, it is believed that due to the difficulties of establishing the validity of assumptions, and to the problem of closed loop stability under repeated applications, the procedure cannot be considered as a reliable method of optimum control.

CHAPTER IV

PREDICTOR UPDATING COMBINED WITH CORRELATION ANALYSIS

4.1. Introduction

The method of repeated application of separation technique analysis developed so far suffers from a restriction on the type of disturbance to which it can be applied. In particular, it has been assumed that the disturbance is of an auto-regressive type in which the number of significant parameters is less than the process time delay which is also assumed a priori known. These restrictions and the doubts concerning convergence do not recommend the procedure as a practically viable method of optimization.

This chapter will present a different approach to optimize the Box and Jenkins control system. The fundamental principle of this technique can be explained as follows. If the process is operated under the Box and Jenkins control system with an added perturbation signal on set-point, then by cross correlating this signal either with the process output y_t or with the signal X_t, the process parameters including the time delay can be consistently estimated. Based on these estimates the control system can be updated by calculating a controller from the inverse of the process model and a predictor from the disturbance model. At this stage, due to the unknown sampling variabilities of estimates for a moderate amount of data, it cannot be claimed that the resulting control system is optimum. However, further improvement can be achieved by the application of repeated predictor updating procedure of [52]. This procedure consists of estimating an auto-regressive model for a set of data $\{X_t;\ t=(i-1)N+1,\ldots,iN\}$ at the i^{th} iteration and updating the predictor at every $(iN)^{th}$ sample using the fitted auto-regressive model of the i^{th} iteration. It has been proved in [52] that this updating procedure converges to an optimum control system if

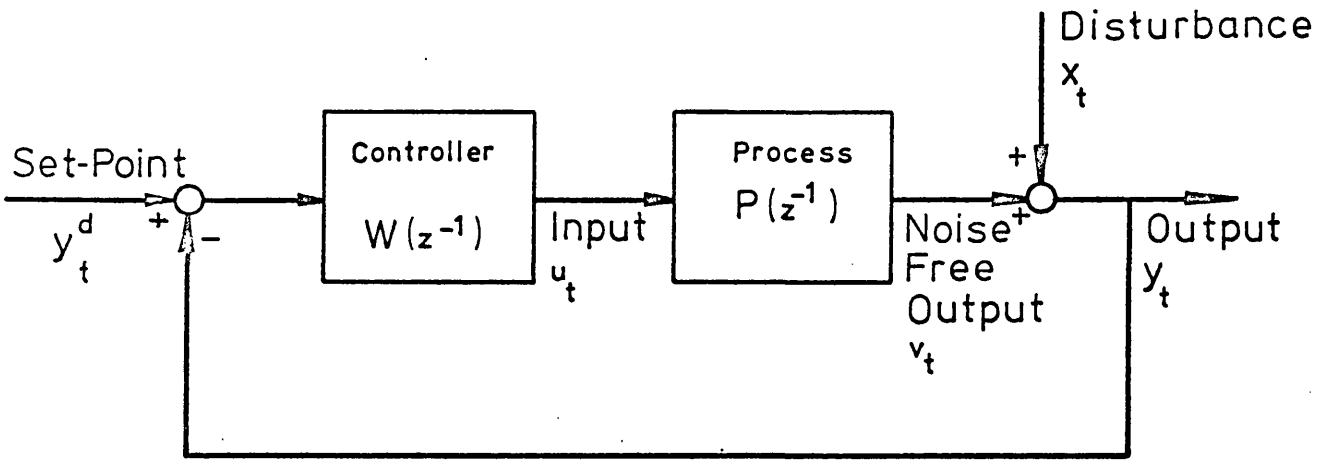

Fig. 4.2.1. CONVENTIONAL CONTROL SYSTEM

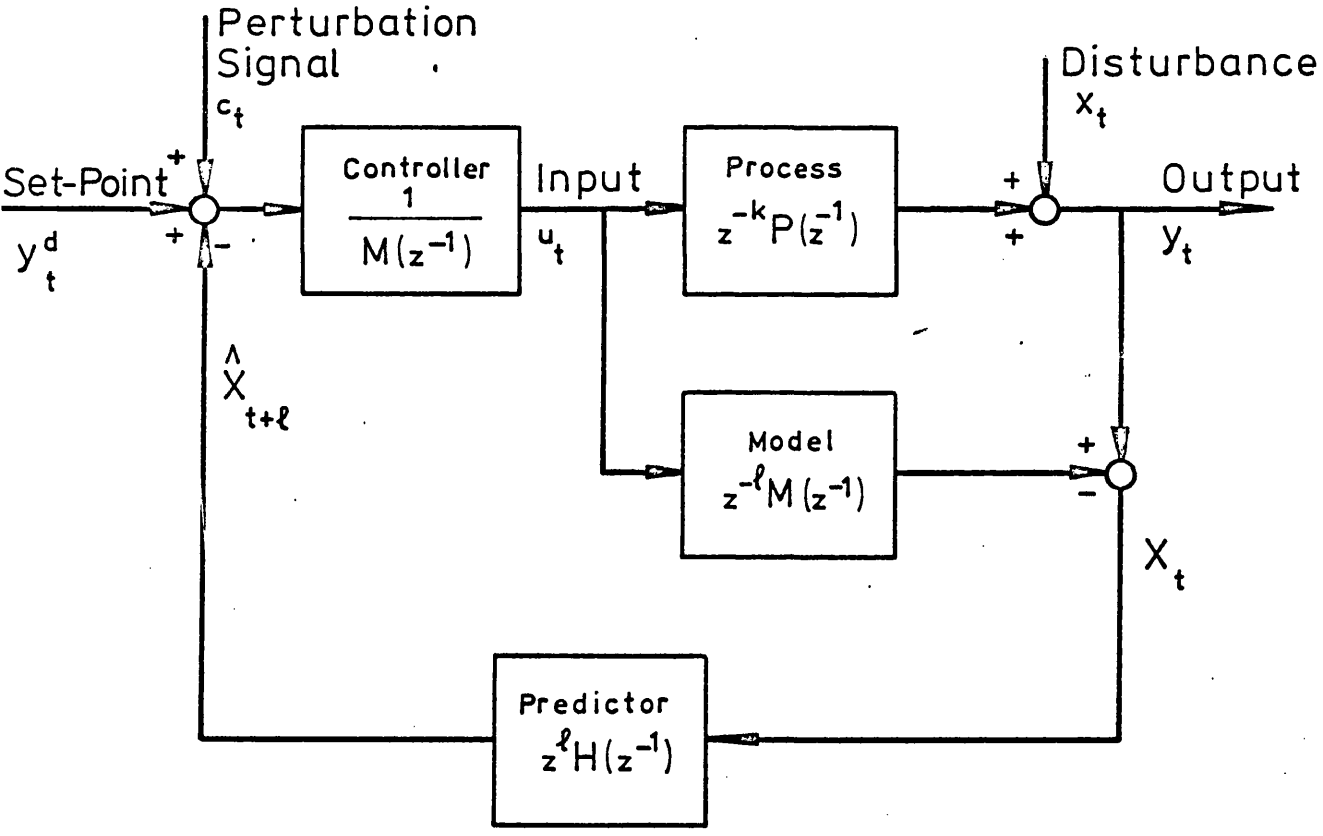

Fig. 4.3.1. THE BOX AND JENKINS CONTROL SYSTEM FOR THE CORRELATION ANALYSIS

the differences between the process and its model are small. The contribution of the proposed correlation analysis of this chapter is to increase the probability of convergence of the predictor updating procedure by providing a fairly good initial process model.

4.2. Correlation analysis as a means of identification in closed loop systems

The identifiability of linear time invariant feedback control systems by correlation analysis has been treated in several papers. Consider the conventional closed loop system shown in Fig.(4.2.1). The purpose of correlation analysis is to estimate the input-output relation between u_t and v_t from the observed series $\{u_t\}$ and $\{y_t\}$. This is accomplished by correlating u_t and y_t with a signal c_t which is independent of the disturbance x_t. As is evident from its definition, the signal c_t plays the role of instrumental variable. In [70], the reference signal y_t^d has been considered as c_t and the correlation analysis has been performed as follows. The system output from Fig.(4.2.1) is

$$y_t = x_t + \sum_{i=0}^{J} p_i \, u_{t-i} \tag{4.2.1}$$

assuming that x_t is a zero mean process then it follows from independence that multiplication of both sides of (4.2.1) by $c_{t-\tau} = y^d_{t-\tau}$; $\tau = 0,1,\ldots,J$ and taking expectations results in

$$E(c_{t-\tau} \, y_t) = \sum_{i=0}^{J} p_i E(c_{t-\tau} \, u_{t-i}) \tag{4.2.2}$$

assuming further that c_t is stationarily correlated with u_t and y_t then the process parameters $\{p_i\}$ can be consistently estimated by substituting cross covariance estimates into (4.2.2).

An alternative choice of c_t has been proposed in [71] and it has been shown that if y_t^d is independent of x_t then under some mild conditions on x_t, the signal $c_t = u_t \, u_{t+\sigma}$ can also be

used instead of y_t^d for a properly selected value of σ. This paper also examines the case $c_t = u_t$ in the context of cross spectral analysis and shows that if u_t contains an additive noise which is independent of x_t then spectral analysis results in the weighted average of the controller and process dynamics where the weights are determined by the unknown properties of disturbance x_t. Similar results for the correlation analysis are also given in [72]. The case where $c_t = u_t$ and u_t contains no noise is well known [13]. In this event the correlation analysis estimates $\{\hat{p}_i\}$ are the parameters of the inverse of the controller transfer function, i.e. $W^{-1}(z^{-1})$.

4.3. Application of correlation analysis to the Box and Jenkins control system

It will be shown in this section that parallel to the conventional feedback control systems, correlation analysis can also be applied to the Box and Jenkins control system if the use of an extra perturbation input signal c_t is allowed.

Consider the signal X_t of the Box and Jenkins control system of Fig. (4.3.1), it can be shown that

$$X_t = \frac{1}{1+V(z^{-1})H(z^{-1})}\, x_t + \frac{V(z^{-1})}{1+V(z^{-1})H(z^{-1})}\,(c_{t-\ell}+y^d_{t-\ell}) \qquad (4.3.1)$$

where $V(z^{-1}) = \dfrac{z^{-k}P(z^{-1})}{z^{-\ell}M(z^{-1})} - 1$, $H(z^{-1}) = \sum_{i=0}^{\infty} h_i z^{-\ell-i}$ and

$z^{-k}P(z^{-1})$, $z^{-\ell}M(z^{-1})$ represent process and model impulse response functions as in the previous chapter. It will be assumed that

i) the reference set-point y_t^d is constant, y^d say,

ii) c_t and x_t are zero mean, mutually independent stationary random processes,

iii) the process model and the predictor are chosen in such a way that the resulting control system is stable and hence X_t is a

stationary process,

iv) the system time delay k is a priori known and hence $\ell=k$.

Then defining,

$$\eta_t = [1+H(z^{-1})V(z^{-1})]^{-1} x_t \tag{4.3.2}$$

and

$$\Omega(z^{-1}) = \sum_{i=0}^{n} \omega_i z^{-i} = [1+H(z^{-1})V(z^{-1})]^{-1} V(z^{-1}) \tag{4.3.3}$$

(4.3.1) can be written as

$$X_t = \eta_t + \sum_{i=0}^{n} \omega_i c_{t-k-i} \tag{4.3.4}$$

where the term associated with y_t^d has been dropped since it is known from section 3.4. that if y_t^d is constant then there is no loss of generality in assuming $y^d=0$. Since c_t and x_t are assumed to be mutually orthogonal then multiplication of (4.3.4) by $c_{t-\tau};\tau=k,k+1,\ldots,k+n$ and taking expectations results in

$$\gamma_{Xc}(\tau)=\omega_o\gamma_{cc}(k-\tau)+\omega_1\gamma_{cc}(k+1-\tau)+\ldots+\omega_n\gamma_{cc}(k+n-\tau) \; ; \; \tau=k,k+1,\ldots k+n \tag{4.3.5}$$

where $\gamma_{Xc}(\cdot)$ and $\gamma_{cc}(\cdot)$ denote cross and auto-covariance coefficients, i.e. $\gamma_{Xc}(j)=E(X_t c_{t-j})$, $\gamma_{cc}(j)=E(c_t c_{t-j})$. If the estimates of covariance coefficients are substituted into (4.3.5) then $\{\omega_i\}$ can now be estimated as the solution of the following linear simultaneous equations

$$\hat{\gamma}_{Xc}(\tau)=\hat{\omega}_o\hat{\gamma}_{cc}(k-\tau)+\hat{\omega}_1\hat{\gamma}_{cc}(k+1-\tau)+\ldots+\hat{\omega}_n\hat{\gamma}_{cc}(k+n-\tau) \; ; \; \tau=k,k+1,\ldots,k+n \tag{4.3.6}$$

where

$$\hat{\gamma}_{Xc}(j) = \frac{1}{N} \sum_{t=j+1}^{N} X_t c_{t-j} \quad , \quad \hat{\gamma}_{cc}(j) = \frac{1}{N} \sum_{t=j+1}^{N} c_t c_{t-j} \tag{4.3.7}$$

Although any stationary sequence $\{c_t\}$ can be used in connection with (4.3.6), considerable simplifications in the solution of (4.3.6) occur if $\{c_t\}$ is chosen as an orthogonal sequence, i.e.

$$\gamma_{cc}(j) = 0 \qquad j \neq 0$$
$$= \sigma_c^2 \qquad j = 0 \tag{4.3.8}$$

then $\hat{\omega}_i$'s are simply obtained as

$$\hat{\omega}_i = \frac{1}{\hat{\sigma}_c^2} \hat{\gamma}_{Xc}(k+i) \qquad i=0,1,\ldots,n \tag{4.3.9}$$

In practice, an orthogonal sequence $\{c_t\}$ can be closely approximated by a pseudo-random binary sequence of sufficiently large period N_1. Denoting $N=iN_1$ as the total number of data where i is a positive integer, and using the following properties of the pseudo-random binary sequence $\{c_t = \pm a\}$

$$\gamma_{cc}(j) = -\frac{a^2}{N_1} \; ; \; j \neq 0 \quad \text{and} \quad \gamma_{cc}(0) = \sigma_c^2 = a^2 \tag{4.3.10}$$

then it can be seen that for a sufficiently large N_1, $\{\hat{\omega}_i\}$ can be solved from (4.3.6) as

$$\hat{\omega}_i = \frac{1}{a^2} \hat{\gamma}_{Xc}(k+i) \qquad i=0,1,\ldots,n \tag{4.3.11}$$

Given $\{\hat{\omega}_i\}$, estimation of process parameters $\{p_i\}$ proceeds as follows. Define analagously to $\Omega(z^{-1})$

$$\hat{\Omega}(z^{-1}) = [1+H(z^{-1})\hat{V}(z^{-1})]^{-1}\hat{V}(z^{-1}) = \sum_{i=1}^{n} \hat{\omega}_i z^{-i} \tag{4.3.12}$$

where $\hat{V}(z^{-1}) = \dfrac{\hat{P}(z^{-1})}{M(z^{-1})} - 1$, solving $\hat{P}(z^{-1})$ from (4.3.12) as

$$\hat{P}(z^{-1}) = M(z^{-1}) \left[1 + \frac{\hat{\Omega}(z^{-1})}{1-H(z^{-1})\hat{\Omega}(z^{-1})}\right] \tag{4.3.13}$$

then $\hat{p}_i$'s are obtained by manipulation of (4.3.13) or by equating coefficients in the expression

$$\left(\sum_{i=0}^{\infty} \hat{p}_i z^{-i}\right)\left[1-\left(\sum_{i=0}^{\infty} h_i z^{-k-i}\right)\left(\sum_{i=0}^{n} \hat{\omega}_i z^{-i}\right)\right] = \left(\sum_{i=0}^{\infty} m_i z^{-i}\right)\left[1+\left(\sum_{i=0}^{n} \hat{\omega}_i z^{-i}\right)\left(1-\sum_{i=0}^{\infty} h_i z^{-k-i}\right)\right]$$

(4.3.14)

where it should be noted that since only (n+1) coefficients $\hat{\omega}_i$ are available it is only possible to estimate the first (n+1) coefficients of $P(z^{-1})$.

To find a model for the disturbance x_t, it will now be assumed in addition to the assumption (ii) that x_t can be adequately described by an auto-regressive process

$$x_t = \frac{1}{1-Q(z^{-1})}\zeta_t \tag{4.3.15}$$

Since the sequence $\{x_t\}$ can be estimated from (4.3.2) as

$$\hat{x}_t = [1+\hat{V}(z^{-1})H(z^{-1})]\hat{n}_t$$

where $\hat{n}_t$ from (4.3.4)

$$\hat{n}_t = X_t - \hat{\Omega}(z^{-1})c_{t-k} \tag{4.3.16}$$

it is then possible to estimate an auto-regressive model for x_t in the form (4.3.15).

The control system can now be updated by replacing $M(z^{-1})$ with $\hat{P}(z^{-1})$ and forming a new predictor based on the auto-regressive model $[1-\hat{Q}(z^{-1})]$ by the method described in section 3.5. However as it will be shown in the next section, although the estimates $\{\hat{p}_i\}$ are consistent, they are in general biased for a moderate amount of data and furthermore the variance of estimates cannot be estimated in terms of known quantities and consequently the resulting updated control system is not, in general, optimum. Nevertheless if a fairly good process model can be found by correlation analysis, it will be shown in section 4.6 that it is possible to optimize the control system by the predictor updating procedure of [52].

Before proceeding to the error analysis it should be noted here that although the above correlation analysis is based on the use of signal X_t, it is equally possible to employ the system output y_t

instead of X_t. As it can be seen from output equation

$$y_t = \frac{1-H(z^{-1})}{1+V(z^{-1})H(z^{-1})} x_t + \frac{1+V(z^{-1})}{1+V(z^{-1})H(z^{-1})} c_{t-k} \qquad (4.3.17)$$

multiplying (4.3.17) with $c_{t-\tau}$; $\tau=k,k+1,\ldots$ and taking expectations results in

$$\gamma_{yc}(\tau) = E\left[\left(\frac{1+V(z^{-1})}{1+V(z^{-1})H(z^{-1})} c_{t-k}\right) c_{t-\tau}\right] \; ; \; \tau=k,k+1,\ldots \qquad (4.3.18)$$

from which $\hat{V}(z^{-1})$ and hence $\hat{P}(z^{-1})$ can be estimated as before. Development using X_t is followed here to parallel the procedure of section 4.6.

4.4. Error Analysis

The first part of this section discusses the statistical properties of estimates $\{\hat{\omega}_i\}$. It has been found that these properties can be conveniently expressed by using a vector-matrix notation and putting the correlation analysis into a least squares framework. To avoid notational complexity, it will be assumed that a single long pseudo-random binary sequence is used instead of several short codes and hence $N=N_1$. Now, for $t=1,2,\ldots,N$ equations (4.3.4) can be collectively written as

$$X = \eta + C\omega \qquad (4.4.1)$$

where $X = (X_1,X_2,\ldots,X_N)^T$, $\eta = (\eta_1,\eta_2,\ldots,\eta_N)^T$, $\omega = (\omega_o,\omega_1,\ldots,\omega_n)^T$, $C = (S^k c, S^{k+1}c,\ldots,S^{k+n}c)$, $c = (c_1,c_2\ldots,c_N)^T$ and S is an NxN shift matrix as defined in Chapter I. Consider the least squares estimate of ω which can be written as

$$\hat{\omega} = \left(\frac{1}{N} C^T C\right)^{-1} \frac{1}{N} C^T X \qquad (4.4.2)$$

from the properties of pseudo-random binary sequences given by (4.3.10), approximately

$$\frac{1}{N} C^T C \simeq a^2 I_{n+1} \qquad \text{for} \qquad N >> k+n \tag{4.4.3}$$

on the other hand, apart from the effect of initial and final terms, elements of $\left(\frac{1}{N} C^T X\right)$ are equivalent to cross covariance estimates between X_t and c_t, i.e.

$$\frac{1}{N} C^T X = \left[\hat{\gamma}_{Xc}(k), \ldots, \hat{\gamma}_{Xc}(k+n)\right]^T \tag{4.4.4}$$

then (4.4.3) and (4.4.4) in connection with (4.4.2) state that for large N, the correlation analysis (4.3.11) and the least squares estimates (4.4.2) of $\{\omega_i\}$ are equivalent. It now follows from (4.4.1) and (4.4.2) that the correlation analysis estimates can be written as

$$\hat{\omega} = \frac{1}{a^2 N} C^T X = \frac{1}{a^2} \left(\frac{1}{N} C^T C\right) \omega + \frac{1}{a^2 N} C^T \eta$$

$$= \omega + \frac{1}{a^2 N} C^T \eta \tag{4.4.5}$$

Since c_t and η_t are mutually independent zero mean stationary random processes then analogously to section 2.4, it can be shown that if

$$|E\eta_t \eta_{t+\tau}| \leq K\tau^{-\delta} \qquad \text{for} \qquad K>0,\ 1>\delta>0,\ \tau \geq 1 \tag{4.4.6}$$

then

$$\lim_{N \to \infty} \frac{1}{N} C^T \eta = \underline{0} \quad \text{with probability one} \tag{4.4.7}$$

and hence $\hat{\omega}$ is a consistent estimate of ω, i.e.

$$\lim_{N \to \infty} \hat{\omega} = \omega \qquad \text{with probability one} \tag{4.4.8}$$

If the condition on the auto-covariance function of η_t given by (4.4.6) is relaxed as

$$\lim_{\tau \to \infty} \tau\ |E\eta_t \eta_{t+\tau}| = 0 \tag{4.4.9}$$

then it follows from section 2.4 that $\hat{\omega}$ is consistent only in probability.

It can also be seen from (4.4.5) that for a given input sequence, $\hat{\omega}$ is unbiased since

$$E\hat{\omega} = \omega + \frac{1}{a^2N} C^T E\eta = \omega \tag{4.4.10}$$

and the covariance matrix is given by

$$E(\hat{\omega}-\omega)(\hat{\omega}-\omega)^T = \frac{1}{a^4N^2} C^T(E\eta\eta^T)C \tag{4.4.11}$$

Since no a priori information is available for the covariance matrix of η, it is, in general, not possible to make statements about the variance of estimates.

Having obtained the statistical properties of $\{\hat{\omega}_i\}$, a similar analysis will now be repeated for that of process model parameters $\{\hat{p}_i\}$. However as eqn.(4.3.13) shows, $\hat{P}(z^{-1})$ is non-linear in $\hat{\Omega}(z^{-1})$. It will be seen, as in the case of separation technique analysis, that this non-linearity will severely limit the information about the errors in estimated parameters. Consistency of $\{\hat{p}_i\}$ immediately follows from substitution of asymptotic and hence true values of $\{\hat{\omega}_i\}$ into (4.3.13). To find expectations of $\{\hat{p}_i\}$, the unbiasedness property of $\hat{\omega}_i$ will be used as

$$E\hat{\Omega}(z^{-1}) = \Omega(z^{-1}) \tag{4.4.12}$$

writing $\hat{\Omega}(z^{-1})$ and $\Omega(z^{-1})$ in terms of $\hat{V}(z^{-1})$ and $V(z^{-1})$ from their defining equations (4.3.3) and (4.3.12), (4.4.12) becomes

$$E\left[\frac{\hat{V}(z^{-1})}{1+\hat{V}(z^{-1})H(z^{-1})}\right] = \frac{V(z^{-1})}{1+V(z^{-1})H(z^{-1})} \tag{4.4.13}$$

which can be rewritten as

$$E\hat{V}(z^{-1}) + \sum_{i=1}^{\infty} (-1)^i H^i(z^{-1})\, E\,\hat{V}^{i+1}(z^{-1}) = V(z^{-1}) + \sum_{i=1}^{\infty} (-1)^i H^i(z^{-1}) V^{i+1}(z^{-1}) \tag{4.4.14}$$

Partitioning $\hat{V}(z^{-1})$ as

$$\hat{V}(z^{-1}) = \sum_{i=0}^{k-1} \hat{v}_i z^{-i} + \sum_{i=k}^{\infty} \hat{v}_i z^{-i} = \sum_{i=0}^{k-1} \hat{v}_i z^{-i} + \hat{V}_T(z^{-1})$$

and similarly $V(z^{-1})$ and noting that the minimum power of z^{-1} of the summed polynomials in (4.4.14) is z^{-k}, it follows that (4.4.14) can be separated into the following two equations

$$\sum_{i=0}^{k-1} E\,\hat{v}_i z^{-i} = \sum_{i=0}^{k-1} v_i z^{-i} \tag{4.4.15}$$

$$E\hat{V}_T(z^{-1}) + \sum_{i=1}^{\infty} (-1)^i H^i(z^{-1}) E\hat{V}^{i+1}(z^{-1}) = V_T(z^{-1}) + \sum_{i=1}^{\infty} (-1)^i H^i(z^{-1}) V^{i+1}(z^{-1}) \tag{4.4.16}$$

then from (4.4.15)

$$E\hat{v}_i = v_i \; ; \quad i=0,1,\ldots,k-1 \tag{4.4.17}$$

which implies

$$E\hat{p}_i = p_i \; ; \quad i=0,1,\ldots,k-1 \tag{4.4.18}$$

that is, the first k coefficients of $\hat{P}(z^{-1})$ are unbiased. This is, in general, the only conclusion with respect to the statistical properties of $\{\hat{p}_i\}$, since it can be seen by writing the bias of estimation from (4.3.3) and (4.3.12) as

$$E\hat{P}(z^{-1}) - P(z^{-1}) = M(z^{-1})\,[E\hat{\Omega}(z^{-1}) - \Omega(z^{-1})] + M(z^{-1}) \sum_{i=1}^{\infty} H^i(z^{-1})\,[E\hat{\Omega}^{i+1}(z^{-1}) - \Omega^{i+1}(z^{-1})] \tag{4.4.19}$$

that $(E\hat{p}_i - p_i)$; $i=k,k+1,\ldots$ are all associated with the second and higher order moments of $\{\hat{\omega}_i\}$ which are unknown. It will also be noted that the same difficulty occurs in the determination of mean squared estimation errors $E(\hat{p}_i - p_i)^2$ for $i \geq 0$.

4.5. Estimation of the process time delay

It has been assumed in the previous sections that the process time delay k is known. This section will show that this is not a restrictive assumption since by a suitable modification to the proposed estimation procedure, the time delay can be consistently estimated. The main aspect of the following technique is to avoid explicit representation of the delay and exploit the effect of this on the polynomial $[z^{-\ell}\Omega(z^{-1})]$. For this purpose, the system representation is to be modified as

$$\bar{P}(z^{-1}) = \sum_{i=0}^{\infty} \bar{p}_i z^{-i} = z^{-k}P(z^{-1}) \qquad (4.5.1)$$

where $\bar{p}_i=0$; $i=0,1,\ldots,k-1$. From (4.3.1), (4.3.2) and (4.3.3)

$$X_t = \eta_t + z^{-\ell}\Omega(z^{-1})c_t \qquad (4.5.2)$$

where now $\Omega(z^{-1}) = \dfrac{V(z^{-1})}{1+V(z^{-1})H(z^{-1})}$ and $V(z^{-1})=z^{\ell}\bar{P}(z^{-1})M^{-1}(z^{-1})-1$.

Defining

$$\beta(z^{-1}) = \sum_{i=0}^{\infty} \beta_i z^{-i} = [1+V(z^{-1})H(z^{-1})]^{-1} \qquad (4.5.3)$$

and

$$\alpha(z^{-1}) = \sum_{i=0}^{\infty} \alpha_i z^{-i} = \bar{P}(z^{-1})M^{-1}(z^{-1})\beta(z^{-1}) \qquad (4.5.4)$$

then $[z^{-\ell}\Omega(z^{-1})]$ of (4.5.2) can be written as

$$z^{-\ell}\Omega(z^{-1}) = \alpha(z^{-1}) - z^{-\ell}\beta(z^{-1}) \qquad (4.5.5)$$

Consider now the following cases

i) $\ell > k-1$

Arranging the powers of z^{-1}, (4.5.5) will be written as

$$z^{-\ell}\Omega(z^{-1}) = \sum_{i=0}^{k-1} \alpha_i z^{-i} + \sum_{i=k}^{\infty} \alpha_i z^{-i} - z^{-\ell}\beta(z^{-1}) \qquad (4.5.6)$$

since $\bar{p}_i=0$; $i=0,1,\ldots,k-1$ and hence, from (4.5.4), $\alpha_i=0$; $i=0,1,\ldots,k-$ then it follows that the first k coefficients of $[z^{-\ell}\Omega(z^{-1})]$ are null.

ii) $\ell \leq k-1$

Writing

$$z^{-\ell}\Omega(z^{-1}) = \left[\alpha_o + \sum_{i=1}^{k-1} \alpha_i z^{-i} - \beta_o z^{-\ell} - \sum_{i=1}^{k-\ell-1} \beta_i z^{-\ell-i}\right] - \sum_{i=k-\ell}^{\infty} \beta_i z^{-\ell-i} + \sum_{i=k}^{\infty} \alpha_i z^{-i} \tag{4.5.7}$$

and considering the first three polynomials and noting that $\alpha_o=0$ and β_o is independent of the coefficients of $V(z^{-1})$, $H(z^{-1})$ and is equal to unity, then it can be seen that the first coefficient of the polynomial given in the parentheses of (4.5.7) is null and the $(\ell+1)^{th}$ coefficient equals -1.

These observations suggest that if correlation analysis is performed by multiplying (4.5.2) by $c_{t-\tau}$; $\tau=0,1,\ldots$ and if resulting $\hat{\omega}_o=0$ and $\hat{\omega}_\ell=-1$ then it can be concluded that $\ell \leq k-1$. Increasing ℓ and checking until $\hat{\omega}_\ell \neq -1$ gives an estimate of the time delay k as the number of first zero coefficients of $\hat{\Omega}(z^{-1})$.

The problem of deciding whether certain values of $\{\hat{\omega}_i\}$ can be assumed insignificantly different from zero does not have a clear answer. An approximate check can be based on a formula given by Bartlett [14] which states that when two normal processes are not correlated and one is white noise then the variance of estimated cross correlation coefficient

$$\hat{\rho}_{Xc}(i) = \frac{\hat{\gamma}_{Xc}(i)}{\sqrt{\hat{\gamma}_{XX}(0)\ \gamma_{cc}(0)}} \tag{4.5.8}$$

is approximately proportional to N^{-1}. Using this result in the comparison of $\hat{\rho}_{Xc}(i)$ with their approximate standard deviation $\frac{1}{\sqrt{N}}$ the number of insignificant parameters of $\{\hat{\omega}_i\}$ or equivalently the time delay can be estimated.

So far it has been shown that if an additional perturbation signal is allowed then the process parameters and time delay can be consistently estimated while the process is operated under the Box and

Jenkins control system. The next section will discuss how the correlation analysis can be used in connection with the predictor updating procedure of [52] to optimize the control system.

4.6. Predictor updating

Consider the Box and Jenkins control system of Fig. (4.3.1) without a perturbation signal, i.e. $c_t=0$. It was proved in [52] that if an auto-regressive model is fitted to a set of data $\{X_t;t=(i-1)N+1,\ldots,iN\}$ at the i^{th} iteration, then successive updating of the predictor at every $(iN)^{th}$ sample based on this fitted auto-regressive model may result in an optimum control system. Since the detailed proof of this property has been given in [52], only a brief discussion will be sufficient here as an easy reference.

Consider the i^{th} iteration where an auto-regressive model is fitted to X_t as

$$X_t = \frac{1}{1-\hat{R}_i(z^{-1})}\ \hat{\zeta}_t \tag{4.6.1}$$

From (4.3.1) and taking $c_t=y_t^d=0$, X_t can also be written as

$$X_t = \frac{1}{[1+V(z^{-1})H_{i-1}(z^{-1})][1-Q(z^{-1})]}\ \zeta_t \tag{4.6.2}$$

where the subscript $(i-1)$ of $H(z^{-1})$ denotes the predictor operating over the interval $t=(i-1)N+1,\ldots,iN$. It is assumed that the auto-regressive parameters of $\{q_i\}$ of the disturbance

$$x_t = \frac{1}{1-Q(z^{-1})}\ \zeta_t \tag{4.6.3}$$

remain constant for all intervals. Then from (4.6.1) and (4.6.2), approximately

$$1-\hat{R}_i(z^{-1}) \simeq [1+V(z^{-1})H_{i-1}(z^{-1})][1-Q(z^{-1})] \tag{4.6.4}$$

where the predictor $H_{i-1}(z^{-1})$ is calculated from the previous auto-regressive model $[1-\hat{R}_{i-1}(z^{-1})]$ by the optimum predictor of eqn. (3.5.18), i.e.

$$H_{i-1}(z^{-1}) = \{[1-\hat{R}_{i-1}(z^{-1})]^{-1}\}_T\ [1-\hat{R}_{i-1}(z^{-1})] \tag{4.6.5}$$

where T denotes k^{th} order front-end truncation operator, then by substituting of $H_{i-1}(z^{-1})$ from (4.6.5) into (4.6.4) and defining

$$1 + \hat{\nu}_i(z^{-1}) = [1-\hat{R}_i(z^{-1})]^{-1} \tag{4.6.6}$$

it can be shown that $V(z^{-1})$ of (4.6.4) can be written as

$$V(z^{-1}) \simeq \frac{[1+\hat{\nu}_{i-1}(z^{-1})]\ \{1-[1-Q(z^{-1})][1+\hat{\nu}_i(z^{-1})]\}}{[1+\hat{\nu}_i(z^{-1})]\ [1-Q(z^{-1})]\ \hat{\nu}_{i-1,T}(z^{-1})} \tag{4.6.7}$$

Now, if it is assumed that the repeated updatings of predictor converge to a stationary solution, then denoting $\hat{\nu}_s(z^{-1})$ as the polynomial $\nu(z^{-1})$ corresponding to the stationary solution, it can be seen that (4.6.7) becomes

$$V(z^{-1}) \simeq \frac{1-[1-Q(z^{-1})]\ [1+\hat{\nu}_s(z^{-1})]}{[1-Q(z^{-1})]\ \hat{\nu}_{s,T}(z^{-1})} \tag{4.6.8}$$

From output eqn. (4.3.17) and taking $c_t=0$ and $H(z^{-1})=H_s(z^{-1})$, the system output in the stationary case can be written as

$$y_{t,s} = \frac{1-H_s(z^{-1})}{[1+V(z^{-1})H_s(z^{-1})][1-Q(z^{-1})]}\ \zeta_t \tag{4.6.9}$$

then substitution of $V(z^{-1})$ from (4.6.8) and using (4.6.5) for the stationary case, $y_{t,s}$ becomes

$$y_{t,s} \simeq [1+\hat{\nu}_s(z^{-1}) - \hat{\nu}_{s,T}(z^{-1})]\ \zeta_t \tag{4.6.10}$$

But from (3.2.17) the optimum output of the Box and Jenkins control system corresponding to the disturbance (4.6.3) is

$$y_t = [1+S(z^{-1}) - S_T(z^{-1})]\ \zeta_t \qquad (4.6.11)$$

where $1+S(z^{-1}) = [1-Q(z^{-1})]^{-1}$. With a completely similar argument to that of section 3.11, it can be shown that

$$1 + \hat{\nu}_s(z^{-1}) - \hat{\nu}_{s,T}(z^{-1}) = 1 + S(z^{-1}) - S_T(z^{-1}) \qquad (4.6.12)$$

then (4.6.12) in connection with (4.6.10) and (4.6.11) proves that if the predictor updating procedure converges to a stationary solution then the resulting Box and Jenkins control system is approximately optimum as far as the disturbances are concerned.

It is also shown in Appendix III that if the modified predictor of eqn. (3.7.11) is used in successive updatings instead of the optimum predictor (3.5.18) then the stationary solution is also optimum in the sense that the stationary output $y_{t,s}$ is equivalent to the prediction error corresponding to the unity steady state gain predictor.

4.7. Convergence of the procedure

A necessary condition for convergence of the predictor updating procedure has been derived in [52] by considering the behaviour of

$$\Delta H_i(z^{-1}) = H_i(z^{-1}) - H_s(z^{-1}) \qquad (4.7.1)$$

From (4.6.4) and (4.6.5) it is possible to write $\Delta H_i(z^{-1})$ in the form

$$\Delta H_i(z^{-1}) = [H^*(z^{-1})-1]\ V(z^{-1})\ \Delta H_{i-1}(z^{-1}) \qquad (4.7.2)$$

where $H^*(z^{-1}) = S_T(z^{-1})[1-Q(z^{-1})]$. Denoting $H_o(z^{-1})$ as the initial predictor, (4.7.2) can also be written as

$$\Delta H_i(z^{-1}) = [H^*(z^{-1})-1]^i\ V^i(z^{-1})\ \Delta H_o(z^{-1}) \qquad (4.7.3)$$

then it follows that the procedure converges if

$$\lim_{i \to \infty} [H^*(z^{-1})-1]^i V^i(z^{-1}) = 0 \qquad (4.7.4)$$

a necessary condition for (4.7.4) to hold has been given in [52] as

$$0 < \frac{p_o}{m_o} < 2 \qquad (4.7.5)$$

Now, consider a situation where prior to the application of the predictor updating procedure, process parameters are estimated by the correlation analysis of section 4.3. The error analysis has shown that

$$E\hat{p}_o = p_o$$

and

$$\lim_{N \to \infty} \hat{p}_o = p_o \qquad \text{with probability one} \qquad (4.7.6)$$

These properties suggest that if the predictor updating procedure is started with a process model $M(z^{-1})$ where m_o is chosen to satisfy the inequality

$$0 < \frac{\hat{p}_o}{m_o} < 2 \qquad (4.7.7)$$

then it may be stated that for this choice of m_o the necessary condition of convergence (4.7.5) will be satisfied with a high probability. However although the above choice of $M(z^{-1})$ with (4.7.7) can provide a solution to the convergence problem, there is a strong incentive here to use $\hat{P}(z^{-1})$, resulting from correlation analysis, as the initial process model $M(z^{-1})$ of predictor updating procedure. This follows from the fact that since $\hat{P}(z^{-1})$ is a consistent estimate then for a sufficiently large number of data $V(z^{-1})$ of (4.7.4) becomes approximately null and hence

$$\lim_{i \to \infty} V^i(z^{-1}) = 0 \qquad (4.7.8)$$

which in turn satisfies the convergence condition (4.7.4). It should however be noted that in practice convergence of the procedure will also be strongly dependent on the behaviour of the first term of (4.7.4), i.e.

TABLE 4.8.1 TIME DELAY ESTIMATION

Disturbance: $x_t = (1-0.4z^{-1})^{-1} \zeta_t$ Perturbation signal: $c_t = \pm 1$ PRBS

$\sigma_\zeta^2 = 1.0$, $N = 255$

Process	Model	Predictor
$P(z^{-1})$	$M(z^{-1})$	$z^{\ell}H(z^{-1})$
$\frac{1}{1-0.5z^{-1}}$	$1-0.4z^{-1}$	0.3

Estimated cross-covariance coefficients

	i	0	1	2	3	4	5
Case 1: ℓ=3,k=5	$\hat{\gamma}_{Xc}(i)$	-0.05	-0.24	-0.12	-1.09	0.04	0.98
Case 2: ℓ=3,k=4	$\hat{\gamma}_{Xc}(i)$	0.01	-0.12	0.04	-0.96	1.10	0.97
Case 3: ℓ=4,k=3	$\hat{\gamma}_{Xc}(i)$	0.09	-0.09	0.03	0.93	0.06	0.57
	$\lvert\hat{\rho}_{Xc}(i)\rvert$	0.05	0.05	0.02	0.52	0.03	0.32

$$\lim_{i \to \infty} [H^*(z^{-1})-1]^i = \lim_{i \to \infty} (-1)^i [1+S(z^{-1})-S_T(z^{-1})]^i [1-Q(z^{-1})]^i \tag{4.7.9}$$

which, in general, tends to infinity unless some restrictive assumptions are satisfied by the disturbance model. Hence it follows that an additional requirement for the convergence is that (4.7.9) must tend to infinity slower than $V^i(z^{-1})$ tends to zero. It is not known how severe this restriction will be in practice. Furthermore, since the convergence criterion (4.7.4) can only be expressed in terms of unknown parameters, the nature of convergence, i.e. monotonic or otherwise, will probably remain an unresolved problem.

In the next section a series of computer simulation results will be presented in order to substantiate the theory given in this chapter.

4.8. Examples

The first part of this section is concerned with the estimation of process time delay as a prelude to the second part which deals with optimizing the Box and Jenkins control system by first applying correlation analysis and then adjusting the resulting control system by the predictor updating procedure.

An example of the system time delay estimation is shown in Table (4.8.1). The Box and Jenkins control system of Fig. (4.3.1) was simulated with an auto-regressive signal as the disturbance x_t, where the white noise ζ_t was normally distributed with unity variance. The perturbation signal c_t was obtained by an eight register pseudo-random binary sequence generator and the period of the sequence was $N=255$. The process $P(z^{-1})$ was taken to be a first order pulse transfer function with the process model as a first order approximation and the the controller was the exact inverse of the process model $M(z^{-1})$.

TABLE 4.8.2 PREDICTOR UPDATING WITH CORRELATION ANALYSIS I

Disturbance: $x_t = (1-0.9z^{-1})^{-1}\zeta_t$ Perturbation signal: $c_t = \pm 1$ PRBS

$\sigma_\zeta^2 = 1.01$, k=2 , N = 255

Process	Model	Predictor
$z^{-k}P(z^{-1})$	$z^{-k}M(z^{-1})$	$z^{k}H(z^{-1})$
$\frac{z^{-2}}{1-0.5z^{-1}}$	z^{-2}	0.36

Correlation analysis estimate $\hat{P}(z^{-1})$

i	0	1	2	3	4	5
p_i	1.00	0.50	0.25	0.12	0.06	0.03
$\hat{p}_i$	0.84	0.30	0.17	0.08	0.09	-0.04

Var $\{y_t\}$ resulted from correlation analysis updating: 2.67

Predictors for $\{X_t\}$

	$\hat{h}_o$	$\hat{h}_1$	$\hat{h}_2$	$\hat{h}_3$	$\hat{h}_4$
$\hat{H}_1$	0.86	-0.20	0.04	0.14	-0.05
$\hat{H}_2$	0.77	-0.27	0.17	0.19	-0.11
$\hat{H}_3$	0.80	-0.21	0.10	0.14	-0.09
$\hat{H}_4$	0.79	-0.24	0.13	0.17	-0.09
$\hat{H}_5$	0.80	-0.23	0.12	0.15	-0.09
$\hat{H}_6$	0.80	-0.24	0.12	0.16	-0.09

Computed values of Var $\{y_t\}$

Iteration	1	2	3	4	5	6
Var $\{y_t\}$	1.96	1.87	1.89	1.87	1.88	1.87

Optimum output variance: 1.83

The simulated control system was operated for 255 sample intervals for three different combinations of process and model time delays k and ℓ. This resulted in a record of $\{X_t, c_t;\ t=1,2,\ldots,255\}$ from which cross covariance coefficients $\hat{\gamma}_{Xc}(i)$ are estimated by

$$\hat{\gamma}_{Xc}(i) = \frac{1}{255} \sum_{t=i+1}^{255} X_t c_{t-i} \qquad i=0,1,\ldots$$

whose 6 values are shown in Table (4.8.1). Following the argument of section 4.5 and noting that in the first two cases $\ell < k-1$ and $\ell = k-1$ respectively, then it should be expected that $\hat{\gamma}_{Xc}(\ell)$ is an estimate of the coefficient -1. Inspection of $\hat{\gamma}_{Xc}(3)$ in the cases 1 and 2 verifies that $\hat{\gamma}_{Xc}(3) \simeq -1$. Increasing the model time delay ℓ such that $\hat{\gamma}_{Xc}(\ell) \neq -1$ corresponds to $\ell > k-1$ and in this case the first k parameters of $\hat{\gamma}_{Xc}(i)$ should be equal to zero. Case 3 is an example of this situation where $k=3$, $\ell=4$ and consequently it was found that $\hat{\gamma}_{Xc}(4) \neq -1$. Comparing cross correlation coefficients $\hat{\rho}_{Xc}(i)$ of the case 3 with their approximate standard deviation $\frac{1}{\sqrt{N}} = 0.06$, it was observed that the first 3 coefficients of $\hat{\gamma}_{Xc}(i)$ can be assumed insignificantly different from zero, that is, the process time delay is $k=3$.

An example of updating the predictor every $N=255$ samples is shown in Table (4.8.2). The initial model and controller parameters of the control system were first updated by applying the correlation analysis of section 4.3. The estimated process parameters are shown at the top of Table (4.8.2). The updated controller was the exact inverse of the estimated process model $\hat{P}(z^{-1})$. The updated process-model and controller parameters were chosen as the initial design parameters prior to the predictor updating procedure. It can be seen, from the resulting output variance, that at this stage the control system was not yet optimum. After the correlation analysis, the perturbation signal c_t was taken as null and each time a record of 255 samples

TABLE 4.8.3. PREDICTOR UPDATING WITH CORRELATION ANALYSIS II

Disturbance: $x_t = (1-0.6z^{-1} + 0.4z^{-2} - 0.2z^{-3})^{-1} \zeta_t$

Perturbation signal: $c_t = \pm 1$ PRBS

$\sigma_\zeta^2 = 1.00$, k=1 , N=510

Process	Model	Predictor
$z^{-k}P(z^{-1})$	$z^{-k}M(z^{-1})$	$z^{k}H(z^{-1})$
$\dfrac{z^{-1}}{(1-0.5z^{-1})(1-0.3z^{-1})}$	$z^{-1}(1+0.5z^{-1})$	$0.85+0.10z^{-1}+0.05z^{-2}$

Correlation analysis estimate $\hat{P}(z^{-1})$

i	0	1	2	3	4	5
p_i	1.00	0.80	0.49	0.28	0.15	0.07
$\hat{p}_i$	0.96	0.68	0.39	0.28	0.17	0.13

	Predictors for $\{X_t\}$					Output Variance
Iteration	h_o	h_1	h_2	h_3	h_4	Var $\{y_t\}$
1	0.54	-0.35	0.17	0.18	-0.10	1.01
2	0.51	-0.38	0.19	-	-	0.96
3	0.55	-0.38	0.19	-	-	1.02
4	0.60	-0.47	0.31	-	-	1.13
5	0.62	-0.38	0.32	0.05	-0.02	1.21
6	0.55	-0.48	0.19	-0.02	-0.06	1.03
7	0.60	-0.47	0.27	0.03	-0.10	1.00
8	0.57	-0.44	0.29	-	-	1.04
9	0.59	-0.38	0.18	0.12	-0.10	1.02
10	0.60	-0.43	0.28	-	-	0.94
11	0.59	-0.43	0.22	-0.02	-0.02	0.97

Optimum output variance: 1.00

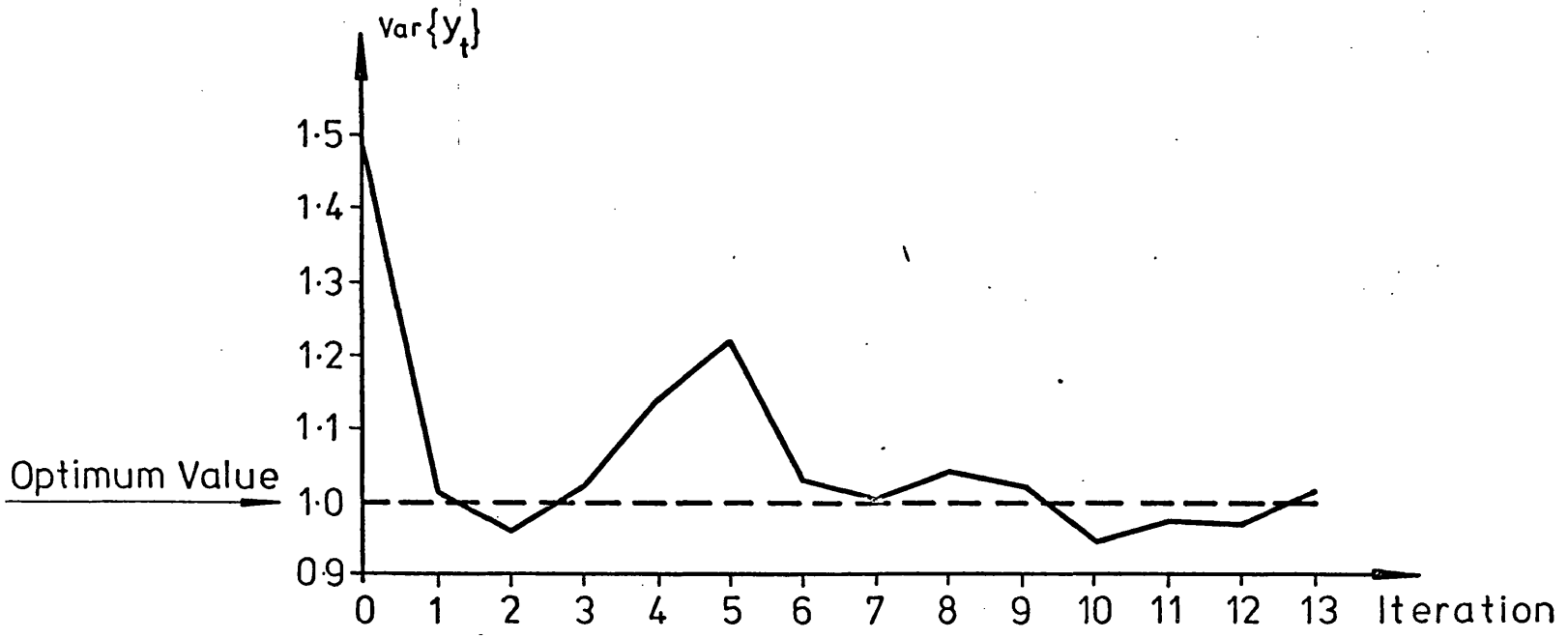

Fig. 4.8.4. EXAMPLE II: SAMPLE VARIANCE OF THE OUTPUT

Var $\{y_t\}_o$ corresponds to the correlation analysis

of $\{X_t\}$ became available, an auto-regressive model was fitted by the least squares algorithm of eqn. (3.4.2). The predictor then was updated by calculating a new optimum predictor from the fitted auto-regressive model as in eqn. (3.5.18). To satisfy the assumption that the disturbance model remained constant, all iterations used the same disturbance sequence $\{x_t\}$. Updated predictor parameters and the sample variance of output are shown at the foot of Table (4.8.2). It can be seen, from the values of Var $\{y_t\}$, that the system converged towards the optimum and in particular that convergence was established by the 5^{th} iteration.

One particular feature of the above example was that the disturbance sequence $\{x_t\}$ was taken to be the same for all iterations of the predictor updating procedure. It was observed in many examples that considerable imagination may be needed to spot a possible convergence if the sequence $\{x_t\}$ changes between updates. This undesirable characteristic of the convergence can be partially explained as follows. In the derivation of optimality of the predictor updating procedure it has been assumed that the disturbance model $[1-Q(z^{-1})]$ remains constant for all iterations. However for a moderate amount of data, different disturbance sequences $\{x_t;\ t=1,2,\ldots N\}$ will, in general, correspond to different auto-regressive models. To illustrate this point, consider the following auto-regressive model fits to $\{x_t;\ t=(i-1)N+1,\ldots,iN\}$ for $i=1,2,\ldots$

$$x_t = [1-\hat{Q}_i(z^{-1})]\ \hat{\zeta}_t$$

it is known from section 3.4 that $\hat{Q}_i(z^{-1})$ remains constant for all i if and only if $N \to \infty$.

A typical example of the results of updating the predictor for different disturbance sequences is shown in Table (4.8.3). Although there was a considerable improvement in the output variance, the convergence of the procedure was found to be difficult to detect. In comparison with the previous example, variations of the estimated

TABLE 4.8.5. PREDICTOR UPDATING WITH CORRELATION ANALYSIS III

Disturbance: $x_t = (1-0.6z^{-1} + 0.4z^{-2} - 0.2z^{-3})^{-1} \zeta_t$

Perturbation signal: $c_t = \pm 1$ PRBS

$\sigma_\zeta^2 = 1.00$, k=1 , N=127

Process	Model	Predictor
$z^{-k}P(z^{-1})$	$z^{-k}M(z^{-1})$	$z^{k}H(z^{-1})$
$\frac{z^{-1}}{1-0.5z^{-1}}$	z^{-1}	$0.7-0.42z^{-1}+0.24z^{-2}$

Correlation analysis estimate $\hat{P}(z^{-1})$

i	0	1	2	3	4	5
p_i	1.00	0.50	0.25	0.12	0.06	0.03
$\hat{p}_i$	0.93	0.52	0.23	0.11	0.03	0.02

Var $\{y_t\}$ resulted from correlation analysis updating : 0.99

Optimum output variance : 1.00

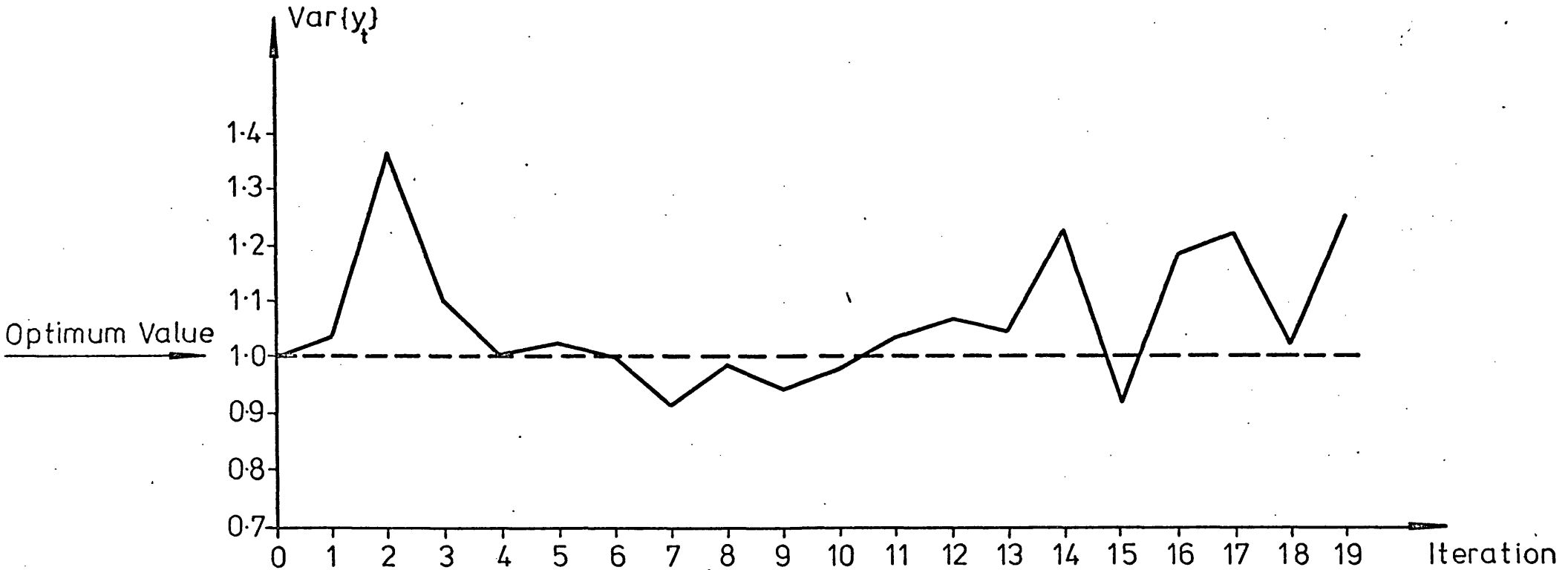

Fig. 4.8.6. EXAMPLE III: SAMPLE VARIANCE OF THE OUTPUT

Var $\{y_t\}_o$ corresponds to the correlation analysis

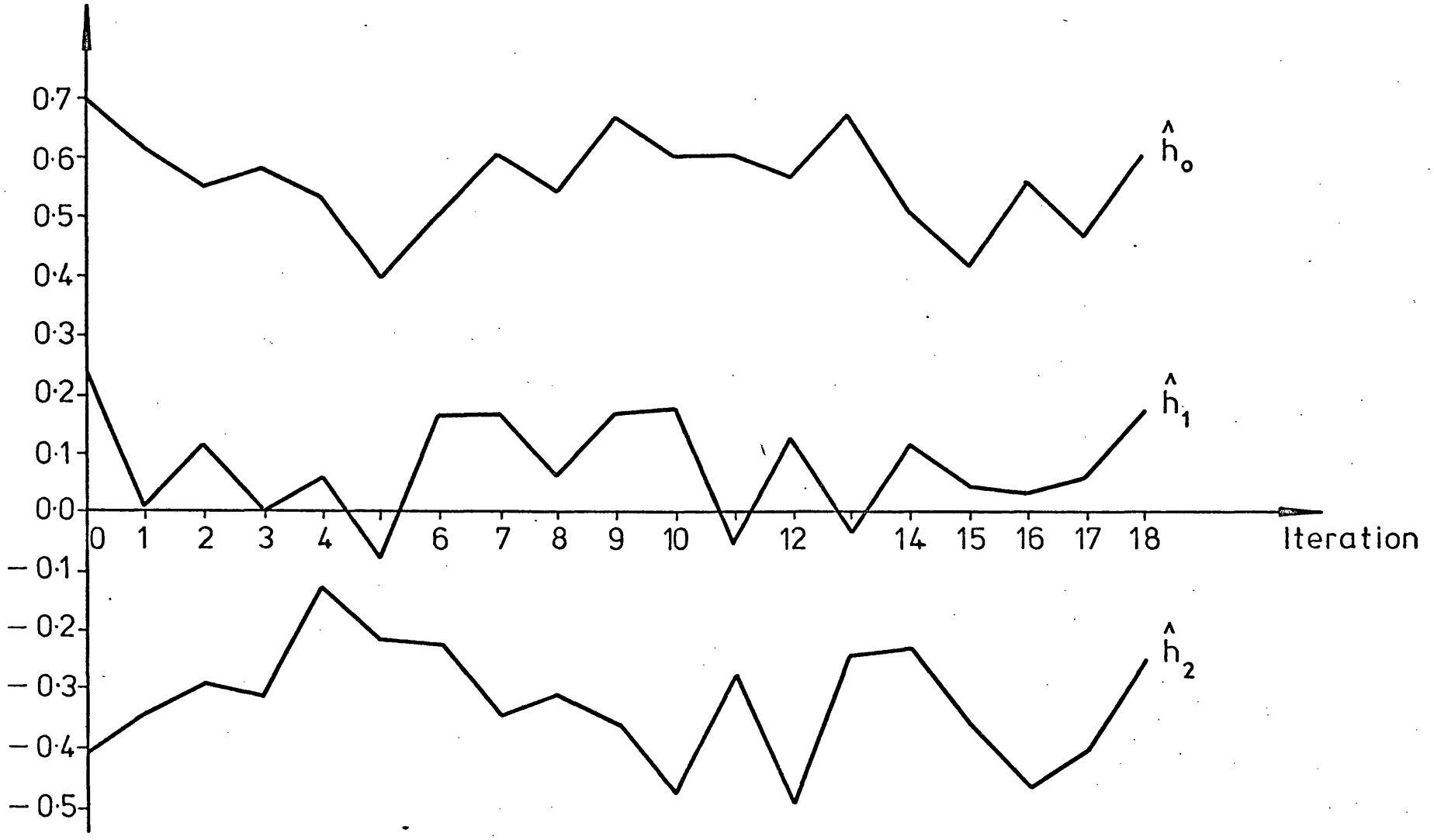

Fig. 4.8.7. EXAMPLE III: PREDICTOR COEFFICIENTS

$\{h_i\}_o$; i=0,1,2 are the initial predictor coefficients

predictor parameters were rather large despite the fact that the data length was twice as much as in example I. Nevertheless as it can be seen from Fig. (4.8.4), the control system became approximately optimum after about the 5^{th} iteration. However since N=510, a total of about 3000 samples was needed to optimize the system - a figure which can be quite unacceptable in practice. Also as is noticeable in this example, the convergence was not found to be monotonic. In fact it was observed that in many cases, where convergence seemed to be established further updatings resulted in large variations of the output variance. An example given in Table (4.8.5) is a typical representation of this situation where the predictor was chosen very close to the optimum predictor and consequently the correlation analysis resulted in a control system which was almost optimum for a data of length N=127. The behaviour of output variance and estimated predictor parameters for the consequent predictor updating procedure are shown in Fig. (4.8.6) and (4.8.7). It can be seen from Fig. (4.8.6) that the procedure could not maintain the control system around optimum after about the 13^{th} iteration because of the large fluctuations in the estimated predictor parameters as shown in Fig. (4.8.7).

In conclusion, the predictor updating procedure combined with the correlation analysis was found to produce optimum control systems if the disturbance model remains almost constant for all iterations. In cases where $[1-\hat{Q}(z^{-1})]$ varied the method proved not entirely satisfactory. A further difficulty was to spot a possible convergence particularly when the sample length was not sufficient. In this case estimated coefficients were subject to rather large errors so that considerable imagination had to be exerted to determine convergence. This difficulty can be partially alleviated if the number of data is increased but this effectively discounts the procedure as a practically viable method of optimization. For example it can be demonstrated that

with a total of about 3000 data which was needed to optimize the control system of example II, a simple correlation analysis with only one step predictor updating results in the same optimum control system.

4.9. Conclusions

This chapter has presented an attempt to facilitate the convergence of the predictor updating procedure of [52] by proposing a correlation analysis prior to the updating procedure. It has been shown that the process parameters including the time delay can be consistently estimated by correlating an additional perturbation signal with the signal X_t of the Box and Jenkins control system. Since the only first k coefficients of the impulse response are unbiased then, in general, the correlation analysis does not lead to an optimum control system. However, based on the results of convergence analysis of the predictor updating procedure given in [52], it was thought that further improvement can be achieved if $\hat{P}(z^{-1})$, resulting from the correlation analysis, is used to design an initial Box and Jenkins control system for the consequent updating procedure. Theoretical analysis has shown that it is possible to satisfy the necessary condition for convergence if the data is plentiful enough. However several computer simulation results have indicated that the proposed combined algorithm does not always perform satisfactorily. In particular, the convergence was not found to be monotonic and in fact in some cases where convergence seemed to be established consequent updatings gave rise to deteriorated performance. This problem was more apparent when the data length was not large enough where it was also difficult to spot convergence due to large errors in the estimated parameters.

CHAPTER V

CONCLUSIONS

This thesis has discussed the problem of identification and self-adaptive control of linear time invariant processes subject to random disturbances.

Chapter I emphasizes the fact that unless assumptions are made about the unknown disturbances the identification problem is from a rigorous viewpoint insoluble. Therefore the main concern of this chapter has been with validity of assumptions and an attempt has been made to classify some current identification techniques from this point of view. Since the disturbance and the test input are at least physically indepedendent, the most plausible assumption is one of statistical independence.

The modified least squares algorithm of Chapter II exploits the statistical independence by assuming that the disturbance and input are linearly independent. This yields a set of assumptions in excess of the minimal requirements for identification, and an attempt has been made to exploit this excess to minimize the estimation errors due to the inevitable violation of assumptions. The resulting algorithm is identical to that of the two stage least squares [30]. Nevertheless it is believed that the effect of assumptions on identification has been shown explicitly.

There are two main problems associated with the modified least squares algorithm. The choice of number of assumptions constitutes the first one and for this a partial answer is available if the process order and the impulse response duration are assumed to be known. The second problem is concerned with the choice of criterion for minimizing the estimation errors. The $(\mathrm{tr}\ \Psi)$ criterion has led to a neat and simple algorithm. Another possible choice is the minimization of the largest eigenvalue of Ψ. This choice has no easy general answer but

nevertheless it has been shown that the chosen algorithm is also one solution to the eigenvalue problem.

There are still unsolved theoretical problems in all phases of identification. Most important among this is estimating the open-loop characteristics of a process without disturbing or breaking the feedback path. In engineering applications this problem arises when a process is open-loop unstable or susceptible to drift. In Chapter III the identifiability problem for a closed loop control system of the Box and Jenkins type has been re-examined to extend the previous investigation of Turtle [52]. A new estimation error equation has shown explicitly that it is possible to estimate the process and disturbance dynamics provided that the disturbance can be described by an auto-regressive model with a restricted number of coefficients. If this identifiability condition is not satisfied then some coefficients of the disturbance model cannot be estimated and hence the resulting process parameter estimates are biased. Nevertheless if repeated applications of the estimation procedure converge to a stationary solution then the resulting control system is optimum as far as the disturbances are concerned.

There are many difficulties associated with this self-adaptive procedure. Firstly, convergence is impossible to demonstrate in terms of practically available quantities. Secondly, and more importantly, apart from the first iteration, the closed loop stability cannot be guaranteed. These doubts added to the restrictive assumptions that the process time delay is known and the disturbance model remains constant, do not recommend the procedure as a practically viable method of optimization.

Chapter IV investigates the situation where an external test signal is applied while the process is operating under the Box and Jenkins control system. It has been shown that if the test signal is independent of the disturbance then a simple correlation analysis can produce consistent estimates of the process parameters. However for a limited amount of data these estimates are, in general, biased and hence do not

lead to an optimum control system. But since a reasonably accurate model of the process is available then the self-adaptive predictor updating procedure of [52] can be applied to improve the control system resulting from the correlation analysis.

The simulation studies have indicated that due to a good initial process model, instability problems are unlikely to arise under repeated applications of predictor updating. However these examples have also shown that parameter estimate variations due to finite sample length can cause the control system to fluctuate around the optimum without actually converging to it. These fluctuations can be reduced by increasing the number of data used in each iteration. However in this situation it is questionable whether to apply the predictor updating procedure since a simple correlation analysis followed by a one step predictor updating may work equally well for a sufficiently large number of data.

The self-adaptive procedures of this thesis are as yet just proposals. Due to doubts about convergence and stability, they are not, at the moment, sufficiently well-developed to be applicable to real-life problems. It appears that difficulties concerning the stability are inherent in the method of solution rather than the problem itself, since there exist self-tuning regulators [55] which can stabilize any system provided that the time delay is known and the model structure satisfies certain assumptions. However, in parallel to the self-adaptive procedures of this thesis, the convergence of these self-tuning regulators cannot always be guaranteed.

APPENDIX I

ASYMPTOTIC PROPERTIES OF THE ERROR EQUATION

Consider the error eqn. (3.9.15) of the separation technique analysis and assume that

i) the disturbance x_t arises from an auto-regressive process of order $(\bar{m}-1)$, and

ii) $(\bar{m}-1)$ coefficients of $R(z^{-1})$ are consistently estimated.

Then

$$\lim_{N \to \infty} E[\hat{P}(z^{-1}) \mid \hat{Q}(z^{-1})] = P(z^{-1}) \tag{A.1.1}$$

i.e. given $\hat{Q}(z^{-1}) = \sum_{i=1}^{\bar{m}-1} \hat{q}_i z^{-i}$, the process model $\hat{P}(z^{-1})$ is asymptotically unbiased.

Proof: Using vector-matrix notation of Chapter I, the set of N equations of (3.9.1) and (3.9.3) can be written as

$$X = (I-\hat{R})^{-1}\hat{\zeta} \tag{A.1.2}$$

$$x = (I+VH)X \tag{A.1.3}$$

where $\hat{R} = \sum_{i=1}^{n} \hat{r}_i S^i$, $V = PM^{-1}-I = \left(\sum_{i=0}^{I} p_i S^i\right)\left(\sum_{j=0}^{I} m_j S^j\right)^{-1} - I$, $H = \sum_{i=0}^{I} h_i S^{k+i}$

and $X, x, \hat{\zeta}$ are N-vectors, for example, $X=(X_1, X_2, \ldots, X_N)^T$.

On substitution of X from (A.1.2), x becomes

$$x = (I+VH)(I-\hat{R})^{-1}\hat{\zeta} \tag{A.1.4}$$

from which $I-G=I-\sum_{i=1}^{m'} g_i S^i$ will be defined analogously to $[1-G(z^{-1})]$ of (3.9.4) as

$$I-G=(I+VH)^{-1}(I-\hat{R}) \tag{A.1.5}$$

Writing (A.1.5) explicitly as

$$\begin{bmatrix} 1 & & & \\ -g_1 & 1 & & \\ \vdots & \ddots & & (0) \\ -g_{m'} & & \ddots & \\ 0 & & & \\ \vdots & & & \\ 0 \ \ldots \ 0 & -g_{m'} & \ldots & -g_1 \ \ 1 \end{bmatrix} = (I+VH)^{-1} \begin{bmatrix} 1 & & & \\ -\hat{r}_1 & 1 & & (0) \\ \vdots & & \ddots & \\ -\hat{r}_n & & \ddots & \\ 0 & & & \\ \vdots & & & \\ 0 \ \ldots \ 0 & -\hat{r}_n & \ldots & -\hat{r}_1 \ \ 1 \end{bmatrix} \qquad \text{(A.1.6)}$$

it can be seen that the first column vectors of the above matrices contain all the parameters of interest. It is then convenient to consider only these vectors and to write from (A.1.6)

$$g \equiv \begin{bmatrix} 1 \\ -g_1 \\ \vdots \\ -g_{m'} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = (I+VH)^{-1} \begin{bmatrix} 1 \\ -\hat{r}_1 \\ \vdots \\ -\hat{r}_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(A.1.7)}$$

Since from (3.9.11)

$$g_i = \hat{q}_i \qquad i=1,2,\ldots,\bar{m}-1 \qquad \text{(A.1.8)}$$

then defining $\hat{q}=(1,-\hat{q}_1,\ldots,-\hat{q}_{\bar{m}-1})^T_{\bar{m}x1}$, $g_T=(-g_{\bar{m}},\ldots,-g_{m'},0,\ldots,0)^T_{(N-\bar{m})x1}$, $\hat{r}=(1,-\hat{r}_1,\ldots,-\hat{r}_n,0,\ldots 0)^T_{Nx1}$ and $F=(I+VH)^{-1}$, it can be seen that (A.1.7) can be written as

$$g = \begin{bmatrix} \hat{q} \\ \hline g_T \end{bmatrix} = F\hat{r} \qquad \text{(A.1.9)}$$

which shows that g and $\hat{r}$ are linearly dependent. Since $\hat{r}$ is asymptotically normally distributed [15], it follows from this linear relation that g is also asymptotically normally distributed. Let $\mu_{\hat{r}}$

and $V_{\hat{r}}$ denote the mean and the covariance of $\hat{r}$ respectively then using (A.1.9), corresponding statistics of g are obtained as

$$\mu_g = \begin{bmatrix} \mu_{\hat{q}} \\ \hline \mu_{g_T} \end{bmatrix} = F\mu_{\hat{r}} \tag{A.1.10}$$

$$V_g = E\left[\begin{array}{c} \hat{q}-\mu_{\hat{q}} \\ \hline g_T-\mu_{g_T} \end{array}\right]\left[\begin{array}{c|c} (\hat{q}-\mu_{\hat{q}})^T & (g_T-\mu_{g_T})^T \end{array}\right]$$

$$= \left[\begin{array}{c|c} V_{\hat{q}\hat{q}} & V_{\hat{q}g_T} \\ \hline V_{g_T\hat{q}} & V_{g_Tg_T} \end{array}\right] = FV_{\hat{r}}F^T \tag{A.1.11}$$

From the properties of the normally distributed random vectors, the conditional expectation of g_T given $\hat{q}$ can now be written as [7]

$$E(g_T|\hat{q}) = \mu_{g_T} + V_{g_T\hat{q}}\, V_{\hat{q}\hat{q}}^{-1}\,(\hat{q}-\mu_{\hat{q}}) \tag{A.1.12}$$

To use this result in the error eqn. (3.9.15), it is convenient to represent (3.9.15) in a vector-matrix form as

$$P - \hat{P} = (P-M)(I-\hat{Q})^{-1}\, G_T + MH^{-1}(I-\hat{Q})^{-1}\, G_T = \Gamma G_T \quad \text{say} \tag{A.1.13}$$

Considering the first column vectors of (A.1.13) and defining $p = (p_o, p_1, \ldots, p_I, 0, \ldots, 0)^T_{Nx1}$ and analogously $\hat{p}$, it is possible to write

$$p - \hat{p} = \Gamma \begin{bmatrix} \underline{0}_{mx1} \\ \hline g_T \\ \hline \underline{0}_{N-m'-1} \end{bmatrix} \tag{A.1.14}$$

where taking conditional expectation and using (A.1.12)

$$E[(p-\hat{p})|\hat{q}] = \Gamma \begin{bmatrix} \underline{0} \\ \hline \mu_{g_T} + V_{g_T\hat{q}} V_{\hat{q}\hat{q}}^{-1} (\hat{q}-\mu_{\hat{q}}) \\ \hline \underline{0} \end{bmatrix} \tag{A.1.15}$$

From this relation it now follows that if

$$\text{a)} \quad \mu_{g_T} = 0 \tag{A.1.16}$$

and

$$\text{b)} \quad \hat{q} = \mu_{\hat{q}} \tag{A.1.17}$$

then given $\hat{q}$, the separation technique analysis estimates are asymptotically unbiased, i.e.

$$E[(p-\hat{p})|\hat{q}] = 0$$

or equivalently

$$E[\hat{P}(z^{-1})|\hat{Q}(z^{-1})] = P(z^{-1}) \tag{A.1.18}$$

The condition (A.1.16) is asymptotically satisfied if the disturbance x_t arises from an auto-regressive process of order $(\bar{m}-1)$, since in this case $g_T = 0$ and hence

$$\mu_{g_T} = 0 \tag{A.1.19}$$

On the other hand, under the assumption that $(\bar{m}-1)$ coefficients of $\hat{R}(z^{-1})$ are consistent

$$\lim_{N \to \infty} \hat{r}_i = \lim_{N \to \infty} E\hat{r}_i \quad i=1,2,\ldots,\bar{m}-1 \tag{A.1.20}$$

But from definition

$$\hat{q}_i = \hat{r}_i \quad i=1,2,\ldots,\bar{m}-1 \tag{A.1.21}$$

and hence

$$\lim_{N \to \infty} \hat{q}_i = \lim_{N \to \infty} E\hat{q}_i$$

or equivalently

$$\hat{q} = \mu_{\hat{q}} \qquad \text{(A.1.22)}$$

thus the second condition (A.1.17) is also satisfied asymptotically. Hence (A.1.19) and (A.1.22) in connection with (A.1.15) demonstrate that if the disturbance arises from an auto-regressive process of order $(\bar{m}-1)$ then the separation technique analysis gives asymptotically unbiased estimates provided that $(\bar{m}-1)$ coefficients of $\hat{R}(z^{-1})$ are consistent.

APPENDIX II

PROPERTY OF THE STATIONARY SOLUTION CORRESPONDING TO THE MODIFIED PREDICTOR

In the case of the modified predictor $\bar{H}(z^{-1})$, the stationary output equation (3.11.3) becomes

$$y_{t,s} = \frac{1-\bar{H}(z^{-1})}{1-\hat{Q}(z^{-1})}\,\hat{\zeta}_t \tag{A.2.1}$$

where, corresponding to the auto-regressive model $[1-\hat{Q}(z^{-1})]$, the modified predictor is given by (3.7.11) as

$$\bar{H}(z^{-1}) = [-\hat{\Phi}(z^{-1}) + \hat{S}_T(z^{-1})][1-\hat{Q}(z^{-1})] \tag{A.2.2}$$

where

$$1 + \hat{S}(z^{-1}) = [1-\hat{Q}(z^{-1})]^{-1}$$

and

$$\hat{\Phi}(z^{-1}) = -\frac{[1+\hat{S}(z^{-1})-\hat{S}_T(z^{-1})]\big|_{z=1}}{\mu} \sum_{i=k}^{k+\mu-1} z^{-i} \tag{A.2.3}$$

Substituting of $\bar{H}(z^{-1})$ into (A.2.1), the output in the stationary case is

$$y_{t,s} = [1+\hat{S}(z^{-1})-\hat{S}_T(z^{-1}) + \hat{\Phi}(z^{-1})]\,\hat{\zeta}_t \tag{A.2.4}$$

Suppose now that the process dynamics were known and the modified predictor was used. In this situation the output of the Box and Jenkins control system would be, from (3.2.17)

$$y_t = [1-\bar{H}^*(z^{-1})]\,x_t \tag{A.2.5}$$

where, for the disturbance model

$$x_t = \frac{1}{1-G(z^{-1})}\,\hat{\zeta}_t \tag{A.2.6}$$

The corresponding modified predictor is

$$\bar{H}^{*}(z^{-1}) = [-\Phi(z^{-1})+S_T(z^{-1})][1-G(z^{-1})] \qquad (A.2.7)$$

where

$$1+S(z^{-1}) = [1-G(z^{-1})]^{-1}$$

and

$$\Phi(z^{-1}) = -\frac{[1+S(z^{-1})-S_T(z^{-1})]\big|_{z=1}}{\mu} \sum_{i=k}^{k+\mu-1} z^{-i} \qquad (A.2.8)$$

Thus using (A.2.6) and (A.2.7)

$$y_t = [1+S(z^{-1})-S_T(z^{-1})+\Phi(z^{-1})]\ \hat{\zeta}_t \qquad (A.2.9)$$

But it is known from (3.11.9) that

$$1+S(z^{-1})-S_T(z^{-1}) = 1+\hat{S}(z^{-1}) - \hat{S}_T(z^{-1}) \qquad (A.2.10)$$

which implies, from (A.2.3) and (A.2.8), that

$$\hat{\Phi}(z^{-1}) = \Phi(z^{-1}) \qquad (A.2.11)$$

Consequently (A.2.10) and (A.2.11) show that (A.2.4) and (A.2.9) are identical equations. This demonstrates the optimum nature of the stationary solution corresponding to the modified predictor.

APPENDIX III

UPDATING THE MODIFIED PREDICTOR

Corresponding to the auto-regressive model $[1-\hat{R}_i(z^{-1})]$ of the i^{th} iteration, the modified predictor $\bar{H}(z^{-1})$ of eqn. (3.7.11) takes the form

$$\bar{H}_i(z^{-1}) = [-\hat{\Phi}_i(z^{-1})+\hat{\nu}_{i,T}(z^{-1})][1-\hat{R}_i(z^{-1})] \tag{A.3.1}$$

where

$$1+\hat{\nu}_i(z^{-1}) = [1-R_i(z^{-1})]^{-1}$$

and

$$\hat{\Phi}_i(z^{-1}) = -\frac{[1+\hat{\nu}_i(z^{-1}) - \hat{\nu}_{i,T}(z^{-1})]\big|_{z=1}}{\mu} \sum_{i=k}^{k+\mu-1} z^{-i} \tag{A.3.2}$$

Substitution of $\bar{H}_i(z^{-1})$ into (4.6.4) and solving for $V(z^{-1})$ gives

$$V(z^{-1}) \simeq \frac{[1+\hat{\nu}_{i-1}(z^{-1})]\{1-[1-Q(z^{-1})][1+\hat{\nu}_i(z^{-1})]\}}{[1+\hat{\nu}_i(z^{-1})][1-Q(z^{-1})][-\hat{\Phi}_{i-1}(z^{-1})+\hat{\nu}_{i-1,T}(z^{-1})]} \tag{A.3.3}$$

and in the stationary case (A.3.3) becomes

$$V(z^{-1}) \simeq \frac{1-[1-Q(z^{-1})][1+\hat{\nu}_s(z^{-1})]}{[1-Q(z^{-1})][-\hat{\Phi}_s(z^{-1})+\hat{\nu}_{s,T}(z^{-1})]} \tag{A.3.4}$$

Under the stationary solution conditions the output can now be written from (4.3.17) as

$$y_{t,s} \simeq \frac{1 - \bar{H}_s(z^{-1})}{1 + V(z^{-1})\,\bar{H}_s(z^{-1})}\, x_t \tag{A.3.5}$$

where

$$\bar{H}_s(z^{-1}) = [-\hat{\Phi}_s(z^{-1})+\hat{\nu}_{s,T}(z^{-1})][1+\hat{\nu}_s(z^{-1})]^{-1} \tag{A.3.6}$$

and $V(z^{-1})$ is given by (A.3.4). Substituting (A.3.4) and (A.3.6) into (A.3.5) and using the disturbance representation

$$x_t = \frac{1}{1-Q(z^{-1})}\,\zeta_t \tag{A.3.7}$$

it can be shown that $y_{t,s}$ becomes

$$y_{t,s} \simeq [1+\hat{\nu}_s(z^{-1}) - \hat{\nu}_{s,T}(z^{-1}) + \hat{\Phi}_s(z^{-1})]\,\zeta_t \tag{A.3.8}$$

Consider now the output corresponding to the modified predictor in the case of known process dynamics. Taking $V(z^{-1})=0$ in (4.3.17) and using (A.3.7), this can be written as

$$y_t = [1-\bar{H}^*(z^{-1})]\,\frac{1}{1-Q(z^{-1})}\,\zeta_t \tag{A.3.9}$$

where

$$\bar{H}^*(z^{-1}) = [-\Phi(z^{-1})+S_T(z^{-1})][1-Q(z^{-1})] \tag{A.3.10}$$

where

$$1+S(z^{-1}) = [1-Q(z^{-1})]^{-1}$$

and

$$\Phi(z^{-1}) = -\frac{[1+S(z^{-1})-S_T(z^{-1})]\big|_{z=1}}{\mu}\sum_{i=k}^{k+\mu-1} z^{-i} \tag{A.3.11}$$

Substituting of $\bar{H}^*(z^{-1})$ from (A.3.10), the output equation (A.3.9) becomes

$$y_t = [1+S(z^{-1})-S_T(z^{-1})+\Phi(z^{-1})]\,\zeta_t \tag{A.3.12}$$

But from (4.6.4), in the stationary case

$$[1+\hat{\nu}_s(z^{-1})]^{-1} = 1-\hat{R}_s(z^{-1}) \simeq [1-Q(z^{-1})][1+\bar{H}_s(z^{-1})V(z^{-1})]$$

or

$$1+\hat{\nu}_s(z^{-1}) \simeq [1+S(z^{-1})][1+\bar{H}_s(z^{-1})\,V(z^{-1})]^{-1} \tag{A.3.13}$$

where applying k^{th} order front-end truncation and subtracting the result from (A.3.13) it can be shown that

$$1+\hat{\nu}_s(z^{-1})-\hat{\nu}_{s,T}(z^{-1}) = 1+S(z^{-1})-S_T(z^{-1}) \qquad (A.3.14)$$

which, from (A.3.2) and (A.3.11), implies

$$\hat{\Phi}_s(z^{-1}) = \Phi(z^{-1}) \qquad (A.3.15)$$

Consequently (A.3.14) and (A.3.15) show that (A.3.8) and (A.3.12) are approximately identical equations. This demonstrates that if the modified predictor is used in successive updatings of the predictor updating procedure then the stationary solution is approximately optimum in the sense that the stationary output $y_{t,s}$ is equivalent to the prediction error corresponding to the modified predictor.

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ABSTRACT

Identification requires assumptions about the unknown disturbance. For an open loop identification experiment, the disturbance and the input are physically independent. Therefore the most reasonable assumption is one of statistical independence.

The estimation technique presented in the first part of the thesis exploits the statistical independence to reduce parameter estimation errors. The resulting algorithm is identical to that of the two stage least squares method [30]. Nevertheless it is believed that the original aspect of the approach is the treatment of the disturbance.

In practice it is often desirable to avoid open loop experimentations due to economic and safety restrictions. In the second part of the thesis, the identifiability problem for the Box and Jenkins feedback control system is re-examined to extend the previous work of Turtle [52]. Based on a new estimation error equation, a self-adaptive optimization procedure is proposed. However due to doubts concerning stability and convergence, the procedure is not, at the moment, sufficiently well-developed for practical applications.

The final part of the thesis investigates the possibility of estimating the process and disturbance dynamics by use of an external perturbation signal while the process is operating under the Box and Jenkins control system. It is shown that a correlation analysis can produce consistent estimates. However for a limited amount of data these estimates are, in general, biased and hence do not always produce an optimum control system. Nevertheless further improvement can be achieved by applying the predictor updating procedure of Turtle [52] after the correlation analysis.