LONG PERIOD AND SEMI-DIURNAL

TIDAL OSCILLATIONS

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by

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CONTENTS

ERRATA

p 12	5th line down delete "and α"	
p 67	Bottom equation	K to read as κ
p 69	Equation (6)	K to read as κ
p 70	Equation (9)	K to read as κ
p 71	Equation (10)	K to read as κ
p 72	Equation (11)	K to read as κ
p 171	Equation (7)	z to read as z^2

Figures 4.2, 4.3 κ to read as $\kappa \times 10^3$

CONTENTS

1 INTRODUCTION

In this study we shall be concerned with the small oscillations of a canal-like body of water under the influence of a periodic disturbance in the gravitational field - such as is produced by the action of the moon. The purpose of the work is to explore the significance of the different parts of the Coriolis force in determining the nature of the solution, and to contrast the findings with those derived from Laplace's equations. We shall therefore find it appropriate to begin this discussion with a review of the assumptions underlying Laplace's equations, and of the subsequent comments which led to the formulation of the present problem.

Laplace considered the tidal oscillations of an ocean of comparatively small depth covering a rotating globe. The motion of this fluid is mainly horizontal, which led him to propose that the vertical acceleration of the fluid particles be ignored. Under these circumstances the corresponding equation of motion is reduced to the hydrostatic pressure law. Consequently, the dynamic pressure may be replaced throughout the equations by the elevation of the free surface ζ, multiplied by a suitable constant.

In the continuity equation it then follows that the vertical velocity is linearly related to the height above the ocean bottom, with the velocity at the free surface, $\frac{\partial \zeta}{\partial t}$, determining the constant of proportionality. Elsewhere, the small vertical component of velocity and all non-linear terms could be regarded as negligible.

By this process, Laplace was led to formulate a set of three linear differential equations in three unknowns; namely the two horizontal velocity components and ζ. These he took as the governing equations for the flow.

Their subsequent application has been widespread to the many particular fluid domains of interest to workers on the dynamical theory of the tides.

However, Laplace's equations were not without criticism. Firstly, the neglect of vertical acceleration will depend on whether significant variations in this direction can occur over a distance shorter than or equal to the depth of the ocean. Such is the case when the motion takes the form of "cellular oscillations", ie periodic changes in the velocity components with increasing depth, including associated reversals of sign. [2] established that such motions are permissible for a tidal constituent of period $2\pi/\sigma$ such that

$$\sigma^2 < 4\omega^2$$

where ω is the angular speed of the earth's rotation. The main force of this criticism referred to the diurnal constituents for which σ^2 is near ω^2.

Furthermore, both [3] and [4], on retaining vertical acceleration in the particular examples which they studied, found that the nature of the motion depended in a fundamental way on the sign of $1 - 4\omega^2/\sigma^2$. This led [3] to propose that solutions differing greatly from those obtained via Laplace's equations would result when $\sigma^2 < 4\omega^2$. Also [4] pointed out that for the semi-diurnal constituent, in which $\sigma^2 = 4\omega^2$, Laplace's equations did not appear to be valid in the case of a flat circular basin of uniform depth.

These various criticisms were taken up by Proudman [5]. Using Solberg's equations [3] or a simplified form of them in cylindrical co-ordinates, he examined the solution for a number of fluid domains. His findings for the different tidal constituents were then contrasted with solutions derived from Laplace's equations. The results of this work indicated that for two particular fluid domains Laplace's equations were not always valid, but that elsewhere the necessary correction would only be slight. The two domains in question were the circular sea of uniform depth near the North Pole and a broad channel of uniform depth near the equator. In the former of these, Proudman found that the case of failure was that of the semi-diurnal constituent as, indeed, had [4]. In the latter it was the long-period constituent, where cellular oscillations were found to occur.

These same two regions are the subject of the present study, with the emphasis on solutions where the necessary correction should only be small. Thus, among other things, Proudman's work has indicated that high accuracy is obtained from Laplace's equations for the semi-diurnal constituent in a broad equatorial canal. A similar statement would hold for a canal near the North Pole when considering the long-period constituent. However, in these two regions, Proudman's remarks were based on simplified equations such that the description of vertical acceleration neglects the vertical part of the Coriolis force. Similarly, the vertical velocity term was also absent from the horizontal part of the Coriolis force. In the present study these two terms are retained, as they were by Solberg, and spherical polar co-ordinates are used throughout. A detailed examination is then made of the long-period constituent near the North Pole and the semi-diurnal constituent near the equator. The findings from these more general equations are then examined to see if Proudman's conclusions can be verified or not.

The conclusions of the present work are distributed at the ends of the relevant sections.

2. THE GOVERNING EQUATIONS

2.1 The Tide-Generating Force

In this section we shall develop an expression for the variations in gravitational potential, at an arbitrary fixed point on the earth's surface, associated with the apparent motions of a neighbouring body such as the moon.

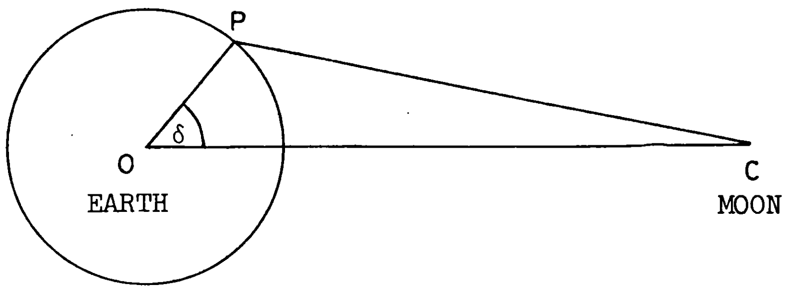


Figure 2.1

Thus, with reference to Fig 2.1, let O and C be the centres of the earth and moon respectively. Let the distance OC be denoted by D and let the radius of the earth be denoted by r. Now the potential at P due to the moon's attraction is given by $-\gamma M/CP$ where M denotes the moon's mass and γ is the gravitational constant. We may rewrite this potential as

$$\frac{-\gamma M}{(D^2 - 2rD\cos\delta + r^2)^{\frac{1}{2}}} \qquad \ldots\ldots (1)$$

However, part of the gravitational force field acts to accelerate the whole mass of the earth parallel to OC. The value of the acceleration is $\gamma M/D^2$ so, evidently, a uniform force field of intensity $\gamma M/D^2$ is the essential part devoted to this motion. The associated potential at P of this force field is

$$\frac{-\gamma M}{D^2} r \cos \delta \qquad \text{..... (2)}$$

As our concern is only with motions relative to the earth's surface, we must subtract this component from (1) whence we obtain an expression for the potential of the relative attraction at P which we shall denote by Ω, ie

$$\Omega = \frac{-\gamma M}{(D^2 - 2rD \cos \delta + r^2)^{\frac{1}{2}}} + \frac{\gamma M}{D^2} r \cos \delta$$

$$= \frac{-\gamma M}{D}\left(1 - \frac{2r}{D} \cos \delta + \frac{r^2}{D^2}\right)^{-\frac{1}{2}} + \frac{\gamma M}{D^2} r \cos \delta$$

$$= \frac{-\gamma M}{D}\left(1 + \frac{r}{D} \cos \delta - \frac{1}{2}\frac{r^2}{D^2} + \frac{3}{2}\frac{r^2}{D^2} \cos^2 \delta - \ldots\right) +$$

$$+ \frac{\gamma M}{D^2} r \cos \delta$$

$$= \frac{-\gamma M}{D}\left(1 - \frac{1}{2}\frac{r^2}{D^2} + \frac{3}{2}\frac{r^2}{D^2} \cos^2 \delta - \cdot\cdot\right)$$

Writing $\Omega_o = \Omega + \gamma M/D$ which is such that zero potential is at the earth's centre, we obtain

$$\Omega_o = \frac{3}{2} \gamma M \frac{r^2}{D^3}\left(\frac{1}{3} - \cos^2 \delta\right) \qquad \text{..... (3)}$$

Now, for the fixed point P on the earth's surface, the angle δ varies with time due to the motions of the earth and the moon. The principal source of variation is, of course, the earth's rotation. However, variations in δ are more precisely related to the moon's hour angle measured from some fixed meridian.

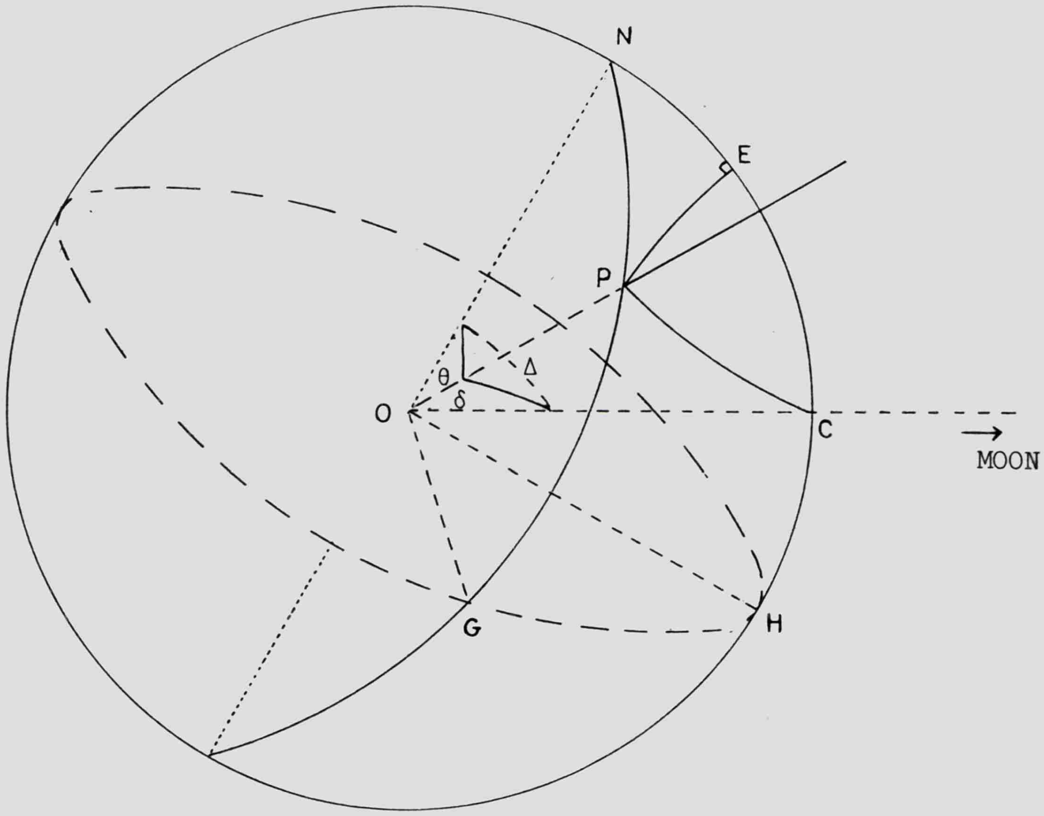


Figure 2.2

To analyse the character of this relationship, let θ be the co-latitude and ϕ the longitude of the point P (where longitude is measured eastward from some fixed meridian). Let Δ be the north-polar distance of the moon (see Fig 2.2) and let α be the hour angle of the moon measured west of the same fixed meridian. Then, in Fig 2.2.

$$N\hat{O}C = \Delta$$

$$H\hat{O}G = \alpha + \phi$$

$$N\hat{O}P = \theta$$

$$P\hat{O}C = \delta$$

Let PE be an arc of the great circle which intersects the longitude arc NC at right angles at E. By this construction we obtain two right spherical triangles PNE and PCE. Call

$$P\hat{O}E = x$$

$$N\hat{O}E = y$$

We now make use of some standard results in spherical trigonometry which can be summarized as follows

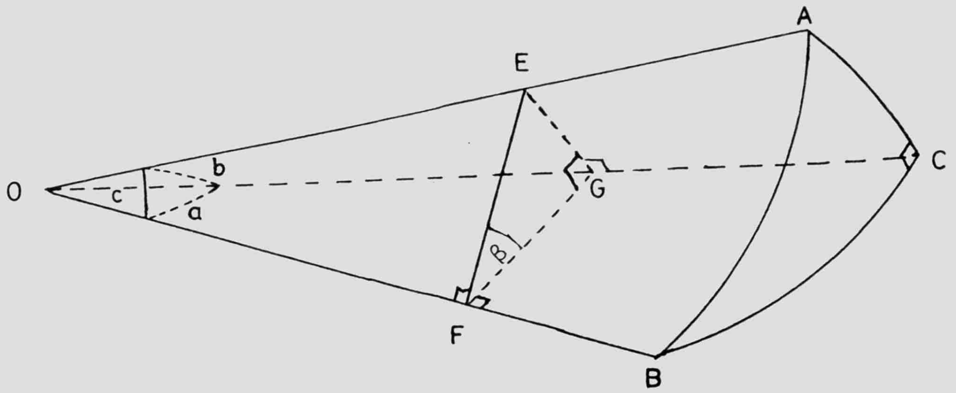


Figure 2.3

In Fig 2.3, ABC is a right spherical triangle on a sphere of arbitrary radius. By considering the right angled triangles EFG, EFO, EGO, GFO, the following results are easily obtained:

$$\left.\begin{aligned}\sin b &= \sin c \sin \beta \\ \cos c &= \cos a \cos b \\ \cos \beta &= \tan a \cot c \\ \sin a &= \tan b \cot \beta\end{aligned}\right\} \qquad \ldots\ldots (4)$$

Thus, consider the right spherical triangle NPE in Fig 2.2. Using the second of the formulae (4) we have

$$\cos \theta = \cos x \cos y \qquad \ldots\ldots (5)$$

and using the fourth formula in (4) we have

$$\sin y = \tan x \cot (\alpha + \phi) \qquad \ldots\ldots (6)$$

Consider, furthermore, the right spherical triangle CPE. Again, from the second of formulae (4) we obtain

$$\begin{aligned}\cos \delta &= \cos x \cos (\Delta - y) \\ &= \cos x (\cos \Delta \cos y + \sin \Delta \sin y) \\ &= \cos \Delta \cos x \cos y + \sin \Delta \sin y \cos x \qquad \ldots\ldots (7)\end{aligned}$$

Then, eliminating cos x in the first of the terms of equation (7) by using (5) we obtain

$$\cos \delta = \cos \Delta \cos \theta + \sin \Delta \sin y \cos x \qquad \ldots\ldots (8)$$

Now eliminating sin y from the second of the terms of equation (8) by using (6) we obtain

$$\cos \delta = \cos \Delta \cos \theta + \sin \Delta \tan x \cot (\alpha + \phi) \cos x$$

$$= \cos \Delta \cos \theta + \sin \Delta \sin x \cot (\alpha + \phi) \qquad \ldots\ldots (9)$$

However, going back to the right spherical triangle NPE and using the first of formulae (4), we have

$$\sin x = \sin \theta \sin (\alpha + \phi) \qquad \ldots\ldots (10)$$

Hence, eliminating sin x from the second of the terms of equation (9) by using (10), we obtain

$$\cos \delta = \cos \Delta \cos \theta + \sin \Delta \sin \theta \cos (\alpha + \phi) \qquad \ldots\ldots (11)$$

Thus δ is expressed in terms of the hour angle α, the north-polar distance Δ, the co-latitude θ and the longitude ϕ. If we substitute from expression (11) for cos δ into our expression (3) for Ω_o we obtain

$$\Omega_o = \frac{3}{2} \gamma M \frac{r^2}{D^3} \left(\frac{1}{3} - [\cos \Delta \cos \theta + \sin \Delta \sin \theta \cos (\alpha + \phi)]^2 \right)$$

$$= \frac{3}{2} \gamma M \frac{r^2}{D^3} \left(\frac{1}{3} - \cos^2 \Delta \cos^2 \theta - \frac{1}{2} \sin 2\Delta \sin 2\theta \cos (\alpha + \phi) - \right.$$

$$\left. - \sin^2 \Delta \sin^2 \theta \cos^2 (\alpha + \phi) \right)$$

$$= \frac{3}{2} \gamma M \frac{r^2}{D^3} \left(\frac{1}{3} - \cos^2 \Delta \cos^2 \theta - \frac{1}{2} \sin 2\Delta \sin 2\theta \cos (\alpha + \phi) - \right.$$

$$\left. - \sin^2 \Delta \sin^2 \theta \left[\frac{\cos 2 (\alpha + \phi) + 1}{2} \right] \right)$$

$$= \frac{3}{2}\,\gamma M\,\frac{r^2}{D^3}\left(\frac{1}{3} - \cos^2\Delta\,\cos^2\theta - \frac{1}{2}\sin 2\Delta\,\sin 2\theta\,\cos(\alpha+\phi) - \right.$$

$$- \frac{1}{2}\sin^2\Delta\,\sin^2\theta\,\cos 2(\alpha+\phi) -$$

$$\left. - \frac{1}{2}(1-\cos^2\Delta)(1-\cos^2\theta)\right)$$

$$= \frac{3}{2}\,\gamma M\,\frac{r^2}{D^3}\left(\frac{1}{3} - \cos^2\Delta\,\cos^2\theta - \frac{1}{2} + \frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\Delta - \right.$$

$$- \frac{1}{2}\cos^2\Delta\,\cos^2\theta - \frac{1}{2}\sin 2\Delta\,\sin 2\theta\,\cos(\alpha+\phi) -$$

$$\left. - \frac{1}{2}\sin^2\Delta\,\sin^2\theta\,\cos 2(\alpha+\phi)\right)$$

Hence, if we let

$$H = -\frac{3}{2}\,\gamma M\,\frac{r^2}{D^3} \qquad \ldots\ldots (12)$$

we obtain

$$\Omega_0 = \frac{3}{2}H\left(\cos^2\Delta - \frac{1}{3}\right)\left(\cos^2\theta - \frac{1}{3}\right) +$$

$$+ \frac{1}{2}H\sin 2\Delta\,\sin 2\theta\,\cos(\alpha+\phi) + \qquad \ldots\ldots (13)$$

$$+ \frac{1}{2}H\sin^2\Delta\,\sin^2\theta\,\cos 2(\alpha+\phi)$$

Each of the terms of (13) may be regarded as representing a partial tide, and the results superposed.

For any given point P, both θ and ϕ will have specific values in the above expression. However, taking into account the detailed motions of the earth and the moon, Δ and α will vary with time in a rather complicated way. Without entering into this complexity, it may be noted that the variations of Δ and α will be of long period, in which case it is clear that the expansion of (13) into a series of simple harmonic functions of time will give rise to terms of three distinct types.

First, we have the tides of long period, for which

$$\Omega_o = K'\left(\cos^2\theta - \frac{1}{3}\right)\cos(\sigma t + \varepsilon) \qquad \ldots\ldots (14)$$

where K' is a constant. Laplace has called these tides the 'Oscillations of the First Species', the most important being the 'lunar fortnightly' where, in degrees per mean solar hour, $\sigma = 1^\circ.098$/hr and the 'solar annual' where $\sigma = 0^\circ.082$/hr.

Secondly, we have the diurnal tides, for which

$$\Omega_o = K''\sin\theta\cos\theta\cos(\sigma t + \phi + \varepsilon) \qquad \ldots\ldots (15)$$

where K'' is a constant and where σ differs but little from the angular velocity of the earth's rotation. Laplace called these tides the "Oscillations of the Second Species" and they include the "lunar diurnal" where $\sigma = 13^\circ.943$/hr and the "solar diurnal" where $\sigma = 14^\circ.959$/hr.

Finally, we have the semi-diurnal tides, for which

$$\Omega_o = K''' \sin^2\theta \cos(\sigma t + 2\phi + \varepsilon) \qquad \text{..... (16)}$$

where K''' is a constant and where σ takes values close to twice the earth's angular velocity. Laplace called these tides the "Oscillations of the Third Species" and they include the 'lunar semi-diurnal' where $\sigma = 28^\circ.984$/hr the 'solar semi-diurnal' $\sigma = 30^\circ$/hr and the 'luni-solar semi-diurnal' where σ is exactly equal to twice the earth's angular velocity, ie $\sigma = 30^\circ.082$/hr.

2.2 The Governing Equations

In association with an arbitrary point P on or above the earth's surface, let (θ^*, ϕ^*, R^*) be co-ordinates representings its co-latitude, longitude and radial distance measured outward from the earth's centre. This then defines a system of spherical polar co-ordinates rotating steadily with the earth's angular velocity about the polar axis.

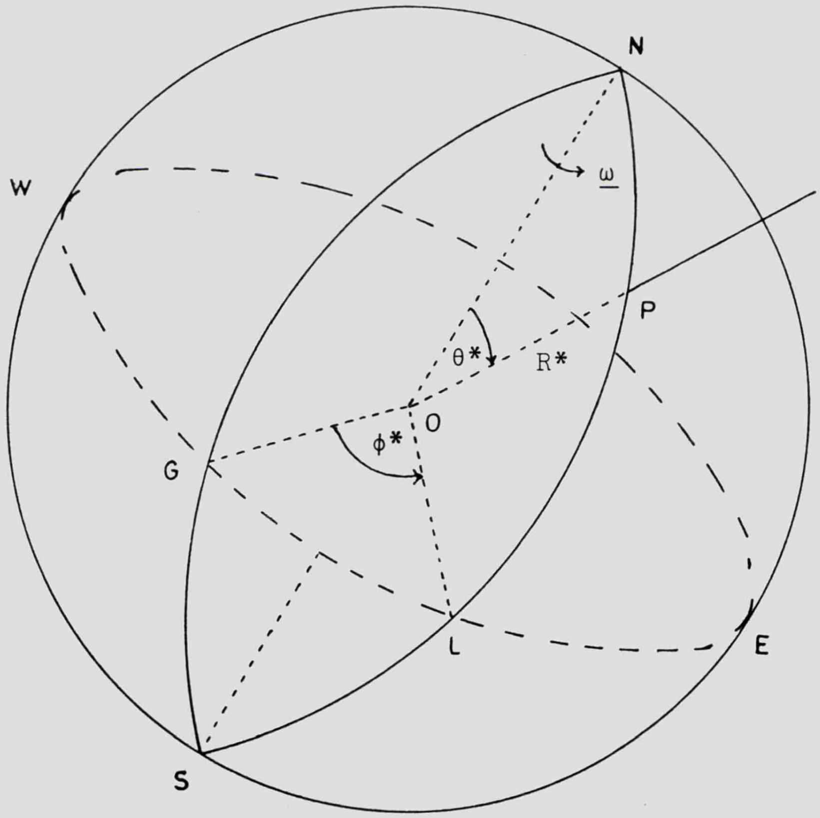


Figure 2.4

Thus, in Fig 2.4, ON is the north polar axis and $\underline{\omega}$ denotes the earth's angular velocity which has magnitude equal to ω and direction $\overrightarrow{ON}$. The meridian NGS is the reference meridian from which longitude is measured eastward.

In this co-ordinate system, the equations governing the motion of an incompressible viscous fluid are those stating the conservation of mass and momentum, ie

$$\nabla \cdot \underline{q}^* = 0 \qquad \qquad ..... (1)$$

$$\frac{\partial}{\partial t^*} \underline{q}^* + \underline{q}^* \cdot \nabla \underline{q}^* + 2\underline{\omega} \times \underline{q}^* + \underline{\omega} \times (\underline{\omega} \times \underline{r}^*)$$
$$= -\frac{1}{\rho} \nabla P^* + \underline{F} - \nu \nabla \times (\nabla \times \underline{q}^*) \qquad \qquad ..... (2)$$

where $\underline{q}^*$ is the particle velocity and $\underline{r}^*$, t^*, P^*, ρ, ν, $\underline{F}$ represent respectively the position vector, time, pressure, density, kinematic viscosity and body force per unit mass. The body force will be assumed to be conservative, taking the form

$$\underline{F} = -\nabla \Omega_o - \nabla(R^* g) \qquad \qquad ..... (3)$$

where Ω_o is the driving potential of one of the partial tides described earlier.

It is convenient to introduce in these equations a modified pressure function $\hat{p}$ defined as follows

$$\hat{p} = P^* + \rho(\Omega_o + R^*g) - \frac{1}{2}\rho(\underline{\omega} \times \underline{r}^*) \cdot (\underline{\omega} \times \underline{r}^*) \qquad \ldots\ldots (4)$$

which allows equation (2) to be written in the form

$$\frac{\partial}{\partial t^*}\underline{q}^* + \underline{q}^* \cdot \nabla \underline{q}^* + 2\underline{\omega} \times \underline{q}^*$$

$$= -\frac{1}{\rho}\nabla\hat{p} - \nu\nabla \times (\nabla \times q^*) \qquad \ldots\ldots (5)$$

The system of equations (1), (3), (4) and (5) then defines the vector form of the governing equations.

Turning to the component form, let us suppose

$$\underline{q}^* = u^*\hat{\underline{\theta}} + v^*\hat{\underline{\phi}} + w^*\hat{\underline{R}} \qquad \ldots\ldots (6)$$

where $\hat{\underline{\theta}}$, $\hat{\underline{\phi}}$, $\hat{\underline{R}}$ are the unit vectors in the directions of increasing θ^*, ϕ^*, R^* respectively. The component form of equation (5) is then

$$\frac{\partial u^*}{\partial t^*} + \frac{u^*}{R^*}\frac{\partial u^*}{\partial \theta^*} + \frac{v^*}{R^* \sin\theta^*}\frac{\partial u^*}{\partial \phi^*} + w^*\frac{\partial u^*}{\partial R^*} + \frac{w^*u^*}{R^*} - \frac{v^{*2}}{R^*}\cot\theta^* -$$

$$- 2\omega v^*\cos\theta^* = -\frac{1}{\rho R^*}\frac{\partial \hat{p}}{\partial \theta^*} + \nu\left(\nabla^2 u^* + \frac{2}{R^{*2}}\frac{\partial w^*}{\partial \theta^*} - \frac{u^*}{R^{*2}\sin^2\theta^*} - \frac{2\cos\theta^*}{R^{*2}\sin^2\theta^*}\frac{\partial v^*}{\partial \phi^*}\right) \qquad \ldots\ldots (7)$$

$$\frac{\partial v^*}{\partial t^*} + \frac{u^*}{R^*}\frac{\partial v^*}{\partial \theta^*} + \frac{v^*}{R^* \sin\theta^*}\frac{\partial v^*}{\partial \phi^*} + w^*\frac{\partial v^*}{\partial R^*} + \frac{w^* v^*}{R^*} + \frac{u^* v^*}{R^*}\cot\theta^* +$$

$$+ 2\omega\, u^* \cos\theta^* + 2\omega\, w^* \sin\theta^* = -\frac{1}{\rho R^* \sin\theta^*}\frac{\partial \hat{p}}{\partial \phi^*} + \qquad \ldots\ldots (8)$$

$$+ \nu\left(\nabla^2 v^* - \frac{v^*}{R^{*2}\sin^2\theta^*} + \frac{2}{R^{*2}\sin\theta^*}\frac{\partial w^*}{\partial \phi^*} + \frac{2\cos\theta}{R^{*2}\sin^2\theta^*}\frac{\partial u^*}{\partial \phi^*}\right)$$

$$\frac{\partial w^*}{\partial t^*} + \frac{u^*}{R^*}\frac{\partial w^*}{\partial \theta^*} + \frac{v^*}{R^* \sin\theta^*}\frac{\partial w^*}{\partial \phi^*} + w^*\frac{\partial w^*}{\partial R^*} - \frac{(u^{*2} + v^{*2})}{R^*} -$$

$$- 2\omega v^* \sin\theta^* = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial R^*} + \nu\left(\nabla^2 w^* - \frac{2}{R^{*2}} w^* - \frac{2}{R^{*2}}\frac{\partial u^*}{\partial \theta^*} - \right. \qquad \ldots\ldots (9)$$

$$- \frac{2}{R^{*2}} u^* \cot\theta^* -$$

$$\left. - \frac{2}{R^{*2}\sin\theta^*}\frac{\partial v^*}{\partial \phi^*}\right)$$

where

$$\nabla^2 \equiv \frac{1}{R^{*2}\sin\theta^*}\frac{\partial}{\partial \theta^*}\left(\sin\theta^*\frac{\partial}{\partial \theta^*}\right) + \frac{1}{R^{*2}\sin^2\theta^*}\left(\frac{\partial^2}{\partial \phi^{*2}}\right) + \qquad \ldots\ldots (10)$$

$$+ \frac{1}{R^{*2}}\frac{\partial}{\partial R^*}\left(R^{*2}\frac{\partial}{\partial R^*}\right)$$

Furthermore, the continuity equation is given by

$$\frac{1}{R^* \sin\theta^*}\frac{\partial}{\partial \theta^*}(u^* \sin\theta^*) + \frac{1}{R^* \sin\theta^*}\frac{\partial v^*}{\partial \phi^*} + \qquad \ldots\ldots (11)$$

$$+ \frac{1}{R^{*2}}\frac{\partial}{\partial R^*}(R^{*2} w^*) = 0$$

We now proceed to simplify these equations as follows. First, let the radius of the earth be denoted by r and the mean height of the ocean be h. Let U and W denote, respectively, the typical horizontal and vertical speeds of flow measured relative to the surface of the earth. We may then introduce a non-dimensional scheme of variables defined by

$$\left.\begin{aligned} R^* &= r + hR & u^* &= Uu \\ \theta^* &= \theta & v^* &= Uv \\ \phi^* &= \phi & w^* &= Ww \\ t^* &= t/\omega & \hat{p} &= \rho\omega Ur\, p \end{aligned}\right\} \qquad \ldots\ldots (12)$$

which allows equations (7), (8), (9) and (11) to be reduced to the dimensionless form

$$\frac{\partial u}{\partial t} + \frac{\varepsilon}{1+\beta R}\left(u\frac{\partial u}{\partial \theta} + \frac{v}{\sin\theta}\frac{\partial u}{\partial \phi} + \lambda_1 (1+\beta R)\, w\frac{\partial u}{\partial R} + \alpha w u - v^2 \cot\theta\right) - 2v\cos\theta$$
$$= -\frac{1}{1+\beta R}\frac{\partial p}{\partial \theta} + E\left(\nabla_1{}^2 u + \frac{2\lambda_2}{(1+\beta R)^2}\frac{\partial w}{\partial \theta} - \frac{\beta^2}{(1+\beta R)^2}\frac{u}{\sin^2\theta} - \frac{2\beta^2\cos\theta}{(1+\beta R)^2\sin^2\theta}\frac{\partial v}{\partial \phi}\right) \qquad \ldots\ldots (13)$$

$$\frac{\partial v}{\partial t} + \frac{\varepsilon}{1+\beta R}\left(u\frac{\partial v}{\partial \theta} + \frac{v}{\sin\theta}\frac{\partial v}{\partial \phi} + \lambda_1 (1+\beta R)\, w\frac{\partial v}{\partial R} + \alpha w v + uv\cot\theta\right) +$$

$$+ 2u \cos\theta + 2\alpha w \sin\theta = -\frac{1}{(1+\beta R)\sin\theta}\frac{\partial p}{\partial\phi} +$$

$$+ E\left(\nabla_1{}^2 v - \frac{\beta^2}{(1+\beta R)^2}\frac{v}{\sin^2\theta} + \right.$$

$$+ \frac{2\lambda_2}{(1+\beta R)^2\sin\theta}\frac{\partial w}{\partial\phi} + \qquad \ldots\ldots (14)$$

$$\left. + \frac{2\beta^2\cos\theta}{(1+\beta R)^2\sin^2\theta}\frac{\partial u}{\partial\phi}\right)$$

$$\alpha\frac{\partial w}{\partial t} + \frac{\alpha\varepsilon}{1+\beta R}\left(u\frac{\partial w}{\partial\theta} + \frac{v}{\sin\theta}\frac{\partial w}{\partial\phi} + \lambda_1(1+\beta R)\,w\frac{\partial w}{\partial R} - \frac{u^2+v^2}{\alpha}\right) -$$

$$- 2v\sin\theta = -\frac{1}{\beta}\frac{\partial p}{\partial R} + E\left(\alpha\,\nabla_1{}^2 w - \frac{2\lambda_2}{(1+\beta R)^2}\,w - \right.$$

$$- \frac{2\beta^2}{(1+\beta R)^2}\frac{\partial u}{\partial\theta} - \frac{2\beta^2\,u\cot\theta}{(1+\beta R)^2} -$$

$$\left. - \frac{2\beta^2}{(1+\beta R)^2\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial\phi} + \frac{\alpha\sin\theta}{\beta(1+\beta R)}\frac{\partial}{\partial R}[(1+\beta R)^2 w] = 0 \qquad \ldots\ldots (16)$$

where

$$\nabla_1{}^2 \equiv \frac{\beta^2}{(1+\beta R)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\beta^2}{(1+\beta R)^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right) + \qquad \ldots\ldots (17)$$

$$+ \frac{1}{(1+\beta R)^2}\frac{\partial}{\partial R}\left[(1+\beta R)^2\frac{\partial}{\partial R}\right]$$

and where α, β, λ_1, λ_2, ε, E are dimensionless quantities defined as follows

$$\left.\begin{aligned} \alpha &= \frac{W}{U} & \beta &= \frac{h}{r} \\ \lambda_1 &= \frac{Wr}{Uh} & \lambda_2 &= \frac{Wh^2}{Ur^2} \end{aligned}\right\} \qquad \ldots\ldots (18)$$

$$\varepsilon = \frac{U}{\omega r} \qquad \ldots\ldots (19)$$

$$E = \frac{\nu}{\omega h^2} . \qquad \ldots\ldots (20)$$

We may regard these as six non-dimensional parameters whose magnitudes indicate the relative importance of the different terms.

The parameter $\varepsilon = U/\omega r$ is known as the Rossby number for the flow and multiplies all non-linear terms. To assess its magnitude we note that U will be typically of order 1 fps whilst, at the earth's surface, the basic rotation speed is of the order 1,000 fps. Hence ε is typically of order 10^{-3}, ie we can assume that

$$\varepsilon \ll 1 \qquad \ldots\ldots (21)$$

Non-linear terms in the governing equations may be neglected as a consequence, the motion being one of almost rigid rotation.

To estimate the magnitude of the Ekman number, E, we note that ν is of the order 10^{-5} ft^2/sec and ωh^2 is of the order 10^3 ft^2/uc. Hence E must be of the order 10^{-8} approximately. However, E multiplies the most highly differentiated terms in the governing equations and, to assess their importance, it is necessary to understand the behaviour within the boundary layers.

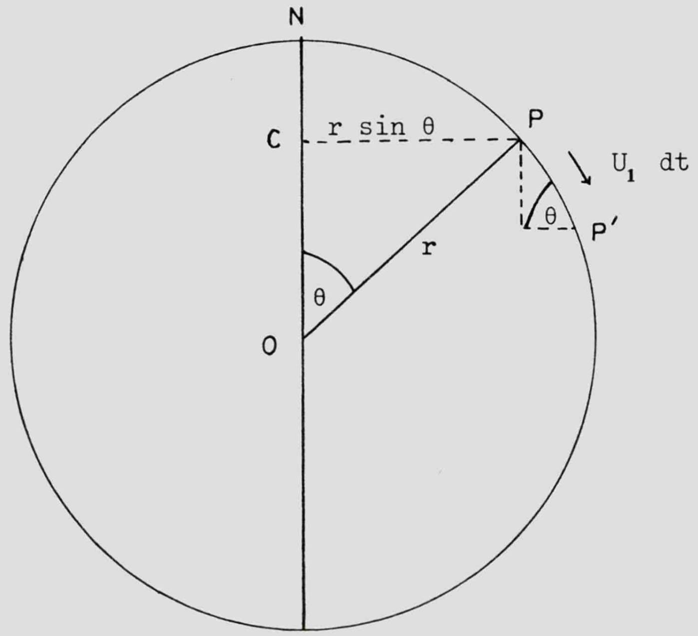


Figure 2.5

Thus, let us consider the Ekman layer at the ocean bottom where the tangential velocity U is brought to its proper boundary value by viscosity. A fluid particle rotating with the boundary layer at P (Fig 2.5) will be thrown outwards along PP' owing to the existence of centrifugal forces. In the process, angular momentum is gained which, ultimately, becomes imparted to the ocean body. This acts to counterbalance angular momentum taken from the ocean by tidal forces and imparted to the orbital motion of the moon.

The Ekman layer therefore acts as a source of angular momentum extracted from the rotating earth. Knowing the rate at which the earth is slowing down we may then estimate the vertical speed of fluid particles leaving the layer. Thus, let δ_L denote the depth of the layer. We shall consider a ring of fluid through P centre C of width $r d\theta$ and depth δ_L. As the ring moves to P' in time dt, its gain in angular momentum is $2\pi\omega\rho\delta_L U_1 r^3 \sin\theta \sin 2\theta \, d\theta \, dt$. Hence, for the entire layer, the total rate of change of angular momentum is

$$\frac{dI_L}{dt} = M_L \omega r U_1 \int_0^{\frac{\pi}{2}} \sin\theta \sin 2\theta \, d\theta = \frac{2}{3} M_L \omega r U_1 \qquad \text{..... (22)}$$

where $M_L = 4\pi\rho r^2 \delta_L$

Now, $d\omega/dt$ denotes the rate of change of the earth's angular velocity, so its rate of loss of angular momentum is given by

$$\frac{dI_E}{dt} = \frac{2}{5} M_E r^2 \frac{d\omega}{dt} \qquad \text{..... (23)}$$

where M_E is the mass of the earth. Equating the expressions (22) and (23) we obtain

$$U_1 \sim \frac{M_E r}{M_L \omega} \frac{d\omega}{dt} \qquad \text{..... (24)}$$

Now, M_E is approximately 10^{25} lbs and we may calculate the value of $4\pi\rho r^2$ to be approximately 10^{17} lbs/ft. From observations of the earth's rotation, the rate of change of angular velocity adds 1/1000 sec to the day every 100 years. Hence $d\omega/dt$ is approximately 10^{-21} rads/sec and therefore $U_1 \sim 1/100\,\delta_L$ fps. Conservation of mass then requires an average vertical velocity of 10^{-9} fps from the layer. If we assume that local variations are not too great, this clearly has a negligible effect on the interior flow.

At the edges of the continents, sidewall boundary layers will exist where the flow patterns can be quite complicated. However, without entering into this complexity, we shall assume that these also have a negligible effect on the interior flow so that terms involving E may be dropped from further consideration.[1]

With the typical dimensions met on earth, an outstanding feature of the flow system is the smallness of h compared with r. In such a system we can expect that a fluid particle will rise through the height h during the time that it takes to travel horizontally through the distance r. Consequently W must be small compared with U and we can write approximately

$$\frac{W}{U} \sim \frac{h}{r} \qquad \text{..... (25)}$$

With $h = 10^3$ ft and $r = 2 \times 10^7$ ft this gives $\alpha = W/U \sim 10^{-4}$ which suggests that vertical motion may be neglected. However, this ignores the effect of the coriolis term $w \sin \theta$ which, near the equator, takes on importance as the $u \cos \theta$ term (equation (14)) diminishes. Accordingly, we shall examine the three-dimensional nature of the motion and will assume the flow conditions are such that these terms may be of significance when

$$\beta \geqslant 10^{-4} \qquad \text{..... (26)}$$

1. However, this assumption is subject to further review in § 4.2 owing to the nature of the solutions obtained there.

for the present purposes.

The simplified equations are therefore as follows

$$\frac{\partial u}{\partial t} - 2v \cos \theta = - \frac{1}{1 + \beta R} \frac{\partial p}{\partial \theta} \qquad \text{..... (27)}$$

$$\frac{\partial v}{\partial t} + 2u \cos \theta + 2\alpha w \sin \theta = - \frac{1}{(1 + \beta R) \sin \theta} \frac{\partial p}{\partial \phi} \qquad \text{..... (28)}$$

$$\frac{\alpha \partial w}{\partial t} - 2v \sin \theta = - \frac{1}{\beta} \frac{\partial p}{\partial R} \qquad \text{..... (29)}$$

$$\frac{\partial}{\partial \theta} (u \sin \theta) + \frac{\partial v}{\partial \phi} + \qquad \text{..... (30)}$$

$$+ \frac{\alpha \sin \theta}{\beta(1 + \beta R)} \frac{\partial}{\partial R} [(1 + \beta R)^2 w] = 0$$

Finally, we need to consider the boundary conditions for the flow. The ocean bottom will be assumed to be the rigid spherical surface $R = 0$. Hence we must have

$$W = 0 \text{ on } R = 0 \qquad \text{..... (31)}$$

From angular momentum considerations we must also have

$$v \to 0 \text{ as } \theta \to 0, \pi \qquad \text{..... (32A)}$$

Furthermore, we shall assume that the ocean is contained by axially-symmetric walls of the type $G(\theta, R) = d$ where d is a constant and $G(\theta, R)$ is a given function of θ and R. The requirement that the normal velocity vanishes on the surface then gives

$$\frac{\beta u}{1 + \beta R} \frac{\partial G}{\partial \theta} + w \frac{\partial G}{\partial R} = 0 , \quad G(\theta, R) = d \qquad \text{.... (32B)}$$

At the free surface we shall assume that the pressure is uniform so that P^* is a constant. Hence

$$p - \Omega - \lambda(1 + \beta R) + \mu(1 + \beta R)^2 \sin^2\theta = \pi_o , \text{ say} \qquad \text{..... (33)}$$

defines the equation of the free surface, $R = R(\theta, \phi, t)$, where the dimensionless parameters λ, μ are given by

$$\left.\begin{aligned} \lambda &= \frac{g}{\omega U} \\ \mu &= \frac{\omega r}{2U} \end{aligned}\right\} \qquad \text{..... (34)}$$

and where Ω is the dimensionless driving potential given by

$$\Omega = \frac{\Omega_o}{\omega U r} \qquad \text{..... (35)}$$

As this surface must always consist of the same fluid particles, the general surface condition $dP^*/dt^* = 0$ holds. Hence we may write

$$\frac{\partial P^*}{\partial t^*} - \rho g w^* = 0$$

ignoring small terms. In the dimensionless form, this last condition becomes

$$\left(\frac{\partial p}{\partial t} - \frac{\partial \Omega}{\partial t} - kw\right)\Bigg|_{R=R(\theta,\phi,t)} = 0 \qquad \ldots\ldots (36)$$

where k is the dimensionless parameter given by

$$k = \frac{gW}{\omega^2 Ur} \qquad \ldots\ldots (37)$$

In the next chapter we shall examine the solution of these equations for the case of the semi-diurnal tide.

3. THE THREE-DIMENSIONAL TIDAL EQUATIONS IN THE CASE OF THE SEMI-DIURNAL TIDE

3.1 The Perturbation Equations

In this chapter we shall examine the solution of equations 2.2 (27)-(30) when the driving potential has a period of approximately 12 hours. Certain features of these equations make the approach a little easier if the period is exactly twice the earth's angular velocity [1]. Such is the case with the luni-solar semi-diurnal tide. However we may regard the period of other "Oscillations of the Third Species" as small departures from this value. Then, formally expanding each of the dependent variables as a perturbation series in the small term, we obtain a sequence of problems where the approach of [1] may be utilized. We present here the analysis of the zeroth-order and first-order equations.

It is convenient to introduce the change of variables

$$\left.\begin{aligned} \beta z &= 1 + \beta R \\ w = \alpha w \qquad \underline{P} &= \frac{1}{\beta} p \end{aligned}\right\} \qquad \ldots\ldots (1)$$

so that the governing equations become

$$\frac{\partial u}{\partial t} - 2v \cos\theta = -\frac{1}{z}\frac{\partial \underline{P}}{\partial \phi} \qquad \ldots\ldots (2)$$

$$\frac{\partial v}{\partial t} + 2u\cos\theta + 2w\sin\theta = -\frac{1}{z\sin\theta}\frac{\partial \underline{P}}{\partial \phi} \qquad \ldots\ldots (3)$$

$$\frac{\partial w}{\partial t} - 2v \sin\theta = -\frac{\partial \underline{P}}{\partial z} \qquad \ldots\ldots (4)$$

$$\frac{\partial}{\partial\theta}(u \sin\theta) + \frac{\partial v}{\partial\phi} + \frac{\sin\theta}{z}\frac{\partial}{\partial z}(z^2 w) = 0 \qquad \ldots\ldots (5)$$

We consider a disturbing potential which is of the form

$$\Omega = 2F(\theta, z) \exp\{-2it + i\varepsilon t + 2i\phi\} \qquad \ldots\ldots (6)$$

where $F(\theta, z)$ is a known function and ε is a small quantity. The problem is to determine the response of the ocean to this external driving potential.

We shall look for a solution of equations (2)-(5) of the form

$$(u,v,w,\underline{P}) = \Big(u(\theta, z), v(\theta, z), w(\theta, z), P(\theta, z)\Big) \times$$
$$\times \exp\{-2it + i\varepsilon t + 2i\phi\} \qquad \ldots\ldots (7)$$

Substituting these expressions for $u,v,w,\underline{P}$ into the above equations we obtain

$$(-2i + i\varepsilon)u - 2v\cos\theta = -\frac{1}{z}\frac{\partial P}{\partial\theta} \qquad \ldots\ldots (8)$$

$$(-2i + i\varepsilon)v + 2u\cos\theta + 2w\sin\theta = -\frac{2i}{z\sin\theta}P \qquad \ldots\ldots (9)$$

$$(-2i + i\varepsilon)w - 2v\sin\theta = -\frac{\partial P}{\partial z} \qquad \ldots\ldots (10)$$

$$\frac{\partial}{\partial\theta}(u \sin\theta) + 2iv + \frac{\sin\theta}{z}\frac{\partial}{\partial z}(z^2 w) = 0 \qquad \ldots\ldots (11)$$

In equations (8), (9), (10), (11) we shall now try to obtain solutions for (u,v,w,P) in the form

$$u = u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots. \qquad \ldots\ldots (12)$$

$$v = v_o + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots. \qquad \ldots\ldots (13)$$

$$w = w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots. \qquad \ldots\ldots (14)$$

$$P = P_o + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots. \qquad \ldots\ldots (15)$$

Hence, substituting these expressions into (8), (9), (10), (11) and equating corresponding powers of ε we obtain.

$$iu_o + v_o \cos\theta = \frac{1}{2z}\frac{\partial P_o}{\partial\theta} \qquad \ldots\ldots (16)$$

$$iv_o - u_o \cos\theta - w_o \sin\theta = \frac{iP_o}{z \sin\theta} \qquad \ldots\ldots (17)$$

$$iw_o + v_o \sin\theta = \frac{1}{2}\frac{\partial P_o}{\partial z} \qquad \ldots\ldots (18)$$

$$\frac{\partial}{\partial\theta}(u_o \sin\theta) + 2iv_o + \frac{\sin\theta}{z}\frac{\partial}{\partial z}(z^2 w_o) = 0 \qquad \ldots\ldots (19)$$

for the zeroth-order coefficients.

Now, the determinant of the coefficients of u_o, v_o, w_o in (16), (17), (18) vanishes which we may exploit in the following way. Solving (16) and (18) for u_o and w_o respectively and substituting in (17) we obtain

$$\cos\theta \left\{ \frac{1}{2z} \frac{\partial P_o}{\partial \theta} - v_o \cos\theta \right\} + v_o +$$

$$+ \sin\theta \left\{ \frac{1}{2} \frac{\partial P_o}{\partial z} - v_o \sin\theta \right\} = \frac{P_o}{z \sin\theta}$$

Hence P_o satisfies

$$\sin\theta \frac{\partial P_o}{\partial z} + \frac{\cos\theta}{z} \frac{\partial P_o}{\partial \theta} = \frac{2P_o}{z \sin\theta} \qquad \ldots\ldots (20)$$

the integral surfaces of which are generated by the integral curves of the equations:

$$\frac{dz}{\sin\theta} = z \frac{d\theta}{\cos\theta} = \frac{z \sin\theta \, dP_o}{2P_o} \qquad \ldots\ldots (21)$$

Thus, the first equation of this set may be written as

$$\frac{dz}{z} = \frac{\sin\theta}{\cos\theta} d\theta$$

which integrates to give

$$z \cos\theta = \text{constant} \qquad \ldots\ldots (22)$$

The second equation of the set can be written as

$$\frac{d\theta}{\sin\theta\cos\theta} = \frac{dP_o}{2P_o}$$

$$\therefore \qquad \frac{\sec^2\theta\, d\theta}{\tan\theta} = \frac{dP_o}{2P_o}$$

which integrates to give

$$P_o \cot^2\theta = \text{constant} \qquad \ldots\ldots (23)$$

From (22), this last relationship may be rewritten as

$$\frac{P_o}{(z\sin\theta)^2} = \text{constant}$$

Hence we may write the general solution of equation (20) as

$$P_o = 2(z\sin\theta)^2\, f(z\cos\theta) \qquad \ldots\ldots (24)$$

where $f(x)$ is an arbitrary function of x.

To obtain an expression for v_o we now substitute for u_o and w_o from (16) and (18) in (19) to give

$$\sin\theta\cos\theta\,\frac{\partial v_o}{\partial\theta} + z\sin^2\theta\,\frac{\partial v_o}{\partial z} + 3v_o$$

$$= \frac{1}{2}\left\{\frac{1}{z}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial P_o}{\partial\theta}\right) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\,\frac{\partial P_o}{\partial z}\right)\right\} \qquad \ldots\ldots (25)$$

Substituting for P_o from (24) we then obtain

$$\sin\theta\cos\theta\,\frac{\partial v_o}{\partial\theta} + z\sin^2\theta\,\frac{\partial v_o}{\partial z} + 3v_o$$

$$= 4z\sin\theta\, f(z\cos\theta) + z^3\sin^3\theta\, f''(z\cos\theta)$$

which has integral surfaces given by the equations

$$\frac{d\theta}{\sin\theta\cos\theta} = \frac{dz}{z\sin^2\theta} =$$

$$= \frac{dv_o}{-3v_o + 4z\sin\theta\, f(z\cos\theta) + z^3\sin^3\theta\, f''(z\cos\theta)}$$

Hence, from the first of these equations

$$\frac{dz}{z} = \frac{\sin\theta\, d\theta}{\cos\theta}$$

$$\therefore \qquad z\cos\theta = c_1 \qquad \text{..... (26)}$$

where c_1 is a constant. The second of the equations then gives

$$\frac{dv_o}{d\theta} + \frac{3v_o}{\sin\theta\cos\theta} = \frac{4c_1\, f(c_1)}{\cos^2\theta} + \frac{c_1^{\,3}\sin^2\theta\, f''(c_1)}{\cos^4\theta}$$

An integrating factor for this equation is $e^{\int \frac{3d\theta}{\sin\theta\cos\theta}} = \tan^3\theta$ so that

$$\frac{d}{d\theta}(v_o \tan^3\theta) = 4c_1 \frac{\sin^3\theta}{\cos^5\theta} f(c_1) + c_1{}^3 \frac{\sin^5\theta}{\cos^7\theta} f''(c_1)$$

$$\text{ie} \quad v_o \tan^3\theta = c_1 f(c_1) \tan^4\theta + \frac{1}{6} c_1{}^3 f''(c_1) + c_2 \qquad \text{..... (27)}$$

where c_2 is a constant. From (26) we may rewrite equation (27) as

$$v_o = z \sin\theta\, f(c_1) + \frac{1}{6}(z \sin\theta)^3 f''(c_1) + \frac{c_3}{(z \sin\theta)^3}$$

Hence, on substituting for c_1 from (26) we obtain the general solution for v_o as follows

$$v_o = z \sin\theta\, f(z \cos\theta) + \frac{1}{6}(z \sin\theta)^3 f''(z \cos\theta) +$$

$$+ \frac{q(z \cos\theta)}{(z \sin\theta)^3} \qquad \text{..... (28)}$$

where $q(x)$ is an arbitrary function of x.

To obtain an expression for w_o we now use equations (18), (24) and (28) which give

$$iw_o = z \sin^2\theta\, f(z\cos\theta) + z^2 \sin^2\theta \cos\theta \times$$

$$\times f'(z\cos\theta) - \frac{1}{6} z^3 \sin^4\theta\, f''(z\cos\theta) -$$

$$- \frac{q(z\cos\theta)}{z(z\sin\theta)^2} \qquad \ldots\ldots (29)$$

For this to satisfy the boundary condition 2.2 (31) we must have

$$\frac{\beta^3 q\left(\frac{1}{\beta}\cos\theta\right)}{\sin^2\theta} \equiv \frac{1}{\beta}\sin^2\theta\, f\left(\frac{1}{\beta}\cos\theta\right) + \frac{1}{\beta^2}\sin^2\theta\cos\theta \times$$

$$\times f'\left(\frac{1}{\beta}\cos\theta\right) - \frac{\sin^4\theta}{6\beta^3} f''\left(\frac{1}{\beta}\cos\theta\right)$$

Hence writing $\eta = \frac{1}{\beta}\cos\theta$ so that $\frac{1}{\beta^2}\sin^2\theta = \frac{1}{\beta^2} - \eta^2$, it follows that

$$q(\eta) \equiv \left(\frac{1}{\beta^2} - \eta^2\right)^2 f(\eta) + \eta\left(\frac{1}{\beta^2} - \eta^2\right)^2 f'(\eta) -$$

$$- \frac{1}{6}\left(\frac{1}{\beta^2} - \eta^2\right)^3 f''(\eta) \qquad \ldots\ldots (30)$$

Accordingly, we can write equation (29) in the form

$$iw_o = z \sin^2\theta\, f(z\cos\theta) + z^2 \sin^2\theta \cos\theta \times$$

$$\times f'(z\cos\theta) - \frac{1}{6} z^3 \sin^4\theta\, f''(z\cos\theta) -$$

$$- \frac{1}{z^3 \sin^2\theta} \left\{ \left(\frac{1}{\beta^2} - z^2\cos^2\theta \right)^2 f(z\cos\theta) + \right.$$

$$+ z\cos\theta \left(\frac{1}{\beta^2} - z^2\cos^2\theta \right)^2 f'(z\cos\theta) - \qquad \text{..... (31)}$$

$$\left. - \frac{1}{6} \left(\frac{1}{\beta^2} - z^2\cos^2\theta \right)^3 f''(z\cos\theta) \right\}$$

which contains one arbitrary function, $f(x)$.

We now consider the conditions on the free surface. From the equations (2), (3) and (4) we note that the pressure is undefined to the extent of an arbitrary constant. Thus,

$$\underline{P} = C + P(\theta, z) \exp\{-2it + i\varepsilon t + 2i\phi\} \qquad \text{..... (32)}$$

where C is a constant. From 2.2 (33), the equation of the free surface is therefore

$$\beta C + \{\beta P(\theta, z) - 2F(\theta, z)\} \exp(-2it + i\varepsilon t + 2i\phi) - \lambda\beta z +$$

$$\text{..... (33)}$$

$$+ \mu\beta^2 z^2 \sin^2\theta = \pi_o$$

Now, at $\theta = 0$, the mean height of this pressure surface is $z = \frac{1}{\beta}(1 + \beta)$, hence

$$\beta C - \lambda(1 + \beta) = \pi_o \qquad \text{..... (34)}$$

Thus, the equation of the free surface is

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\mu\beta}{\lambda} z^2 \sin^2\theta + \frac{1}{\lambda\beta}\{\beta P(\theta, z) - 2F(\theta, z)\} \times$$

$$\times \exp(-2it + i\varepsilon t + 2i\phi) \qquad \ldots\ldots (35)$$

The mean position of this surface will be given by

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\mu\beta}{\lambda} z^2 \sin^2\theta$$

and this can be written with sufficient accuracy in the form

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\mu}{\lambda\beta}(1 + \beta)^2 \sin^2\theta$$

or

$$z = \frac{1}{\beta}(1 + \beta)\left\{1 + \varepsilon_o \sin^2\theta\right\},$$

$$\varepsilon_o = \frac{\mu}{\lambda}(1 + \beta) = O(10^{-3}) \qquad \ldots\ldots (36)$$

We shall now consider the kinematic condition 2.2 (36). As the last term of (35) is only $O(10^{-6})$ we may take this condition to be satisfied on the mean surface (36). Hence we may write

$$2\{\beta P_o(\theta, z) - 2F(\theta, z)\} - kiw_o = 0,$$

$$z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_o \sin^2\theta\right) \qquad \ldots\ldots (37)$$

Substituting for P_o and w_o from (24) and (31) respectively, we obtain

$$\frac{2}{k}\{2\beta(z \sin\theta)^2 f(z\cos\theta) - 2F(\theta, z)\} - z\sin^2\theta\, f(z\cos\theta) -$$

$$- z^2\sin^2\theta\cos\theta\, f'(z\cos\theta) + \frac{1}{6} z^3\sin^4\theta\, f''(z\cos\theta) +$$

$$+ \frac{1}{z^2\sin^2\theta}\left[\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f(z\cos\theta) + z\cos\theta \times \right. \qquad \ldots\ldots (38)$$

$$\times \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f'(z\cos\theta) -$$

$$\left. - \frac{1}{6}\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f''(z\cos\theta)\right] = 0$$

with $z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_o\sin^2\theta\right)$. Accordingly, when this value of z is inserted in (38) we have a linear second order differential equation for the determination of the function f. The function $F(\theta, z)$ has been discussed in 2.1 and it is clear that we have to study the case of

$$F(\theta, z) = \kappa\sin^2\theta \qquad \ldots\ldots (39)$$

Thus the differential equation for $f(\eta)$ is

$$\frac{4\beta}{k} z(z^2 - \eta^2)^2 f(\eta) - \left[(z^2 - \eta^2)^2 - \left(\frac{1}{\beta^2} - \eta^2 \right)^2 \right] \times$$

$$\times \; (f(\eta) + \eta f'(\eta)) + \frac{1}{6} \left[(z^2 - \eta^2)^3 - \left(\frac{1}{\beta^2} - \eta^2 \right)^3 \right] f''(\eta) \qquad \ldots\ldots (40)$$

$$= \frac{4\kappa}{kz} (z^2 - \eta^2)^2$$

where

$$z = \frac{1}{\beta} (1 + \beta) \left(1 + \varepsilon_o \sin^2 \theta \right) = \frac{\eta}{\cos \theta}$$

We may now consider the remaining boundary conditions. Now, the nature of the function v_o implies that condition 2.2 (32A) cannot be satisfied for all z on $\theta = 0, \pi$. Hence our attention will be restricted to canal-like regions between two surfaces of the form

$$\beta z \cos \theta = d \qquad \ldots\ldots (41)$$

where d is a given constant.

Figure follows

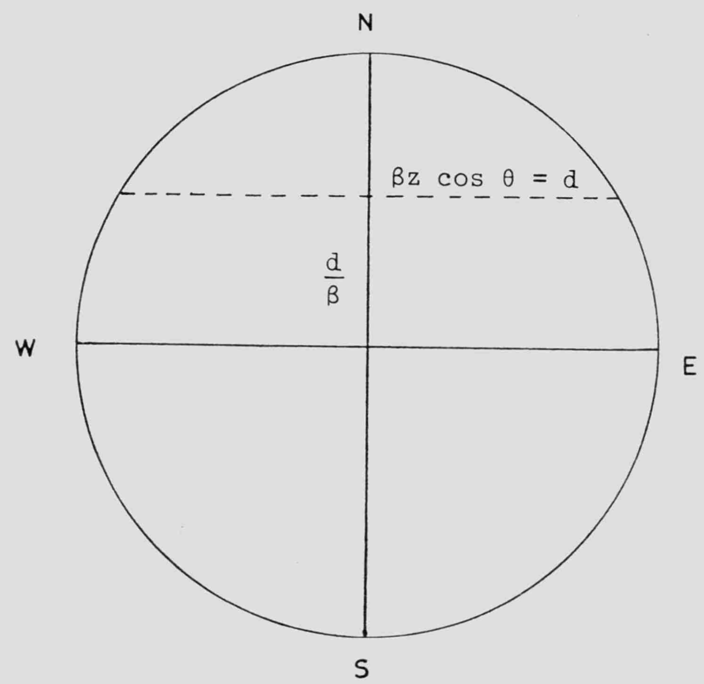


Figure 3.1

Physically, these surfaces are planes parallel to the equatorial plane and distance d from it. It follows from the geometry that

$$-1 \leqslant d \leqslant 1 \qquad \ldots\ldots (42)$$

Now, from (16), (24), (28) and (30) we have

$$iu_o = z \sin\theta \cos\theta\, f(z\cos\theta) - \sin\theta\,(z\sin\theta)^2\, f'(z\cos\theta) -$$

$$- \frac{1}{6}\cos\theta\,(z\sin\theta)^3\, f''(z\cos\theta) - \frac{\cos\theta}{z^3\sin^3\theta} \times$$

$$\times \left\{ \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f(z\cos\theta) + z\cos\theta \times \right. \qquad \ldots\ldots (43)$$

$$\times \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 \; f'(z\cos\theta) - \frac{1}{6}\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f''(z\cos\theta) \Bigg\}$$

Hence, applying 2.2 (32B) we obtain

$$w_o \cos\theta - u_o \sin\theta = 0 \quad , \quad \beta z \cos\theta = d$$

which, from (31) and (41) leads to the condition

$$f'\left(\frac{d}{\beta}\right) = 0 \qquad \ldots\ldots (44)$$

consequently two such surface conditions fully determine the function f.

The case where the two values of d are given by

$$d = \pm\delta \quad , \quad \delta \text{ small}$$

is of particular interest. Under these circumstances we may treat the sidewalls as vertical and the physical domain becomes the "equatorial canal" of width $2\delta/\beta$. The foregoing analysis describes the tidal theory in the canal when there is an appreciable vertical velocity.

The first order coefficients in (8), (9), (10) and (11) yield the equations

$$iu_1 + v_1 \cos\theta = \frac{1}{2z}\frac{\partial P_1}{\partial\theta} + \frac{i}{2}u_o \qquad \ldots\ldots (45)$$

$$iv_1 - u_1\cos\theta - w_1\sin\theta = \frac{i}{z\sin\theta}P_1 + \frac{i}{2}v_o \qquad \ldots\ldots (46)$$

$$iw_1 + v_1\sin\theta = \frac{1}{2}\frac{\partial P_1}{\partial z} + \frac{i}{2}w_o \qquad \ldots\ldots (47)$$

$$\frac{\partial}{\partial\theta}(u_1\sin\theta) + 2iv_1 + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2 w_1\right) = 0 \qquad \ldots\ldots (48)$$

As with the zeroth-order equations we find that the determinant of the coefficients of u_1, v_1, w_1 in (45), (46), (47) vanishes. Thus, from equation (45)

$$u_1 = -\frac{i}{2z}\frac{\partial P_1}{\partial\theta} + iv_1\cos\theta + \frac{1}{2}u_o$$

and, using (16), we then obtain

$$u_1 = -\frac{i}{2z}\frac{\partial P_1}{\partial\theta} + iv_1\cos\theta - \frac{1}{2}\left(\frac{i}{2z}\frac{\partial P_o}{\partial\theta} - iv_o\cos\theta\right) \qquad \ldots\ldots (49)$$

Also, from (47)

$$w_1 = -\frac{i}{2}\frac{\partial P_1}{\partial z} + iv_1\sin\theta + \frac{1}{2}w_o$$

which, using (18), becomes

$$w_1 = -\frac{i}{2}\frac{\partial P_1}{\partial z} + iv_1 \sin\theta - \frac{1}{2}\left(\frac{i}{2}\frac{\partial P_o}{\partial z} - iv_o \sin\theta\right) \qquad \ldots\ldots (50)$$

Now substituting for u_1 and w_1 in equation (46) using the expressions (49) and (50) we obtain

$$\cos\theta\left\{\left[\frac{i}{2z}\frac{\partial P_1}{\partial\theta} - iv_1\cos\theta + \frac{1}{2}\left(\frac{i}{2z}\frac{\partial P_o}{\partial\theta} - iv_o\cos\theta\right)\right]\right\} +$$

$$+ iv_1 + \sin\theta\left\{\left[\frac{i}{2}\frac{\partial P_1}{\partial z} - iv_1\sin\theta + \frac{1}{2}\left(\frac{i}{2}\frac{\partial P_o}{\partial z} - iv_o\sin\theta\right)\right]\right\}$$

$$= \frac{i}{z\sin\theta}\, P_1 + \frac{i}{2}\, v_o$$

ie putting $\qquad \bar{p} = P_1 + \frac{1}{2} P_o \qquad \ldots\ldots (51)$

we obtain

$$\sin\theta\frac{\partial\bar{p}}{\partial z} + \frac{\cos\theta}{z}\frac{\partial\bar{p}}{\partial\theta} = \frac{2\bar{p}}{z\sin\theta} + \left(2v_o - \frac{P_o}{z\sin\theta}\right) \qquad \ldots\ldots (52)$$

Using (24), (28) and (30) we may express this in terms of the function $f(z\cos\theta)$ as follows

$$\sin\theta\frac{\partial\bar{p}}{\partial z} + \frac{\cos\theta}{z}\frac{\partial\bar{p}}{\partial\theta} = \frac{2\bar{p}}{z\sin\theta} + \frac{1}{3}(z\sin\theta)^3\, f''(z\cos\theta) +$$

$$+ \frac{2}{(z \sin \theta)^3} \left\{ \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^2 f(z \cos \theta) + z \cos \theta \times \right.$$

$$\times \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^2 f'(z \cos \theta) - \qquad \ldots\ldots (53)$$

$$\left. - \frac{1}{6} \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^3 f''(z \cos \theta) \right\}$$

Now it may be observed that a particular integral of the equation

$$\sin \theta \frac{\partial y}{\partial R} + \frac{\cos \theta}{R} \frac{\partial y}{\partial \theta} - \frac{my}{R \sin \theta} = a(R \sin \theta)^s \; G(R \cos \theta)$$

in which a, m and s are constant with $s \neq m - 1$ is

$$y = \frac{a(R \sin \theta)^{s+1} \; G(R \cos \theta)}{s + 1 - m}$$

Hence a particular integral of (53) is

$$\bar{p}_I = \frac{1}{6} (z \sin \theta)^4 \; f''(z \cos \theta) - \frac{2}{4(z \sin \theta)^2} \times$$

$$\left\{ \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^2 f(z \cos \theta) + z \cos \theta \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^2 \times \right. \qquad \ldots\ldots (54)$$

$$\left. \times f'(z \cos \theta) - \frac{1}{6} \left(\frac{1}{\beta^2} - z^2 \cos^2 \theta \right)^3 f''(z \cos \theta) \right\}$$

To this must be added the complementary function which is of the same form as for the zeroth-order analysis. Hence

$$\bar{p} = 2(z \sin\theta)^2 \; g(z\cos\theta) + \bar{p}_I \qquad \ldots\ldots (55)$$

where $g(x)$ is an arbitrary function of x. Using the relationships (24) and (51) we obtain

$$\begin{aligned} P_1 = {} & 2(z\sin\theta)^2 \; g(z\cos\theta) + \frac{1}{6}(z\sin\theta)^4 \; f''(z\cos\theta) - \\ & (z\sin\theta)^2 \; f(z\cos\theta) - \frac{1}{2(z\sin\theta)^2} \times \\ & \times \left\{ \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f(z\cos\theta) + z\cos\theta \times \right. \\ & \times \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f'(z\cos\theta) - \\ & \left. - \frac{1}{6}\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f''(z\cos\theta) \right\} \end{aligned} \qquad \ldots\ldots (56)$$

The solution of v_1 may now be obtained by substituting for u_1 and w_1 from (49) and (50) in equation (48). Thus we have

$$\sin\theta\cos\theta \frac{\partial v_1}{\partial\theta} + z\sin^2\theta \frac{\partial v_1}{\partial z} + 3v_1 = \frac{1}{2}\left\{\frac{1}{z}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P_1}{\partial\theta}\right) + \right.$$

$$+ \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\frac{\partial P_1}{\partial z}\right)\Bigg\} +$$

$$+ \frac{1}{4}\left\{\frac{1}{z}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_o}{\partial\theta}\right) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\frac{\partial P_o}{\partial z}\right)\right\} -$$

$$- \frac{1}{2}\sin\theta\cos\theta\frac{\partial v_o}{\partial\theta} - \frac{z}{2}\sin^2\theta\frac{\partial v_o}{\partial z} - \frac{1}{2}v_o$$

or, writing $\bar{v} = v_1 + \frac{1}{2}v_o$ ..... (57)

the above becomes

$$\sin\theta\cos\theta\frac{\partial\bar{v}}{\partial\theta} + z\sin^2\theta\frac{\partial\bar{v}}{\partial z} + 3\bar{v} = \frac{1}{2}\left\{\frac{1}{z}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\bar{p}}{\partial\theta}\right) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\frac{\partial\bar{p}}{\partial z}\right)\right\} + v_o$$

ie, on dividing throughout by $z\sin\theta$

$$\sin\theta\frac{\partial\bar{v}}{\partial z} + \frac{\cos\theta}{z}\frac{\partial\bar{v}}{\partial\theta} + \frac{3\bar{v}}{z\sin\theta} = \frac{1}{2z\sin\theta}\left\{\frac{1}{z}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\bar{p}}{\partial\theta}\right) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\frac{\partial\bar{p}}{\partial z}\right)\right\} + \frac{v_o}{z\sin\theta} \quad \text{..... (58)}$$

In equation (58) we note that if h(z cos θ) is any function of z cos θ then

$$\frac{1}{z}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial}{\partial\theta}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}\right]+\frac{\sin\theta}{z}\times$$

$$\times\frac{\partial}{\partial z}\left[z^{2}\ \frac{\partial}{\partial z}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}\right]$$

$$\equiv\ \frac{\cos\theta}{z}\frac{\partial}{\partial\theta}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}\ +$$

$$+\ \frac{\sin\theta}{z}\frac{\partial^{2}}{\partial\theta^{2}}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}\ +$$

$$+\ 2\sin\theta\,\frac{\partial}{\partial z}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}\ +$$

$$+\ z\sin\theta\,\frac{\partial^{2}}{\partial 2^{2}}\{(z\sin\theta)^{s}\ h(z\cos\theta)\}$$

$$\equiv\ \frac{\cos\theta}{z}\Big(sz^{3}\sin^{s-1}\theta\cos\theta\ h(z\cos\theta)\ -\ (z\sin\theta)^{s+1}\ \times$$

$$\times\ h'(z\cos\theta)\Big)\ +\ \frac{\sin\theta}{z}\times$$

$$\times\ \Big(s(s-1)\ z^{s}\sin^{s-2}\theta\cos^{2}\theta\ h(z\cos\theta)\ -$$

$$- sz^s \sin^s\theta\ h(z\cos\theta) - sz^{s+1}\sin^s\theta\cos\theta\ h'(z\cos\theta) -$$

$$- (s+1)z^{s+1}\ \sin^s\theta\cos\theta\ h'(z\cos\theta) + (z\sin\theta)^{s+2}h''(z\cos\theta)\Big) +$$

$$+ 2\sin\theta\Big(sz^{s-1}\sin^2\theta\ h(z\cos\theta) + \cos\theta\ (z\sin\theta)^s h'(z\cos\theta)\Big) +$$

$$+ z\sin\theta\Big(s(s-1)z^{s-2}\sin^s\theta\ h(z\cos\theta) + sz^{s-1}\sin^s\theta\cos\theta\ \times$$

$$\times\ h'(z\cos\theta) + sz^{s-1}\sin^s\theta\cos\theta\ h'(z\cos\theta) +$$

$$+ \cos^2\theta\ (z\sin\theta)^s\ h''(z\cos\theta)\Big)$$

$$\equiv\ s^2 z^{s-1}\ \sin^{s-1}\theta\ h(z\cos\theta) + z^{s+1}\ \sin^{s+1}\theta\ h''(z\cos\theta)$$

Using the above result, we find that on substituting in the RHS of (58) for $\bar{p}$ and v_o we obtain

$$\sin\theta \frac{\partial\bar{v}}{\partial z} + \frac{\cos\theta}{z}\frac{\partial v}{\partial\theta} + \frac{3\bar{v}}{z\sin\theta} =$$

$$= \frac{1}{2z\sin\theta}\left\{8(z\sin\theta)\, g(z\cos\theta) + 2(z\sin\theta)^3 \times\right.$$

$$\times\, g''(z\cos\theta) + \frac{8}{3}(z\sin\theta)^3\, f''(z\cos\theta) +$$

$$+ \frac{1}{6}(z\sin\theta)^5\, f''''(z\cos\theta) -$$

$$\left. - \frac{1}{2}\frac{q''(z\cos\theta)}{(z\sin\theta)}\right\} +$$

$$+ \frac{1}{z\sin\theta}\left\{(z\sin\theta)\, f(z\cos\theta) + \frac{1}{6}(z\sin\theta)^3 \times\right.$$

$$\left.\times\, f''(z\cos\theta)\right\}$$

We have already observed that a particular integral is of the form $a(z\sin\theta)^{s+1}\, G(z\cos\theta)$ for each term and so we may write down the solution for $\bar{v}$ as

$$\bar{v} = (z\sin\theta)\, g(z\cos\theta) + \frac{1}{6}(z\sin\theta)^3\, g''(z\cos\theta) +$$

$$+ \frac{2}{9}(z\sin\theta)^3\, f''(z\cos\theta) + \frac{1}{96}(z\sin\theta)^5\, f''''(z\cos\theta) -$$

$$- \frac{1}{8}\frac{q''(z \cos\theta)}{z \sin\theta} + \frac{1}{4}(z \sin\theta)\, f(z \cos\theta) +$$

$$+ \frac{1}{36}(z \sin\theta)^3\, f''(z \cos\theta) + \frac{Q(z \cos\theta)}{(z \sin\theta)^3} \qquad \ldots\ldots (59)$$

where $Q(z \cos\theta)$ is an arbitrary function of $z \cos\theta$. Hence from equation (57)

$$v_1 = (z \sin\theta)\, g(z \cos\theta) + \frac{1}{6}(z \sin\theta)^3\, g''(z \cos\theta) +$$

$$+ \frac{2}{9}(z \sin\theta)^3\, f''(z \cos\theta) + \frac{1}{96}(z \sin\theta)^5\, f''''(z \cos\theta) -$$

$$- \frac{1}{8}\frac{q''(z \cos\theta)}{z \sin\theta} + \frac{1}{4}(z \sin\theta)\, f(z \cos\theta) +$$

$$+ \frac{1}{36}(z \sin\theta)^3\, f''(z \cos\theta) + \frac{Q(z \cos\theta)}{(z \sin\theta)^3} - \qquad \ldots\ldots (60)$$

$$- \frac{1}{2}(z \sin\theta)\, f(z \cos\theta) -$$

$$- \frac{1}{12}(z \sin\theta)^3\, f''(z \cos\theta) - \frac{q(z \cos\theta)}{2(z \sin\theta)^3}$$

We may now obtain the solution for w_1 by using equations (47), (56) and (60) together with equation (30). Thus

$$
\begin{aligned}
iw_1 = {} & 2z \sin^2\theta\, g(z\cos\theta) + z^2\sin^2\theta\cos\theta\, g'(z\cos\theta) + \\
& + \frac{1}{3} z^3 \sin^4\theta f''(z\cos\theta) + \frac{1}{12} z^4 \sin^4\theta \cos\theta\, f'''(z\cos\theta) - \\
& - z\sin^2\theta f(z\cos\theta) - \frac{1}{2} z^2\sin^2\theta\cos\theta\, f'(z\cos\theta) + \\
& + \frac{q(z\cos\theta)}{2z^3\sin^2\theta} - \frac{\cos\theta\, q'(z\cos\theta)}{4z^2\sin^2\theta} - z\sin^2\theta\, g(z\cos\theta) - \\
& - \frac{1}{6} z^3\sin^4\theta\, g''(z\cos\theta) - \frac{2}{9} z^3\sin^4\theta\, f''(z\cos\theta) - \\
& - \frac{1}{96} z^5 \sin^6\theta\, f''''(z\cos\theta) + \frac{q''(z\cos\theta)}{8z} - \\
& - \frac{1}{4} z\sin^2\theta\, f(z\cos\theta) - \frac{1}{36} z^3\sin^4\theta\, f''(z\cos\theta) - \\
& - \frac{Q(z\cos\theta)}{z^3\sin^2\theta} + \frac{1}{2} z\sin^2\theta\, f(z\cos\theta) + \\
& + \frac{1}{12} z^3\sin^4\theta\, f''(z\cos\theta) + \frac{q(z\cos\theta)}{2z^3\sin^2\theta} + \\
& + \frac{1}{2} z\sin^2\theta\, f(z\cos\theta) + \frac{1}{2} z^2\sin^2\theta\cos\theta \times \\
& \times f'(z\cos\theta) - \frac{1}{12} z^3\sin^4\theta\, f''(z\cos\theta) -
\end{aligned}
$$

$$- \frac{q(z \cos \theta)}{2z^3 \sin^2 \theta}$$

ie we have

$$iw_1 = z \sin^2\theta \, g(z \cos \theta) + z^2 \sin^2\theta \cos \theta \, g'(z \cos \theta) -$$

$$- \frac{1}{6} z^3 \sin^4\theta \, g''(z \cos \theta) - \frac{Q(z \cos \theta)}{z^3 \sin^2 \theta} -$$

$$- \frac{3}{4} z \sin^2\theta \, f(z \cos \theta) - \frac{1}{2} z^2 \sin^2\theta \cos \theta \times$$

$$\times f'(z \cos \theta) + \frac{2}{9} z^3 \sin^4\theta \, f''(z \cos \theta) +$$

$$\qquad \ldots\ldots (61)$$

$$+ \frac{1}{12} z^4 \sin^4\theta \cos \theta \, f''' (z \cos \theta) -$$

$$- \frac{1}{96} z^5 \sin^6\theta \, f'''' (z \cos \theta) + \frac{q(z \cos \theta)}{z^3 \sin^2 \theta} -$$

$$- \frac{\cos \theta \, q'(z \cos \theta)}{4z^2 \sin^2 \theta} + \frac{q''(z \cos \theta)}{8z} +$$

$$+ \frac{1}{2} z \sin^2\theta \, f(z \cos \theta) + \frac{1}{2} z^2 \sin^2\theta \cos \theta \times$$

$$\times\ f'(z\cos\theta) - \frac{1}{12} z^3 \sin^4\theta\ f''(z\cos\theta) - \frac{q(z\cos\theta)}{2z^3\sin^2\theta}$$

We can now deal with the boundary conditions. From condition 2.2 (31) we must have

$$\beta^3\ \frac{Q\left(\frac{1}{\beta}\cos\theta\right)}{\sin^2\theta} \equiv \frac{1}{\beta}\sin^2\theta\ g\left(\frac{1}{\beta}\cos\theta\right) + \frac{1}{\beta^2}\sin^2\theta\cos\theta\ \times$$

$$\times\ g'\left(\frac{1}{\beta}\cos\theta\right) - \frac{\sin^4\theta}{6\beta^3}\ g''\left(\frac{1}{\beta}\cos\theta\right) -$$

$$- \frac{3}{4\beta}\sin^2\theta\ f\left(\frac{1}{\beta}\cos\theta\right) - \frac{1}{2\beta^2}\sin^2\theta\ \times$$

$$\times\ \cos\theta\ f'\left(\frac{1}{\beta}\cos\theta\right) + \frac{2}{9\beta^3}\sin^4\theta\ \times \quad \ldots\ldots \quad (62)$$

$$\times\ f''\left(\frac{1}{\beta}\cos\theta\right) + \frac{1}{12\beta^4}\sin^4\theta\cos\theta\ \times$$

$$\times\ f'''\left(\frac{1}{\beta}\cos\theta\right) - \frac{1}{96\beta^5}\sin^6\theta\ \times$$

$$\times\ f''''\left(\frac{1}{\beta}\cos\theta\right) + \frac{\beta^3 q\left(\frac{1}{\beta}\cos\theta\right)}{\sin^2\theta}$$

$$- \frac{\beta^2 \cos\theta \, q'\left(\frac{1}{\beta}\cos\theta\right)}{4\sin^2\theta} + \frac{\beta}{8} q''\left(\frac{1}{\beta}\cos\theta\right)$$

Hence, writing $\eta = \frac{1}{\beta}\cos\theta$ so that $\frac{1}{\beta^2}\sin^2\theta = \frac{1}{\beta^2} - \eta^2$, the relationship (62) can be written as

$$\begin{aligned}
Q(\eta) \equiv & \left(\frac{1}{\beta^2} - \eta^2\right)^2 g(\eta) + \eta\left(\frac{1}{\beta^2} - \eta^2\right)^2 g'(\eta) - \\
& - \frac{1}{6}\left(\frac{1}{\beta^2} - \eta^2\right)^3 g''(\eta) - \frac{3}{4}\left(\frac{1}{\beta^2} - \eta^2\right)^2 f(\eta) - \\
& - \frac{1}{2}\eta\left(\frac{1}{\beta^2} - \eta^2\right)^2 f'(\eta) + \frac{2}{9}\left(\frac{1}{\beta^2} - \eta^2\right)^3 f''(\eta) + \\
& + \frac{1}{12}\eta\left(\frac{1}{\beta^2} - \eta^2\right)^3 f'''(\eta) - \frac{1}{96}\left(\frac{1}{\beta^2} - \eta^2\right)^4 \times \\
& \times f''''(\eta) + q(\eta) - \frac{1}{4}\eta\, q'(\eta) + \frac{1}{8}\left(\frac{1}{\beta^2} - \eta^2\right) q''(\eta)
\end{aligned} \qquad \ldots\ldots (63)$$

Accordingly, we may write (61) in the form

$$iw_1 = z\sin^2\theta\, g(z\cos\theta) + z^2\sin^2\theta\cos\theta\, g'(z\cos\theta) -$$

$$- \frac{1}{6} z^3 \sin^4\theta \; g''(z\cos\theta) - \frac{3}{4} z \sin^2\theta \; f(z\cos\theta) - \frac{1}{2} z^2 \sin^2\theta \times$$

$$\times \cos\theta \; f'(z\cos\theta) + \frac{2}{9} z^3 \sin^4\theta \; f''(z\cos\theta) + \frac{1}{12} z^4 \sin^4\theta \; \times$$

$$\times \cos\theta \; f'''(z\cos\theta) - \frac{1}{96} z^5 \sin^6\theta \; f''''(z\cos\theta) -$$

$$- \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2}{z^3\sin^2\theta} g(z\cos\theta) - \frac{z\cos\theta\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 g'(z\cos\theta)}{z^3\sin^2\theta} +$$

$$+ \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 g''(z\cos\theta)}{6z^3\sin^2\theta} + \frac{3\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f(z\cos\theta)}{4z^3\sin^2\theta} +$$

$$+ \frac{z\cos\theta\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f'(z\cos\theta)}{2z^3\sin^2\theta} -$$

$$- \frac{2\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f''(z\cos\theta)}{9z^3\sin^2\theta} -$$

$$- \frac{z\cos\theta\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f'''(z\cos\theta)}{12\, z^3\sin^2\theta} +$$

$$+ \frac{1}{96} \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^4 f''''(z\cos\theta)}{z^3\sin^2\theta} - \frac{q(z\cos\theta)}{z^3\sin^2\theta} +$$

$$+\frac{z\cos\theta\, q'(z\cos\theta)}{4z^3\sin^2\theta}-\frac{\left(\frac{1}{\beta^2}-z^2\cos^2\theta\right)q''(z\cos\theta)}{8z^3\sin^2\theta}+\frac{q(z\cos\theta)}{z^3\sin^2\theta}-$$

$$-\frac{\cos\theta\, q'(z\cos\theta)}{4z^2\sin^2\theta}+\frac{q''(z\cos\theta)}{8z\sin\theta}+\frac{1}{2}z\sin^2\theta\, f(z\cos\theta)+$$

$$+\frac{1}{2}z^2\sin^2\theta\cos\theta\, f'(z\cos\theta)-\frac{1}{12}z^3\sin^4\theta\, f''(z\cos\theta)-$$

$$-\frac{q(z\cos\theta)}{2z^3\sin^2\theta}$$

which contains one arbitrary function $g(x)$.

We may rewrite this equation as follows

$$i\left(w_1-\frac{1}{2}w_o\right)=\Big\{g(z\cos\theta)+z\cos\theta\, g'(z\cos\theta)-$$

$$-\frac{1}{6}z^2\sin^2\theta\, g''(z\cos\theta)\Big\}\times$$

$$\times\left\{z\sin^2\theta-\frac{\left(\frac{1}{\beta^2}-z^2\cos^2\theta\right)^2}{z^3\sin^2\theta}\right\}-$$

$$-\Big\{\frac{3}{4}f(z\cos\theta)+\frac{1}{2}z\cos\theta\, f'(z\cos\theta)-$$

$$- \frac{1}{12} z^2 \sin^2\theta \, f''(z\cos\theta)\Bigg]\Bigg\{ z\sin^2\theta - \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2}{z^3\sin^2\theta} \Bigg\} +$$

$$+ \Bigg\{ \frac{5}{36} f''(z\cos\theta) + \frac{1}{12} z\cos\theta \, f'''(z\cos\theta) - \frac{1}{96} z^2\sin^2\theta \, \times$$

$$\times \, f''''(z\cos\theta)\Bigg\} \Bigg\{ z^3\sin^4\theta - \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3}{z^3\sin^2\theta} \Bigg\} +$$

$$+ \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2\left(\frac{1}{\beta^2} - z^2\right) g''(z\cos\theta)}{6z^3\sin^2\theta} - \qquad \ldots\ldots (64)$$

$$- \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2\left(\frac{1}{\beta^2} - z^2\right) f''(z\cos\theta)}{12z^3\sin^2\theta} +$$

$$+ \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3\left(\frac{1}{\beta^2} - z^2\right) f''''(z\cos\theta)}{96z^3\sin^2\theta} -$$

$$- \frac{\left(\frac{1}{\beta^2} - z^2\right) q''(z\cos\theta)}{8z^3\sin^2\theta}$$

At the free surface we require the kinematic condition 2.2 (36) to be satisfied. Now from (1), (7) and (15) we have

$$\frac{\partial p}{\partial t} = \beta(-2i + i\varepsilon)(P_o(\theta, z) + \varepsilon P_1(\theta, z)) \exp\{-2it + i\varepsilon t + 2i\phi\} \quad \ldots\ldots (65)$$

to first order in ε. Hence substituting (6), (14) and (65) into 2.2 (36) and equating corresponding powers of ε we obtain

$$2\beta P_1(\theta, z) - kiw_1 = \beta P_o(\theta, z) - 2F(\theta, z) \qquad \ldots\ldots (66)$$

for the first-order coefficients. Equations (37) and (66) may be combined to give the equation

$$2\beta P_1(\theta, z) - ki\left(w_1 - \frac{1}{2}w_o\right) = 2(\beta P_o(\theta, z) - 2F(\theta, z)) \qquad \ldots\ldots (67)$$

where $$z = \frac{1}{\beta}(1 + \beta)(1 + \varepsilon_o \sin^2\theta)$$

Substituting for $P_1(\theta, z)$, $i\left(w_1 - \frac{1}{2}w_o\right)$ and $P_o(\theta, z)$ we obtain

$$\frac{4\beta}{k}(z\sin\theta)^2\, g(z\cos\theta) - \left\{g(z\cos\theta) + z\cos\theta\, g'(z\cos\theta) - \right.$$

$$\left. - \frac{1}{6}z^2\sin^2\theta\, g''(z\cos\theta)\right\} \times$$

$$\times \left\{z\sin^2\theta - \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3}{z^3\sin^2\theta}\right\} -$$

$$- \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2\left(\frac{1}{\beta^2} - z^2\right) g''(z\cos\theta)}{6z^3\sin^2\theta}$$

$$= \frac{4}{k}\left\{\beta(z\sin\theta)^2\, f(z\cos\theta) - F(\theta, z)\right\} -$$

$$- \frac{\beta}{3k}(z \sin\theta)^4 \, f''(z\cos\theta) + \frac{2\beta}{k}(z\sin\theta)^2 \, f(z\cos\theta) +$$

$$+ \frac{\beta}{k(z\sin\theta)^2}\left[\left\{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f(z\cos\theta) + z\cos\theta \times\right.\right.$$

$$\times \left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2 f'(z\cos\theta) -$$

$$\left.\left. - \frac{1}{6}\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3 f''(z\cos\theta)\right\}\right] -$$

$$- \left[\frac{3}{4} f(z\cos\theta) + \frac{1}{2} z\cos\theta \, f'(z\cos\theta) - \frac{1}{12} z^2\sin^2\theta \times\right.$$

$$\left.\times f''(z\cos\theta)\right]\left[z\sin^2\theta - \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^2}{z^3\sin^2\theta}\right] +$$

$$+ \left[\frac{5}{36} f''(z\cos\theta) + \frac{1}{12} z\cos\theta \, f'''(z\cos\theta) -\right.$$

$$\left. - \frac{1}{96} z^2\sin^2\theta \, f''''(z\cos\theta)\right]\left[z^3\sin^4\theta - \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3}{z^3\sin^2\theta}\right] +$$

$$+ \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)^3\left(\frac{1}{\beta^2} - z^2\right) f''''(z\cos\theta)}{96 z^3\sin^2\theta} -$$

$$- \frac{\left(\frac{1}{\beta^2} - z^2\cos^2\theta\right)\left(\frac{1}{\beta^2} - z^2\right) f''(z\cos\theta)}{12z^3\sin^2\theta} - \frac{\left(\frac{1}{\beta^2} - z^2\right) q''(z\cos\theta)}{8z^3\sin^2\theta}$$

with $z = \frac{1}{\beta}(1+\beta)\left(1 + \varepsilon_o \sin^2\theta\right)$

Accordingly, when this value of z is inserted into the above we obtain a linear second order ordinary differential equation for the determination of the function g. Thus, with (39), the differential equation for $g(\eta)$ is

$$\frac{4\beta}{k} z(z^2 - \eta^2)^2 g(\eta) - \left[(z^2 - \eta^2)^2 - \left(\frac{1}{\beta^2} - \eta^2\right)^2\right] \times$$

$$\times \left(g(\eta) + \eta g'(\eta)\right) + \frac{1}{6}\left[(z^2 - \eta^2)^3 - \left(\frac{1}{\beta^2} - \eta^2\right)^3\right] g''(\eta)$$

$$= -\frac{4\kappa}{kz}(z^2 - \eta^2)^2 + \frac{4\beta}{k} z(z^2 - \eta^2)^2 f(\eta) - \qquad \text{.....} \; (68)$$

$$- \frac{\beta}{3k} z(z^2 - \eta^2)^3 f''(\eta) + \frac{2\beta}{k} z(z^2 - \eta^2)^2 f(\eta) +$$

$$+ \frac{\beta z}{k}\left\{\left(\frac{1}{\beta^2} - \eta^2\right)^2 f(\eta) + \eta\left(\frac{1}{\beta^2} - \eta^2\right)^2 f'(\eta) - \right.$$

$$
- \frac{1}{6}\left(\frac{1}{\beta^2} - \eta^2\right)^3 f''(\eta)\Bigg\}\Bigg] - \Bigg[\Bigg\{\frac{3}{4} f(\eta) + \frac{1}{2}\eta f'(\eta) -
$$

$$
- \frac{1}{12}(z^2 - \eta^2)\ f''(\eta)\Bigg\}\Bigg] \times
$$

$$
\times \left[\left\{(z^2 - \eta^2)^2 - \left(\frac{1}{\beta^2} - \eta^2\right)^2\right\}\right] + \Bigg[\Bigg\{\frac{5}{36} f''(\eta) + \frac{1}{12}\eta f'''(\eta) -
$$

$$
- \frac{1}{96}(z^2 - \eta^2)\ f''''(\eta)\Bigg\}\Bigg] \times
$$

$$
\times \left[\left\{(z^2 - \eta^2)^3 - \left(\frac{1}{\beta^2} - \eta^2\right)^3\right\}\right] + \frac{1}{96}\left(\frac{1}{\beta^2} - \eta^2\right)^3\left(\frac{1}{\beta^2} - z^2\right) \times
$$

$$
\times f''''(\eta) - \frac{1}{12}\left(\frac{1}{\beta^2} - \eta^2\right)^2\left(\frac{1}{\beta^2} - z^2\right) f''(\eta) -
$$

$$
- \frac{1}{8}\left(\frac{1}{\beta^2} - z^2\right)\ q''(\eta)
$$

where $\quad z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_o \sin^2\theta\right) = \frac{\eta}{\cos\theta}$

At the surfaces $\beta z \cos\theta = \pm d$ the boundary condition 2.2 (32B) requires that

$$w_1 \cos\theta - u_1 \sin\theta = 0 \ , \ \beta z \cos\theta = \pm d$$

Hence, from (45) and (47) we must have

$$\frac{\sin\theta}{z}\frac{\partial P_1}{\partial\theta} - \cos\theta \frac{\partial P_1}{\partial z} = 0 \ , \ \beta z \cos\theta = \pm d \qquad \ldots\ldots (69)$$

Now, using (56) and (30)

$$\frac{\partial P_1}{\partial\theta} = 4(z\sin\theta)\, z\cos\theta\, g(z\cos\theta) - 2(z\sin\theta)^2\, z\sin\theta\, g'(z\cos\theta) +$$

$$+ \frac{2}{3}(z\sin\theta)^3 z\cos\theta\, f''(z\cos\theta) -$$

$$- \frac{1}{6}(z\sin\theta)^4\, z\sin\theta\, f'''(z\cos\theta) -$$

$$- 2(z\sin\theta)\, z\cos\theta\, f(z\cos\theta) +$$

$$+ (z\sin\theta)^2\, z\sin\theta\, f'(z\cos\theta) +$$

$$+ (z\sin\theta)^{-3}\, z\cos\theta\, q(z\cos\theta) +$$

$$+ \frac{1}{2} (z \sin \theta)^{-2} z \sin \theta \; q'(z \cos \theta)$$

and

$$\frac{\partial P_1}{\partial z} = 4(z \sin \theta) \sin \theta \; g(z \cos \theta) + 2(z \sin \theta)^2 \cos \theta \; g'(z \cos \theta) +$$

$$+ \frac{2}{3} (z \sin \theta)^3 \sin \theta \; f''(z \cos \theta) + \frac{1}{6} (z \sin \theta)^4 \cos \theta \times$$

$$\times \; f'''(z \cos \theta) - 2(z \sin \theta) \sin \theta \; f(z \cos \theta) -$$

$$-(z \sin \theta)^2 \cos \theta \; f'(z \cos \theta) + (z \sin \theta)^{-3} \sin \theta$$

$$q(z \cos \theta) - \frac{1}{2} (z \sin \theta)^{-2} \cos \theta \; q'(z \cos \theta)$$

Hence (69) gives

$$2(z \sin \theta)^2 \; g'(z \cos \theta) + \frac{1}{6} (z \sin \theta)^4 \; f'''(z \cos \theta) -$$

$$- (z \sin \theta)^2 \; f'(z \cos \theta) - \frac{1}{2} (z \sin \theta)^{-2} \; q'(z \cos \theta) = 0$$

on $\beta z \cos \theta = \pm d$. Thus, eliminating θ

$$\left.\begin{aligned}
&2\left(z^2 - \frac{d^2}{\beta^2}\right) g'\left(\frac{d}{\beta}\right) + \frac{1}{6}\left(z^2 - \frac{d^2}{\beta^2}\right)^2 f'''\left(\frac{d}{\beta}\right) - \\
&\quad -\left(z^2 - \frac{d^2}{\beta^2}\right) f'\left(\frac{d}{\beta}\right) - \frac{1}{2}\left(z^2 - \frac{d^2}{\beta^2}\right)^{-1} q'\left(\frac{d}{\beta}\right) = 0 \\
&\text{and} \\
&2\left(z^2 - \frac{d^2}{\beta^2}\right) g'\left(-\frac{d}{\beta}\right) + \frac{1}{6}\left(z^2 - \frac{d^2}{\beta^2}\right)^2 f'''\left(-\frac{d}{\beta}\right) - \\
&\quad -\left(z^2 - \frac{d^2}{\beta^2}\right) f'\left(-\frac{d}{\beta}\right) - \frac{1}{2}\left(z^2 - \frac{d^2}{\beta^2}\right)^{-1} q'\left(-\frac{d}{\beta}\right) = 0
\end{aligned}\right\} \quad \ldots\ldots (70)$$

Now, for the case of the equatorial canal we may take

$$z = \frac{1}{\beta}(1 + \beta)(1 + \varepsilon_o) = \text{constant} \quad \ldots\ldots (71)$$

neglecting terms of the second order in small quantities. In this case (70) and (71) give two boundary conditions which, in conjunction with equation (65), fully determine $g(\eta)$.

As d increases, (71) becomes less accurate, eventually, for values of $d >> 0.05$ the error becomes of the first order in ε_o. The value of z is then no longer a constant but, to first order in ε_o, is dependent on η. However, $g(\eta)$ itself is the first order term in the perturbation series and, as such, the net contribution of the error is of order $\varepsilon\varepsilon_o$. Thus, if ε and ε_o are of comparable magnitude the net error is still of the second order. Accordingly, we shall take (70) and (71) to apply throughout the range of values of d.

3.2 The Differential Equations

It has been shown that the problem of determining the functions $\{u_o, u_1, v_o, v_1, w_o, w_1, P_o, P_1\}$ reduces to one of solving the differential equations 3.1.40 and 3.1.68. We now turn to this aspect of the problem, beginning the analysis by re-stating 3.1.40 together with its boundary conditions. Thus, we have

$$\frac{4\beta}{k} z(z^2 - \eta^2)^2 f(\eta) - \left[(z^2 - \eta^2)^2 - \left(\frac{1}{\beta^2} - \eta^2\right)^2\right]\Big(f(\eta) + \eta f'(\eta)\Big) + \qquad \ldots\ldots (1)$$

$$+ \frac{1}{6}\left[(z^2 - \eta^2)^3 - \left(\frac{1}{\beta^2} - \eta^2\right)^3\right] f''(\eta) = \frac{4\kappa}{kz}(z^2 - \eta^2)^2$$

$$\text{where} \quad z = \frac{1}{\beta}(1 + \beta)\Big(1 + \varepsilon_o \sin^2\theta + O\left(\varepsilon_o^{\,2}\right)\Big) = \frac{\eta}{\cos\theta} \qquad \ldots\ldots (2)$$

Equation (1) holds throughout the region $-d \leqslant \beta\eta \leqslant d$, where d is a constant satisfying the inequality $0 < d \leqslant 1$. At the boundaries we have

$$f'(d/\beta) = 0$$
$$\ldots\ldots (3)$$
$$f'(-d/\beta) = 0$$

Now, from (2)

$$z = \frac{1}{\beta}(1+\beta)\left(1+\varepsilon_o - \varepsilon_o\cos^2\theta + 0\left(\varepsilon_o{}^2\right)\right)$$

and

$$\cos\theta = \frac{\beta\eta}{(1+\beta)}\left(1+0(\varepsilon_o)\right)$$

$$\therefore \quad z = \frac{1}{\beta}(1+\beta)\left[1+\varepsilon_o - \frac{\varepsilon_o\beta^2\eta^2}{(1+\beta)^2} + 0\left(\varepsilon_o{}^2\right)\right]$$

$$\therefore \quad z = \frac{1}{\beta}(1+\beta)\left[1+\varepsilon_o\left(1-\frac{\beta^2\eta^2}{(1+\beta)^2}\right) + 0\left(\varepsilon_o{}^2\right)\right] \qquad \ldots\ldots (4)$$

Thus

$$z^2 - \eta^2 = \frac{1}{\beta^2}(1+\beta)^2\left[1+2\varepsilon_o\left(1-\frac{\beta^2\eta^2}{(1+\beta)^2}\right) + 0\left(\varepsilon_o{}^2\right)\right] - \eta^2$$

$$= \left[\frac{1}{\beta^2}(1+\beta)^2 - \eta^2\right] + \frac{1}{\beta^2}(1+\beta)^2 \times$$

$$\times \left[\frac{2\varepsilon_o\beta^2}{(1+\beta)^2}\left(\frac{1}{\beta^2}(1+\beta)^2 - \eta^2\right) + 0\left(\varepsilon_o{}^2\right)\right]$$

ie, to sufficient accuracy, we have

$$z^2 - \eta^2 = (1 + 2\varepsilon_o)\left[\frac{1}{\beta^2}(1 + \beta)^2 - \eta^2\right] \qquad \ldots\ldots (5)$$

Now using (4) and (5) we may eliminate z in (1) and obtain

$$\frac{4\beta}{k}(1 + 2\varepsilon_o)^2\left\{\frac{1}{\beta}(1 + \beta) + \frac{1}{\beta}(1 + \beta)\,\varepsilon_o\left(1 - \frac{\beta^2\eta^2}{(1 + \beta)^2}\right)\right\} \times$$

$$\times \left[\frac{1}{\beta^2}(1 + \beta)^2 - \eta^2\right]^2 f(\eta) - \left\{(1 + 2\varepsilon_o)^2\left[\frac{1}{\beta^2}(1 + \beta)^2 - \eta^2\right]^2 - \right.$$

$$\left. - \left(\frac{1}{\beta^2} - \eta^2\right)^2\right\}\Big(f(\eta) + \eta f'(\eta)\Big) +$$

$$+ \frac{1}{6}\left\{(1 + 2\varepsilon_o)^3\left[\frac{1}{\beta^2}(1 + \beta)^2 - \eta^2\right]^3 - \left(\frac{1}{\beta^2} - \eta^2\right)^3\right\} f''(\eta)$$

$$= \frac{4K(1 + 2\varepsilon_o)^2\left[\frac{1}{\beta^2}(1 + \beta)^2 - \eta^2\right]^2}{k\left\{\frac{1}{\beta}(1 + \beta) + \frac{1}{\beta}(1 + \beta)\,\varepsilon_o\left(1 - \frac{\beta^2\eta^2}{(1 + \beta)^2}\right)\right\}}$$

ie multiplying through by $\left\{\frac{1}{\beta}(1 + \beta) + \frac{1}{\beta}(1 + \beta)\,\varepsilon_o\left(1 - \frac{\beta^2\eta^2}{(1 + \beta)^2}\right)\right\}$

and bringing together corresponding powers of η, we obtain

$$\frac{4\beta}{k}\left[(1 + 6\varepsilon_o)\,\frac{(1 + \beta)^6}{\beta^6} - (2 + 14\varepsilon_o)\,\frac{(1 + \beta)^4}{\beta^4}\,\eta^2 + (1 + 10\varepsilon_o)\,\times\right.$$

$$\left.\times\;\frac{(1 + \beta)^2}{\beta^2}\,\eta^4 - 2\varepsilon_o\eta^6\right] f(\eta) -$$

$$-\left\{\left[1 + 5\varepsilon_o - \frac{1}{(1 + \beta)^4} - \frac{\varepsilon_o}{(1 + \beta)^4}\right]\frac{(1 + \beta)^5}{\beta^5} -\right.$$

$$-\left[2 + 11\varepsilon_o - \frac{2}{(1 + \beta)^2} - \frac{\varepsilon_o}{(1 + \beta)^4} - \frac{2\varepsilon_o}{(1 + \beta)^2}\right]\frac{(1 + \beta)^3}{\beta^3}\,\eta^2 +$$

$$\left.+\;\varepsilon_o\left[6 - \frac{2}{(1 + \beta)^2}\right]\frac{(1 + \beta)}{\beta}\,\eta^4\right\}\Big(f(\eta) + f'(\eta)\Big) + \qquad \ldots\ldots (6)$$

$$+\;\frac{1}{6}\left\{\left[1 + 7\varepsilon_o - \frac{1}{(1 + \beta)^6} - \frac{\varepsilon_o}{(1 + \beta)^6}\right]\frac{(1 + \beta)^7}{\beta^7} -\right.$$

$$-\left[3 + 22\varepsilon_o - \frac{3}{(1 + \beta)^4} - \frac{\varepsilon_o}{(1 + \beta)^6} - \frac{3\varepsilon_o}{(1 + \beta)^4}\right]\times$$

$$\times\;\frac{(1 + \beta)^5}{\beta^5}\,\eta^2 + \left[3 + 24\varepsilon_o - \frac{3}{(1 + \beta)^2} -\right.$$

$$\left. - \frac{3\varepsilon_o}{(1 + \beta)^4} - \frac{3\varepsilon_o}{(1 + \beta)^2}\right]\frac{(1 + \beta)^3}{\beta^3}\,\eta^4 -$$

$$- \varepsilon_o \left[9 - \frac{3}{(1 + \beta)^2} \right] \frac{(1 + \beta)}{\beta} \, \eta^6 \Bigg] \quad f''(\eta)$$

$$= \frac{4K}{k} \left[(1 + 4\varepsilon_o) \frac{(1 + \beta)^4}{\beta^4} - 2(1 + 4\varepsilon_o) \frac{(1 + \beta)^2}{\beta^2} \eta^2 + \right.$$

$$\left. + (1 + 4\varepsilon_o) \, \eta^4 \right]$$

The coefficient multiplying $f''(\eta)$ has no zeros in the interval $-\frac{1}{\beta} \leqslant \eta \leqslant \frac{1}{\beta}$ as shown in Fig 3.12. Furthermore, in this interval, all of the coefficients of the equation are finite, one-valued and continuous. Accordingly, we look for a solution of the form

$$f(\eta) = a_o + a_1 \eta + a_2 \eta^2 + \ldots + a_r \eta^r + \ldots \qquad \ldots\ldots (7)$$

Expression (7) is substituted into (6) and the corresponding powers of η are equated. The remaining arbitrariness is then resolved from (3). However, in this process it is clear that we will find

$$a_1 = a_3 = a_5 = \ldots = 0 \qquad \ldots\ldots (8)$$

For the even power terms we obtain

$$a_o\left\{\frac{4\beta}{k}(1 + 6\varepsilon_o)\frac{(1+\beta)^6}{\beta^6} - \left[1 + 5\varepsilon_o - \frac{1}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^4}\right] \times\right.$$

$$\left.\times\ \frac{(1+\beta)^5}{\beta^5}\right\} + \frac{a_2}{3}\left[1 + 7\varepsilon_o - \frac{1}{(1+\beta)^6} - \frac{\varepsilon_o}{(1+\beta)^6}\right]\frac{(1+\beta)^7}{\beta^7} \qquad \ldots\ldots (9)$$

$$= \frac{4K}{k}(1 + 4\varepsilon_o)\frac{(1+\beta)^4}{\beta^4}$$

$$a_o\left\{\left[2 + 11\varepsilon_o - \frac{2}{(1+\beta)^2} - \frac{\varepsilon_o}{(1+\beta)^4} - \frac{2\varepsilon_o}{(1+\beta)^2}\right]\frac{(1+\beta)^3}{\beta^3} - \right.$$

$$\left. - \frac{4\beta}{k}(2 + 14\varepsilon_o)\frac{(1+\beta)^4}{\beta^4}\right\} -$$

$$- a_2\left\{\frac{1}{3}\left[3 + 22\varepsilon_o - \frac{3}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^6} - \frac{3\varepsilon_o}{(1+\beta)^4}\right]\frac{(1+\beta)^5}{\beta^5} - \right. \qquad \ldots\ldots (10)$$

$$- \frac{4\beta}{k}(1 + 6\varepsilon_o)\frac{(1+\beta)^6}{\beta^6} + 3\left[1 + 5\varepsilon_o - \frac{1}{(1+\beta)^4} - \right.$$

$$\left.\left.\frac{\varepsilon_o}{(1+\beta)^4}\right]\frac{(1+\beta)^5}{\beta^5}\right\} +$$

$$+ 2a_4 \left[1 + 7\varepsilon_o - \frac{1}{(1+\beta)^6} - \frac{\varepsilon_o}{(1+\beta)^6} \right] \frac{(1+\beta)^7}{\beta^7}$$

$$= - \frac{8K}{k} (1 + 4\varepsilon_o) \frac{(1+\beta)^2}{\beta^2}$$

$$a_o \left\{ \frac{4\beta}{k} (1 + 10\varepsilon_o) \frac{(1+\beta)^2}{\beta^2} - \varepsilon_o \left[6 - \frac{2}{(1+\beta)^2} \right] \frac{(1+\beta)}{\beta} \right\} +$$

$$+ a_2 \left\{ \frac{1}{3} \left[3 + 24\varepsilon_o - \frac{3}{(1+\beta)^2} - \frac{3\varepsilon_o}{(1+\beta)^4} - \frac{3\varepsilon_o}{(1+\beta)^2} \right] \frac{(1+\beta)^3}{\beta^3} - \right.$$

$$- \frac{4\beta}{k} (2 + 14\varepsilon_o) \frac{(1+\beta)^4}{\beta^4} + 3 \left[2 + 11\varepsilon_o - \frac{2}{(1+\beta)^2} - \right.$$

$$\left. \left. - \frac{\varepsilon_o}{(1+\beta)^4} - \frac{2\varepsilon_o}{(1+\beta)^2} \right] \frac{(1+\beta)^3}{\beta^3} \right\} - \quad \ldots\ldots (11)$$

$$- a_4 \left\{ 2 \left[3 + 22\varepsilon_o - \frac{3}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^6} - \right. \right.$$

$$\left. - \frac{3\varepsilon_o}{(1+\beta)^4} \right] \frac{(1+\beta)^5}{\beta^5} -$$

$$- \frac{4\beta}{k}(1 + 6\varepsilon_o)\frac{(1+\beta)^6}{\beta^6} + 5\left[1 + 5\varepsilon_o - \frac{1}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^4}\right] \times$$

$$\times \left.\frac{(1+\beta)^5}{\beta^5}\right] + 5a_6\left[1 + 7\varepsilon_o - \frac{1}{(1+\beta)^6} - \frac{\varepsilon_o}{(1+\beta)^6}\right]\frac{(1+\beta)^7}{\beta^7} =$$

$$= \frac{4K}{k}(1 + 4\varepsilon_o)$$

and then, for the subsequent even powered terms:

$$-8\varepsilon_o \frac{\beta}{k} a_{2(r-3)} - a_{2(r-2)}\left\{\frac{1}{6}(2r-4)(2r-5)\,\varepsilon_o\left[9 - \frac{3}{(1+\beta)^2}\right] \times\right.$$

$$\times \frac{(1+\beta)}{\beta} - \frac{4\beta}{k}(1 + 10\varepsilon_o)\frac{(1+\beta)^2}{\beta^2} +$$

$$\left. + (2r-3)\,\varepsilon_o\left[6 - \frac{2}{(1+\beta)^2}\right]\frac{(1+\beta)}{\beta}\right] +$$

$$+ a_{2(r-1)}\left\{\frac{1}{6}(2r-2)(2r-3)\left[3 + 24\varepsilon_o - \frac{3}{(1+\beta)^2} - \frac{3\varepsilon_o}{(1+\beta)^4} -\right.\right.$$

$$\left. - \frac{3\varepsilon_o}{(1+\beta)^2}\right]\frac{(1+\beta)^3}{\beta^3} -$$

$$- \frac{4\beta}{k}(2 + 14\varepsilon_o)\frac{(1+\beta)^4}{\beta^4} + (2r-1)\left[2 + 11\varepsilon_o - \frac{2}{(1+\beta)^2} - \frac{\varepsilon_o}{(1+\beta)^4} - \right.$$

$$\left. - \frac{2\varepsilon_o}{(1+\beta)^2}\right]\frac{(1+\beta)^3}{\beta^3}\Bigg\}\Bigg] -$$

$$- a_{2r}\left\{\frac{1}{6}2r(2r-1)\left[3 + 22\varepsilon_o - \frac{3}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^6} - \right.\right.$$

$$\left.\frac{3\varepsilon_o}{(1+\beta)^4}\right]\frac{(1+\beta)^5}{\beta^5} - \frac{4\beta}{k}(1 + 6\varepsilon_o)\frac{(1+\beta)^6}{\beta^6} + \qquad \ldots\ldots \quad (12)$$

$$\left. + (2r+1)\left[1 + 5\varepsilon_o - \frac{1}{(1+\beta)^4} - \frac{\varepsilon_o}{(1+\beta)^4}\right]\frac{(1+\beta)^5}{\beta^5}\right\} +$$

$$+ \frac{1}{6}a_{2(r+1)}\,(2r+2)(2r+1)\left[1 + 7\varepsilon_o - \frac{1}{(1+\beta)^6} - \right.$$

$$\left. - \frac{\varepsilon_o}{(1+\beta)^6}\right]\frac{(1+\beta)^7}{\beta^7} = 0$$

From equations (9)-(12) we may determine all of the coefficients in terms of a_o. Furthermore, for sufficiently large N, (12) gives:

$$\beta\left[\frac{a_{N-2}}{\beta^3}-\frac{2a_N}{\beta^5}+\frac{a_{N+2}}{\beta^7}\right]-\varepsilon_o\left[\frac{a_{N-4}}{\beta}-\frac{3a_{N-2}}{\beta^3}+\frac{3a_N}{\beta^5}-\frac{a_{N+2}}{\beta^7}\right] = 0$$

approximately. Thus, as $N \to \infty$, $\frac{a_N}{a_{N-2}} \to \beta^2$ ..... (13)

From (3) and (7) we obtain

$$2a_2 + 4a_4\frac{d^2}{\beta^2} + 6a_6\frac{d^4}{\beta^4} + \ldots = 0 \qquad \text{..... (14)}$$

which resolves a_o.

The complete set of equations (9)-(13) have been processed on the computer and the results are shown in Fig 3.2 (a)-(h) and Fig 3.3 (a)-(e). These two figures correspond to the two canal depths given by $\beta = 0.0001$ (22,000 ft) and $\beta = 0.001$ (40,000 ft) respectively. Various values of the canal semi-width, d, were taken as shown on the figures. The value of the constant k is given from 2.2.37 in which the value of W/U was estimated as the value of β. This gave $k = 0.03$ for the $\beta = 0.0001$ case (Fig 3.2) and $k = 0.3$ for the $\beta = 0.001$ case (Fig 3.3). The constant κ was estimated from 2.1.12, 2.1.13 and 2.2.35. This constant, and hence the results, was scaled by the value $2\mu \times 10^3$ (which, from 2.2.34, is approximately 1.4584×10^6) for display purposes.

To obtain an understanding of the shape of the curves we may observe, from (1), that the complementary function is approximately given by

$$(\text{COEFF}) \times f''(\eta) + \frac{4f(\eta)}{k\beta^5} = 0$$

where the value of COEFF can be obtained from Fig 3.12. Thus, for the case $\beta = 0.0001$ and a canal semi-width $\frac{d}{\beta} = \frac{0.2}{\beta}$ we have

$$\frac{58.66}{6\beta^6} f''(\eta) + \frac{400}{3\beta^5} f(\eta) = 0$$

$$\therefore \qquad f''(\eta) = -13.64\beta \, f(\eta)$$

The wavelength of these waves is $2\pi/\sqrt{13.64\beta}$. Thus, the number of cycles in a distance of $\frac{0.2}{\beta}$ is

$$\frac{0.2}{\beta \times \text{WAVELENGTH}} = \frac{0.2\sqrt{13.64\beta}}{2\pi\beta} \sim \underline{11.76 \text{ cycles}}$$

and this result may be compared with Fig 3.2(h) which shows the computed value for the same distance.

Similarly, for the case $\beta = 0.001$ and the same value of d we have

$$\frac{10.90}{6\beta^6} f''(\eta) + \frac{40}{3\beta^5} f(\eta) = 0$$

$$\therefore \qquad f''(\eta) = 7.34\beta\, f(\eta)$$

The wavelength is now $2\pi/\sqrt{7.34\beta}$. Thus, the number of cycles in a distance of $\frac{0.2}{\beta}$ is

$$\frac{0.2\sqrt{7.34\beta}}{2\pi\beta} \sim \underline{2.73 \text{ cycles}}$$

and this result may be compared with Fig 3.3(c).

The above gives us a simple check on the results obtained. It also shows us that the wavelength is associated with the value $\sqrt{\beta}$ with the longer waves corresponding to the deeper canal.

For the values of d considered, Fig 3.12 shows us that the wavelength is approximately independent of d. The apparent change in wavelength in the successive curves of Fig 3.2 or 3.3 is due to the way the results are displayed. In fact the horizontal axis distance scale is increasing so that the length of the axis always represents the distance from the equator to the canal edge. Bearing this in mind, an examination of the curves of Fig 3.2 or 3.3 shows us that, indeed, the wavelength is approximately constant for a given value of β.

A similar analysis has been made of the second differential equation 3.1.68. The results are displayed in Fig 3.4(a)-(e) and Fig 3.5(a)-(f). As the complementary function is given by the same equation the above comments will again apply. Thus Fig 3.5 (f) shows the case for $d = 0.2$, $\beta = 0.001$ and, again, the number of cycles is approximately 2.5.

The phenomenon of resonance is displayed and for this reason, it was not possible to compute $g(\eta)$ for the larger values of d.

With the computed values of $f(\eta)$ and $g(\eta)$ it becomes possible to determine the velocity components u, v, w for both the zeroth-order and first-order terms of the perturbation series. The results are displayed in Figs 3.6-3.11. Each of these components must be multiplied by a time-dependent term, namely $\cos(2\omega t + \varepsilon\omega t + 2\phi)$ for the v component and $\sin(2\omega t + \varepsilon\omega t + 2\phi)$ for the u and w components. For display purposes, all of the components are multiplied by the constant 10^3.

It is of interest of compare the solutions obtained here with that of [6]. In the latter the moon is assumed to describe an orbit in the plane of the equator and the motion of any particular fluid particle is assumed to be confined to a plane parallel to the equatorial plane. The elevation of the free surface above the undisturbed state is then found to be given by

$$E = \frac{1}{2} \frac{c^2 H}{c^2 - n^2 a^2} \cos 2(nt + \phi + \varepsilon) \qquad \ldots\ldots (15)$$

where n is the angular velocity of the moon relative to a fixed meridian, $c = \sqrt{gh}$ where h is the undisturbed height, a is the radius of the earth and $H = \frac{3}{2} \gamma M a^2 / g D^3$ where D is the distance between the centres of the earth and moon, M is the moon's mass and γ is the gravitational constant. From (15), the velocity component v can be found to be given by

$$v = -\frac{1}{2} \frac{gn\ Ha}{c^2 - n^2 a^2} \cos 2(nt + \phi + \varepsilon) \qquad \ldots\ldots (16)$$

Thus v is independent of the vertical co-ordinate. Turning to Figures 3.7 and 3.10 it can be seen that the variations in v are indeed small in the vertical direction, with the most significant changes occurring near the free surface. The amplitude of expression (16) is 2×10^{-2} ft/sec which is comparable to the values for v shown in Figures 3.7 and 3.10 namely

$$v \sim 30 \times 10^{-3} = 3 \times 10^{-2} \text{ ft/sec}$$

From (16) and the continuity equation, an expression may be derived for w, namely

$$w = \beta R \frac{gn\,Ha}{c^2 - n^2 a^2} \sin 2(nt + \phi + \varepsilon) \qquad \ldots\ldots (17)$$

where R is a vertical co-ordinate which is such that R = 0 at the canal bottom and R = 1 on the spherical surface of radius a + h. Thus w is proportional to the height above the bottom. Referring to Figures 3.8 and 3.11 we see that this is indeed the case throughout the domain. The amplitude of (17) is $2\beta R \times$ (amplitude of (16)). Again this agrees well with the findings presented in Figures 3.8 and 3.11 where the amplitude at eg 20,000 ft when $\beta = 0.001$ is approximately 6×10^{-5} ft/sec (from Figures 3.11) which is $2\beta \times 1 \times v$.

Turning to the elevation of the tide we see from 3.1.35 that the amplitude of the periodic disturbance is given by

$$\frac{1}{\lambda\beta}(\beta P - 2F) \times \beta a$$

ie

$$\frac{2\omega U}{g\beta}(\beta z^2 f - \kappa) \sin^2\theta \times \beta a$$

Thus, taking the case of Figure 3.3(a) and taking $z = \frac{1}{\beta}(1 + \beta) \times (1 + \varepsilon \sin^2\theta)$ the maximum value of the above expression is 1.0×10^{-3} ft compared with one of 2.5×10^{-1} ft in expression (15).

In both theories the time of minimum elevation occurs when the moon is overhead, ie the tide is inverted. To see if this remains the case as β increases, the progression of Figures 3.2(a) and 3.3(a) may be followed by putting f', $f'' = 0$ in equation (1). Ignoring terms in η and above we then obtain

$$\left(\frac{\beta z}{k} - \beta - \varepsilon_o\right) f = \frac{\kappa}{kz}$$

a formula which, incidentally, provides a means of checking the magnitudes of the computed values of f shown in the figures.

The value of k varies with β. From equation 2.2(37) $k = 300\beta$ approximately, so the value of the expression on the LHS is zero when the critical point $\beta \sim 0.06$ is reached. Above this point the sign of f changes and the tides become direct. This compares with the critical point $\beta \sim 0.003$ in [6]. Of course at the critical point the approximation f', $f'' = 0$ in equation (1) no longer holds. The form of equation (1) predicts no singularity at this point. However further investigation would be necessary in order to find a solution which satisfies the boundary conditions.

Thus, to summarize, the description of the tidal elevation is qualitatively the same as that of [6] but differs in quantitative terms. However in other respects the two theories agree fairly precisely. As [6] is in accordance with Laplace's assumptions this finding supports Proudman's work.

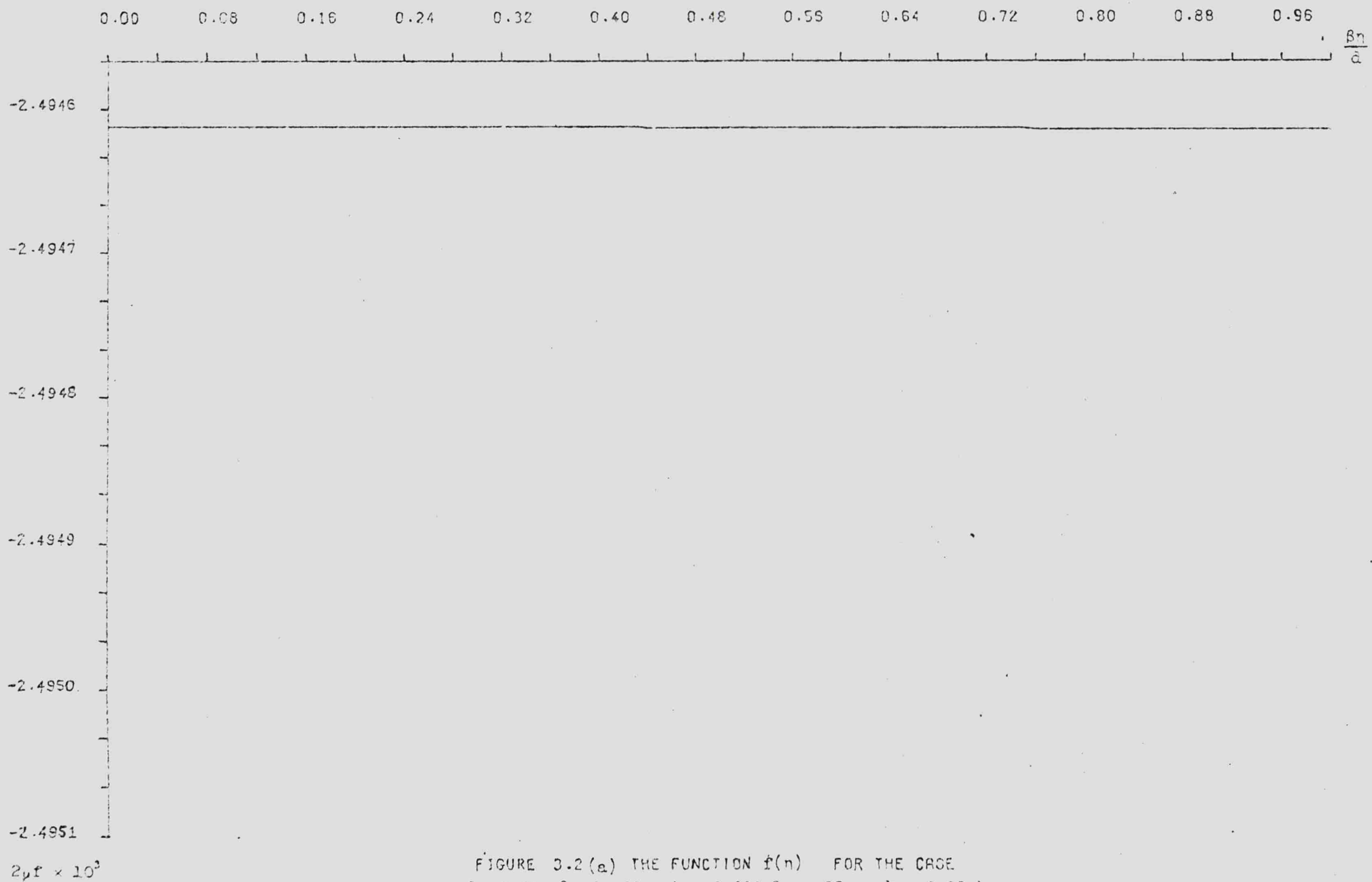
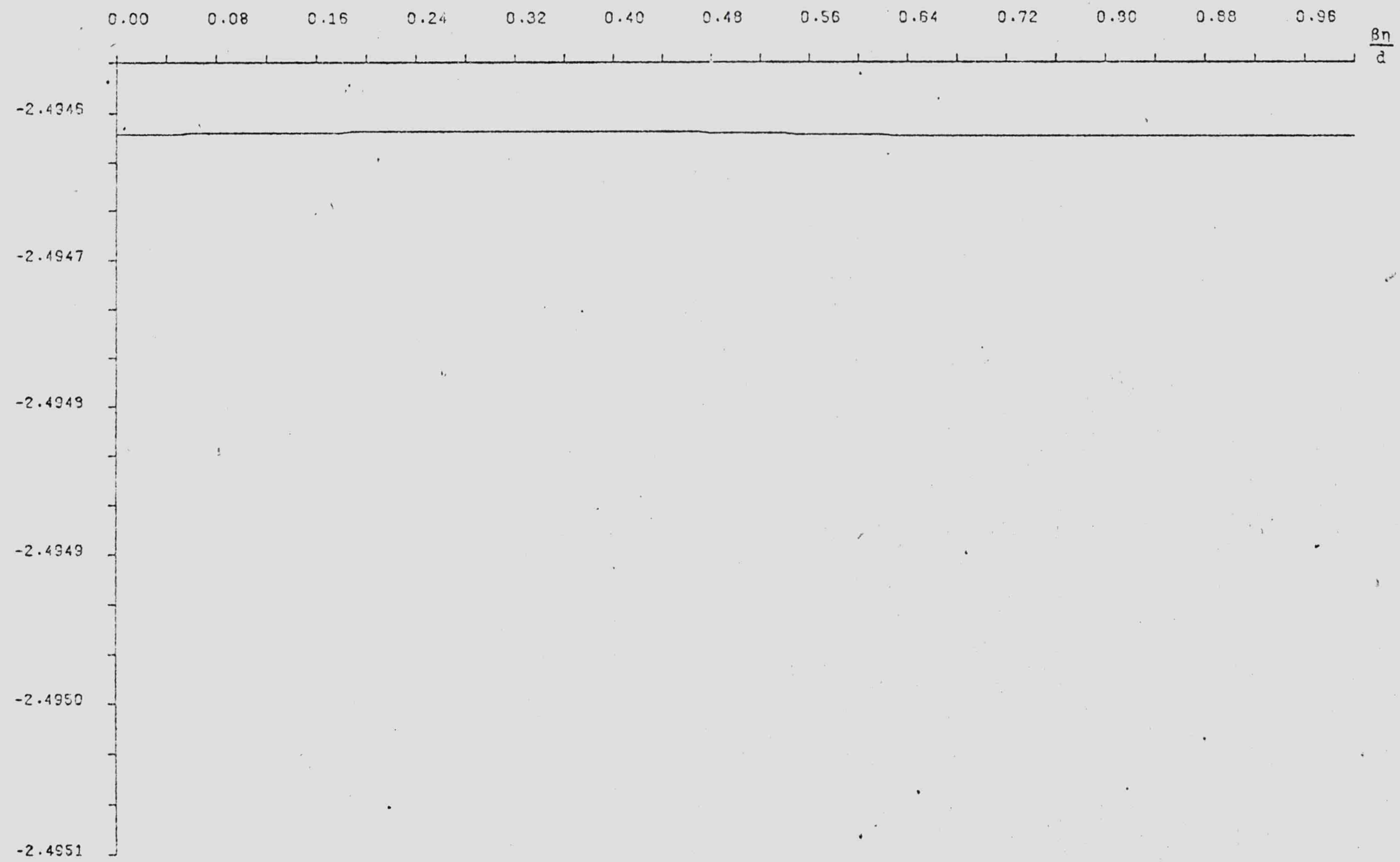


FIGURE 3.2(a) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.01$, $\beta = 0.0001$ ($\epsilon = 0.001, 2\mu\kappa = -25$, $k = 0.03$)



$2\mu f \times 10^3$

FIGURE 3.2 (b) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.025, \beta = 0.0001$ ($\varepsilon = 0.001, 2\mu\kappa = -25$, $k = 0.03$)

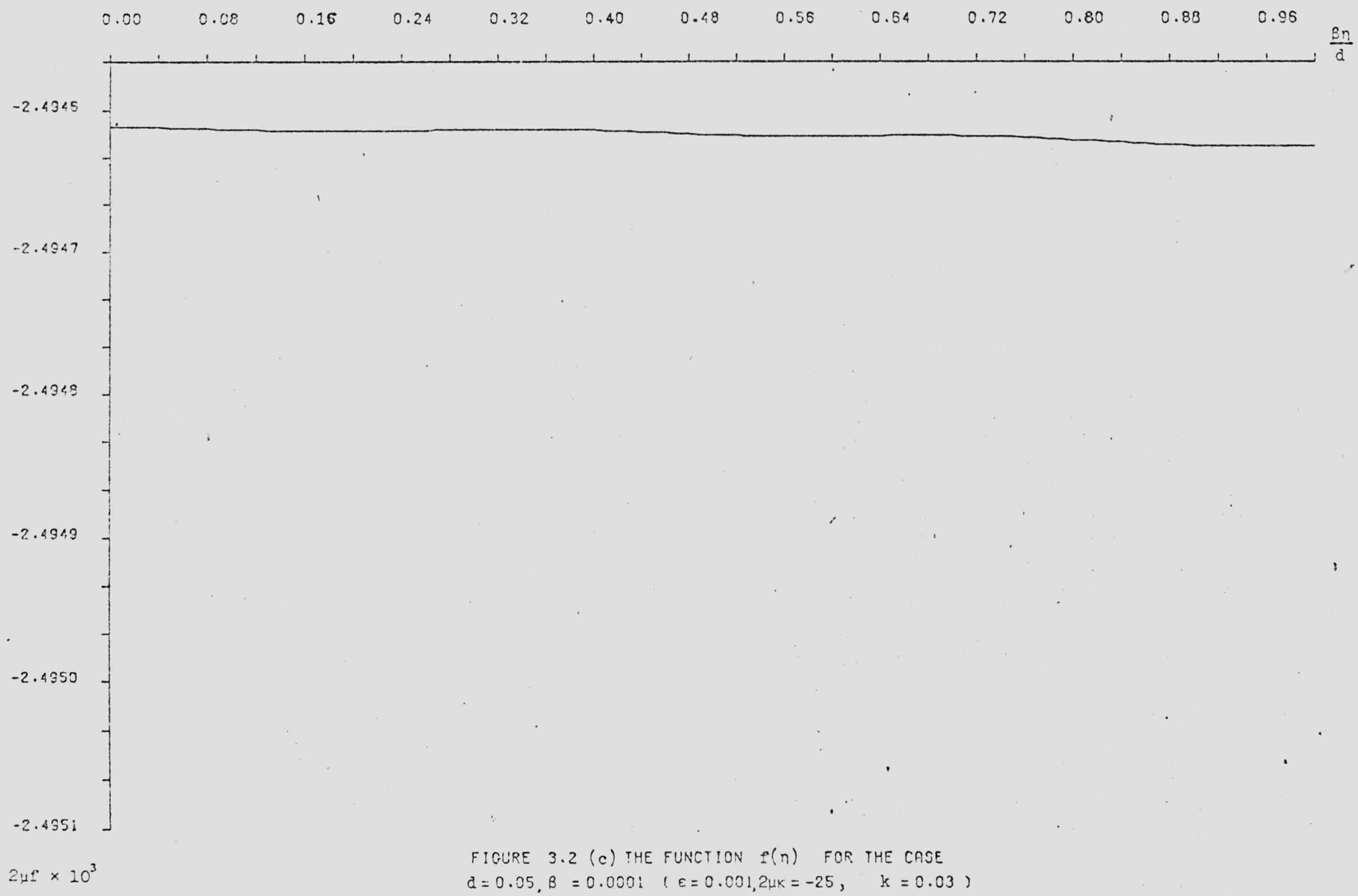


FIGURE 3.2 (c) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.05$, $\beta = 0.0001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

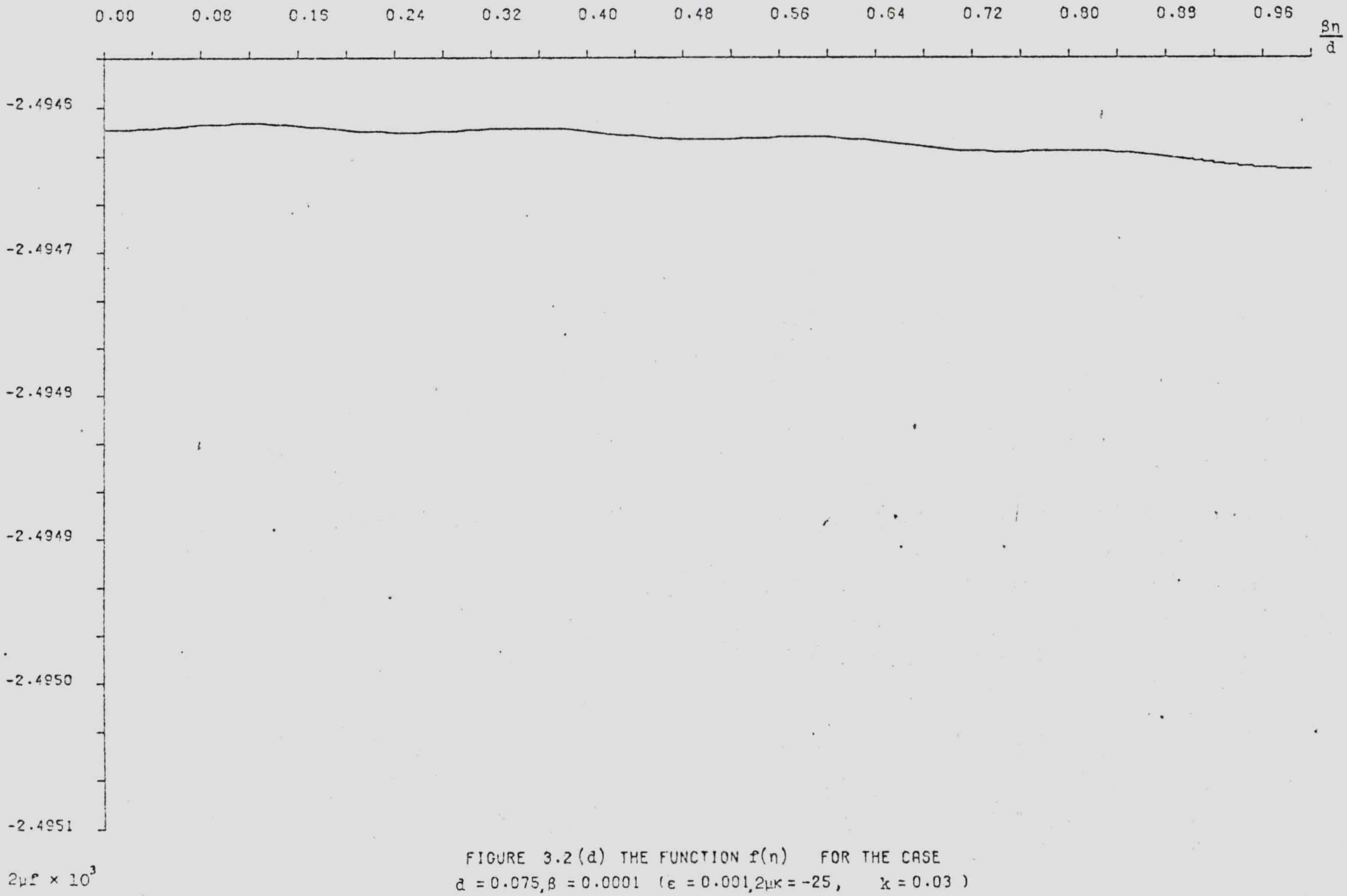


FIGURE 3.2(d) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.075, \beta = 0.0001$ ($\epsilon = 0.001, 2\mu\kappa = -25$, $\chi = 0.03$)

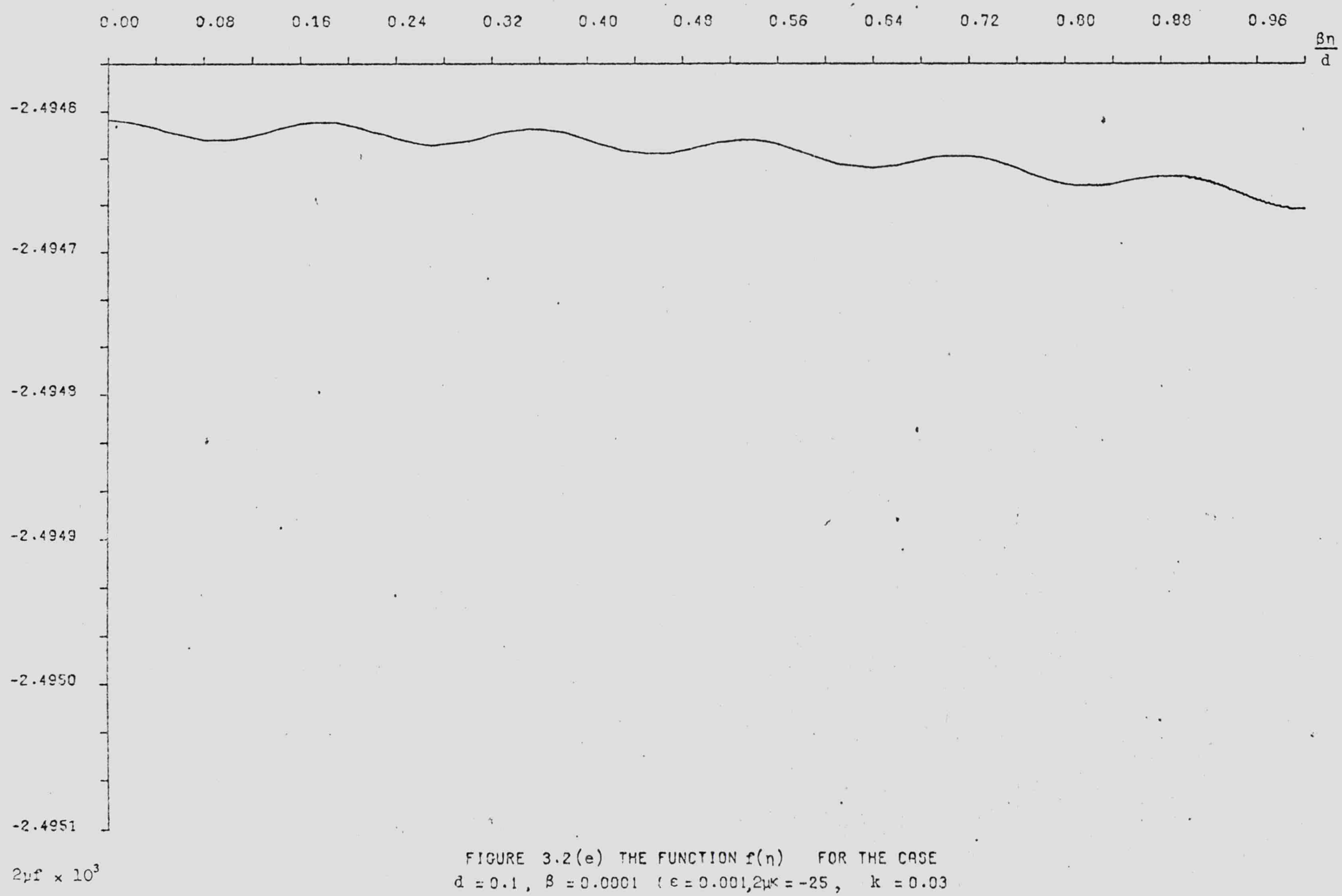


FIGURE 3.2(e) THE FUNCTION $f(\eta)$ FOR THE CASE
$d = 0.1$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$

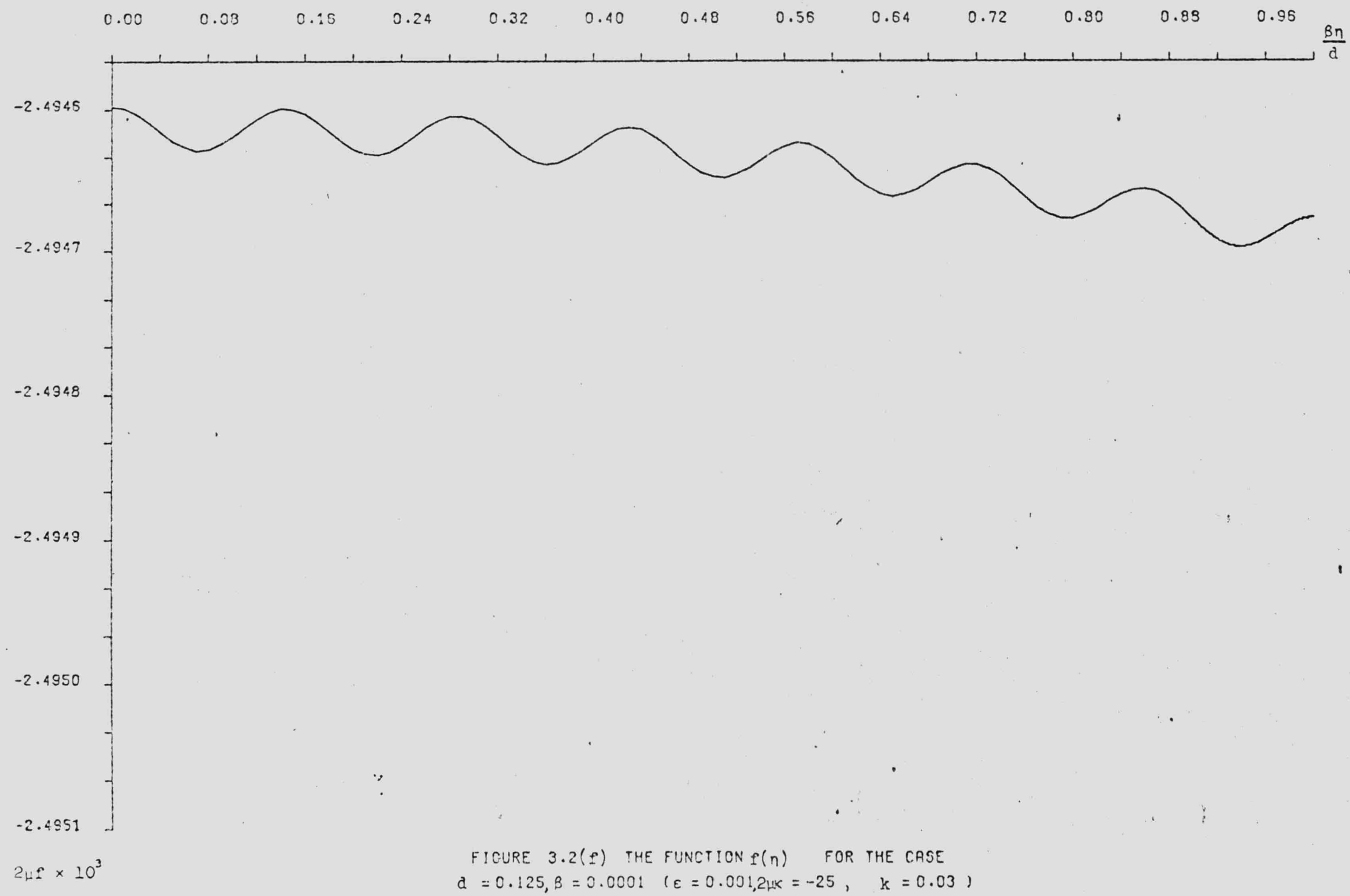


FIGURE 3.2(f) THE FUNCTION $f(\eta)$ FOR THE CASE
$d = 0.125, \beta = 0.0001$ ($\varepsilon = 0.001, 2\mu\kappa = -25$, $k = 0.03$)

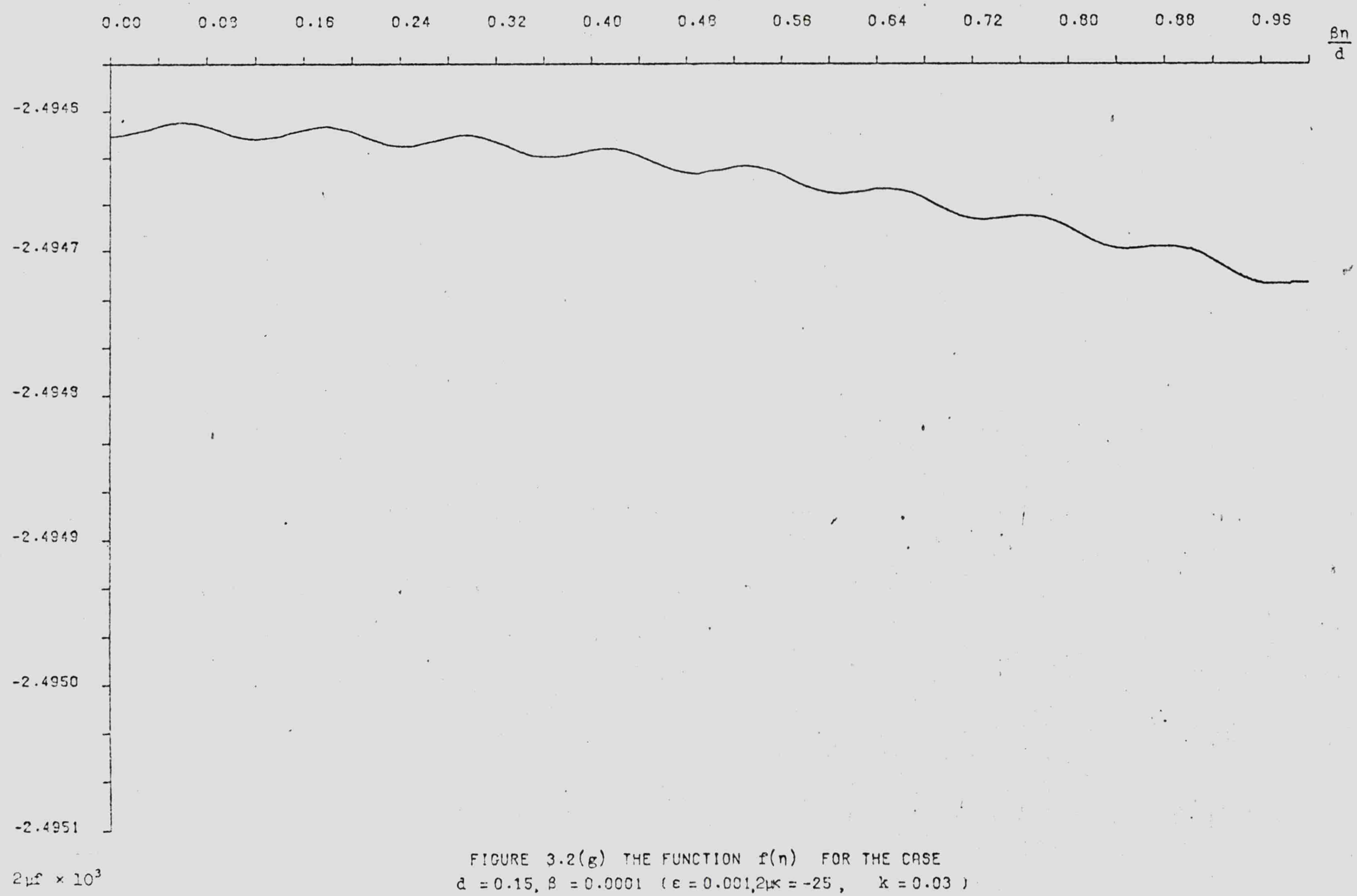


FIGURE 3.2(g) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.15$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

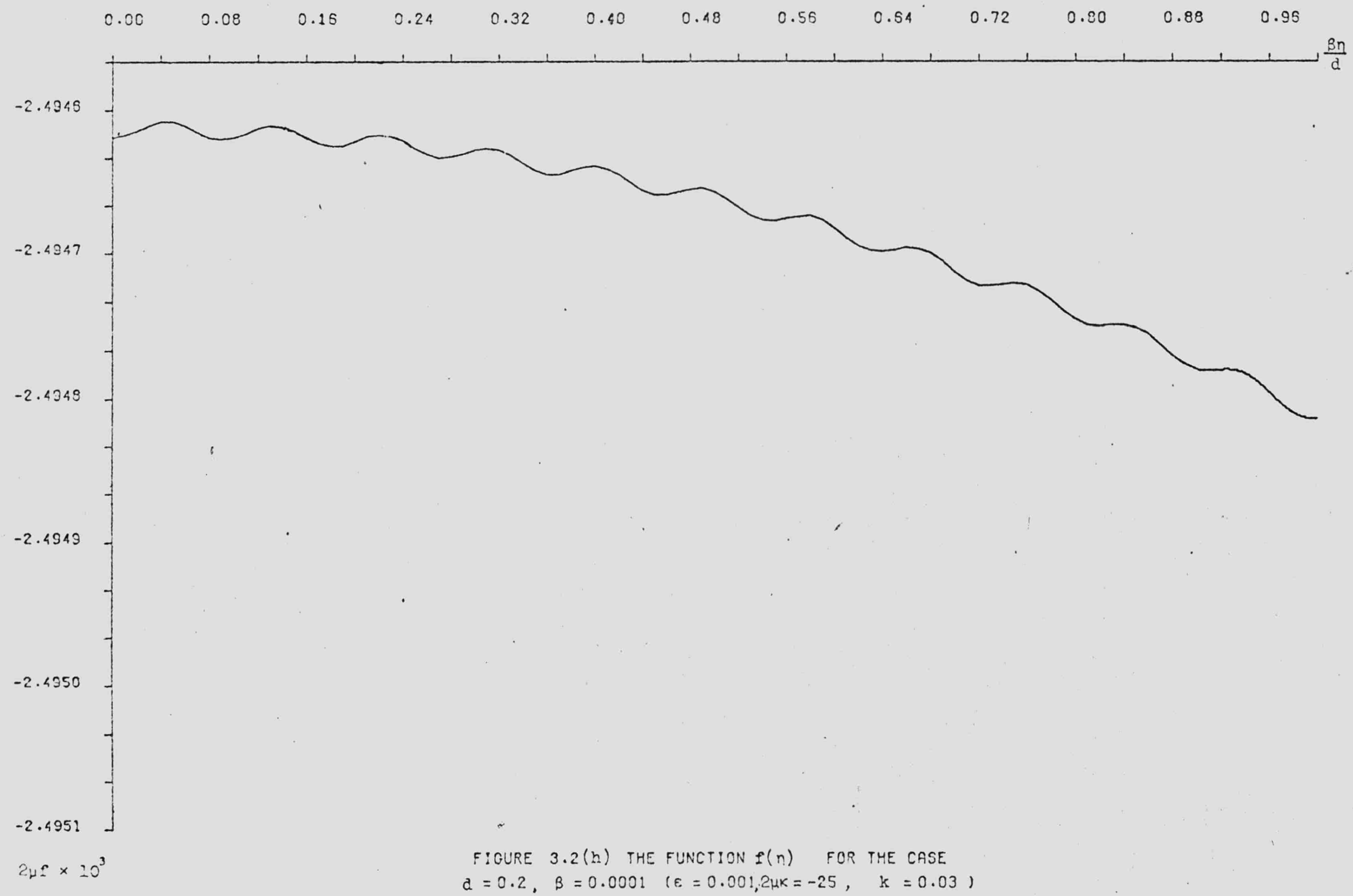


FIGURE 3.2(h) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.2$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

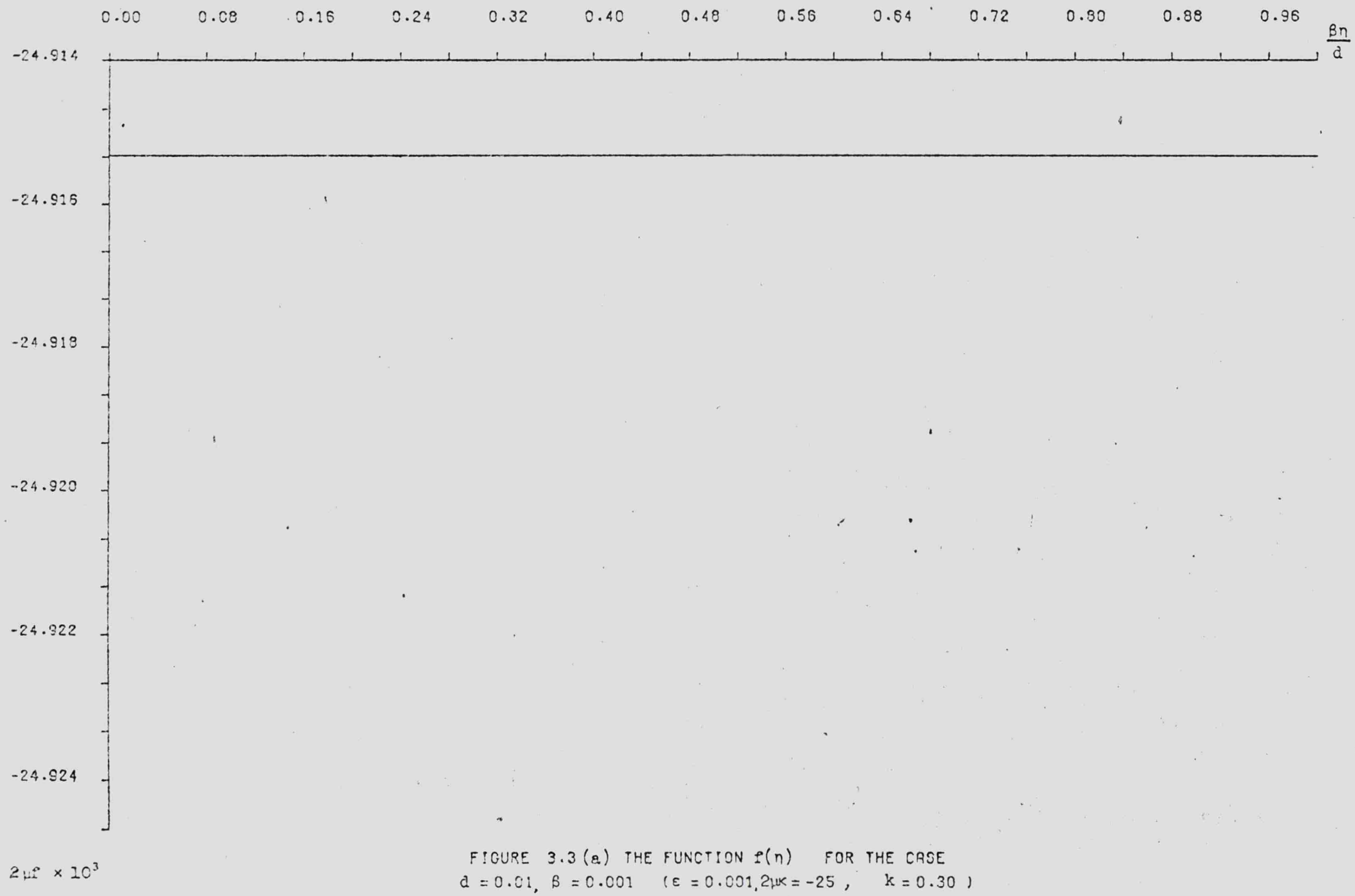


FIGURE 3.3 (a) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.01$, $\beta = 0.001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

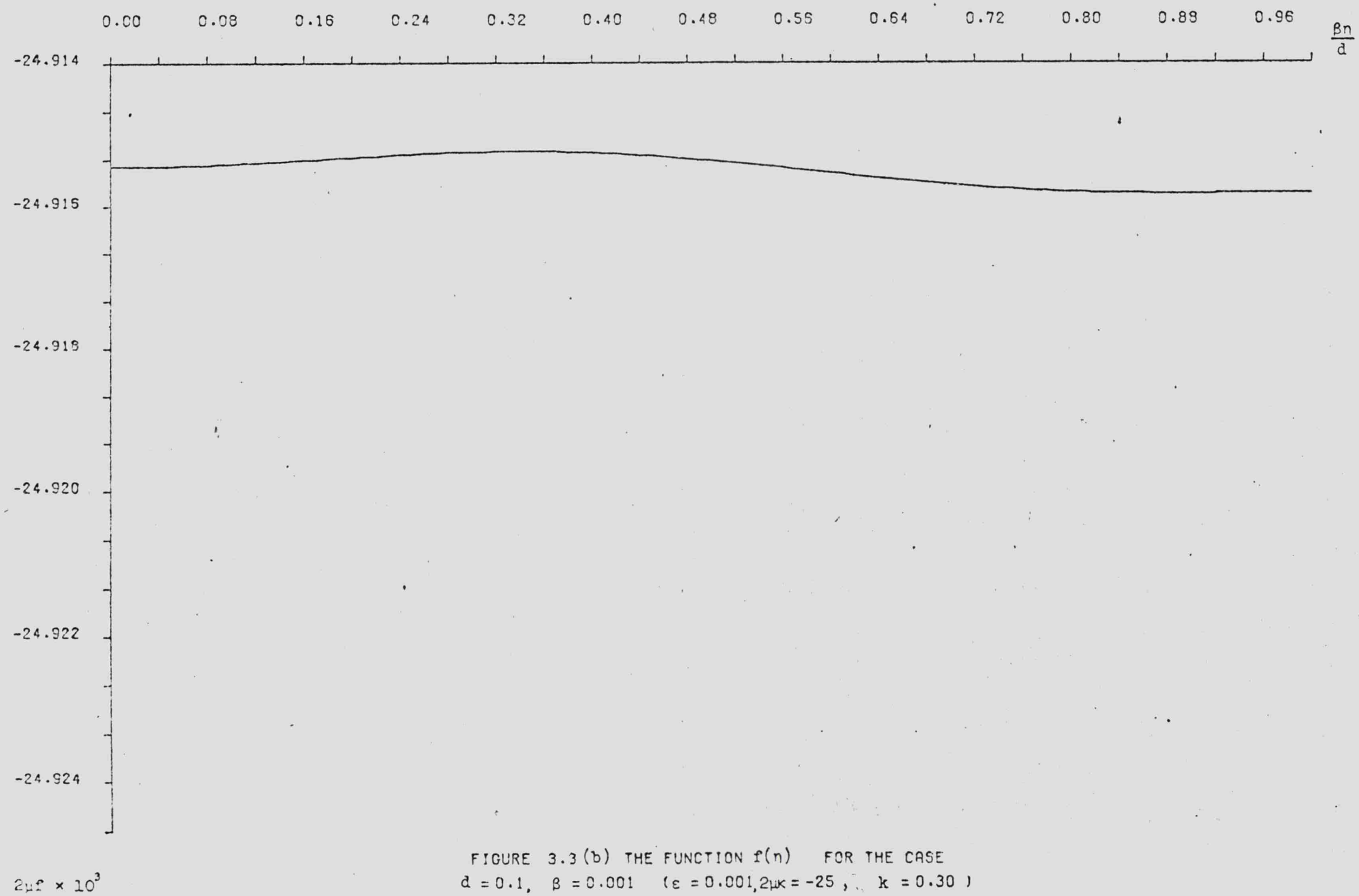


FIGURE 3.3 (b) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.1$, $\beta = 0.001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

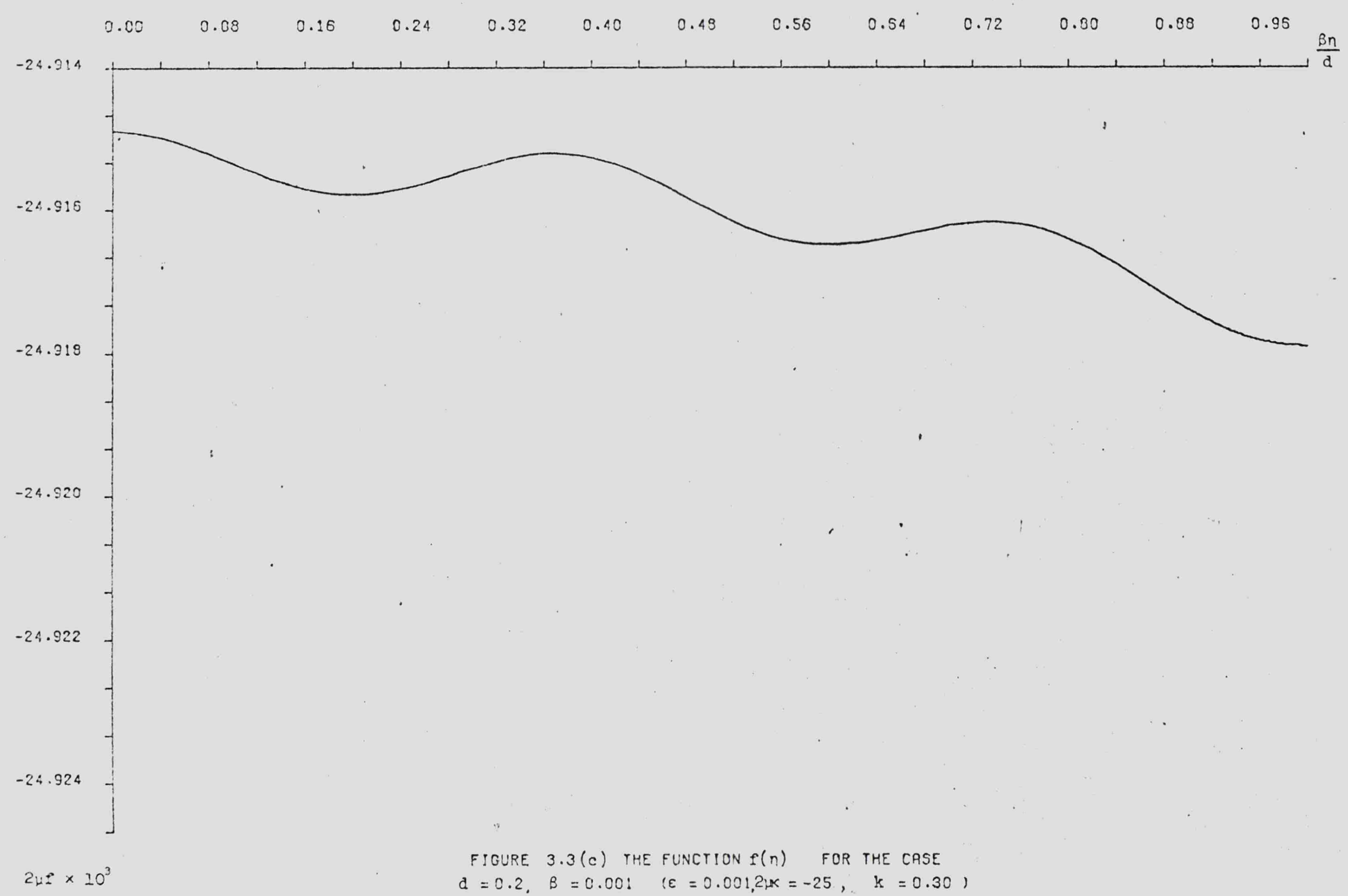


FIGURE 3.3(c) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.2$, $\beta = 0.001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

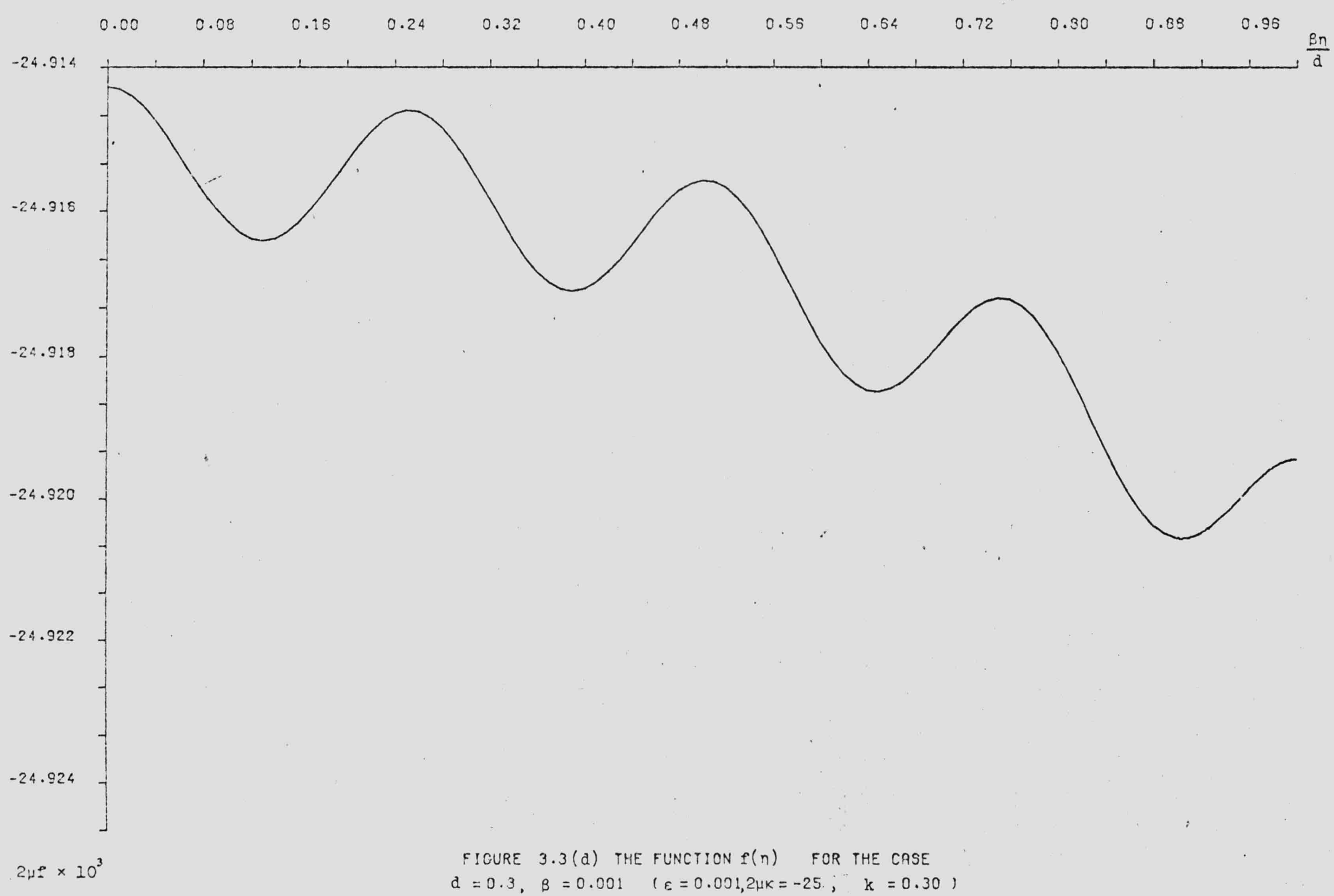


FIGURE 3.3(d) THE FUNCTION $f(n)$ FOR THE CASE
$d = 0.3$, $\beta = 0.001$ ($\varepsilon = 0.001, 2\mu\kappa = -25.$, $k = 0.30$)

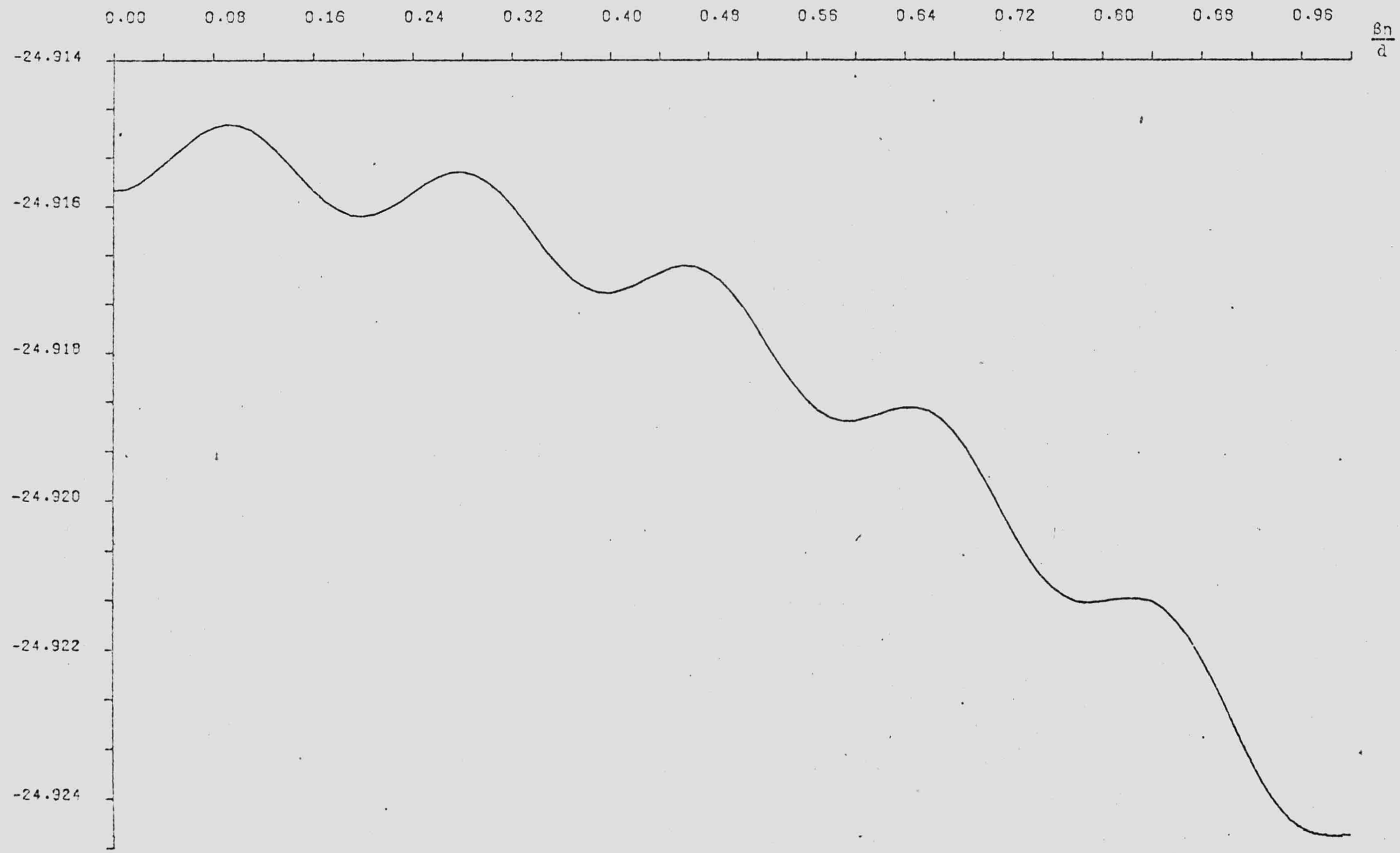


FIGURE 3.3(e) THE FUNCTION $f(\eta)$ FOR THE CASE
$d = 0.4$, $\beta = 0.001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

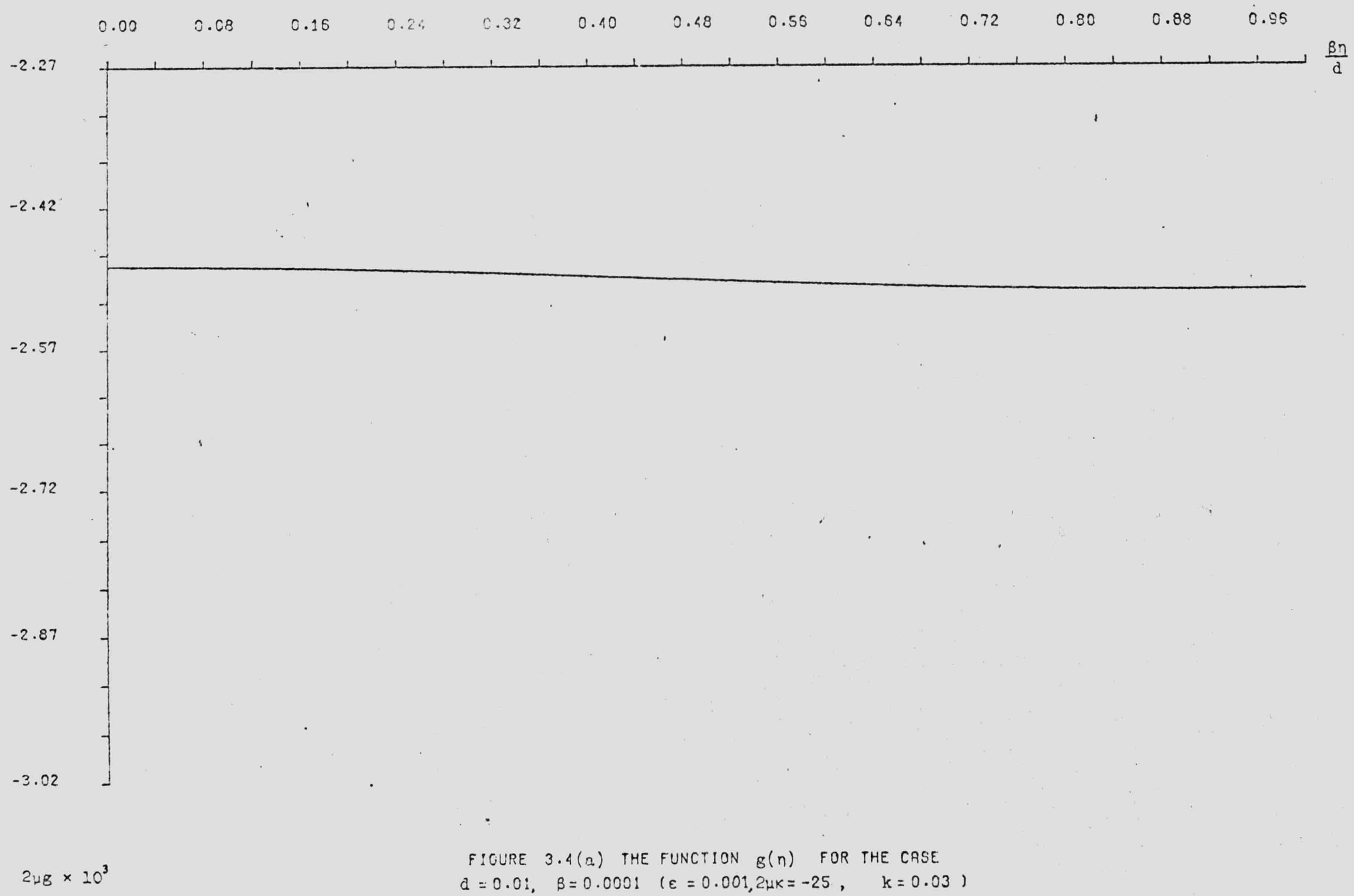


FIGURE 3.4(a) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.01$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

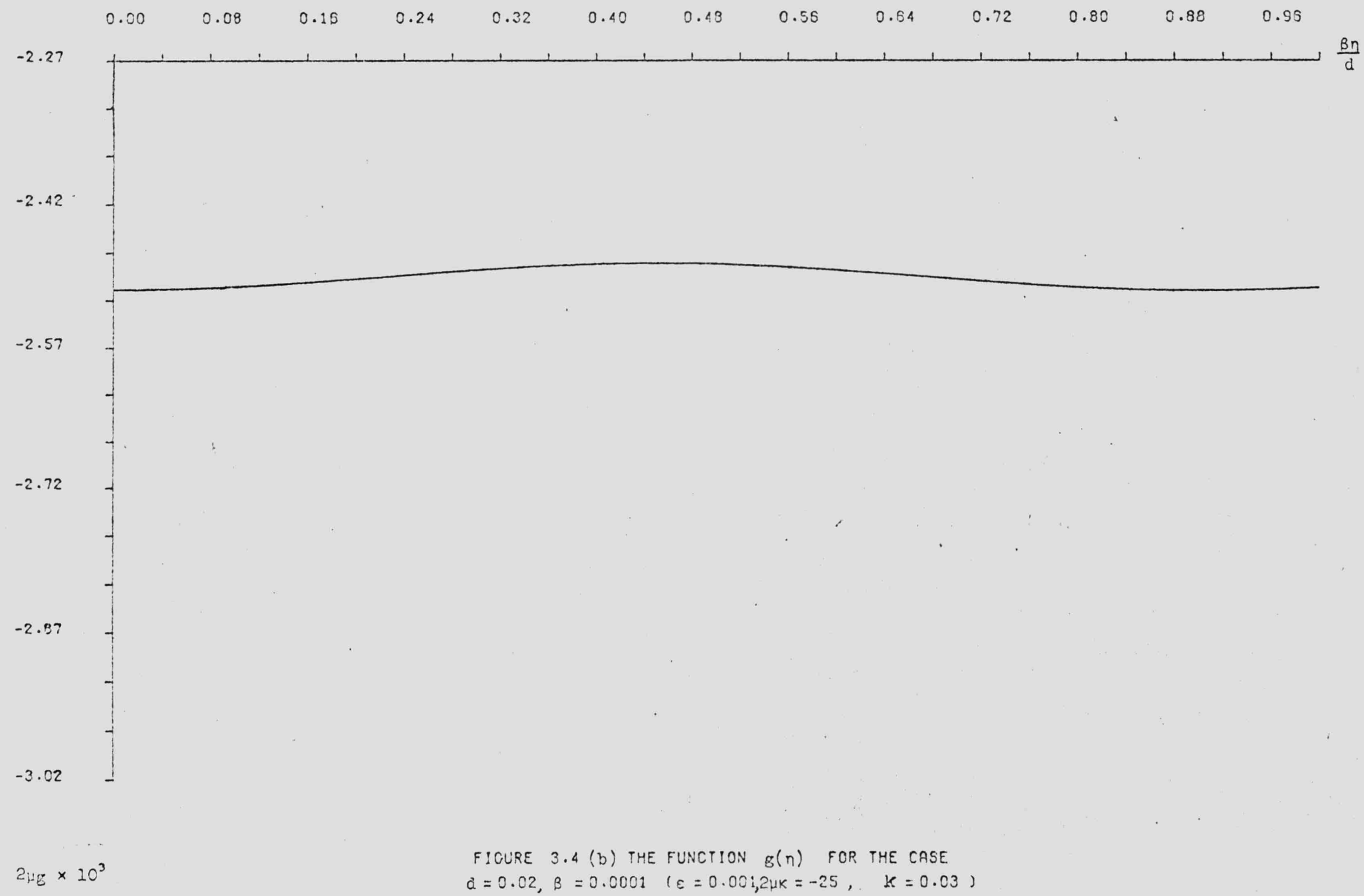


FIGURE 3.4 (b) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.02$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $K = 0.03$)

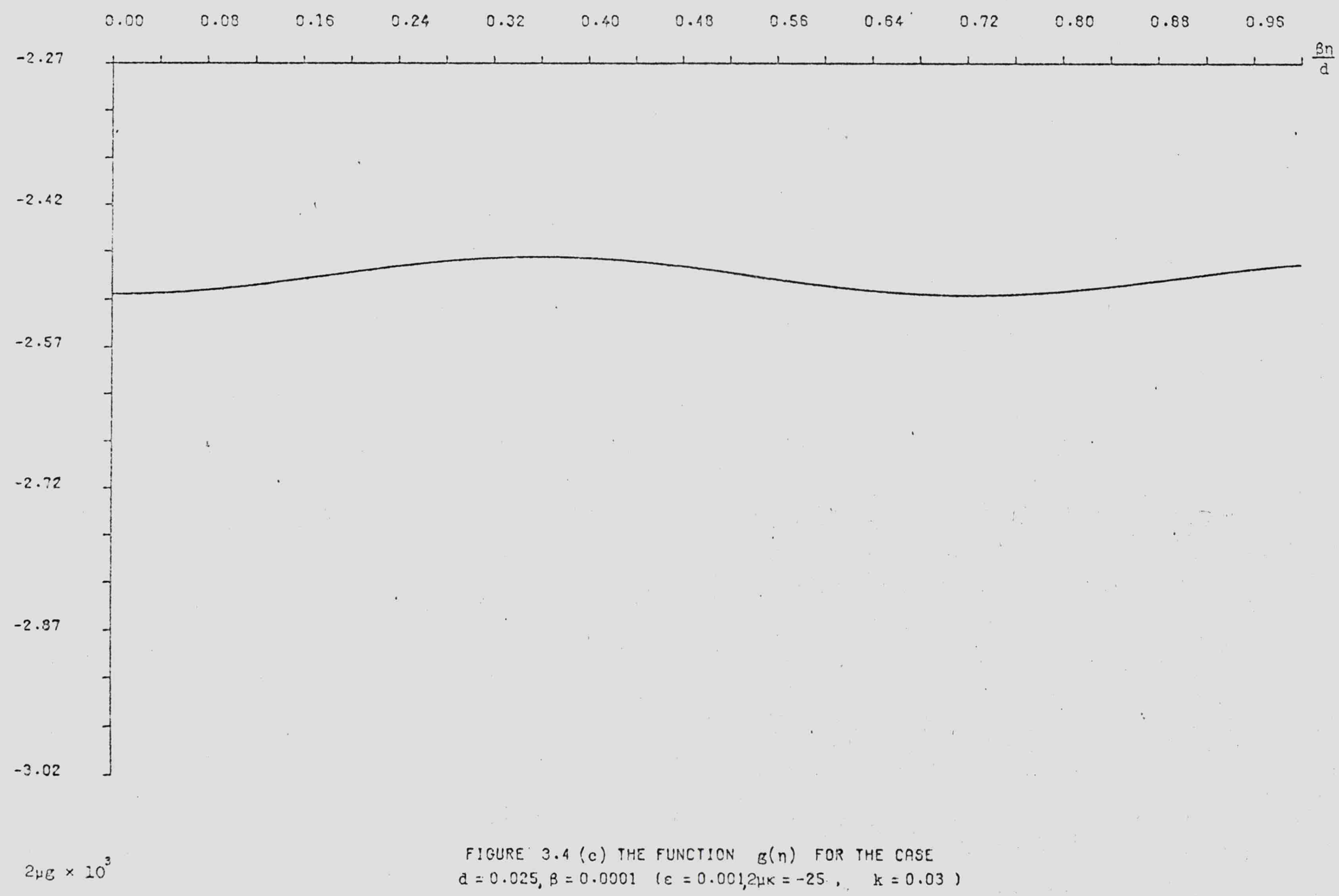


FIGURE 3.4 (c) THE FUNCTION $g(n)$ FOR THE CASE
$d = 0.025$, $\beta = 0.0001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

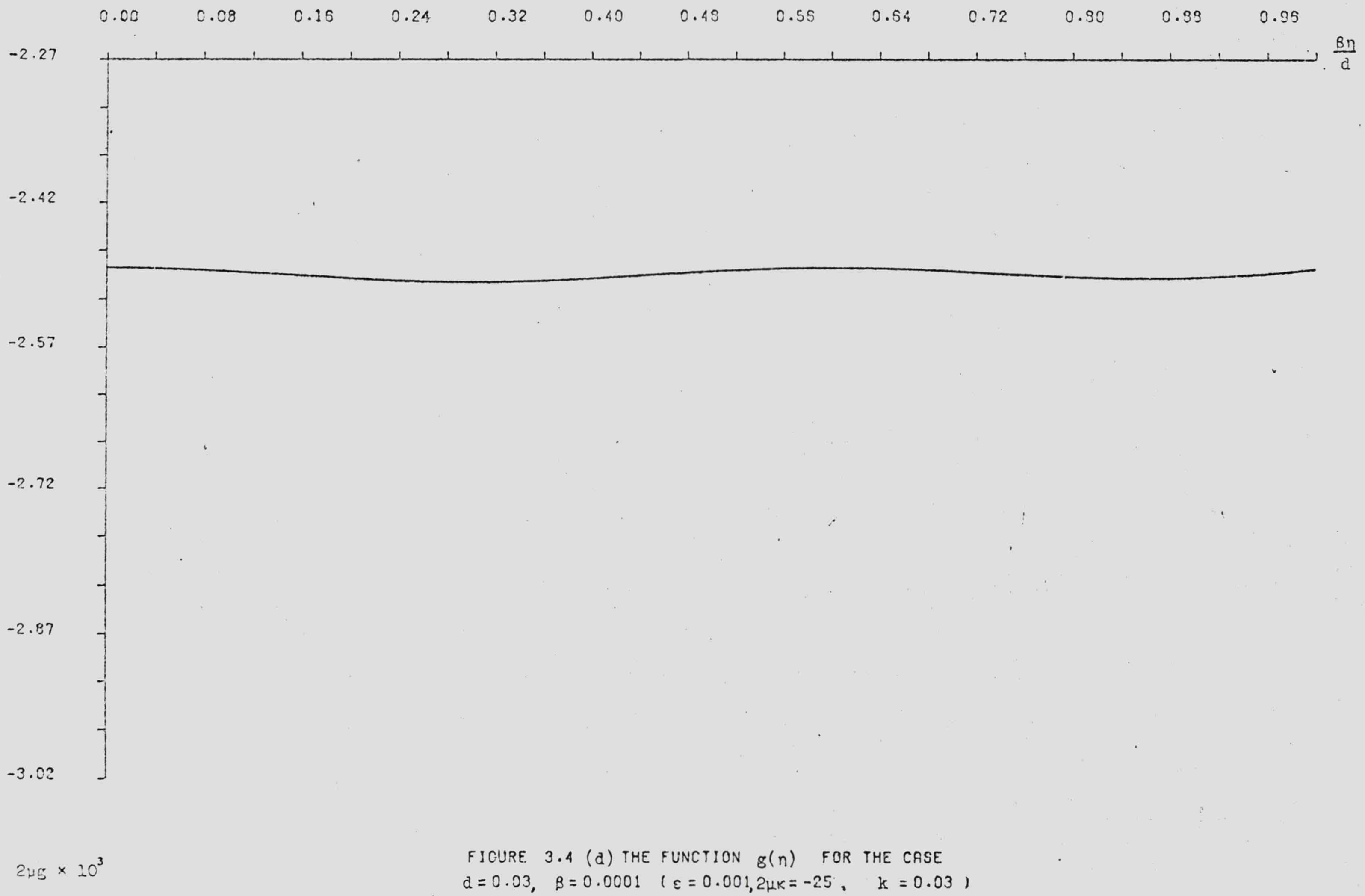


FIGURE 3.4 (d) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.03$, $\beta = 0.0001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

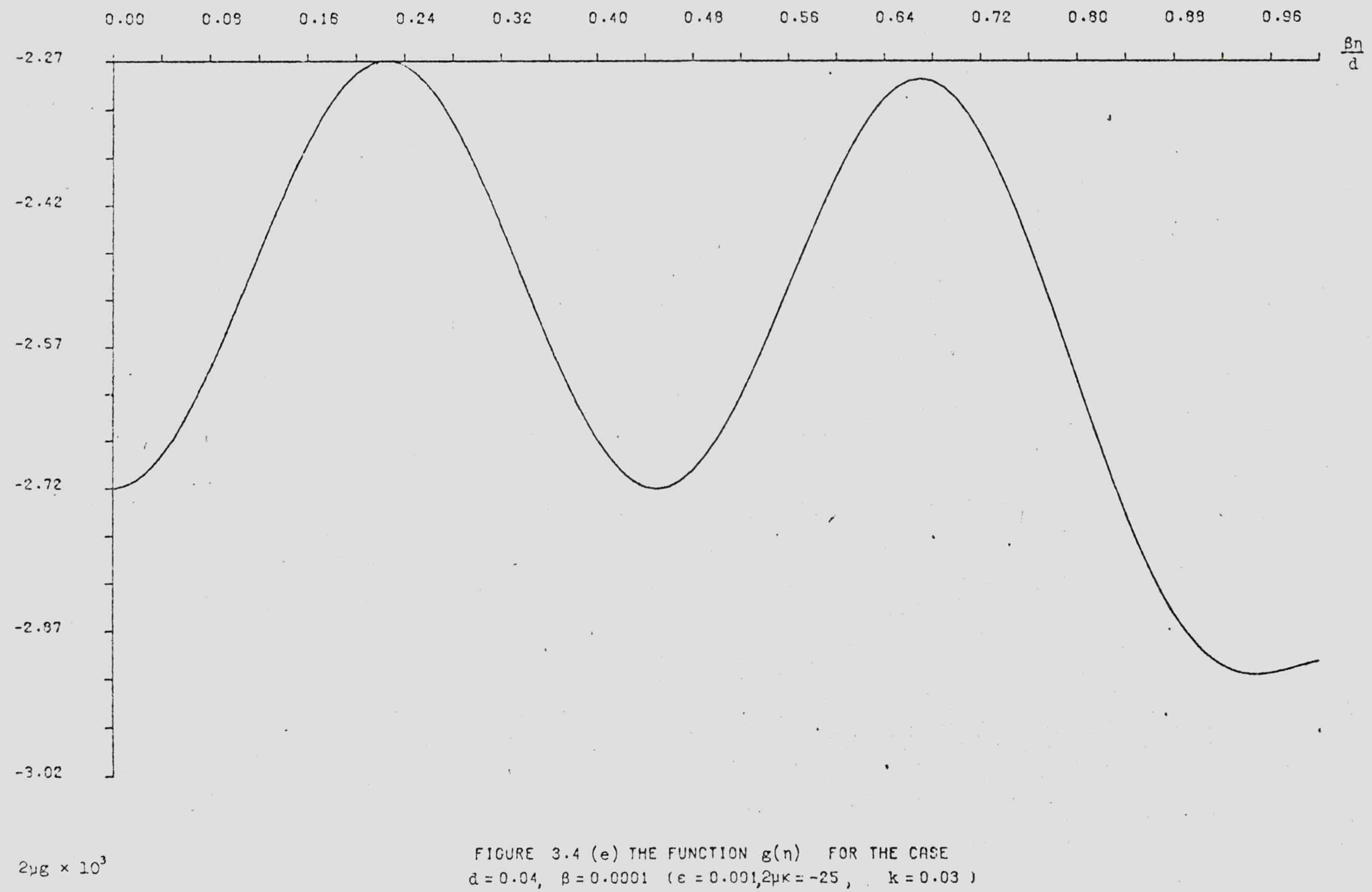


FIGURE 3.4 (e) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.04$, $\beta = 0.0001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.03$)

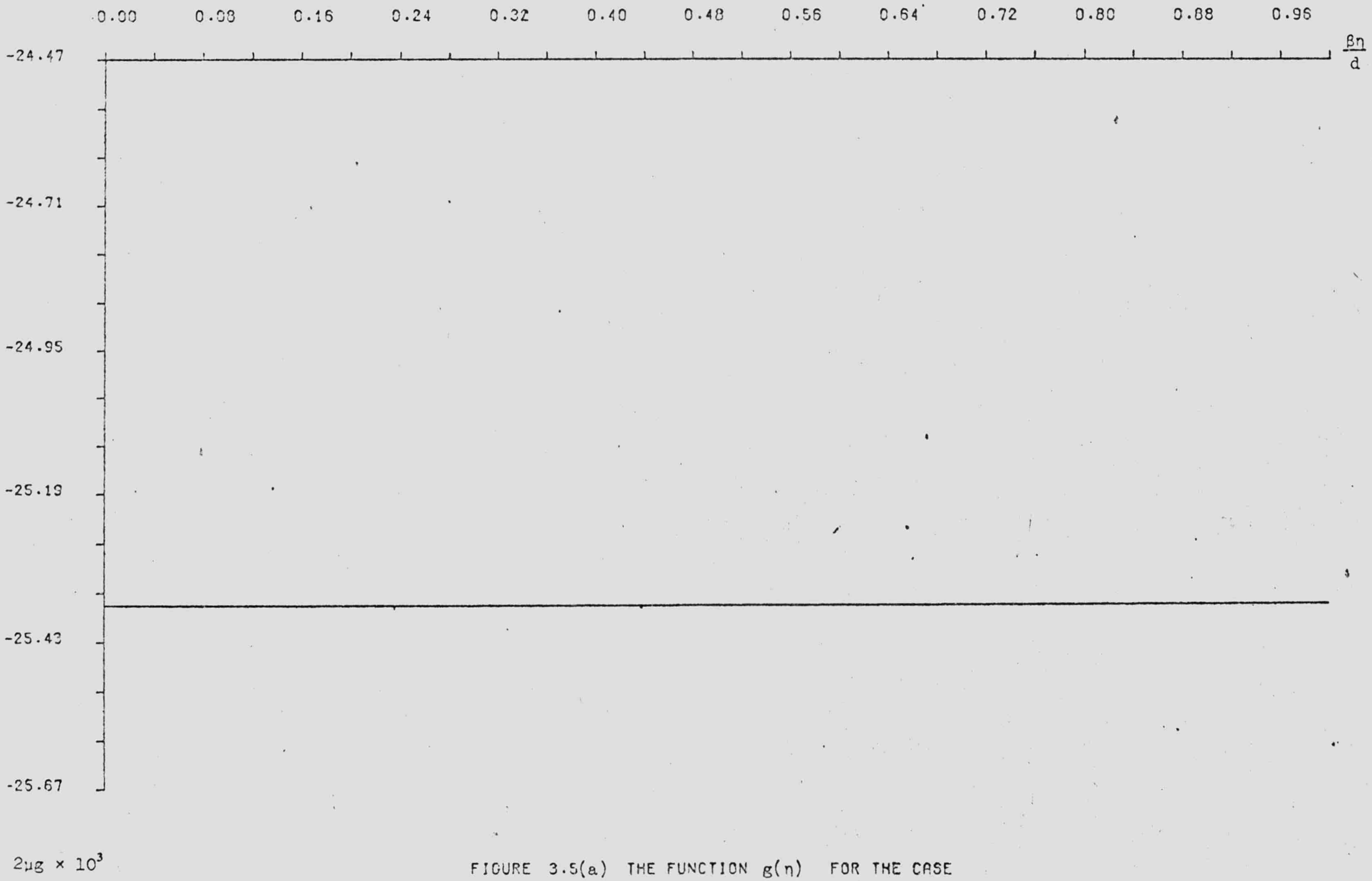


FIGURE 3.5(a) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.01$, $\beta = 0.001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

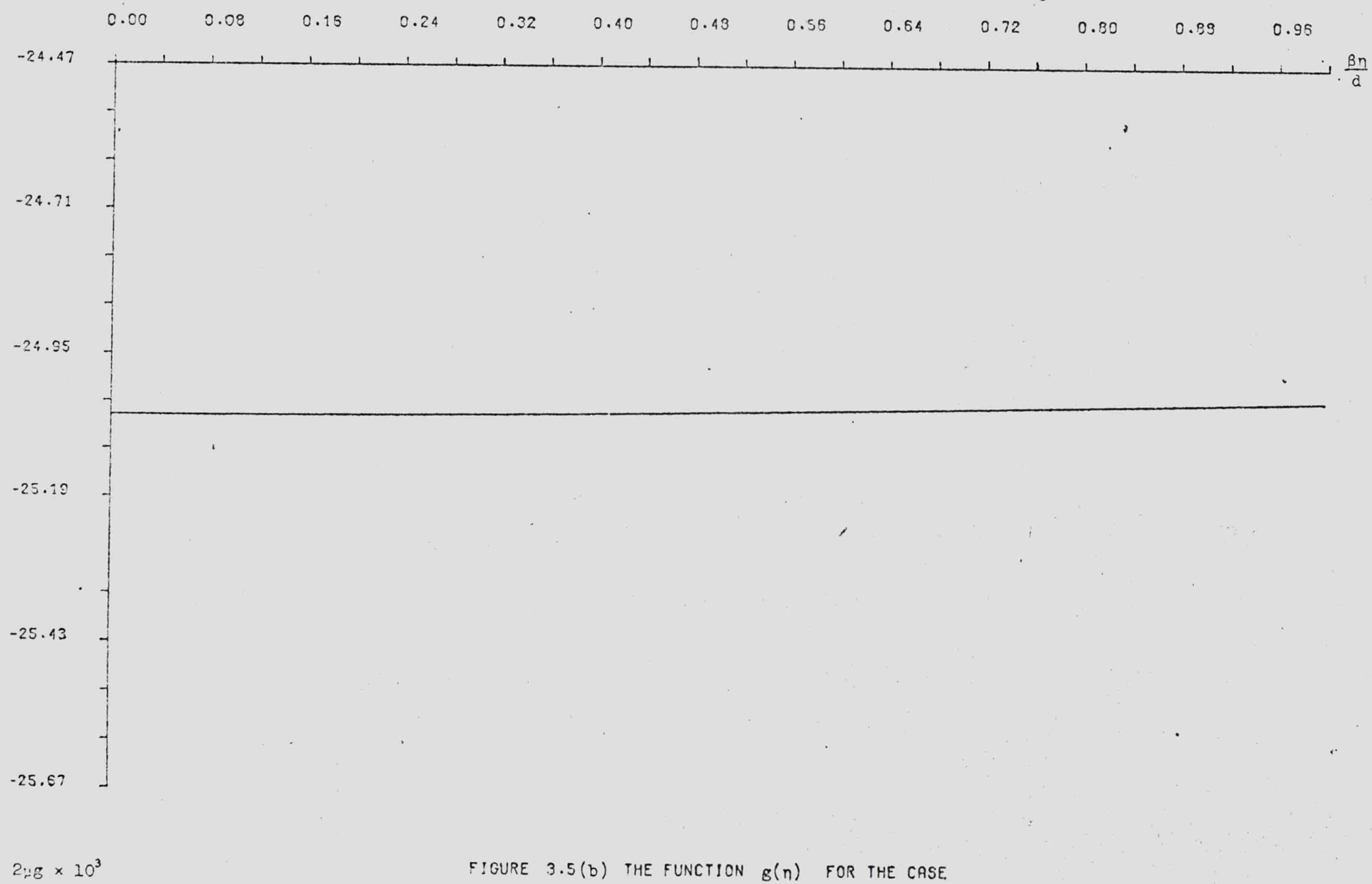


FIGURE 3.5(b) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.02$, $\beta = 0.001$ ($\epsilon = 0.001, 2\mu\kappa = -25$, $k = 0.30$)

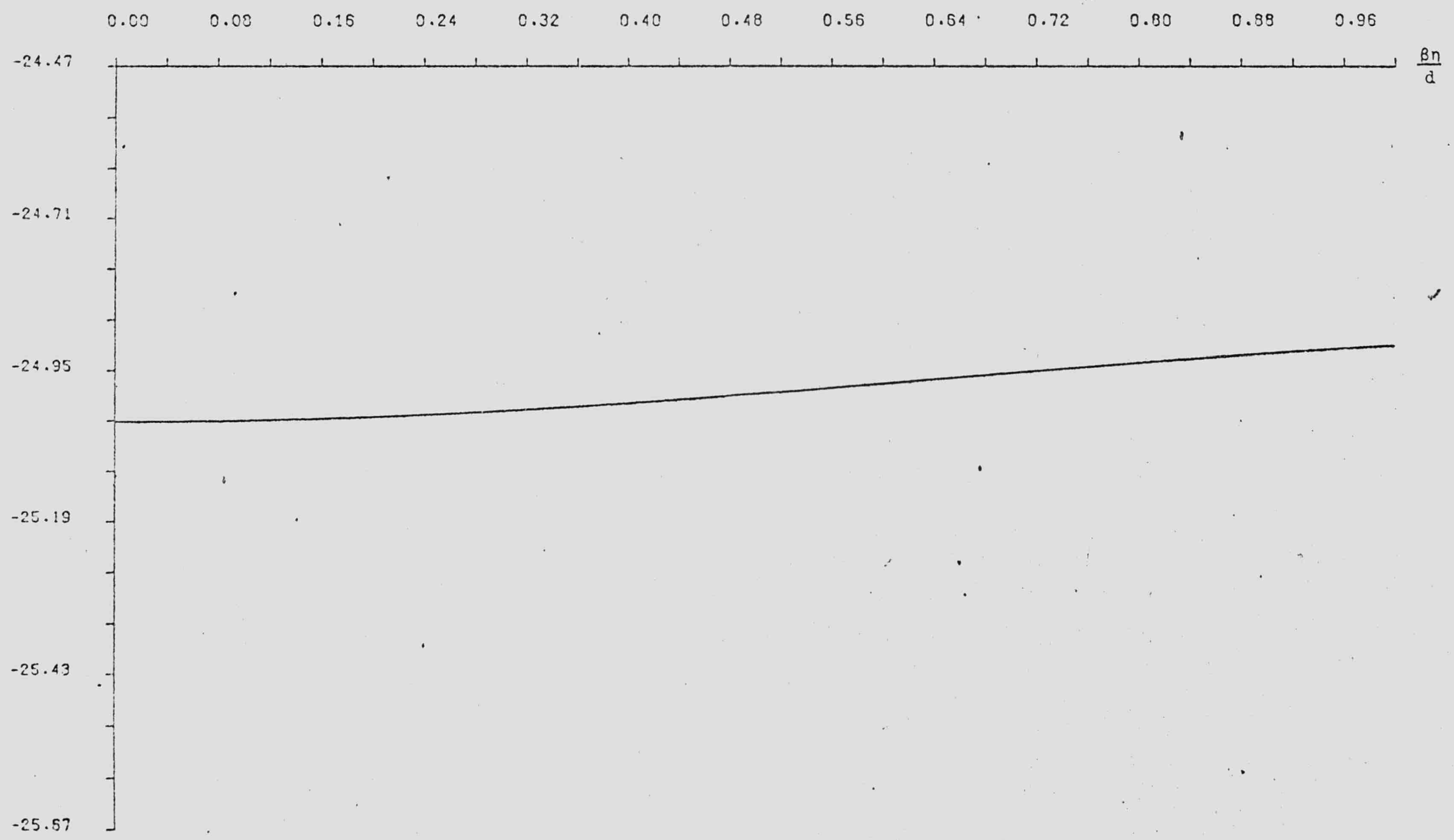


FIGURE 3.5(c) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.03$, $\beta = 0.001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

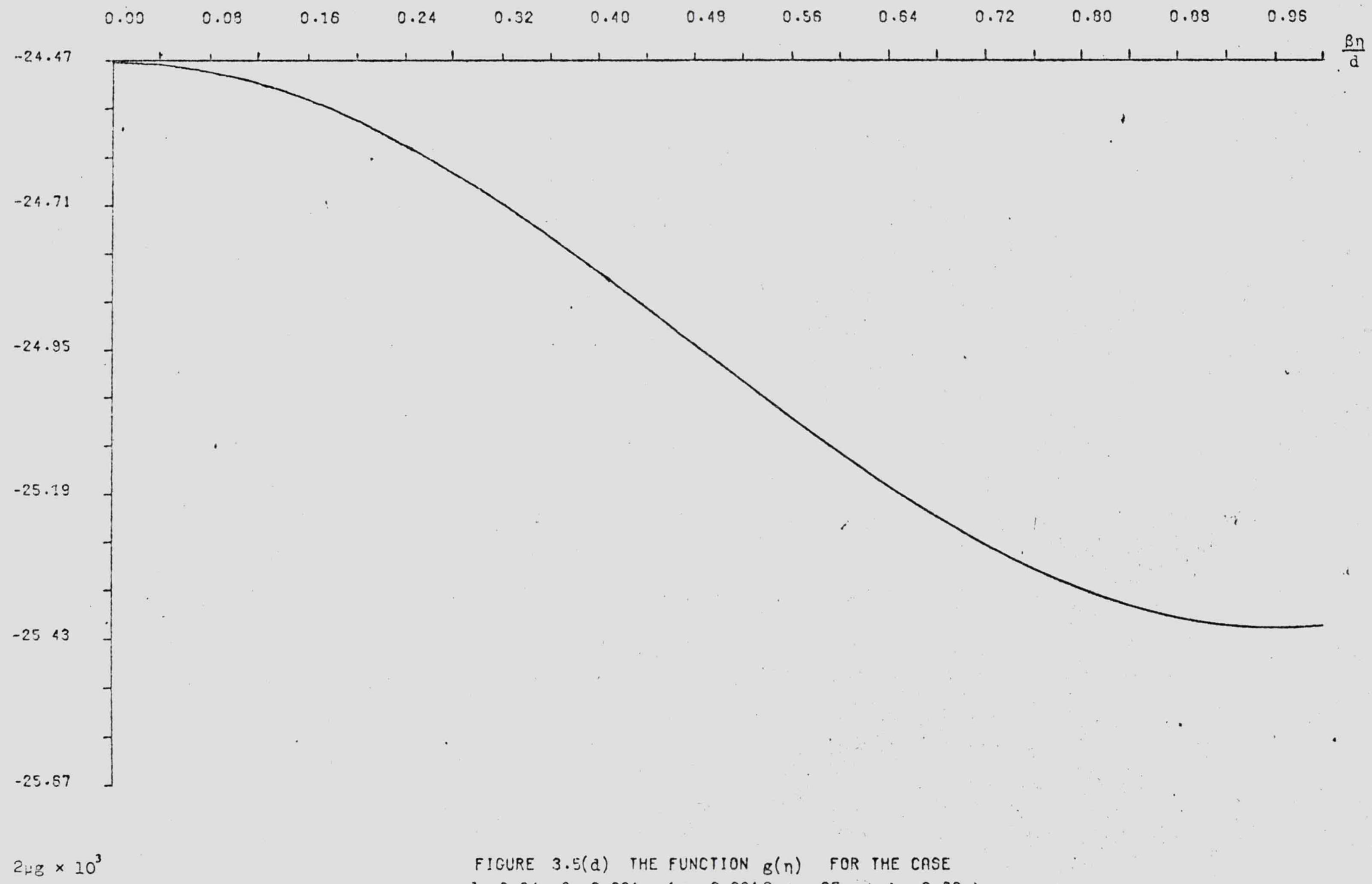


FIGURE 3.5(d) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.04$, $\beta = 0.001$ ($\epsilon = 0.001, 2\mu\kappa = -25$, $k = 0.30$)

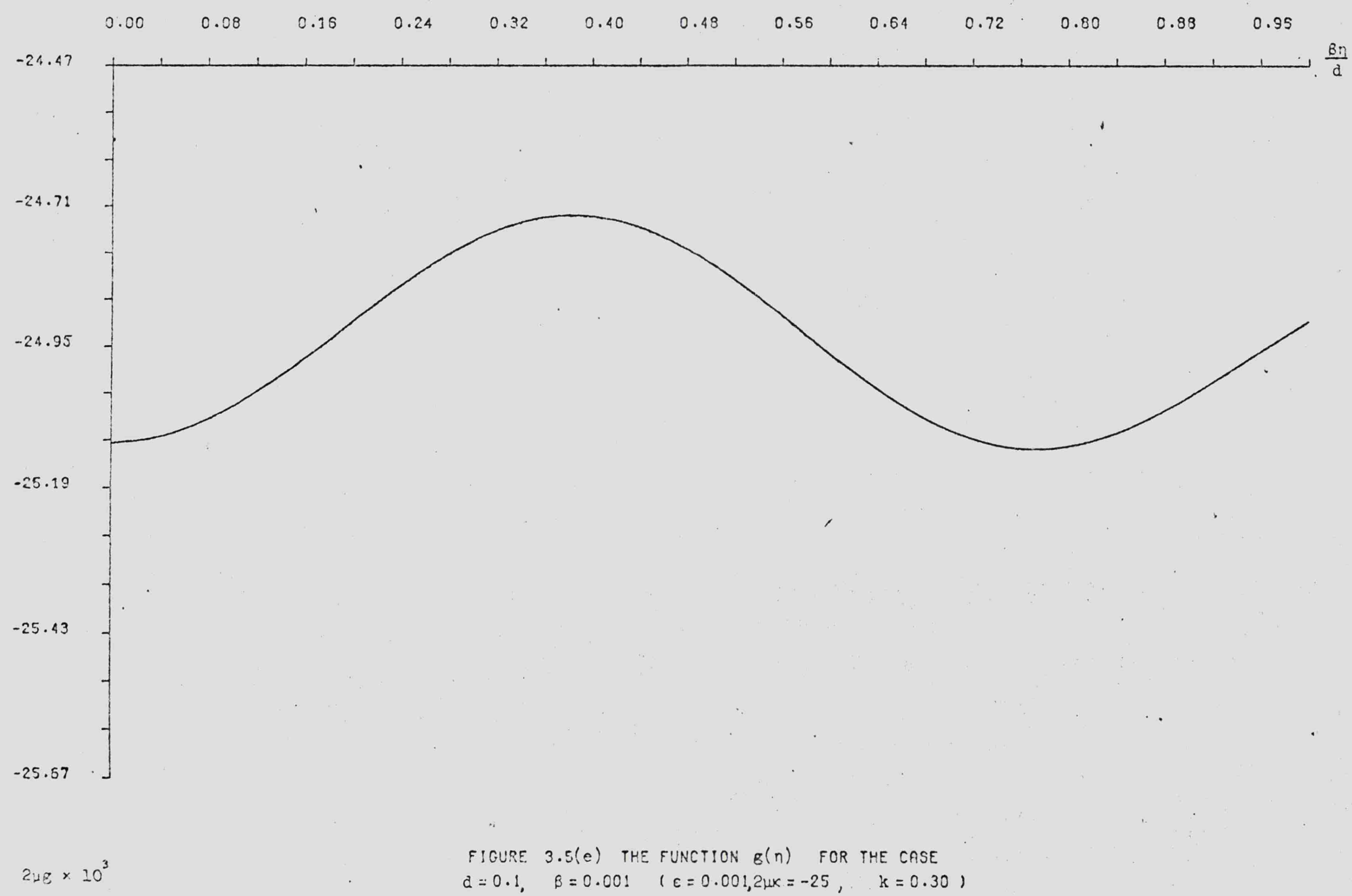


FIGURE 3.5(e) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.1$, $\beta = 0.001$ ($\epsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

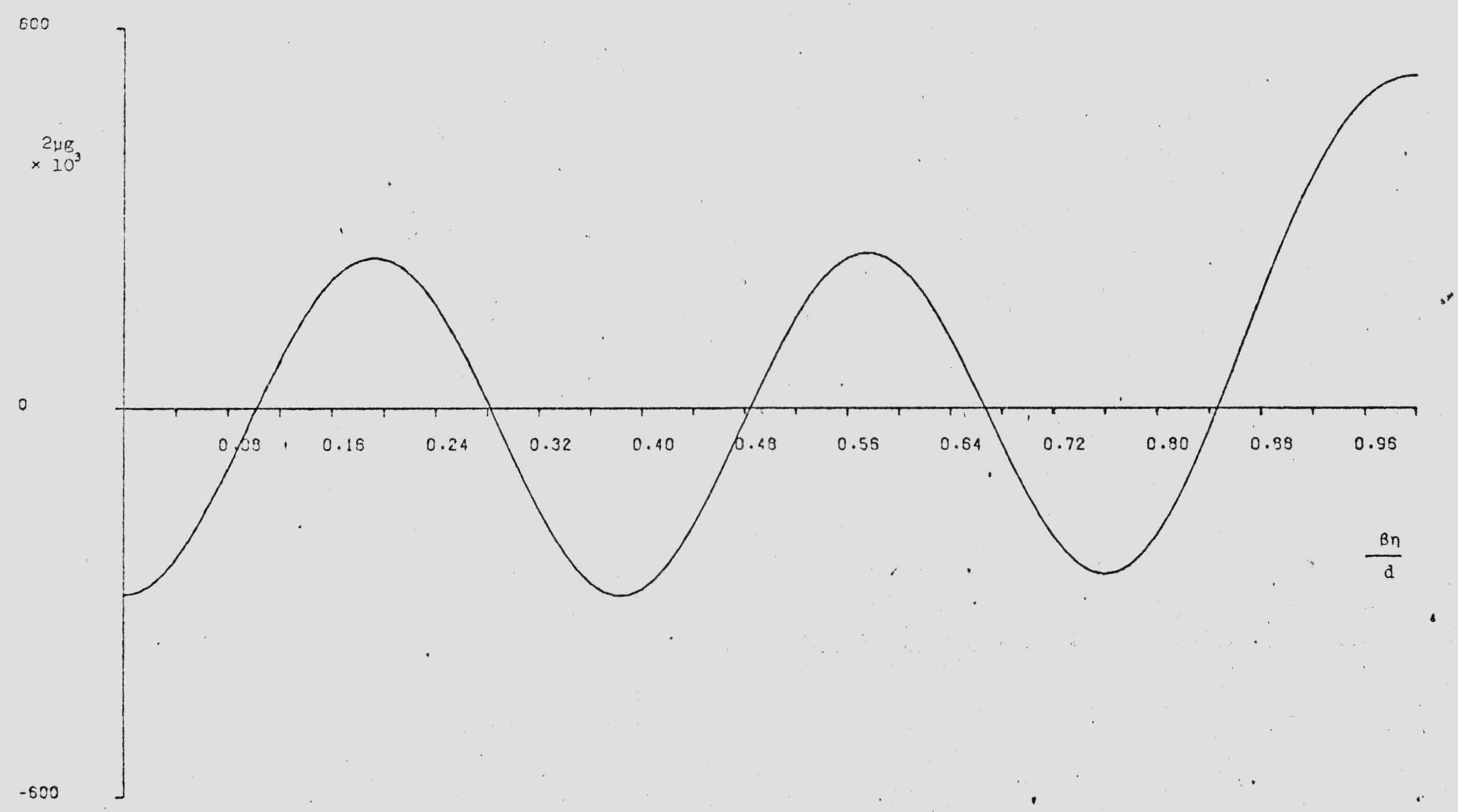


FIGURE 3.5(f) THE FUNCTION $g(\eta)$ FOR THE CASE
$d = 0.2$, $\beta = 0.001$ ($\varepsilon = 0.001$, $2\mu\kappa = -25$, $k = 0.30$)

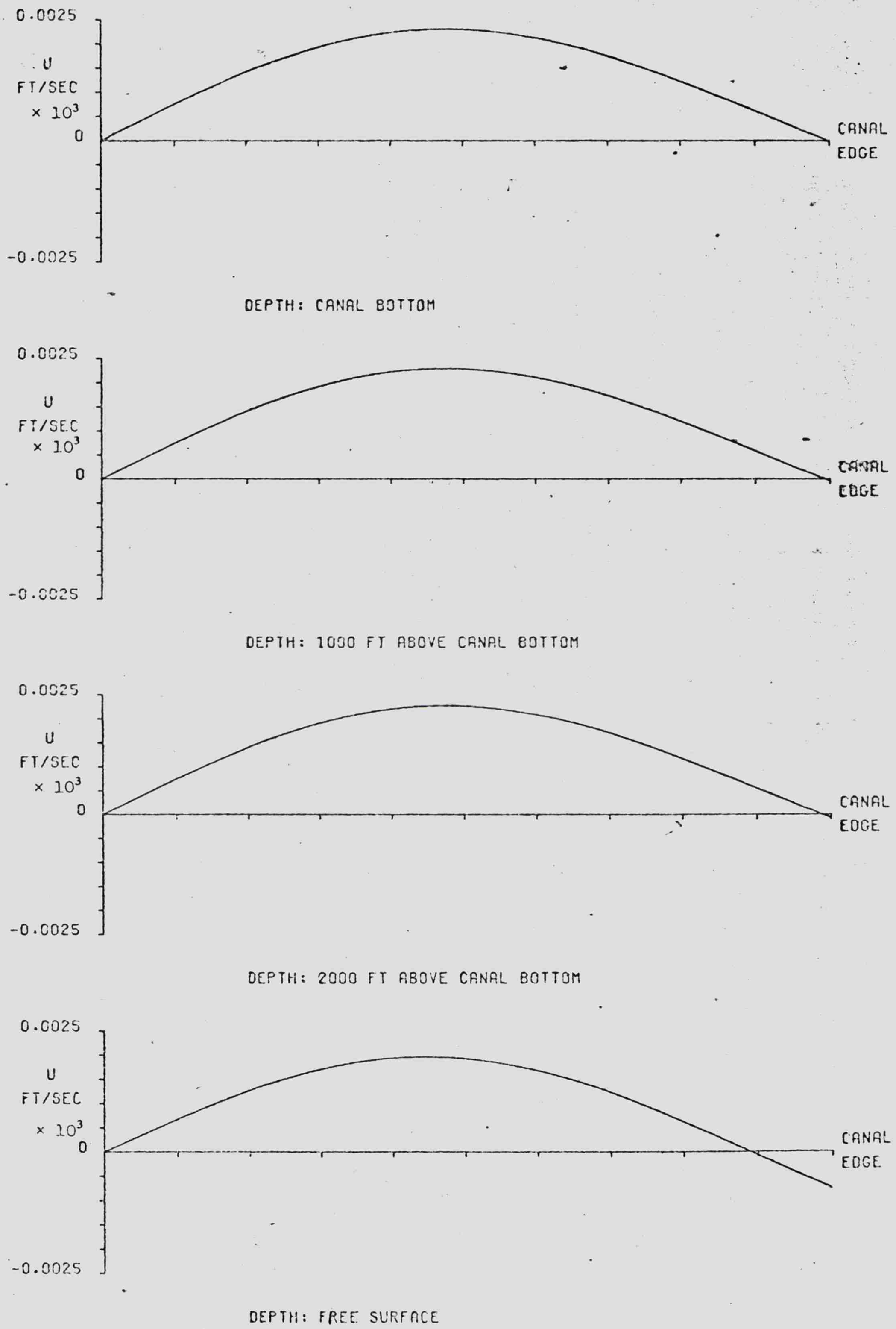
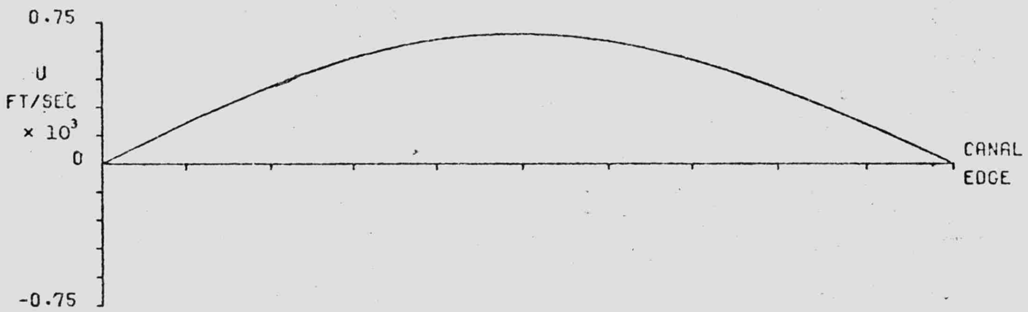
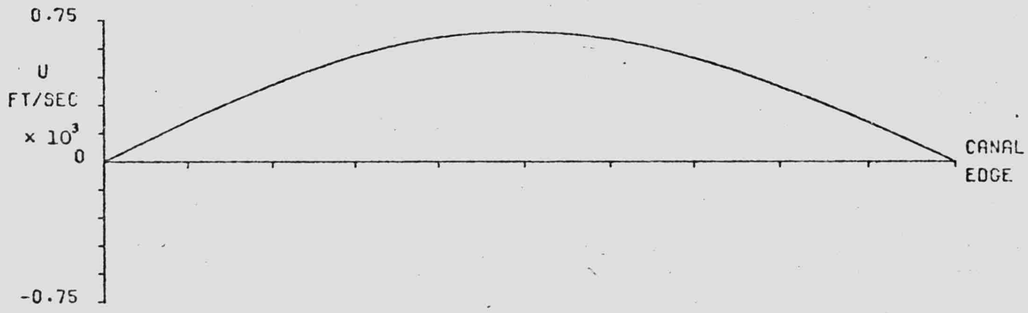


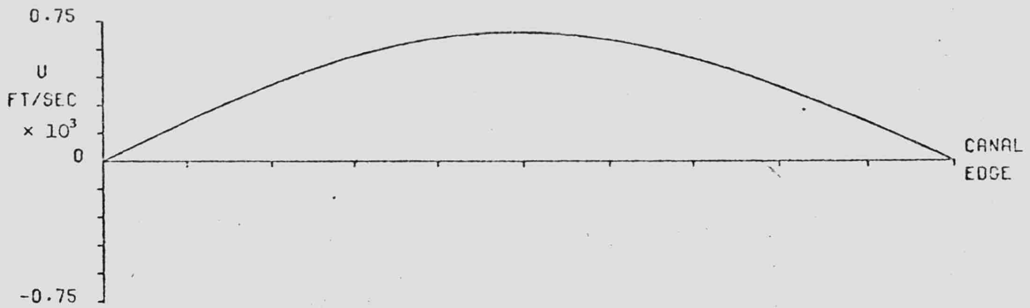
FIG 3.6 U FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
A THE DISTANCE FROM EQUATOR TO CANAL EDGE
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$



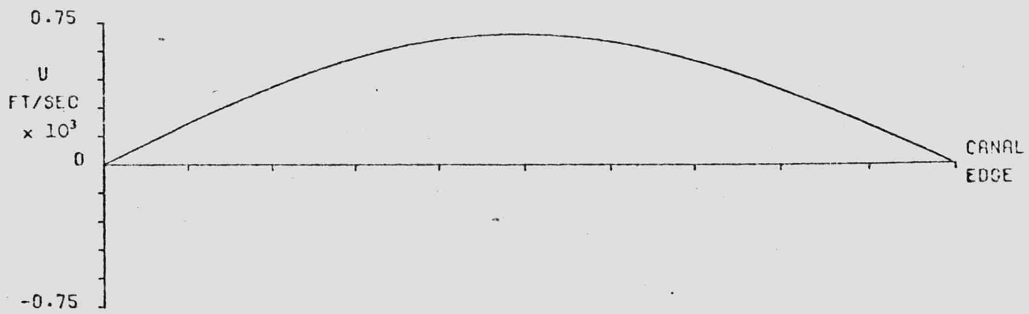
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



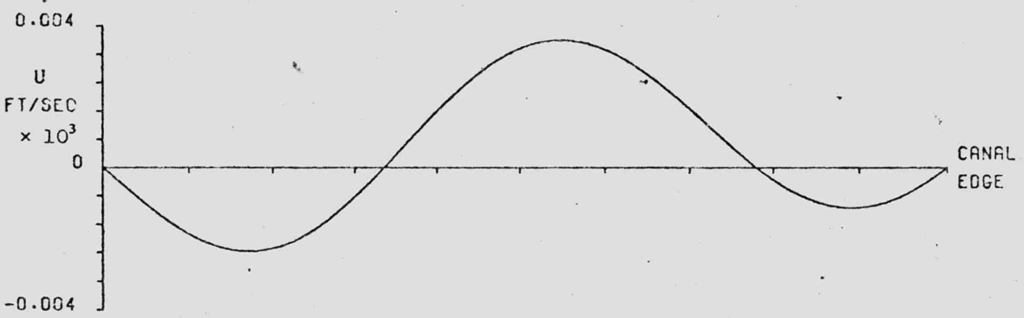
DEPTH: 2000 FT ABOVE CANAL BOTTOM



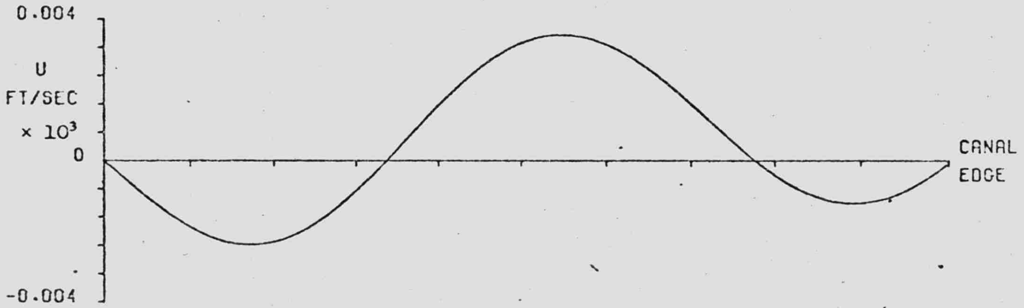
DEPTH: FREE SURFACE

FIG 3.6B U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

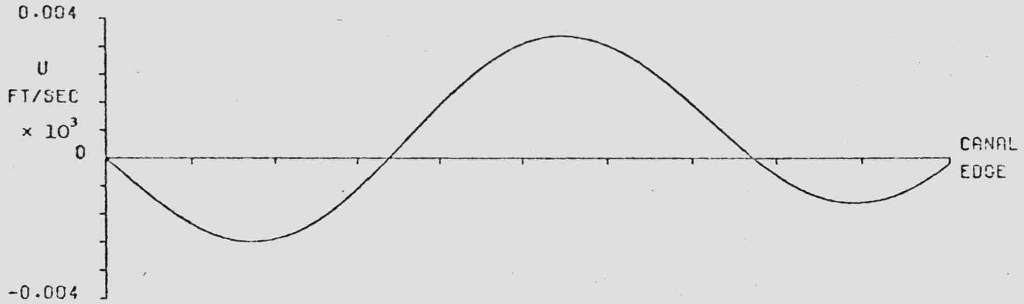
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$



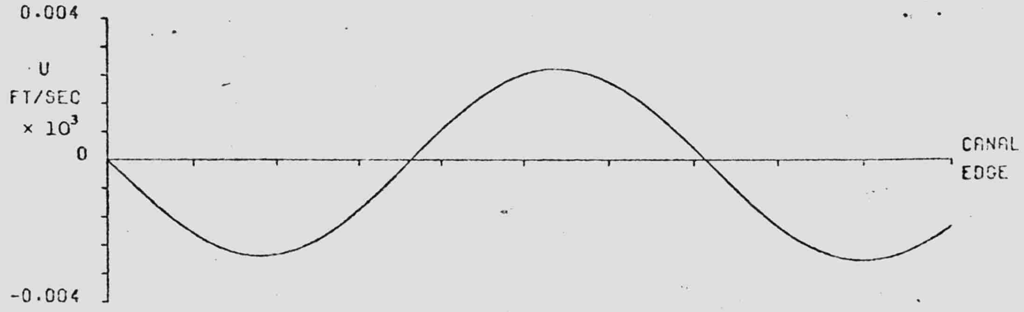
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



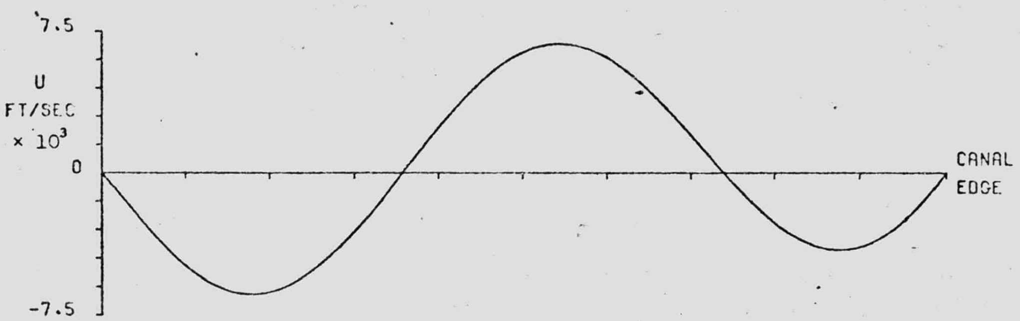
DEPTH: 2000 FT ABOVE CANAL BOTTOM



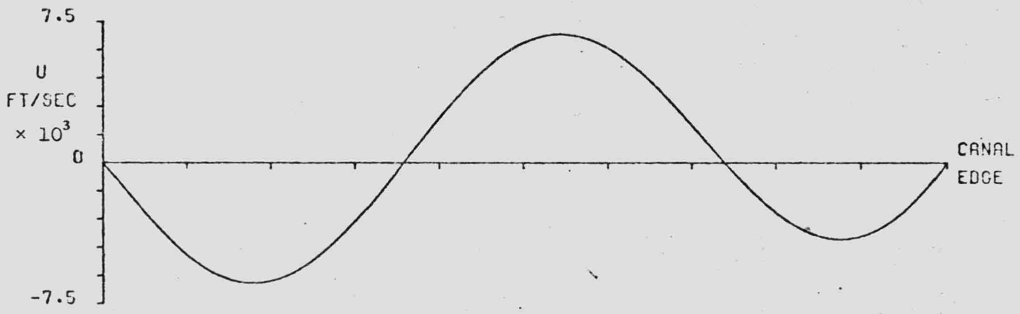
DEPTH: FREE SURFACE

FIG 3.6c U FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

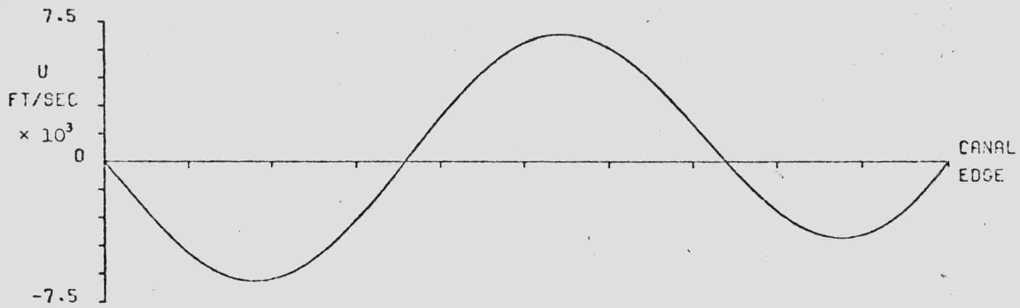
CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) β = 0.0001



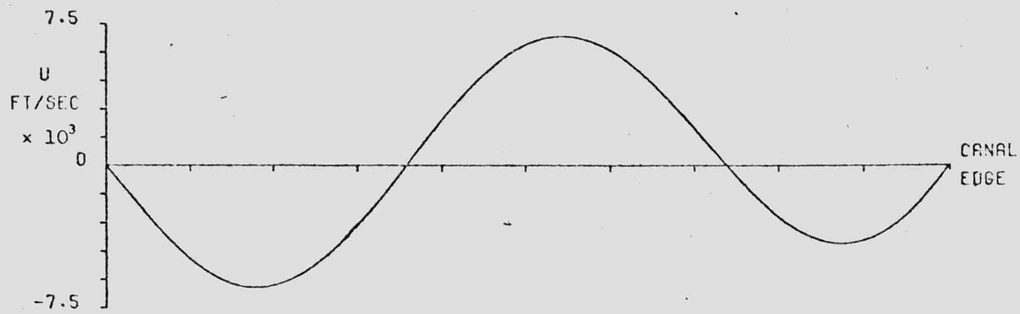
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



DEPTH: 2000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.6 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

D

CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) β = 0.0001

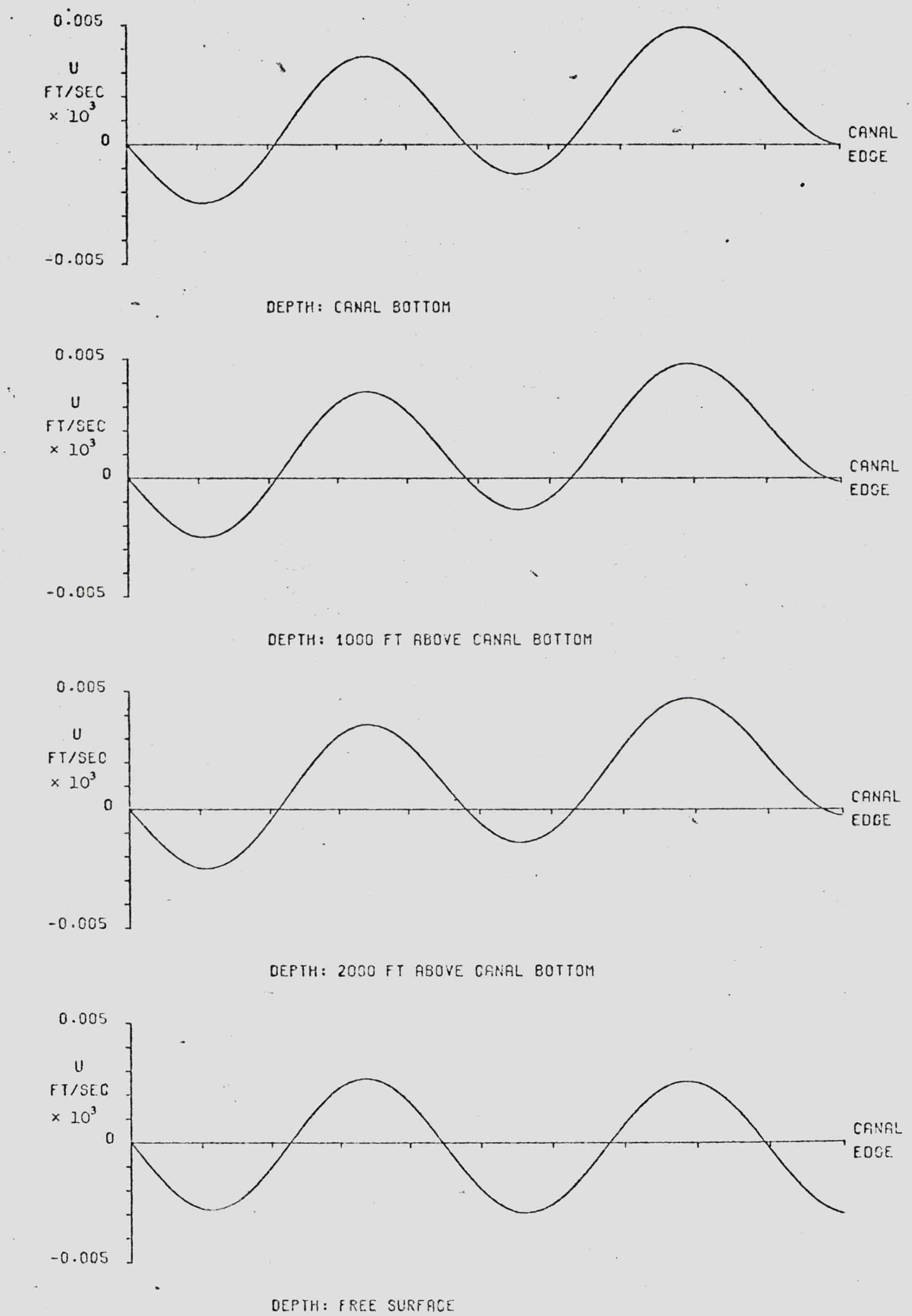


FIG 3.6 U_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) $\delta = 0.0001$

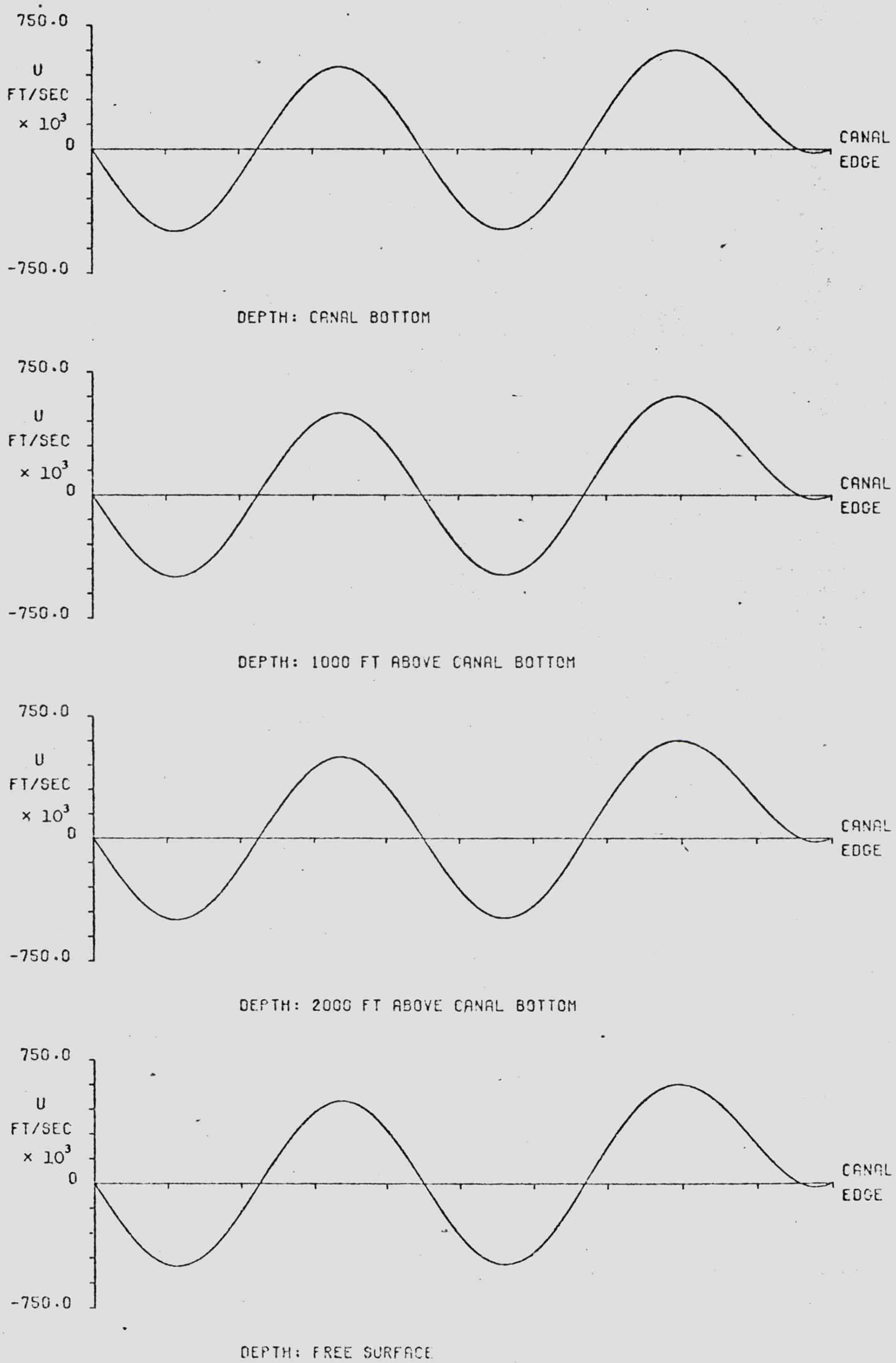


FIG 3.6 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) β = 0.0001

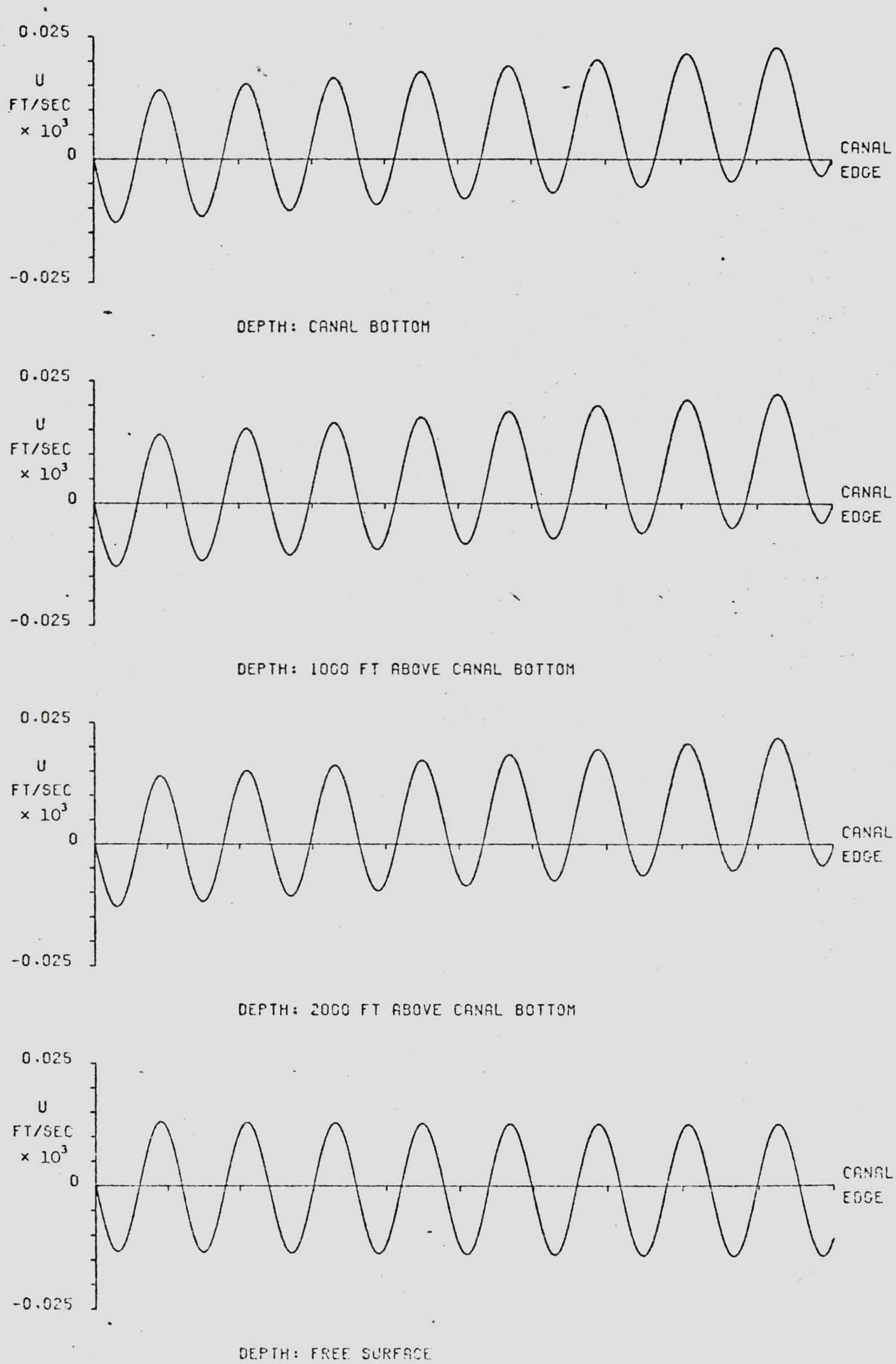
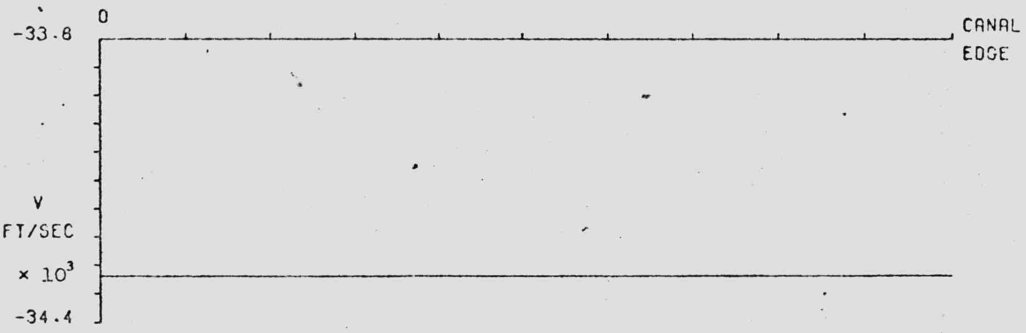
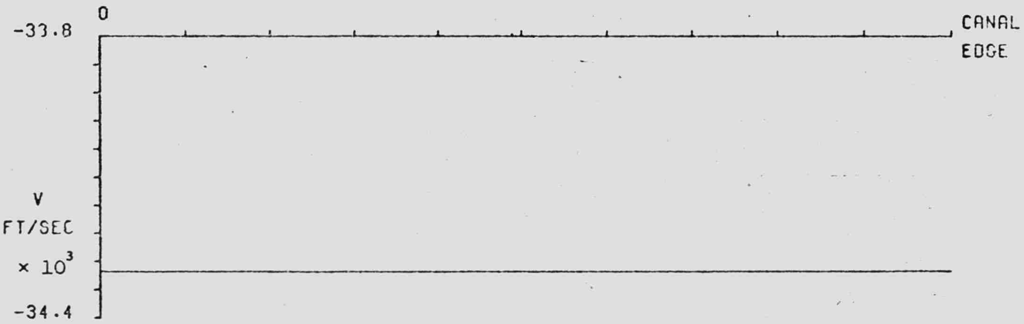


FIG 3.6 U FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE
G

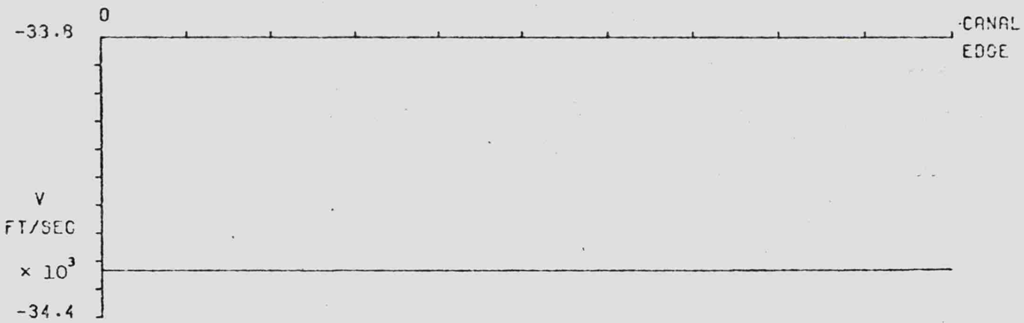
CANAL SEMI-WIDTH= 3×10^6 FT (568.2 MLS) $\beta = 0.0001$



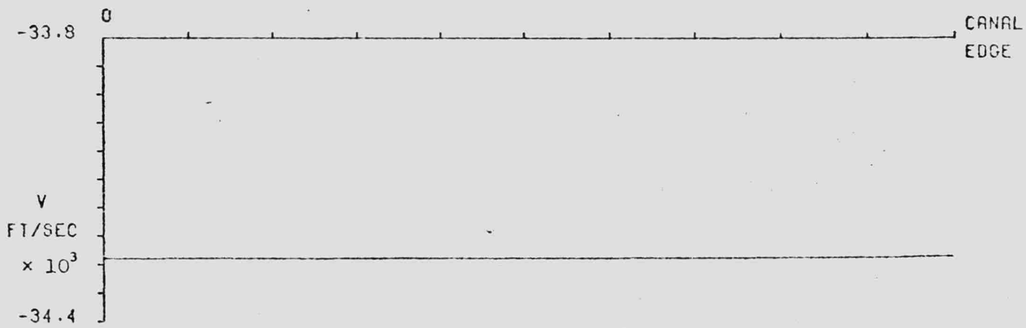
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



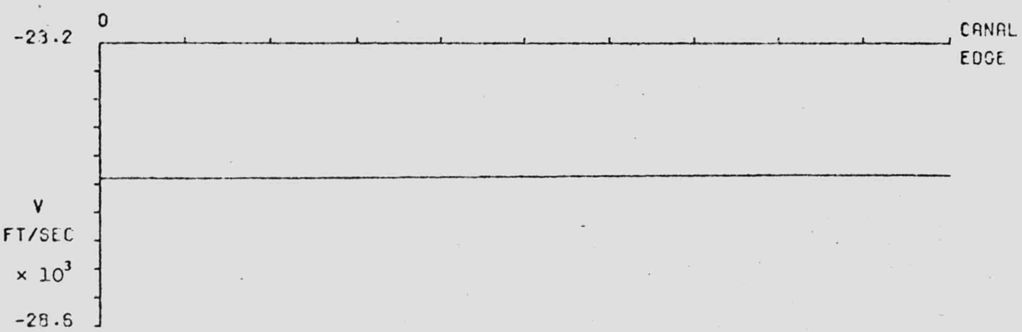
DEPTH: 2000 FT ABOVE CANAL BOTTOM



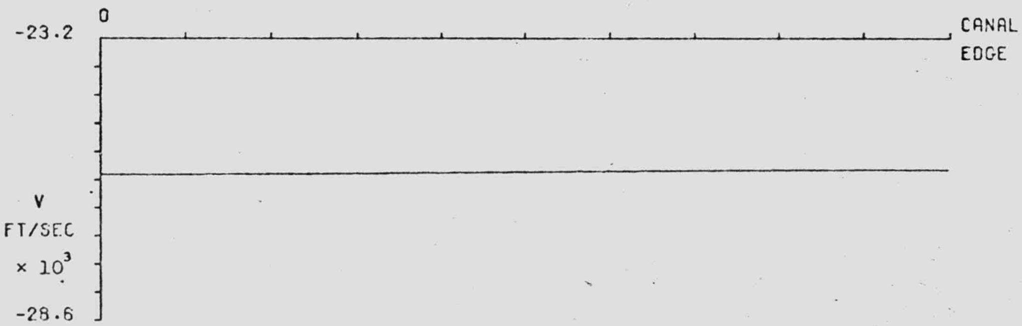
DEPTH: FREE SURFACE

FIG 3.7 V_A FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

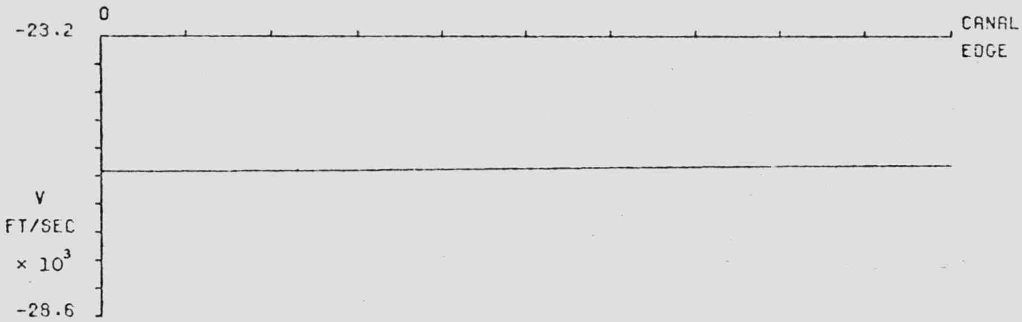
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$



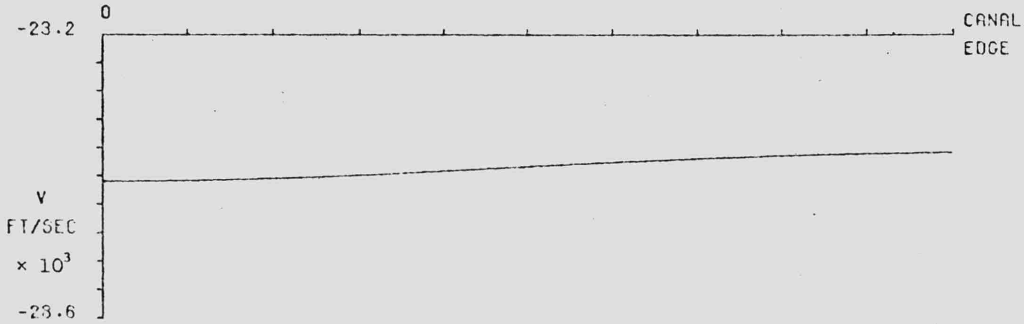
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



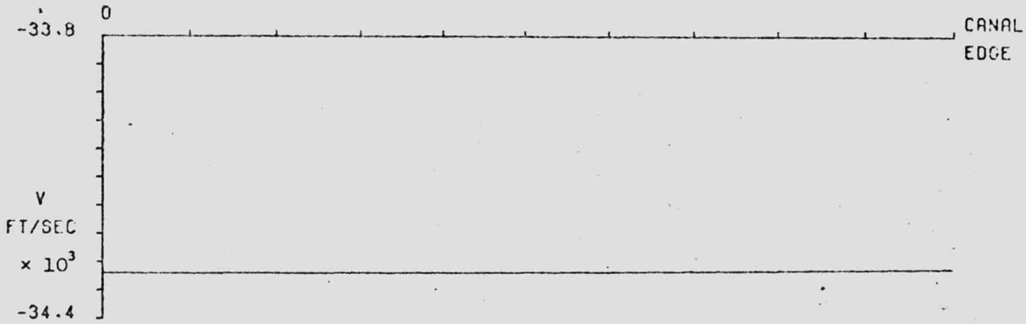
DEPTH: 2000 FT ABOVE CANAL BOTTOM



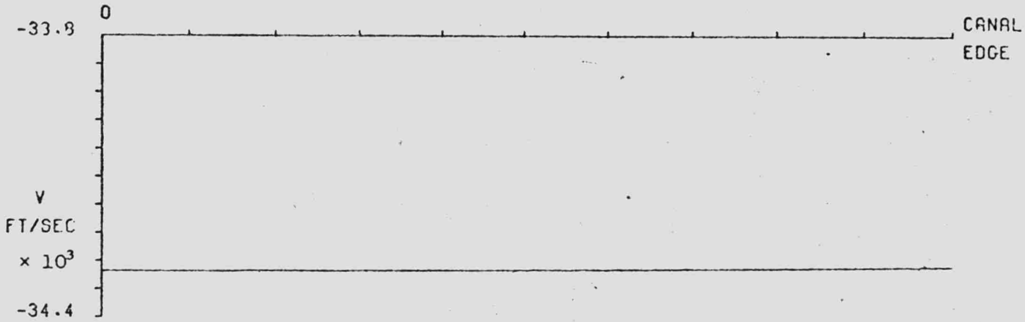
DEPTH: FREE SURFACE

FIG 3.7 V FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
B THE DISTANCE FROM EQUATOR TO CANAL EDGE

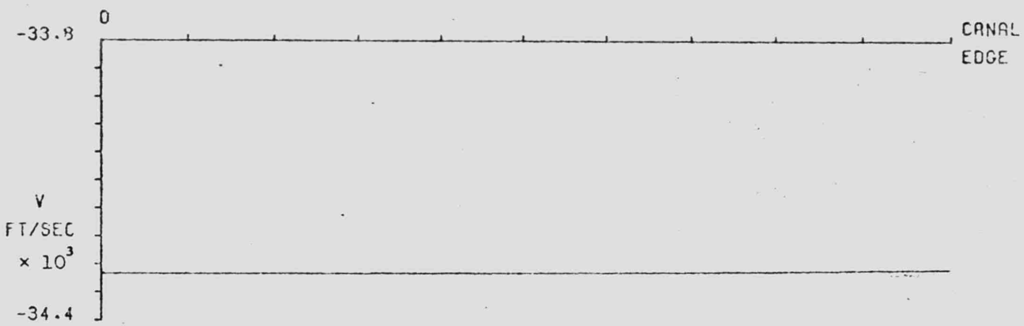
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$



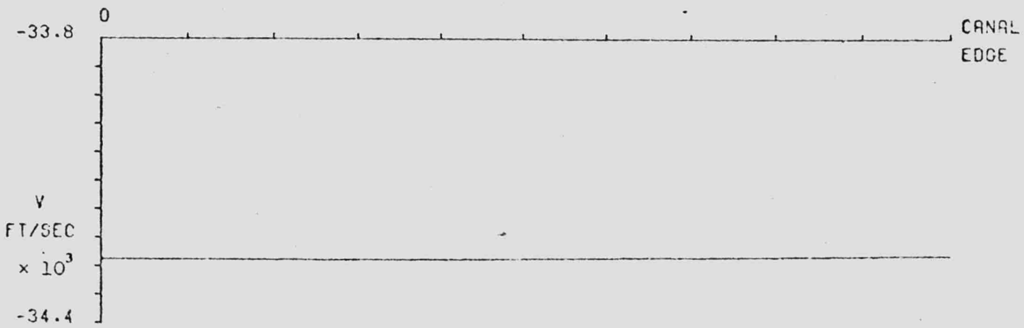
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



DEPTH: 2000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.7 V_C FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) $\beta = 0.0001$

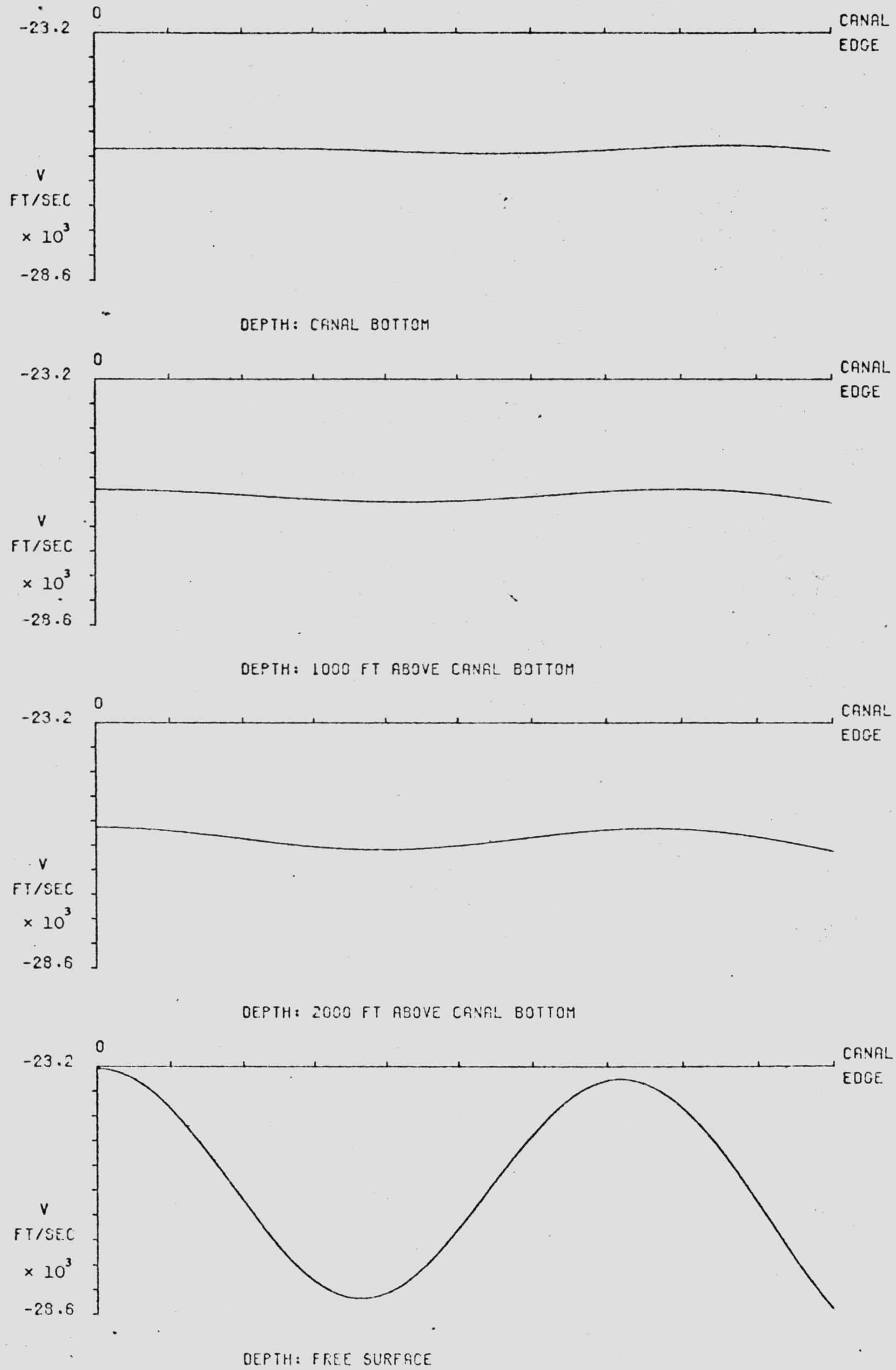


FIG 3.7 V_D FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) β = 0.0001

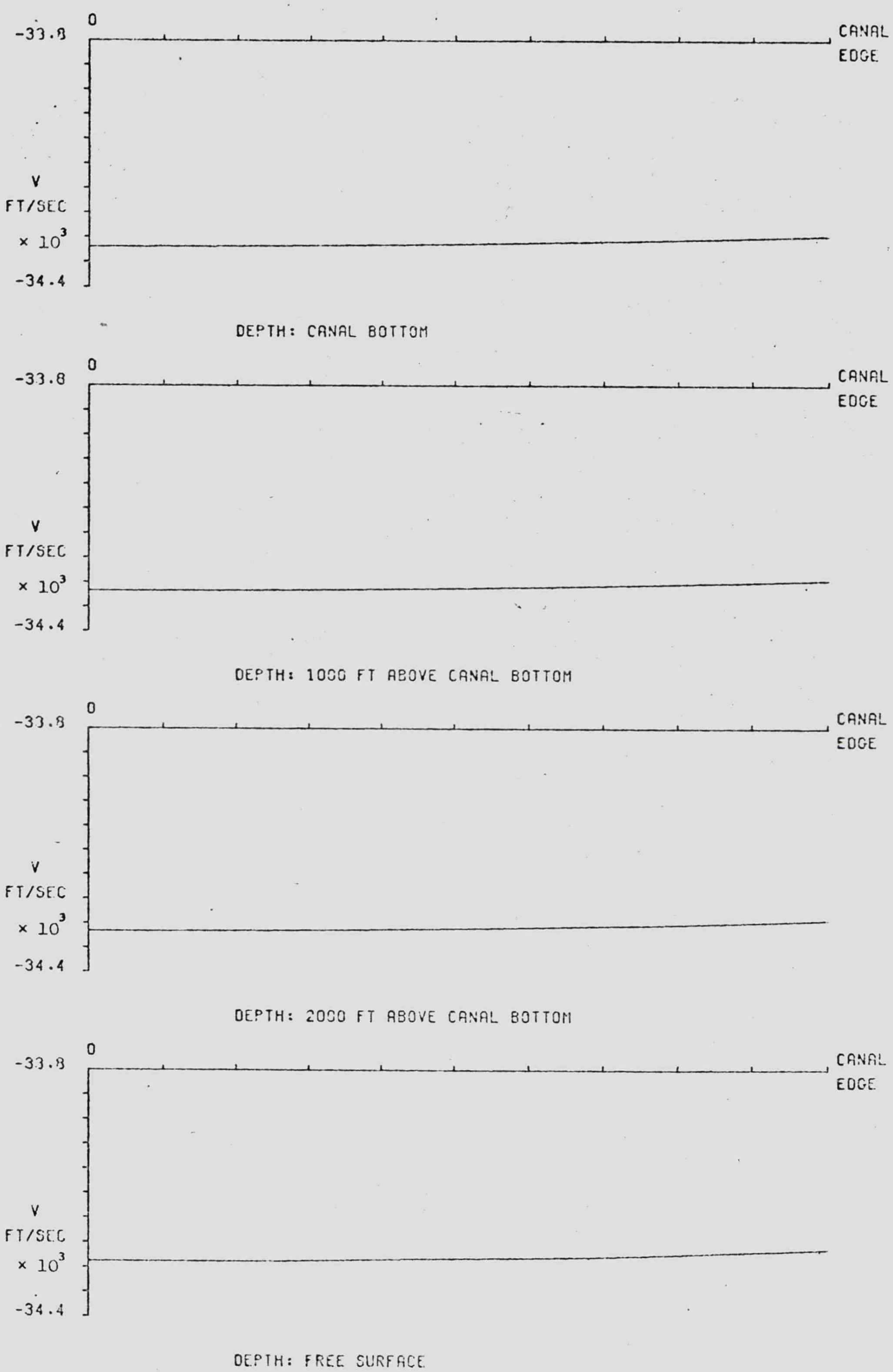
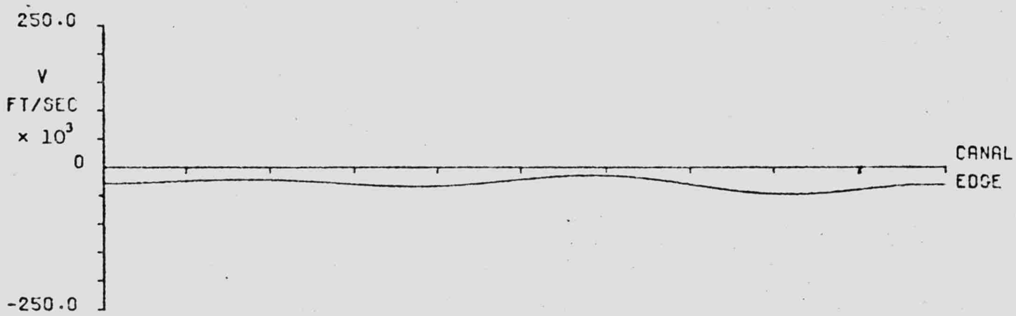
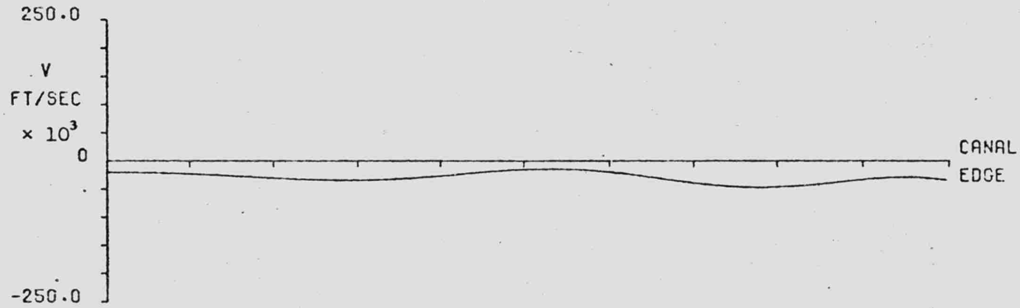


FIG 3.7 V_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

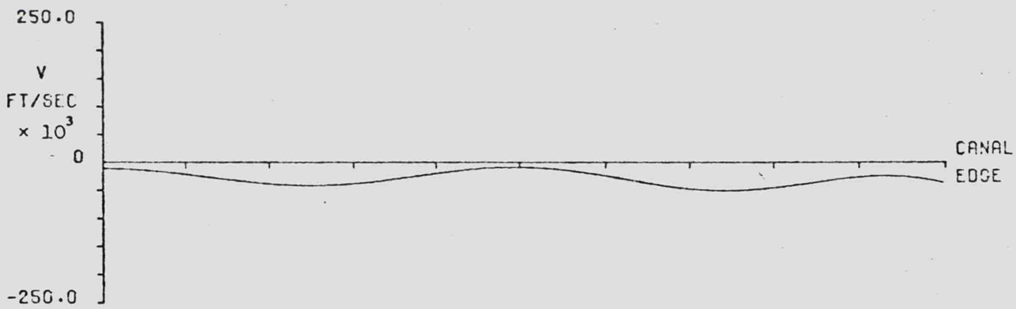
CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) $\beta = 0.0001$



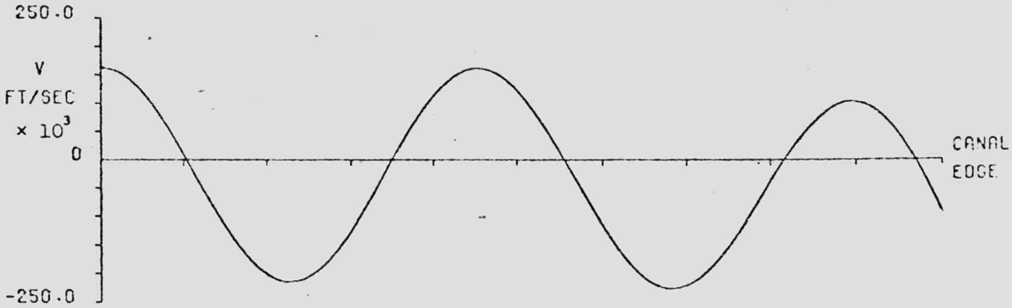
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



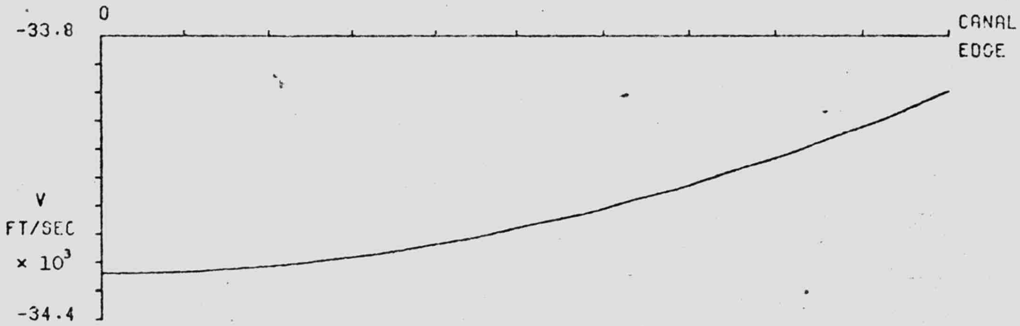
DEPTH: 2000 FT ABOVE CANAL BOTTOM



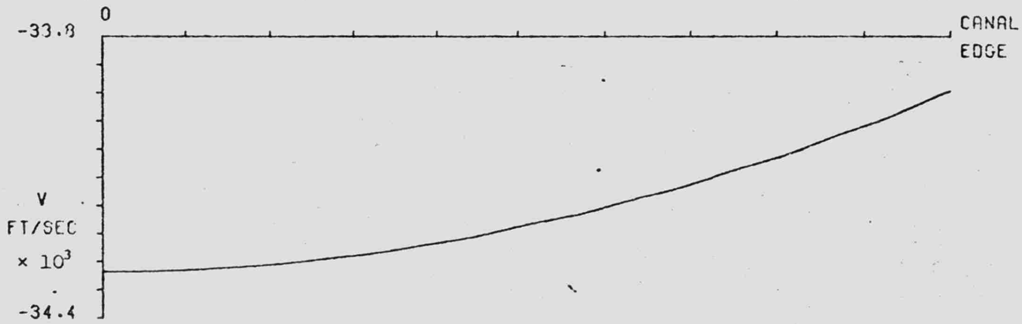
DEPTH: FREE SURFACE

FIG 3.7 V_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

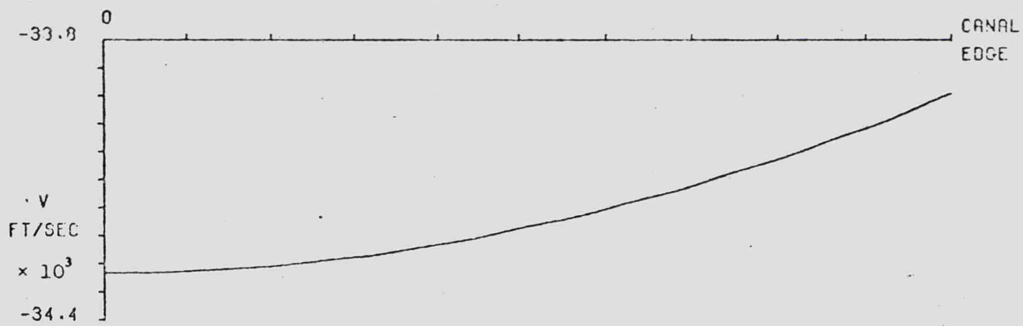
CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) $\beta = 0.0001$



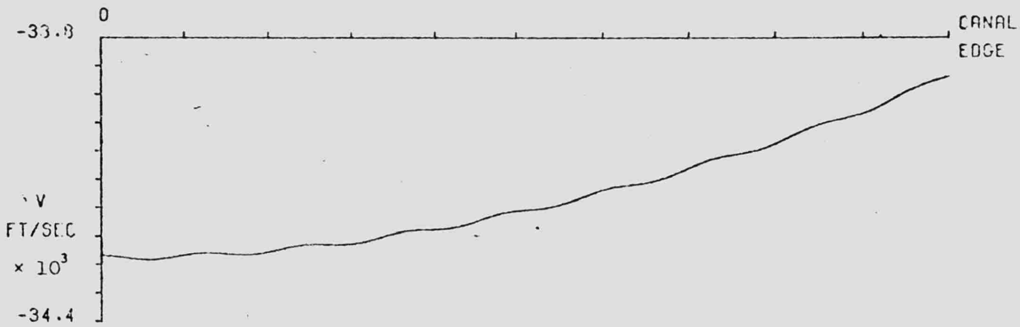
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM

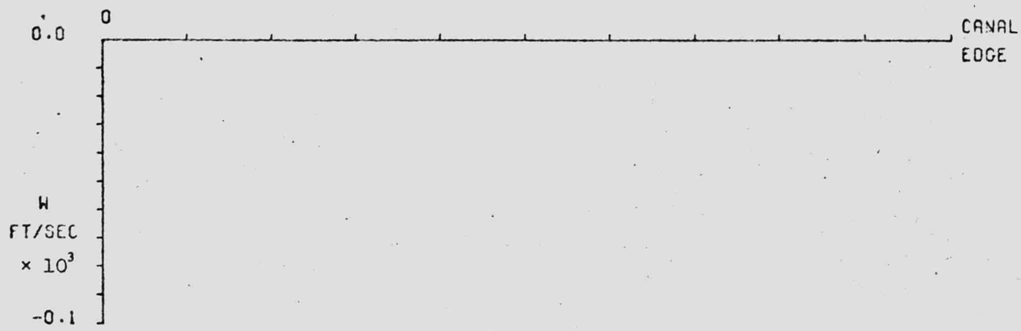


DEPTH: 2000 FT ABOVE CANAL BOTTOM

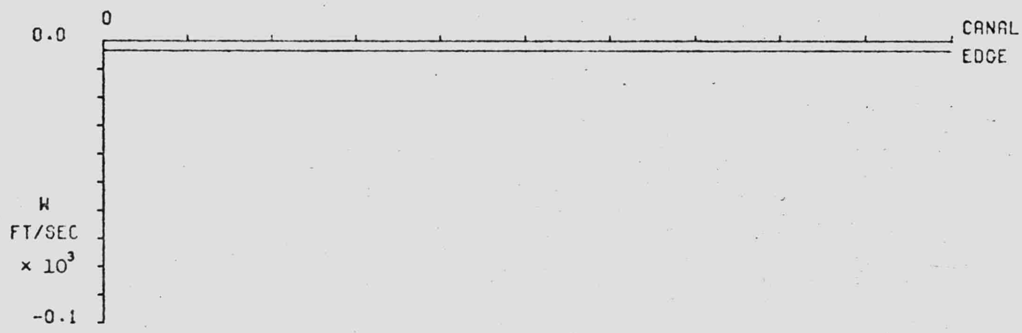


DEPTH: FREE SURFACE

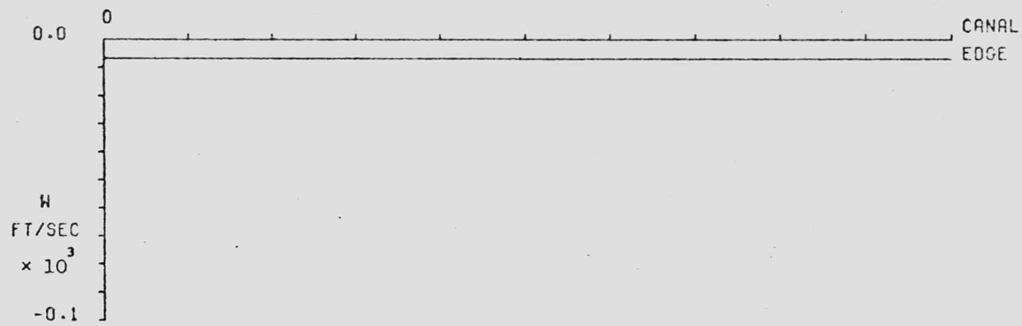
FIG 3.7 V_G FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE.
CANAL SEMI-WIDTH= 3×10^6 FT (568.2 MLS) $\beta = 0.0001$



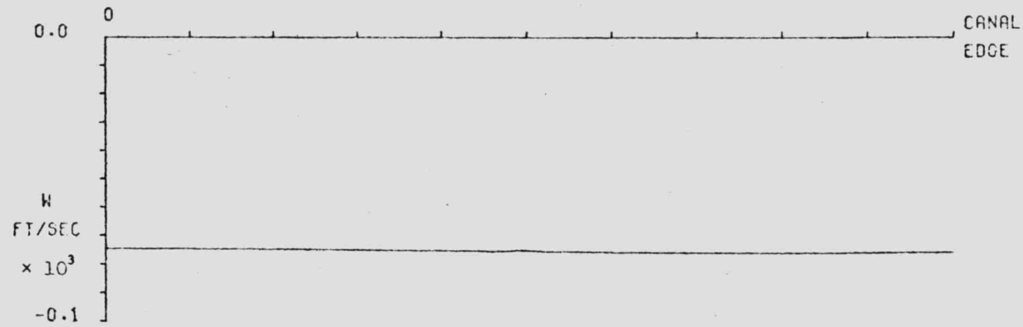
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



DEPTH: 2000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.8 A W FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$

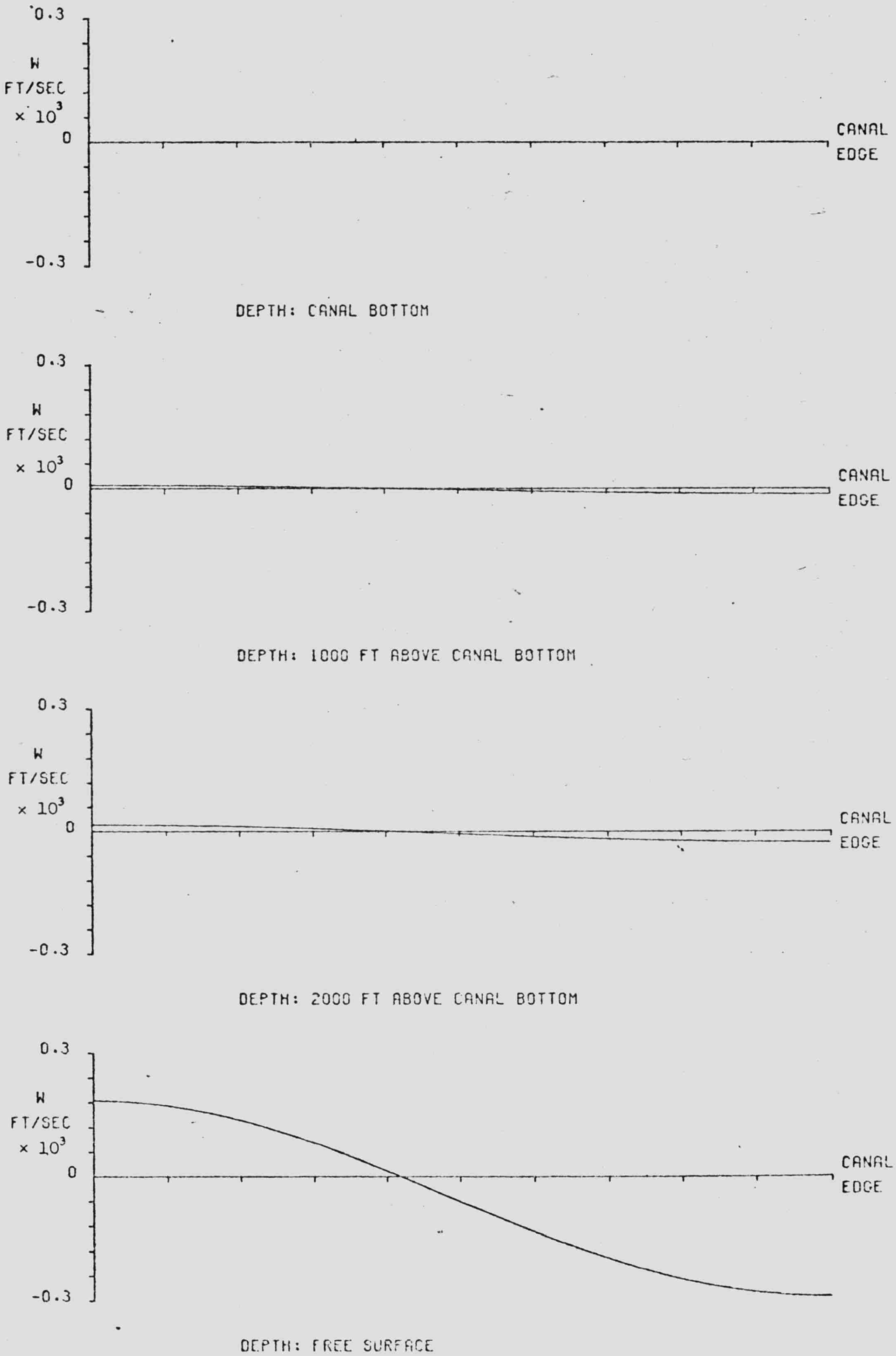
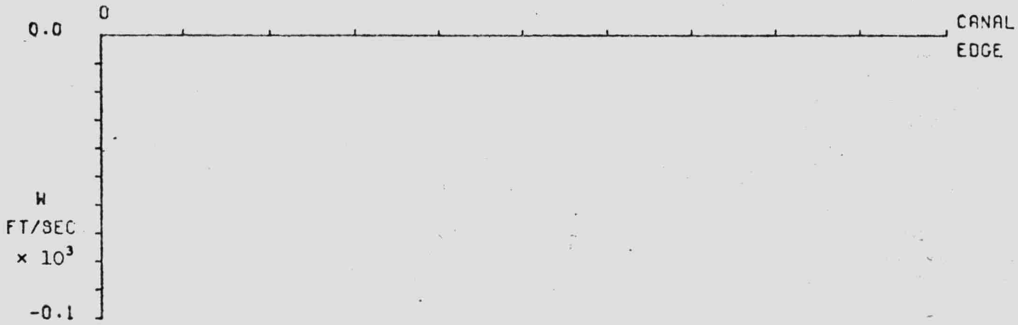
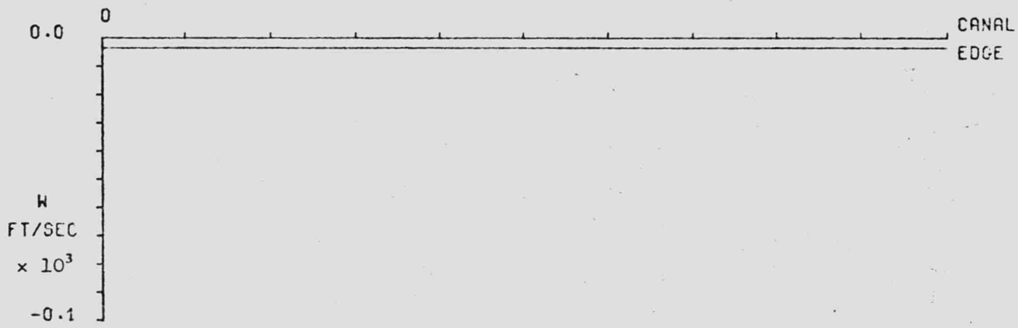


FIG 3.8 B W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

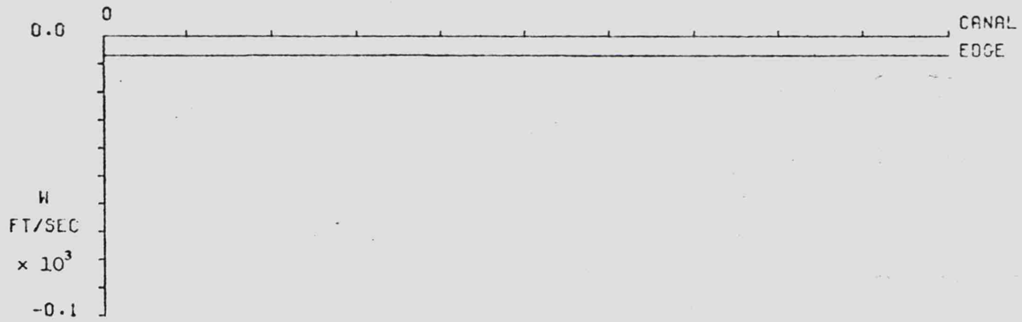
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.0001$



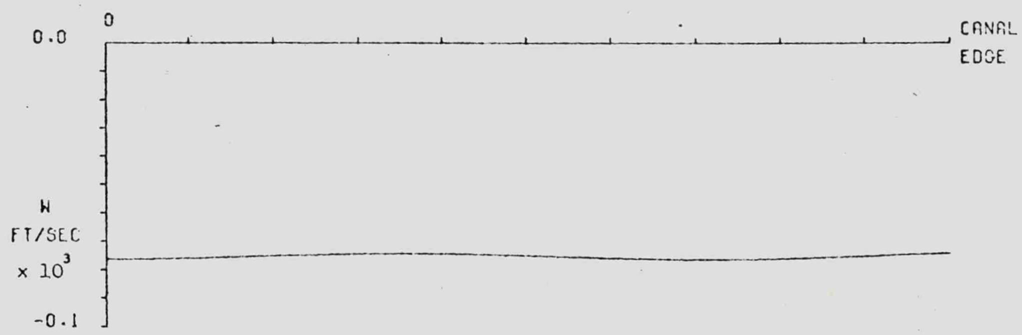
DEPTH: CANAL BOTTOM



DEPTH: 1000 FT ABOVE CANAL BOTTOM



DEPTH: 2000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.8_C W FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) $\delta = 0.0001$

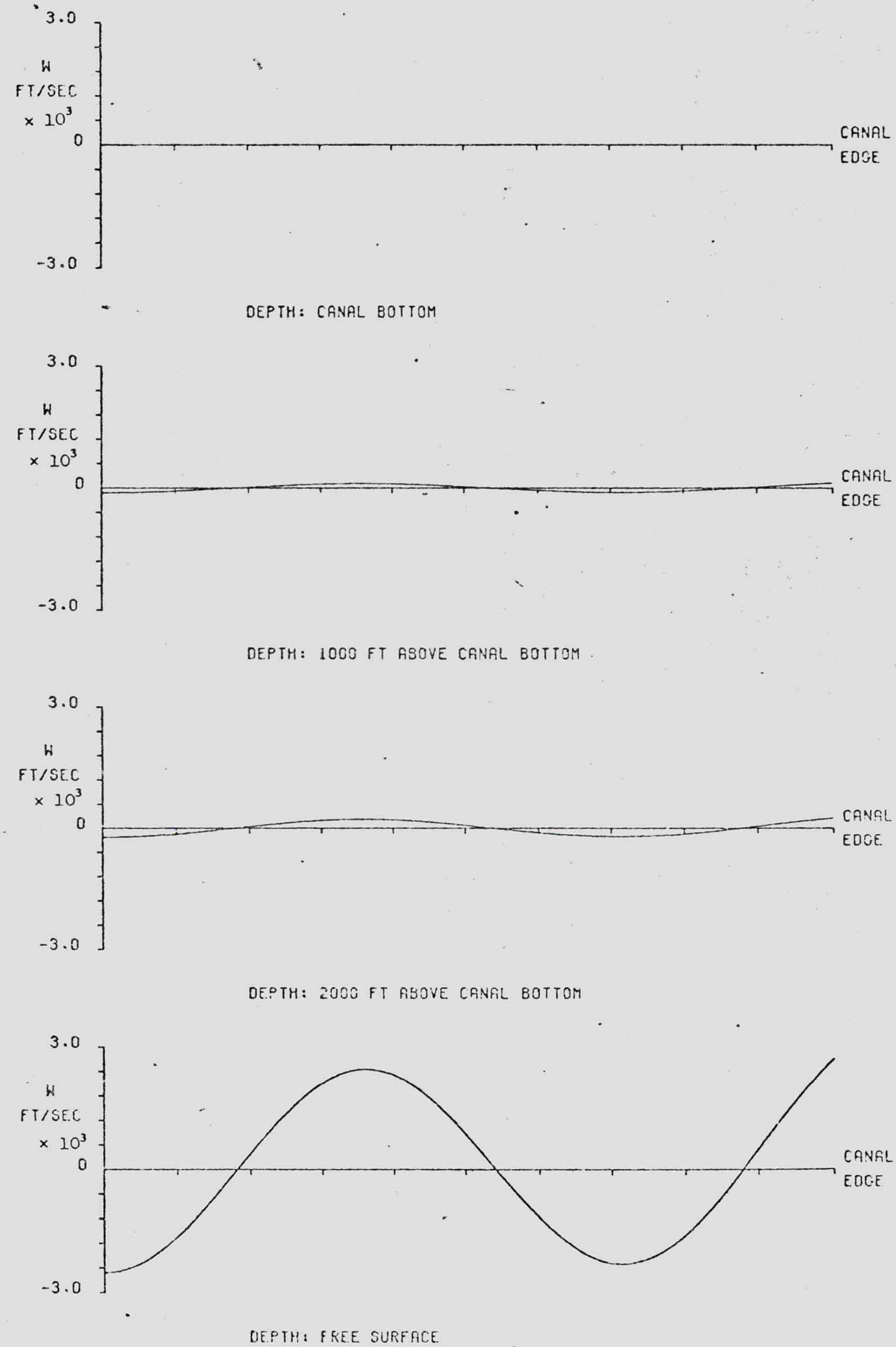


FIG 3.8$_D$ W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 5×10^5 FT (94.7 MLS) β = 0.0001

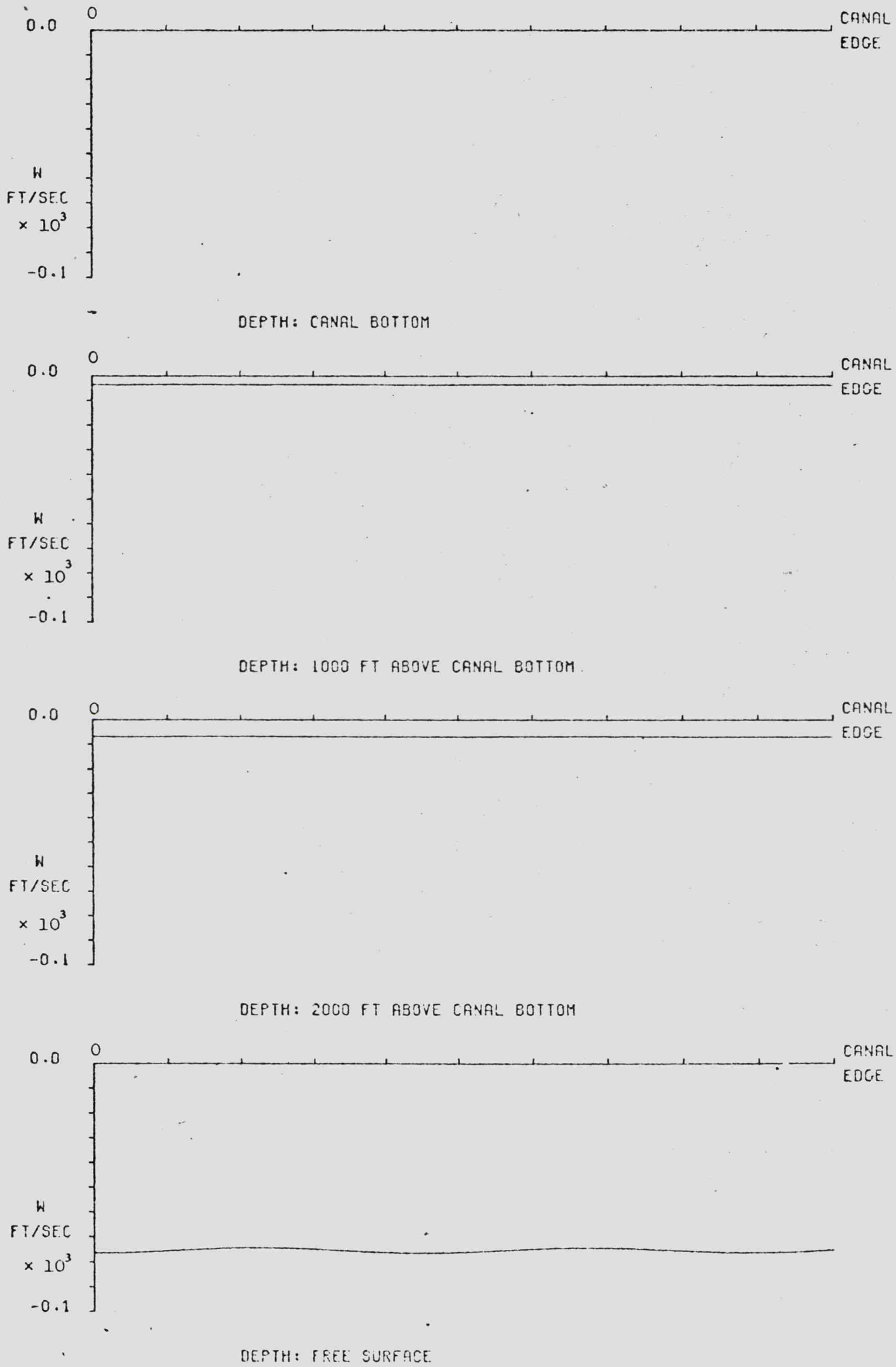


FIG 3.8 W_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) $\beta = 0.0001$

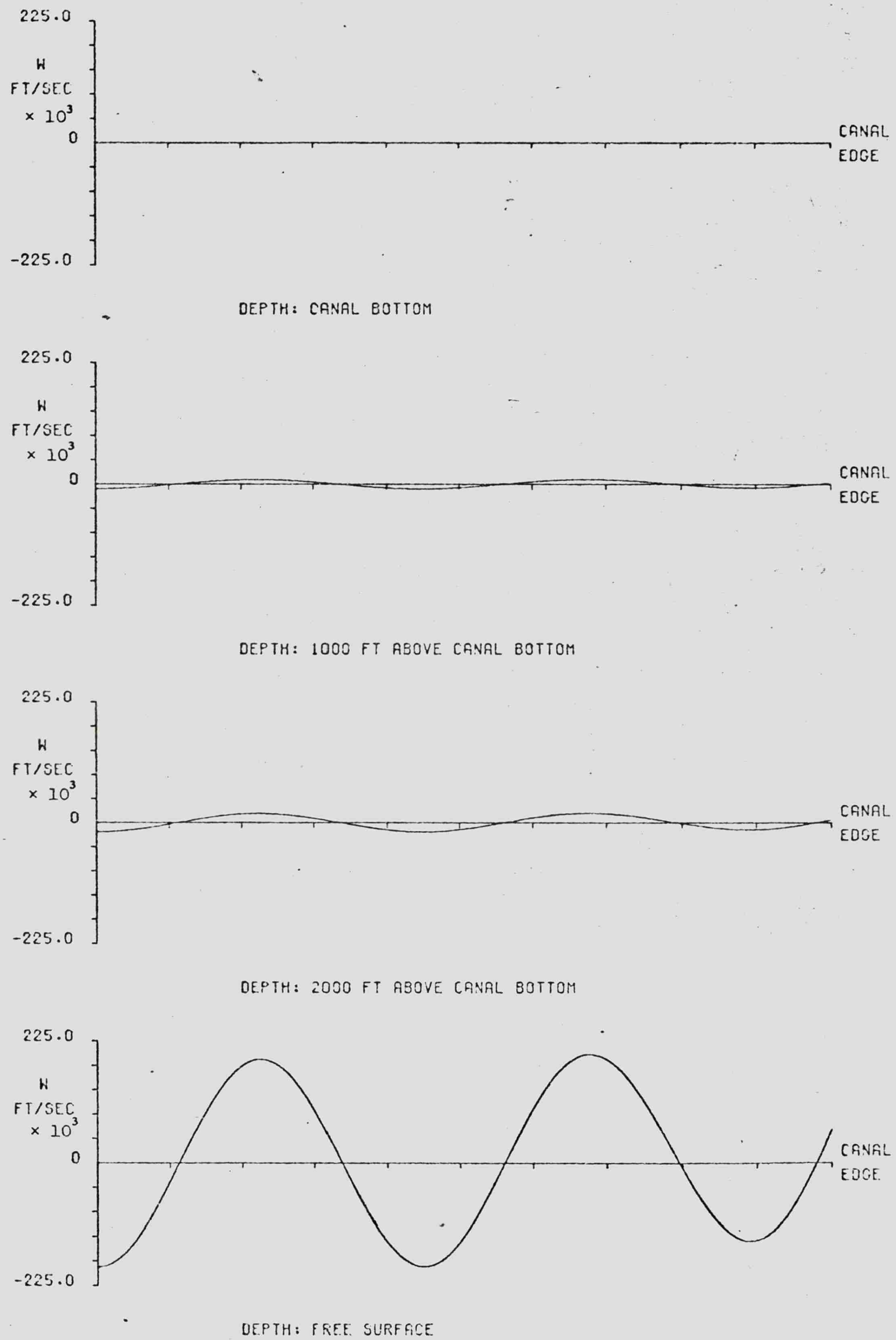


FIG 3.8 W_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 8×10^5 FT (151.5 MLS) $\beta = 0.0001$

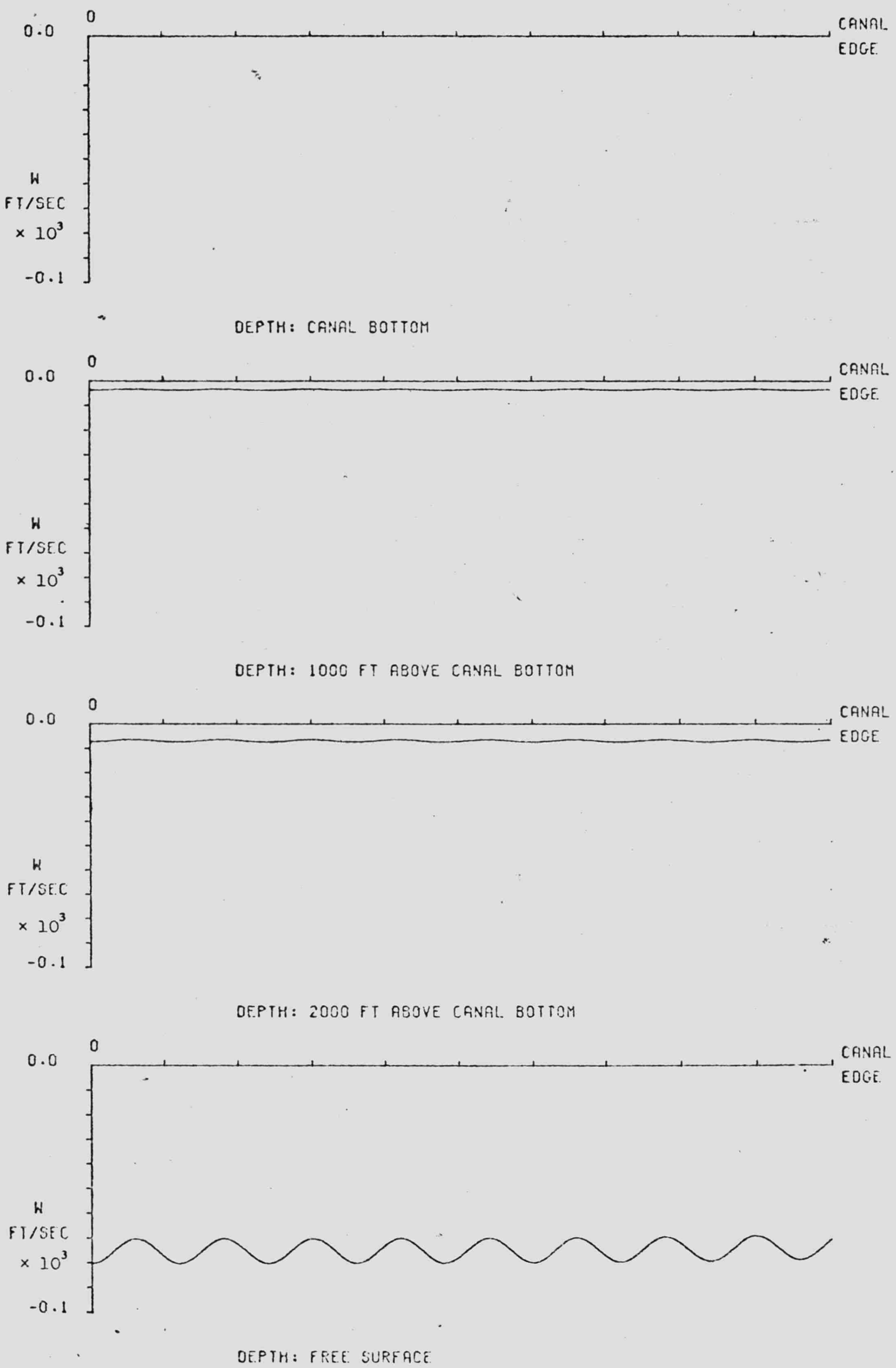


FIG 3.8G W FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 3×10^6 FT (568.2 MLS) $\beta = 0.0001$

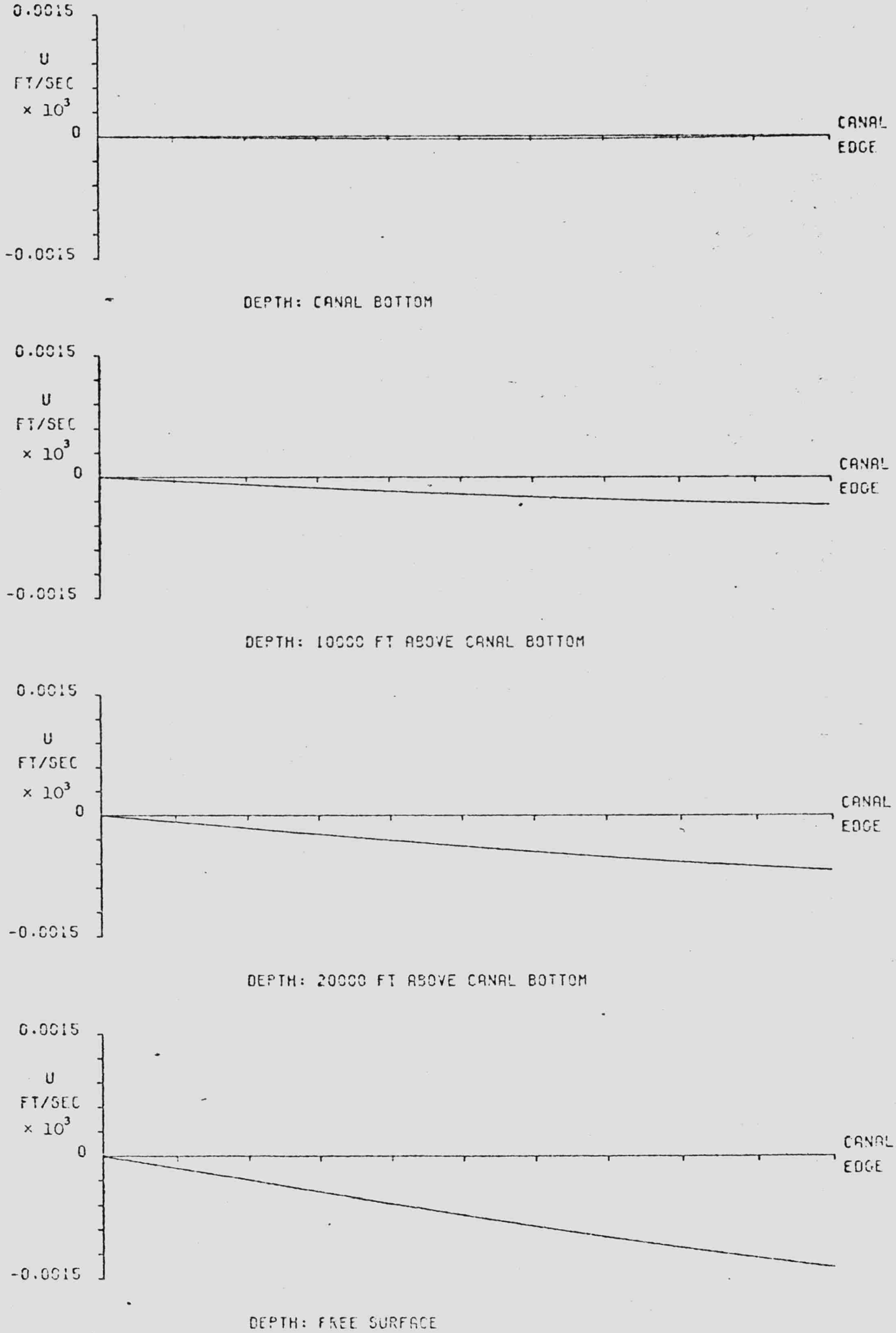
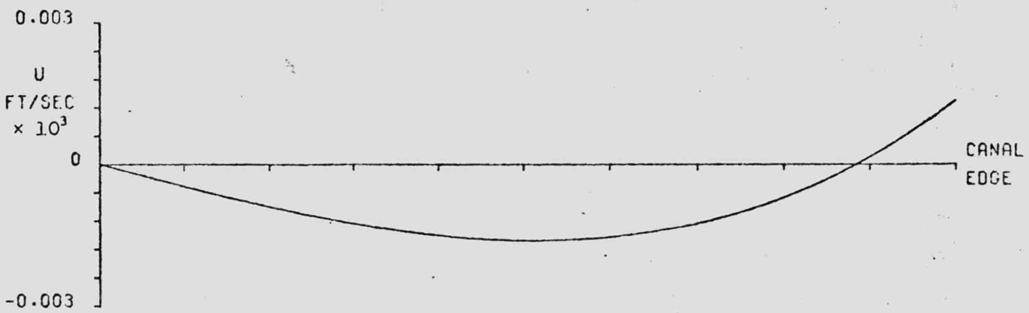
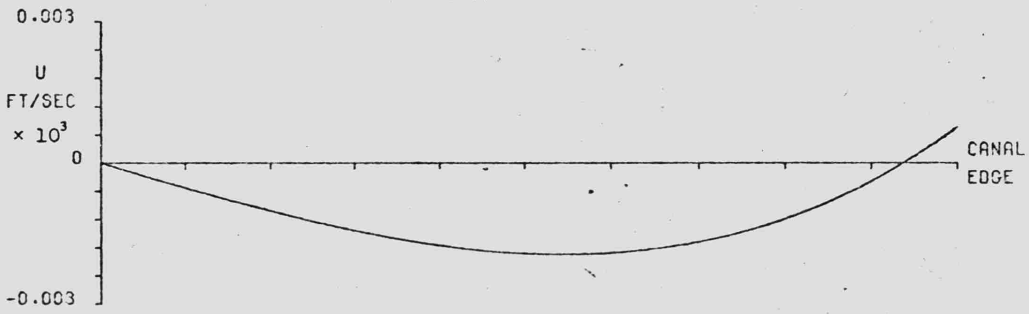


FIG 3.9 U FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
A THE DISTANCE FROM EQUATOR TO CANAL EDGE

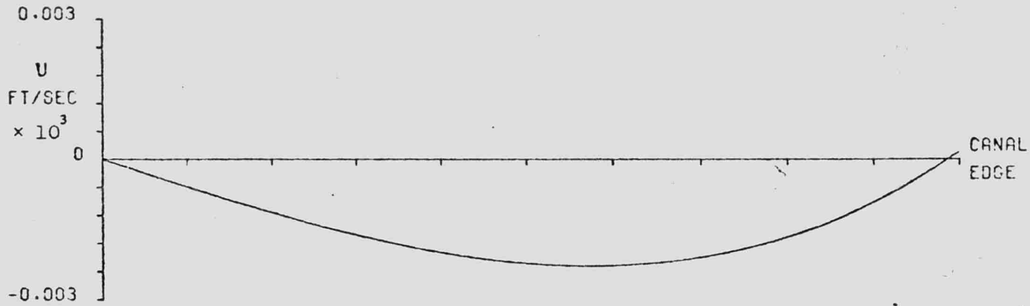
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.001$



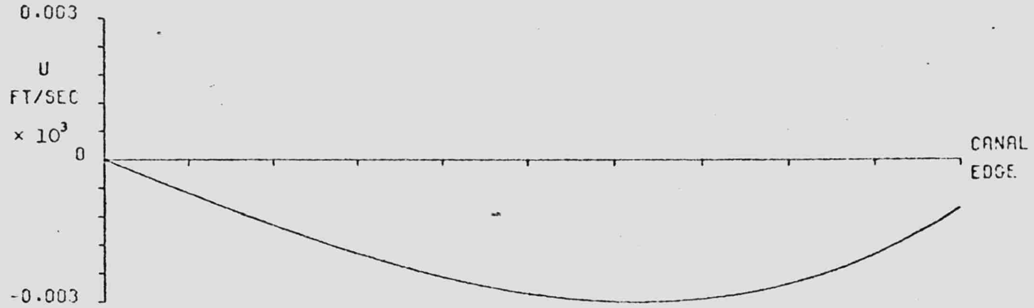
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.9 U_B FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.001$

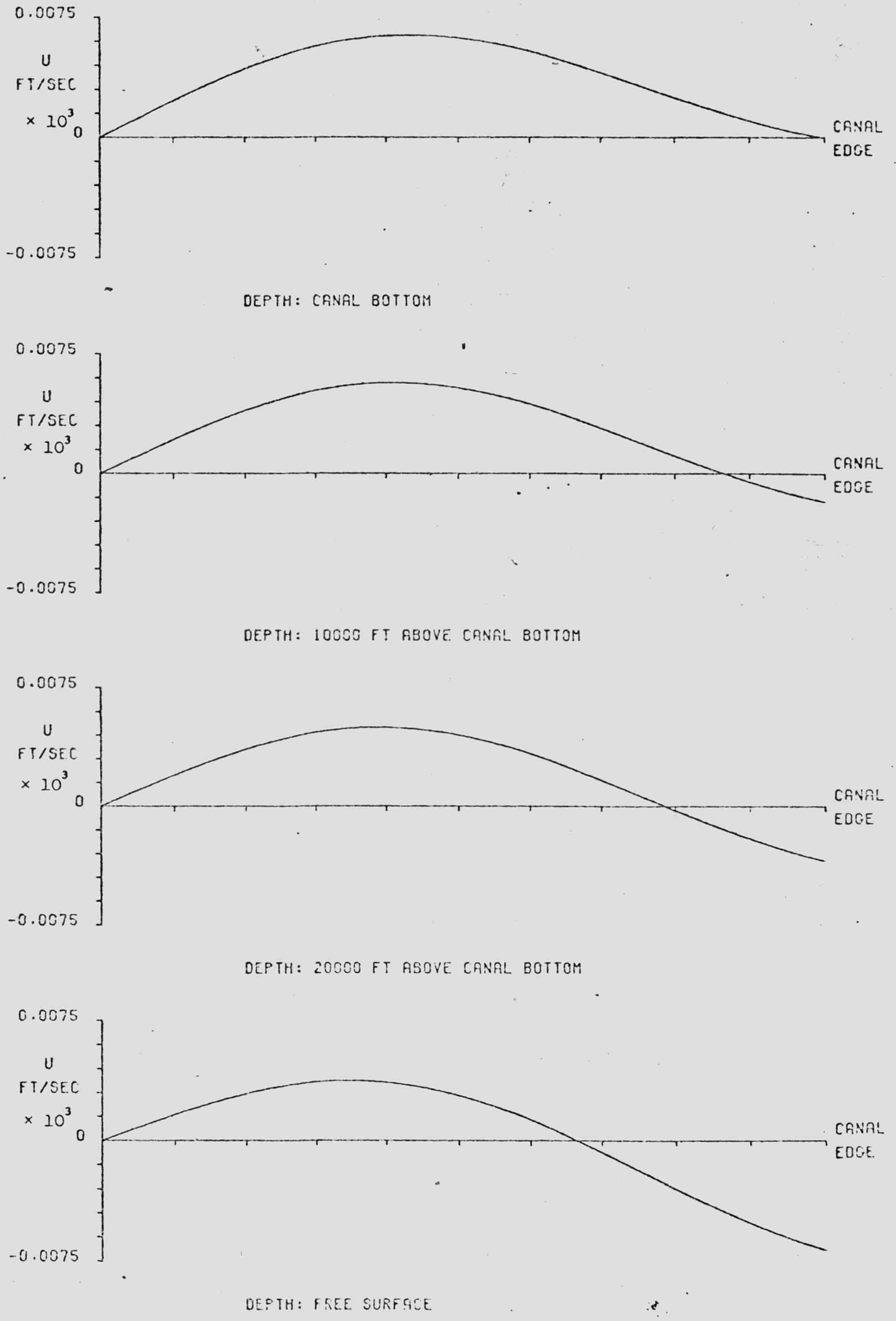
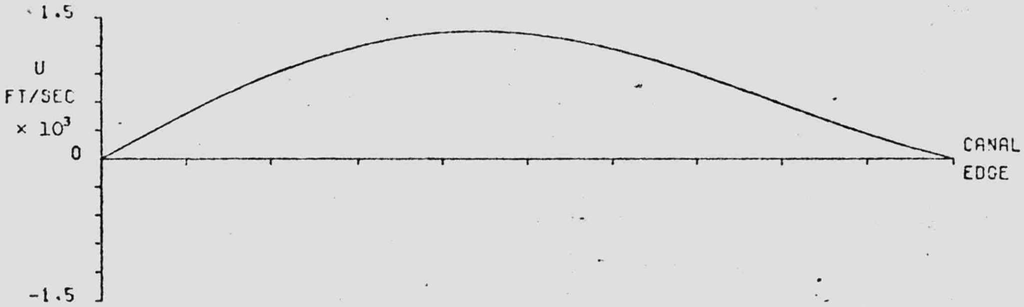
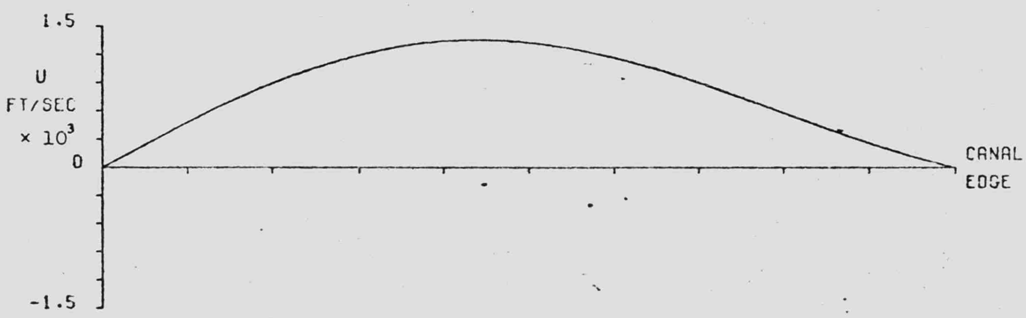


FIG 3.9 U_C FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

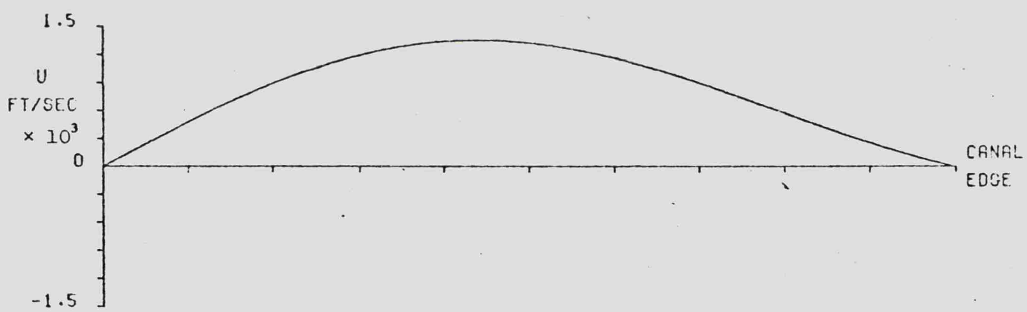
CANAL SEMI-WIDTH= 1×10^6 FT (189.4 MLS) $\beta = 0.001$



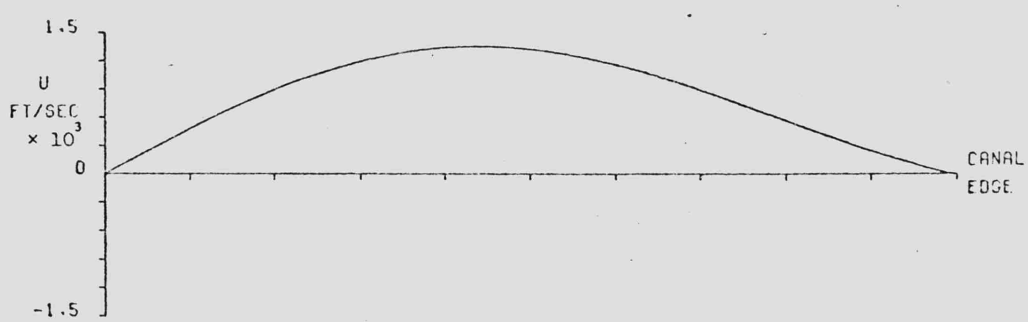
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.9 U_D FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 1×10^6 FT (189.4 MLS) $\beta = 0.001$

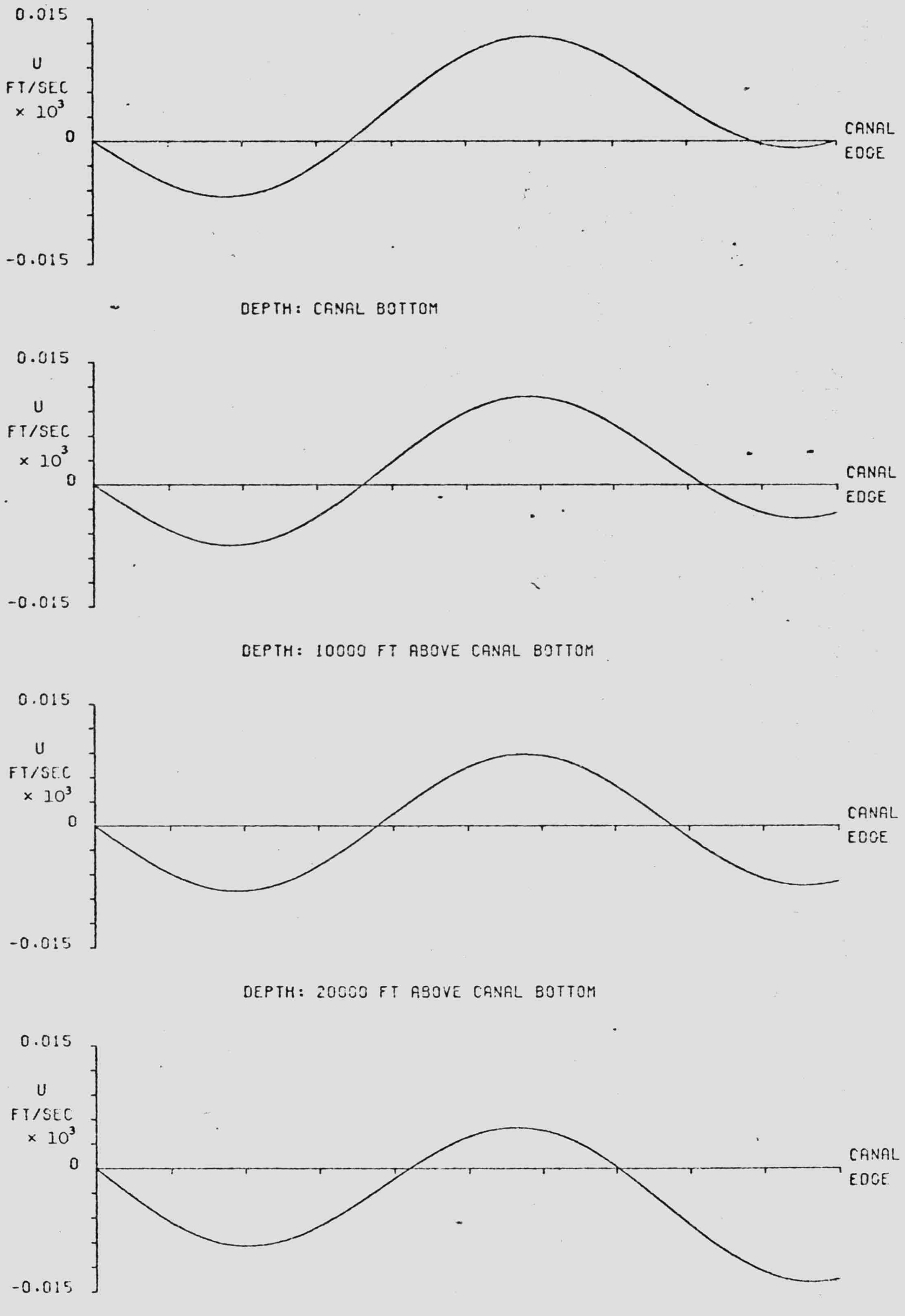


FIG 3.9 U_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^6 FT (378.9 MLS) $\beta = 0.001$

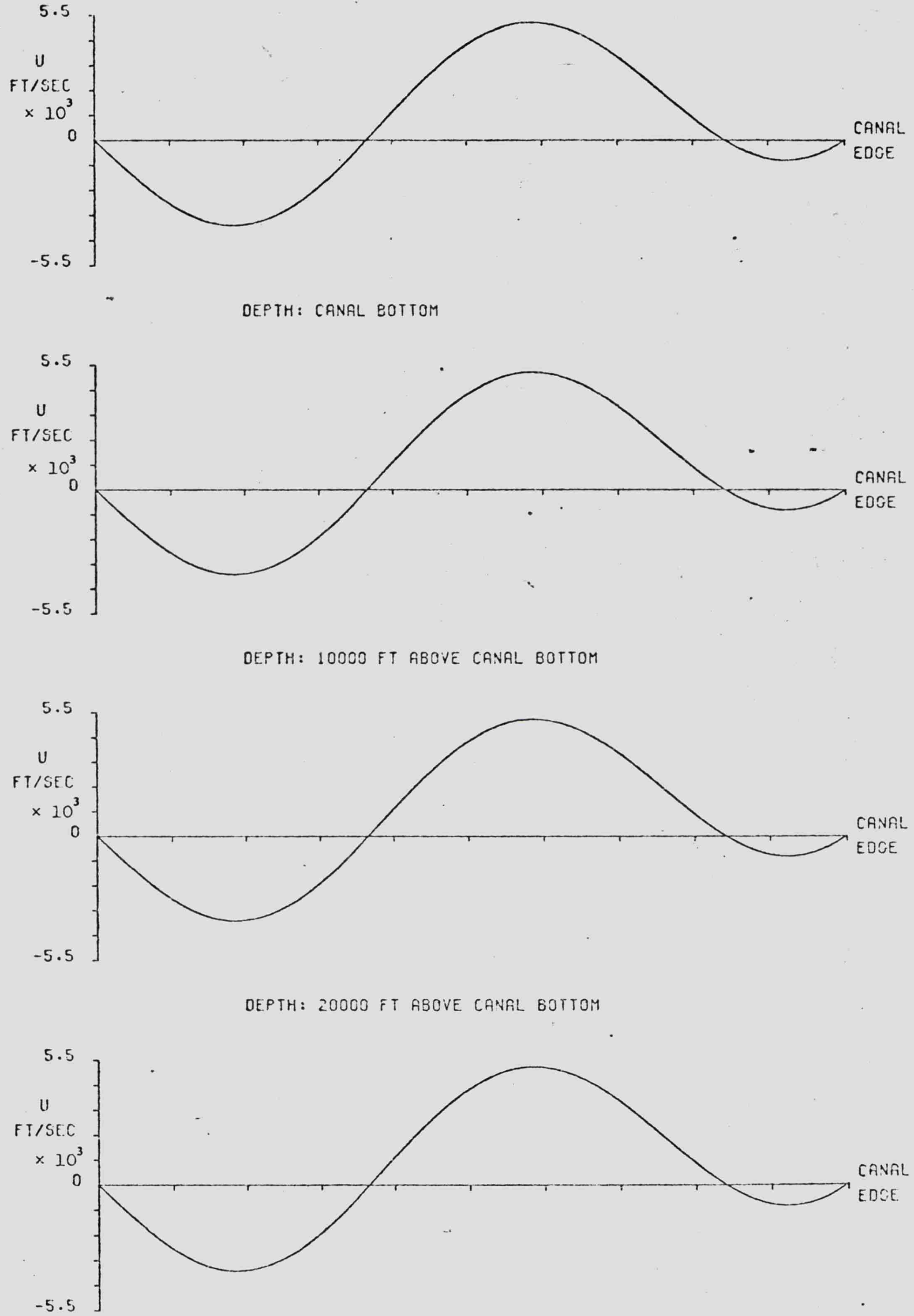


FIG 3.9 U_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH = 2×10^6 FT (378.8 MLS) $\beta = 0.001$

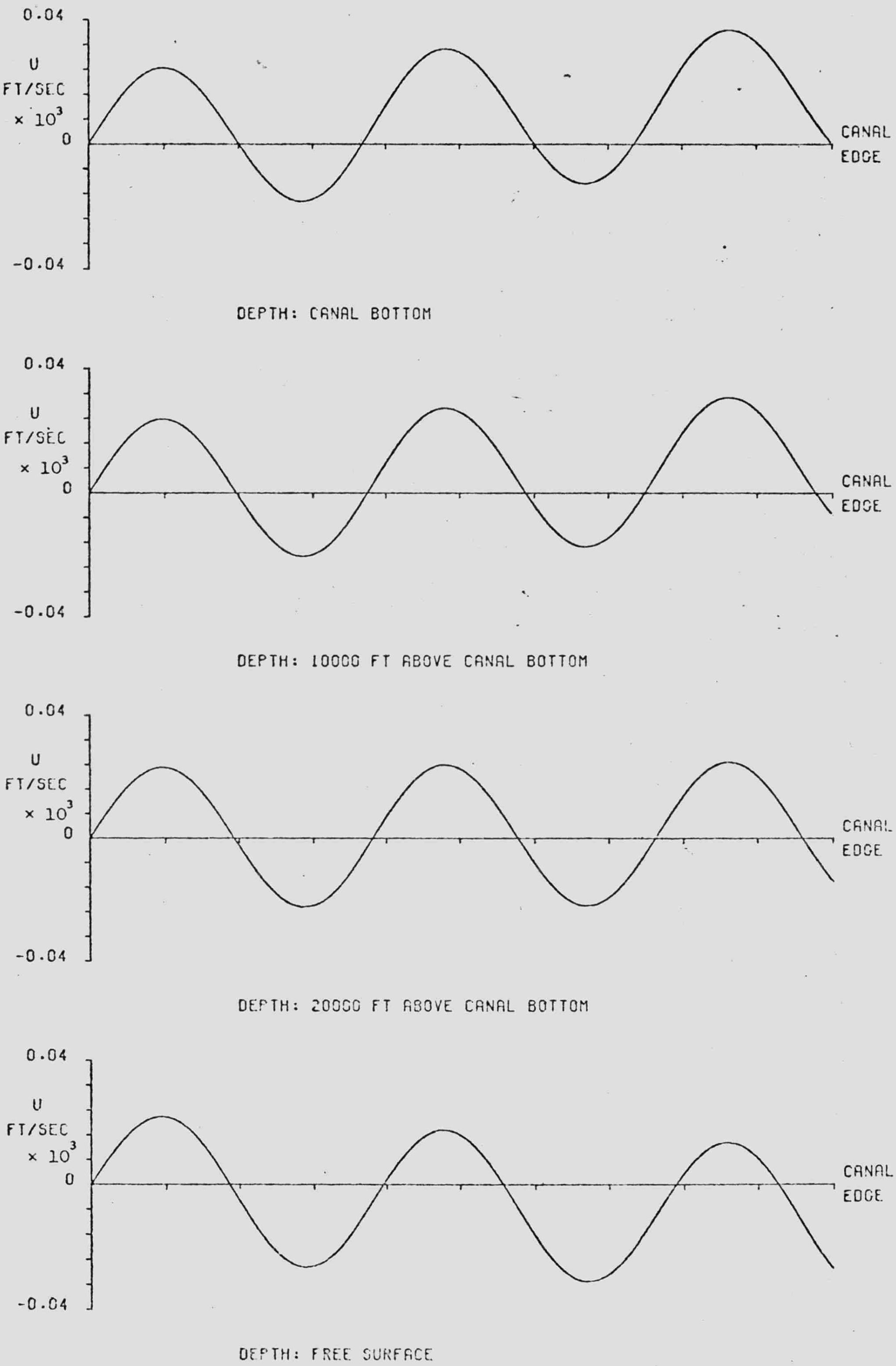


FIG 3.9 U FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
G THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 4 x 10^6 FT (757.6 MLS) $\beta = 0.001$

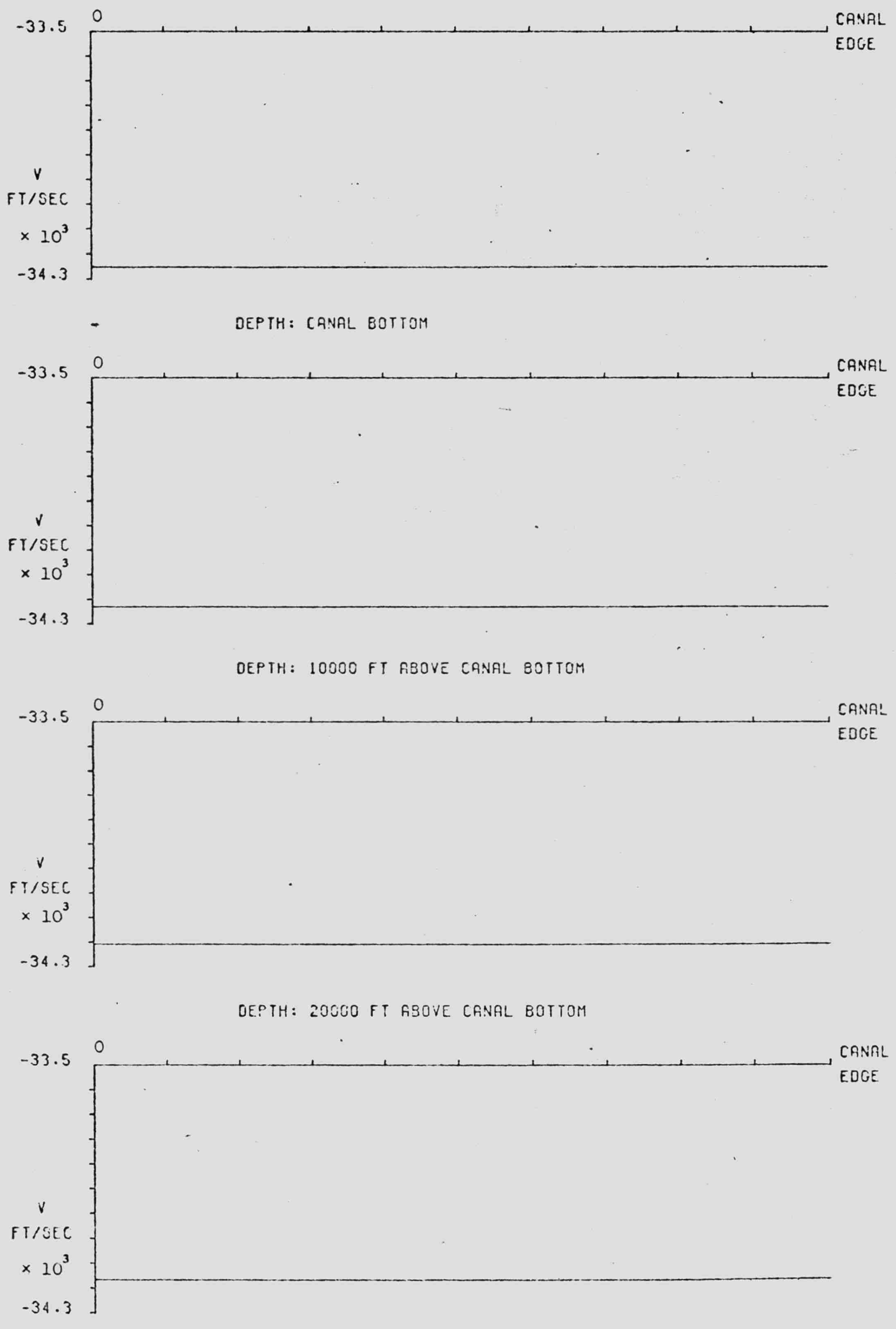


FIG 3.10 V_A FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.001$

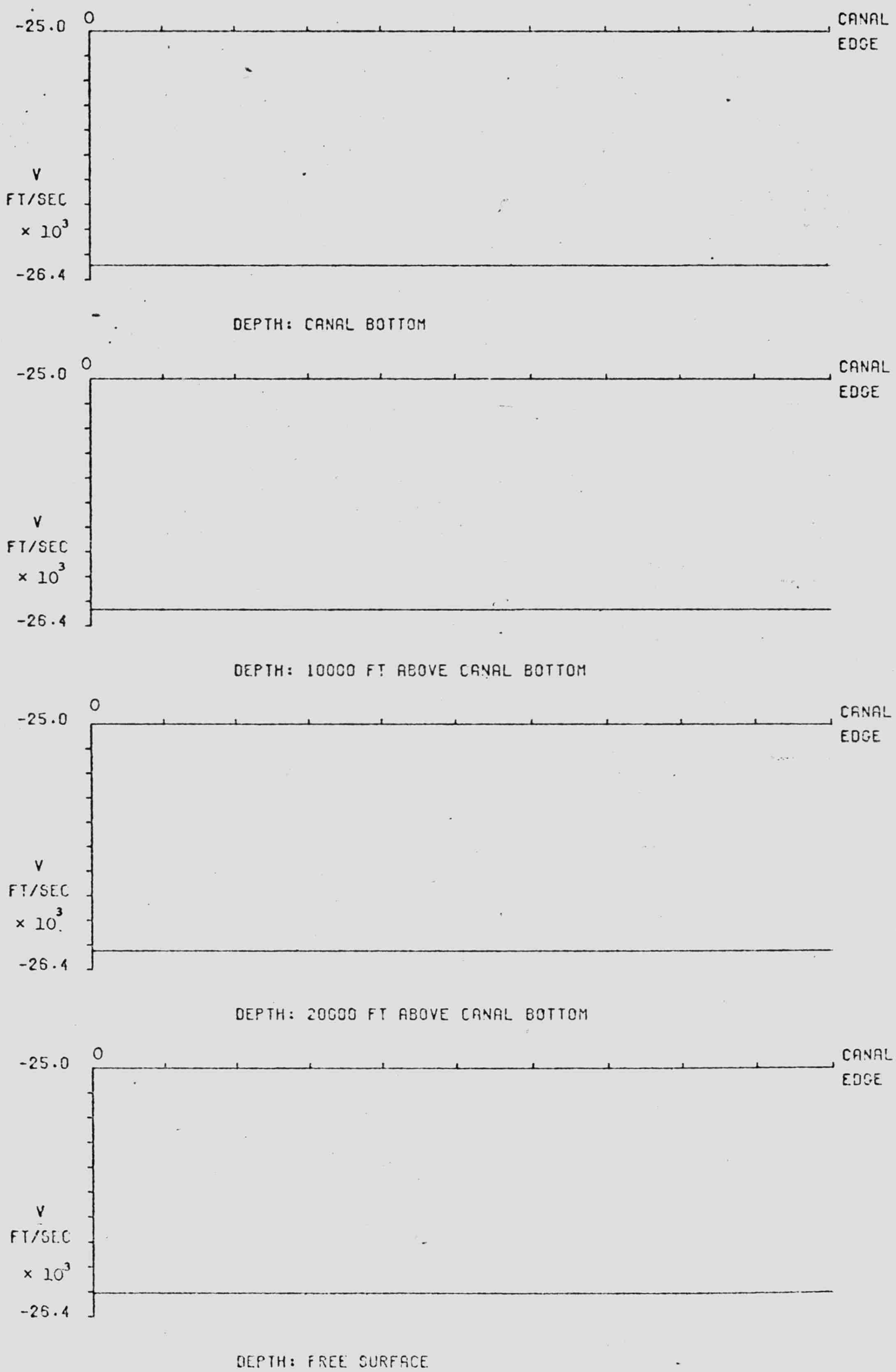
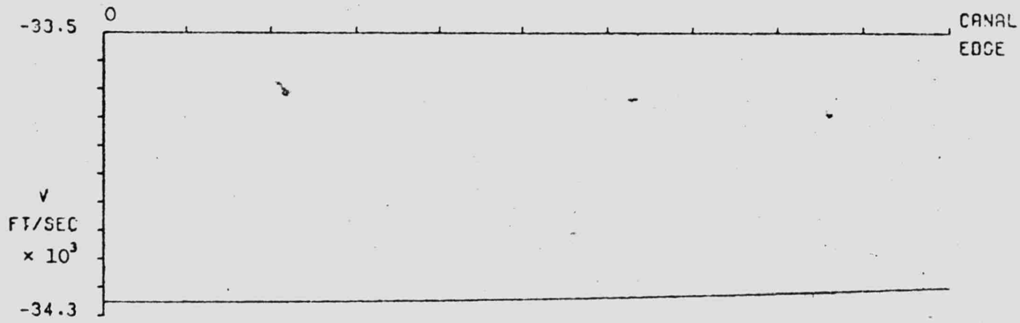


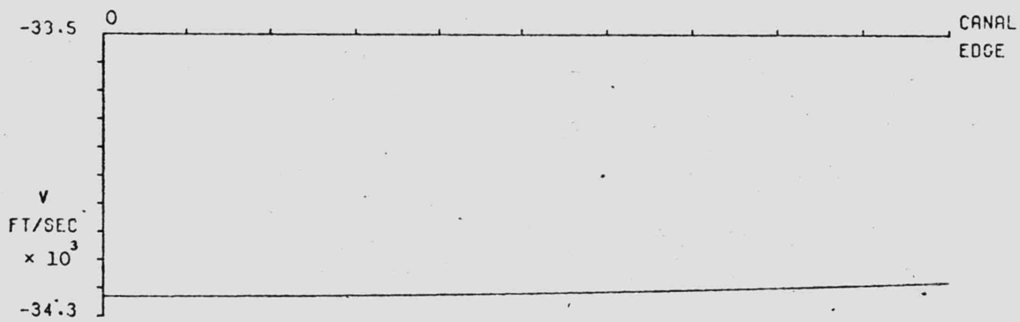
FIG 3.10 V FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

B

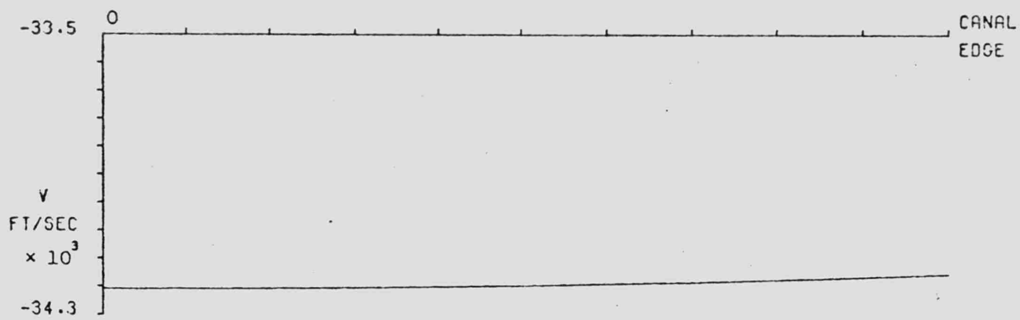
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) $\beta = 0.001$



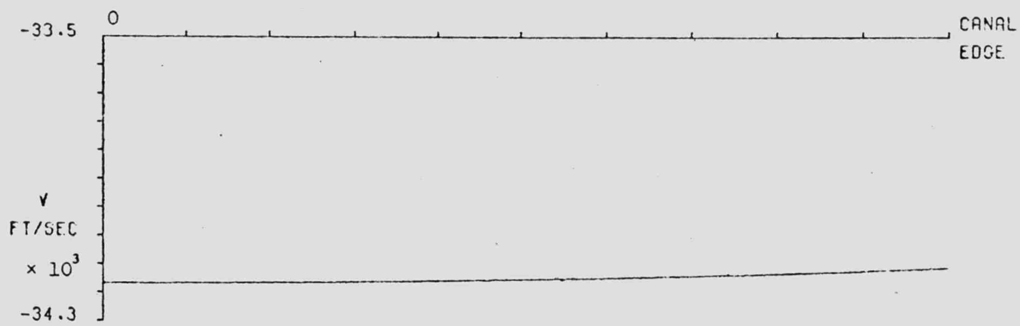
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.10 V_C FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE.

CANAL SEMI-WIDTH= 1×10^6 FT (189.4 MLS) $\beta = 0.001$

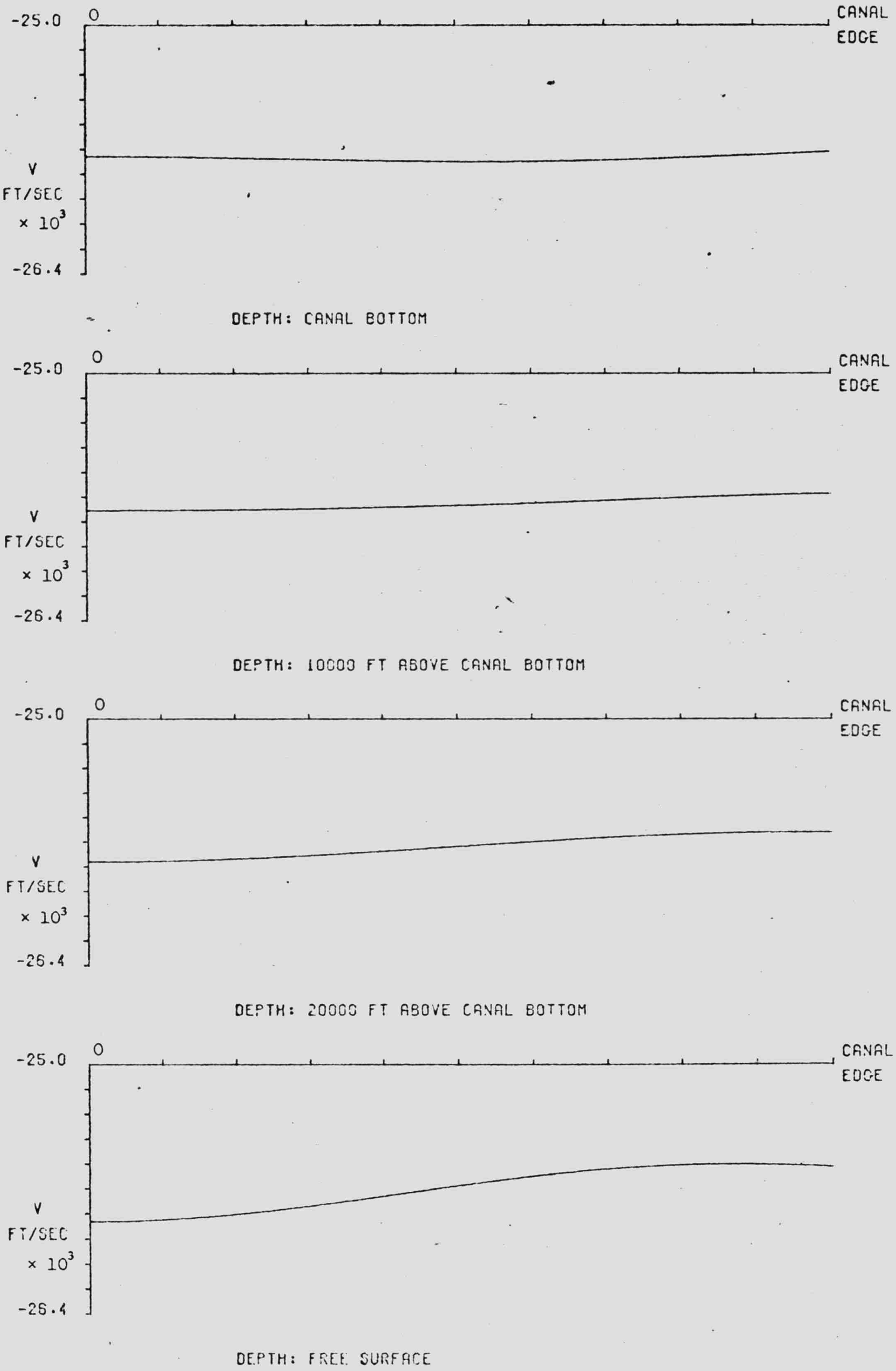
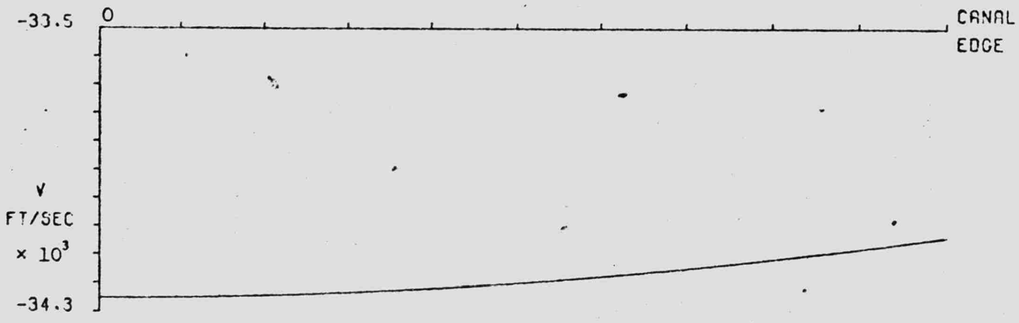
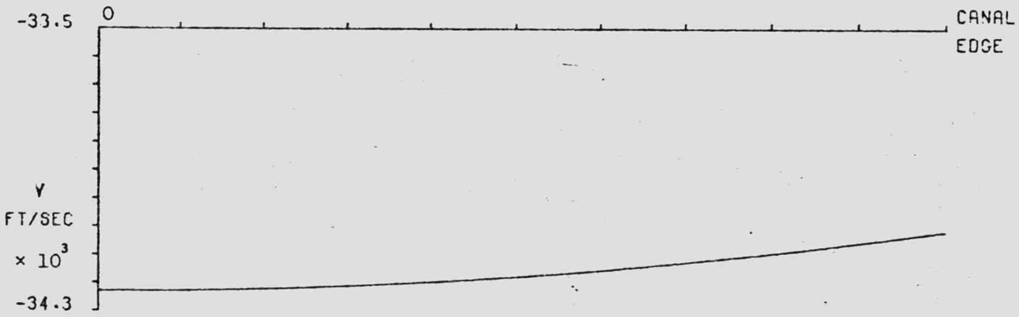


FIG 3.10 V FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE
D

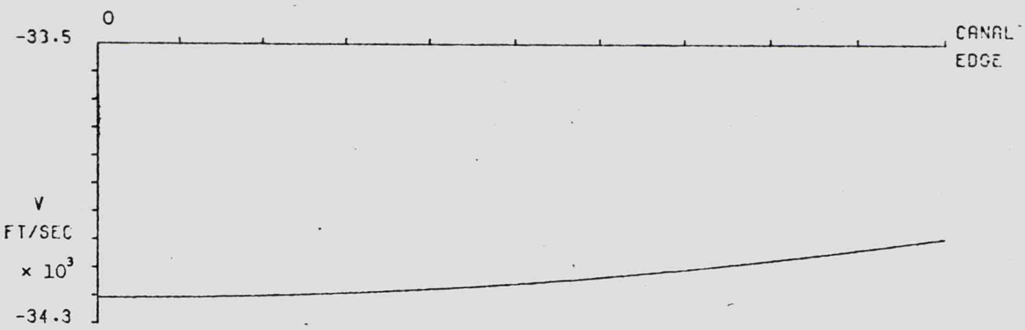
CANAL SEMI-WIDTH= 1×10^6 FT (189.4 MLS) $\beta = 0.001$



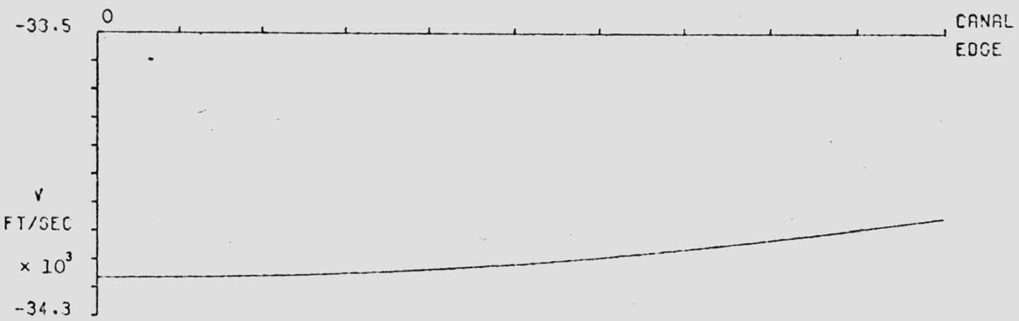
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



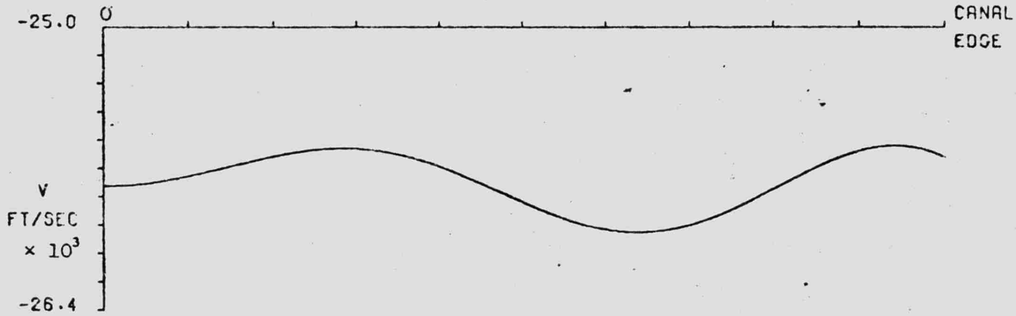
DEPTH: 20000 FT ABOVE CANAL BOTTOM



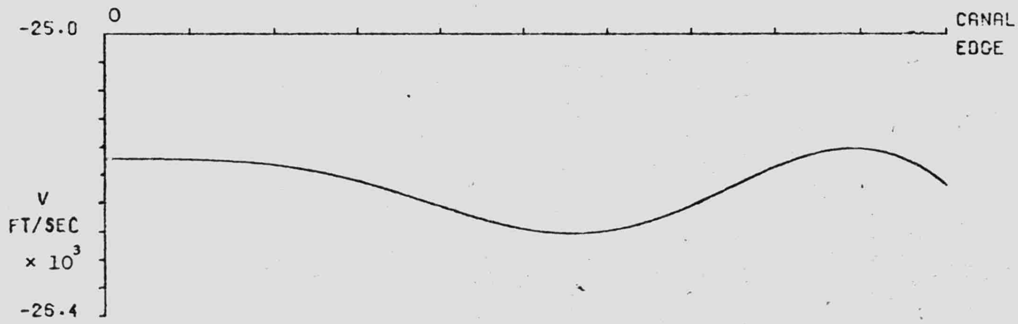
DEPTH: FREE SURFACE

FIG 3.10 V_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

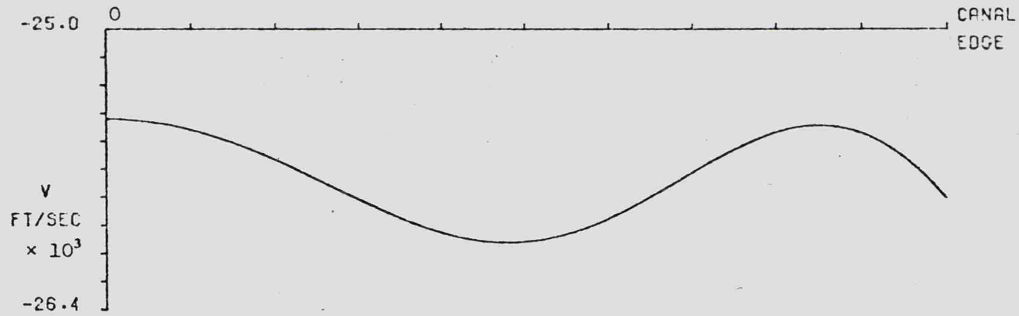
CANAL SEMI-WIDTH= 2×10^6 FT (378.8 MLS) $\beta = 0.001$



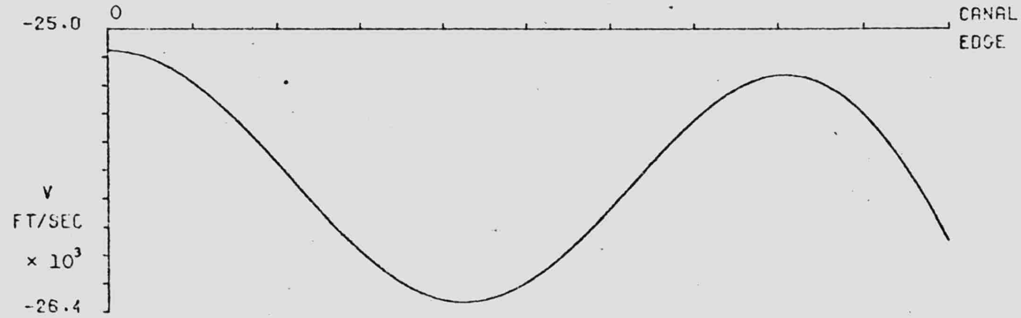
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.10 V FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

F

CANAL SEMI-WIDTH= 2×10^6 FT (378.8 MLS) $\beta = 0.001$

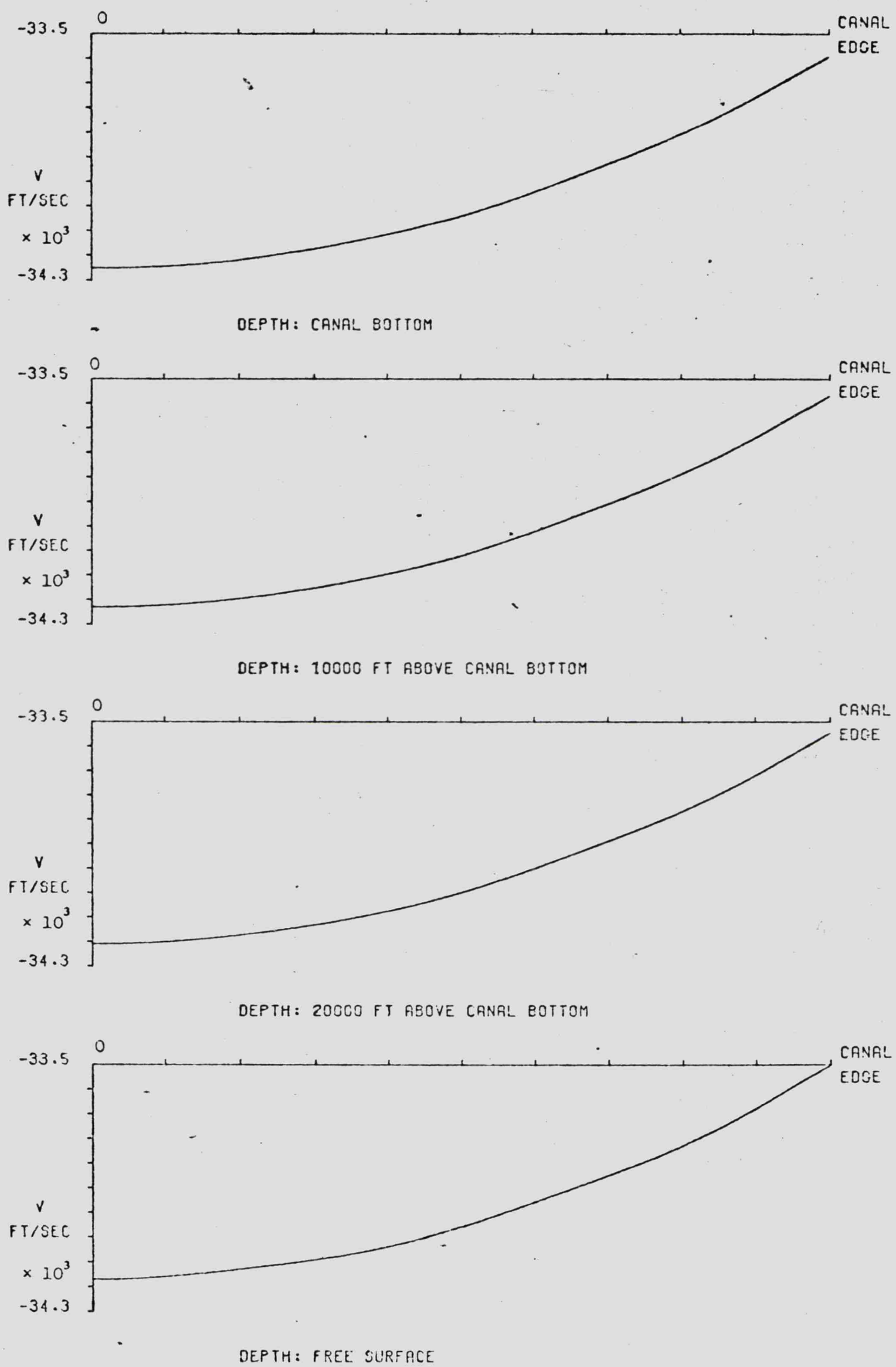


FIG 3.10 V_G FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 4×10^5 FT (757.6 MLS) $\beta = 0.001$

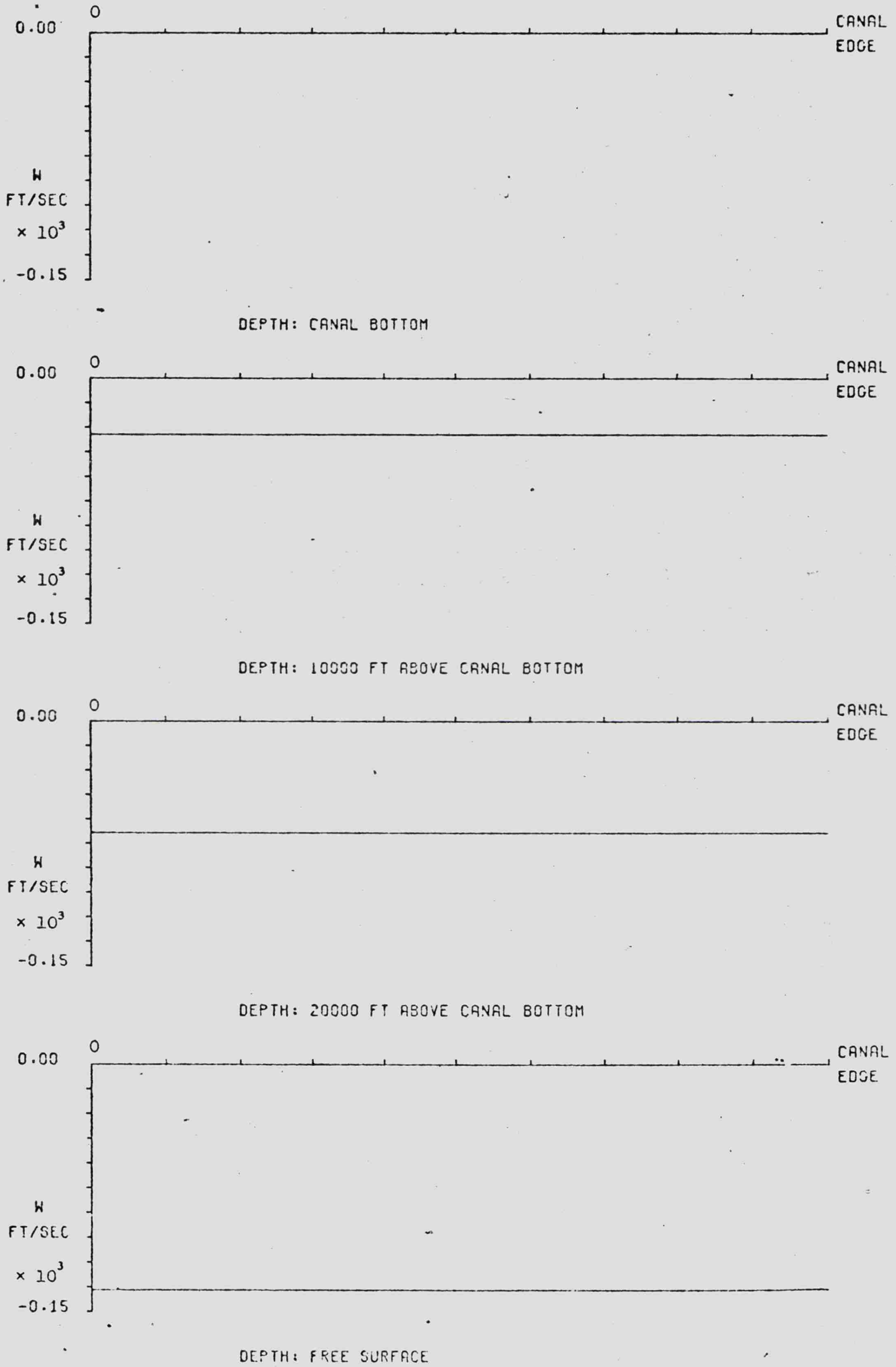


FIG 3.11 W FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
A THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) δ = 0.001

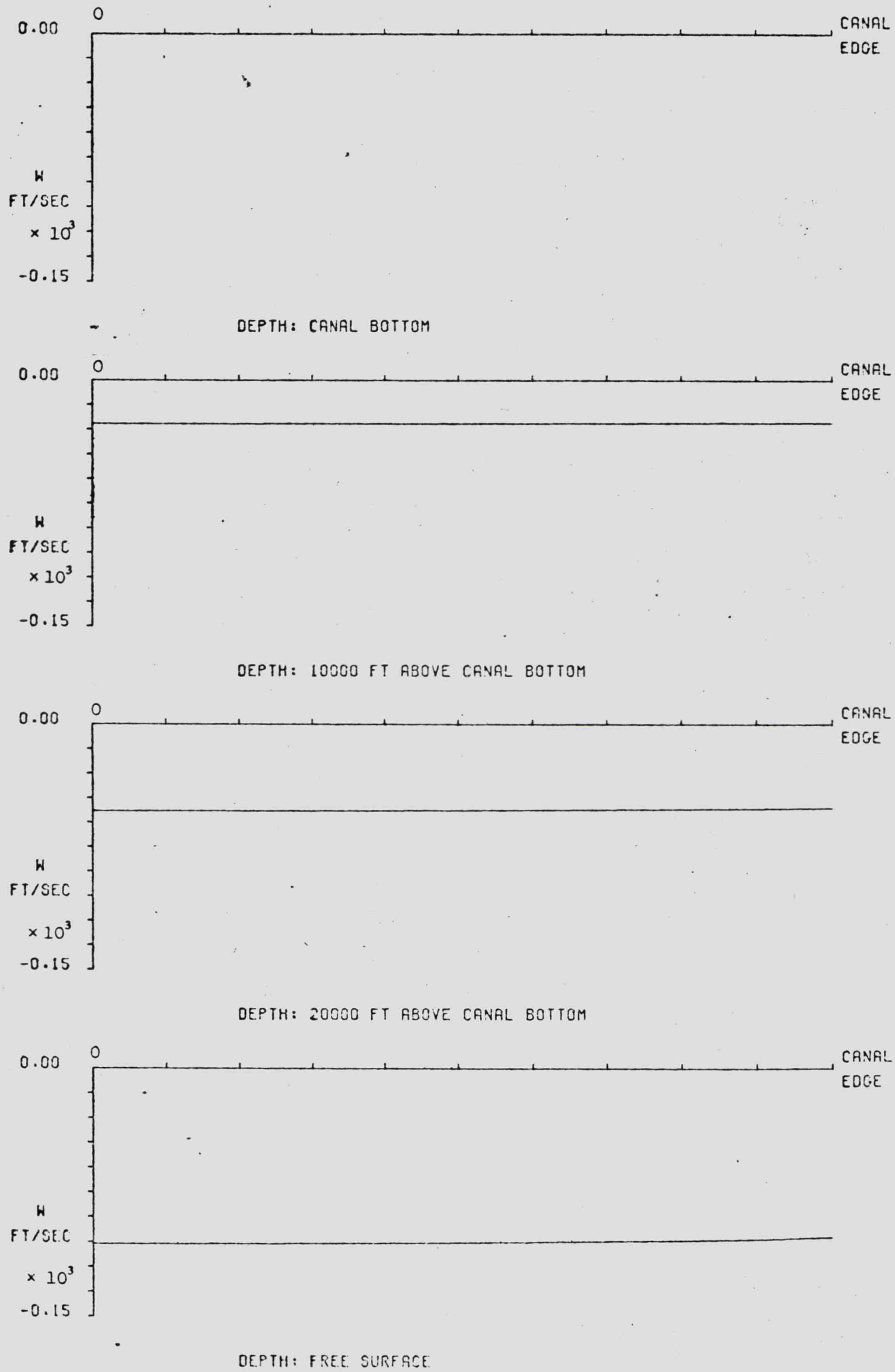
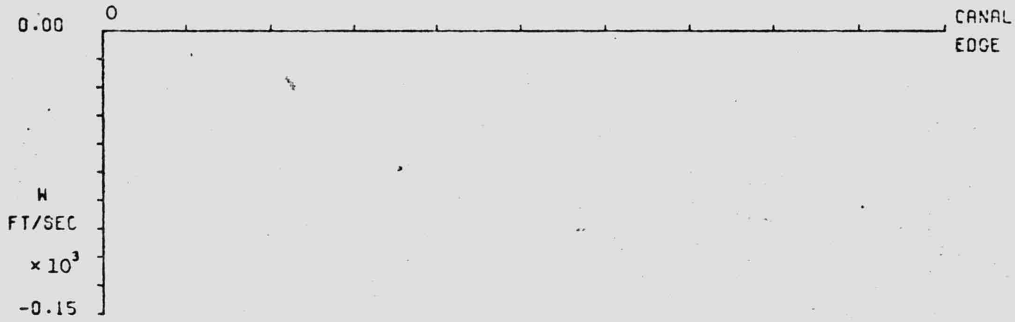
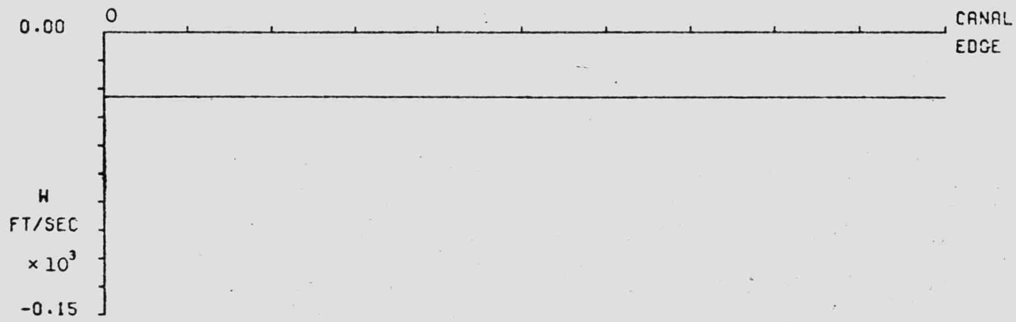


FIG 3.11 W_B FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

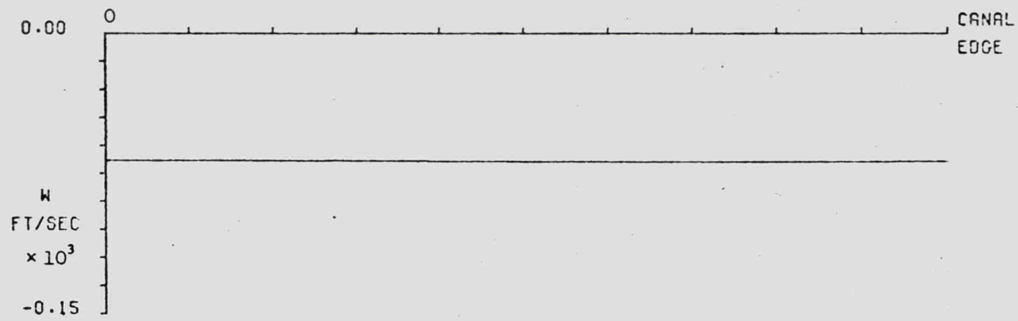
CANAL SEMI-WIDTH= 2×10^5 FT (37.9 MLS) β = 0.001



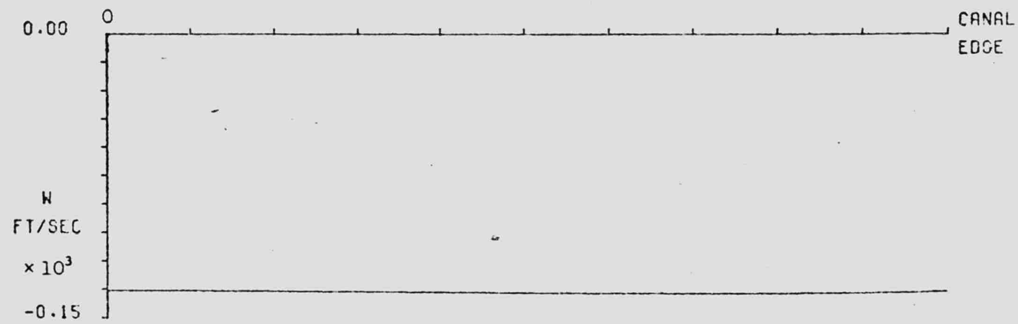
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 3.11 W_C FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 1 × 10^6 FT (189.4 MLS) β = 0.001

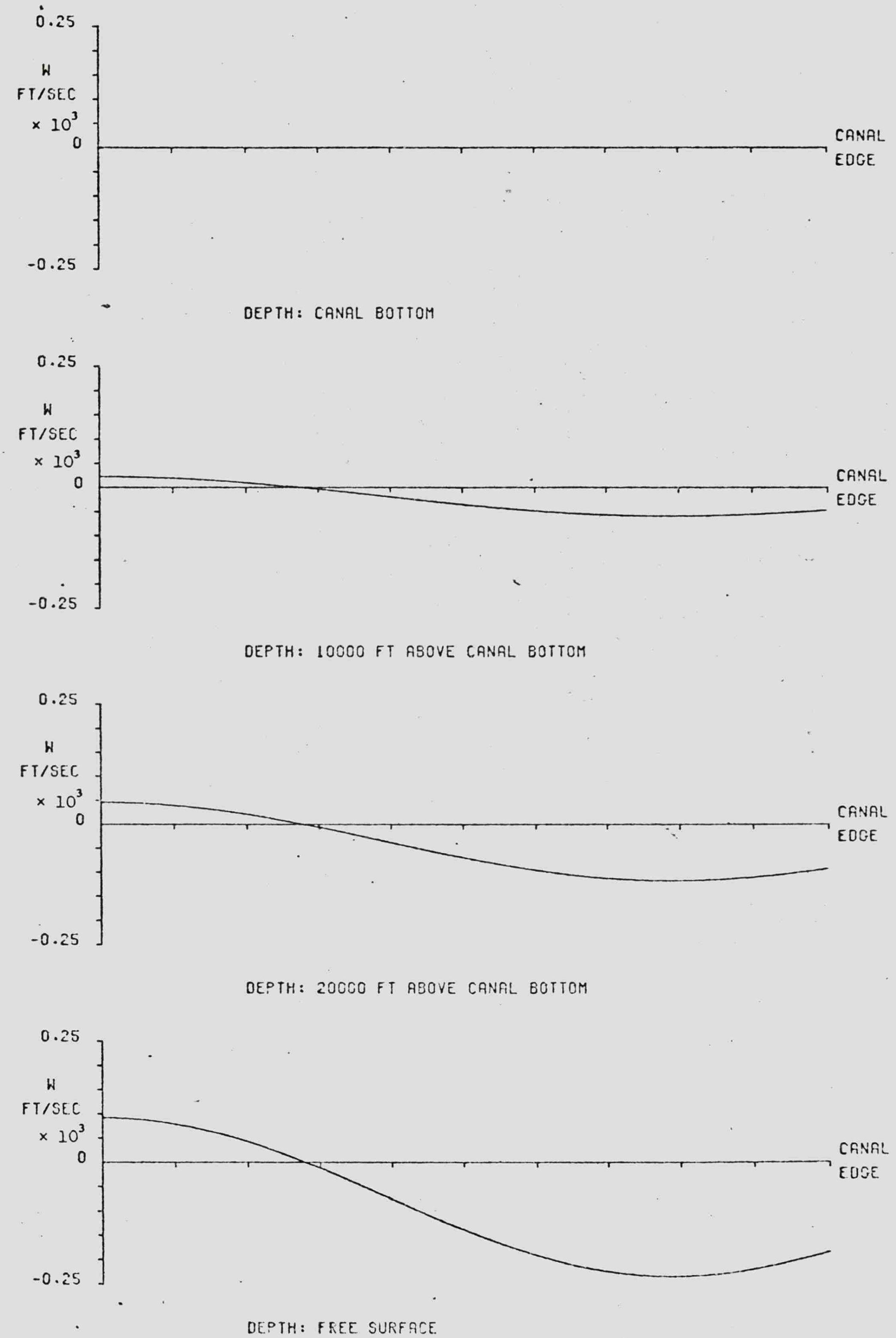


FIG 3.11 W_D FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 1×10^6 FT (189.4 MLS) $\beta = 0.001$

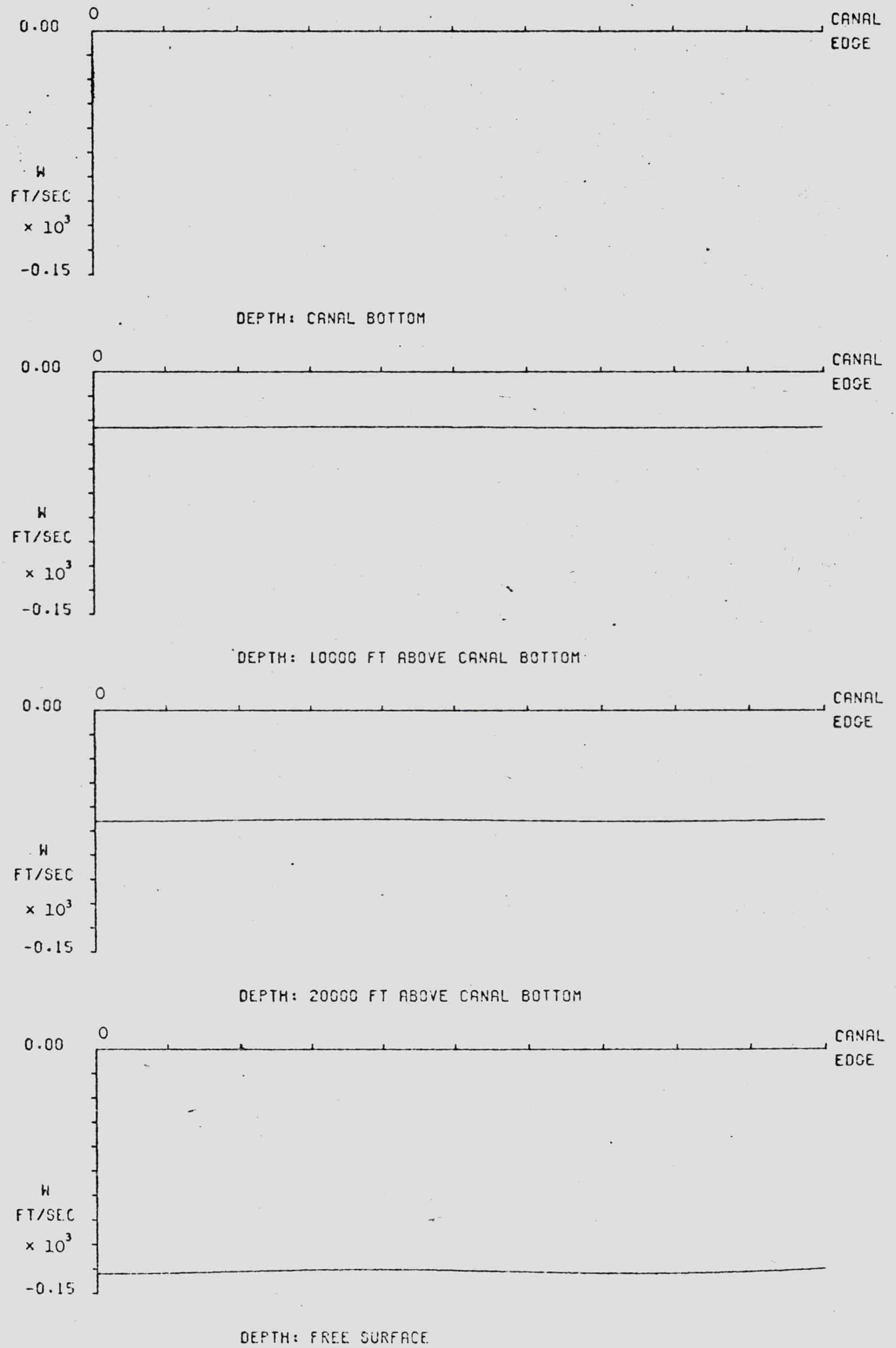


FIG 3.11 W_E FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^6 FT (378.8 MLS) $\beta = 0.001$

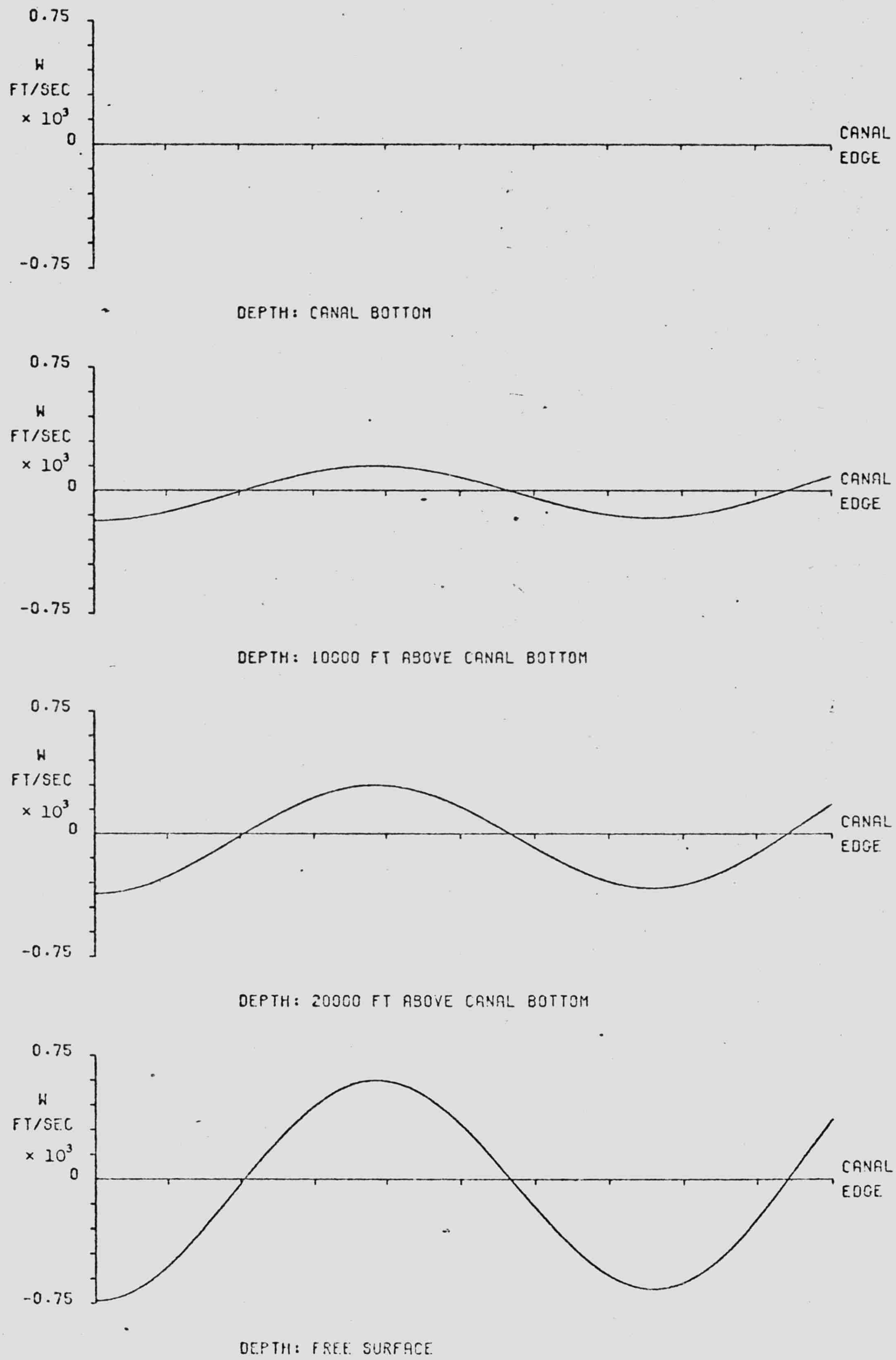


FIG 3.11 W_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 2×10^5 FT (378.9 MLS) $\beta = 0.001$

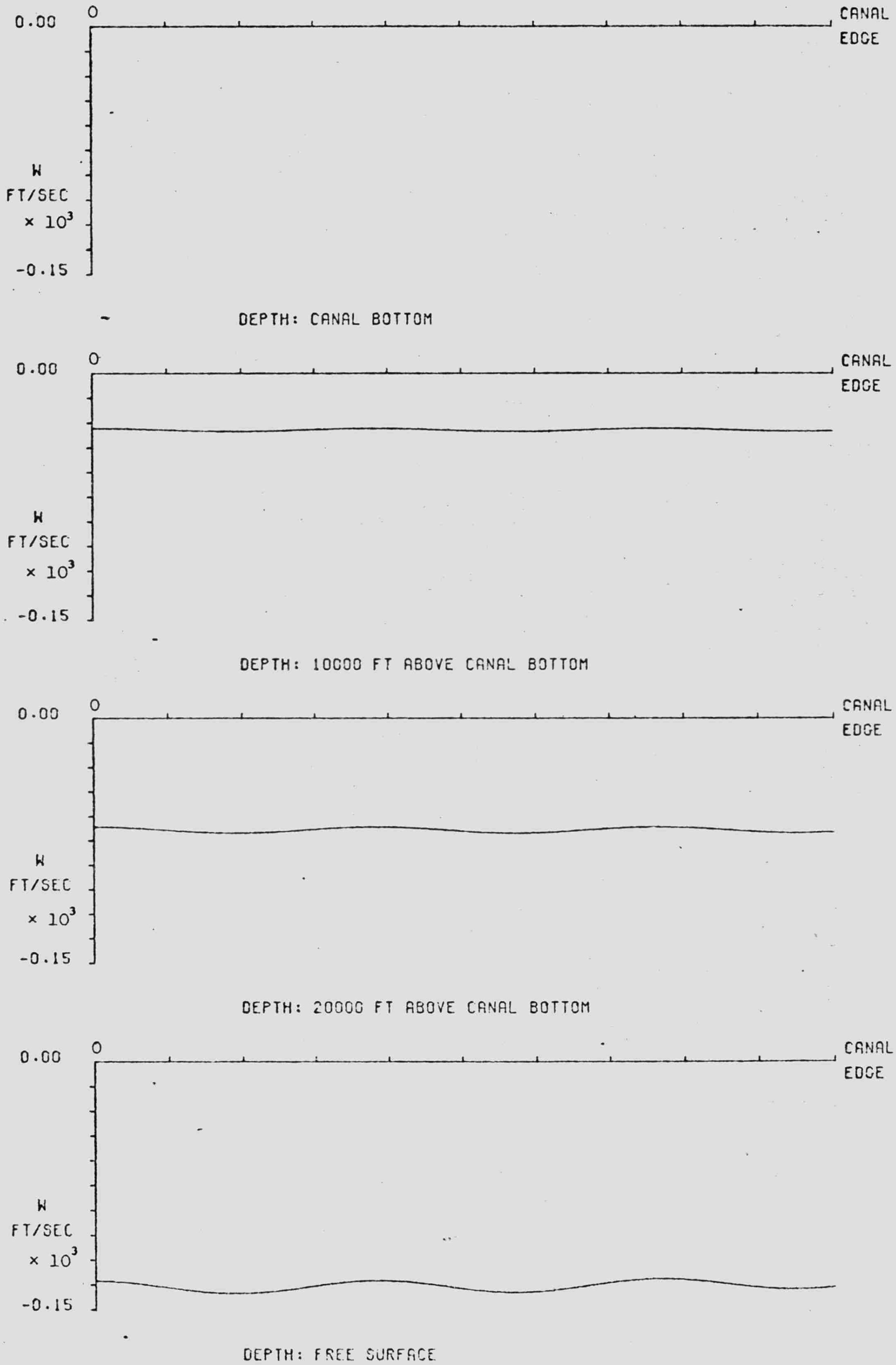


FIG 3.11 W_G FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM EQUATOR TO CANAL EDGE

CANAL SEMI-WIDTH= 4×10^5 FT (757.6 MLS) $\beta = 0.001$

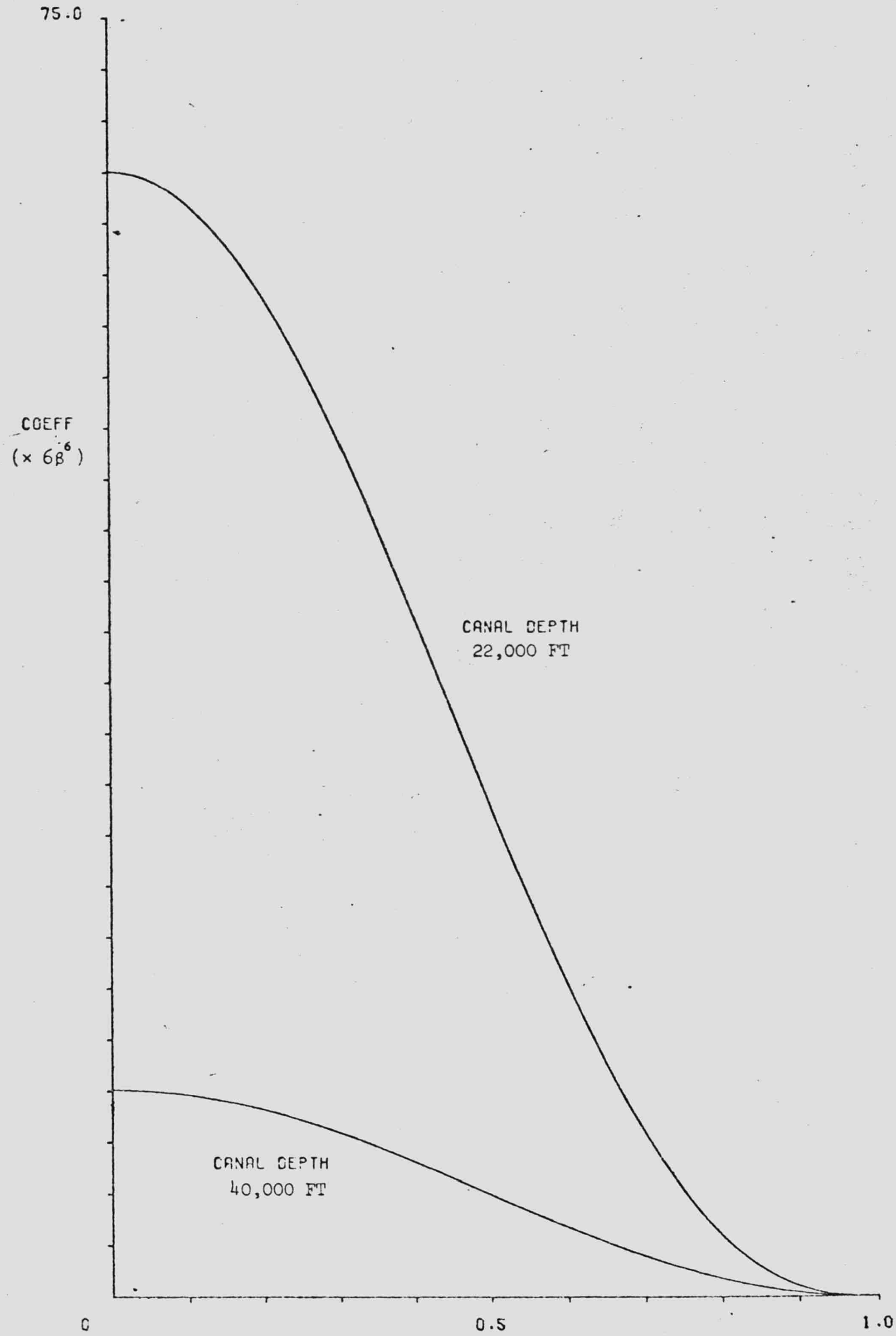


FIG 3.12 THE SECOND ORDER COEFFICIENT

4. THE THREE-DIMENSIONAL TIDAL EQUATIONS IN THE CASE OF FORCED OSCILLATIONS WITH VERY LONG PERIOD

4.1 The Perturbation Equations

In this section we examine Laplace's "tidal oscillations of the first species" where the disturbing potential has the form

$$\Omega = \kappa\left(\cos^2\theta - \frac{1}{3}\right)\cos\sigma t$$

in which θ represents colatitude and σ takes values close to zero. Where $\sigma = 0$, simplifications in the governing equations would once again allow the approach of [1] to be adopted. This suggests that, for small σ, we express the solution as a perturbation series in the small term and then utilize [1] in each member of the problem sequence. Following this approach we present below the development of the solutions for the zeroth-order and first-order terms.

The linearised form of the governing equations may be written as follows

$$\frac{\partial u}{\partial t} - 2v\cos\theta = -\frac{1}{z}\frac{\partial P}{\partial\theta} \qquad \text{..... (1)}$$

$$\frac{\partial v}{\partial t} + 2u\cos\theta + 2w\sin\theta = -\frac{1}{z\sin\theta}\frac{\partial P}{\partial\phi} \qquad \text{..... (2)}$$

$$\frac{\partial w}{\partial t} - 2v\sin\theta = -\frac{\partial P}{\partial z} \qquad \text{..... (3)}$$

$$\frac{\partial}{\partial\theta}(u \sin\theta) + \frac{\partial v}{\partial\phi} + \frac{\sin\theta}{z}\frac{\partial}{\partial z}(z^2 w) = 0 \qquad \ldots\ldots (4)$$

where

$$\left.\begin{aligned} \beta z &= 1 + \beta R \\ w = \alpha w \quad \underline{P} &= \frac{1}{\beta} p \end{aligned}\right\} \qquad \ldots\ldots (5)$$

We consider a disturbing potential of the form

$$\Omega = 2F(\theta, z) \exp(i\varepsilon t) \qquad \ldots\ldots (6)$$

where $F(\theta, z)$ is a known function and ε is a small quantity. The problem, once again, is to determine the response of the ocean to this external driving potential.

We look for a solution to equations (1), (2), (3) and (4) of the form

$$(u, v, w, \underline{P}) = \{u(\theta, z), v(\theta, z), w(\theta, z), P(\theta, z)\} \exp(i\varepsilon t) \qquad \ldots\ldots (7)$$

Hence, substituting these expressions for u, v, w, $\underline{P}$ into the above equations we obtain

$$i\varepsilon u - 2v \cos\theta = -\frac{1}{z}\frac{\partial P}{\partial\theta} \qquad \ldots\ldots (8)$$

$$i\varepsilon v + 2u \cos\theta + 2w \sin\theta = 0 \qquad \ldots\ldots (9)$$

$$i\varepsilon w - 2v \sin\theta = -\frac{\partial P}{\partial z} \qquad \ldots\ldots (10)$$

$$\frac{\partial}{\partial\theta}(u \sin\theta) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}(z^2 w) = 0 \qquad \ldots\ldots (11)$$

In equations (8), (9), (10) and (11) we shall now express the solutions for (u, v, w, P) in the form

$$u = u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \qquad \ldots\ldots (12)$$

$$v = v_o + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \qquad \ldots\ldots (13)$$

$$w = w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \qquad \ldots\ldots (14)$$

$$P = P_o + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots \qquad \ldots\ldots (15)$$

Substituting these expressions into the above equations and equating corresponding powers of ε we obtain

$$v_o \cos\theta = \frac{1}{2z}\frac{\partial P_o}{\partial\theta} \qquad \ldots\ldots (16)$$

$$u_o \cos\theta + w_o \sin\theta = 0 \qquad \ldots\ldots (17)$$

$$v_o \sin\theta = \frac{1}{2}\frac{\partial P_o}{\partial z} \qquad \ldots\ldots (18)$$

$$\frac{\partial}{\partial\theta}(u_o \sin\theta) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2 w_o\right) = 0 \qquad \ldots\ldots (19)$$

for the zeroth-order coefficients. As in the previous section we find that the determinant of the coefficients of u_o, v_o, w_o in (16), (17) and (18) vanishes. Thus, from equations (16) and (18) we may eliminate v_o and obtain

$$\frac{1}{z\cos\theta}\frac{\partial P_o}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial P_o}{\partial z} \qquad \ldots\ldots (20)$$

This has the general solution

$$P_o = 2f\,(z\sin\theta) \qquad \ldots\ldots (21)$$

where $f(x)$ is an arbitrary function of x

From (18) we therefore find that

$$v_o = f'(z\sin\theta) \qquad \ldots\ldots (22)$$

Turning to the solution for w_o we see that, from equations (17) and (19)

$$\frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2 w_o\right) - \frac{\partial}{\partial\theta}\left(w_o\frac{\sin^2\theta}{\cos\theta}\right) = 0$$

$$\therefore z \sin\theta \frac{\partial w_o}{\partial z} - \frac{\sin^2\theta}{\cos\theta}\frac{\partial w_o}{\partial \theta} + 2\sin\theta\, w_o -$$

$$- w_o\left(2\sin\theta + \frac{\sin^2\theta}{\cos^2\theta}\sin\theta\right) = 0$$

$$\text{ie} \quad z\cos\theta \frac{\partial w_o}{\partial z} - \sin\theta \frac{\partial w_o}{\partial \theta} - \frac{\sin^2\theta}{\cos\theta} w_o = 0 \qquad \ldots\ldots (23)$$

The integral surfaces of this equation are generated by the integral curves of the equations

$$\frac{dz}{z\cos\theta} = \frac{d\theta}{-\sin\theta} = \frac{\cos\theta\, dw_o}{\sin^2\theta\, w_o} \qquad \ldots\ldots (24)$$

The first equation of this set can be written as

$$\frac{dz}{z} = -\frac{\cos\theta}{\sin\theta}d\theta$$

which integrates to give

$$z\sin\theta = c_1 \qquad \ldots\ldots (25)$$

where c_1 is a constant. The second equation of (24) can be written as

$$\frac{dw_o}{w_o} = -\frac{\sin\theta}{\cos\theta}\,d\theta$$

which integrates to give

$$\frac{w_o}{\cos\theta} = c_2 \qquad \ldots\ldots (26)$$

where c_2 is a second constant. Hence the general solution of (23) is

$$w_o = \cos\theta\; q(z\sin\theta) \qquad \ldots\ldots (27)$$

where $q(z \sin\theta)$ is an arbitrary function of $z \sin\theta$.

We may now turn to the boundary conditions. From 2.2 (31) we require that $w_o = 0$ at $z = \frac{1}{\beta}$. Hence we obtain

$$q\left(\frac{1}{\beta}\sin\theta\right) \equiv 0 \qquad \text{for all } \theta$$

so that

$$w_o \equiv 0 \qquad \ldots\ldots (28)$$

Furthermore, from (17)

$$u_o \equiv 0 \qquad \text{..... (29)}$$

We consider now the condition on the free-surface. Once again, from equations (1), (2) and (3) we note that the pressure is undefined to the extent of an arbitrary constant. Hence we may write

$$\underline{P} = C + P(\theta, z)\exp(i\varepsilon t) \qquad \text{..... (30)}$$

where C is a constant. Hence, from 2.2 (33) we obtain

$$\beta C + \{\beta P(\theta, z) - 2F(\theta, z)\}\exp(i\varepsilon t) - \lambda\beta z + \qquad \text{..... (31)}$$
$$+ \mu\beta^2 z^2 \sin^2\theta = \pi_o$$

At $\theta = 0$ the mean height of the ocean is $z = \frac{1}{\beta}(1 + \beta)$. Hence

$$\beta C - \lambda(1 + \beta) = \pi_o \qquad \text{..... (32)}$$

Thus the equation of the free-surface is

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\beta\mu}{\lambda} z^2 \sin^2\theta + \frac{1}{\lambda\beta}\{\beta P(\theta, z) - \qquad \text{..... (33)}$$
$$- 2F(\theta, z)\}\exp(i\varepsilon t)$$

and its mean position will be given by

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\beta\mu}{\lambda} z^2 \sin^2\theta$$

This can be written with sufficient accuracy in the form

$$z = \frac{1}{\beta}(1 + \beta) + \frac{\beta\mu}{\lambda} \frac{(1 + \beta)^2}{\beta^2} \sin^2\theta$$

ie $$z = \frac{1}{\beta}(1 + \beta)(1 + \varepsilon_o \sin^2\theta) \quad ,$$

$$\varepsilon_o = \frac{\mu}{\lambda}(1 + \beta) = 0\ (10^{-3}) \qquad \ldots\ldots (34)$$

We next consider the kinematic condition 2.2 (36). Thus, at the free surface, we obtain from (14), (15) and 2.2 (36)

$$\beta\varepsilon(P_o + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots) - 2\varepsilon F(\theta, z) + ik(w_o + \varepsilon w_1 + $$

$$+ \varepsilon^2 w_2 + \ldots)$$

$$= 0$$

Equating corresponding powers of ε we then get

$$w_o = 0 \qquad \ldots\ldots (35)$$

and $$\beta P_0 - 2F(\theta, z) + ikw_1 = 0 \qquad \ldots\ldots (36)$$

The zeroth-order condition is satisfied by virtue of (28). We will return to the second condition after we have developed appropriate expressions for the first-order variables of the solution.

From equations (8)-(15) we have

$$v_1 \cos\theta = \frac{1}{2z}\frac{\partial p_1}{\partial\theta} + \frac{i}{2}u_0 \qquad \ldots\ldots (37)$$

$$u_1 \cos\theta + w_1 \sin\theta + \frac{i}{2}v_0 = 0 \qquad \ldots\ldots (38)$$

$$v_1 \sin\theta = \frac{1}{2}\frac{\partial P_1}{\partial z} + \frac{i}{2}w_0 \qquad \ldots\ldots (39)$$

$$\frac{\partial}{\partial\theta}(u_1 \sin\theta) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2\ w_1\right) = 0 \qquad \ldots\ldots (40)$$

for the first-order coefficients. Once again, the determinant of the coefficients of u_1, v_1, w_1 in equations (37), (38) and (39) vanishes. Also u_0 and w_0 are identically zero by virtue of (28) and (29). Thus, eliminating v_1 between (37) and (39) we obtain

$$\frac{1}{z\cos\theta}\frac{\partial P_1}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial P_1}{\partial z} \qquad \ldots\ldots (41)$$

which has the general solution

$$P_1 = 2g(z \sin \theta) \qquad \ldots\ldots (42)$$

where g is an arbitrary function.

Hence from (37) we obtain

$$v_1 = g'(z \sin \theta) \qquad \ldots\ldots (43)$$

Furthermore, to obtain an expression for w_1, we see that, from equations (38) and (40)

$$\frac{\sin \theta}{z} \frac{\partial}{\partial z}\left(z^2 \; w_1\right) - \frac{\partial}{\partial \theta}\left(w_1 \; \frac{\sin^2 \theta}{\cos \theta}\right)$$

$$= \frac{i}{2} \frac{\partial}{\partial \theta}\left(v_o \; \frac{\sin \theta}{\cos \theta}\right)$$

ie $$z \cos \theta \frac{\partial w_1}{\partial z} - \sin \theta \frac{\partial w_1}{\partial \theta} - \frac{\sin^2 \theta}{\cos \theta} w_1$$

$$= \frac{i}{2} \frac{\partial v_o}{\partial \theta} + \frac{i}{2} \frac{v_o}{\sin \theta \cos \theta}$$

Using (22), this last equation then becomes

$$z \cos\theta \frac{\partial w_1}{\partial z} - \sin\theta \frac{\partial w_1}{\partial \theta} - \frac{\sin^2\theta}{\cos\theta} w_1 = \frac{i}{2} z \cos\theta\, f''(z \sin\theta) + \frac{if'(z \sin\theta)}{2 \sin\theta \cos\theta} \qquad \ldots\ldots (44)$$

In this case, the auxiliary equations are

$$\frac{dz}{z \cos\theta} = \frac{d\theta}{-\sin\theta} = \frac{dw_1}{\frac{\sin^2\theta}{\cos\theta} w_1 + \frac{i}{2} z \cos\theta\, f'' + \frac{if'}{2 \sin\theta \cos\theta}} \qquad \ldots\ldots (45)$$

The first equation of this set is similar to that of equations (24) and gives us

$$z \sin\theta = c_1 \qquad \ldots\ldots (46)$$

where c_1 is a constant. The second equation of (45) may be written as

$$\frac{dw_1}{d\theta} = -\left(\frac{\sin^2\theta}{\cos\theta} w_1 + \frac{ic_1 \cos\theta\, f''(c_1)}{2 \sin\theta} + \frac{if'(c_1)}{2 \sin\theta \cos\theta} \right) \frac{1}{\sin\theta}$$

$$\text{ie} \quad \frac{dw_1}{d\theta} + \frac{\sin\theta}{\cos\theta} w_1 = -\frac{ic_1 f''(c_1)\cos\theta}{2\sin^2\theta} - \frac{if'(c_1)}{2\sin^2\theta\cos\theta} \qquad \ldots\ldots (47)$$

An integrating factor for this equation is $e^{\int \frac{\sin\theta}{\cos\theta} d\theta} = \frac{1}{\cos\theta}$ so that

$$\frac{d}{d\theta}\left(\frac{w_1}{\cos\theta}\right) = -\frac{ic_1 f''(c_1)}{2\sin^2\theta} - \frac{2i\, f'(c_1)}{\sin^2 2\theta}$$

which integrates to give

$$\frac{w_1}{\cos\theta} = \frac{ic_1}{2} f''(c_1)\cot\theta + if'(c_1)\cot 2\theta + c_2 \qquad \ldots\ldots (48)$$

where c_1 is a second constant. Thus, on substituting for c_1 from (46), we obtain the general solution of (44) as follows

$$w_1 = \frac{i}{2} z\cos^2\theta\, f''(z\sin\theta) + \frac{i\cos 2\theta\, f'(z\sin\theta)}{2\sin\theta} + \qquad \ldots\ldots (49)$$

$$+ \cos\theta\, Q(z\sin\theta)$$

where $Q(z \sin\theta)$ is an arbitrary function of $z \sin\theta$.

The boundary condition 2.2 (31) then gives us the following identity in θ.

$$Q\left(\frac{1}{\beta}\sin\theta\right) \equiv -\frac{i}{2\beta}\cos\theta\, f''\left(\frac{1}{\beta}\sin\theta\right) - i\cot 2\theta\, f'\left(\frac{1}{\beta}\sin\theta\right)$$

and thus, if we write $\eta = \frac{1}{\beta}\sin\theta$ and note that

$$\frac{1}{\beta^2}\cos^2\theta = \frac{1}{\beta^2} - \eta^2 \quad , \cot 2\theta = \frac{\frac{1}{\beta^2} - 2\eta^2}{2\eta\sqrt{\frac{1}{\beta^2} - \eta^2}}$$

it follows that

$$Q(\eta) \equiv -\frac{i}{2}\sqrt{\frac{1}{\beta^2} - \eta^2}\, f''(\eta) - - i\left(\frac{\frac{1}{\beta^2} - 2\eta^2}{2\eta\sqrt{\frac{1}{\beta^2} - \eta^2}}\right) f'(\eta) \qquad \text{..... (50)}$$

Accordingly, we can write (49) in the form

$$w_1 = \frac{i}{2} z\cos^2\theta\, f''(z\sin\theta) + \frac{i\cos 2\theta\, f'(z\sin\theta)}{2\sin\theta} - - \frac{i\cos\theta}{2}\left(\sqrt{\frac{1}{\beta^2} - z^2\sin^2\theta}\, f''(z\sin\theta) + \right.$$

$$
+\left(\frac{\frac{1}{\beta^2} - 2z^2\sin^2\theta}{z\sin\theta\sqrt{\frac{1}{\beta^2} - z^2\sin^2\theta}}\right) f'(z\sin\theta)\Bigg) \qquad \ldots\ldots (51)
$$

which contains one arbitrary function, $f(z \sin \theta)$.

We now consider condition (36) at the free-surface. From (21), (51) we obtain

$$
\frac{4}{k}\{\beta f(z\sin\theta) - F(\theta, z)\} - z\cos^2\theta\, f''(z\sin\theta) -
$$

$$
-\frac{\cos 2\theta\, f'(z\sin\theta)}{\sin\theta} + \cos\theta\left(\sqrt{\frac{1}{\beta^2} - z^2\sin^2\theta}\; f''(z\sin\theta) + \right. \qquad \ldots\ldots (52)
$$

$$
+\left(\frac{\frac{1}{\beta^2} - 2z^2\sin^2\theta}{z\sin\theta\sqrt{\frac{1}{\beta^2} - z^2\sin^2\theta}}\right) \times
$$

$$
\left. \times\, f'(z\sin\theta)\right) = 0
$$

with $z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_o \sin^2\theta\right)$. Thus, when this value of z is inserted into equation (52) we have a linear second-order ordinary differential equation for the determination of the function f. As indicated in the introduction, $F(\theta, z)$ takes the form.

$$
F(\theta, z) = \kappa\left(\cos^2\theta - \frac{1}{3}\right), \quad \kappa \text{ constant} \qquad \ldots\ldots (53)
$$

Hence the differential equation for $f(\eta)$ is

$$\frac{4\beta f(\eta)}{k} - \frac{(z^2 - \eta^2)f''(\eta)}{z} - \frac{(z^2 - 2\eta^2)\ f'(\eta)}{z\eta} +$$

$$+ \frac{1}{z}\sqrt{(z^2 - \eta^2)\left(\frac{1}{\beta^2} - \eta^2\right)}\ f''(\eta) + \frac{1}{z}\sqrt{z^2 - \eta^2} \times \qquad ..... (54)$$

$$\times \left(\frac{\frac{1}{\beta^2} - 2\eta^2}{\eta\sqrt{\frac{1}{\beta^2} - \eta^2}}\right) f'(\eta) = \frac{4\kappa}{3kz^2}\ (2z^2 - 3\eta^2)$$

where $z = \frac{1}{\beta}\ (1 + \beta)\left(1 + \varepsilon_o \sin^2\theta\right) = \frac{\eta}{\sin\theta}$

Turning to the remaining boundary conditions, the nature of the solutions suggests that we consider axially symmetric boundary surfaces of the form

$$\beta z \sin\theta = d \qquad ..... (55)$$

where d is a constant satisfying $-1 \leqslant d \leqslant 1$.

Figure follows

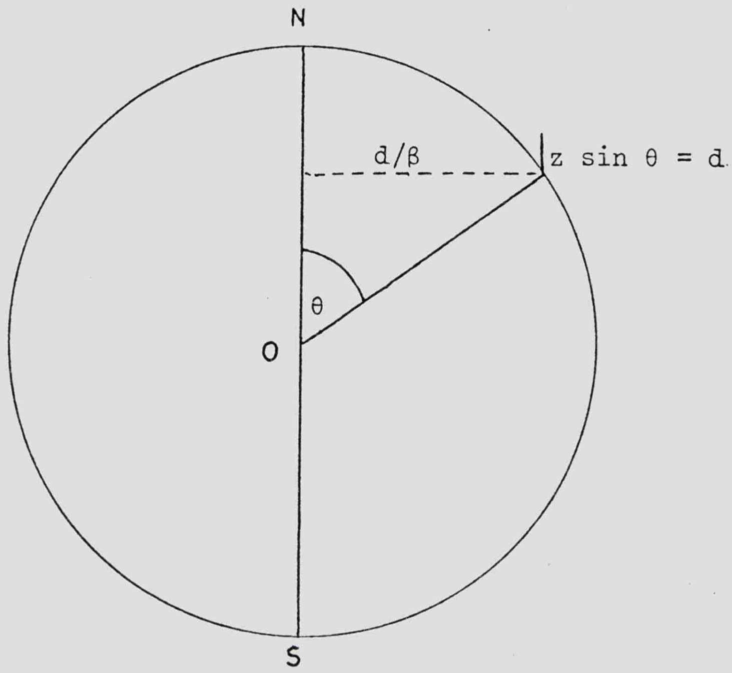


Figure 4.1

These surfaces are the right circular cylinders of radius $\frac{d}{\beta}$.
Condition 2.2 (33B) then gives

$$u \cos\theta + w \sin\theta = 0 \quad , \quad \beta z \sin\theta = d$$

for the vanishing of the normal velocity. Using (12) and (14) this becomes

$$(u_0 + \varepsilon u_1 + \ldots) \cos\theta + (w_0 + \varepsilon w_1 + \ldots) \sin\theta = 0 \qquad \ldots\ldots (56)$$

on $\beta z \sin\theta = d$.

In (56), the zeroth-order terms vanish by virtue of equation (17). Furthermore, from (38), the first-order terms amount to the condition:

$$v_o = 0 \quad , \quad \beta z \sin\theta = d$$

which, from (22), yields

$$f'\left(\frac{d}{\beta}\right) = 0 \qquad \ldots\ldots (57)$$

Two such conditions then fully determine the function f.

Of particular interest is the case

$$\left.\begin{aligned} f'\left(\frac{\delta_1}{\beta}\right) &= 0 \\ f'\left(\frac{\delta_2}{\beta}\right) &= 0 \end{aligned}\right\} \qquad \ldots\ldots (58)$$

where δ_1 and δ_2 are both small and are both of the same sign. This describes a canal-like region around an ice-cap. Furthermore, by virtue of the smallness of δ_1 and δ_2, its sidewalls may be considered to be approximately vertical.

The expressions for v_1 and p_1 involve the function $g(z \sin\theta)$ which, at the moment, remains arbitrary. In order to resolve this further we must examine equations (8)-(15) for the second-order coefficients. Thus we have

$$v_2 \cos\theta = \frac{1}{2z}\frac{\partial P_2}{\partial\theta} + \frac{i}{2} u_1 \qquad \ldots\ldots (59)$$

$$u_2 \cos\theta + w_2 \sin\theta + \frac{i}{2} v_1 = 0 \qquad \ldots\ldots (60)$$

$$v_2 \sin\theta = \frac{1}{2}\frac{\partial P_2}{\partial z} + \frac{i}{2} w_1 \qquad \ldots\ldots (61)$$

$$\frac{\partial}{\partial\theta}(u_2 \sin\theta) + \frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2 w_2\right) = 0 \qquad \ldots\ldots (62)$$

Now, from equations (60) and (62) we obtain

$$\frac{\sin\theta}{z}\frac{\partial}{\partial z}\left(z^2 w_2\right) - \frac{\partial}{\partial\theta}\left(w_2 \frac{\sin^2\theta}{\cos\theta}\right) = \frac{i}{2}\frac{\partial}{\partial\theta}\left(v_1 \frac{\sin\theta}{\cos\theta}\right)$$

ie

$$z\cos\theta \frac{\partial w_2}{\partial z} - \sin\theta \frac{\partial w_2}{\partial\theta} - \frac{\sin^2\theta}{\cos\theta} w_2 = \frac{i}{2}\frac{\partial v_1}{\partial\theta} + \frac{i v_1}{2\sin\theta\cos\theta} \qquad \ldots\ldots (63)$$

Hence, using (43), we obtain

$$z\cos\theta \frac{\partial w_2}{\partial z} - \sin\theta \frac{\partial w_2}{\partial\theta} - \frac{\sin^2\theta}{\cos\theta} w_2 = \frac{i}{2} z\cos\theta\, g''(z\sin\theta) + \frac{i g'(z\sin\theta)}{2\sin\theta\cos\theta} \qquad \ldots\ldots (64)$$

which is identical in form to equation (44). Hence we may write down the solution to (64) as follows

$$w_2 = \frac{i}{2} z \cos^2\theta\, g''(z \sin\theta) + \frac{i \cos 2\theta\, g'(z \sin\theta)}{2 \sin\theta} -$$

$$- \frac{i}{2} \cos\theta \left(\sqrt{\frac{1}{\beta^2} - z^2 \sin^2\theta}\; g''(z \sin\theta) + \right. \qquad \text{..... (65)}$$

$$\left. + \left(\frac{\frac{1}{\beta^2} - 2z^2 \sin^2\theta}{z \sin\theta \sqrt{\frac{1}{\beta^2} - z^2 \sin^2\theta}} \right) g'(z \sin\theta) \right)$$

where we have satisfied condition 2.2 (31). The above expression only contains the one arbitrary function $g(z \sin\theta)$.

Returning, once again, to the kinematic condition 2.2 (36) at the free-surface, the second-order condition yields

$$\beta P_1 + ikw_2 = 0 \qquad \text{..... (66)}$$

on $z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_0 \sin^2\theta\right)$

Thus, from (42) and (66) we obtain

$$\frac{4\beta}{k} g(z \sin\theta) - z \cos^2\theta\, g''(z \sin\theta) - \frac{\cos 2\theta}{\sin\theta}\; g'(z \sin\theta) +$$

$$+ \cos\theta \sqrt{\frac{1}{\beta^2} - z^2\sin^2\theta}\; g''(z\sin\theta) + \cos\theta \times$$

$$\times \left(\frac{\dfrac{1}{\beta^2} - 2z^2\sin^2\theta}{z\sin\theta\sqrt{\dfrac{1}{\beta^2} - z^2\sin^2\theta}} \right) g'(z\sin\theta) = 0 \qquad \ldots\ldots (67)$$

with $z = \frac{1}{\beta}(1+\beta)\left(1 + \varepsilon_o \sin^2\theta\right)$. Again, when this value of z is inserted into (67) we have a linear second order ordinary differential equation for the determination of the function g. The differential equation for $g(\eta)$ can be written in the form

$$\frac{4\beta}{k} g(\eta) - \frac{(z^2 - \eta^2)\, g''(\eta)}{z} - \frac{(z^2 - 2\eta^2)\, g'(\eta)}{z\eta} +$$

$$+ \frac{1}{z}\sqrt{(z^2 - \eta^2)\left(\frac{1}{\beta^2} - \eta^2\right)}\;\; g''(\eta) + \frac{1}{z}\sqrt{z^2 - \eta^2} \times \qquad \ldots\ldots (68)$$

$$\times \left(\frac{\dfrac{1}{\beta^2} - 2\eta^2}{\eta\sqrt{\dfrac{1}{\beta^2} - \eta^2}} \right) g'(\eta) = 0$$

where $z = \frac{1}{\beta}(1+\beta)\left(1 + \varepsilon_o \sin^2\theta\right) = \eta/\cos\theta$.

On a surface $\beta z \;\; \sin\theta = d$ we must have

$$u_2 \cos\theta + w_2 \sin\theta = 0$$

Hence, from (60), this implies

$$v_1 = 0 \quad , \ \beta z \sin\theta = d$$

which, from (43), yields

$$g'\left(\frac{d}{\beta}\right) = 0 \qquad \ldots\ldots (69)$$

The function g is then fully determined by two conditions of this type. Clearly, the solution is

$$g \equiv 0 \qquad \ldots\ldots (70)$$

From (42) and (43) we therefore have

$$\left.\begin{aligned} P_1 &= 0 \\ v_1 &= 0 \end{aligned}\right\} \qquad \ldots\ldots (71)$$

4.2 The Differential Equation

It has been shown that the functions $\{u_o, w_o, P_1, v_1\}$ vanish and that the problem of determining the remaining functions reduces to one of solving the differential equation 4.1.54. In order to proceed further, we shall begin by restating the differential equation together with its boundary conditions. Thus, we have

$$\frac{4\beta}{k} f(\eta) - \frac{(z^2 - \eta^2)}{z} f''(\eta) - \frac{(z^2 - 2\eta^2)}{z\eta} f'(\eta) +$$

$$+ \frac{\sqrt{(z^2 - \eta^2)\left(\frac{1}{\beta^2} - \eta^2\right)}}{z} f''(\eta) + \frac{\sqrt{z^2 - \eta^2}}{z\eta} \times \qquad \text{..... (1)}$$

$$\times \left(\frac{\frac{1}{\beta^2} - 2\eta^2}{\sqrt{\frac{1}{\beta^2} - \eta^2}}\right) f'(\eta) = \frac{4\kappa}{3kz^2}(2z^2 - 3\eta^2)$$

where $z = \frac{1}{\beta}(1 + \beta)\left(1 + \varepsilon_o \sin^2\theta\right) = \frac{\eta}{\sin\theta}$ ..... (2)

Equation (1) holds throughout the region $\delta_1 \leqslant \beta\eta \leqslant \delta_2$ where δ_1 and δ_2 are two constants of the same sign satisfying the inequalities.

$$\left.\begin{array}{l} 0 \leqslant |\delta_1| << 1 \\ 0 \leqslant |\delta_2| << 1 \end{array}\right\} \qquad \text{..... (3)}$$

At the boundaries we have

$$\left.\begin{aligned} f'\left(\frac{\delta_1}{\beta}\right) &= 0 \\ f'\left(\frac{\delta_2}{\beta}\right) &= 0 \end{aligned}\right\} \qquad \ldots\ldots (4)$$

Now, from (2)

$$\sin\theta = \frac{\beta\eta}{(1+\beta)}\left(1 + O(\varepsilon_o)\right)$$

$$\therefore \qquad z = \frac{1}{\beta}(1+\beta)\left[1 + \varepsilon_o \frac{\beta^2\eta^2}{(1+\beta)^2} + O\left(\varepsilon_o^{\,2}\right)\right] \qquad \ldots\ldots (5)$$

Also

$$\left[(z^2 - \eta^2)\left(\frac{1}{\beta^2} - \eta^2\right)\right]^{\frac{1}{2}} \equiv \left\{(z^2 - \eta^2)^2\left[1 + \frac{\frac{1}{\beta^2} - z^2}{z^2 - \eta^2}\right]\right\}^{\frac{1}{2}}$$

$$= (z^2 - \eta^2)\left[1 + \frac{\frac{1}{\beta^2} - z^2}{2(z^2 - \eta^2)} + O(\beta^2)\right] \qquad \ldots\ldots (6)$$

and

$$\left[\frac{z^2 - \eta^2}{\dfrac{1}{\beta^2} - \eta^2}\right]^{\frac{1}{2}} \equiv \left\{1 + \frac{z^2 - \dfrac{1}{\beta^2}}{\dfrac{1}{\beta^2} - \eta^2}\right\}^{\frac{1}{2}}$$

$$= 1 + \frac{z - \dfrac{1}{\beta^2}}{2\left(\dfrac{1}{\beta^2} - \eta^2\right)} + O(\beta^2) \qquad \ldots\ldots (7)$$

Hence, using (6) and (7) the differential equation becomes

$$\frac{4\beta}{k} f(\eta) - \frac{\left(z^2 - \dfrac{1}{\beta^2}\right)}{2\beta^2 z\eta\left(\dfrac{1}{\beta^2} - \eta^2\right)} f'(\eta) - \frac{\left(z^2 - \dfrac{1}{\beta^2}\right)}{2z} f''(\eta)$$

$$\ldots\ldots (8)$$

$$= \frac{4\kappa}{3kz^2}(2z^2 - 3\eta^2)$$

where z is given by expression (5). Thus, multiplying (8) throughout by $2z^2\eta\left(\dfrac{1}{\beta^2} - \eta^2\right)$ we obtain

$$\frac{8\beta}{k} z^2\eta\left(\frac{1}{\beta^2} - \eta^2\right) f(\eta) - \frac{z}{\beta^2}\left(z^2 - \frac{1}{\beta^2}\right) f'(\eta) -$$

$$- z\eta\left(\frac{1}{\beta^2} - \eta^2\right)\left(z^2 - \frac{1}{\beta^2}\right) f''(\eta)$$

$$= \frac{8\kappa}{3k}\,\eta\left(\frac{1}{\beta^2} - \eta^2\right)(2z^2 - 3\eta^2)$$

On substituting for z from (5) and bringing together corresponding powers of η we then have

$$\frac{8\beta}{k}\left[\frac{\eta}{\beta^4}(1+\beta)^2 - \left(\frac{(1+\beta)^2}{\beta^2} - \frac{2\varepsilon_o}{\beta^2}\right)\eta^3 - 2\varepsilon_o\eta^5\right] f(\eta) -$$

$$- \frac{1}{\beta^2}\left[\left(\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3}\right) + \varepsilon_o\left(\frac{3(1+\beta)}{\beta} - \frac{1}{\beta(1+\beta)}\right)\eta^2\right] \times$$

$$\times\ f'(\eta) - \left[\left(\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3}\right)\frac{\eta}{\beta^2} - \qquad \ldots\ldots (9)$$

$$- \left(\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3} - \frac{3\varepsilon_o(1+\beta)}{\beta^3} +\right.$$

$$\left. + \frac{\varepsilon_o}{\beta^3(1+\beta)}\right)\eta^3 -$$

$$- \varepsilon_o \left(\frac{3(1 + \beta)}{\beta} - \frac{1}{\beta(1 + \beta)} \right) \eta^5 \Bigg] \; f''(\eta)$$

$$= \frac{8K}{3k} \left[\frac{2}{\beta^4} (1 + \beta)^2 \eta - \left(\frac{2(1 + \beta)^2}{\beta^2} + \frac{3}{\beta^2} - \frac{4\varepsilon_o}{\beta^2} \right) \eta^3 + \right.$$

$$\left. + (3 - 4\varepsilon_o) \eta^5 \right]$$

The coefficient multiplying $f''(\eta)$ has a zero at $\eta = 0$. Thus considering, for the moment,,the complementary function let us look for a solution of the form

$$CF = \eta^c \left(a_o + a_1 \eta + a_2 \eta^2 + \ldots \right) \qquad \ldots\ldots (10)$$

The indicial equation then gives us

$$[c(c - 1) + c]a_o = 0$$

$$\text{ie} \qquad c^2 = 0 \qquad \ldots\ldots (11)$$

Thus the indicial equation for c has a repeated root. Furthermore, it is clear that, on substituting (10) into (9):

$$a_1 = a_3 = a_5 = \ldots = 0$$

For terms in η^{c+1}, η^{c+3}, ... we obtain

$$\frac{8}{k\beta^3}(1+\beta)^2 a_o - \frac{(c+2)^2}{\beta^2}\left[\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3}\right] a_2 = 0 \qquad \ldots\ldots (12)$$

$$- \frac{8}{k\beta}\left[(1+\beta)^2 - 2\varepsilon_o\right] a_o + \left\{\frac{8(1+\beta)^2}{k\beta^3} - \frac{(c+2)}{\beta^2}\varepsilon_o \times \right.$$

$$\times \left[\frac{3(1+\beta)}{\beta} - \frac{1}{\beta(1+\beta)}\right] +$$

$$+ (c+2)(c+1) \times$$

$$\times \left[\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3} - \right. \qquad \ldots\ldots (13)$$

$$- 3\varepsilon_o \frac{(1+\beta)}{\beta^3} +$$

$$\left.\left. + \frac{\varepsilon_o}{\beta^3(1+\beta)}\right]\right\} a_2 -$$

$$- \frac{(c + 4)^2}{\beta^2} \left[\frac{(1 + \beta)^3}{\beta^3} - \frac{(1 + \beta)}{\beta^3} \right] a_4 = 0$$

and for subsequent terms

$$- \frac{16\varepsilon_o \beta}{k} a_{2(r-2)} + \Bigg\{ (c + 2r - 2)(c + 2r - 3)\, \varepsilon_o \times$$

$$\times \left[\frac{3(1 + \beta)}{\beta} - \frac{1}{\beta(1 + \beta)} \right] -$$

$$- \frac{8}{k\beta} \left[(1 + \beta)^2 - 2\varepsilon_o \right] \Bigg\} a_{2(r-1)} +$$

$$+ \Bigg\{ \frac{8}{k\beta^3} (1 + \beta)^2 - \frac{(c + 2r)\varepsilon_o}{\beta^2} \left[\frac{3(1 + \beta)}{\beta} \; \frac{1}{\beta(1 + \beta)} \right] + \qquad \ldots\ldots (14)$$

$$+ (c + 2r)(c + 2r - 1) \Bigg[\frac{(1 + \beta)^3}{\beta^3} - \frac{(1 + \beta)}{\beta^3} -$$

$$- 3\varepsilon_o \frac{(1 + \beta)}{\beta^3} +$$

$$+ \frac{\varepsilon_o}{\beta^3 (1 + \beta)} \Bigg] \Bigg\} a_{2r} -$$

$$-\frac{(c+2r+2)^2}{\beta^2}\left[\frac{(1+\beta)^3}{\beta^3}-\frac{(1+\beta)}{\beta^3}\right]a_{2(r+1)} = 0$$

where r = 2, 3, 4, ...

Equations (12)-(14) represent a solution of c = 0. A second solution may be obtained by differentiating (10) with respect to c and putting c = 0. Moverover, a particular solution to equation (9) may be obtained by replacing the RHS of equations (12)-(14) by the coefficients of the RHS of (9). Thus, the general solution of $f(\eta)$ may be generated.

For sufficiently large N, (14) gives

$$\varepsilon_o\left[\frac{3(1+\beta)}{\beta}-\frac{1}{\beta(1+\beta)}\right]\left(a_{2(N-1)}-\frac{1}{\beta^2}a_{2N}\right)+$$

$$+\left[\frac{(1+\beta)^3}{\beta^3}-\frac{(1+\beta)}{\beta^3}\right]\left(a_{2N}-\frac{1}{\beta^2}a_{2(N+1)}\right) = 0$$

approximately. Thus, as $N \to \infty$, $a_N/a_{N-2} \to \beta^2$ ..... (15)

The complete set of equations (12)-(14) together with the boundary conditions (4) have been processed on the computer and the results are shown in Figures 4.2 and 4.3. These figures show f for various values of the boundary parameters δ_1 and δ_2 and for two values of the depth parameter β.[1]

In order to check the results it may be observed that equation (1) is approximately

$$A\eta f - Bf' - B\eta f'' = a\eta + b\eta^3 \qquad \ldots\ldots (16)$$

where A, B, a, b are constant given by

$$\left.\begin{aligned} a &= \frac{8\kappa}{3k}\left(\frac{2(1+\beta)^2}{\beta^4}\right) \qquad b = -\frac{8\kappa}{3k}\left(\frac{2(1+\beta)^2}{\beta^2} + \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{3}{\beta^2} - \frac{4\varepsilon_o}{\beta^2}\right) \\ A &= \frac{8(1+\beta)^2}{k\beta^3} \qquad B = \frac{1}{\beta^2}\left(\frac{(1+\beta)^3}{\beta^3} - \frac{(1+\beta)}{\beta^3}\right) \end{aligned}\right\} \qquad \ldots\ldots (17)$$

For a particular solution, try

$$f_{PS} = c_1 + c_2\eta^2$$

$$f'_{PS} = 2c_2\eta$$

1 The horizontal axis covers the distance from the north canal edge to the south edge. The northerly and southerly bounds are the corresponding bounds of $\beta\eta$

$$f''_{PS} = 2c_2$$

$$\therefore \quad Ac_1\eta + Ac_2\eta^3 - 2Bc_2\eta - 2Bc_2\eta = a\eta + b\eta^3$$

$$\therefore \quad Ac_1 - 4Bc_2 = a$$

and $$Ac_2 = b$$

$$\therefore \quad \left.\begin{aligned} c_2 &= \frac{b}{A} \\ c_1 &= \frac{a}{A} + \frac{4bB}{A^2} \end{aligned}\right\} \qquad \text{..... (18)}$$

For the complementary function, try

$$f_{cF} = a_o\eta^c + a_1\eta^{c+2} + \ldots$$

$$\therefore \quad f'_{cF} = ca_o\eta^{c-1} + (c+2)a_1\eta^{c+1} + \ldots$$

$$f''_{cF} = c(c-1)a_o\eta^{c-2} + (c+2)(c+1)a_1\eta^c + \ldots$$

The indicial equation then gives

$$c^2 = 0 \qquad \ldots\ldots (19)$$

and, on equating corresponding terms in the higher powers, we obtain

$$Aa_o - B(c + 2)a_1 - B(c + 2)(c + 1)a_1 = 0$$

$$\therefore \qquad Aa_o - B(c + 2)^2 a_1 = 0 \qquad \ldots\ldots (20)$$

$$Aa_1 - B(c + 4)a_2 - B(c + 4)(c + 3)a_2 = 0$$

$$\therefore \qquad Aa_1 - B(c + 4)^2 a_2 = 0 \qquad \ldots\ldots (21)$$

etc

Because of the repeated root in equation (19), a second function may be obtained by differentiating the first with respect to c and putting c = 0. Thus

$$\frac{\partial a_1}{\partial c} = \frac{2A\beta}{\beta(c + 2)^3} \qquad \ldots\ldots (22)$$

$$\frac{\partial a_2}{\partial c} = -\frac{2A^2}{B^2}\left[\frac{1}{(c+2)^3\ (c+4)^2} + \frac{1}{(c+2)^2\ (c+4)^3}\right] \qquad \ldots\ldots (23)$$

etc

Bringing together these results we obtain the general solution for (16) namely

$$f = c_1 + c_2\eta^2 + \alpha\left[1 + \frac{A}{B}\times\frac{\eta^2}{2^2} + \frac{A^2}{B^2}\times\frac{\eta^4}{2^2 4^2} + \ldots\right]$$

$$+ \beta\left[1 + \frac{A}{B}\times\frac{\eta^2}{2^2} + \frac{A^2}{B^2}\times\frac{\eta^4}{2^2 4^2} + \ldots\right]\log\eta\ - \qquad \ldots\ldots (24)$$

$$- 2\beta\left[\frac{A}{B}\times\left(\frac{1}{2}\right)\times\frac{\eta^2}{2^2} + \frac{A^2}{B^2}\times\frac{\left(\frac{1}{2}+\frac{1}{4}\right)\eta^4}{2^2 4^2} + \right.$$

$$\left. + \frac{A^3}{B^3}\times\frac{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)\eta^6}{2^2 4^2 6^2} + \ldots\right]$$

ie

$$f = c_1 + c_2\eta^2 + \alpha\, I_o\left(\sqrt{\frac{A}{B}}\times\eta\right) - \beta\, K_o\left(\sqrt{\frac{A}{B}}\times\eta\right)$$

where I_o and K_o are Modified Bessel functions. The relatively simple form of this solution makes it a convenient means of checking the results obtained in Figure 4.2 and 4.3 and results compared favourably.

Furthermore, the solution (24) shows the presence of a logarithmic singularity in the function f, which is located at the pole.

Figures 4.4, 4.5 and 4.6 show the computed velocity components u, v, w.[1] The component v must be multiplied by $\cos(\varepsilon\omega t)$ and u and w by $\varepsilon \sin(\varepsilon\omega t)$. All three components are scaled by the factor 500 for display purposes. From Figures 4.5 it can be seen that v is independent of depth (cf § 3.2) but not of the canal width. The symmetrical pattern exhibited in the narrow canals gives way to one with large velocity changes near the southern canal boundary. Indeed, the latter point is a feature of u as well, and there are also large changes in w at the southern boundary for the wider canals. These are all associated with a similar large variation in P, as demonstrated in Figures 4.3 and 4.2 (g), (h), (j), (k). In practice, however, these solutions will be modified by non-linear and viscous forces operating in this region next to the southern edge.

From Figures 4.6 it can be seen that the vertical velocity is proportional to the height above the canal bottom (cf § 3.2). However, the ratio of $w/\varepsilon v$ is, at maximum, much bigger than β and remains roughly in the proportion 1:4.

Turning to the elevation of the tide we see from 4.1.33 that the amplitude of the periodic disturbance is given by

$$\frac{1}{\lambda\beta}(\beta P - 2F) \times \beta a$$

1. The quantity NEP shown in the figures is the distance between the canal's north edge to the pole, measured along a meridian.

ie $$\frac{2\omega U}{g\beta}(\beta F - \kappa \sin^2\theta) \times \beta a$$

Thus for the case shown in Figure 4.2(a) we have an amplitude of 1:1 ft. The time of maximum elevation occurs when the moon is overhead ie the tide is direct (cf § 3.2). This result compares favourably with Laplace's theory for the long-period constituent [6].

As for the other features of the motion, they may be contrasted with Proudman's solution for a flat circular sea of uniform depth at the North Pole [5]. Using cylindrical polar co-ordinates z, x, ϕ, where z is in the direction of the axis of the earth, he found

$$\left.\begin{aligned}
\frac{P}{gA} &= J_o(Kx)\cos\lambda Kz \cos\varepsilon t \\
\frac{U}{\varepsilon A} &= \frac{Kg\lambda^2}{\varepsilon^2} J_o{}'(Kx)\cos\lambda Kz \sin\varepsilon t \\
\frac{V}{A} &= \frac{2\omega Kg\lambda^2}{\varepsilon^2} J_o{}'(Kx)\cos\lambda Kz \cos\varepsilon t \\
\frac{W}{\varepsilon A} &= \frac{Kg\lambda}{\varepsilon^2} J_o(Kx)\sin\lambda Kz \sin\varepsilon t
\end{aligned}\right\} \quad \ldots\ldots (25)$$

where $\lambda = \left(\frac{4\omega^2}{\varepsilon^2} - 1\right)^{-\frac{1}{2}}$ and J_o is Bessel's function of zero order. Also, from the conditions at the free surface

$$\frac{\Omega}{gA} = -\left(\frac{\lambda Kg}{\varepsilon^2}\sin\lambda Kh + \cos\lambda Kh\right)J_o(Kx)\cos\varepsilon t$$

and the condition for u = 0 on the circular boundary x = c gives

$$J_o{}'(Kc) = 0 \qquad \text{..... (26)}$$

Thus K is inversely related to the length c.

It can be seen that, depending on the choice of root in (26), λK may be either small or large. In the latter case cellular oscillations will occur. However this arbitrariness is absent in the solutions we have obtained owing to the non-cyclical nature of the functions Io and Ko.

When λKh is small it follows from (25) that

$$\frac{v}{A'} = -2\omega\, J_1(Kx)\cos\varepsilon t$$

where

$$\frac{1}{A'} = g\left(Kh + \frac{4\omega^2 - \varepsilon^2}{Kg}\right)\Big/\Omega$$

Thus for $h = 2\times10^4$ ft and $c = 10^6$ ft, v is approximately 3×10^{-3} ft/sec which is similar to the value 1/500 ft/sec shown in Fig 4.5(B) for approximately the same width.

The components u and w in (25) are both of the first order in ε, as they are in the solution we have obtained. But the magnitude of the components is much larger in Proudman's case. Also, no large gradients are present in any of the solutions (25) so that viscous and non-linear terms would be negligible there.

Thus we conclude that significant differences in the nature of the solutions u, v, w are obtained when all parts of the Coriolis force are present, as compared to the solutions (25) derived by Proudman using the simplifications discussed in Section 1.

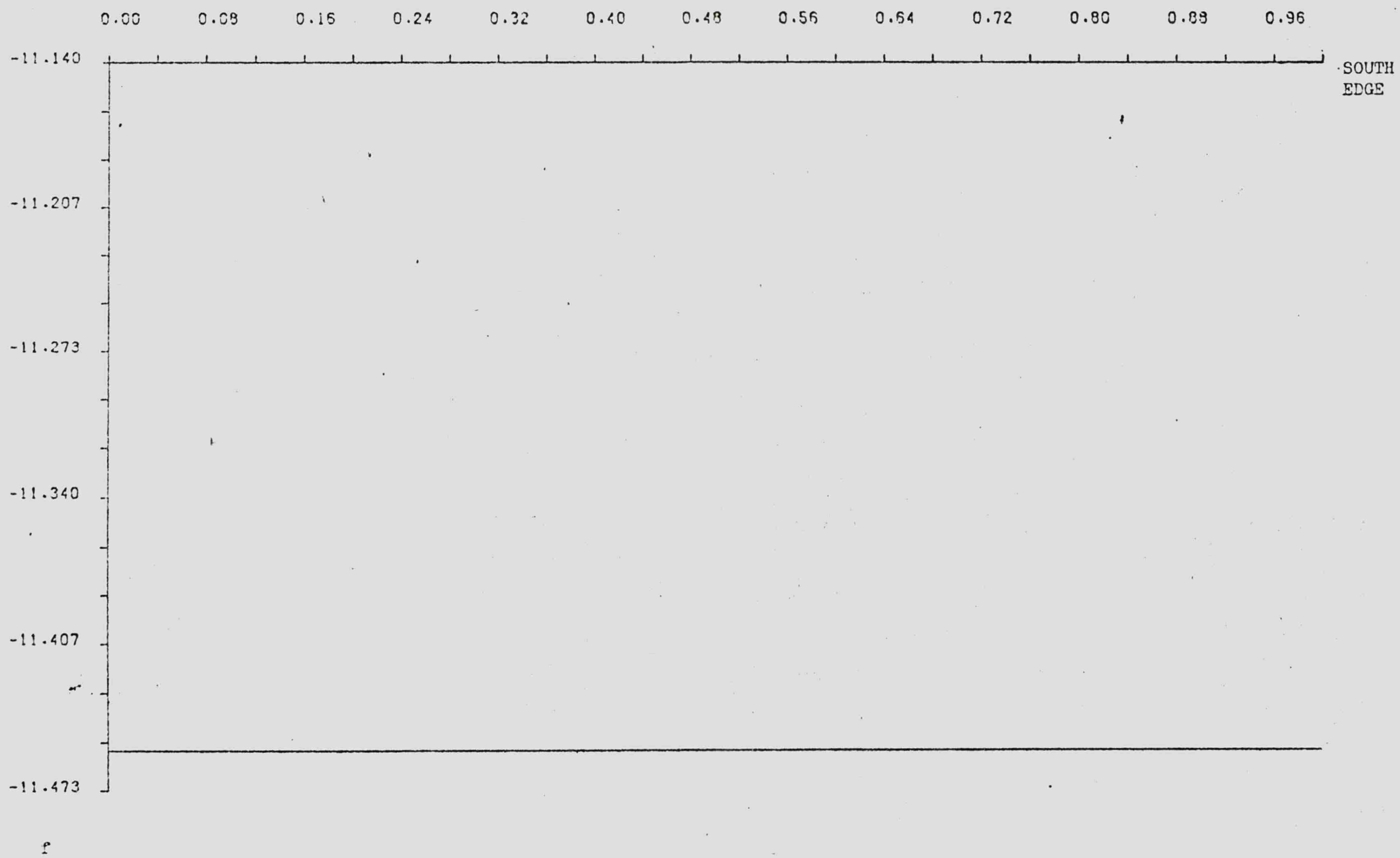


FIGURE 4.2(a) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.02 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

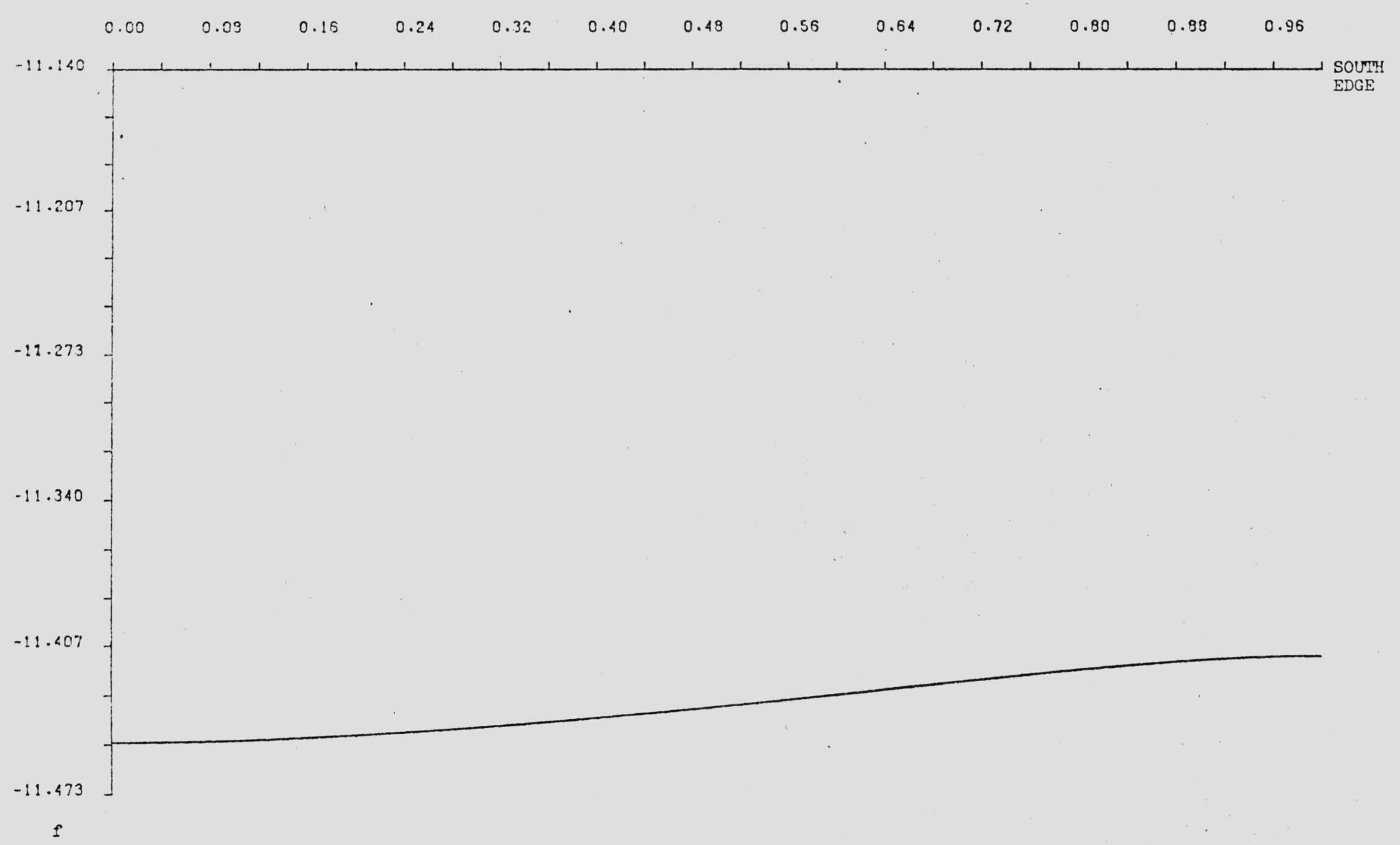


FIGURE 4.2(b) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.06 β = 0.001 (ε = 0.001 κ = -17.19 k = 0.30)

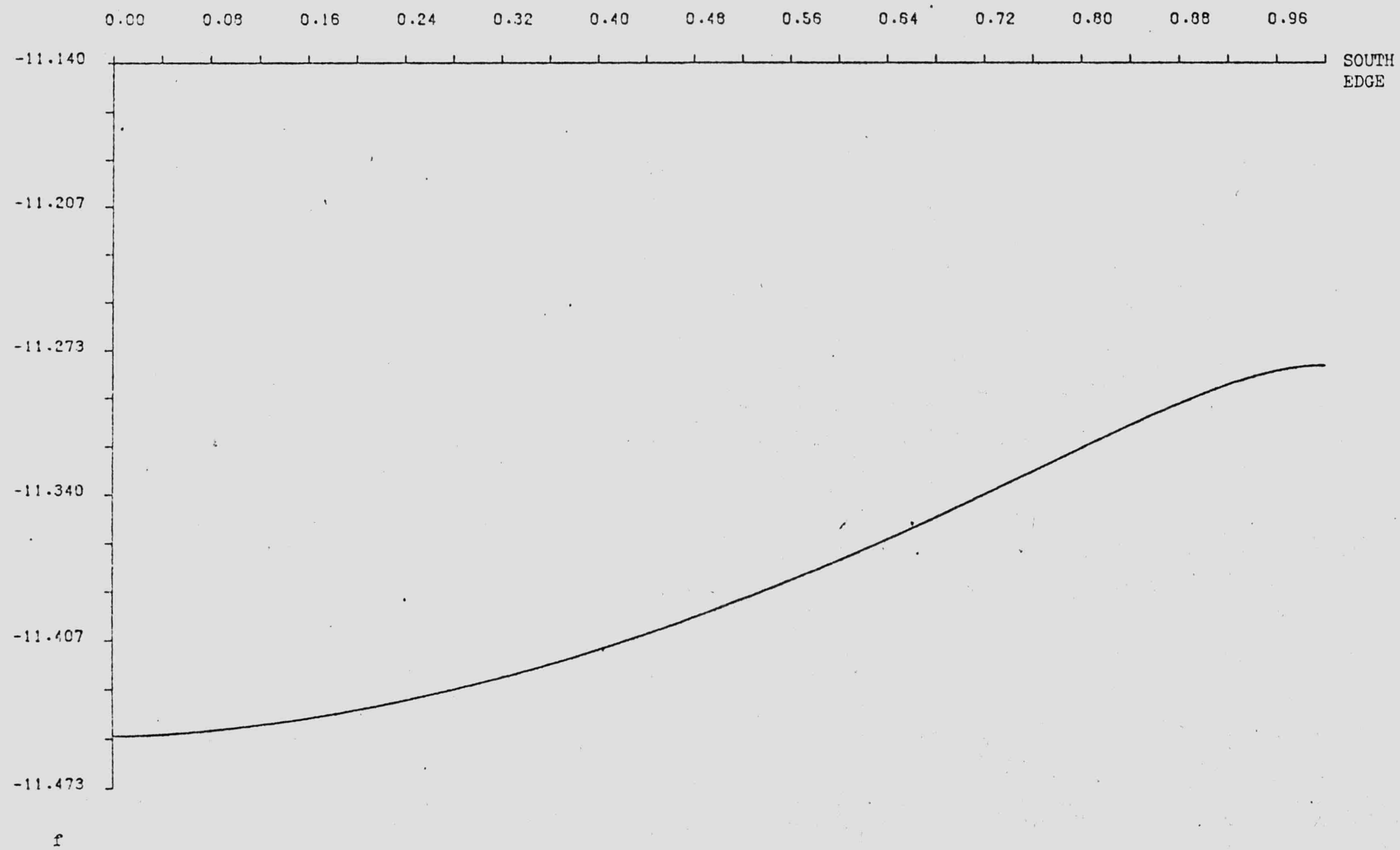


FIGURE 4.2(c) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.11 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

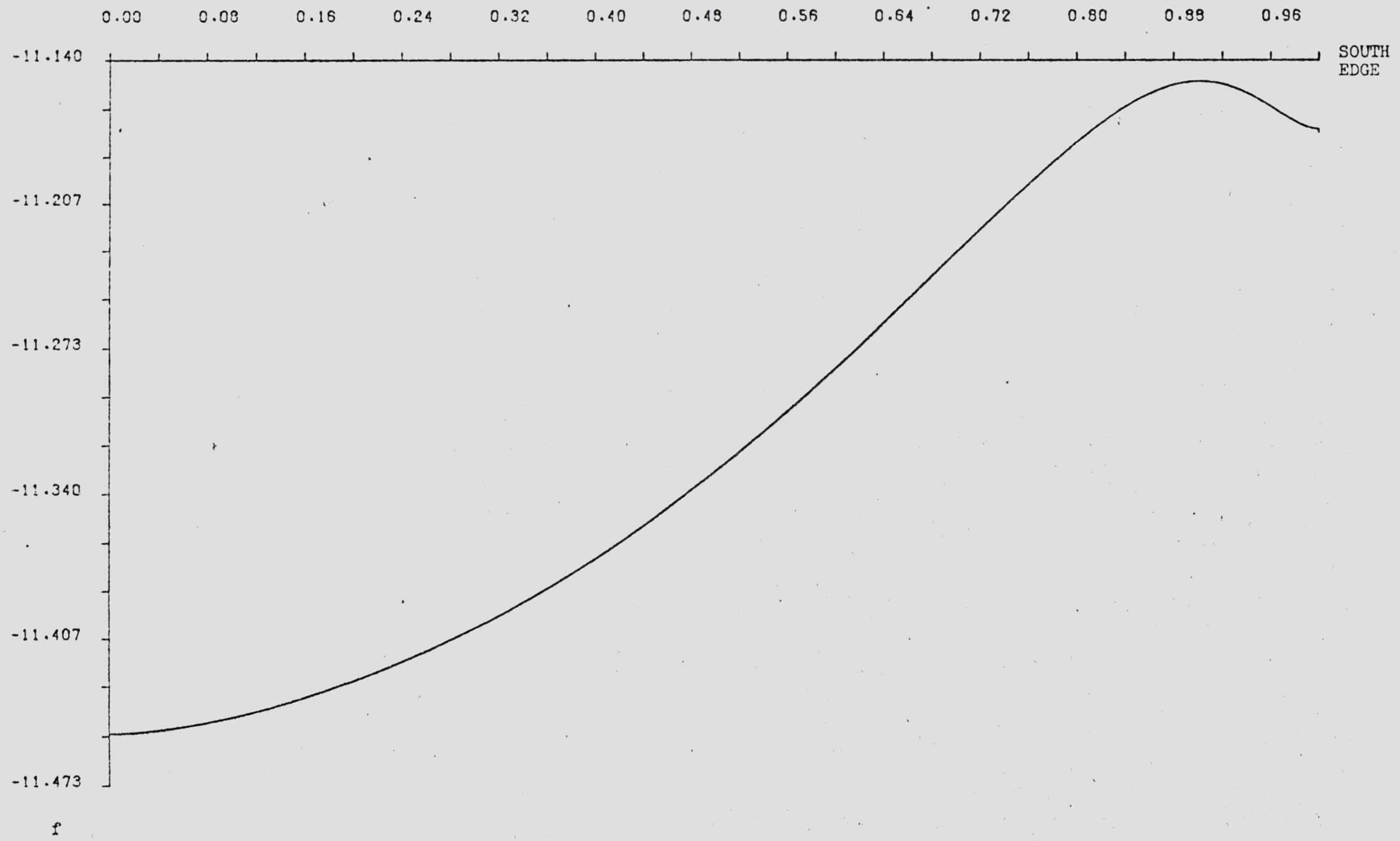


FIGURE 4.2(d) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.16 β= 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

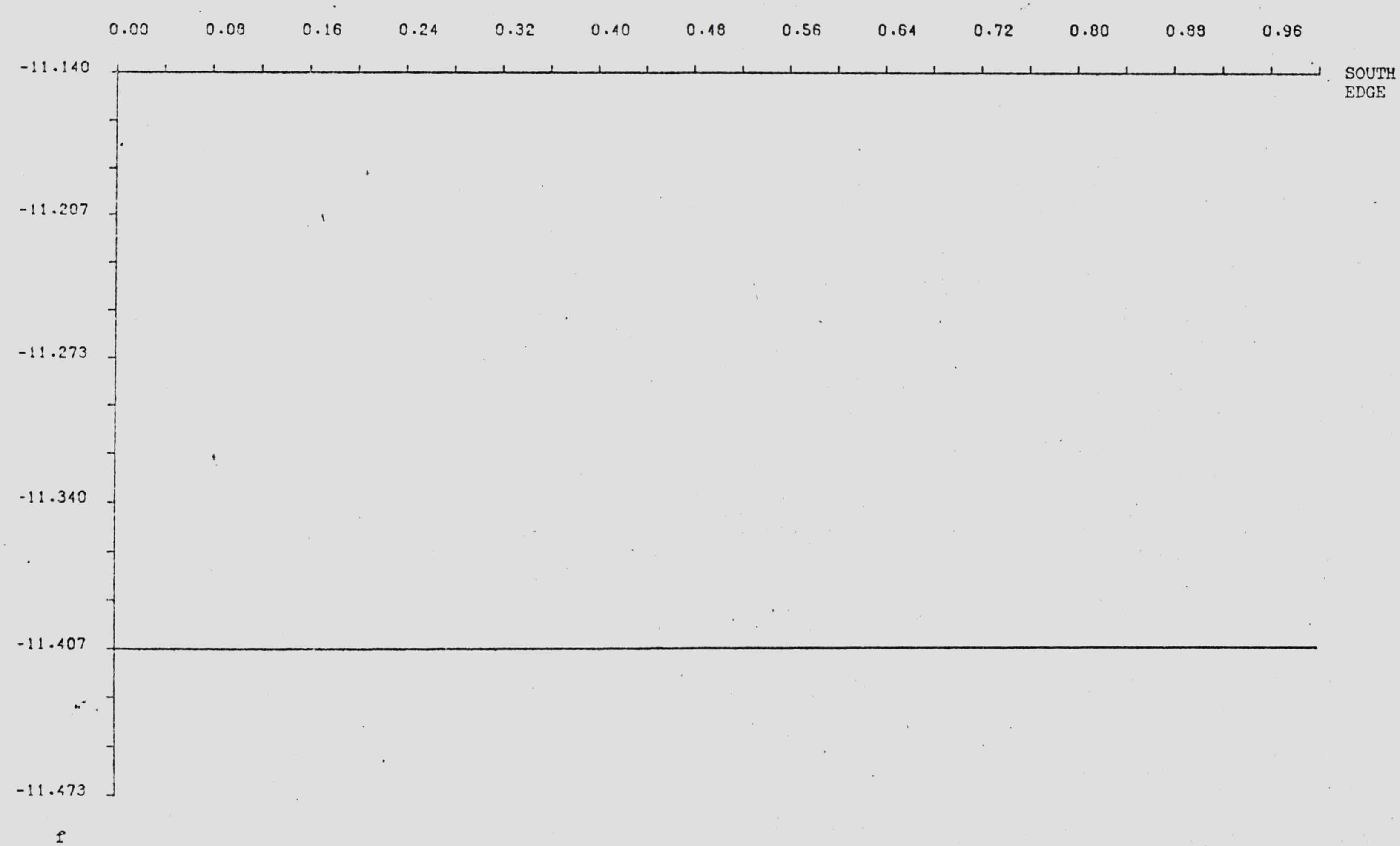


FIGURE 4.2(e) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.06 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

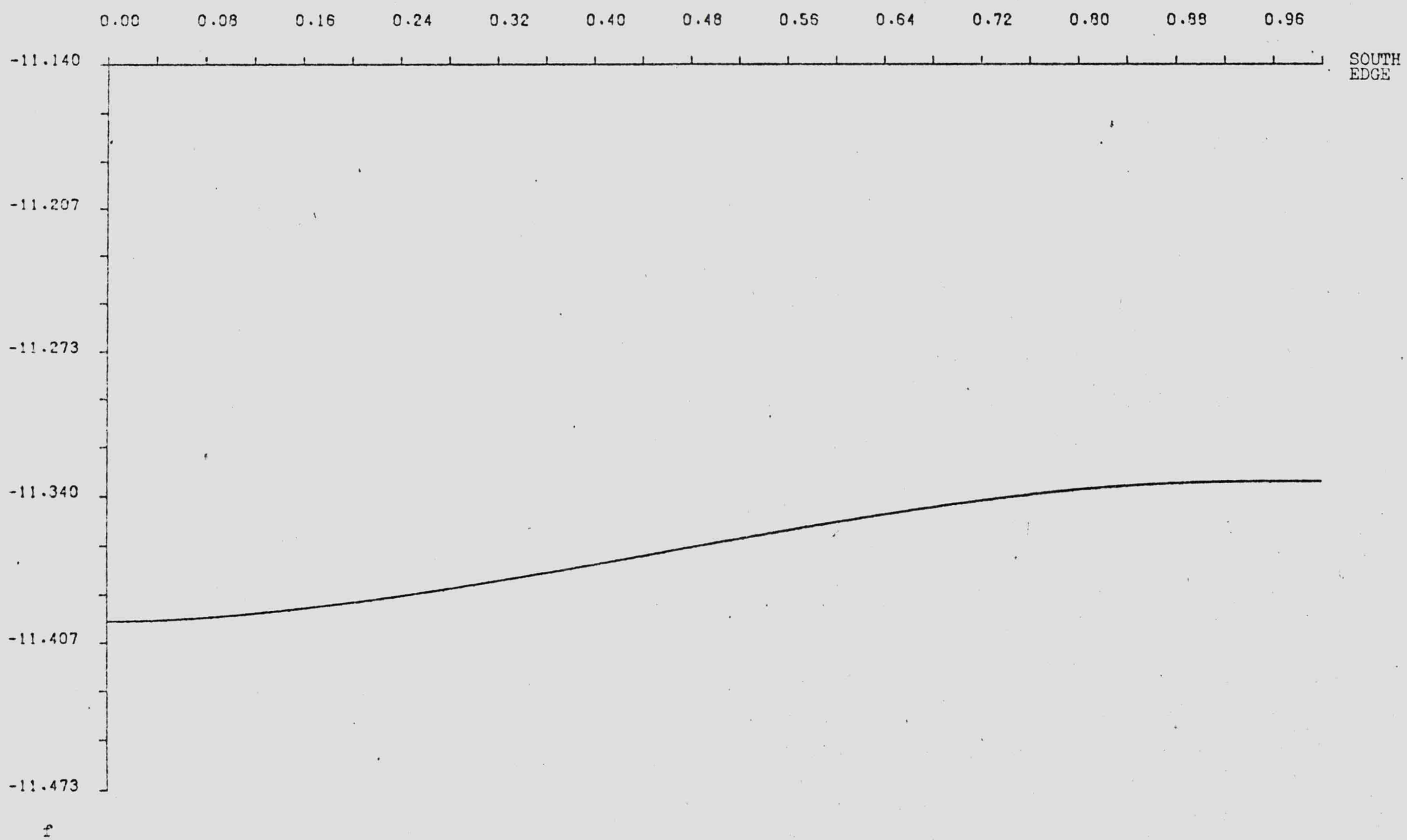


FIGURE 4.2(f) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.10 β= 0.001 (ε = 0.001 κ = -17.19 k = 0.30)

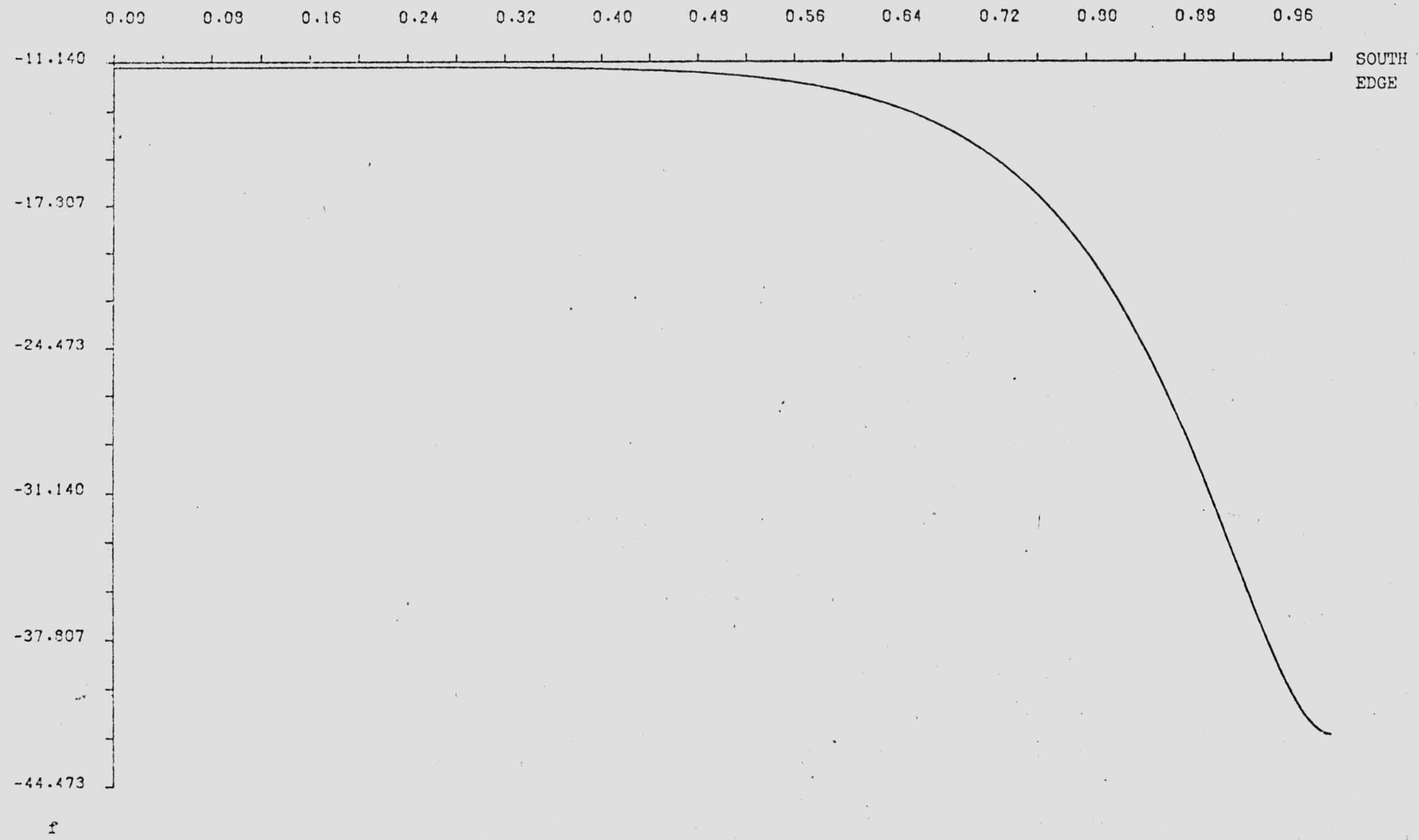


FIGURE 4.2(g) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.15 β = 0.001 (ε = 0.001 κ = -17.19 k = 0.30)

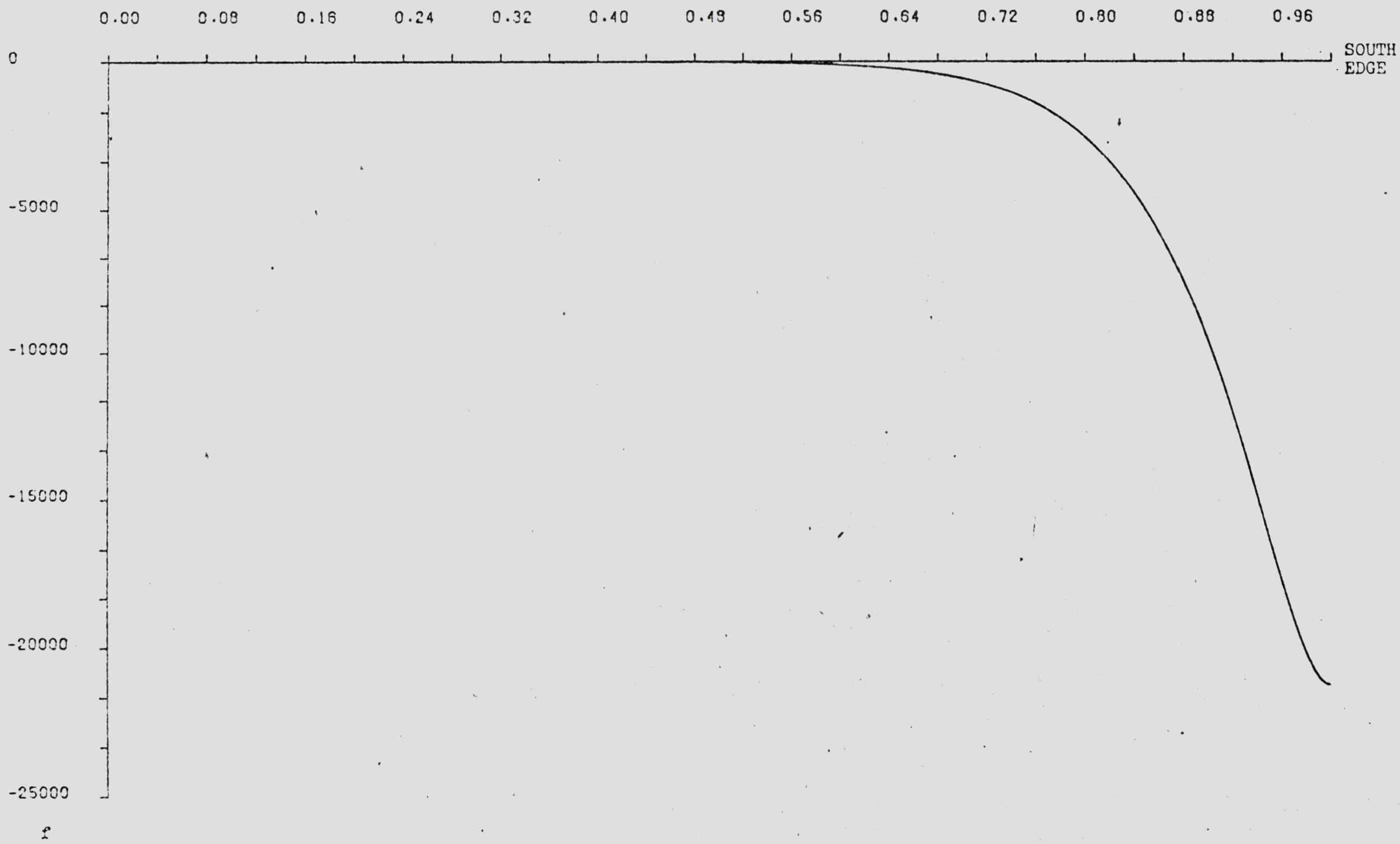


FIGURE 4.2(h) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.20 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

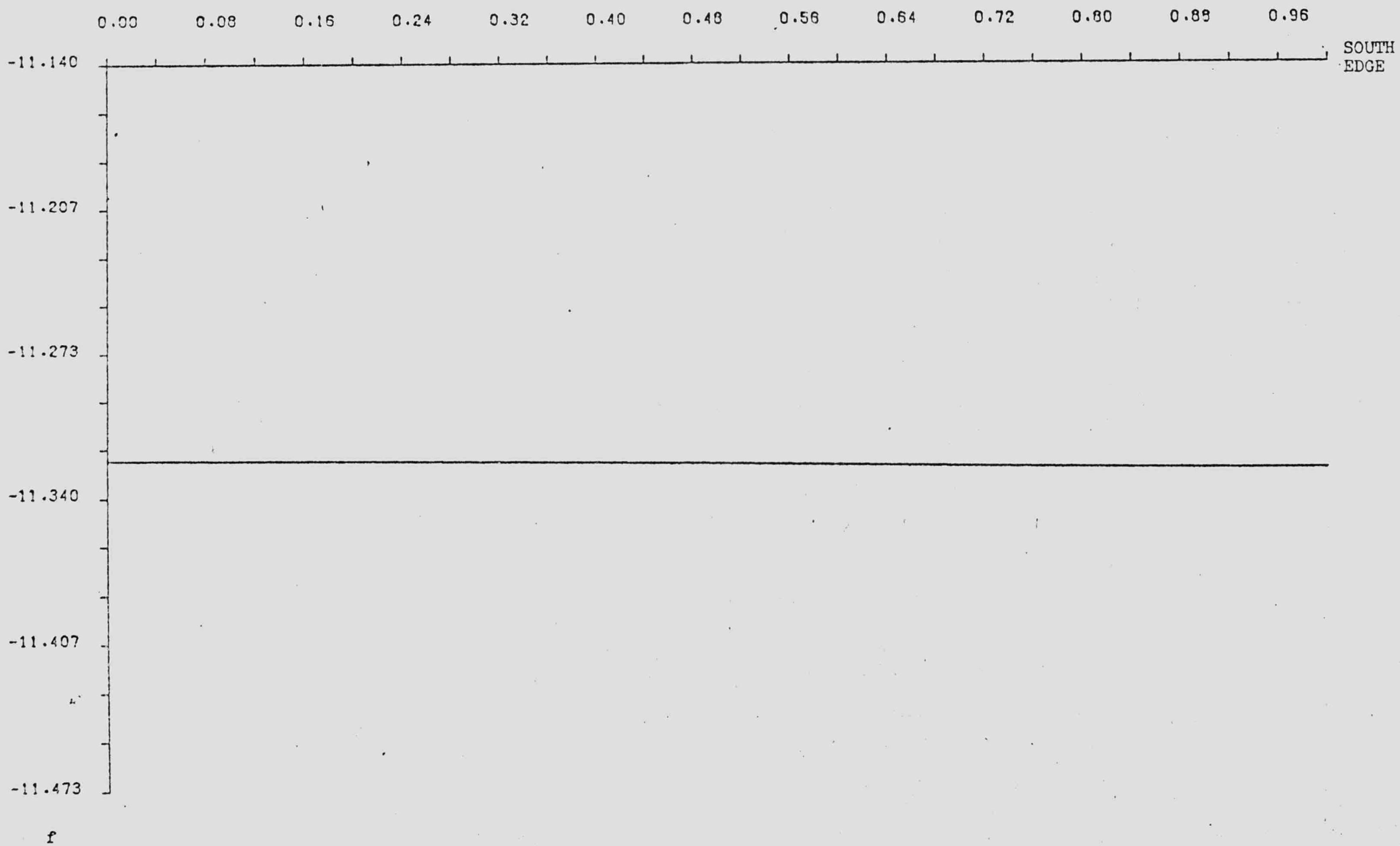


FIGURE 4.2(i) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.10 SOUTHERLY BOUND = 0.11 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

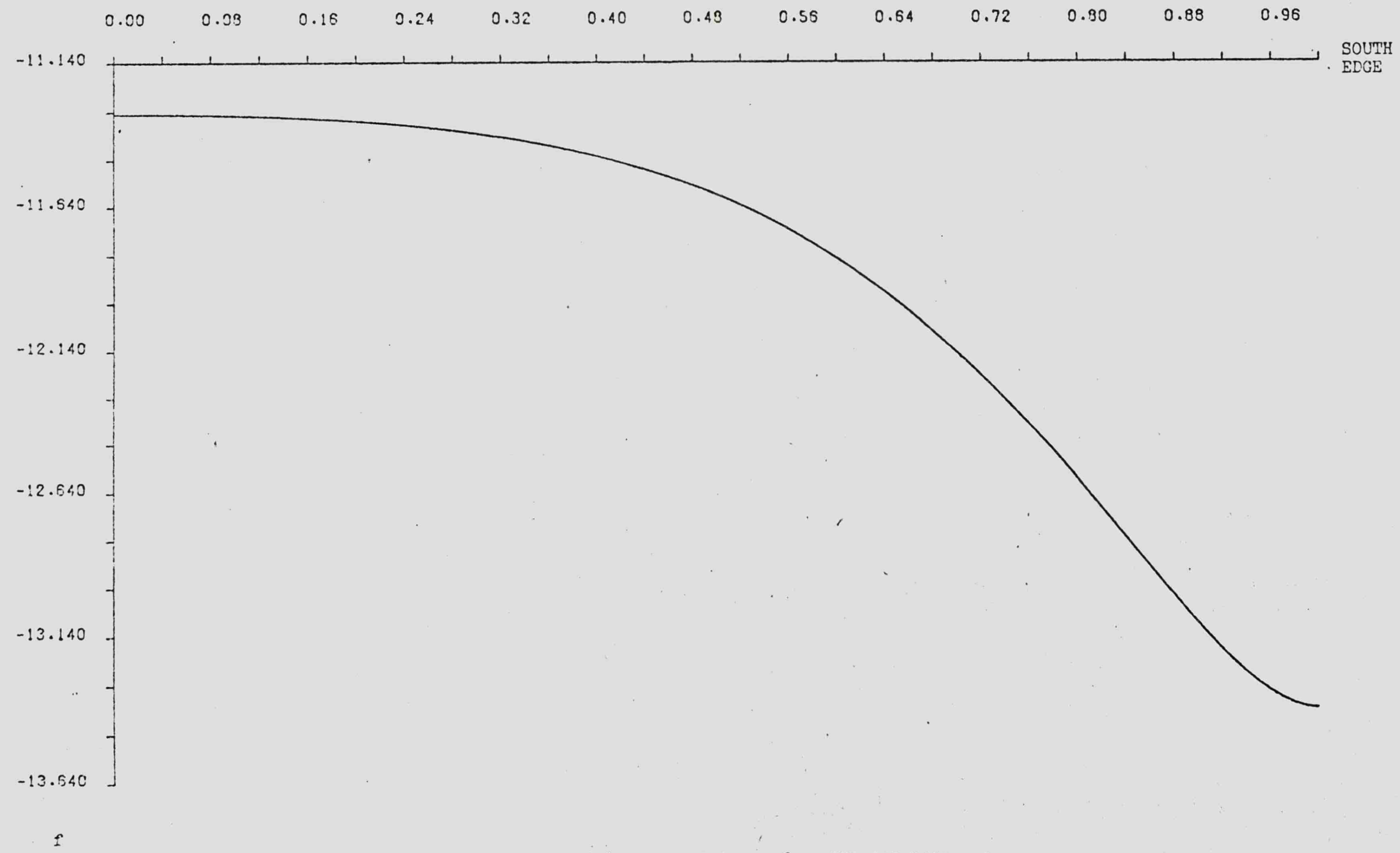


FIGURE 4.2(j) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.10 SOUTHERLY BOUND = 0.15 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

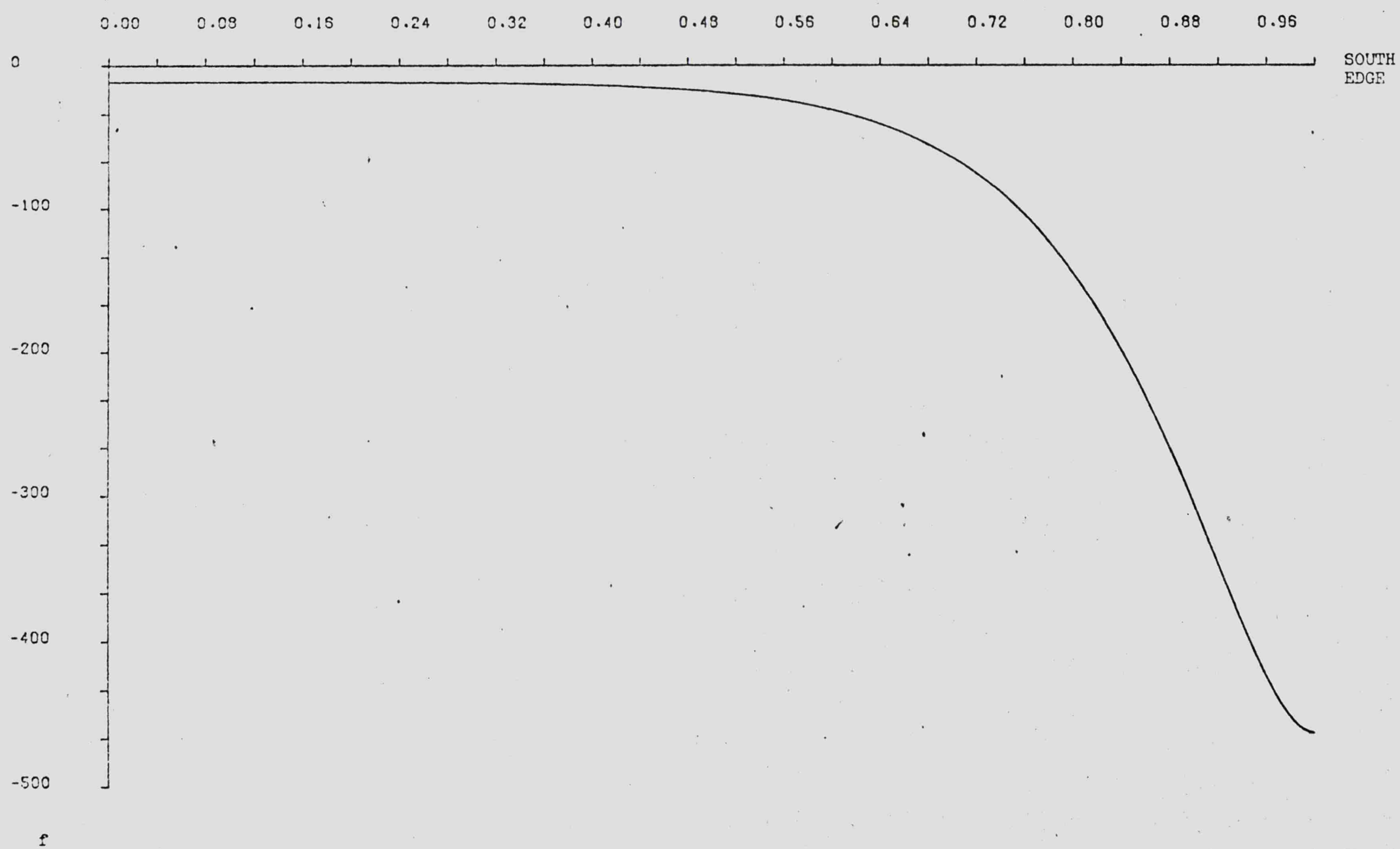


FIGURE 4.2(k) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.10 SOUTHERLY BOUND = 0.20 β = 0.001 (ϵ = 0.001 κ = -17.19 k = 0.30)

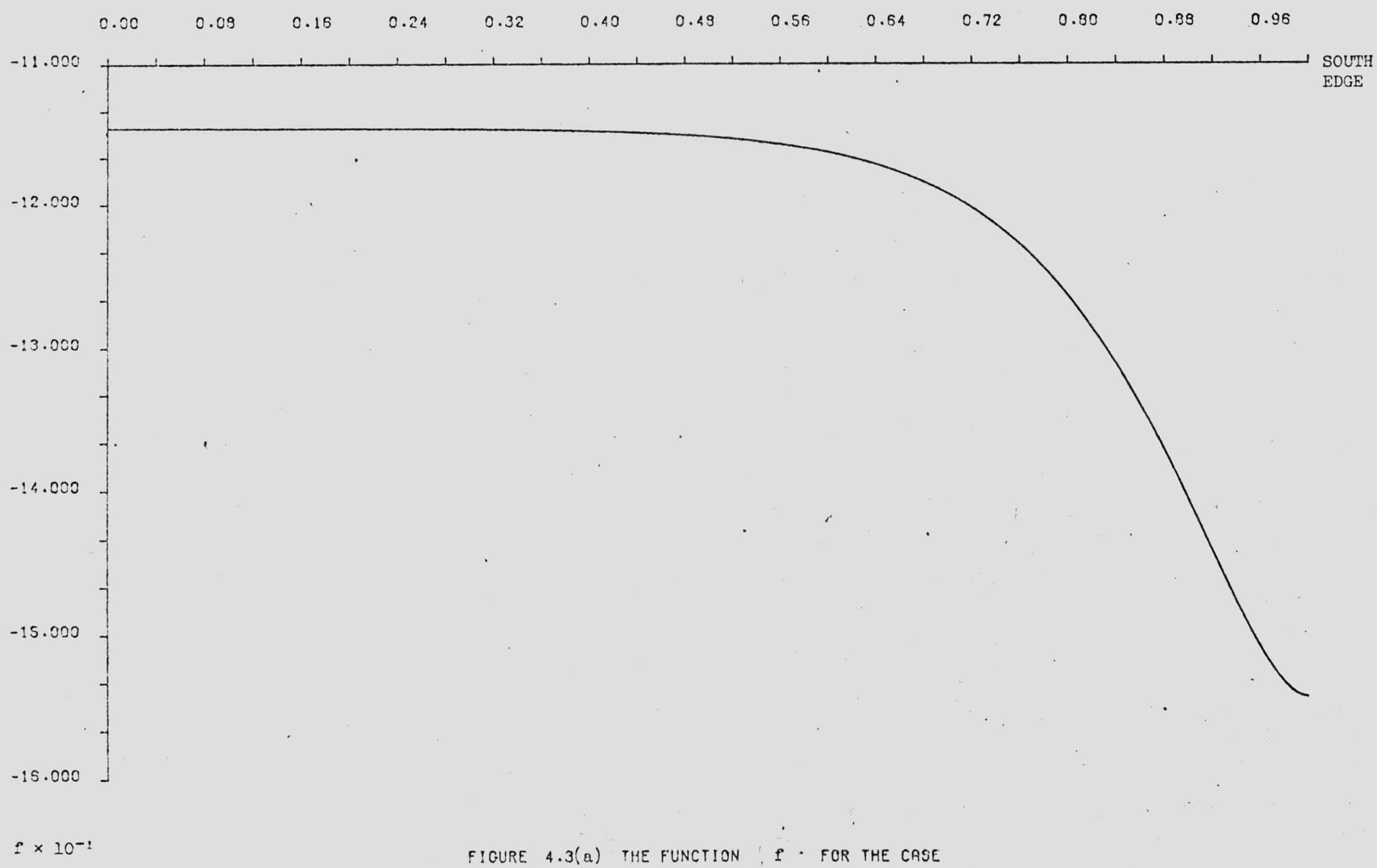


FIGURE 4.3(a) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.0200 β = 0.0001 (ε = 0.001 κ = -17.19 k = 0.03)

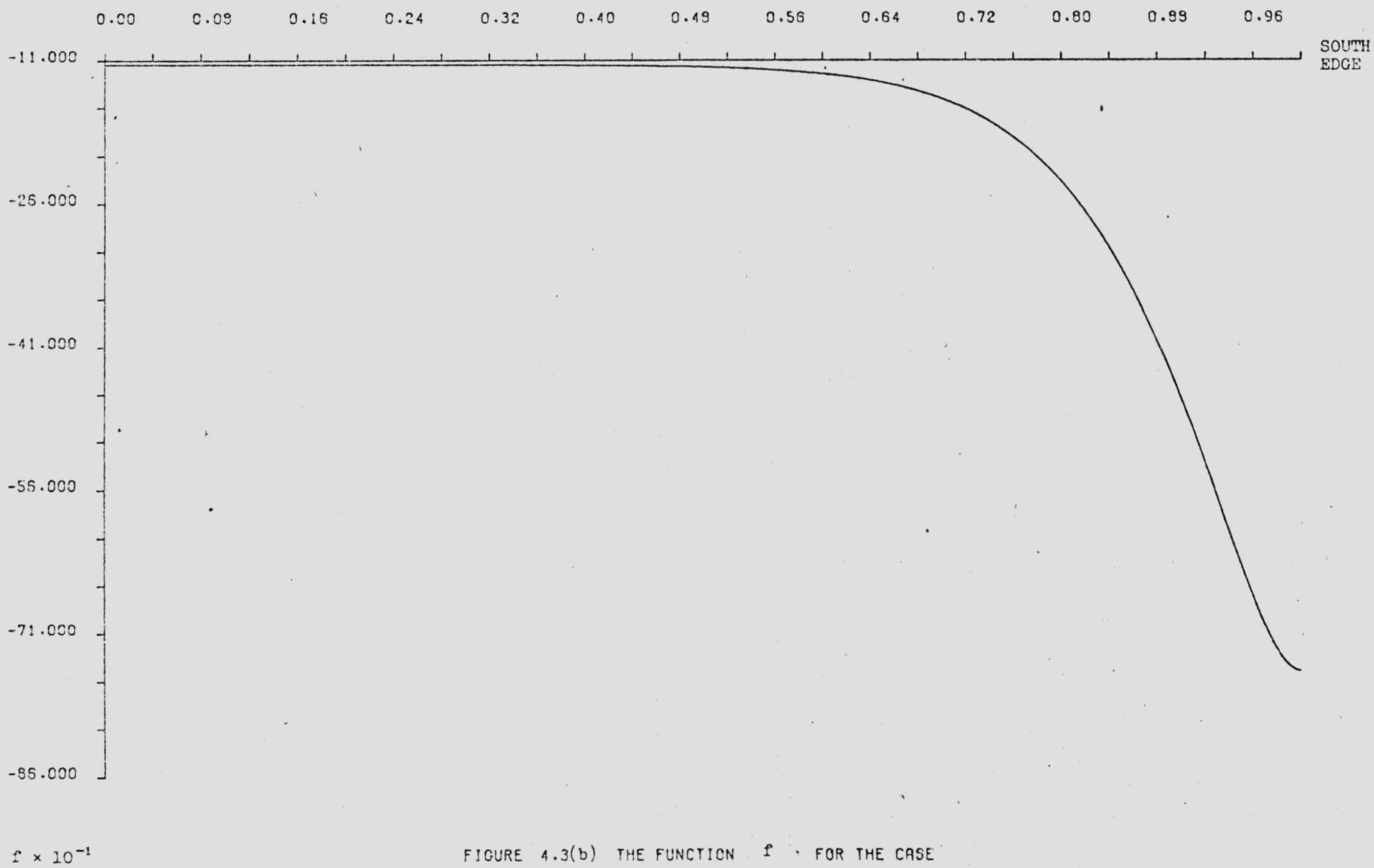


FIGURE 4.3(b) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.0225 β = 0.0001 (ϵ = 0.001 κ = -17.19 k = 0.03)

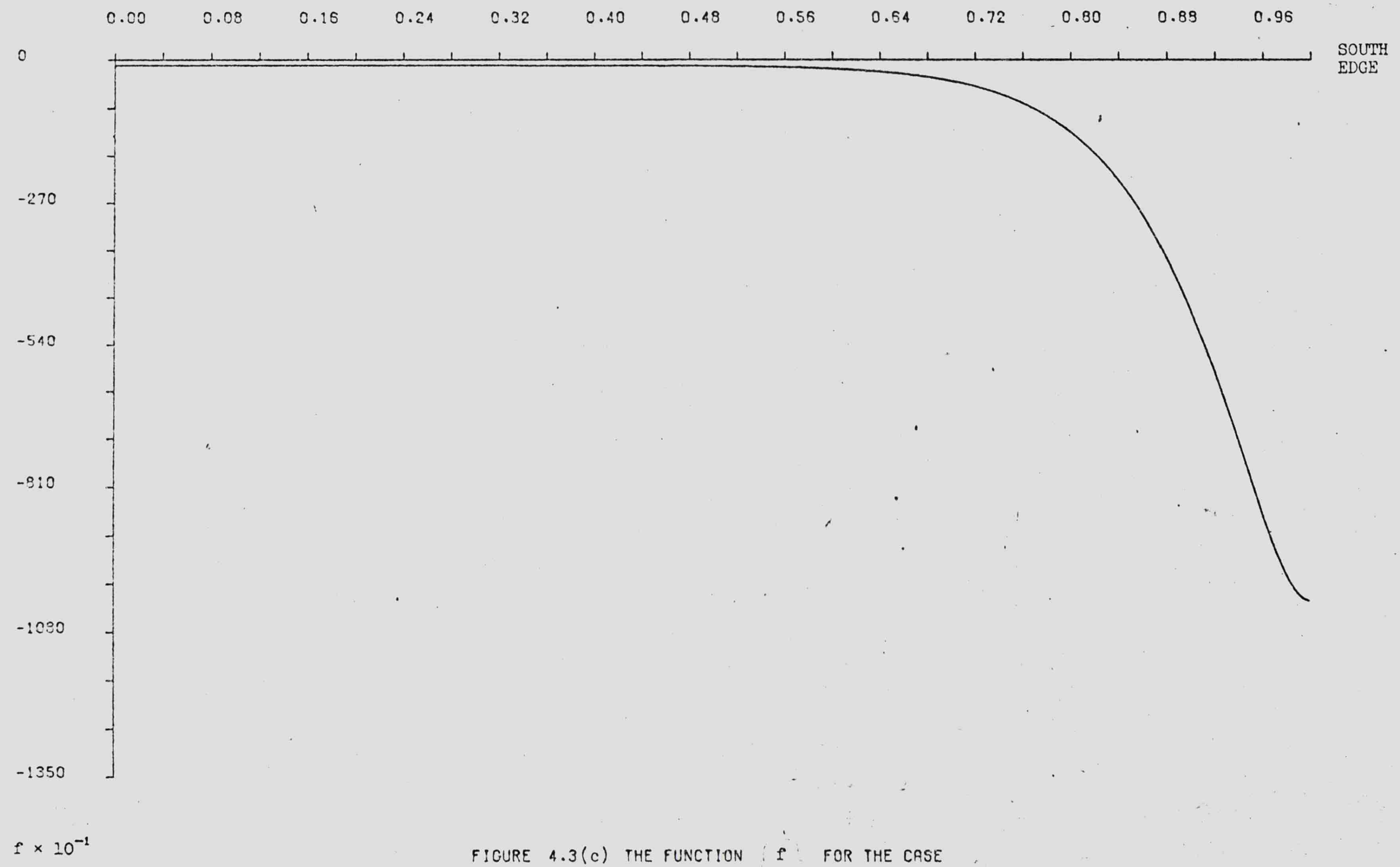


FIGURE 4.3(c) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.01 SOUTHERLY BOUND = 0.0250 β = 0.0001 (ε = 0.001 κ = -17.19 k = 0.03)

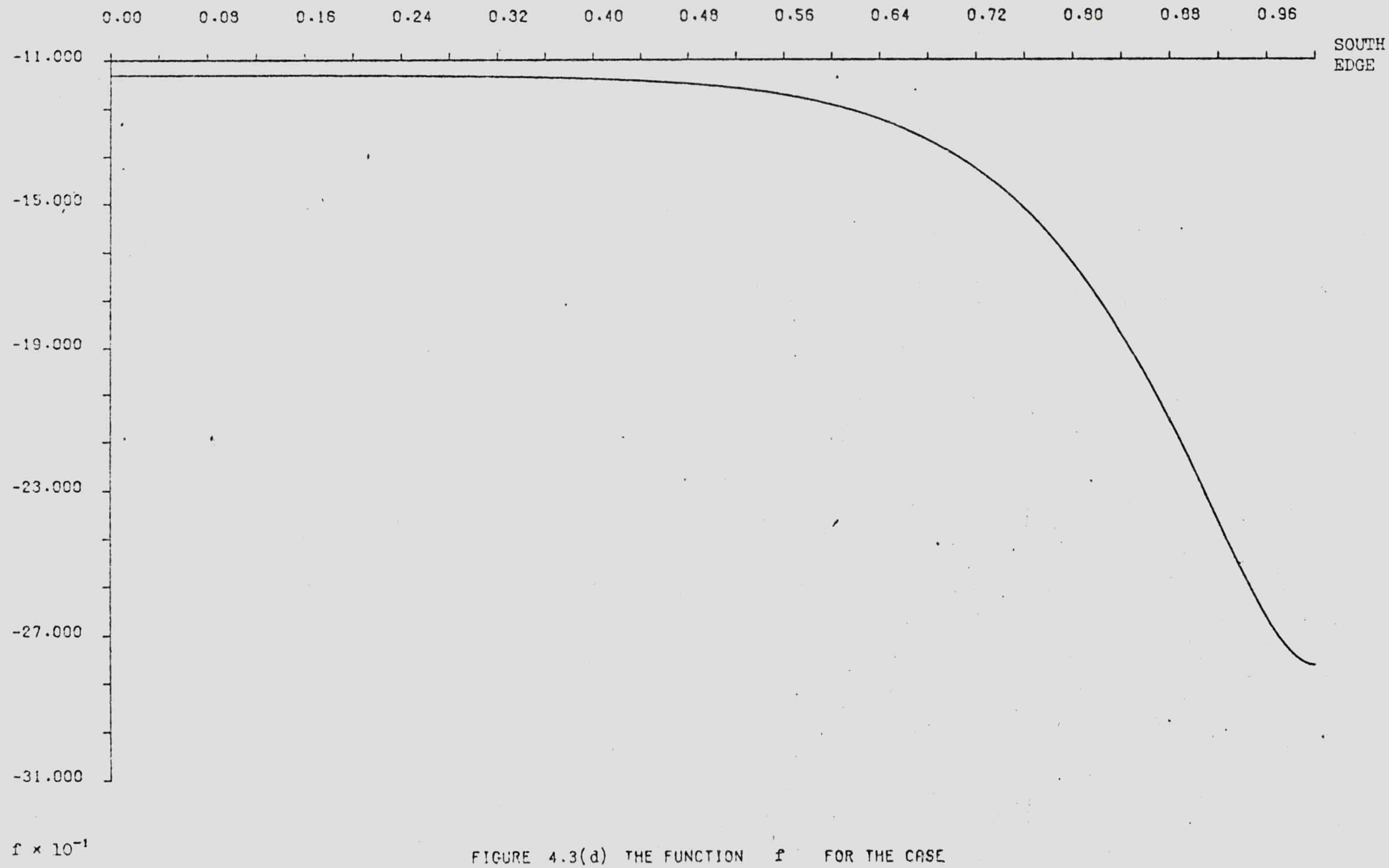


FIGURE 4.3(d) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.0600 β= 0.0001 (ε = 0.001 κ = -17.19 k = 0.03)

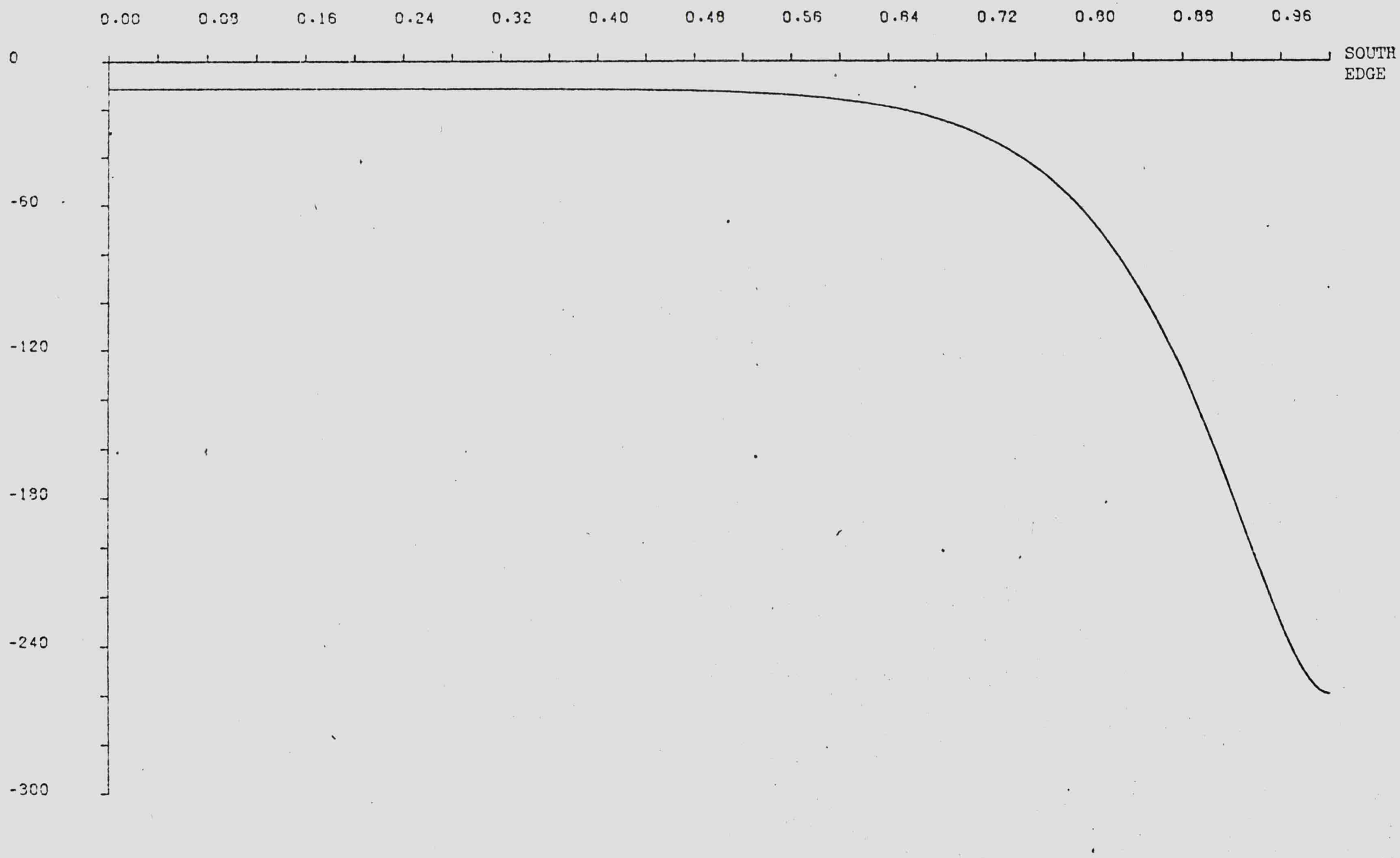
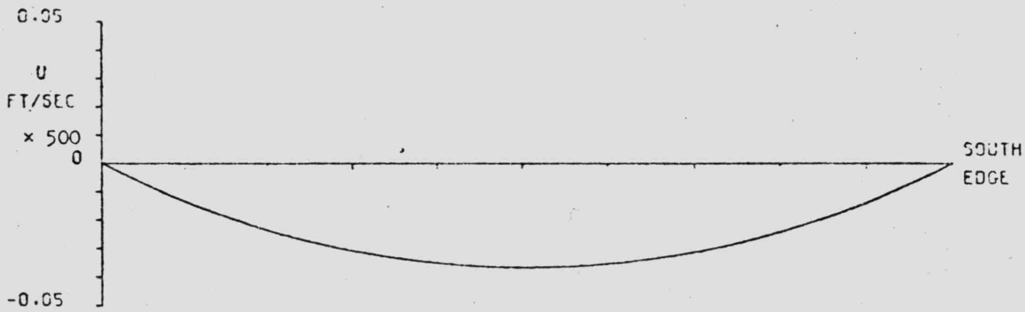
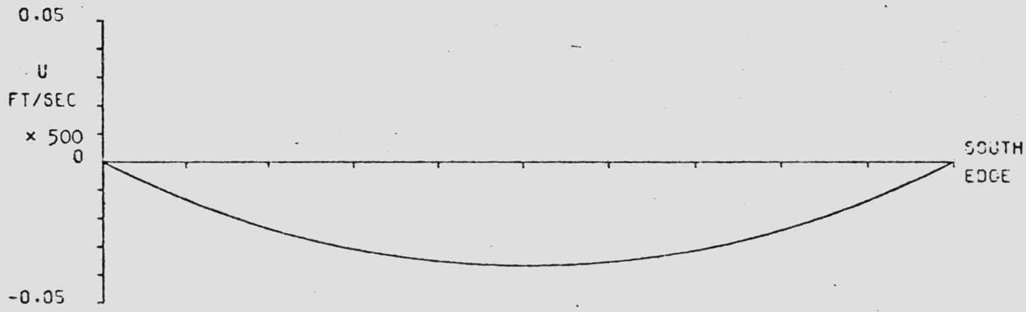


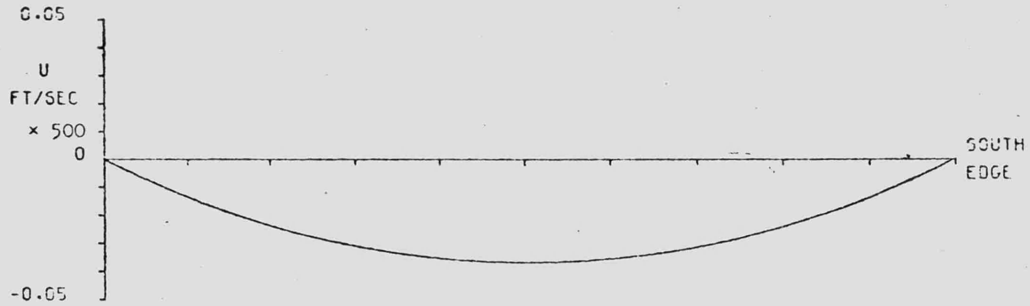
FIGURE 4.3(e) THE FUNCTION f FOR THE CASE
NORTHERLY BOUND = 0.05 SOUTHERLY BOUND = 0.0625 β= 0.0001 (ϵ = 0.001 κ = -17.19 k = 0.03)



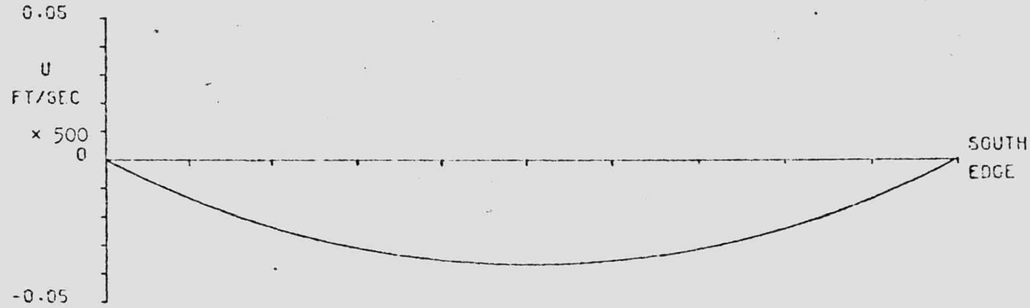
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



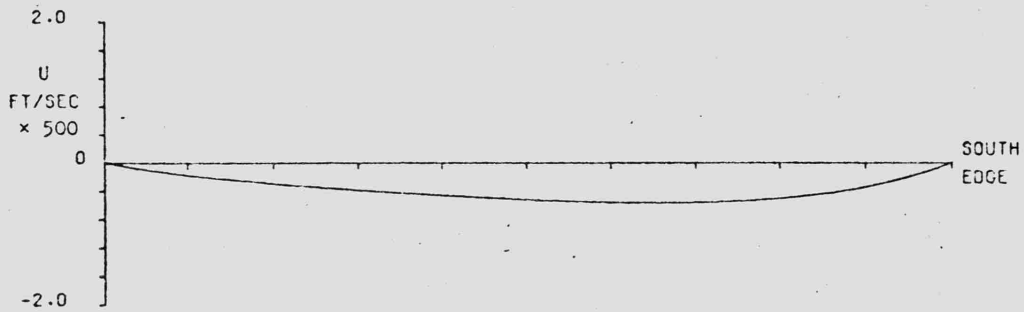
DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

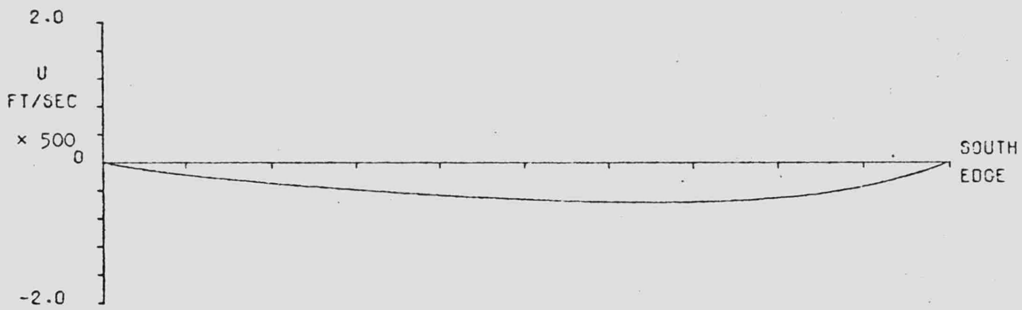
FIG 4.4 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
Λ THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 37.9 MLS.NEP=37.9 MLS β = 0.001

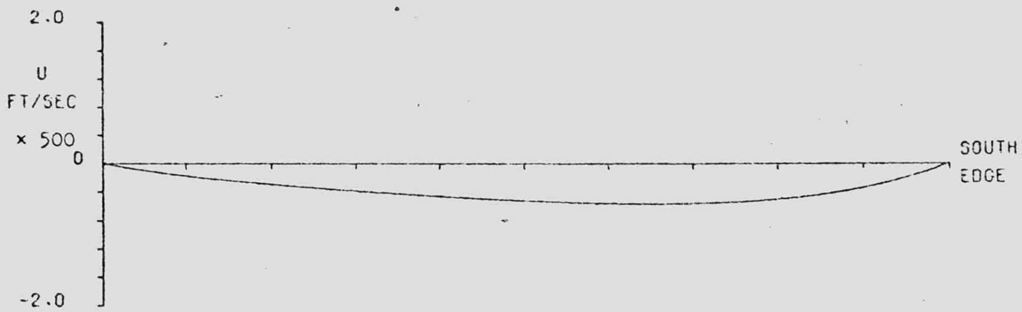


DEPTH: CANAL BOTTOM

DEPTH: 10000 FT ABOVE CANAL BOTTOM



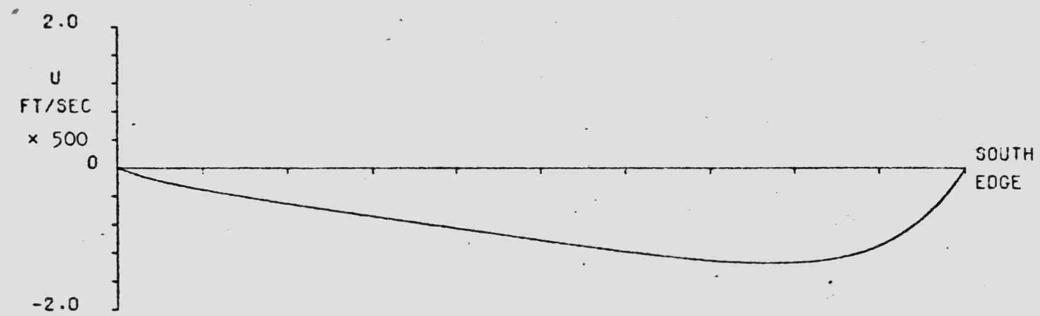
DEPTH: 20000 FT ABOVE CANAL BOTTOM



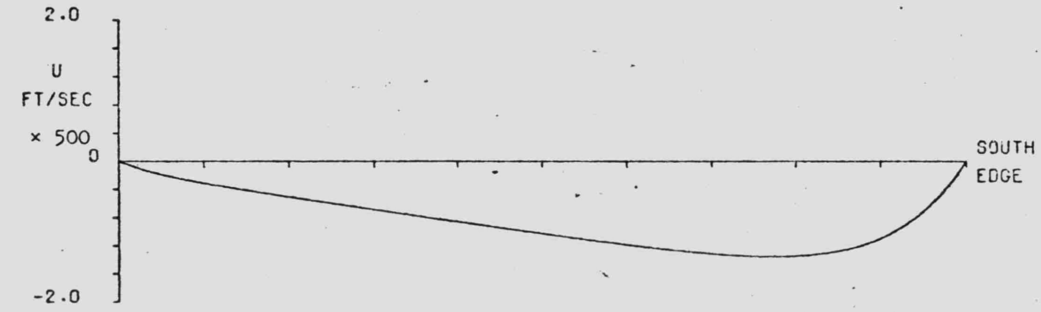
DEPTH: FREE SURFACE

FIG 4.4 B U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

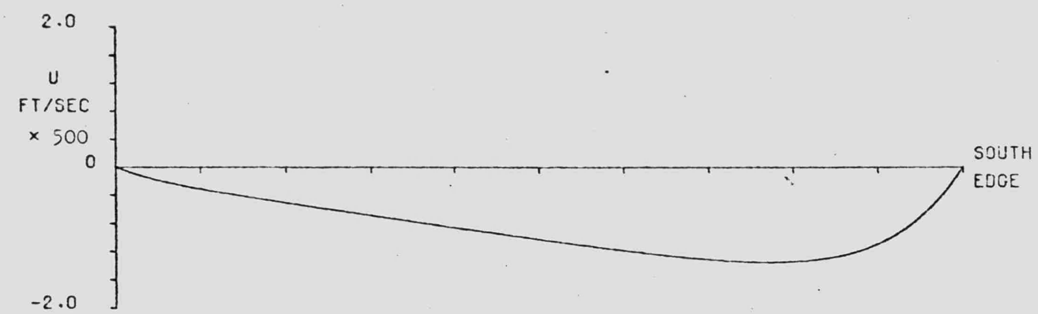
CANAL WIDTH= 199.5 MLS.NEP=37.9 MLS $\beta = 0.001$



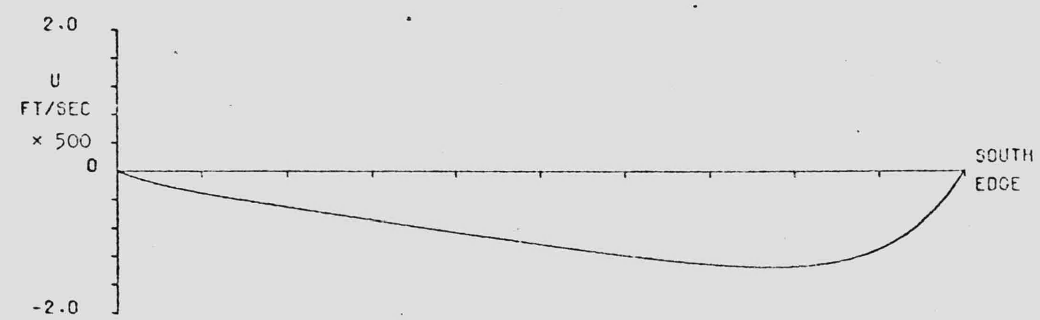
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



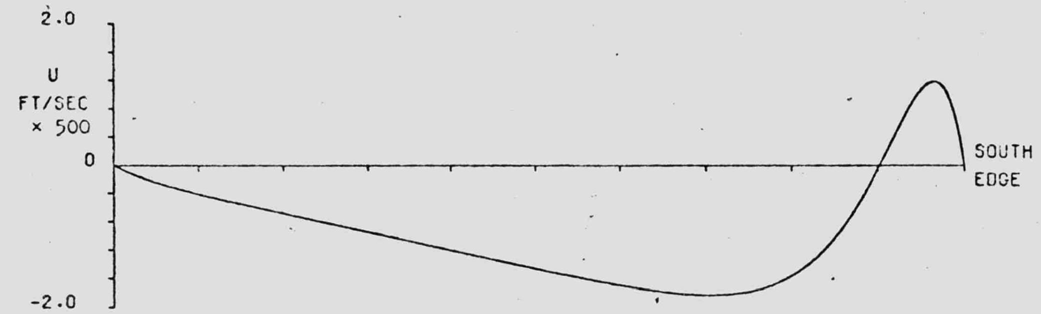
DEPTH: 20000 FT ABOVE CANAL BOTTOM



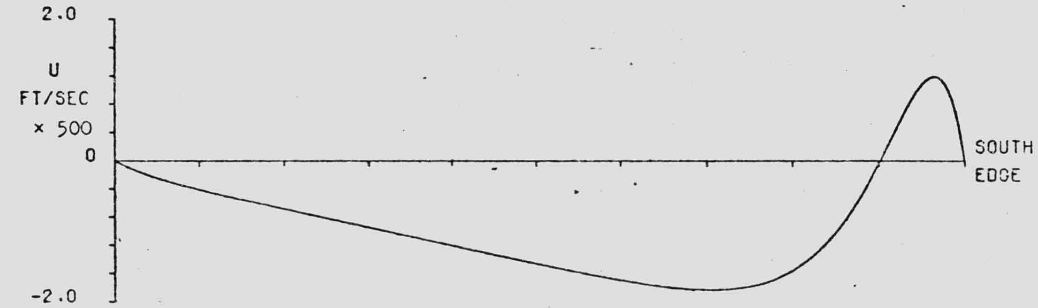
DEPTH: FREE SURFACE

FIG 4.4 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
C
THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

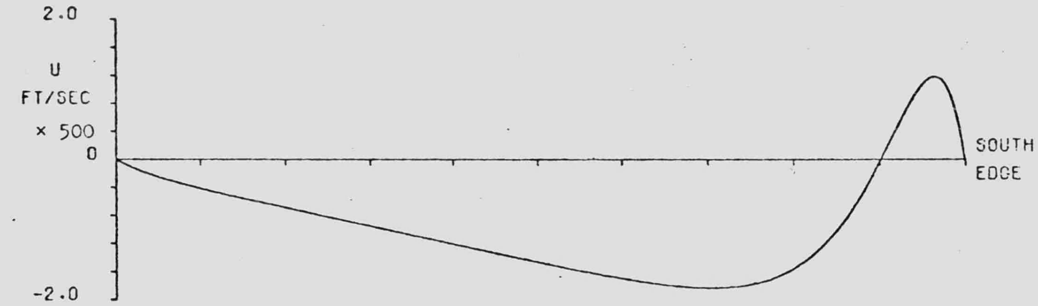
CANAL WIDTH= 379.6 MLS.NEP=37.9 MLS β = 0.001



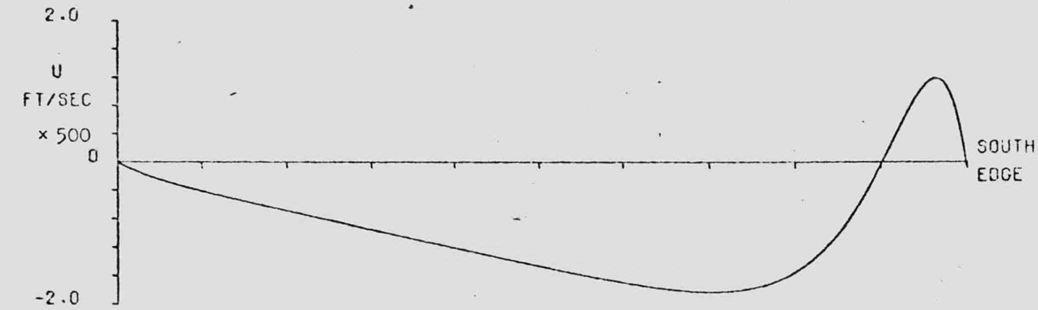
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM.



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 4.4 U_D FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 570.8 MLS.NEP=37.9 MLS β = 0.001

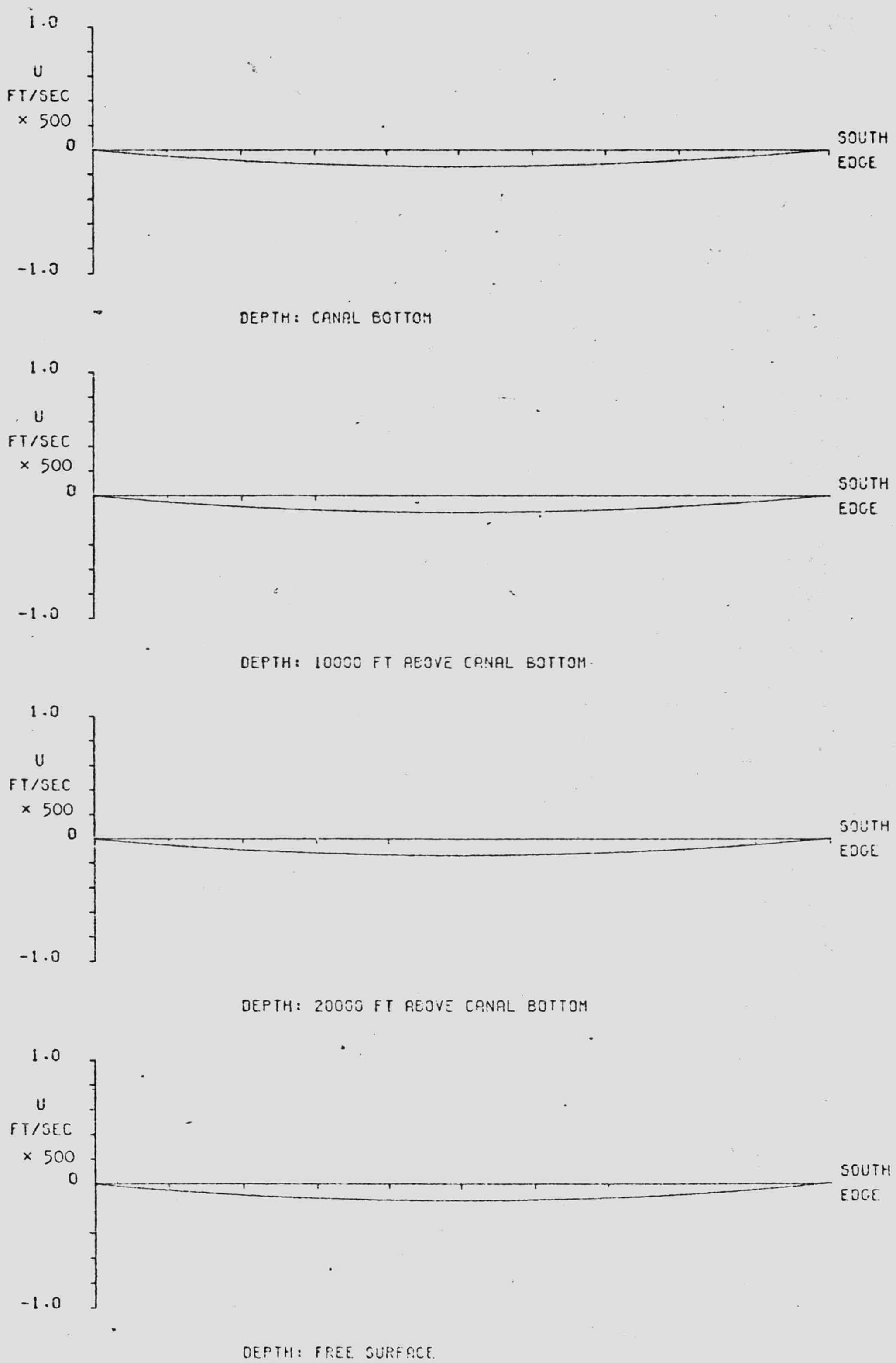


FIG 4.4E U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 37.9 MLS.NEP=189.5 MLS β = 0.001

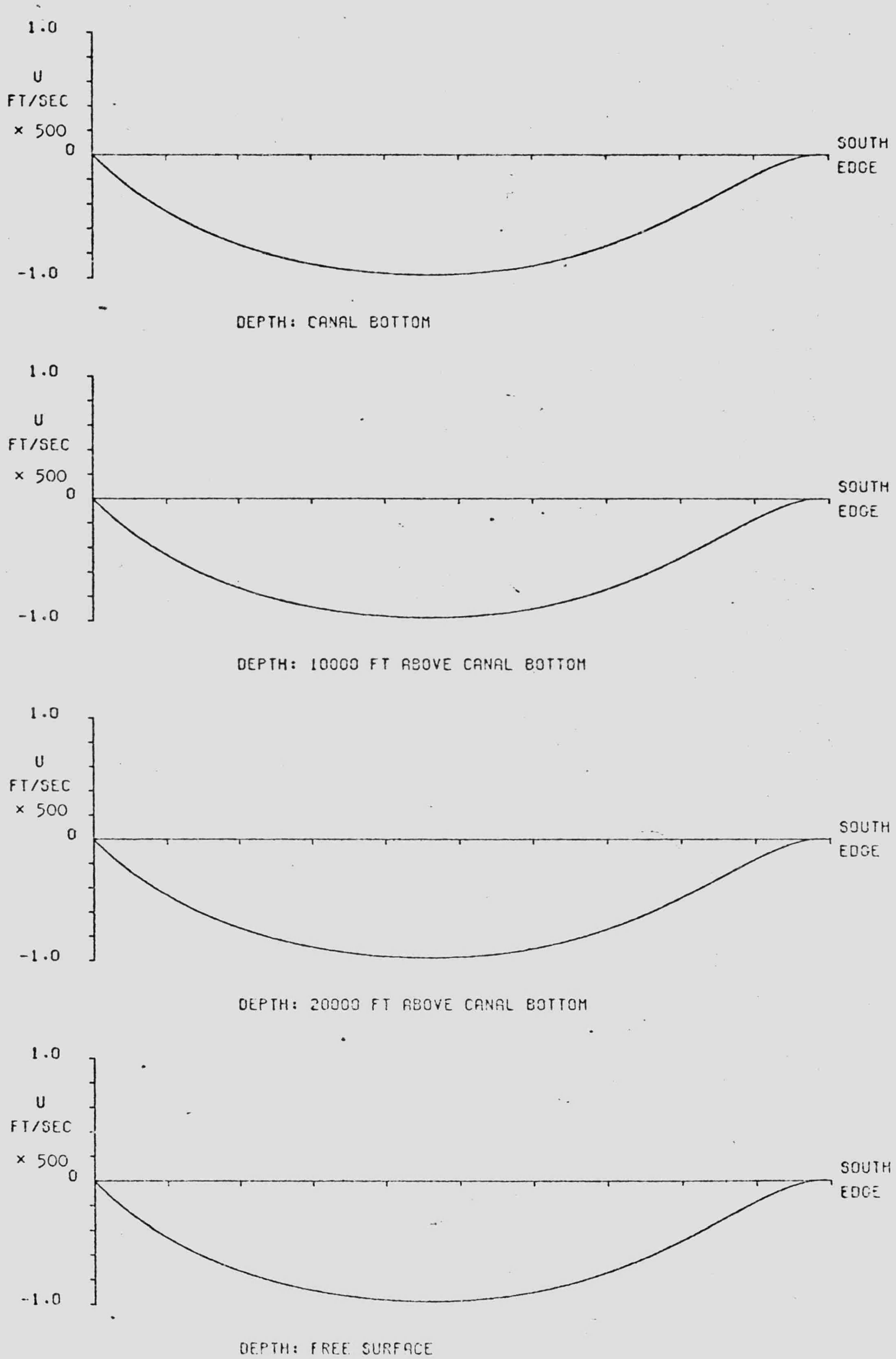


FIG 4.4 U_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 189.9 MLS.NEP=189.5 MLS β= 0.001

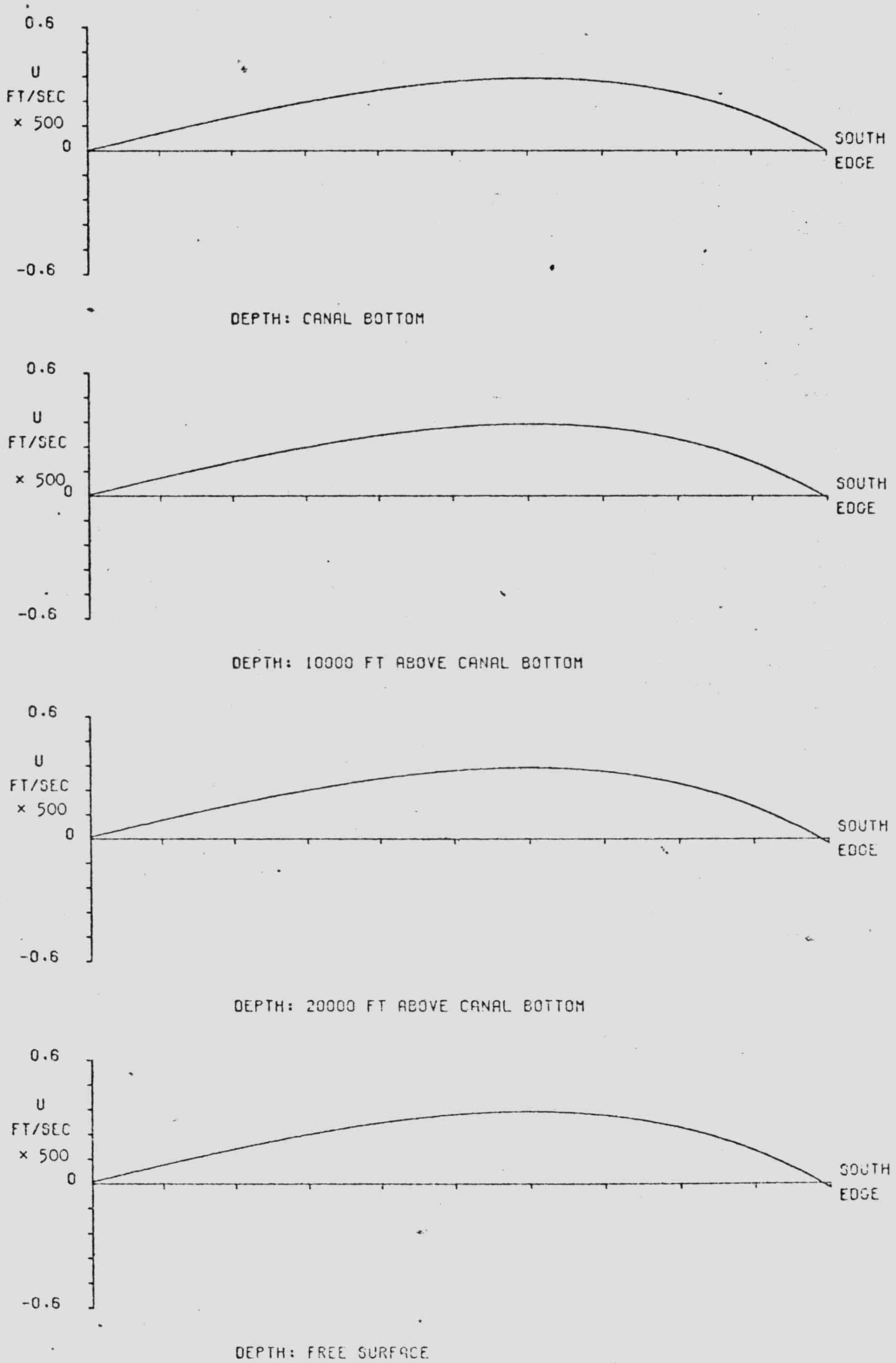
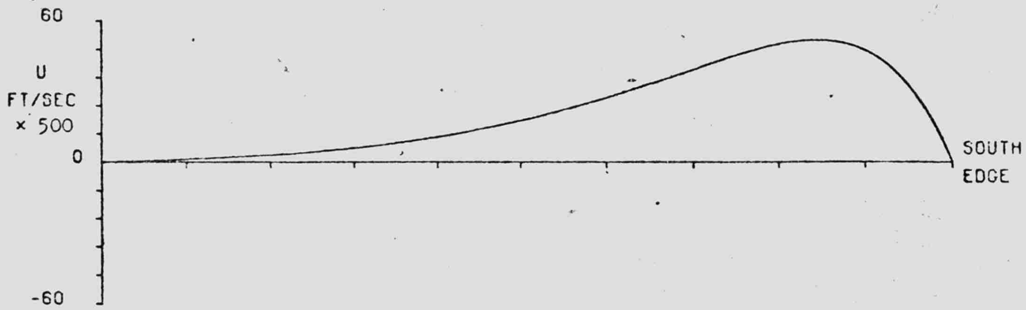
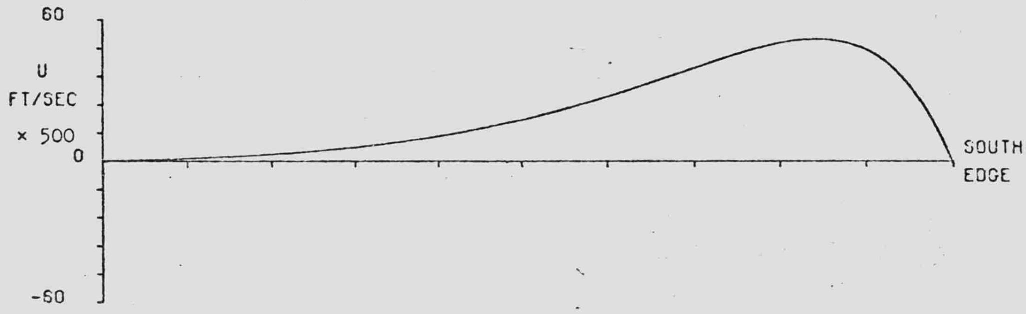


FIG 4.4 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
G THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

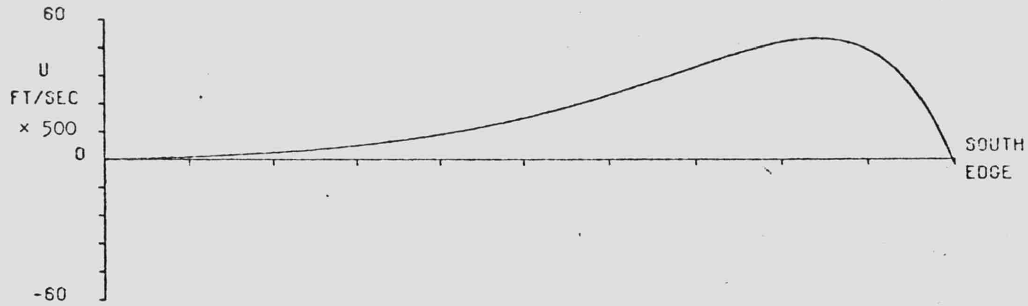
CANAL WIDTH= 38.1 MLS.NEP=379.4 MLS β = 0.001



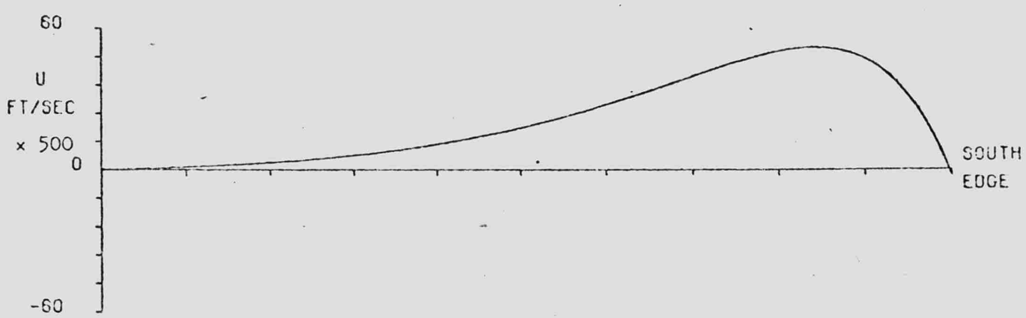
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 4.4 U FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
H THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE
CANAL WIDTH= 190.9 MLS.NEP=379.4 MLS β= 0.001

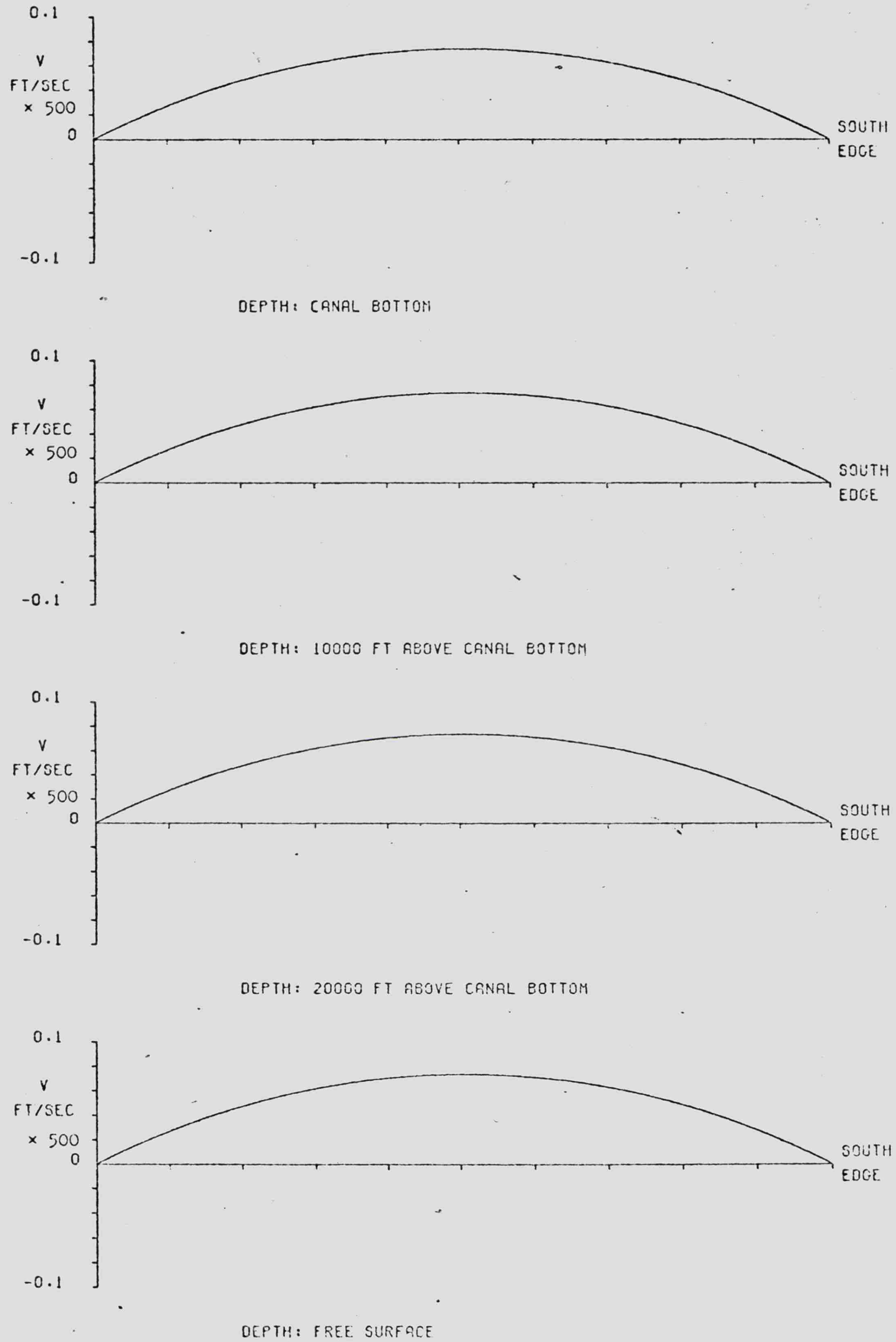
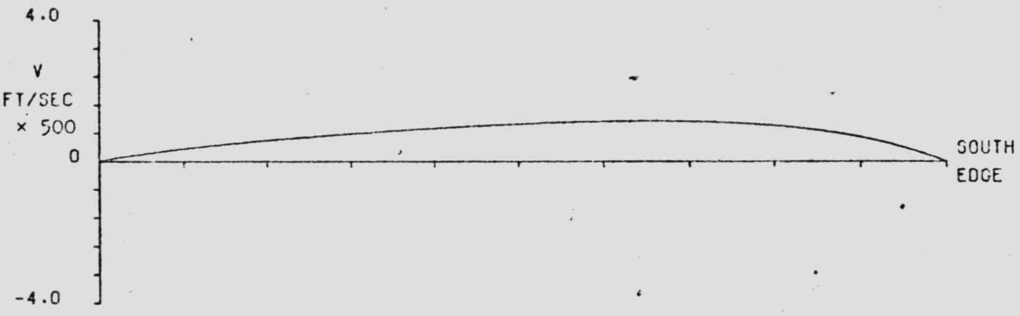
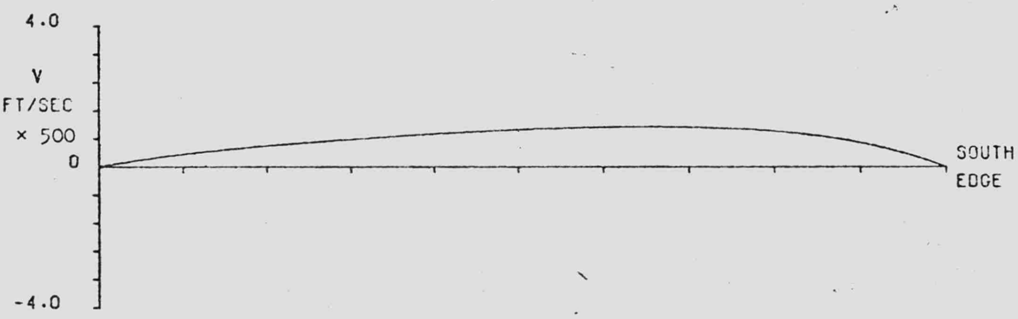


FIG 4.5 A   V FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

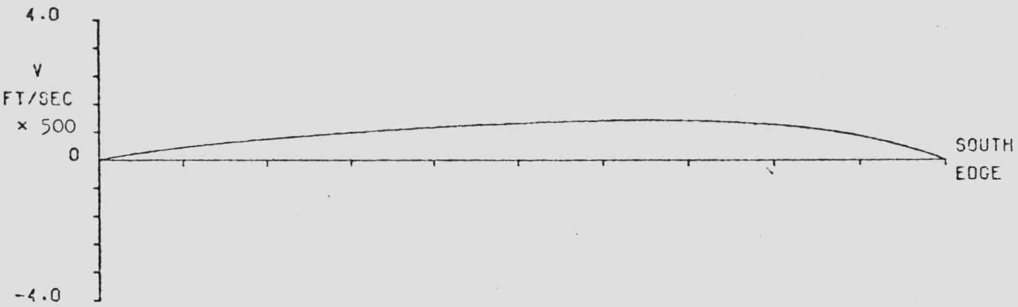
CANAL WIDTH= 37.9 MLS.NEP=37.9 MLS β = 0.001



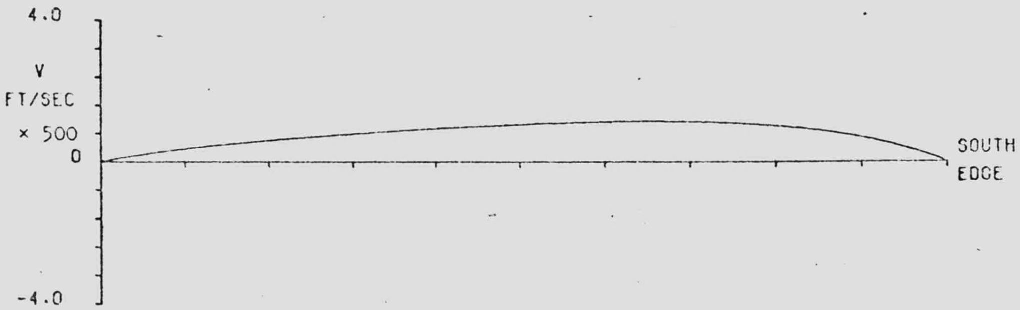
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



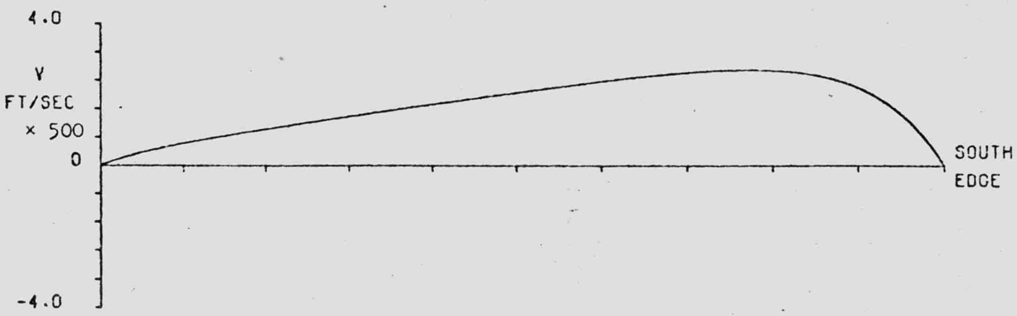
DEPTH: 20000 FT ABOVE CANAL BOTTOM



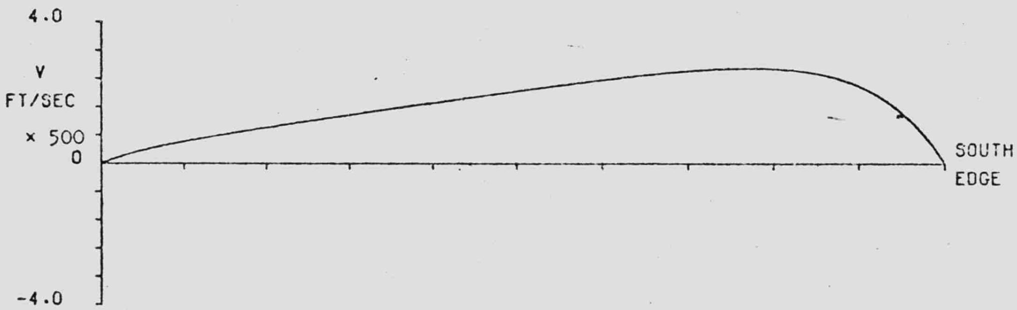
DEPTH: FREE SURFACE

FIG 4.5_B V FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

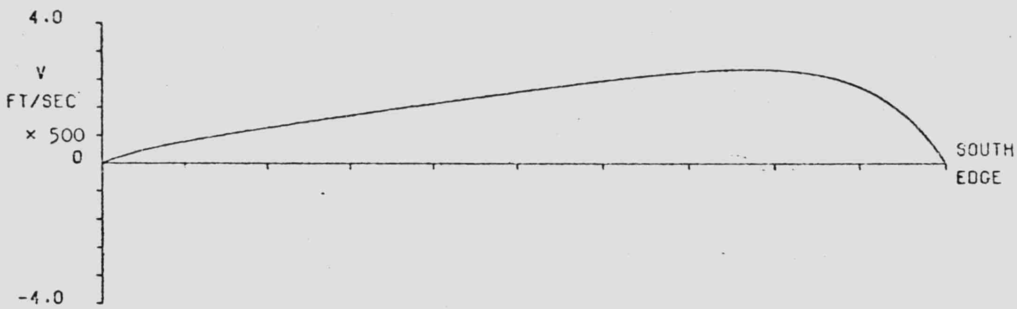
CANAL WIDTH= 189.5 MLS,NEP=37.9 MLS β = 0.001



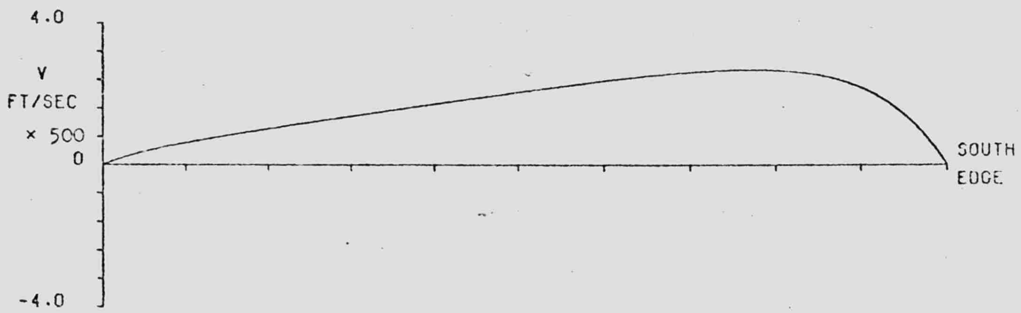
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



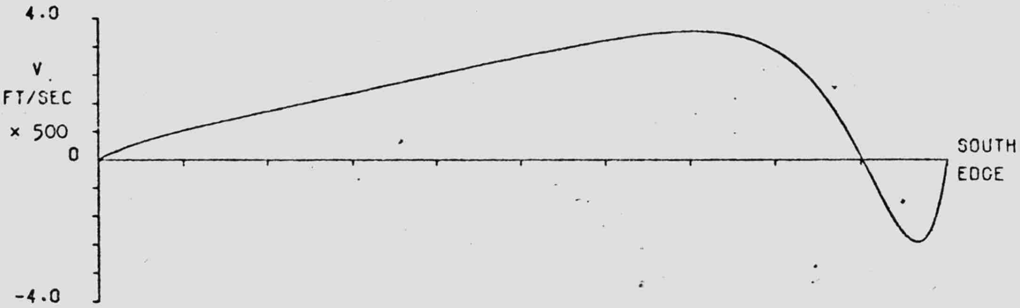
DEPTH: 20000 FT ABOVE CANAL BOTTOM



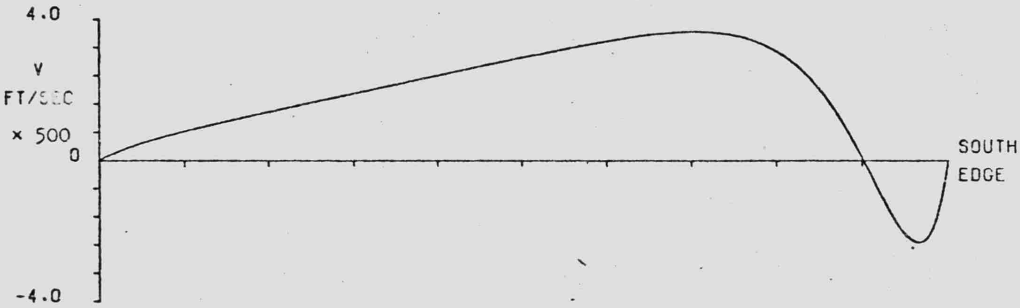
DEPTH: FREE SURFACE

FIG 4.5 V_C FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

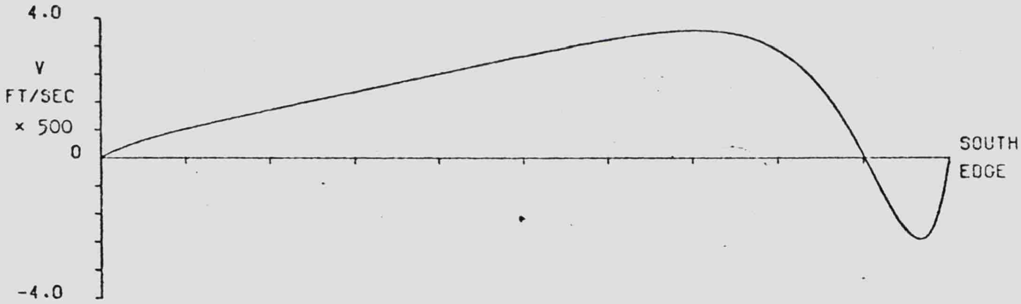
CANAL WIDTH= 379.6 MLS.NEP=37.9 MLS δ = 0.001



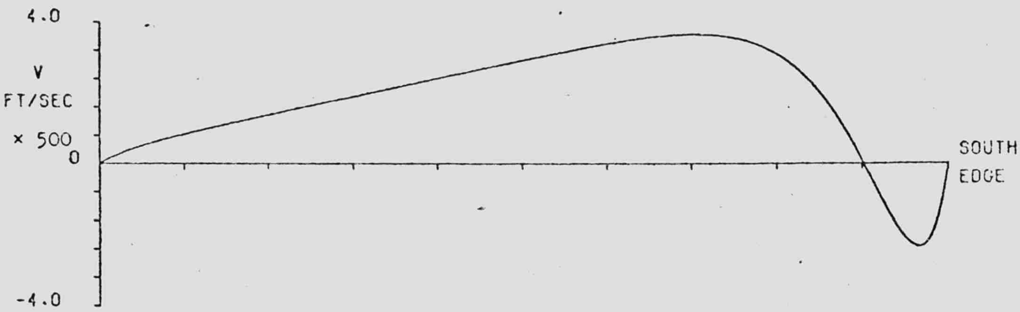
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



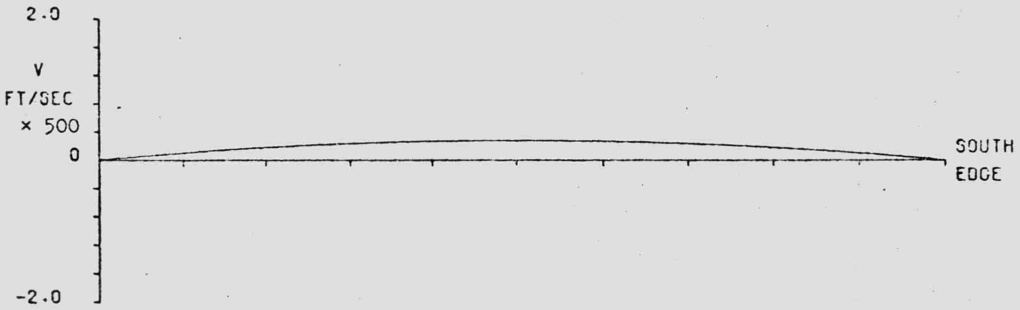
DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 4.5 V FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
D THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

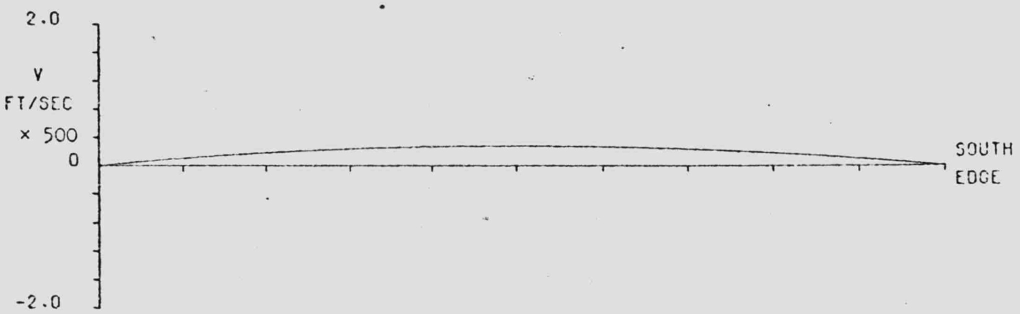
CANAL WIDTH= 570.8 MLS.NEP=37.9 MLS β = 0.001



DEPTH: CANAL BOTTOM

DEPTH: 10000 FT ABOVE CANAL BOTTOM

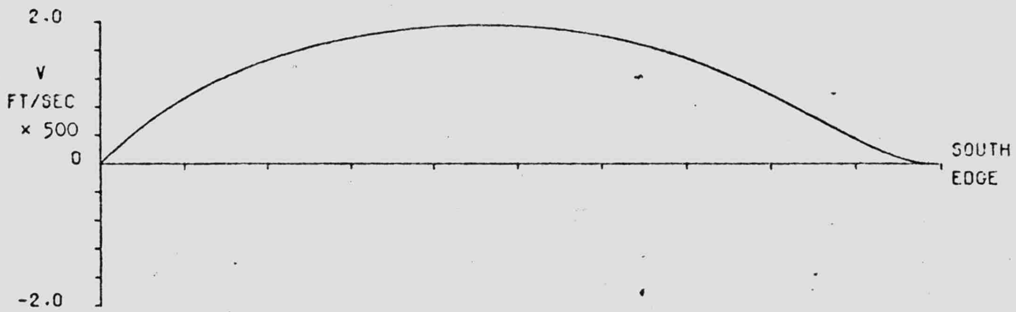
DEPTH: 20000 FT ABOVE CANAL BOTTOM



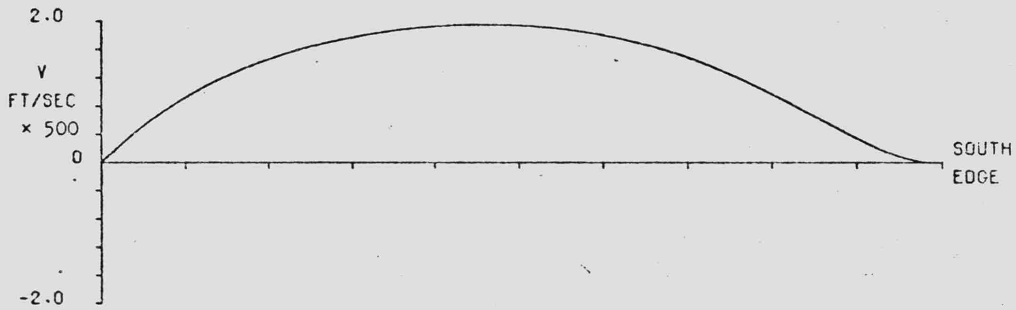
DEPTH: FREE SURFACE

FIG 4.5E V FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

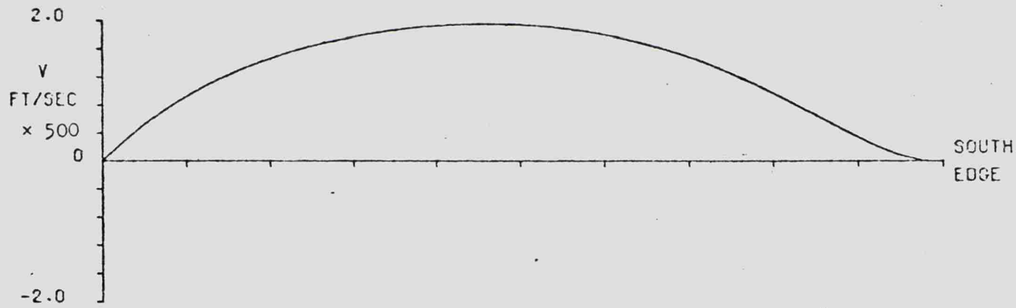
CANAL WIDTH= 37.9 MLS.NEP=199.5 MLS β = 0.001



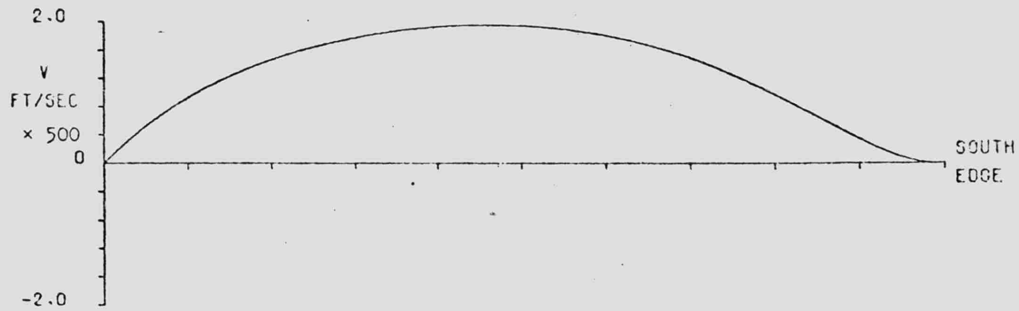
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 4.5 V FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
F THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 189.9 MLS.NEP=189.5 MLS $\beta = 0.001$

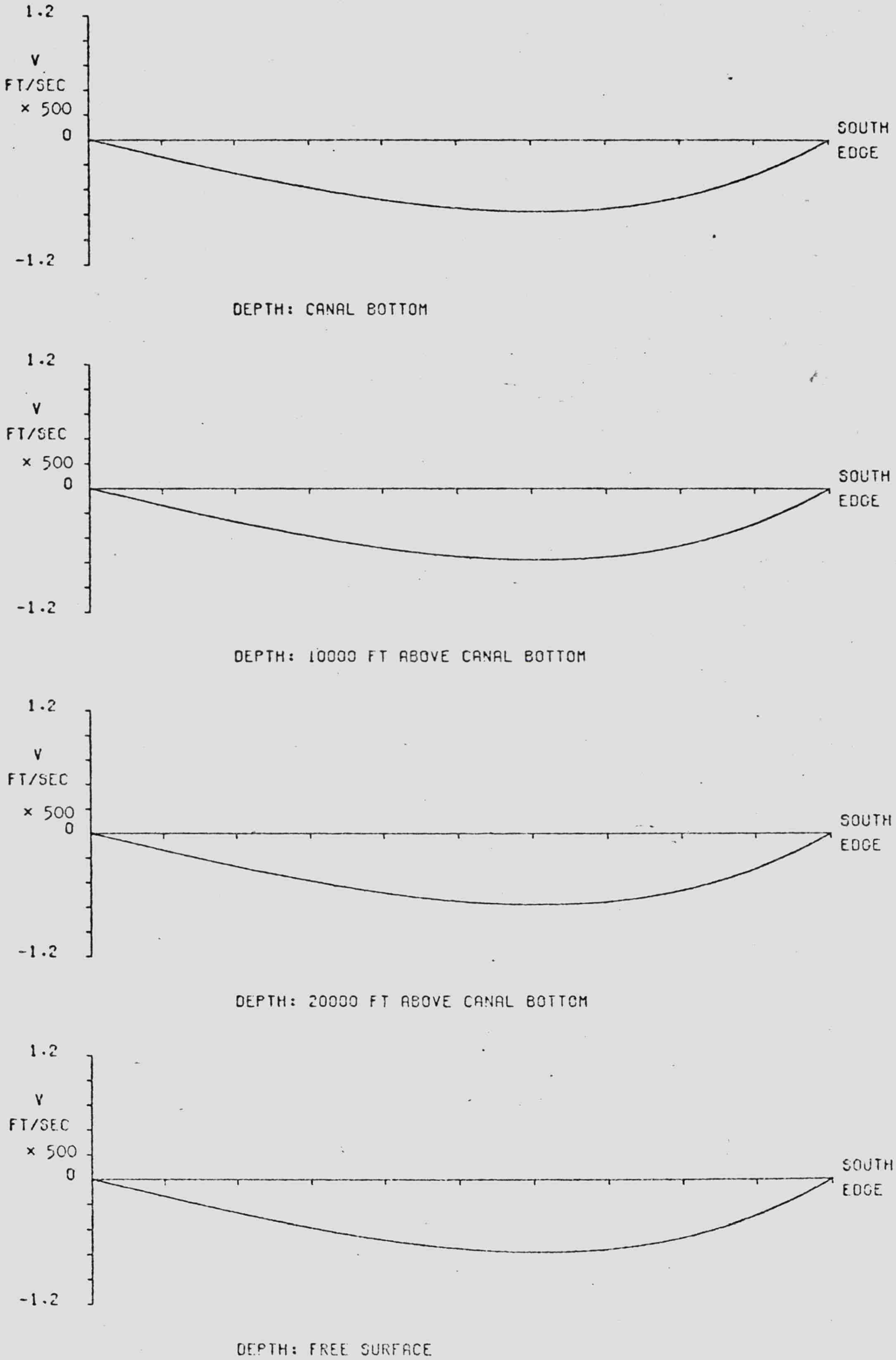


FIG 4.5 V_G FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 38.1 MLS.NEP=379.4 MLS β = 0.001

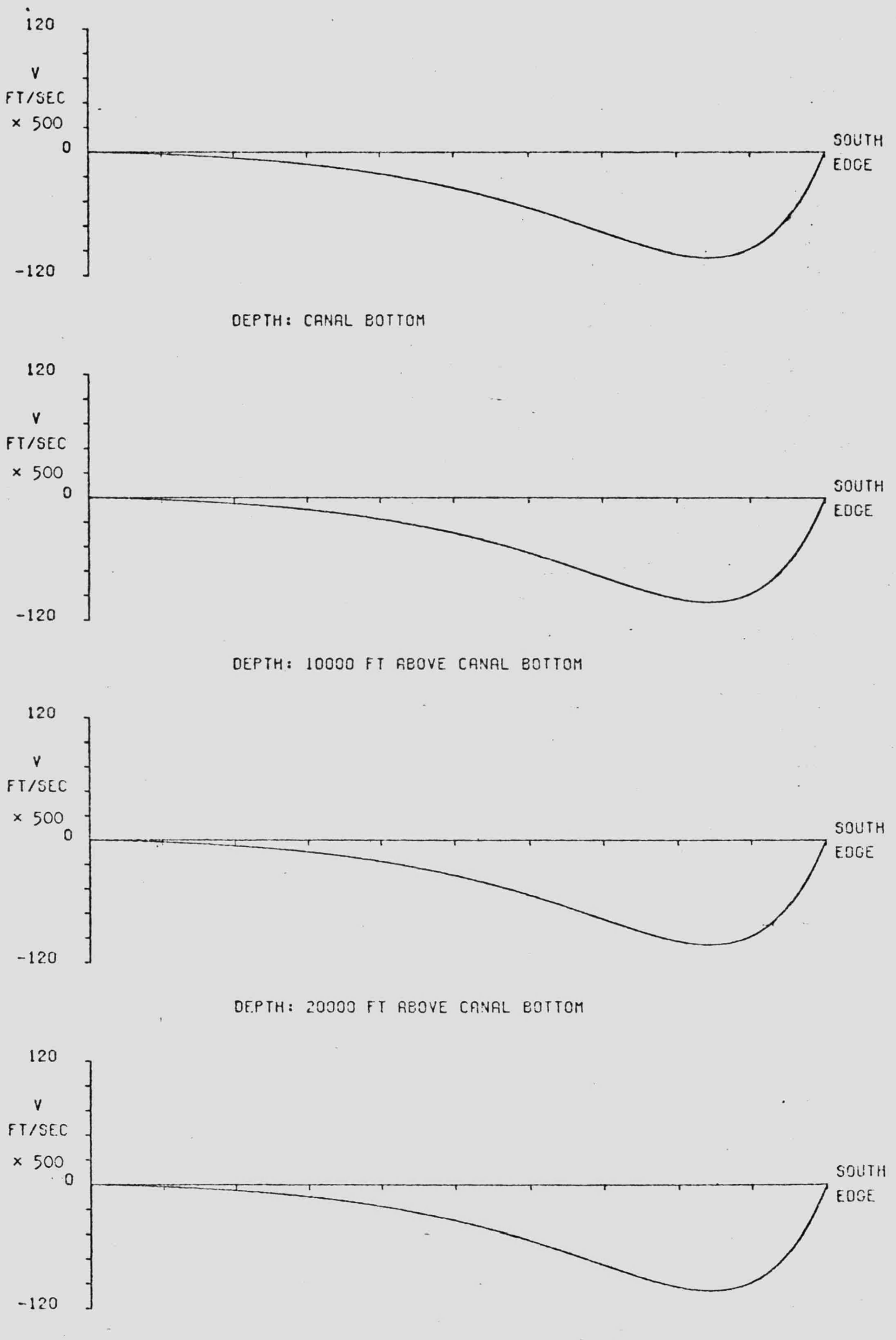


FIG 4.5 V_H FT/SEC (ZEROTH ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 190.9 MLS.NEP=379.4 MLS $\beta = 0.001$

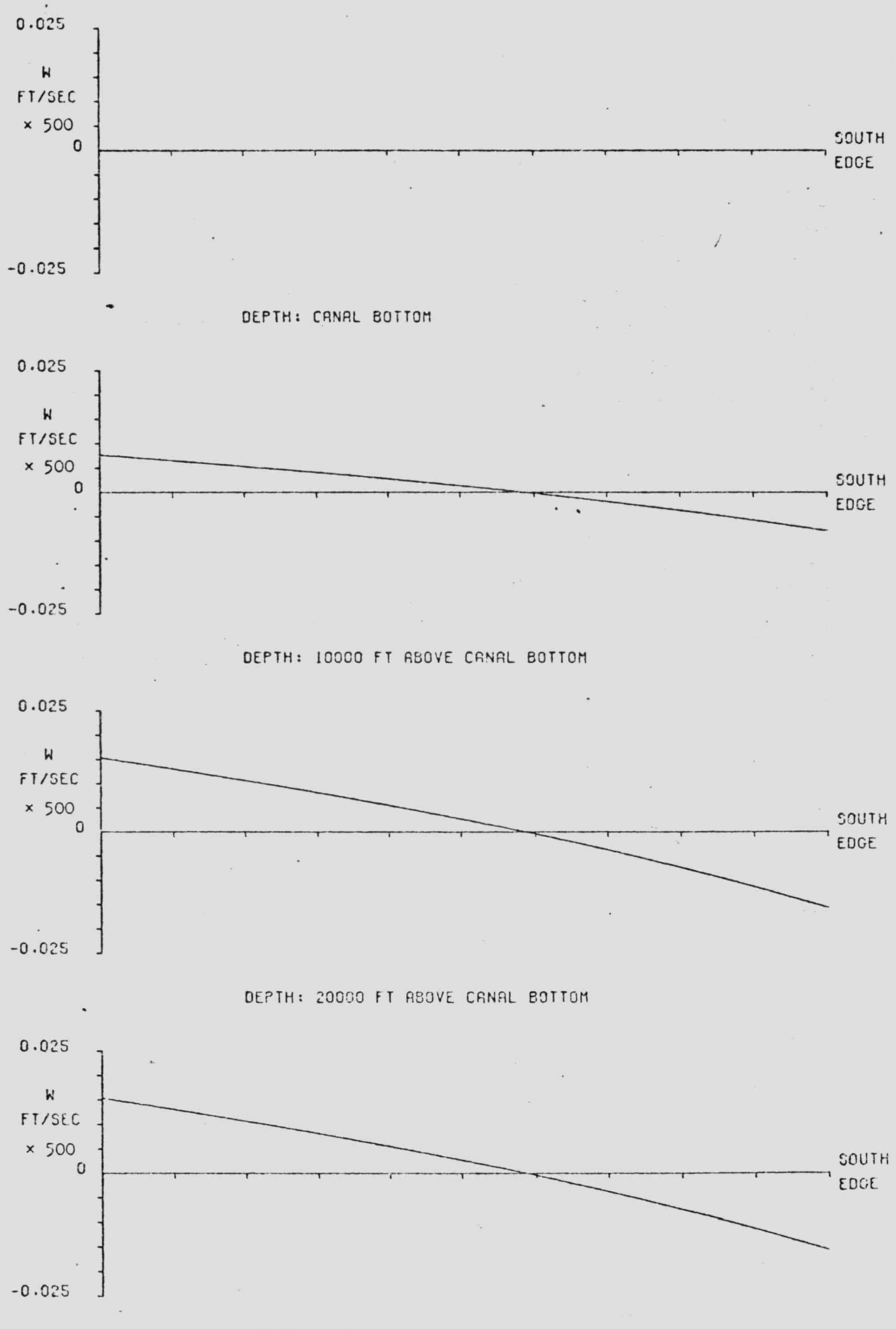
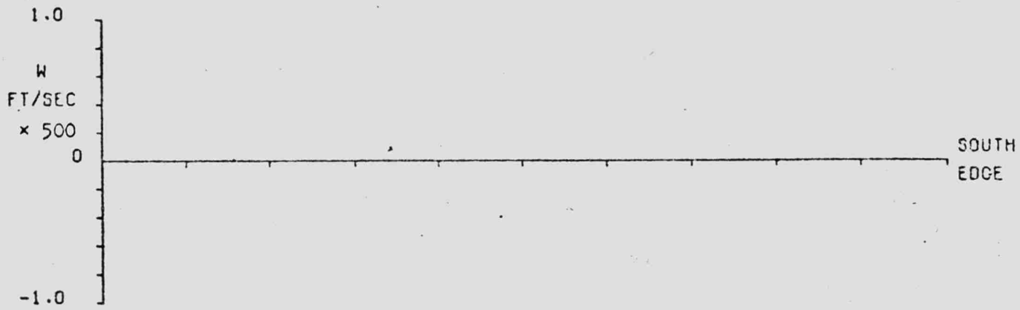
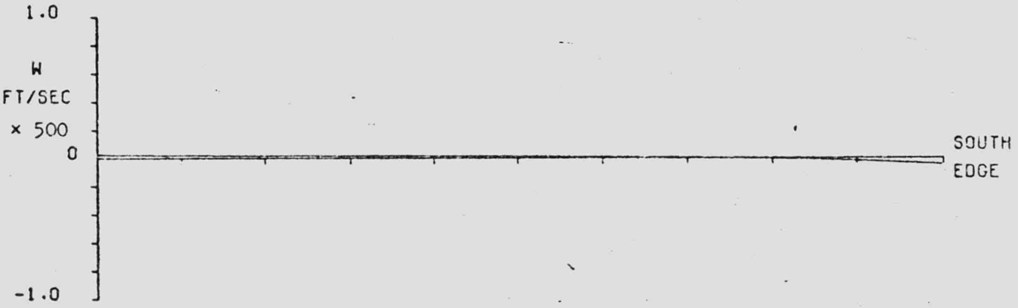


FIG 4.6A W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 37.9 MLS, NEP=37.9 MLS β = 0.001

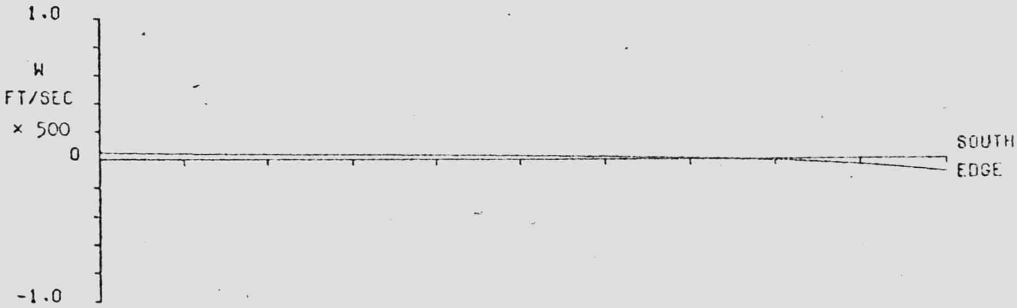


DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM

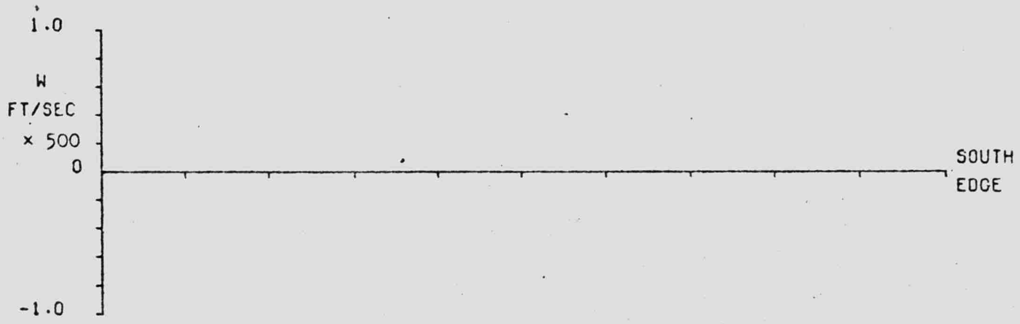
DEPTH: 20000 FT ABOVE CANAL BOTTOM



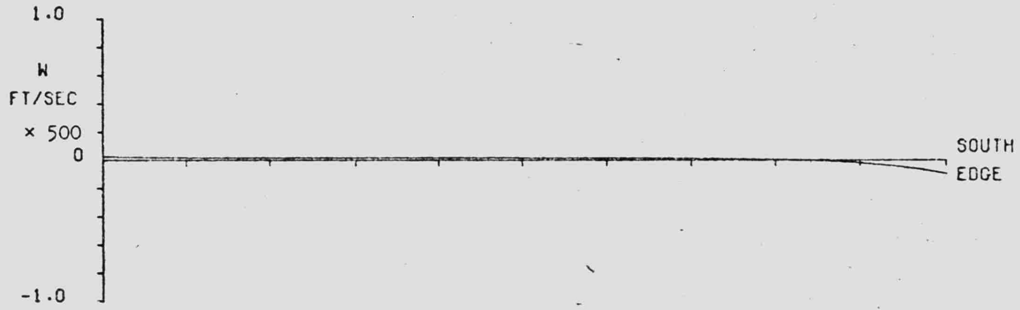
DEPTH: FREE SURFACE

FIG 4.6B W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

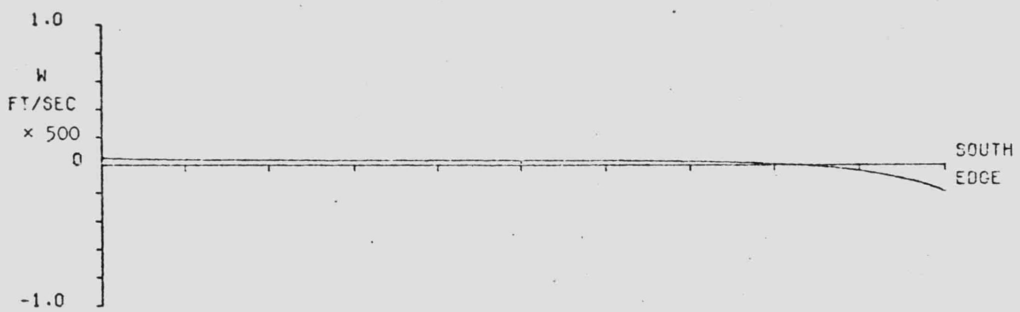
CANAL WIDTH= 199.5 MLS.NEP=37.9 MLS $\beta = 0.001$



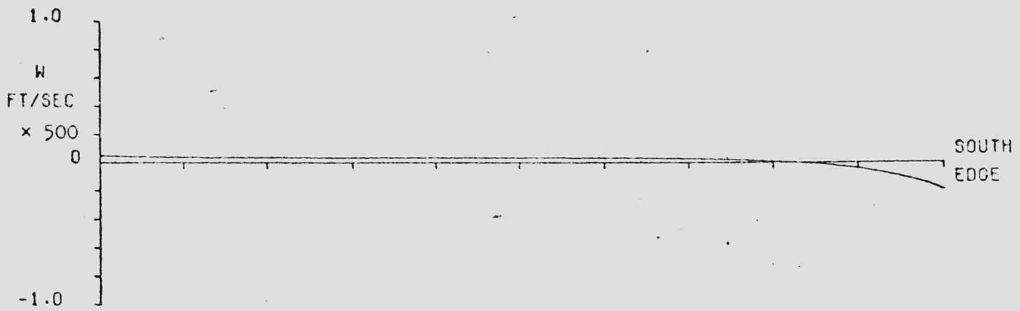
DEPTH: CANAL BOTTOM



DEPTH: 10000 FT ABOVE CANAL BOTTOM



DEPTH: 20000 FT ABOVE CANAL BOTTOM



DEPTH: FREE SURFACE

FIG 4.6 W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF
C THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE
CANAL WIDTH= 379.6 MLS.NEP=37.9 MLS β = 0.001

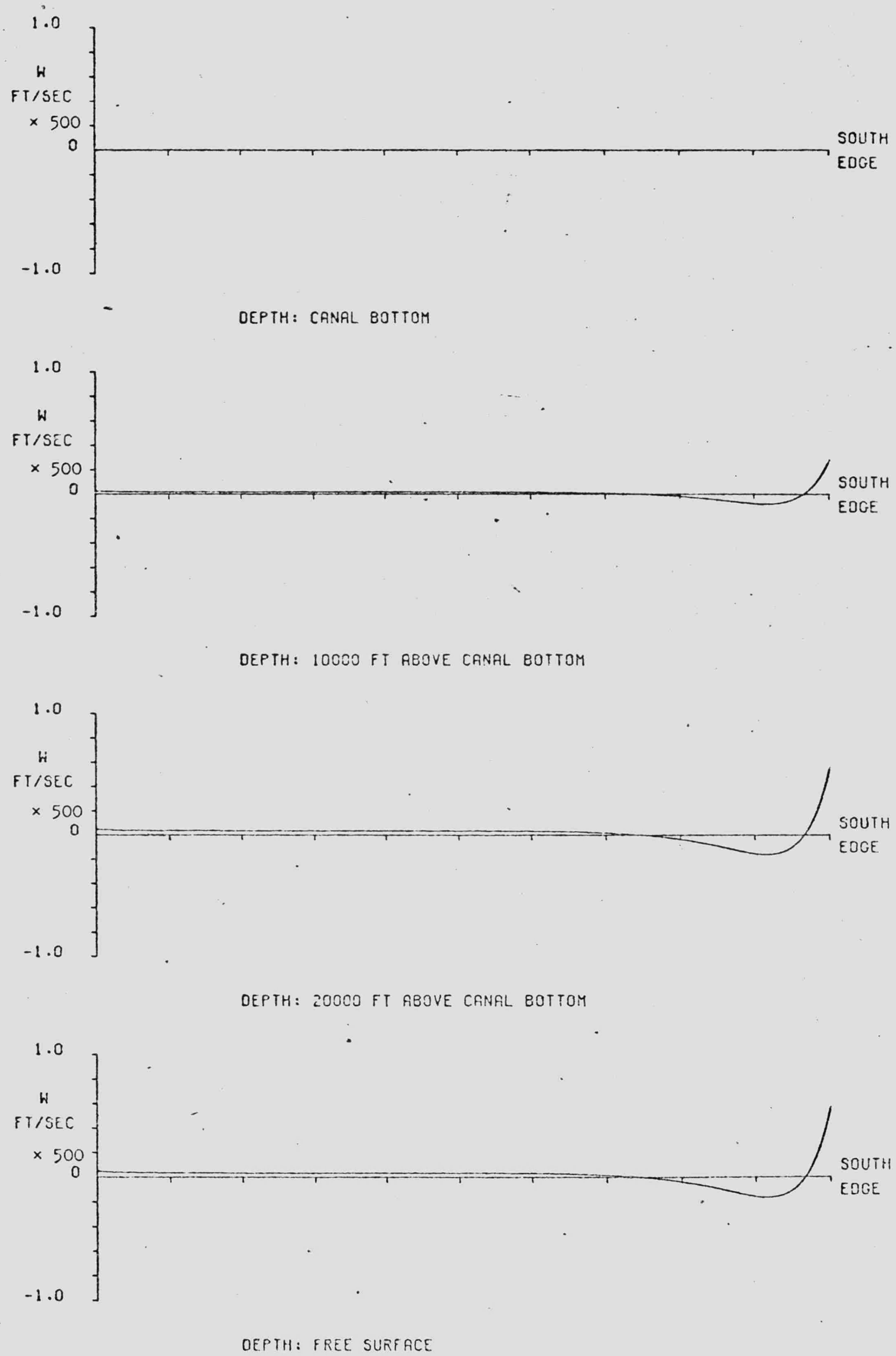


FIG 4.6 W_D FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 570.8 MLS, NEP=37.9 MLS $\beta = 0.001$

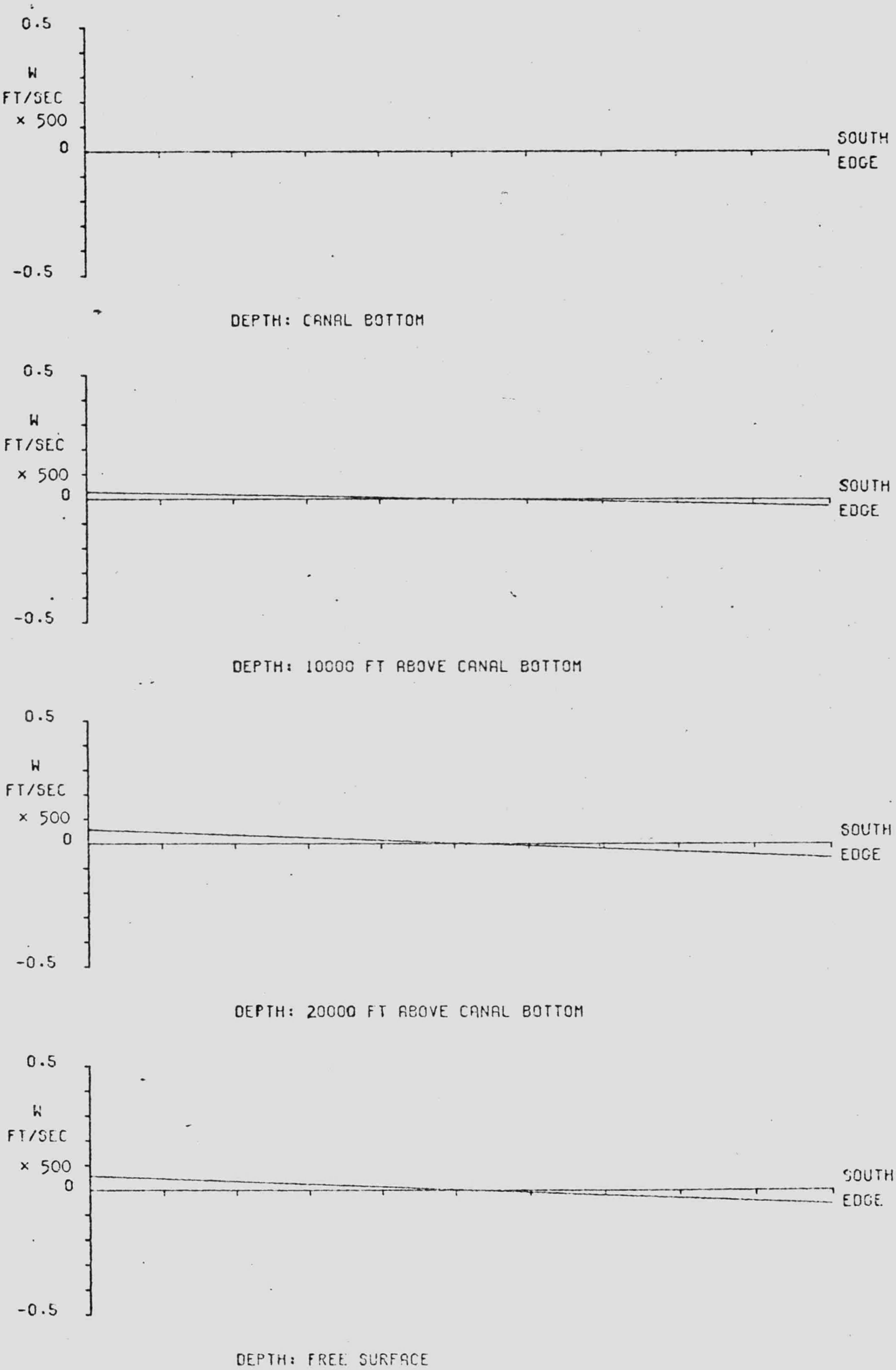


FIG 4.6 W_E FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 37.9 MLS.NEP=139.5 MLS β= 0.001

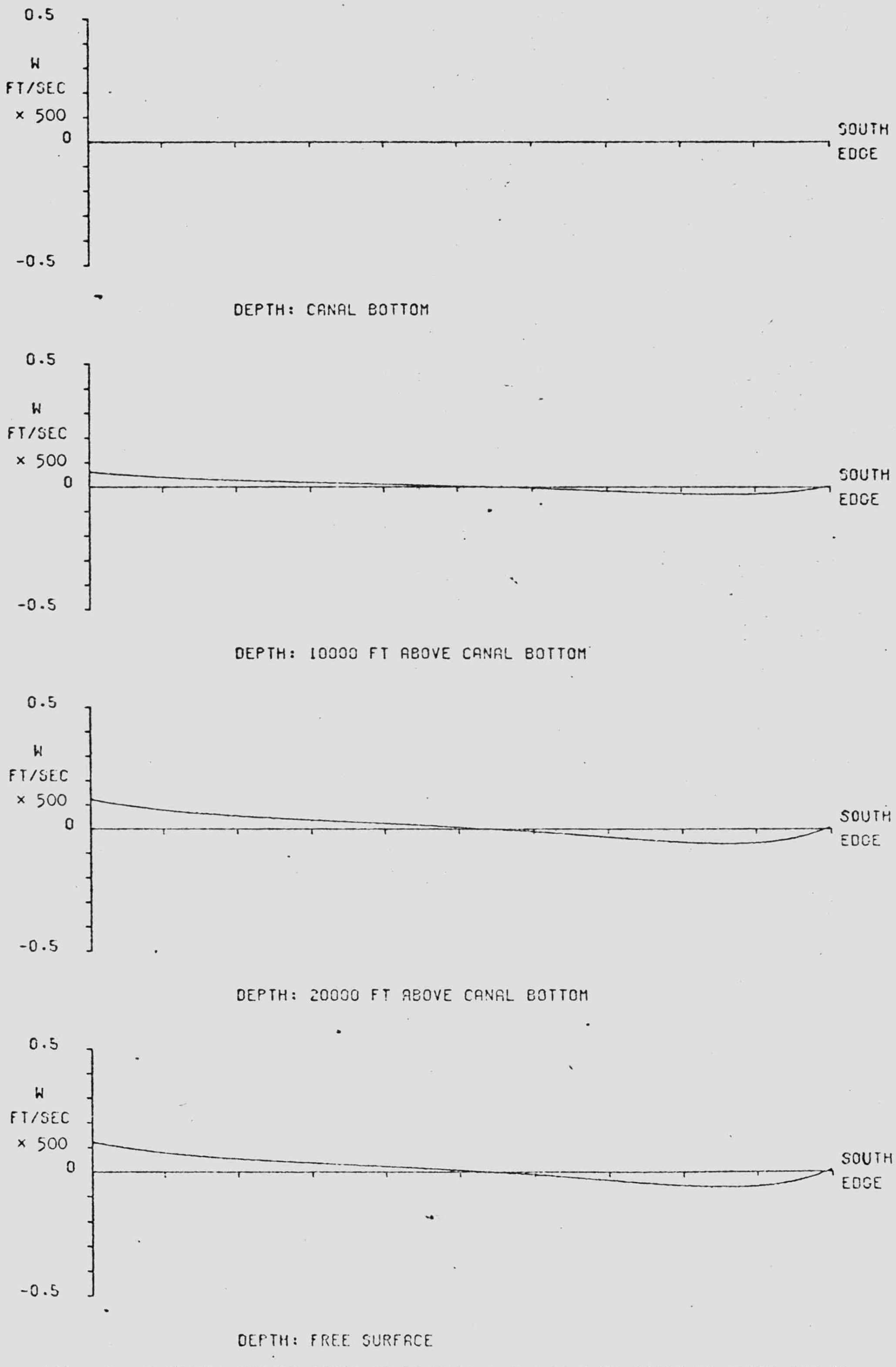


FIG 4.6 W_F FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH=   189.9 MLS.NEP=189.5 MLS β= 0.001

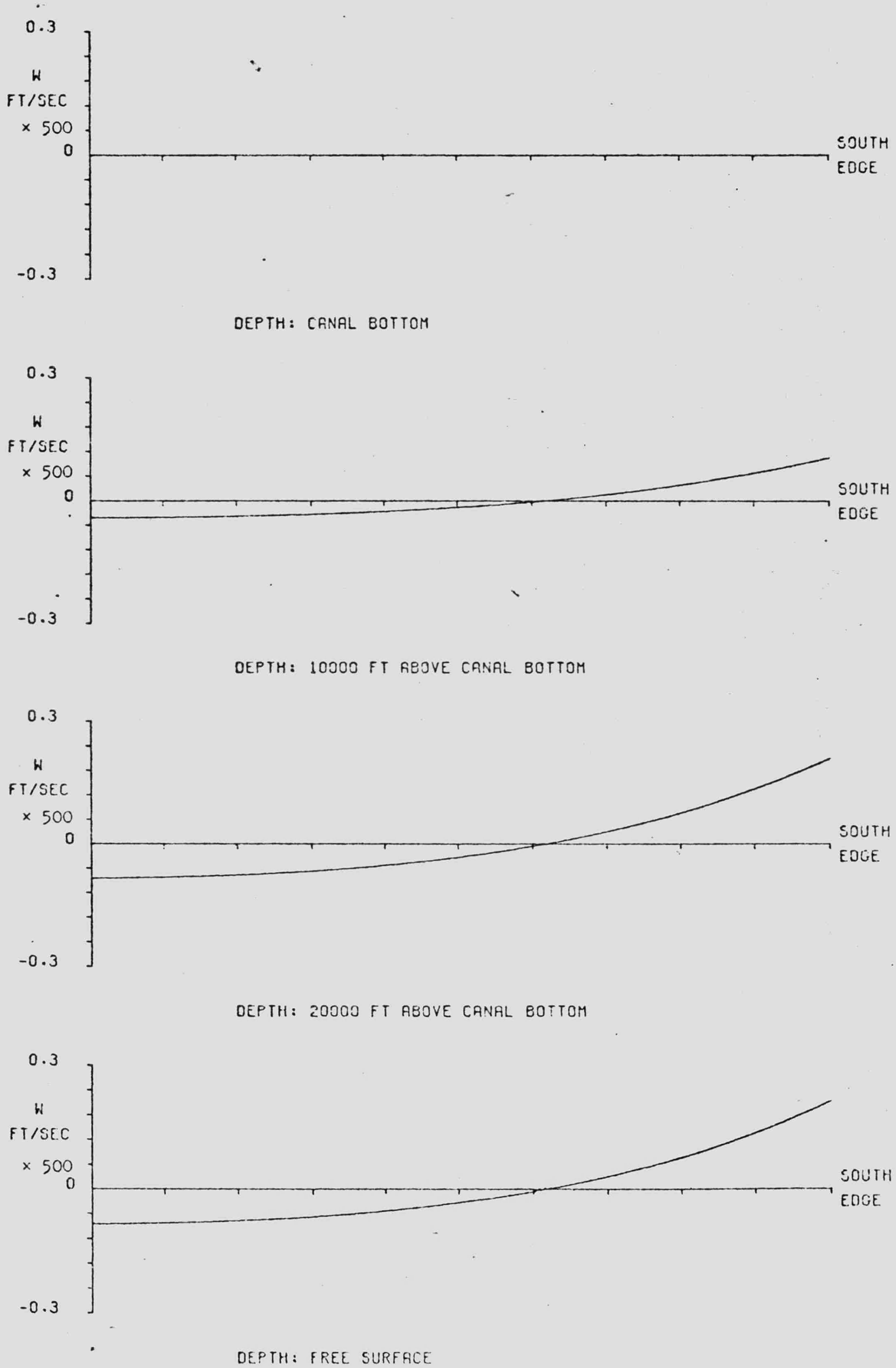


FIG 4.6$_G$ W FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 38.1 MLS.NEP=379.4 MLS $\beta = 0.001$

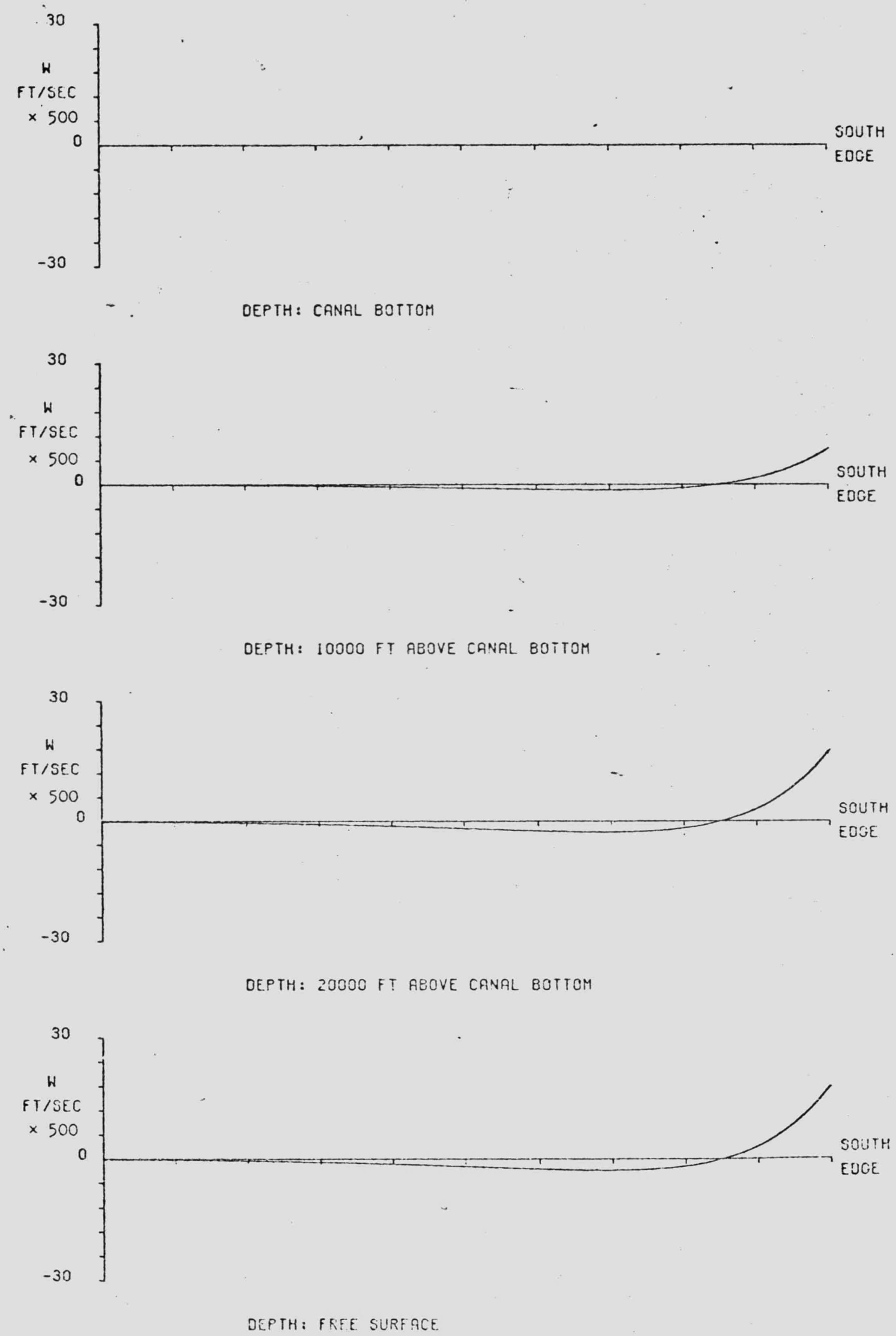


FIG 4.6 W_H FT/SEC (FIRST ORDER COMPONENT) SHOWN AS A FUNCTION OF DEPTH AND OF THE DISTANCE FROM NORTH EDGE TO SOUTH EDGE

CANAL WIDTH= 199.9 MLS.NEP=379.4 MLS β = 0.001

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J.L.ADAMS Ph.D. THESIS 1979

LONG PERIOD AND SEMI-DIURNAL TIDAL OSCILLATIONS

by

J L Adams

Abstract

A brief review is made of Laplace's equations governing tidal oscillations and of the subsequent claims and counter-claims on their validity. The purpose of this study is to investigate these claims further, with regard to long period and semi-diurnal oscillations.

As the underlying assumptions are of importance, these are considered first in some depth. A set of equations is thereby formulated which differ from Laplace's equations in that extra terms of the Coriolis force are retained. These equations are taken as the basis from which a comparison is made with the previous findings.

Taking the semi-diurnal constituent first, a solution is derived in the Equatorial Canal. Graphs are produced showing the velocity components as functions of canal depth and width. These compare favourably with Laplace's theory. However, whilst the description of the tidal elevation is qualitatively the same as before, there are significant quantitative differences. In particular tides become direct only in a much deeper ocean than previously predicted.

Using a similar approach a solution is derived for the long period constituent in a canal-like region near the North Pole. Whereas Laplace's theory for this region gives a solution involving Bessel functions, these become Modified Bessel functions in the derived solution. Arising from this, some different effects are noted in the velocity components.