ON THE REPRESENTATION DIMENSION AND FINITISTIC DIMENSION OF SPECIAL MULTISERIAL ALGEBRAS

SIBYLLE SCHROLL

Dedicated to Ed Green on the occasion of his 70th birthday

ABSTRACT. For monomial special multiserial algebras, which in general are of wild representation type, we construct radical embeddings into algebras of finite representation type. As a consequence, we show that the representation dimension of monomial and self-injective special multiserial algebras is less than or equal to three. This implies that the finitistic dimension conjecture holds for all special multiserial algebras.

INTRODUCTION

Many of the important open conjectures in representation theory of Artin algebras are of a homological nature, such as the finitistic dimension conjecture, Nunke's condition and Nakayma's conjectures. Amongst these conjectures there is a logical hierarchy, in that if the finitistic dimension conjecture holds then Nunke's condition holds which in turn implies the Nakayama conjectures; for an overview, see, for example [8, 11, 14].

The finitistic dimension conjecture states that for any Artin algebra A, the supremum of the projective dimensions of the finitely generated right A-modules of finite projective dimension is finite. This conjecture was originally posed as a question by Rosenberg and Zelinsky and then published by Bass in 1960 [1].

Although the finitistic dimension conjecture is open in general, there has been much related work in recent years reducing the problem to simpler classes of algebras [12, 13]. There are many classes of algebras where the conjecture has been shown to hold [2, 9]. For classes of algebras of mostly wild representation type, the two most prominent examples where the finitistic dimension conjecture is known to hold are the monomial algebras [3, 10] and the radical cubed zero algebras [7].

In this paper, we will show that the finitistic dimension conjecture holds for special multiserial algebras, a large class of mostly wild algebras, containing many other important and well-studied classes of algebras such as, for example, special biserial algebras, symmetric radical cubed zero algebras and almost gentle algebras [4, 5].

It is well known that most finite dimensional algebras are of wild representation type implying that their representation theory is at least as complicated as the representation

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theory of the free associative algebra in two variables. Special multiserial algebras form a class of mostly wild finite dimensional algebras. It was shown in [4] that the radical of their indecomposable modules is a sum of uniserial modules whose pairwise intersection is either a simple module or zero. This is an indication that uniserial modules play an important role in the study of their representation theory. In this paper, we show that for a monomial special multiserial algebra A of infinite representation type, the direct sum of all uniserial submodules of A gives rise to an Auslander generator of A.

In order to show this, we construct radical embeddings from monomial special multiserial algebras to a direct product of representation finite string algebras whose quivers are linearly oriented Dynkin diagrams of type $\mathbb{A}$ and cyclically oriented Dynkin diagrams of type $\tilde{\mathbb{A}}$. Therefore by [2] we obtain that the representation dimension of a monomial special multiserial algebra is less or equal to three.

We further show that for any special multiserial algebra A, a relation is either monomial or is a linear combination of elements in the socle of A. We then apply the results in [2] in combination with our results on monomial special multiserial algebras, to show that the representation dimension of self-injective special multiserial algebras is less or equal to three.

To summarise, in this paper we show the following:

Theorem 1. *Let A be a monomial special multiserial algebra. Then there exists a radical embedding $f : A \to B$ where B is an algebra of finite representation type.*

Corollary 2. *Let A be a monomial special multiserial algebra. Then repdim$(A) \leq 3$.*

In [6] Brauer configuration algebras are defined as generalisations of Brauer graph algebras. Brauer configuration algebras are symmetric algebras, so in particular they are self-injective and it follows from the next result that their representation dimension is less or equal to 3.

Corollary 3. *Let A be a self-injective special multiserial algebras. Then repdim$(A) \leq 3$. In particular, the representation dimension of a Brauer configuration algebra is less or equal to* 3.

Corollary 4. *Let A be a special multiserial algebra. Then the finitistic dimension of A is finite.*

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1. Background

Let K be an algebraically closed field. A quiver $Q = (Q_0, Q_1, s, e)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1 and maps $s, e : Q_1 \to Q_0$ where, for $a \in Q_1$,

$s(a)$ denotes the vertex at which a starts and $e(a)$ denotes the vertex at which a ends. For $a, b \in Q_1$, such that $e(a) = s(b)$, we write ab for the element in KQ given by the concatenation of a and b. For $v \in Q_0$, denote by e_v the associated idempotent. We call an element $x \in KQ$ uniform if there exists $v, w \in Q_0$ such that $e_v x e_w = x$. All modules considered are finitely generated right modules and for A a finite dimensional K-algebra, we denote by mod A the category of finitely generated right A-modules. Furthermore, set $D(A) = \mathrm{Hom}_K(A, K)$ and denote by J_A the Jacobson radical of A. We call a finite dimensional K-algebra basic, if $A = KQ/I$ for I an admissible ideal in KQ.

From now on, whenever we write $A = KQ/I$, we assume that I is admissible.

Recall that $\mathrm{gldim}(A) = \sup\{\mathrm{pd}(M) \mid M \in \mathrm{mod}\, A\}$ and that M is a generator-cogenerator of A if $A \oplus D(A) \in \mathrm{add}\, M$ where add M is the subcategory of mod A generated by direct sums of direct summands of M. Then

$$\mathrm{repdim}(A) = \inf\{\mathrm{gldim}(\mathrm{End}_A(M)^{op}) \mid M \text{ is a generator-cogenerator of } A\ \}.$$

Moreover, M is an *Auslander generator* of A if $\mathrm{gldim}(End_A(M)^{op}) = \mathrm{repdim}(A)$. The *finitistic dimension of* A is given by

$$\mathrm{findim}(A) = \sup\{\mathrm{pd}(M) \text{ for all } M \text{ such that } \mathrm{pd}(M) < \infty\}.$$

Let $A = KQ/I$, we say that condition (S) holds for A if the following holds:

(S) For all $a \in Q_1$ there exists at most one arrow $b \in Q_1$ such that $ab \notin I$ and there exists at most one arrow $c \in Q_1$ such that $ca \notin I$.

Definition 5. A finite dimensional algebra A is *special multiserial* if it is Morita equivalent to an algebra KQ/I such that (S) holds.

We recall the following results and definitions from [2]. For $v \in Q_0$, set $S(v)$ to be the subset of Q_1 consisting of arrows starting at v and set $E(v)$ to be the set of arrows of Q_1 ending at v. Note that if there is a loop a at v then $a \in E(v) \cap S(v) \neq \emptyset$.

Suppose $S(v) = S_1 \sqcup S_2$ and $E(v) = E_1 \sqcup E_2$ are disjoint unions. The collection $Sp = (S_1, S_2, E_1, E_2)$ is a *splitting datum at* v (for I) if

(1) $ab \in I$, for all $a \in E_i$ and $b \in S_j$ with $i \neq j$,
(2) $I = \langle \rho \rangle$ where ρ is a set of relations of the form $\sum \lambda apb$ such that none of the a are in E_1 or none of the a are in E_2 and such that none of the b are in S_1 or none of the b are in S_2.

An algebra KQ/I is called *monomial* if I is monomial, that is if I is generated by paths. Remark that condition (2) always holds if I is monomial.

Let $Sp = (S_1, S_2, E_1, E_2)$ be a splitting datum at v. Then we define a new quiver

$$Q^{Sp} = (Q_0^{Sp}, Q_1^{Sp}, s^{Sp}, e^{Sp})$$

by setting

$$Q_0^{Sp} = \{v_1, v_2\} \cup Q_0 \setminus \{v\}$$

and

$$Q_1^{Sp} = Q_1.$$

The map $s^{Sp}: Q_1^{Sp} \to Q_0^{Sp}$ is given by

$$s^{Sp}(a) = \begin{cases} v_i & \text{if } a \in S_i, i = 1, 2, \\ s(a) & \text{otherwise.} \end{cases}$$

The map $e^{Sp}: Q_1^{Sp} \to Q_0^{Sp}$ is given by

$$e^{Sp}(a) = \begin{cases} v_i & \text{if } a \in E_i, i = 1, 2, \\ e(a) & \text{otherwise.} \end{cases}$$

We define $A^{Sp} = KQ^{Sp}/I^{Sp}$ for $I^{Sp} = \langle \rho^{Sp} \rangle$ where

$$\rho^{Sp} = \rho \setminus (\{ab | a \in E_i \text{ and } b \in S_j, \text{ for } i \neq j\}$$

A *radical embedding* $f : A \to B$ is an algebra monomorphism such that $f(J_A) = J_B$. It is shown in [2] that a splitting datum gives rise to a radical embedding.

Proposition 6. [2] *Let $A = KQ/I$ with I admissible. Let $Sp = (E_1, E_2, S_1, S_2)$ be a splitting datum at some vertex v of Q. Then there exists a radical embedding $f : A \to A^{Sp}$.*

Also recall the following results from [2].

Theorem 7. [2] *Let A and B be basic algebras.*

1. *If $f : A \to B$ is a radical embedding with B a representation finite algebra then $repdim(A) \leq 3$.*
2. *Let P be an indecomposable projective-injective A-module and set $A/soc(P)$. Then $repdim(A) \leq 3$ if $repdim(A/soc(P)) \leq 3$.*

2. Some results on special multiserial algebras

In the following proposition we show that the relations in a special multiserial algebra are of a particular form.

Recall first that the socle of A as an A-A-bimodule is given by $\mathrm{soc}({}_AA_A) = \mathrm{soc}(A_A) \cap \mathrm{soc}({}_AA)$ where $\mathrm{soc}(A_A)$ is the socle of A as a right A-module and $\mathrm{soc}({}_AA)$ is the socle of A as a left A-module.

Proposition 8. *Let $A = KQ/I$ be a special multiserial algebra satisfying condition (S). Let $r = \sum \lambda_p p \in I$ be uniform with $\lambda_p \in K$ and where each p is a path in Q. Then either r is a path or every p is in $\mathrm{soc}({}_AA_A)$.*

Proof. We will start by showing that the result holds for the socle of A as a right A-module. Suppose there exists a unique $\lambda_p \neq 0$, then $r = p$.

Suppose that $r = \lambda_p p - \lambda_q q$ with $p, q \notin I$. Then without loss of generality we can assume that $\lambda_p = 1$.
Now suppose that $p \notin \mathrm{soc}(A_A)$ and that $q \in \mathrm{soc}(A_A)$. Then there exists $a \in Q_1$ such that $pa \notin I$ but since $q \in \mathrm{soc}(A_A)$, we have $qa \in I$ and this a contradiction.

Suppose now that $p, q \notin \operatorname{soc}(A_A)$. Then there exist $a, b \in Q_1$ such that $pa, qb \notin I$. Since $p - \lambda_q q \in I$, by condition (S) we have $a = b$. Therefore if $p = p'c$ and $q = q'd$ for $c, d \in Q_1$ then $ca, da \notin I$. This implies by condition (S) that $c = d$ and hence $(p' - \lambda_q q')c \in I$. Moreover, $p'c, q'c \notin I$. Now let $p' = p''c'$ and $q' = q''d'$ which implies that $c'c, d'c \notin I$ and $c' = d'$. Continuing in this way, we see that $p = q$.

Suppose now that $r = \sum \lambda_p p$ and suppose that $\lambda_q \neq 0$ and $q \notin \operatorname{soc} A$. Then there exists $a \in Q_1$ such that $qa \notin I$ and therefore there exists q' with $\lambda_{q'} \neq 0$ and $q'a \notin I$. Since A is special, this implies $q = q'$. Inductively it then follows that $pa \notin I$ for any p such that $\lambda_p \neq 0$ and using that A satisfies condition (S) it follows that $r = q$.

Note that we have only used specialness on the right side. Using specialness on the left side, we obtain the result for the socle of A as a left A-module. $\square$

The following follows directly from condition (S).

Lemma 9. *Let $A = KQ/I$ be monomial special multiserial and let $Sp = (S_1, S_2, E_1, E_2)$ be a splitting datum at some vertex v in Q.*

(1) Suppose that $S_1 = \{b\}$, for $b \in Q_1$. Then E_1 consists of the unique arrow a such that $ab \notin I$ if such an arrow a exists, otherwise E_1 is empty.

(2) Suppose that $E_1 = \{c\}$, for $c \in Q_1$ then $S_1 = \{d\}$ where d is the unique arrow such that $cd \notin I$ if such an arrow d exits and S_1 is empty otherwise.

Moreover, for Sp as in (1) or (2) above, A^{Sp} is monomial special multiserial.

3. Proof of Theorem 1

We show that for any monomial special multiserial algebra $A = KQ/I$ there is a radical embedding of A into a disjoint union of representation finite string algebras whose underlying quiver is either a linearly oriented quiver of type $\mathbb{A}$ and or a cyclically oriented quiver of type $\widetilde{\mathbb{A}}$.

Proof of Theorem 1 and Corollary 2: Let $A = KQ/I$ be a monomial special multiserial algebra such that I is generated by paths. Define $c(A) = |\{v \in Q_0 | S(v) > 1\}| + |\{v \in Q_0 | E(v) > 1\}|$.

If $c(A) = 0$ then Q is a disjoint union of quivers where each quiver is either a linearly oriented quiver of type $\mathbb{A}$ or a cyclically oriented quiver of type $\widetilde{\mathbb{A}}$. So A is a product of representation finite string algebras, and it therefore is of finite representation type.

Suppose that $c(A) \geq 1$. Let $v \in Q_0$ such that $|S(v)| > 1$ or $|E(v)| > 1$. Suppose that $S(v) = \{b_1, \ldots, b_n\}$ with $n \in \mathbb{N}, n > 1$. Set

$$\begin{array}{rcl} S_1 & = & \{b_1\}, \\ S_2 & = & \{b_2, \ldots, b_n\}, \\ E_2 & = & \{a \in E(v) | ab_1 \in I\}, \\ E_1 & = & E(v) \setminus E_2. \end{array}$$

Note that E_1 consists of the unique arrow $a \in Q_1$ such that $ab_1 \notin I$ if such an arrow exists. That $Sp = (S_1, S_2, E_1, E_2)$ is a splitting datum at v follows directly (S) and from the fact that I is monomial.

By Lemma 9, A^{Sp} is again a monomial special multiserial algebra and $c(A^{Sp}) \leq c(A) - 1$.

We treat the case $|E(v)| > 1$ in a similar way.

Repeating this a finite number of times and setting $A = A_1$ and $A_2 = A^{Sp}$, we obtain by Proposition 6 a sequence of radical embeddings $A_1 \to A_2 \to \cdots A_k = B$ such that B is a string algebra with $c(B) = 0$ and B is therefore representation finite. Then it follows from Theorem 7 (1) that $repdim(A) \leq 3$. □

Let $f : A \to B$ be the radical embedding constructed in the proof of Theorem 1 above. By the proof of Theorem 1.1 in [2] an Auslander generator of A is given by $A \oplus D(A) \oplus N$ where N is the direct sum of isomorphism class representatives of the indecomposable B-modules considered as A-modules. Therefore in the case of a monomial special multiserial algebra N is given by the direct sum of all uniserial submodules of A.

Proof of Corollary 3: Since A is self-injective, every projective is injective. Applying Proposition 8 we obtain that iteratively factoring out the socles of the projective injective indecomposable modules gives rise to a monomial special multiserial algebra. Thus the result follows from Theorem 1 and by the successive application of Theorem 7 (2). □

Proof of Corollary 4: It follows from Proposition 8 that $A/\operatorname{soc}(A)$ is a monomial special multiserial algebra and the result follows from [13, 4.3] and Theorem 1. □

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Department of Mathematics, University of Leicester, University Road, Leicester, LE1 7RH, United Kingdom.

E-mail address: `schroll@leicester.ac.uk`