Differentiable Stacks and Equivariant Cohomology

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Never mistake knowledge for wisdom. One helps you make a living; the other helps you make a life.

Eleanor Roosevelt.

Abstract

In this thesis, we are interested in the study of cohomology of differentiable stacks and we want to provide a good notion of equivariant cohomology for differentiable stacks. For this we describe in detail some of the cohomology theories found in the literature and give some relations between them. As we want a notion of equivariant cohomology, we discuss the notion of an action on a stack by a Lie group G and how to define the quotient stack for this action. We find that this quotient stack is a differentiable stack and we describe its homotopy type. We provide a Cartan model for equivariant cohomology and we show that it coincides with the cohomology of the quotient stack previously defined. We prove that the G-equivariant cohomology can be expressed in terms of a T-equivariant cohomology for T a maximal torus of G and its Weyl group. Finally we construct some spectral sequences that relate the cohomology of the nerve associated to the Lie groupoid of the quotient stack with the previously described equivariant cohomology.

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Contents

Abbreviations

Cat = Category of (small) categories and functors

Top = Category of topological spaces and continuous maps

Diff = Category of smooth manifolds and smooth maps

Grpds = Category of groupoids and groupoid morphisms

LieGrpds = Category of Lie groupoids and Lie groupoids morphisms

St = 2-category of stacks

DiffSt = 2-category of differentiable stacks

G-**St** = 2-category of G-stacks

G-**DiffSt** = 2-category of differentiable G-stacks

To my family

Chapter 1

Introduction

This thesis is concerned with equivariant cohomology theories of differentiable stacks. Historically, the concept of stacks as algebraic stacks was introduced by Grothendieck in algebraic geometry for the study of moduli problems and non-abelian cohomology in algebraic geometry and it can be found in Giraud's work [25]. Later, Deligne and Mumford used algebraic stacks in [17] to study moduli spaces of algebraic curves and their irreducibility. Meanwhile in 1974, Artin in [3] introduced a generalisation of Deligne-Mumford stacks; which is now called an Artin stack to study moduli problems, quotient spaces and generalised global deformations. Recently, stacks were also introduced in other areas like algebraic topology, differential geometry and mathematical physics. For instance, in algebraic topology the notion of a topological stack was introduced by Noohi in [51] to study quotient spaces and cohomology theories. In differential geometry and mathematical physics the notion of a differentiable stack was introduced to study objects that are not smooth manifolds like for example orbifolds, which were independently introduced by Satake [54] and Thurston [59], classifying spaces of compact Lie groups or quotients of Lie group actions on smooth manifolds that are not necessarily free actions as in [5]. Some general approaches to the notion of a differentiable stack were given by Behrend in [5, 6] and Heinloth in [28]. In particular they initiated to study de Rham cohomology of diffferentiable stacks and Lie groupoids.

The notion of a stack is based on the language of 2-categories as we are here considering a stack as a pseudo functor from the opposite category of smooth manifolds to the category of groupoids, $\mathcal{M} : \mathbf{Diff}^{op} \to \mathbf{Grpds}$ (described for example in [28, 50]), with conditions for gluing objects and morphisms. If X is a smooth manifold, the stack associated to it will be given by the pseudo-functor $\mathrm{Hom}(\_, X)$, the set of all smooth maps with codomain in X. A stack $\mathcal{M}$ is then a differentiable stack if there exists a morphism of stacks $X \to \mathcal{M}$ between the stack associated to a smooth manifold and $\mathcal{M}$ which in addition is a surjective submersion, which means that we have the following commutative diagram

$$\begin{tikzcd} T \times_{\mathcal{M}} X \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & \mathcal{M} \end{tikzcd}$$

where $T \times_{\mathcal{M}} X$ is a differentiable manifold and $T \times_{\mathcal{M}} X \to T$ is a submersion for any morphism of stacks $T \to \mathcal{M}$, with T a smooth manifold. The main examples of differentiable stacks are the ones given by a smooth manifold X, that is the pseudo-functor $\mathrm{Hom}(\_, X)$, the classifying stack $\mathcal{B}G$ and the quotient stack $[X/G]$ associated to a Lie group action of G on a smooth manifold X such that for any smooth manifold T this gives a groupoid of pairs $\langle E \xrightarrow{p} T, E \xrightarrow{f} X \rangle$, where p is a principal G-bundle and f is a G-equivariant map. If X is a point, we recover the classifying stack $\mathcal{B}G$, which classifies principal G-bundles and plays a similar role then the classifying space in algebraic topology.

If we have a differentiable stack $\mathcal{M}$ with an atlas $X \to \mathcal{M}$, we can consider the smooth manifold $X \times_{\mathcal{M}} X$ and this helps us to get a Lie groupoid $(X \times_{\mathcal{M}} X \rightrightarrows X)$ associated to $\mathcal{M}$. Conversely, we can get for a Lie groupoid $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ a differentiable stack built by principal Γ-bundles. This relation between differentiable stacks and Lie groupoids gives an equivalence of categories if we keep in mind that two Lie groupoids define the same differentiable stack if and only if they are Morita equivalent. Now we can use the nerve of the associated Lie groupoid which is a simplicial smooth manifold to get different cohomologies for differentiable stacks as sheaf cohomology, cohomology of the associated Lie groupoid, de Rham cohomology and hypercohomology (see for these [4, 28]) and also a notion of Čech cohomology that uses coverings of simplicial smooth manifolds as in [20]. Dupont's work on de Rham cohomology for simplicial smooth manifold in [19] shows that the fat geometric realisation $\| X_\bullet \|$ gives in fact the homotopy type for a given differentiable stack with atlas $X \to \mathcal{M}$.

It is possible to define principal G-bundles over a differentiable stack $\mathcal{M}$ with atlas $X \to \mathcal{M}$ as a principal G-bundle over X. If we consider the classifying stack $\mathcal{B}G$, we get that the collection of morphism of stacks from a differentiable stack $\mathcal{M}$ into

$\mathcal{B}G$ is equivalent to the collection of all principal G-bundles on $\mathcal{M}$, that is

$$\mathrm{Maps}_{St}(\mathcal{M}, \mathcal{B}G) \cong \mathcal{B}un_G(\mathcal{M}).$$

The aim in this thesis is to develop a good notion of equivariant cohomology for differentiable stacks with an action of a Lie group G. For this we consider a compact Lie group G and we follow the classical point of view that consists of associating a Borel model and a Cartan model to this equivaraint situation. In order to pursue this we will first define an action of a Lie group G on a differentiable stack $\mathcal{M}$ as first described in general terms by Romagny in [53] and Ginot-Noohi in [24] in the algebraic geometric context. We then prove that the quotient stack $\mathcal{M}/G$ is in fact a differentiable stack and its cohomology coincides with the cohomology of $EG\times_G \parallel X_\bullet \parallel$; such that when $\mathcal{M}$ is the stack associated to a smooth manifold we recover the classical Borel model. For the Cartan model version of a differentiable G-stack we follow a construction given by Meinreken in [46] for simplicial smooth manifolds and we verify that actually its cohomology is equivalent to the cohomology of $EG\times_G \parallel X_\bullet \parallel$. Therefore, we conclude that both models coincide in cohomology like in the classical result on smooth manifolds as first proven by Cartan in [14].

We can also restrict the Lie group G to a closed subgroup and we prove that the G-equivariant cohomology can be expressed in terms of the equivariant cohomology of a maximal torus T of G and the Weyl group W of this torus as follows by the ring of invariants

$$H^*_G(X_\bullet, \mathbb{R}) \cong H^*_T(X_\bullet, \mathbb{R})^W.$$

This generalises a result by Guillemin and Sternberg [27], when we have a G-atlas $X \to \mathcal{M}$.

In the last part of this thesis we get some comparison results via spectral sequences for the equivariant cohomology of differentiable stacks that generalises previous results. For instance, the group cohomology of each level in the simplicial smooth manifold is related to the equivariant cohomology by a general spectral sequence

$$E_2^{p,r} = H^p(G, H^r(\mathcal{M}, \mathfrak{F})) \Rightarrow H^{p+r}(\mathcal{M}/G, \mathfrak{F}).$$

generalising a result by Felder et al., in [20]. In a similar way, we prove that there is a spectral sequence

$$E_2^{k,p} = \mathrm{Tot}_k \bigoplus_{q+n=k} H^p(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes S^t(\mathfrak{g})) \Rightarrow H_{dR}^{k+p}(\mathcal{M}/G)$$

which is a more general version of the spectral sequence

$$E_2^{t,p} = H^p(G; S^t(\mathfrak{g})) \Rightarrow H^{t+p}(BG, \mathbb{R})$$

first described by Bott in [9] and by Stasheff in [55], when $\mathcal{M}$ is a point. Related spectral sequences in the manifold case was also developed more recently by Abad-Uribe [1] and García-Compeán-Paniagua-Uribe [22]. These are also concerned with general actions of non-compact Lie groups on smooth manifolds using Getzler's model [23]. We aim to address also non-compact Lie group actions in the future.

As a consequence of the results of this thesis, it should be noted that the notion of equivariant cohomology and the two given models allow us to relate the quotient stack with its smooth structure given by the atlas with equivariant differential forms. It also gives us the versatility to calculate different properties such as those seen previously that occur in the classic case. Last but not least, the concepts and results of this thesis allow us to think about a possible classification of K-equivariant principal $\mathbb{T}$-bundles over a differentiable stack X via the equivariant cohomology group $H^1(EG \times_G X, \mathbb{T})$ similar as was discussed in the smooth manifold case by Brylinsky in [13]. This is currently another part of my research in progress.

Thesis outline

The content of this thesis is subdivided into three chapters after this introduction and is organised as follows:

Chapter 2 introduces basic concepts on categories, 2-categories and simplicial sets. It reviews some basic definitions in the category of smooth manifolds and smooth maps with special attention on principal G-bundles, simplicial smooth manifolds and the two classical models of equivariant cohomology, the Borel model and the Cartan model. It recalls the definition of groupoids and Lie groupoids with several examples and the definition of a principal Γ-bundle where Γ is a Lie groupoid. Finally it introduces the concept of spectral sequences and some important properties of them.

Chapter 3 is concerned with the general theory of differentiable stacks. We define what is a stack and we give a Yoneda lemma version for stacks. We define fibered products for stacks and using this we define the notion of an atlas for a stack. From this we get the notion of a differentiable stack. We provide the main examples of differentiable stacks as the ones associated to a smooth manifold and the classifying stack $\mathcal{B}G$ of a Lie group. For a Lie group G, we define then principal G-bundles and we show their relation with the classifying stack $\mathcal{B}G$. We discuss the relation between differentiable stacks and Lie groupoids and we prove that both categories are equivalent up to Morita equivalences. We give the definition of a sheaf on a differentiable stack and define sheaf cohomology. We use the nerve of the Lie groupoid associated to a differentiable stack to define the cohomology of the associated groupoid, the de Rham cohomology, singular homology, singular cohomology, hypercohomology and Čech cohomology. We check that these cohomologies are well defined, that is they are Morita invariant. Also we prove a de Rham theorem version for differentiable stacks. We show that hypercohomology and de Rham cohomology are equivalent. We get a similar result for de Rham cohomology and Čech cohomology under the condition of the existence of an acyclic covering.

Chapter 4 defines general Lie group actions on differentiable stacks and also defines quotient stacks for this kind of actions. We check that these quotient stacks are in fact differentiable stacks and we describe their homotopy types. Then we introduce the concept of equivariant cohomology for differentiable stacks with G-actions. We define a Cartan model for differentiable stacks and we show that this model has the same cohomology as the equivariant cohomology described before. We then restrict the group in the equivariant cohomology to a subgroup and we get a relation between the equivariant cohomology of the Lie group and of its maximal torus. Finally, we derive some general spectral sequences that relate the cohomology of the nerve of the Lie groupoid associated to the differentiable stack with the equivariant cohomology of the differentiable stack. We will analyse them in particular situations and discuss their homological properties.

Chapter 2

Preliminaries

We begin with the necessary preliminaries for our discussion and, with them, we will conceive the equivariant cohomology for differentiable stacks and their fundamental properties. In most of the cases the technicalities can be found in the references provided, and we will only give them when necessary.

2.1 Categories

Categories and 2-categories are going to unify the language in our discussion, since they provide a rigorous definition for stacks being examples of pseudo-functors between 2-categories. Hence, it is necessary to recall some aspects about them. For this, we follow [32, 40].

Definition 2.1.1. A *category* $\mathcal{C}$ consists of:

- a collection of objects $\mathrm{Ob}(\mathcal{C})$,
- a collection of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$ for two objects $x, y \in \mathrm{Ob}(\mathcal{C})$,

with the following properties:

1. For $x, y, z \in \mathrm{Ob}(\mathcal{C})$, there exists a function

$$\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$$

$$(g, f) \mapsto g \circ f$$

called a *composition*.

2. For each element $x \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $1_x \in \mathrm{Mor}_{\mathcal{C}}(x,x)$ called *identity* of x, such that for each $f \in \mathrm{Mor}_{\mathcal{C}}(x,y)$, $f \circ 1_x = 1_y \circ f$.

3. For each $f \in \mathrm{Mor}_{\mathcal{C}}(x,y)$, $g \in \mathrm{Mor}_{\mathcal{C}}(y,z)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(z,w)$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Remark 2.1.2. If the collections of objects and morphism are sets, we say that the category is a *small category*.

Example 2.1.3.

1. The category of sets and functions is denoted by **Sets**.

2. The category of topological spaces with morphism being continuous functions is denoted by **Top**.

3. The category of smooth manifolds with smooth maps as morphisms is denoted by **Diff**.

4. The category of groups with group homomorphisms, is denoted by **Gr**. If we ask for only abelian groups the category is going to be denoted by **Ab**.

Definition 2.1.4. For each category $\mathcal{C}$, there exists a category called *the opposite category*, $\mathcal{C}^{op}$, that consists of:

- objects of $\mathcal{C}^{op}$ the same objects as $\mathcal{C}$,

- morphisms of $\mathcal{C}^{op}$ are arrows f^{op} in one-one correspondence with arrows f of $\mathcal{C}$ such that if $f^{op} : x \to y$ then $f : y \to x$.

The composition in $\mathcal{C}^{op}$ is defined as $f^{op} \circ g^{op} = (g \circ f)^{op}$.

We can relate two categories in the following functorial way:

Definition 2.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,

- for each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(x,y)$, there is a morphism $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(x), F(y))$, such that:

1. $F(g \circ f) = F(g) \circ F(f)$.
2. $F(1_x) = 1_{F(x)}$ for any $x \in \mathrm{Ob}(C)$.

Definition 2.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- for each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$, there is a morphism $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(y), F(x))$,

such that:

1. $F(g \circ f) = F(f) \circ F(g)$.
2. $F(1_x) = 1_{F(x)}$ for any $x \in \mathrm{Ob}(C)$.

In our work it will be necessary to consider higher categories as well, so we recall some aspects of the theory of 2-categories, for this we follow [50, 2.1].

Definition 2.1.7. A 2-*category* consists of the following:

- a collection of objects, denoted by $\mathrm{ob}(\mathcal{C})$,
- a collection of 1-*morphisms* of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any two objects $X, Y \in \mathrm{ob}(\mathcal{C})$,
- a collection of 2-*morphisms* α of $\mathrm{Hom}_{\mathcal{C}}(f, g)$ between two 1-morphism $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. An element $\alpha \in \mathrm{Hom}_{\mathcal{C}}(f, g)$ will be illustrated as arrows in the following way:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

with the conditions

1. Objects and 1-morphisms form a category.
2. For fixed objects X and Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and the collection of 2-morphisms on it, form a category under the operation known as *vertical composition*:

$$\begin{tikzcd} X \arrow[r, bend left=40, "f"] \arrow[r] \arrow[r, bend right=40, "h"'] \arrow[r, phantom, shift left=2.5, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5, "\Downarrow\beta"] & Y \end{tikzcd}$$

This vertical composition is denoted by $\beta \cdot \alpha : f \Rightarrow h$ and its identities $1_f : f \Rightarrow f$.

3. There is a *horizontal composition*:

$$\begin{tikzcd} X \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left=40, "u"] \arrow[r, bend right=40, "v"'] \arrow[r, phantom, "\Downarrow\gamma"] & Z \end{tikzcd} \qquad = \qquad \begin{tikzcd} X \arrow[r, bend left=40, "uf"] \arrow[r, bend right=40, "vg"'] \arrow[r, phantom, "\Downarrow\gamma*\alpha"] & Z \end{tikzcd}$$

The horizontal composition is denoted by $\gamma * \alpha : uf \Rightarrow vg$. With this composition, the 2-morphisms form a category with identities:

$$\begin{tikzcd} X \arrow[r, bend left=40, "1_X"] \arrow[r, bend right=40, "1_X"'] \arrow[r, phantom, "\Downarrow 1_{1_X}"] & X \end{tikzcd}$$

4. If there is:

$$\begin{tikzcd} X \arrow[r, bend left=40, "f"] \arrow[r] \arrow[r, bend right=40, "h"'] \arrow[r, phantom, shift left=2.5, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5, "\Downarrow\beta"] & Y \arrow[r, bend left=40, "v"] \arrow[r] \arrow[r, bend right=40, "w"'] \arrow[r, phantom, shift left=2.5, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2.5, "\Downarrow\delta"] & Z \end{tikzcd}$$

then $(\delta * \beta) \cdot (\gamma * \alpha) = (\delta \cdot \gamma) * (\beta \cdot \alpha)$.

5. And if there is:

$$\begin{tikzcd} X \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & Y \arrow[r, bend left=40, "u"] \arrow[r, bend right=40, "u"'] \arrow[r, phantom, "\Downarrow 1_u"] & Z \end{tikzcd}$$

then $1_u * 1_f = 1_{uf}$.

Definition 2.1.8. Let X and Y be objects in a 2-category $\mathcal{C}$. They are equivalent if there exist two 1-morphisms $f : X \to Y$, $g : Y \to X$ and two 2-isomorphisms, that is, $\alpha : g \circ f \xrightarrow{\cong} id_X$ and $\beta : f \circ g \xrightarrow{\cong} id_Y$.

Definition 2.1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a *pseudo-functor* if the following data is given:

- For every object $X \in \mathcal{C}$ there is an object $\mathcal{F}(X) \in \mathcal{D}$.
- For each 1-morphism $f : X \to Y$ in $\mathcal{C}$ there is a 1-morphism $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ in $\mathcal{D}$.

- For each 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$ there is a 2-morphism $\mathcal{F}(\alpha) : \mathcal{F}(f) \Rightarrow \mathcal{F}(g)$ in $\mathcal{D}$.

such that:

1. $\mathcal{F}$ respects identity 1-morphisms, that is $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$,
2. $\mathcal{F}$ respects identity 2-morphisms, that is $\mathcal{F}(f) = id_{\mathcal{F}(f)}$,
3. $\mathcal{F}$ respects composition of 1-morphism up to 2-isomorphism, that is for any diagram of the form
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
there is a 2-isomorphism $\varepsilon_{g,f} : \mathcal{F}(g) \circ \mathcal{F}(f) \Rightarrow \mathcal{F}(g \circ f)$ such that:
   (a) $\varepsilon_{f,id_X} = \varepsilon_{id_Y,f} = id_{\mathcal{F}(f)}$.
   (b) ε is associative, that is
$$\begin{tikzcd} \mathcal{F}(h) \circ \mathcal{F}(g) \circ \mathcal{F}(f) \arrow[r, Rightarrow, "\varepsilon_{h,g} * id_{\mathcal{F}(f)}"] \arrow[d, Rightarrow, "id_{\mathcal{F}(h)} * \varepsilon_{g,f}"] & \mathcal{F}(h \circ g) \circ \mathcal{F}(f) \arrow[d, Rightarrow, "\varepsilon_{h \circ g, f}"] \\ \mathcal{F}(h) \circ \mathcal{F}(g \circ f) \arrow[r, Rightarrow, "\varepsilon_{h, g \circ f}"] & \mathcal{F}(h \circ g \circ f) \end{tikzcd}$$
4. $\mathcal{F}$ respects vertical composition of 2-morphisms, that is, for every pair of 2-morphisms $\alpha : f \to f'$ and $\beta : g \to g'$ then
$$\mathcal{F}(\beta \circ \alpha) = \mathcal{F}(\beta) \circ \mathcal{F}(\alpha),$$
5. $\mathcal{F}$ respects horizontal composition of 2-morphisms, that is, for every pair of 2-morphism $\alpha : f \to f'$ and $\beta : g \to g'$ the following diagram
$$\begin{tikzcd} \mathcal{F}(g) \circ \mathcal{F}(f) \arrow[r, Rightarrow, "\mathcal{F}(\beta) * \mathcal{F}(\alpha)"] \arrow[d, Rightarrow, "\varepsilon_{g,f}"] & \mathcal{F}(g') \circ \mathcal{F}(f') \arrow[d, Rightarrow, "\varepsilon_{g',f'}"] \\ \mathcal{F}(g \circ f) \arrow[r, Rightarrow, "\mathcal{F}(\beta * \alpha)"] & \mathcal{F}(g' \circ f'). \end{tikzcd}$$
commutes.

Definition 2.1.10. Consider two 2-categories $\mathcal{C}$ and $\mathcal{D}$. A 2-*functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a correspondence that takes objects to objects, 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms, such that all compositions and all identities are preserved.

The following notion of a site of a category will be important to define sheaf cohomology later.

Definition 2.1.11. If $\mathcal{C}$ is a category then $\mathcal{C}_J$ is a *site*, where J is a Grothendieck topology, and J is a *Grothendieck topology* if:

1. (Existence of fibered products). $X \to Y$, $Z \to Y$ in J then $X \times_Y Z \to X$.
2. (Stability under base change). For any $Y \to X$ and for every $\{X_\alpha \to X\}$ covering, $\{X_\alpha \times_X Y \to Y\}$ is a covering.
3. (Local character). $\{X_\alpha \to X\}$ in J and $\{X_{\beta\alpha} \to X_\alpha\}$ in J then $\{X_{\beta\alpha} \to X_\alpha \to X\}$ is in J.
4. (Isomorphism). If $Y \to X$ is an isomorphism, then $\{Y \to X\}$ is in J.

Example 2.1.12.

1. In **Top** a site can be considered with all the topological spaces with open coverings.
2. In **Diff** a site can be considered with all the smooth manifolds and local diffeomorphisms.

2.2 Simplicial sets

Some basic concepts of simplicial homotopy theory are relevant for the study of stacks and their cohomology. Most of this information can be found in [26, 43, 52].

Definition 2.2.1. A *simplicial set* is a graded set $K_\bullet$ such that K_q is a set for each $q = 0, 1, \ldots$ together with face maps $\partial_j : K_q \to K_{q-1}$, $0 \leq i \leq q$, and degeneracy maps $\sigma_i : K_q \to K_{q+1}$, $0 \leq i \leq q$, which satisfy the following identities:

1. $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i < j$,
2. $\sigma_i \sigma_j = \sigma_{j+1} \sigma_i$ if $i \leq j$,
3. $\partial_i \sigma_j = \sigma_{j-1} \partial_i$ if $i < j$,

   $\partial_j \sigma_j = id = \partial_{i+1} \sigma_j$,

$\partial_i \sigma_j = \sigma_j \partial_{i-1}$ if $i > j + 1$.

Remark 2.2.2. Consider Δ the category of finite ordinal numbers. That is, if $[n] \in \Delta$ we have that $[n] = (0 < 1 < \ldots < n)$ and morphisms given by $\theta : [m] \to [n]$ order-preserving set functions. We can simply say that a simplicial set is a contravariant functor $X_\bullet : \Delta^{op} \to \mathbf{Sets}$.

Definition 2.2.3. A *simplicial map* $f : K_\bullet \to L_\bullet$ is a map, such that for each degree q there is $f_q : K_q \to L_q$ and these functions commute with the face and degeneracy maps. That is,

$$f_q \partial_i = \partial_i f_{q+1}$$

$$f_q \sigma_i = \sigma_i f_{q-1}.$$

Example 2.2.4. Let $\Delta^n = \{(t_0, \ldots, t_n) \mid 0 \leq t_i \leq 1, \sum t_i = 1\} \subset \mathbb{R}^{n+1}$, the *standard n-simplex.* By a singular n-simplex of a topological space X we mean a continuous function $f : \Delta^n \to X$. We consider $S_n(X)$, namely the set of singular n-simplices of X, and the graded set $S(X)$ of all singular simplices is called the *total singular complex* of X. $S(X)$ is a simplicial set if we define face and degeneracy maps by:

$$(\partial^i f)(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and

$$(\sigma^i f)(t_0, \ldots, t_{n+1}) = f(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}).$$

Definition 2.2.5. For a simplicial set $X_\bullet$, its *geometric realisation* is the quotient space

$$|X_\bullet| = |p \mapsto X_p| = \bigcup_{p \in \mathbb{N}} \Delta^p \times X_p / \sim$$

with the identifications $(\partial^i t, x) \sim (t, \partial_i x)$ and $(\sigma^j t, x) \sim (t, \sigma_j x)$ for any $x \in X_p$, $t \in \Delta^{p-1}, i, j = 0, \ldots, n$ and p.

If we only use the identifications $(\partial^i t, x) \sim (t, \partial_i x)$, the resulting set is called *fat geometric realisation* and it is denoted by $\|X_\bullet\|$ or $\|p \mapsto X_p\|$.

Definition 2.2.6. A *bisimplicial set* $X_{\bullet,\bullet}$ is a simplicial object in the category of simplicial sets or as a functor given by:

$$X_{\bullet,\bullet} : \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}.$$

The set of $X_{m,n}$ has *bidegree* (m, n), where m is the horizontal degree and n the vertical degree.

Remark 2.2.7. In this way, a bisimplicial set has vertical and horizontal face maps, as vertical and horizontal degeneracy maps that commute with each other.

2.3 Smooth manifolds

In this section we give some definition and relevant results on the de Rham complex, Borel model and Cartan model for equivariant cohomology. A good overview on this subject can be found in [10, 12, 61].

2.3.1 Smooth maps

The category of smooth manifolds and smooth maps is denoted by **Diff**. We recall some important morphisms in this category.

Definition 2.3.1. A smooth map $f : M \to N$ is a *diffeomorphism* if there exists an inverse smooth map for it, and a smooth map $f' : M \to N$ is a *local diffeomorphism* if every $m \in M$ has an open set U such that $f'(U)$ is open in N and $f' \mid_U : U \to f'(U)$ is a diffeomorphism.

Definition 2.3.2. Let M be a smooth manifold and $\gamma : \mathbb{R} \to M$ a smooth curve with $\gamma(0) = p$ (γ need only be defined in a neighborhood of 0.) Let $f : U \to \mathbb{R}$ be smooth where U is an open neighborhood of p. Then the *directional derivative* of f along γ at p is

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t)) \mid_{t=0} .$$

The operator D_γ is called the *tangent vector* to γ at p. For two such curves γ and γ' we regard $D_\gamma = D_{\gamma'}$ if they have the same value at p on each such function f.

Definition 2.3.3. If M is a smooth manifold and $p \in M$, T_pM denotes the vector space of all tangent vectors to M at p.

Definition 2.3.4. If $f : M \to N$ is a smooth map between two smooth manifolds then we define the *differential* of f at $p \in M$ to be the function

$$f_* : T_pM \to T_{f(p)}N$$

given by $f_*(D_\gamma) = D_{f\circ\gamma}$.

Definition 2.3.5. A *vector field* on a smooth manifold M is a function ξ on M, such that $\xi(p) \in T_pM$ and which is smooth in the following sense: Given local coordinates $x_1, \ldots, x_n$ near $p \in M$, one can write

$$\xi(p); \sum_{i=1}^{n} a_i(p)\partial/\partial x_i$$

and smoothness of ξ means that the a_i are smooth functions.

For a smooth manifold M of dimension n put $TM = \bigsqcup_{p\in M} T_pM$. This is the set of all ordered pairs (p, ξ) where $\xi \in T_pM$. There is a projection $\pi : TM \to M$. Let $\phi : U \to U' \subset \mathbb{R}^n$ be a chart giving the local coordinates $x_1, \ldots, x_n$ near p.Then any tangent vector at a point of U is of the form $\sum_i a_i\partial/\partial_i$. Therefore $\pi^{-1}(U) \cong U \times \mathbb{R}^n \cong U \times \mathbb{R}^n$ and a specific map is

$$(\phi \circ \pi) \times \phi_* : \pi^{-1}(U) \to U' \times \mathbb{R}^n$$

taking $v \in T_pM$ to $(\phi(\pi(v)), \phi_*(v)) = (\phi(p), \phi_*(v))$. We can take this as a chart to TM. If $\psi : V \to \mathbb{R}^n$ is another chart on M so that $\theta = \psi\phi^{-1} : \phi(U\cap V) \to \psi(U\cap V)$ is the transition function, then the corresponding transition function on TM is $\theta \times \theta_*$. This makes TM into a smooth $2n$-manifold, called the *tangent bundle* of M. A *vector field* ξ in M is then just a smooth section of this bundle. That is, it is a smooth map $\xi : M \to TM$ such that $\pi \circ \xi = Id_M$.

Definition 2.3.6. A smooth map $f : M \to N$ is called a *submersion* if $f_* : T_mM \to T_{f(m)}N$ is surjective for each point $m \in M$. If f_* is injective, f is called an *immersion.*

Example 2.3.7.

1. If $M_1, \ldots, M_n$ are smooth manifolds, each projection $\pi_i : M_1 \times \cdots \times M_n \to M_i$ is a submersion.
2. If f is a local diffeomorphism then f is a submersion and an immersion. Thus smooth covering maps are submersions and immersions.

We recall some useful properties of a submersion map, since these properties are a first approach for the concept of an atlas of a differentiable stack. These are the following:

Proposition 2.3.8. *[38, 7.16]*

Let $f : M \to N$ be a submersion. Then the following holds:

1. *f is an open map.*

2. *Every point of M is in the image of a local section of f.*

3. *If f is surjective then it is a quotient map.*

4. *For any smooth map $g : P \to N$ the pullback space $g^*(P) = \{(p, m) \in P \times M \mid g(p) = f(m)\}$ induced by g is a smooth manifold.*

Remark 2.3.9. The last property follows because a submersion is transversal to every smooth map. For further details see [12, 15.2].

2.3.2 De Rham complex

The de Rham complex of a manifold allows us to get information about the topology and geometry of a smooth manifold via differentiable forms. For example, we can compute its cohomology directly using the calculus of differentiable forms. In fact, singular cohomology (with real coefficients) and de Rham cohomology are the same via the classical de Rham theorem (see for example [12, V.9.1]). For this section we recall some results on the de Rham complex and we follow [10].

For a complete construction of a differential p-form and the exterior derivative on smooth manifolds, we follow [10, 12, 38, 61]. With these concepts, we get that:

Let M be a smooth manifold.

Definition 2.3.10. A *differential p-form* ω on M is a differentiable function which assigns to each point $x \in M$, an element in $\omega_x \in A^p(T_xM)$ where $A^p(T_xM)$ is the vector space of all alternating multilinear p-forms on T_xM.

Remark 2.3.11. We denote the vector space of all differentiable p-forms on M by $\Omega^p(M)$. Note that $\Omega^0(M)$ is the space of all real valued smooth functions on M.

Definition 2.3.12. If $f : M \to N$ is smooth then the *pullback map* $f^* : \Omega^p(N) \to \Omega^p(M)$ is defined by

$$f^*(\omega)(X_{1_x}, \ldots, X_{p_x}) = \omega(f_*(X_{1_x}), \ldots, f_*(X_{p_x}))$$

where $X_{1_x}, \ldots, X_{p_x}$ are tangent vectors to M at x. On functions $f^*(g) = g \circ f$.

Definition 2.3.13. The *exterior derivative* on a smooth manifold M of dimension n is an operator $d_{dR} : \Omega^p(M) \to \Omega^{p+1}(M)$ defined as

1. If $f \in \Omega^0(M)$ then it is locally defined as

$$d_{dR} f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(M).$$

2. If $\omega \in \Omega^p(M)$ with a local form given by $\omega = \sum f_I dx_I$ then

$$d_{dR}\omega = \sum df_I \wedge dx_I \in \Omega^{p+1}(M).$$

Proposition 2.3.14. *[12, V.2.2] For the exterior derivative one has that*

$$d_{dR} \circ d_{dR} = 0.$$

Definition 2.3.15. Let M be a smooth manifold of dimension n. The *de Rham complex* consists of the collection $\Omega^*(M)$ of differential forms on M and can be graded as

$$\Omega^*(M) = \bigoplus_{p=0}^{n} \Omega^p(M)$$

where $\Omega^p(M)$ is the collection of differential p-forms with the exterior derivative d_{dR} on smooth manifolds such that

$$d_{dR} : \Omega^p(M) \to \Omega^{p+1}(M)$$

for each p.

We recall some functorial properties of the de Rham complex.

Proposition 2.3.16. *[61, 2.23] Let $f : M \to N$ be a smooth map. The pullback map $f^* : \Omega^q(N) \to \Omega^q(M)$ commutes with the differential operator d_{dR}, for each q and Ω^* is a contravariant functor over* **Diff***.*

Based on [61, Chapter 2], we recall the following useful definitions and properties.

Definition 2.3.17. [61, 2.21] The *interior multiplication* with $X \in TM$, where TM is the tangent space of M, is an operator $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ whose value at $x \in M$ is given by

$$(\iota_X \omega(Y_2, \ldots, Y_k)) \mid_x = \iota_{X|_x} \omega_x(Y_2 \mid_x, \ldots, Y_k \mid_x) = \omega_x(X \mid_x, Y_2 \mid_x, \ldots, Y_k \mid_x).$$

Proposition 2.3.18. *For each $X, Y \in TM$, $\iota_X \circ \iota_Y = -\iota_Y \circ \iota_X$. Hence $\iota_X \circ \iota_X = 0$.*

Also, it is important to know the relation between the interior multiplication and the Lie derivative.

Definition 2.3.19. The *Lie derivative* is defined as

$$\mathcal{L}_X \omega = \frac{d}{dt}\bigg|_{t=0} \varphi_t^* \omega$$

where $X \in TM$, and $\varphi : U \subset \mathbb{R} \times M \to M$ is the flow of the vector field X, that is, $\varphi'(t,x) = X(\varphi(t,x))$ which is uniquely defined on an open neighborhood of $0 \in \mathbb{R}$ for any $x \in M$, see [36, IV.1].

Some of the Lie derivative properties are useful for us later and given by the next proposition.

Proposition 2.3.20. *[61, 2.25] Let $X \in TM$ then*

1. $\mathcal{L}_X(f) = df(X)$, *for* $f \in C^\infty(M)$.
2. $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$.

2.3.3 Lie groups and Lie algebras

As we will be concerned with actions of Lie groups on differentiable stacks, it is convenient to present some properties related to Lie groups and Lie algebras. More details about this discussion can be found in [61].

Definition 2.3.21. A *Lie group* is a smooth manifold G, which has the structure of a group and the map

$$G \times G \to G$$

$$(g,h) \mapsto g \cdot h^{-1}$$

is a smooth map.

Definition 2.3.22. A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space with a bilinear function $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

1. $[x,y] = -[y,x]$, anti-commutativity.

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$, Jacobi identity.

The important relation between Lie group and Lie algebras is based on the fact that we can assign to each Lie group G a Lie algebra $\mathfrak{g}$.

Definition 2.3.23. The Lie algebra $\mathfrak{g}$ associated to a Lie group G is defined as the Lie algebra of left invariant vector fields of G.

Remark 2.3.24. The Lie algebra associated to a Lie group can be considered as the tangent space at the identity of the Lie group. See [61, 3.8].

Definition 2.3.25. A *Lie group action* on a smooth manifold M is an action of a group G on M such that $\mu : G \times M \to M$ is a smooth map. The action is *free* if for all $m \in M$, $gm = m$ implies $g = e$, where e is identity in the group G.

Remark 2.3.26. We usually write $\mu(g, m)$ as $g \cdot m$ when the context is clear.

Definition 2.3.27. Let $\mu : G \times M \to M$ and $\nu : G \times N \to N$ be two Lie group actions on the smooth manifolds M and N, respectively. A map $f : M \to N$ is called an *G-equivariant map* if $f(\mu(g, m)) = \nu(g, f(m))$.

Let us also consider the next result concerned with the case of simply connected Lie groups.

Theorem 2.1. *[29, II.3.1.5][33, I.4] Let G be a Lie group. For every $X \in \mathfrak{g}$ there is a unique smooth homomorphism $exp_X : \mathbb{R} \to G$ such that $d(exp_X)(0) = X$.*

So we can define

Definition 2.3.28. The *exponential map* is defined as

$$\exp : \mathfrak{g} \to G$$

$$X \mapsto \exp(X) = \exp_X(1).$$

It is good to remember that

Theorem 2.2. *[61, 3.32] If $\varphi : H \to G$ is a homomorphism, then the following diagram*

$$\begin{tikzcd} H \arrow[r, "\varphi"] & G \\ \mathfrak{h} \arrow[u, "\exp"] \arrow[r, "d\varphi"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes.

Consider the action of G on itself, which is given by conjugation, that is $\mu : G \times G \to G$ and defined by $\mu(g,h) = ghg^{-1}$. Then the identity is a fixed point of this action. We recall

Definition 2.3.29. The *adjoint representation* is given by

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

$$g \mapsto d\mu_e \Big|_{T_eG \cong \mathfrak{g}}$$

and the *adjoint action* is given by

$$G \times \mathfrak{g} \to \mathfrak{g}$$

$$(g, X) \mapsto \mathrm{Ad}_g(X)$$

Definition 2.3.30. Let G be a Lie group and M a smooth manifold. Suppose that G acts on $m \in M$ and $X \in \mathfrak{g}$. The *fundamental vector field* of X on M is defined as

$$X^\sharp(m) = \frac{d}{dt}\Big|_{t=0} (\exp(tX) \cdot m) \in T_m M.$$

2.3.4 Principal G-bundles

The category of principal G-bundles provides some of the main examples of differentiable stacks (see example 3.1.4 and definition 3.1.11), and the existence of universal principal G-bundles and the classifying space is important in the construction of the Borel model for equivariant cohomology and also in the general theory of characteristic cohomology classes. For this, we mainly follow [31] and [56].

Definition 2.3.31. Let E be a smooth manifold with a Lie group action $\mu : G \times E \to E$. Consider M a smooth manifold and a smooth map $\pi : E \to M$. (E, π, G) is a *principal G-bundle* over M if:

1. There is an open covering $\{U_i \mid i \in I\}$ of M and G-equivariant homeomorphisms $\phi_{U_i} : G \times U_i \to \pi^{-1}U_i$ such that the diagram

$$
\begin{tikzcd}
\pi^{-1}U_i \arrow[d] & U_i \times G \arrow[l, "\phi_U"'] \arrow[dl] \\
U_i &
\end{tikzcd}
$$

commutes.

2. $\pi(g \cdot f) = \pi(f)$ for $g \in G$ and $f \in E$.

3. If $f, f' \in \pi^{-1}(m)$ then there exists a unique $g \in G$ such that $f \cdot m = f'$.

The pair (U_i, ϕ_i) is called a *local trivialisation*. The collection of all local trivialisation $\{(U_i, \phi_i)\}$ on M is called an *atlas* of (E, π, G).

For any atlas (U_i, ϕ_i) of a principal G-bundle (E, π, G) over M, we restrict ϕ_i and ϕ_j to $U_i \cap U_j$ and we get a unique map $g_{i,j} : U_i \cap U_j \to G$, the so-called *transition function*, such that $\phi_j(b, y) = \phi_i(b, g_{i,j}(b)y)$ for $(b, y) \in U_i \cap U_j \times M$. The functions $g_{i,j}$ on $U_i \cap U_j$ have the following properties:

- For each $b \in U_i \cap U_j \cap U_k$ we have the relation $g_{i,k}(b) = g_{i,j}(b)g_{j,k}(b)$.
- For each $b \in U_i$ we have $g_{i,i} = id_G$.
- For each $b \in U_i \cap U_j$ we have $g_{i,j}(b) = g_{j,i}(b)^{-1}$.

Definition 2.3.32. Two systems of transition functions $\{g_{i,j}\}$ and $\{g'_{i,j}\}$ relative to the same open covering (U_i, ϕ_i) of a smooth manifold M are equivalent if there exist maps $\tau_i : U_i \to G$ satisfying the relation $g'_{i,j}(b) = \tau_i(b)^{-1}g_{i,j}(b)\tau_j(b)$.

Theorem 2.3. *[31, 5.2.7] Let (E, π, G) and (E', π', G) be two principal G-bundles over M. If there is an atlas (U_i, ϕ_i) for π with transition functions $\{g_{i,j}\}$ and an atlas (U_i, ϕ'_i) for π' with transition functions $\{g'_{i,j}\}$, then π and π' are isomorphic over M if and only if $\{g_{i,j}\}$ and $\{g'_{i,j}\}$ are equivalent systems of transition functions.*

Definition 2.3.33. Consider (E, π, G) and (E', π', G) two principal G-bundles over M and M', respectively. A *morphism of principal G-bundles* is a pair (u, f) of two smooth maps, where $u : E \to E'$ is an equivariant map and $f : M \to M'$ is a smooth map such that the diagram

$$
\begin{tikzcd}
E \arrow[r, "u"] \arrow[d] & E' \arrow[d] \\
M \arrow[r, "f"] & M'
\end{tikzcd}
$$

commutes.

Remark 2.3.34. The category of principal G-bundles over M will be denoted by $\mathcal{B}G(M)$.

Example 2.3.35. If we consider the action $\mu : G \times G \times M \to G \times M$ given by $\mu(g, g', m) = (g \cdot g', m)$, where M is a smooth manifold and G is a Lie group, then the second projection $\pi_2 : G \times M \to M$ forms a principal G-bundle. This principal G-bundle is called the *product principal G-bundle*.

Example 2.3.36. Let H be closed subgroup of a Lie group G. If we consider the action $\mu : H \times G \to G$ such that $\mu(h, g) = hg$, then the smooth map $\pi : G \to G/H$ forms a principal H-bundle (G, π, H) over G/H.

The next example is known as the *quotient manifold theorem*. To see more details follow [31, 4.4.1] and [38, 9.16].

Example 2.3.37. Let M be a smooth manifold and G a Lie group that acts on M freely and properly. We get that M/G is a smooth manifold and the map $p : M \to M/G$ is a submersion and (M, p, G) is a principal G-bundle over M/G.

Definition 2.3.38. A principal G-bundle (E, π, G) over M is a *trivial principal G-bundle* if it is isomorphic to the product principal G-bundle.

Definition 2.3.39. Let (E, π, G) be a principal G-bundle over M. A *local section* of the principal bundle π is a smooth map $s : U \to E$, where U is an open set of M such that $\pi \circ s = id_U$. If $U = M$, the section is *global*.

Proposition 2.3.40. *[31, 4.8.3] A principal G-bundle (E, π, G) is trivial if and only if it admits a global section.*

Theorem 2.4. *[31, 4.3.2] Every morphism in $\mathcal{B}G(M)$ is an isomorphism.*

Example 2.3.41. Let (E, π, G) be a principal G-bundle over M such that π is a submersion and $f : N \to M$ a smooth map, then we can get a principal G-bundle called *pullback of f* given by $f^* : f^*(E) \to N$ where $f^*(E) = \{(e, n) \in E \times N \mid \pi(e) = f(n)\}$ and $f^*(e, n) = n$. Since π is a submersion for transversality, $f^*(E)$ is a smooth manifold and π^* is a smooth map. G is acting on $f^*(E)$ by the induced map of the action of G in the first component. Notice that we have that the diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\pi_2} & E \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

commutes.

Proposition 2.3.42. *[31, 2.5.5. & 4.4.2] Consider a principal G-bundle (E, π, G) over M, a smooth map $f : N \to M$ and the canonical morphism of principal G-bundles (π_2, f) between the pullback f^* and π. If there is a principal G-bundle (E', π', G) over N with a morphism of principal G-bundle from π' to π, then there exists a morphism of principal G-bundle (σ, id_N) from π' to f^* such that $f^*\sigma = \pi'$ and the principal G-bundle given by π' and f^* are isomorphic. Finally $f^* : \mathcal{B}G(M) \to \mathcal{B}G(N)$ is a functor.*

Definition 2.3.43. A principal G-bundle (E, π, G) over B is *numerable* if there is a numerable cover $\{U_i\}_{i\in I}$ of B such that $\pi\mid_U : p^{-1}(U) \to U$ is trivial for each $i \in I$.

Definition 2.3.44. Let (E, π, G) be a numerable principal G-bundle over B. π is a *universal principal G-bundle* if for each principal G-bundle (E, p, G) over M there exists a unique up to homotopy continuous map $f : M \to B$ such that p is isomorphic to the pullback principal G-bundle induced by f.

Remark 2.3.45. Universal principal G-bundles can be characterized as a principal G-bundle (E, π, G) over B, where G acts freely on E and E is contractible. See [44, 23.8].

We recall that a Hausdorff space B is paracompact if and only if each open covering is numerable. As every smooth manifold is a Hausdorff space and paracompact, we have that any principal G-bundle over a smooth manifold is numerable, so we have the next result by Milnor [48].

Theorem 2.5. *(Milnor)[48, Section 3] Let G be a Lie group then there exists a universal principal G-bundle (EG, p, G) over BG, where G is acting freely on EG.*

Remark 2.3.46.

1. BG is called a *classifying space*, since each principal G-bundle over M is isomorphic to a pullback principal G-bundle of a map from M to BG up to homotopy.

2. There are several models for EG and BG. The most relevant model for our current discussion makes use of simplicial smooth manifolds, it is going to be stated later in the example 2.4.20 and also can be found in [19, Ch.5 Example 3].

2.3.5 Simplicial smooth manifolds

Due to the interplay between simplicial smooth manifolds, Lie groupoids and differentiable stacks, as we will explore in sections 3.3 and 3.4, it will be necessary to recall here some properties of the homotopy theory of simplicial smooth manifolds. For this, we follow [18] and [19].

Definition 2.3.47. A *simplicial smooth manifold* is a simplicial set $X_\bullet$, where every X_q is a smooth manifold, and all face and degeneracy maps are smooth maps. A *simplicial smooth map* between simplicial smooth manifolds is a simplicial map such that f_q is a smooth map for each q.

Remark 2.3.48. We can see a *simplicial smooth manifold* as a functor

$$X_\bullet : \Delta^{op} \to \mathbf{Diff}$$

Example 2.3.49. Let G a Lie group and $\mu : G \times M \to M$ a Lie group action. We consider the set of smooth manifolds given by $\{G^p \times M\}_{p \geq 0}$ and $G^p = G \times \cdots \times G$ the Cartesian product of p copies of G. We take the face maps as

$$\partial_0(g_1, \dots, g_p, x) = (g_2, \dots, g_p, x)$$

$$\partial_i(g_1, \dots, g_p, x) = (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_p, x) \text{ for } 1 \leq i \leq p-1$$

$$\partial_p(g_1, \dots, g_p, x) = (g_1, \dots, g_{p-1}, g_p \cdot x)$$

and degeneracy maps as

$$\sigma^i(g_1, \dots, g_p, x) = (g_1, \dots, g_i, e, g_{i+1}, \dots g_p, x)$$

so they are smooth maps and satisfy the conditions that make $\{G^p \times M\}_{p \geq 0}$ a simplicial manifold. An important property for us occurs with the face maps with domain $G \times M$, because ∂_0 is the projection to the second component and ∂_1 is the action μ.

In the same way, as in simplicial sets we can get the geometric realisation and the fat geometric realisation for simplicial smooth manifolds. These geometric realisations are quotient spaces.

Example 2.3.50. The fat geometric realisation of the simplicial smooth manifold $\{G^p \times M\}$ is a model for $EG \times M/G$. If M is a single point, we have that the fat geometric realisation is a model for BG and in the case $M = G$ we get a model for EG. Further details can be found in [19, Ch.5 Example 3].

There exists an analogue version for group actions on simplicial smooth manifold.

Definition 2.3.51. Let G be a Lie group and $X_\bullet$ a smooth simplicial smooth manifold. A simplicial map

$$\mu_\bullet : G \times X_\bullet \to X_\bullet$$

is a *smooth action* of G on $X_\bullet$, if $\mu_n : G \times X_n \to X_n$ is an action for the smooth manifold X_n.

For a simplicial smooth manifold we have a notion of covering as well. Compare with [20, A.2].

Definition 2.3.52. A *covering* $\mathcal{V}$ of a simplicial smooth manifold $X_\bullet$ is a collection of open coverings $\mathcal{V}_n = \{V_{n,\alpha}\}_{\alpha\in I}$ for every smooth manifold X_n such that it is compatible with face maps, that is, if $x \in V_{n,\alpha}$ then $\partial_i(x) \in V_{n-1,d_i(\alpha)}$ for $0 \leq i \leq n$, where $d_i : I \to I$ with the same compatible conditions as face maps on a simplicial set.

For simplicial smooth manifolds, we can associate a de Rham complex in the following way.

Definition 2.3.53. Let $X_\bullet$ be a simplicial smooth manifold. The *de Rham complex for a simplicial manifold* is the complex given by

$$C^k = \bigoplus_{p+n=k} \Omega^p(X_n)$$

with differential operator $D = d_{dR} + (-1)^p\partial$, where d_{dR} is the exterior derivative and $\partial = \sum_i (-1)^i \partial_i^*$ the alternate sum of pullback of face maps. The cohomology defined by this complex is the *simplicial de Rham cohomology* of $X_\bullet$, and it is denoted by $H^*_{dR}(X_\bullet)$.

The relation between simplicial de Rham cohomology and singular cohomology is given as:

Theorem 2.6. *[18, 2.8] There exists an isomorphism such that*

$$H^*_{dR}(X_\bullet) \cong H^*(\|X_\bullet\|, \mathbb{R}).$$

Definition 2.3.54. A *bisimplicial smooth manifold* is a functor

$$X_{\bullet,\bullet} : \Delta^{op} \times \Delta^{op} \to \mathbf{Diff}.$$

Example 2.3.55. Let G be a Lie group acting on a simplicial smooth manifold $X_\bullet$. Consider the set $\{G^p \times X^n\}$. This can be considered as a bisimplicial smooth manifold with the followings face horizontal maps

$$\partial_0^H(g_1, \ldots, g_p, x) = (g_2, \ldots, g_n, x)$$

$$\partial_i^H(g_1, \ldots, g_p, x) = (g_1, \ldots g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n, x) \text{ for } 1 \leq i \leq p$$

$$\partial_p^H(g_1, \ldots, g_p, x) = (g_1, \ldots, g_{n-1}, g_n x)$$

and degeneracy horizontal maps

$$\sigma^i_H(g_1, \ldots, g_p, x) = (g_1, \ldots, g_i, e, g_{i+1}, \ldots g_p, x)$$

Meanwhile, face and degeneracy vertical maps are the induced maps by the face and degeneracy maps of $X_{\bullet,\bullet}$.

Definition 2.3.56. Let $X_{\bullet,\bullet}$ be a bisimplicial smooth manifold. Its *fat geometric realisation* is the quotient space

$$\| X_{\bullet,\bullet} \| = \coprod_{m,n\geq 0} (\Delta^m \times \Delta^n \times X_{m,n})/\sim$$

where the equivalence relation is given by

$$(\partial^i \times id \times id)(t, s, x) \sim (id \times id \times \partial_i)(t, s, x)$$

for any $(t, s, x) \in \Delta^{m-1} \times \Delta^n \times X_{m,n}$ and

$$(id \times \partial^i \times id)(t, s, x) \sim (id \times id \times \partial'_i)(t, s, x)$$

for any $(t, s, x) \in \Delta^m \times \Delta^{n-1} \times X_{m,n}$.

In [52] it is shown that the geometric realisation for a bisimplicial smooth manifold can be calculated, considering the diagonal geometric realisation or working with any horizontal or vertical degree first. The next result is going to be fundamental for our current discussion, since this allows us to relate the fat geometric realisation of a simplicial smooth manifold with the one of a bisimplicial smooth manifold. Thus we can use bisimplicial smooth manifolds to get results for the homotopy theory of simplicial smooth manifolds.

Proposition 2.3.57. *[52, Lemma p.86] There are homeomorphisms*

$$\| X_{\bullet,\bullet} \| \cong \| p \mapsto X_{p,p} \| \cong \| p \mapsto \| q \mapsto X_{p,q} \| \| \cong \| q \mapsto \| p \mapsto X_{p,q} \| \| .$$

Example 2.3.58. If we consider the bisimplicial smooth manifold $\{G^\bullet \times X_\bullet\}$ defined in the example 2.3.55, we get that its fat geometric realisation is given by

$$\| p \mapsto \| n \mapsto G^p \times X_n \| \| \cong \| p \mapsto G^p \times \| X_\bullet \| \| \cong EG \times \| X_\bullet \| /G.$$

In the same way, as for simplicial smooth manifolds, we can define a de Rham complex for a bisimplicial smooth manifold.

Definition 2.3.59. Let $X_{\bullet,\bullet}$ be a bisimplicial smooth manifold. The *de Rham complex* $C^\bullet$ is defined as

$$C^k = \bigoplus_{m+n+p=k} \Omega^p(X_{m,n})$$

with differential operator $D = d_{dR} + \partial + \partial'$, where d_{dR} is the exterior derivative, ∂ is the alternate sum of horizontal face maps, and ∂' is the alternate sum of vertical face maps. This cohomology is called *de Rham cohomology* for the bisimplicial smooth manifold $X_{\bullet,\bullet}$, and denoted by $H^*_{dR}(X_{\bullet,\bullet})$.

In addition, we have a similar result as for simplicial smooth manifolds.

Theorem 2.7. *[21, p.40] There exists an isomorphism such that*

$$H^*_{dR}(X_{\bullet,\bullet}) \cong H^*(\|X_{\bullet,\bullet}\|, \mathbb{R}).$$

2.3.6 Equivariant cohomology

The idea of equivariant cohomology is important when looking for a good notion of cohomology for the quotient space M/G, when G is a Lie group acting on a smooth manifold M. We know that this quotient is not always a smooth manifold and the cohomology might not be well-defined. For our current discussion we recall two models. The first one is the Borel model, built from topological properties, and the second is the Cartan model, built from geometric properties. We follow here [27, Ch.1],[34], [35], [7, Ch.7], [39] and [23].

Let G be a compact Lie group and M a smooth manifold with a smooth action

$$\mu : G \times M \to M.$$

The idea of Borel model is trying to get a space E with the following two conditions:

- E is contractible.
- There is a free action $\nu : G \times E \to E$.

So, we can consider the action:

$$\theta : G \times E \times M \to E \times M$$

$$(g, f, m) \mapsto \theta(g, f, m) = (\nu(g, f), \mu(g, m))$$

which is a free action, because if $(f, m) = \theta(g, f, m) = (\nu(g, f), \mu(g, m))$ this implies that $g = e$ is the identity in G. Thus, we have the smooth manifold $E \times M/G$ and we denote it also as $E \times_G M$.

Proposition 2.3.60. *Let G be a Lie group acting freely on a smooth manifold M. If there are two contractible spaces E, E' with free actions by G, then*

$$H^*(E \times_G M, \mathbb{R}) = H^*(E' \times_G M, \mathbb{R})$$

Proof. Since E and E' are contractible, we can consider the universal principal G-bundles (E, p, G) over E/G and (E', q, G) over E'/G. Thus we get G-equivariant isomorphisms $\phi : E \to E'$ and $\psi : E' \to E$ such that $\phi \circ \psi \simeq id_{E'}$ and $\psi \circ \phi \simeq id_E$, that is, they are homotopic between each other. As cohomology is invariant under homotopy equivalence, we set the result. $\square$

The Borel model for equivariant cohomology is independent of the choice of E. In the subsection 2.3.4, we get the space EG, which is related with the universal G-bundle $p : EG \to BG$, and this space has the properties required for the Borel model. So we say that

Definition 2.3.61. The *equivariant cohomology* of a smooth manifold is defined as

$$H_G^*(M) = H^*(E \times_G M, \mathbb{R}).$$

This model is called the *Borel model.*

Example 2.3.62. Let G be a Lie group with a free smooth action on the smooth manifold M. As EG is contractible, we get

$$H_G^*(M) = H^*(M/G).$$

The next model is the Cartan model and is formed by equivariant forms, see [14], [34, 2.4]. For this, let G be a compact Lie group and a smooth action on M given by $\mu : G \times M \to M$.

Definition 2.3.63. An *equivariant form* of a smooth manifold M acted on by the compact Lie group G is a polynomial function $\alpha : \mathfrak{g} \to \Omega^*(M)$ such that

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\alpha"] \arrow[d, "\mathrm{Ad}_g"] & \Omega^*(M) \arrow[d, "g"] \\ \mathfrak{g} \arrow[r, "\alpha"] & \Omega^*(M) \end{tikzcd}$$

that is, $\alpha(\mathrm{Ad}_g X) = g\alpha(X)$. Here $\mathfrak{g}$ is the Lie algebra of G and $g\alpha$ is given by the pullback in differential forms of the smooth map $\mu_g : M \to M$ such that $\mu_g(m) = g \cdot m$. The set of equivariant forms is denoted as $\mathrm{Map}(\mathfrak{g}, \Omega^*(M))$.

Remark 2.3.64. α is a $\Omega^*(M)$-valued polynomial function from $\mathfrak{g}$ to $\Omega^*(M)$. This construction comes from the Weyl algebras as it is done in [27, 3], [47, 5].

We can induce an action on $\mathrm{Map}(\mathfrak{g}, \Omega^*(M))$ in the following way

$$\nu : G \times \mathrm{Map}(\mathfrak{g}, \Omega^*(M)) \to \mathrm{Map}(\mathfrak{g}, \Omega^*(M))$$

$$(g, \alpha)(X) \mapsto \nu(g, \alpha)(X) = g\alpha(\mathrm{Ad}_{g^{-1}} X)$$

for any $X \in \mathfrak{g}$.

Proposition 2.3.65. *Consider the action*

$$\nu : G \times Map(\mathfrak{g}, \Omega^*(M)) \to Map(\mathfrak{g}, \Omega^*(M)).$$

α is an equivariant form if and only if α is invariant for the action ν.

Proof. We suppose that α is an equivariant form then $\nu(g,\alpha)(X) = g\alpha(\mathrm{Ad}_{g^{-1}}X) = g(g^{-1})\alpha(X) = \alpha(X)$, then α is invariant.

If we have that α is invariant then $\nu(g^{-1},\alpha)(X) = g^{-1}\alpha(\mathrm{Ad}_g X) = \alpha(X)$, that is, $\alpha(\mathrm{Ad}_g X) = g\alpha(X)$. Then α is equivariant. □

Now we consider the morphism

$$d_G : \mathrm{Map}(\mathfrak{g}, \Omega^*(M)) \to \mathrm{Map}(\mathfrak{g}, \Omega^*(M))$$

given by $d_G(\alpha)(X) = d_{dR}(\alpha(X)) - \iota_{X^\#}\alpha(X)$, where d_{dR} is the exterior derivative of differential forms, and $\iota_{X^\#}$ is the interior multiplication by the fundamental vector field $X^\#$.

Proposition 2.3.66. *The morphism d_G is well defined and $d_G^2 = 0$*

Proof. To see that d_G is well defined, we need to check that $d_G(\alpha(X))$ is equivariant form. Firstly, it is a polynomial function since d_{dR} is only applied in the elements of $\Omega^*(M)$ and $i_X^\#$ is applied as well in the elements of $\Omega^*(M)$ but in this case we get elements with a one degree least in $\Omega^*(M)$ but with one degree plus as polynomial function, compared with the one at the beginning.

Secondly, to check that is equivariant we take $g \in G$,

$$d_G\alpha(\mathrm{Ad}_g X) = d_{dR}(\alpha(\mathrm{Ad}_g X)) - \iota_{(\mathrm{Ad}_g X)^\#}\alpha(\mathrm{Ad}_g X)$$

$$= d_{dR}(g\alpha(X)) - g\iota_{X^\#}g^{-1}g\alpha(X) = gd_{dR}(\alpha) - g\iota_{X^\#}\alpha(X)$$

as we want.

On the other hand,

$$d_G^2(\alpha)(X) = d_{dR}^2(\alpha)(X) - (d_{dR}\iota_{X^\#}(\alpha)(X) + \iota_{X^\#}d_{dR}(\alpha)(X)) + \iota_{X^\#}^2(\alpha)(X)$$

$$= \mathcal{L}_X\alpha$$

since $d_{dR}^2 = 0$, $\iota_{X\#}^2 = 0$ and the Lie derivative $\mathcal{L}_X\alpha = d_{dR}\iota_{X\#} + \iota_{X\#}d_{dR}$. As $\alpha(X)$ is invariant we have that $\mathcal{L}_X\alpha = 0$. Therefore $d_G^2 = 0$. $\square$

As an equivariant form $\alpha : \mathfrak{g} \to \Omega^*(M)$ is a polynomial, we can consider it as an element in $(S^*(\mathfrak{g}^\vee)\otimes\Omega^*(M))^G$, where $\mathfrak{g}^\vee$ is the dual of the Lie algebra $\mathfrak{g}$ and $S^*(\mathfrak{g}^\vee)$ its symmetric algebra. The algebra of equivariant n-forms can be expressed as the algebra of invariant forms:

$$\Omega_G^n(M) = \bigoplus_{2k+i=n} (S^k(\mathfrak{g}^\vee) \otimes \Omega^i(M))^G.$$

We notice that the degree of an equivariant form is twice the degree of the polynomial plus the degree of the differential form. In this way, the morphism d_{dR} increases the degree by one. Meanwhile, $\iota_{X\#}$ increases the degree of the polynomial by one and the degree of the form decreases by one. Therefore the degree of d_G increases by one, and $(\Omega_G^n(M), d_G)$ is a complex.

Example 2.3.67. If we consider the trivial action of the Lie group G on the point pt, we get that

$$H_G^k(pt) = S^{2k}(\mathfrak{g}^\vee).$$

The Borel model and the Cartan model have a strong interplay. The next result is due to Cartan in [14].

Theorem 2.8. *(Cartan)[14] If G is a compact Lie group acting on a smooth compact manifold M, then the complex of equivariant forms computes the equivariant cohomology in the Borel model, i.e., the Cartan and Borel model give isomorphic cohomologies.*

2.4 Lie groupoids

The category of groupoids play an important role in the definition of stacks that we are going to provide later. In this section we give definitions of Lie groupoids and their morphisms following [41] and [42].

A *groupoid* is a small category in which every morphism is an *isomorphism*. However, we can provide a more structural definition, as follows:

Definition 2.4.1. A *groupoid* consists of two sets Γ_1 and Γ_0, and some functions $s, t : \Gamma_1 \to \Gamma_0$, the *source* and the *target map*, respectively. Also, there is a map

$$1 : \Gamma_0 \to \Gamma_1$$

$$m \mapsto 1_m$$

called *inclusion map.* There exists a *multiplication*

$$\Gamma_1 \times_{s,t} \Gamma_1 \to \Gamma_1$$

$$(h, g) \mapsto hg$$

where $\Gamma_1 \times_{s,t} \Gamma_1 = \{(h, g) \in \Gamma_1 \times \Gamma_1 | s(h) = t(g)\}$ such that:

1. $s(hg) = s(g)$ and $t(hg) = t(h)$ for any $(h, g) \in \Gamma_1 \times_{s,t} \Gamma_1$.
2. $j(hg) = (jh)g$ for any $j, h, g \in \Gamma_1$ such that $s(j) = t(h)$ and $s(h) = t(g)$.
3. $s(1_x) = t(1_x)$, for all $x \in \Gamma_0$.
4. $g1_{s(g)} = g$ and $1_{t(g)}g = g$ for all $g \in \Gamma_1$.
5. For each $g \in \Gamma_1$ there is an inverse g^{-1} such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$ and $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$.

Remark 2.4.2.

1. A groupoid Γ_1 over Γ_0 is denoted by $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$.
2. Elements in Γ_1 are called *arrows* and elements in Γ_0 are called *objects.*
3. The arrow 1_x is called the *identity of* x, for any $x \in \Gamma_0$.

Proposition 2.4.3. *[42, 1.1.2] Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a groupoid over Γ_0. Consider $g \in \Gamma_1$ with $s(g) = x$ and $t(g) = y$.*

1. *If $h \in \Gamma_1$ with $s(h) = y$ and $hg = g$ then $h = 1_y$. If $j \in \Gamma_1$ with $t(j) = x$ and $gj = g$ then $j = 1_x$.*
2. *If $h \in \Gamma_1$ with $s(h) = y$ and $hg = 1_x$ then $h = g^{-1}$. If $j \in G$ with $t(j) = x$ and $gj = 1_y$ then $j = g^{-1}$.*

Example 2.4.4. Let M be a set. Consider $\Gamma_1 = M \times M$ with $s(a,b) = a$, $t(a,b) = b$ and $\mu : \Gamma_1 \times_{s,t} \Gamma_1 \to \Gamma_1$ such that $((a,b),(b,c)) = (a,c)$. In this case, we have $1 : M = \Gamma_0 \to \Gamma_1$ given by $1_x = (x,x)$.

Definition 2.4.5. A *Lie groupoid* $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ is a groupoid where Γ_1 and Γ_0 are smooth manifolds such that s, t are surjective submersions. 1 and the multiplication are required to be smooth.

Remark 2.4.6. $\Gamma_1 \times_{s,t} \Gamma_1 = (s \times t)^{-1}(\Delta_{\Gamma_0})$ is a closed embedded smooth submanifold of $\Gamma_1 \times \Gamma_1$ where Δ_{Γ_0} is the diagonal of $\Gamma_1 \times \Gamma_1$, since s and t are submersions. Compare with [12, II.15.2].

Remark 2.4.7. We can define a topological groupoid $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ where Γ_1 and Γ_0 are topological spaces and the source, target and multiplication maps are continuos.

Proposition 2.4.8. *[42, 1.1.5] Let* $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ *be a Lie groupoid over* Γ_0*. The inverse function* $i(g) = g^{-1}$ *is a diffeomorphism.*

Example 2.4.9.

1. We can consider a smooth manifold M as a Lie groupoid $(M \rightrightarrows M)$ over itself with $s = t = id_M$. Here, every element is a identity. A groupoid, where every element is a identity, will be called the *base groupoid.* Observe that $M \times_{s,t} M = \Delta_M = \{(x,y) \in M \times M \mid x = y\}$.

2. Let be M a smooth manifold and G a Lie group. Consider $(M \times G \times M \rightrightarrows M)$ as a Lie groupoid with

$$s = \pi_3 : M \times G \times M \to M$$

$$(m, g, n) \mapsto n$$

$$t = \pi_1 : M \times G \times M \rightrightarrows M$$

$$(m, g, n) \mapsto m$$

$$1 : M \to M \times G \times M$$

$$m \mapsto (m, 1, m)$$

and multiplication given by $(z, h, y')(y, g, x) = (z, hg, x)$ defined if and only if $y = y'$. The inverse of (y, g, x) is (x, g^{-1}, y). We call this groupoid the *trivial groupoid* over M with group G.

3. Every cartesian product $M \times M$ is a groupoid over M and it is called the *pair groupoid.*

4. Let $q : M \to Q$ be a surjective submersion. Then

$$R(q) = M \times_Q M = \{(x, y) \in M \times M | q(x) = q(y)\}$$

is a Lie groupoid over M with respect to the restriction of the structure of the pair groupoid. This Lie groupoid is called the *banal groupoid* induced by q.

5. Let M be a smooth manifold. Then, the set $\Pi(M)$ of homotopy classes $< \gamma >$ of relative endpoints of smooth paths $\gamma : [0, 1] \to M$ is a groupoid on M with respect to $\alpha(< \gamma >) = \gamma(0)$, $\beta(< \gamma >) = \gamma(1)$, $1 : M \to \Pi(M)$ $m \mapsto 1_m =< C_m >$, where C_m is the constant path in m. The multiplication is the concatenation, and the inverse element is the reverse of the path. This groupoid is called *fundamental groupoid* and it is a Lie groupoid, see [49, 5.1.6].

6. Let (E, q, M) be a vector bundle over M. Let $\Psi(E)$ denote the set of all the isomorphism of vector spaces $\eta : E_x \to E_y$ for $x, y \in M$. Then $(\Psi(E) \rightrightarrows M)$ with the structure maps $\alpha(\eta) = x$, $\beta(\eta) = y$ and $1(x) = 1_x = id_{E_x}$ is a groupoid. If $\xi : E_y \to E_z$ then the multiplication is given by $\xi \circ \eta$ and the inverse of η is η^{-1}. With this structure $\Psi(E)$ is called the *frame groupoid* of (E, q, M).

   The smooth structure on $\Psi(E)$ is induced by E. Consider an atlas $\{\psi_i : U \times V \to E_U\}$ for E. For each i and j, it is defined

$$\overline{\psi_i^j} : U_j \times GL(V) \times U_i \to \Psi(E)_{U_i}^{U_j}$$

$$(y, A, x) \mapsto \psi_{j,y} \circ A \circ \psi_{i,x}^{-1}$$

   where each $\overline{\psi_i^j}$ is a bijection and any $(\overline{\psi_k^l})^{-1} \circ \overline{\psi_i^j}$ is a diffeomorphism.

Definition 2.4.10. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ and $\Gamma' = (\Gamma'_1 \rightrightarrows \Gamma'_0)$ be two groupoids. A *morphism* between Γ and Γ' is pair (F, f) of maps $F : \Gamma_1 \to \Gamma'_1$, $f : \Gamma_0 \to \Gamma'_0$ such that $s' \circ F = f \circ s$, $t' \circ F = f \circ t$ and $F(hg) = F(h)F(g)$ for any $(h, g) \in \Gamma_1 \times_{s,t} \Gamma_1$.

Remark 2.4.11. A morphism of groupoids is also a functor $\Gamma \to \Gamma'$.

Definition 2.4.12. Let $(F, f) : \Gamma \to \Gamma'$ and $(G, g) : \Gamma \to \Gamma'$ be morphism of groupoids. A 2-*morphism* of groupoids $\theta : (F, f) \Rightarrow (G, g)$ is a map $\theta : \Gamma_0 \to \Gamma'_1$

such that θ has the same properties as a natural transformation of categories, that is, for any element in Γ_1 with form $x_k \xrightarrow{\phi_k+1} x_{k+1}$ the following diagram

$$\begin{tikzcd} f(x_k) \arrow[r, "F(\phi_{k+1})"] \arrow[d, "\theta(x_k)"'] & f(x_{k+1}) \arrow[d, "\theta(x_{k+1})"] \\ g(x_k) \arrow[r, "G(\phi_{k+1})"'] & g(x_{k+1}) \end{tikzcd}$$

commutes.

Remark 2.4.13.

1. The category constituted by groupoids and groupoid morphisms, is denoted by **Grpds**.
2. If Γ and Γ' are Lie groupoids then (F, f) is a morphism of Lie groupoids if (F, f) is a morphism of groupoids, and both F and f are smooth. The category of Lie groupoids and Lie groupoid morphisms is denoted by **LieGrpds**. For a 2-morphism θ of Lie groupoids, $\theta : \Gamma_0 \to \Gamma'_1$ is a smooth map.

Proposition 2.4.14. *[42, 1.2.2] Let $(F, f) : \Gamma \to \Gamma'$ be morphism of groupoids. Then $F(1_m) = 1_{f(m)}$ and $F(g^{-1}) = F(g)^{-1}$ for all $m \in \Gamma_0$ and $g \in \Gamma_1$.*

Definition 2.4.15. A morphism (F, f) between groupoids $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ and $\Gamma' = (\Gamma'_1 \rightrightarrows \Gamma'_0)$, is an *isomorphism of Lie groupoids* if F and f are diffeomorphisms.

Example 2.4.16. A morphism of trivial groupoids

$$F : M \times G \times M \to M' \times G' \times M'$$

where $F(y, g, x) = (f(y), \theta(y)l(g)\theta(x)^{-1}, f(x))$ with $l : G \to G'$ a morphism of Lie groups and $\theta : M \to G'$ a smooth function.

Definition 2.4.17. The *nerve of a Lie groupoid* $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ is the simplicial smooth manifold $(\Gamma_n)_{n\in\mathbb{N}}$ where

$$\Gamma_n = \{(g_1, g_2, \ldots, g_n) \,|\, s(g_i) = t(g_{i+1}) \text{ and } g_i \in \Gamma_1, \forall_{i=1,\ldots,n}\}$$

with face maps

$$\partial^0(g_1, g_2, \ldots, g_n) = (g_2, \ldots, g_n),$$

$$\partial^i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_i{\cdot}g_{i+1}, \ldots, g_n) \text{ for } 0 < i < n,$$

$$\partial^n(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_{n-1})$$

and degeneracy maps

$$\sigma^i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_i, 1_{s(g_i)}, g_{i+1}, \ldots, g_n).$$

For a Lie groupoid we can induce a covering for its nerve, as it was defined for simplicial smooth manifolds in the definition 2.3.52. This was provided by [20, A.2].

Example 2.4.18. Let $\Gamma_\bullet$ be the nerve of the Lie groupoid $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$. If we have an open covering $\mathcal{V}_0 = \{V_i\}$ of Γ_0, we can induce a covering $\mathcal{V}_n$ of Γ_n with the following open sets

$$V_{i_0 i_1 \cdots i_n} \coloneqq \{(g_1, g_2, \ldots, g_n) : t(g_1) \in V_{i_0}, s(g_1) \in V_{i_1}, \ldots, s(g_n) \in V_{i_n}\}$$

and maps for indices given by $d_k(i_0 \cdots i_n) = i_0 \cdots \widehat{i_k} \cdots i_n$. If $(g_1, g_2, \ldots, g_n) \in V_\alpha$, we have

$$\partial^0((g_1, g_2, \ldots, g_n)) = (g_2, \ldots, g_n) \in V_{n-1, d_0(\alpha)}$$

$$\partial^i((g_1, g_2, \ldots, g_n)) = (g_1, \ldots, g_i{\cdot}g_{i+1}, \ldots, g_n) \in V_{n-1, d_i(\alpha)}$$

$$\partial^n((g_1, g_2, \ldots, g_n)) = (g_1, , \ldots, g_{n-1}) \in V_{n-1, d_n(\alpha)}.$$

2.4.1 Two important Lie groupoids

The first important example for us is the following Lie groupoid:

Definition 2.4.19. Let $\phi : G \times M \to M$ be a smooth action of a Lie group on a smooth manifold M. $G \times M \rightrightarrows M$ is a Lie groupoid in the following way

$$s : G \times M \to M$$

$$(g, m) \mapsto m$$

$$t : G \times M \to M$$

$$(g, m) \mapsto \phi(g, m) = gm$$

$$1 : M \to G \times M$$

$$m \mapsto (1, m)$$

with multiplication $(g_2, m)(g_1, n) = (g_2 g_1, n)$ if and only if $m = g_1 n$, The inverse of (g, m) is (g^{-1}, gx). This groupoid is called the *action groupoid* or *transformation groupoid*.

Example 2.4.20. Consider the action groupoid $(G \times M \rightrightarrows M)$ where G is a Lie group and M is a smooth manifold. The nerve is given by $(G \times M)_n$ where an element in $(G \times M)_n$ has the form $(g_1, g_2 g_3 \cdots g_n m_n, g_2, g_3 g_4 \cdots g_n m_n, \ldots, g_n, m_n)$. Thus we can define a diffeomorphism between $(G \times M)_n$ and $G \times \ldots \times G \times M$ with n copies of G given by

$$\Phi_n : (G \times M)_n \to G \times \ldots \times G \times M$$

$$(g_1, g_2 g_3 \cdots g_n m_n, g_2, g_3 g_4 \cdots g_n m_n, \ldots, g_n, m_n) \mapsto (g_1, \ldots, g_n, m_n).$$

If we consider this diffeomorphism, the face maps are given by

$$\partial^0 (g_1, g_2, \ldots, g_n, m) = (g_2, \ldots, g_n, m),$$

$$\partial^i (g_1, g_2, \ldots, g_n, m) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, m) \text{ for } 0 < i < n,$$

$$\partial^n (g_1, g_2, \ldots, g_n, m) = (g_1, \ldots, g_{n-1}, g_n m)$$

and degeneracy maps by

$$\sigma^i (g_1, g_2, \ldots, g_n, m) = (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n, m).$$

Thus, the nerve of this groupoid coincides with the simplicial smooth manifold defined on the example 2.3.49, and its fat geometric realisation is a model for $EG \times M/G$, when M is a point this is a model for the classifying space BG. Compare with [19, Ch.5 Example 3].

Our next example was provided in the subsection 2.3.4.

Definition 2.4.21. The category of principal G-bundles over a smooth manifold M is a groupoid, since each morphism between principal bundles is an isomorphism. The multiplication for the groupoid is the composition of morphisms and the inverse function is given by taking the inverse morphism. This groupoid is denoted by $\mathcal{B}G(M)$.

Remark 2.4.22. This example will give us later one of the most important differentiable stacks and it is going to be related with the action groupoid. See section 3.1.

2.4.2 Action of Lie groupoids and principal Γ-bundles

The following definitions can be found in [49, 5.7].

Definition 2.4.23. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie groupoid and M a smooth manifold. A *left action of* Γ *on* M *along a smooth map* $\epsilon : M \to \Gamma_0$ is given by a smooth map

$$\mu : \Gamma_1 \times_{\Gamma_0} M \to M$$

$$(g, y) \mapsto \mu(g, y) = gy$$

with $\Gamma_1 \times_{\Gamma_0} M = \{(g, y) \in \Gamma_1 \times M \mid s(g) = \epsilon(y)\}$ which satisfies the following identities:

- $\epsilon(gy) = t(g)$,
- $1_{\epsilon(y)} y = y$,
- $g'(gy) = (g'g)y$,

for $g', g \in \Gamma_1$ and $y \in M$ with $s(g') = t(g)$ and $s(g) = \epsilon(y)$.

Definition 2.4.24. A *right action* of the Lie groupoid $\Gamma' = (\Gamma'_1 \rightrightarrows \Gamma'_0)$ on a smooth manifols M along a smooth map $\epsilon' : M \to \Gamma'_0$ is given by

$$\mu : M \times_{\Gamma'_0} \Gamma'_1 \to M$$

with $M \times_{\Gamma'_0} \Gamma'_1 = \{(y, g) \in M \times \Gamma'_1 \mid t(g) = \epsilon(y)\}$ which satisfies the following identities:

- $\epsilon(yg) = s(g)$,
- $y 1_{\epsilon(y)} = y$,
- $(yg)g' = y(gg')$,

for $g', g \in \Gamma_1$ and $y \in M$ with $t(g') = s(g)$ and $t(g) = \epsilon(y)$.

Definition 2.4.25. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie groupoid. A *principal right* Γ-*bundle* over a smooth manifold M is a smooth manifold P equipped with a map $\pi : P \to M$ and a smooth right Γ-action μ on P along $\epsilon : P \to \Gamma_0$ which is fibrewise with respect to π, that is $\pi(pg) = \pi(p)$ for any $p \in P$ and any $g \in \Gamma_1$ with $\epsilon(p) = t(g)$ such that

1. π is surjective submersion,
2. $(pr_1, \mu) : P \times_{\Gamma_0} \Gamma_1 \to P \times_M P$ given by $(p, g) \mapsto (p, pg)$ is a diffeomorphism.

Remark 2.4.26. We can talk about a principal left Γ'-bundle over a smooth manifold beginning with a left action of a Lie groupoid Γ' over M by changing the details in the previous definition.

Definition 2.4.27. An *equivariant map* between principal right Γ-bundles $\pi : P \to M$ and $\pi' : P' \to M$ is a smooth map $f : P \to P'$ which commutes with all the structure maps, that is $\pi'(f(p)) = \pi(p)$, $\epsilon'(f(p)) = \epsilon(p)$ and $f(pg) = f(p)g$ for any $p \in P$, $g \in \Gamma_1$ with $\epsilon(p) = t(g)$.

Remark 2.4.28. Also we get a definition for equivariant map for principal left Γ'-bundles.

Definition 2.4.29. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ and $\Gamma' = (\Gamma'_1 \rightrightarrows \Gamma'_0)$ be Lie groupoids. A Γ-Γ' *bibundle* is a right principal Γ'-bundle $E \to \Gamma_0$ over Γ_0 and a smooth right Γ'-action on E along $\epsilon' : E \to \Gamma'_0$ together with a left Γ-action along $\epsilon : E \to \Gamma_0$ such that the two actions commute.

Example 2.4.30.

1. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie groupoid. We have that Γ_1 is a principal Γ-bundle over Γ_0 with projection given by the target map along the source map. We call this the *unit bundle* of Γ.
2. Let $P \to M$ be a principal Γ-bundle. If we have a smooth map $f : N \to M$ then the pullback $N \times_M P$ is a principal Γ-bundle over N.
3. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie groupoid. If we have a smooth map $\alpha : M \to \Gamma_0$, there is a principal Γ-bundle that is the pullback of α for the unit bundle of Γ. This bundle is called the *trivial bundle.*

Proposition 2.4.31. *[49, 5.34.4] Let* $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ *be a Lie groupoid over* Γ_0. *Any principal* Γ*-bundle is locally trivial.*

2.5 Spectral sequences

When we try to find some properties that relate different cohomology theories or properties that show some facts about a specific cohomology theory via successive approximations, spectral sequences are a very useful tool to work with. In this way, we recall some ideas from [27, Chapter 6] and [45, Chapter 2].

Let
$$C = \bigoplus_{p,q\in\mathbb{Z}} C^{p,q}$$
be a double complex of vector spaces with differential operators

$$d : C^{p,q} \to C^{p,q+1}, \delta : C^{p,q} \to C^{p+1,q}$$

satisfying
$$d^2 = 0, \delta^2 = 0 \text{ and } d\delta + \delta d = 0.$$
The associated total complex is defined by $C^n = \bigoplus_{p+q=n} C^{p+q}$ with differential $d+\delta : C^n \to C^{n+1}$. Hence we can look at the filtration $C^n_k = \bigoplus_{p+q=n,p\geq k} C^{p,q}$ of C^{n+1} with

$$Z^n_k = \{z \in C^n_k | (d+\delta)z = 0\} \text{ and } B^n = (d+\delta)C^{n-1}$$

Then the map
$$Z^n_k \to Z^n_k/(B^n \cap Z^n_k)$$
gives a decreasing filtration

$$\ldots \subset H^n_{k+1} \subset H^n_k \subset H^n_{k-1} \subset \ldots$$

of $H^n(C, d+\delta)$. Thus, we denote the successive quotients by $H^{k,n-k} = H^n_k/H^n_{k+1}$ and $\mathrm{gr}H^n = \bigoplus_k H^{k,n-k}$ as the associated graded vector space.

Definition 2.5.1. Let $Z^{p,q}$ be the set of elements $a \in C^{p,q}$ such that the following system of equations has a solution

$$da = 0$$

$$\delta a = -da_1$$

$$\delta a_1 = -da_2$$

$$\delta a_2 = -da_3$$

$$\vdots$$

where $a_i \in C^{p+i,q-i}$.

Definition 2.5.2. Let $B^{p,q} \subset C^{p,q}$ be the set of all b such that the following system of equation has a solution

$$dc_0 + \delta c_{-1} = b$$

$$dc_{-1} + \delta c_{-2} = 0$$

$$dc_{-2} + \delta c_{-3} = 0$$

$$\vdots$$

where $c_i \in C^{p-i,q+i-1}$.

Remark 2.5.3. The systems of equations in the definitions 2.5.1 and 2.5.2 are solvable if $C^{i,j} = 0$ for $i + j = p + q$, $|i - j| > m_l$, for some m_l and for each i the systems of equations are solvable for a bounded range of i.

Proposition 2.5.4. *[27, p.64] $H^{p,q}$ can be also described as*

$$H^{p,q} = Z^{p,q}/B^{p,q}.$$

To describe this, we define $Z_r^{p,q}$ as the set of elements in $C^{p,q}$ such that the equation system in the definition 2.5.1 has a solution for the first $r-1$ equations, and $B_r^{p,q}$ the set of all $b \in C^{p,q}$ with a solution for the system in the definition 2.5.2, with $c_i = 0$ for $i \geq r$. Then we get:

Theorem 2.9. *[27, 6.1.1] Let be $a \in Z_r^{p,q}$. Then*

$$a \in Z_{r+1}^{p,q} \Leftrightarrow \delta a_{r-1} \in B_r^{p_r+1,q_r}$$

for any solutions $(a_1, \ldots, a_r)$ of the first $r-1$ equations in the system of the definition 2.5.1.

We define $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$ and since $\delta a_{r-1} \in B_{r+1}^{p_r+1,q_r} \subset Z^{p_r+1,q_r} \subset Z_r^{p_r+1,q_r}$, we can see the element δa_{r-1} as an element $\delta_r a \in E_r^{p_r+1,q_r}$. In this way, we define $\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Proposition 2.5.5. *[27, p.66] The sequence of complexes (E_r, δ_r) has $H(E_r, \delta_r) = E_{r+1}$.*

Remark 2.5.6.

1. If we have the condition in the remark 2.5.3, the spectral sequence (E_r, δ_r) stabilises for some r, that is, $E_r^{p,q} = E_k^{p,q}$ for all $k > r$, and we can consider its limit as $E_\infty^{p,q} = \lim_r E_r^{p,q} = H^{p,q}$.

2. A spectral sequence converges if for every p, q, if r is sufficiently large then δ_r vanishes on $E_r^{p,q}$ and $E_r^{p+r,q-r+1}$.

It is important to know the behaviour of the spectral sequences when we have two double complexes (C, d, δ), (C', d', δ') and a morphism $\rho : C \to C'$ of double complexes of bidegree (m, n) such that $\rho d = d'\rho$, and $\rho\delta = \delta'\rho$. This gives rise to a cochain map

$$\rho : (C, d + \delta) \to (C', d' + \delta')$$

of degree $m + n$. It induces a map in the total cohomology

$$\rho_\# : H(C, d + \delta) \to H(C', d' + \delta')$$

of degree $m + n$. Similarly, ρ maps the cochain complex $(C^{p,*}, d)$ into the cochain complex $((C')^{p+m,*}, d')$ and hence ρ induces a map on cohomology

$$\rho_1 : E_1 \to E_1'$$

of bidegree (m, n) with $\rho_1\delta_1 = \delta_1'\rho_1$. Inductively, we get maps

$$\rho_r : (E_r, \delta_r) \to (E_r', \delta_r').$$

Here ρ_{r+1} is the map on cohomology induced from ρ_r where $E_{r+1} = H(E_r, \delta_r)$.

Theorem 2.10. *[27, 6.4.1] If the two spectral sequences converge, then*

$$\lim_r \rho_r = gr\rho_\#$$

where $gr\rho_\#$ is the induced morphism in the total complex.

Theorem 2.11. *[27, 6.4.2] If ρ_r is an isomorphism for some $r = r_0$ then there is an isomorphism for all $r > r_0$, and so, if both spectral sequences converge, then $\rho_\#$ is an isomorphism.*

Chapter 3

Stacks and cohomology

In this chapter we will discuss the category of differentiable stacks, cohomology theories in this category, and the different interplays between these cohomology theories. For a first view on this category, we take as a main reference [28]. Thus, we consider a stack as a pseudo-functor between the category of smooth manifolds **Diff** and **Grpds**, instead of using the fibered categories approach, as in [4], but always bearing in mind that both approaches are equivalent. To study cohomology theories in the category of differentiable stacks and some techniques to work on it, we use [4, 6, 20, 28].

In the first three sections, we focus our efforts on discussing the category of differentiable stacks and on how we can consider a geometrical environment for them. Also, we discuss the relation between a Lie groupoid and a differentiable stack. In particular, we discuss how a Morita equivalence for Lie groupoids gives rise to the same differentiable stack. The fourth section is devoted to different cohomology theories for differentiable stacks and how the theories interplay with each other.

3.1 Differentiable stacks

In this chapter, we consider **Diff** with the big site of smooth manifolds and local diffeomorphisms as it was defined previously in 2.1.11 and 2.1.12. Compare with [28, 1.2.6] and [6, 2]. We begin with a preparatory definition.

Definition 3.1.1. A *prestack* over **Diff** is a pseudo-functor

$$\mathcal{M} : \mathbf{Diff}^{op} \to \mathbf{Grpds}.$$

Remark 3.1.2. If $\mathcal{M}$ is a prestack with $P \in \mathcal{M}(U)$ and $f : U' \to U$ a smooth map in **Diff**, we will denote f^*P in the groupoid $\mathcal{M}(U')$ as $P|_{U'}$. If we have a covering $\{U_i \xrightarrow{i} U\}_{i\in I}$ and $P \in \mathcal{M}(U)$ we denote by $P|_{U_i}$ the pullback i^*P. We will denote by $P_i|_{U_{ij}}$ the pullback $i^*_{ij,i}P_i$ given by the smooth map $i^*_{ij,i} : U_{ij} \to U_i$ for $P_i \in \mathcal{M}(U_i)$.

Definition 3.1.3. A *stack* $\mathcal{M}$ over **Diff** is a prestack

$$\mathcal{M} : \mathbf{Diff}^{op} \to \mathbf{Grpds} \subset \mathrm{Cat}$$

such that:

1. One can glue objects: given a covering $\{U_i \to X\}_{i\in I}$, objects $P_i \in \mathcal{M}(U_i)$ and isomorphisms $\phi_{ij} : P_i|_{U_{ij}} \to P_j|_{U_{ij}}$ which satisfy the cocycle condition on threefold products $\phi_{jk} \circ \phi_{ij} = \phi_{ik}|_{U_{ijk}}$ there is an object $P \in \mathcal{M}(X)$ together with isomorphisms $\phi_i : P|_{U_i} \to P_i$ such that $\phi_{ij} = \phi_j \circ \phi_i^{-1}$.
2. One can glue morphisms: given objects $P, P' \in \mathcal{M}(X)$, a covering $\{U_i \to X\}_{i\in I}$ and isomorphisms $\phi_i : P|_{U_i} \to P'|_{U_i}$ such that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$, then there is a unique $\phi : P \to P'$ such that $\phi_i = \phi|_{U_i}$.

Example 3.1.4.

1. For any smooth manifold $X \in \mathbf{Diff}$ we can associate a stack given by

$$\underline{X} = \mathrm{Map}(\_, X) : \mathbf{Diff}^{op} \to \mathbf{Grpds}$$

which takes a $Y \in \mathbf{Diff}$ and associates the set of all smooth morphism between Y and X, $\mathrm{Map}(Y, X)$. The morphisms in $\mathrm{Map}(Y, X)$ as a groupoid, are identity maps.

(a) We can glue objects: let $\{U_i \to Y\}_{i\in I}$ be a covering of Y. If there exists $U_i \xrightarrow{p_i} X \in \mathrm{Map}(U_i, X)$ for each i such that $p_i|_{U_{ij}} : U_{ij} \to X$ and $p_j|_{U_{ij}} : U_{ij} \to X$ are isomorphic, then $p_i|_{U_{ij}} = p_j|_{U_{ij}} \circ id_{U_{ij}} = p_i|_{U_{ij}} = p_j|_{U_{ij}}$. Therefore we can define $p : Y = \sqcup_i U_i \to X$ with $p(z) = p_i(z)$ if $z \in U_i$. This map is well defined since the maps p_i coincide on U_{ij} and they are all smooth maps.

(b) We can glue morphisms: Let $Y \xrightarrow{p} X$, $Y \xrightarrow{p'} X$ be elements in $\underline{X}(Y)$. If we have a covering $\{U_i \to Y\}_{i\in I}$ and isomorphisms $\phi_i : p|_{U_i} \to p'|_{U_i}$ such that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$, we know that $p|_{U_i}$ is the element $U_i \xrightarrow{i} Y \xrightarrow{p} X$

and by ϕ_i, we know that $U_{ij} \to U_i \xrightarrow{i} Y \xrightarrow{p} X$ is the same map as $U_{ij} \to U_i \xrightarrow{i} Y \xrightarrow{p'} X$. Since this is true for each U_{ij}, we get that $p = p'$ as we want.

2. Let G be a Lie group. Consider the functor $\mathcal{B}G : \mathbf{Diff}^{op} \to \mathbf{Grpds}$ which assigns to any smooth manifold the category of principal G-bundles over the smooth manifold, as in theorem 2.4.

   (a) We can glue objects: let $\{U_i \to X\}_{i \in I}$ be a covering of the smooth manifold X. If we have principal G-bundles (P_i, p_i, G) over U_i and isomorphism $\phi_{ij} : P_i|_{U_{ij}} \to P_j|_{U_{ij}}$, we can get an open cover $\{V_l\}_{l \in L}$ of local trivialisations for p_i and transition functions $\{g_{m,n}\}$. As $\phi_{ij} : P_i|_{U_{ij}} \to P_j|_{U_{ij}}$ with $\phi_{jk} \circ \phi_{ij} = \phi_{ik}|_{U_{ijk}}$, we get a principal G-bundle (P, p, G) over X by theorem 2.3, such that $P|_{U_i} \to U_i$ is isomorphic to $P_i \to U_i$ via an isomorphism $\phi : P|_{U_i} \to U_i$ and $\phi_{ij} = \phi_j \circ \phi_i^{-1}$.

   (b) We can glue morphisms: let (P, p, G) and (P', p', G) be elements in $\mathcal{B}G(X)$. If we have $\{U_i \to X\}_{i \in I}$ a covering of X with isomorphisms $\phi_i : P|_{U_i} \to P'|_{U_i}$ such that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$, we can consider an open covering $\{V_k\}_{k \in K}$ of local trivialisation for $P|_{U_i} \xrightarrow{p} U_i$ with transition functions $\{g_{m,n}\}$ and $P'|_{U_i} \xrightarrow{p'} U_i$ with transition functions $\{g'_{m,n}\}$. We can consider the principal G-bundles that come from the transition functions, and we get an isomorphism between these principal G-bundles by theorem 2.3, and as these principal G-bundles are isomorphic to (P, p, G) and (P', p', G), so we get finally an isomorphism of principal G-bundle $\phi : P \to P'$ with the required properties.

Definition 3.1.5. 1-morphism and 2-morphisms on stacks are defined in the following way:

1. A 1-*morphism of stacks* between two stacks $\mathcal{M}$ and $\mathcal{N}$ is given by a natural transformation of functors of 2-categories $F : \mathcal{M} \to \mathcal{N}$, that is:

   - for every smooth manifold $X \in \mathbf{Diff}$, a functor $F_X : \mathcal{M}(X) \to \mathcal{N}(X)$,
   - for every morphism $f : X \to Y$ in $\mathbf{Diff}$, an invertible natural transformation $F_f : \mathcal{N}(f) \circ F_Y \Rightarrow F_X \circ \mathcal{M}(f)$ which is compatible with the natural transformations
   $$\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*,$$
   that is, there exists the following commutative diagram for F_f

$$\begin{tikzcd} \mathcal{M}(Y) \arrow[r, "F_Y"] \arrow[d, "\mathcal{M}(f)"'] & \mathcal{N}(Y) \arrow[d, "\mathcal{N}(f)"] \arrow[dl, Rightarrow, "F_f"'] \\ \mathcal{M}(X) \arrow[r, "F_X"] & \mathcal{N}(X) \end{tikzcd}$$

such that:

i. if $f = id_X$ then $F_{id_X} = id_{F_X}$.

ii. if $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $F_{g\circ f}$ is the composite of the commutative diagrams by F_f and F_g further composed with the composition of the pullback isomorphisms $\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$ for $\mathcal{M}$ and $\mathcal{N}$.

2. A 2-*morphism of stacks* $\phi : F \to G$ between two 1-morphisms $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{M} \to \mathcal{N}$ is given by the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{N} \end{tikzcd}$$

such that for any $X \in$ **Diff** there are invertible natural transformations $\phi_X : F_X \to G_X$ of the form

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, bend left, "F_X"] \arrow[r, bend right, "G_X"'] \arrow[r, phantom, "\Downarrow\phi_X"] & \mathcal{N}(X). \end{tikzcd}$$

As we build stacks and its different kinds of morphisms in the previous way, we get the next property:

Proposition 3.1.6. *The category of stacks over* **Diff** *with* 1*-morphisms and* 2*-morphisms form a* 2*-category. This* 2*-category is denoted by* **St**.

Remark 3.1.7. In the same way, we can consider the 2-category of pre-stacks and we denote it by pre-**St**.

The next proposition relates pre-stacks with stacks and it is known as *stackification* of a pre-stack.

Proposition 3.1.8. *[15, 8.8.1] Let $\mathcal{M}$ be a pre-stack. Then there exists a morphism of prestacks $\sigma : \mathcal{M} \to \tilde{\mathcal{M}}$, with $\tilde{\mathcal{M}}$ a stack, such that for every stack $\mathcal{N}$, the functor $Hom_{\mathbf{St}}(\tilde{\mathcal{M}}, \mathcal{N}) \xrightarrow{\sigma^*} Hom_{pre-\mathbf{St}}(\mathcal{M}, \mathcal{N})$ is an equivalence of categories.*

The next result is a version for stacks of the classical Yoneda lemma in category theory, and it is going to allow us to consider elements of a stack as morphisms of stacks and vice-versa. We are going to refer to this result as 2-*Yoneda lemma*.

Lemma 3.1.9. *(2-Yoneda lemma) Let $\mathcal{M}$ be a stack and $X \in$ **Diff**, a smooth manifold. Then there is a canonical equivalence of categories $\mathcal{M}(X) \cong Mor_{St}(\underline{X}, \mathcal{M})$.*

Proof. We define the morphism $\Phi : \mathcal{M}(X) \to \mathrm{Mor}_{St}(\underline{X}, \mathcal{M})$, such that for any element P of $\mathcal{M}(X)$ and we assign $\Phi(P) = F_P(f) = f^*P$ with $Y \xrightarrow{f} X$, and for any isomorphism $\phi : P \to P'$ between element P, P' in $\mathcal{M}(X)$ is defined as a natural transformation $\Phi_\phi : F_P \to F_{P'}$ by $f^*P \to f^*P'$.

Also, we define $\Psi : \mathrm{Mor}_{St}(\underline{X}, \mathcal{M}) \to \mathcal{M}(X)$ given by $\Psi(F) = F(id_X) = P_F$ where $F \in \mathrm{Mor}_{St}(\underline{X}, \mathcal{M})$. Thus we have that:

1. $\Psi \circ \Phi(P) = \Psi(F_P) = F_P(id_X) = id^*(P) = P$ so $\Psi \circ \Phi = id_{\mathcal{M}(X)}$.
2. $\Phi \circ \Psi(F) = \phi(P_F) = \Phi(F(id_X)) = F_{P_F}$. We observe that for $f \in \underline{X}(Y)$ we have $F_{P_F}(f) = f^*P_F = f^*(F(id_X))$. Since F is a natural transformation we get $f^*(F_X(id_X)) = F((id_X) \circ f)$, that is $f^*(F(id_X)) = F(f)$. Then $F_{P_F} = F$ and $\Phi \circ \Psi = id_{\mathrm{Mor}_St(\underline{X}, \mathcal{M})}$.

Since the compositions are equal to the identities, it is enough to show that ψ is natural.

If we fix $\mathcal{M}$ and consider $Y \xrightarrow{f} X$ a morphism in **Diff**, we have that $F(f)(id_Y) = F(f) = f^*(F)$ and then the diagram commutes

$$\begin{tikzcd} \mathrm{Mor}_{St}(\underline{X}, \mathcal{M}) \arrow[r, "\_\circ f"] \arrow[d, "\psi_X"] & \mathrm{Mor}_{St}(\underline{Y}, \mathcal{M}) \arrow[d, "\psi_Y"] \\ \mathcal{M}(X) \arrow[r, "f*"] & \mathcal{M}(Y) \end{tikzcd}$$

Let $X \in$ **Diff** be. If we consider $\mathcal{M}, \mathcal{M}'$ stacks and a natural transformation $\theta : \mathcal{M} \Rightarrow \mathcal{M}'$, then the following diagram

$$\begin{tikzcd} \mathrm{Mor}_{St}(\underline{X}, \mathcal{M}) \arrow[r, "\theta\circ\_"] \arrow[d, "\Psi"] & \mathrm{Mor}_{St}(\underline{X}, \mathcal{M}') \arrow[d, "\Psi"] \\ \mathcal{M}(X) \arrow[r, "\theta_X"] & \mathcal{M}'(X) \end{tikzcd}$$

commutes, since for $F : \underline{X} \to \mathcal{M}$ we know that $\theta_X \circ F_X = (\theta \circ F)_X$. $\square$

Remark 3.1.10. We will write X for the stack $\underline{X}$, and for the diagram $X \to \mathcal{M}$ we understand a stack 1-morphism between the stacks $\underline{X}$ and $\mathcal{M}$.

Definition 3.1.11. Let G be a Lie group acting on a smooth manifold X via $\mu : G \times X \to X$. Then the *quotient stack* $[X/G]$ is defined by

$$[X/G](Y) := \langle (P \xrightarrow{p} Y, P \xrightarrow{f} X) \mid p \text{ forms a principal } G\text{-bundle, } f \text{ is } G\text{-equivariant} \rangle$$

Morphisms in this groupoid are G-equivariant isomorphisms, that means, if we have $(P \xrightarrow{p} Y, P \xrightarrow{f} X), (P' \xrightarrow{p'} Y, P' \xrightarrow{f'} X) \in [X/G]$ then a morphism between them consists of $\phi : P \to P'$ such that $p' \circ \phi = p$ and $f' \circ \phi = f$ with ϕ a G-equivariant morphism. Since it is possible to glue principal G-bundles, it follows $[X/G]$ is a stack.

On the other hand, if Y, Y' are smooth manifolds and $h : Y \to Y'$ then the morphism induced by the stack is

$$[X/G](h) : [X/G](Y') \to [X/G](Y)$$

$$(p, f) \mapsto (h^*p, qf)$$

where

$$\begin{tikzcd} h^*P \arrow[r, "q"] \arrow[d, "h^*p"] & P \arrow[r, "f"] \arrow[d, "p"] & X \\ Y \arrow[r, "h"] & Y' & \end{tikzcd}$$

Remark 3.1.12. For G acting trivially on $X = pt$ the quotient $[pt/G]$ is the stack $\mathcal{B}G$ classifying principal G-bundles. To check this, let Y be a smooth manifold then

$$[pt/G](Y) = \langle (P \xrightarrow{p} Y, P \xrightarrow{c} pt) \rangle = \langle (P \xrightarrow{p} Y) \rangle = \mathcal{B}G(Y)$$

and if $Y' \xrightarrow{h} Y$ is a morphism in **Diff** then

$$[pt/G](h) = h^*p : h^*P \to Y'.$$

Hence $[pt/G] = \mathcal{B}G$.

Proposition 3.1.13. *If the action of G is proper and free on X, the quotient stack $[X/G]$ and the smooth manifold X/G define the same stack.*

Proof. As the action is proper and free, $X \to X/G$ is a G-bundle. Let us consider $f : Y \to X/G$ a smooth morphism, so we can assign the pair $(f^*X \xrightarrow{f^*} Y, f^*X \xrightarrow{q} X)$ in the stack. Moreover, if we have the pair $(P \xrightarrow{p} Y, P \xrightarrow{f} X)$, we can define $\phi_f : Y \to X/G$ as follows, for each $w \in P$ then $\phi_f(p(w)) = \pi(f(w)) = f(w)G$.

Thus, we define $\Psi : X/G \to [X/G]$ for $Y \in \mathbf{Diff}$ as

$$X/G(Y) \xrightarrow{\Psi} [X/G](Y)$$

$$f \mapsto (f^*X \xrightarrow{f^*} Y, f^*X \xrightarrow{q} X)$$

and $\Phi : [X/G] \to X/G$ as

$$[X/G](Y) \xrightarrow{\Phi} X/G(Y)$$

$$(P \xrightarrow{p} Y, P \xrightarrow{f} X) \mapsto \phi_f$$

since $\Psi \circ \Phi \cong id_{[X/G](Y)}$ and $\Phi \circ \Psi = id_{X/G(Y)}$, so $[X/G] \cong X/G$ as stacks. $\square$

3.2 Fibered products

In order to get geometric properties of morphisms like embedding morphisms, open morphisms and others, it is necessary to define the fibered product stack, which allows us to relate differentiable stacks with Lie groupoids as well.

Definition 3.2.1. Let $F : \mathcal{M} \to \mathcal{N}$ and $F' : \mathcal{M}' \to \mathcal{N}$ be morphisms of stacks, so there is a diagram of morphisms of stacks given by

$$\begin{tikzcd} & \mathcal{M} \arrow[d, "F"] \\ \mathcal{M}' \arrow[r, "F'"] & \mathcal{N} \end{tikzcd}$$

and the *fibered product* $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ is defined as

$$\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'(X) = \langle (f, f', \phi) \mid X \xrightarrow{f} \mathcal{M}, X \xrightarrow{f'} \mathcal{M}', \phi : F \circ f \Rightarrow F' \circ f' \rangle.$$

We will show below that this definition gives us a stack where a morphism $(f, f', \phi) \to (g, g', \psi)$ is given by a pair of morphisms

$$(\Phi_{f,g} : f \to g, \Phi_{f',g'} : f' \to g')$$

such that

$$\psi \circ F(\Phi_{f,g}) = F'(\Phi_{f',g'}) \circ \phi.$$

Remark 3.2.2. We note that if $f, g \in \mathrm{Mor}_{\mathrm{St}}(X, \mathcal{M}) \cong \mathcal{M}(X)$ then $\Phi_{f,g}$ is a morphism and hence an isomorphism. Moreover, ψ is an isomorphism as well, because it is a morphism in $\mathrm{Mor}_{\mathrm{St}}(X, \mathcal{N}) \cong \mathcal{N}(X)$.

Proposition 3.2.3. *The fibered product is a stack.*

Proof.

1. Consider $\{U_i\}$ a covering of X and $(f_i, f_i', \phi_i) \in \mathcal{M} \times_{\mathcal{N}} \mathcal{M}'(U_i)$ with $(\Phi_{ij}, \Phi_{ij}') : (f_i, f_i', \phi_i)|_{U_{ij}} \to (f_j, f_j', \phi_j)|_{U_{ij}}$ which satisfy the cocycle condition. Then, there are $f : X \to \mathcal{M}(X)$ and $f' : X \to \mathcal{M}'(X)$, such that there exist $\Phi : f|_{U_i} \to f_i$ and $\Phi' : f'|_{U_i} \to f_i'$ isomorphisms with $\Phi|_{U_{ij}} = \Phi_{ij}$ and $\Phi'|_{U_{ij}} = \Phi_{ij}'$.

2. We consider $(f, f', \phi), (g, g', \psi) \in \mathcal{M} \times_{\mathcal{N}} \mathcal{M}'(X)$ and $\{U_i\}_{i \in I}$ a open covering of X with isomorphisms $(\Phi_i, \Phi_i') : (f, f', \phi)|_{U_i} \to (g, g', \psi)|_{U_i}$ such that $\Phi_i|_{U_{ij}} = \Phi_j|_{U_{ij}}$ and $\Phi_i'|_{U_{ij}} = \Phi_j'|_{U_{ij}}$, then there exist $\Phi : f \to g$ and $\Phi' : f' \to g'$ because they are morphisms in $\mathcal{M}(X)$ and $\mathcal{M}'(X)$. Besides, the condition $\psi_i \circ (F_i \circ \Phi_i) = F_i' \circ \Phi_i(\phi_i)$ holds because it glues like an object in $\mathcal{N}(X)$. $\square$

Example 3.2.4. Consider $pt \to \mathcal{B}G$ and $X \to \mathcal{B}G$. By the 2-Yoneda lemma, there is a one-to-one correspondence between $\mathrm{Mor}_{\mathrm{St}}(X, \mathcal{B}G)$ and $\mathcal{B}G(X)$. So if $P \in \mathcal{B}G(X)$ is a principal G-bundle on X, there exists F_P in $\mathrm{Mor}_{\mathrm{St}}(X, \mathcal{B}G)$ such that

$$pt \times_{\mathcal{B}G} X(Y) = \langle (f, g, \phi) \mid Y \xrightarrow{f} pt, Y \xrightarrow{g} X, \phi : F_P \circ g \Rightarrow F' \circ f \rangle.$$

Since $\mathrm{Mor}_{\mathrm{St}}(Y, X) \cong X(Y)$, g could be considered as an element in $X(Y)$ and

$$pt \times_{\mathcal{B}G} X(Y) \cong \langle (g, \phi) \mid Y \xrightarrow{g} X, \phi : g^*P \to F' \circ f \rangle$$

as we know that $\mathrm{Mor}_{\mathrm{St}}(Y, pt) \cong pt(Y)$ then $F' \circ f = f^*pt$. That means

$$pt \times_{\mathcal{B}G} X(Y) \cong \langle g \mid Y \xrightarrow{g} X, g^*P \cong f^*pt \cong G \times Y \rangle$$

$$\cong \langle g \mid Y \xrightarrow{g} X, s : Y \to g^*P \text{ a local section } \rangle$$

so if we could consider $\overline{g} = g^* \circ s$ then we get $pt \times_{\mathcal{B}G} X(Y) \cong \langle \overline{g} : Y \to P \rangle = P(Y)$. The last equation is because if $f : Y \to P$, then $\pi \circ f : Y \to X$ and, by the 2-Yoneda lemma, $\pi \circ f$ can be considered as an element in $\mathrm{Mor}_{\mathrm{St}}(Y, X)$.

Proposition 3.2.5. *[15, 8.8.4] Let $\mathcal{P}$ a pre-stack. If there are two morphisms of stacks $F : \mathcal{M} \to \mathcal{N}$, $F' : \mathcal{M}' \to \mathcal{N}$ and the following 2-commutative diagram of pre-stacks*

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\ \mathcal{M}' \arrow[r, "F'"] & \mathcal{N} \end{tikzcd}$$

then the stack $\tilde{\mathcal{P}}$ is isomorphic to the fibered product $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$.

We can now define the notion of a differentiable stack, see [28, 1.1] and [50, 2.10].

Definition 3.2.6. A stack $\mathcal{M}$ is called a *differentiable stack* if there is a smooth manifold X and a morphism of stacks $p : X \to \mathcal{M}$ between the stack associated to X, as in example 3.1.4 and the stack $\mathcal{M}$ such that:

1. For all morphism of stacks $Y \to \mathcal{M}$ the stack associated to $X \times_{\mathcal{M}} Y$ is isomorphic to a smooth manifold.
2. p is a submersion, i.e., for all $Y \to \mathcal{M}$ the projection $X \times_{\mathcal{M}} Y \to Y$ is a submersion.

The map $X \to \mathcal{M}$ is then called an *atlas* or a *presentation of* $\mathcal{M}$.

The first property in the definition above is going to be used to get morphisms with geometric properties.

Definition 3.2.7. A morphism of stacks $F : \mathcal{M} \to \mathcal{N}$ is called *representable*, if for any morphism of stacks $Y \to \mathcal{N}$, where Y is a smooth manifold, the fibered product $\mathcal{M} \times_{\mathcal{N}} Y$ is a stack, which is equivalent to a smooth manifold.

Example 3.2.8.

1. The map $pt \to \mathcal{B}G$ gives that $pt \times_{\mathcal{B}G} X \cong P$ by example 3.2.4. That is, $pt \to \mathcal{B}G$ is representable.
2. The map $F : \mathcal{B}G \to pt$ is not representable. Consider $Y = pt$, $F' : pt \to pt$ and Z a smooth manifold, then

$$\mathcal{B}G \times_{pt} pt(Z) = \langle (f, f', \phi) \mid Z \xrightarrow{f} \mathcal{B}GZ \xrightarrow{p} t, \phi : F' \circ f' \Rightarrow F \circ f \rangle$$

$$= \langle (f, \phi) \mid Z \xrightarrow{f} \mathcal{B}G, \phi : C_{pt} \Rightarrow F \circ f \rangle$$

where $C_{pt} : Z \to pt$. Since

$$\mathcal{B}G \times_{pt} pt(Z) \cong \mathrm{Mor}_{\mathrm{St}}(Z, \mathcal{B}G) \cong \mathcal{B}G(Z)$$

and $\mathcal{B}G$ is not a smooth manifold, therefore $F : \mathcal{B}G \to pt$ is not representable.

Proposition 3.2.9. *The quotient stack $[X/G]$ is a differentiable stack if X and G are smooth.*

Proof. We are going to check that $X \to [X/G]$ is an atlas, defined by the trivial G-bundle $G \times X \xrightarrow{q} X$ and by the action map $G \times X \xrightarrow{a} X$, where a is considered as a G-equivariant map.

1. $X \xrightarrow{F_q} [X/G]$ is representable. Let be Y, T smooth manifolds and $Y \xrightarrow{F_P} [X/G]$ then we have

   $$X \times_{[X/G]} Y(T) = \langle (f, f', \phi) \mid T \xrightarrow{f} X, T \xrightarrow{f'} Y, \phi : F_P f' \Rightarrow F_q f \rangle$$

   Consider $(p, h) \in [X/G](Y) \cong \mathrm{Mor}_{\mathrm{St}}(Y, [X/G])$ then, by the 2-Yoneda Lemma, we have $F_p(f') = ((f')^*P \xrightarrow{p^*} T, (f')^*P \xrightarrow{hf^*} X)$ and $F_q(f) = (f^*(G \times X) \xrightarrow{q^*} T, f^*(G \times X) \xrightarrow{af^*} X)$. As q is the trivial principal G-bundle, we know that $f^*(G \times X) \cong G \times T$. Since a morphism ϕ between $F_p(f')$ and $F_q(f)$ is an isomorphism we have $(f')^*P \cong G \times T$, thus $X \times_{[X/G]} Y(T) = \langle (f', \phi) \mid T \xrightarrow{f'} Y, (f')^*P \cong G \times T \rangle$. Hence

   $$X \times_{[X/G]} Y(T) \cong \langle (f', s) \mid T \xrightarrow{f'} Y, T \xrightarrow{s} (f')^*P \text{ is a section } \rangle$$

   and we can consider $\overline{f} = (f')^* \circ s : T \to P$ and $X \times_{[X/G]} Y(T) \cong P(T)$. Therefore $X \times_{[X/G]} Y(T) \cong P(T)$. That is, $X \to [X/G]$ is representable.

2. $X \to [X/G]$ is a submersion. We have to show that $P(T) \cong X \times_{[X/G]} Y(T) \to Y(T)$ is a submersion, but this is already given because $P \to Y$ is a submersion. □

Lemma 3.2.10.

1. *(Composition) If $F : \mathcal{K} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{N}$ are representable, then $F \circ G$ is representable.*

2. *(Pull-back) If $F : \mathcal{M} \to \mathcal{N}$ is representable, and $G : \mathcal{M}' \to \mathcal{N}$ is arbitrary, then the projection $\mathcal{M}' \times_{\mathcal{N}} \mathcal{M} \to \mathcal{M}'$ is representable.*

Proof.

1. For this, we need to check that $Y \times_{\mathcal{N}} \mathcal{K} \cong (Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{K}$. If we have a morphism $V : Y \to \mathcal{N}$ and T a smooth manifold, then

$$Y \times_{\mathcal{N}} \mathcal{K}(T) = \langle (f', f, \phi) \mid f' : T \to Y, f : T \to \mathcal{K}, \phi : Vf' \Rightarrow (G \circ F)f \rangle$$

and for $W : Y \times_{\mathcal{N}} \mathcal{M} \to \mathcal{M}$ we get

$$(Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{K}(T) = \langle (h', f, \psi) \mid h' : T \to Y \times_{\mathcal{N}} \mathcal{M}, f : T \to \mathcal{K}, \psi : Wh' \Rightarrow Ff \rangle.$$

Then, by the 2-Yoneda Lemma $h' \in \mathrm{Mor}_{\mathrm{St}}(T, Y \times_{\mathcal{N}} \mathcal{M}) \cong Y \times_{\mathcal{N}} \mathcal{M}(T)$, that is $h' = (f', Ff, \phi)$ with $\phi : Vf' \to G(Ff)$.

Thus, we get the next equivalence,

$$(Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{K}(T) \cong$$

$$\langle (f', Ff, \phi, f, \psi) \mid T \xrightarrow{f} Y, T \xrightarrow{Ff} \mathcal{M}, \phi : Vf' \Rightarrow G(Ff), T \xrightarrow{f} \mathcal{K}, \psi : Wh' \Rightarrow Ff \rangle$$

$$\cong \langle (f', Ff, \phi, f, \psi) \mid T \xrightarrow{f'} Y, T \xrightarrow{Ff} \mathcal{M}, \phi : Vf' \Rightarrow G(Ff), T \xrightarrow{f} \mathcal{K}, \psi : Ff \Rightarrow Ff \rangle$$

$$\cong \langle (f', \phi, f) \mid T \xrightarrow{f'} Y, \phi : Vf' \Rightarrow G(Ff), T \xrightarrow{f} \mathcal{K}, \rangle.$$

$$= Y \times_{\mathcal{N}} \mathcal{K}(T)$$

2. We need to note that $Y \times_{\mathcal{M}'} (\mathcal{M}' \times_{\mathcal{N}} \mathcal{M}) \cong Y \times_{\mathcal{N}} \mathcal{M}$.

First, we observe that $Y \times_{\mathcal{M}'} \mathcal{M}'(T) \cong Y(T)$ for $T \in \textbf{Diff}$ and $Y \xrightarrow{W} \mathcal{M}'$, because

$$Y \times_{\mathcal{M}'} \mathcal{M}'(T) = \langle (f', f, \zeta) | T \xrightarrow{f'} Y, T \xrightarrow{f} \mathcal{M}', \zeta : Wf' \Rightarrow f \rangle \cong Y(T)$$

Second, $Y \times_{\mathcal{M}'} (\mathcal{M}' \times_{\mathcal{N}} \mathcal{M}) \cong (Y \times_{\mathcal{M}'} \mathcal{M}') \times_{\mathcal{N}} \mathcal{M}$

$$(Y \times_{\mathcal{M}'} \mathcal{M}') \times_{\mathcal{N}} \mathcal{M}(T)$$

$$= \langle (u', u, \phi) | T \xrightarrow{u'} Y \times_{\mathcal{M}'} \mathcal{M}', T \xrightarrow{u} M, \phi : G \circ W \circ \pi_1(u') \Rightarrow Fu \rangle$$

$$\langle (v', v, \theta, u, \phi) | T \xrightarrow{v'} Y, T \xrightarrow{v} \mathcal{M}', \theta : v \Rightarrow Wv', T \xrightarrow{u} M, \phi : G \circ Wv' \Rightarrow Fu \rangle$$

Here, we are using the 2-Yoneda lemma for $u' \in \mathrm{Mor}_{\mathrm{St}}(T, Y \times_{\mathcal{M}'} \mathcal{M}') \cong Y \times_{\mathcal{M}'} \mathcal{M}'(T)$, thus $u' = (v', v, \theta)$. On the other hand,

$$Y \times_{\mathcal{M}'} (\mathcal{M}' \times_{\mathcal{N}} \mathcal{M})(T) = \langle (f', h, \psi) | T \xrightarrow{f'} Y, T \xrightarrow{h} \mathcal{M}' \times_{\mathcal{N}} \mathcal{M}, \psi : W \circ f' \Rightarrow \pi_1 \circ h \rangle$$

$$\cong \langle (f', g', f, \mu, \psi) | T \xrightarrow{f'} Y, T \xrightarrow{g'} \mathcal{M}', T \xrightarrow{f} \mathcal{M}, \psi : W \circ f' \Rightarrow g', \mu : Gg' \Rightarrow Ff \rangle$$

$$\cong \langle (f', g', f, \mu, \psi) | T \xrightarrow{f'} Y, T \xrightarrow{g'} \mathcal{M}', T \xrightarrow{f} \mathcal{M}, \psi : W \circ f' \Rightarrow g', \mu : G \circ W f' \Rightarrow Ff \rangle$$

Therefore, $Y \times_{\mathcal{M}'} (\mathcal{M}' \times_{\mathcal{N}} \mathcal{M}) \cong Y \times_{\mathcal{N}} \mathcal{M}$. $\square$

Definition 3.2.11. A representable morphism $\mathcal{M} \to \mathcal{N}$ is an *open embedding*, if for an atlas $Y \to \mathcal{N}$ the map $\mathcal{M} \times_{\mathcal{N}} Y \to Y$ is an open embedding.

Remark 3.2.12. The previous definition can be used in the same way for different properties such as *closed embedding*, *submersion* or *proper* and it does not depend of the atlas [28, 2], [50, 2.2]. We can see this independence in the following example.

Example 3.2.13. If $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds, we have that any submersion map is representable because transversality and the usual notion of open embedding agree with the representable notion. To check this:

1. We assume that $M \xrightarrow{f} N$ is an open embedding and consider $Y \xrightarrow{g} N$ an atlas, then the map $M \times_N Y = \{(m, y) \in M \times Y \mid f(m) = g(y)\} \xrightarrow{\pi_2} Y$ has image equal to $g^{-1}(f(M))$. Thus, if $f(M)$ is open then $\pi_2(M) = g^{-1}(f(M))$ is also open.

   We need to check that π_2 is diffeomorphic to its image. It is injective because if $\pi_2(m, a) = \pi_2(n, b)$ implies that $a = b$ so $f(m) = g(a) = g(b) = f(n)$ since f is injective then $m = n$. Therefore π_2 is injective.

   We know that there is a well-defined map $\pi_2^{-1} : \pi_2(M \times_N Y) \to M \times_N Y$ given by $y \mapsto (f^{-1}(g(y)), y)$ and because g and f^{-1} are smooth, then π_2 is also smooth.

   Therefore π_2 is an open embedding.

2. Now if $Y \xrightarrow{g} N$ is an atlas then $M \times_N Y \xrightarrow{\pi_2} Y$ is an open embedding. Therefore $\pi_2(M) = g^{-1}(f(M))$ is open. But g is an atlas, so g is surjective and open, hence $g(g^{-1}(f(M))) = f(M)$.

   We have that f is injective, because if $f(m) = f(m')$ then there are y, y' such that $f(m) = g(y)$ and $f(m') = g(y')$. Therefore $(m, y), (m', y) \in M \times_N Y$ and thus, $\pi_2(m', y) = \pi_2(m, y)$. Since π_2 is injective $m' = m$ and f is injective.

As π_2^{-1} is smooth and $\pi_2^{-1}(y) = (f^{-1}(g(y)), y)$ then we have that $f^{-1}g$ and g are smooth. Besides we know that g is a quotient map and f^{-1} is smooth.

Definition 3.2.14. A morphism $\mathcal{M} \to \mathcal{N}$ of differentiable stacks is *smooth* if for an atlas $X \to \mathcal{M}$ the composition $X \to \mathcal{N}$ is smooth, i.e., for an atlas $Y \to \mathcal{N}$ the fibered product $X \times_{\mathcal{N}} Y \to Y$ is smooth.

Definition 3.2.15. A *principal G-bundle over a stack* $\mathcal{M}$ is given by a principal G-bundle (P_X, π_X, G) over X, where $X \to \mathcal{M}$ is an atlas for $\mathcal{M}$, together with an isomorphism of the two pullbacks of $\phi_{12} : p_1^* P_X \to p_2^* P_X$ on $X \times_{\mathcal{M}} X$ satisfying the cocyle condition on $X \times_{\mathcal{M}} X \times_{\mathcal{M}} X$, that is $\phi_{12} \circ \phi_{23} = \phi_{13}$.

The groupoid of principal G-bundles over $\mathcal{M}$ is denoted by $\mathcal{B}un_G(\mathcal{M})$.

Remark 3.2.16. The same definition can be applied to vector bundles. Since each rank n vector bundle can be seen as a principal Gl_n-bundle where GL_n is the general n-linear group. See [31, I.5.3.2].

Example 3.2.17. For each principal G-bundle P_X over a differentiable stack $\mathcal{M}$ with an atlas $X \to \mathcal{M}$, there exists a differentiable stack given by

$$\mathcal{P} : \mathbf{Diff}^{op} \to \mathbf{Grpds}$$

such that $\mathcal{P}(T) = \langle (f : T \to \mathcal{M}, s : T \to P_{T,f} \text{ a local section }) \rangle$ where $P_{T,f}$ comes for gluing.

Remark 3.2.18. For any $f : T \to \mathcal{M}$ we can define a principal G-bundle $(P_{T,f}, \pi, G)$ over T, since $X \times_{\mathcal{M}} T \to T$ has local sections. We consider $\{U_i \hookrightarrow T\}_{i \in I}$ a covering then there exist sections $U_i \xrightarrow{s_i} X \times_{\mathcal{M}} T$. For each of these sections we notice that the arrow $U_i \xrightarrow{s_i} X \times_{\mathcal{M}} T \xrightarrow{p_1} X$ which allows us to pullback with P_X. The gluing of this collection of pullbacks gives us the principal G-bundle. Therefore, this automatically defines a differentiable stack $\mathcal{P} \xrightarrow{p} \mathcal{M}$ where

$$\mathcal{P}(T) = \langle (f : T \to \mathcal{M}, s : T \to P_{T,f} \text{ a local section }) \rangle$$

An atlas of this stack is given by $(\mathcal{P}_X, \mathcal{P}_X \xrightarrow{\Delta} \mathcal{P}_X \times_X \mathcal{P}_X)$. This shows that universal bundles on stacks classifying principal G-bundles exist.

Example 3.2.19.

1. $pt \to [pt/G]$ gives a G-bundle over pt and it is, in fact, the trivial bundle.

2. We have the atlas $pt \to \mathcal{B}G$, so a G-bundle associated to $\mathcal{B}G$ is given by

$$\begin{tikzcd} G \times pt \arrow[d, "p"] & \\ pt \arrow[r] & \mathcal{B}G \end{tikzcd}$$

By the above remark, we have that the associated stack is

$$\mathcal{E}G(T) =< T \xrightarrow{f} \mathcal{B}G, T \xrightarrow{s} P_{T,f} \text{ a section } >$$

and an atlas for $\mathcal{E}G$ is given by $\alpha : G \times pt \to \mathcal{E}G$ such that the diagram

$$\begin{tikzcd} G \times pt \arrow[r, "\alpha"] \arrow[d, "p"] & \mathcal{E}G \arrow[d] \\ pt \arrow[r] & \mathcal{B}G \end{tikzcd}$$

commutes.

Theorem 3.2.20. *There exists an equivalence of categories given by*

$$Maps_{St}(\mathcal{M}, \mathcal{B}G) \cong \mathcal{B}un_G(\mathcal{M})$$

Proof. Let G be a Lie group and the classifying stack $\mathcal{B}G$ with its atlas $pt \to \mathcal{B}G$. By The 2-Yoneda Lemma we have that for $X \in \mathbf{Diff}$ we get

$$\text{Maps}_{St}(X, \mathcal{B}G) \xrightarrow{\phi} \mathcal{B}G(X)$$

If we have $\Theta \in \text{Maps}_{St}(\mathcal{M}, \mathcal{B}G)$ and atlas $X \xrightarrow{a} \mathcal{M}$, we get $\Theta \circ a \in \text{Maps}_{St}(X, \mathcal{B}G)$ and so $\phi(\Theta \circ a) \in \mathcal{B}G(X)$. Hence, we get $\phi(\Theta \circ a)$ a principal G-bundle over X, that is, an element in $\mathcal{B}un_G(\mathcal{M})$. Observe that this principal G-bundle comes via the 2-diagram

$$\begin{tikzcd} X \times_{\mathcal{B}G} G \arrow[rr] \arrow[dd] & & G \arrow[d] \\ & & pt \arrow[d] \\ X \arrow[r, "a"] & \mathcal{M} \arrow[r, "\Theta"] & \mathcal{B}G \end{tikzcd}$$

On the other hand, if we have a principal G-bundle $P_X \xrightarrow{p} X \to \mathcal{M}$, we can define a morphism between $\mathcal{M}$ and $\mathcal{B}G$ considering the following 2-diagram

$$\begin{array}{ccccc}
T \times_{\mathcal{M}} P_X & \longrightarrow & P_X & \longrightarrow & G \\
\Big\downarrow f^* & & \Big\downarrow p & & \Big\downarrow \\
 & & X & & pt \\
 & & \Big\downarrow a & & \Big\downarrow \\
T & \xrightarrow{f} & \mathcal{M} & \xrightarrow{\Psi_p} & \mathcal{B}G
\end{array}$$

where Ψ_p is defined as $\Psi_p(f) = f^*$.

Due to the last two diagrams, we can notice that $\Psi_{\phi(\Theta \circ a)}(f) \cong \Theta(f)$ and $\phi(\Psi_p \circ a) \cong p$. Hence, we get the result. $\square$

3.3 Morita equivalence

Let $X \to \mathcal{M}$ be an atlas of the differentiable stack $\mathcal{M}$. We add another structure which allow to define a Lie groupoid associated to $\mathcal{M}$ with the next structure, the two projections $p_1, p_2 : X \times_{\mathcal{M}} X \to X$ as the source and target map respectively, and $1 = \Delta : X \to X \times_{\mathcal{M}} X$ the identity of the Lie groupoid. The multiplication is given by

$$(X \times_{\mathcal{M}} X) \times_X (X \times_{\mathcal{M}} X)(T) \cong X \times_{\mathcal{M}} X \times_{\mathcal{M}} X(T) \to X \times_{\mathcal{M}} X(T)$$

$$((f, f', \eta), (g, g', \xi), \varphi) \mapsto (f, g', \varphi \circ \eta \circ \xi)$$

for any smooth manifold T. Besides, if (f, f', η), its inverse is (f', f, η^{-1}). This Lie groupoid will be denoted by $(X_1 \rightrightarrows X_0) = (X \times_{\mathcal{M}} X \rightrightarrows X)$.
Conversely, for any groupoid $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ we can define a differentiable stack

$$[\Gamma_0/\Gamma_1](Y) = \langle (P \xrightarrow{p} Y, P \xrightarrow{f} \Gamma_0) \text{ where } p \text{ is a principal } \Gamma\text{-bundle along } f \ \rangle$$

as in section 2.4.2.

Lemma 3.3.1. *The unit Γ-bundle $\Gamma_1 \to \Gamma_0$ induces a map $\Gamma_0 \xrightarrow{\pi} [\Gamma_0/\Gamma_1]$ which is an atlas for $[\Gamma_0/\Gamma_1]$, the map π is the map given by the pullback Γ-bundle over $[\Gamma_0/\Gamma_1]$. The groupoids Γ and $\Gamma_{0,\bullet}$ are canonically isomorphic.*

Proof. Let $Y \xrightarrow{f_P} [\Gamma_0/\Gamma_1]$ be given by a bundle P in lemma 3.1.9. Then

$$(\Gamma_0 \times_{[\Gamma_0/\Gamma_1]} Y)(T) \cong \langle (T \xrightarrow{f} Y, T \xrightarrow{g} \Gamma_0), \varphi : f_P \circ f \xrightarrow{\cong} \pi \circ g \rangle$$

$$\cong \langle (f, g, \varphi : f^*P \cong g^*\Gamma_1) \rangle$$

$$\cong \langle (f, T \xrightarrow{\overline{f}} P) | Pr_Y \circ \overline{f} = f \rangle$$

$$\cong \{\overline{f} : T \to P\} = P(T).$$

The groupoids Γ and $\Gamma_{0,\bullet}$ are canonically isomorphic because $\Gamma_0 \xrightarrow{\pi} [\Gamma_0/\Gamma_1]$ is an atlas, then $(\Gamma_0 \times_{[\Gamma_0/\Gamma_1]} \Gamma_0 \rightrightarrows \Gamma_0)$ is the groupoid $\Gamma_{0,\bullet}$. Therefore, $\Gamma_0 \times_{[\Gamma_0/\Gamma_1]} \Gamma_0(T) \cong \Gamma_1(T)$, for any smooth manifold T. $\square$

Definition 3.3.2. Let Γ and Γ' be Lie groupoids. A morphism of groupoids $\phi = (\phi_1, \phi_0) : \Gamma \to \Gamma'$ is called a *Morita morphism*, if:

1. $\phi_0 : \Gamma_0 \to \Gamma'_0$ is a surjective submersion.
2. the diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "s\times t"] \arrow[d, "\phi_1"] & \Gamma_0 \times \Gamma_0 \arrow[d, "\phi_0\times\phi_0"] \\ \Gamma'_1 \arrow[r, "s'\times t'"] & \Gamma'_0 \times \Gamma'_0 \end{tikzcd}$$

   commutes.

We say that a Morita morphism $\phi_\bullet = (\phi_1, \phi_0) : \Gamma \to \Gamma'$ *admits a section* if there exists $s : \Gamma'_0 \to \Gamma_0$ such that $s \circ \phi_0 = id_{\Gamma'_0}$.

Definition 3.3.3. Two Lie groupoids Γ and Γ' are called *Morita equivalent*, if there exists a third Lie groupoid Γ'' and Morita morphisms $\Gamma'' \to \Gamma$ and $\Gamma'' \to \Gamma'$. That is, there exists a diagram

$$\begin{tikzcd} & \Gamma'' \arrow[dl] \arrow[dr] & \\ \Gamma & & \Gamma' \end{tikzcd}$$

of Lie groupoids.

Theorem 3.1. *Let Γ and Γ' be Lie groupoids. Let $[\Gamma_0/\Gamma_1]$ and $[\Gamma'_0/\Gamma'_1]$ be the associated differentiable stacks. Then the following are equivalent:*

i. The differentiable stacks $[\Gamma_0/\Gamma_1]$ and $[\Gamma'_0/\Gamma'_1]$ are isomorphic.

ii. The Lie groupoids Γ and Γ' are Morita equivalent.

iii. *There exists a smooth manifold M with two smooth maps $p : M \to \Gamma_0$ and $p' : M \to \Gamma'_0$, and actions of Γ_1 and Γ'_1 such that M is a left Γ-principal bundle over Γ'_0 by p', and a right Γ'-principal bundle over Γ_0 by p. Such M is called a Γ-Γ'-principal bundle.*

Proof. 1. *i.* implies *iii.* Consider the atlases $\Gamma_0 \xrightarrow{\pi} [\Gamma_0/\Gamma_1]$ and $\Gamma'_0 \xrightarrow{\pi'} [\Gamma'_0/\Gamma'_1]$ and suppose we have an isomorphism

$$\psi : [\Gamma'_0/\Gamma'_1] \to [\Gamma_0/\Gamma_1]$$

then $\Gamma'_0 \xrightarrow{\psi\circ\pi'} [\Gamma_0/\Gamma_1]$ is an atlas. Hence, it is possible to consider the smooth manifold $\Gamma_0 \times_{[\Gamma_0/\Gamma_1]} \Gamma'_0$. This smooth manifold is a right Γ'-principal bundle over Γ_0 with the following structure

$$\begin{tikzcd} \Gamma_0 \times_{[\Gamma_0/\Gamma_1]} \Gamma'_0 \arrow[r, "p_2"] \arrow[d, "p_1"] & \Gamma'_0 \\ \Gamma_0 & \end{tikzcd}$$

2. *iii.* implies *ii.* Let M be a smooth manifold with the following right Γ'-principal bundle

$$\begin{tikzcd} M \arrow[r, "a'"] \arrow[d, "p"] & \Gamma'_0 \\ \Gamma_0 & \end{tikzcd}$$

and the left Γ-principal bundle

$$\begin{tikzcd} M \arrow[r, "a"] \arrow[d, "p'"] & \Gamma_0 \\ \Gamma'_0 & \end{tikzcd}$$

Consider $M_1 = \Gamma'_1 \times_{\Gamma'_0,s'} M \times_{\Gamma_0,t} \Gamma_1$. This smooth manifold is a groupoid $\overline{M}$ over M with structure given by $\overline{s} = \overline{t} = p_2$, that is, the source and the target map are the same projection in the second component. The identity map is $\overline{1}(m) = (1'(p'(m)), m, 1(p(m))$ where 1 and $1'$ are the identity maps of Γ and Γ' respectively. The multiplication is

$$M_1 \times_M M_1 \to M_1$$

$$((h', m'h), (g', m, g)) \mapsto (h'g', m, gh)$$

where every function here is smooth, so $(M_1 \rightrightarrows M)$ is a Lie groupoid. A Morita equivalence between $\overline{M}$ and Γ is given by $\varphi = (p_3, p \times p) : \overline{M} \to \Gamma$, and a Morita equivalence between $\overline{M}$ and Γ' is $\varphi' = (p_3, p' \times p') : \overline{M} \to \Gamma'$.

3. *ii.* implies *iii.* To prove this, it is necessary to show two things. First, if $\phi : \Gamma'' \to \Gamma$ is a Morita morphism, then $M = \Gamma''_0 \times_{\Gamma_0, t} \Gamma_1$ is a Γ''-Γ-principal bundle over Γ''_0 with the following structure of a Γ-principal bundle

$$\begin{array}{ccc} M & \xrightarrow{t \circ p_2} & \Gamma_0 \\ \Big\downarrow{\scriptstyle p_1} & & \\ \Gamma''_0 & & \end{array}$$

and as a Γ'-principal bundle

$$\begin{array}{ccc} M & \xrightarrow{p_1} & \Gamma''_0 \\ \Big\downarrow{\scriptstyle t \circ p_2} & & \\ \Gamma_0 & & \end{array}$$

Second, if M is a Γ-Γ''-principal bundle and $\widehat{M}$ is Γ''-Γ'-principal bundle then

$$M \wedge^{\Gamma''_0} \widehat{M} = M \times_{\Gamma''_0} \widehat{M} / \sim$$

is a Γ-Γ'-principal bundle, where $\sim$ is the equivalence relation defined by

$$(\mu''(u, \psi''), v) \sim (u, \overline{\mu''}(\psi'', v))$$

where μ'' and $\overline{\mu''}$ are right and left Γ''-actions on M and $\widehat{M}$, respectively. For more details about the smooth structure of $M \wedge^{\Gamma''_0} \widehat{M}$ we follow [8, 2.11]. If M is a left Γ-principal bundle with

$$\begin{array}{ccc} M & \xrightarrow{a} & \Gamma_0 \\ \Big\downarrow{\scriptstyle p''} & & \\ \Gamma''_0 & & \end{array}$$

then the structure of a left Γ-principal bundle of $M \wedge^{\Gamma''_0} \widehat{M}$ is given by

$$\begin{array}{ccc} M \wedge^{\Gamma''_0} \widehat{M} & \xrightarrow{a \circ p_1} & \Gamma_0 \\ \Big\downarrow{\scriptstyle \pi \circ p_2} & & \\ \Gamma'_0 & & \end{array}$$

with the action

$$\Theta : \Gamma_1 \times_{\Gamma_0,s} M \wedge^{\Gamma_0''} \widehat{M} \to M \wedge^{\Gamma_0''} \widehat{M}$$

$$(\psi, [u, v]) \mapsto [\mu(\psi, u), v],$$

where μ is the left Γ-action on M.

In the same way, a structure of a right Γ'-principal bundle can be given to $M \wedge^{\Gamma_0''} \widehat{M}$.

4. *iii*. implies *i*. Given a Γ-principal bundle F over U. If we consider $E = M \wedge^{\Gamma_0} F$ then E is a Γ'-principal bundle over U, where the equivariant map is $E \xrightarrow{a} \Gamma_0'$ given by $a([m, f]) = p'(m)$ and the action is $[m, f]g' = [mg', f]$. Therefore, for any element in $[\Gamma_0/\Gamma_1](U)$, we can obtain an element in $[\Gamma_0'/\Gamma_1'](U)$. Besides, any morphism between elements of $[\Gamma_0/\Gamma_1](U)$ induces a morphism between elements in $[\Gamma_0'/\Gamma_1']$.

   This functor gives an equivalence of categories. $\square$

3.4 Cohomology theories for differentiable stacks

3.4.1 Sheaf cohomology

Before we can define a sheaf for a differentiable stack we need to define the site $\mathcal{M}_s$. See [6, 3.1] and [50, 3.1].

Definition 3.4.1. Let $\mathcal{M}$ be a differentiable stack. The site $\mathcal{M}_s$ on $\mathcal{M}$ is defined as the following category:

1. The objects are given as pairs (U, u), where U is a smooth manifold and $u : U \to \mathcal{M}$ is a morphism of stacks.
2. The morphisms are given as pairs $(\phi, \alpha) : (U, u) \to (V, v)$, where $\phi : U \to V$ is a local diffeomorphism and $\alpha : u \Rightarrow v \circ \phi$ is a 2-isomorphism, i.e. there is a 2-commutative diagram of the form

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "u"'] \arrow[rr, phantom, shift right=3ex, "\Rightarrow\alpha"] & & V \arrow[dl, "v"] \\ & \mathcal{M} & \end{tikzcd}$$

3. The coverings of an object (U, u) are families of morphisms

$$\{(\phi_i, \alpha_i) : (U_i, u_i) \to (U, u)\}_{i \in I}$$

such that the morphism

$$\bigsqcup_{i \in I} \phi_i : \bigsqcup_{i \in I} U_i \to U$$

is surjective.

Remark 3.4.2. For the covering $\{(\phi_i, \alpha_i) : (U_i, u_i) \to (U, u)\}_{i \in I}$, we use the notation $\{U_i \to U\}$.

The notion of sheaf on stacks can be expressed in two ways. We consider both and we check that these two approaches are equivalent.

Definition 3.4.3. Let $\mathcal{M}_s$ be the site of the stack $\mathcal{M}$. $\mathcal{F} : \mathcal{M}_s \to \mathbf{Ab}$ is a *sheaf* over the site J in abelian groups $\mathbf{Ab}$ if:

1. (Presheaf). $\mathcal{F}$ is a contravariant functor.
2. (a) If $\{U_i \to U\}$ covering of U and $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} \cong t|_{U_i}$ for all i, then $s \cong t$.
   (b) If $\{U_i \to U\}$ covering of U and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{ij}} \cong s_j|_{U_{ij}}$ then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} \cong s_i$, for all i.

Definition 3.4.4. A *sheaf* $\mathcal{F}$ *on a stack* $\mathcal{M}$ is a collection of sheaves $\mathcal{F}_{X \to \mathcal{M}}$ in $\mathbf{Ab}$ for any $X \to \mathcal{M}$ such that for any triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] \arrow[rr, phantom, shift right=3, "\overset{\phi}{\Rightarrow}"] & & Y \arrow[dl, "g"] \\ & \mathcal{M} & \end{tikzcd}$$

there is a morphism of sheaves $\Phi_{\phi,f} : f^*\mathcal{F}_{Y \to \mathcal{M}} \to \mathcal{F}_{X \to \mathcal{M}}$ such that for

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] \arrow[r, phantom, shift right=3, "\overset{\phi}{\Rightarrow}"] & Y \arrow[r, "g"] \arrow[d] \arrow[r, phantom, shift right=3, "\overset{\psi}{\Rightarrow}"] & Z \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

it holds $\Phi_{\phi,f} \circ f^*\Phi_{\psi,g} = \Phi_{\phi \circ f^*\psi, f \circ g}$.

Remark 3.4.5. The sheaf $\mathcal{F}$ is called *Cartesian* if $\Phi_{\phi,f}$ are isomorphisms.

Proposition 3.4.6. *Both definitions for a sheaf on a stack are equivalent.*

Proof. 3.4.3 $\Rightarrow$ 3.4.4. If we have a sheaf defined on $\mathcal{M}$, $\mathcal{F} : \mathcal{M}_s \to \mathbf{Ab}$, we can define for each $X \to \mathcal{M}$ a sheaf by

$$\mathcal{F}_{X\to\mathcal{M}} : X_s \to \mathbf{Ab}$$

$$(U \to X) \mapsto \mathcal{F}_{X\to\mathcal{M}}(U) = \mathcal{F}(U)$$

This is a sheaf since we are considering local diffeomorphism and the properties of $\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & \mathcal{M} & \end{tikzcd}$$

We get that there exists a morphism of sheaves $\Phi_{\phi,f} : f^*\mathcal{F}_{Y\to\mathcal{M}} \to \mathcal{F}_{X\to\mathcal{M}}$ given by the usual induced morphism of sheaves and it gives the required property.

3.4.4 $\Rightarrow$ 3.4.3. If we have a sheaf for any $X \to \mathcal{M}$, we define $\mathcal{F} : \mathcal{M}_s \to \mathbf{Ab}$ as

$$(U \to \mathcal{M}) \mapsto \mathcal{F}(U) = \mathcal{F}_{U\to\mathcal{M}}(U)$$

As our site is given by local diffeomorphisms and $\mathcal{F}_{U\to\mathcal{M}}$ is a sheaf, we get the result. $\square$

Definition 3.4.7. A *morphism of sheaves* $h : \mathcal{F} \to \mathcal{F}'$ on $\mathcal{M}_s$ is a collection of morphisms of sheaves $h_{X,x} : \mathcal{F}_{X,x} \to \mathcal{F}'_{X,x}$ on X for any morphism $X \to \mathcal{M}$ with X a smooth manifold, i.e. for all $(X,x) \in \mathcal{M}_s$ which are compatible with $\Phi_{\phi,f}$ and $\Phi'_{\phi,f}$ in the following way

$$\begin{tikzcd} f^*\mathcal{F}_{Y,y} \arrow[r, "f^*h_{X,x}"] \arrow[d, "\Phi_{\phi,f}"] & f^*\mathcal{F}'_{Y,y} \arrow[d, "\Phi'_{\phi,f}"] \\ \mathcal{F}_{X,x} \arrow[r, "h_{X,x}"] & \mathcal{F}'_{X,x} \end{tikzcd}$$

Remark 3.4.8. We denote by $\mathbf{Sh}/\mathcal{M}$ the category of sheaves of Abelian groups over the differentiable stack $\mathcal{M}$.

Example 3.4.9. Let $\mathcal{M}$ be a differentiable stack. We can define the q-th De Rham sheaf Ω^q_{DR} over $\mathcal{M}$ setting that for $U \to \mathcal{M}$ in the site. Let $(\Omega^q_{DR})_U = \Omega^q_U$ be the sheaf of q-forms on U. Observe that if there is $f : U \to V$ such that

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "h"'] & & V \arrow[dl, "g"] \\ & \mathcal{M} & \end{tikzcd}$$

there exists an isomorphism $\Phi_f : f^*\Omega^q_V \to \Omega^q_U$, since f is a local diffeomorphism. Hence, the sheaf Ω^q_{DR} is Cartesian.

Definition 3.4.10. Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$. Let $\{f_i : U_i \to U\}_{i\in I}$ be an covering in the site $\mathcal{M}_s$. A *descent datum* $(\mathcal{F}_i, \phi_{i,j})$ for sheaves $\mathcal{F}_i$ on U_i is a collection of sheaf morphisms $\phi_{i,j} : p_i^*\mathcal{F}_i \to p_j^*\mathcal{F}_j$ on U_{ij} with $p_i : U_{ij} \to U_i$ and $p_j : U_{ij} \to U_j$, satisfying the cocycle condition

$$\phi_{i,j}|_{U_{ijk}} \circ \phi_{j,k}|_{U_{ijk}} = \phi_{i,k}|_{U_{ijk}}$$

with U_{ijk} for all $i, j, k \in I$.

The next result allows us to consider a sheaf $\mathcal{F}$ over an atlas $X \to \mathcal{M}$, as a way to get an atlas on the differentiable stack $\mathcal{M}$. Moreover, it states that if we have a sheaf on the atlas $X \to \mathcal{M}$, we can define a sheaf on the nerve of the Lie groupoid. For further details, we refer to [28, 4.3] and [37, 12.4.5].

Proposition 3.4.11. *A cartesian sheaf $\mathcal{F}$ is the same as a sheaf $\mathcal{F}_X$ on some atlas $X \to \mathcal{M}$ together with a descent datum, that is, $\Phi : pr_1^*\mathcal{F}_X \to pr_2^*\mathcal{F}_X$ on $X \times_{\mathcal{M}} X$, which satisfies the cocycle condition on $X^{\times^3_{\mathcal{M}}}$.*

Remark 3.4.12. The sheaf induced on $X_\bullet$ by the sheaf $\mathcal{F}$ on $\mathcal{M}$ will be denoted by $\mathcal{F}_\bullet$.

Definition 3.4.13. The *global sections of a Cartesian sheaf on* $\mathcal{M}$ can be defined as

$$\Gamma(\mathcal{M}, \mathcal{F}) = \mathrm{Ker}(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X \times_{\mathcal{M}} X, \mathcal{F}))$$

Lemma 3.4.14. *For a cartesian sheaf $\mathcal{F}$ on $\mathcal{M}$ the group $\Gamma(\mathcal{M}, \mathcal{F})$ does not depend on the choice of the atlas.*

Proof. Let $X \to \mathcal{M}$ be an atlas for the differentiable stack $\mathcal{M}$. We consider an atlas $X' \to \mathcal{M}$, which factors as $X' \xrightarrow{f} X \to \mathcal{M}$ such that f has local sections.

Since $\mathcal{F}$ is a cartesian sheaf and we have $f \circ s = id_{X'}$, we have the isomorphism $\Phi_{\phi, f\circ s}\mathcal{F}(X') \to (f \circ s)^*(X) = id^*_{X'}\mathcal{F}(X) = \mathcal{F}(X)$. Thus, any global section on X' induces one on X and vice-versa. $\square$

For a not necessarily Cartesian sheaf $\mathcal{F}$ over the differentiable stack $\mathcal{M}$ we have the next definition. See [28, 4] and [50, 3.10].

Definition 3.4.15. The *set of global sections of a sheaf on* $\mathcal{M}$ is defined as

$$\Gamma(\mathcal{M}, \mathcal{F}) = \lim_{\leftarrow} \Gamma(X, \mathcal{F}_{X \to \mathcal{M}})$$

This limit is taken over all atlases $X \to \mathcal{M}$.

Remark 3.4.16. $\Gamma(\mathcal{M}, \mathcal{F})$ is an inverse limit.

If $\{\mathcal{F}_{X \to \mathcal{M}}\}_{X \to \mathcal{M}}$ is a collection of sheaves over all the atlases of $\mathcal{M}$ related to the sheaf $\mathcal{F}$ on $\mathcal{M}$, then we can express this inverse limit as

$$\Gamma(\mathcal{M}, \mathcal{F}) = \left\{ (a_{x'}) \in \prod \Gamma(X', \mathcal{F}_{X' \to \mathcal{M}}) \mid \Phi_{f,\phi}(a_{x'}) = a_x \text{ with } X \xrightarrow{f} X' \right\}.$$

Lemma 3.4.17. *For a Cartesian sheaf* $\mathcal{F}$ *on a stack* $\mathcal{M}$ *the two notions of global sections coincide.*

Proof. We define two functions

$$\begin{tikzcd} \lim\limits_{\leftarrow} \Gamma(X', \mathcal{F}_{X' \to \mathcal{M}}) \arrow[r, bend left, "q"] & \mathrm{Ker}(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X \times_{\mathcal{M}} X, \mathcal{F})) \arrow[l, bend left, "p"] \end{tikzcd}$$

For $(a_{x'}) \in \lim\limits_{\leftarrow} \Gamma(X', \mathcal{F}_{X' \to \mathcal{M}})$, we can consider the atlas $X \to \mathcal{M}$ and so $a_x \in \Gamma(X, \mathcal{F}_{X \to \mathcal{M}})$. Then we define $q(a_{x'}) = a_x$ since $X \times_{\mathcal{M}} X \rightrightarrows X$ then $\Phi_{p_1,\phi}(a_x) = \Phi_{p_2,\phi}(a_x)$. Therefore $a_x \in \mathrm{Ker}(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X \times_{\mathcal{M}} X, \mathcal{F}))$.

On the other hand, by lemma 3.4.14, we define $p(a_x) = (a_{x'})$ for each $a_x \in \mathrm{Ker}(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X \times_{\mathcal{M}} X, \mathcal{F}))$ and where $a'_x \in \mathrm{Ker}(\Gamma(X', \mathcal{F}) \rightrightarrows \Gamma(X' \times_{\mathcal{M}} X', \mathcal{F}))$ is the induced element by a_x for any atlas $X' \to \mathcal{M}$. We then get that q and p are mutually inverse since both maps are defined in terms of the isomorphisms in the proof of lemma 3.4.14. $\square$

Remark 3.4.18.

1. As limits are left exact functors, then the functor of global section is left exact [12, D.4].

2. The category of all sheaves of Abelian groups on the stack $\mathcal{M}$, $\mathbf{Sh}/\mathcal{M}$ is an abelian category with enough injectives [2, 2.1.1.i].

Hence, for any $\mathcal{F} \in Sh/\mathcal{M}$, we can choose an injective resolution

$$0 \to \mathcal{F} \to I^{\bullet}.$$

As the global section functor is left exact we can apply it to this sequence and, after that, we can use the derived functor such that we can define the *sheaf cohomology* of the differentiable stack $\mathcal{M}$ by

$$H^*_{Sh}(\mathcal{M}, \mathcal{F}) := H^*(\Gamma(\mathcal{M}, \mathcal{I}^{\bullet})) = R^*\Gamma(\mathcal{F}).$$

Example 3.4.19. We know that an injective resolution for $\mathbb{R}$ is given by the de Rham complex of differential forms and we get $H^*_{dR}(\mathcal{M}) \cong H^*_{Sh}(\mathcal{M}, \mathbb{R})$.

3.4.2 Cohomology of Lie groupoids

We consider the Lie groupoid $X = (X_1 \rightrightarrows X_0)$ associated to the differentiable stack $\mathcal{M}$ with atlas $X \to \mathcal{M}$. We can consider the following simplicial smooth manifold via iterated pullbacks

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_1 \;\rightrightarrows\; X_0$$

where $X_n = X \times_{\mathcal{M}} X \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} X$, the n-times product of X with the atlas $X \to \mathcal{M}$. Let Ω^q be the sheaf of q-forms, so we get

$$\Omega^q(X_0) \;\rightrightarrows\; \Omega^q(X_1) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \Omega^q(X_2) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

We can associate a complex

$$\Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \Omega^q(X_2) \xrightarrow{\partial} \ldots$$

where $\partial : \Omega^q(X_{p-1}) \to \Omega^q(X_p)$ is given by $\partial = \sum_{i=0}^{p}(-1)^i \partial_i^*$ and $\partial^2 = 0$. Hence we can talk about $H^k(X_\bullet, \Omega^q)$, the cohomology groups of the associated Lie groupoid.

Proposition 3.4.20. *Let $f, g : X \to Y$ be a morphism of Lie groupoids $X = (X_1 \rightrightarrows X_0)$, $Y = (Y_1 \rightrightarrows Y_0)$. If $\theta : f \Rightarrow g$ a 2-morphism of Lie groupoids, then f^*, g^* induce maps in the de Rham complex*

$$\partial\theta^* + \theta^*\partial = g^* - f^*.$$

Proof. We have that f induces a map $f^* : \Omega^*(Y_p) \to \Omega^*(X_p)$ with $f^*(\omega)(\phi_1 \dots \phi_p) = \omega(f(\phi_1) \dots f(\phi_p))$ in the same way as g. Also, we get that $\theta^* : f^* \Rightarrow g^*$ defines a map

$$\theta^* : \Omega^q(Y_{p+1}) \to \Omega^q(X_p)$$

given by

$$\theta^*(\omega)(\phi_1 \dots \phi_p) = \sum_{i=0}^{p} \omega(f(\phi_1) \dots f(\phi_i)\theta(x_i)g(\phi_{i+1}) \dots g(\phi_p))$$

with $f(\phi_i)\theta(x_i)g(\phi_{i+1}) = \theta(x_0)g(\phi_1)$ for $i = 0$ and $f(\phi_i)\theta(x_i)g(\phi_{i+1}) = f(\phi_p)\theta(x_p)$ for $i = p$. Let ω be an element in $\Omega^q(Y_p)$ and an element in X_p with form

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} x_2 \xrightarrow{\phi_3} \cdots\cdots\cdots \xrightarrow{\phi_p} x_p$$

then we have that the sum $\theta^*\partial\omega(\phi_1\phi_2 \dots \phi_p)$ has $p^2 + 3p + 2$ summands, and each summand will be denoted by $(-1)^{k+n}C_{k,n}$ with $0 \leq k \leq p$, $0 \leq n \leq p+1$, and $C_{k,n}$ means that this summand comes from applying ∂_n^* to k-th summand of θ^*. There are some properties about $C_{k,n}$:

1. $C_{0,0} = g^*\omega$, $C_{p,p+1} = f^*\omega$. $C_{p,p+1}$ has negative sign.

2. $C_{k,k} = C_{k-1,k}$ for $1 \leq k \leq p$. Since

$$C_{k,k} = \omega(f(\phi_1) \dots f(\phi_{k+1}) * \theta(x_{k+1}) \dots g(\phi_p))$$

$$C_{k-1,k} = \omega(f(\phi_1) \dots \theta(x_k) * g(\phi_{k+1}) \dots g(\phi_p))$$

and

$$\begin{tikzcd} f(x_k) \arrow[r, "f(\phi_{k+1})"] \arrow[d, "\theta(x_k)"'] & f(x_{k+1}) \arrow[d, "\theta(x_{k+1})"] \\ g(x_k) \arrow[r, "g(\phi_{k+1})"'] & g(x_{k+1}) \end{tikzcd}$$

3. The pair of elements $C_{k,k} = C_{k-1,k}$ add zero to $\theta^*\partial\omega(\phi_1\phi_2\ldots\phi_p)$ because they have different signs. These are $2p$ summands that come to zero.

On the other hand, we have the sum $\partial\theta^*\omega(\phi_1\phi_2\ldots\phi_p)$ with p^2+p summands, and each summand will be denoted by $(-1)^{l+m}D_{l,m}$ with $0 \leq l \leq p-1$, $0 \leq m \leq p$. Hence, we understand by $D_{l,m}$ as the l-th summand of θ^* applying on $\partial_m^*\omega$. $D_{l,m}$ are related with $C_{k,n}$ in the following way:

- $D_{k,n-1} = C_{k,n}$ for any $0 \leq k \leq p$, $0 \leq n \leq p+1$ such that $k \neq n$ and $k \neq n-1$. If we have that $k < n-1$

$$D_{k,n-1} = \omega(f(\phi_1)\ldots f(\phi_l)\theta(x_l)g(\phi_{l+1})\ldots g(\phi_m * \phi_{m+1})\ldots g(\phi_p))$$

$$= \omega(f(\phi_1)\ldots f(\phi_l)\theta(x_l)g(\phi_{l+1})\ldots g(\phi_m) * g(\phi_{m+1})\ldots g(\phi_p)) = C_{k,n}$$

If we have that $k \geq n-1$

$$D_{k,n-1} = \omega(f(\phi_1)\ldots f(\phi_m * \phi_{m+1})\ldots f(\phi_l)\theta(x_l)g(\phi_{l+1})\ldots g(\phi_p))$$

$$= \omega(f(\phi_1)\ldots f(\phi_m) * f(\phi_{m+1})\ldots f(\phi_l)\theta(x_l)g(\phi_{l+1})\ldots g(\phi_p)) = C_{k,n}$$

$D_{k,n-1}$ and $C_{k,n}$ have different signs, so its addition in $(\partial\theta^* + \theta^*\partial)\omega$ is zero. As we start with the sum $\theta^*\partial\omega(\phi_1\phi_2\ldots\phi_p)$ with p^2+3p+2 summands, we add the pair elements $C_{n,n} = C_{n-1,n}$ as zero, we keep in mind that there are $2p$ elements. Thus, we get p^2+p+2 summands in this addition. When we compare this with $D_{l,m}$, we add zero and we consider p^2+p summands. As a result, we only have 2 elements in $(\partial\theta^* + \theta^*\partial)\omega$, and those are $g^*\omega$, $-f^*\omega$, so we get the final result. □

Proposition 3.4.21.

1. *Lie groupoid morphisms induce homomorphisms on cohomology groups of the groupoid.*

2. *2-isomorphic Lie groupoid morphisms induce identical homomorphisms on cohomology groups of the Lie groupoid.*

3. *A Morita morphism with a section induces isomorphisms on cohomology groups of the Lie groupoid.*

Proof.

1. For a morphism f of Lie groupoids we induce a morphism in the nerve of the groupoid and then we consider f^* the induced map in Ω^q.

2. For proposition 3.4.20, we have that θ^* is a chain homotopy for ∂.

3. If we have a Morita morphism $f : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ with a section $s : Y_0 \to X_0$, we can define $\theta : X_0 \to X_1$ as $\theta = \psi \circ s \circ f$ where ψ is a local section for the source map in $(X_1 \rightrightarrows X_0)$ and for proposition 3.4.20, $\theta^*\partial + \partial\theta^* = (s \circ f)^* - (id_X)^*$. We have the result because $(f \circ s)^* = id_Y$ and $(s \circ f)^* = id_X$ in cohomology. □

Proposition 3.4.22. *If $X_1 \rightrightarrows X_0$ is the banal Lie groupoid associated to a surjective submersion of smooth manifolds $X_0 \to Y$, then the cohomology groups $H^k(X, \Omega^q)$ vanish for all $k > 0$ and all $q \geq 0$. Moreover, $H^0(X, \Omega^q) = \Gamma(Y, \Omega^q)$.*

Proof. We verify the following cases:

1. We consider $\{U_i\}_{i \in I}$ an open covering of Y and the surjective submersion is given by $\coprod_{i \in I} U_i \to Y$. If its associated Lie groupoid is $\coprod U_i \cap U_j \rightrightarrows \coprod_{i \in I} U_i$, this is a result for the usual cohomology on smooth manifolds given in [10, 8.5 & 8.8].

2. Suppose we have a Morita morphism $\phi : (X_1 \rightrightarrows X_0) \to (Y \rightrightarrows Y)$ with a section, where $(Y \rightrightarrows Y)$ is the pair groupoid. If we use the proposition 3.4.21, we get the result.

3. In general, if we have the Lie groupoid $X_1 \rightrightarrows X_0$ and $\{U_i\}_{i \in I}$ an open cover of Y such that $X \to Y$ admits local sections with the Lie groupoid $V_1 = \coprod U_i \cap U_j \rightrightarrows V_0 = \sqcup U_i$, we define the bisimplicial smooth manifold $W_{m,n} = X_m \times_Y V_n$. If we apply Ω^q to this bisimplicial smooth manifold, we get a double complex with rows

$$\Omega^q(V_n) \to \Omega^q(X_0 \times_Y V_n) \to \Omega^q(X_1 \times_Y V_n) \to \Omega^q(X_2 \times_Y V_n) \to \dots$$

such that the morphisms are the differentials induced by the simplicial smooth manifold $X_\bullet$, denoted by ∂_X, and the morphism given by the surjective submersion $X_0 \times_Y V_n \to V_n$. We observe that the groupoid $(X_1 \times_Y V_n \rightrightarrows X_0 \times_Y V_n)$ is in fact the banal groupoid associated to $X_0 \times_Y V_n \to V_n$, because

$$(X_0 \times_Y V_n) \times_{V_n} (X_0 \times_Y V_n) \cong X_0 \times_Y X_0 \times_Y V_n = X_1 \times_Y V_n$$

Moreover, we can define the Morita morphism

$$\phi : (X_1 \times_Y V_n \rightrightarrows X_0 \times_Y V_n) \to (V_n \rightrightarrows V_n)$$

given by the projection on V_n, and a section $s = (s', id_{V_n}) : V_n \to X_0 \times_Y V_n$ where s' is the global section of $X_0 \to V_n$ coming from all the local section of $X \to Y$. On the other hand, we have that the columns are

$$\Omega^q(X_m) \to \Omega^q(X_m \times_Y V_0) \to \Omega^q(X_m \times_Y V_1) \to \Omega^q(X_m \times_Y V_2) \to \dots$$

with morphisms ∂_V induced by the one in $V_\bullet$ and the one induced by the surjective submersion $X_m \times_Y V_0 \to X_m$. We observe that the groupoid $(X_m \times_Y V_1 \rightrightarrows X_m \times_Y V_0)$ is the banal Lie groupoid of $X_m \times_Y V_0 \to X_m$ since

$$(X_m \times_Y V_0) \times_{X_m} (X_m \times_Y V_0) \cong X_m \times_Y V_0 \times_Y V_0 = X_m \times_Y V_1.$$

In this way, we have a homomorphism between complexes

$$(\Omega^q(V_\bullet), \partial_V) \to \left(\bigoplus_{m+n=\bullet} \Omega^q(W_{mn}), \partial_X + (-1)^m \partial_V \right)$$

Since we have that the Morita morphism ϕ has a section, and the previous case in this proof says that we have an isomorphism in cohomology then each column in $\Omega^q(W_{mn})$ given by

$$\Omega^q(X_m) \to \Omega^q(X_m \times_Y V_0) \to \Omega^q(X_m \times_Y V_1) \to \Omega^q(X_m \times_Y V_2) \to \dots$$

is exact.
Also, there is a homomorphism

$$(\Omega^q(X_\bullet), \partial_X) \to \left(\bigoplus_{m+n=\bullet} \Omega^q(W_{mn}), \partial_X + (-1)^m \partial_V \right)$$

and as each row is exact, because we are in case 1 of this proof. Thus we get an isomorphism in cohomology.

Therefore, we get the commutative diagram

$$\begin{tikzcd} H^*(W,\Omega^q) \arrow[r, "\text{case 2}", "\cong"'] \arrow[d, "\text{case 1}"', "\cong"] & H^*(V,\Omega^q) \arrow[d, "\cong"', "\text{case 1}"] \\ H^*(X,\Omega^q) \arrow[r] & H^*(Y \rightrightarrows Y,\Omega^q) \end{tikzcd}$$

and so, we get the result. $\square$

Corollary 3.4.23. *Any Morita morphism of Lie groupoids*

$$f = (f_1, f_0) : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$$

induces an isomorphism on cohomology groups $f^* : H^k(Y_\bullet, \Omega^q) \to H^k(X_\bullet, \Omega^q)$.

Proof. Let $\mathcal{M} = [Y_0/Y_1]$ be the differentiable stack given by $(Y_1 \rightrightarrows Y_0)$. We have that $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{p} \mathcal{M}$ is an atlas for $\mathcal{M}$ since f_0 is a surjective submersion.
We consider the bisimplicial smooth manifold given by $Z_{mn} = X_m \times_{\mathcal{M}} Y_n$ and we see that the rows of this bisimplicial smooth manifold are given by the nerve of the Lie groupoid

$$(X_1 \times_{\mathcal{M}} Y_n \rightrightarrows X_0 \times_{\mathcal{M}} Y_n)$$

which is the banal Lie groupoid of the surjective submersion $X_0 \times_{\mathcal{M}} Y_n \to Y_n$ since $X_0 \times_{\mathcal{M}} Y_n \times_{Y_n} X_0 \times_{\mathcal{M}} Y_n \cong X_1 \times_{\mathcal{M}} Y_n$ and the columns are given by the nerve of the Lie groupoid

$$(X_m \times_{\mathcal{M}} Y_1 \rightrightarrows X_m \times_{\mathcal{M}} Y_0)$$

which is the banal Lie groupoid of the surjective submersion $X_m \times_{\mathcal{M}} Y_0 \to X_m$ because $X_m \times_{\mathcal{M}} Y_0 \times_{X_m} X_m \times_{\mathcal{M}} Y_0 \cong X_m \times_{\mathcal{M}} Y_1$. Therefore for proposition 3.4.22, we get two quasi-isomorphism in the following way

$$(\Omega^q(X_\bullet), \partial_X) \to \left(\bigoplus_{m+n=\bullet} \Omega^q(Z_{mn}), \partial_X + (-1)^m \partial_Y \right)$$

$$(\Omega^q(Y_\bullet), \partial_Y) \to \left(\bigoplus_{m+n=\bullet} \Omega^q(Z_{mn}), \partial_X + (-1)^m \partial_Y \right)$$

and so we get the result. $\square$

From theorem 3.1 and as a consequence of the previous corollary, we know that this cohomology is well-defined for differentiable stacks and we define:

Definition 3.4.24. Let $\mathcal{M}$ be a differentiable stack, then the cohomology of the associated Lie groupoid for $\mathcal{M}$ is

$$H^k(X_\bullet, \Omega^q)$$

where $X \to \mathcal{M}$ is an atlas.

3.4.3 De Rham cohomology

3.4.3.1 The de Rham complex

Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$. We have that the exterior derivative $d_{dR} : \Omega^q \to \Omega^{q+1}$ connects the complexes Ω^* in the previous construction. Thus, we get a commutative diagram

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & \\
\Omega^2(X_0) & \xrightarrow{\partial} & \Omega^2(X_1) & \xrightarrow{\partial} & \Omega^2(X_2) & \longrightarrow & \dots \\
{\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & \\
\Omega^1(X_0) & \xrightarrow{\partial} & \Omega^1(X_1) & \xrightarrow{\partial} & \Omega^1(X_2) & \longrightarrow & \dots \\
{\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & {\scriptstyle d_{dR}}\uparrow & & \\
\Omega^0(X_0) & \xrightarrow{\partial} & \Omega^0(X_1) & \xrightarrow{\partial} & \Omega^0(X_2) & \longrightarrow & \dots
\end{array}$$

since $\partial \circ d_{dR} = d_{dR} \circ \partial$, because d_{dR} commutes with pullbacks.
If we consider $D = \partial + (-1)^p d_{dR}$ and $\Omega^n_{DR}(X_\bullet) = \bigoplus_{p+q=n} \Omega^q(X_p)$, we set the complex

$$(\Omega^n_{dR}(X_\bullet), D)$$

because $D^2 = 0$.

Definition 3.4.25. The cohomology given by the complex $(\Omega^n_{dR}(X_\bullet), D)$ is called the *de Rham cohomology of* $X_\bullet$, and it is denoted by $H^n_{dR}(X_\bullet)$.

Proposition 3.4.26. $H^n_{DR}(X_\bullet)$ *is invariant under Morita equivalence.*

Proof. Let $f = (f_1, f_0) : (X_1 \rightrightarrows X_0) \to (Y_1 \rightrightarrows Y_0)$ be a Morita morphism and $\mathcal{M} = [Y_0/Y_1]$ be the differentiable stack given by the Lie groupoid Y. We consider the atlas $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{p} \mathcal{M}$ for $\mathcal{M}$ and the bisimplicial smooth manifold $Z_{mn} = X_m \times_{\mathcal{M}} Y_n$, then for corollary 3.4.23 we know that the rows and columns can be considered as the nerve of the of banal groupoids induced by $X_0 \times_{\mathcal{M}} Y_n \to Y_n$ and $X_m \times_{\mathcal{M}} Y_0 \to X_m$, respectively. Therefore, we have the quasi-isomorphisms

$$(\Omega^\bullet_{DR}(X), D) \to \big(\mathrm{Tot}_{q+m+n=\bullet}\Omega^q(Z_{mn}), \partial_X + (-1)^m\partial_Y + (-1)^{m+n}d_{dR}\big)$$

$$(\Omega^\bullet_{DR}(Y), D) \to \big(\mathrm{Tot}_{q+m+n=\bullet}\Omega^q(Z_{mn}), \partial_X + (-1)^m\partial_Y + (-1)^{m+n}d_{dR}\big)$$

and this is what we want. □

Thus, this cohomology is well-defined for differentiable stacks. We define now the de Rham cohomology of a differentiable stack:

Definition 3.4.27. Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$, its *de Rham cohomology* is given as:

$$H^n_{DR}(\mathcal{M}) \coloneqq H^n_{DR}(X_\bullet).$$

We have that the de Rham cohomology of $X_\bullet$ is related by theorem 2.6 with the one of its fat geometric realisation.

Definition 3.4.28. The homotopy type of the differential stack $X \to \mathcal{M}$ is given by the homotopy type of the fat geometric realisation $\| X_\bullet \|$.

Remark 3.4.29. We observe that the same construction can be done for a complex $\mathcal{L}^\bullet$ of cartesian sheaves of abelian groups on the differentiable stack $\mathcal{M}$ with atlas $X \to \mathcal{M}$. The cohomology that we get, will be denoted by

$$H^*(\mathcal{M}, \mathcal{L}^\bullet) = H^*(X_\bullet, \mathcal{L}^\bullet).$$

Compare with [16, 3.4.27], where for this cohomology is shown that the homotopy type is the fat geometric realisation $\| X_\bullet \|$.

3.4.4 Singular homology and cohomology

In the same way as working with the cohomology of a Lie groupoid and its de Rham complex, we can consider the simplicial set given by a differentiable stack $\mathcal{M}$ and its atlas $X \to \mathcal{M}$

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_1 \;\rightrightarrows\; X_0$$

If we apply $C_\bullet$, the singular chain complex, we get

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; C_\bullet(X_2) \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; C_\bullet(X_1) \;\rightrightarrows\; C_\bullet(X_0)$$

We define $\partial = \sum_{i=0}^{p}(-1)^i\partial_i$ where $\partial : C_\bullet(X_p) \to C_\bullet(X_{p-1})$. Thus

$$\begin{tikzcd}
\cdots \arrow[r] & C_0(X_2) \arrow[r, "\partial"] & C_0(X_1) \arrow[r, "\partial"] & C_0(X_0) \\
\cdots \arrow[r] & C_1(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & C_1(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & C_1(X_0) \arrow[u, "d"] \\
\cdots \arrow[r] & C_2(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & C_2(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & C_2(X_0) \arrow[u, "d"] \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

We define the associated total complex as $C_\bullet(X) = \bigoplus_{p+q=n} C_q(X_p)$ with differential $\delta : C_n \to C_{n-1}$ given by

$$\delta(\gamma) = (-1)^{p+q}\partial(\gamma) + (-1)^q d(\gamma)$$

if $\gamma \in C_q(X_p)$ and $\delta^2 = 0$.

Definition 3.4.30. The complex $(C_\bullet(X), \delta)$ is called a *singular chain complex* of the topological groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$. Its homology groups $H_n(X, \mathbb{Z})$ are called the singular homology groups of $X_\bullet$.

Remark 3.4.31. For a definition of a topological groupoid, see the definition 2.4.7.

This cohomology is invariant under Morita equivalence and hence, we can define it for a differentiable stack.

Theorem 3.2. *The homology is invariant under Morita equivalence.*

Remark 3.4.32. If we follow the same argument as in proposition 3.4.26, we get the result.

For a topological groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$, we denote the dual of the complex $C_\bullet(X)$ by $C^\bullet(X)$ where

$$C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z}).$$

In the same way as above, we can define the singular cohomology groups of the stack $\mathcal{M}$ with atlas $X \to \mathcal{M}$.

$$H^n(\mathcal{M}, \mathbb{Z}) = H^n(X, \mathbb{Z})$$

Definition 3.4.33. If A is an arbitrary abelian group, the singular homology for A is defined by

$$H_k(\mathcal{M}, A) = h_k(C_\bullet(X) \otimes_{\mathbb{Z}} A)$$

and in a similar way, the singular cohomology

$$H^k(\mathcal{M}, A) = h^k(C^\bullet(X) \otimes_{\mathbb{Z}} A).$$

3.4.4.1 De Rham theorem for differentiable stacks

Theorem 3.3. *[4, p.28] Let $\mathcal{M}$ be a differentiable stack.*

$$H^*_{DR}(\mathcal{M}) = H^*(\mathcal{M}, \mathbb{R}).$$

Proof. Consider the Lie groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$ associated to the atlas $X \to \mathcal{M}$ and consider its singular cohomology. We define a pairing given by

$$\Omega^\bullet_{DR}(X_\bullet) \bigotimes C_\bullet(X_\bullet) \to \mathbb{R}$$

$$\omega \otimes \gamma \mapsto \int_\gamma \omega$$

We note that $\int_\gamma \omega = 0$ unless $p = p'$ and $q = q'$. We get an homomorphism of complexes since the pairing vanishes on coboundaries of total degree zero by the chain rule and Stokes' theorem. This pairing induces one paring given by

$$H^k_{DR}(\mathcal{M}) \bigotimes H_k(\mathcal{M}, \mathbb{R}) \to \mathbb{R}$$

for any differentiable stack $\mathcal{M}$. This pairing can be used to define when a de Rham cohomology class $[\omega]$ is integral, namely by requiring

$$\int_\gamma \omega \in \mathbb{Z}$$

for all $[\gamma] \in H_k(\mathcal{M})$. The first pairing also give rise to a homomorphism of complexes

$$\Omega^\bullet_{DR}(X_\bullet) \xrightarrow{\Psi^\bullet} C^\bullet(X_\bullet) \otimes \mathbb{R}$$

Since for each p, q

$$\Omega^p(X_q) \xrightarrow{\Psi^{p,q}} C^p(X_q)$$

we have an isomorphism on smooth manifolds

$$H^p_{DR}(X_q) \to H^p(X_q, \mathbb{R})$$

then for spectral sequence, theorem 2.11, we have that Ψ induces an isomorphism on cohomology, as we want. $\square$

3.4.5 Hypercohomology and de Rham cohomology

Let $\mathcal{L}^\bullet$ be a complex of sheaves of abelian groups on $\mathcal{M}$. We consider the injective resolution $0 \to \mathcal{L}^\bullet \to I^{\bullet,q}$ and

$$K = \bigoplus_{p,q} K^{p,q} = \bigoplus_{p,q} \Gamma(\mathcal{M}, I^{p,q})$$

with differential from the resolution

$$\delta : K^{p,q} \to K^{p+1,q}$$

and the differential from the complex

$$d : K^{p,q} \to K^{p,q+1}$$

d is defined considering that $d : \mathcal{L}^q \to \mathcal{L}^{q+1}$ induces a morphism in $I^{\bullet,q} \to I^{\bullet,q+1}$ because $I^{\bullet,q}$ is injective, [11, II.3.4] and the differentials commutes.

Definition 3.4.34. The *hypercohomology* $\mathbb{H}^*(\mathcal{M}, \mathcal{L}^\bullet)$ of the complex $\mathcal{L}^\bullet$ is the total cohomology of the double complex, that is

$$K^\bullet = \bigoplus_k K^k = \bigoplus_k \bigoplus_{p+q=k} K^{p,q}$$

with differential $D = \delta + (-1)^p d$.

Theorem 3.4. *Let $\mathcal{M}$ be a differentiable stack and $\mathcal{L}^\bullet$ a complex of Cartesian sheaves of abelian groups on $\mathcal{M}$ such that $\mathcal{L}^q$ is acyclic, then there is an isomorphism between $\mathbb{H}^*(\mathcal{M}, \mathcal{L}^\bullet) \cong H^*(\mathcal{M}, \mathcal{L}^\bullet)$, where $H^*(\mathcal{M}, \mathcal{L}^\bullet)$ is the cohomology associated to the complex of sheaves $\mathcal{L}^\bullet$ applied to the nerve $X_\bullet$, see remark 3.4.29.*

Proof. Consider the triple complex

$$N^{p,q,r} = I^{r,q}(X_p)$$

with the three differentials: d_X the simplicial differential, d_I the resolution differential and $d_{\mathcal{L}}$ the differential of the complex of sheaves. Consider the double complex

$$\overline{N}^{p,l} = \bigoplus_{q+r=l} N^{p,q,r}$$

with differential $\delta' = d_X$ and $d' = d_{\mathcal{L}} + (-1)^q d_I$. Then considering the l-th row of $\overline{N}^{\bullet,\bullet}$ is

$$0 \to \bigoplus_{q+r=l} I^{r,q}(X_0) \to \cdots \to \bigoplus_{q+r=l} I^{r,q}(X_p) \to \bigoplus_{q+r=l} I^{r,q}(X_{p+1}) \to \cdots$$

with differential $\delta' = d_X$. As $I^{r,q}$ is injective then it is flabby by [11, II.5.3]. Therefore, the functions are surjective. Hence each row of the complex is exact except in the zero-th column and we get a spectral sequence with

$$E_1^{p,l} = \begin{cases} M^l & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \tag{3.1}$$

where $M^l = \bigoplus_{q+r=l} \Gamma(\mathcal{M}, I^{r,q})$, since $I^{r,q}$ is Cartesian. So $E_2 \cong \mathbb{H}^*(\mathcal{M}, \mathcal{L}^\bullet)$ and the spectral sequence degenerates, that is, $E_\infty \cong E_2$. $\square$

Proposition 3.4.35. *The hypercohomology of $\mathcal{M}$ with respect to the sheaf complex $\mathcal{L}_\bullet$ is Morita invariant.*

Proof. Considering the complex $N^{p,q,r} = I^{r,q}(X_p)$ and filter $N^{p,q,r}$ by $\bigoplus_{q+r\geq n} N^{p,q,r}$. Then we get the spectral sequence $E_0^{r,q} = N^{r,q} = \Gamma(\mathcal{M}, I^{r,q})$ and filtering this by $\bigoplus_{q\geq n} N^{r,q}$ we get $E_1^{r,q} = H^r(\mathcal{M}, \mathcal{L}^q)$ the sheaf cohomology of $\mathcal{L}^q$, which is Morita invariant by corollary 3.4.23. Therefore, the hypercohomology is Morita invariant. □

As the de Rham sheaf is Cartesian, we have:

Proposition 3.4.36. *Consider the de Rham complex of sheaves $\Omega^\bullet$. Then de Rham cohomology is the same as the hypercohomology of the complex $\Omega^\bullet$.*

Proof. We consider an injective resolution $0 \to \Omega^\bullet \to I^{\bullet,q}$ and as below, the triple complex $N^{p,q,r} = I^{r,q}(X_p)$. We are going to work with the associated double complex given by $N^{k,r} = \bigoplus_{p+q=k} N^{p,q,r}$. Hence we have that the k-th column of the complex $N^{k,r}$ is given by:

$$0 \to \bigoplus_{q+r=k} I^{0,q}(X_p) \to \cdots \to \bigoplus_{q+r=k} I^{r,q}(X_p) \to \bigoplus_{q+r=k} I^{r+1,q}(X_p) \to \cdots$$

Since $\Omega^\bullet$ is acyclic, this column is exact except in the zero-th row and we get a spectral sequence with

$$E_1^{k,q} = \begin{cases} K^k & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases} \tag{3.2}$$

where $K^k = \bigoplus_{q+r=k} \Omega^q(X_p)$ and the differential is $\delta = d_X + (-1)^p d_\Omega$. Thus $E_2 \cong H^*_{DR}(\mathcal{M})$ and it degenerates at E_2.

Therefore we have that $H^*_{DR}(\mathcal{M}) = \mathbb{H}^*(\mathcal{M}, \Omega^\bullet)$. □

Example 3.4.37.

1. Consider the atlas $pt \to \mathcal{B}G$ by example 3.2.8, and the nerve associated

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; pt \times_{\mathcal{B}G} pt \times_{\mathcal{B}G} pt \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; pt \times_{\mathcal{B}G} pt \rightrightarrows pt$$

We have that $pt \times_{\mathcal{B}G} pt \cong G$ and $pt \times_{\mathcal{B}G} G \cong G \times G$ hence, the nerve can be written as

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times G \times G \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \; \rightrightarrows \; pt$$

Therefore, the nerve associated to $\mathcal{B}G$ is the same as the nerve usually associated to the classifying space BG. If we apply the cohomology we get $H^*(\mathcal{B}G, \mathbb{R}) \cong H^*(BG, \mathbb{R})$

2. Considering the example 3.2.19, we have an atlas $G \to \mathcal{E}G$ so the nerve associated to $\mathcal{E}G$ is

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times_{\mathcal{E}G} G \times_{\mathcal{E}G} G \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times_{\mathcal{E}G} G \; \rightrightarrows \; G$$

and using the example 3.2.8, we get that the nerve can be seen as

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times G \times G \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times G \; \rightrightarrows \; G$$

This is the usual nerve for the simplicial construction of the universal space EG. Therefore if we find the cohomology we got that $H^*(\mathcal{E}G) \cong H^*(EG) \cong H^*(pt)$, that is, $\mathcal{E}G$ is contractible.

3.4.6 Čech cohomology for differentiable stacks

Let $\mathcal{M}$ be a differentiable stack with an atlas $X \to \mathcal{M}$ and its nerve $X_\bullet$. If $\mathcal{F}$ is a sheaf on $\mathcal{M}$, we get the collection of induced sheaves $\{\mathcal{F}_n\}$ on X_n for proposition 3.4.11.

If there exists a covering $\mathcal{V}$ for the simplicial smooth manifold $X_\bullet$, we can consider

$$C^{n,k}(\mathcal{V}, \mathcal{F})(X_n) := \{s|_{\bigcap_{i=0}^k V_{n,\alpha_i}} : s \in \mathcal{F}_n(X_n) \text{ for any } \bigcap_{i=0}^k V_{n,\alpha_i} \text{ in } \mathcal{V}_n\}$$

that is, the sections in $\mathcal{F}_n(X_n)$ restricted to $k+1$ intersections of elements in $\mathcal{V}_n$. With morphisms given by

$$\check{\delta} : C^{n-1,k}(\mathcal{V}, \mathcal{F})(X_n) \to C^{n-1,k+1}(\mathcal{V}, \mathcal{F})(X_n)$$

the Čech differential is $\check{\delta}(s)|_{\bigcap_{i=0}^{k+1} V_{n-1,\alpha_i}} = \sum_{j=0}^{k+1} (-1)^j s|_{\bigcap_{i=0, i\neq j}^{k} V_{n-1,\alpha_i}}$, as given in [10, II.10] and

$$\delta : C^{n-1,k}(\mathcal{V}, \mathcal{F})(X_n) \to C^{n,k}(\mathcal{V}, \mathcal{F})(X_n)$$

such that $\delta(s)|_{\bigcap_{i=0}^{k} V_{n,\alpha_i}} = \sum_{l=0}^{n}(-1)^{l+n}(\partial_l^*(s|_{\bigcap_{i=0}^{k} V_{n-1,d_l(\alpha_i)}}))$.

Proposition 3.4.38. *The following holds:*

1. $\delta^2 = 0$.
2. $\delta \circ \check{\delta} = \check{\delta} \circ \delta$.

Proof.

1. If $s \in \mathcal{F}(U)$ then:

$$\delta^2(s)|_{\bigcap_{i=0}^{k} V_{n+1,\beta_i}} = \delta(\delta(s|_{\bigcap_{i=0}^{k} V_{n-1,\alpha_i}}))$$

$$= \delta\left(\sum_{l=0}^{n}(-1)^{l+n}(\partial_l^*(s)|_{\bigcap_i V_{n,d_l(\alpha_i)}})\right) = \sum_{l=0}^{n}(-1)^{l+n}\delta(\partial_l^*(s)|_{\bigcap_i V_{n,d_l(\alpha_i)}})$$

$$= \sum_{l=0}^{n}(-1)^{l+n}\sum_{j=0}^{n+1}(-1)^{j+n+1}\partial_j^*(\partial_l^*(s))|_{\bigcap_i V_{n+1,d_j d_l(\alpha_i)}}$$

$$= \sum_{j\le l}(-1)^{l+j+1}\partial_j^*(\partial_l^*(s))|_{\bigcap_i V_{n+1,d_j d_l(\alpha_i)}} + \sum_{j>l}(-1)^{l+j+1}\partial_j^*(\partial_l^*(s))|_{\bigcap_i V_{n+1,d_j d_l(\alpha_i)}}$$

$$= \sum_{j\le l}(-1)^{l+j+1}\partial_j^*(\partial_l^*(s))|_{\bigcap_i V_{n+1,d_j d_l(\alpha_i)}} + \sum_{j>l}(-1)^{l+j+1}\partial_{l-1}^*(\partial_j^*(s))|_{\bigcap_i V_{n+1,d_{l-1} d_j(\alpha_i)}}$$

$$= \sum_{h\le f}(-1)^{f+h+1}\partial_h^*(\partial_f^*(s))|_{\bigcap_i V_{n+1,d_h d_l(\alpha_i)}} + \sum_{h\le f}(-1)^{f+h}\partial_h^*(\partial_f^*(s))|_{\bigcap_i V_{n+1,d_h d_f(\alpha_i)}}$$

$$= 0.$$

2. If $s \in \mathcal{F}(U)$ then also:

$$\delta \circ \check{\delta}(s)|_{\bigcap_{i=0}^{k+1} V_{n,\beta_i}} = \delta\left(\sum_{j=0}^{k}(-1)^j s|_{\bigcap_{i=0,i\neq j}^{k} V_{n-1,\alpha_i}}\right)$$

$$= \sum_{j=0}^{k}(-1)^j\sum_{l=0}^{n}(-1)^{l+n}(\partial_l^* s)|_{\bigcap_{i=0,i\neq j}^{k} V_{n-1,d_l(\alpha_i)}})$$

$$= \sum_{l=0}^{n}(-1)^{l+n}\sum_{j=0}^{k}(-1)^{j}(\partial_l^* s)|_{\bigcap_{i=0,i\neq j}^{k} V_{n-1,d_l(\alpha_i)}})$$

$$= \sum_{l=0}^{n}(-1)^{l+n}\check{\delta}(\partial_l^* s)|_{\bigcap_{i=0}^{k} V_{n-1,d_l(\alpha_i)}}) = \check{\delta} \circ \delta(s)|_{\bigcap_{i=0}^{k+1} V_{n,\beta_i}} \qquad \square$$

Definition 3.4.39. The *Čech cohomology of the cover $\mathcal{V}$ with values in the sheaf $\mathcal{F}$* as the cohomology of the double complex given by

$$(C^{n,k}(\mathcal{V},\mathcal{F})(X_n), \delta, \check{\delta})$$

and it is denoted by

$$H^*_{\mathcal{V}}(X_\bullet, \mathcal{F}).$$

Definition 3.4.40. Let $\mathcal{M}$ be a differentiable stack with an atlas $X \to \mathcal{M}$. For a sheaf $\mathcal{F}$ on $\mathcal{M}$, define the sheaf

$$C^{n,k}(\mathcal{V},\mathcal{F})(U) := \{s|_{\bigcap_{i=0}^k V_{n,\alpha_i}} : s \in \mathcal{F}_n(U) \text{ for any } \bigcap_{i=0}^k V_{n,\alpha_i} \text{ in } \mathcal{V}_n\}$$

for any open set U of X_n, where $\mathcal{F}_n$ is the sheaf induced on X_n by $\mathcal{F}$. We call $C^{n,k}(\mathcal{V},\mathcal{F})$ a *Čech resolution* associated to $\mathcal{V}$ of $X_\bullet$ with values in $\mathcal{F}_\bullet$.

Remark 3.4.41. We have the complex of sheaves

$$0 \to \mathcal{F}_n \to C^{n,0}(\mathcal{V},\mathcal{F}) \to C^{n,1}(\mathcal{V},\mathcal{F}) \to \cdots$$

Definition 3.4.42. A covering $\mathcal{V}$ is *acyclic* if

$$H^l(\bigcap_{j=0,\dots,k} V_{n,j}, \mathcal{F}_n) = 0$$

for $l > 0$ and for all n.

Proposition 3.4.43. *The covering $\mathcal{V}$ of $X_\bullet$ is acyclic if and only if $C^{n,0}(\mathcal{V},\mathcal{F})$ is an acyclic resolution of $\mathcal{F}_\bullet$.*

Proof. Since we have that $H^*(\bigcap_{j=0,\dots,k} V_{n,j}, \mathcal{F}_n) = 0$ if and only if

$$H^*(X_n, C^{n,k}(\mathcal{V},\mathcal{F})) = H^*\left(\bigsqcup \bigcap_{j=0,\dots,k} V_{n,j}, \mathcal{F}_n\right) = 0. \quad \square$$

We have that every resolution of $\mathcal{F}_\bullet$ maps to an injective resolution of $\mathcal{F}_\bullet$. Therefore, we have an induced map $H^*_{\mathcal{V}}(X_\bullet, \mathcal{F}) \to H^i(X_\bullet, \mathcal{F}_\bullet)$, for further details check [11, II.3.4].

Proposition 3.4.44. *[20, A.2] Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$ and nerve $X_\bullet$. If $\mathcal{V}$ is an acyclic covering of the nerve $X_\bullet$, then*

$$H^*_{\mathcal{V}}(X_\bullet, \mathcal{F}) \cong H^*(X_\bullet, \mathcal{F}_\bullet) \cong H^*_{Sh}(\mathcal{M}, \mathcal{F}).$$

Proof. As $\mathcal{V}$ is acyclic then the associated Čech resolution is acyclic and the map $H^*_{\mathcal{V}}(X_\bullet, \mathcal{F}) \to H^i(X_\bullet, \mathcal{F}_\bullet)$ is an isomorphism. □

Definition 3.4.45. Let $\mathcal{V}$ and $\mathcal{U}$ be coverings of the simplicial smooth manifold $X_\bullet$. We say that $\mathcal{V}$ is *finer than* $\mathcal{U}$ if for all $V_{n,\alpha} \in \mathcal{V}_N$, there is a $U_{n,\beta} \in \mathcal{U}_n$ such that $V_{n,\alpha} \subset U_{n,\beta}$ for all n and the maps $\varphi : \alpha \to \beta$ are compatible with face maps. When $\mathcal{V}$ is finer than $\mathcal{U}$ it will be denoted by $\mathcal{V} \prec \mathcal{U}$.

The collection of maps φ induces a map

$$\varphi^* : C^{n,k}(\mathcal{U}, \mathcal{F}) \to C^{n,k}(\mathcal{V}, \mathcal{F})$$

This map commutes with $\check{\delta}$ as it is shown in [10, 10.4.1], and with δ, since φ is compatible with face maps. Moreover, we get a well-defined map in

$$H^*_{\mathcal{U}}(X_\bullet, \mathcal{F}) \to H^*_{\mathcal{V}}(X_\bullet, \mathcal{F})$$

and thus, a direct system given by $\{H^*_{\mathcal{V}}(X_\bullet, \mathcal{F})\}_{\mathcal{V}}$.

Definition 3.4.46. Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$ and nerve $X_\bullet$. If $\mathcal{F}$ is a sheaf on $\mathcal{M}$ then the limit

$$\check{H}^*(X_\bullet, \mathcal{F}) := \lim_{\to} H^*_{\mathcal{V}}(X_\bullet, \mathcal{F})$$

is the *Čech cohomology of $X_\bullet$ with values in $\mathcal{F}$*.

As we have $H^*_{\mathcal{V}}(X_\bullet, \mathcal{F}) \to H^i(X_\bullet, \mathcal{F}_\bullet)$ for any covering $\mathcal{V}$ we have a map

$$\check{H}^*(X_\bullet, \mathcal{F}) \to H^*(X_\bullet, \mathcal{F}_\bullet) \cong H^*(\mathcal{M}, \mathcal{F})$$

and this map is an isomorphism if there exists an acyclic covering.

Chapter 4

Equivariant cohomology for differentiable stacks

In this chapter we study the notion of an action by a Lie group G on a stack and the 2-category of stacks with an action by G, denoted as $G - \mathbf{St}$, the 2-category of G-stacks. We discuss the concept of a quotient stack $\mathcal{M}/G$ and we provide the idea of a G-atlas for a stack in $G - \mathbf{St}$, such that this G-atlas allows us to consider $\mathcal{M}/G$ as a differentiable stack. Then we devote our efforts to get the homotopy type of $\mathcal{M}/G$ and to understand how this homotopy type is related to a bisimplicial smooth manifold given by $G^\bullet \times X_\bullet$, where G^p is the Cartesian p-product and X_n is the n-th element of the nerve associated to the Lie groupoid given by the G-atlas $X \to \mathcal{M}$. Consequently, we get a notion of equivariant cohomology that generalises the one in smooth manifolds. We provide a Cartan model of equivariant cohomology based on Meinreken's work [46] and we compare this notion with the one found through the G-atlas. Finally, we get some spectral sequences that converge to the equivariant cohomology and which generalise some results by Felder et al. in [20] and by Stasheff in [55].

The first section is devoted to the notion of a group action on a stack, the category of G-stacks, as in [53] and [24]. In addition, we discuss how a G-stack can be considered a differentiable stack. In the next section, we provide the notion of a quotient stack $\mathcal{M}/G$, some properties of this concept and we check that this stack is a differentiable stack. In the third section we give the notion of equivariant cohomology working with the homotopy type of $\mathcal{M}/G$ and we provide a Cartan model that coincides in cohomology with the notion of equivariant cohomology previously defined. In the fourth section we construct some spectral sequences that are conceived from the study of the homotopy type of the quotient stack.

4.1 Group actions on a stack

Let G be a Lie group and $\mathcal{M}$ a differentiable stack with atlas $X \to \mathcal{M}$.

Definition 4.1.1. A morphism of stacks $\mu : G \times \mathcal{M} \to \mathcal{M}$ is called an *action on* $\mathcal{M}$ if for each $T \in \mathbf{Diff}$ the following diagrams

$$\begin{tikzcd} G \times G \times \mathcal{M}(T) \arrow[r, "m \times id_{\mathcal{M}(T)}"] \arrow[d, "id_G \times \mu_T"'] & G \times \mathcal{M}(T) \arrow[d, "\mu_T"] \\ G \times \mathcal{M}(T) \arrow[r, "\mu_T"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{M}(T) \end{tikzcd}$$

and

$$\begin{tikzcd} G \times \mathcal{M}(T) \arrow[r, "\mu_T"] & \mathcal{M}(T) \\ \mathcal{M}(T) \arrow[u, "e \times id_{\mathcal{M}(T)}"] \arrow[ur, "id_{\mathcal{M}(T)}"'] \arrow[u, phantom, shift left=6, "\Leftarrow \mathfrak{a}"] & \end{tikzcd}$$

are 2-commutative, that is, for every $T \in \mathbf{Diff}$ the following holds:

1. $(g \cdot \alpha^x_{h,k})\alpha^x_{g,hk} = \alpha^{k \cdot x}_{g,h}\alpha^x_{gh,k}$, for all $g, h, k \in G$ and $x \in \mathcal{M}(T)$.
2. $(g \cdot \mathfrak{a}^x)\alpha^x_{g,e} = 1_{g \cdot x} = \mathfrak{a}^{g \cdot x}\alpha^x_{e,g}$ for every $g \in G$, $x \in \mathcal{M}(T)$ and with e the identity in G.

where the dot is denoting the action μ. Meanwhile, $\alpha^x_{g,h} : g \cdot (h \cdot x) \to (gh) \cdot x$ and $\mathfrak{a}^x : x \to e \cdot x$ in $\mathcal{M}(T)$.

Definition 4.1.2. The pair $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$ is a *G-stack* if μ is an action of G on $\mathcal{M}$.

Definition 4.1.3. A *morphism of G-stacks* between $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$ and $(\mathcal{N}, \nu, \beta, \mathfrak{b})$ is a morphism of stacks $F : \mathcal{M} \to \mathcal{N}$ together with a 2-morphism σ with the following 2-commutative diagram

$$\begin{tikzcd} G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "id_G \times f"'] & \mathcal{M} \arrow[d, "f"] \arrow[dl, Rightarrow, "\sigma"] \\ G \times \mathcal{N} \arrow[r, "\nu"] & \mathcal{N} \end{tikzcd}$$

such that, for every $T \in \mathbf{Diff}$

1. $\sigma_g^{h\cdot x}(g \cdot \sigma_h^x)\beta_{g,h}^{F(x)} = F(\alpha_{g,h}^x)\sigma_{gh}^x$, for every $g, h \in G$ and $x \in \mathcal{M}(T)$.
2. $F(\mathfrak{a}^x)\sigma_e^x = \mathfrak{b}^{F(x)}$, for every object $x \in \mathcal{M}(T)$ and e the identity element of G.

where $\sigma_g^x : F(g \cdot x) \to g \cdot F(x)$ in $\mathcal{N}(T)$.

Definition 4.1.4. A *2-morphism of G-stacks* between 1-morphism of G-stacks, (F, σ) and (F', σ'), is a 2-morphism of stacks $\phi : F \Rightarrow F'$ such that

* $(\sigma_g^x)(g \cdot \phi_x) = (\phi_{g\cdot x})(\sigma_g^{\prime x})$ for every $g \in G$ and $x \in \mathcal{M}(T)$.

Here $\phi_x : F(x) \to F'(x)$ is the 2-morphism ϕ applied to $x \in \mathcal{M}(T)$.

Remark 4.1.5. In this way, we define a 2-category of G-stacks denoted by G-**St**.

Example 4.1.6. The definition of an action of G on a smooth manifold M coincide with the one above, where the diagrams are strictly commutative. In the same way, the notion of G equivariant smooth maps in **Diff** coincides with the one of morphism of G-stacks.

4.2 Quotient stacks

Let G be a Lie group acting on a differentiable stack $\mathcal{M}$.

Definition 4.2.1. Consider the pseudo-functor

$$\mathcal{M}/G : \mathbf{Diff}^{op} \to \mathbf{Grpds}$$

such that for each $T \in \mathbf{Diff}$, an element in $\mathcal{M}/G(T)$ is a triple $t = (p, f, \sigma)$ such that $p : E \to T$ forms a principal G-bundle and an equivariant morphism $(f, \sigma) : E \to \mathcal{M}$. The arrows in $\mathcal{M}/G(T)$ are pairs (u, α) with a G-morphism $u : E \to E'$ and a 2-commutative diagram of G-stacks given by

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "{(f,\sigma)}"'] \arrow[rr, phantom, shift right=3ex, "\Rightarrow\alpha"] & & E' \arrow[dl, "{(f',\sigma')}"] \\ & \mathcal{M} & \end{tikzcd}$$

If there is a smooth map $T \xrightarrow{h} S$, then there exists a morphism $\mathcal{M}/G(S) \to \mathcal{M}/G(T)$ given by the pullback as in the following commutative diagram

$$\begin{tikzcd} T \times_S E \arrow[r] \arrow[d, "h^*"] & E \arrow[r, "{(f,\sigma)}"] \arrow[d, "p"] & \mathcal{M} \\ T \arrow[r, "h"] & S & \end{tikzcd}$$

where $\mathcal{M}/G(h) = h^*$.

Proposition 4.2.2. *Let G be a Lie group with an action on $\mathcal{M}$. The pseudo-funtor $\mathcal{M}/G$ is a stack.*

Proof. Since it is possible to glue principal G-bundles, the gluing conditions in the definition of a stack hold. Therefore the quotient $\mathcal{M}/G$ is a stack. $\square$

Example 4.2.3. Let M be a smooth manifold and an action by G on M. We have that the usual quotient stack $[M/G]$, given in definition 3.1.11, is the same as the quotient $M/G = \mathrm{Hom}(\_, M)/G$ defined as in the definition above.

Another way to consider this stack, is defining a prestack $\mathcal{P}$ such that for $T \in \mathbf{Diff}$ we have $\mathcal{P}(T) = \mathcal{M}(T)$ and morphisms between x and y in $\mathcal{M}(T)$ are pairs (g, φ) with $g \in G$ and $\varphi : g.x \to y$ a morphism in $\mathcal{M}(T)$. If we use stackification as in the proposition 3.1.8, we get the stack $(\mathcal{M}/G)^* = \tilde{\mathcal{P}}$ associated to $\mathcal{P}$.

Proposition 4.2.4. *The stacks $\mathcal{M}/G$ and $(\mathcal{M}/G)^*$ are isomorphic.*

Proof. Consider the morphism $\Phi : \mathcal{P} \to \mathcal{M}/G$ such that for each $T \xrightarrow{x} \mathcal{M}$ we get as $\Phi(x)$ the following principal G-bundle

$$\begin{tikzcd} G \times T \arrow[r, "\mu\circ(id\times x)"] \arrow[d, "p_2"] & \mathcal{M} \\ T & \end{tikzcd}$$

where μ is the action on $\mathcal{M}$.

We recall that a morphism $(g, \varphi) : x \to y$ in $\mathcal{P}$, is a morphism $\varphi : g \cdot x \to y$. We define $\Phi(g, \varphi)$ as the 2-morphism in the following commutative diagram

$$\begin{tikzcd} G \times T \arrow[rr, "id_{G\times T}"] \arrow[dr, "\mu\circ(id\times g\cdot x)"'] \arrow[rr, phantom, shift right=3ex, "\xRightarrow{id_G\times\varphi}"] & & G \times T \arrow[dl, "\mu\circ(id\times y)"] \\ & \mathcal{M} & \end{tikzcd}$$

where φ is the morphism described by the commutative diagram

$$\begin{tikzcd} T \arrow[rr, "id_T"] \arrow[dr, "g\cdot x"'] \arrow[rr, phantom, shift right=3ex, "\xRightarrow{\varphi}"] & & T \arrow[dl, "y"] \\ & \mathcal{M} & \end{tikzcd}$$

Φ is a fully faithful morphism, for this we consider $T \in \mathbf{Diff}$ and

$$\mathrm{Hom}_{\mathcal{P}(T)}(x, y) \xrightarrow{\Phi_{x,y}} \mathrm{Hom}_{\mathcal{M}/G(T)}(\Phi(x), \Phi(y))$$

then a morphism in $\mathrm{Hom}_{\mathcal{M}/G(T)}(\Phi(x), \Phi(y))$ is given by $h \times id_T : G \times T \to G \times T$, where $g_1 \cdot gx \cong h(g_1) \cdot y$, that means $h^{-1}(g_1) \cdot (g_1 \cdot gx) \cong y$, but as $y \cong g \cdot x$ we have that $h^{-1}(g_1) \cdot g_1 \cong e$, the identity element of G. Therefore $h = id_G$, $\Phi_{x,y}$ is a bijection and Φ is fully faithful. We observe that this morphism is locally essentially surjective since its image are the trivial bundles. For stackification in the proposition 3.1.8, this morphism extends to an isomorphism of stacks $\Phi' : (\mathcal{M}/G)^* \to \mathcal{M}/G$ as we want. $\square$

Remark 4.2.5. We observe that $\mathcal{M}(T)$ can be considered as a subcategory of $\mathcal{M}/G(T)$ where to each element $x \in \mathcal{M}(T)$, by the 2-Yoneda Lemma we can take its morphism $T \xrightarrow{x} \mathcal{M}$ and assign the element in $\mathcal{M}/G(T)$ given by the diagram

$$\begin{tikzcd} G \times T \arrow[r, "\mu\circ(id\times x)"] \arrow[d, "p_2"] & \mathcal{M} \\ T & \end{tikzcd}$$

that is the trivial principal bundle over T.

Example 4.2.6. For any stack $\mathcal{N}$, we can associate a G-stack given by (N, pr_2, id, id), where $pr_2 : G \times \mathcal{N} \to \mathcal{N}$ is the projection to the second component. This is a G-stack, since we have the following 2-commutative diagrams

$$\begin{tikzcd} G \times G \times \mathcal{N}(T) \arrow[r, "m \times id_{\mathcal{N}(T)}"] \arrow[d, "id_G \times pr_2"'] & G \times \mathcal{N}(T) \arrow[d, "pr_2"] \\ G \times \mathcal{N}(T) \arrow[r, "pr_2"] \arrow[ur, phantom, "\Nearrow id"] & \mathcal{N}(T) \end{tikzcd}$$

and

$$\begin{tikzcd} G \times \mathcal{N}(T) \arrow[r, "pr_2"] & \mathcal{N}(T) \\ \mathcal{N}(T) \arrow[u, "1 \times id_{\mathcal{N}(T)}"] \arrow[ur, "id_{\mathcal{N}(T)}"'] \arrow[u, phantom, shift left=4ex, "\Leftarrow id"] & \end{tikzcd}$$

with:

1. $(\alpha^x_{h,k})\alpha^x_{g,hk} = \alpha^x_{g,h}\alpha^x_{gh,k}$, for all $g, h, k \in G$ and $x \in \mathcal{M}(T)$.
2. $(\mathfrak{a}^x)\alpha^x_{g,e} = 1_x = \mathfrak{a}^x\alpha^x_{e,g}$ for every $g \in G$, $x \in \mathcal{M}(T)$ and with e the identity in G,

because α and $\mathfrak{a}$ are identities. Then (N, pr_2, id, id) is a G-stack.

We can consider the 2-functor $\iota : \mathbf{St} \to G - \mathbf{St}$ such that

1. for $\mathcal{N} \in \mathbf{St}$, we have $\iota(\mathcal{N}) = (\mathcal{N}, pr_2, id, id)$.
2. for a 1-morphism of stacks $\mathcal{M} \xrightarrow{F} \mathcal{N}$, we have $\iota(\mathcal{M}) \xrightarrow{\iota(F)} \iota(\mathcal{N})$ where $\iota(F) = (F, id)$.
3. For a 2-morphism of stacks $F \xrightarrow{\phi} F'$, we have $(F, id) \xrightarrow{\phi} (F', id)$.

This is a 2-functor since all the identities provided by ι preserve all identities and all compositions.

Proposition 4.2.7. *The stack $\mathcal{M}/G$ 2-represents the 2-functor $\mathbf{St} \to \mathbf{Cat}$ defined by*

$$F(\mathcal{N}) = Hom_{G-\mathbf{St}}(\mathcal{M}, \iota(\mathcal{N}))$$

Proof. Let $f \in Hom_{G-\mathbf{St}}(\mathcal{M}, \iota(\mathcal{N}))$ be a morphism of G-stacks. If we consider the prestack $\mathcal{P}_{\mathcal{N}}$ associated to $\mathcal{N}$, we can see that any element in $\mathcal{P}_{\mathcal{N}}(T)$ is given by

$$\begin{tikzcd} G \times T \arrow[r, "pr_2 \circ (id \times x)"] \arrow[d, "p_2"] & \mathcal{M} \\ T & \end{tikzcd}$$

as the action is pr_2 we have that any element in $\mathcal{P}_{\mathcal{N}}(T)$ is in bijective correspondence with the elements in $\mathcal{N}(T)$. Therefore if we use stackification we get that $\mathcal{N}/G \cong \mathcal{N}$. Hence we get an element in $Hom(\mathcal{M}/G, \mathcal{N})$ and we have that $Hom_{G-\mathbf{St}}(\mathcal{M}, \iota(\mathcal{N})) \cong Hom(\mathcal{M}/G, \mathcal{N})$ by proposition 3.1.8. □

Proposition 4.2.8. *The projection $\mathcal{M} \xrightarrow{q} \mathcal{M}/G$ has local sections.*

Proof. Firstly we need to check how q is defined. Let V be a smooth manifold then

$$\mathcal{M}(V) \xrightarrow{q} \mathcal{M}/G(V)$$
$$V \xrightarrow{f} \mathcal{M} \mapsto (G \times V \xrightarrow{pr_2} V, G \times V \xrightarrow{\mu \circ (id_G \times f)} \mathcal{M})$$

We need to check that in the following diagram

$$\begin{tikzcd} V \times_{\mathcal{M}/G} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "q"] \\ V \arrow[r, "p"] & \mathcal{M}/G \end{tikzcd}$$

there exist local sections on $V \times_{\mathcal{M}/G} \mathcal{M} \to V$ that makes the diagram commutes. Now we consider a covering $\{U_i \xrightarrow{i} V\}$ such that U_i are local trivialisation of $E \to V$. Then we have the diagram

$$\begin{tikzcd} G \times U_i \arrow[r, "n"] \arrow[d] & E \arrow[r, "h"] \arrow[d, "p"] & \mathcal{M} \\ U_i \arrow[r, "i"] & T & \end{tikzcd}$$

If we consider the section $s_i : U_i \to G \times U_i$ then the section for q is given by $s = h \circ n \circ s_i \in \mathcal{M}(U_i)$. □

Definition 4.2.9. A stack $\mathcal{M}$ is called a *differentiable G-stack* if there is a smooth manifold X with an action by G and a 1-morphism of G-stacks $p : X \to \mathcal{M}$ such that:

1. p is representable.
2. p is a submersion.

The map $X \to \mathcal{M}$ is then called an *G-atlas of* $\mathcal{M}$.

Remark 4.2.10. We denote the 2-category of differentiable G-stacks by $G-\mathbf{DiffSt}$.

Proposition 4.2.11. *Let $\mathcal{M}$ be a differentiable G-stack with G-atlas given by $X \xrightarrow{p} \mathcal{M}$. If σ is the smooth action by G on X, this action induces a simplicial smooth action in the associated nerve of the Lie groupoid $(X \times_{\mathcal{M}} X \rightrightarrows X)$.*

Proof. If we consider the fibered n-product $X_n = X \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} X$, we define

$$\sigma_{n,T} : G \times X_n(T) \to X_n(T)$$

such that

$$\sigma_{n,T}(g, (x_1, x_2, \ldots, x_n; p(x_1) \Rightarrow \ldots \Rightarrow p(x_n)))$$
$$= (g \cdot x_1, g \cdot x_2, \ldots, g \cdot x_n; p(g \cdot x_1) \Rightarrow \ldots \Rightarrow p(g \cdot x_n))$$

this morphism is well-defined since there exists $g \cdot p(x_1) \Rightarrow \ldots \Rightarrow g \cdot p(x_n)$ and, $g \cdot p(z) \cong p(g \cdot z)$ for any $z \in X(T)$ because p is a 1-morphism of G-stacks. We observe that we have the following diagrams:

$$\begin{tikzcd} G \times G \times X_n(T) \arrow[r, "m \times id_{X_n(T)}"] \arrow[d, "id_G \times \sigma_{n,T}"'] & G \times X_n(T) \arrow[d, "\sigma_{n,T}"] \\ G \times X_n(T) \arrow[r, "\sigma_{n,T}"'] \arrow[ur, phantom, "\Rightarrow\alpha"] & X_n(T) \end{tikzcd}$$

and

$$\begin{tikzcd} G \times X_n(T) \arrow[r, "\sigma_{n,T}"] & X_n(T) \\ X_n(T) \arrow[u, "e \times id_{X_n(T)}"] \arrow[ur, "id_{X_n(T)}"'] \arrow[u, phantom, shift left=6, "\Leftarrow\mathfrak{a}"] & \end{tikzcd}$$

where α and $\mathfrak{a}$ are identities. By construction the collection of these morphisms is a simplicial map and it is a smooth simplicial map since makes the following diagram

$$\begin{tikzcd} G \times X_n(T) \arrow[r, "\sigma_{n,T}"] \arrow[d] & X_n(T) \arrow[d] \\ G \times X(T) \arrow[r, "\sigma"] & X(T) \end{tikzcd}$$

commutes, where the vertical maps are given by the different compositions of face maps of $X_\bullet$. $\square$

Proposition 4.2.12. *Let $X \to \mathcal{M}$ be a G-atlas. Then there is an atlas for the quotient stack given by $X \to \mathcal{M} \to \mathcal{M}/G$.*

Proof. We get that p and q have local sections, so it remains to check that $q \circ p$ is representable. Hence if we consider the coverings $\{U_i \to T\}$ and $\{U_{ij} \to U_i\}$ such that the first one is the local sections for q and the second one for p. Hence we get the following commutative diagram

$$\begin{tikzcd} & & U_{ij} \times_{\mathcal{M}} X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & & U_i \times_{\mathcal{M}/G} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "q"] \\ U_{ij} \arrow[r] \arrow[uurr] & U_i \arrow[r] \arrow[ur] & T \arrow[r] & \mathcal{M}/G \end{tikzcd}$$

Thus we glue every $U_{ij} \times_{\mathcal{M}} X$ with this local section, we get a smooth manifold and as the diagram commutes we have that $T \times_{\mathcal{M}/G} X$ is a smooth manifold. $\square$

4.3 Equivariant cohomology

4.3.1 Homotopy type of the differentiable stack $\mathcal{M}/G$

Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with a G-atlas $X \xrightarrow{p} M$. We denote the action on $\mathcal{M}$ by G with $\mu : G \times \mathcal{M} \to \mathcal{M}$ and the action on X by G with $\sigma : G \times X \to X$. Then by proposition 4.2.12, we get an atlas $X \xrightarrow{p} \mathcal{M} \xrightarrow{q} \mathcal{M}/G$ and we recall that the homotopy type for the quotient stack is given by the fat geometric realisation of the nerve of the Lie groupoid $(X \times_{\mathcal{M}/G} X \rightrightarrows X)$ as we discussed in 3.4.28. In this section we will now devote our effors to get the homotopy type for $\mathcal{M}/G$.

In order to find the homotopy type of $\mathcal{M}/G$ we use the following result by Ginot-Noohi, in [24, 4] and as a consequence of proposition 3.2.5.

Proposition 4.3.1. *The following diagram is a 2-commutative diagram*

$$\begin{tikzcd} G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "pr_2"] & \mathcal{M} \arrow[d, "q"] \\ \mathcal{M} \arrow[r, "q"] & \mathcal{M}/G \end{tikzcd}$$

Therefore, we can consider the 2-commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\mu_1"] & G \times X \arrow[r, "\sigma"] \arrow[d, "id_G \times p"] & X \arrow[d, "p"] \\ G \times X \arrow[r, "id_G \times p"] \arrow[d, "Pr_2"] & G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "Pr_2"] & \mathcal{M} \arrow[d, "q"] \\ X \arrow[r, "p"] & \mathcal{M} \arrow[r, "q"] & \mathcal{M}/G \end{tikzcd}$$

and we can conclude that:

Proposition 4.3.2. *An equivalence of stacks is given by*

$$X \times_{\mathcal{M}/G} X \cong (G \times X) \times_{\mathcal{M}} X \cong G \times (X \times_{\mathcal{M}} X)$$

Proof. We have the first equivalence thanks to the diagram above $X \times_{\mathcal{M}/G} X \cong (G \times X) \times_{\mathcal{M}} X$. For the equivalence $(G \times X) \times_{\mathcal{M}} X \cong G \times (X \times_{\mathcal{M}} X)$, we consider $T \in \mathbf{Diff}$ and we see that any element in $(G \times X) \times_{\mathcal{M}} X(T)$ has the form $(g \times x, y; g \cdot p(x) \Rightarrow p(y))$, where $x, y \in X(T)$ and $g \in G$. In the same way, an element in $G \times (X \times_{\mathcal{M}} X)(T)$ has the form $(g, (x, y; p(x) \Rightarrow p(y)))$. Then we define the morphism

$$\eta_T : (G \times X) \times_{\mathcal{M}} X(T) \to G \times (X \times_{\mathcal{M}} X)(T)$$

such that $\eta(g \times x, y; g \cdot p(x) \Rightarrow p(y)) = (g, (x, g^{-1} \cdot y; p(x) \Rightarrow p(g^{-1} \cdot y)))$, where $g^{-1} \cdot y = \sigma(g^{-1}, y)$ and the morphism

$$\xi_T : G \times (X \times_{\mathcal{M}} X)(T) \to (G \times X) \times_{\mathcal{M}} X(T)$$

such that $\xi(g,(x,y;p(x)\Rightarrow p(y)))=(g\times x, g\cdot y; g\cdot p(x)\Rightarrow p(g\cdot y))$ Then we get that $\eta_T\circ\xi_T=id_{G\times(X\times_{\mathcal{M}}X)(T)}$ and $\xi_T\circ\eta_T=id_{(G\times X)\times_{\mathcal{M}}X(T)}$, as we want. $\square$

If we iterate the last proposition, we get that the $(n+1)$-product

$$X\times_{\mathcal{M}/G}X\times_{\mathcal{M}/G}\ldots\times_{\mathcal{M}/G}X\cong G^n\times X_{n+1}$$

where G^n is the cartesian n-product of G and $X_{n+1}=X\times_{\mathcal{M}}X\times_{\mathcal{M}}\ldots\times_{\mathcal{M}}X$ is the fibered $(n+1)$-product.

If we consider $T\in\mathbf{Diff}$ and the face maps in the nerve associated to $X\xrightarrow{q\circ p}\mathcal{M}/G$, we get the face maps for the simplicial smooth manifold $(G^n\times X_{n+1})_{n\geq 0}$

$$\partial_i: G^{n+1}\times X_{n+2}(T)\to G^n\times X_{n+1}(T)$$

such that

$$\partial_i(g_1,g_2,\ldots,g_{n+1},(x_1,x_2,\ldots,x_{n+2};p(x_1)\Rightarrow\ldots\Rightarrow p(x_{n+2})))$$

is equal to

$$(g_2,g_3,\ldots,g_{n+1},\pi_1(x_1,x_2,\ldots,x_{n+2};p(x_1)\Rightarrow\ldots\Rightarrow p(x_{n+2}))) \text{ if } i=0,$$

$$(g_1,\ldots,g_i{\cdot}g_{i+1},\ldots,g_{n+2},\pi_i(x_1,x_2,\ldots,x_{n+2};p(x_1)\Rightarrow\ldots\Rightarrow p(x_{n+2}))) \text{ if } 0<i<n,$$

$$(g_1,g_2,\ldots,g_n,g_{n+1}\cdot\pi_{n+2}(x_1,x_2,\ldots,x_{n+2};p(x_1)\Rightarrow\ldots\Rightarrow p(x_{n+2}))) \text{ if } i=0,$$

where $\pi_j: X_{n+2}\to X_{n+1}$ is the i-face map of the nerve of the simplicial smooth manifold $X_\bullet$ and $g_{n+1}\cdot\pi_{n+2}$ is the action induce by σ in X_{n+1}.

Theorem 4.3.3. *Let $\mathcal{M}$ be a differentiable stack and G a Lie group with a G-atlas $X\to\mathcal{M}$. Then*

$$H^*(\mathcal{M}/G,\mathbb{R})\cong H(EG\times_G\parallel X_\bullet\parallel,\mathbb{R}).$$

Proof. If we consider the bisimplicial smooth manifold given by $\{G^p\times X_n\}_{p\geq 0,n>0}$, where G^p is the cartesian p-product of G and $X_n=X\times_{\mathcal{M}}X\times_{\mathcal{M}}\ldots\times_{\mathcal{M}}X$ is the fibered n-product. With face vertical maps given by the faces in the simplicial manifold $X_\bullet$ and face horizontal maps given by

$$\partial_0^H(g_1,\ldots,g_p,z)=(g_2,\ldots,g_n,z)$$

$$\partial_i^H(g_1,\dots,g_p,z) = (g_1,\dots g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n,z) \text{ for } 1 \leq i \leq p$$

$$\partial_p^H(g_1,\dots,g_p,z) = (g_1,\dots,g_{n-1},g_n \cdot z)$$

where $z \in X_n$ for some n. Then by proposition 2.3.57 we get that $\parallel G^\bullet \times X_\bullet \parallel \cong EG \times_G \parallel X_\bullet \parallel$ as we want. $\square$

Example 4.3.4. If $\mathcal{M} = X$ is a smooth manifold, we have that the below definition of equivariant cohomology for stacks coincide with usual equivariant cohomology for smooth manifolds. Since

$$(G \times X) \times_X X \simeq G \times X$$

and the maps of the Lie groupoid is given by the action $\mu : G \times X \to X$ and projection $pr_2 : G \times X \to X$, hence this Lie groupoid coincide with the transformation groupoid and its fat geometric realisation is $EG \times_G X$. Then de Rham cohomology of X/G is $H^*_{dR}(X/G) = H^*(EG \times_G X, \mathbb{R})$, which is the Borel model as defined in 2.3.61. So when we have a differentiable stack associated to a smooth manifold the equivariant stack cohomology coincides with the classical equivariant cohomology of the given smooth manifold.

Since we note that in the classical case the cohomology of a quotient stack coincide with the one given by the Borel model we give the following notion:

Definition 4.3.5. Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with a G-atlas $X \xrightarrow{p} M$. The *equivariant cohomology* of $\mathcal{M}$, $H^*_G(\mathcal{M})$, is given by

$$H^*_G(\mathcal{M}) = H^*(\mathcal{M}/G, R)$$

for R any commutative ring with identity.

Remark 4.3.6. Since we are interested in the comparison with de Rham cohomology, in our current work we focus in singular cohomology with real coefficients, see subsection 3.4.4.1.

Remark 4.3.7. By remark 3.4.29, this definition of equivariant cohomology can be done for any cartesian sheaf or complex of cartesian sheaves.

4.3.2 Cartan model for differentiable stacks

In this subsection, we apply the Cartan model for simplicial smooth manifold as described in appendix A.

Let $\mathcal{M}$ be a differentiable G-stack with G-atlas $X \to \mathcal{M}$ for a compact Lie group G. We denote the action on X by σ and the action on the stack $\mathcal{M}$ by μ. Then we can consider the simplicial smooth action $\sigma_\bullet$ induced by σ in $X_\bullet$, as in proposition 4.2.11.

Then we can consider the complex of simplicial equivariant forms

$$C^{2p+m} = (\bigoplus_{q+r=m} (S^p(\mathfrak{g}^\vee) \otimes \Omega^q(X_r)^G), D - \iota)$$

as in the subsection A.0.1. Now we can consider its cohomology $H^*_G(X_\bullet)$. If we apply the theorem A.0.1 to this complex, then:

Proposition 4.3.8. *The cohomology of the complex $C^\bullet$ holds that*

$$H^*_G(X_\bullet) \cong H^*(EG \times_G \parallel X_\bullet \parallel, \mathbb{R})$$

with $\parallel X_\bullet \parallel$ is the fat geometric realisation of the simplicial smooth manifold $X_\bullet$.

If we compare this result with the theorem 4.3.3, we get that this Cartan model and the cohomology of the quotient stack $\mathcal{M}/G$ coincide, that is:

$$H^*_G(X_\bullet) \cong H^*(\mathcal{M}/G, \mathbb{R}).$$

Let G be a compact connected Lie group and K a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and the Lie algebra of K, respectively. $\mathfrak{g}^\vee$ is the dual of the Lie algebra $\mathfrak{g}$ and $S(\mathfrak{g}^\vee)$ its symmetric algebra. Then by subsection A.0.3 we get that:

Proposition 4.3.9. *Suppose that the restriction map*

$$S(\mathfrak{g}^\vee)^G \to S(\mathfrak{k}^\vee)^K$$

is an isomorphism of algebras. Then the restriction map

$$H_G(X_\bullet) \to H_K(X_\bullet)$$

in equivariant cohomology is an isomorphism.

By theorem A.0.6, we get that if have a differentiable G-stack $\mathcal{M}$ with G-atlas $X \to \mathcal{M}$, then:

Theorem 4.3.10. *Let G be a connected compact Lie group, T a maximal torus and W its Weyl group. Then*

$$H_G^*(X_\bullet) \cong H_T^*(X_\bullet)^W.$$

4.4 Some results via spectral sequences

We state some results on spectral sequences for the equivariant cohomology of a differentiable G-stack.

4.4.1 Sheaf cohomology on quotient stacks

Let G be a Lie group, $\mathcal{M}$ a G-stack and $\mathfrak{F}$ a cartesian sheaf on the quotient stack $\mathcal{M}/G$ with atlas $X \to \mathcal{M} \to \mathcal{M}/G$, where $X \to \mathcal{M}$ is a G-atlas for $\mathcal{M}$. So we can get an induced sheaf $\mathfrak{F}$ on $\mathcal{M}$ and also a sheaf $\mathfrak{F}_\bullet$ on the associated bisimplicial smooth manifold $(G^p \times X^n)_{n>0}$. We present two spectral sequences under the previous assumptions.

Theorem 4.4.1. *There exists a spectral sequence such that*

$$E_1^{r,n} = H^r([X^n/G], \mathfrak{F}) \Rightarrow H_G^{r+n}(\mathcal{M}, \mathfrak{F}).$$

Proof. We take an acyclic equivariant resolution K^q of $\mathfrak{F}$. Then we can consider the induced sheaves $K^{p,n,q}$ on $G^p \times X^n$. Then we get the triple complex $\Gamma(G^p \times X^n, K^{p,n,q})$. We know that the geometric resolution of bisimplical smooth manifold accomplishes

$$\| G^p \times X^{p+1} \| \cong \| p \to \| n \to \| G^p \times X^n \| \| \| \| .$$

So the double complex $\Gamma(G^p \times X^{p+1}, K^{p,q})$ calculates the same cohomology. We know that the resolution is acyclic and computing first with respect to q we get the complex given by $\Gamma(G^p \times X^{p+1}, \mathfrak{F})$ and that is $H_G^*(\mathcal{M}, \mathfrak{F})$. On the other hand if we consider the triple complex $\Gamma(G^p \times X^n, K^{p,q})$ with differential d_K given by the resolution and a second differential δ the one provided by the bisimplicial smooth manifold making the double complex $C^{r,n} = \bigoplus_{r=p+q} \Gamma(G^p \times X^n, K^{p,q})$, we get

$$E_0^{r,n} = \bigoplus_{r=p+q} \Gamma(G^p \times X^n, K^{p,q})$$

and

$$E_1^{r,n} = H^r([X^n/G], \mathfrak{F}).$$

Therefore we get the result. $\square$

Theorem 4.4.2. *If G is a discrete group, there exists a spectral sequence such that*

$$E_2^{p,r} = H^p(G, H^r(\mathcal{M}, \mathfrak{F})) \Rightarrow H_G^{p+r}(\mathcal{M}, \mathfrak{F}).$$

Proof. We can consider that the sheaf induced by $\mathfrak{F}$ on $\mathcal{M}$ with the same $\mathfrak{F}$ and also a sheaf $\mathfrak{F}_\bullet$ on the associated bisimplicial smooth manifold $(G^p \times X^n)_{n>0}$. Consider an acyclic resolution $K^\bullet$ of $\mathfrak{F}$ and the induced sheaves in the bisimplicial smooth manifold, then we get the triple complex $\Gamma(G^p \times X^n, K^{p,n,q})$. As we know that the geometric resolution of bisimplical smooth manifold accomplishes

$$\| G^p \times X^{p+1} \| \cong \| p \to \| n \to \| G^p \times X^n \| \| \| \|$$

we have again that the double complex $\Gamma(G^p \times X^{p+1}, K^{p,q})$ calculates the same cohomology. We know that the resolution is acyclic and computing first with respect to q we get the complex given by $\Gamma(G^p \times X^{p+1}, \mathfrak{F})$ and that is $H_G^*(\mathcal{M}, \mathfrak{F})$. If we consider the triple complex $\Gamma(G^p \times X^n, K^{p,n,q})$ where each element can be seen as a map $G^p \to \Gamma(X^n, K^{n,q})$ since G is discrete and for each $f(\overline{g}, \overline{x}) \in \Gamma(G^p \times X^n, K^{p,n,q})$ we can define a map $\overline{f}(\overline{x})(\overline{g}) = f(\overline{g}, \overline{x}) \in \Gamma(X^n, K^{n,q})$. We are going to consider this triple complex as a double complex with a first differential given by $\delta = d_q + (-1)^n d_n$ and d_p, where d_q is the differential given by the resolution, d_n given by the simplicial structure of $X_\bullet$ and d_p by the simplicial structure on $G_\bullet$. Besides there exists an action ϕ of G on $H^n(\mathcal{M}, \mathfrak{F})$ given by

$$\phi : G \times H^n(\mathcal{M}, \mathfrak{F}) \to H^n(\mathcal{M}, \mathfrak{F})$$

$$(g, [f]) \mapsto \phi(g, [f]) = [f(\overline{g}, g \cdot \overline{x})]$$

where $f \in \Gamma(G^n \times X^{n+1}, \mathfrak{F})$. Hence, if we apply the differential δ first, we get

$$E_1^{p,r} = C^p(G, H^r(\mathcal{M}, \mathfrak{F}))$$

and as $d_1 = \delta_G$ is the differential induced by d_p then

$$E_2^{p,r} = H^p(G, H^r(\mathcal{M}, \mathfrak{F})). \quad \square$$

Example 4.4.3. If $\mathcal{M} = X$ is a smooth manifold with an action of a discrete group G, we have that the previous result generalises the spectral sequence

$$E_2^{p,n} = H^p(G, H^n(X, \mathfrak{F})) \Rightarrow H^{p+n}([X/G], \mathfrak{F})$$

which is given by Felder et al., in [20, A.4].

4.4.2 Hypercohomology of a quotient stack

Let $\mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m$ be a complex of Cartesian sheaves of abelian groups over $\mathcal{M}/G$ with an atlas given by $X \to \mathcal{M} \to \mathcal{M}/G$, where $X \to \mathcal{M}$ is a G-atlas. Consider $\mathfrak{F}_r$ the sheaf associated on $\mathcal{M}$ and for the bisimplicial smooth manifold $G^p \times X^n$ the bisimplicial sheaf $\mathfrak{F}_{r,\bullet}$, for any r.

Theorem 4.4.4. *There exists a spectral sequence such that*

$$E_1^{s,n} = H^s([X^n/G], \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m) \Rightarrow H_G^{s+n}(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \cdots \to\to \mathfrak{F}_m).$$

Proof. Let K_r^q be an acyclic equivariant resolution of $\mathfrak{F}_r$ and we denote by $K_r^{p,n,q}$ the sheaves generated on $G^p \times X^n$. Then we have the quadruple complex

$$C^{p,n,q,r} = \Gamma(G^p \times X^n, K_r^{p,n,q}).$$

As below we can consider the complex $C^{p,q,r} = \Gamma(G^p \times X^{p+1}, K_r^{p,q})$ and we have that $K_r^{p,q}$ is acyclic. That is if we compute with respect to q we get the double complex $C^{p,r} = \Gamma(G^p \times X^{p+1}, \mathfrak{F}_r^p)$. Hence the quadruple complex has as cohomology $H_G^*(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)$.

Besides we can consider the quadruple complex as a double complex if we consider the complex given by $C^{s,n} = \bigoplus_{s=p+q+r} \Gamma(G^p \times X^n, K_r^{p,q})$ with its respective differential and as a second differential the one given on $X_\bullet$. Then

$$E_0^{s,n} = \bigoplus_{s=p+q+r} \Gamma(G^p \times X^n, K_r^{p,q})$$

and

$$E_1^{s,n} = H^s([X^n/G], \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m). \quad \square$$

Theorem 4.4.5. *If G is a discrete group, there exists a spectral sequence such that*

$$E_2^{p,s} = H^p(G, H^s(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)) \Rightarrow H_G^{p+s}(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \cdots \to \mathfrak{F}_m).$$

Proof. Let K_r^q be an acyclic equivariant resolution of $\mathfrak{F}_r$ and we denote by $K_r^{p,n,q}$ the sheaves generated on $G^p \times X^n$. Then we have the quadruple complex

$$C^{p,n,q,r} = \Gamma(G^p \times X^n, K_r^{p,n,q}).$$

As below we can consider the complex $C^{p,q,r} = \Gamma(G^p \times X^{p+1}, K_r^{p,q})$. If we compute with respect to the resolution we get the double complex $C^{p,r} = \Gamma(G^p \times X^{p+1}, \mathfrak{F}_r^p)$. Hence the quadruple complex has as cohomology $H_G^*(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)$.

Besides we can consider the quadruple complex $C^{p,n,q,r}$ as a double complex given by the differential for n, q, r and as second differential the one of the simplicial structure of $G_\bullet$. Moreover if we consider the elements in $C^{p,n,q,r}$ as maps $G^p \to \Gamma(X^n, K_r^q)$ then

$$E_0^{p,s} = C^p(G, \bigoplus_{n+q+r=s} \Gamma(X^n, K_r^q))$$

$$E_1^{p,s} = C^p(G, H^s(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m))$$

and

$$E_2^{p,s} = H^p(G, H^s(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)). \quad \square$$

Example 4.4.6. If $\mathcal{M} = X$ is smooth manifold with an action of a discrete group G, we have that the previous result generalises the spectral sequence

$$E_2^{p,n} = H^p(G, H^n(X, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)) \Rightarrow H^{p+n}([X/G], \mathfrak{F}_0 \to \mathfrak{F}_1 \cdots \to \mathfrak{F}_m)$$

which is given by Felder et al., in [20, A.7].

4.4.3 Spectral sequence of group cohomology for compact Lie group actions

Let us finally now deal with the case that the group G is a compact Lie group. We then get the following general spectral sequence:

Theorem 4.4.7. *There exists a spectral sequence such that*

$$E_1^{k,p} = \bigoplus_{q+n=k} H^p(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes S^t(\mathfrak{g}^\vee))$$

$$E_2^{k,p} = Tot_k \bigoplus_{q+n=k} H^p(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes S^t(\mathfrak{g}^\vee)) \Rightarrow H_G^{k+p}(\mathcal{M}, \mathbb{R}).$$

Proof. Let $(\Omega^q(G^p \times X_n), d_{dR}, \partial_G, \partial_X)$ be the triple complex which computes the cohomology $H^*_{dR}(\mathcal{M}/G)$. We can consider the complex of smooth functions

$$C_\infty(G^p, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes \wedge^t(\mathfrak{g}^{\vee p}))$$

with differentials induced for d_{dR}, ∂_G and ∂_X. So if we take the function

$$\Psi : \Omega^q(G^p \times X_n) \to C_\infty(G^p, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes \wedge^t(\mathfrak{g}^{\vee p}))$$

$$\sum \omega_i(\vec{g}, \vec{x}) dg_I dx_J \mapsto \vec{g} \xrightarrow{\omega} (\omega_i(\vec{g}, \vec{x}) dx_J \otimes dg_I)$$

which commutes with the differentials and is a bijection, we have that the complex of smooth functions computes the same cohomology as $\Omega^q(G^p \times X_n)$.

Moreover if we consider the double complex

$$C^{k,p} = \bigoplus_{q+n=k} C_\infty(G^p, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes \wedge^t(\mathfrak{g}^{\vee p}))$$

Then $E_0^{k,p} = C^{k,p}$,

$$E_1^{k,p} = \bigoplus_{q+n=k} H^p_\infty(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes H^p(\wedge^t(\mathfrak{g}^{\vee p})))$$

$$= \bigoplus_{q+n=k} H^p_\infty(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes S^t(\mathfrak{g}^\vee))$$

Here H^p_∞ means the group cohomology for smooth cochains, however these cochains can be approximated by continuous cochains [30, 5], [55, 6], [60, I]; so we can consider this cohomology as the continuous version H^p. Hence we get for the second page

$$E_2^{k,p} = \mathrm{Tot}_k \bigoplus_{q+n=k} H^p(G, \bigoplus_{s+t=q} \Omega^s(X_n) \otimes S^t(\mathfrak{g}^\vee)). \quad \square$$

Example 4.4.8. If $\mathcal{M} = pt$, this result generalises the spectral sequence

$$E^{t,p} = H^p(G, S^t(\mathfrak{g}^\vee)) \Rightarrow H^{t+p}(BG, \mathbb{R})$$

which is given by Bott in [9] and by Stasheff in [55].

Appendix A

The Cartan model for simplicial smooth manifolds

In this appendix, we give the construction of equivariant forms for simplicial smooth manifolds, following [46, C.1], [57, 2.2] and [58]. We provide a discussion about the restriction of the group of equivariance based on [27, 6.5 & 6.8].

A.0.1 Cartan model

Let $M_\bullet$ be a simplicial smooth manifold and a simplicial smooth action $\mu_\bullet : G \times M_\bullet \to M_\bullet$ with G a compact Lie group.

We can define the complex

$$C^{2p+m} = \left(\bigoplus_{q+r=m} (S^p(\mathfrak{g}^\vee) \otimes \Omega^q(M_r)^G), D - \iota \right)$$

where $D = d_{dR} + (-1)^q \partial$ is the operator of the simplicial de Rham complex defined in the subsection 2.3.6 and ι the operator defined by the interior multiplication in the usual Cartan model, as in the definition 2.3.53. Then the cohomology of this complex its denoted by $H^*_G(M_\bullet, \mathbb{R})$, compare with [46, C.1] and [58, 4.2].

Proposition A.0.1. *[58, 4.2],[57, 4.1] The cohomology of $C^\bullet$ holds that*

$$H^*_G(M_\bullet, \mathbb{R}) \cong H^*(EG \times_G \parallel M_\bullet \parallel, \mathbb{R})$$

A.0.2 Cartan model as a double complex

Consider the double complex

$$C^{p,q} = \left(S^p(\mathfrak{g}^\vee) \otimes \left(\bigoplus_{s+r=q-p} \Omega^s(M_r) \right) \right)^G$$

with horizontal operator D and horizontal ι.

Theorem A.0.2. *The E_1 term in the spectral sequence of $C^{p,q}$ is*

$$(S^p(\mathfrak{g}^\vee) \otimes H^{q-p}(M_\bullet, \mathbb{R}))^G.$$

Proof. The complex $C^{p,q}$ with the boundary D sits inside the complex $Z = (S^*(\mathfrak{g}^\vee) \otimes (\bigoplus \Omega^*(M_\bullet)))$ and the cohomology groups of $C^{p,q}$ are just the G-invariant components of the cohomology which are appropriately graded components of $S^*(\mathfrak{g}^\vee) \otimes H(M_\bullet, \mathbb{R})$. $\square$

To compute E_1 we are going to use that $\partial_i^* \iota_X \omega = \iota_X \partial_i^* \omega$ where ∂_i is a face in the simplicial manifold. We get

$$\partial_i^*(\iota(X)\omega)_p(X_1, \ldots X_{k-1}) = \partial_i^* \omega_p(X(p), X_1, \ldots, X_{k-1})$$

$$= \omega_{\partial_i(p)}(X(\partial_i(p)), d_{\partial_{i_p}} X_1(p), \ldots, d_{\partial_{i_p}} X_{k-1}(p)) = \iota(X)\omega_{\partial_i(p)}(d_{\partial_{i_p}} X_1(p), \ldots, d_{\partial_{i_p}} X_{k-1}(p))$$

$$= \iota(X)(\partial_i^* \omega)_p(X_1, \ldots, X_{k-1})$$

and we set the following proposition

Proposition A.0.3. *The connected component of the identity in G acts trivially on $H^*(M_\bullet, \mathbb{R})$.*

Proof. We consider

$$\iota D + D\iota$$

$$= \iota(d + (-1)^{q+1}\partial) + (d + (-1)^q \partial)\iota = \iota d + d\iota = L_\alpha$$

So the Lie derivative L_α is chain homotopic to 0 in $\Omega^q(M_r)$. As in the Lie derivative the action acts trivially ia connected component of the identity, we get the result. $\square$

Theorem A.0.4. *If G is connected, then $E_1^{p,q} = S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(M_\bullet, \mathbb{R})$*

Proof. By proposition A.0.3, we get that G acts trivial on $H^*(M_\bullet, \mathbb{R})$ and if we apply this in the theorem A.0.2, we get the result. $\square$

A.0.3 Restricting the acting group

Let G be a compact connected Lie group and K a closed subgroup of G (not necessarily connected). We then get an injection of Lie algebras

$$\mathfrak{k} \to \mathfrak{g}$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ is the Lie algebra of G. Also we get an injection in the dual spaces

$$\mathfrak{g}^\vee \to \mathfrak{k}^\vee$$

which extends to the symmetric algebras

$$S(\mathfrak{g}^\vee) \to S(\mathfrak{k}^\vee)$$

and then to

$$(S(\mathfrak{g}^\vee) \otimes \bigoplus \Omega^*(M_\bullet))^G \to S(\mathfrak{k}^\vee \otimes \bigoplus \Omega^*(M_\bullet))^K$$

Thus we get a restriction map

$$H_G(M_\bullet, \mathbb{R}) \to H_K(M_\bullet, \mathbb{R})$$

and also a restriction morphism at each stage of the corresponding spectral sequences. Since G acts trivially on $H^*(M_\bullet, \mathbb{R})$ and K is a subgroup of G then K acts trivially as well. So by the theorem A.0.2, we get a morphism on E_1

$$S(\mathfrak{g}^\vee)^G \otimes \bigoplus \Omega^*(M_\bullet) \to S(\mathfrak{k}^\vee)^K \otimes \bigoplus \Omega^*(M_\bullet)$$

Then we get:

Theorem A.0.5. *Suppose that the restriction map*

$$S(\mathfrak{g}^\vee)^G \to S(\mathfrak{k}^\vee)^K$$

is an isomorphism. Then the restriction map

$$H_G(M_\bullet, \mathbb{R}) \to H_K(M_\bullet, \mathbb{R})$$

in equivariant cohomology is an isomorphism.

Let T be a maximal torus of G and let $K = N(T)$ be its normalizer. The quotient group $W = K/T$ is the Weyl group. It is a finite group so the Lie algebra of K is the same as the Lie algebra of T. Since T is abelian its action on $\mathfrak{t}^\vee$ and on $S(\mathfrak{t}^\vee)$ is trivial. So we get

$$S(\mathfrak{k}^\vee)^K = S(\mathfrak{t}^\vee)^K = S(\mathfrak{t}^\vee)^W$$

According to *Chevalley restriction theorem* the restriction morphism

$$S(\mathfrak{g}^\vee)^G \to S(\mathfrak{t}^\vee)^W$$

is an isomorphism, see [62, 2.1.5.1], so we can apply the theorem A.0.5. Besides considering the inclusion $T \to K$ we get a morphism of double complexes

$$C_K(M_\bullet) \to C_T(M_\bullet)^W$$

which induces a morphism

$$H_K(M_\bullet) \to H_T(M_\bullet)^W$$

and a morphism at each level of the spectral sequences. At E_1-level we get the identity morphism

$$S(\mathfrak{t}^\vee)^W \otimes H^*(M_\bullet, \mathbb{R}) \to S(\mathfrak{t}^\vee)^W \otimes H^*(M_\bullet, \mathbb{R})$$

then by theorem A.0.5 we get $H^*_K(M_\bullet, \mathbb{R}) = H^*_T(M_\bullet, \mathbb{R})^W$. And we conclude:

Theorem A.0.6. *Let G be a connected compact Lie group, T a maximal torus and W its Weyl group. Then:*

$$H^*_G(M_\bullet, \mathbb{R}) \cong H^*_T(M_\bullet, \mathbb{R})^W$$

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