Dynamical Systems, Cocycles and Cohomology of Action Groupoids

Thesis submitted for the degree of Doctor of Philosophy at the University of Leicester

by

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Abstract

In this thesis, we study general cocycles of dynamical systems in topological, measurable and smooth (differentiable) settings. Dynamical systems are viewed here as given by actions of a discrete, topological, measurable or Lie group on a set, topological space, measurable space or smooth manifold respectively depending on the given geometrical setting to be considered. We will mostly concentrate on the topological and smooth settings in this thesis, but will comment about the necessary alterations in the discrete and measurable setting. Cocycles are functions on the Cartesian product of the spaces and groups involved with values in an abelian group depending again on the given geometrical settings. A main task of this thesis is to interpret these cocycles as general cohomology classes of certain action groupoids, which decode the dynamical system. Similarly, we show that cohomology classes of action groupoids associated to dynamical systems can be viewed as cocycles. The action groupoids which arise out of the dynamical systems and the given geometrical setting are discrete groupoids, topological groupoids, measurable groupoids or Lie groupoids. We will introduce a very general groupoid cohomology and homology theory with values in vector bundles and discuss its basic properties generalising group cohomology and singular cohomology. Furthermore, we will study extensions of dynamical systems via cocycles and interpret these as low-dimensional cohomology classes. Some low-dimensional homology and cohomology groups are calculated. Finally, we interpret cocycle cohomology classes as cohomological obstructions for extending dynamical systems following a suggestion by Tao.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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This dissertation is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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Chapter 1

Introduction

1.1 Groupoids and Dynamical Systems

In this thesis we discuss cocycles and the general cohomology of action groupoids to study topological and smooth dynamical systems.

Groupoids were introduced and studied first by the German mathematician Brandt [6] in 1926 to analyse the composition of quadratic forms in four variables. They soon turned out to be useful when studying more complicated symmetries of geometric structures than groups allow. Groups are normally enough to describe symmetries of homogeneous structures, but there are plenty of geometric structures including dynamical systems, which only admit few or non-trivial automorphisms and in order to describe their multi-object symmetries algebraically it is more useful to use the language of groupoids. Groupoids can formally be defined as categories such that their morphisms are isomorphisms, but for most applications also here in our work we define them intrinsically as ordered pairs with structure maps which some axioms naturally axioms for compositions in the same spirit as originally emphasised by Brandt. A groupoid can also be seen as giving a family of symmetries collecting symmetries of different objects like fibres of fibre bundles. Ehresmann [13] added further structures (topological and differentiable as well as algebraic) to groupoids, thereby introducing groupoids as a tools to study symmetries in algebraic topology and differential geometry. An important example of a topological groupoid is for example the fundamental groupoid, which encodes symmetries of homotopy classes of paths in a topological spaces without taking into account choices of base points in contrast to the fundamental group as first introduced by Poincaré [7]. Later, Grothendieck [15] revolutionising algebraic geometry used groupoids extensively and, in particular, showed how they could be used successfully to tame the unruly equivalence relations and high number of automorphisms which arise in the construction of moduli spaces for classifying geometrical structures, thus giving rise to algebraic stacks (see also [30]). Later Connes [9] used groupoids in his framework of non-commutative geometry for a unified study of operator theory, C^*-algebras, foliations and index theory. In using groupoids associated to C^*-algebras, groupoids also find their ways into the theory of dynamical systems. Here versions of groupoids in the category of smooth manifolds play a prominent role. These Lie groupoids (sometimes also called smooth or differentiable groupoids) were first studied in the 1950s by Ehresmann as an alternative to principal bundles and connection theory in differential geometry (see also [4]). Similarly as Lie groups have associated Lie algebras to study infinitesimal properties and their symmetries, Lie groupoids possess associated Lie algebroids, introduced by Pradines in 1967 and which take care of the multi-object infinitesimal symmetries of Lie groupoids [31].

Dynamical systems can be studied in a discrete, topological, measurable and smooth geometrical setting, but essentially they can be understand as particular examples as group actions on a spaces [20]. Most of these actions are highly complicated and therefore need good algebraic invariants to study, classify and distinguish them. A direct cohomologcal invariant arises with the construction of cocycles for the group action with values normally in an abelian group. Cocycles play a prominent role in the study of dynamical systems for example when dealing with autonomous dynamical systems as indicated by Oseledets theorem and their generalisations as in the more recent work of Ruelle and others [36]. In effect, whenever one wants to study multiplicative properties in dynamics cocycles arise. They also give naturally rise to cohomology classes and this is what we are studying for general topological and smooth dynamical systems, which arise via group actions. Group actions give rise to so-called action groupoids, which basically model nice quotient spaces for non-free actions (see [9], [7]). They are special cases of topological or Lie groupoids and have very recently been studied also in the context of dynamical systems [8].

Cocycles are closed cochains and used in algebraic topology to express obstructions (for example, to integrating a differential equation on a closed manifold). They are also used in group cohomology as explicit constructible cohomology classes. Here, they arise as cohomology classes of group actions on spaces underlying dynamical systems, which in the special case of a group action on a point space recovers the analogue notions from group cohomology.

Our main aim in this thesis is to study dynamical systems in the topological and smooth geometrical setting via group actions and action groupoids using explicit constructions of cocycles as cohomology classes. We will construct a general cohomology theory for topological and Lie groupoids, which in the case of associated action groupoids will be used for the cohomological study of dynamical systems. We will show the equivalence of working with cocycles and general cohomology classes when studing dynamical systems from a homological point of view. Finally, following a suggestion by Tao we will also show how higher cocycles or cohomology classes can be interpreted as obstructions for extending dynamical systems. We intend to pursue this research further in the future looking into non-abelian cohomology and higher categorical aspects of cohomology for the construction of higher and refined algebraic invariants suitable for dynamical systems.

1.2 Structure of the Thesis

This thesis is organised as follows:

Chapter 1 comprises the introduction with an overview and background of the material covered, in particular concerning groupoids, cocycles, dynamical systems and their interplay.

In chapter 2, we recall the notion of group actions which is our essential set-up to study dynamical systems. Topological dynamics refers to the study of continuous actions of topological groups on a topological space. So we begin with the definition of topological groups and discuss some basic examples. We study topological transformation groups and properties of their actions such as transitivity and freeness with some examples from topology. Applications to dynamical systems focus either on the topological group of the real numbers to define a continuous flow or on the topological group of integers to define a discrete flow. Another class of dynamical systems arises as smooth dynamical systems which considers Lie group actions on smooth manifolds.

This requires to recall the definition of a Lie group with examples. Also some information on ergodic theory is given which is important to discuss group actions on measure spaces. This material provides a basis to study measurable groupoids in chapter 3. Afterwards we review some aspects of the general theory of fibre bundle and define a particular class of fibre bundles, namely principal bundles and vector bundles to be uses later in the definition of general groupoid cohomology. Then we introduces and discusses the important concept of a cocycle of a dynamical system along with illustrating examples such as the Radon-Nikodam derivative from measure theory. We also provide basic but important properties from the theory of cocycles and their appearance in different geometrical settings. A dynamical system can be extended by construction from a given cocycle. Dynamical system extensions via cocycles are defined and studied, generalising group extensions. Finally, two examples of cocycles of dynamical systems are studied as natural ways in which cocycles arise , namely the derivative cocycle and the time change cocycle. Then the general concept of cohomology of dynamical system is studied using methods and concepts from algebraic topology. We discuss alternative approaches and basic properties of cohomology of dynamical systems. The last section of chapter 2 gives some basic calculations of cohomology groups, in particular concerning transitive and free group actions.

In chapter 3, we start with reviewing the main aspects of the general theory of groupoids in different settings. We begin with a discussion of discrete groupoids, i.e. groupoids in the category of sets. Transformations between groupoids are defined afterwards. Topological, Lie and measurable groupoids are defined as groupoids in the category of topological spaces, smooth manifolds and measure spaces. We give many examples of groupoids from these different geometrical settings. In the next section we study the relationship between dynamical systems and action groupoids, which is the main concept here. We then discuss ways of how to construct new groupoids out of given ones, namely we introduce induced groupoids and strong and weak pullbacks of groupoids. Next we discuss the several notions of equivalences of Lie groupoids, namely strong, weak and Morita equivalences. In the following sections representation of Lie groupoids are defined as a particular type of groupoid action generalising similar concepts for Lie groups. Then a general Lie groupoid cohomology theory is introduced and fundamental examples and properties are studied. In the final section of this thesis we interpret cocycles of a dynamical system as action groupoid cohomology classes and vice versa. Finally, following a suggestion of Tao we study cocycle extensions of dynamical systems using cohomology classes as obstructions.This allows for an interpretation of the second cohomology group of dynamical systems and in particular how to see low dimensional cohomology classes as extension classes for dynamical system.

Chapter 2

Dynamical Systems, Cocycles and Cohomology

The theory of topological dynamics deals with the study of continuous actions of any topological group G on a topological space X [17]. A particular topological dynamical system is concerned with group actions of the set of real numbers $\mathbb{R}$ or the set of integers $\mathbb{Z}$ on topological (smooth or measurable) spaces. In this chapter, the concept of a group action is defined in the different contexts with examples. The smooth versions of topological dynamical systems deal with smooth actions of Lie groups on differentiable (smooth) manifolds. Again a main example is given by actions of the real numbers $\mathbb{R}$. Some preliminaries on fibre bundles are recalled preparing for the next chapter. Cocycles of dynamical systems will be defined as continuous (smooth or measurable) functions with values in an abelian group satisfying the cocycle equation. Finally, notions of cohomology for dynamical systems are introduced based on the general theory of cocycles and singular cochains of topological spaces.

2.1 Transformation Groups, Actions and Dynamical Systems

Some of the content of this section can be found in several texts on dynamics and algebraic topology, but for the material presented here we mainly recommend [1], [20] and [17].

Let us begin with group actions (in an abstract setting) without any extra structure, so the basic set-up in the study of dynamical systems is that a group G is acting on a set X.

Definition 2.1. *Suppose G is a group and X is a set. A* ***left action of G on X*** *(or X is said to be a* ***left G-set****) is a map $\phi : G \times X \longrightarrow X$, written $(g, x) \mapsto \phi(g, x) = gx$, with the following properties:*

(i) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ for all $x \in X$ and all $g_1, g_2 \in G$,

(ii) $\phi(e, x) = x$ for all $x \in X$ and e is the identity of G.

Remark 2.1. (1) Similarly, a **right action of G on X** can be defined as a map $\phi : X \times G \to X$, written $(x, g) \mapsto xg$ with the same properties, but the composition of maps works in reverse: $((x, g_1)\phi, g_2)\phi = (x, g_1 g_2)\phi$.

(2) Any right action determines a left action, and vice versa, by using the following correspondence $\phi(g, x) = (x, g^{-1})\phi$. And according to this bijective correspondence between left action and right action it will therefore be sufficient to study only one of these type of actions.

Definition 2.2. *Let X be a left G-set. For each $x \in X$, the set $G_x = \{g \in$*

*$G : gx = x\}$ is a subgroup of G, called the **isotropy group** or the **stabilizer group** of x.*

2.1.1 Examples

Let H be a subgroup of a group G and $h \in H$, $g \in G$. Then H acts on G as follows in two different ways $\phi : H \times G \longrightarrow G$:

(1) $\phi(h, g) = hg$ (left translation), and

(2) $\phi(h, g) = hgh^{-1}$ (conjugation by h).

(3) Let G be the group of the nonzero real numbers with multiplication and S be the set of all vectors in the three-dimensional space $\mathbb{R}^3$. Then G acts on S by scalar multiplication. This means, $\phi(g, (a, b, c)) = (ga, gb, gc)$ if g is a nonzero real number. Now for any vector v, we have $\phi(1, v) = v$. If $g, h \in G$, then $\phi(gh, v) = (gha, ghb, ghc) = g(ha, hb, hc) = g(h, v) = \phi(g, \phi(h, v))$.

2.1.2 Topological Transformation Groups

In practice, the set X normally has many additional geometrical structures, and the action by the group G is compatible with these structures. For example, if X is a topological space, then one can study these systems when G has a topological structure acting via continuous maps.

Definition 2.3. *A **topological group** is a group G endowed with a topology such that the maps $m : G \times G \longrightarrow G$ and $i : G \longrightarrow G$ given by $m(g_1, g_2) = g_1 g_2$, $i(g) = g^{-1}$ are continuous.*

Each of the following is a topological group:

(i) set of the real numbers $\mathbb{R}$ with additive group structure and Euclidean topology.

(ii) the **general linear group** $GL(n, \mathbb{R})$, which is the set of $n \times n$ invertible real matrices under matrix multiplication, with the induced topology obtained from $\mathbb{R}^{n^2}$.

(iii) A **discrete group** G which is any group with the discrete topology. This includes also finite groups.

Definition 2.4. *Let G be a topological group acting on a topological space X by a map $\phi : G \times X \longrightarrow X$. The action is said to be **continuous** if the map ϕ is continuous. The action is called an **action by homeomorphisms** if for each $g \in G$, the map $\phi^g : X \longrightarrow X$, defined by, $x \mapsto gx$ is a homeomorphism of X.*

The next proposition explains the relationship between the two concepts in the last definition.

Proposition 2.1. *Let G be a topological group acting on a topological space X by a map $\phi : G \times X \longrightarrow X$.*

(i) If the action ϕ is continuous, then it is an action by homeomorphisms.

(ii) If the group G has the discrete topology, then the action ϕ is continuous if and only if it is an action by homeomorphisms .

Proof. See [24] p.79.

□

Remark 2.2. *The triple (G, X, ϕ) is called a* ***topological transformation group*** *and X is called a* ***G-space****.*

Definition 2.5. *Let (G, X, ϕ) be a topological transformation group. For any $x \in X$, the set $G \cdot x = \{\phi(g, x) : g \in G\} \subseteq X$ is called the* ***orbit of x****. The action is said to be* ***transitive*** *if for every pair of points $x, y \in X$, there is a group element g such that $\phi(g, x) = y$, that is, the orbit of each point of X covers all the space X. The action is said to be* ***free*** *if the only element of the group G that fixes any point in X is the identity ; that is, $\phi(g, x) = x$ for some x implies $g = e$.*

Definition 2.6. *Let X be a G-space, we define an equivalence relation $\sim$ by saying $x \sim y$ if there is an element $g \in G$ such that $\phi(g, x) = y$. The* ***orbit space*** *is the quotient space denoted by X/G provided with the finest topology such that the quotient map $\gamma : X \longrightarrow X/G$ is continuous.*

2.1.3 Examples

(1) For any topological space X and any topological group G, the **trivial** topological transformation group is defined by $\phi(g, x) = x$. Every orbit is a singleton set $\{x\}$.

(2) The two-element discrete group $G = (\{\pm 1\}, \cdot)$ acts continuously on S^n by the scalar multiplication $\pm 1 \cdot x = \pm x$. This action is free and each orbit has two points $\{x, -x\}$.

(3) If $G = (\mathbb{R}^*, \cdot)$ and $X = \mathbb{R}^n \backslash \{0\}$, then G acts continuously by multiplication. This action is free, and the orbits are lines through the origin point.

(4) Any topological group G acts continuously, freely, and transitively on itself by left translation $L_g(g^{'}) = gg^{'}$, see [24], p.80.

2.1.4 Preliminaries on Ergodic Theory

Ergodic theory is the study of group actions on measure spaces. Historically this meant the study of integer actions [22]. The main definitions of this subsection are from [14].

In general, we will assume that X is an infinite set, and $\mathbb{P}(X)$, the *power set of* X is the collection of all subsets of X.

Definition 2.7. *A set $\mathscr{A} \subseteq \mathbb{P}(X)$ is called a **semi-algebra** if the following three axioms are satisfied:*

(1) $\phi \in \mathscr{A}$,

(2) if $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$, and

(3) if $A \in \mathscr{A}$, then the complement of A is a finite union of pairwise disjoint elements in $\mathscr{A}$,

also if

*(4) $A \in \mathscr{A}$ implies that complement of A is an element in $\mathscr{A}$ then it is called an **algebra**. If $\mathscr{A}$ satisfies the following property*

*(5) $A_1, A_2, ... \in \mathscr{A}$ implies that $\bigcup\limits_{i=1}^{\infty} A_n \in \mathscr{A}$, then $\mathscr{A}$ is called a σ**-algebra**.*

Example 2.1. *The set of intervals in $[0, 2]$ is a semi-algebra (see [14]).*

Definition 2.8. *A **measure space** $(X, \mathscr{A}, \mu)$ consists of a non-empty collection σ- algebra $\mathscr{A}$ of subsets of a set X and a non-negative real-valued monotone function μ defined on $\mathscr{A}$.*

Remark 2.3. *(i) Being a σ-algebra means that $\mathscr{A}$ is closed under complements and countable unions (or intersections) of subsets of the set X. Every subset of X is called **measurable set**.*

*(ii) $\mathscr{A}$ is **complete**, means if $A \in \mathscr{A}$ and $\mu(A) = 0$, then $B \in \mathscr{A}$ for every $B \subset A$.*

*(iii) The measure function μ is said to be **finite** if $\mu(X) < \infty$, and σ**-finite** if we can write $X = \bigcup_{i \in \mathbb{N}} X_i$, where $\mu(X_i) < \infty$.*

*(iv) A set of measure zero is called **null set** (for example, the empty set ϕ and the rational numbers $\mathbb{Q}$).*

Definition 2.9. *Let X be a topological space and $\mathscr{U}$ be the smallest σ- algebra which contains all the open sets of X. Then $\mathscr{U}$ is called a **Borel** σ**-algebra** and any measure μ defined on $\mathscr{U}$ is a **Borel measure** if the measure for any compact set is finite.*

Definition 2.10. *A one-point set is called **atom** if it has positive measure. A **Lebesgue measure space** or **Lebesgue space** is a space with finite measure and it is isomorphic to the union of an interval $[0, a]$ (with Lebesgue measure).*

Example 2.2. *The unit square $[0, 1] \times [0, 1]$ with the Lebesgue measure is a Lebesgue space.*

Definition 2.11. *Let G be a group which has a locally compact second count-*

able topology. Let (X, μ) be a measure space with a σ-finite measure μ. A ***measurable action*** *of a locally compact topological group G on a Lebesgue measure space $(X, \mathscr{S}, \mu)$ is a measurable map $\phi : G \times X \to X$ where, $G \times X$ has the product measurable structure and ϕ has the usual properties of a group action:*

(i) $\phi(e, x) = x$ for all $x \in X$,

(ii) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Remark 2.4. *(1) The measure μ for the last definition is* ***invariant*** *if $\mu(gA) = \mu(A)$ for all $A \subseteq X$, $g \in G$,*

(2) Two measures are said to be in the same ***measure class*** *if they have the same null sets.*

Definition 2.12. *A* ***measurable group*** *is a σ- finite measure space $(G, \mathscr{S}, \mu)$ such that*

(i) μ is not identically zero,

(ii) G is a group,

(iii) the σ- algebra $\mathscr{S}$ and the measure μ are invariant under the left translations, and

(iv) the transformation $T : G \times G \to G \times G$ defined by $T(g, h) = (g, (gh))$ is measure preserving.

Definition 2.13. *Let $(X, \mathscr{S}, \mu)$ and (Y, Ψ, ν) be measurable spaces. A function $f : X \to Y$ is called*

(i) ***measurable*** *if the pre-image of a measurable set $A \subset Y$ is measurable*

set in X.

(ii) ***non-singular*** *if the pre image of null set of Y is a null set in X.*

(iii) ***measure-preserving*** *if $\mu(f^{-1}(B)) = \nu(B)$ for every $B \in \Psi$.*

2.1.5 Topological and Smooth Dynamical Systems

In the subsection 2.1.2, we defined a topological transformation group (G, X, ϕ) as an action of any topological group G on a topological space X by a continuous map ϕ. We will especially study dynamical systems as examples defined by an action of the real numbers $\mathbb{R}$ or integers $\mathbb{Z}$ on a space X.

Topological Dynamical Systems

Let G be the additive topological group $\mathbb{R}$ of real numbers with its standard topology or the additive topological group $\mathbb{Z}$ with its discrete topology.

Definition 2.14. *A* ***topological dynamical system*** *on a topological space X is a continuous map $\phi : G \times X \longrightarrow X$ such that, for all $x \in X$ and for all $s, t \in G$,*

(1) $\phi(t + s, x) = \phi(s, \phi(t, x))$,

(2) $\phi(0, x) = x$.

Remark 2.5. *(i) For each $t \in G$, $\phi^t : X \longrightarrow X$, defined by $\phi^t(x) = \phi(t, x)$ is a homeomorphism and Homeo $(X) = \{\phi^t : X \longrightarrow X | \phi^t(x) = \phi(t, x)\}$ is a topological group if one defines the group product of any two homeomorphisms f and g to be the composite map fg.*

(ii) The function $\theta : G \longrightarrow Homeo(X)$ is a topological group homomorphism defined by $\theta(t) = \phi^{^t}$, for each $t \in G$.

*(iii) If $G = \mathbb{R}$ then the dynamical system ϕ is called a **flow** on X, or a **one-parameter group of homeomorphisms** of X. If $G = \mathbb{Z}$, then the dynamical system ϕ is called a **discrete dynamical system** or **discrete flow**.*

Proposition 2.2. *Let (G, X, ϕ) be a topological dynamical system. For all $t \in G$, ϕ^t is a homeomorphism and if ϕ is a $C^{^r}$-differentiable map, then ϕ^t is a $C^{^r}$-diffeomorphism.*

Proof. see [17] p.13.

$\square$

Remark 2.6. *The relationship between group actions and dynamical systems is obvious, where any map $\theta : G \to$ Homeo(X) gives a map $\phi : G \times X \longrightarrow X$ by $\phi(t, x) = \theta^{^t}(x)$.*

On the other hand, if ϕ is a continuous map satisfying the conditions of a group action, then the previous proposition guarantees that the map θ : $G \longrightarrow$Homeo(X) defined by $\theta(t) = \phi^{^t}$ and θ is a group homomorphism.

Smooth Dynamical Systems

The smooth version of dynamical system requires that G is a Lie group, X is smooth (differentiable) manifold and ϕ is a smooth or C^∞-function. Let us recall the definition of a Lie group.

Definition 2.15. *A **Lie group** is a smooth manifold G that is also a group in the algebraic sense with the property that the multiplication $* : G \times G \to G$ and inversion map $inv : G \to G$, given by $*(g,h) = gh$, $inv(g) = g^{-1}$ are both smooth.*

Each of the following examples is a Lie group. For more details, see also [25], chapter 7.

(1) Clearly, $\mathbb{R}^n$ has an n-dimensional smooth manifold structure, besides it satisfies the conditions of a Lie group where $* : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $(x,y) \mapsto x + y$; and the inversion map $inv : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $x \mapsto -x$ are both smooth maps.

(2) The space of all $n \times n$ matrices with real entries, denoted $M(n,\mathbb{R})$ is a Lie group by making the identification $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(3) The space of all $n \times n$ invertible matrices with real entries, called **general linear group** and denoted $GL(n,\mathbb{R})$ is an open subset of the vector space $M(n,\mathbb{R})$, and thus it is a smooth manifold. In addition, the operations of matrix multiplication $(A,B) \mapsto AB$ and inversion $A \mapsto A^{-1}$ defined on $GL(n,\mathbb{R})$ are polynomials in each component. This means that these are smooth maps, and therefore $GL(n,\mathbb{R})$ is a Lie group.

(4) Let V be a real or complex vector space and $GL(V)$ the set of all isomorphisms from V to itself. It is a group under composition. If V has finite dimension n, then any basis from V determines an isomorphism of $GL(V)$ with $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$, therefore $GL(V)$ is a Lie group.

(5) The unit circle $S^1 \subseteq \mathbb{C}^*$ is a Lie group, called the **circle group**, by

identifying it with the set of nonzero complex numbers of norm equal to one. Furthermore, the group operations are exactly the addition $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ and inversion $\theta \mapsto -\theta$ inherited from the complex numbers $\mathbb{C}$.

We will now look at smooth versions of dynamical systems. This version of dyanmical system requires that X is now a differentiable manifold, ϕ is a smooth or $C^\infty-$ function and G is a Lie group (which need not to be abelian) satisfying the multiplication axioms of Definition 2.1., that is, $\phi(st, x) = \phi(s, \phi(t, x))$ and $\phi(e, x) = x$.

Remark 2.7. *One can define a smooth Lie group action as a smooth homomorphism from G into $Diff(X)$, the group of all smooth diffeomorphisms of the smooth manifold X, but the difference with the continuous setting is that Diff(X) has a Lie group structure only if it has infinite dimension.*

2.1.6 Examples

The next examples give smooth versions of dynamical systems:

(i) If $G = X = \mathbb{R}$ and $\phi(t, x) = t + x$, then this flow has only one orbit which is the manifold $\mathbb{R}$.

(ii) Every Lie group G acts smoothly on itself by conjucation: $\phi(g, h) = ghg^{-1}$, where $g, h \in G$.

(iii) The Lie group $G = GL(n, \mathbb{R})$ acts on the left on the manifold $X = \mathbb{R}^n$ via matrix multiplication, presenting each vector $x \in X$ as a column matrix and satisfying the group action conditions. It is a smooth action because the components of Ax depend polynomially on the matrix entries of $A \in G$ and the components of x. This smooth dynamical system has two orbits $\{0\}$ and $\mathbb{R}^n \backslash \{0\}$.

Some of the orbit spaces resulting from Lie group actions on smooth manifolds are manifolds, while other orbit spaces are not. The following examples indicate both situations:

Examples

(1) The orbit space M/G of a trivial Lie group action on a smooth manifold has only one-point sets as orbits. This means $M/G = M$, so it is a smooth manifold.

(2) Suppose G is the unit circle group S^1 which acts on the manifold $M = \mathbb{C}$ by complex multiplication $\phi(z, w) = zw$. The orbit space $\mathbb{C}/S^1$ gives all circles centred at the origin and the singleton set $\{0\}$. Now the quotient map $f : \mathbb{C} \longrightarrow [0, \infty)$ which is defined by $f(z) = |z|$ makes the same identifications as the projection map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/S^1$. According to the theorem of uniqueness of orbit spaces (see [25] p.606) the orbit space $\mathbb{C}/S^1$ is homeomorphic to $[0, \infty)$ and it is not a manifold.

Definition 2.16. *If G is a Lie group and M is a smooth manifold, then a continuous left action by G on M is called a* ***proper action*** *if the map $\pi : G \times M \longrightarrow M \times M$ which is defined by $(g, x) \longmapsto (gx, x)$ is a proper map. That is, for every compact set $K \subseteq M \times M$, the preimage $\pi^{-1}(K)$ is compact set in $G \times M$.*

Now we state the following proposition which is related to the orbit space arising from a Lie group action on a smooth manifold:

Proposition 2.3. *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimensiomn equal to* $\dim M - \dim G$*, and has a unique smooth structure with the property that the quotient map $\theta : M \longrightarrow M/G$ is a smooth submersion.*

Proof. See [25] p.544 □

2.2 Fiber Bundles

We need some notions and examples from algebraic topology and recommend [1] and [25] as some resources for review.

Definition 2.17. *A **bundle** is a triple $\zeta = (E, p, B)$ consisting of topological spaces E,B called the **total** and **base** space respectively and a surjective continuous map p called the **projection map** of the bundle. For each $b \in B, E_b = p^{-1}(b)$ is called the **fibre** with the induced topology by the inclusion in E.*

Definition 2.18. *A **bundle morphism** or a **fibre map** between two bundles $\zeta = (X, p, B)$ and $\eta = (Y, q, A)$, denoted $(f, g) : \zeta \to \eta$ is a pair of continuous maps $f : X \to Y$ and $g : B \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ B \arrow[r, "g"] & A \end{tikzcd}$$

i.e., $q \circ f = g \circ p$.

Definition 2.19. *A bundle morphism $(f, g) : \zeta \to \eta$ is said to be a* ***bundle isomorphism*** *if both the maps $f : X \to Y$ and $g : B \to A$ are homeomorphisms.*

Definition 2.20. *Let $\zeta = (X, p, B)$ and $\eta = (Y, q, B)$ be two bundles over the same base space B. If $f : X \to Y$ is a homeomorphism, then f is called a B-**isomorphism**. In addition, f is called **locally isomorphism** if for each point $b \in B$, there is an open neighbourhood U_b of b and an U_b- isomorphism between the restricted bundles $\zeta \mid U_b$ and $\eta \mid U_b$.*

Definition 2.21. *A **fibre bundle** is a quadruple $\zeta = (E, \pi, B, F)$, consisting of*

(i) a total space E, a base space B, a fiber F, and a projection $p : E \to B$,

*(ii) there is a family $\{(V_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$, the **local trivialization** of the bundle, such that $\{V_\alpha\}$ is an open cover of B, and for all $\alpha \in \Lambda$, $\phi_\alpha : p^{-1}(V_\alpha) \to V_\alpha \times F$ is an homeomorphism that makes the following diagram commutative:*

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"] & V_\alpha \times F \arrow[dl, "p"] \\ V_\alpha & \end{tikzcd}$$

Given a collection of local trivializations $\{(V_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ for which V_α is an open cover of the space B, one has:$(\phi_\alpha \circ \phi_\beta^{-1})(x, y) = (x, \phi_{\alpha\beta}(x, y))$, where

*$\phi_{\alpha\beta} : (V_\alpha \cap V_\beta) \times F \longrightarrow F$ is continuous and $y \longmapsto \phi_{\alpha\beta}(x, y)$ is a homeomorphism of F. The functions $\phi_{\alpha\beta}(x) := (x,)$ are called **transition func-***

tions. *They satisfy the following cocycle equation:* $\phi_{\alpha\gamma}(x) \circ \phi_{\gamma\beta}(x) = \phi_{\alpha\beta}(x)$, $X \in V_\alpha \cap V_\beta \cap V_\gamma$, *and* $\phi_{\alpha\alpha}(x)$ *is the identity of* F *for* $x \in V\alpha$.

Let $A \subset B$. *A* ***section*** *of a fibre bundle* $\zeta = (E, p, B, F)$ *over* A *is a continuous map* $\boldsymbol{s} : A \to E$ *such that* $p \circ \boldsymbol{s} = 1_A$. *A* ***smooth fibre bundle*** *is a fibre bundle with* E, B *and* F *smooth manifolds, and* p *a smooth map. A quadruple* (E, p, B, F) *is a* ***measurable fibre bundle*** *if* F *is a smooth manifold with Borel measurable structure,* E *and* B *are measurable spaces, and there exists a measurable isomorphism* $\phi : E \to B \times F$, *called a* ***measurable trivialization*** *which preserves the fibres.*

Example 2.3. *Let* M *be a smooth manifold.* $\zeta = (TM, \pi, M, \mathbb{R}^n)$ *is a smooth fibre bundle, where* $TM = \bigsqcup_{p \in M} T_pM$ *is the total space,* $\pi : v \longmapsto p$ *if* $v \in T_pM$, *and* T_pM *denotes the tangent space of* M *at a point* $p \in M$.

Definition 2.22. *Let* B *be a topological space. A* ***(real) vector bundle of rank k over B*** *is a topological space* E *together with a surjective continuous map* $\pi : E \longrightarrow B$ *such that each fibre* $E_b = \pi^{-1}(b)$ *is a* k*-dimensional* $\mathbb{R}$-*vector space satisfying the following condition:*

To each point $b \in B$, *there is an open neighbourhood* U *of* b *in* B *and a homeomorphism* $\Theta : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ *(called a* ***local trivialization of*** E ***over*** U*), satisfying the following two conditions:*

(1) $\pi_U \circ \Theta = \pi$ *where* $\pi_U : U \times \mathbb{R}^k \longrightarrow U$ *is the projection, and*

(2) for each $q \in U$, *the restriction of* Θ *to* E_q *is a vector space isomorphism from* E_q *to* $q \times \mathbb{R}^k \cong \mathbb{R}^k$.

Example 2.4. *Let* F *denote the field of real numbers* $\mathbb{R}$ *or complex numbers* $\mathbb{C}$.

(i) For any topological space B, the ***trivial bundle*** $(B \times F^n, p, B, F^n)$ *is an n-dimensional F-vector bundle.*

(ii) For $n \geq 1$, the ***normal bundle*** $N(S^n)$ *over the n-sphere S^n is the fibre bundle $\zeta = (E, p, S^n, \mathbb{R})$, where $E = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|x\| = 1, y = rx, r \in \mathbb{R}\}$ and $p : E \to B, (x, y) \mapsto x$.*

Now we define $\phi : \mathbb{R} \times S^n \to E, (r, x) \mapsto (rx, x)$ and $\psi : E \to \mathbb{R} \times S^n, (x, y) \mapsto (< x, y >, x)$.

Then ϕ is a homeomorphism with inverse ψ and $N(S^n)$ is a 1-dimensional real trivial bundle.

2.3 Principal G-bundles for Lie Group Actions

This section defines principal G-bundles over smooth manifolds when G is a Lie group. Similarly there are topological versions of these.

Definition 2.23. *Let G be a Lie group. A bundle (E, p, M) is said to be a* ***(smooth) G-bundle*** *if the bundles (E, p, M) and $(E, p_E, E/G)$ are isomorphic for some G-space structure on E by an isomorphism $(1_d, f) : (E, p_E, E/G) \to (E, p, M)$ i.e., there exists a diffeomorphism $f : E/G \to M$ making the following diagram commutative:*

$$\begin{array}{ccc} E & \xrightarrow{1_d} & E \\ \downarrow{\scriptstyle p_E} & & \downarrow{\scriptstyle p} \\ E/G & \xrightarrow{f} & M \end{array}$$

Example 2.5. *The tangent bundle $TM \to M$ of an n-dimensional manifold M is an example of a $GL(n, \mathbb{R})$- bundle. In this case the fibre is $\mathbb{R}^n$ with the linear action of $GL(n, \mathbb{R})$ and the transition functions are the Jacobian matrices of coordinate changes.*

Definition 2.24. *A* ***principal (smooth) G-bundle*** *is a triple (E, p, M) such that $p : E \to M$ is a smooth map of smooth manifolds. In addition, E is given a smooth left G-action such that the following conditions hold:*

(i) $E_x = p^{-1}(x), x \in M$ are the orbits for the G-action.

(ii) For each $x \in M$, there exists an open neighbourhood U and a diffeomorphism $\psi : p^{-1}(U) \to G \times U$ such that the diagram below commutes,

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\psi"] \arrow[d, "p"] & G \times U \arrow[dl, "q"] \\ U & \end{tikzcd}$$

i.e., $\psi_x = \psi \mid E_x$ maps to $G \times \{x\}$; and ψ is equivariant i.e., $\psi(gx) = g\psi(x)$, $\forall x \in p^{-1}(U), g \in G$, where G acts on $G \times U$ by $g(h, x) = (gh, x)$.

Example 2.6. *(i) $p : \mathbb{R} \to S^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$ is a principal $\mathbb{Z}$-bundle: $r \longmapsto e^{2\pi i r}$. The group $\mathbb{Z}$ acts on $\mathbb{R}$ by translations: $rn = r + n$ for $n \in \mathbb{Z}$ and $r \in \mathbb{R}$. A set of local trivializations is $U_1 = \{e^{2i\theta} \mid \theta \in (0, 2\pi)\}$ and $U_2 = \{e^{2i\theta} \mid \theta \in (-\pi, \pi)\}$ with trivialization maps*

$$\phi_1 : p^{-1}(U_1) \to \mathbb{Z} \times U_1, \text{ where } x \in (0, 1),$$

$$r = x + n \mapsto (n, e^{2\pi i x})$$

$$\phi_2 : p^{-1}(U_2) \to \mathbb{Z} \times U_2, \ \textit{where}\ x \in \Big(-\frac{1}{2}, \frac{1}{2}\Big),$$

$$r = x + n \mapsto (e^{2\pi i x}, n)$$

The trivializations and the projection p are $\mathbb{Z}$-equivariant.

(ii) Let G be any Lie group and M be a smooth manifold. Then $(G \times M, p, M)$ with p the projection onto the first factor, is a principal G-bundle called the ***product bundle****,see [1],p.240 .*

2.4 Cocycles and Dynamical Systems

One of the most important algebraic tools to study dynamics directly related to the group action is the concept of a cocycle. Many questions in dynamics can be solved by determining whether two cocycles are equivalent or not. These questions are:

• Given an action of a group G on a space X, is there a measure on the space X so that it is invariant under this action?

• Are two given group actions on a space X isomorphic (conjugate)?

Cocycles allow to study group actions in an extended action of a fibred space (the abelian group) and the information resulting from the extended action will help us to analyse the original action. In addition, from these cocycles one can construct a new dynamical system that inherits the properties from the given one [21]. So group actions play a central role here in order to describe dynamical systems.

Definition 2.25. *An* ***isomorphism*** *or* ***conjugacy*** *between two actions of* *a group G, say $\phi : G \times X \to X$ and $\psi : G \times Y \to Y$, is a bijection $f : X \to Y$ that preserves the structures (homeomorphism, diffeomorphism, measure-preserving, etc) such that $f(\phi(g, x)) = \psi(g, f(x))$ for all $g \in G, x \in X$.*

Example 2.7. *$f : [0, 1] \to [-2, 2]$ is topological conjugate for $Q(x) = x^2 - 2, x \in [-2, 2]$ and $P(x) = \{2x, x \in 0 \leq x \leq 1/2\}$ defined by $f(\theta) = 2 \cos \pi\theta$.*

2.4.1 Standard Definitions of Cocycles

One of the main goals in this thesis is to answer the relationship between groupoid cohomology of action groupoids and cocycles for dynamical systems given by group actions. The basic definitions of cocycles in the topological, smooth and measurable settings are similar (see [21]). We define it here in the contexts of continuous and measurable cocycles for topological and measurable group actions and discuss examples from different dynamical systems.

The following definitions can be found in [3] and [20].

Definition 2.26. *(Topological cocycles)*
Let X be a topological space. Let $\phi : G \times X \to X$ be a continuous (or discrete) action of a topological (or discrete) group G on X. If U is a topological group, then a ***cocycle*** *for the action ϕ with values in the topological group U is a continuous map $\rho : G \times X \to U$ satisfying $\rho(gh, x) = \rho(g, hx)\rho(h, x)$, where $g, h \in G, x \in X$.*

Remark 2.8. *(i) The condition in Definition 2.26. is called* ***cocycle identity****. This equation implies that the cocycle is independent of the variable x,* *so it is suitable to think of the cocycle identity as a topological homomorphism $\alpha : G \to U$.*

(ii) If U is an abstract (discrete) group we can give it the discrete topology and recover the classical cocycle definition for abstract group actions on sets.

(iii) For a cocycle $\rho : G \times X \to U$ to be a ***smooth cocycle****, we require that the cocycle ρ be a smooth map, X be a smooth manifold and that G and U are Lie groups.*

(iv) A cocycle whose value is the identity element in the group U is called ***trivial cocycle****, while a homomorphism $\theta : G \to U$, which satisfies the cocycle condition such that $\theta(g, x) = \theta(x)$ is called a* ***constant cocycle*** *(not depending on x).*

Let us state for completeness the definition of cocycles for the measurable and smooth context accordingly (see also [20], [21]).

Definition 2.27. *(Measurable cocycles)*
Given a left group action G on a measurable space (X, μ), a cocycle with values in the measurable group U is a measurable map $\rho : G \times X \longrightarrow U$ satisfying the following cocycle conditions:

$$\rho(gh, x) = \rho(g, hx)\rho(h, x)$$

for all $g, h \in G$ and $x \in X$.

Definition 2.28. *(Smooth cocycles)*
Let M be a smooth manifold. Let $\phi : G \times M \to M$ be a smooth action of a Lie group G on M. If U is a Lie group, then a ***cocycle*** *for the action ϕ*

with values in the Lie group G is a smooth map $\rho : G \times X \to U$ satisfying $\rho(gh, x) = \rho(g, hx)\rho(h, x)$, where $g, h \in G, x \in X$.

Here are some illuminating examples of cocycles from different settings:

Example 2.8. *(1) (Measure theory)*

*Let ϕ be an action of a group G on a measurable space (X, μ) which preserves measure classes represented by a not necessary invariant measure μ. Then we can obtain a cocycle called **Radon-Nikodym derivative** which is defined by : $J : G \times X \to \mathbb{R}^*$; $J(g, x) = d(g_*^{-1}\mu)(x)/d\mu$ where $(g * \mu)(A) = \mu(g^{-1}A)$ for any measurable set $A \subset X$. Let f be a measurable bounded function on X.*

$$
\begin{aligned}
\int f(x)J(g_1, g_2x)J(g_2, x)d\mu(x) &= \int f(x)J(g_1, g_2x)d(g_2^{-1})*\mu)(x) \\
&= \int f(g_2^{-1}x)J(g_1, x)d\mu(x) \\
&= \int f(g_2^{-1}x)d(g_*^{-1}\mu)(x) \\
&= \int f(g_2^{-1}(g_1^{-1}))d\mu(x) \\
&= \int f((g_1g_2)^{-1}(x))d\mu(x) \\
&= \int f(x)J(g_1g_2, x)d\mu(x).
\end{aligned}
$$

Since f is any measurable function, we obtain almost everywhere that

$$J(g_1g_2, x) = J(g_1, (g_2x))J(g_2, x).$$

(2) (Differential topology)

Let $f : M \to M$ be a diffeomorphism on a smooth n-dimensional manifold M and $\pi : TM \to M$ be the tangent bundle. Then the derivative Df is the *smooth map $Df : M \to GL(n, \mathbb{R})$ from M to the Lie group $GL(n, \mathbb{R})$, which by the Chain Rule, $D_x f^{n+m} = Df^n_{f^m(x)} \cdot D_x f^m$ satisfies the cocycle identity.*

As we are especially interested in topological dynamics here, we will from now on mostly work in the topological and smooth setting, though it is clear how to amend the notions and constructions from the topological or smooth setting to the measurable context.

As we want to study cohomological properties we are mostly interested in cocycles with values in abelian groups, i.e. $U = (A, +)$, which will be written as usual in additive notation, which we state for completeness again:

Definition 2.29. *(Topological cocycles)*
Let X be a topological space. Let $\phi : G \times X \to X$ be a continuous (or discrete) action of a topological (or discrete) group G on X. If $A = (A, +)$ is an abelian topological (or discrete) group, then a ***cocycle*** *for the action ϕ with values in the abelian group A is a continuous map $\rho : G \times X \to A$ satisfying $\rho(gh, x) = \rho(g, hx) + \rho(h, x)$, where $g, h \in G, x \in X$. Similarly, we have measurable and smooth cocycles with values in an abelian measurable, respectively abelian Lie groups A.*

Let us now introduce another type of cocycles which plays an important role in this thesis. For more information and the following definitions, see also [20].

Definition 2.30. *$A = (A, +)$ is an abelian topological (or smooth) group. A cocycle $\rho : G \times X \to A$ is called a* ***coboundary*** *if there is a continuous (smooth) map $F : X \to A$, called a* ***transfer function*** *or* ***transfer map****,* *such that for each $g \in G$ and $x \in X$, we have:*

$$\rho(g, x) = F(gx) - F(x).$$

Remark 2.9. *(i) The coboundary equation in the previous definition implies the cocycle identity, showing that every coboundary is in fact a cocycle, because we have:*

$$\begin{aligned}\rho(gh, x) - \rho(g, hx) &= F((gh)x) - F(x) - [F(g(hx)) - F(hx)] \\ &= F(hx) - F(x) \\ &= \rho(h, x)\end{aligned}$$

and therefore: $\rho(gh, x) = \rho(h, x) + \rho(g, hx)$.
(ii) While every coboundary is a cocycle as we saw above, the converse is not always true. For example, if $X = \{x\}$ is a point, then $\rho(g, x) = F(gx) - F(x) = 0$. This means the only coboundary in this situation is given by the trivial function.

(iii) In general, for a given dynamical system there can be more cocycles than coboundaries. Only in very special situations cocycles will also be coboundaries, for example if X is a finite set and G a finite group with a free action on X (see [38]). In fact, the failure of a cocycle being a coboundary for the dynamical system is measured by the first cohomology group of the system as we will introduce and review in details below.

The classification problem of cocycles for a given dynamical system provides us with an important insight into the structure of the dynamical system and its dynamics as describes by the group action and its orbits. Among the different algebraic possibilities of introducing an equivalence relation among cocycles in order to classify them, we shall here mainly deal with the **cohomologous relation**.

Definition 2.31. *Two cocycles ρ_1 and ρ_2 with values in A are called* ***cohomologous*** *if there exists a continuous (smooth map) $F : X \to A$ satisfying the* ***cohomology equation*** *:*

$$\rho_2(g, x) = F(gx) + \rho_1(g, x) - F(x)$$

for all $g \in G$ and $x \in X$.

Proposition 2.4. *The cohomologous relation is an equivalence relation.*

Proof. To see that the relation of cohomologous between cocycles is an equivalence relation, we have to show that it satisfies reflexivity, symmetry and transitivity. This relation is reflexive because each cocycle is cohomologous to itself according to the cohomology equation $\rho(g, x) = F(gx) + \rho(g, x) - F(x)$. Now if ρ_1 cohomologous to ρ_2 then $\rho_1(g, x) - \rho_2(g, x)$ is a couboundary i.e.

$$\rho_1(g, x) - \rho_2(g, x) = F(gx) - F(x)$$

for some function $F : X \longrightarrow A$, then

$$\rho_2(g, x) - \rho_1(g, x) = -[\rho_1(g, x) - \rho_2(g, x)] = -[F(gx) - F(x)],$$

which is also a coboundary. This means ρ_2 is cohomologous to ρ_1, so symmetry follows. The cohomologous relation is transitive as

$$\rho_2(g, x) = H(gx) + \rho_3(g, x) - H(x)$$

for some functions $F : X \longrightarrow A$ and $H : X \longrightarrow A$. Then

$$\rho_1(g, x) = F(g, x) + H(gx) + \rho_3(g, x) - H(x) - F(x) = K(gx) + \rho_3(g, x) - K(x)$$

where $K : X \longrightarrow A$ and $K = F + H$, so ρ_1 is cohomologous to ρ_3 and we have proven transitivity. $\square$

We can now directly define the abelian groups of cocycles and coboundaries associated to a dynamical system (G, X, ϕ) and an abelian topological (or discrete) group $A = (A, +)$.

Definition 2.32. *Let (G, X, ϕ) be a dynamical system and $A = (A, +)$ an abelian topological (or discrete) group. The* **group of cocycles** $Z^1(G, X, A)$ *and the* **group of coboundaries** $B^1(G, X, A)$ *of (G, X, ϕ) are given as:*

$$Z^1(G, X, A) = \{\rho : G \times X \to A : \rho \text{ is a cocycle}\}$$

$$B^1(G, X, A) = \{\rho : G \times X \to A : \rho \text{ is a coboundary}\}.$$

The **first cohomology group** *of (G, X, ϕ) is given as the quotient of cocycles modulo coboundaries:*

$$H^1(G, X; A) = Z^1(G, X, A)/B^1(G, X, A).$$

Remark 2.10. *(i) It is clear from the definitions that $Z^1(G, X, A)$ is an abelian group, with the group structure given by defining the addition $\rho + \rho'$ of two cocycles ρ, ρ' using the abelian group A:*

$$(\rho + \rho')(g, x) := \rho(g, x) + \rho'(g, x)$$

and the neutral cocycle ε is given by setting:

$$\varepsilon(g, x) := 0,$$

where 0 is the neutral element of A. The group axioms can easily be checked and it follows from the above that $B^1(G, X, A)$ is a subgroup of $Z^1(G, X, A)$ and the cohomology group $H^1(G, X; A)$ is the quotient group.

(ii) If the action of G on X is free, it follows from the considerations above immediately that we have

$$H^1(G, X; A) \cong 0$$

as every cocycle is a coboundary.

(iii) This definition can also be amended for the non-commutative setting, i.e. where we use a general group U as above and use the machinery of nonabelian cohomology to define $H^1(G, X; U)$ as a quotient space of cocycles modulo coboundaries, but we will not pursue this here. See for example [20] [21] for the use of nonabelian cohomology in dynamics. There is scope for the development of a full-blown theory of nonabelian cohomology for dynamical systems (G, X, ϕ) in particular for applications in Number Theory [21], [14].

(iv) We will later see how all this fits into the general framework of cohomology as used in Algebraic Topology, which allows for a systematic cohomological interpretation and use of cocycles as cohomology classes of spaces and action groupoids.

2.4.2 Construction of Dynamical Systems via Cocycles and Cocycle Extensions

In this section we use the notion of a cocycle ρ defined for the group action $\phi : G \times X \to X$ to construct an important particular kind of (abelian) extension or skew product dynamical system denoted by $X \times_\rho A$ of a given dynamical system (G, X, ϕ), which generalises the Cartesian product [16].

Definition 2.33. *Suppose we have a dynamical system (G, X, ϕ), $A = (A, +)$ is an abelian topological group and a cocycle $\rho : G \times X \to A$. The* ***extension of*** (G, X, ϕ) ***by*** A ***associated to the cocycle*** ρ *is the dynamical system $(G, X \times_\rho A, \psi)$ where $X \times_\rho A$ is the Cartesian product $X \times A$ with the induced* *group action $\psi : G \times X \times A \to X \times A$ defined by:*

$$\psi[g,(x,a)] = [\phi(g,x), a + \rho(g,x)].$$

Let us write down the dynamical system conditions for this extension:

$$\begin{aligned}(1)\psi[g,\psi[h,(x,a)]] &= \psi[g,\phi(h,x), a+\rho(h,x)] \\ &= (\phi(g,\phi(h,x), a+\rho(h,x)+\rho(g,\phi(h,x)))) \\ &= (\phi(gh,x), a+\rho(gh,x)) \\ &= [gh,(x,a)].\end{aligned}$$

(2) $\rho(0,x) = \rho(0+0,x) = \rho(0,x)+\rho(0,\phi(0,x)) = \rho(0,x)+\rho(0,x)$, and therefore $\rho(0,x)=0$. Now

$$\begin{aligned}\psi[0,(x,a)] &= [\phi(0,x), a+\rho(0,x)] \\ &= (x,a).\end{aligned}.$$

This shows that $(G, X \times_\rho A, \psi)$ is a dynamical system.

Definition 2.34. *Let (G,X,ϕ) be a dynamical system and A be an abelian topological group. We define a new dynamical system $(G, X\times A, \psi)$. For every $a \in A$ we define an* ***automorphism*** *R_a of $(G, X \times A, \psi)$ by $R_a(x,b) = (x, a+b)$ for every $(x,a) \in X \times A$ and $R_aR_b = R_{a+b}$ for every $a,b \in A$.*

Proposition 2.5. *Let $\rho_1 : G \times X \longrightarrow A$, $\rho_2 : G \times X \longrightarrow A$ be two cohomologous cocycles. Then $(G, X \times_{\rho_1} A, \psi_1)$ and $(G, X \times_{\rho_2} A, \psi_2)$ are conjugate.*

Proof. Let $F : X \longrightarrow A$ be a continuous map with $\rho_1(g,x) - \rho_2(g,x) = F(gx) - F(x)$ for every $g \in G$ and $x \in X$.

Now we define an automorphism θ_F of $X \times A$ by $\theta_F(x, a) = (x, a + F(x))$. We have:

$$\begin{aligned}
\rho_1\theta_F(x, a) &= \rho_1((x, a + F(x))) \\
&= (gx, a + F(x) + \rho_1(g, x)). \\
\theta_F\rho_2(x, a) &= \theta_F(gx, a + \rho_2(g, x)) \\
&= (gx, a + \rho_2(g, x) + F(gx)) \\
&= (gx, a + \rho_1(g, x) - F(gx) + F(x) + F(gx)) \\
&= (gx, a + \rho_1(g, x) + F(x)).
\end{aligned}$$

□

Definition 2.35. *Depending on the cohomologous relation in Definition 2.31.and Proposition 2.5., we can say that a cocycle is a coboundary if it is cohomologous to the trivial cocycle. Then the extension $X \times_\rho A$ by a coboundary $\rho(g, x) = F(gx) - F(x)$ can be conjugated to the extension by the trivial function $X \times_0 A$ by using the form $(x, a) \mapsto (x, a - F(x))$. Also a cocycle is an* ***almost coboundary*** *if it is cohomologous to a constant cocycle.*

2.5 Some Natural Ways for Cocycles to Arise

In this section we will discus two important examples of cocycles arising in the theory of dynamical systems and topology. They are crucial to describe the geometrical context in local terms. They are also examples for cocycles taking values in non-abelian groups.

2.5.1 The Derivative Cocycle

Let $\phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth n-manifold M. Then ϕ gives an action $\hat{\phi}$ of G on the tangent bundle

$$\hat{\phi}^g(v) = d(\phi^g)_x v$$

for all $x \in M$ and $v \in T_x M$.

We can obtain a trivialization of TM as follows: Suppose that we choose an identification of $T_x M$ with $\mathbb{R}^n$ for each $x \in M$. This is given by a choice of an isomorphism defined by $\eta(x) : \mathbb{R}^n \to T_x M$, for each $x \in M$. Now the derivative map $d(\phi^g)_x : T_x M \to T_{\phi^g(x)} M$ can be represented by an automorphism of $\mathbb{R}^n$. Let us denote the automorphism by $c(g,x) \in GL(n,\mathbb{R})$, so we have

$$c(g,x) := \eta(\phi^g(x))^{-1} \circ d(\phi^g)_x \circ \eta(x).$$

This means the map $c : G \times M \to GL(n,\mathbb{R})$ as just defined is a cocycle for the action ϕ taking values in the linear general group of dimension n.

2.5.2 Orbit Equivalence and Time Change

Cocycles can also appear in connection with the orbit equivalence of dynamical systems [21]. Let (G, X, ϕ) and (H, Y, ψ) be two dynamical systems (with the same structure). **Orbit equivalence** is a bijection map $\alpha : X \to Y$ that takes orbits of X under G to orbits of Y under H and preserves the structures. This means there exists a map $\beta : G \times X \to H$ such that $\alpha(\phi(g,x)) = \psi[\beta(g,\alpha(x)),\alpha(x)]$. The last condition shows that the map β is a cocycle because:

$$
\begin{aligned}
\psi[\beta(g_2g_1,\alpha(x)),\alpha(x)] &= \alpha(\phi(g_2g_1,x)) \\
&= \alpha(\phi(g_2,\phi(g_1,x))) \\
&= \alpha(\phi(g_2,\alpha^{-1}(\psi\beta[g_2,\alpha(x),\alpha(x)])) \\
&= \psi[\beta(g_2,\alpha(\phi(g_1,x),\alpha(x))),\psi[\beta(g_1,\alpha(x),\alpha(x))] \\
&= \psi[\beta(g_2,\phi(g_1,x))),\psi[\beta(g_1,\alpha(x)),\alpha(x)]]
\end{aligned}
$$

which gives

$$\beta(g_2g_1,x) = \beta(g_2,\phi(g_1,x))\beta(g_1,x).$$

In the simplest cases, we have a flow on the smooth manifold X. Then, an orbit equivalence that preserves orbits is called a **time change**, and yields a cocycle $\beta : \mathbb{R} \times X \to \mathbb{R}$. In this case all smooth cocycles are cohomologically constant and will tell us that all time changes are of a particular type: they arise by taking some smooth homomorphism $\mathbb{R} \to \mathbb{R}$ and a smooth transfer function map $X \to \mathbb{R}$, and defining the cocycle according to the cohomology equation in Definition 2.31. Thus the cocycle condition and the cohomology equation gives a way of studying orbit equivalences of dynamical systems.

2.6 Cohomology of Dynamical Systems

This section deals with one of the main concepts of this thesis which is the cohomology of dynamical systems. We will fit the concepts of cocycles into a general setting of cohomology. As it is said in the introduction of this chapter the definitions of this section are analoguous to the similar concepts of singular homology and cohomology in algebraic topology. See [3] and [33], chapter 4 and 12 to compare.

Now let us fix, G to be a topological group, X a topological space and (G, X, ϕ) a topological dynamical system.

Definition 2.36. *(i) For $n \in \mathbb{Z}$ and $n \geq 0$, a **singular n-simplex** is an element $(g_1, g_2, ..., g_n, x)$ in $G^n \times X$, where $g_1, g_2, ..., g_n \in G$ and $x \in X$.*

*(ii) We denote by $C_n(G, X)$ the free abelian group with basis all singular n-simplexes in $G^n \times X$, any element of $C_n(G, X)$ is called a **singular n-chain**.*

(iii) For $n < 0$, we define $C_n(G, X) = 0$ to be the trivial group.

Remark 2.11. *(i) A singular n-chain is an oriented simplex connecting the $n+1$ points $x, g_n x, g_{n-1}g_n x, g_{n-2}g_{n-1}g_n x, \ldots, g_1 g_2 \cdots g_n x$. In particular, an element of $C_0(G, X)$ is a finite formal linear combination of elements of X with $\mathbb{Z}$-coefficients. This means a 0-chain is of the form $\sum_{i=1}^m c_i x_i$ of points and any 1-chain is a formal integer linear combination $\sum_{i=1}^m c_i(g_i, x_i)$ containing line segments from x_i to $g_i x_i$ with integer coefficients and so on.*
(ii) If G is not a discrete group, i.e. a general topological group the definition of the singular chain complex needs modification. We will understand by $C_n(G, X)$ the space of continuous singular cochains i.e. such singular cochains whose restriction to the space of simplices (endowed with the compact-open topology) define continuous functions as defined in general for example in [41].
Definition 2.37. *Let (G, X, ϕ) be a dynamical system, then (i) For $n \in \mathbb{Z}$ and $n > 0$, we define the **boundary map***

$$\partial_n : C_n(G, X) \to C_{n-1}(G, X)$$

to be the unique group homomorphism given by

$$
\begin{aligned}
\partial_n(g_1, ..., g_n, x) = &(g_1, ..., g_{n-1}, g_n x)+ \\
&+ \sum_{k=1}^{n-1} (-1)^{n-k} (g_1, ..., g_{k-1}, g_k g_{k+1}, g_{k+2}, ..., g_n, x)+ \\
&+ (-1)^n (g_2, ..., g_n, x).
\end{aligned}
$$

(ii) For $n \leq 0$, we define $\partial_n : C_n(G,X) \to C_{n-1}(G,X)$ to be the zero map.

Example 2.9. *For $n = 0$, $\partial_0 : C_0(G,X) \to C_{-1}(G,X)$, $\partial_0(x) = 0$,*

$n = 1$, $\partial_1 : C_1(G,X) \to C_0(G,X)$, $\partial_1(g,x) = gx - x$,

$n = 2$, $\partial_2 : C_2(G,X) \to C_1(G,X)$, $\partial_2(g,h,x) = (g,hx) - (gh,x) + (h,x)$,

$n = 3$, $\partial_3 : C_3(G,X) \to C_2(G,X)$, $\partial_3(g,h,k,x) = (g,h,kx) - (g,hk,x) + (gh,k,x) - (h,k,x)$.

Definition 2.38. *The **singular chain complex** of a dynamical system (G,X,ϕ) is the sequence of free abelian groups $C_n(G,X)$ and homomorphisms ∂_n*

$$
\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(G,X) \xrightarrow{\partial_{n+1}} C_n(G,X) \xrightarrow{\partial_n} C_{n-1}(G,X) \xrightarrow{\partial_{n-1}} \cdots
$$

such that $\partial_n \partial_{n+1} = 0$ for each $n \in \mathbb{Z}$.

Remark 2.12. *(i) We will use the notation $(C_\bullet(G,X), \partial)$ or more simply $C_\bullet$ to refer to the singular chain complex of a dynamical system (G,X,ϕ).*

(ii) The condition $\partial_n \partial_{n+1} = 0$ is equivalent to im $\partial_{n+1} \subset$ ker ∂_n.

Definition 2.39. *Let (G,X,ϕ) be a dynamical system and $C_\bullet$ be its chain complex. Then ker ∂_n is called the group of (singular) **n-cycles** and is denoted by $Z_n(C_n(G,X), \partial)$ and im ∂_{n+1} is called the group of (singular)*

n-boundaries *and is denoted by* $B_n(C_n(G,X),\partial)$. *The* ***n-the homology group*** *of this chain complex is* $H_n(C_\bullet,\partial) = Z_n(C_\bullet,\partial)/B_n(C_\bullet,\partial)$. *We will also write* $H_n(G,X,\phi)$ *or* $H_n(G,X;A)$ *depending on the emphasis and group of coefficients. If* $A = \mathbb{Z}$ *we normally omit it from the notation.*

Remark 2.13. *If* $\partial^2 = 0$, *then every a n-boundary is an n-cycle. For example the 1-chain* $(gh,x) - (h,x) - (g,hx)$ *is both 1-boundary and 1-cycle. However if* g *is not the trivial element in* G *which fixes* x *and* G *is an abelian group, then the 1-chain* (g,x) *is an 1-cycle* $(\partial(g,x) = gx - x = 0)$ *but not a 1-boundary. Also* $Z_n(G,X)$ *and* $B_n(G,X)$ *are abelian groups and* $B_n(G,X)$ *is contained in* $Z_n(G,X)$.

Now we calculate the homology groups for some simple dynamical systems:

Proposition 2.6. *Let* G *be a topological group and* X *be a topological space.*

(i) If $X = \{x_0\}$ *a point, then* $H_0(G,X) \cong \mathbb{Z}$.

(ii) If G *acts transitively on* X, *then* $H_0(G,X) \cong \mathbb{Z}$. *Similarly, in the topological context of dynamical systems transitivity corresponds to minimal and in the measurable or ergodic context, transitivity corresponds to ergodicity (see [20]).*

(iii) If G *acts freely on X, then* $H_0(G,X) \cong \mathbb{Z}$ *while* $H_n(G,X) \cong 0$ *is trivial for* $n > 0$.

(iv) If X *is a point, then* $H_1(G,X) = G/[G,G]$ *is the abelianisation of* G.

Proof. (i) By definition of the zero homology group $H_0(G,X) = ker\partial_0 = C_0(G,X)/im\partial_1 = \partial_1(C_1(G,X)) = \mathbb{Z}X\diagup\mathbb{Z} < x - x_0 | \forall x \in X > \cong \mathbb{Z} < x_0 >$.

(ii) Since G acts transitively on the topological space X, for each $x, y \in X$, there exists a $g \in G$ such that $y = gx$. This means X has only one orbit (one path component).

(iii) Since G acts freely on X, then $gx = x$ implies that $g = 0$. We know that all points of $C_0(G, X) = X$ are cycles and boundaries $im\ C_1(G, X)$ are linear combinations of (g, x) such that $gx - x = 0$. This means $gx = x$ and $H_0(G, X) = \mathbb{Z}$.

(iv) This follows immediately from the definitions and basic properties as we have seen already before. $\square$

We have seen that the concept of singular homology groups can be derived from a singular chain complex of chain groups for a dynamical system (G, X, ϕ) with some maps $\partial_n \colon C_n(G, X) \longrightarrow C_{n-1}(G, X)$ where $n \in \mathbb{Z}$.

Now if A is an abelian (topological) group, then we can define the dual cochain group of a dynamical system denoted by $C^n(G, X, A)$ depending on a contravariant functor $Hom(-, A)$.

Definition 2.40. *Let (G, X, ϕ) be a dynamical system and A be an abelian (topological) group. We define the abelian group $C^n(G, X, A)$ to be all (continuous) group homomorphisms between $C_n(G, X)$ and the group A, i.e.,*

$$C^n(G, X, A) = Hom(C_n(G, X), A).$$

Remark 2.14. *(i) Elements of $C^n(G, X, A)$ are called **singular n-cochains** and a singular n-cochain θ is a group homomorphism i.e.*

$$\theta : C_n(G, X) \longrightarrow A.$$

If A is a not necesarily non-discrete topological group we understand by

$C^n(G,X,A)$ the space of continuous cochains $G^n \times X \to A$, i.e. those singular cochains whose restriction to the space of simplices define continuous functions to the topological group A and we will make the necessary modifications whenever needed without explicitily refering to them (compare [41]).

(ii) Since $C_n(G,X)$ is a free abelian group and the collection of n-simplices is a basis for $C_n(G,X)$, then an n-cochain is determined by the values it assigns to n-simplices. We therefore identify singular n-cochains with functions of the form $G^n \times X \longrightarrow A$.

Definition 2.41. *Let (G,X,ϕ) be a dynamical system and A an abelian topological group. We define the* ***coboundary map***

$$\delta : C^{n-1}(G,X,A) \longrightarrow C^n(G,X,A)$$

by setting $(\delta F)(c) = F(\partial c)$ where $F \in C^{n-1}(G,X,A)$ and $c \in C_n(G,X)$.

Explicitly, for $F \in C^{n-1}(G,X,A)$ and an n-simplex $(g_1,...,g_n,x) \in G^n \times X$, we have:

$$\begin{aligned}(\delta F)(g_1,...,g_n,x) &= F(\partial(g_1,...,g_n,x)) \\ &= F(g_1,...,g_{n-1},g_n x)+ \\ &+ \textstyle\sum_{k=1}^{n-1}(-1)^{n-k}F(g_1,...,g_{k-1},g_k g_{k+1},g_{k+2},...,g_n,x)+ \\ &+ (-1)^n F(g_2,...,g_n,x).\end{aligned}$$

Example 2.10. *Let (G,X,ϕ) be a dynamical system and A be an abelian topological group. If $F \in C^0(G,X,A)$ i.e. F is a singular 0-cochain, $F : X \longrightarrow A$, then $\delta F(g,x) = F(\partial(g,x)) = F(gx - x) = F(gx) - F(x)$.*

Now if $\rho \in C^1(G,X,A)$ i.e. ρ is an singular 1-cochain, $\rho : G \times X \longrightarrow A$, then $\delta\rho(g,h,x) = \rho(g,hx) - \rho(gh,x) + \rho(h,x)$.

Remark 2.15. *To define the singular cochain complex we need to satisfy the condition $\delta^2 = 0$ between $C^n(G, X, A)$ for each $n \in \mathbb{Z}$. So let $c \in C_n(G, X)$ and $F \in C^{n-2}(G, X, A)$ such that $L = \delta F$, then*

$$(\delta^2 F)(c) = \delta(\delta F)(c) = (\delta L)(c) = L(\partial c) = (\delta F)(\partial c) = F(\partial^2 c) = F(0) = 0.$$

This means that $\delta^2 = 0$ and depending on this result we can define the following concept.

Definition 2.42. *If A is an abelian topological group, then the* ***singular cochain complex*** *of a dynamical system (G, X, ϕ) which is denoted by $(C^\bullet(G, X), \delta)$ or simply $C^\bullet$ is a sequence of $C^n(G, X, A)$ with the coboundary maps $\delta : C^n(G, X, A) \longrightarrow C^{n+1}(G, X, A)$ for each $n \in \mathbb{Z}$.*

Remark 2.16. *(i) We denote the kernel of $\delta : C^n(G, X, A) \longrightarrow C^{n+1}(G, X, A)$ by $Z^n(G, X, A)$, and elements of $Z^n(G, X, A)$ are called* ***n-cocycles****.*

(ii) We denote the image of $\delta : C^{n-1}(G, X, A) \longrightarrow C^n(G, X, A)$ by $B^n(G, X, A)$, and elements of $B^n(G, X, A)$ are called ***n-coboundaries****.*

(iii) Since $\delta^2 = 0$, an n-coboundary is a cocycle. Also $Z^n(G, X, A)$ and $B^n(G, X, A)$ are abelian groups with $B^n(G, X, A) \subseteq Z^n(G, X, A)$, and thus we write $H^n(C^\bullet(G, X, \phi), \delta) = Z^n(C^\bullet(G, X, \phi), \delta)/B^n(C^\bullet(G, X, \phi), \delta)$, which we call the ***n-th cohomology group****. We will also write $H^n(G, X, \phi)$ or $H^n(G, X; A)$ depending on the emphasis and group of coefficients. If $A = \mathbb{Z}$ we normally omit it from the notation.*

(iv) We identify the group $C^1(G, X, A)$ with functions from $G \times X \longrightarrow A$. This means if ρ is a 1-cochain, then to be a 1-cocycle is equivalent to that for any $(g, h, x) \in G^2 \times X$, we have $(\delta\rho)(g, h, x) = 0$, therefore $\rho(g, hx) - \rho(gh, x) + \rho(h, x) = 0$ and this implies that $\rho(gh, x) = \rho(h, x) + \rho(g, hx)$. Also

to say ρ is a 1-coboundary means that there is a continuous function $X \to A$, a 0-cochain F such that $\rho = \delta F$ i.e., for any $(g, x) \in G \times X$, then

$$\rho(g, x) = (\delta F)(g, x) = F(\partial(g, x)) = F(gx) - F(x).$$

Thus $\rho(g, x) = F(gx) - F(x)$. This means in particular also, that we now recover the groups of cocycles and coboundaries as defined directly before.

2.7 Some Basic Cohomology Calculations for Dynamical Systems

We will look at some general examples describing particular situations in which group actions govern dynamical systems (G, X, ϕ). These have many particular incarnations in the literature (see e.g. [14], [20]), but are studied here in purely algebraic-topological terms.

Example 2.11. *(1) If G is a topological group acting transitively on a topological space X and A is an abelian group, then for the zeroth cohomology group we have: $H^0(G, X; A) \cong A$.*

Now we will consider $H^0(G, X; A)$ to be the kernel of δ^0 of the cochain complex for the dynamical system (G, X, ϕ). This is the set of continuous maps F from X into A. Now $\delta^0 F$ is a continuous map from $G \times X$ into A given by $\delta^0 F((g, x)) = F(\partial_1(g, x)) = F(gx - x) = F(gx) - F(x) = 0$, for all $g \in G$ and $x \in X$. This means $F(gx) = F(x)$ for all $g \in G$ and $x \in X$. So we see that the zero cohomology group $H^0(G, X; A)$ corresponds to the set of all elements $a = F(x)$ of A fixed by G i.e., $H^0(G, X; A) \cong A$.

(2) Let (G, X, ϕ) a dynamical system and A a (topological) abelian group, then

$Z^1(G, X, \phi) = \{\rho : G \times X \longrightarrow A \mid \rho(gh, x) = \rho(h, x) + \rho(g, hx)\}$.

$B^1(G, X, \phi) = \{\rho : G \times X \longrightarrow A \mid \rho(g, x) = F(gx) - F(x)$ *for some function* $F : X \longrightarrow A\ \}$.

If $\rho = (\delta^0 F)(g, x) = F(gx) - F(x)$*, then*

$$\begin{aligned}\delta^1\delta^0(F)(g, x) &= \delta^1(\rho)(g, h, x)\\ &= \rho(g, hx) - \rho(gh, x) + \rho(h, x)\\ &= \rho(g, hx) - \rho(h, x) - \rho(g, hx) + \rho(h, x) = 0.\end{aligned}$$

Now if the action of G on X is trivial, then

$$Z^1(G, X, \phi) = \{\rho : G \times X \longrightarrow A \mid \rho(gh, x) = \rho(h, x) + \rho(g, hx)\}$$

and $B^1(G, X, \phi) = 0$ implying that we get:

$$H^1(G, X, \phi) = Z^1(G, X, \phi).$$

Remark 2.17. *These examples can also be looked at for dynamical systems (G, X, ϕ) in the measurable or smooth context, where similar homological considerations can be pursued, independent of the geometric or topological nature of the system itself.*

Chapter 3

Groupoids, Cohomology and Dynamical Systems

The aim of this chapter is first to give an introduction to the basic concepts of groupoid theory with an emphasis on topological and Lie groupoids. We will discuss the concept of a groupoid in the discrete, topological, measurable and smooth setting. Morphisms, equivalences and natural constructions for constructing new groupoids out of given ones are discussed with details. The main part is then concerned with the development of a general concept of cohomology for Lie groupoids with particular applications to cohomology of action groupoids associated to a given dynamical system (G, X, ϕ). We will interprete cocycle cohomology classes of dynamical systems as action groupoid cohomology classes and look at some examples. Representation of groupoids are introduced as a concept related to groupoid actions used for our approach to define general groupoid cohomology. Many of these concepts generalises notions from group representations and group cohomology. In the last section of this chapter, following a suggestion of Tao [38], we will discuss how to interpret the cohomology classes defined via cocycle functions as action groupoid cohomology classes and in general how to see low-dimensional cohomology classes as extension classes for dynamical systems. In this chapter, we will be mostly interested in topological and Lie groupoids and in particular in action groupoids associated to a dynamical system (G, X, ϕ) arising from an action of a topological or Lie group. We concentrate on the Lie groupoid case, but it it is clear from the constructions how to amend things in the discrete, topological and measurable setting.

3.1 Glimpses of Groupoid Theory: Discrete, Topological, Measurable, Lie Groupoids

This section reviews the general theory of groupoids in four settings, namely groupoids in sets, in topological spaces, in measure spaces and in smooth manifolds. We will recall the main definitions and basic facts of the theory of groupoids and study transformations and equivalences between them, illustrated with specific examples. In general, we recommend [29] and [27] for this section. A good overview about the general use of groupoids to describe symmetries can be found in [40].

3.1.1 Discrete Groupoids

Let us first define groupoids in the category **Sets** of sets, which are of a purely categorical nature.

Definition 3.1. *A **groupoid** is a (small) category such that all morphisms* *are invertible. As with the familiar conventions, we will denote a groupoid by* $(\mathcal{G}, X, s, t, m, id, i)$, *or* $\mathcal{G} \rightrightarrows X$, *or* $\mathcal{G}$, *subject to the following internal description:*

(1) $\mathcal{G}$ *is the set of morphisms (arrows or morphism space) and* X *is the set of objects (base or object space);*

(2) $s : \mathcal{G} \longrightarrow X$ *is the source (domain) of a morphism and* $t : \mathcal{G} \longrightarrow X$ *is the target (range) of a morphism;*

(3) $m : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ *is the multiplication map, defined on the subset of the composable morphisms* $\mathcal{G}^{(2)} = \mathcal{G}_s \times_t \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$;

(4) $id : X \longrightarrow \mathcal{G}, x \mapsto 1_x$ *gives the identity morphisms ;*

(5) $i : \mathcal{G} \longrightarrow \mathcal{G}, g \mapsto g^{-1}$ *gives the inverse morphisms.*

These maps satisfy the following identities:

(i) $s(hg) = s(g), t(hg) = t(h)$,

(ii) $k(hg) = (kh)g$,

(iii) $1_{t(g)}g = g = g1_{s(g)}$, *and*

(iv) $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$

for any $k, h, g \in \mathcal{G}$ *with* $s(k) = t(h)$ *and* $s(h) = t(g)$. *We sometimes refer to* ***structure maps*** *of a groupoid* $(\mathcal{G}, X, s, t, m, id, i)$ *to mean* s ,t ,m ,id , *or* i *and we talk of a* ***groupoid*** $\mathcal{G}$ ***over*** X.

Definition 3.2. *A* ***functor*** *between two groupoids* $\mathcal{G} \rightrightarrows X$ *and* $\mathcal{H} \rightrightarrows Y$ *consists of two functions* $\varphi_0 : X \longrightarrow Y$ *and* $\varphi_1 : \mathcal{G} \longrightarrow \mathcal{H}$ *that respect all structure maps. We also call a functor between groupoids a* ***morphism of***

groupoids.

Definition 3.3. *Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are said to be* ***isomorphic*** *if there are morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ and $\psi : \mathcal{H} \longrightarrow \mathcal{G}$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity morphisms of $\mathcal{H}$ and $\mathcal{G}$ respectively. In this case φ and ψ are called* ***isomorphisms***.

The following follows directly from the definitions:

Proposition 3.1. *Groupoids and morphisms between groupoids form a category, the category of groupoids, denoted by* ***Gpd***.

Example 3.1. *(1) Let X be a set, then it can be described as the* ***trivial groupoid*** *or* ***unit groupoid*** *where the only morphisms are identities. The source and target maps are both the identity map 1_X, and the multiplication is only defined between a morphism and itself: $xx = x$ where $x \in X$.*

(2) Any (abstract) group G is a groupoid with one object $*$ and the morphisms are the elements of G. The composition of morphisms is given by the group operation of the group G.

(3) Any set X gives rise to the ***pair groupoid*** *of X. The set of objects is X, and the set of morphisms or arrows is $X \times X$, so we have $X \times X \rightrightarrows X$. The source and target maps are the first and second projection maps. Multiplication is defined as follows: $(x, y)(y, z) = (x, z)$.*

Proposition 3.2. *A groupoid $\mathcal{G}$ gives rise to an equivalence relation $\sim$ on the object space X as follows: for $x, y \in X$, $x \sim y$ if there is a morphism $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$.*

Proof. (1) $\sim$ is reflexive: $x \sim x$ for any $x \in X$, because there exists $1_x : x \rightarrow$

x,

(2) $\sim$ is symmetric: $x \sim y \Rightarrow y \sim x$ because each $g : x \longrightarrow y$ has an inverse $g^{-1} : y \longrightarrow x$,

(3) $\sim$ is transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$, because $h : y \longrightarrow z$ can be composed with $g : x \longrightarrow y$, giving $hg : x \longrightarrow z$. $\square$

3.1.2 Topological Groupoids

Until now, we have only defined a groupoid $\mathcal{G} \rightrightarrows X$ in sets, i.e. groupoids in which both object space and morphism space are sets. But in the most interesting geometric contexts, these groupoids have more structure. For example, they could be topological spaces and such groupoids are called topological groupoids and belong to **TopGpd**, the category of topological groupoids as defined below (see [7], [10] for the basic theory of topological groupoids).

Definition 3.4. *A* ***topological groupoid*** $\mathcal{G} \rightrightarrows X$ *is a groupoid where*

(1) the arrow (morphisms) space $\mathcal{G}$ *is a second-countable, locally compact Hausdorff topological space,*

(2) the object space X *is a Hausdorff, second-countable topological space;*

(3) the source map $s : \mathcal{G} \longrightarrow X$ *and the target map* $t : \mathcal{G} \longrightarrow X$ *are continuous maps;*

(3) the multiplication map $m : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ *defined on the topological subspace of the composable morphisms* $\mathcal{G}^{(2)} = \mathcal{G}_s \times_t \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$*; is a continuous map;*

(4) the identity map $id : X \longrightarrow \mathcal{G}, x \longmapsto 1_x$ *is a continuous map and gives*

the identity morphisms;

(5) the inverse map $i : \mathcal{G} \longrightarrow \mathcal{G}, g \longmapsto g^{-1}$ is a homeomorphism and gives the inverse morphisms.

Remark 3.1. *Let $(\mathcal{G}, X, s, t, m, id, i)$ be a topological groupoid and $x \in X$.*

*(1) The **isotropy group** G_x of x is considered as a closed topological space of $\mathcal{G}$ and contains all morphisms $g : x \longrightarrow x$. G_x has a topological group structure.*

*(2) A **foliation groupoid** is a topological groupoid such that all the isotropy groups G_x are discrete, $x \in X$.*

*(3) If the source map s and the target map t are local homeomorphism, then $\mathcal{G}$ is called an **étale groupoid**.*

*(4) A topological groupoid $\mathcal{G}$ is **proper** if the map $(s, t) : \mathcal{G} \longrightarrow X \times X$ is proper (i.e. the pre-image of any compact set is also a compact set).*

Definition 3.5. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be two topological groupoids, a **homomorphism between topological groupoids** $\mathcal{G}$ and $\mathcal{H}$ is a functor Φ given by a continuous map $\varphi_1 : \mathcal{G} \longrightarrow \mathcal{H}$ on arrows and a continuous map $\varphi_0 : M \longrightarrow N$ on objects, which together preseve the topological groupoid structure i.e.we have:*

$$\varphi_0(s(g)) = s(\varphi_1(g)), \varphi_0(t(g)) = t(\varphi_1(g)), \varphi_1(1_p) = 1_{\varphi_0(p)}$$

and

$$\varphi_1(hg) = \varphi_1(h)\varphi_1(g),$$

*which implies also $\varphi_1(g^{-1}) = \varphi(g)^{-1}$, for any $g, h \in \mathcal{G}$ with $s(h) = t(g)$ and any $p \in M$. If $M = N$ and $\varphi_0 = id_M$ we say that Φ is **homomorphism***

over *M, or that φ_1 is a* ***base-preserving homomorphism****.*

Proposition 3.3. *Topological groupoids and homomorphisms between topological groupoids form a category, the category of topological groupoids, denoted by* ***TopGpd****.*

Definition 3.6. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be two topological groupoids. We say that $\mathcal{G}$ and $\mathcal{H}$ are* ***isomorphic*** *if there are homomorphisms $\Phi : \mathcal{G} \longrightarrow \mathcal{H}$ and $\Psi : \mathcal{H} \longrightarrow \mathcal{G}$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity homomorphisms of $\mathcal{G}$ and $\mathcal{H}$ respectively. In this case Φ and Ψ are called* ***isomorphisms****. A* ***natural transformation*** *between functors $\tau : \varphi \Rightarrow \psi : \mathcal{G} \longrightarrow \mathcal{H}$ is a function $\tau : X \longrightarrow \mathcal{H}$ such that $\tau(y) \circ \varphi(g) = \psi(g) \circ \tau(x)$ for every arrow $g : x \longrightarrow y$ in $\mathcal{G}$. This natural transformation is invertible because every arrow in a groupoid is invertible.*

Examples

(1) Any topological space X gives two topological groupoids. One of them has the space of morphisms $\mathcal{G}$ and the space of objects X being the same ($\mathcal{G} = X \rightrightarrows X$) that is a groupoid with identity morphisms only. The second arises when the space of objects is X and ($\mathcal{G} = X \times X \rightrightarrows X$), is a groupoid with for any two elements $x, y \in X$ there is exactly one morphism (y, x) from x to y.

(2) A topological group is a topological groupoid. The same is true for a disjoint union of topological groups which is also topological group.

(3) If $\mathfrak{U} := \{U_i\}_{i \in I}$ is an open cover of a topological space X, then the **cover groupoid** of the cover, which we denote $\mathcal{G}_{\mathfrak{U}}$, is defined as follows:

$\mathcal{G}_{\mathfrak{U}} := (\coprod_{i,j \in I} U_{ij} \rightrightarrows \coprod_{i \in I} U_i)$ where $U_{ij} = U_i \cap U_j$ for $i, j \in I$. The source and target maps are the inclusion maps $U_{i,j} \hookrightarrow U_j$ and $U_{i,j} \hookrightarrow U_i$ respectively. The multiplication $U_{i,j} \times_X U_{j,k} \longrightarrow U_{i,k}$ is defined as $(x, y) \longmapsto x = y$. The identity map as $U_i \longrightarrow U_{ii} = U_i$. The inversion map is the identity map $U_{i,j} \longrightarrow U_{j,i}$.

(4) The **fundamental groupoid** ΠX of a topological space X is the groupoid with object space X and with arrow space $\mathcal{G}$ consisting of homotopy classes $[\gamma]$ (rel. end points) of continuous paths $\gamma : [0, 1] \longrightarrow X$. Multiplication is composition of paths. The source is given by the starting point $(s[\gamma] = \gamma(0))$ and the target by the end point $(t[\gamma] = \gamma(1))$. The inverse is defined by inverting the direction of the path $(i([\gamma]) = [t \longrightarrow \gamma(1 - t)])$.

3.1.3 Lie Groupoids

Now we will discuss the smooth category to define Lie groupoids which we are most interested in this work (see also [29]).

Definition 3.7. *A **Lie groupoid** is a groupoid $(\mathcal{G}, M, s, t, m, id, i)$ such that*

(1) M is a Hausdorff, second-countable smooth manifold;

(2) $\mathcal{G}$ is a second-countable smooth manifold, (not necessarily Hausdorff);

(3) the source s and the target t are surjective submersions;

(4) the multiplication m and the identity id are smooth;

(5) the inversion map $i : \mathcal{G} \longrightarrow \mathcal{G}$ is a diffeomorphism.

Remark 3.2. *Let $(\mathcal{G}, M, s, t, m, id, i)$ be a Lie groupoid and $x \in M$.*

(1) The ***isotropy group*** G_x *of* x *is considered as a closed submanifold of* $\mathcal{G}$ *and contains all morphisms* $g : x \longrightarrow x$. G_x *has a Lie group structure.*

(2) A ***foliation Lie groupoid*** *is a Lie groupoid such that all the isotropy groups* G_x *are discrete,* $x \in M$.

(3) If the source map s *and the target map* t *are local diffeomorphism, then* $\mathcal{G}$ *is called an* ***étale Lie groupoid****.*

(4) A Lie groupoid $\mathcal{G}$ *is* ***proper*** *if the map* $(s, t) : \mathcal{G} \longrightarrow M \times M$ *is proper (i.e. the pre-image of any compact set is also a compact set).*

Definition 3.8. *Let* $\mathcal{G} \rightrightarrows M$ *and* $\mathcal{H} \rightrightarrows N$ *be two Lie groupoids, a* ***homomorphism between Lie groupoids*** $\mathcal{G}$ *and* $\mathcal{H}$ *is a functor* Φ *given by a smooth map* $\varphi_1 : \mathcal{G} \longrightarrow \mathcal{H}$ *on arrows and a smooth map* $\varphi_0 : M \longrightarrow N$ *on objects, which together preseve the Lie groupoid structure i.e.*

$$\varphi_0(s(g)) = s(\varphi_1(g)), \varphi_0(t(g)) = t(\varphi_1(g)), \varphi_1(1_p) = 1_{\varphi_0(p)}$$

and

$$\varphi_1(hg) = \varphi_1(h)\varphi_1(g),$$

which implies also $\varphi_1(g^{-1}) = \varphi(g)^{-1}$, *for any* $g, h \in \mathcal{G}$ *with* $s(h) = t(g)$ *and any* $p \in M$. *If* $M = N$ *and* $\varphi_0 = id_M$ *we say that* Φ *is* ***homomorphism over*** M, *or that* φ_1 *is a* ***base-preserving homomorphism****.*

Proposition 3.4. *Lie groupoids and homomorphisms between Lie groupoids form a category, the category of Lie groupoids, denoted by* ***LieGpd****.*

Definition 3.9. *Let* $\mathcal{G} \rightrightarrows M$ *and* $\mathcal{H} \rightrightarrows N$ *be two Lie groupoids. We say that* $\mathcal{G}$ *and* $\mathcal{H}$ *are* ***isomorphic*** *if there are homomorphisms* $\Phi : \mathcal{G} \longrightarrow \mathcal{H}$

*and $\Psi : \mathcal{H} \longrightarrow \mathcal{G}$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity homomorphisms of $\mathcal{G}$ and $\mathcal{H}$ respectively. In this case Φ and Ψ are called **isomorphisms**. A **natural transformation** between functors $\tau : \varphi \Rightarrow \psi : \mathcal{G} \longrightarrow \mathcal{H}$ is a smooth function $\tau : X \longrightarrow \mathcal{H}$ such that $\tau(y) \circ \varphi(g) = \psi(g) \circ \tau(x)$ for every arrow $g : x \longrightarrow y$ in $\mathcal{G}$. This natural transformation is invertible because every arrow in a groupoid is invertible.*

Examples

(1) Let M be a smooth manifold and let $p : E \longrightarrow M$ be a vector bundle. Let $GL(E)$ be the set of all linear vector space isomorphisms between fibres E_x for each $x \in M$. Now each fibre E_x always has the identity automorphism id_{E_x}, and this gives an inclusion map $M \hookrightarrow GL(E)$. Also there is a multiplication map defined on $GL(E)$, which is defined by composition of maps, whenever possible, that is, $E_x \xrightarrow{\varphi} E_y \xrightarrow{\psi} E_z$ for each $x, y, z \in M$. Since all maps $E_x \xrightarrow{\varphi} E_y$ are isomorphisms this implies they are invertible and $E_y \xrightarrow{\varphi^{-1}} E_x$ is also in $GL(E)$. Above any given $x \in M$, the set of isomorphisms $E_x \longrightarrow E_x$ is a Lie group which is isomorphic to $GL(V)$, where V is a typical fibre of the vector bundle $p : E \longrightarrow M$. We call $GL(E)$ the **general linear groupoid** of the bundle E.

(2) A Lie groupoid $\mathcal{G} \rightrightarrows M = \{pt\}$ is a Lie group. A functor between two Lie groupoids $\mathcal{G} \rightrightarrows M = \{pt\}$ and $\mathcal{H} \rightrightarrows N = \{pt\}$ is given by a Lie group homomorphism $\mathcal{G} \longrightarrow \mathcal{H}$. A natural transformation between these functors is given by an element of $\mathcal{H}$ conjugating between the two homomorphisms.

(3) If G is a Lie group acting smoothly from the left on a smooth manifold M, then associated to this action is the **action** or **translation groupoid** $G \ltimes M$. The object space is M and the arrow space is the manifold $G \times M$,

where an orderd pair $(g, x) \in G \times M$ corresponds to an arrow $x \longrightarrow g \cdot x$ with the obvious composition rule. Now if the action is almost free (each isotropy group G_x is discrete), then $G \ltimes M$ is a foliation groupoid. If the group G is discrete, then $G \ltimes M$ is an étale Lie groupoid. If we suppose the action of G is proper, then the action groupoid is proper. Action groupoids for right actions are defined similarly. If $P \longrightarrow M$ is a left principal G -bundle, then $P \longrightarrow M$ induces a Lie groupoid functor $G \ltimes M \longrightarrow M$, where M is the trivial groupoid on M.

(4) For any manifold M, the **pair groupoid** of M is denoted by $\mathcal{G} = Pair(M) = M \times M$, where the source and target maps are the first and second projection.

(5) Let M be a smooth manifold and $\Pi M = \{(x, [\sigma], y) \mid x, y \in M, [\sigma]$ is the homotopy class of paths $\sigma(0) = x, \sigma(1) = y\}$. Then ΠM is a groupoid on M with these rules: $s(x, [\sigma], y) = x, t(x, [\sigma], y) = y, m((x, [\sigma], y)(y', [\tau], z)) = (x, [\sigma \circ \tau], z)$ iff $y = y'$, where $\sigma \circ \tau$ is the concatenation of paths σ and τ , $id(x) = (x, [constant], x)$ and $i(x, [\sigma], y) = (y, [\sigma^{-1}], x)$ where $\sigma^{-1}(t) = \sigma(1-t)$, for each $t \in [0, 1]$. If ΠM is equipped with the quotient topology for the compact open topology on the space of paths of M, then $s \times t : \Pi M \longrightarrow M \times M$ is a covering map. It follows that ΠM is a Lie groupoid and called again the **fundamental groupoid** of M, whose underlying topological groupoid we have already met in the last subsection. Its isotropy groups are the fundamental groups $\pi_1(M, x)$ for each $x \in M$.

(6) Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The **opposite Lie groupoid** for $\mathcal{G}$ denoted by $\mathcal{G}^{op}$ has the same object space and arrow space, but all of the arrows are inverted. The inversion map $i : \mathcal{G} \longrightarrow \mathcal{G}$ induces a Lie groupoid isomorphism from $\mathcal{G}$ onto $\mathcal{G}^{op}$.

(7) Let $f : M \longrightarrow N$ be a surjective submersion. The associated **kernel groupoid** over M, denoted by $ker(f)$, is the Lie groupoid with object space M, arrow space $M \times_N M$, $s(x, y) = y$, $t(x, y) = x$, $id(x) = (x, x)$, $i(x, y) = (y, x)$, and $m((x, y), (y, z)) = (x, z)$. If $f : M \longrightarrow pt$, then we get the pair groupoid $\mathcal{G} = Pair(M) = M \times M$. When f is the identity map $M \longrightarrow M$, we get the trivial groupoid on M, denoted M. This induces an embedding of categories **Mfd** $\hookrightarrow$ **LieGpd**, $M \longmapsto M$. For any $f : M \longrightarrow N$, there is a Lie groupoid functor $\tau(f) : ker(f) \longrightarrow N$ induced by f.

3.1.4 Measurable Groupoids

Measurable groupoids come with a Borel structure. A **Borel space** is defined as an ordered pair of a set X and σ-algebra $\mathscr{B}(X)$ (see Definition 2.9.) of subsets of the set X in which every one is called a **Borel set**. A map from a Borel space into a another Borel space is called **Borel map** if the inverse image of every Borel set is a Borel set. A bijective Borel map in both directions is called **Borel isomorphism**.

Definition 3.10. *A* ***measurable groupoid*** *is a groupoid* $(\mathcal{G}, X, , m, s, t, id, i)$ *such that the underlying space is endowed with a Borel structure* $\mathscr{B}$ *and all of the structure maps are Borel:*

(1) $i : \mathcal{G} \longrightarrow \mathcal{G}$; $g \longmapsto g^{-1}$;

(2) $m : \mathcal{G}^2 \longrightarrow \mathcal{G}$; $(g_1, g_2) \longmapsto g_1g_2$;

(3) $s, t : \mathcal{G} \longrightarrow X$.

Here $\mathcal{G} \times \mathcal{G}$ *has the product Borel structure,* $X \subset \mathcal{G}$ *and* $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ *the induced Borel structure.*

Definition 3.11. *Two measurable groupoids $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ are said to be* ***isomorphic*** *if there exists a Borel isomorphism Φ of $\mathcal{G}$ onto $\mathcal{H}$ such that Φ and Φ^{-1} are algebraically inverse homomorphisms.*

3.2 Dynamical Systems and Action Groupoids

A dynamical system (G, M, ϕ) is defined by a group G acting on a space M. In example (3) of subsection 3.1.3 an action groupoid $G \ltimes M$ is constructed depending on a group action. The following two examples refer to dynamical systems out of given action groupoids explaining the relationship between dynamical systems and action groupoids.(See also [2]).

Example 3.2. *(1) Let $M = \{0, 1\}$ and G be the cyclic group of order 2 i.e. $G = C_2 = \{1, c\}$. If $ob(G \ltimes M) = M$ and $mor(G \ltimes M) = \{(1, 0) : 0 \longrightarrow 0, (1, 1) : 1 \longrightarrow 1, (c, 0) : 0 \longrightarrow 1, (c, 1) : 1 \longrightarrow 0\}$, then the dynamical system related to the action groupoid $G \ltimes M$ is given by the map ϕ defined on $G \times M$ interchanging 0 and 1 by c and fixing 0 and 1 by 1.*

(2) For any vector field X defined on a smooth manifold M, the domain $D(X)$ of the flow ϕ_t of X, $D(X) = \{(t, x) \in \mathbb{R} \times M\}$ such that ϕ_t is defined can be seen as an action groupoid $G \ltimes M$ with the source and target as follows $s(t, x) = x$ and $t(t, x) = \phi(t, x)$. The composition of this groupoid is $m(t', \phi(t, x), (t, x)) = (t' + t, x)$. Now if X is a complete vector field (all its integral curves extend over the whole $\mathbb{R}$), then $D(X)$ is the action groupoid $\mathbb{R} \ltimes M$ related to the global flow of the vector field X interpreted as an action of the real numbers $\mathbb{R}$ on the smooth manifold M.

For the next sections, groupoid will always mean Lie groupoid if not otherwise stated as it is a main subject for this thesis.

3.3 Constructions of Groupoids

In this section, we give some general construction methods for Lie groupoids, including induced Lie groupoids, strong pullbacks and weak pullbacks. This can be found partially also in [29], chapter 5.

3.3.1 Induced Groupoids

Definition 3.12. *Let ($\mathcal{G} \rightrightarrows \mathcal{G}_0$) be a groupoid and $f : M \longrightarrow \mathcal{G}_0$ be a smooth map. The* ***induced groupoid*** *$f^*(\mathcal{G})_1$ over M is defined as follows:*

$f^*(\mathcal{G}) = M \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} M = \{(x, g, y) \in M \times \mathcal{G} \times M \mid t(g) = f(x), s(g) = f(y)\}$, *the sourse and target maps $s(x, g, y) = y, t(x, g, y) = x$ the multiplication given by $(x, h, z)(z, g, y) = (x, hg, y)$, the identity $id(x) = (x, id_{f(x)}, x)$ and the inverse $i(x, g, y) = (y, g^{-1}, x)$.*

Remark 3.3. *We note that there is a functor τ from $f^*(\mathcal{G})_1$ to $\mathcal{G}$ where $\tau = f : M \longrightarrow \mathcal{G}$, and $\tau : f^*(\mathcal{G}) \longrightarrow \mathcal{G}, \tau(x, g, y) = g$.*

Proposition 3.5. *The induced groupoid $f^*(\mathcal{G})$ is a Lie groupoid provided that $t \circ pr_1 : \mathcal{G} \times_{\mathcal{G}_0} M \longrightarrow \mathcal{G}_0$ is a surjective submersion.*

Proof. We observe that $f^*(\mathcal{G})_1 = M \times_{\mathcal{G}_0} \times \mathcal{G}_1 \times_{\mathcal{G}_0} M$ can be constructed by two pullbacks given by the diagram

$$\begin{tikzcd}
f^*(\mathcal{G})_1 \arrow[rr] \arrow[d] & & M \arrow[d, "f"] \\
\mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[r, "pr_1"] \arrow[d] & \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{G}_0 \\
M \arrow[r, "f"] & \mathcal{G}_0 &
\end{tikzcd}$$

Since the composite map $\mathcal{G}_1 \times_{\mathcal{G}_0} M \overset{pr_1}{\longrightarrow} \mathcal{G}_1 \overset{t}{\longrightarrow} \mathcal{G}_0$ is a surjective submersion, $f^*(\mathcal{G})_1$ is representable and $f^*(\mathcal{G})_1 \longrightarrow \mathcal{G}_0$ is a surjective submersion. Therefore the diagram below is a pullback square in **Mfd**:

$$\begin{tikzcd}
f^*(\mathcal{G})_1 \arrow[r] \arrow[d, "{(s,t)}"] & \mathcal{G}_1 \arrow[d, "{(s,t)}"] \\
M \times M \arrow[r] & \mathcal{G}_0 \times \mathcal{G}_0
\end{tikzcd}$$

Thus $f^*(\mathcal{G})_1$ is a Lie groupoid and f induces a Lie groupoid functor $\tau : f^*(\mathcal{G}) \longrightarrow \mathcal{G}$. $\square$

3.3.2 Strong Pullbacks

For this and the following constructions, we suppose $(\mathcal{G} \rightrightarrows M)$, $(\mathcal{H} \rightrightarrows N)$ and $(\mathcal{K} \rightrightarrows L)$ are groupoids.

Definition 3.13. *If $\varphi : \mathcal{G} \longrightarrow \mathcal{K}$ and $\psi : \mathcal{H} \longrightarrow \mathcal{K}$ are two groupoid morphisms, the **strong pullback** groupoid $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ is a groupoid with object space $M \times_L N = \{(x, y) \in M \times N, \varphi_0(x) = \psi_0(y)\}$, morphism space $\mathcal{G} \times_L \mathcal{H} = \{(g, h) \in \mathcal{G} \times \mathcal{H}, \varphi_1(g) = \psi_1(h)\}$ and the composition defined componentwise.*

Remark 3.4. *In general, the groupoid $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ is not a Lie groupoid. However, if (φ_0, ψ_0) and (φ_1, ψ_1) are both transversal, then $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ is a Lie groupoid. This means the strong pullback satisfies the usual universality property for pullbacks in the category **LieGpd**.*

Example 3.3. *The induced groupoid can be written as a strong pullback. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $f : L \longrightarrow M$ be a smooth map. The diagram below is a strong pullback.*

$$\begin{tikzcd} f^*(\mathcal{G}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ P(L) \arrow[r] & P(M) \end{tikzcd}$$

Here $P(L)$ and $P(M)$ are pair groupoids, the Lie groupoid functor $\mathcal{G} \longrightarrow P(M)$ is determined by $(s, t) : \mathcal{G} \longrightarrow M \times M$, and $P(L) \longrightarrow P(M)$ is induced by f.

3.3.3 Weak Pullbacks

We will use the notion of comma category (see [26], section II.6).

Definition 3.14. *Given two functors $\varphi : \mathcal{G} \longrightarrow \mathcal{K}$ and $\psi : \mathcal{H} \longrightarrow \mathcal{K}$, the **weak pullback** groupoid $\mathcal{G} \times^w_{\mathcal{K}} \mathcal{H}$ is the comma category $(\varphi \downarrow \psi)$. Explicitly,* *objects of* $\mathcal{G} \times^w_{\mathcal{K}} \mathcal{H}$ *are triples* $(x, k, y), x \in M, y \in N$ *and* $k : \varphi_0(x) \longrightarrow \psi_0(y)$ *is an arrow in* $\mathcal{K}$*. Arrows from* (x_1, k_1, y_1) *to* (x_2, k_2, y_2) *are pairs* (g, h) *of arrows* $g \in \mathcal{G}$ *and* $h \in \mathcal{H}$ *where* $g : \varphi_0(x_1) \longrightarrow \varphi_0(x_2)$ *and* $h : \psi_0(x_1) \longrightarrow \psi_0(x_2)$ *are arrows in* $\mathcal{K}$ *such that* $h \circ k_1 = k_2 \circ g$*. The composition is given componentwise.*

Proposition 3.6. *The weak pullback* $\mathcal{G} \times^w_{\mathcal{K}} \mathcal{H}$ *defined above is a Lie groupoid if* $t \circ pr_2 : M \times_{L,s} \mathcal{K} \longrightarrow L$ *or* $s \circ pr_2 : N \times_{L,t} \mathcal{K} \longrightarrow L$ *is a surjective submersion.*

Proof. The object space of the groupoid $\mathcal{G} \times^w_{\mathcal{K}} \mathcal{H}$ is the limit of the following diagram:

$$\begin{tikzcd}
\mathcal{G} \times_{s,L} \mathcal{K} \times_{t,L} \times \mathcal{H} \arrow[rr] \arrow[d] & & \mathcal{H} \arrow[d, "s"] \\
\mathcal{G} \times_{s,L} \mathcal{K} \times N \arrow[r] \arrow[d] & M \times \mathcal{K} \times N \arrow[r, "pr_2"] \arrow[d, "pr_1"] & N \\
\mathcal{G} \arrow[r, "s"] & M &
\end{tikzcd}$$

which is representable if $t \circ pr_2 : M \times_{L,s} \mathcal{K} \longrightarrow L$ is a surjective submersion. In a similar way, it is representable if $s \circ pr_2 : N \times_{L,t} \mathcal{K} \longrightarrow L$ is a surjective submersion. In both cases, it is easy to show that the space of arrows $\mathcal{G} \times_{t \circ \varphi_1, L, s} \mathcal{K} \times_{t, L, s \circ \varphi_1} \mathcal{H}$ is also a manifold, and the source and target maps are surjective submersions. This proves the claim. □

3.4 Equivalences of Lie Groupoids

The classical notion of equivalence of categories can be applied to groupoids provided that no additional structures such as topology or smoothness are involved [29]. In this part we discuss the notion of equivalence of groupoids. There exist several different notions of equivalence for Lie groupoids (see also [29]).

3.4.1 Strong Equivalences

Definition 3.15. *Let ($\mathcal{G} \rightrightarrows M$) and ($\mathcal{H} \rightrightarrows N$) be two Lie groupoids. We say that $\mathcal{G}$ and $\mathcal{H}$ are* ***strongly equivalent*** *if there exist two homomorphisms, $\Phi : \mathcal{G} \longrightarrow \mathcal{H}$ and $\Psi : \mathcal{H} \longrightarrow \mathcal{G}$ and, together with smooth natural transformations $S : \Psi \circ \Phi \longrightarrow id_{\mathcal{G}}$, $T : \Phi \circ \Psi \longrightarrow id_{\mathcal{H}}$.*

3.4.2 Weak Equivalences

Definition 3.16. *A Lie groupoid functor φ from ($\mathcal{G} \rightrightarrows M$) to ($\mathcal{H} \rightrightarrows N$) is a* ***weak equivalence*** *(or* ***essential equivalence****) if it is*

(1) essentially surjective; that is, the map $t \circ pr_1 : \mathcal{H} \times_N M \longrightarrow N$ which sends a pair (h, x) with $s(h) = \varphi(x)$ to $t(h)$ is a surjective submersion, and

(2) fully faithful; that is, the diagram below is a pullback sqare in ***Mfd****.*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\varphi_1} & \mathcal{H} \\ \Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle (s,t)} \\ M \times M & \xrightarrow{\varphi_0 \times \varphi_0} & N \times N \end{array}$$

Remark 3.5. *According to Proposition 3.3., the first condition ensures that the pullback in the the second condition exists. Now if* $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ *is a weak equivalence, then there is an isomorphism between* $\mathcal{G}$ *and the induced groupoid* $\varphi_0^*(\mathcal{H})$.

In general, the strong equivalences between Lie groupoids are rare in comparison with the weak equivalences.

Example 3.4. *(1) Let* M *be a smooth manifold and* $\mathcal{G} = Pair(M) = M \times M$ *its pair groupoid. If* $\mathcal{G}'$ *is a trivial groupoid consisting of one point* $\{pt\}$ *and only identity morphism, then the functor* $\varphi : \mathcal{G} \longrightarrow \mathcal{G}'$ *is a strong and a weak equivalence.*

(2) If $(\mathcal{G} \rightrightarrows M)$ *is a Lie groupoid, then the smooth map* $f : L \longrightarrow M$ *in Proposition 3.5. induces a weak equivalence* $\tau : f^*(\mathcal{G}) \longrightarrow \mathcal{G}$*, see [29], Example 5.10(4).*

Proposition 3.7. *Every strong equivalence of Lie groupoids is a weak equivalence.*

Proof. Let $\phi : G \longrightarrow H$ be a strong equivalence, with $\psi : H \longrightarrow G$ and S and T as in the definition of strong equivalence above. We prove first that the map $t \circ pr_1 : H_1 \times_{H_0} G_0 \longrightarrow H_0$ of the definition of weak equivalence is a surjective submersion. Clearly it is surjective because any $y \in H_0$ is the image of $(T(y), \psi(y))$. To see that it is a submersion, we prove that it has a local section through any point $(h_0 : \phi(x_0 \longrightarrow y_0, x_0))$ of $H_1 \times_{H_0} G_0$. To this end, consider the arrow $T(y_0)^{-1}h_0 : \phi(x_0 \longrightarrow \phi(\psi(y_0)))$ in H. Since ϕ is an equivalence of categories, there is a unique arrow $g_0 : x_0 \longrightarrow \psi(y_0)$ in G with $\phi(g_0) = T(y_0)^{-1}h_0$. Let $\lambda : U \longrightarrow G_1$ be a local bisection through g_0 in G, and let $\tilde{\lambda} = t \circ \lambda : U \longrightarrow G_0$ be the associated diffeomorphism onto an open neighbourhood V of $\psi(y_0)$. Let $\kappa : \psi^{-1}(V) \longrightarrow H_1 \times_{H_0} G_0$ be the map $\kappa(y) = (T(y)\phi(\lambda(\tilde{\lambda})^{-1}(\psi(y)))), \tilde{\lambda}^{-1}(\psi(y)))$. Then κ is a section of $t \circ pr_1$ through the given point (h_0, x_0). This proves that $t \circ pr_1$ is a surjective submersion. In particular, the fibred product $G_0 \times_{H_0} H_1 \times_{H_0} G_0$ of $t \circ pr_1$ along $\phi : G_0 \longrightarrow H_0$ is a manifold, which fits into a pull-back diagram

$$\begin{tikzcd}
G_0 \times_{H_0} H_1 \times_{H_0} G_0 \arrow[r, "pr_2"] \arrow[d, "(pr_3, pr_1)"] & H_1 \arrow[d, "(s,t)"] \\
G_0 \times G_0 \arrow[r, "\phi \times \phi"] & H_0 \times H_0
\end{tikzcd}$$

Since ϕ is an equivalence of categories, the map $G_1 \longrightarrow G_0 \times_{H_0} H_1 \times_{H_0} G_0$ sending g to $(s(g), \phi(g), t(g))$ is a bijection. $\square$

Remark 3.6. *The following two examples show that the converse is not true.*

Example 3.5. *(1) Let $f : M \longrightarrow N$ be a surjective submersion, and let $ker(f)$ be the corresponding kernel groupoid. Then $c(f) : ker(f) \longrightarrow N$ is a weak equivalence. Let $g : N \longrightarrow ker(f)$ be a quasi-inverse of $c(f)$. Then $c(f) \circ f$ is the identity map because N is a trivial groupoid. Hence g_0 is a section of $f : M \longrightarrow N$, but f need not admit a section in general.*

(2) Let G be a Lie group and $P \longrightarrow M$ a left principal G -bundle. The functor $G \ltimes P \longrightarrow M$ is a weak equivalence. $P \longrightarrow M$ being a surjective submersion implies essential surjectivity. Since the action is principal, the diagram

$$\begin{array}{ccc} P \times G & \longrightarrow & M \\ \downarrow & & \downarrow \\ P \times P & \longrightarrow & M \times M \end{array}$$

is a pullback square, hence implies fully faithful. For a similar reason as in the previous example, $G \ltimes P \longrightarrow M$ need not be a strong equivalence.

Weak equivalences of Lie groupoids satisfy the following properties:

Proposition 3.8. *Let $(\mathcal{G} \rightrightarrows M)$, $(\mathcal{H} \rightrightarrows N)$ and $(\mathcal{K} \rightrightarrows L)$ be Lie groupoids.*

(i) For two functors $\varphi, \psi : \mathcal{G} \longrightarrow \mathcal{H}$, if there is a natural transformation $\tau : \varphi \Rightarrow \psi$, then φ is a weak equivalence if and only ψ is.

(ii) The composite of weak equivalence is a weak equivalence.

(iii) For a weak equivalence $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ and a functor $\psi : \mathcal{K} \longrightarrow \mathcal{H}$, their weak pullback P exists. Moreover, $P \longrightarrow \mathcal{K}$ is a weak equivalence and the map from object space of P to L is a surjective submersion.

(iv) For functors $\varphi, \psi : \mathcal{G} \longrightarrow \mathcal{H}$ and a weak equivalence $\tau : \mathcal{H} \longrightarrow \mathcal{K}$, if there is a natural transformation $\tau \circ \varphi \Rightarrow \tau \circ \psi$ then there is a natural transformation $\varphi \Rightarrow \psi$.

Proof. The statements (i) and (ii) are straightforward implications from the definitions. For (iii) see [29], Proposition 5.12. For (iv) see [32], Section 4.1. $\square$

3.4.3 Morita Equivalences

Morita equivalence is the smallest equivalence relation between Lie groupoids whenever there exists a weak equivalence between them.

Definition 3.17. *Two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are **Morita equivalent** ($\mathcal{G} \overset{M}{\sim} \mathcal{H}$) if there exist two weak equivalences $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ and $\psi : \mathcal{K} \longrightarrow \mathcal{H}$ for a third Lie groupoid $\mathcal{K} \rightrightarrows L$.*

Now depending on the Proposition 3.8. (iii) we have the following proposition:

Proposition 3.9. *Morita equivalence is the smallest equivalence relation containing weak equivalence.*

Proof. The proposition is clear as soon as we have asserted that $\overset{M}{\sim}$ is an equivalence relation. We only need to prove transitivity of $\overset{M}{\sim}$. Suppose $G \overset{M}{\sim} H \overset{M}{\sim} K$. Then there is a diagram $G \longleftarrow L' \longrightarrow H \longleftarrow L'' \longrightarrow K$, where each arrow is an strong essential equivalence . The arrow and object part of an essential equivalence are surjective submersions, and therefore the strict pullback L of L' and L'' over H is well defined. By symmetry in (L', G) and (L'', K), it is suffices to show that the natural map $L \longrightarrow L'' \longrightarrow G$ is an essential equivalence. We check both conditions:

(i) It is immediate that $L_0 \longrightarrow L'_0$ is an surjective submersion since it is the pullback of $L''_0 \longrightarrow H_0$.

(ii) To prove that the composed rectangle in the diagram

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d] & L'_1 \arrow[r] \arrow[d] & G_1 \arrow[d] \\
L_0 \times L_0 \arrow[r] & L'_0 \times L'_0 \arrow[r] & G_0 \times G_0
\end{tikzcd}$$

is a pullback, it suffices to show that the left hand square is a pullback since the right hand square is. But we know that the composed rectangle

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d] & L_0 \times L_0 \arrow[r] \arrow[d] & L''_0 \times L_0\text{”} \arrow[d] \\
L'_1 \arrow[r] & L'_0 \times L'_0 \arrow[r] & H_0 \times H_0
\end{tikzcd}$$

is a pullback, and thus the left hand square is also a pullback. □

3.4.4 Examples

(1) Let M be a smooth manifold. Consider ΠM, the fundamental groupoid of M and $\pi_1(M,p)$ the fundamental group at p of M. There is a Morita equivalence $\Pi M \overset{M}{\sim} \pi_1(M,p)$ for $p \in M$, where we regard the fundamental group $\pi_1(M,p)$ as a groupoid over the singleton $\{p\}$. In this case, we can see this as the following

$$\Pi M \longleftarrow \pi_1(M,p) \xrightarrow{id} \pi_1(M,p).$$

(2) Let G be a Lie group acting freely and properly on a smooth manifold M. Consider the action groupoid $G \ltimes M$ over M. There is a Morita equivalence $G \ltimes M \overset{M}{\sim} M/G$, where we regard the quotient space M/G as the unit

groupoid. So,

$$G \ltimes M \longleftarrow M/G \xrightarrow{id} M/G \ .$$

(3) Two Lie groups G and $G^{'}$ are Morita equivalent if and only if they are isomorphic.

(4) The general linear groupoid $GL(E) \rightrightarrows M$ associated to the vector bundle $p : E \longrightarrow M$ is Morita equivalent to the Lie group $GL(E_x)$, for any $x \in M$. In addition, the natural inclusion $GL(E_x) \longrightarrow GL(E)$ is a weak equivalence.

3.5 Groupoid Actions and Representations

The theory of representations of Lie groupoids gives a unified view for the study of vector bundles on singular geometric spaces like orbifolds, spaces of leaves of foliations or orbits of Lie group actions or dynamical systems. Also, this theory has a direct relationship with the theory of representations of Lie groups. In addition, it is also used to describe symmetries of fibre bundles in a similar way as in the case for Lie groups, which are used to study symmetries of manifolds [19]. In this section, we introduce Lie groupoid actions and representations of Lie groupoids. We recommend [10] and [27] as additional references for some of the material presented here.

3.5.1 Lie groupoid actions

Definition 3.18. *Let ($\mathcal{G} \rightrightarrows M$) be a Lie groupoid, N a smooth manifold and $\epsilon : N \longrightarrow M$ a smooth map called* ***moment map****. A smooth* ***left action*** *of $\mathcal{G}$ on N* ***along*** *ϵ is given by a smooth map $\mu : \mathcal{G} \times_M N \longrightarrow N$, $\mu(g, y) = gy$ defined on the pullback manifold $\mathcal{G} \times_M N = \{(g, y) \in \mathcal{G} \times N \mid s(g) = \epsilon(y)\}$, which* *satisfies the following identities:* $\epsilon(gy) = t(g)$, $1_{\epsilon(y)}y = y$ *and* $g'(gy) = (g'g)y$, *for any* $g', g \in \mathcal{G}$ *and* $y \in N$ *with* $s(g') = t(g)$ *and* $s(g) = \epsilon(y)$.

Remark 3.7. *That the pullback in the previous definition is a smooth manifold comes from the fact that the morphism* s *of the the groupoid is a submersion (see [29],p.122).*

Example

Let $(\mathcal{G} \rightrightarrows M)$ be a Lie groupoid and $(M \times F, p, M)$ be the trivial vector bundle. Then, a left action of the Lie groupoid $\mathcal{G}$ along p can be defined by the smooth map $\mu : \mathcal{G} \times_M M \times F \longrightarrow M \times F$ as follows : $\mu(g, (s(g), a)) = (t(g), a)$, for $g \in \mathcal{G}$, $a \in F$.

Given two left actions (ϵ_1, μ_1) on N and (ϵ_2, μ_2) on L respectively, a map $f : N \longrightarrow L$ is called **equivariant** if $\epsilon_1 = \epsilon_2 \circ f$ and $f(gy)$ for $(g, y) \in \mathcal{G} \times_M N$.

Remark 3.8. *(1) A right action is defined analogously. Given a left action of* $\mathcal{G}$ *on a manifold* N *along* ϵ*, then* $yg = g^{-1}y$ *defines a right action on* N *along* ϵ *and vise versa.*

(2) An action μ *realizes the arrows of the groupoid* $(\mathcal{G} \rightrightarrows M)$ *as symmetries of the collection of fibres of the moment map, i.e. for each arrow* $x \xrightarrow{g} y$ *we have a diffeomorphism* $N_x \xrightarrow{\mu_g} N_y$.

(3) In the previous definition, if M *is a one-point manifold, then* $\mathcal{G}$ *is a Lie group, and we recover the action of a Lie group on a manifold as defined in chapter 2.*

An action of a Lie groupoid ($\mathcal{G} \rightrightarrows M$) on a manifold N along a map ϵ induces an equivalence relation on N. The quotient space $N/\mathcal{G}$ is usually not a manifold even for a Lie group action (Proposition 2.3.).

Definition 3.19. *Given a right action (μ_1, ϵ_1) of a Lie groupoid ($\mathcal{G} \rightrightarrows M$) on a manifold N. A **semi-direct product groupoid** $N \rtimes \mathcal{G}$ is a Lie groupoid with object space N and morphism space $N \times_{M,t} \mathcal{G}$. The source map is μ, the target map is $N \times_{M,t} \mathcal{G} \longrightarrow N$ and the composition is given by the composition in $\mathcal{G}$. There is a Lie groupoid functor $\pi : N \rtimes \mathcal{G} \longrightarrow \mathcal{G}$ given by $\epsilon : N \longrightarrow M$ and $pr_2 : N \times_{M,t} \mathcal{G} \longrightarrow \mathcal{G}$.*

Example 3.6. *(1) Let ($\mathcal{G} \rightrightarrows M$) be a Lie groupoid. There is a natural right action on M with moment map $id : M \longrightarrow M$ and with action map $s : \mathcal{G} \cong M \times_{id,M,t} \mathcal{G} \longrightarrow M$. The semi-direct product groupoid of this action is $\mathcal{G}$. This action may be regarded as a universal action. Given a right action (μ, ϵ) on N, then $\epsilon : N \longrightarrow M$ is an equivariant map.*

(2) Let ($\mathcal{G} \rightrightarrows M$) be a Lie groupoid acting on manifolds N and L. Suppose that either $N \longrightarrow M$ or $L \longrightarrow M$ is a surjective submersion. Then there is an action on $N \times_M L$ given by $(x, y).g = (x.g, y.g)$. This is the product in the category of spaces with groupoid action.

3.5.2 Representations of Lie Groupoids

We study now particular types of Lie groupoid actions, which are generalising representations of Lie groups.

Definition 3.20. *Let ($\mathcal{G} \rightrightarrows M$) be a Lie groupoid and $p : E \longrightarrow M$ a smooth* *vector bundle. A* ***representation*** *of the groupoid* $\mathcal{G}$ *on* E *or* $\mathcal{G}$***-module*** *is a smooth left action* $\rho : \mathcal{G} \times_{_M} E \longrightarrow E$*, denoted by* $\rho(g, v) = gv$*, of* $\mathcal{G}$ *on* E *along the bundle projection* $p : E \longrightarrow M$*, such that for any arrow* g *between* $x, y \in M$*, the induced map between fibers* $g_{_*} : E_x \longrightarrow E_y$ *,* $v \longmapsto gv$*, is a linear isomorphism.*

Example 3.7. *(1) Representations of the unit groupoid related to a smooth manifold M correspond precisely to smooth real vector bundles over M.*

(2) Let $\mathcal{G}$ *be a point groupoid with only one object, i.e.* $\mathcal{G}$ *is a Lie group* K*. The representation of* $\mathcal{G}$ *then coincides with representation of the Lie group* K *on finite dimensional real vector spaces.*

(3) The pair groupoid $\mathcal{G} = M \times M$ *over a smooth manifold* M *has both projections as source and target maps and multiplication defined in a natural way. Every representation of* $\mathcal{G}$ *on a vector bundle* $p : E \longrightarrow M$ *amounts to a natural identification of all the fibres of* E *and is thus isomorphic to a trivial representation.*

(4) Let $\mathcal{G} = K \ltimes M$ *be the action groupoid of a smooth left action of a Lie group* K *on a smooth manifold* M*. In this case the representations of the groupoid* $\mathcal{G}$ *correspond to K-equivariant vector bundles over M (see also [37]).*

(5) When $\mathcal{G} = Pair(M) = M \times M$*, then a representation of* $\mathcal{G}$ *on* E *is equivalent to a trivializtion of E. For this example, if the vector bundle E is nontrivializable, then there is not any representation of the pair groupoid* $\mathcal{G}$ *on* E *which is equivalent to a trivializtion of* E*.*

3.6 Lie Groupoid Cohomology

The (smooth) Eilenberg-MacLane cohomology of a Lie group can be generalized to the cohomology of a Lie groupoid (see [28]). In this section, we give a definition of Lie groupoid cohomology in the category of smooth manifolds **Mfd**.

Definition 3.21. *Let ($\mathcal{G} \rightrightarrows M$) be a Lie groupoid and $\pi : E \longrightarrow M$ be a vector bundle. Let $\theta : \mathcal{G} \times_M E \longrightarrow E, (g, v) \longmapsto g.v$ be a Lie groupoid action. Let $\mathcal{G}^{(n)}$ denote the manifold of all n- tuples $(g_1, ..., g_n) \in \mathcal{G}^{(n)}$ such that the product of any two successive morphisms is defined, i.e.,*

$$\mathcal{G}^{(n)} = \{(g_1, ..., g_n) \in \mathcal{G} \times \mathcal{G} \times ... \times \mathcal{G} \mid s(g_i) = t(g_{i+1}), i = 1, 2, ..., n-1\}.$$

*The **Lie groupoid cohomology** of a $\mathcal{G}$-module E is the cohomology $H^\bullet(\mathcal{G}, E)$ of the cochain complex*

$$C^0(\mathcal{G}, E) \xrightarrow{\delta^0} C^1(\mathcal{G}, E) \xrightarrow{\delta^1} C^2(\mathcal{G}, E) \xrightarrow{\delta^2} C^3(\mathcal{G}, E) \xrightarrow{\delta^3} \cdots,$$

where:

(1) For $n = 0$, let $C^0(\mathcal{G}, E)$ be the set $\Gamma(E)$ of smooth sections σ of the vector bundle $\pi : E \longrightarrow M$;

(2) for all $n \in \mathbb{N}^*$, $C^n(\mathcal{G}, E)$ is the set of smooth functions F from $\mathcal{G}^{(n)}$

to E, such that $F(g_1, ..., g_n) \in E_{t(g_1)}$, where $E_{t(g_1)}$ is the fibre at the point $t(g_1) \in M$, for all $(g_1, ..., g_n) \in \mathcal{G}^{(n)}$;

(3) for all $\sigma \in \Gamma(E)$, $\delta^0\sigma$ is the smooth function of $C^1(\mathcal{G}, E)$ defined on $\mathcal{G}$ into E by $\delta^0\sigma(g) = g \cdot \sigma(s(g)) - \sigma(t(g))$ for all $g \in \mathcal{G}^{(1)} = \mathcal{G}$;

(4) for all $n \in \mathbb{N}$ and all $F \in C^{n}(\mathcal{G}, E)$, $\delta^n F$ is the element of $C^{n+1}(\mathcal{G}, E)$ defined by

$$\delta^n F(g_0, ..., g_n) = g_0 \cdot F(g_1, ..., g_n) + \sum_{i=1}^{n} (-1)^{i} F(g_0, ..., g_i.g_{i+1}, ..., g_n) +$$
$$+ (-1)^{n+1} F(g_0, ..., g_{n-1})$$

for all $(g_0, ..., g_n) \in \mathcal{G}^{(n+1)}$.

Remark 3.9. *(i) The first cohomology group denoted by $H^1(\mathcal{G}, E)$, is given by 1-cocycles as functions F from $\mathcal{G}$ to E such that $F(g) \in E_{t(g)}$ for all $g \in \mathcal{G}$ and satisfying the* ***cocycle definition*** *$F(g_1.g_2) = g_1.F(g_2) + F(g_1)$ for all $g_1, g_2 \in \mathcal{G}$. But 1-coboundaries are E-valued functions on $\mathcal{G}$ of the form $F(g) = g \cdot \sigma(s(g)) - \sigma(t(g))$ for some smooth section $\sigma \in \Gamma(E)$.*

(ii) The space $C^{n}(\mathcal{G}, E)$ can also be defined as the space of sections of the vector bundle $t^*E \longrightarrow \mathcal{G}^{(n)}$, where $t : \mathcal{G}^{(n)} \longrightarrow M$ represents the map given by $(g_1, ..., g_n) \longmapsto t(g_1)$.

(iii) Each dynamical system (G, X, ϕ) gives rise to an action groupoid and in particular, we recover the cohomology of a dynamical system (G, X, ϕ) as a special case of this more general groupoid cohomology of the associated action groupoid with constant coefficients A (see the last section below). It is interesting to study these groupoid cohomology groups with more general coefficients in the case of dynamical systems, which will be part of future research.

Example 3.8. *(1) Let $\mathcal{G}$ be an action groupoid resulting from an action of a group G on a one-point space $\{*\}$ i.e., $\mathcal{G} = G \ltimes \{*\} \rightrightarrows \{*\}$. For $\sigma \in C^0(\mathcal{G}, E)$ we have $(\delta^0 \sigma)(g) = g.\sigma(s(g)) - \sigma(t(g))$ and so ker δ^0 is the set $\{\sigma \in \Gamma(E) \ |g.\sigma(s(g)) - \sigma(t(g)) = 0\}$ for all $g \in \mathcal{G}$ i.e., $Z^0(\mathcal{G}, E) = \Gamma_{\mathcal{G}}(E)$*

and so $H^0(\mathcal{G}, E) = \Gamma_{\mathcal{G}}(E)$ *which is invariant* $\mathcal{G}$*- functions defined on the one-point space* $\{*\}$.

(2) Suppose that $G = \{e\}$ *is the trivial group acting on the topological space* X. *Then the action groupoid* $\mathcal{G}$ *takes the form* $\mathcal{G} = \{(g, x) \ |x \in X\}$. *Then* $\mathcal{G}^n = \{(e, x), ..., (e, x) \ |x \in X\}$ *is also trivial groupoid, so* $f \in C^n(\mathcal{G}, E)$ *is determined by* $f((e, x), ..., (e, x)) = a \in E$. *Identifying* $f = a$ *we obtain* $C^n(\mathcal{G}, E) = \Gamma(E)$ *for all* $n \geq 0$. *Then, if* $f \in \Gamma(E)$, *then* $\delta = 0$ *if* n *is even, while* $\delta^n = 1$ *if* n *is odd. Therefore* $H^0(\mathcal{G}, E) = \Gamma_{\mathcal{G}}(E) = E$ *and* $H^n(\mathcal{G}, E) = 0$ *for all* $n \geq 1$.

(3) Let ($\mathcal{G} = M \rightrightarrows M$*) be the trivial groupoid of a smooth manifold* M, *which has a Lie groupoid structure. Since* $M^{(p)} \cong M$, *then any smooth vector bundle* $E \longrightarrow M$ *is a representation for the manifold* M. *Therefore* $C^n(\mathcal{G}, E)$ *can be identified with* $\Gamma^\infty(E)$, *the space of smooth sections of the vector bundle* $E \longrightarrow M$. *Now one can compute* $\delta^i = 0$ *if* i *is even and* $\delta^i = id_E$ *if* i *is odd. Then* $H^0(\mathcal{G}, E) = \Gamma^\infty(E)$ *and* $H^n(\mathcal{G}, E) = 0$ *for* $n \in \mathbb{N}$.

3.7 Cocycles of Dynamical Systems and Lie Groupoid Cohomology

The aim of this section is to represent cocycles of a dynamical system as action groupoid cohomology classes and to understand how cohomology classes of an action groupoid associated to a dynamical system appear as cocycles of the given dynamical system. This follows some suggestions due to Tao and related comments by Kim [38] to study dynamical systems from a purely homological algebra point of view. As before, we will work in the discrete or topological situation, i.e. in the topological framework all group actions and maps involved need to be assumed to be continuous.

Let (G, X, ϕ) be a dynamical system and A an abelian (topological) group. The function $\rho : G \times X \longrightarrow A$ is a cocycle of G with values in A if the following condition is satisfied: $\rho(gh, x) = \rho(h, x) + \rho(g, hx)$ for all $g, h \in G$ and $x \in X$.

Also, out of this data we can construct an associated action groupoid $\mathcal{G} = G \ltimes X \rightrightarrows X$, i.e. $\mathcal{G} = \{(g, x) : g \in G, x \in X\}$, where $s((g, x)) = x$ and $t((g, x)) = \phi(g, x) = gx$.

Firstly, let us write the cocycle function as an action groupoid cohomology class. We expect that this class should be in the first cohomology group of the action groupoid. The cohomology class of the action groupoid is denoted by $[c] \in H^1(\mathcal{G}, A)$.

As we know, $\mathcal{G}^2 = \mathcal{G} \times \mathcal{G} = \{(\gamma_1, \gamma_2) : t(\gamma_1) = s(\gamma_2)\}$ where $\gamma_1 = (g, x)$ and $\gamma_2 = (h, y)$, $y = \phi(g, x) = gx = t((g, x))$.

The cocycle function of the dynamical system $\rho(gh, x) = \rho(h, x) + \rho(g, hx)$ can be written as a groupoid cohomology class using the function $F : \mathcal{G} \longrightarrow A$, which satisfies the cocycle condition $F(\gamma_1 \cdot \gamma_2) = F(\gamma_1) + \gamma_1 \cdot F(\gamma_2)$ added to $\delta^0\sigma(\gamma) = \gamma \cdot \sigma(s(\gamma)) - \sigma(t(\gamma))$.

This means the following:

$$\begin{aligned}\rho(gh, x) = \rho(h, x) + \rho(g, hx) &= F(m((g, x), (h, y))) + \delta^0\sigma((g, x)) \\ &= F(gh, x) + \gamma_* \cdot \sigma(x) - \sigma(gx).\end{aligned}$$

Secondly, if we have a cohomology class of the action groupoid associated to a dynamical system, we can also interpret it as a cohomology class being a cocycle of the dynamical system in the following way:

$$c = F((g,x),(h,gx)) + \delta^0\sigma((g,x)) = F(gh,x) + \sigma(x) - \sigma(gx)$$
$$= \rho(gh,x) + \rho(g,x).$$

We end this thesis with a theorem which written without details in [38] concerning the interpretation of the higher cohomology groups $H^n(G,X;A)$ of a given dynamical system (G,X,ϕ). We will give details of this problem and will restrict ourselves to the two-dimensional case i.e. we will be looking for an interpretation of $H^2(G,X;V)$ here. It is a suggestion of Kim in regard to Tao comments in [38] to study the role of this second cohomology group in detail. This group is directly related to the problem of extending a dynamical system and will be interpreted here in homological terms.

Theorem 3.1. *Let (G,X,ϕ) be a dynamical system and A an abelian (topological) group. If for the second cohomology group $H^2(G,X,V) = 0$, then it will be an obstruction group to extending a dynamical system and in general we have a long exact sequence in cohomology:*

$$\cdots \to H^1(G,X;V) \to H^1(G,X;\tilde{A}) \to H^1(G,X;A) \to H^2(G,X;V) \to \cdots$$

Proof. Suppose we have given a short exact sequence of abelian groups

$$0 \to V \xrightarrow{\iota} \tilde{A} \xrightarrow{\pi} A \to 0 \ .$$

By the definition of a group extension, one can view $\tilde{A}$ as the set of ordered pairs $\{(a,v) \mid a \in A, v \in V\}$ equipped with an additive group law:

$$(a,v) \oplus (a',v') := (a+a', v+v' + B(a,a')) \qquad (2)$$

for some function $B : A \times A \to V$ (see [35]).

According to Theorem 21, p.802, [12] we will get a natural long exact sequence in cohomology of the form:

$$\cdots \to H^1(G,X;V) \to H^1(G,X;\tilde{A}) \to H^1(G,X;A) \to H^2(G,X;V) \to \cdots \quad (3)$$

Thus $H^2(G,X;V)$ is an obstruction group for detecting if an A-extension of a given dynamical system (G,X,ϕ) can be lifted to an $\tilde{A}$-extension.

We will deduce this in several steps by constructing the maps involved: The second map in (3) from $H^1(G,X;\tilde{A})$ to $H^1(G,X;A)$ is obvious where the projection map π in the short exact sequence from $\tilde{A}$ to A induces a projection group homomorphism between their first cohomology groups. That is, it sends each cohomology class $\tilde{\rho} + B(G,X;\tilde{A}) \in H^1(G,X,\tilde{A})$ to the cohomology class $\rho + B(G,X;A) \in H^1(G,X;A)$, where $\tilde{\rho} : G \times X \longrightarrow \tilde{A}$ is an 1-cocycle and thus the group homomorphism transverses the first cohomology group $H^1(G,X;\tilde{A})$ to $H^1(G,X;A)$.

The third map in (3) is given as follows: Suppose we have given a cohomology class which is represented by $\rho : G \times X \to A \in H^1(G,X,A)$ and want to lift it to a cohomology class which is represented by $\tilde{\rho} : G \times X \to \tilde{A} \in H^1(G,X,\tilde{A})$ by using the projection map from $H^1(G,X,\tilde{A})$ to $H^1(G,X,A)$. Now by writing elements of $H^1(G,X,\tilde{A})$, as a direct sum from $H^1(G,X,A)$ and $H^1(G,X,V)$ then we can write $\tilde{\rho} = (\rho,\sigma)$ for some 1-cocycle $\sigma : G\times X \to V$. By using the cocycle equation $\rho(gh,x) = \rho(h,x)+\rho(g,hx)$ for all $g,h \in G$, $x \in X$ and the additive group law in (2), we get

$$(\rho,\sigma)(gh,x) = (\rho,\sigma)(h,x) + (\rho,\sigma)(g,hx)$$

$$= (\rho(h,x) + \rho(g,hx), \sigma(h,x) + \sigma(g,hx) + B(\rho(h,x) + \rho(g,hx)).$$

This means we need to find a 1-cocycle σ such that the following holds:

$$\sigma(gh,x) = \sigma(h,x) + \sigma(g,hx) + B(\rho(h,x), \rho(g,hx)).$$

We can write this as follows:

$$\sigma(gh,x) - \sigma(h,x) - \sigma(g,hx) = B(\rho(h,x), \rho(g,hx))$$

The left side gives a 2-cocycle in $H^2(G,X,V)$ and implies that there is a map defined as $\Phi : H^1(G,X,A) \longrightarrow H^2(G,X,V)$ such that Φ sends the cohomology class $[\rho]$ representing the 1-cocycle $\rho : G \times X \to A$ to the cohomology class in $H^2(G,X,V)$. $B(\rho(h,x) + \rho(g,hx))$ is its coboundary and $\Phi(\rho) : (g,h,x) \mapsto B(\rho(h,x), \rho(g,hx))$ is a V-valued 2-coboundary. If we suppose that σ is a trivial cocycle, then there is a map sending $(g,h,x) \mapsto (0, \Phi(\rho))$, which gives a $\tilde{A}$-valued 2-coboundary. But every coboundary is a cocycle, so it is a $\tilde{A}$-valued 2-cocycle, and $\Phi(\rho)$ is a V-valued 2-cocycle. This means the map $\rho \mapsto \Phi(\rho)$ maps 1-cocycles $\rho : G \times X \to A$ to 2-cocycles $\Phi(\rho) : G \times G \times X \to V$. Analogously, if we are given two 1-cocycles $\rho, \rho' : G \times X \to A$, it follows that $(\rho + \rho', 0)$ differs from $(\rho, 0) + (\rho,' 0)$ by some V-valued 1-cochain which is $B(\rho, \rho')$ and taking derivatives, we see that $\Phi(\rho + \rho')$ differs from $\Phi(\rho) + \Phi(\rho')$ by some 2-coboundary. Therefore, it follows that Φ is a linear map modulo 2-coboundaries and in fact a group homomorphism.

Finally, as we know that each cohomology class is a cocycle modulo a coboundary, we will take ρ as an A-valued 1-coboundary, then depending on assuming that cocycles in $H^1(G,X,\tilde{A})$ are ordered pairs as defined before, thus $(\rho, 0)$ is the sum of an $\tilde{A}$-valued 1-coboundary and a V-valued 1-cochain, and so taking derivatives again, we see that Φ maps 1-coboundaries from $H^1(G,X,\tilde{A})$ to 2-coboundaries in $H^2(G,X,V)$. Therefore we finally get an induced map from $H^1(G,X;A)$ to $H^2(G,X;V)$, and therefore the sequence (3) is a short exact sequence. $\square$

It is an interesting question to ask for an interpretation also for the higher cohomology groups $H^n(G, X; V)$ and $H^n(G, X; A)$, which we plan to look into in the future. It looks possible to develop a whole theory of cocycle extensions for dynamical systems (G, X, ϕ) using methods from homological algebra and higher category theory. It can be expected that also an adequate version of Hochschild cohomology will enter the game, which is a natural tool to study extensions and deformations of this type.

Bibliography

[1] Adhikari, M. R., Basic Algebraic Topology and its Applications. Springer, 2016.

[2] Bak, A., Brown, R., Minian, G. and Porter, T., Global Actions, Groupoid Atlases and Applications, Journal of Homotopy and Related Structures, vol. 1(1), (2006):101-167.

[3] Bell, J., The Gottschalk-Hedlund theorem, cocycles, and small divisors, Lecture Notes, University of Toronto, 2014.

[4] Biswas, I. and Neumann, F., Atiyah sequences, connections and characteristic forms for principal bundles over groupoids and stacks. C. R. Acad. Sci. Sér. I Math. Paris 352 (2015), 59-64.

[5] Bos, R., Lecture Notes on Groupoid Cohomology, Intituto Superior Técnico, Lisbon, 2007.
www.math.tecnico.ulisboa.pt/rbos/GpdCoh.pdf

[6] Brandt, H., Über eine Verallgemeinerung des Gruppenbegriffes, Mathematische Annalen 96, no.1 (1927): 360–366.

[7] Brown, R., From groups to groupoids: a brief survey, Bulletin of the London Mathematical Society 19, no.2 (1987): 113–134.

[8] Cabrera, A., del Hoyo, M., and Pujals, E., Discrete dynamics and differentiable stacks, arXiv preprint arXiv:1804.00220 (2018)

[9] Connes, A., Noncommutative Geometry, Academic Press, San Diego, 1994.

[10] Daenzer, C., A groupoid approach to noncommutative T-duality. Communications in Mathematical Physics 288, no.1 (2009): 55-96.

[11] Daenzer, C., Noncommutative Geometry and T-duality, Lecture Notes, Mathematisches Insititut, Göttingen

[12] Dummit, D. S., Foote, R. M. Abstract Algebra (Vol.3). Hoboken: Wiley. 2004.

[13] Ehresmann, C., Oeuvres complètes et commentées, (A. C. Ehresmann, ed.) Suppl. Cahiers Top. Géom. Diff., Amiens, 1980-1984.

[14] Einsiedler, M. and Ward, T., Ergodic theory with a view towards Number Theory, Graduate Texts in Mathematics, Vol. 259, Springer, 2010.

[15] Grothendieck, A., Techniques de construction et théorèmes d?existence en géométrie algébrique III: préschemas quotients, Séminaire Bourbaki 13e année 1960/61, no. 212 (1961).

[16] Hasselblatt, B. and Katok, A., Principal structures. In: Handbook of Dynamical Systems, vol. 1. Elsevier Science, 2002: 1-203.

[17] Irwin, M. C., Smooth Dynamical Systems. Advanced Series in Nonlinear Dynamics: Vol. 17. World Scientific, 2001.

[18] Kalisnik, J., Representations of orbifold groupoids, Topology and its Applications 155, no. 11 (2008): 1175-1188.

[19] Kalisnik, J., Groupoid representations and modules over the convolution algebras, arXiv preprint arXiv:0806.1832 (2008).

[20] Katok, A. and Hasselblatt B., Introduction to the Modern Theory of Dynamical Systems. Encyclopedia of Mathematics and its Applications Vol. 54. Cambridge University Press, 1997.

[21] Katok, A. and Nitica, V., Rigidity in Higher Rank Abelian Group Actions. Volume 1. Introduction and Cocycle Problem, Cambridge Tracts in Mathematics vol. 185., Cambridge University Press, 2011.

[22] Kerr, D., and Li, H. Ergodic theory. Independence and Dichotomies, Springer Monographs in Mathematics, Springer Verlag, 2016.

[23] Laurent-Gengoux, C. and Voglaire Y., Invariant connections and PBW theorem for Lie groupoid pairs, arXiv preprint arXiv:1507.01051 (2015).

[24] Lee, J. M., Introduction to Topological Manifolds, Graduate Texts in Mathematics, Vol. 202, Second Edition, Springer, 2010.

[25] Lee, J. M., Introduction to Smooth Manifolds, Graduate Texts in Mathematics, Vol. 218. Second Edition, Springer, 2012.

[26] MacLane, S., Categories for the Working Mathematician, Graduate Texts in Mathematics, Vol. 5, Second Edition, Springer, 1998.

[27] Mackenzie, K. CH., General theory of Lie Groupoids and Lie Algebroids. LMS Lecture Note Series, Vol. 213, Cambridge University Press, 2005.

[28] Mehta, R. A., Modular classes of Lie groupoid representations up to homotopy, arXiv preprint arXiv:1502.06253 (2015).

[29] Moerdijk, I., and Mrcun, J., Introduction to Foliations and Lie Groupoids, Cambridge Studies in Advanced Mathematics, Vol. 91, Cambridge University Press, 2003.

[30] Neumann, F., Algebraic Stacks and Moduli of Vector Bundles. Publicaçoes Matematicas, Research Monograph, IMPA Rio de Janeiro 2009 (2nd ed. 2011).

[31] Pradines, J., Troisième théorème de Lie sur les groupoides différentiables, C. R. Acad. Sci. Sér. I Math. Paris 267 (1968), 21-23.

[32] Pronk, D. A., Etendues and stacks as bicategories of fractions, Compositio Mathematica 102, no. 3 (1996): 243-304.

[33] Rotman, J. J., An introduction to Algebraic Topology, Graduate Texts in Mathematics, Vol. 119, Springer, 2010.

[34] Rotman, J. J., An Introduction to the Theory of Groups, Graduate Texts in Mathematics, Vol. 148, Springer, 2010.

[35] Rotman, J. J., An Introduction to Homological Algebra, Universitext, Second Edition, Springer, 2008.

[36] Ruelle, D., Ergodic theory of differentiable dynamic systems. IHES Publ. Math. 50 (1) (1979): 27-58.

[37] Segal, G., Equivariant K-theory, IHES Publ. Math. 34, (1) (1968): 129-151.

[38] Tao, T., What's New, Web Blog, https://terrytao.wordpress.com/

[39] van der Laan, P., Smooth Groupoids, Lecture Notes, University of Utrecht Lecture Notes, based on lectures by I. Moerdijk and J. Mrcun, University of Utrecht, 2000.

[40] Weinstein, A., Groupoids: Unifying Internal and External Symmetry-A Tour through Some Examples, Notices of the AMS, Vol. 43 (7), 1996.

[41] Wagemann, F., Wockel, C., A cocycle model for topological and Lie group cohomology. Trans. Amer. Math. Soc. 367 (2015), no. 3, 1871 - 1909.