

Discrete and Continuous Ito-Malliavin Type Calculus  
with Illustration in Finance

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*This work is dedicated to my family, my friends and my supervisor professor Utev for always supporting me in whatever dreams I pursue.*

# Abstract

As reliable mathematical methods for finance, various concepts of the stochastic calculus are discussed in detail in this thesis such as the Ito integral, the (continuous and discrete) Malliavin calculus and the Stratonovich integral. The derivative of a natural number and the quantum calculus are also illustrated in this thesis.

The Stroock lemma and the duality formula are two methods when the Malliavin calculus is applied to calculate the preceding quantities. To extend the range of application of these rules is a crucial purpose of this thesis. Solving certain equations based on the Ito integral and the Malliavin calculus has also been introduced and analysed in this thesis. This equation, which is also a kind of stochastic differential equation, can be treated as an inverse application of the Malliavin derivative. Finally, the product rule for other derivative operators is extensively introduced and analysed throughout the whole thesis, since this rule in the stochastic calculus or the quantum calculus is sometimes different from the traditional infinitesimal calculus.

To explore the idea of differential dynamics with the non-standard and new types of differentiation, the differential operators discussed and introduced

in this thesis, such as the continuous and the discrete Malliavin derivative operator and the  $q$ -derivation operator are applied as transforms on some state spaces, such as measurable space and the space constructed by the finite fields.

# Chapter 1

## Introduction

This thesis consists of three main parts. The three parts are loosely connected. The two main joint points in all modelling and analysis are trying to identify conditions when the Lie-bracket or the Stroock type relationship and the duality property are satisfied. This relationship can be simply represented by

$$D\delta = I + \delta D,$$

or equivalently,

$$[D, \delta] = I,$$

where  $[\cdot, \cdot]$  is the Lie bracket.

From the pure mathematics point of view, both parts of this thesis are interesting as mathematical questions. Bernt Oksendal [38] [40] [41] and David Nualart [34] [36] are partly treated some stochastic differential equations which are defined as the Ito-Malliavin type equations in this thesis. The  $q$ -derivative,

or Jackson derivative, which first introduced by Frank Hilton Jackson [20] has been widely applied in number theory such as the sums of two and of four squares, the sums of two and of four triangular numbers [21]. It is interesting to investigate how to apply  $q$ -derivation operator on the finite field.

From the applied mathematics point of view, both the  $q$ -derivation operators and the solutions to the Ito-Malliavin type equation give another way to the financial modelling. These notions lead to the non-standard financial modelling. There are more details in the paper written by X. Ma et al. [32], the paper written by Rukiye Samci Karadeniz et al. [22] and the Ph.D. thesis of Wenyan Hao [16].

The inspiration of the Ito-Malliavin type equation comes from the application of the Stroock lemma ( or called "a fundamental theorem of calculus" by Giulia Di Nunno et al. [38] ) in the Malliavin calculus and some Gaussian processes such as Brownian bridge. The adapted solutions of all three Ito-Malliavin type equations illustrated in this thesis have been given.

The Stroock lemma is an important notion in the stochastic calculus. Three different approaches of generalization of this lemma such as the discrete Malliavin calculus, the traditional derivative and the arithmetic derivative are given in the second part. Here in this part a generalised duality formula via a bilinear map is also given in the first part.

In general, the  $q$ -derivation operator and the  $q$ -type derivation operator introduced in this thesis are non-commutative. These operators, different from the semi-derivation operators and the derivation operators ( see Chapter 10 ), satisfy the  $q$ -product rule. In this thesis, these operators are applied to a vector space of  $\mathbb{F}_p$ -valued functions where  $\mathbb{F}_p$  is a finite field. It gives another way to construct the famous Cox-Ross-Rubinstein model.

## Main structure of the thesis

Before the main contents, all commonly used notations are listed in Chapter 2 for convenience.

- The first part of this thesis is certain equations based on the Ito integral and the Malliavin calculus which are simply called the Ito-Malliavin type equations in this thesis. These equations can be simply represented as

$$D_t X_u = F(t, W_t, u, W_u).$$

Furthermore, the Stratonovich-Malliavin type equations are also introduced and analysed in this part.

All terminologies and methods can be found in Chapter 3. Several simple lemmas and examples are illustrated in Chapter 4. The specific definitions and methods of preceding equations are shown in Chapter 5.

- The second part of this thesis is about the general Stroock lemma. To find different forms of this lemma, several concepts, such as the discrete Malliavin calculus and the derivative of a natural number are introduced in this part.

All terminologies and methods can be found in Chapter 6. Several simple lemmas and examples are illustrated in Chapter 7. Various topics about the general Stroock lemma are shown in Chapter 8.

- The third part of this thesis is a discrete differential dynamics. The  $q$ -derivation operator is applied to  $\mathbb{F}_p^n$ -valued vectors to investigate how differential dynamics works.

All terminologies and methods can be found in Chapter 9. Several simple lemmas and examples are illustrated in Chapter 10. The discrete differential dynamics of  $q$ -derivation and the characterization of  $q$ -derivation are shown in Chapter 11. The application of the  $q$ -derivation operator to Cox-Ross-Rubinstein model is demonstrated in Chapter 12.

Results: There are too many to state. Chapter 4, 5, 7, 8, 10, 11 and 12 are based on my results.

# Chapter 2

## Notations

In this chapter, some commonly used notations of the whole thesis are illustrated here.

$\{X_t : t \geq 0\}$ ,  $X_t$ ,  $X_{t,z}$ : stochastic process

$\{X_t^{(m)} : t \geq 0\}$ ,  $X_t^{(m)}$ : a sequence of stochastic processes

$\{W_t : t \geq 0\}$ ,  $W_t$ : Wiener process

$\{B_t : t \geq 0\}$ ,  $B_t$ : standard Brownian motion

$\{\mathcal{F}_t : t \geq 0\}$ ,  $\mathcal{F}_t$ : filtration

$\Delta$ ,  $\tilde{\Delta}$ : delta operator, shift-equivariant linear operator

$L^2([0, T]^n)$ : standard space of square integrable Borel real functions on  $[0, T]^n$

$\tilde{L}^2([0, T]^n)$ : standard space of symmetric square integrable Borel real functions on  $[0, T]^n$

$I_n$ :  $n$ -fold iterated Ito integrals

$L^2(P)$ : space of square integrable random variables

$(g, h)_{L^2([0, T]^n)}$ : the inner product of  $L^2([0, T]^n)$

$f_n(t_1, \dots, t_n, t) = f_{n,t}(t_1, \dots, t_n) = f_n(\cdot, t), n = 1, 2, \dots$ : functions of  $n + 1$  variables, that is  $(t_1, \dots, t_n) \in [0, T]^n$  and the parameter  $t \in [0, T]$

$\tilde{f}_n(t_1, \dots, t_n, t_{n+1}) = \tilde{f}_n, n = 1, 2, \dots$ : symmetric functions derived from  $f_n(\cdot, t), n = 1, 2, \dots$

$\delta(u)$ : Skorohod integral of the stochastic process  $u_t$

$D_t$ : Malliavin derivative at time  $t$

$\mathbb{D}_{1,2}$ : subspace of  $L^2(P)$  of which function  $F$  satisfies  $\|F\|_{\mathbb{D}_{1,2}}^2 = \sum_{n=1}^{\infty} nn! \|f_n\|_{L^2([0, T]^n)}^2 < \infty$

$\mathbb{D}_{1,2}^0$ : the set of all  $F \in L^2(P)$  whose chaos expansion has only finitely many terms

$G(t, X_t)$ : function of the process  $X_t$  and the time  $t$

$\int_0^T G(t, X_t) \circ dW_t$ : Stratonovich integral of the function  $G(t, X_t)$

$H_n(\lambda, x)$ : Hermite polynomial of degree  $n$  and parameter  $\lambda > 0$

$(\Omega, \mathcal{F}, P)$ : a complete probability space

$\mathbb{R}$ : the set of real numbers

$\mathbb{R}_0$ :  $\mathbb{R} \setminus \{0\}$

$\mathbb{N}$ : the set of natural numbers

$\Lambda$ : discrete time set

$\omega$ : Bernoulli random variable

$\Omega$ : set of  $\omega$

$\mathfrak{L}^2(\Omega, P)$ : discrete version of the Wiener space with respect to the uniform probability measure  $P$

$\mathfrak{B}, \mathfrak{B}_t$ : discrete Brownian motion

$\mathfrak{X}_s, (\mathfrak{X}_s)_{s \in \Lambda}$ : discrete stochastic process

$\mathfrak{X}_n(t_1, \dots, t_n)$ : symmetric function  $\mathfrak{X}_n$  on  $\Lambda^n$

$\mathfrak{X}_{n,s}(t_1, \dots, t_n; s)$ : coefficient function with respect to the process variable  $s$

$\mathbf{B}$ : bilinear map

$\mathbb{F}_p$ : finite field with  $p$  elements

$\mathbb{F}_{p^n}$ : finite field with  $p^n$  elements and characteristic  $p$

$D_q$ :  $q$ -derivation operator

$\mathcal{A}_{n,p} = \{(f_1, \dots, f_n)^T : f_i \in \mathbb{F}_p\}$ : a vector space of  $\mathbb{F}_p$ -valued functions

$O_{\mathcal{A}_{n,p} \rightarrow \mathcal{A}_{n,p}}$ : a class of matrices from  $\mathcal{A}_{n,p}$  to  $\mathcal{A}_{n,p}$

## Part 1 :

# Ito-Malliavin Equations

Main definitions from the Ito integral, the Stratonovich integral and the Malliavin calculus are introduced in Chapter 3. Chapter 4 contains several examples and lemmas on the Ito integral, the Stratonovich integral and Malliavin calculus. The Ito-Malliavin type equations, the Stratonovich-Malliavin type equations and related main results are in Chapter 5 based on [9].

# Chapter 3

## Terminology and Methods

In this chapter, all crucial notations and methods are introduced in details including definitions and properties. These concepts will be applied throughout part 1. Several lemmas with proofs are also given here.

### 3.1 Ito integral

Ito stochastic integral is a stochastic generalization of the RiemannStieltjes integral. This notion has a lot of applications in financial mathematics and stochastic differential equations.

This section follows Rogers Chris et al. [5] and Revuz Daniel et al. [8].

#### 3.1.1 Definition

Let  $\{t_0, t_1, \dots, t_k\}$  be a partition of the interval  $[0, T]$ , that is,

$$0 = t_0 < t_1 < \dots < t_k = T.$$

Assume that an adapted stochastic process  $\{X_t : t \geq 0\}$  is constant in  $t$  on each subinterval  $[t_i, t_{i+1}), i = 0, 1, \dots, k-1$ , such that for each  $t \in [0, T]$ ,  $E_p [(X_t)^2] < +\infty$ . This kind of process is often called a simple process. Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  be a filtered probability space. The Ito integral can be defined as

$$\int_0^t X_s dW_s, t \in [0, T],$$

where  $\{W_t : t \geq 0\}$  is an adapted Wiener process with respect to a natural filtration  $\{\mathcal{F}_t : t \geq 0\}$  which is a sub- $\sigma$ -algebra of  $\mathcal{F}$ . Then, for  $t \in [t_i, t_{i+1}]$ ,

$$\begin{aligned} I(t) &= \int_0^t X_s dW_s \\ &= \sum_{j=0}^{i-1} X_{t_j} (W_{t_{j+1}} - W_{t_j}) + X_{t_i} (W_t - W_{t_i}). \end{aligned}$$

Then, consequently, the definition of the Ito integral is given as follow.

**Definition 3.1.1.** (*Ito integral*)

Suppose  $\{W_t : t \geq 0\}$  is a Wiener process and  $\{X_t : t \geq 0\}$  is an adapted stochastic process with respect to a filtration  $\{\mathcal{F}_t : t \geq 0\}$ . Then the Ito integral is defined as

$$\int_0^T X_t dW_t = \lim \sum_i X_{t_i} (W_{t_{i+1}} - W_{t_i}), i = 0, 1, \dots, k-1,$$

where the symbol of limit represents that the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  of  $[0, T]$  tends to 0.

### 3.1.2 Properties

Since the definition of Ito integral is given, some important properties which will be used in this thesis are illustrated in this section.

As a simple application of discretization method, the Ito isometry is proved in details.

**Lemma 3.1.1.** (*Ito isometry*)

*The Ito integral satisfies*

$$E \left[ \left| \int_0^t X_s dW_s \right|^2 \right] = E \left[ \int_0^t |X_s|^2 ds \right].$$

**Lemma 3.1.2.** (*Convergence*)

Let  $\{X_t^{(m)} : t \geq 0, m = 1, 2, \dots\}$  be a sequence of adapted stochastic processes with respect to a filtration  $\{\mathcal{F}_t : t \geq 0\}$  which satisfies

$$X_t^{(m)} \rightarrow X_t$$

in  $L^2(P)$  which is a space of square integrable random variables. Then,

$$\int_0^T X_s^{(m)} dW_s \rightarrow \int_0^T X_s dW_s$$

in  $L^2(P)$  with respect to  $m \rightarrow \infty$ .

*Proof.*

For each  $m$ ,

$$E \left[ \left( \int_0^T X_s^{(m)} dW_s - \int_0^T X_s dW_s \right)^2 \right] = E \left[ \left( \int_0^T (X_s^{(m)} - X_s) dW_s \right)^2 \right].$$

Through Ito isometry ( Lemma 3.1.1 ),

$$E \left[ \left( \int_0^T (X_s^{(m)} - X_s) dW_s \right)^2 \right] = E \left[ \int_0^T (X_s^{(m)} - X_s)^2 ds \right].$$

From the condition that  $X_t^{(m)} \rightarrow X_t$  in  $L^2(P)$ ,

$$\lim_{m \rightarrow \infty} \|X_t^{(m)} - X_t\|^2 = 0.$$

Therefore,

$$E \left[ \int_0^T (X_s^{(m)} - X_s)^2 ds \right] \rightarrow 0$$

in  $L^2(P)$  with respect to  $m \rightarrow \infty$ . This completes the proof.  $\square$

### 3.1.3 Ito process

**Definition 3.1.2.** (*Ito process*)

Let  $\{\mathcal{F}_t : t \geq 0\}$  be a filtration and let  $\{W_t : t \geq 0\}$  be an adapted Wiener process. An Ito process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s,$$

where  $X_0$  is deterministic and  $\{\mu_t : t \geq 0\}$  and  $\{\sigma_t : t \geq 0\}$  are adapted stochastic processes, which satisfy

$$E \left[ \int_0^t |\mu_s|^2 ds \right] < +\infty$$

and

$$E \left[ \int_0^t |\sigma_s|^2 ds \right] < +\infty.$$

The former definition is the integral form of the Ito process. The differential form of the Ito process is

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

**Lemma 3.1.3.** (*Uniqueness*)

Suppose  $\{X_u : u \geq 0\}$  and  $\{Y_u : u \geq 0\}$  are two Ito processes defined in Definition 3.1.2 which can be represented as

$$dX_u = \mu_u du + \sigma_u dW_u$$

and

$$dY_u = \bar{\mu}_u du + \bar{\sigma}_u dW_u.$$

If these two processes satisfy

$$X_t + \int_t^u \mu_s ds + \int_t^u \sigma_s dW_s = Y_t + \int_t^u \bar{\mu}_s ds + \int_t^u \bar{\sigma}_s dW_s,$$

then  $\mu_s = \bar{\mu}_s$  and  $\sigma_s = \bar{\sigma}_s$  almost surely, and therefore  $X_t = Y_t$  almost surely.

## 3.2 Wiener-Ito chaos expansion

This section follows Giulia Di Nunno et al. [38].

Let  $L^2([0, T]^n)$  be the standard space of square integrable Borel real functions on  $[0, T]^n$  such that

$$\|g\|_{L^2([0, T]^n)}^2 = \int_{[0, T]^n} g^2(t_1, \dots, t_n) dt_1 \cdots dt_n < \infty.$$

Let  $\tilde{L}^2([0, T]^n) \subset L^2([0, T]^n)$  be the space of symmetric square integrable Borel real functions on  $[0, T]^n$ .

**Definition 3.2.1.** (*n-fold iterated Ito integrals*)

If  $g \in \tilde{L}^2([0, T]^n)$ , the *n-fold iterated Ito integrals* can be defined as

$$\begin{aligned} I_n(g) &= \int_{[0, T]^n} g(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n}. \end{aligned}$$

The function  $g$  in Definition 3.2.1 is defined to be symmetric because it is convenient to explain the Wiener-Ito chaos and the Malliavin calculus through symmetric expansion, as opposed to the time-ordered expansion. Through the definition of the *n-fold iterated Ito integrals*  $I_n$ , the following Wiener-Ito chaos expansion is given here as a theorem.

**Lemma 3.2.1.** (*The Wiener-Ito chaos expansion*)

Let  $\xi$  be an  $\mathcal{F}_t$  - measurable random variable in  $L^2(P)$ , that is the space of square integrable random variables. Then there exists a unique sequence  $\{f_n : n = 1, 2, \dots, \infty\}$  of functions  $f_n \in \tilde{L}^2([0, T]^n)$  such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in  $L^2(P)$ . Moreover, the isometry can be written as

$$\|\xi\|_{L^2(P)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2([0, T]^n)}^2.$$

The proof of Lemma 3.2.1 can be found in section 1.3 of Chapter 1 in the book "Malliavin Calculus for Levy Processes with Applications to Finance" written by Giulia Di Nunno et al. [38].

If  $g \in \tilde{L}^2([0, T]^m)$  and  $h \in \tilde{L}^2([0, T]^n)$ , remark that the following relations always hold:

$$E[I_m(g) I_n(h)] = \begin{cases} 0, & n \neq m \\ (g, h)_{L^2([0, T]^n)}, & n = m \end{cases} \quad (m, n = 1, 2, \dots),$$

where

$$\begin{aligned} (g, h)_{L^2([0, T]^n)} &= \int_{L^2([0, T]^n)} g(t_1, \dots, t_n) h(t_1, \dots, t_n) dt_1 \cdots dt_n \\ &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1, \dots, t_n) h(t_1, \dots, t_n) dt_1 \cdots dt_n \end{aligned}$$

is the inner product of  $L^2([0, T]^n)$ .

### 3.3 Skorohod integral

The Skorohod integral was introduced for the first time by A. Skorohod in 1975. This stochastic integral is connected to the Malliavin calculus which will be introduced in the next part.

This section follows Giulia Di Nunno et al. [38].

#### 3.3.1 Definition

Let  $f_{n,t} = f_{n,t}(t_1, \dots, t_n)$ ,  $(t_1, \dots, t_n) \in [0, T]^n$ ,  $n = 1, 2, \dots$  be the symmetric functions and  $f_{n,t} \in \tilde{L}^2([0, T]^n)$ . Since the functions  $f_{n,t}$ ,  $n = 1, 2, \dots$  depend on the parameter  $t \in [0, T]$ ,

$$f_n(t_1, \dots, t_n, t_{n+1}) = f_n(t_1, \dots, t_n, t) = f_{n,t}(t_1, \dots, t_n)$$

and  $f_n$  can be regarded as a function of  $n + 1$  variables. The symmetrization  $\tilde{f}_n$  of  $f_n$  is given by

$$\begin{aligned} \tilde{f}_n(t_1, \dots, t_n, t_{n+1}) &= \frac{1}{n+1} [f_n(t_1, \dots, t_n, t_{n+1}) \\ &\quad + f_n(t_2, \dots, t_{n+1}, t_1) + \dots + f_n(t_1, \dots, t_{n-1}, t_{n+1}, t_n)]. \end{aligned}$$

**Definition 3.3.1.** (*The Skorohod integral*)

Let  $u_t$ ,  $t \in [0, T]$ , be a measurable stochastic process such that for all  $t \in [0, T]$  the random variable  $u_t$  is  $\mathcal{F}_T$ -measurable and  $E \left[ \int_0^T u_t^2 dt \right] < \infty$ . Let its Wiener-Ito chaos expansion be

$$u_t = \sum_{n=0}^{\infty} I_n (f_{n,t}) = \sum_{n=0}^{\infty} I_n (f_n (\cdot, t)).$$

Then the Skorohod integral of  $u$  can be defined by

$$\delta (u) = \int_0^T u_t \delta W_t = \sum_{n=0}^{\infty} I_{n+1} (\tilde{f}_n)$$

when convergent in  $L^2 (P)$ . Here  $\tilde{f}_n$ ,  $n = 1, 2, \dots$ , are the symmetric functions derived from  $f_n (\cdot, t)$ ,  $n = 1, 2, \dots$ .  $u$  is Skorohod integrable if  $\sum_{n=0}^{\infty} I_{n+1} (\tilde{f}_n)$  converges in  $L^2 (P)$ , which can be represented as  $u \in \text{Dom} (\delta)$ .

By isometry in Lemma 3.2.1 a stochastic process  $u$  belongs to  $\text{Dom} (\delta)$  if and only if

$$E [\delta (u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L^2([0,T]^{n+1})}^2 < \infty.$$

### 3.3.2 The connection between the Skorohod integral and the Ito integral

The Skorohod integral is an extension of Ito integral. The two integrals are the same elements of  $L^2 (P)$  if the integrand  $u$  is  $\mathcal{F}$  adapted. This will be proved in this section.

**Lemma 3.3.1.** *Let  $u = u_t$ ,  $t \in [0, T]$ , be a measurable stochastic process such that, for all  $t \in [0, T]$ , the random variable  $u_t$  is  $\mathcal{F}_T$ -measurable and  $E [u_t^2] < \infty$ . Let*

$$u_t = \sum_{n=0}^{\infty} I_n (f_n (\cdot, t))$$

be its Wiener-Ito chaos expansion. Then  $u$  is  $\mathcal{F}$  adapted if and only if

$$f_n (t_1, \dots, t_n, t) = 0$$

if  $t < \max_{1 \leq i \leq n} t_i$ . The above equality is meant almost everywhere in  $[0, T]^n$  with respect to Lebesgue measure.

*Proof.*

First note that for any  $g \in \tilde{L}^2 ([0, T]^n)$ ,

$$\begin{aligned} E [I_n (g) | \mathcal{F}_t] &= E \left[ \int_{[0, T]^n} g (t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} | \mathcal{F}_t \right] \\ &= n! E \left[ \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g (t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} | \mathcal{F}_t \right] \\ &= n! \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} g (t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &= I_n (g (t_1, \dots, t_n) \cdot \chi_{\{\max t_i < t\}}). \end{aligned}$$

Now,  $u$  is  $\mathcal{F}$  adapted if and only if  $E [u_t | \mathcal{F}_t] = u_t$ . Namely, if and only if  $\sum_{n=0}^{\infty} I_n (f_n (\cdot, t)) = \sum_{n=0}^{\infty} E [I_n (f_n (\cdot, t)) | \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n (f_n (\cdot, t) \cdot \chi_{\{\max t_i < t\}})$ . And thus if and only if  $f_n (t_1, \dots, t_n, t) \cdot \chi_{\{\max t_i < t\}} = f_n (t_1, \dots, t_n, t)$  almost everywhere in  $[0, T]^n$  with respect to Lebesgue measure. By uniqueness of the sequence of deterministic functions on the Wiener-Ito chaos expansion ( Lemma 3.2.1 ), this lemma is proved.  $\square$

**Lemma 3.3.2.** *Let  $u = u_t$ ,  $t \in [0, T]$ , be a measurable  $\mathcal{F}$  adapted stochastic process such that*

$$E \left[ \int_0^T u_t^2 dt \right] < \infty.$$

*Then  $u \in \text{Dom}(\delta)$  and its Skorohod integral coincides with the Ito integral*

$$\int_0^T u_t \delta W_t = \int_0^T u_t dW_t.$$

*Proof.*

Let  $u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$  be the chaos expansion of  $u_t$ . By Lemma 3.3.1,

$$\tilde{f}_n(t_1, \dots, t_n, t_{n+1}) = \frac{1}{n+1} f_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}, t_j),$$

where

$$j = \operatorname{argmax}_{1 \leq i \leq n+1} t_i.$$

Consider the set

$$S_n = \{(t_1, \dots, t_n) \in [0, T]^n : 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}.$$

Hence

$$\begin{aligned}
& \|\tilde{f}_n\|_{L^2([0,T]^{n+1})}^2 \\
&= (n+1)! \int_{S_{n+1}} \tilde{f}_n^2(t_1, \dots, t_{n+1}) dt_1 \cdots dt_{n+1} \\
&= \frac{(n+1)!}{(n+1)^2} \int_{S_{n+1}} f_n^2(t_1, \dots, t_{n+1}) dt_1 \cdots dt_{n+1} \\
&= \frac{n!}{n+1} \int_0^T \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f_n^2(t_1, \dots, t_n, t) dt_1 \cdots dt_n dt \\
&= \frac{n!}{n+1} \int_0^T \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n^2(t_1, \dots, t_n, t) dt_1 \cdots dt_n dt \\
&= \frac{1}{n+1} \int_0^T \|f_n(\cdot, t)\|_{L^2([0,T]^n)}^2 dt,
\end{aligned}$$

again by using Lemma 3.3.1. Hence, by isometry in Lemma 3.2.1,

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L^2([0,T]^{n+1})}^2 &= \sum_{n=0}^{\infty} n! \int_0^T \|f_n(\cdot, t)\|_{L^2([0,T]^n)}^2 dt \\
&= \int_0^T \sum_{n=0}^{\infty} n! \|f_n(\cdot, t)\|_{L^2([0,T]^n)}^2 dt \\
&= E \left[ \int_0^T u_t^2 dt \right] < \infty.
\end{aligned}$$

This proves that  $u \in \text{Dom}(\delta)$ . Finally,

$$\begin{aligned}
& \int_0^T u_t dW_t \\
= & \sum_{n=0}^{\infty} \int_0^T I_n(f_n(\cdot, t)) dW_t \\
= & \sum_{n=0}^{\infty} \int_0^T n! \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} f_n(t_1, \dots, t_n, t) dW_{t_1} \dots dW_{t_n} dW_t \\
= & \sum_{n=0}^{\infty} \int_0^T n!(n+1) \int_{0 \leq t_1 \leq \dots \leq t_n \leq t_{n+1}} \tilde{f}_n(t_1, \dots, t_n, t_{n+1}) dW_{t_1} \dots dW_{t_n} dW_{t_{n+1}} \\
= & \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \\
= & \int_0^T u_t \delta W_t
\end{aligned}$$

By this the proof is complete.  $\square$

### 3.4 Malliavin calculus for Brownian motion

Paul Malliavin is the first one who introduces the Malliavin calculus, an infinite-dimensional differential calculus on the Wiener space, to the public in the 1970s. Many new applications of this calculus have appeared as a result of the development of this theory. In this thesis, the Malliavin derivative operator has been applied to both the Ito integral and the Stratonovich integral.

In those famous researches did by Fournie E. et al. [13] [14], the Malliavin calculus has been used to calculate Delta, Gamma and Vega for no jump process. Their approach is based on the integration-by-parts formula which is also called the duality formula somewhere. Such calculations can be also found in the lecture notes of Eulalia Nualart [37] and the book written by Paul

Malliavin et al. [44]. Then Fournie E. et al. [13] [14] did numerical experiments by using Monte Carlo simulations after calculating those Greeks. On the other hand, Evangelia Petrou [45] gives her calculation of the Greeks for a general stochastic volatility model with jumps both in the underlying and the volatility. These works show that the Malliavin calculus has widespread applications in finance. Considering the practicability of the Malliavin calculus, it plays an important part in hedging and related topics.

This section follows Peter K. Friz [15], Giulia Di Nunno et al. [38], Bernt Oksendal [40], David Nualart [34] and Eulalia Nualart [37]. There are more details in the book written by Paul Malliavin himself [44].

### 3.4.1 Definition

**Definition 3.4.1.** (*Malliavin derivative*)

Let  $F \in L^2(P)$  be  $\mathcal{F}_T$ -measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where  $f_n \in \tilde{L}^2([0, T]^n)$ ,  $n = 1, 2, \dots$ .

(1)  $F \in \mathbb{D}_{1,2}$  if

$$\|F\|_{\mathbb{D}_{1,2}}^2 = \sum_{n=1}^{\infty} nn! \|f_n\|_{L^2([0, T]^n)}^2 < \infty.$$

(2) If  $F \in \mathbb{D}_{1,2}$ , the Malliavin derivative  $D_t F$  of  $F$  at time  $t$  as the expansion can be defined as

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), t \in [0, T],$$

where  $I_{n-1}(f_n(\cdot, t))$  is the  $(n-1)$ -fold iterated integral of  $f_n(t_1, \dots, t_{n-1}, t)$  with respect to the first  $n-1$  variables  $t_1, \dots, t_{n-1}$  and  $t_n = t$  left as parameter.

**Lemma 3.4.1.** *(Closability of the Malliavin derivative)*

Suppose  $F \in L^2(P)$  and  $F_k \in \mathbb{D}_{1,2}$ ,  $k = 1, 2, \dots$ , such that

(1)  $F_k \rightarrow F$ ,  $k \rightarrow \infty$ , in  $L^2(P)$

(2)  $\{D_t F_k\}_{k=1}^\infty$  converges in  $L^2(P \times \lambda)$ .

Then  $F \in \mathbb{D}_{1,2}$  and  $D_t F_k \rightarrow D_t F$ ,  $k \rightarrow \infty$ , in  $L^2(P \times \lambda)$ .

The proof can be found in the book written by Giulia Di Nunno et al. [38].

### 3.4.2 Properties

Let  $\mathbb{D}_{1,2}^0$  be the set of all  $F \in L^2(P)$  whose chaos expansion has only finitely many terms.

**Proposition 3.4.1.** *(Product rule)*

Suppose  $F_1, F_2 \in \mathbb{D}_{1,2}^0$ . Then  $F_1, F_2 \in \mathbb{D}_{1,2}$  and also  $F_1 F_2 \in \mathbb{D}_{1,2}$  with

$$D_t(F_1 F_2) = F_1 D_t F_2 + F_2 D_t F_1.$$

**Proposition 3.4.2.** *(Sum rule)*

Suppose  $F_1 F_2 \in \mathbb{D}_{1,2}$ . and  $\alpha$  and  $\beta$  be two constants. Then,

$$D_t(\alpha F_1 + \beta F_2) = \alpha D_t F_1 + \beta D_t F_2.$$

**Proposition 3.4.3.** (*Chain rule*)

Let  $F \in \mathbb{D}_{1,2}$  and  $f$  is a function with bounded derivative. Then  $f(F) \in \mathbb{D}_{1,2}$  and

$$D_t f(F) = f'(F) D_t F.$$

Here  $f'(x) = df(x)/dx$ .

**Lemma 3.4.2.** Let  $G \subseteq [0, T]$  be a Borel set and  $v = v_t$ ,  $t \in [0, T]$ , be a stochastic process such that

(1) for all  $t$ ,  $v_t$  is measurable with respect to  $\mathcal{F}_t \cap \mathcal{F}_G$

(2)  $E \left[ \int_0^T v_t^2 dt \right] < \infty$ .

Then

$$\int_G v_t dW_t$$

is  $\mathcal{F}_G$ -measurable.

**Proposition 3.4.4.** Let  $u = u_s$ ,  $s \in [0, T]$ , be an  $\mathcal{F}$  adapted stochastic process and assume that  $u_s \in \mathbb{D}_{1,2}$  for all  $s$ . Then

(1)  $D_t u_s$ ,  $s \in [0, T]$ , is  $\mathcal{F}$  adapted for all  $t$ ;

(2)  $D_t u_s = 0$ , for  $t > s$ .

All proofs provided by Giulia Di Nunno et al. [38].

**Proposition 3.4.5.** (*Constant rule*)

Let  $C_t$  be a function with respect to  $t$ . Then, for all  $t$ ,

$$D_t C_s = 0.$$

This proposition is a direct result of the definition of the Malliavin derivative.

### 3.4.3 Duality formula

This section follows Giulia Di Nunno et al. [38]. The Malliavin derivative operator is the adjoint operator of the Skorohod integral, which will be shown as the following lemma. Note that the following lemma may be defined as the integration-by-parts formula in some papers ( Fournie E. et al. [13],[14] ).

**Lemma 3.4.3.** (*Duality formula*)

Let  $F \in \mathbb{D}_{1,2}$  be  $\mathcal{F}_T$ -measurable and let  $u$  be a Skorohod integrable stochastic process. Then

$$E \left[ F \int_0^T u_t \delta W_t \right] = E \left[ \int_0^T u_t D_t F dt \right].$$

*Proof.*

Let  $F = \sum_{n=0}^{\infty} I_n(f_n)$  and, for all  $t$ ,  $u_t = \sum_{k=0}^{\infty} I_k(g_k(\cdot, t))$  be the chaos expansions of  $F$  and  $u_t$ , respectively. Then

$$\begin{aligned}
E \left[ F \int_0^T u_t \delta W_t \right] &= E \left[ \sum_{n=0}^{\infty} I_n(f_n) \int_0^T \sum_{k=0}^{\infty} I_k(g_k(\cdot, t)) \delta W_t \right] \\
&= E \left[ \sum_{n=0}^{\infty} I_n(f_n) \sum_{k=0}^{\infty} I_{k+1}(\tilde{g}_k) \right] \\
&= E \left[ \sum_{k=0}^{\infty} I_{k+1}(f_{k+1}) I_{k+1}(\tilde{g}_k) \right] \\
&= \sum_{k=0}^{\infty} (k+1)! \int_{[0, T]^{k+1}} f_{k+1}(x) \tilde{g}_k(x) dx \\
&= \sum_{k=0}^{\infty} (k+1)! (f_{k+1}, \tilde{g}_k)_{L^2([0, T]^{k+1})},
\end{aligned}$$

where  $\tilde{g}_k$  is the symmetrization of  $g_k(x_1, \dots, x_n, t)$  as a function of  $n+1$  variables. On the other side,

$$\begin{aligned}
E \left[ \int_0^T u_t D_t F dt \right] &= E \left[ \int_0^T \left( \sum_{k=0}^{\infty} I_k(g_k(\cdot, t)) \right) \left( \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \right) dt \right] \\
&= \int_0^T \sum_{k=0}^{\infty} E [(k+1) I_k(g_k(\cdot, t)) I_k(f_{k+1}(\cdot, t))] dt \\
&= \int_0^T \sum_{k=0}^{\infty} (k+1) k! (f_{k+1}(\cdot, t), g_k(\cdot, t))_{L^2([0, T]^k)} dt \\
&= \sum_{k=0}^{\infty} (k+1)! (f_{k+1}, g_k)_{L^2([0, T]^{k+1})}.
\end{aligned}$$

Now

$$\begin{aligned}
(f_{k+1}, \tilde{g}_k)_{L^2([0, T]^{k+1})} &= \int_0^T (f_{k+1}(\cdot, t), \tilde{g}_k(\cdot, t))_{L^2([0, T]^k)} dt \\
&= \frac{1}{k+1} \sum_{j=1}^{k+1} \int_0^T (f_{k+1}(\cdot, t_j), g_k(\cdot, t_j))_{L^2([0, T]^k)} dt_j \\
&= \int_0^T (f_{k+1}(\cdot, t), g_k(\cdot, t))_{L^2([0, T]^k)} dt \\
&= (f_{k+1}, g_k)_{L^2([0, T]^{k+1})}.
\end{aligned}$$

This finishes the proof.  $\square$

Note that if  $u$  is an  $\mathcal{F}$  adapted process with

$$E \left[ \int_0^T u_t^2 dt \right] < \infty.$$

Then

$$E \left[ F \int_0^T u_t dW_t \right] = E \left[ \int_0^T u_t D_t F dt \right].$$

**Lemma 3.4.4.** (*Integration by parts*)

Let  $u_t, t \in [0, T]$ , be a Skorohod integrable stochastic process and  $F \in \mathbb{D}_{1,2}$  such that the product  $Fu_t, t \in [0, T]$ , is Skorohod integrable. Then

$$F \int_0^T u_t \delta W_t = \int_0^T Fu_t \delta W_t + \int_0^T u_t D_t F dt.$$

*Proof.*

Assume that  $F \in \mathbb{D}_{1,2}^0$ . Choose  $G \in \mathbb{D}_{1,2}^0$ . By product rule ( Proposition 3.4.1 ) and duality formula ( Lemma 3.4.3 ),

$$\begin{aligned} E \left[ G \int_0^T Fu_t \delta W_t \right] &= E \left[ \int_0^T Fu_t D_t G dt \right] \\ &= E \left[ GF \int_0^T u_t \delta W_t \right] - E \left[ G \int_0^T u_t D_t F dt \right]. \end{aligned}$$

Since the set of all  $G \in \mathbb{D}_{1,2}^0$  is dense in  $L^2(P)$ , it follows that

$$F \int_0^T u_t \delta W_t = \int_0^T Fu_t \delta W_t + \int_0^T u_t D_t F dt, P - a.s.$$

Then the result follows for general  $F \in \mathbb{D}_{1,2}$  by approximating  $F$  by  $F^{(n)} \in \mathbb{D}_{1,2}^0$  such that  $F^{(n)} \rightarrow F$  in  $L^2(P)$  and  $D_t F^{(n)} \rightarrow D_t F$  in  $L^2(P \times \lambda)$ , for  $n \rightarrow \infty$ .

□

The duality formula can be also used to prove the following important result.

**Lemma 3.4.5.** (*Closability of the Skorohod integral*)

Suppose that  $u_t^{(n)}$ ,  $t \in [0, T]$ ,  $n = 1, 2, \dots$ , is a sequence of Skorohod integrable stochastic processes and that the corresponding sequence of Skorohod integrals

$$\delta(u^{(n)}) = \int_0^T u_t^{(n)} \delta W_t, n = 1, 2, \dots$$

converges in  $L^2(P)$ . Moreover, suppose that

$$\lim_{n \rightarrow \infty} u^{(n)} = 0 \quad \text{in} \quad L^2(P \times \lambda).$$

Then

$$\lim_{n \rightarrow \infty} \delta(u^{(n)}) = 0 \quad \text{in} \quad L^2(P).$$

*Proof.*

By duality formula ( Lemma 3.4.3 ),

$$(\delta(u^{(n)}), F)_{L^2(P)} = (u^{(n)}, DF)_{L^2(P \times \lambda)} \rightarrow 0, n \rightarrow \infty,$$

for all  $F \in \mathbb{D}_{1,2}$ . Then, conclude that  $\delta(u^{(n)}) \rightarrow 0$  weakly in  $L^2(P)$ . Since  $\{\delta(u^{(n)}) : n = 0, 1, \dots, \infty\}$  is convergent in  $L^2(P)$ , it can be seen that  $\delta(u^{(n)}) \rightarrow 0$  in  $L^2(P)$ . □

### 3.4.4 A fundamental theorem of calculus

This section follows Giulia Di Nunno et al. [38].

The relation between differentiation and Skorohod integration is given as the next lemma.

**Lemma 3.4.6.** *(The fundamental theorem of calculus)*

Let  $u = u_s$ ,  $s \in [0, T]$ , be a stochastic process such that

$$E \left[ \int_0^T u_s^2 ds \right] < \infty$$

and assume that, for all  $s \in [0, T]$ ,  $u_s \in \mathbb{D}_{1,2}$  and that, for all  $t \in [0, T]$ ,  $D_t u \in \text{Dom}(\delta)$ . Assume also that

$$E \left[ \int_0^T (\delta(D_t u))^2 dt \right] < \infty.$$

Then  $\int_0^T u_s \delta W_s$  is well-defined and belongs to  $\mathbb{D}_{1,2}$  and

$$D_t \left( \int_0^T u_s \delta W_s \right) = \int_0^T D_t u_s \delta W_s + u_t.$$

*Proof.*

First assume that

$$u_s = I_n(f_n(\cdot, s)),$$

where  $f_n(t_1, \dots, t_n, s)$  is symmetric with respect to  $t_1, \dots, t_n$ . Then

$$\int_0^T u_s \delta W_s = I_{n+1} [\tilde{f}_n],$$

where

$$\tilde{f}_n(x_1, \dots, x_{n+1}) = \frac{1}{n+1} [f_n(\cdot, x_1) + \dots + f_n(\cdot, x_{n+1})]$$

is the symmetrization of  $f_n$  as a function of all its  $n+1$  variables. Hence

$$D_t \left( \int_0^T u_s \delta W_s \right) = (n+1) I_n [\tilde{f}_n(\cdot, t)],$$

where

$$\tilde{f}_n(\cdot, t) = \frac{1}{n+1} [f_n(t, \cdot, x_1) + \dots + f_n(t, \cdot, x_n) + f_n(\cdot, t)].$$

Then, by linearity and former results, it is clear that

$$\begin{aligned} D_t \left( \int_0^T u_s \delta W_s \right) &= I_n [f_n(t, \cdot, x_1) + \dots + f_n(t, \cdot, x_n) + f_n(\cdot, t)] \\ &= I_n [f_n(t, \cdot, x_1) + \dots + f_n(t, \cdot, x_n)] + u(t). \end{aligned} \quad (1)$$

Consider

$$\begin{aligned} \delta(D_t u) &= \int_0^T D_t u_s \delta W_s \\ &= \int_0^T n I_{n-1} [f_n(\cdot, t, s)] \delta W_s \\ &= n I_n [\hat{f}_n(\cdot, t, \cdot)], \end{aligned}$$

where

$$\hat{f}_n(x_1, \dots, x_{n-1}, t, x_n) = \frac{1}{n} [f_n(t, \cdot, x_1) + \dots + f_n(t, \cdot, x_n)]$$

is the symmetrization of  $f_n(x_1, \dots, x_{n-1}, t, x_n)$  with respect to  $x_1, \dots, x_n$ .

Then,

$$\int_0^T D_t u_s \delta W_s = I_n [f_n(t, \cdot, x_1) + \dots + f_n(t, \cdot, x_n)]. \quad (2)$$

After comparing (1) and (2), the result is clear. Next, consider the general case when

$$u_s = \sum_{n=0}^{\infty} I_n [f_n(\cdot, s)].$$

Define

$$u_s^{(m)} = \sum_{n=0}^m I_n [f_n(\cdot, s)], \quad m = 1, 2, \dots$$

By  $E \left[ \int_0^T u_s^2 ds \right] < \infty$ ,  $\|u - u^{(m)}\|_{L^2(P \times \lambda)}^2 \rightarrow 0$ ,  $m \rightarrow \infty$ . Then, for all  $m$ ,

$$D_t (\delta (u^{(m)})) = \delta (D_t u^{(m)}) + u_t^{(m)}.$$

By  $\delta (D_t u) = n I_n [\hat{f}_n(\cdot, t, \cdot)]$ ,  $E \left[ \int_0^T (\delta (D_t u))^2 dt \right] < \infty$  is equivalent to saying that

$$\begin{aligned} E \left[ \int_0^T (\delta (D_t u))^2 dt \right] &= \sum_{n=1}^{\infty} n^2 n! \int_0^T \|\hat{f}_n(\cdot, t, \cdot)\|_{L^2([0, T]^n)}^2 dt \\ &= \sum_{n=1}^{\infty} n^2 n! \|\hat{f}_n\|_{L^2([0, T]^{n+1})}^2 < \infty, \end{aligned} \quad (3)$$

since  $D_t u \in \text{Dom}(\delta)$ . Hence, for  $m \rightarrow \infty$ ,

$$\|\delta (D_t u) - \delta (D_t u^{(m)})\|_{L^2(P \times \lambda)}^2 = \sum_{n=m+1}^{\infty} n^2 n! \|\hat{f}_n\|_{L^2([0, T]^{n+1})}^2 \rightarrow 0. \quad (4)$$

Therefore,

$$D_t (\delta (u^{(m)})) \rightarrow \delta (D_t u) + u (t), \quad m \rightarrow \infty,$$

in  $L^2 (P \times \lambda)$ . Note that

$$(n+1) \tilde{f}_n (\cdot, t) = n \hat{f}_n (\cdot, t, \cdot) + f_n (\cdot, t)$$

and hence

$$(n+1)! \|\tilde{f}_n\|_{L^2([0, T]^{n+1})}^2 \leq \frac{2n^2 n!}{n+1} \|\hat{f}_n\|_{L^2([0, T]^{n+1})}^2 + \frac{2n!}{n+1} \|f_n\|_{L^2([0, T]^{n+1})}^2.$$

Therefore,

$$\begin{aligned} \|\delta (u)\|_{\mathbb{D}_{1,2}}^2 &= \sum_{n=0}^{\infty} (n+1) (n+1)! \|\tilde{f}_n\|_{L^2([0, T]^{n+1})}^2 \\ &\leq \sum_{n=0}^{\infty} \left[ 2n^2 n! \|\hat{f}_n\|_{L^2([0, T]^{n+1})}^2 + 2n! \|f_n\|_{L^2([0, T]^{n+1})}^2 \right] \\ &\leq 2 \|\delta (D_t u)\|_{L^2(P \times \lambda)}^2 + 2 \|u\|_{L^2(P \times \lambda)}^2 < \infty, \end{aligned}$$

by (3) and  $E \left[ \int_0^T u_s^2 ds \right] < \infty$ . Then  $\delta (u)$  is well-defined and belongs to  $\mathbb{D}_{1,2}$ .

Similarly,

$$\begin{aligned} &\|D_t \left( \int_0^T u_s \delta W_s \right) - D_t \left( \int_0^T u_s^{(m)} \delta W_s \right)\|_{L^2(P \times \lambda)}^2 \\ &= \left\| \sum_{n=m+1}^{\infty} (n+1) I_n \left( \tilde{f}_n (\cdot, t) \right) \right\|_{L^2(P \times \lambda)}^2 \\ &= \int_0^T \sum_{n=m+1}^{\infty} (n+1)^2 n! \|\tilde{f}_n (\cdot, t)\|_{L^2([0, T]^n)}^2 dt \\ &\leq 2 \sum_{n=m+1}^{\infty} \left[ n^2 n! \|\hat{f}_n\|_{L^2([0, T]^{n+1})}^2 + n! \|f_n\|_{L^2([0, T]^{n+1})}^2 \right], \quad (5) \end{aligned}$$

which vanishes when  $m \rightarrow \infty$ . Hence given (4) and (5),

$$D_t(\delta(u)) = \delta(D_t u) + u(t),$$

by letting  $m \rightarrow \infty$  in  $D_t(\delta(u^{(m)})) = \delta(D_t u^{(m)}) + u_t^{(m)}$ .  $\square$

**Lemma 3.4.7.** (*Stroock lemma*)

Let  $u$  be as in Lemma 3.4.6 and assume in addition that  $u_s, s \in [0, T]$  is  $\mathcal{F}$  adapted. Then

$$D_t\left(\int_0^T u_s dW_s\right) = \int_t^T D_t u_s dW_s + u_t.$$

*Proof.*

This lemma can be proved directly by Lemma 3.3.2 and Lemma 3.4.6.  $\square$

## 3.5 Hermite polynomials

This section follows Pierre-Simon Laplace [29].

The definition of Hermite polynomials are given here, since it will be used many times in the following examples.

**Definition 3.5.1.** (*Hermite polynomials*)

The Hermite polynomials of degree  $n$  and parameter  $\lambda > 0$  can be defined by

$$H_n(\lambda, x) = (-\lambda)^n e^{\frac{x^2}{2\lambda}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2\lambda}}, n \geq 1, x \in \mathbb{R},$$

where  $H_0(\lambda, x) = 1$ .

Moreover, the following properties hold:

**Proposition 3.5.1.**

$$\frac{\partial}{\partial x} H_n(\lambda, x) = nH_{n-1}(\lambda, x), n \geq 1.$$

**Proposition 3.5.2.**

$$H_{n+1}(\lambda, x) = xH_n(\lambda, x) - n\lambda H_{n-1}(\lambda, x), n \geq 1.$$

**Proposition 3.5.3.**

$$H_n(\lambda, -x) = (-1)^n H_n(\lambda, x), n \geq 1.$$

**Proposition 3.5.4.**

$$\frac{\partial}{\partial \lambda} H_n(\lambda, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(\lambda, x), n \geq 1.$$

Note that Hermite polynomials satisfy the following lemma.

**Lemma 3.5.1.** *Let  $H_n(\lambda, x)$  be the Hermite polynomials defined by Definition 3.5.1. Then,*

$$\exp\left(tx - \frac{t^2\lambda}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\lambda, x).$$

## 3.6 Stratonovich integral

Stratonovich integral was developed by Ruslan L. Stratonovich and D. L. Fisk simultaneously, which is a stochastic integral. The history of Stratonovich integral is explored by Jarrow Robert et al. [48]. The application of Stratonovich integral exists in applied mathematics and physics. This part is motivated by Ruslan L. Stratonovich [52] and D. L. Fisk [12]. One can find more details in Chapter 3 of the book 'Stochastic Differential Equations, An Introduction with Applications' written by Bernt Oksendal [41].

### 3.6.1 Definition

**Definition 3.6.1.** (*Stratonovich integral*)

Suppose  $W_t$  is a Wiener process and  $\{X_t : t \geq 0\}$  is an adapted stochastic process according to the nature filtration  $\{\mathcal{F}_t : t \geq 0\}$ . Then the Stratonovich integral is defined as

$$\int_0^T G(t, X_t) \circ dW_t = \lim \sum_i \frac{G(t_i, X_{t_{i+1}}) + G(t_i, X_{t_i})}{2} (W_{t_{i+1}} - W_{t_i}),$$

$$i = 0, 1, \dots, k - 1,$$

where the symbol of limit represents that the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  of  $[0, T]$  tends to 0 and  $G(t, \cdot)$  is a  $\mathcal{F}_t$  measurable function.

There is an another way to define the Stratonovich integral. The Stratonovich integral can be also defined as

$$\int_0^T G(t, X_t) \circ dW_t = \lim \sum_i G\left(t_i, \frac{X_{t_{i+1}} + X_{t_i}}{2}\right) (W_{t_{i+1}} - W_{t_i}),$$

$$i = 0, 1, \dots, k-1,$$

where the symbol of limit represents that the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  of  $[0, T]$  tends to 0 and  $G(t, \cdot)$  is a  $\mathcal{F}_t$ -measurable function according to Suresh P. Sethi et al. [50].

### 3.6.2 Properties

Next lemma shows the relationship between the Ito integral and the Stratonovich integral. This lemma can be found at page 101 of the book written by Suresh P. Sethi et al. [26] where also proof is provided.

**Lemma 3.6.1.** (*Conversion between Ito integral and Stratonovich integral*)

Suppose  $f(t, W_t)$  is any continuously differentiable function of two variables  $W_t$  and  $t$ , then

$$\int_0^T f(t, W_t) \circ dW_t = \frac{1}{2} \int_0^T \frac{\partial f}{\partial W_t}(t, W_t) dt + \int_0^T f(t, W_t) dW_t.$$

**Example 3.6.1.**

Let  $B_t$  be a standard Brownian motion. By Definition 3.6.1,

$$B_T = \int_0^T 1 \circ dB_t.$$

For  $n \geq 1$ ,

$$(n+1) \int_0^T B_t^n \circ dB_t = (n+1) \int_0^T B_t^n dB_t + \frac{n(n+1)}{2} \int_0^T B_t^{n-1} dt$$

by Lemma 3.6.1. According to the Ito representation,

$$B_T^{n+1} = B_0 + (n+1) \int_0^T B_t^n dB_t + \frac{n(n+1)}{2} \int_0^T B_t^{n-1} dt.$$

Therefore,

$$B_T^{n+1} = (n+1) \int_0^T B_t^n \circ dB_t, n \geq 1.$$

Finally, concluding the former results,

$$B_T^{n+1} = (n+1) \int_0^T B_t^n \circ dB_t, n \in \mathbb{N}. \quad \square$$

### 3.7 Stochastic differential equation

A stochastic differential equation is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process.

The Ito stochastic differential equation in the form of

$$dX_t = f(t, X_t) dt + g(t, X_t) dW_t, X_{t_0} = 0$$

is one of the classical stochastic differential equations which has been used in many fields, such as physics and finance. This kind of stochastic differential equations will be considered many times in this thesis.

The Brownian bridge introduced next is an important example. This Gaussian process is the inspiration of the Equation B and Equation C in Chapter 5. This example can be also found in page 75 of the book 'Stochastic Differential Equations' written by Bernt Oksendal [41].

**Example 3.7.1.** (*Brownian bridge*)

For fixed  $a, b \in \mathbb{R}$ , consider the following equation

$$dY_u = \frac{b - Y_u}{1 - u} du + dB_u, 0 \leq u < 1, Y_0 = a.$$

Turn the differential form of the former process  $Y_u$  to the integral form

$$Y_u = Y_0 + \int_0^u \left( \frac{b - Y_s}{1 - s} \right) ds + \int_0^u dB_s.$$

For fix  $t$ , applying Malliavin derivative operator  $D_t$  to the process  $Y_u$ ,

$$\begin{aligned} D_t Y_u &= \int_t^u D_t \left( \frac{b - Y_s}{1 - s} \right) ds + 1 \\ &= \int_t^u \frac{-D_t Y_s}{1 - s} ds + 1. \end{aligned}$$

Let  $A_s = D_t Y_s$ , therefore,

$$A_u = \int_t^u \frac{-A_s}{1 - s} ds + 1.$$

The derivative of  $A_u$  is

$$A_u' = \frac{-A_u}{1 - u}.$$

Solve former equation and get

$$\begin{aligned}\ln A_u &= - \int \frac{1}{1-u} du \\ &= \ln(1-u) + \ln C,\end{aligned}$$

and then,

$$A_u = C(1-u)$$

where  $C$  is a constant. Since  $A_t = 1$ ,

$$C = \frac{1}{1-t}.$$

Conclude former results,

$$A_u = D_t Y_u = \frac{1-u}{1-t}$$

where  $0 \leq t < u < 1$ .

There is another form of Brownian bridge, which is

$$Y_u = a(1-u) + bu + (1-u) \int_0^u \frac{dB_s}{1-s}, 0 \leq u < 1.$$

For fix  $t$ , applying derivative operator  $D_t$  to the process  $Y_u$ ,

$$\begin{aligned}D_t Y_u &= D_t \left[ a(1-u) + bu + (1-u) \int_0^u \frac{dB_s}{1-s} \right] \\ &= (1-u) D_t \int_0^u \frac{dB_s}{1-s} \\ &= \frac{1-u}{1-t} + \int_0^u D_t \frac{dB_s}{1-s} \\ &= \frac{1-u}{1-t}\end{aligned}$$

where  $0 \leq t < u < 1$ .  $\square$

# Chapter 4

## Simple Results

In this chapter, numerous examples and lemmas are treated as simple results here. These results may help to understand not only several lemmas given before but the inspiration of this part as well. Note that these lemmas given here will be applied in following chapters.

### 4.1 Some related results of Malliavin calculus

The following lemma gives another way to prove the Stroock lemma which follows Professor Utev.

**Lemma 4.1.1.** *(Direct proof of Stroock lemma)*

Let  $F = \int_0^T \sigma_u dW_u$  in which  $\sigma_u$  is  $\mathcal{F}_u$  adapted and  $W_u$  is the Wiener process.

Then, for fixed  $t$ ,

$$D_t F = \int_0^T D_t \sigma_u dW_u + \sigma_t.$$

*Proof.*

This equation holds if proven for

$$\sigma_t = \begin{cases} \eta_a, & a < t \leq b, \\ 0, & \text{else,} \end{cases}$$

where  $\eta_a$  is a random variable and  $a, b \in [0, T]$ . Then,

$$\begin{aligned} \sigma_t &= \eta_a I(a < t \leq b) \\ &= \eta_a [I(a < t \leq T) - I(b < t \leq T)]. \end{aligned}$$

Without loss of generality, it is equivalent to proof

$$\sigma_t = \eta_a I(a < t \leq T).$$

According to the definition of the Ito integral,

$$\begin{aligned} F &= \int_0^T \sigma_u dW_u \\ &= \sum_j \eta_{t_j} (W_{t_{j+1}} - W_{t_j}) \\ &= \eta_a (W_T - W_a) \\ &= \eta_a W_h, \end{aligned}$$

where  $W_h = \int_0^T h(s) dW_s$  and

$$h(t) = \begin{cases} 1, & a < t \leq T, \\ 0, & \text{else.} \end{cases}$$

Applying Proposition 3.4.1 and the fact that

$$D_t W_h = h(t),$$

the left-hand side of the equation is

$$\begin{aligned} D_t F &= D_t (\eta_a W_h) \\ &= \eta_a D_t W_h + W_h D_t \eta_a \\ &= \eta_a I(a < t \leq T) + (W_T - W_a) D_t \eta_a \\ &= \eta_a h(t) + (W_T - W_a) D_t \eta_a. \end{aligned}$$

Meanwhile, applying the fact that  $D_t h = 0$ ,

$$\begin{aligned} D_t \sigma_u &= D_t \eta_a I(a < t \leq T) \\ &= D_t \eta_a h(t) \\ &= h(t) D_t \eta_a, \end{aligned}$$

and then the first term of the right-hand side of the equation is

$$\begin{aligned} \int_0^T D_t \sigma_u dW_u &= D_t \sum_j \eta_{t_j} (W_{t_{j+1}} - W_{t_j}) \\ &= D_t \eta_a (W_T - W_a) \end{aligned}$$

since  $D_t \sigma_u$  is  $\mathcal{F}_u$  adapted because  $t$  is fixed and  $\sigma_u$  is  $\mathcal{F}_u$  adapted. The second term of the right-hand side of the equation is

$$\sigma_t = \eta_a I(a < t \leq T) = \eta_a h(t).$$

In terms of the aforementioned results,

$$D_t F = \int_0^T D_t \sigma_u dW_u + \sigma_t. \quad \square$$

**Lemma 4.1.2.** *Let  $f$  be a function with respect to  $u$  and Wiener process  $W_u$ , such function  $f$  and its first and second derivative are bounded. Then,*

$$D_t D_u f(u, W_u) = D_u D_t f(u, W_u).$$

*Proof.*

By Proposition 3.4.3, from left hand side of the former equation,

$$\begin{aligned} D_t D_u f(u, W_u) &= D_t \{f'_{W_u}(u, W_u)\} \\ &= f''_{W_u W_u}(u, W_u) I(u > t). \end{aligned}$$

Similarly, from right hand side of the former equation,

$$\begin{aligned} D_u D_t f(u, W_u) &= D_u \{f'_{W_u}(u, W_u) I(u > t)\} \\ &= f''_{W_u W_u}(u, W_u) I(u > t). \end{aligned}$$

Therefore, it is clear that

$$D_t D_u f(u, W_u) = D_u D_t f(u, W_u). \quad \square$$

Remark that

$$D_t D_u f(s, W_s) \neq D_u D_t f(s, W_s).$$

Without loss of generality, assume that  $s > t > u$ . Then, from left hand side of the former equation,

$$\begin{aligned}
D_t D_u f(s, W_s) &= D_t \{f'_{W_s}(s, W_s) I(s > u)\} \\
&= \{f''_{W_s W_s}(s, W_s) I(s > u)\} I(u > t) \\
&= 0.
\end{aligned}$$

From right hand side of the former equation,

$$\begin{aligned}
D_u D_t f(s, W_s) &= D_u \{f'_{W_s}(s, W_s) I(s > t)\} \\
&= \{f''_{W_s W_s}(s, W_s) I(s > t)\} I(t > u) \\
&= f''_{W_s W_s}(s, W_s) I(s > u).
\end{aligned}$$

Therefore,  $D_t D_u f(s, W_s)$  is not always equal to  $D_u D_t f(s, W_s)$ . Lemma 4.1.2 is just a special case.

## 4.2 Some related results of Stratonovich integral

The following lemma gives another way to represented the relationship between the Ito integral and the Stratonovich integral.

**Lemma 4.2.1.** *Suppose  $\{X_u : u \geq 0\}$  is  $\mathcal{F}_u$  adapted. Then*

$$\int_0^T X_u \circ dW_u = \frac{1}{2} \int_0^T \sigma_u du + \int_0^T X_u dW_u$$

where  $dX_u = \sigma_u dW_u + \mu_u du$ .

*Proof.*

According to the Definition 3.6.1, the Stratonovich integral can be represented as

$$\begin{aligned}
\int_0^T X_u \circ dW_u &= \lim \sum_j \left( \frac{X_{u_j} + X_{u_{j+1}}}{2} \right) (W_{u_{j+1}} - W_{u_j}) \\
&= \lim \sum_j X_{u_j} (W_{u_{j+1}} - W_{u_j}) \\
&\quad + \frac{1}{2} \lim \sum_j (X_{u_{j+1}} - X_{u_j}) (W_{u_{j+1}} - W_{u_j}). \quad (\star)
\end{aligned}$$

The first term of the right hand side of  $(\star)$  is equal to  $\int_0^T X_u dW_u$ . Since  $X_u$  is an Ito process and the differential form of this process is  $dX_u = \sigma_u dW_u + \mu_u du$ , it can be seen that

$$X_{u_{j+1}} - X_{u_j} = \int_{u_j}^{u_{j+1}} \sigma_u dW_u + \int_{u_j}^{u_{j+1}} \mu_u du.$$

Then the second term of the right hand side of  $(\star)$  can be divided into two parts. The first part of this term is

$$\sum_j \left( \int_{u_j}^{u_{j+1}} \mu_u du \right) (W_{u_{j+1}} - W_{u_j}) \rightarrow 0$$

in  $L^2$  since  $\Delta u \Delta W_u \rightarrow 0$  in  $L^2$  ( see Appendix ). The second part of this term is

$$\sum_j \left( \int_{u_j}^{u_{j+1}} \sigma_u dW_u \right) (W_{u_{j+1}} - W_{u_j}) \rightarrow \int_0^T \sigma_u du$$

in  $L^2$  since  $\Delta W_u \Delta W_u \rightarrow \Delta u$  in  $L^2$  ( see Appendix ). To sum up the former results,

$$\lim \sum_j (X_{u_{j+1}} - X_{u_j}) (W_{u_{j+1}} - W_{u_j}) = \int_0^T \sigma_u du. \quad \square$$

The next lemma explains the uniqueness of the representation of the Stratonovich integral.

**Lemma 4.2.2.** (*Uniqueness*)

Suppose  $\{X_u : u \geq 0\}$  and  $\{Y_u : u \geq 0\}$  are two adapted processes, which can be represented as

$$dX_u = \left( \mu_u - \frac{1}{2} (\sigma_u)'_{W_u} \right) du + \sigma_u dW_u$$

and

$$dY_u = \left( \bar{\mu}_u - \frac{1}{2} (\bar{\sigma}_u)'_{W_u} \right) du + \bar{\sigma}_u dW_u.$$

If these two processes satisfy

$$X_u = Y_u,$$

and therefore

$$\begin{cases} X_t = Y_t & a.s. \\ \mu_p = \bar{\mu}_p & a.s. \\ \sigma_p = \bar{\sigma}_p & a.s.. \end{cases}$$

*Proof.*

From the definition of Stratonovich integral and Lemma 4.2.1, the left-hand side of former equation can be illustrated as

$$X_t + \int_t^u \left( \mu_p - \frac{1}{2} (\sigma_p)'_{W_p} \right) dp + \int_t^u \sigma_p \circ dW_p = X_t + \int_t^u \mu_p dp + \int_t^u \sigma_p dW_p$$

and the right-hand side can be similarly illustrated as

$$Y_t + \int_t^u \left( \bar{\mu}_p - \frac{1}{2} (\bar{\sigma}_p)'_{W_p} \right) dp + \int_t^u \bar{\sigma}_p \circ dW_p = Y_t + \int_t^u \bar{\sigma}_p dW_p + \int_t^u \bar{\mu}_u du$$

Then, according to Lemma 3.1.3, the result is clear.  $\square$

## 4.3 Malliavin calculus for Stratonovich integral

The Stroock lemma can be extended to the Stratonovich integral by following lemma.

**Lemma 4.3.1.** (*Stroock lemma for Stratonovich integral*)

Assume that  $f(t, W_t)$  is any continuously differentiable function of two variables  $W_t$  and  $t$ . Then

$$D_t \left( \int_0^T f(u, W_u) \circ dW_u \right) = \int_0^T D_t f(u, W_u) \circ dW_u + f(t, W_t).$$

*Proof.*

Considering Lemma 4.1.1 and Lemma 3.6.1,

$$\begin{aligned}
& D_t \left( \int_0^T f(u, W_u) \circ dW_u \right) \\
&= D_t \left( \int_0^T f(u, W_u) dW_u \right) + \frac{1}{2} D_t \left( \int_0^T \frac{\partial f(u, W_u)}{\partial W_u} du \right) \\
&= \int_0^T D_t f(u, W_u) dW_u + f(t, W_t) + \frac{1}{2} \int_0^T D_t \frac{\partial f(u, W_u)}{\partial W_u} du \\
&= \int_0^T D_t f(u, W_u) \circ dW_u - \frac{1}{2} \int_0^T \frac{\partial D_t f(u, W_u)}{\partial W_u} du + \\
&\quad + \frac{1}{2} \int_0^T D_t \frac{\partial f(u, W_u)}{\partial W_u} du + f(t, W_t) \\
&= \int_0^T D_t f(u, W_u) \circ dW_u - \frac{1}{2} \int_0^T D_u D_t f(u, W_u) du + \\
&\quad + \frac{1}{2} \int_0^T D_t D_u f(u, W_u) du + f(t, W_t).
\end{aligned}$$

According to Lemma 4.1.2,

$$\int_0^T D_u D_t f(u, W_u) du = \int_0^T D_t D_u f(u, W_u) du.$$

This finishes the proof.  $\square$

**Lemma 4.3.2.** *Assume that  $X_t$  is an adapted process which can be defined as  $dX_t = \mu_t dt + \sigma_t dW_t$  where  $W_t$  is a Wiener process and  $D_t X_t$  is also an adapted process where  $D$  is the differential operator. Then*

$$D_t \left( \int_0^T X_u \circ dW_u \right) = \int_0^T D_t X_u \circ dW_u + X_t.$$

*Proof.*

Considering Lemma 4.1.1 and Lemma 4.2.1,

$$\begin{aligned}
& D_t \left( \int_0^T X_u \circ dW_u \right) \\
&= D_t \left( \int_0^T X_u dW_u \right) + \frac{1}{2} D_t \left( \int_0^T \sigma_u du \right) \\
&= \int_0^T D_t X_u dW_u + X_t + \frac{1}{2} \int_0^T D_t \sigma_u du \\
&= \int_0^T D_t X_u \circ dW_u - \frac{1}{2} \int_0^T \sigma_v (D_t X_u) dv + \frac{1}{2} \int_0^T D_t (\sigma_u) du + X_t
\end{aligned}$$

where  $\sigma_v$  is a operator which lets  $d(D_t X_u)_v = \mu_v (D_t X_u) dv + \sigma_v (D_t X_u) dW_v$ .

The process  $X_u$  in the former equation can be represented as

$$X_u = X_0 + \int_0^u \mu_v dv + \int_0^u \sigma_v dW_v.$$

Therefore,

$$D_t X_u = \int_0^u D_t (\mu_v) dv + \int_0^u D_t (\sigma_v) dW_v + \sigma_t.$$

Applied the operator  $\sigma_v$ ,

$$\sigma_v (D_t X_u) = D_t (\sigma_v). \quad \square$$

Next, some examples are introduced here to see how Statonovich integral works.

### Example 4.3.1.

Apply the derivative operator  $D_t$  to the Stratonovich integral  $\int_0^T B_u \circ dB_u$ .

Then,

$$\begin{aligned}
D_t \left( \int_0^T B_u \circ dB_u \right) &= D_t \left( \int_0^T B_u dB_u \right) + \frac{1}{2} D_t \left( \int_0^T 1 du \right) \\
&= \int_0^T D_t B_u dB_u + B_t \\
&= \int_0^T D_t B_u \circ dB_u - \frac{1}{2} \int_0^T \frac{\partial D_t B_u}{\partial B_u} du + B_t \\
&= \int_0^T D_t B_u \circ dB_u + B_t. \quad \square
\end{aligned}$$

**Example 4.3.2.**

Apply the derivative operator  $D_t$  to the Stratonovich integral  $\int_0^T H_n(u, B_u) \circ dB_u$  where  $H_n(t, B_t)$  is Hermite polynomial defined by Definition 3.5.1. Then,

$$\begin{aligned}
&D_t \left( \int_0^T H_n(u, B_u) \circ dB_u \right) \\
&= D_t \left( \int_0^T H_n(u, B_u) dB_u \right) + \frac{1}{2} D_t \left( \int_0^T n H_{n-1}(u, B_u) du \right) \\
&= \int_0^T D_t H_n(u, B_u) dB_u + H_n(t, B_t) + \frac{1}{2} \int_0^T n D_t H_{n-1}(u, B_u) du \\
&= \int_0^T D_t H_n(u, B_u) \circ dB_u - \frac{1}{2} \int_0^T \frac{\partial D_t H_n(u, B_u)}{\partial B_u} du + \\
&\quad + \frac{n}{2} \int_0^T D_t H_{n-1}(u, B_u) du + H_n(t, B_t) \\
&= \int_0^T D_t H_n(u, B_u) \circ dB_u - \frac{n}{2} \int_0^T \frac{\partial H_{n-1}(u, B_u) I(u > t)}{\partial B_u} du + \\
&\quad + \frac{n(n-1)}{2} \int_t^T H_{n-2}(u, B_u) du + H_n(t, B_t) \\
&= \int_0^T D_t H_n(u, B_u) \circ dB_u + H_n(t, B_t).
\end{aligned}$$

In this example,  $n$  is larger than 1 because there exists a slight difference between  $n = 1$  and  $n > 1$  in specific calculation. The situation of  $n = 1$  has been already illustrated in the example 4.3.1.  $\square$

**Example 4.3.3.**

Let  $S_t = S_0 e^{at + \sigma B_t}$ , where  $a = \mu - \sigma^2/2$ . Apply the derivative operator  $D_t$  again to the Stratonovich integral  $\int_0^T S_u \circ dB_u$ . Then,

$$\begin{aligned}
& D_t \left( \int_0^T S_u \circ dB_u \right) \\
&= D_t \left( \int_0^T S_u dB_u \right) + \frac{1}{2} D_t \left( \int_0^T \sigma S_u du \right) \\
&= \int_0^T D_t S_u dB_u + S_t + \frac{1}{2} \int_0^T D_t \sigma S_u du \\
&= \int_0^T D_t S_u \circ dB_u - \frac{1}{2} \int_0^T \frac{\partial D_t S_u}{\partial B_u} du + \frac{1}{2} \int_0^T D_t \sigma S_u du + S_t \\
&= \int_0^T D_t S_u \circ dB_u - \frac{1}{2} \int_0^T \frac{\partial \sigma S_u I(u > t)}{\partial B_u} du + \frac{1}{2} \int_t^T \sigma^2 S_u du + S_t \\
&= \int_0^T D_t S_u \circ dB_u + S_t. \quad \square
\end{aligned}$$

**Lemma 4.3.3.** (*Duality formula for Stratonovich integral*)

Let  $u$  be a Skorohod integrable stochastic process and  $Y_t$  is defined by

$$Y_T = \int_0^T X_t \circ dW_t = \int_0^T X_t dW_t + \frac{1}{2} \int_0^T \sigma_t dt$$

(Stratonovich integrable), where

$$dX_t = \sigma_t dW_t + \mu_t dt$$

is adapted. Note that  $\sigma_t$  and  $\mu_t$  are also adapted. Suppose that

$$\int_0^T X_t dW_t \in \mathbb{D}_{1,2}$$

and

$$\int_0^T \sigma_t dt \in \mathbb{D}_{1,2}.$$

Then,

$$E \left[ Y_T \int_0^T u(t) \delta W_t \right] = E \left[ \int_0^T u(t) D_t Y_T dt \right].$$

*Proof.*

Since  $Y_T$  is defined by

$$\int_0^T X_t \circ dW_t,$$

it is equal to prove

$$\begin{aligned} & E \left[ \left( \int_0^T X_t \circ dW_t \right) \left( \int_0^T u(t) \delta W_t \right) \right] \\ &= E \left[ \int_0^T u(t) D_t \left( \int_0^T X_t \circ dW_t \right) dt \right]. \end{aligned}$$

The left hand side of the former equation is

$$\begin{aligned} & E \left[ \left( \int_0^T X_t \circ dW_t \right) \left( \int_0^T u(t) \delta W_t \right) \right] \\ &= E \left[ \left( \int_0^T X_t dW_t + \frac{1}{2} \int_0^T \sigma_t dt \right) \int_0^T u(t) \delta W_t \right] \\ &= E \left[ \left( \int_0^T X_t dW_t \right) \left( \int_0^T u(t) \delta W_t \right) \right] + \frac{1}{2} E \left[ \left( \int_0^T \sigma_t dt \right) \left( \int_0^T u(t) \delta W_t \right) \right] \end{aligned}$$

by linearity, and the right hand side of this equation is

$$\begin{aligned}
& E \left[ \int_0^T u(t) D_t \left( \int_0^T X_t \circ dW_t \right) dt \right] \\
&= E \left[ \int_0^T u(t) \left( \int_0^T D_t X_u dW_u + X_t + \frac{1}{2} \int_0^T D_t \sigma_u du \right) dt \right] \\
&= E \left[ \int_0^T u(t) \left( \int_0^T D_t X_u dW_u \right) dt \right] + E \left[ \int_0^T u(t) X_t dt \right] \\
&\quad + \frac{1}{2} E \left[ \int_0^T u(t) \left( \int_0^T D_t \sigma_u du \right) dt \right] \\
&= E \left[ \int_0^T u(t) D_t \left( \int_0^T X_u dW_u \right) dt \right] + \frac{1}{2} E \left[ \int_0^T u(t) \left( \int_0^T D_t \sigma_u du \right) dt \right]
\end{aligned}$$

by Lemma 4.3.2, linearity and Lemma 3.4.7. According to Lemma 3.4.3,

$$E \left[ \left( \int_0^T X_t dW_t \right) \left( \int_0^T u(t) \delta W_t \right) \right] = E \left[ \int_0^T u(t) D_t \left( \int_0^T X_u dW_u \right) dt \right]$$

and

$$E \left[ \left( \int_0^T \sigma_t dt \right) \left( \int_0^T u(t) \delta W_t \right) \right] = E \left[ \int_0^T u(t) \left( \int_0^T D_t \sigma_u du \right) dt \right].$$

This completes the proof.  $\square$

## 4.4 Stratonovich stochastic differential equation

Consider the Ito stochastic differential equation,

$$dX_t = f(t, X_t) dt + G(t, X_t) dW_t, X_{t_0} = 0$$

where  $G(t, x)$  is differentiable in  $x$ .

For Definition 3.6.1,

$$\begin{aligned} & \frac{1}{2} (G(t_i, X_{t_{i+1}}) + G(t_i, X_{t_i})) (W_{t_{i+1}} - W_{t_i}) = \\ & \frac{1}{2} (G(t_i, X_{t_{i+1}}) - G(t_i, X_{t_i})) (W_{t_{i+1}} - W_{t_i}) + G(t_i, X_{t_i}) (W_{t_{i+1}} - W_{t_i}), \end{aligned}$$

where  $t_{i+1}$  and  $t_i$  are time points of the partition  $0 = t_0 < t_1 < \dots < t_k = T$ .

The second term of this equation is

$$G(t_i, X_{t_i}) (W_{t_{i+1}} - W_{t_i}) \rightarrow G(t, X_t) dW_t$$

as the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  tends to 0 in  $L^2(P)$ .

Then, rewrite the first term, which is

$$\frac{1}{2} \frac{G(t_i, X_{t_{i+1}}) - G(t_i, X_{t_i})}{X_{t_{i+1}} - X_{t_i}} (X_{t_{i+1}} - X_{t_i}) (W_{t_{i+1}} - W_{t_i}).$$

Applying mean value theorem and the fact that  $X_t$  is an Ito process, it can be seen that

$$\frac{1}{2} \frac{G(t_i, X_{t_{i+1}}) - G(t_i, X_{t_i})}{X_{t_{i+1}} - X_{t_i}} \rightarrow \frac{1}{2} G_x(t, X_t)$$

and

$$X_{t_{i+1}} - X_{t_i} \rightarrow f(t, X_t) dt + G(t, X_t) dW_t$$

as the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  tends to 0 in  $L^2(P)$ .

Thus, using the fact that  $dt \cdot dW_t = 0$  and  $dW_t \cdot dW_t = dt$  ( see Appendix ),

$$\frac{1}{2} G_x(t, X_t) (f(t, X_t) dt + G(t, X_t) dW_t) dW_t = \frac{1}{2} G_x(t, X_t) G(t, X_t) dt.$$

For another definition of the Stratonovich integral,

$$\begin{aligned} & G\left(t_i, \frac{X_{t_{i+1}} + X_{t_i}}{2}\right) (W_{t_{i+1}} - W_{t_i}) \\ = & \left( G\left(t_i, \frac{X_{t_{i+1}} + X_{t_i}}{2}\right) - G(t_i, X_{t_i}) \right) (W_{t_{i+1}} - W_{t_i}) \\ & + G(t_i, X_{t_i}) (W_{t_{i+1}} - W_{t_i}), \end{aligned}$$

where  $t_{i+1}$  and  $t_i$  are time points of the partition  $0 = t_0 < t_1 < \dots < t_k = T$ .

The second term of this equation has the same representation as the former one. Then, rewrite the first term, which is

$$\frac{G\left(t_i, \frac{X_{t_{i+1}} + X_{t_i}}{2}\right) - G(t_i, X_{t_i})}{\frac{X_{t_{i+1}} + X_{t_i}}{2} - X_{t_i}} \left( \frac{X_{t_{i+1}} + X_{t_i}}{2} - X_{t_i} \right) (W_{t_{i+1}} - W_{t_i}).$$

Applying mean value theorem and the fact that  $X_t$  is an Ito process again, it can be seen that

$$\frac{G\left(t_i, \frac{X_{t_{i+1}} + X_{t_i}}{2}\right) - G(t_i, X_{t_i})}{\frac{X_{t_{i+1}} + X_{t_i}}{2} - X_{t_i}} \rightarrow G_x(t, X_t)$$

and

$$\frac{X_{t_{i+1}} + X_{t_i}}{2} - X_{t_i} = \frac{1}{2} (X_{t_{i+1}} - X_{t_i}) \rightarrow \frac{1}{2} (f(t, X_t) dt + G(t, X_t) dW_t).$$

as the mesh of the partition  $0 = t_0 < t_1 < \dots < t_k = T$  tends to 0 in  $L^2(P)$ .

Thus, using the fact that  $dt \cdot dW_t = 0$  and  $dW_t \cdot dW_t = dt$  ( see Appendix ),

$$\frac{1}{2} (f(t, X_t) dt + G(t, X_t) dW_t) G_x(t, X_t) dW_t = \frac{1}{2} G_x(t, X_t) G(t, X_t) dt.$$

Therefore, for both two definitions, the representation of the Stratonovich stochastic differential equation is

$$dX_t = \left( f(t, X_t) - \frac{1}{2} G_x(t, X_t) G(t, X_t) \right) dt + G(t, X_t) \circ dW_t, X_{t_0} = 0.$$

# Chapter 5

## Ito-Malliavin Type Equations

Since Paul Malliavin introduces Malliavin calculus in the 1970s, it has been wildly applied in finance. For instance, Fournie E. et al. use Malliavin calculus to calculate "Greeks" in their famous paper [13] ( There are similar calculations of "Greeks" in the lecture notes of Eulalia Nualart [37]. ) and Bernt Oksendal applies Malliavin calculus to the Black and Scholes formula as well [40].

The concept of stochastic differential equations (SDEs) is a big topic both in physics and mathematical finance. SDEs can be driven by different processes such as the Wiener process, the pure jump process and the Levy process. Many SDEs ( stochastic differential equations ) are already solved [31], such as first-order Ito equations. However, Malliavin calculus can be also applied to solving stochastic differential equations. For more details there are many references, such as the book written by Giulia Di Nunno et al. [38] and the paper written by David Nualart et al. [36].

The Ito-Malliavin type equations or simply called Ito-Malliavin equations can be treated as the inverse application of the Malliavin derivative. The aim of

introducing Ito-Malliavin type equations and other related equations is to find the unknown original process in these equations. In this chapter, concepts of Ito-Malliavin type equations and Stratonovich-Malliavin type equations are illustrated as a special series of SDEs ( stochastic differential equations ). Some related examples are also given to explain how to solve these equations.

## 5.1 Ito-Malliavin type equations

**Definition 5.1.1.** (*Ito-Malliavin type equations*)

Let  $X_u$  be an unknown  $\mathcal{F}_u$  adapted Gaussian process and  $X_u \in \mathbb{D}_{1,2}$ .  $F$  is a function with respect to the time  $t$ ,  $u$  and the Wiener processes  $W_t$ ,  $W_u$  and  $F(t, W_t, u, W_u)$  is  $\mathcal{F}_u$  adapted.  $D_t$  is the Malliavin derivative operator. Then, equations which have the form of

$$D_t X_u = F(t, W_t, u, W_u)$$

are called Ito-Malliavin type equations.

Notice that  $u > t$  is the only case considered in this chapter, since it is clear that  $D_t X_u = 0$  for all  $\mathcal{F}_u$  adapted Gaussian processes  $X_u$  when  $u < t$  according to Proposition 3.4.4. This is also the reason why  $F(t, W_t, u, W_u)$  is  $\mathcal{F}_u$  adapted in Definition 5.1.1.

Next, some specific Ito-Malliavin type equations are introduced here and these equations will be solved in the next section. The Equation A is an application of the Stroock lemma. The inspiration of the Equation B and the Equation C comes from the Brownian bridge which is a Gaussian process.

**Equation A:**

Find all Gaussian processes  $X_t$  which satisfy the equation A

$$D_t \int_0^T X_u dW_u = X_t + \int_0^T g(u, W_u) dW_u$$

where  $g(u, W_u)$  is a known  $\mathcal{F}_u$  measurable function with respect to  $u$  and  $W_u$ .

The initial value of the process  $X_0 = 0$ .

**Equation B:**

Find all Gaussian processes  $X_t$  which satisfy the equation B

$$D_t X_u = f(u)$$

where  $f(u)$  is a known continuous function of  $u$  with continuous derivatives up to order one.

**Equation C:**

Find all Gaussian processes  $X_t$  which satisfy the equation C

$$D_t X_u = f_1(u) f_2(t)$$

where  $f_1(u)$  is a known continuous function of  $u$  and  $f_2(t)$  is a known continuous function of  $t$ .

## 5.2 Solving different kinds of Ito-Malliavin type equations

In this section, each part illustrates one of the aforementioned Ito-Malliavin type equations as well as some related examples.

### 5.2.1 Equation A

Before solving this kind of Ito-Malliavin type equations, it is necessary to simplify this equation first. Applying Lemma 3.4.7 to the Equation A defined in 5.1, this equation becomes

$$\int_0^T D_t X_u dW_u = \int_0^T g(u, W_u) dW_u.$$

Then, this Ito-Malliavin type equation is equivalent to

$$D_t X_u = g(u, W_u)$$

where

$$g(u, W_u) = g(t, W_t) + \int_t^u \sigma_m dm + \int_t^u \mu_m dW_m.$$

Assume that

$$X_u = X_t + \int_t^u p_m dm + \int_t^u q_m dW_m.$$

Applying Lemma 3.4.7 again to  $X_u$ ,

$$D_t X_u = \int_t^u D_t p_m dm + \int_t^u D_t q_m dW_m + q_t.$$

Comparing  $g(u, W_u)$  and  $D_t X_u$  through using Lemma 3.1.3, solving this Ito-Malliavin type equation is equivalent to solving the set

$$\begin{cases} g(t, W_t) & = q_t \quad a.s. \\ \sigma_m & = D_t p_m \quad a.s.. \end{cases}$$

**Example 5.2.1.** *Solve the Ito-Malliavin type equation*

$$D_t \int_0^T X_u dB_u = X_t + B_T^2 - T,$$

where  $t$  is fixed. The initial value of this Gaussian process is  $X_0 = 0$ .

To simplify this Ito-Malliavin type equation, aforementioned method shows that

$$\int_0^T D_t X_u dB_u = 2 \int_0^T B_u dB_u$$

since  $B_T^2 - T = 2 \int_0^T B_u dB_u$ , and then

$$D_t X_u = 2B_u$$

which can be also represented as

$$D_t X_u = 2B_t + 2 \int_t^u dB_m.$$

Assume that

$$X_u = X_0 + \int_0^u p_m dm + \int_0^u q_m dB_m.$$

Then, combining all former results,

$$\begin{aligned} D_t X_u &= q_t + \int_t^u D_t p_m dm + \int_t^u D_t q_m dB_m \\ &= 2B_t + 2 \int_t^u dB_m. \end{aligned}$$

Therefore, this Ito-Malliavin type equation is equivalent to

$$\begin{cases} D_t p_m = 0 & a.s. & (1) \\ q_t = 2B_t & a.s. & (2). \end{cases}$$

Leave equation (1) here, it will be solved later.

The next example is more general than the former one.  $\square$

**Example 5.2.2.** *Solve the Ito-Malliavin type equation*

$$D_t \int_0^T X_u dB_u = X_t + H_n(T, B_T),$$

where  $H_n(T, B_T)$ ,  $n \geq 2$  is Hermite polynomials defined by Definition 3.5.1 and  $t$  is fixed. The initial value of this Gaussian process is  $X_0 = 0$ .

Simplify this equation to get that

$$\int_0^T D_t X_u dB_u = n \int_0^T H_{n-1}(u, B_u) dB_u$$

since  $H_n(T, B_T) = n \int_0^T H_{n-1}(u, B_u) dB_u$ , and then

$$D_t X_u = n H_{n-1}(u, B_u)$$

which can be also represented as

$$D_t X_u = n H_{n-1}(t, B_t) + n(n-1) \int_t^u H_{n-2}(m, B_m) dB_m.$$

Assume that

$$X_u = X_0 + \int_0^u p_m dm + \int_0^u q_m dB_m.$$

Then, combining all former results, it can be seen that

$$\begin{aligned} D_t X_u &= q_t + \int_t^u D_t p_m dm + \int_t^u D_t q_m dB_m \\ &= nH_{n-1}(t, B_t) + n(n-1) \int_t^u H_{n-2}(m, B_m) dB_m. \end{aligned}$$

Therefore, this Ito-Malliavin type equation is equivalent to

$$\begin{cases} D_t p_m = 0 & a.s. & (1) \\ q_t = nH_{n-1}(t, B_t) & a.s. & (2). \end{cases}$$

Similarly, here left the equation (1) where  $t$  is fixed, which will be solved later.

□

**Example 5.2.3.** *Solve the Ito-Malliavin type equation*

$$D_t \int_0^T X_u dB_u = X_t + S_T,$$

where  $S_T = S_0 e^{(-\sigma^2/2)T + \sigma B_T}$  and  $t$  is fixed. The initial value of this Gaussian process is  $X_0 = 0$ .

Simplify former equation to get that

$$\int_0^T D_t X_u dB_u = S_T = \int_0^T \sigma S_u dB_u,$$

and then

$$D_t X_u = \sigma S_u$$

which can be also illustrated as

$$D_t X_u = \sigma S_t + \sigma^2 \int_t^u S_m dB_m.$$

Assume that

$$X_u = X_0 + \int_0^u p_m dm + \int_0^u q_m dB_m.$$

Then, combining all former results, it can be seen that

$$\begin{aligned} D_t X_u &= q_t + \int_t^u D_t p_m dm + \int_t^u D_t q_m dB_m \\ &= \sigma S_t + \sigma^2 \int_t^u S_m dB_m. \end{aligned}$$

Therefore, this Ito-Malliavin type equation is equivalent to

$$\begin{cases} D_t p_m = 0 & a.s. & (1) \\ q_t = \sigma S_t & a.s. & (2). \end{cases}$$

Now, all three former examples have left the same problem which is how to solve

$$D_t p_m = 0,$$

where  $t$  is fixed and  $t < m < u$ . Split  $p_m$  of the former equation to two terms, which can be shown as

$$D_t \left( p_t + \int_t^m dp_v \right) = 0.$$

Since the operator  $D_t$  is linear, this problem becomes to solving

$$D_t p_t = 0$$

and

$$D_t \int_t^m dp_v = 0.$$

Here  $dp_v$  can be represented as

$$dp_v = \alpha_v dB_v + \beta_v dv,$$

where  $\alpha_v$  and  $\beta_v$  are two unknown processes, and therefore

$$\begin{aligned} D_t \int_t^m dp_v &= D_t \left( \int_t^m \alpha_v dB_v + \int_t^m \beta_v dv \right) \\ &= D_t \int_t^m \alpha_v dB_v + D_t \int_t^m \beta_v dv \\ &= \alpha_t + \int_t^m D_t \alpha_v dB_v + \int_t^m D_t \beta_v dv. \end{aligned}$$

Combine all aforementioned results,

$$\begin{cases} \alpha_t = 0 & a.s. \\ D_t \beta_v = 0 & a.s.. \end{cases}$$

It is meaningless to repeat solving the second equation of the former equation set, which means the equation  $D_t p_m = 0$  cannot be solved by this method.

Come back to the original problem and let

$$p_m(w) = \begin{cases} f(w) + h(w, B_w), & w < t, \\ f(w), & t \leq w < m, \end{cases}$$

where  $h$  is an arbitrary function with respect to  $w$  and  $B_w$ . For  $w < t$ ,

$$\begin{aligned}
D_t p_m(w) &= D_t (f(w) + h(w, B_w)) \\
&= D_t f(w) + D_t h(w, B_w) \\
&= 0 + h'_{B_w} I(w > t) \\
&= 0,
\end{aligned}$$

since  $I(w > t) = 0$ . For  $t \leq w < m$ ,

$$D_t p_m(w) = D_t f(w) = 0.$$

Here  $p_m(w)$  is a solution of the equation  $D_t p_m = 0$  for some fixed  $t$ . Then, for arbitrary  $t$  the solution of this equation becomes

$$p_m(w) = f(w).$$

Next, some crucial properties of this solution will be explained.

**Lemma 5.2.1.** (*Linearity*)

Assume that  $X_u$  and  $Y_u$  are respective solutions of

$$D_t \int_0^T X_u dW_u = X_t + f(W_T, T) \quad (1)$$

where  $f(W_t, t)$  is a known  $\mathcal{F}_t$  measurable function of  $t$  and  $W_t$  and

$$D_t \int_0^T Y_u dW_u = Y_t + g(W_T, T) \quad (2)$$

where  $g(W_t, t)$  is a known  $\mathcal{F}_t$  measurable function of  $t$  and  $W_t$ . Then,  $Z_u = \alpha X_u + \beta Y_u$  is the solution of

$$D_t \int_0^T Z_u dW_u = Z_t + \alpha f(W_T, T) + \beta g(W_T, T),$$

where  $\alpha$  and  $\beta$  are two constants.

*Proof.*

The equation  $\alpha \cdot (1) + \beta \cdot (2)$  is

$$\begin{aligned} & \alpha \left( D_t \int_0^T X_u dW_u \right) + \beta \left( D_t \int_0^T Y_u dW_u \right) \\ &= \alpha (X_t + f(W_T, T)) + \beta (Y_t + g(W_T, T)). \end{aligned}$$

By linearity of  $D_t$ , the left hand side of  $\alpha \cdot (1) + \beta \cdot (2)$  equals to

$$D_t \left( \alpha \int_0^T X_u dW_u + \beta \int_0^T Y_u dW_u \right),$$

and then, by linearity of the Ito integral, the former equation is equivalent to

$$D_t \left( \int_0^T (\alpha X_u + \beta Y_u) dW_u \right).$$

Therefore,  $Z_u = \alpha X_u + \beta Y_u$  satisfies the equation

$$D_t \int_0^T Z_u dW_u = \alpha X_t + \beta Y_t + \alpha f(W_T, T) + \beta g(W_T, T). \quad \square$$

**Lemma 5.2.2.** (*Uniqueness*)

Assume that  $X_u^{(1)}$  and  $X_u^{(2)}$  are two solutions of

$$D_t \int_0^T X_u dW_u = X_t + f(W_T, T)$$

where  $f(W_t, t)$  is a known  $\mathcal{F}_t$  measurable function of  $t$  and  $W_t$ . Suppose that  $t$  is not fixed. Then,

$$X_u^{(1)} = X_u^{(2)} \quad a.s..$$

*Proof.*

Let  $Y_u = X_u^{(1)} - X_u^{(2)}$ . According to Lemma 5.2.1, it is sufficient to prove that  $Y_u = 0$  is the almost surely solution of

$$D_t \int_0^T Y_u dW_u = 0. \quad (\star)$$

Then, using Lemma 4.1.1,

$$D_t \int_0^T Y_u dW_u = Y_t + \int_t^T D_t Y_u dW_u = 0.$$

Solving this Ito-Malliavin equation,  $Y_t = 0$  almost surely and

$$\int_t^T D_t Y_u dW_u = 0.$$

Therefore,

$$D_t Y_u = 0.$$

By Proposition 3.4.5,  $Y_u = C(u)$  for all  $u > t$ . Since  $t$  is not fixed,  $Y_u = 0$  is the almost surely solution of the  $(\star)$ . This completes the proof.  $\square$

**Lemma 5.2.3.** (*Convergence*)

Let  $\{X_t^{(m)} : t \geq 0, m \rightarrow \infty\}$  be a sequence of adapted stochastic processes with respect to a filtration  $\{\mathcal{F}_t : t \geq 0\}$  which satisfies

$$X_t^{(m)} \rightarrow X_t$$

in  $L^2(P)$  and

$$D_t X_u^{(m)} \rightarrow D_t X_u$$

in  $L^2(P \times \lambda)$ . Then,

$$D_t \int_0^T X_u^{(m)} dW_u \rightarrow D_t \int_0^T X_u dW_u$$

in  $L^2(P)$  with respect to  $m \rightarrow \infty$ .

*Proof.*

According to Lemma 3.1.2

$$\int_0^T X_u^{(m)} dW_u \rightarrow \int_0^T X_u dW_u$$

in  $L^2(P)$ . Let

$$Y_t^{(m)} = \int_0^t X_u^{(m)} dW_u.$$

Applying  $D_t$  to  $Y_T^{(m)}$ ,

$$D_t Y_T^{(m)} = X_t^{(m)} + \int_t^T D_t X_u^{(m)} dW_u$$

by Lemma 3.4.6. Similarly,

$$D_t Y_T = X_t + \int_t^T D_t X_u dW_u$$

Then,

$$D_t Y_T^{(m)} \rightarrow D_t Y_T$$

in  $L^2(P \times \lambda)$  with respect to  $m \rightarrow \infty$ , since  $D_t X_u^{(m)} \rightarrow D_t X_u$  in  $L^2(P \times \lambda)$ .

□

**Example 5.2.4.** (*Example on Lemma 5.2.3*)

Let  $X_t^{(m)}$  be as in Lemma 5.2.3 (excluding all solutions which do not satisfy  $D_t X_u^{(m)} \rightarrow D_t X_u$  in  $L^2(P \times \lambda)$ ). Consider a sequence of Ito-Malliavin type equations, which has the form of

$$D_t \int_0^T X_u^{(m)} dB_u = X_t^{(m)} + \frac{\alpha^m}{m!} H_m(T, B_T), m = 0, 1, \dots, n,$$

where  $H_m(T, B_T)$  are Hermite polynomials defined by Definition 3.5.1. Then, summing both sides of the equations,

$$D_t \int_0^T \sum_{m=0}^n X_u^{(m)} dB_u = \sum_{m=0}^n X_t^{(m)} + \sum_{m=0}^n \frac{\alpha^m}{m!} H_m(T, B_T).$$

Assume that

$$Y_t^n = \sum_{m=0}^n X_t^{(m)} \rightarrow Y_t = \lim_{n \rightarrow \infty} \sum_{m=0}^n X_t^{(m)}$$

in  $L^2(P)$ . It can be seen that

$$\sum_{m=0}^n \frac{\alpha^m}{m!} H_m(T, B_T) \rightarrow \exp \left\{ \alpha T - \frac{\alpha^2}{2} B_T \right\}$$

in  $L^2(P)$  when  $n \rightarrow \infty$  due to Lemma 3.5.1. According to Lemma 5.2.3,

$$\begin{aligned} D_t \int_0^T Y_u^{(n)} dB_u &= D_t \int_0^T \sum_{m=0}^n X_u^{(m)} dB_u \\ \rightarrow D_t \int_0^T Y_u dB_u &= \lim_{n \rightarrow \infty} D_t \int_0^T \sum_{m=0}^n X_u^{(m)} dB_u \end{aligned}$$

in  $L^2(P \times \lambda)$ . Let

$$\begin{cases} S_0 = 1, \\ \alpha = \mu - \frac{\sigma^2}{2}, \\ -\frac{\alpha^2}{2} = \sigma. \end{cases}$$

Then,

$$D_t \int_0^T Y_u dW_u = Y_t + S_0 \left\{ \left( \mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\}.$$

Note that this kind of Ito-Malliavin type equations is exactly the one introduced in Example 5.2.3 when  $\mu = 0$ , which has been solved.

## 5.2.2 Equation B

Note that the method solving the Equation A in 5.2.1 cannot be applied here to solve the Equation B defined in 5.1. To see this, first assume that

$$X_u = X_0 + \int_0^u \alpha_s ds + \int_0^u \beta_s dB_s,$$

where  $\alpha_s$  and  $\beta_s$  are  $\mathcal{F}_s$  adapted. Then, applying Lemma 3.4.7 to  $X_u$ ,

$$D_t X_u = \beta_t + \int_t^u D_t \beta_s dB_s + \int_t^u D_t \alpha_s ds.$$

Let  $D_t X_u = M_u$  for all  $u > t$ . Then,  $M_t = \beta_t$ . The differential form of  $M_u$  is

$$dM_u = D_t \beta_u dB_u + D_t \alpha_u du$$

for all  $u > t$ . Let  $f(u) = Q_u$  which is a function of  $u$  and a constant with respect to  $W_u$ , and then

$$dQ_u = f'(u) du.$$

Since the representation of the Equation B is  $D_t X_u = f(u)$ ,

$$\begin{cases} D_t \alpha_u = f'(u) & a.s. \\ D_t \beta_u = 0 & a.s. \\ \beta_t = M_t = Q_t = f(t) & a.s.. \end{cases}$$

Then, former set is equivalent to

$$\begin{cases} X_u = X_0 + \int_0^u \alpha_s ds + \int_0^u f(s) dB_s & a.s. \\ D_t \alpha_u = f'(u) & a.s.. \end{cases}$$

It can be seen that this method is complicated in solving equation B if the higher-order derivative of  $f(u)$  is still differentiable with respect to  $u$ . It even produces a recurrence in solving the Equation B if  $f(u)$  are exponential functions. Next example is a case which can be solved directly by this method, since  $f'(u) = 0$ .

**Example 5.2.5.** *Solve the Ito-Malliavin type equation*

$$D_t X_u = C$$

where  $C$  is a constant.

Assume that

$$X_u = X_0 + \int_0^u \alpha_s ds + \int_0^u \beta_s dB_s,$$

where  $\alpha_s$  and  $\beta_s$  are  $\mathcal{F}_s$  adapted. Then, for  $u > t$ ,

$$D_t X_u = \beta_t + \int_t^u D_t \beta_s dB_s + \int_t^u D_t \alpha_s ds = M_u.$$

Then,

$$\begin{cases} D_t \alpha_u = 0 & a.s. \\ D_t \beta_u = 0 & a.s. \\ \beta_t = C & a.s.. \end{cases}$$

Therefore,

$$\begin{aligned} X_u &= X_0 + \phi_u + C B_u \\ &= \widetilde{\phi}_u + C B_u \end{aligned}$$

where  $\widetilde{\phi}_u$  is an arbitrary non-random function of  $u$ .  $\square$

The Equation B can be solved by applying former method  $n$  times if  $f(u)$  is a known continuous function of  $u$  with continuous derivative up to order  $n$  and  $f^{(n)}(u) = 0$ . Next, an improved method is introduced to solve a more general case.

Assume that

$$X_u = g(u) + h(u) \widetilde{X}_u,$$

where  $g(u)$  and  $h(u) \neq 0$  are non-random function and  $\widetilde{X}_u$  is a continuous martingale. By the chain rule of the Malliavin derivative ( Proposition 3.4.3 ) and the constant rule of the Malliavin derivative ( Proposition 3.4.5 )

$$D_t X_u = h(u) D_t \widetilde{X}_u.$$

Since  $\widetilde{X}_u$  is a continuous martingale, assume that

$$\widetilde{X}_u = \widetilde{X}_0 + \int_0^u \widetilde{\beta}_s dB_s$$

where  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. The entire reason of setting  $dt$  term to zero can be found from Chapter 7 Section 5 of "A First Course in Stochastic Processes" [23] and Chapter 15 Section 12 of "A Second Course in Stochastic Processes" [24]. Applying Lemma 3.4.7 to  $\widetilde{X}_u$ ,

$$D_t \widetilde{X}_u = \widetilde{\beta}_t + \int_t^u D_t \widetilde{\beta}_s dB_s.$$

Then,

$$D_t X_u = h(u) \widetilde{\beta}_t + h(u) \int_t^u D_t \widetilde{\beta}_s dB_s = f(u).$$

Now, it is equivalent to solve the equation set

$$\begin{cases} D_t \widetilde{\beta}_s = 0 & a.s. \\ h(u) \widetilde{\beta}_t = f(u) & a.s.. \end{cases}$$

Therefore,  $\widetilde{\beta}_t = f(u)/h(u)$  for all  $t < u$ . According to the constant rule of the Malliavin derivative ( Proposition 3.4.5 ),  $\widetilde{\beta}_t = C$  where  $C$  is a constant since  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. Then,  $h(u) = Cf(u)$ . Finally, after substitute all preceding results into  $X_u$ ,

$$\begin{aligned}
X_u &= g(u) + h(u) \widetilde{X}_0 + h(u) \int_0^u \frac{f(s)}{h(s)} dB_s \\
&= g(u) + C f(u) \widetilde{X}_0 + f(u) B_u \\
&= \phi_u + f(u) B_u,
\end{aligned}$$

where  $\phi_u$  is an arbitrary non-random function of  $u$ .

**Example 5.2.6.** *Solve the Ito-Malliavin type equation*

$$D_t X_u = e^u.$$

According to the aforementioned method, assume that

$$X_u = g(u) + h(u) \widetilde{X}_u,$$

where  $g(u)$  and  $h(u) \neq 0$  are non-random function and

$$\widetilde{X}_u = \widetilde{X}_0 + \int_0^u \widetilde{\beta}_s dB_s,$$

where  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. Then,

$$D_t \widetilde{X}_u = \widetilde{\beta}_t + \int_t^u D_t \widetilde{\beta}_s dB_s,$$

and therefore

$$\begin{aligned}
D_t X_u &= h(u) \widetilde{X}_u \\
&= h(u) \widetilde{\beta}_t + h(u) \int_t^u D_t \widetilde{\beta}_s dB_s \\
&= e^u.
\end{aligned}$$

Finally, it can be seen that

$$X_u = \phi_u + e^u B_u$$

where  $\phi_u$  is an arbitrary non-random function of  $u$ .  $\square$

Similarly, some crucial properties of  $X_u$  are illustrated next.

**Lemma 5.2.4.** (*Linearity*)

*Assume that  $X_u$  and  $Y_u$  are respective solutions of*

$$D_t X_u = f(u) \quad (1)$$

*where  $f(u)$  is a continuous function of  $u$  and*

$$D_t Y_u = g(u) \quad (2)$$

*where  $g(u)$  is a continuous function of  $u$ . Then,  $\alpha X_u + \beta Y_u$  is the solution of*

$$D_t Z_u = \alpha f(u) + \beta g(u)$$

*where  $\alpha$  and  $\beta$  are two constants.*

*Proof.*

The equation  $\alpha \cdot (1) + \beta \cdot (2)$  is

$$\alpha (D_t X_u) + \beta (D_t Y_u) = \alpha f(u) + \beta g(u).$$

By linearity of  $D_t$ , the left hand side of  $\alpha \cdot (1) + \beta \cdot (2)$  equals to

$$D_t(\alpha X_u + \beta Y_u).$$

Therefore,  $Z_u = \alpha X_u + \beta Y_u$  satisfies the equation

$$D_t Z_u = D_t \alpha f(u) + \beta g(u). \quad \square$$

**Lemma 5.2.5.** (*Uniqueness*)

Assume that  $X_u^{(1)}$  and  $X_u^{(2)}$  are two solutions of

$$D_t X_u = f(u)$$

where  $f(u)$  is a continuous function of  $u$ . Then,

$$X_u^{(1)} = X_u^{(2)} + C(u) \quad a.s.$$

where  $C(u)$  is a function of  $u$ .

*Proof.*

Let  $Y_u = X_u^{(1)} - X_u^{(2)} = C(u)$ . By Lemma 5.2.4, it suffices to prove that

$Y_u = C(u)$  is the almost surely solution of

$$D_t Y_u = 0.$$

Then, by constant rule of the Malliavin derivative ( Proposition 3.4.5 ),

$$Y_u = C(u).$$

This completes the proof.  $\square$

**Lemma 5.2.6.** (*Convergence*)

Let  $\{X_t^{(m)} : t \geq 0, m \rightarrow \infty\}$  be sequence of adapted stochastic processes with respect to a filtration  $\{\mathcal{F}_t : t \geq 0\}$  which satisfies

$$X_t^{(m)} \rightarrow X_t$$

in  $L^2(P)$ . Then,

$$D_t X_u^{(m)} \rightarrow D_t X_u$$

in  $L^2(P \times \lambda)$  with respect to  $m \rightarrow \infty$ .

*proof.*

This is a direct result of Lemma 3.4.1.  $\square$

### 5.2.3 Equation C

The method to solve the Equation C defined in 5.1 is similar to the second method of the Equation B introduced in 5.2.2.

First, assume that

$$X_u = g(u) + h(u) \widetilde{X}_u$$

where  $g(u)$  and  $h(u) \neq 0$  are non-random function and  $\widetilde{X}_u$  is a continuous martingale. By the chain rule of the Malliavin derivative ( Proposition 3.4.3 ) and the constant rule of the Malliavin derivative ( Proposition 3.4.5 )

$$D_t X_u = h(u) \widetilde{X}_u.$$

Since  $\widetilde{X}_u$  is a continuous martingale, assume that

$$\widetilde{X}_u = \widetilde{X}_0 + \int_0^u \widetilde{\beta}_s dB_s$$

where  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. Applying Lemma 3.4.7 to  $\widetilde{X}_u$ ,

$$D_t \widetilde{X}_u = \widetilde{\beta}_t + \int_t^u D_t \widetilde{\beta}_s dB_s.$$

Then,

$$D_t X_u = h(u) \widetilde{\beta}_t + h(u) \int_t^u D_t \widetilde{\beta}_s dB_s = f_1(u) f_2(t).$$

Now, it is equivalent to solve

$$\begin{cases} D_t \widetilde{\beta}_s = 0 & a.s. \\ h(u) \widetilde{\beta}_t = f_1(u) f_2(t) & a.s.. \end{cases}$$

Therefore,  $\widetilde{\beta}_t = (f_1(u) f_2(t)) / h(u)$  for all  $t < u$ . According to the constant rule of the Malliavin derivative ( Proposition 3.4.5 ),  $\widetilde{\beta}_t = C f_2(t)$  where  $C$  is a constant since  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. Then,  $h(u) = C f_1(u)$ . Finally, after substitute all preceding results into  $X_u$ ,

$$\begin{aligned} X_u &= g(u) + h(u) \widetilde{X}_0 + h(u) \int_0^u \frac{f_1(u) f_2(s)}{h(u)} dB_s \\ &= g(u) + C f_1(u) \widetilde{X}_0 + f_1(u) \int_0^u f_2(s) dB_s \\ &= \phi_u + f_1(u) \int_0^u f_2(s) dB_s \end{aligned}$$

where  $\phi_u$  is an arbitrary non-random function of  $u$ .

The next example is related to the Example 3.7.1 which is about the Brownian bridge.

**Example 5.2.7.** *Solve the Ito-Malliavin type equation*

$$D_t Y_u = \frac{1-u}{1-t}$$

for all  $u > t$ .

Assume that

$$Y_u = f(u) + g(u) \widetilde{Y}_u$$

where  $f(u)$  and  $g(u) \neq 0$  are non-random function and

$$\widetilde{Y}_u = \widetilde{Y}_0 + \int_0^u \widetilde{\beta}_s dB_s$$

where  $\widetilde{\beta}_s$  is  $\mathcal{F}_s$  adapted. Then,

$$D_t \widetilde{Y}_u = \widetilde{\beta}_t + \int_t^u D_t \widetilde{\beta}_s dB_s,$$

and therefore

$$\begin{aligned} D_t Y_u &= g(u) D_t \widetilde{Y}_u \\ &= g(u) \widetilde{\beta}_t + g(u) \int_t^u D_t \widetilde{\beta}_s dB_s \\ &= \frac{1-u}{1-t}. \end{aligned}$$

It can be seen that

$$Y_u = \phi_u + (1-u) \int_0^u \frac{1}{1-s} dB_s$$

where  $\phi_u$  is an arbitrary non-random function of  $u$  by aforementioned method.

□

Note that the Brownian bridge introduced in Example 3.7.1 is a special case of the solution of former Ito-Malliavin type equation. This is the inspiration of the Equation C comes from.

Next, some crucial properties of  $X_u$  are given here.

**Lemma 5.2.7.** (*Linearity*)

*Assume that  $X_u$  and  $Y_u$  are respective solutions of*

$$D_t X_u = f_1(u) f_2(t) \quad (1)$$

*where  $f_1(u)$  is a continuous function of  $u$  and  $f_2(t)$  is a continuous function of  $t$  and*

$$D_t Y_u = g_1(u) g_2(t) \quad (2)$$

*where  $g_1(u)$  is a continuous function of  $u$  and  $g_2(t)$  is a continuous function of  $t$ . Then,  $\alpha X_u + \beta Y_u$  is the solution of*

$$D_t Z_u = \alpha f_1(u) f_2(t) + \beta g_1(u) g_2(t)$$

*where  $\alpha$  and  $\beta$  are two constants.*

*Proof.*

The equation  $\alpha \cdot (1) + \beta \cdot (2)$  is

$$\alpha (D_t X_u) + \beta (D_t Y_u) = \alpha f_1(u) f_2(t) + \beta g_1(u) g_2(t).$$

By linearity of  $D_t$ , the left hand side of  $\alpha \cdot (1) + \beta \cdot (2)$  equals to

$$D_t(\alpha X_u + \beta Y_u).$$

Therefore,  $Z_u = \alpha X_u + \beta Y_u$  satisfies the equation

$$D_t Z_u = \alpha f_1(u) f_2(t) + \beta g_1(u) g_2(t). \quad \square$$

**Lemma 5.2.8.** (*Uniqueness*)

Assume that  $X_u^{(1)}$  and  $X_u^{(2)}$  are two solutions of

$$D_t X_u = f_1(u) f_2(t)$$

where  $f(u)$  is a continuous function of  $u$ . Then

$$X_u^{(1)} = X_u^{(2)} + C(u) \quad a.s.$$

where  $C(u)$  is a function of  $u$ .

*Proof.*

Let  $Y_u = X_u^{(1)} - X_u^{(2)} = C(u)$ . By Lemma 5.2.7 it suffices to prove that

$Y_u = C(u)$  is the almost surely solution of

$$D_t Y_u = 0.$$

Then, by constant rule of the Malliavin derivative ( Proposition 3.4.5 )

$$Y_u = C(u).$$

This completes the proof.  $\square$

**Lemma 5.2.9.** (*Convergence*)

Let  $\{X_t^{(m)} : t \geq 0, m \rightarrow \infty\}$  be a sequence of adapted stochastic processes with respect to a filtration  $\{\mathcal{F}_t : t \geq 0\}$  which satisfies

$$X_t^{(m)} \rightarrow X_t$$

in  $L^2(P)$ . Then,

$$D_t X_u^{(m)} \rightarrow D_t X_u$$

in  $L^2(P \times \lambda)$  with respect to  $m \rightarrow \infty$ .

*Proof.*

This property is exactly the same as Lemma 5.2.6.  $\square$

### 5.3 Stratonovich-Malliavin type equations

As an extension of the Ito-Malliavin type equations, the definition of the Stratonovich-Malliavin type equations is given here.

**Definition 5.3.1.** (*Stratonovich-Malliavin type equations*)

Assume that  $X_t$  is an unknown adapted Gaussian process and  $g(u, W_u)$  is a known  $\mathcal{F}_u$  measurable function where  $W_u$  is a Wiener process. Then equations of the form

$$D_t \int_0^T X_u \circ dW_u = X_t + \int_0^T g(u, W_u) \circ dW_u$$

are called the Stratonovich-Malliavin type equations.

According to Lemma 4.3.2,

$$D_t \int_0^T X_u \circ dW_u = X_t + \int_0^T D_t X_u \circ dW_u.$$

By Definition 5.3.1, compare the former equation with the Stratonovich-Malliavin type equations and obtain

$$\int_0^T D_t X_u \circ dW_u = \int_0^T g(u, W_u) \circ dW_u.$$

Then, by Lemma 4.2.2, the Stratonovich-Malliavin type equations are equivalent to

$$D_t X_u = g(u, W_u).$$

To solve these equations, use exactly the same method that one solve the Equation A of the Ito-Malliavin type equations.

Here is an example of the Stratonovich-Malliavin type equations.

**Example 5.3.1.** *Solve the Stratonovich-Malliavin type equation*

$$D_t \int_0^T X_u \circ dB_u = X_t + \int_0^T nH_{n-1}(u, B_u) \circ dB_u$$

where  $H_n(T, B_T)$ ,  $n \geq 2$  is the Hermite polynomial defined by Definition 3.5.1 and  $t$  is fixed. The initial value of the Gaussian process is  $X_0 = 0$ .

By Lemma 4.3.2 and Definition 5.3.1,

$$\int_0^T D_t X_u \circ dW_u = \int_0^T nH_{n-1}(u, B_u) \circ dB_u.$$

Then, by Lemma 4.2.2,

$$D_t X_u = n H_{n-1}(u, B_u).$$

Assume that

$$X_u = X_0 + \int_0^u p_m dm + \int_0^u q_m dB_m.$$

Combining former results, it can be seen that

$$\begin{aligned} D_t X_u &= q_t + \int_t^u D_t p_m dm + \int_t^u D_t q_m dB_m \\ &= n H_{n-1}(t, B_t) + n(n-1) \int_t^u H_{n-2}(m, B_m) dB_m. \end{aligned}$$

Therefore, this Stratonovich-Malliavin type equation is equivalent to

$$\begin{cases} D_t p_m &= 0 \quad a.s. \\ q_t &= n H_{n-1}(t, B_t) \quad a.s. \\ D_t q_m &= n(n-1) H_{n-2}(m, B_m), \end{cases}$$

which has been solved in Example 5.2.2.

## Part 2 :

# A General Stroock Lemma

Main definitions from the discrete Malliavin calculus, the generating function and a derivative of a nature number are introduced in Chapter 6. Chapter 7 contains several examples and lemmas on delta operator and the discrete Stratonovich integral. Three types of the Stroock lemma are the main results in Chapter 8 based on [10].

# Chapter 6

## Terminology and Methods

In this chapter, all crucial notations and methods are introduced in details including definitions and properties. These concepts will be applied throughout this part. Numerous important lemmas with proofs are also given here.

### 6.1 Discrete Malliavin calculus

In a real situation, the discrete version of Malliavin calculus along with the binomial tree can be applied to calculating Greeks according to Yoshifumi Muroi et al. in their paper [35].

In this section, basic knowledge about the discrete Malliavin calculus is given. This section follows Martin Leitz-Martini [33] and H. Holden et al. [18] [19]. There are more interesting details in papers written by Martin Leitz-Martini [33] and Nicolas Privault [46].

### 6.1.1 Definitions, notations and propositions

Let  $N \in \mathbb{N}$  and  $\Delta t = 1/N$ . Then

$$\Lambda = \{0, \Delta t, \dots, (N-1) \Delta t\}$$

is a discrete time set. The uniform counting measure assigns  $\mu(A) = |A|/N$  for  $A \subset \Lambda$ . The discrete version of the Lebesgue space  $([0, 1], \mathcal{B}, \lambda)$  is  $(\Lambda, \mathcal{P}(\Lambda), \mu)$ .

According to Martin Leitz-Martini, each element in the set

$$\Omega = \{\omega | \omega : \Lambda \rightarrow \{-1, 1\}\}$$

can be treated as a Bernoulli random variable. On  $\mathcal{P}(\Omega)$  the uniform probability measure  $P$  for every subset  $S \subset \Omega$  is given by  $P(S) = |S|/|\Omega| = |S|/2^N$ . The space  $\mathfrak{L}^2(\Omega, P)$  with the inner product

$$\langle \mathfrak{x}, \mathfrak{y} \rangle_{\mathfrak{L}^2} = \sum_{\omega \in \Omega} \mathfrak{x}(\omega) \mathfrak{y}(\omega) P(\omega)$$

is the discrete version of the Wiener space with respect to  $P$ . The dimension of the space  $\mathfrak{L}^2(\Omega, P)$  is  $2^N$ .

**Definition 6.1.1.** For  $A \in \mathcal{P}(\Lambda)$ , the functions  $\chi_A : \Omega \rightarrow \mathbb{R}$  are defined by

$$\chi_A(\omega) = \prod_{s \in A} \omega(s).$$

**Proposition 6.1.1.** The set  $\{\chi_A\}_{A \in \mathcal{P}(\Lambda)}$  is a basis for  $\mathfrak{L}^2(\Omega, P)$ .

To simplify notations, set

$$\mathcal{P}_n = \{A \in \mathcal{P}(\Lambda) : |A| = n\}$$

and

$$\mathcal{P} = \mathcal{P}(\Lambda) = \dot{\cup}_n \mathcal{P}_n,$$

where  $\dot{\cup}_n \mathcal{P}_n = \cup_n \mathcal{P}_n - \sum_{i,j=1,\dots,n;i \neq j} \mathcal{P}_i \cap \mathcal{P}_j$ .

**Definition 6.1.2.** (*Walsh decomposition*)

For  $\mathfrak{X} \in L^2(\Omega, P)$ , the Walsh decomposition of  $\mathfrak{X}$  is

$$\mathfrak{X} = \sum_{A \in \mathcal{P}} \mathfrak{X}(A) \chi_A = \sum_n \sum_{A \in \mathcal{P}_n} \mathfrak{X}(A) \chi_A.$$

**Definition 6.1.3.** (*Wick product*)

Let  $\mathfrak{X} = \sum_{A \in \mathcal{P}} \mathfrak{X}(A) \chi_A$  and  $\mathfrak{Y} = \sum_{B \in \mathcal{P}} \mathfrak{Y}(B) \chi_B$  be random variables. Then the Wick product  $\mathfrak{X} \diamond \mathfrak{Y}$  is defined by

$$\mathfrak{X} \diamond \mathfrak{Y} = \sum_{C \in \mathcal{P}} \left( \sum_{A \dot{\cup} B = C} \mathfrak{X}(A) \mathfrak{Y}(B) \right) \chi_C,$$

where  $A \dot{\cup} B = A \cup B - A \cap B$ .

Note that

$$\chi_A \diamond \chi_B = \begin{cases} \chi_{A \dot{\cup} B} = \chi_A \cdot \chi_B, & \text{if } A \cap B = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 6.1.4.** (*discrete analogue*)

(1) A discrete stochastic process is a family of random variables  $(\mathfrak{X}_s)_{s \in \Lambda}$ , i.e. a map  $\mathfrak{X} : \Omega \times \Lambda \rightarrow \mathbb{R}$  such that for each fixed  $s \in \Lambda$  the map  $\mathfrak{X}(\cdot, s)$  is in  $\mathfrak{L}^2(\Omega, P)$ .

(2) The discrete Brownian motion  $\mathfrak{B}$  is the random walk

$$\mathfrak{B} : \Omega \times \Lambda \rightarrow \mathbb{R}, \mathfrak{B}(\omega, t) = \sum_{s < t} \omega(s) \sqrt{\Delta t}.$$

(3) The forward increment of  $\mathfrak{B}$  is defined by

$$\Delta \mathfrak{B}_t = \Delta \mathfrak{B}(\omega, t) = \mathfrak{B}(\omega, t + \Delta t) - \mathfrak{B}(\omega, t) = \omega(t) \sqrt{\Delta t}.$$

(4) Let  $(\mathfrak{X}_s)_{s \in \Lambda}$  be an adapted discrete stochastic process. Then the Ito integral is defined by

$$\int \mathfrak{X} d\mathfrak{B} = \int \mathfrak{X}_s d\mathfrak{B}_s = \sum_s \mathfrak{X}_s \cdot \Delta \mathfrak{B}_s.$$

To compare with the continuous time theory, Martin Leitz-Martini defines the symmetric functions to be zero on diagonals since he points out that the diagonals do not have measure zero in a discrete measure space in his work [33].

Let  $\mathfrak{X} = \sum_{A \in \mathcal{P}} \mathfrak{X}(A) \chi_A$  be the Walsh decomposition of  $\mathfrak{X}$ . For  $n > 0$  the symmetric function  $\mathfrak{X}_n$  on  $\Lambda^n$  is defined by

$$\mathfrak{X}_n(t_1, \dots, t_n) = \begin{cases} (\Delta t^{\frac{n}{2}} n!)^{-1} \mathfrak{X}(\{t_1, \dots, t_n\}), & \text{if } t_i \neq t_j \text{ for } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

where  $\mathfrak{X}(\{t_1, \dots, t_n\})$  is the Walsh component to  $A = \{t_1, \dots, t_n\}$ . For  $n = 0$ ,  $\mathfrak{X}_0 = \mathfrak{X}(\emptyset) = E[\mathfrak{X}]$ . According to these definitions,

$$\begin{aligned}
\mathfrak{X} &= \sum_{A \in \mathcal{P}} \mathfrak{X}(A) \chi_A \\
&= \sum_n \sum_{A \in \mathcal{P}_n} \mathfrak{X}(A) \chi_A \\
&= \sum_n \sum_{\{t_1, \dots, t_n\} \in \mathcal{P}_n} \mathfrak{X}(\{t_1, \dots, t_n\}) \omega(t_1) \cdots \omega(t_n) \\
&= \sum_n \sum_{\substack{(t_1, \dots, t_n) \in \Lambda^n \\ t_1 < \dots < t_n}} n! \mathfrak{X}_n(\{t_1, \dots, t_n\}) \Delta t^{\frac{n}{2}} \omega(t_1) \cdots \omega(t_n) \\
&= \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^n} \mathfrak{X}_n(\{t_1, \dots, t_n\}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n)
\end{aligned}$$

which is the discrete version of the Wiener-Ito chaos decomposition for random variables  $\mathfrak{X} \in \mathcal{L}^2(\Omega, P)$ .

### 6.1.2 Conditional expectations

For  $B \subset \Lambda$ ,  $\mathcal{F}_B$  is the  $\sigma$ -algebra on  $\Omega$  generated by the random variables  $\{\omega(s) : s \in B\}$ .

**Proposition 6.1.2.** *Let  $\mathfrak{X} = \sum_{A \subset \Lambda} \mathfrak{X}(A) \chi_A$  and  $\mathcal{F}_B$  be given. Then the conditional expectation of  $\mathfrak{X}$  with respect to  $\mathcal{F}_B$  is given by*

$$E[\mathfrak{X} | \mathcal{F}_B] = \sum_{A \subset B} \mathfrak{X}(A) \chi_A.$$

**Proposition 6.1.3.** *Let  $A, B \subset \Lambda$  and  $\mathfrak{X}, \mathfrak{Y} \in \mathcal{L}^2(\Omega, P)$ . Assume  $A \cap B = \emptyset$  and that  $\mathfrak{X}$  is  $\mathcal{F}_A$ -measurable and  $\mathfrak{Y}$  is  $\mathcal{F}_B$ -measurable. Then*

$$\mathfrak{X} \diamond \mathfrak{Y} = \mathfrak{X} \cdot \mathfrak{Y}.$$

Martin Leitz-Martini introduces the  $\sigma$ -algebras which constitute the discrete filtration in his definition. In these algebras, the information of the present is excluded.

**Definition 6.1.5.** For  $t \in \Lambda$  the past algebra is defined by

$$\begin{aligned} \mathcal{F}_t &= \sigma - \text{alg} [\{\omega(s) \mid s < t\}] \\ &= \sigma - \text{alg} [\{\{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = 1\}\} \mid s < t] \end{aligned}$$

A random variable  $\mathfrak{X}$  is said to be  $\mathcal{F}_t$ -adapted if

$$E[\mathfrak{X} \mid \mathcal{F}_t] = \mathfrak{X}.$$

This means that the Walsh decomposition of  $\mathfrak{X}$  has the form

$$\mathfrak{X} = \sum_{A \subset [0, t)} \mathfrak{X}(A) \chi_A$$

with

$$[0, t) = \{s \in A : s < t\}.$$

A discrete stochastic process  $(\mathfrak{X}_t)_{t \geq 0}$  is adapted if the random variable  $\mathfrak{X}_t$  is  $\mathcal{F}_t$ -adapted for each  $t \in \Lambda$ .

**Proposition 6.1.4.** For every process  $(\mathfrak{X}_t)_{t \in \Lambda}$  with Walsh decomposition  $\mathfrak{X}_t = \sum_{A \subset \Lambda} \mathfrak{X}(A; t) \chi_A$ ,

$$E[\mathfrak{X}_t | \mathcal{F}_t] = \sum_{A \subset [0, t)} \mathfrak{X}(A; t) \chi_A = \sum_{\substack{A \subset \Lambda \\ \max A < t}} \mathfrak{X}(A; t) \chi_A.$$

### 6.1.3 Discrete Skorohod integral

**Definition 6.1.6.** (discrete Skorohod integral)

Let  $\mathfrak{X} : \Omega \times \Lambda \rightarrow \mathbb{R}$  be a discrete stochastic process. The Skorohod integral of  $\mathfrak{X}$  with respect to the discrete Brownian motion  $\mathfrak{B}$  is defined by

$$\int \mathfrak{X} \delta \mathfrak{B} = \int \mathfrak{X}_s \delta \mathfrak{B}_s = \sum_{s \in \Lambda} \mathfrak{X}_s \diamond \Delta \mathfrak{B}_s = \sum_{s \in \Lambda} \mathfrak{X}_s \diamond \chi_{\{s\}} \sqrt{\Delta t}.$$

**Proposition 6.1.5.** Let

$$\mathfrak{X}_s = \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^n} \mathfrak{X}_{n,s}(t_1, \dots, t_n) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n)$$

be the discrete Wiener-Ito decomposition of the discrete process  $\mathfrak{X}_s$ .

(1) If the discrete stochastic process  $\mathfrak{X}_s$  is adapted then the Skorohod integral reduces to the Ito integral.

(2)

$$\int \mathfrak{X}_s \delta \mathfrak{B}_s = \sum_n \sum_{(t_1, \dots, t_{n+1}) \in \Lambda_{n+1}} \hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{n+1})$$

whereby  $\hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1})$  is the symmetrization of the coefficient function  $\mathfrak{X}_{n,s}(t_1, \dots, t_n)$  with respect to the process variable  $s$ .

*Proof.*

(1) Take  $A < s$  as notation for  $\max A < s$ . Since  $\mathfrak{X}_s$  is adapted it has the Walsh decomposition  $\mathfrak{X}_s = \sum_{A < s} \mathfrak{X}(A; s) \chi_A$ . Hence  $A$  and  $\{s\}$  are disjoint,

$$\begin{aligned}
\int \mathfrak{X}_s \delta \mathfrak{B}_s &= \sum_s \sum_{A < s} \mathfrak{X}(A; s) \chi_A \diamond \chi_{\{s\}} \sqrt{\Delta t} \\
&= \sum_s \sum_{A < s} \mathfrak{X}(A; s) \chi_A \cdot \chi_{\{s\}} \sqrt{\Delta t} \\
&= \sum_s \mathfrak{X}_s \cdot \Delta \mathfrak{B}_s \\
&= \int \mathfrak{X}_s d\mathfrak{B}_s
\end{aligned}$$

by proposition 6.1.3.

(2)

$$\begin{aligned}
&\int \mathfrak{X} \delta \mathfrak{B} \\
&= \sum_s \mathfrak{X}_s \diamond \chi_{\{s\}} \sqrt{\Delta t} \\
&= \sum_s \left( \sum_n \sum_{A \in \mathcal{P}_n} \mathfrak{X}(A; s) \chi_A \right) \diamond \chi_{\{s\}} \sqrt{\Delta t} \\
&= \sum_s \left( \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^n} \mathfrak{X}_{n,s}(t_1, \dots, t_n) \chi_{\{t_1, \dots, t_n\}} \Delta t^{\frac{n}{2}} \right) \diamond \chi_{\{s\}} \sqrt{\Delta t}
\end{aligned}$$

with  $\mathfrak{X}_{n,s}(\cdot)$  the symmetric functions in the Wiener-Ito decomposition of  $\mathfrak{X}_s$ .

Let  $s = t_{n+1}$  and the symmetric functions  $\hat{\mathfrak{X}}_{n+1}$  of  $n+1$  arguments by

$$\hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1}) = 0$$

if  $t_i = t_j$  for some  $i \neq j$  and

$$\begin{aligned}
\hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1}) &= \frac{1}{n+1} \left( \sum_{k=1}^{n+1} \mathfrak{X}_{n,t_k}(t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_{n+1}) \right) \\
&= \frac{1}{n+1} \left( \sum_{k=1}^{n+1} \mathfrak{X}_{n+1}(t_1, \dots, t_{k-1}, t_{n+1}, t_{k+1}, \dots, t_n) \right)
\end{aligned}$$

otherwise. Then, changing the sum over  $s$  inside,

$$\begin{aligned}
&\int \mathfrak{X} \delta \mathfrak{B} \\
&= \sum_n \sum_{(t_1, \dots, t_{n+1}) \in \Lambda^{n+1}} \hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1}) \chi_{\{t_1, \dots, t_{n+1}\}} \Delta t^{\frac{n+1}{2}} \\
&= \sum_n \sum_{(t_1, \dots, t_{n+1}) \in \Lambda^{n+1}} \hat{\mathfrak{X}}_{n+1}(t_1, \dots, t_{n+1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{n+1}). \quad \square
\end{aligned}$$

#### 6.1.4 Discrete Malliavin derivative

For  $s \in \Lambda$  and  $\omega \in \Omega$ ,  $\omega_s^+$  and  $\omega_s^-$  are defined by

$$\omega_s^\pm(t) = \begin{cases} \omega(t), & \text{for } t \neq s, \\ \pm 1, & \text{for } t = s. \end{cases}$$

**Definition 6.1.7.** (*discrete Malliavin derivative*)

For every random variable  $\mathfrak{X} \in \mathfrak{L}^2(\Omega, P)$  the Malliavin derivative  $(D_t \mathfrak{X})_{t \geq 0}$  are defined by the family  $(D_t)_{t \geq 0}$  of operators on  $\mathfrak{L}^2(\Omega, P)$ :

$$D_t \mathfrak{X}(\omega) = \frac{\mathfrak{X}(\omega_t^+) - \mathfrak{X}(\omega_t^-)}{2\sqrt{\Delta t}}.$$

This family of operators can be seen as an operator

$$D : \mathfrak{L}^2(\Omega, P) \rightarrow \mathfrak{L}^2(\Omega \times \Lambda, P \times \mu).$$

**Proposition 6.1.6.** *Let*

$$\mathfrak{X} = \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^n} \mathfrak{X}_n(t_1, \dots, t_n) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n).$$

*Then*

$$D_t \mathfrak{X} = \sum_n \sum_{(t_1, \dots, t_{n-1}) \in \Lambda^{n-1}} n \mathfrak{X}_{n,t}(t_1, \dots, t_{n-1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{n-1}).$$

*I.e. the Malliavin derivative acts on the discrete Wiener-Ito decomposition as multiplication by the level number  $n$  and then just leaving aside the integration over  $\Delta \mathfrak{B}(t_n)$ .*

*Proof.*

Clearly,  $(D_t)_{t \geq 0}$  is linear. Then, by Definition 6.1.7 and Definition 6.1.1,

$$\begin{aligned}
D_t \mathfrak{X}(\omega) &= D_t \left( \sum_n \sum_{(t_1 < \dots < t_n) \in \Lambda^n} n! \mathfrak{X}_n(t_1, \dots, t_n) \Delta t^{\frac{n}{2}} \chi_{\{t_1, \dots, t_n\}}(\omega) \right) \\
&= \sum_n \sum_{(t_1 < \dots < t_n) \in \Lambda^n} n! \mathfrak{X}_n(t_1, \dots, t_n) \Delta t^{\frac{n}{2}} \\
&\quad \frac{\chi_{\{t_1, \dots, t_n\}}(\omega_t^+) - \chi_{\{t_1, \dots, t_n\}}(\omega_t^-)}{2\sqrt{\Delta t}} \\
&= \sum_n \sum_{(t_1 < \dots < t_n) \in \Lambda^n} n! \mathfrak{X}_n(t_1, \dots, t_n) \Delta t^{\frac{n-1}{2}} \cdot \\
&\quad \frac{1}{2} \left( \prod_{s \in \{t_1, \dots, t_n\}} \omega_t^+(s) - \prod_{s \in \{t_1, \dots, t_n\}} \omega_t^-(s) \right) \\
&= \sum_n \sum_{\substack{(t_1 < \dots < t_n) \in \Lambda^n \\ t \in \{t_1, \dots, t_n\}}} n! \mathfrak{X}_n(t_1, \dots, t_n) \Delta t^{\frac{n-1}{2}} \chi_{\{t_1, \dots, t_n\} \setminus \{t\}}(\omega) \\
&= \sum_n \sum_{(t_1 < \dots < t_{n-1}) \in \Lambda^{n-1}} n! \mathfrak{X}_{n,t}(t_1, \dots, t_{n-1}) \Delta t^{\frac{n-1}{2}} \chi_{\{t_1, \dots, t_{n-1}\}}(\omega) \\
&= \sum_n \sum_{(t_1, \dots, t_{n-1}) \in \Lambda^{n-1}} n \mathfrak{X}_{n,t}(t_1, \dots, t_{n-1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{n-1}).
\end{aligned}$$

Privault points out the convergence of  $D\mathfrak{X}$  [46]. He defines the  $\mathfrak{L}^2$  domain of  $D$  as the space of functional  $\mathfrak{X}$  such that

$$E \left[ \|D\mathfrak{X}\|_{\mathfrak{L}^2(\Omega \times \Lambda)}^2 \right] < \infty,$$

or equivalently

$$\sum_n n n! \|\mathfrak{X}_n\|_{\mathfrak{L}^2(\Lambda^n)}^2 < \infty$$

if  $\mathfrak{X} = \sum_n \mathfrak{X}_n$ .  $\square$

## 6.2 A derivative of a natural number

This section follows Victor Ufnarovski et al. [53] where also proofs are provided.

### 6.2.1 Definition

Let  $n$  be a positive integer. Victor Ufnarovski et al. define a derivative  $n'$  of  $n$  which ignore linearity and use Leibnitz rule only [53]. This derivative is defined by using two natural rules:

- (1)  $p' = 1$  for any prime  $p$ ,
- (2)  $(mn)' = m'n + mn'$  for any  $a, b \in \mathbb{N}$  (Leibnitz rule).

**Lemma 6.2.1.** (*Well-defined*)

*The derivative  $n'$  can be well-defined as follows: if  $n = \prod_{i=1}^k p_i^{n_i}$  is a factorization in prime powers, then*

$$n' = n \sum_{i=1}^k \frac{n_i}{p_i}.$$

*This is the only way to define  $n'$  that satisfies desired properties. [21]*

Note that  $1' = (1 \cdot 1)' = 1' \cdot 1 + 1 \cdot 1' = 2 \cdot 1'$ . It is clear that  $1' = 0$ . Let  $n = \prod_{i=1}^{k_1} p_i^{n_i}$  and  $m = \prod_{i=1}^{k_2} q_i^{m_i}$ . Then according to lemma 6.2.1 the Leibnitz rule looks as

$$mn \left( \sum_{i=1}^{k_1} \frac{n_i}{p_i} + \sum_{i=1}^{k_2} \frac{m_i}{q_i} \right) = \left( m \sum_{i=1}^{k_2} \frac{m_i}{q_i} \right) n + m \left( n \sum_{i=1}^{k_1} \frac{n_i}{p_i} \right).$$

## 6.2.2 Properties

Some useful properties of the derivative of a nature number defined by Victor Ufnarovski et al. [53] are given in this section.

**Lemma 6.2.2.** *Let  $p$  be a prime and  $a = p + 2$ . Then  $2p$  is a solution for the equation  $n' = a$ .*

Corollary 6.2.1 is a direct result of lemma 6.2.2.

**Corollary 6.2.1.** *Let  $p, q$  be two primes and  $b = p + q$ . Then  $pq$  is a solution for the equation  $n' = b$ .*

**Lemma 6.2.3.** *The differential equation  $n' = 1$  in nature numbers has only primes as solutions.*

# Chapter 7

## Simple Results

In this chapter, numerous examples and lemmas are treated as simple results here.

### 7.1 Delta operator

The idea of this entire section follows Professor Utev. In this section, another approach of Ito integral and Stratonovich integral is introduced here through defining a delta operator. It is necessary to mention that this approach is different to the discrete Ito integral.

#### 7.1.1 Definition

**Definition 7.1.1.** (*Delta Operator*)

*If  $f$  is a continuous function of  $x$ , then the delta operator can be defined as*

$$\Delta f(x) = f(x + \Delta x) - f(x).$$

According to Definition 7.1.1, if  $f$  is a continuous function of a standard Brownian motion  $B_t$ , then

$$\Delta f(B_t) = f(B_{t+\Delta t}) - f(B_t).$$

Next example shows how delta operator works on random variable.

**Example 7.1.1.**

Let  $B_t$  be a standard Brownian motion. By definition of the delta operator ( Definition 7.1.1 ),

$$\begin{aligned} \Delta B_t^2 &= B_{t+\Delta t}^2 - B_t^2 \\ &= (B_{t+\Delta t} - B_t + B_t)^2 - B_t^2 \\ &= (B_{t+\Delta t} - B_t)^2 + 2(B_{t+\Delta t} - B_t)B_t + B_t^2 - B_t^2. \end{aligned}$$

Since

$$\Delta B_t = B_{t+\Delta t} - B_t,$$

the former equation becomes

$$\Delta B_t^2 = (\Delta B_t)^2 + 2B_t \Delta B_t.$$

Note that

$$\begin{aligned} dB_t^2 &= (dB_t)^2 + 2B_t dB_t \\ &= dt + 2B_t dB_t \end{aligned}$$

by Ito formula.  $\square$

**Example 7.1.2.**

Consider a more general case  $B_t^m$  where  $m \in \mathbb{N}$ . According to Definition 7.1.1,

$$\begin{aligned}\Delta B_t^m &= B_{t+\Delta t}^m - B_t^m \\ &= (B_{t+\Delta t} - B_t + B_t)^m - B_t^m \\ &= (\Delta B_t + B_t)^m - B_t^m \\ &= \sum_{k=1}^m \binom{m}{k} (\Delta B_t)^k B_t^{m-k}.\end{aligned}$$

Let  $O [(\Delta B_t)^3]$  represents the infinitesimal asymptotics of  $(\Delta B_t)^3$ . Therefore,

$$\Delta B_t^m = mB_t^{m-1}\Delta B_t + \frac{m(m-1)}{2}B_t^{m-2}(\Delta B_t)^2 + O [(\Delta B_t)^3].$$

Note that

$$\begin{aligned}dB_t^m &= \frac{m(m-1)}{2}B_t^{m-2}(dB_t)^2 + mB_t^{m-1}dB_t \\ &= \frac{m(m-1)}{2}B_t^{m-2}dt + mB_t^{m-1}dB_t\end{aligned}$$

by Ito formula.  $\square$

Let  $N_t$  be a Poisson process. Then, by similar calculation of  $\Delta B_t^m$ ,

$$\Delta N_t^m = \sum_{k=1}^m \binom{m}{k} (\Delta N_t)^k N_t^{m-k}.$$

Note that  $(\Delta N_t)^k \rightarrow 0, k \in \mathbb{N}$  in  $L^2$ .

### 7.1.2 Another delta operator

Let  $\Delta$  be the delta operator defined in Definition 7.1.1 and  $f$  is a continuous function. According to the Taylor series,

$$f(B_{t+\Delta t}) = f(B_t) + f'(B_t)\Delta B_t + \frac{1}{2}f''(B_t)(\Delta B_t)^2 + O[(\Delta B_t)^3]$$

where  $O[(\Delta B_t)^3]$  represents the infinitesimal asymptotics of  $(\Delta B_t)^3$ . Since  $(\Delta B_t)^2 = \Delta t + O(\Delta t^{3/2})$ ,

$$f(B_{t+\Delta t}) = f(B_t) + f'(B_t)\Delta B_t + \frac{1}{2}f''(B_t)\Delta t + O(\Delta t^{\frac{3}{2}}).$$

Then, by the definition of the delta operator ( Definition 7.1.1 ),

$$\Delta f(B_t) = f'(B_t)\Delta B_t + \frac{1}{2}f''(B_t)\Delta t + O(\Delta t^{\frac{3}{2}}).$$

Now define a new delta operator as

$$\tilde{\Delta}f(B_t) = \Delta f(B_t) - \frac{1}{2}f''(B_t)\Delta t.$$

Furthermore,

$$\tilde{\Delta}f(B_t) = f'(B_t)\Delta B_t + O(\Delta t^{\frac{3}{2}}).$$

Next example shows how this operator works on random variable.

**Example 7.1.3.**

Let  $B_t$  be a standard Brownian motion. By Example 7.1.1,

$$\begin{aligned}\Delta B_t^2 &= (\Delta B_t)^2 + 2B_t\Delta B_t \\ &= \Delta t + 2B_t\Delta B_t + O\left(\Delta t^{\frac{3}{2}}\right).\end{aligned}$$

Then, by the definition of  $\tilde{\Delta}$ ,

$$\begin{aligned}\tilde{\Delta}B_t^2 &= \Delta B_t^2 - \Delta t \\ &= 2B_t\Delta B_t + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= 2B_t\tilde{\Delta}B_t + O\left(\Delta t^{\frac{3}{2}}\right).\end{aligned}$$

Note that

$$\begin{aligned}B_t^2 &= \int_0^t 2B_s dB_s + \int_0^t dt \\ &= \int_0^t 2B_s \circ dB_s\end{aligned}$$

by Stratonovich integral.  $\square$

# Chapter 8

## A General Stroock Lemma

The Stroock lemma is a fundamental concept of the Malliavin calculus. This lemma can be used in solving various equations. In this chapter, this lemma is illustrated through several different cases.

### 8.1 Discrete Stroock lemma

The idea of the discrete Stroock lemma, which can be treated as the counterpart of the Stroock lemma in discrete Malliavin calculus, is inspired by the corresponding lemma in continuous Malliavin calculus. Comparing with the continuous Malliavin calculus, the discrete version of Stroock lemma is slightly different.

#### 8.1.1 Discrete Stroock lemma

Before introducing the discrete version of the Stroock lemma, an important property is given first.

For each fixed  $t$ , the discrete Wiener-Ito decomposition

$$\mathfrak{X}_{s,t} = \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^{n-1}} n \mathfrak{X}_{n-1,s,t}(t_1, \dots, t_{n-1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n)$$

equals to zero for  $s = t$  by default.

**Lemma 8.1.1.** (*Discrete Stroock lemma*)

Let

$$\mathfrak{X}_s = \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^n} \mathfrak{X}_{n,s}(t_1, \dots, t_n) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n)$$

and

$$\mathfrak{X}_{s,t} = D_t \mathfrak{X}_s = \sum_n \sum_{(t_1, \dots, t_n) \in \Lambda^{n-1}} n \mathfrak{X}_{n-1,s,t}(t_1, \dots, t_{n-1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_n)$$

be the discrete Wiener-Ito decomposition of  $\mathfrak{X}_s$  and  $\mathfrak{X}_{s,t}$  respectively.  $(D_t)_{t \geq 0}$  are the derivative operators defined in Definition 6.1.7. Assume that  $\mathfrak{X}_{n,s}$  is a symmetric function of  $n$  arguments  $t_1, \dots, t_n$ . Assume also that

$$E \left[ \|\mathfrak{X}_s\|_{\mathfrak{L}^2(\Omega \times \Lambda)}^2 \right] < \infty$$

and

$$E \left[ \|\mathfrak{X}_{s,t}\|_{\mathfrak{L}^2(\Omega \times \Lambda^2)}^2 \right] < \infty.$$

Then,

$$D_t \left( \int \mathfrak{X}_s \delta \mathfrak{B}_s \right) = \mathfrak{X}_t + \int D_t \mathfrak{X}_s \delta \mathfrak{B}_s.$$

Note that the condition ' $\mathfrak{X}_{n,s}$  is a symmetric function of  $n$  arguments  $t_1, \dots, t_n$ ' is necessary in former lemma. The reason will be shown by the proof of Lemma 8.1.1 in the next section.

### 8.1.2 Proof of Lemma 8.1.1

The proof of lemma 8.1.1 consists of three parts.

(1) For  $n = 1$ , let

$$\mathfrak{Y}_s^{(1)} = \sum_{t_1 \in \Lambda} \mathfrak{Y}_{1,s}(t_1) \Delta \mathfrak{B}(t_1).$$

According to Proposition 6.1.5 and Proposition 6.1.6,

$$\begin{aligned} & D_t \left( \int \mathfrak{Y}_s^{(1)} \delta \mathfrak{B}_s \right) \\ &= D_t \left( \sum_{(t_1, t_2) \in \Lambda^2} \frac{1}{2} [\mathfrak{Y}_{1,t_2}(t_1, t_2) + \mathfrak{Y}_{1,t_1}(t_2, t_1)] \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2) \right) \\ &= D_t \left( \sum_{(t_1, t_2) \in \Lambda^2} \hat{\mathfrak{Y}}_2(t_1, t_2) \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2) \right) \\ &= \sum_{t_1 \in \Lambda} 2\mathfrak{Y}_{1,t}(t_1) \Delta \mathfrak{B}(t_1), \end{aligned}$$

where  $\hat{\mathfrak{Y}}_2(t_1, t_2)$  is the symmetrization of  $\mathfrak{Y}_{1,s}(t_1)$  and

$$\begin{aligned} \int D_t \mathfrak{Y}_s^{(1)} \delta \mathfrak{B}_s &= \int \mathfrak{Y}_{0,t,s}(\emptyset) \delta \mathfrak{B}_s \\ &= \sum_{t_1 \in \Lambda} \frac{1}{2} [\mathfrak{Y}_{1,t_1}(t, t_1) + \mathfrak{Y}_{1,t}(t_1, t)] \Delta \mathfrak{B}(t_1) \\ &= \sum_{t_1 \in \Lambda} \mathfrak{Y}_{1,t}(t_1) \Delta \mathfrak{B}(t_1). \end{aligned}$$

Concluding all former results,

$$D_t \left( \int \mathfrak{Y}_s^{(1)} \delta \mathfrak{B}_s \right) = \mathfrak{Y}_t^{(1)} + \int D_t \mathfrak{Y}_s^{(1)} \delta \mathfrak{B}_s.$$

(2) For  $n = 2$ , let

$$\mathfrak{Y}_s^{(2)} = \sum_{(t_1, t_2) \in \Lambda^2} \mathfrak{Y}_{2,s}(t_1, t_2) \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2).$$

According to Proposition 6.1.5 and Proposition 6.1.6,

$$\begin{aligned} & D_t \left( \int \mathfrak{Y}_s^{(2)} \delta \mathfrak{B}_s \right) \\ &= D_t \left( \sum_{(t_1, t_2, t_3) \in \Lambda^3} \frac{1}{3} [\mathfrak{Y}_3(t_1, t_2, t_3) + \mathfrak{Y}_3(t_1, t_3, t_2) + \mathfrak{Y}_3(t_3, t_1, t_2)] \right. \\ &\quad \left. \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2) \Delta \mathfrak{B}(t_3) \right) \\ &= D_t \left( \sum_{(t_1, t_2, t_3) \in \Lambda^3} \hat{\mathfrak{Y}}_3(t_1, t_2, t_3) \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2) \Delta \mathfrak{B}(t_3) \right) \\ &= \sum_{(t_1, t_2) \in \Lambda^2} 3\mathfrak{Y}_{2,t}(t_1, t_2) \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2), \end{aligned}$$

where  $\hat{\mathfrak{Y}}_3(t_1, t_2, t_3)$  is the symmetrization of  $\mathfrak{Y}_{2,s}(t_1, t_2)$  and

$$\begin{aligned} \int D_t \mathfrak{Y}_s^{(2)} \delta \mathfrak{B}_s &= \int \left( \sum_{t_1 \in \Lambda} 2\mathfrak{Y}_{1,t,s}(t_1) \Delta \mathfrak{B}(t_1) \right) \delta \mathfrak{B}_s \\ &= \sum_{(t_1, t_2) \in \Lambda^2} 2 \cdot \frac{1}{2} [\mathfrak{Y}_{2,t}(t_1, t_2) + \mathfrak{Y}_{2,t}(t_2, t_1)] \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2) \\ &= \sum_{(t_1, t_2) \in \Lambda^2} 2\mathfrak{Y}_{2,t}(t_1, t_2) \Delta \mathfrak{B}(t_1) \Delta \mathfrak{B}(t_2). \end{aligned}$$

Concluding all former results,

$$D_t \left( \int \mathfrak{Y}_s^{(2)} \delta \mathfrak{B}_s \right) = \mathfrak{Y}_t^{(2)} + \int D_t \mathfrak{Y}_s^{(2)} \delta \mathfrak{B}_s.$$

(3) For  $n = k$ ,

$$\mathfrak{Y}_s^{(k)} = \sum_{(t_1, \dots, t_k) \in \Lambda^k} \mathfrak{Y}_{k,s}(t_1, \dots, t_k) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_k).$$

According to Proposition 6.1.5 and Proposition 6.1.6,

$$\begin{aligned} & D_t \left( \int \mathfrak{Y}_s^{(k)} \delta \mathfrak{B}_s \right) \\ = & D_t \left( \sum_{(t_1, \dots, t_{k+1}) \in \Lambda^{k+1}} \frac{\mathfrak{Y}_{k+1}(t_1, \dots, t_n, t_{n+1}) + \cdots + \mathfrak{Y}_{k+1}(t_{n+1}, t_1, \dots, t_n)}{k+1} \right. \\ & \left. \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{k+1}) \right) \\ = & D_t \left( \sum_{(t_1, \dots, t_{k+1}) \in \Lambda^{k+1}} \hat{\mathfrak{Y}}_{k+1}(t_1, \dots, t_n, t_{n+1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{k+1}) \right) \\ = & \sum_{(t_1, \dots, t_k) \in \Lambda^k} (k+1) \mathfrak{Y}_{k,t}(t_1, \dots, t_k) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_k), \end{aligned}$$

where  $\hat{\mathfrak{Y}}_{k+1}(t_1, \dots, t_n, t_{n+1})$  is the symmetrization of  $\mathfrak{Y}_{k,s}(t_1, \dots, t_k)$  and

$$\begin{aligned} & \int D_t \mathfrak{Y}_s^{(k)} \delta \mathfrak{B}_s \\ = & \int \left( \sum_{(t_1, \dots, t_{k-1}) \in \Lambda^{k-1}} k \mathfrak{Y}_{k-1,t,s}(t_1, \dots, t_{k-1}) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_{k-1}) \right) \delta \mathfrak{B}_s \\ = & \sum_{(t_1, \dots, t_k) \in \Lambda^k} k \cdot \frac{1}{k} [\mathfrak{Y}_{k,t}(t_k, t_1, \dots, t_{k-1}) + \cdots + \mathfrak{Y}_{k,t}(t_1, \dots, t_{k-1}, t_k)] \cdot \\ & \cdot \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_k) \\ = & \sum_{(t_1, \dots, t_k) \in \Lambda^k} k \mathfrak{Y}_{k,t}(t_1, \dots, t_k) \Delta \mathfrak{B}(t_1) \cdots \Delta \mathfrak{B}(t_k) \end{aligned}$$

Concluding all former results,

$$D_t \left( \int \mathfrak{Y}_s^{(k)} \delta \mathfrak{B}_s \right) = \mathfrak{Y}_t^{(k)} + \int D_t \mathfrak{Y}_s^{(k)} \delta \mathfrak{B}_s.$$

In general, let

$$\mathfrak{X}_s = \sum_n \mathfrak{Y}_s^{(n)}$$

and

$$\mathfrak{X}_{s,t} = \sum_n D_t \mathfrak{Y}_s^{(n)} = D_t \sum_n \mathfrak{Y}_s^{(n)} = D_t \mathfrak{X}_s.$$

Since  $E \left[ \|\mathfrak{X}_{s,t}\|_{\mathfrak{L}^2(\Omega \times \Lambda^2)}^2 \right] < \infty$  and  $E \left[ \|\mathfrak{X}_s\|_{\mathfrak{L}^2(\Omega \times \Lambda)}^2 \right] < \infty$ ,

$$D_t \left( \int \mathfrak{X}_s \delta \mathfrak{B}_s \right) = \mathfrak{X}_t + \int D_t \mathfrak{X}_s \delta \mathfrak{B}_s.$$

## 8.2 A generalised Stroock lemma

A generalised Stroock lemma and the related duality lemma are introduced in this section.

### 8.2.1 A Generalised Stroock lemma

Consider two linear operators  $D$  and  $\delta$ . Define  $D_t$  as the original derivative operator, which means

$$D_t F = F'(t),$$

where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is an arbitrary differentiable function. Let  $u_h(t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  be a non-negative integrable function with respect to  $tin\mathbb{R}$  and  $h$ . Suppose that  $D_t$  satisfies the Stroock lemma

$$D_t (\delta (u_h (t))) = \delta (D_t (u_h (t))) + u_h (t).$$

After given all definitions and conditions, the first problem is what kind of operator  $\delta$  satisfies the former equation. To solve this problem, it is necessary to introduce the following lemma.

**Lemma 8.2.1.** *Let  $u_h(t) = e^{ht}$ . Define the derivative operator  $D_t$  as before.  $D_t$  satisfies the Stroock lemma*

$$D_t (\delta (u_h (t))) = \delta (D_t (u_h (t))) + u_h (t).$$

*Then*

$$\delta (u_h (t)) = (t + C_h) u_h (t)$$

*where  $C_h$  is a function of  $h$ .*

*Proof.*

Applying  $D_t$  to  $u_h(t)$ ,

$$D_t (u_h (t)) = h e^{ht}.$$

Then, by aforementioned assumption,

$$\begin{aligned}
D_t(\delta(u_h(t))) &= \delta(D_t(u_h(t))) + u_h(t) \\
&= \delta(hu_h(t)) + u_h(t) \\
&= h\delta(u_h(t)) + u_h(t) \\
&= h\delta(e^{ht}) + e^{ht}.
\end{aligned}$$

Fix  $h$  and define  $\delta(u_h(t)) = \delta(e^{ht}) = X(t)$ , the former equation becomes

$$X'(t) = hX(t) + e^{ht},$$

which is an ordinary differential equation. Solve this equation and the result is

$$\begin{aligned}
X(t) &= (t + C_h) e^{ht} \\
&= (t + C_h) u_h(t) \\
&= \delta(u_h(t)),
\end{aligned}$$

which completes the proof.  $\square$

Here consider  $\delta(u_h(t)) = tu_h(t)$  only, since  $C_h$  is a constant with respect to  $t$ .

Applying  $D_t$  to  $\delta(u_h(t))$ ,

$$D_t(\delta(u_h(t))) = D_t(tu_h(t)) = u_h(t) + tu_h'(t).$$

Then, applying  $\delta$  to  $D_t(u_h(t))$ , one can obtain

$$\delta(D_t(u_h(t))) = tD_t(u_h(t)) = tu_h'(t).$$

After concluding all former results,  $\delta$  still satisfies

$$D_t(\delta(u_h(t))) = \delta(D_t(u_h(t))) + u_h(t)$$

which is a generalised Stroock lemma.

**Lemma 8.2.2.** (*A generalised Stroock lemma*)

Let  $u_h(t)$  be a non-negative integrable function with respect to  $t$ . Define the operator  $D_t$  by

$$D_t(u_h(t)) = u_h'(t)$$

and the operator  $\delta$  by

$$\delta(u_h(t)) = tu_h(t).$$

Then,

$$D_t(\delta(u_h(t))) = \delta(D_t(u_h(t))) + u_h(t).$$

## 8.2.2 A Generalised duality lemma

The second problem is what kind of duality formula do those aforementioned  $D_t$  and  $\delta$  satisfy. To deduce such kind of duality formula, it is necessary to consider a bilinear map  $\mathbf{B}$ . Then, for all non-negative integrable functions  $u_h(t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  and  $\phi_g(t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ , the duality formula can be represented as

$$\mathbf{B}(D_t(\delta(u_h(t))), \phi_g(t)) = \mathbf{B}(u_h(t), \delta(D_t(\phi_g(t)))).$$

By linearity of the bilinear map  $\mathbf{B}$  and the generalised Stroock lemma ( Lemma 8.2.2 ), the left hand side of the former equation can be simplified as

$$\begin{aligned}
\mathbf{B}(D_t(\delta(u_h(t))), \phi_g(t)) &= \mathbf{B}((tu_h(t))', \phi_g(t)) \\
&= \mathbf{B}(tu_h'(t) + u_h(t), \phi_g(t)) \\
&= \mathbf{B}(tu_h'(t), \phi_g(t)) + \mathbf{B}(u_h(t), \phi_g(t)), \quad (1)
\end{aligned}$$

and the right hand side of this equation is

$$\mathbf{B}(u_h(t), \delta(D_t(\phi_g(t)))) = \mathbf{B}(u_h(t), t\phi_g'(t)).$$

Now, this problem becomes to find that what kind of bilinear map  $\mathbf{B}$  satisfy

$$\mathbf{B}(tu_h'(t), \phi_g(t)) + \mathbf{B}(u_h(t), \phi_g(t)) = \mathbf{B}(u_h(t), t\phi_g'(t)). \quad (2)$$

The following two lemmas show that the duality formula in this case is not as simple as before.

**Lemma 8.2.3.** *Suppose that  $D_t$  and  $\delta$  satisfy (2). Assume that the bilinear map  $\mathbf{B}$  satisfies the commutative law, which means*

$$\mathbf{B}(x(t), y(t)) = \mathbf{B}(y(t), x(t)).$$

*Then, for all  $x(t)$  and  $y(t)$ ,*

$$\mathbf{B}(x(t), y(t)) = 0.$$

*Proof.*

By the commutative law of the bilinear map  $\mathbf{B}$ ,

$$\mathbf{B}(tu'_h(t), \phi_g(t)) = \mathbf{B}(\phi_g(t), tu'_h(t)).$$

Then, according to equation (2),  $\mathbf{B}(u_h(t), t\phi'_g(t))$  is equivalent to

$$\mathbf{B}(\phi_g(t), tu'_h(t)) + \mathbf{B}(u_h(t), \phi_g(t)).$$

Applying (2) to the first term of the former polynomial,

$$\mathbf{B}(\phi_g(t), tu'_h(t)) = \mathbf{B}(t\phi'_g(t), u_h(t)) + \mathbf{B}(\phi_g(t), u_h(t)).$$

Therefore, by the commutative law of the bilinear map  $\mathbf{B}$  again,

$$\begin{aligned} & \mathbf{B}(u_h(t), t\phi'_g(t)) \\ &= \mathbf{B}(t\phi'_g(t), u_h(t)) + \mathbf{B}(\phi_g(t), u_h(t)) + \mathbf{B}(u_h(t), \phi_g(t)) \\ &= \mathbf{B}(u_h(t), t\phi'_g(t)) + 2\mathbf{B}(u_h(t), \phi_g(t)). \end{aligned}$$

After comparing former equation and the right hand side of (2),

$$\mathbf{B}(u_h(t), \phi_g(t)) = 0,$$

which completes the proof.  $\square$

Lemma 8.2.3 shows that the generalised duality formula does not satisfy the commutative law in general.

**Lemma 8.2.4.** *Let  $D_t$  and  $\delta$  satisfy (2). If  $\mathbf{B}(x(t), y(t)) \neq 0$ , there is no such a constant  $a$  that*

$$\mathbf{B}(x(t), y(t)) = a\mathbf{B}(y(t), x(t))$$

for all  $x(t)$  and  $y(t)$ .

*Proof.*

Suppose such kind of  $a$  exists. Therefore,

$$\mathbf{B}(tu'_h(t), \phi_g(t)) = a\mathbf{B}(\phi_g(t), tu'_h(t)).$$

Then, the left hand side of (2) is equivalent to

$$a\mathbf{B}(\phi_g(t), tu'_h(t)) + \mathbf{B}(u_h(t), \phi_g(t)).$$

Applying (2) to the first term of the former polynomial,

$$\mathbf{B}(\phi_g(t), tu'_h(t)) = \mathbf{B}(t\phi'_g(t), u_h(t)) + \mathbf{B}(\phi_g(t), u_h(t)).$$

Then,

$$\begin{aligned} & a\mathbf{B}(\phi_g(t), tu'_h(t)) + \mathbf{B}(u_h(t), \phi_g(t)) \\ &= a(\mathbf{B}(t\phi'_g(t), u_h(t)) + \mathbf{B}(\phi_g(t), u_h(t))) + \mathbf{B}(u_h(t), \phi_g(t)) \\ &= a\mathbf{B}(t\phi'_g(t), u_h(t)) + a^2\mathbf{B}(u_h(t), \phi_g(t)) + \mathbf{B}(u_h(t), \phi_g(t)) \\ &= a^2\mathbf{B}(u_h(t), t\phi'_g(t)) + (a^2 + 1)\mathbf{B}(u_h(t), \phi_g(t)). \end{aligned}$$

After comparing former equation and the right hand side of (2),

$$\begin{cases} a^2 = 1, \\ a^2 + 1 = 0, \end{cases}$$

since  $\mathbf{B}(x(t), y(t)) \neq 0$  for all  $x(t)$  and  $y(t)$ . This equation set has no possible solution, which completes the proof by contradiction.  $\square$

To find a possible generalised duality formula, consider a basis  $\{t^k\}_{k=0}^{\infty}$  on the algebra of polynomials. Let  $u_h(t) = u = t^k$  and  $\phi_g(t) = \phi = t^m$ . Then, the left hand side of (2) becomes

$$\mathbf{B}(kt^k, t^m) + \mathbf{B}(t^k, t^m) = k\mathbf{B}(t^k, t^m) + \mathbf{B}(t^k, t^m)$$

and the right hand side of (2) becomes

$$\mathbf{B}(t^k, mt^m) = m\mathbf{B}(t^k, t^m),$$

since  $\mathbf{B}$  is a bilinear map. Let  $\mathbf{B}(t^k, t^m) = \beta_{k,m}$ . Therefore, solving (2) is equivalent to solving

$$\beta_{k,m}(k+1-m) = 0 \quad (3)$$

under this basis. Here omit the solution  $\beta_{k,m} \equiv 0$ , because this is a meaningless solution of (3). Hence, the remaining solution of (3) is  $k+1-m=0$  and

$$\begin{cases} \beta_{k,k+1} \neq 0, \\ \beta_{j,i} = 0, \quad i \neq j+1. \end{cases} \quad (4)$$

Then, by (4),

$$\beta_{k,k+1} = \begin{pmatrix} & k-1 & k & k+1 & \\ \dots & 0 & 1 & 0 & \dots \end{pmatrix} (\beta_{i,j})_{i,j=0,1,\dots} \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

where

$$(\beta_{i,j})_{i,j=0,1,\dots} = \begin{matrix} & 0 & 1 & 2 & 3 & \dots \\ 0 & \begin{pmatrix} 0 & \beta_{0,1} & 0 & 0 & \dots \end{pmatrix} \\ 1 & \begin{pmatrix} 0 & 0 & \beta_{1,2} & 0 & \dots \end{pmatrix} \\ 2 & \begin{pmatrix} 0 & 0 & 0 & \beta_{2,3} & \dots \end{pmatrix} \\ \vdots & \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{matrix}.$$

Now consider a more general situation. Let  $u_h(t) = u = \sum_{j=0}^M x_j t^j$  and  $\phi_g(t) = \phi = \sum_{i=0}^N y_i t^i$  be two sequences of partial sums associated to two series. Then,

$$\mathbf{B}(u, \phi) = \begin{pmatrix} x_0 & x_1 & \dots & x_M & 0 & \dots \end{pmatrix} (\beta_{i,j})_{i,j=0,1,\dots} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \\ 0 \\ \vdots \end{pmatrix}.$$

Without loss of generality, suppose that  $M \geq N$ . Therefore,

$$\begin{aligned}
\mathbf{B}(u, \phi) &= \mathbf{B}\left(\sum_{j=0}^M x_j t^j, \sum_{i=0}^N y_i t^i\right) \\
&= \sum_{j=0}^M \sum_{i=0}^N (x_j y_i) \mathbf{B}(t^j, t^i) \\
&= \sum_{j=0}^M (x_j y_{j+1}) \mathbf{B}(t^j, t^{j+1}) \\
&= \sum_{j=0}^M (x_j y_{j+1}) \beta_{j,j+1},
\end{aligned}$$

by (4). Let  $x_j = h^j/j!$  and  $y_i = g^i/i!$ . Then, the left hand side of (2) is

$$\begin{aligned}
&\mathbf{B}\left(t \sum_{j=1}^M \frac{h^j}{j!} j t^{j-1}, \sum_{i=0}^N \frac{g^i}{i!} t^i\right) + \mathbf{B}\left(\sum_{j=0}^M \frac{h^j}{j!} t^j, \sum_{i=0}^N \frac{g^i}{i!} t^i\right) \\
&= \mathbf{B}\left(\sum_{j=0}^M j \frac{h^j}{j!} t^j, \sum_{i=0}^N \frac{g^i}{i!} t^i\right) + \mathbf{B}\left(\sum_{j=0}^M \frac{h^j}{j!} t^j, \sum_{i=0}^N \frac{g^i}{i!} t^i\right) \\
&= \sum_{j=0}^M j (x_j y_{j+1}) \beta_{j,j+1} + \sum_{j=0}^M (x_j y_{j+1}) \beta_{j,j+1} \\
&= \sum_{j=0}^M (j+1) (x_j y_{j+1}) \beta_{j,j+1}
\end{aligned}$$

and the right hand side of (2) is

$$\begin{aligned}
\mathbf{B}\left(\sum_{j=0}^M \frac{h^j}{j!} t^j, t \sum_{i=1}^N \frac{g^i}{i!} i t^{i-1}\right) &= \mathbf{B}\left(\sum_{j=0}^M \frac{h^j}{j!} t^j, \sum_{i=0}^N \frac{g^i}{i!} t^i\right) \\
&= \sum_{j=0}^M (j+1) (x_j y_{j+1}) \beta_{j,j+1},
\end{aligned}$$

since  $i = j + 1$  from (4). Assume that  $\lim_{M \rightarrow \infty} \sum_{j=0}^M (j+1) (x_j y_{j+1}) \beta_{j,j+1}$  exists. Then,

$$\mathbf{B} \left( \sum_{j=0}^{\infty} \frac{h^j}{j!} t^j, \sum_{i=0}^{\infty} \frac{g^i}{i!} t^i \right) = \mathbf{B} (e^{ht}, e^{gt}),$$

and therefore,

$$\begin{aligned} & \mathbf{B} \left( t \sum_{j=1}^{\infty} \frac{h^j}{j!} j t^{j-1}, \sum_{i=0}^{\infty} \frac{g^i}{i!} t^i \right) + \mathbf{B} \left( \sum_{j=0}^{\infty} \frac{h^j}{j!} t^j, \sum_{i=0}^{\infty} \frac{g^i}{i!} t^i \right) \\ &= \mathbf{B} \left( \sum_{j=0}^{\infty} \frac{h^j}{j!} t^j, t \sum_{i=1}^{\infty} \frac{g^i}{i!} i t^{i-1} \right) \\ &= \lim_{M \rightarrow \infty} \sum_{j=0}^M (j+1) (x_j y_{j+1}) \beta_{j,j+1}. \end{aligned}$$

According to the former equation, the bilinear map  $\mathbf{B}$  which satisfies (2) exists under the aforementioned assumption. Let  $\beta_{j,j+1} = j!$ . Then,

$$\begin{aligned} \lim_{M \rightarrow \infty} \sum_{j=0}^M (j+1) (x_j y_{j+1}) \beta_{j,j+1} &= \lim_{M \rightarrow \infty} \sum_{j=0}^M (j+1) (x_j y_{j+1}) j! \\ &= \lim_{M \rightarrow \infty} \sum_{j=0}^M (j+1) \frac{h^j}{j!} \cdot \frac{g^{j+1}}{(j+1)!} j! \\ &= \lim_{M \rightarrow \infty} \sum_{j=0}^M \frac{g (gh)^j}{j!} \\ &= g e^{gh}. \end{aligned}$$

Hence  $\mathbf{B} (e^{ht}, e^{gt}) = g e^{gh}$ . To find a more general bilinear map  $\mathbf{B}$ , it is necessary to use inverse Laplace transform.

**Lemma 8.2.5.** *(A generalised duality lemma)*

Define the bilinear operator  $\mathbf{B}$  by  $\mathbf{B} (e^{ht}, e^{gt}) = g e^{gh}$ . Let  $\mathcal{S}$  be a space of all functions which have the inverse Laplace transform. If  $F \in \mathcal{S}$ ,

$$\mathbf{B}(e^{ht}, f(t)) = f'_h(h),$$

where  $f(t)$  be a piecewise-continuous and exponentially-restricted real function which is the inverse Laplace transform of  $F$ .

*Proof.*

The inverse Laplace transform of the function  $F$  is

$$f(t) = \mathcal{L}^{-1}\{F(g)\}(t) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{gt} F(g) dg$$

where  $\gamma$  is a vertical contour in the complex plane chosen so that all singularities of  $F$  are to the left of it. It is equivalent to prove that

$$\begin{aligned} \mathbf{B}(e^{ht}, \mathcal{L}^{-1}\{F(g)\}(t)) &= \mathbf{B}\left(e^{ht}, \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{gt} F(g) dg\right) \\ &= (\mathcal{L}^{-1}\{F(g)\}(h))'_h \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} g e^{gh} F(g) dg, \end{aligned}$$

since  $F \in \mathcal{S}$ . Then, by the definition of the bilinear operator  $\mathbf{B}$ ,

$$\begin{aligned} \mathbf{B}(e^{ht}, \mathcal{L}^{-1}\{F(g)\}(t)) &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} \mathbf{B}(e^{ht}, e^{gt}) F(g) dg \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} g e^{gh} F(g) dg, \end{aligned}$$

which completes the proof.  $\square$

### 8.3 Stroock lemma for the derivative of a natural number

Consider a stochastic process  $\{X_t : t \geq 0\}$ . Let  $X_t \equiv C$  for  $t \geq 0$  where  $C$  is a constant, whereupon this process is not random. Then, it is interesting to introduce a new concept introduced by Victor Ufnarovski et al. [53] which is the derivative of a natural number.

Now, consider the derivative given by Lemma 6.2.1 which is

$$D(n) = n' = n \sum_{i=1}^k \frac{n_i}{p_i}$$

where  $n = \prod_{i=1}^k p_i^{n_i}$  is a factorization in prime powers. To seek a proper operator  $\delta$  in the related Stroock lemma

$$D\delta = I + \delta D,$$

it is necessary to use Lemma 6.2.2 and Corollary 6.2.1 given before.

Note that  $p = q$  is possible in Corollary 6.2.1 due to  $(p^2)' = 2p$ . Since  $p, q$  are two primes, one can obtain

$$D(pq) = (pq)' = p'q + pq' = p + q$$

by using Leibnitz rule and the fact that  $p' = 1$  for any prime  $p$ .

Corollary 6.2.1 gives an idea to define a linear operator  $\delta_p(q) = pq$  where  $p$  is a fixed prime and  $q$  is an arbitrary prime. This operator together with the derivative of the natural number satisfy

$$D\delta = I + \delta D$$

which is the Stroock lemma (  $I$  is the identity operator ).

**Lemma 8.3.1.** (*Stroock lemma for the derivative of a prime number*)

Let  $p$  be a fixed prime and  $q$  be an arbitrary prime. The operator  $D$  is given by Lemma 6.2.1 and the linear operator  $\delta_p$  is defined as  $\delta_p(q) = pq$ . Then,

$$D\delta_p(q) = I(q) + \delta_p D(q).$$

*Proof.*

The left hand side of the former equation is

$$\begin{aligned} D\delta_p(q) &= D(pq) \\ &= p'q + pq' \\ &= q + p. \end{aligned}$$

The right hand side of the former equation is

$$\begin{aligned} I(q) + \delta_p D(q) &= q + \delta_p(q') \\ &= q + p. \end{aligned}$$

Note that  $p' = 1$  and  $q' = 1$ . Comparing two sides of the equation, the proof is finish.  $\square$

The definition of  $q$  in Lemma 8.3.2 can be extended to all natural number.

This leads to the following corollary.

**Corollary 8.3.1.** *Let  $n$  be a positive integer and  $p$  be a fixed prime. The operator  $D$  is given by Lemma 6.2.1 and the linear operator  $\delta_p$  is defined as  $\delta_p(n) = pn$ . Then,*

$$D\delta_p(n) = I(n) + \delta_p D(n).$$

*Proof.*

The left hand side of the former equation is

$$\begin{aligned} D\delta_p(n) &= D(pn) \\ &= p'n + pn' \\ &= n + pn'. \end{aligned}$$

The right hand side of the former equation is

$$\begin{aligned} I(n) + \delta_p D(n) &= n + \delta_p(n') \\ &= n + pn'. \end{aligned}$$

Note that  $p' = 1$ . Comparing two sides of the equation, the proof is finish.

□

The next question is whether the restriction of choosing  $p$  as a prime number in Lemma 8.3.1 and following corollary is necessary or not. Lemma 6.2.3 is the key point to answer this.

**Lemma 8.3.2.** (*Stroock lemma for the derivative of a natural number*)

Let  $n$  be a positive integer and  $m$  be a fixed positive integer. The operator  $D$  is given by Lemma 6.2.1 and the linear operator  $\delta_m$  is defined as  $\delta_m(n) = mn$ . If these two operators satisfy the Stroock lemma

$$D\delta_m(n) = I(n) + \delta_m D(n),$$

the fixed positive number  $m$  must be a prime.

*Proof.*

The left hand side of the former equation is

$$\begin{aligned} D\delta_m(n) &= D(mn) \\ &= m'n + mn'. \end{aligned}$$

The right hand side of the former equation is

$$\begin{aligned} I(n) + \delta_m D(n) &= n + \delta_m(n') \\ &= n + mn'. \end{aligned}$$

Comparing the former two equations, it is clear that  $D$  and  $\delta$  satisfy

$$D\delta_m(n) = I(n) + \delta_m D(n)$$

if and only if  $m' = 1$ . Lemma 6.2.3 shows that  $m$  must be a prime.  $\square$

## Part 3 :

# A Discrete Differential Dynamics

Main definitions from the finite fields and the  $q$ -derivative are introduced in Chapter 9. Chapter 10 contains several examples and lemmas on semi-derivation on finite commutative algebras over finite fields and partial  $q$ -derivative. A discrete differential dynamics and application to the Cox-Ross-Rubinstein model are the main results in Chapter 11 and Chapter 12 based on [11].

# Chapter 9

## Terminology and Methods

In this chapter, all crucial notations and methods are introduced in details including definitions and properties. These concepts will be applied throughout this part. Numerous important lemmas with proofs are also given here.

### 9.1 Finite fields

The definition of finite fields is given in this section, which will be used many times later. There are more details in the book written by Pierre Samuel [49] and the materials of Gilberto Bini et al. [2] and other books about number theory.

Let  $K$  be a field. There is a unique ring homomorphism  $\phi : \mathbb{Z} \rightarrow K$  ( defined by  $\phi(n) = 1 + 1 + \cdots + 1$ ,  $n$  times, for  $n \geq 0$  and by  $\phi(-n) = -\phi(n)$  ). If  $\phi$  is not injective, its kernel is an ideal  $p\mathbb{Z}$  where  $p > 0$ ; then  $\mathbb{Z}/p\mathbb{Z}$  is identified with a subring of  $K$ ; thus  $\mathbb{Z}/p\mathbb{Z}$  is a field from which it follows that  $p$  is a prime number.  $K$  is of characteristic  $p$ . Such  $K$  is a finite field. The subfield,

$\mathbb{Z}/p\mathbb{Z}$ , of  $K$  is the smallest subfield of  $K$ ; it is called the prime subfield of  $K$ . Here, write  $\mathbb{F}_p$  for  $\mathbb{Z}/p\mathbb{Z}$ .

A finite field is defined on a finite set with four operations multiplication, addition, subtraction and division (excluding division by zero). In other words, these four operations are well-defined on this finite set. Since the calculations of multiplication, addition and subtraction for  $\mathbb{F}_p$  are obvious, division is the only operation explained specifically in this part. To define the operation division for  $\mathbb{F}_p$  is equivalent to define the multiplicative inverse for  $\mathbb{F}_p$ . For instance, the following four tables give the multiplicative inverse for each nonzero element  $a$  of  $\mathbb{F}_3$ ,  $\mathbb{F}_5$ ,  $\mathbb{F}_7$  and  $\mathbb{F}_{11}$ . Generally, the multiplicative inverse for an element  $a$  of a finite field can be calculated by many different ways such as Brute-force search, extended Euclidean algorithm, subtraction of logarithms and so on. There are more details about division of nonzero elements of a finite field in Chapter 10 of the book written by Rudolf Lidl et al. [30].

a	1	2
1/a	1	2

Table 9.1: The multiplicative inverse for an element  $a$  of  $\mathbb{F}_3$ .

a	1	2	3	4
1/a	1	3	2	4

Table 9.2: The multiplicative inverse for an element  $a$  of  $\mathbb{F}_5$ .

a	1	2	3	4	5	6
1/a	1	4	5	2	3	6

Table 9.3: The multiplicative inverse for an element  $a$  of  $\mathbb{F}_7$ .

Next lemma shows that the characteristic of a finite field  $\mathbb{F}_p$  can only be prime number.

a	1	2	3	4	5	6	7	8	9	10
1/a	1	6	4	3	9	2	8	7	5	10

Table 9.4: The multiplicative inverse for an element  $a$  of  $\mathbb{F}_{11}$ .

**Lemma 9.1.1.** *The ring  $\mathbb{F}_p$  is a field if and only if  $p$  is a prime number.*

The entire proof of Lemma 9.1.1 can be found in the book written by Ian Stewart [51]. Here quotes how to find the multiplicative inverse for a given element only.

Putting  $n \in p\mathbb{Z}$ . Let  $n + r$  be a non-zero element of  $\mathbb{Z}/p\mathbb{Z}$ . Since  $r$  and  $p$  are coprime then by Bezout's lemma of  $\mathbb{Z}$  there exist integers  $a$  and  $b$  such that  $ar + bp = 1$ . Then

$$(n + a)(n + r) = (n + 1) - (n + p)(n + b) = n + 1$$

and similarly

$$(n + r)(n + a) = n + 1.$$

Since  $n + 1$  is the identity element of  $\mathbb{Z}/p\mathbb{Z}$ , there exists a multiplicative inverse for the given element  $n + r$ . Thus every non-zero element of  $\mathbb{Z}/p\mathbb{Z}$  has an inverse.

Next lemma shows some possible methods to find the multiplicative inverse for some particular elements in a finite field of order  $p$ .

**Lemma 9.1.2.** *Let  $\mathbb{F}_p$  be a finite field of order  $p$ . The following four statements are valid for all primes  $p > 2$ :*

1) *The multiplicative inverse of  $p - 1$  in  $\mathbb{F}_p$  is itself.*

2) There exists a non-negative integer  $n$  such that  $p - 2 | 1 + np$  in  $\mathbb{Z}$  and the multiplicative inverse of  $p - 2$  in  $\mathbb{F}_p$  is  $n + 1$ ;

3) If  $m$  is the multiplicative inverse of  $n$  in  $\mathbb{F}_p$ , then  $p - m$  is the multiplicative inverse for  $p - n$ ;

4) The multiplicative inverse of any non-zero element  $n$  in  $\mathbb{F}_p$  is  $n^{p-2}$ .

*Proof.*

1) The proof is clear since  $(p - 1) \cdot (p - 1) = 1$  in  $\mathbb{F}_p$ .

2) Rewrite  $1 + np$  by  $1 + np = n(p - 2) + 2n + 1$ . Since all primes  $p > 2$  are odd, there exists a non-negative integer  $n$  such that  $p - 2 = 2n + 1$  and  $n = (p - 3) / 2$ . Therefore,  $(p - 2) \cdot (n + 1) = 1 + np$  in  $\mathbb{Z}$  and  $(p - 2) \cdot (n + 1) = 1$  in  $\mathbb{F}_p$ .

3) Since  $m$  is the the multiplicative inverse for  $n$  of  $\mathbb{F}_p$ ,  $m \cdot n = 1$ . Therefore,  $(p - m) \cdot (p - n) = m \cdot n = 1$ .

4) To prove this, the following lemma is introduced here first.

**Lemma 9.1.3.** *If  $\mathbb{F}$  is a finite field with  $m$  elements, then every  $n \in \mathbb{F}$  satisfies  $n^m = n$ . [30]*

By Lemma 9.1.3, for each non-zero element  $n$  of  $\mathbb{F}_p$ ,  $n^{p-1} = 1$ . The proof is clear.  $\square$

Note that Lemma 9.1.3 is famous and has various statements. There are some connections between this lemma and Fermat's little theorem, but this is irrelevant to the main topic and is not discussed here.

Generally, consider a finite field  $\mathbb{F}_{p^n}$  of order  $p^n$  with prime number  $p$  and positive integer  $n$ . Note that all fields of this order are isomorphic. Given  $m = p^n$  with  $n > 1$ , the finite field  $\mathbb{F}_m$  for non-prime  $m$  can be constructed by the quotient ring  $\mathbb{F}_m = \mathbb{F}_p(x)/(P)$  where  $P$  is an irreducible polynomial in  $\mathbb{F}_p(x)$  of degree  $n$ . W. H. Bussey illustrated tables of some non-prime finite fields in his papers [3], [4].

Next example is given here to show that how to construct a specific non-prime finite field.

**Example 9.1.1.**

According to the preceding concept of  $\mathbb{F}_{p^n}$ ,  $\mathbb{F}_4 = \mathbb{F}_2(x)/(P)$  where  $P = x^2 + x + 1$  since this is the only irreducible polynomial of degree 2 over  $\mathbb{F}_2$ . Let  $a$  be a root of  $P = 0$  in  $\mathbb{F}_4$ . The set  $\{0, 1, a, 1 + a\}$  along with addition, subtraction, multiplication and division defined on it forms  $\mathbb{F}_4$ . Note that  $a$  is the generator of the cyclic group formed by the non-zero element of  $\mathbb{F}_4$  along with multiplication.  $\square$

**Proposition 9.1.1.** *The sum of all elements of a finite field is 0 except for  $\mathbb{F}_2$*

*Proof.*

Let  $f : \mathbb{F} \rightarrow \mathbb{F}$  be a bijection such that  $\sum_{x \in \mathbb{F}} x = \sum_{x \in \mathbb{F}} f(x)$ . Since  $\mathbb{F}$  has more than two elements, pick  $\alpha \in \mathbb{F} \setminus \{0, 1\}$  and  $x \mapsto \alpha x$  is such a bijection. Therefore,  $(1 - \alpha) \sum_{x \in \mathbb{F}} x = 0$  and  $1 - \alpha \neq 0$ , i.e.  $\sum_{x \in \mathbb{F}} x = 0$ .  $\square$

## 9.2 $q$ -Derivative

This section follows by Victor Kac et al. [21] and J. Koekoek et al. [27]. There are more details from the book written by A. Aral et al. [1].

Since there is no non-zero semi-derivation  $D$  on finite commutative algebra over finite field which satisfies 'ordinary' product rule ( see 12.1 ), it is interesting to investigate if there exists another kind of derivation satisfying other special kind of product rule. This is the reason why  $q$ -derivation is taken into consideration.

The definitions of the  $q$ -differential and the  $q$ -derivative of the function  $f(x)$  are given below.

**Definition 9.2.1.** ( *$q$ -derivative*)

*Consider an arbitrary function  $f(x)$ . Its  $q$ -differential is*

$$d_q f(x) = f(qx) - f(x).$$

*The following expression,*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x},$$

*is called the  $q$ -derivative of the function  $f(x)$ .*

Note that

$$\lim_{q \rightarrow 1} D_q f(x) = \frac{df(x)}{dx}$$

if  $f(x)$  is differentiable.

The  $q$ -derivative  $D_q$  has the linear property that for any constants  $a$  and  $b$ ,

$$D_q (af(x) + bg(x)) = aD_q f(x) + bD_q g(x).$$

The  $q$ -derivative  $D_q$  also have the  $q$ -product rule that

$$D_q (f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x).$$

By symmetry,

$$D_q (f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x).$$

Note that there does not exist a general chain rule for  $q$ -derivatives, though some special cases may exist.

The higher order  $q$ -derivative is illustrated here. The second  $q$ -derivative of the arbitrary function  $f(x)$  is

$$D_q^2 f(x) = D_q (D_q f(x)) = \frac{f(q^2x) - (q+1)f(qx) + qf(x)}{q(q-1)^2 x^2}.$$

**Lemma 9.2.1.** *If  $q_1 \neq q_2$ , then  $D_{q_1}D_{q_2}f(x) \neq D_{q_2}D_{q_1}f(x)$  for arbitrary function  $f(x)$ .*

*Proof.*

According to the Definition 9.2.1, one has

$$D_{q_1}D_{q_2}f(x) = D_{q_1} (D_{q_2}f(x)) = \frac{f(q_1q_2x) - f(q_1x) - q_1f(q_2x) + q_1f(x)}{q_1(q_1-1)(q_2-1)x^2}$$

and

$$D_{q_2} D_{q_1} f(x) = D_{q_2} (D_{q_1} f(x)) = \frac{f(q_1 q_2 x) - f(q_2 x) - q_2 f(q_1 x) + q_2 f(x)}{q_2 (q_1 - 1) (q_2 - 1) x^2}.$$

Since  $q_1 \neq q_2$ , the result is clear.  $\square$

# Chapter 10

## Simple Results

In this chapter, numerous examples and lemmas are treated as simple results here. These results may help to understand not only several lemmas given before but the inspiration of this part as well. Note that these lemmas given here will be applied in following chapters.

### 10.1 Semi-derivation on finite commutative algebras over finite fields

**Definition 10.1.1.** (*semi-derivation and derivation*)

Let  $\mathfrak{R}$  be a ring. A mapping  $\delta : r \in \mathfrak{R} \rightarrow r' \in \mathfrak{R}$  is a semi-derivation if

$$(r_1 r_2)' = r_1 r_2' + r_1' r_2$$

for all  $r_1, r_2 \in \mathfrak{R}$ . This is the Leibniz rule or product rule. A semi-derivation is a derivation if

$$(r_1 + r_2)' = r_1' + r_2'$$

for all  $r_1, r_2 \in \mathfrak{R}$  also holds. This is additivity or the additivity rule. [6]

An example is given here to explain the existence of derivation on finite commutative algebras.

**Example 10.1.1.**

Consider a pair  $(x, \alpha)$  where  $x, \alpha \in \mathbb{F}_p$ . The operation "+" and "." are defined by

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$

and

$$(x, \alpha) \cdot (y, \beta) = (\alpha y + \beta x, \alpha \beta),$$

respectively. Since  $(x, \alpha)(0, 1) = (x, \alpha)$ ,  $\{(x, \alpha) : x, \alpha \in \mathbb{F}_p\}$  along with two operations "+" and "." forms a finite commutative algebra. Let  $D$  be the derivation on the preceding finite commutative algebra which can be represented by

$$D(x, \alpha) = (Dx, 0).$$

It is sufficient to check that  $D$  satisfies the Leibniz rule according to the following statement:

$$D[(x, \alpha)(y, \beta)] = (\alpha Dy + \beta Dx, 0) = (x, \alpha) D(y, \beta) + (y, \beta) D(x, \alpha). \quad \square$$

Therefore, the following lemma is clear.

**Lemma 10.1.1.** *In general, there exists non-zero derivation on finite commutative algebras.*

The preceding lemma gives a general result to this topic. The counterexample shows that the non-zero derivation satisfying the Leibniz rule can be defined on the finite commutative algebra with some special definitions of operation. In other words, there exists non-zero derivation on several typical finite commutative algebras.

Consider a set  $\mathcal{A}_{n,p} = \{(f_1, \dots, f_n)^T : f_i \in \mathbb{F}_p\}$  together with vector addition and scalar multiplication which is a vector space over a finite field  $\mathbb{F}_p$  spanned by the standard basis. The operation ' $\otimes$ ' is the tensor product defined by

$$f \otimes g = (f_1 \cdot g_1, \dots, f_n \cdot g_n)^T$$

where  $f, g \in \mathcal{A}_{n,p}$ . The set  $\mathcal{A}_{n,p}$  along with the operation ' $\otimes$ ' forms a commutative monoid with identity element  $\mathbf{1} = (1, \dots, 1)^T$ . This set together with two operations '+' and ' $\otimes$ ' forms integral domain (or non-zero commutative ring) since it satisfies

$$\begin{aligned} f \otimes (g + h) &= (f_1 \cdot (g_1 + h_1), \dots, f_n \cdot (g_n + h_n))^T \\ &= (f_1 \cdot g_1, \dots, f_n \cdot g_n)^T + (f_1 \cdot h_1, \dots, f_n \cdot h_n)^T \\ &= f \otimes g + f \otimes h \end{aligned}$$

where  $f, g, h \in \mathcal{A}_{n,p}$  (distributivity of multiplication over addition). Note that  $f \otimes f$  is simply denoted by  $f^2$ .

Let  $D$  be a semi-derivation operator which can be represented as a  $n \times n$  matrix. Assume that this operator satisfies the Leibniz rule (or called product rule) which means that it satisfies the chain rule as well. Next lemma gives a general result which is also a crucial reason that  $q$ -derivative is applied here in this part.

**Lemma 10.1.2.** *Let  $\mathcal{A}_{n,p}$  along with ' $\otimes$ ' be a finite commutative algebra over the finite field  $\mathbb{F}_p$  ( $p > 2$ ) with unity and  $D$  be a semi-derivation on  $\mathcal{A}_{n,p}$  defined before. Then,  $D \equiv 0$ .*

*Proof.*

For every  $f \in \mathcal{A}_{n,p}$ , applying the semi-derivation  $D$  to  $f^p$ , one has

$$Df^p = f \otimes Df^{p-1} + f^{p-1} \otimes Df$$

due to the product rule. Then, by induction and the property of  $\mathbb{F}_p$ , it is equivalent to prove that  $D \equiv 0$  is the only solution of the equation

$$Df^p = pf^{p-1} \otimes Df \equiv 0.$$

According to Lemma 9.1.3 and Lemma 9.1.1, it is clear that  $f_i^p = f_i$  for every  $f_i \in \mathbb{F}_p$  and  $p$  can only be a prime number or a finite power of a prime number. Therefore, one has  $f^p = f$  and  $Df^p = Df$ . Concluding all former results, one can discover  $Df \equiv 0$ , and thus  $D \equiv 0$  on account of arbitrariness of  $f$ .  $\square$

## 10.2 Partial $q$ -derivative

On the basis of  $q$ -derivative, the partial  $q$ -derivative is defined by following definition.

**Definition 10.2.1.** (*partial  $q$ -derivative*)

Let  $f(x, y)$  be an arbitrary function of  $x$  and  $y$ . The partial  $q$ -derivative with respect to  $x$  is

$$D_{q_x} f(x, y) = \frac{f(q_x x, y) - f(x, y)}{(q_x - 1)x}.$$

The cross partial  $q$ -derivative with respect to  $x$  and  $y$  is

$$D_{q_x q_y} f(x, y) = D_{q_x} (D_{q_y} f(x, y))$$

by taking the partial  $q$ -derivative of  $f$  with respect to  $y$ , and then taking the partial  $q$ -derivative of the result with respect to  $x$ .

**Lemma 10.2.1.** *The cross partial  $q$ -derivative is unaffected by which variable the partial  $q$ -derivative is taken with respect to first and which is taken second.*

*That is,*

$$D_{q_x q_y} f(x, y) = D_{q_y q_x} f(x, y).$$

*Proof.*

By Definition 10.2.1,

$$\begin{aligned}
& D_{q_x q_y} f(x, y) \\
= & \frac{f(q_x x, q_y y) - f(q_x x, y) - f(x, q_y y) + f(x, y)}{(q_y - 1) y (q_x - 1) x} \\
= & D_{q_y q_x} f(x, y). \quad \square
\end{aligned}$$

# Chapter 11

## A Discrete Differential Dynamics

For a given polynomial  $F(x) \in \mathbb{F}_{p^n}(x)$  the definition of the derivative at  $a \in \mathbb{F}_{p^n}(x) \setminus \{0\}$  given by E. Pasalic et al [43]. is  $D_a F(x) = F(x+a) - F(x)$ . Here the concept of  $q$ -derivative gives another possible method to define the derivation on finite fields. It describes the relations between some different states since there is no non-zero semi-derivation on finite commutative algebra over finite field except some specific multiplicative operator satisfying classic product rule. A discrete differential dynamics system can be created by choosing some special  $q$ -derivation operators. Here consider the vector space with dimension  $p^n$  where  $p$  is a prime number and  $n$  is a positive integer due to the definition of finite fields.

### 11.1 Lemmas

All main results of this part are demonstrated in this section.

Let  $\mathbb{F}_p$  be a finite field and  $n$  be a positive integer elements,  $\mathcal{A}_{n,p} = \{(f_1, \dots, f_n)^T : f_i \in \mathbb{F}_p\}$  be a vector space of  $\mathbb{F}_p$ -valued functions and  $O_{\mathcal{A}_{n,p} \rightarrow \mathcal{A}_{n,p}}$  be a class of matrices from  $\mathcal{A}_{n,p}$  to  $\mathcal{A}_{n,p}$ . More exactly, the dynamics of  $q$ -type derivation operators in  $O_{\mathcal{A}_{n,p} \rightarrow \mathcal{A}_{n,p}}$  are analysed. Here define  $D_q \tilde{f} = 0$  where  $D_q$  is a  $q$ -type derivation operator,  $\tilde{f} = (f_0, f_0, \dots, f_0)^T$  and  $\tilde{f} \in \mathcal{A}_{n,p}$ . Note that all  $q$ -derivation operators are  $q$ -type derivation operators.

### 11.1.1 Differential dynamics of $q$ -derivations on $\mathbb{F}_p$

For general prime  $p$  ( $p > 2$ ), the following lemmas are true:

**Lemma 11.1.1.** *Let  $q = p - 1$ ,  $n = q/2$  and  $m_i = 1/(2i)$  for all  $1 \leq i \leq n$ .*

*Then, the  $q$ -derivation operator  $D_q$  in  $O_{\mathcal{A}_{p,p} \rightarrow \mathcal{A}_{p,p}}$  is given by*

$$D_q = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{D}_q \end{pmatrix}_{p \times p}$$

where the submatrices of  $\widetilde{D}_q$  are

$$D_i = \begin{matrix} & i & p-i \\ i & \begin{pmatrix} m_i & p-m_i \\ m_i & p-m_i \end{pmatrix} & \\ p-i & & \end{matrix}, 1 \leq i \leq n.$$

Moreover,  $D_q^2 = 0$ .

**Lemma 11.1.2.** *Let  $q = 0$ ,  $r = p - 1$ ,  $n = r/2$  and  $m_i = 1/i$  for all  $1 \leq i \leq n$ .*

*Then, the  $q$ -derivation operator  $D_0$  in  $O_{\mathcal{A}_{p,p} \rightarrow \mathcal{A}_{p,p}}$  is given by*

$$D_0 = (a_{ij})_{p \times p}, 0 \leq i, j \leq p-1$$

where

$$\begin{cases} a_{00} = 0; \\ a_{0i} = m_i, 1 \leq i \leq n; \\ a_{0i} = p - m_{r+1-i}, n+1 \leq i \leq r; \\ a_{ii} = p - m_i, 1 \leq i \leq n; \\ a_{ii} = m_{r+1-i}, n+1 \leq i \leq r; \\ a_{ij} = 0, i \neq 0, i \neq j. \end{cases}$$

### 11.1.2 Characterization of $q$ -type derivations on $\mathbb{F}_p$ -valued vectors

Since  $q = 1$  is meaningless for  $q$ -derivative, the  $q$ -type derivation operator  $D_1$  for  $\mathbb{F}_p$  is not taken into account. Next lemma ( which is a special case of Lemma 10.1.2 ) shows that there is no such derivation operator  $D$  satisfying the  $q$ -product rule of  $q$ -derivation when  $q = 1$  unless  $D \equiv 0$ .

**Lemma 11.1.3.** *Let  $r = p - 1$  and  $D = (a_{ij})_{p \times p}, 0 \leq i, j \leq r$ . Then, there is no non-zero derivation operator  $D$  satisfying*

$$D(f \otimes g) = f \otimes (Dg) + g \otimes (Df)$$

where  $f, g \in \mathcal{A}_{p,p}$ .

For general prime  $p$  ( $p > 2$ ), the following lemmas are true:

**Lemma 11.1.4.** *Let  $q = p - 1$  and  $D_{(q=p-1)} = (a_{ij})_{p \times p}, 0 \leq i, j \leq q$  in  $O_{\mathcal{A}_{p,p} \rightarrow \mathcal{A}_{p,p}}$ . For general  $\mathbb{F}_p$ , there are  $p^q - 1$  possible  $q$ -type derivation operators satisfying*

$$D_{(q=p-1)}(f \otimes g) = (D_{(q=p-1)}f) \otimes g + \bar{f} \otimes (D_{(q=p-1)}g)$$

where  $\bar{f} = (f_0, f_q, \dots, f_{q^2})^T$  valid for all  $f, g \in \mathcal{A}_{p,p}$ . Let  $n = q/2$ . These  $p^q - 1$  possible  $q$ -type derivation operators have the following representation:

$$D_{(q=p-1)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}_{(q=p-1)} \end{pmatrix}_{p \times p}.$$

The submatrices of  $\tilde{D}_{(q=p-1)}$  are

$$D_k = \begin{matrix} & k & p-k \\ k & \begin{pmatrix} a_{i,i} & a_{i,p-i} \\ a_{j,p-j} & a_{j,j} \end{pmatrix} & \\ p-k & & \end{matrix}, 1 \leq k \leq n$$

where  $a_{i,i} + a_{i,p-i} = 0, 1 \leq i \leq n$  and  $a_{j,p-j} + a_{j,j} = 0, n+1 \leq j \leq q$ . Moreover, any power of the  $q$ -type derivation operator  $D_{(q=p-1)}$  is included in aforementioned  $p^q - 1$  cases plus zero matrix.

**Lemma 11.1.5.** *Let  $q = 0, r = p - 1$  and  $D_{(q=0)} = (a_{ij})_{p \times p}, 0 \leq i, j \leq r$  in  $O_{\mathcal{A}_{p,p} \rightarrow \mathcal{A}_{p,p}}$ . For general  $\mathbb{F}_p$ , there are  $p^r - 1$  possible  $q$ -type derivation operators satisfying*

$$D_{(q=0)}(f \otimes g) = (D_{(q=0)}f) \otimes g + \bar{f} \otimes (D_{(q=0)}g)$$

where  $\bar{f} = (f_0, f_0, \dots, f_0)^T$  valid for all  $f, g \in \mathcal{A}_{p,p}$ . These  $p^r - 1$  possible  $q$ -type derivation operators have the following representation:

$$D_{(q=0)} = (a_{ij})_{p \times p}, 0 \leq i, j \leq p-1$$

where

$$\begin{cases} a_{00} = 0; \\ a_{0i} + a_{ii} = 0, 1 \leq i \leq p-1; \\ a_{ij} = 0, i \neq 0, i \neq j. \end{cases}$$

### 11.1.3 Differential dynamics of $q$ -derivations on $\mathbb{F}_{p^n}$

Given a non-prime finite field  $\mathbb{F}_{p^n} = \mathbb{F}_p/(P)$  with prime  $p$  and integer  $n \geq 2$  where  $P$  is an irreducible polynomial in  $\mathbb{F}_p(x)$  of degree  $n$ . Let  $a$  be a root of  $P = 0$  in  $\mathbb{F}_{p^n}$ . Then, the following lemma is true:

**Lemma 11.1.6.** *Let  $m = p^n$  and  $q = a^k$  for integer  $1 \leq k \leq m-2$ . Then, the  $q$  derivation operator  $D_q$  in  $O_{\mathcal{A}_{m,m} \rightarrow \mathcal{A}_{m,m}}$  is given by*

$$D_q = (d_{ij})_{m \times m}, 0 \leq i, j \leq m-1$$

where

$$\begin{cases} d_{i0} = d_{0j} = 0, 0 \leq i, j \leq m-1; \\ d_{ij} = \frac{1}{(a^k - 1) a^{i-1}}, 1 \leq i \leq m-1-k, j = i+k; \\ d_{ij} = \frac{1}{(a^k - 1) a^{i-1}}, m-1-k < i \leq m-1, j = i+k-(m-1); \\ d_{ij} = \frac{p-1}{(a^k - 1) a^{i-1}}, 1 \leq i \leq m-1, j = i; \\ d_{ij} = 0, \text{ other.} \end{cases}$$

**Corollary 11.1.1.** *The trace of the  $q$ -derivation operator  $D_q = (d_{ij})_{m \times m}, 0 \leq i, j \leq m-1$  derived in lemma 11.1.6 is 0.*

### 11.1.4 Characterization of $q$ -type derivations on $\mathbb{F}_{p^n}$ -valued vectors

Consider a non-prime finite field  $\mathbb{F}_{p^n} = \mathbb{F}_p/(P)$  with prime  $p$  and integer  $n \geq 2$  where  $P$  is an irreducible polynomial in  $\mathbb{F}_p(x)$  of degree  $n$ . For the root  $a$  of  $P = 0$  in  $\mathbb{F}_{p^n}$ , the following lemma is true:

**Lemma 11.1.7.** *Let  $m = p^n$  and  $D_{(q=a^k)} = (d_{ij})_{m \times m}, 0 \leq i, j \leq m$  in  $O_{\mathcal{A}_{m,m} \rightarrow \mathcal{A}_{m,m}}$ . For general  $\mathbb{F}_{p^n}$ , there are  $m^{m-1} - 1$   $q$ -type derivation operators satisfying*

$$D_{(q=a^k)}(f \otimes g) = (D_{(q=a^k)}f) \otimes g + \bar{f} \otimes (D_{(q=a^k)}g)$$

where  $\bar{f} = (f_0, f_q, \dots, f_{q^{(m-1)}})^T$  valid for all  $f, g \in \mathcal{A}_{m,m}$ . These  $m^{(m-1)} - 1$  possible  $q$ -type derivation operators have the following representation:

$$D_{(q=a^k)} = (d_{ij})_{m \times m}, 0 \leq i, j \leq m - 1$$

where

$$\begin{cases} d_{i0} = d_{0j} = 0, 0 \leq i, j \leq m - 1; \\ d_{ii} + d_{ij} = 0, 1 \leq i \leq m - 1 - k, j = i + k; \\ d_{ii} + d_{ij} = 0, m - k \leq i \leq m - 1, j = i + k - (m - 1); \\ d_{ij} = 0, \text{ other.} \end{cases}$$

## 11.2 Proofs and properties

All proofs of lemmas introduced before and several properties are given in this section.

### 11.2.1 Differential dynamics of $q$ -derivations on $\mathbb{F}_p$

Proof of Lemma 11.1.1:

Consider the finite field  $\mathbb{F}_p$  where  $p$  is a prime and  $p > 2$ . Let  $f$  be a function in  $\mathcal{A}_{p,p}$ . Suppose  $q$  equals to  $p - 1$ . Applying the  $q$ -derivation operator  $D_q (= D_{p-1})$  to  $f$  in  $\mathcal{A}_{p,p}$ . Let  $D_q f = g = (g_0, g_1, \dots, g_q)^T$ . By assumption,  $g_0 = 0$ . Let  $n = q/2$ . For each  $1 \leq i \leq n$ ,

$$g_i = \frac{1}{2i} f_i + \frac{1}{p-2i} f_{p-i}$$

and for  $n+1 \leq j \leq q$ ,

$$g_j = \frac{1}{2(p-j)} f_{p-j} + \frac{1}{2j-p} f_j.$$

Let  $m_i = 1/(2i)$ . By Lemma 9.1.2 3),  $p-m_i = 1/(p-2i)$ . Since  $1 \leq p-j \leq n$ , the proof is clear. For all submatrices  $D_i$ ,  $1 \leq i \leq n$ ,  $D_i^2 = 0$ , and therefore  $D_q^2 = 0$ .

Since multiplication, addition, subtraction and division (or multiplicative inverse) for  $\mathbb{F}_p$  are well-defined, the  $q$ -derivation operator  $D_q$  is well-defined as well.

Let  $m_i = 1/(2i)$  for  $1 \leq i \leq n$ . Note that

$$\{m_i : 1 \leq i \leq n\} \cup \{p-m_i : 1 \leq i \leq n\} = \{1, 2, \dots, q\}.$$

Proof of Lemma 11.1.2:

Suppose  $q$  equals to 0. Applying the  $q$ -derivation operator  $D_0$  to  $f$  in  $\mathcal{A}_{p,p}$ . Let  $D_q f = g = (g_0, g_1, \dots, g_q)^T$ . By assumption,  $g_0 = 0$ . Let  $r = p - 1$ . For each  $1 \leq i \leq r$ ,

$$g_i = \frac{f_0}{p-i} + \frac{f_i}{i}.$$

Let  $m_i = 1/(p-i)$ . By Lemma 9.1.2 3),  $p-m_i = 1/i$ . Note that for  $r/2+1 \leq j \leq r$ ,  $1 \leq p-j \leq r/2$ . Then the proof is clear.  $\square$

Note that  $D_0^p = D_0$ .

### 11.2.2 Characterization of $q$ -type derivations on $\mathbb{F}_p$ -valued vectors

Proof of Lemma 11.1.3:

It is equivalent to prove that  $a_{ki} \equiv 0, i = 0, 1, \dots, r$  ( $r = p - 1$ ) are the only solutions of

$$\sum_{i=0}^r a_{ki} f_i g_i = f_k \sum_{i=0}^r a_{ki} g_i + g_k \sum_{i=0}^r a_{ki} f_i \quad (18)$$

for all  $k \in \{0, 1, \dots, r\}$ . The the right hand side of equation (18) equals to

$$\sum_{i=0}^r (f_k a_{ki} g_i + g_k a_{ki} f_i).$$

Then, comparing both sides of equation (18),

$$\sum_{i=0}^r a_{ki} (f_i g_i - f_k g_i - f_i g_k) = 0.$$

To find the value of each  $a_{ki}$ , set  $f_i = g_i = 1$  for fixed  $i$  and others zero since  $f$  and  $g$  are arbitrary functions in  $\mathcal{A}_{p,p}$ . Therefore,  $a_{ki} \equiv 0$  where  $i = 0, 1, \dots, r$ .

□

Proof of Lemma 11.1.4:

Let  $q = p - 1$ . The  $q$ -product rule is equivalent to

$$\sum_{i=0}^q a_{ki} f_i g_i = f_{(qk)} \sum_{i=0}^q a_{ki} g_i + g_k \sum_{i=0}^q a_{ki} f_i \quad (19)$$

for all  $k \in \{0, 1, \dots, q\}$ . The right hand side of equation (19) equals to

$$\sum_{i=0}^q (f_{(qk)} a_{ki} g_i + g_k a_{ki} f_i).$$

Let  $i' = qk$  and  $i'' = k$ . Note that  $i' = p - k$ . Then, comparing both sides of equation (19),

$$\sum_{i \neq i', i''} a_{ki} f_i g_i = \sum_{i \neq i', i''} (f_{(qk)} a_{ki} g_i + g_k a_{ki} f_i) + g_k a_{ki'} f_{i'} + f_{i'} a_{ki''} g_{i''}$$

for each  $k$ . Therefore,

$$\sum_{i \neq i', i''} a_{ki} (f_i g_i - f_{(qk)} g_i - g_k f_i) = 0$$

and

$$g_k a_{k,p-k} f_{p-k} + f_{p-k} a_{kk} g_k = 0.$$

To find the value of each  $a_{ki}$  which  $i \neq i', i''$ , set  $f_i = g_i = 1$  for fixed  $i$  and others zero since  $f$  and  $g$  are arbitrary functions in  $\mathcal{A}_{p,p}$ . Thus,  $a_{ki} \equiv 0$  if  $i \neq k, p - k$  and  $a_{k,p-k} + a_{kk} = 0$  for each  $k \in \{0, 1, \dots, q\}$ .

For any prime  $p > 2$  there are  $p$  different ordered pairs  $(x, y)$  in  $\mathbb{F}_p$  satisfying  $x + y = 0$  and  $D_{(q=p-1)} \neq 0$ . These are the reasons why there are  $p^q - 1$  possible  $q$ -type derivation operators satisfying the product rule.

Let  $n = q/2$ . Without loss of generality, consider a submatrix of one particular  $q$ -type derivation operator  $D_{(q=p-1)}$  out of aforementioned  $p^q - 1$  cases,

$$D_k = \begin{matrix} & k & p - k \\ k & \begin{pmatrix} a_{ii} & p - a_{ii} \\ p - a_{jj} & a_{jj} \end{pmatrix} & \\ p - k & & \end{matrix}, 1 \leq k \leq n$$

where  $1 \leq i \leq n$  and  $n + 1 \leq j \leq q$ . Then,

$$D_k^2 = \begin{pmatrix} a_{ii}^2 + a_{ii}a_{jj} & -a_{ii}^2 - a_{ii}a_{jj} \\ -a_{jj}a_{ii} - a_{jj}^2 & a_{jj}a_{ii} + a_{jj}^2 \end{pmatrix}$$

which is also a submatrix of one particular  $q$ -type derivation operator  $D_{(q=p-1)}$  out of aforementioned  $p^q - 1$  cases since

$$a_{ii}^2 + a_{ii}a_{jj} - a_{ii}^2 - a_{ii}a_{jj} = 0$$

and

$$-a_{jj}a_{ii} - a_{jj}^2 + a_{jj}a_{ii} + a_{jj}^2 = 0.$$

This shows that any power of the  $q$ -type derivation operator  $D_{(q=p-1)}$  can be found in aforementioned  $p^q - 1$  cases plus zero matrix.  $\square$

Proof of Lemma 11.1.5:

Let  $r = p - 1$ . The  $q$ -product rule is equivalent to

$$\sum_{i=0}^r a_{ki} f_i g_i = f_{(0)} \sum_{i=0}^r a_{ki} g_i + g_k \sum_{i=0}^r a_{ki} f_i$$

for all  $k \in \{0, 1, \dots, r\}$ . The right hand side of preceding equation equals to

$$\sum_{i=0}^r (f_{(0)} a_{ki} g_i + g_k a_{ki} f_i).$$

By similar way in proving Lemma 11.1.4, the designing results are clear.  $\square$

### 11.2.3 Differential dynamics of $q$ -derivations on $\mathbb{F}_{p^n}$

Proof of Lemma 11.1.6:

Consider the finite field  $\mathbb{F}_m = \mathbb{F}_{p^n}$  where  $p$  is a prime and the integer  $n \geq 2$ . Note that  $P$  is an irreducible polynomial in  $\mathbb{F}_p(x)$  of degree  $n$ ,  $a$  is a root of  $P = 0$  in  $\mathbb{F}_m$  and  $a^{m-1} = 1$ . Let  $f$  be a function in  $\mathcal{A}_{m,m}$ . Suppose  $q$  equals to  $a^k$ ,  $1 \leq k \leq m-2$ . Applying the  $q$ -derivation operator  $D_q = D_{a^k}$  to  $f$ . Let  $D_q f = g = (g_0, g_1, \dots, g_{m-1})^T$ . By assumption,  $g_0 = 0$ . Then, for  $1 \leq i \leq m-1$ ,

$$g_i = \frac{p-1}{(a^k-1)^i} f_i + \frac{1}{(a^k-1)^i} f_{a^k i}.$$

According to the definition of the non-prime finite field,  $a^r = a^{r-(m-1)}$  for any  $m-1 < r < 2(m-1)$ . The proof is clear.  $\square$

Proof of Corollary 11.1.1:

For the  $q$ -derivation operator  $D_q = (d_{ij})_{m \times m}$ ,  $0 \leq i, j \leq m-1$  given in Lemma 11.1.6, to prove  $tr(D_q) \equiv 0$  is equivalent to prove  $\sum_{i=1}^{m-1} d_{ii} \equiv 0$  since  $d_{00} \equiv 0$ . According to Lemma 11.1.6,

$$\begin{aligned}
\sum_{i=1}^{m-1} d_{ii} &= \sum_{i=1}^{m-1} \frac{p-1}{(a^k-1) a^{i-1}} \\
&= \frac{p-1}{(a^k-1) a^{m-2}} \sum_{i=1}^{m-1} a^{m-i-1} \\
&= \frac{p-1}{(a^k-1) a^{m-2}} \sum_{i=0}^{m-2} a^i.
\end{aligned}$$

Note that the sum of all elements of a finite field is 0 except for  $\mathbb{F}_2$  ( Proposition 9.1.1 ). Thus,  $\sum_{i=0}^{m-2} a^i \equiv 0$ .  $\square$

#### 11.2.4 Characterization of $q$ -type derivations on $\mathbb{F}_{p^n}$ -valued vectors

Proof of Lemma 11.1.7:

Let  $m = q^n$ . The  $q$ -product rule is equivalent to

$$\begin{aligned}
& d_{00}f(0)g(0) + \sum_{j=1}^{m-1} d_{0j}f(a^{j-1})g(a^{j-1}) \\
&= f(0)d_{00}g(0) + g(0)d_{00}f(0) + f(0)\sum_{j=1}^{m-1} d_{0j}g(a^{j-1}) \\
& \quad + g(0)\sum_{j=1}^{m-1} d_{0j}f(a^{j-1}) \quad (20)
\end{aligned}$$

and

$$\begin{aligned}
& d_{i0}f(0)g(0) + \sum_{j=1}^{m-1} d_{ij}f(a^{j-1})g(a^{j-1}) \\
= & f(0)d_{i0}g(0) + g(0)d_{i0}f(0) + f(a^{k+i-1}) \sum_{j=1}^{m-1} d_{ij}g(a^{j-1}) \\
& + g(a^{i-1}) \sum_{j=1}^{m-1} d_{ij}f(a^{j-1}) \quad (21)
\end{aligned}$$

for all  $i \in \{1, \dots, m-1\}$ . By comparing both sides of equation (20) and (21),  $d_{i0} = d_{0j} = 0, 0 \leq i, j \leq m-1$ . Therefore, equation (21) equals to

$$\begin{aligned}
& \sum_{j=1}^{m-1} d_{ij}f(a^{j-1})g(a^{j-1}) \\
= & f(a^{k+i-1}) \sum_{j=1}^{m-1} d_{ij}g(a^{j-1}) + g(a^{i-1}) \sum_{j=1}^{m-1} d_{ij}f(a^{j-1})
\end{aligned}$$

for all  $i \in \{1, \dots, m-1-k\}$  and

$$\begin{aligned}
& \sum_{j=1}^{m-1} d_{ij}f(a^{j-1})g(a^{j-1}) \\
= & f(a^{k+i-m}) \sum_{j=1}^{m-1} d_{ij}g(a^{j-1}) + g(a^{i-1}) \sum_{j=1}^{m-1} d_{ij}f(a^{j-1})
\end{aligned}$$

for all  $i \in \{m-k, \dots, m-1\}$  since  $a^{m-1} = 1$  in  $\mathbb{F}_m$ . By similar way in proving Lemma 11.1.4,

$$\begin{cases} d_{ii} + d_{ij} = 0, 1 \leq i \leq m-1-k, j = i+k; \\ d_{ii} + d_{ij} = 0, m-k \leq i \leq m-1, j = i+k-(m-1); \\ d_{ij} = 0, \text{ other.} \end{cases}$$

For any prime  $p$  and positive integer  $n \geq 2$ , there are  $m$  different ordered pairs  $(x, y)$  in  $\mathbb{F}_m$  satisfying  $x + y = 0$  and  $D_{(q=a^k)} \neq 0$ . Note that for each non-zero

element  $a^k, 0 \leq k \leq m - 2$  the inverse element of  $a^k$  is unique. These shows that there are  $m^{m-1} - 1$  possible  $q$ -type derivation operators satisfying the  $q$ -product rule.  $\square$

## 11.3 Examples

In this section, several examples are given here to explain preceding lemmas.

### 11.3.1 Differential dynamics of $q$ -derivations on $\mathbb{F}_p$ with small $p$

Differential dynamics of  $D_{p-1}$  on  $\mathbb{F}_p$  with small  $p$

#### Example 11.3.1.

Consider a finite field  $\mathbb{F}_3$ . Let  $f = (f_0, f_1, f_2)^T$  be a function in  $\mathcal{A}_{3,3}$ . Suppose  $q$  equals to 2. Applying the  $q$ -derivation operator  $D_2$  in  $O_{\mathcal{A}_{3,3} \rightarrow \mathcal{A}_{3,3}}$  to  $f$ . Then,

$$D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

The next step is to investigate whether the  $q$ -product rule of  $D_2$  is satisfied in  $\mathbb{F}_3$ . To see this, it is sufficient to check

$$D_2(f \otimes g) = (D_2 f) \otimes g + \bar{f} \otimes (D_2 g), \quad (1)$$

where the function  $g$  has the same form as  $f$  and  $\bar{f} = (f_0, f_2, f_1)^T$ . By standard calculation,  $D_2$  does satisfy (1) and  $D_2^2 = 0$ .  $\square$

**Example 11.3.2.**

Consider a finite field  $\mathbb{F}_5$ . Suppose  $q$  equals to 4. By similar calculation, the  $q$ -derivation operator  $D_4$  in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$  has the following form:

$$D_4 = \begin{pmatrix} 0 & & 0 \\ & 3 & 2 \\ & & 4 & 1 \\ & & & 4 & 1 \\ 0 & 3 & & & 2 \end{pmatrix}.$$

This operator  $D_4$  has the  $q$ -product rule,

$$D_4(f \otimes g) = (D_4 f) \otimes g + \bar{f} \otimes (D_4 g),$$

where  $\bar{f} = (f_0, f_4, f_3, f_2, f_1)^T$ ,  $f = (f_0, f_1, f_2, f_3, f_4)^T$  and  $g \in \mathcal{A}_{5,5}$ . Note that  $D_4^2 = 0$ .  $\square$

**Example 11.3.3.**

As  $\mathbb{F}_3$  and  $\mathbb{F}_5$  given before, other cases for small primes are given below. For  $\mathbb{F}_7$ ,

$$D_6 = \begin{pmatrix} 0 & & & & & 0 \\ & 4 & & & & 3 \\ & & 2 & & & 5 \\ & & & 6 & 1 & \\ & & & 6 & 1 & \\ & & & & & 5 \\ 0 & 4 & & & & 3 \end{pmatrix}$$

in  $O_{\mathcal{A}_{7,7} \rightarrow \mathcal{A}_{7,7}}$  and  $D_6^2 = 0$ . For  $\mathbb{F}_{11}$ ,

$$D_{10} = \begin{pmatrix} 0 & & & & & & & & & 0 \\ & 6 & & & & & & & & 5 \\ & & 3 & & & & & & & 8 \\ & & & 2 & & & & & & 9 \\ & & & & 7 & & & & & 4 \\ & & & & & 10 & 1 & & & \\ & & & & & 10 & 1 & & & \\ & & & & & & & 7 & & 4 \\ & & & & & & & & 2 & 9 \\ & & & & & & & & & & 3 \\ & & & & & & & & & & & 8 \\ 0 & 6 & & & & & & & & & & 5 \end{pmatrix}$$

in  $O_{\mathcal{A}_{11,11} \rightarrow \mathcal{A}_{11,11}}$  and  $D_{10}^2 = 0$ . Here omit all calculations for former matrices because these specific processes are complicated.  $\square$

## Differential dynamics of $D_0$ on $\mathbb{F}_p$ with small $p$

### Example 11.3.4.

Consider a finite field  $\mathbb{F}_3$ . Let  $f = (f_0, f_1, f_2)^T$  be a function in  $\mathcal{A}_{3,3}$ . Suppose  $q$  equals to 0. Applying the  $q$ -derivation operator  $D_0$  in  $O_{\mathcal{A}_{3,3} \rightarrow \mathcal{A}_{3,3}}$  to  $f$ . Then,  $D_0$  has the form

$$D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

This operator  $D_0$  has the  $q$ -product rule,

$$D_0(f \otimes g) = (D_0 f) \otimes g + \tilde{f} \otimes (D_0 g),$$

where  $\tilde{f} = (f_0, f_0, f_0)^T$ ,  $f = (f_0, f_1, f_2)^T$  and  $g \in \mathcal{A}_{3,3}$ . Note that  $D_0^3 = D_0$ .

□

### Example 11.3.5.

Consider a finite field  $\mathbb{F}_5$ . Suppose  $q$  equals to 0. By similar calculation, the  $q$ -derivation operator  $D_0$  in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$  has the form

$$D_0 = \begin{pmatrix} 0 & & & & 0 \\ 4 & 1 & & & \\ 2 & & 3 & & \\ 3 & & & 2 & \\ 1 & & & & 4 \end{pmatrix}.$$

This operator  $D_0$  has the  $q$ -product rule,

$$D_0(f \otimes g) = (D_0f) \otimes g + \tilde{f} \otimes (D_0g),$$

where  $\tilde{f} = (f_0, f_0, f_0, f_0, f_0)^T$ ,  $f = (f_0, f_1, f_2, f_3, f_4)^T$  and  $g \in \mathcal{A}_{5,5}$ . Note that  $D_0^5 = D_0$ .  $\square$

**Example 11.3.6.**

As  $\mathbb{F}_3$  and  $\mathbb{F}_5$  given before, other cases for small primes are given below. For  $\mathbb{F}_7$ ,

$$D_0 = \begin{pmatrix} 0 & & & & & & 0 \\ 6 & 1 & & & & & \\ 3 & & 4 & & & & \\ 2 & & & 5 & & & \\ 5 & & & & 2 & & \\ 4 & & & & & 3 & \\ 1 & & & & & & 6 \end{pmatrix}$$

in  $O_{\mathcal{A}_{7,7} \rightarrow \mathcal{A}_{7,7}}$  and  $D_0^7 = D_0$ . Here omit all calculations for former matrices because these specific processes are complicated.  $\square$

**Differential dynamics of other possible  $q$ -derivation operators on  $\mathbb{F}_p$  with small  $p$**

**Example 11.3.7.**

Consider a finite field  $\mathbb{F}_5$ . Suppose  $q$  equals to 2. By the method given before, the  $q$ -derivation operator  $D_2$  in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$  has the form

$$D_2 = \begin{pmatrix} 0 & & 0 \\ & 4 & 1 \\ & & 2 & 3 \\ & 2 & & 3 \\ 0 & & 4 & 1 \end{pmatrix}.$$

This operator  $D_2$  has the  $q$ -product rule,

$$D_2(f \otimes g) = (D_2f) \otimes g + \bar{f} \otimes (D_2g),$$

where  $\bar{f} = (f_0, f_2, f_4, f_1, f_3)^T$ ,  $f = (f_0, f_1, f_2, f_3, f_4)^T$  and  $g \in \mathcal{A}_{5,5}$ . Note that  $D_2^4 = 0$ . Suppose  $q$  equals to 3. By similar calculation, the  $q$ -derivation operator  $D_3$  in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$  has the form

$$D_3 = \begin{pmatrix} 0 & & 0 \\ & 2 & 3 \\ & 4 & 1 \\ & & 4 & 1 \\ 0 & 2 & & 3 \end{pmatrix}.$$

This operator  $D_3$  has the  $q$ -product rule,

$$D_3(f \otimes g) = (D_3f) \otimes g + \bar{f} \otimes (D_3g),$$

where  $\bar{f} = (f_0, f_3, f_1, f_4, f_2)^T$ ,  $f = (f_0, f_1, f_2, f_3, f_4)^T$  and  $g \in \mathcal{A}_{5,5}$ . Note that  $D_3^4 = 0$ .  $\square$

**Example 11.3.8.**

As  $\mathbb{F}_5$  given before, other cases for small primes are given below. For  $\mathbb{F}_7$  and  $q = 2$ ,

$$D_2 = \begin{pmatrix} 0 & & & 0 \\ & 6 & 1 & \\ & & 3 & 4 \\ & & & 2 & 5 \\ & 2 & & 5 & \\ & & 3 & 4 & \\ 0 & & & 6 & 1 \end{pmatrix}$$

in  $O_{\mathcal{A}_{7,7} \rightarrow \mathcal{A}_{7,7}}$  and  $D_2^3 = 0$ . For  $\mathbb{F}_7$  and  $q = 3$ ,

$$D_3 = \begin{pmatrix} 0 & & & 0 \\ & 3 & 4 & \\ & & 5 & 2 \\ & & 6 & 1 \\ & & & 6 & 1 \\ & 5 & & 2 & \\ 0 & & 3 & 4 \end{pmatrix}$$

in  $O_{\mathcal{A}_{7,7} \rightarrow \mathcal{A}_{7,7}}$  and  $D_3^6 = 0$ . Here omit all calculations for former matrices because these specific processes are complicated.  $\square$

### 11.3.2 Characterization of $q$ -type derivations on $\mathbb{F}_p$ -valued vectors with small $p$

#### Characterization of $q$ -type derivations on $\mathbb{F}_3$ -valued vectors

##### Example 11.3.9.

Consider a finite field  $\mathbb{F}_3$ . Let  $D = (a_{ij})_{3 \times 3}$ ,  $0 \leq i, j \leq 2$  be a  $q$ -type derivation operator in  $O_{\mathcal{A}_{3,3} \rightarrow \mathcal{A}_{3,3}}$ . Suppose this operator satisfies the  $q$ -product rule which can be represented as

$$D(f \otimes g) = (Df) \otimes g + \bar{f} \otimes (Dg), \quad (2)$$

where  $f = (f_0, f_1, f_2)^T$ ,  $\bar{f} = (f_0, f_2, f_1)^T$  and  $g \in \mathcal{A}_{3,3}$ . It is interesting to find how many such kind of  $q$ -type derivation operators exist in  $\mathbb{F}_3$  and some related properties of these operators if they exist.

The left hand side of equation (2) is equivalent to

$$h_0 = a_{00}f_0g_0 + a_{01}f_1g_1 + a_{02}f_2g_2, \quad (3)$$

$$h_1 = a_{10}f_0g_0 + a_{11}f_1g_1 + a_{12}f_2g_2 \quad (4)$$

and

$$h_2 = a_{20}f_0g_0 + a_{21}f_1g_1 + a_{22}f_2g_2, \quad (5)$$

where  $D(f \otimes g) = h = (h_0, h_1, h_2)^T$ . The right hand side of equation (2) is equivalent to

$$\bar{f}_0 g_0 + f_0 \bar{g}_0 = (a_{00} f_0 + a_{01} f_1 + a_{02} f_2) g_0 + (a_{00} g_0 + a_{01} g_1 + a_{02} g_2) f_0, \quad (6)$$

$$\bar{f}_1 g_1 + f_2 \bar{g}_1 = (a_{10} f_0 + a_{11} f_1 + a_{12} f_2) g_1 + (a_{10} g_0 + a_{11} g_1 + a_{12} g_2) f_2 \quad (7)$$

and

$$\bar{f}_2 g_2 + f_1 \bar{g}_2 = (a_{20} f_0 + a_{21} f_1 + a_{22} f_2) g_2 + (a_{20} g_0 + a_{21} g_1 + a_{22} g_2) f_1, \quad (8)$$

where  $\bar{f} = (f_0, f_2, f_1)^T$ ,  $g = (g_0, g_1, g_2)^T$ ,  $\bar{f} = (\bar{f}_0, \bar{f}_1, \bar{f}_2)^T$  and  $\bar{g} = (\bar{g}_0, \bar{g}_1, \bar{g}_2)^T$ . Comparing (3) and (6), (4) and (7), (5) and (8). Then,  $a_{00} = a_{01} = a_{02} = 0$ ,  $a_{11} + a_{12} = 0$ ,  $a_{10} = 0$ ,  $a_{21} + a_{22} = 0$ ,  $a_{20} = 0$ .

Concluding all former results, there are 8  $q$ -type derivation operators ( $D \neq 0$ ) satisfying the  $q$ -product rule (equation (2)), which are

$$D_{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, D_{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, D_{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix},$$

$$D_{(4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, D_{(5)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, D_{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_{(7)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}, D_{(8)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Note that  $D_{(3)}$  is the  $q$ -derivation operator defined by  $q$ -derivation.

Now consider the dynamics of the former 8  $q$ -type derivation operators in  $\mathcal{A}_{3,3}$ .

It is can be seen that

$$D_{(3)}^2 = 0 = D_{(3)}^n, n \geq 2$$

and

$$D_{(4)}^2 = 0 = D_{(4)}^n, n \geq 2.$$

For  $D_{(1)}$ , the second order derivation is

$$D_{(1)}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = D_{(1)} = D_{(1)}^n, n \in \mathbb{N}.$$

Then,

$$\begin{aligned} e^{zD_{(1)}} &= I + \sum_{n=1}^{\infty} D_{(1)}^n \frac{z^n}{n!} \\ &= I + D_{(1)} \left( \sum_{n=1}^{\infty} \frac{z^n}{n!} \right) \\ &= I + D_{(1)} (e^z - 1). \end{aligned}$$

Similarly, for  $D_{(6)}$  and  $D_{(7)}$ , the second order derivation are

$$D_{(6)}^2 = D_{(6)} = D_{(6)}^n, n \in \mathbb{N}$$

and

$$D_{(7)}^2 = D_{(7)} = D_{(7)}^n, n \in \mathbb{N}$$

respectively. Then,

$$e^{zD_{(6)}} = I + D_{(6)}(e^z - 1)$$

and

$$e^{zD_{(7)}} = I + D_{(7)}(e^z - 1).$$

For  $D_{(2)}$ , the second order derivation is

$$D_{(2)}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = D_{(1)} = D_{(2)}^{2k}, k \in \mathbb{N}$$

and the third order derivation is

$$D_{(2)}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = D_{(2)} = D_{(2)}^{2k-1}, k \in \mathbb{N}.$$

Here  $D_{(2)}^3 = D_{(2)}(D_{(2)}D_{(2)}) = (D_{(2)}D_{(2)})D_{(2)}$  since  $D_{(2)}$  is symmetric. Note that the multiplication of two matrices  $A = (a_{ij})_{n \times n}$  and  $B = (b_{ij})_{n \times n}$  satisfying  $AB = BA$  (law of commutation) if and only if  $\sum_{i,j,k} a_{ij}b_{jk} = \sum_{i,j,k} b_{ij}a_{jk}$ . Then,

$$\begin{aligned}
e^{zD_{(2)}} &= I + \sum_{n=1}^{\infty} D_{(2)}^n \frac{z^n}{n!} \\
&= I + D_{(2)} \left( \sum_{k=1}^{\infty} \frac{z^{2k-1}}{(2k-1)!} \right) + D_{(1)} \left( \sum_{k=1}^{\infty} \frac{z^{2k}}{(2k)!} \right) \\
&= I + D_{(2)} (i \sin(iz)) + D_{(1)} (\cos(iz) - 1).
\end{aligned}$$

Similarly, for  $D_{(5)}$ , the second order derivation is

$$D_{(5)}^2 = D_{(6)} = D_{(5)}^{2k}, k \in \mathbb{N}$$

and the third order derivation is

$$D_{(5)}^3 = D_{(5)} = D_{(5)}^{2k-1}, k \in \mathbb{N}.$$

For  $D_{(8)}$ , the second order derivation is

$$D_{(8)}^2 = D_{(7)} = D_{(8)}^{2k}, k \in \mathbb{N}$$

and the third order derivation is

$$D_{(8)}^3 = D_{(8)} = D_{(8)}^{2k-1}, k \in \mathbb{N}.$$

Then,

$$e^{zD_{(5)}} = I + D_{(5)} (i \sin(iz)) + D_{(6)} (\cos(iz) - 1)$$

and

$$e^{zD_{(8)}} = I + D_{(8)} (i \sin(iz)) + D_{(7)} (\cos(iz) - 1). \quad \square$$

## Characterization of $q$ -type derivations on $\mathbb{F}_5$ -valued vectors

### Example 11.3.10.

Consider a finite field  $\mathbb{F}_5$ . Let  $D = (a_{ij})_{5 \times 5}$ ,  $0 \leq i, j \leq 4$  be a  $q$ -type derivation operator in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$ . Suppose that this operator has the  $q$ -product rule which can be represented as

$$D(f \otimes g) = (Df) \otimes g + \bar{f} \otimes (Dg), \quad (9)$$

where  $f = (f_0, f_1, f_2, f_3, f_4)^T$ ,  $\bar{f} = (f_0, f_4, f_3, f_2, f_1)^T$  and  $g \in \mathcal{A}_{5,5}$ . Similarly, it is interesting to find how many such kind of  $q$ -type derivation operators exist in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$  and some related properties of these operators if they exist.

Similarly,

$$\begin{cases} a_{00} = a_{01} = a_{02} = a_{03} = a_{04} = 0, \\ a_{10} = a_{12} = a_{13} = 0, a_{11} + a_{14} = 0 & (10), \\ a_{20} = a_{21} = a_{24} = 0, a_{22} + a_{23} = 0 & (11), \\ a_{30} = a_{31} = a_{34} = 0, a_{32} + a_{33} = 0 & (12), \\ a_{40} = a_{42} = a_{43} = 0, a_{41} + a_{44} = 0 & (13). \end{cases}$$

Concluding all former results, there are four pairs of non-zero integers satisfying equation (10), (11), (12) and (13) respectively, which are (4, 1), (1, 4), (2, 3) and (3, 2). Then, there are 624 ( $= 5^4 - 1$ ) ( $D \neq 0$ ) possible  $q$ -type derivation operators satisfying the product rule (equation (9)) including the  $q$ -derivation operator defined by  $q$ -derivation.

Now consider the dynamics of these possible  $q$ -type derivation operators in  $O_{\mathcal{A}_{5,5} \rightarrow \mathcal{A}_{5,5}}$ . Since there are 624 possible  $q$ -type derivation operators, only four examples are given here to explain some typical cases. The first one is exactly

the same  $q$ -derivation operator defined by  $q$ -derivation as mentioned in last section, which is

$$D_{(1)} = \begin{pmatrix} 0 & & 0 \\ & 3 & 2 \\ & & 4 & 1 \\ & & & 4 & 1 \\ 0 & 3 & & & 2 \end{pmatrix}.$$

It is clear that

$$D_{(1)}^2 = 0 = D_{(1)}^n, n \geq 2.$$

Note that there are 24 ( $= 5^2 - 1$ ) out of the 624 cases owning similar property.

The second one is

$$D_{(2)} = \begin{pmatrix} 0 & & 0 \\ & 3 & 2 \\ & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 2 & & & 3 \end{pmatrix}.$$

For  $D_{(2)}$ , the second order derivation is

$$D_{(2)}^2 = D_{(2)} = D_{(2)}^n, n \in \mathbb{N}.$$

Then,

$$e^{zD_{(2)}} = I + D_{(2)}(e^z - 1).$$

Note that there are at least 9 out of the 624 cases owning similar property.

The third one is

$$D_{(3)} = \begin{pmatrix} 0 & & 0 \\ & 2 & 3 \\ & & 0 & 0 \\ & & 0 & 0 \\ 0 & 3 & & 2 \end{pmatrix}.$$

For  $D_{(3)}$ , the second order derivation is

$$D_{(3)}^2 = D_{(2)} = D_{(3)}^{2k}, k \in \mathbb{N}$$

and the third order derivation is

$$D_{(3)}^3 = D_{(3)} = D_{(3)}^{2k-1}, k \in \mathbb{N}.$$

Here  $D_{(3)}^3 = D_{(3)} (D_{(3)} D_{(3)}) = (D_{(3)} D_{(3)}) D_{(3)}$  since  $D_{(3)}$  is symmetric. Then,

$$e^{zD_{(3)}} = I + D_{(3)} (i \sin (iz)) + D_{(2)} (\cos (iz) - 1).$$

Note that there are at least 9 out of the 624 cases owning similar property.

The fourth one is

$$D_{(4)} = \begin{pmatrix} 0 & & 0 \\ & 4 & 1 \\ & & 0 & 0 \\ & & 0 & 0 \\ 0 & 1 & & 4 \end{pmatrix}.$$

Let

$$D_{(5)} = \begin{pmatrix} 0 & & 0 \\ & 1 & 4 \\ & & 0 & 0 \\ & & 0 & 0 \\ 0 & 4 & & 1 \end{pmatrix}.$$

For  $D_{(4)}$ , the second order derivation is

$$D_{(4)}^2 = D_{(3)} = D_{(4)}^{4k-2}, k \in \mathbb{N},$$

the third order derivation is

$$D_{(4)}^3 = D_{(5)} = D_{(4)}^{4k-1}, k \in \mathbb{N},$$

the fourth order derivation is

$$D_{(4)}^4 = D_{(2)} = D_{(4)}^{4k}, k \in \mathbb{N},$$

and the fifth order derivation is

$$D_{(4)}^5 = D_{(4)} = D_{(4)}^{4k-3}, k \in \mathbb{N}.$$

Here  $D_{(4)}^5 = D_{(4)} (D_{(4)} (D_{(4)} (D_{(4)} D_{(4)}))) = (((D_{(4)} D_{(4)}) D_{(4)}) D_{(4)}) D_{(4)}$  since  $D_{(4)}$  is symmetric. Then,

$$\begin{aligned}
e^{zD_{(4)}} &= I + \sum_{n=1}^{\infty} D_{(4)}^n \frac{z^n}{n!} \\
&= I + D_{(4)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-3}}{(4k-3)!} \right) + D_{(3)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-2}}{(4k-2)!} \right) \\
&\quad + D_{(5)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-1}}{(4k-1)!} \right) + D_{(2)} \left( \sum_{k=1}^{\infty} \frac{z^{4k}}{(4k)!} \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
e^{zD_{(5)}} &= I + D_{(5)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-3}}{(4k-3)!} \right) + D_{(3)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-2}}{(4k-2)!} \right) \\
&\quad + D_{(4)} \left( \sum_{k=1}^{\infty} \frac{z^{4k-1}}{(4k-1)!} \right) + D_{(2)} \left( \sum_{k=1}^{\infty} \frac{z^{4k}}{(4k)!} \right).
\end{aligned}$$

Note that there are at least 18 out of the 624 cases owning similar property.

□

### Other situations

As  $\mathbb{F}_3$  and  $\mathbb{F}_5$  given before, for  $\mathbb{F}_7$  and  $\mathbb{F}_{11}$ , there are  $7^6 - 1$  and  $11^{10} - 1$  possible  $q$ -type derivation operators satisfying the following  $q$ -product rule:

$$D(f \otimes g) = (Df) \otimes g + \bar{f} \otimes (Dg),$$

where  $f = (f_0, f_1, \dots, f_q)^T$ ,  $\bar{f} = (f_0, f_q, \dots, f_{q^2})^T$  and  $g \in \mathcal{A}_{7,7}$  (Here  $q = 6$  for  $p = 7$  and  $q = 10$  for  $p = 11$ ).

### 11.3.3 Differential dynamics of $q$ -derivations on $\mathbb{F}_{p^n}$ with small $p$ and $n$

#### Differential dynamics of $D_a$ on $\mathbb{F}_{p^2}$ with small $p$

##### Example 11.3.11.

Consider a finite field  $\mathbb{F}_4$ . The element  $a$  is a root of  $x^2 + x + 1 = 0$  in  $\mathbb{F}_4$ . Let  $f = (f_0, f_1, f_a, f_{a^2})^T = (f_0, f_1, f_a, f_{1+a})^T$  be a function in  $\mathcal{A}_{4,4}$  and  $D = (a_{ij})_{4 \times 4}$ ,  $0 \leq i, j \leq 3$  be a  $q$ -derivation operator in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$ . Suppose  $q$  equals to  $a$ . Applying the  $q$ -derivation operator  $D_a$  in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$  to  $f$ . Then,  $D_a$  has the form

$$D_a = \begin{pmatrix} 0 & & & 0 \\ & a & a & \\ & & 1 & 1 \\ 0 & 1+a & & 1+a \end{pmatrix}.$$

This operator  $D_a$  satisfies the  $q$ -product rule,

$$D_a(f \otimes g) = (D_a f) \otimes g + \bar{f} \otimes (D_a g),$$

where  $\bar{f} = (f_0, f_a, f_{a^2}, f_1)^T$  and  $g \in \mathcal{A}_{4,4}$ . Note that  $D_a^3 = 0$ .  $\square$

##### Example 11.3.12.

Consider a finite field  $\mathbb{F}_9$ . The element  $a$  is a root of  $x^2 = x + 1$  in  $\mathbb{F}_9$ . Let

$$\begin{aligned} f &= (f_0, f_1, f_a, f_{a^2}, f_{a^3}, f_{a^4}, f_{a^5}, f_{a^6}, f_{a^7})^T \\ &= (f_0, f_1, f_a, f_{1+a}, f_{1+2a}, f_2, f_{2a}, f_{2+2a}, f_{2+a})^T \end{aligned}$$

be a function in  $\mathcal{A}_{9,9}$ . By similar calculation, the  $q$ -derivation operator  $D_a$  in  $O_{\mathcal{A}_{9,9} \rightarrow \mathcal{A}_{9,9}}$  has the form

$$D_a = \begin{pmatrix} 0 & & & & & & & & 0 \\ & 2a & a & & & & & & \\ & & 2 & 1 & & & & & \\ & & & 2a+1 & a+2 & & & & \\ & & & & a+1 & 2a+2 & & & \\ & & & & & a & 2a & & \\ & & & & & & 1 & 2 & \\ & & & & & & & a+2 & 2a+1 \\ 0 & a+1 & & & & & & & 2a+2 \end{pmatrix}_{9 \times 9}.$$

This operator  $D_a$  satisfies the  $q$ -product rule,

$$D_a(f \otimes g) = (D_a f) \otimes g + \bar{f} \otimes (D_a g),$$

where  $\bar{f} = (f_0, f_a, f_{a^2}, f_{a^3}, f_{a^4}, f_{a^5}, f_{a^6}, f_{a^7}, f_1)^T$  and  $g \in \mathcal{A}_{9,9}$ .  $\square$

### Differential dynamics of $D_a$ on $\mathbb{F}_8$

#### Example 11.3.13.

Consider a finite field  $\mathbb{F}_8$ . The element  $a$  is a root of  $x^3 = x + 1$  in  $\mathbb{F}_8$ . Let

$$\begin{aligned} f &= (f_0, f_1, f_a, f_{a^2}, f_{a^3}, f_{a^4}, f_{a^5}, f_{a^6})^T \\ &= (f_0, f_1, f_a, f_{a^2}, f_{1+a}, f_{a+a^2}, f_{1+a+a^2}, f_{1+a^2})^T \end{aligned}$$

be a function in  $\mathcal{A}_{8,8}$ . By similar calculation, the  $q$ -derivation operator  $D_a$  in  $O_{\mathcal{A}_{8,8} \rightarrow \mathcal{A}_{8,8}}$  has the form

$$D_a = \begin{pmatrix} 0 & & & & & & & 0 \\ & a^2 + a & a^2 + a & & & & & \\ & & a + 1 & a + 1 & & & & \\ & & & a^2 & a^2 & & & \\ & & & & a & a & & \\ & & & & & 1 & 1 & \\ & & & & & & a^2 + 1 & a^2 + 1 \\ 0 & a^2 + a + 1 & & & & & & a^2 + a + 1 \end{pmatrix}_{8 \times 8}.$$

This operator  $D_a$  satisfies the  $q$ -product rule,

$$D_a(f \otimes g) = (D_a f) \otimes g + \bar{f} \otimes (D_a g),$$

where  $\bar{f} = (f_0, f_a, f_{a^2}, f_{a^3}, f_{a^4}, f_{a^5}, f_{a^6}, f_1)^T$ .  $\square$

### Differential dynamics of $D_{1+a}$ on $\mathbb{F}_4$

#### Example 11.3.14.

Let  $\mathbb{F}_4$  be the finite field given before. Suppose  $q$  equals to  $1 + a$ . Applying the  $q$ -derivation operator  $D_{1+a}$  in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$  to  $f$ . Then  $D_{1+a}$  has the following form:

$$D_{1+a} = \begin{pmatrix} 0 & & 0 \\ & 1+a & 1+a \\ & a & a \\ 0 & & 1 & 1 \end{pmatrix}.$$

This operator  $D_{1+a}$  has the  $q$ -product rule,

$$D_{1+a}(f \otimes g) = (D_{1+a}f) \otimes g + \bar{f} \otimes (D_{1+a}g),$$

where  $f = (f_0, f_1, f_a, f_{1+a})^T$  and  $\bar{f} = (f_0, f_{1+a}, f_1, f_a)^T$ . Note that  $D_{1+a}^3 = 0$ .

□

### Differential dynamics of $D_0$ on $\mathbb{F}_4$

#### Example 11.3.15.

Let  $\mathbb{F}_4$  be the finite field given before. Suppose  $q$  equals to 0. Applying the  $q$ -derivation operator  $D_0$  in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$  to  $f$ . Then,  $D_0$  has the following form:

$$D_0 = \begin{pmatrix} 0 & & 0 \\ 1 & 1 & \\ 1+a & & 1+a \\ a & & a \end{pmatrix}.$$

This operator  $D_0$  satisfies the  $q$ -product rule,

$$D_0(f \otimes g) = (D_0f) \otimes g + \tilde{f} \otimes (D_0g),$$

where  $f = (f_0, f_1, f_a, f_{1+a})^T$  and  $\tilde{f} = (f_0, f_0, f_0, f_0)^T$ . Note that  $D_0^4 = D_0$ .

□

### 11.3.4 Characterization of $q$ -type derivations on $\mathbb{F}_{p^n}$ -valued vectors with small $p$ and $n$

#### Characterization of $q$ -type derivations on $\mathbb{F}_4$ -valued vectors

##### Example 11.3.16.

Consider a finite field  $\mathbb{F}_4$ . Let  $D = (a_{ij})_{4 \times 4}$ ,  $0 \leq i, j \leq 3$  be a  $q$ -type derivation operator in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$ . Suppose that this operator satisfies the  $q$ -product rule which can be represented as

$$D(f \otimes g) = (Df) \otimes g + \bar{f} \otimes (Dg), \quad (14)$$

where  $f = (f_0, f_1, f_a, f_{1+a})^T$ ,  $\bar{f} = (f_0, f_a, f_{1+a}, f_1)^T$  and  $g \in \mathcal{A}_{4,4}$ . Similarly, it is interesting to find how many such kind of  $q$ -type derivation operators exist in  $O_{\mathcal{A}_{4,4} \rightarrow \mathcal{A}_{4,4}}$  and some related properties of these operators if they exist.

Then,

$$\begin{cases} a_{00} = a_{01} = a_{02} = a_{03} = 0, \\ a_{10} = a_{13} = 0, a_{11} + a_{12} = 0 & (15), \\ a_{20} = a_{21} = 0, a_{22} + a_{23} = 0 & (16), \\ a_{30} = a_{32} = 0, a_{31} + a_{33} = 0 & (17). \end{cases}$$

Concluding all former results, there are three pairs of non-zero elements satisfying equation (15), (16) and (17) respectively, which are  $(1, 1)$ ,  $(a, a)$  and  $(1 + a, 1 + a)$ . Then, there are  $63 (= 4^3 - 1)$  ( $D \neq 0$ ) possible  $q$ -type derivation operators satisfying the product rule (equation (14)) including the  $q$ -derivation operator defined by  $q$ -derivation.

Similarly, there are  $63 (= 4^3 - 1)$  ( $D \neq 0$ ) possible  $q$ -type derivation operators satisfying the following product rule:

$$D(f \otimes g) = (Df) \otimes g + \bar{f} \otimes (Dg),$$

where  $f = (f_0, f_1, f_a, f_{1+a})^T$ ,  $\bar{f} = (f_0, f_{1+a}, f_1, f_a)^T$  and  $g \in \mathcal{A}_{4,4}$ . By standard calculation,

$$\begin{cases} a_{00} = a_{01} = a_{02} = a_{03} = 0, \\ a_{10} = a_{12} = 0, a_{11} + a_{13} = 0, \\ a_{20} = a_{23} = 0, a_{21} + a_{22} = 0, \\ a_{30} = a_{31} = 0, a_{32} + a_{33} = 0. \quad \square \end{cases}$$

# Chapter 12

## Application to Cox-Ross-Rubinstein Model

An effective way to simulate a realistic situation is replacing continuous model by discrete model through letting the time period sufficient small like Yoshifumi Muroi et al. [35] did in their paper. They calculate Delta, Gamma and Vega by means of binomial tree and discrete Malliavin calculus.

The discrete differential dynamics is an important topic in this thesis. This concept can be applied to the binomial options pricing model. In original Cox-Ross-Rubinstein model the commutativity of the random steps makes sure that the share price at each point does not depend on its path. If this proposition does not work in this model, it gives an opportunity to investigate the distinction. For instance the application of non commutative derivation operator in high dimensional Cox-Ross-Rubinstein model has been introduced in part 3. Beside the  $q$ -derivation on finite fields introduced in this thesis, there are many other possible derivative such as the derivative defined by E. Pasalic et al. [43] in their paper. Matsumura Hideyuki even gives another product rule of derivation in his research [17].

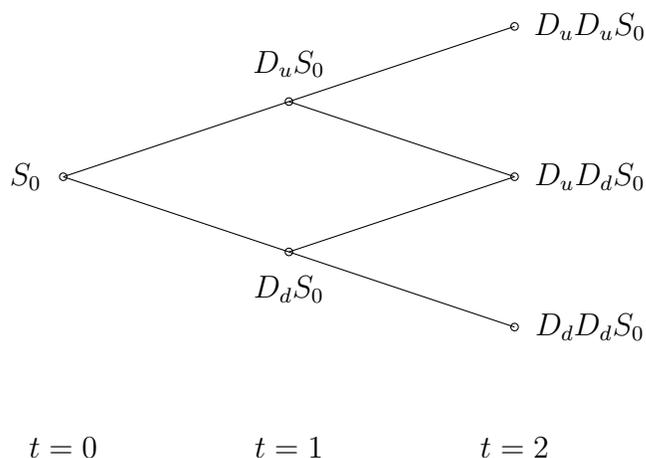
## 12.1 Cox-Ross-Rubinstein model via derivation operator

The original Cox-Ross-Rubinstein model is introduced by Cox J. et al. in their paper [7], which is a simple discrete-time model for valuing options. This simple but powerful model is commonly used to clarify option pricing. In this section, the binomial model is derived into matrix form.

Consider a share  $S_n$  defined by the geometric random walk

$$S_n = D_n S_{n-1} = D_n (D_{n-1} (\cdots (D_1 S_0)))$$

where  $D_i \in \{D_u, D_d\}$ . Let the initial value of this share be  $S_0 = (s_0, \alpha)$  which is the pair defined in 10.1.1. Let  $D_y$  be the derivation operator defined in the same section and  $D_y(s_0, \alpha) = (s_0 y, 0)$ . Suppose  $y$  takes two values  $u$  and  $d$  with probability  $p$  and  $1 - p$  respectively. Note that the operator  $D_y$  is commutative. The first two steps of this geometric random walk are given below.



Therefore, by Markov property the time 0 option price of the European call option claim  $S_1$  is

$$OP(S_1|0) = (1+r)^{-1} [tr(D_u S_0) q_u + tr(D_d S_0) q_d].$$

where  $r$  is the fix interest rate and  $tr(\cdot)$  is the trace of the matrix. By the no-arbitrage condition  $OP(S_1|0) = s_0$ , the  $Q$ -probability  $q_u$  and  $q_d$  are

$$q_u = \frac{1+r-d}{u-d}, q_d = 1 - q_u.$$

Note that the  $Q$ -probability exists if  $d < 1+r < u$ .

## 12.2 Cox-Ross-Rubinstein model via $q$ -derivation operator

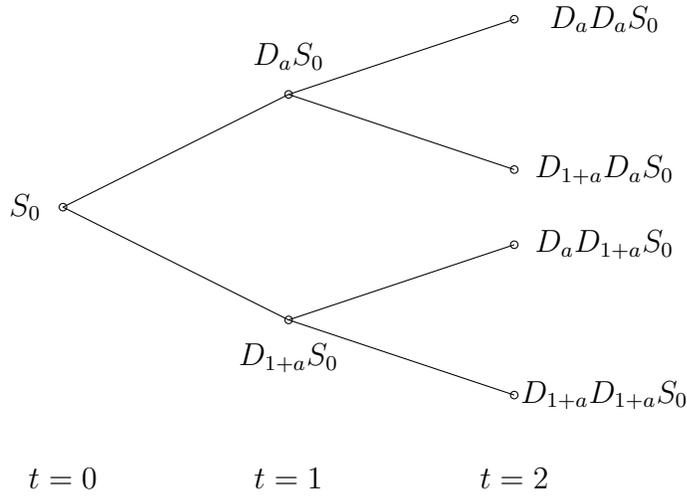
In former section, the original Cox-Ross-Rubinstein model can be represented into matrix form. The operator  $D_i \in \{D_u, D_d\}$  is commutative. It is interesting to consider the situation of the non-commutative operator such as the  $q$ -derivation operator.

Replace  $D_y$  by the  $q$ -derivation operator  $D$ . For any aforementioned particular finite field  $\mathbb{F}_{p^n}$  the set of all  $p^n - 1$  possible  $q$ -derivation operator ( $q \neq 1$ ) along with multiplication forms a semigroupoid. This semigroupoid is non-commutative by Lemma 9.2.1.

Take  $q$ -derivation operator  $D$  for  $\mathbb{F}_4$  as an example. First let  $D_y \in \{D_a, D_{1+a}\}$ . From preceding results,

$$D_a D_{1+a} = \begin{pmatrix} 0 & & & 0 \\ & a & 1+a & 1 \\ & a & 1+a & 1 \\ 0 & a & 1+a & 1 \end{pmatrix}, D_{1+a} D_a = \begin{pmatrix} 0 & & & 0 \\ & 1+a & 1 & a \\ & 1+a & 1 & a \\ 0 & 1+a & 1 & a \end{pmatrix}$$

and  $D_y^3 \equiv 0$ . Therefore this geometric random walk only has two meaningful steps. Note that  $a$  is merely a symbol and can be replaced by any real number in this model. Suppose  $y$  takes two values  $a$  and  $1+a$  with probability  $p_1$  and  $1-p_1$  respectively. Let the initial value of the share be  $s_0$  and  $S_0 = s_0 \cdot I$  where  $I$  is the identity matrix. The first two steps of this geometric random walk are given below.



Let the weight matrix be  $\rho = (\rho_{ij}), 0 \leq i, j \leq 3, \rho_{ij} = 0$  if  $i \neq j$  and  $\sum_{i=0}^3 \rho_{ii} \equiv 1$  by default. Then, by Markov property the time 0 option price of the European call option claim  $S_1$  is

$$OP(S_1|0) = (1+r)^{-1} s_0 [tr(\rho D_a) q_1 + tr(\rho D_{1+a}) (1-q_1)]$$

where  $r$  is the fix interest rate. To search the  $Q$ -probability, let  $OP(S_1|0) = s_0$  by no-arbitrage condition. Therefore,

$$q_1 = \frac{1 + r - tr(\rho D_{1+a})}{tr(\rho D_a) - tr(\rho D_{1+a})}.$$

Note that the  $q_1$  exists if

$$\max\{tr(\rho D_{1+a}), tr(\rho D_a)\} < 1 + r < \min\{tr(\rho D_{1+a}), tr(\rho D_a)\}.$$

The time 0 option price of option claim  $S_2$  is

$$\begin{aligned} & OP(S_2|0) \\ = & (1 + r)^{-2} s_0 [tr(\rho D_a^2) q_2^2 + (tr(\rho D_{1+a} D_a) + tr(\rho D_a D_{1+a})) q_2 (1 - q_2) \\ & + tr(\rho D_{1+a}^2) (1 - q_2)^2]. \end{aligned}$$

By letting  $OP(S_2|0) = s_0$ , the  $Q$ -probability can be found by solving

$$\begin{aligned} 0 = & q_2^2 [tr(\rho D_a^2) - tr(\rho D_{1+a} D_a) - tr(\rho D_a D_{1+a}) + tr(\rho D_{1+a}^2)] \\ & + q_2 [tr(\rho D_{1+a} D_a) + tr(\rho D_a D_{1+a}) - 2tr(\rho D_{1+a}^2)] \\ & - (1 + r)^2 + tr(\rho D_{1+a}^2) \end{aligned}$$

and discard the negative results according to the choice of  $a$  and  $\rho$ . If  $q_2$  exists,  $q_1 \neq q_2$  in general.

Now take the  $q$ -derivation operator  $D_0$  for  $\mathbb{F}_4$  into consider. This means  $D_y \in \{D_0, D_a, D_{1+a}\}$  and  $D_y^3 \neq 0$ . This is a much more complicated case than previous one. The  $Q$ -probabilities in calculating the option price of the share  $S_n$  at different time spots are different since  $D_y$  is not commutative, which makes the no-arbitrage condition more complicated as well.

# Appendix

## Ito multiplication table

The following Ito multiplication table has been used many times as known condition through this article. The proof of this table is given here in this section. Some clues of the proof can be found in Chapter 7 of "A First Course in Stochastic Processes" written by Samuel Karlin and Howard M. Taylor [23] [24] and more details in [25].

	$dt$	$dB_t$
$dt$	0	0
$dB_t$	0	$dt$

The proof of the Ito multiplication table here is separated into two parts, which are  $(dB_t)^2 = dt$  and  $(dB_t)(dt) = 0$  respectively.

(1).  $(dB_t)^2 = dt$ .

*Proof.*

Let  $B_t$  be standard Brownian motion. For every fixed  $t > 0$ , it is sufficient to show that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ B\left(\frac{k}{2^n}t\right) - B\left(\frac{k-1}{2^n}t\right) \right]^2 = t.$$

Let  $\Delta_{nk} = B(kt/2^n) - B((k-1)t/2^n)$ ,  $k = 1, 2, \dots, 2^n$  and  $X_{nk} = \Delta_{nk}^2 - t/2^n$ ,  $k = 1, 2, \dots, 2^n$ . Therefore, to show

$$\sum_{k=1}^{2^n} \Delta_{nk}^2 \rightarrow t$$

is equivalent to show

$$\sum_{k=1}^{2^n} X_{nk} \rightarrow 0.$$

For each  $n$ , it is clear that  $\{X_{nk} : k = 1, 2, \dots, 2^n\}$  are independent, identically distributed random variables, and

$$E[X_{nk}] = E[\Delta_{nk}^2] - \frac{t}{2^n} = 0.$$

Then, the second moment is

$$\begin{aligned} E[X_{nk}^2] &= E\left[\left(\Delta_{nk}^2 - \frac{t}{2^n}\right)^2\right] \\ &= E\left[\Delta_{nk}^4 + \frac{t^2}{4^n} - \frac{2t\Delta_{nk}^2}{2^n}\right] \\ &= \frac{2t^2}{4^n}. \end{aligned}$$

Since  $E[X_{nk}X_{nj}] = 0$  if  $j \neq k$ ,

$$\begin{aligned}
E \left[ \left\{ \sum_{k=1}^{2^n} X_{nk} \right\}^2 \right] &= \sum_{k=1}^{2^n} E [X_{nk}^2] \\
&= 2^n \frac{2t^2}{4^n} \\
&= \frac{2t^2}{2^n} \rightarrow 0
\end{aligned}$$

as  $n \rightarrow \infty$ . This immediately shows that  $\sum_{k=1}^{2^n} X_{nk}$  converges to 0 in mean square sense.

$$(2). (dB_t)(dt) = 0.$$

*Proof.*

Since  $(dB_t)^2 = dt$  has already been proved in former part, it can be seen that

$$(dB_t)(dt) = (dB_t)^3.$$

Let  $B_t$  be standard Brownian motion. For every fixed  $t > 0$ , it is sufficient to show that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ B\left(\frac{k}{2^n}t\right) - B\left(\frac{k-1}{2^n}t\right) \right]^3 = 0.$$

Let  $\Delta_{nk} = B(kt/2^n) - B((k-1)t/2^n)$ ,  $k = 1, 2, \dots, 2^n$  and  $X_{nk} = \Delta_{nk}^3$ ,  $k = 1, 2, \dots, 2^n$ . For each  $n$ , it is clear that  $\{X_{nk} : k = 1, 2, \dots, 2^n\}$  are independent, identically distributed random variables, and

$$E[X_{nk}] = E[\Delta_{nk}^3] = 0.$$

Then, the second moment is

$$\begin{aligned}
E[X_{nk}^2] &= E[\Delta_{nk}^6] \\
&= 5 \cdot 3 \cdot \left(\frac{t}{2^n}\right)^3 \\
&= \frac{15t^3}{2^{3n}}.
\end{aligned}$$

Since  $E[X_{nk}X_{nj}] = 0$  if  $j \neq k$ ,

$$\begin{aligned}
E\left[\left\{\sum_{k=1}^{2^n} X_{nk}\right\}^2\right] &= \sum_{k=1}^{2^n} E[X_{nk}^2] \\
&= 2^n \frac{15t^3}{2^{3n}} \\
&= \frac{15t^3}{4^n} \rightarrow 0
\end{aligned}$$

as  $n \rightarrow \infty$ . This immediately shows that  $\sum_{k=1}^{2^n} X_{nk}$  converges to 0 in mean square sense.

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