Consistent Preference Similarity Network Clustering and Influence Based Consensus Group Decision Making

Thesis submitted for the degree of

Doctor of Philosophy

at the

University of Leicester

by

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2019

"You never fail, until you stop trying."

Albert Einstein

Abstract

In this thesis, we introduce a novel consensus-based group decision making (CGDM) model by integrating the notions of Social Network Analysis (SNA), clustering and Social Influence Network (SIN). Four main contributions are presented in order to handle a number of issues in CGDM.

In dealing with the issue of the consistency of preferences, we introduce a consistency operator and construct a consistency control module for the purpose of securing the correctness of expert preferences. The proposed work guarantees a sufficient preference consistency level for each expert. In the case of inconsistent experts, only minimum changes of preferences are required for them to be consistent, depending on their personal level of inconsistency.

The second area of interest focuses on consensus modeling. We develop a novel consensus model by firstly defining the preference similarity network based on the structural equivalence concept. Structurally equivalent experts are partitioned into clusters, thus intra-clusters' experts are high in density and inter-clusters' experts are rich in sparsity. A measure of consensus is defined and the consensus degree of a group of experts obtained reflects the overall agreed solution.

A feedback mechanism is presented in dealing with insufficient consensus. We introduce the influence-based feedback system by incorporating the influence score measure in nominating a network leader. Our proposed procedure positively influenced the experts with low consensus contribution to change their preferences closer to each other, by following recommendations from a network influencer. This work guarantees a sufficient consensus level with better clustering solution.

Lastly, a procedure of aggregating preferences is laid out whereby the influence function is used in defining a new fusion operator, which helps to aggregate all individual expert preferences into a collective one. This is necessary to ensure that all the properties contained in all the individual preferences are summarized and appropriately taken into considerations.

Acknowledgements

First and foremost, I would like to express my deepest appreciation and sincere gratitude to my main supervisors, Professor Francisco Chiclana (from De Montfort University) and Professor Jeremy Levesley for their virtuous and valuable guidance, advice, support and intellectual experiences. Thanks should go to my second supervisor, Professor Alexander Gorban for his brilliant ideas on the clustering methodology.

I am very grateful to my husband (Mohd Azlan Mohd Yusof), father (Kamis Saad), mother (Fiatimah Awang), mother-in-law (Salmah Mat Zain), my siblings (Mat Rozi, Rosli, Salmiza, Ashila, Kamaruzzaimi, Rozalina, Sarah, Nabil and Azhari) and my nephews and nieces (Rizman, Bella, Adam, Iffa, Zinnirah and Haura) for their everlasting loves, prayers and support in making my dream came true.

I offer my heartfelt thanks to my fellow members of Faculty of Computer and Mathematical Sciences, UiTM Shah Alam, especially to the dean (Prof. Dr. Hajah Azlinah Mohamed), Mathematics' heads of department, my mentor (Prof. Dr. Daud Mohamad and Dr. Nor Hashimah Sulaiman) and colleagues for their professional support and great encouragement.

Special thanks should go to my close friends in Malaysia (Ella, Amy, Dayah, Fariez, etc.) and UK (Kak Eva, Zati, Adi, Sha, Ayuni, etc.), who have always been great open-minded listeners, indefatigable supporters and source of inspiration. I also offer my thanks to my fellow PhD colleagues (Asma', Ruaa, Jehan, Awaz, Weam, etc.) who have made my time at the University of Leicester a meaningful and memorable friendship.

To the internal examiner (Dr. Bogdan Grechuk) and external examiner (Associate Professor Dr. Joshua Ignatius from University of Warwick), thanks a lot for the great viva session and all suggestions for the thesis improvement.

Finally, I am very thankful to the Ministry of Higher Education Malaysia and Universiti Teknologi MARA (UiTM) for the financial support throughout my PhD study.

Publications Arising From This Work

1. N. H. Kamis, F. Chiclana, J. Levesley, Preference similarity network structural equivalence clustering based consensus group decision making model, Applied Soft Computing 67 (2018) 706-720.

2. N. H. Kamis, F. Chiclana, J. Levesley, Geo-uninorm consistency control module for preference similarity network hierarchical clustering based consensus model, Knowledge-Based Systems 162 (2018) 103-114.

3. N. H. Kamis, F. Chiclana, J. Levesley, An Influence-Driven Feedback System For Preference Similarity Network Clustering Based Consensus Group Decision Making Model, Information Fusion 52 (2019) 257-267.

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Abbreviations

GDM Group Decision Making

CGDM Consensus Group Decision Making

FST Fuzzy Set Theory

FPR Fuzzy Preference Relations

RPR Reciprocal Preference Relations

RFPR Reciprocal Fuzzy Preference Relations

IPV Intensity Preference Vector

SNA Social Network Analysis

FSNA Fuzzy Social Network Analysis

SIN Social Influence Network

SSIN Similarity Social Influence Network

RIM Regular Increasing Monotone

RDM Regular Decreasing Monotone

OWA Ordered Weighted Averaging

IOWA Induced Ordered Weighted Averaging

σ**-IOWA** σ-Induced Ordered Weighted Averaging

QGDD Quantifier Guided Dominance Degree

QGNDD Quantifier Guided Non-dominance Degree

To my lovely husband, parents & siblings

Part I

INTRODUCTION AND BACKGROUND

Chapter 1

Overview

1.1 Introduction

A social network is an association of individuals drawn together by interpersonal relationships, such as family, friends, work or hobby. This term has become very popular after the successful development of various internet-based social networking websites such as Twitter, Instagram and Facebook. Hundreds of the millions of users have embraced these sites for communication, business, entertainment and educational purposes. This online platform allows users to share similar personal or career activities, knowledge, news and opinions. A related area of study, named Social Network Analysis (SNA) has emerged in order to model, analyse and visualise these social network structures and interaction patterns.

A network of interactions between users (experts) also appears in the group decision-making (GDM) environment. GDM theory is concerned with the description and analysis of the process by which individual opinions are aggregated into a decision of the group as a whole. This is necessary to ensure that agreement (consensus) is achieved before a final decision is executed by them.

In order to develop more accurate and realistic consensus based GDM (CGDM) models, researchers have come up with the idea of integrating SNA concepts and properties in this framework. However, this work is least explored and need to be investigated further.

From the CGDM perspective, another important theory linked to SNA is the Social Influence Network (SIN). Both SNA and SIN have been developed to achieve a sufficient consensus level, where all individual expert opinions are appropriately considered in obtaining a final decision. However, SIN focuses on the interpersonal influence process. In this context, conflicting influential opinions can be effectively managed, revised and induced to be closer to the others, thus a sufficient consensus state is achieved.

It is common to exchange opinion through interaction in a network, but it will be difficult to get a high level of consensus when it involves a large number of users. The clustering technique is one of the promising tools in partitioning a large number of objects into small groups based on the similarity of certain criterion, for example opinion, feeling, expertise and motivation. This advantage suggests the application of the clustering technique in different area of studies, such as SNA and GDM methodologies.

In this thesis, we place great attention on bridging the gap between CGDM, SNA, clustering and SIN. Integration of these concepts encouraged us to develop a theoretical framework of Consistent Preference Similarity Network Clustering and Influence Based Consensus Group Decision Making model.

1.2 Motivation

From the existing literature on the CGDM processes, we raise several interesting subjects to be discussed further. We focus on five main situations or problems, described in the following.

1. Consistency of expert preferences.

    - In decision making, experts give their evaluations (preferences) over a set of alternatives. Consistent preferences reflect *correctness* of expert evaluations, therefore misleading decisions can be avoided. It is questionable on how to define and measure consistency of expert preferences. We believe that a special consistency operator, that satisfies certain desirable properties enable us to handle this problem.

    - Inconsistency of expert preferences cannot be avoided. In this case, the inconsistent experts need to be identified and a specific recommendation system has to be activated. This situation has prompted our development of a promising consistency control module, which guarantees the consistency degree of the inconsistent experts are improved.

2. Measurement and visualisation of experts' preference similarities.
   One of the important measures in presenting consensus of a group of experts is their similarity of preferences. Difficulty occurs when they are unable to envision the degree of their similarity/dissimilarity of preferences to each other. They cannot clearly visualise the actual distance that they had in their network connections. We are interested in investigating the development of a relevant similarity function to measure the similarity of expert preferences. This information is expected to be used in the construction and visualisation of the expert network structure.

3. Consensus in GDM.
   It is common to have groups of experts and multi-criteria/alternatives in decision making process, but it is questionable on how to obtain an overall agreed solution. There is a possibility that some experts may not accept the decision made because their individual preferences have not considered appropriately. Thus, it is worth to suggest that experts should engage in a consensus process where they can discuss and change their preferences to make them closer to each other in order to obtain a sufficient level of group agreement. This procedure is expected to be cumbersome and time-consuming due to diverse opinions from groups of experts. Their preferences need to be extracted and effectively managed, thus the decision making process becomes less expensive. Therefore, the development of a new CGDM model with capability to handle this situation is necessary. A possible approach to be explored is a clustering algorithm.

4. Feedback mechanism and generation of advice.

   - If the group consensus level is insufficient, it needs to be revised after identifying a specific expert that disagrees or is against the decision. It is unreasonable to iterate the consensus process without identifying the correct person to be advised. This situation will affect the efficiency of the decision making process and waste other experts' time.
   - If a moderator's help is needed to improve the consensus, experts or the system itself should determine who is eligible to be a moderator and what criteria are required for that chosen person.

- After specific experts are recognized as inefficient contributors to the consensus, a special procedure has to be carried out for the revision objective. What can a moderator do to influence the identified experts in changing their preferences towards higher consensus level? How does one generate advice so that experts willingly accept those recommendations?
- As listed above, several problems arised when the consensus state is insufficient. Thus, an effective feedback mechanism and advice generation needs to be introduced. We focus on the SIN theory because SIN is capable to influence inefficient or conflicting experts toward achieving consensus.

5. Aggregation (fusion) of expert preferences.
    - When multi-experts give their evaluations towards alternatives in the decision making process, there exists a challenge on how to fuse all expert individuals preferences into a collective one. This issue motivates us to introduce a new fusion operator, inspired by the well-known proposal of IOWA-based aggregation operators [1].

1.3 Objectives

The main purpose of this research is to propose a novel CGDM model with integration of the notions from SNA, clustering and SIN frameworks. The Consistent Preference Similarity Network Clustering and Influence Based CGDM model is developed, aiming to achieve the following objectives.

1. To introduce a consistency operator, which verifies desirable properties of consistent preferences.
2. To develop a personalized consistency control module, which will be activated when the consistency level is insufficient.
3. To measure the similarity of expert preferences and visualise them in an informative network structure.
4. To propose a new CGDM model with clustering capability.

5. To develop a new feedback mechanism and advice generation procedure by incorporating the SIN theory.

6. To present a new fusion operator in order to aggregate all individual expert preferences into a collective one.

1.4 Thesis outline

This thesis comprises five parts.

PART I consists of two chapters. Chapter 1 introduces the research, and details its main motivation and objectives. A literature review is presented in Chapter 2.

PART II contains Chapter 3, which sets out the research framework.

PART III presents the novel knowledge contributions, which are discussed in detail in four chapters. The proposal of the geo-uninorm consistency control module is presented in Chapter 4. Chapter 5 discusses on the preference similarity network clustering based consensus model. The influence-driven feedback mechanism and the resolution process appear in Chapter 6 and Chapter 7, respectively.

PART IV consists of two chapters. Chapter 8 focuses on the complexity computation of the proposed model and its relevant real-life applications. Chapter 9 comparing our work with other methodologies.

PART V contains Chapter 10, which summarises the main conclusions of the thesis and discusses possible future work.

Chapter 2

State of the Art

2.1 Consensus Group Decision Making

The decision making process begins with defining and analysing the problem. Then, all possible alternatives/criteria need to be identified and the individuals/experts will evaluate the alternatives based on the considered criteria before the best alternative will be chosen [2]. This decision making process seems undemanding to an individual, however when groups of people/experts are involved and multi-criteria/alternatives have to be taken into accounts, it becomes complex, cumbersome and time-consuming.

On the contrary, social psychologists in group performance research suggested that decisions made by groups tend to be more effective than individuals [3] because they can discuss systematically and obtain the final decision as representation of the whole groups' agreement.

Group decision making (GDM) formally defined as a process with possible alternatives to choose from and a group of experts with different background and level of knowledge who give preferences about them and where they aim to obtain an overall agreed solution [4, 5].

There exists two categories of GDM problems [6]; homogeneous, where experts' opinions are equally assessed, and heterogeneous, when the opinions are treated differently according to their importance. The heterogeneity occurs when experts have distinct backgrounds, levels of knowledge, experience and expertise about the issue/problem. As proposed by Yager [7], this situation can be handled by assigning a weight value to each expert reflecting his/her importance or knowledge degree about the problem. One way of generating weight is by utilizing experts opinion as an input in measuring their centrality, which reflects their individual *standard*, *position* or *status* with respect to the opinion of the group [8].

A GDM procedure involves multi-experts interacting with each other in order to reach a final decision. An important issue that has to be taken into account is the level of group agreement before making the decision because some experts may not accept the decision made because their individual preferences have not considered appropriately [9, 10]. Thus, it is worth to suggest that experts should engaged in a *consensus* process where they can discuss and change their preferences to make them closer to each other with the purpose of obtaining a high level of group agreement.

As defined by Saint and Lawson [10] and Herrera-Viedma et al. [11], consensus is a mutual relationship agreement within a group of experts, where the feasible solution gives satisfaction to the entire group. The pioneer research on consensus reaching algorithm from a mathematical perspective were done by Coch and French [12] and French [13].

Conventionally, consensus is considered as a full and unanimity agreement [14], by assuming values in the interval [0,1], with 0 as no consensus, 1 as full consensus and the rest of assessments in (0,1) as partial consensus degrees. However, this objective is unreachable and might not be desirable nor necessary in real situations, thus the concept of achieving full unanimous consensus was replaced by the introduction of milder definitions [9], which considered unanimity minus number of individuals whose disagree on the decision.

Since GDM is human centered, imprecision and vagueness of opinions in consensus reaching process exists. In order to handle this situation, the consensus based GDM (CGDM) model in fuzzy environment was first addressed by Regade [15]. Otherwise, people are also willing to accept the consensus when most experts agree on the preferences associated to the most relevant alternatives. This *soft* definition of consensus makes use of fuzzy linguistic quantifier [16], e.g., *most* leads the proposal of soft consensus decision making models presented by Kacprzyk and Fedrizzi [17, 18, 19].

An insufficient consensus state may occur if some experts reject the final decision made by assuming that their individual preferences are not appropriately taken into consideration [9, 10]. Therefore, it is suggested that an appropriate feedback mechanism procedure need to be activated, where experts can discuss and modify their preferences closer to each other for the purpose of achieving enough consensus level.

The appointment of a consensus moderator as an advisor in giving advices on how to change the expert preferences with low contribution to consensus toward the collective one is necessary, thus group consensus state is improved [20]. Recently, advanced models based on interactive tools or systems that substitute the role of a consensus moderator are then introduced [21, 22, 23, 24, 25, 26, 27], enabling the consensus reaching processes are automatically carried out.

Generally, the CGDM procedure involves six main stages, described as follows:

1. *Define, analyse the problem and identify the alternatives*: The problem to be solved is presented to the experts, along with the possible alternatives they have to choose the best one.

2. *Experts' discussion*: Experts may discuss and share their knowledge, expertise or experience about the problem and alternatives for the purpose of assisting the process of latterly providing their opinions.

3. *Experts preferences*: Experts evaluate the alternatives in the form of some preference representation formats. Uniformity of information procedure need to be done here if different formats are expressed by the individual experts.

4. *Measure of consensus*: The moderator receives all the experts' preferences and computes some consensus measures, such as consensus degree, proximity degree and consistency measure. The moderator then identifies whether the consensus/consistency level is enough or not. If an enough consensus/-consistency state has been achieved, the consensus process ends and the selection procedure begins with two continuous phases; fusion (aggregation) and exploitation. Otherwise, next step should be carried out.

5. *Feedback mechanism*: Based on collection of all the information (experts' preferences, consensus criteria, consistency level), the moderator identifies those experts and preference values that are contributing less to consensus.

6. *Generation of advice*: The moderator generates recommendation, guidance or advice in order to help the experts in changing their opinions so as facilitating the reach of consensus. The advice provided to the experts will be

used in the next discussion process to approach their points of view (Step 2).

2.2 Fuzzy Set Theory

When discussing human capability as decision makers, it is common that people tend to use words or language during conversations. For example, a student tells his/her friends about the classroom's temperature, it makes more sense to say *this classroom is cold* than *this classroom temperature is* 5^oC. Similarly, we prefer to say *the bag is heavy* rather than *the bag is 20 kg weight*. Words like *cold* and *heavy* make our conversation more meaningful and close to human perception and intuition.

However, words used in human thought processes are found uncertain, vague and imprecise in many ways. Vagueness and imprecision in any notions exist when its *meaning* is not fixed by sharp boundaries [28] and difficult to categorize into a specific class. For instance, flowers, trees, vegetables etc. can clearly categorized as the class of plants and exclude its members such objects like cats, mountains, rivers, etc. However, it is ambiguous to classify algae and fungi into plants' class because they do not precisely verify membership criteria of plants.

Due to this lack of information in modeling language and human reasoning, Lotfi A. Zadeh [29] first introduced fuzzy set theory (FST). From mathematical point of view, FST is an extension of classical set theory. The boundary of a classical set is sharp or well-defined, meaning that each element either belongs to (member) or does not belong (non-member) to a given set. A classical set assigns a membership of 1 to elements which are members of a set, and 0 to those which are not. In other words, if $x \in A$ is true, then $x \notin A$ is false.

For example, by using the classical way of evaluating social relationship, a network can only be considered as *has relation* or *no relation*. This situation shows that every set must be precise and well-defined. However, words like *strong* and *weak* relationship are normally used in real life, expressing that they are not precise in nature and carry a certain amount of fuzziness.

Fuzzy sets, though, capture this vagueness by characterizing the set membership to some degree, represented by a membership function as presented in below definition.

Definition 2.2.1. A *fuzzy set* B of a universe of discourse X is characterized by its membership function

$$\mu_B(x) : X \to [0, 1]$$

where $\mu_B(x)$ is the degree of membership of an element x in B for each $x \in X$ [29].

In fuzzy set, the value of an element is defined in terms of 0 to 1 interval degrees. If 0 is false, a value approaching 0 means the value is becoming 'false'. If 1 is denoted as true, a value approaching 1 means the value is approaching 'true'. For instance, if $\mu_{relationship}(x)$ is the membership function of *relationship* and x is the measurement to indicate social network relations, then the closer the value of $\mu_{relationship}(x)$ is to 1, the more x belongs to *relationship* and the closer $\mu_{relationship}(x)$ is to 0, the less x belongs to *relationship*.

Human preferences may not be appropriately represented by crisp values. Therefore, a linguistic variable which is expressed in terms of fuzzy numbers could be more realistic approach in handling this problem. Linguistic variables such as *very weak*, *weak*, *fair*, *strong* and *very strong* are normally used to measure social network relationship regarding certain conditions or situations. According to Asai [30], the linguistic variable can be defined as:

Definition 2.2.2. A *linguistic variable* is characterized by a quintuple

$$(x, T(x), U, G, M)$$

where x is the name of linguistic variable, $T(x)$ is the term set of x, that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U, U is the universal set or universe of discourse, G is a syntactic rule for generating the names of value of x and M is a semantic rule for associating each value with its meaning.

As an example, the linguistic variable, *social relation* in a network can take linguistic terms *very weak*, *weak*, *fair*, *strong* and *very strong* as its linguistic values. Thus,

$$T\,(social\ relation) = \{very\ weak, weak, fair, strong, very\ strong\}.$$

Fig. 2.1 illustrates the linguistic variable on the universal set, U.

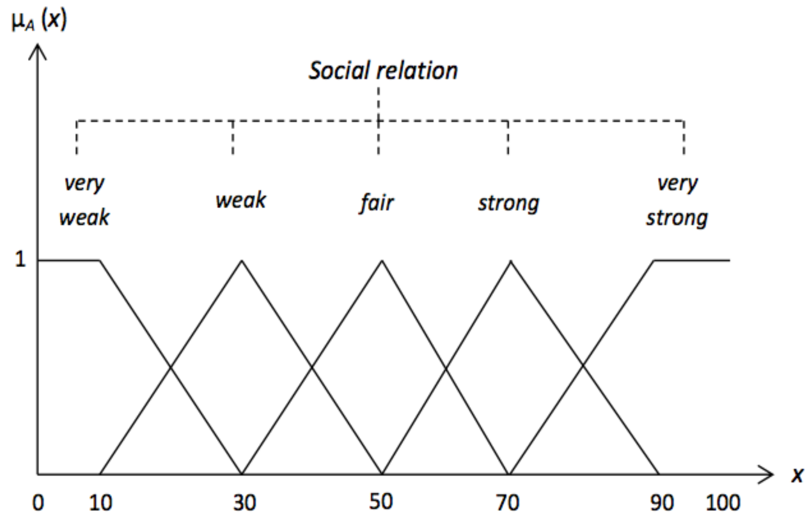

FIGURE 2.1: The linguistic variable on the universal set, U

It describes the graph of the function representing the linguistic value for the linguistic variable, $T(\textit{social relation})$ in a universe of discourse $U = [0, 100]$. The syntactic rule, G generates *very weak* for *scores obtained below 10*, *weak* for *scores between 10 to 30*, *fair* for *scores between 30 to 70*, *strong* for *scores between 50 to 90* and *very strong* for *scores above 70.* The semantic rule, M is the linguistic hedges, defined by the terms of *very weak*, *weak*, *fair*, *strong* and *very strong.*

2.3 Elementary of Preferences

In decision making, there exist various preference representation formats that can be used by experts in expressing their opinions over a set of alternatives. These preferences have their own characteristics, which are able to perform their consistencies and similarities based on the certain functions or measures.

2.3.1 Preference Representation Formats

In real life, experts may have different backgrounds, motivation, levels of knowledge and expertise in certain areas. Thus, it is reasonable to assume that they might express their opinions or preferences over alternatives using different formats of representation, whether in numerical or linguistic forms.

Numerically, non-fuzzy preferences may be expressed in terms of the set of preferred alternatives (choice set), preference relations or utility functions (cardinal) [31]. For fuzzy preferences, it can be considered as fuzzy choice sets, fuzzy preference relations and fuzzy utility functions, as defined in the following:

- *Preference ordering (choice set)*: The alternatives are ranked from the best to the worst, without other additional information will be given.
- *Fuzzy preference relations (FPR)*: Binary relation over the set of alternatives is given by the expert, by means of, to what degree an alternative is preferred to another (pair-wise comparison).
- *Utility function*: An expert provides a real evaluation (physical or monetary value) for each alternative, i.e., a function that conjoins each alternative with a real number implying the performance of that alternative based on his/her judgement. The utility values can be presented in various ways, such as ordinal, ratio, interval and difference.

Another representation format introduced in recent years is reciprocal preference relation (RPR). The concept of RPR, that represents the intensities of preferences was proposed by Bezdek et al. [32], comprehensively interpreted in Nurmi [33] and broadly studied in [34, 35, 36, 37, 38, 39, 40, 41].

The RPR is related to a fuzzy binary relation and the definitions are stated as follows:

Definition 2.3.1. Let a finite set of alternatives, $X = \{x_1, x_2, \ldots, x_n\}$ $(n > 2)$ be a non empty set. A fuzzy binary relation R on X is a fuzzy subset of the Cartesian product $X \times X$ characterized by a membership function $\mu_R : X \times X \longrightarrow [0, 1]$, where $\mu_R(x_i, x_j) = r_{ij}$ represents the strength of the relation between x_i and x_j.

Definition 2.3.2. A RPR on X is a fuzzy binary relation P where the preference intensity of alternative x_i over alternative x_j, $\mu_P(x_i, x_j) = p_{ij}$, verifies $\mu_P(x_i, x_i) = 0.5$ $\forall x_i \in X$ and $p_{ij} + p_{ji} = 1$, $\forall x_i, x_j \in X$.

According to Definition 2.3.2, an expert not only declares his/her preference on alternative x_i over x_j, but also establishes the intensity of preference by providing the value of p_{ij}. The higher p_{ij}, the higher the preference intensity of alternative

x_i over x_j. The associated semantic for the unit interval of a RPR is assumed to be as follows:

$$p_{ij} = \begin{cases} 0 & \text{if } x_j \text{ is definitely preferred to } x_i \\ p_{ij} \in [0, 0.5] & \text{if } x_j \text{ is preferred to } x_i \\ 0.5 & \text{if } x_i \text{ and } x_j \text{ are equally preferred (indifference)} \\ p_{ij} \in [0.5, 1] & \text{if } x_i \text{ is preferred to } x_j \\ 1 & \text{if } x_i \text{ is definitely preferred to } x_j \end{cases}$$

Let $\mathbb{P}_{n\times n}$ denotes the set of $n \times n$ matrices P constructed from all RPR on X:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

verifying: $0 \leq p_{ij} \leq 1$ and $p_{ij} + p_{ji} = 1$ for $i, j \in \{1, 2, \ldots, n\}$.

The RPR can also be mathematically represented by means of a vector known as the *intensity preference vector* (IPV) [42].

Definition 2.3.3. The intensity preference vector (IPV) of a RPR $P = (p_{ij})_{n\times n} \in \mathbb{P}_{n\times n}$ is the vector of dimension $n(n-1)/2$, $V \in \mathbb{R}^{n(n-1)/2}$, with elements above its main diagonal:

$$\begin{aligned} V &= \left(p_{12}, p_{13}, \ldots, p_{1n}, p_{23}, \ldots, p_{2n}, \ldots, p_{(n-1)n}\right) \\ &= \left(v_1, v_2, \ldots, v_r, \ldots, v_{n(n-1)/2}\right). \end{aligned}$$

The reciprocity property [43] allows the use of the preference values below the main diagonal of P as components of its intensity preference vector:

$$V_{lower} = \left(p_{21}, p_{31}, \ldots, p_{n1}, p_{32}, \ldots, p_{n2}, \ldots, p_{n(n-1)}\right).$$

Notice that the representation of RPR in terms of preference intensities in fuzzy set theory is referred as reciprocal fuzzy preference relations (RFPR), which are a particular case of (weakly) complete FPR, i.e. FPR satisfying $p_{ij} + p_{ji} \geq 1, \forall i, j$.

2.3.2 Consistency of Preferences

Consistency of preferences becomes one of the necessary consensus criterion in GDM, where its appropriate integration ables to avoid misleading solutions and to estimate unknown or missing information [44, 45, 46, 47]. When experts give their evaluation towards alternatives, there exist three hierarchical levels of rationality assumption;

Level 1. *Indifference* – indifference when comparing an alternative x_i and itself x_i;
Level 2. *Asymmetry* – an expert cannot prefer alternative x_i to alternative x_j and alternative x_j to x_i simultaneously;
Level 3. *Transitivity* – if an expert prefers alternative x_i to alternative x_j, and prefers alternative x_j to alternative x_k then this expert should prefer alternative x_i to alternative x_k.

Reciprocity property in the pairwise comparison of two alternatives is verified at Level 1 and 2, as declared by Saaty [43] as a 'reasonable assumption' for pairing purpose. In the case of preference relations, it is classified as consistent when Level 3 is satisfied, in such a way that the transitivity in the pairwise comparison among any three alternatives is guaranteed, commonly known as consistency property.

Particularly, transitivity explains that 'if alternative x_i is preferred to alternative x_j ($p_{ij} \geq 0.5$) and this one to x_k ($p_{jk} \geq 0.5$) then alternative x_i should be preferred to x_k ($p_{ik} \geq 0.5$)'. This kind of transitivity is known as *weak stochastic transitivity* [48].

Other models under transitivity of RPR have been introduced and widely used in the literature, such as min transitivity, moderate stochastic transitivity, max transitivity, strong stochastic transitivity, additive transitivity and multiplicative transitivity.

- Min transitivity [31]:
  $p_{ik} \geq min \ \{p_{ij}, p_{jk}\} \ \ \forall i, j, k.$
- Moderate stochastic transitivity [31, 48, 49]:
  $min\,\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq min\,\{p_{ij}, p_{jk}\} \ \ \forall i, j, k.$
- Max transitivity [31]:
  $p_{ik} \geq max\,\{p_{ij}, p_{jk}\} \ \ \forall i, j, k.$

- Strong stochastic transitivity [31, 48, 49]:
  $min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq max\{p_{ij}, p_{jk}\}\ \ \forall i, j, k.$

- Additive transitivity [48]:
  $(p_{ij} - 0.5) + (p_{jk} - 0.5) = p_{ik} - 0.5\ \forall i, j, k.$

- Multiplicative transitivity [31, 48, 49]:
  $p_{ij}, p_{jk}, p_{ki} \notin \{0, 1\} \Rightarrow p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}\ \forall i, j, k.$

Related to reciprocity property, Chiclana et al. [34] revealed that max transitivity is only feasible for equally preferred alternatives and additive transitivity is conflicting with the unit interval scale, which makes both properties irrelevant to be utilized in modeling consistency of RPR.

Literally, when consistency of preferences is designed as the 'cardinal consistency in the strength of preferences', which was stated by Saaty in [43, page 7] as

> "not merely the traditional requirement of the transitivity of preferences [...], but the actual intensity with which the preference is expressed transits through the sequence of objects in comparison,"

Chiclana et al. [34] argued that consistency of RPR can be theoretically modeled by means of a functional equation,

$$p_{ik} = f\left(p_{ij}, p_{jk}\right)\ \ \forall i, j, k. \tag{2.1}$$

being f a function $f : [0, 1] \times [0, 1] \longrightarrow [0, 1]$.

The function of f imposed the following properties [34]:

- Monotonicity:
  $f(x, y) \geq f\left(x', y'\right)$ if $x \geq x'$ and $y \geq y'$.

- Associativity:
  $f(f(x, y), z) = f(x, f(y, z))\ \ \forall i, j, k.$

- Reciprocity:
  $f(x, y) + f(1 - y, 1 - x) = 1\ \forall (x, y) \in [0, 1] \times [0, 1] \setminus \{(0, 1), (1, 0)\}.$

- Identity element:
  $f(0.5, x) = f(x, 0.5) = x\ \forall x \in [0, 1].$

- Continuity:
  f is continuous in $[0,1] \times [0,1] \setminus \{(0,1), (1,0)\}$.

- Cancellative:
  $f(x,y) = f(x,z) \bigwedge f(y,x) = f(z,x) \ \ \forall x \in]0,1[\Rightarrow y = z.$

Equation 2.1 is proven having the set of self-dual representable uninorms [50] as its solution.

Proposition 2.1. *Let $P = (p_{ij})$ be a RPR. A function $f : [0,1] \times [0,1] \longrightarrow [0,1]$ verifying $p_{ik} = f(p_{ij}, p_{jk})$ $(\forall i,j,k)$ is self-dual, i.e.*

$$f(x,y) + f(1-y, 1-x) = 1 \ \forall x,y \in [0,1].$$

Yager and Rybalov [50] firstly introduced uninorms as a generalization of the t-norm and t-conorm, where it supported by almost associativity, continuity and monotonicity conditions.

A t-norm, T [50], is a mapping, $T : [0,1] \times [0,1] \longrightarrow [0,1]$ having the following properties:

- Commutativity:
  $T(x,y) = T(y,x)$.

- Monotonicity:
  $T(x,y) \geq T\left(x', y'\right)$ if $x \geq x'$ and $y \geq y'$.

- Associativity:
  $T\left(x, T\left(y, x'\right)\right) = T\left(T(x,y), x'\right)$.

- Boundary:
  $T(x,1) = x$.

Examples of t-norm functions, satisfying the above properties are Łukasiewicz t-norm, G̈odel t-norm, *Product t-norm* and *Drastic-Product t-norm* [51].

A t-conorm, S [50], is a mapping, $S : [0,1] \times [0,1] \longrightarrow [0,1]$ having the following properties:

- Commutativity:
  $S(x,y) = S(y,x)$.

- Monotonicity:
  $S(x,y) \geq S(x',y')$ if $x \geq x'$ and $y \geq y'$.
- Associativity:
  $S(x, S(y,x')) = S(S(x,y), x')$.
- Boundary:
  $S(x,0) = x$.

Bounded sum, *Maximum* and *Probabilistic sum* are some of the t-conorm functions [51].

A uninorm, U [50], is a mapping, $U : [0,1] \times [0,1] \longrightarrow [0,1]$ having the following properties:

- Commutativity:
  $U(x,y) = U(y,x)$.
- Monotonicity:
  $U(x,y) \geq U(x',y')$ if $x \geq x'$ and $y \geq y'$.
- Associativity:
  $U(x, U(y,x')) = U(U(x,y), x')$.
- Identity:
  There exists some elements $e \in [0,1]$ called the *identity* element such that for all $x \in [0,1]$, $U(x,e) = x$.

Other than *cross-ratio uninorm*, the *Least* and *Greatest Uninorms* [51] are also considered as uninorm functions, which satisfy above properties.

Consistency of preferences are measured from the initial evaluations provided by experts over alternatives in order to avoid misleading solution in the decision making process [38, 52] and guarantee the final decision solution is acceptable by a group of experts as a whole. For the purpose of achieving both objectives (consistency and consensus), the following consecutive steps must be applied:

(1) achieving a minimum threshold of expert's preference consistency level; and
(2) achieving a minimum threshold of experts group consensus level .

The threshold value represents a minimum satisfied level of consistency or consensus, set up by the group of experts and this particular value must be high enough (the closer to one the better).

If low consistency level is obtained i.e., it is below the minimum threshold level of consistency, the consistency control system need to be implemented, in such a way that a sufficient consistency state is achieved [53, 54]. This step is strictly applied before securing consensus to avoid divergence of the previously agreed consensus position and leading to rejection of the final solution.

2.3.3 Similarity/dissimilarity of Preferences

Measurement of preference similarity/dissimilarity in consensus reaching process is purposely conducted to determine how close experts' preferences with each other and between experts in a group with the collective ones [55, 56]. General definitions of dissimilarity (distance) and similarity functions are provided below:

Definition 2.3.4. Let B be a set. A function $D : B \times B \to R$ is called a distance (or disimilarity) on B if, for all $x, y \in B$, there holds properties of non-negativity ($D(x, y) \geq 0$), symmetry ($D(x, y) = D(y, x)$) and reflexivity ($D(x, x) = 0$) [57].

Definition 2.3.5. Let B be a set. A function $S : B \times B \to R$ is called a similarity on B if S is non-negative, symmetric, and if $S(x, y) \leq S(x, x)$ holds for all $x, y \in B$, with equality if and only if $x = y$ [57].

The transformation of dissimilarity (distance) and similarity bounded by 1 [57] can be done by:

$$D = 1 - S; D = \frac{1 - S}{S}; D = \sqrt{1 - S}; D = \sqrt{2(1 - S)^2}; D = \arccos S; D = -\ln S.$$

The most common dissimilarity functions implemented in consensus processes are Manhattan, Euclidean, Cosine, Dice and Jaccard. Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be two vectors of real numbers.

- Manhattan, d_M:

$$d_M (u, v) = \sum_{i=1}^{n} \mid u_i - v_i \mid .$$

- Euclidean, d_E:

$$d_E\left(u,v\right)=\sqrt{\sum_{i=1}^{n}\mid u_i-v_i\mid^2}.$$

- Cosine, d_C:

$$d_C\left(u,v\right)=\frac{\sum\limits_{i=1}^{n}u_i\cdot v_i}{\sqrt{\sum\limits_{i=1}^{n}u_i^2}\cdot\sqrt{\sum\limits_{i=1}^{n}v_i^2}}.$$

- Dice, d_D:

$$d_D\left(u,v\right)=\frac{2\cdot\sum\limits_{i=1}^{n}u_i\cdot v_i}{\sum\limits_{i=1}^{n}u_i^2+\sum\limits_{i=1}^{n}v_i^2}.$$

- Jaccard, d_J:

$$d_J\left(u,v\right)=\frac{\sum\limits_{i=1}^{n}u_i\cdot v_i}{\sum\limits_{i=1}^{n}u_i^2+\sum\limits_{i=1}^{n}v_i^2-\sum\limits_{i=1}^{n}u_i\cdot v_i}.$$

The above functions were analysed in three different levels of expert evaluations: *pairs of alternatives*, *alternatives* and *relations* [20]. Rather than the similarity/dissimilarity functions, alternative measurements have been proposed in determining the closeness of experts' preferences and the consensus state level, such as in [58, 59, 60].

2.4 Feedback System and Advice Generation

The feedback mechanism and generation of advice needs to be activated when the consensus level is insufficient. Formulations of feedback mechanism in consensus reaching processes are categorized into two approaches: traditional and moderator-based methods. On the one hand, matrix calculus or Markov chains has been used to model the time evolution of changes of preferences toward the consensus [12, 13, 61]. The second, consensus process supported by a moderator, act as an advisor to the experts in modifying their preferences and controlling the consensus state.

Several consensus models based on feedback mechanism (with a moderator) can be found in [17, 62, 63, 64, 20]. However, difficulty will happen if experts decide not to accept the recommendations from the moderator. Medium to persuade the experts to follow the moderator advices need to be taken into account, so that enough consensus level can be achieved.

A recent study that considers the weapon of influence in psychology into negotiation was carried out to change the individual attitudes, beliefs or the behaviour, voluntarily without any force or pressure. Perez et al. [65] developed a feedback mechanism, where points of references were obtained for the purpose of urging the discussion towards optimal and consensus solution that limits the gap between experts' positions.

Since a moderator is a human, thus there exist some subjectivity in the consensus control process, such as he/she might include bias on giving the advice or recommendation to the experts. Therefore, advanced consensus approaches have been developed in order to substitute the roles of moderator, in such a way that the consensus process can be carried out automatically.

Herrera-Viedma et al. [66] designed a consensus support system for guiding the consensus process based on the consensus measure and assisting feedback mechanism using the proximity measure. Herrera-Viedma et al. [67] presented an automated consensus tool based on a multi-granular linguistic methodology, consensus degree and proximity measure as consensus criteria and guidance advice system in feedback mechanism phase. Other similar automated consensus control systems can be referred further in Herrera-Viedma et al. [21] and Cabrerizo et al. [22].

It is known that the moderator will act as a leader to bring experts' opinions closer to the consensus, which means the moderator knows the actual level of group agreement, which experts are contributed less to consensus and which experts need more advice than others. Since the moderator is not essential in automated consensus systems, experts cannot visualise their current state of agreement and they might have problems to identify who is not contributing towards the consensus or who is not giving full cooperation during discussion.

Wu and Chiclana [24] proposed the information feedback mechanism in terms of automated visualization to investigate the corresponding experts and alternative preferences that less contribute to the consensus. The simulation provides recommendation towards better consensus direction and presented simulated future consensus scenarios using graphical representations if the suggested changes were fully implemented. By providing more information to the experts, this model practised voluntarily mechanism in changing opinion and revisits the decision without any pressure.

2.5 Resolution Process

When group consensus is sufficient, it means that experts individual preferences have been considered appropriately and the feasible solution obtained in the *resolution process* will give satisfaction to the entire group. The resolution process involves two main procedures: Fusion phase and ranking of alternatives. Experts' opinions will be collectively aggregated in the fusion phase and a final ranking of alternatives will be derived afterwards.

2.5.1 Fusion of Preferences

Fusion or aggregation phase in CGDM process consists of combination of the experts' individual preferences into a collective one, thus all the properties contained in all the individual preferences are summarized and appropriately taken into considerations. There exist two (2) situations where fusion of information step is needed:

1. Heterogeneous preference representation formats - In real life, experts are allowed to give evaluations over alternatives in various ways, such as ordering, utility functions or preference relations. These preferences must be transformed into one particular preference structure as the aggregation base, thus all preferences are homogeneous. Several successful proposals in handling this problem were presented in [68, 69, 70, 71] and a brief survey on the fusion process for heterogeneous preference structures was done by [72].

2. Heterogeneous experts' importance weights - Due to different background, motivations, knowledge and expertise, experts are having different importance weights. Other than that, different experts' weights can be assigned depending on their behaviours or positions with respect to the group's opinion position [73]. Therefore, appropriate aggregation operators need to be utilized in order to properly handle the weights assigned to each of them.

One of the powerful tools proposed in the literature is *OWA operator*, introduced by Yager in 1988 [74] and the evolution of the OWA-based fusion studies are progressively emerged.

Definition 2.5.1. An OWA operator of dimension n is a function $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ that has associated a set of weights or weighting vector, $W = (\omega_1, \ldots, \omega_n)$ to it, so that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and is defined to fuse a list of values $\{p_1, \ldots, p_n\}$ according to the following expression,

$$\phi\left(p_1, \ldots, p_n\right) = \sum_{i=1}^{n} \omega_i \cdot p_{\sigma(i)}$$

being $\sigma\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ a permutation such that $p_{\sigma(i)} \geq p_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1, i.e.,$ $p_{\sigma(i)}$ is the i-th highest value in the set $\{p_1, \ldots, p_n\}$ [74].

It is questionable in the definition of the OWA operator on how to get the associated weighting vector and Yager [74] provided two solutions on this matter. Firstly, by utilization of some kind of learning mechanism using some sample data. Secondly, by giving some semantics or meaning to the weights, which leads to the application of *quantifier guided aggregations* [75].

Quantifiers can be applied to express the amount of items satisfying a given predicate, where in decision making context, it indicates that the proportion of satisfied criteria 'necessary for a good solution' [76]. Yager linked the concept of *fuzzy majority* in OWA-based fusion functions, represented by natural language expressions such as *'most of'*, *'all'*, *'many'*, *'at most one'*, *'few'* and many more. These quantifiers are categorized into two: regular increasing monotone (RIM), such as *'most of'*, *'all'*, *'many'* and regular decreasing monotone (RDM), such as *'at most one'* and *'few'*.

Experts' weights vector are generated through the mathematical formulation of the fuzzy majority concept via an appropriate quantifier membership function Q, known as *quantifier guided linguistic OWA operators*. In this research, we use fuzzy linguistic quantifier *most of* represented by the parameterized family of RIM quantifiers [76],

$$Q\left(r\right) = r^{\frac{1}{2}}. \tag{2.2}$$

This type of RIM function guarantees that all experts contribute to the final fusion value because it is a strictly increasing function [6]. In addition, the higher the ranking of a value, the higher the weighting value associated to it.

The OWA weights generated based on a regular increasing monotone (RIM) quantifier, $Q\colon\ [0,1] \rightarrow [0,1]$ such that $Q(0) = 0, Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$ represent the proportion of criteria satisfied by an alternative [74]:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), i = 1, \ldots, n. \tag{2.3}$$

This concept was improved by providing the evaluation procedure of the overall satisfaction of 'Q important criteria or experts' (μ_k or e_k) by an alternative x_j [76] using the following expression:

$$w_i = Q\left(\frac{\sum\limits_{k=1}^{i} \mu_\sigma(k)}{D}\right) - Q\left(\frac{\sum\limits_{k=1}^{i-1} \mu_\sigma(k)}{D}\right) \tag{2.4}$$

where $D = \sum\limits_{k=1}^{n} \mu_k$ is the total sum of importance and σ is the permutation applied to obtain the ordering of the values to be fused.

The extension work of OWA operator, *Induced OWA* (IOWA) had been done by Yager and Filev [1], where the reordering step of the argument variable is induced upon the magnitude of an additional variable, known as the order inducing variable. The formulation of IOWA operator is presented as:

Definition 2.5.2. An IOWA operator of dimension n is a function $\Phi_W\colon\ (\mathbb{R} \times \mathbb{R})^n \longrightarrow \mathbb{R}$, to which a set of weights or weighting vector is associated $W = (\omega_1, \ldots, \omega_n)$ such that $\omega_i \in [0,1]$ and $\sum\limits_i \omega_i = 1$, and it is defined to fuse the set of second arguments of a list of n 2-tuples $\{\langle\mu_1, p_1\rangle, \ldots, \langle\mu_n, p_n\rangle\}$ according to the following expression,

$$\Phi_W\{\langle\mu_1, p_1\rangle, \ldots, \langle\mu_n, p_n\rangle\} = \sum_{i=1}^{n} \omega_i \cdot p_{\sigma(i)}$$

being $\sigma\colon\ \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ a permutation such that $\mu_{\sigma(i)} \geq \mu_{\sigma(i+1)},\ \forall i = 1, \ldots, n-1,\quad i.e.,\ \left\langle\mu_{\sigma(i)}, p_{\sigma(i)}\right\rangle$ is the 2-tuple with $\mu_{\sigma(i)}$ the i-th highest value in the set $\{\mu_1, \ldots, \mu_n\}$.

Based on above definition, the reordering of the set of values to fuse, $\{p_1, \ldots, p_n\}$ is induced by the reordering of the set of associated values $\{\mu_1, \ldots, \mu_n\}$, where it is based upon their magnitude. Specifically, the set of values $\{\mu_1, \ldots, \mu_n\}$ is known as an order inducing variable and $\{p_1, \ldots, p_n\}$ is the value of the argument variable [1].

It is clear that the OWA operator is different with IOWA operator in terms of the reordering step of the argument variable. The reordering of the OWA operator is based upon the magnitude of the values to be fused. For IOWA operator, an order inducing variable has to be used as the criterion to induce that reordering. Notice that the IOWA operator is reduced to the OWA operator if the order inducing variable is the argument variable [6].

Since reordering of the argument to be aggregated by means of additional order inducing variables is a foundation feature of IOWA-based operators, many researchers are motivated to propose extended versions of it [77, 78, 79, 6, 80, 81, 82]. It seems reliable to introduce some semantic meaning in these techniques, therefore the fusion phase can be effectively controlled.

2.5.2 Ranking of Alternatives

Ranking of alternatives, also known as an *exploitation procedure* can be carried out after the individual expert preferences are successfully fused into a collective form. The objective of implementing this step is to select the *best* alternative(s) that acceptable by the majority Q of the most important experts.

By means of the OWA operator guided by the linguistic quantifier Q, the *Quantifier Guided Dominance Degree* (QGDD) or *Quantifier Guided Non-Dominance Degree* (QGNDD) [68] can be utilized. Formal definitions of QGDD and QGNDD are stated as follows:

Definition 2.5.3. Given a preference relation $P^c = \left(p_{ij}^c\right)$ on a finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, the quantifier guided dominance degree, $QGDD\left(x_i\right)$, quantifies the dominance that an alternative x_i has over all the other alternatives in a fuzzy majority sense as:

$$QGDD\left(x_i\right) = \Phi_Q\left(p_{ij}^c, j = 1, \ldots, n, j \neq i\right)$$

where Φ_Q is an OWA operator guided by the linguistic quantifier Q representing the fuzzy majority concept.

Definition 2.5.4. The quantifier guided non-dominance degree (QGNDD) quantifies the degree to which the alternative x_i is not dominated by a fuzzy majority of the remaining alternative as:

$$QGNDD\left(x_i\right) = \Phi_Q\left(1 - p^s_{ji},\ j = 1, \ldots, n,\ j \neq i\right)$$

where $p^s_{ji} = max\left\{p_{ji} - p_{ij}, 0\right\}$ represents the degree to which x_i is strictly dominated by x_j.

The final ranking solution is then determined according to the following expressions, where the elements of the set,

$$X^{QGDD} = \left\{x \mid x \in X,\ QGDD\left(x\right) = sup_{x \in X}\ QGDD\left(x\right)\right\} \tag{2.5}$$

are called the maximum dominance elements of the fuzzy majority of X quantified by Q

and,

$$X^{QGNDD} = \left\{x \mid x \in X,\ QGNDD\left(x\right) = sup_{x \in X}\ QGNDD\left(x\right)\right\} \tag{2.6}$$

are called maximum non-dominated elements by the fuzzy majority of X quantified by Q.

2.6 New Direction to Consensus

Most CGDM models have been dealing with few number of experts, because normally important decisions are only made by professional, skillful and authorized person in the companies, administrations or organizations. However, current electronic technology and society demands lead us to a large-scale group decision making paradigms, like *social networks* (*Facebook*, *Instagram*, *Twitter*, etc.) and *Web 2.0* (*Wikipedia*, *Amazon* online store, blogs, forums, etc.).

Some characteristics of these technologies [23] make CGDM processes complicated, difficult to manage and may require high cost to carried out, such as:

- *Large user base*: These technologies might have thousands of users, thus a large and diverse amount of opinions are unmanageable, difficult to be extracted and utilized in obtaining a final decision about certain issue/problem.
- *Heterogeneous user base*: Users may come from different backgrounds, experiences and level of knowledge, so it is unfair to treat each opinion equally in the decision process. Other than that, heterogeneity also may exist in the form of different representation formats provided by the web. For example, some users prefer to use numerical ratings, others may like linguistic assessments.
- *Unstable contribution rates*: Due to a large number of users, it is impossible to ensure all users will take part in the decision process. Some of them may collaborate half-way and others continue their contributions. These situations develop unstable, temporarily and partial collaboration rates to the whole decision procedures.
- *Real time communications*: These technologies mostly support real time communications among its members, such as voting events or surveys. Time given to the users is also limited and it is not easy to run second round of consensus if there exist a problem in first one.
- *Difficulty to form trust relations*: Normally, most users on the web do not know each other personally, therefore trust is difficult to build around them. Trust is one of the necessary elements in obtaining high consensus or agreement among groups of users.
- *Effect of external factors (influence)*: Online stores mostly implement the recommendation system to their users about related/similar products that might interest them to buy. Rather than this automated recommendation, there exist external factors, which will influence the users to/not to buy a certain product, such as peer-reviews, reduction price in physical store, trial kit service and so on.

2.6.1 Social Network Analysis

Social Network Analysis (SNA) in social science focuses on the relationships, structures and patterns of social bodies and it has become popular in different areas including but not limited to politics, economics, environment, health and finance. Barnes [83] first introduced the use of the *social network* term and Harary [84][85] and Lorrain and White [86] developed pioneer theories on mathematical social networks perspectives. Early contributions of SNA in group problem solving and community elite decision making have been investigated in Bavelas [87], Leavitt [88], Laumann et al. [89] and Laumann and Pappi [90].

Essential focus on SNA is correlated to the structures and patterns formed by the nodes (actors or group) connected by the links that represent their relationships or flow. Previously, SNA considered crisp binary relation of the links between nodes, expressed by 1 (exists relation) or 0 (no relation).

In order to cater to the vagueness in the network that exists in reality, such as *strong* or *weak* relation, this *hard* definition is not suitable. Therefore, fuzzy set theory introduced by Zadeh [29] is used in defining several social network terms linguistically, such as fuzzy links, fuzzy relation, fuzzy connection, fuzzy network and so on. Fuzzy Social Network Analysis (FSNA) is then introduced and some related studies were proposed, including fuzzy social relational networks [91], fuzzy technology innovation networks [92] and directed fuzzy social networks [93].

Another interesting area of study related to SNA methodology is network interactions between individual and group of users/experts, that contributes to sufficient agreement level in decision making processes. Thus, integration of SNA properties, such as centrality, adjacency, trust statements, prestige and structural balance in the construction of consensual decision making models are recently presented [35, 94, 95, 96, 97, 98, 99, 100, 101, 26, 102, 103, 104, 105].

Brief discussion and informative overview on consensus reaching decision making within social network environments can be found in Herrera-Viedma et al. [106]. The challenges of huge number of users, heterogeneous user background with different importance weights, low contribution and involvement rates, intermittent contributions and real time communication in societal technologies were successfully clarified by [23, 107, 108].

2.6.2 Clustering Algorithms

One way of managing large number of users in social network environment is by utilizing clustering techniques [109, 110, 111, 112, 103]. Users will be partitioned into subgroups, thus complex interactions of opinion exchange in a network are effectively treated.

In the context of consensus decision making, clustering approach is able to partition a finite set of individual experts into *clusters* (subgroups) by means of preferences similarity measures. This concept allocates experts with similar preferences in the same cluster, which closer to each other and being far from other experts with different preferences and cluster(s). Recent models development based on clustering methodologies in decision making frameworks have been done by Perony et al. [113], Garcia-Lapresta and Perez-Roman [114, 115, 116], Abel et al. [117], Li et al. [118] and Palomares et al. [119].

2.6.3 Social Influence Network

Social influence network (SIN) has been developed continuously since the 1950s by French [13], Harary [61], DeGroot [120] and Friedkin and Johnsen [121] and it is one of the important fields that is directly linked to CGDM and SNA. The attention of the researchers is quite similar, which is to achieve the global consensus that represents solid agreement of a group. However, SIN focuses on interpersonal influence process, where experts are able to manage the conflicting influential opinions by revising it and then, they can induce the others to behave in a similar way.

Friedkin and Johnsen [122] pointed out two main problems in SIN that effect attitudes and opinions of the experts toward group agreement. First, how experts change their opinions and influence the others and second, how to develop structure of social influence models with better configuration and strengthen the interpersonal influence processes.

For the case of GDM, social influence is clarified by changes of attitudes, thoughts, feelings, characters or behaviors of expert(s), if there exist interaction from another expert in a group [123]. It is believed that experts' preferences are possible to be modified due to the social influence factor during interactions, discussions or opinion exchange in a network.

In recent times, influence-based consensus decision making models proposed several measurements of social influence, including the normalization of network's tie strength [123], normalization of numerical trust level [35, 105] and estimation from initial experts' opinions over alternatives [95, 104].

Perez et al. [95] and Capuano et al. [104] modelled influence processes in GDM by utilization of the recursive definition from [122]. Let a set of experts, E having an $m \times m$ fuzzy adjacency matrix $W = (w_{ij})$ as their relative importance over other preferences including themself and y^1 be the initial experts' preferences. After t iterations, the generated preferences is presented as:

$$y^t = AWy^{(t-1)} + (I - A)\, y^1, \tag{2.7}$$

where $A = diag\,(a_{11}, a_{22}, \ldots, a_{mm})$ is the susceptibility of experts to interpersonal influences and I is the $m \times m$ identity matrix.

The above expression is able to estimate the evolution of experts' preferences iteratively until the process reaches an equilibrium, i.e. $y^\infty = lim_{t \longrightarrow \infty} y^t$ exist, then:

$$y^\infty = (I - AW)^{-1}\,(I - A)\, y^1. \tag{2.8}$$

Based on a predefined SIN, Capuano et al.'s [104] work provided a greater flexibility compared to the results obtained by Perez et al. [95]. Capuano's proposed model proved that the complication of defining a numerical level expressing the susceptibility of experts towards influence can be avoided, estimation of missing preferences can be successfully operated and the convergence of experts' preferences can be achieved.

An alternative influence measure, known as *alpha-centrality*, was introduced by Bonancich and Llyod [124]. The alpha-centrality, denoted here by x, is an eigenvector-like measure that determine expert's status in a network by considering their influence pattern, and it is expressed as follows:

$$x = \left(I - \Upsilon \boldsymbol{A}^{T}\right)^{-1} \boldsymbol{e} \tag{2.9}$$

where $\boldsymbol{A} = (a^{ij})$ is an adjacency matrix that represents the group's influence pattern, i.e. a^{ij} represents the degree of influence of expert i by expert j; Υ is a scalar describing the relative importance degree of endogenous (internal) versus exogenous (external) effects of experts in a group. Bonacich and Llyod suggested that Υ should approach $1/\lambda_1$ from below in order to achieve convergence of the solution, with λ_1 being the largest eigenvalue of $\boldsymbol{A}$. Notice that the endogenous factor emerges from connections in the network itself while the exogenous factor, $\boldsymbol{e}$, is external to the network of experts, such as from third party involvements, and that it can effect or change the experts' status.

Variable $\boldsymbol{e}$ was introduced in order to allow some *status* over experts, rather than dependent only on their connections in a network. The *status*, also known as *centrality* of an expert is characterized as a function of an expert to whom his/her is connected.

For example, the selection of the most popular student will involve schoolmate reviews, which can be presented as a relationship network in a school and this criteria is considered as an endogenous effect. In some situation, the selection process involves evaluation from teachers and principles, which can be seen an exogenous factor in that nomination process.

Part II

PROPOSED METHODOLOGY

Chapter 3

Research Framework

This chapter provides the framework of the consistent preference similarity network clustering and influence based consensus group decision making model. Specifically, the model consists of four main stages: (1) the development of the geo-uninorm consistency control module; (2) the preference similarity network clustering based consensus; (3) the influence-based feedback mechanism; and (4) the influence-driven resolution process. Based on Figure 3.1, the descriptions of all stages are presented as follows.

- *STAGE 1 – Geo-uninorm Consistency Control Module (Chapter 4)*:

  In decision making process, experts are allowed to discuss on their evaluations over a finite set of alternatives. They can expressed their evaluations by means of reciprocal preference relations (RPR) (Definition 2.3.2 on page 13). These preferences are extracted as the intensity preference vector (IPV) (Definition 2.3.3 on page 14). In order to generate consistent preferences, the geo-uninorm consistency operator (Equation 4.5 on page 42) is introduced. The IPV of the geo-uninorm consistent preferences are then extracted. The cosine-consistency degree (Definition 4.1.2 on page 45) of each expert preferences is obtained by measuring the similarity between the initial IPV with the constructed geo-uninorm consistent IPV. This measure presents the consistency level of expert preferences, which is necessary to be carried out in order to secure the *correctness* of information, thus misleading solution in decision making process can be avoided.

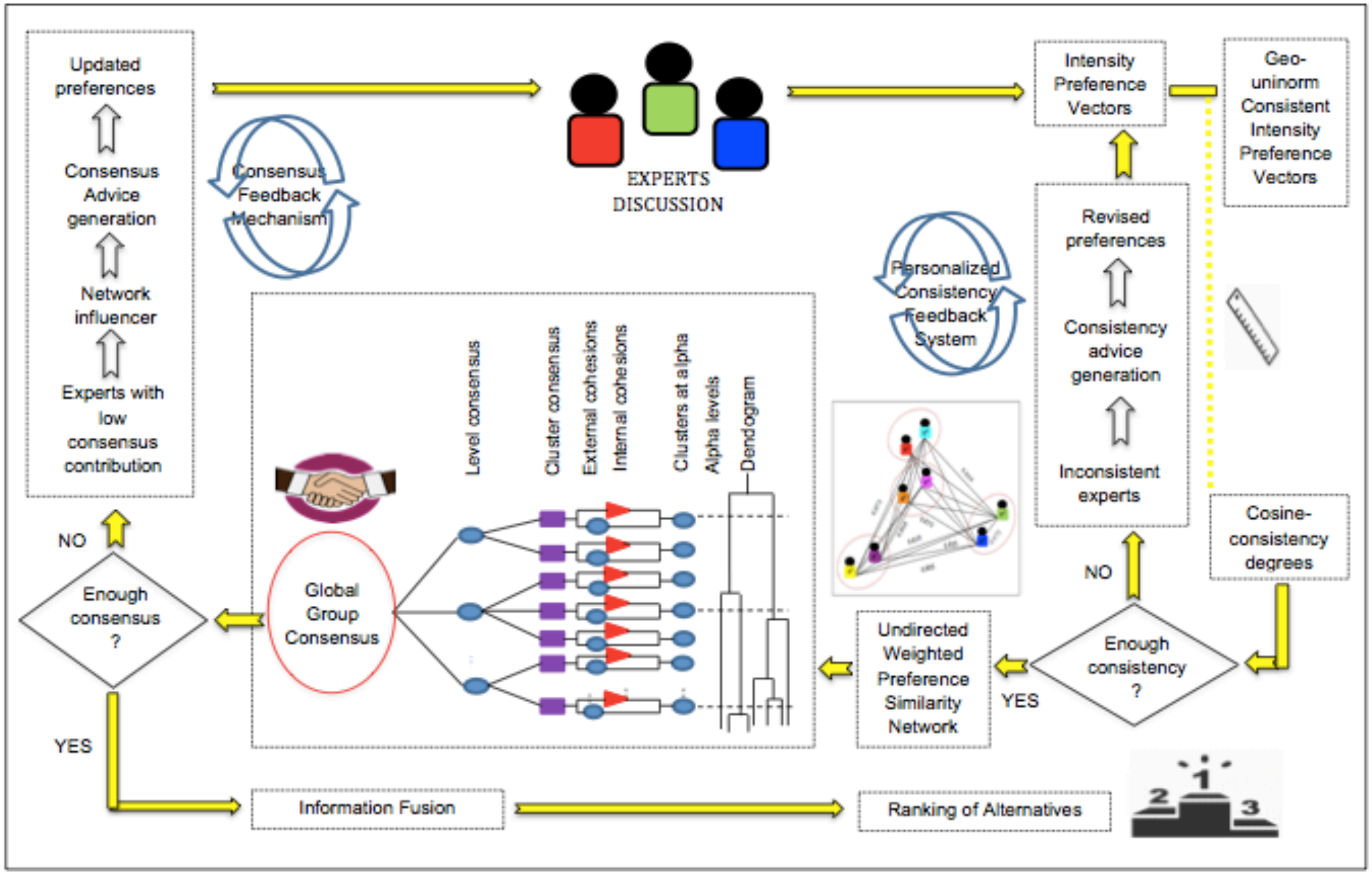

FIGURE 3.1: A framework of Consistent Preference Similarity Network Clustering and Influence Based Consensus Group Decision Making model

The personalized consistency feedback system (Section 4.2 on page 49) is activated if expert's consistency level is insufficient (lower than the predetermined consistency threshold). This procedure begins with the identification of inconsistent expert(s). These identified expert(s) will be personally advised to change their initial preferences closer to the constructed geo-uninorm consistent preferences. Different experts will be treated differently, depending on their inconsistency levels. The revised preferences will be utilized in the next consistency measure, in such a way that the consistency level is improved. The proposed consistency feedback system guarantees a sufficient consistency state of each expert preferences if all inconsistent experts accept those recommendations.

- *STAGE 2 – Preference Similarity Network Clustering Based Consensus (Chapter 5)*:

The RPR from the previous stage are now considered as *consistent* and their consistent IPVs will be used in measuring the similarity of preferences. By utilization of the cosine similarity function (Equation 5.1 on page 54), the obtained similarity degrees express the strength of experts' connections sharing most similar preferences, rely on a concept of structural equivalence relation (refer Page 53). This measure is then presented in terms of the preference similarity matrix and visualises as an undirected weighted preference similarity network (Definition 5.1.1 on page 55). The proposed network consists of a set of experts nodes, connected to each other by links with a unique weight (preference similarity degree) attached to them.

The undirected weighted preference similarity network is used as an input in the agglomerative hierarchical clustering methodology (Algorithm 2 on page 59). Experts are categorized into subgroups based on their preference similarities and the clustering solution is displayed by the generated dendogram at all α-levels (Figure 5.3 on page 60).

Since experts are structurally equivalence, the clustering result performs homogeneity, where the experts are strongly connected with each other within the cluster, compared to the outsiders. We defined the cluster internal (Definition 5.2.1 on page 60) and external cohesions (Definition 5.2.2 on page 60), the α-level cluster consensus (Equation 5.2 on page 61) and the α-level cluster consensus degree of the group of experts (Definition 5.2.4 on page 61). The highest of all the α-level cluster group consensus degree is considered as the global cluster consensus degree of the group at the optimal clustering level (Definition 5.2.7 on page 62). This network clustering consensus measure guarantees the homogeneity of experts' clusters, which indirectly perform cohesiveness of preferences (consensus).

- *STAGE 3 – Influence-based Feedback Mechanism (Chapter 6)*:

  For the case of insufficient consensus state, a feedback mechanism need to be activated. First effort to overcome this situation is identification of experts with low contribution to consensus (Section 6.1 on page 68). They are having less cluster consensus degree than the average of the cluster consensus degrees of all clusters at the optimal clustering α-level. The network influencer (Section 6.2 on page 69) is determined using the σ-centrality measure (Definition 6.2.2 on page 70). The influence scores produce from the σ-centrality measure presents the experts' status (importance) weights in the similarity network. The expert with maximum influence score is nominated as a network influencer, who acts as a leader in giving recommendations or advice on how to change their preferences closer to each other. The updated IPV (Equation 6.6 on page 74) are then used in the second consensus round (Section 6.4 on page 78). At this stage, the optimum revised global cluster consensus degree achieves a sufficient consensus state with better clustering solution.

- *STAGE 4 – Influence-driven Resolution Process (Chapter 7)*:

  After a satisfactory consensus level is obtained, the individuals expert preferences need to be fused into a collective form. A new IOWA operator, known as σ-IOWA (Definition 7.1.1 on page 82) is introduced as a fusion function, related on the set of influence scores from σ-centrality as a set of order inducing aggregation variable, associating with *most of* RIM fuzzy liguistic quantifier (Section 2.5.1 on page 23).

It is necessary to have satisfaction of preference of one alternative towards another for *most of* the more influential experts in the network. The QGDD concept (Section 2.5.2 on page 25) is then implemented in order to rank the alternatives by quantifying the dominance that one alternative has over all the other alternatives in the fuzzy majority (Page 23) sense. It is finalized that the first rank of alternative is the best alternative chosen by the group of experts, representing their individual preferences are appropriately taken into considerations and agreed as a whole.

Part III

NOVEL KNOWLEDGE CONTRIBUTIONS

Chapter 4

Geo-uninorm Consistency Control Module

The idea of having a consistency control module before the consensus decision making process has been carried out is necessary to ensure the *correctness* of information, thus misleading solutions can be avoided. For the purpose of meeting this challenge, we construct the geo-uninorm consistency control module [1] and it descriptions are presented in two sections: (1) the geo-uninorm based consistency measure; and (2) the personalized consistency feedback system.

4.1 Geo-uninorm based Consistency

A review on the consistency of preferences is previously presented in Section 2.3.2 on page 15. In this chapter, we focus on the development of consistency of RPR under the transitivity property (Page 15) and the uninorm operator concept (Page 18).

It is realized that experts are assumed to be able to quantify their preferences in the form of $[0, 1]$, instead of $\{0, 1\}$, in such a way that they can make a selection from an infinite set of possible alternatives. This assumption leads to the positive outcome, saying that the consistency of RPR can be mathematically modelled via a functional equation [34] (Equation 2.1 on page 16). Under the conditions of associativity, almost continuity and monotonicity, via Aczél's theorem [126, page 107],

[1]The content of this chapter has been published in [125].

the set of self-dual representable uninorms [50, 127] (Proposition 2.1 on page 17) is proved by Chiclana et al. [34] as the consistency functional equation of RPR solution. For the purpose of modeling transitivity of RPR, Tanino's multiplicative transitivity property under reciprocity becomes

$$U(x,y)=\begin{cases} 0, & (x,y)\in\{(0,1),(1,0)\},\\ \dfrac{xy}{xy+(1-x)(1-y)}, & \text{otherwise.}\end{cases} \tag{4.1}$$

This is the cross ratio uninorm; a conjunctive self-dual representable uninorm (with identity element 0.5), alternatively known as the symmetric sum [128]. Associativity of uninorms allow the extension of them into m arguments. The cross-ratio uninorm becomes the three $\prod$ operator [50]:

$$U(x_1,x_2,\ldots,x_m)=\begin{cases} 0, & \text{if } \exists i,j:(x_i,x_j)\in\{(0,1),(1,0)\},\\ \dfrac{\prod\limits_{i=1}^{m} x_i}{\prod\limits_{i=1}^{m} x_i+\prod\limits_{i=1}^{m}(1-x_i)}, & \text{otherwise.}\end{cases} \tag{4.2}$$

Uninorm operators can be differentiated from mean operators by the property of reinforcement. Given a set of input values, an operator is a *reinforcement type operator* if the output is above the maximum of the input values when all input values are 'high' and below the minimum of the input values when all input values are 'low'. Therefore, mean operators are not categorized as of the reinforcement type operators, since they are located between the minimum and maximum of their input values. On the other hand, uninorm operators are reinforcement operators because all input values are above or below their identity element.

Associativity and idempotency are incompatible properties for operators which are not minimum and maximum operators [129]. Yager [130] stated that the reinforcement property was not considered essential for the case when the input values correspond to criteria measuring the same property, and he added

> "[...] at a meta-level the use of mean type operators is appropriate in situations in which the values being aggregated are essentially multiple manifestations of the same variable. In this environment, the mean operator is acting like a *smoothing operator* to unify the different manifestations of the same concept."

Therefore, associativity is not important in the aggregation of individual information into a collective form, when mean operators (simple or weighted averages) are required, especially when the input values to aggregate measure the same property [131].

We further explore Yager's [130] extended mean operator concept, particularly with regard to special class of extended mean operators that include the well-known classical average operators, named *classical mean operators.* These operators satisfy *commutativity, idempotency, monotonicity* and *self identity* properties. Because all of the input values of the cross-ratio uninorm refer to the same property (preference modeling), we introduce a classical average operator application, by means of the geometric mean operator, accordingly with the cross-ratio uninorm operator (consistency modeling). This is relevant to work with because the cross ratio uninorm is a particular type of the more general class of operators [132, 133],

$$PI\left(x_1, \cdots, x_m\right) = \begin{cases} 0, & \text{if } \exists i, j : \left(x_i, x_j\right) \in \{(0,1),(1,0)\}, \\ \dfrac{\prod_{i=1}^{m} M\left(x_i\right)}{\prod_{i=1}^{m} M\left(x_i\right) + \prod_{i=1}^{m} M\left(1-x_i\right)}, & \text{otherwise}, \end{cases} \tag{4.3}$$

where M is a non-negative, increasing 'it generating' function. When the generating function $M\left(x\right) = x^{\frac{1}{m}}$, the following operator is obtained:

$$G_U\left(x_1, x_2, \cdots, x_m\right) = \begin{cases} 0, & \text{if } \exists i, j : \left(x_i, x_j\right) \in \{(0,1),(1,0)\}, \\ \dfrac{\prod_{i=1}^{m} x_i^{\frac{1}{m}}}{\prod_{i=1}^{m} x_i^{\frac{1}{m}} + \prod_{i=1}^{m} \left(1-x_i\right)^{\frac{1}{m}}}, & \text{otherwise}. \end{cases} \tag{4.4}$$

We name this the geo-uninorm operator, which can be rewritten as

$$G_U(x_1, x_2, \cdots, x_m) = \begin{cases} 0, & \text{if } \exists i, j : (x_i, x_j) \in \{(0,1), (1,0)\}, \\ \dfrac{1}{1 + \prod_{i=1}^{m} \left(\dfrac{1}{x_i} - 1\right)^{\frac{1}{m}}}, & \text{otherwise.} \end{cases} \tag{4.5}$$

The geo-uninorm operator is a classical mean operator and the following results are proved.

Proposition 4.1. *The geo-uninorm operator verifies the following properties: (1)* ***Commutativity*** *(2)* ***Idempotency*** *(3)* ***Monotonicity*** *(4)* ***Self identity****.*

Proof. Let $G(x_1, x_2, \cdots, x_m) = \prod_{i=1}^{m} x_i^{\frac{1}{m}}$ be the geometric mean operator.

Commutativity. Because G satisfies commutativity it is clear that G_U does so also.

Idempotency. Because G satisfies idempotency, then for all $x \in [0,1]$ $G(x, \cdots, x) = x$, and $G(1-x, \cdots, 1-x) = 1-x$. Consequently,

$$G_U(x, \cdots, x) = \frac{x}{x + (1-x)} = x.$$

Monotonicity. Let us assume we have $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ such that: $0 < y_i \leq x_i \leq 1$ $\forall i$. The case when some of the y_i are zero is evident because then

$$0 = G_U(y_1, \cdots, y_m) \leq G_U(x_1, \cdots, x_m).$$

In this case, we have that

$$\forall i: \ 1 \leq \frac{1}{x_i} \leq \frac{1}{y_i} \iff 0 \leq \frac{1}{x_i} - 1 \leq \frac{1}{y_i} - 1 \iff 0 \leq \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}} \leq \left(\frac{1}{y_i} - 1\right)^{\frac{1}{m}}.$$

This implies that

$$0 \leq \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}} \leq \prod_{i=1}^{m} \left(\frac{1}{y_i} - 1\right)^{\frac{1}{m}} \iff G_U(y_1, \cdots, y_m) \leq G_U(x_1, \cdots, x_m).$$

Self identity. We prove that

$$G_U(x_1, \cdots, x_m, G_U(x_1, \cdots, x_m)) = G_U(x_1, \cdots, x_m).$$

Again, if one of the x_i is zero then the above is evident as both left and right hand sides of the equation are zero. In all other cases, we have by definition:

$$G_U(x_1, \cdots, x_m, G_U(x_1, \cdots, x_m)) = \frac{1}{1 + \left(\frac{1}{G_U(x_1, \cdots, x_m)} - 1\right)^{\frac{1}{m+1}} \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m+1}}}$$

However,

$$G_U(x_1, x_2, \cdots, x_m) = \frac{1}{1 + \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}}} \iff \frac{1}{G_U(x_1, \cdots, x_m)} - 1 = \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}}.$$

Therefore,

$$\begin{aligned} G_U(x_1, \cdots, x_m, G_U(x_1, \cdots, x_m)) &= \frac{1}{1 + \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m(m+1)}} \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m+1}}} \\ &= \frac{1}{1 + \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m+1} + \frac{1}{m(m+1)}}}. \end{aligned}$$

Because $\frac{1}{m+1} + \frac{1}{m(m+1)} = \frac{m+1}{m(m+1)} = \frac{1}{m}$, we conclude that

$$G_U(x_1, \cdots, x_m, G_U(x_1, \cdots, x_m)) = \frac{1}{1 + \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}}} = G_U(x_1, x_2, \cdots, x_m).$$

□

As explained above, it is obvious that the geo-uninorm operator does not satisfy the reinforcement property because it is a mean operator. In spite of that, it satisfies weaker reinforcement properties which are desirable for transitivity of preferences. Literally, denoting by $x_* = \min\{x_1, x_2, \cdots, x_m\}$ and $x^* = \max\{x_1, x_2, \cdots, x_m\}$, monotonicity of G_U implies that

$$G_U(x_*, x_*, \cdots, x_*) \leq G_U(x_1, x_2, \cdots, x_m) \leq G_U(x^*, x^*, \cdots, x^*).$$

Idempotency of G_U results in

$$\min\{x_1, x_2, \cdots, x_m\} \leq G_U(x_1, x_2, \cdots, x_m) \leq \max\{x_1, x_2, \cdots, x_m\}.$$

Proposition 4.2. *The geo-uninorm operator satisfies:*

- ***Weak stochastic transitivity:*** $x_i \geq 0.5\ \forall i \implies G_U(x_1, x_2, \cdots, x_m) \geq 0.5$;
- ***Min transitivity:*** $G_U(x_1, x_2, \cdots, x_m) \geq \min\{x_1, x_2, \cdots, x_m\}\ \forall x_i$;
- ***Moderate stochastic transitivity:*** $x_i \geq 0.5\ \forall i \implies G_U(x_1, x_2, \cdots, x_m) \geq \min_i x_i$.

Moreover, since the function $f(x) = \log\left(\frac{1}{x} - 1\right)$ is concave on $[0.5, 1)$ $(f''(x) \leq 0)$ and convex on $(0, 0.5]$ $(f''(x) \geq 0)$, the following reinforcement properties were proved in [132]:

(a) When $x_i \in [0.5, 1)\ \forall i \implies f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \geq \sum_{i=1}^{m} \alpha_i f(x_i)$ subject to $\sum_{i=1}^{m} \alpha_i = 1$. Taking $\alpha_i = \frac{1}{m}$ we have

$$\log\left(\frac{1}{\frac{1}{m}\sum_{i=1}^{m} x_i} - 1\right) \geq \sum_{i=1}^{m} \frac{1}{m} \log\left(\frac{1}{x_i} - 1\right) = \log \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}}.$$

Monotonicity of f implies that

$$\frac{1}{\frac{1}{m}\sum_{i=1}^{m} x_i} - 1 \geq \prod_{i=1}^{m} \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}} \implies \frac{1}{m}\sum_{i=1}^{m} x_i \leq G_U(x_1, x_2, \cdots, x_m).$$

(b) The case when $x_i \in (0, 0.5]\ \forall i$ is derived similarly to the above case by changing the inequality symbols.

Proposition 4.3. *The geo-uninorm operator satisfies mean reinforcement properties:*

- $x_i \geq 0.5 \ \forall i \implies G_U\left(x_1, x_2, \cdots, x_m\right) \geq \frac{1}{m}\sum_{i=1}^{m} x_i$;
- $x_i \leq 0.5 \ \forall i \implies G_U\left(x_1, x_2, \cdots, x_m\right) \leq \frac{1}{m}\sum_{i=1}^{m} x_i$.

Hence, the geo-uninorm operator captures properties from both the geometric mean operator and the cross ratio uninorm operator. This means this operator is appropriate to be used in modelling transitivity, which is related to the consistency of fuzzy preferences.

In the following we propose the geo-uninorm consistency property of a RPR.

Definition 4.1.1. A reciprocal fuzzy preference relation, $P = (p_{ij})$, on a finite set of alternatives, $\mathbb{A} = \{A_1, A_2, \cdots, A_m\}$, is geo-uninorm consistent when

$$p_{ij} = G_U\left(p_{ik}, p_{kj}\right) \ \forall i, j, k, \text{ such that } \left(p_{ik}, p_{kj}\right) \notin \left\{(0,1), (1,0)\right\}.$$

Referring to Chiclana et al.'s work on construction of the uninorm-based consistent RPR [34], we define the geo-uninorm consistent RPR, $C = (c_{ij})$, based on the set of $(m-1)$ RPR values $P = \{p_{i(i+1)}; i = 1, \ldots, m-1\}$ as:

1. *Case 1*: For (i, j) such that $j > (i+1)$: $c_{ij} = G_U\left(p_{i(i+1)}, p_{(i+1)(i+2)}, \cdots, p_{(j-1)j}\right)$;
2. *Case 2*: For (i, j) such that $j < i$: $c_{ij} = 1 - c_{ji}$.

We further describe the consistency measure of RPR by firstly constructing the associated geo-uninorm consistent RPR. These consistent preferences are then utilized for the purpose of measuring the similarity between the initial preferences and the consistent ones. This similarity degree represents the consistency level of each expert preferences, defined using the cosine similarity measure between the experts' initial IPVs (Definition 2.3.3 on page 14) and the associated geo-uninorm consistent IPVs.

Formal definition of the cosine-consistency degree of expert h, $CCD\left(e^h\right)$, is presented as:

Definition 4.1.2. The cosine-consistency degree of expert h, $CCD\left(e^h\right)$, is the similarity degree between the IPV, $VP^h = \left(vp_k^h\right)$, and the geo-uninorm consistent

IPV, $VC^h = \left(vc_k^h\right)$,

$$CCD\left(e^h\right) = \frac{\sum\limits_{k=1}^{m(m-1)/2} \left(vp_k^h \cdot vc_k^h\right)}{\sqrt{\sum\limits_{k=1}^{m(m-1)/2} \left(vp_k^h\right)^2} \cdot \sqrt{\sum\limits_{k=1}^{m(m-1)/2} \left(vc_k^h\right)^2}}.$$

Example 4.1. *A committee of eight (8) heads of department in ABC company, also known as experts $E = \{e^1, e^2, \ldots, e^8\}$, give their evaluations over a set of six (6) potential candidates, $\mathbb{A} = \{A_1, A_2, \ldots, A_6\}$ for nomination of the best employer of the year 20XX, by means of RPR (Definition 2.3.2 on page 13).*

For instance, the evaluation matrix from Expert 1, P^1 is given below. These values are transformed in terms of IPV (Definition 2.3.3 on page 14). VP^1 is formed from the boldfaced elements of P^1:

$$P^1 = \begin{bmatrix} 1 & \mathbf{0.4} & \mathbf{0.6} & \mathbf{0.9} & \mathbf{0.7} & \mathbf{0.8} \\ 0.6 & 1 & \mathbf{0.7} & \mathbf{1} & \mathbf{0.8} & \mathbf{0.9} \\ 0.4 & 0.3 & 1 & \mathbf{0.8} & \mathbf{0.6} & \mathbf{0.7} \\ 0.1 & 0 & 0.2 & 1 & \mathbf{0.3} & \mathbf{0.4} \\ 0.3 & 0.2 & 0.4 & 0.7 & 1 & \mathbf{0.6} \\ 0.2 & 0.1 & 0.3 & 0.6 & 0.4 & 1 \end{bmatrix}$$

$$VP^1 = (0.4, 0.6, 0.9, 0.7, 0.8, 0.7, 1, 0.8, 0.9, 0.8, 0.6, 0.7, 0.3, 0.4, 0.6).$$

The rest of the experts, $\{e^2, \ldots, e^8\}$, also expressed their evaluations towards alternatives and their respective IPVs are extracted as follows:

$VP^2 = (0.7, 0.8, 0.6, 1, 0.9, 0.6, 0.4, 0.8, 0.7, 0.3, 0.7, 0.6, 0.9, 0.8, 0.4)$;

$VP^3 = (0.69, 0.12, 0.2, 0.36, 0.9, 0.06, 0.1, 0.2, 0.8, 0.64, 0.8, 0.98, 0.69, 0.97, 0.94)$;

$VP^4 = (0.1, 0.36, 0.69, 0.16, 0.26, 0.84, 0.95, 0.62, 0.76, 0.8, 0.25, 0.39, 0.08, 0.14, 0.66)$;

$VP^5 = (0.34, 0.25, 0.82, 0.75, 0.87, 0.25, 0.18, 0.82, 0.91, 0.94, 0.91, 1, 0.34, 0.75, 0.82)$;

$VP^6 = (0.13, 0.18, 0.34, 0.75, 0.09, 0.66, 0.82, 0.91, 0.25, 0.75, 0.87, 0.82, 0.75, 0.91, 0.97)$;

$VP^7 = (0.55, 0.45, 0.25, 0.7, 0.3, 0.7, 0.85, 0.4, 0.8, 0.65, 0.7, 0.6, 0.95, 0.6, 0.85)$;

$VP^8 = (0.7, 0.75, 0.95, 0.6, 0.85, 0.55, 0.8, 0.4, 0.65, 0.7, 0.6, 0.45, 0.85, 0.4, 0.75)$.

The geo-uninorm consistent RPR, C^h, associated to P^h is then determined based on the proposed geo-uninorm operator (Equation 4.4 on page 41) and the construction procedure of the geo-uninorm consistent RPR (Page 45).

For instance, using the preference values of $P^1 : \{p_{12} = 0.4, p_{23} = 0.7, p_{34} = 0.8, p_{45} = 0.3, p_{56} = 0.6\}$, upper diagonal elements for C^1 (Case 1 on page 45) are calculated and presented as follows:

$$
\begin{aligned}
c_{13} &= \frac{(p_{12} \times p_{23})^{\frac{1}{2}}}{(p_{12} \times p_{23})^{\frac{1}{2}} + [(1 - p_{12}) \times (1 - p_{23})]^{\frac{1}{2}}} \\
&= \frac{(0.4 \times 0.7)^{\frac{1}{2}}}{(0.4 \times 0.7)^{\frac{1}{2}} + [(1 - 0.4) \times (1 - 0.7)]^{\frac{1}{2}}} = 0.555;
\end{aligned}
$$

$$
\begin{aligned}
c_{14} &= \frac{(p_{12} \times p_{23} \times p_{34})^{\frac{1}{3}}}{(p_{12} \times p_{23} \times p_{34})^{\frac{1}{3}} + [(1 - p_{12}) \times (1 - p_{23}) \times (1 - p_{34})]^{\frac{1}{3}}} \\
&= \frac{(0.4 \times 0.7 \times 0.8)^{\frac{1}{3}}}{(0.4 \times 0.7 \times 0.8)^{\frac{1}{3}} + [(1 - 0.4) \times (1 - 0.7) \times (1 - 0.8)]^{\frac{1}{3}}} = 0.648.
\end{aligned}
$$

The remaining values (lower diagonal entries) are determined using the reciprocity property (Case 2 on page 45), resulting in:

$c_{31} = 1 - c_{13} = 1 - 0.555 = 0.445;$

$c_{41} = 1 - c_{14} = 1 - 0.648 = 0.352.$

After complete computations are carried out, the geo-uninorm consistent RPR matrix, C^1 is constructed as follows:

$$
C^1 = \begin{bmatrix}
1 & 0.4 & 0.555 & 0.648 & 0.561 & 0.569 \\
0.6 & 1 & 0.7 & 0.753 & 0.614 & 0.610 \\
0.445 & 0.3 & 1 & 0.8 & 0.567 & 0.578 \\
0.352 & 0.247 & 0.2 & 1 & 0.3 & 0.445 \\
0.439 & 0.387 & 0.433 & 0.7 & 1 & 0.6 \\
0.431 & 0.389 & 0.422 & 0.555 & 0.4 & 1
\end{bmatrix}
$$

From C^1, the corresponding geo-uninorm consistent IPV, VC^1 is:

$$
\begin{aligned}
VC^1 = (&0.4, 0.555, 0.648, 0.561, 0.569, 0.7, 0.753, 0.614, 0.610, 0.8, 0.567, 0.578, \\
&0.3, 0.445, 0.6).
\end{aligned}
$$

Similarly, the rest of the experts' geo-uninorm consistent IPVs are computed and listed as follows:

$$
\begin{aligned}
VC^2 &= (0.7, 0.652, 0.534, 0.657, 0.608, 0.6, 0.445, 0.642, 0.584, 0.3, 0.663, 0.578,\\
&\quad 0.9, 0.710, 0.4);\\
VC^3 &= (0.69, 0.274, 0.387, 0.464, 0.607, 0.06, 0.252, 0.387, 0.585, 0.64, 0.666, 0.798,\\
&\quad 0.69, 0.855, 0.94);\\
VC^4 &= (0.1, 0.433, 0.570, 0.402, 0.454, 0.84, 0.821, 0.55, 0.578, 0.8, 0.371, 0.467,\\
&\quad 0.08, 0.291, 0.66);\\
VC^5 &= (0.34, 0.293, 0.582, 0.520, 0.591, 0.25, 0.696, 0.582, 0.652, 0.94, 0.740, 0.769,\\
&\quad 0.34, 0.605, 0.82);\\
VC^6 &= (0.13, 0.35, 0.488, 0.560, 0.708, 0.66, 0.707, 0.722, 0.830, 0.75, 0.75, 0.869,\\
&\quad 0.75, 0.908, 0.97);\\
VC^7 &= (0.55, 0.628, 0.635, 0.76, 0.781, 0.7, 0.676, 0.813, 0.823, 0.65, 0.856, 0.854,\\
&\quad 0.95, 0.912, 0.85);\\
VC^8 &= (0.7, 0.628, 0.653, 0.713, 0.720, 0.55, 0.628, 0.717, 0.725, 0.7, 0.784, 0.773,\\
&\quad 0.85, 0.805, 0.75).
\end{aligned}
$$

As per Definition 4.1.2 (Page 45), the cosine-consistency degree for each expert, $CCD\left(e^h\right)$ is measured. The computation for $CCD\left(e^1\right)$ is shown below:

Previously, $VP^1 = (0.4, 0.6, 0.9, 0.7, 0.8, 0.7, 1, 0.8, 0.9, 0.8, 0.6, 0.7, 0.3, 0.4, 0.6)$ *and* $VC^1 = (0.4, 0.555, 0.648, 0.561, 0.569, 0.7, 0.753, 0.614, 0.610, 0.8, 0.567, 0.578, 0.3, 0.445, 0.6)$.

$$
\begin{aligned}
CCD\left(e^1\right) &= \frac{(0.4 \times 0.4) + (0.6 \times 0.555) + \ldots + (0.6 \times 0.6)}{\left(\sqrt{(0.4)^2 + (0.6)^2 + \ldots + (0.6)^2}\right) \times \left(\sqrt{(0.4)^2 + (0.555)^2 + \ldots + (0.6)^2}\right)}\\
&= 0.989.
\end{aligned}
$$

Therefore, all $CCD\left(e^h\right)$ values are:

$$CCD\left(e^2\right) = 0.989;\ CCD\left(e^3\right) = 0.977;\ CCD\left(e^4\right) = 0.976;\ CCD\left(e^5\right) = 0.962;$$

$$CCD\left(e^6\right) = 0.941;\ CCD\left(e^7\right) = 0.960;\ CCD\left(e^8\right) = 0.962.$$

4.2 Personalized Consistency Feedback System

If the cosine-consistency degree of expert is insufficient, which is lower than the pre-defined consistency threshold, the consistency feedback mechanism needs to be activated. We proposed a procedure of the *personalized consistency feedback system* in order to improve the consistency level of the inconsistency experts. In this process, different inconsistent expert(s) will have different recommendations depending on the optimal control parameter values, making sure that only minimum changes are needed.

This feedback procedure begins with the identification of inconsistent expert(s), e_{low} who have cosine-consistency degree less than the consistency threshold, η. It is formulated as:

$$e_{low} = \left\{ e_{low}^{h} \mid CCD\left(e^{h}\right) < \eta \right\}. \tag{4.6}$$

These identified experts are then recommended to change their initial IPVs, VP^h to be closer to the geo-uninorm consistent IPVs, VC^h based on the following linear combination with personalized consistency control parameter, γ:

$$VP_{\gamma}^{h} = (1-\gamma) \cdot VP^{h} + \gamma \cdot VC^{h}. \tag{4.7}$$

This step is called *consistency advice generation.*

For the sake of simplicity, we suggest that the consistency control parameter, γ are selected from the discrete set $\{0.1, 0.2 \ldots, 0.9, 1\}$. The revised cosine-consistency degree, $CCD_{\gamma}\left(e^{h}\right)$, is computed between $VP_{\gamma}^{h} = \left\{ vp_{\gamma 1}^{h}, \ldots, vp_{\gamma k}^{h}, \ldots, vp_{\gamma_{\frac{m(m-1)}{2}}}^{h} \right\}$ and $VC^{h} = \left\{ vc_{1}^{h}, \ldots, vc_{k}^{h}, \ldots, vc_{\frac{m(m-1)}{2}}^{h} \right\}$ by:

$$CCD_{\gamma}(e^{h}) = \frac{\sum_{k=1}^{m(m-1)/2} \left(vp_{\gamma k}^{h} \cdot vc_{k}^{h} \right)}{\sqrt{\sum_{k=1}^{m(m-1)/2} \left(vp_{\gamma k}^{h} \right)^{2}} \cdot \sqrt{\sum_{k=1}^{m(m-1)/2} \left(vc_{k}^{h} \right)^{2}}}. \tag{4.8}$$

The value of γ controls the amount of change required for an expert to be consistent. The larger the value of γ, the closer VP_{γ}^{h} will be to VC^{h}, and concurrently increase the value of $CCD_\gamma(e^h)$. Notice that when $\gamma = 0$, $VP_0^h = VP^h$ and $CCD_0(e^h) = CCD(e^h)$ and when $\gamma = 1$, $VP_1^h = VC^h$ and $CCD_1(e^h) = 1$.

The optimal control parameter, $\hat{\gamma}$ corresponds to the sufficient consistency degree, $CCD_{\hat{\gamma}}(e^h) = \eta$ is chosen as an optimize solution of the change cost (difference between the initial preference values and the revised personalized consistent preference values) for an inconsistent expert to be classed as consistent.

In summary, the personalized consistency feedback system is presented in Algorithm 1 below:

Data: The IPV of expert e^h, $VP^h = \left\{vp_1^h, vp_2^h, \cdots, vp_{(m-1)m/2}^h\right\}$;
the geo-uninorm consistent IPV of expert e^h,
$VC^h = \left\{vc_1^h, vc_2^h, \cdots, vc_{(m-1)m/2}^h\right\}$;
the cosine-consistency degrees of all experts,
$\overline{CCD} = \{CCD(e^1), CCD(e^2), \cdots, CCD(e^m)\}$;
the personalized consistency control parameter, $\gamma \in [0, 1]$;
the consistency threshold, η.

begin
- Construct the geo-uninorm consistent RPR, $C = (c_{ij})$, and extract its respective IPV, VC^h ;
- Compute the cosine-consistency degree of expert h, $CCD\left(e^h\right)$;
- **if** $CCD\left(e^h\right) < \eta$ **then**
  - activate personalized consistency control module;
  - **begin**
    - Identify inconsistent expert(s), e_{low} ;
    - Recommend e_{low}^h to change his/her VP^h closer to VC^h ;
    - Compute the revised cosine-consistency degree, $CCD_\gamma\left(e^h\right)$;
    - Choose the optimal control parameter, $\hat{\gamma}$: $CCD_{\hat{\gamma}}(e^h) = \eta$;
  - **end**
- **else**
  - proceed to consensus process (Chapter 5) ;
- **end**

end

Algorithm 1: Personalized consistency feedback system

Example 4.2 (Continuation of Examples 4.1)**.** *Let the consistency threshold value be as $\eta = 0.962$. Referring to $CCD\left(e^h\right)$ values in the Example 4.1 on page 48, the inconsistent experts are e^6 and e^7, where both of them have smaller CCD* *values than η. Thus, e^6 and e^7 are required to get advice on how to change their preferences in order to be consistent.*

As suggested in Section 4.2 (Page 49), we use discrete values of γ from the set $\{0.1, 0.2, \ldots, 0.9, 1\}$ in order to generate the revised preferences for each of them. Each γ value produces a distinct revised consistency degree for each inconsistent expert, meaning that the consistency control module is personally carried out depending on their inconsistency level.

The computational example for inconsistent expert e^7 is presented as follows:

Previously,

$$\begin{aligned} VP^7 &= (0.55, 0.45, 0.25, 0.7, 0.3, 0.7, 0.85, 0.4, 0.8, 0.65, 0.7, 0.6, 0.95, 0.6, 0.85) \\ &\quad \textit{and} \\ VC^7 &= (0.55, 0.628, 0.635, 0.76, 0.781, 0.7, 0.676, 0.813, 0.823, 0.65, 0.856, 0.854, \\ &\quad 0.95, 0.912, 0.85). \end{aligned}$$

Begin with $\gamma = 0.1$,

$$VP^7_{0.1,1} = (1 - 0.1) \cdot VP^7_1 + 0.1 \cdot VC^7_1 = (0.9 \times 0.55) + (0.1 \times 0.55) = 0.55.$$

After complete calculation is carried out for all IPV elements of VP^7 and VC^7, the revised preferences of e^7, $VP^7_{0.1}$, are,

$$\begin{aligned} VP^7_{0.1} &= (0.55, 0.468, 0.289, 0.706, 0.348, 0.7, 0.833, 0.441, 0.802, 0.65, 0.716, 0.625, \\ &\quad 0.95, 0.631, 0.85). \end{aligned}$$

The revised cosine-consistency degree, $CCD_{0.1}(e^7)$ is then obtained by measuring the cosine-consistency degree between $VP^7_{0.1}$ and VC^7.

$$\begin{aligned} CCD_{0.1}\left(e^7\right) &= \frac{(0.55 \times 0.55) + (0.468 \times 0.628) + \ldots + (0.85 \times 0.85)}{\left(\sqrt{(0.55)^2 + (0.468)^2 + \ldots + (0.85)^2}\right) \times \left(\sqrt{(0.55)^2 + (0.628)^2 + \ldots + (0.85)^2}\right)} \\ &= 0.969. \end{aligned}$$

Table 4.1 displayed the revised cosine-consistency degrees, CCD_γ, of the inconsistent experts e^6 and e^7 at each discrete γ value from the set $\{0.1, 0.2, \ldots, 0.9, 1\}$.

TABLE 4.1: Revised cosine-consistency degrees, CCD^h_γ, for inconsistent experts e^6 and e^7 at all discrete γ values

γ	**0.1**	**0.2**	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$CCD_\gamma(e^6)$	0.952	**0.962**	0.971	0.979	0.986	0.991	0.995	0.998	0.999	1
$CCD_\gamma(e^7)$	**0.969**	0.976	0.982	0.988	0.992	0.995	0.997	0.998	0.999	1

According to Table 4.1, Expert e^6 *achieves consistency at* $\gamma = 0.2$ *and Expert* e^7 *only requires a smaller* γ*-value,* 0.1 *to be consistent. It is because the amount of changes in the consistency control module is depending on the inconsistency level of each expert. Since the consistency threshold is* 0.962*, Expert* e^7 *(previously has* $CCD\,(e^7) = 0.960$*) just need a small change to attain the threshold value, rather than Expert* e^6 *(previously has* $CCD\,(e^6) = 0.941$*).*

Chapter 5

Preference Similarity Network Clustering Based Consensus

The expert preferences from the previous chapter (Chapter 4) are now considered as 'consistent' and relevant to be utilized in the consensus decision making model. We introduce the preference similarity network clustering based consensus process [2] and it descriptions are provided in two parts: (a) the construction of preference similarity network; and (b) the network clustering consensus methodology.

5.1 Preference Similarity Network

Our proposed preference similarity network is derived based on the *structural equivalence* concept [86], as appeared in the Social Network Analysis (SNA) framework. Structural equivalence can be interpreted as: 'two experts are structurally equivalent if both of them are connected to the same experts or having the same neighbors, making them share similar characteristics in their own network environments' [134].

In our case, the structural equivalence concept represents the idea of experts having similar preferences with other experts and relies on the application of a similarity function over the set of IPVs representing their preferences.

[2] The content of this chapter has been appeared in [8].

We build our structural equivalence preference similarity network by measuring the cosine similarity degree, S^{cd}, between the pair of consistent IPVs from Expert e^c, $VC^c = \left\{ vc_1^c, \ldots, vc_k^c, \ldots, vc_{\frac{m(m-1)}{2}}^c \right\}$ and Expert e^d, $VC^d = \left\{ vc_1^d, \ldots, vc_k^d, \ldots, vc_{\frac{m(m-1)}{2}}^d \right\}$, as the following expression:

$$S^{cd} = S\left(VC^c, VC^d\right) = \frac{\sum\limits_{k=1}^{m(m-1)/2} \left(vc_k^c \cdot vc_k^d\right)}{\sqrt{\sum\limits_{k=1}^{m(m-1)/2} \left(vc_k^c\right)^2} \cdot \sqrt{\sum\limits_{k=1}^{m(m-1)/2} \left(vc_k^d\right)^2}}. \tag{5.1}$$

The higher value of S^{cd} indicates that the greater similarity of the preferences of Experts e^c and e^d, thus the more strongly connected they are in the preference similarity network.

Generally, *structural equivalence* is represented using a particular similarity function from three common groups of vector-based distances [57]:

(1) Linear correlation based distances – focus on the strength and direction of association patterns, instead of the mean and variance as criteria of expert similarities [135];
(2) Euclidean-based distances – insensitive to linear association (relationship);
(3) Exact matches-based distances (Jaccard, Hamming, etc.) – emphasis on exact matches of vectors.

We pointed out that the first and second distance groups are normally utilized for both binary and valued data (the degree of relationship), whereas the third one is specifically applied to binary data, which is irrelevant for our preference representation framework.

The cosine similarity function is observed as a special type of Pearson correlation coefficient for the case when the mean of both vectors are considered as zero, i.e. it is a type of Pearson correlation coefficient that is insensitive to the mean value.

In the consensus modeling context, the cosine similarity measure is shown as a stable function regardless of the number of experts involved, compared to the Euclidean distance [20]. Based of this justification, we use the cosine similarity function in measuring the similarity of the preference network, by means of the structural equivalence relation. Notice that other similarity functions (the Euclidean distance or correlation based) can also be applied and similar results to the ones presented in this thesis will be produced.

We exploit the consistent IPVs in order to develop an undirected weighted preference similarity network, $\mathbb{N}$, which consists of undirected complete links, T, that connected to the expert nodes, E, with a set of similarity of preference weights, $\mathbb{S}$, attached to each pair of them. The similarity network is classed as undirected pattern because of the symmetrical property on the similarity function, i.e, the preference similarity of experts e^c and e^d coincides with the preference similarity of e^d and e^c.

The mathematical formulation of an undirected weighted preference similarity network, $\mathbb{N}$, is given below and its general graphical representation is displayed in Fig. 5.1.

Definition 5.1.1. Let E be a set of experts and $\mathbb{VC} = \{VC^1, VC^2, \ldots, VC^n\}$ be the corresponding set of geo-uninorm consistent IPVs on a set of alternatives $\mathbb{A}$. Let S be the IPVs of cosine preference similarity function, i.e., a reflexive and symmetric function $S : \mathbb{V}(\mathbb{A}) \times \mathbb{V}(\mathbb{A}) \to [0, 1]$. Then, the set of experts, E, can be connected by a set of links, $T = \left\{t^{12}, \ldots, t^{1m}, t^{23}, \ldots, t^{2m}, \ldots, t^{(m-1)m}\right\}$, with the following set of preference similarity weights attached, $\mathbb{S} = S(\mathbb{VC} \times \mathbb{VC}) = \left\{S^{12}, \ldots, S^{1m}, S^{23}, \ldots, S^{2m}, \ldots, S^{(m-1)m}\right\}$. The resulting undirected weighted preference similarity network will be denoted by $\mathbb{N} = \langle E, T, \mathbb{S} \rangle$.

Example 5.1 (Continuation of Examples 4.2)**.** *At this stage, all experts preferences are considered as consistent, consisting of the revised preferences of Expert* e^6 *at* $\gamma = 0.2$ $(VP^6_{0.2})$*, Expert* e^7 *at* $\gamma = 0.1$ $(VP^7_{0.1})$ *(Page 51) and the initial preferences of the rest of experts (*$VP^1, VP^2, VP^3, VP^4, VP^5, VP^8$*).*

These IPVs are then utilized in order to develop an undirected weighted preference similarity network, $\mathbb{N}$*, by means of structural equivalence relations. Structurally equivalent experts in the network,* $\mathbb{N}$*, can be represented by measuring their similarities between all pairs of expert preferences.*

From Example 4.1 on page 46,

$$VP^1 = (0.4, 0.6, 0.9, 0.7, 0.8, 0.7, 1, 0.8, 0.9, 0.8, 0.6, 0.7, 0.3, 0.4, 0.6)$$

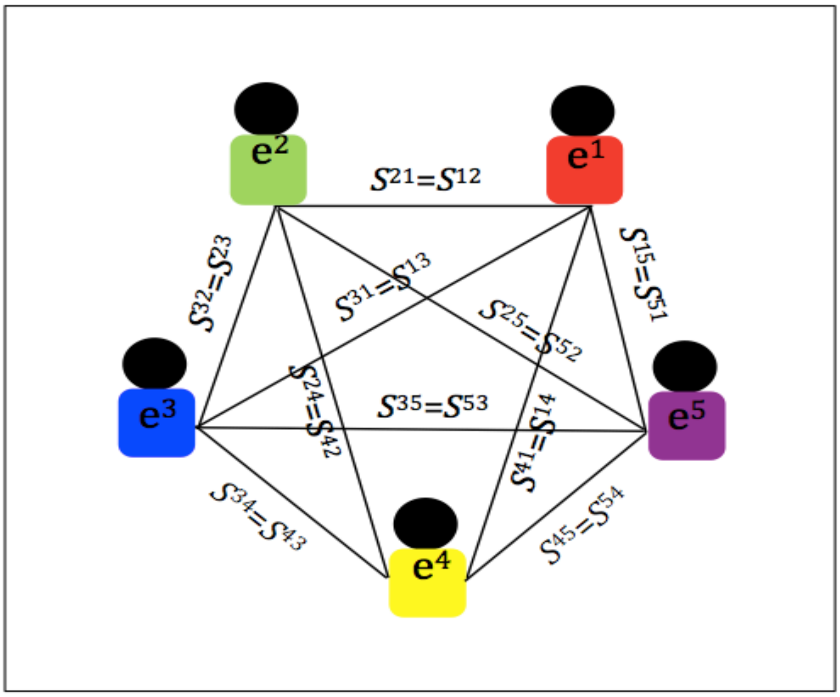

Figure 5.1: A general structure of undirected weighted preference similarity network, $\mathbb{N}$ consisting 5 experts' nodes

and

$$VP^2 = (0.7, 0.8, 0.6, 1, 0.9, 0.6, 0.4, 0.8, 0.7, 0.3, 0.7, 0.6, 0.9, 0.8, 0.4).$$

We measure the cosine similarity degree between VP^1 and VP^2, S^{12}, using Equation 5.1 (page 54) as:

$$S^{12} = \frac{(0.4 \times 0.7) + (0.6 \times 0.8) + \ldots + (0.6 \times 0.4)}{\left(\sqrt{(0.4)^2 + (0.6)^2 + \ldots + (0.6)^2}\right) \times \left(\sqrt{(0.7)^2 + (0.8)^2 + \ldots + (0.4)^2}\right)}$$
$$= 0.896.$$

After a complete computation for all pairs of experts is carried out, their cosine similarity degrees are then presented in terms of a symmetric preference similarity matrix, $\mathbb{S}$, as follows:

$$\mathbb{S} = \begin{bmatrix} 1 & 0.896 & 0.769 & 0.929 & 0.909 & 0.872 & 0.905 & 0.945 \\ 0.896 & 1 & 0.829 & 0.711 & 0.878 & 0.850 & 0.909 & 0.926 \\ 0.769 & 0.829 & 1 & 0.610 & 0.906 & 0.817 & 0.855 & 0.814 \\ 0.929 & 0.711 & 0.610 & 1 & 0.777 & 0.812 & 0.828 & 0.839 \\ 0.909 & 0.878 & 0.906 & 0.777 & 1 & 0.879 & 0.859 & 0.868 \\ 0.872 & 0.850 & 0.817 & 0.812 & 0.879 & 1 & 0.938 & 0.843 \\ 0.905 & 0.909 & 0.855 & 0.828 & 0.859 & 0.938 & 1 & 0.929 \\ 0.945 & 0.926 & 0.814 & 0.839 & 0.868 & 0.843 & 0.929 & 1 \end{bmatrix}.$$

From this matrix, the undirected weighted preference similarity network, $\mathbb{N}$*, is constructed and visualised in the following figure (Figure 5.2). Notice that only a few link weights, S are displayed for the sake of simplicity.*

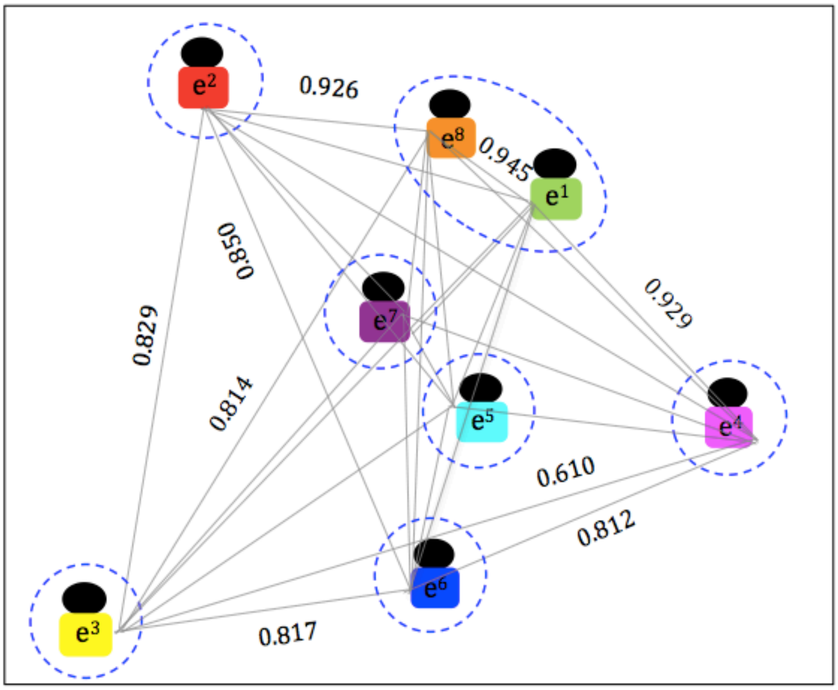

FIGURE 5.2: The undirected weighted preference similarity network, $\mathbb{N}$ of 8 experts

As referred to matrix $\mathbb{S}$ *and Figure 5.2, the highest similarity degree is* 0.945*, comes from Expert* e^1 *and* e^8*. This value represents Experts* e^1 *and* e^8 *are very closely connected in the network,* $\mathbb{N}$*. Contrary, Expert* e^3 *and* e^4 *has lowest similarity degree* (0.610)*, meaning that their preferences is far away from each other.*

5.2 Network Clustering Consensus

After the undirected weighted preference similarity network, $\mathbb{N}$, has been successfully constructed, we partition all experts into clusters using an agglomerative hierarchical clustering algorithm. This algorithm is chosen because it has the ability to partition objects discretely into groups, while providing an explicit procedure and a clear interpretation [136]. The formation of clusters in our context is referred to as a collection of experts, having similar preferences among them and dissimilar preferences to the outsider experts (different clusters).

Since the agglomerative hierarchical clustering method has no predetermined number of clusters, a set of distinct α-levels needs to be provided. Let $\mathbb{L} = \{\alpha_l \ : \ l = 2, \ldots, n-1\}$. Notice that level α_1 shows the extreme case of having a single cluster with all experts inside and level α_n places each member into its own cluster as the initial partition of the agglomerative hierarchical clustering solution. Practically, no clustering technique effectively applies for these two extreme levels (α_1 and α_n).

In the clustering methodology, pairs of objects (experts) have to be connected in close proximity using *linkage functions*. Complete link, average link, single link and centroid link are some of the common linkage functions used in this procedure. In this study, we apply a *complete linkage function* because it provides more homogeneous and stable clusters, compared with the others and it is less susceptible to noise and outliers [137].

The clustering procedure based on the agglomerative hierarchical clustering with complete linkage, by means of structural equivalence preference similarity network environment, is presented in Algorithm 2 below.

Data: A profile of IPVs, $VP = \{vp^1, vp^2, \ldots, vp^n\}$, expressed by a set of experts, $E = \{e^1, \ldots, e^n\}$, towards a set of alternatives, $\mathbb{A} = \{A_1, \ldots, A_m\}$.
Result: A hierarchical sequence of clustering solution: $P^n, P^{n-1}, \ldots, P^1$.
begin
    Start the clustering with partition $P^n = \{K_1, \ldots, K_n\}$ *where each cluster* K_p *has exactly one element* e^p*:*
    $P^n = \{\{e^1\}, \{e^2\}, \ldots, \{e^n\}\} = \{\{K_1\}, \{K_2\}, \ldots, \{K_n\}\}$;
    $i \longleftarrow n$;
    while $i > 1$ **do**
        Identify clusters K_c and K_d in $P^i = \{K_1, \ldots, K_i\}$ with maximal distance $\left(D^{cd}\right)$ (complete link);
        Merge clusters K_c and K_d to cluster K_k ;
        Build new partition P^{i-1} by removing K_c and K_d and adding cluster K_k ;
        $i \longleftarrow i - 1$;
    end
end

Algorithm 2: Agglomerative hierarchical clustering procedure with complete linkage function

The hierarchical sequence of clustering solution, referred to as a *dendogram* is generated after Algorithm 2 is implemented (Figure 5.3). This convenient graphical visualization is horizontally cut at a certain α-level equivalent to the chosen number of clusters at the level of the preference similarity matrix, $\mathbb{S}$.

Referring to this clustering result, we develop a procedure for measuring consensus based on the concept of cluster homogeneity, which is reasonable to be implemented to achieve cohesiveness of preferences (consensus). Furthermore, experts who are grouped according to their structural equivalence relations are expected to have strong connections within their cluster's members, compared to the outsider experts in different clusters [8].

Let $K_l = \{K_{lr} : \ r = 1, \ldots, l\}$ be the set of clusters at level α_l and $\sharp K_{lr}$ be the cardinality of K_{lr}. For the purpose of measuring consensus with capability of structural equivalent relations presented by the agglomerative hierarchical clustering, the respective definitions are given.

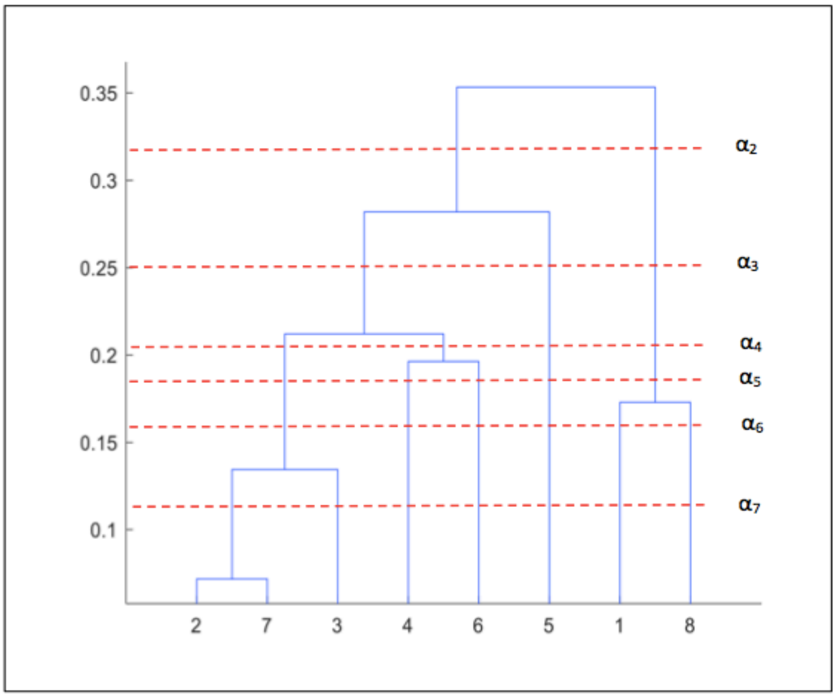

FIGURE 5.3: Dendogram, a hierarchical sequence of clustering solutions for 8 experts

Definition 5.2.1. The α_l-level cluster internal cohesion degree of cluster K_{lr} is

$$\delta_{int}\left(K_{lr}\right) = \frac{\sum\limits_{i \in K_{lr}} \sum\limits_{j \in K_{lr}} S^{ij}}{\left(\sharp K_{lr}\right)^2},$$

where S^{ij} is preference similarity degree between expert i and j in the cluster K_{lr}.

Definition 5.2.2. The α_l-level cluster external cohesion degree of cluster K_{lr} is

$$\delta_{ext}\left(K_{lr}\right) = \frac{\sum\limits_{i \in K_{lr}} \sum\limits_{j \notin K_{lr}} S^{ij}}{\sharp K_{lr}\left(n - \sharp K_{lr}\right)},$$

where $n = \sharp E$ (the total number of experts) and S^{ij} is preference similarity degree between expert i in the cluster K_{lr} and expert j outside the cluster K_{lr}.

Definition 5.2.3. The α_l-level cluster consensus degree of cluster K_{lr}, $\delta_{CC}\left(K_{lr}\right)$, is computed as:

$$\delta_{CC}\left(K_{lr}\right) = \frac{\sharp K_{lr} \cdot \delta_{int}\left(K_{lr}\right)}{n} + \frac{\left(n - \sharp K_{lr}\right) \cdot \delta_{ext}\left(K_{lr}\right)}{n}.$$

The α_l-level cluster consensus degree of cluster K_{lr} can be re-written as:

$$\delta_{CC}\left(K_{lr}\right) = \frac{\sharp K_{lr}\left(\delta_{int}\left(K_{lr}\right) - \delta_{ext}\left(K_{lr}\right)\right)}{n} + \delta_{ext}\left(K_{lr}\right). \tag{5.2}$$

The following situations are observed:

Case 1: $\delta_{int}\left(K_{lr}\right) > \delta_{ext}\left(K_{lr}\right)$. Similarities of expert preferences are higher internally, meaning that they are closer within their group members than the outsider experts, and they are high in homogeneity.

Case 2: $\delta_{int}\left(K_{lr}\right) < \delta_{ext}\left(K_{lr}\right)$. Similarities of expert preferences are higher externally, meaning that they are closer with outside members compared to their own group members, and they are low in homogeneity.

Since experts are grouped according to their similarity of preferences, it is expected that $\delta_{int}\left(K_{lr}\right) > \delta_{ext}\left(K_{lr}\right)$ will be satisfied in this consensus framework. They are categorized as a very homogeneous group, meaning that the similarities between experts within a cluster are greater internally because of their strong connections with each other, rather than with experts in different clusters.

In order to measure the group consensus degree at each clustering α-level, all clusters' associated consensus degrees are combined to obtain the collective cluster consensus degree at that cluster level. This can be formally defined by:

Definition 5.2.4. The α_l-level cluster consensus degree of the group of experts E, $\delta_{LC}\left(l\right)$, is computed as:

$$\delta_{LC}\left(l\right) = \frac{\sum_{r=1}^{l} \delta_{CC}\left(K_{lr}\right)}{l}.$$

$\delta_{LC}\left(l\right)$ expresses the homogeneity degree of experts' preferences at each clustering α-level, correlating with the measurement of consensus degree between the experts at that cluster level. By aiming to have high consensus state, the maximum of all the α_l-level cluster consensus degrees is chosen as the criterion to select the optimal agglomerative hierarchical clustering α_l-level solution.

Definition 5.2.5. The optimal agglomerative hierarchical clustering level, $\alpha_{\hat{l}}$-level, is the solution to the following optimization problem

$$\max_{\alpha_l \in \mathbb{L}} \ \delta_{LC}\left(l\right).$$

We pointed out that the above optimization problem is solvable, i.e. there is a solution in all cases because L has finite cardinality. However, the solution might not be unique, i.e. there might be more than one α_l-level with same maximum cluster consensus degree.

For this kind of situation, an additional criterion is needed to discriminate further between these α_l-levels. It happens in statistical scenario where two sample distributions with different size have the same average value, thus one of the statistical measures that effectively compare these distributions is the coefficient of variation (CV). CV considers the standard deviation or dispersion of data with respect to the mean value. For equal average values, lower CV is desired as homogeneity of data will be higher.

Relating CV concept in our consensus framework, $\alpha_{\hat{l}}$-level amongst all the α_l-levels with maximum cluster consensus degree will be the one with lowest cluster consensus coefficient of variation, $CCV_{LC}(l)$. This is formally defined as:

Definition 5.2.6. The α_l-level cluster consensus coefficient of variation, $CCV_{LC}\left(l\right)$ is defined as:

$$CCV_{LC}\left(l\right) = \frac{CSD_{LC}\left(l\right)}{\delta_{LC}\left(l\right)},$$

where

$$CSD_{LC}\left(l\right) = \sqrt{\frac{\sum_{r=1}^{l}\ \left[\delta_{CC}\left(K_{lr}\right) - \delta_{LC}\left(l\right)\right]^2}{l}}.$$

In the situation where there are two or more α_l-levels with same maximum cluster level consensus degree and same cluster consensus coefficient of variation, the lowest α_l-level value will be chosen for the reason of having the lowest number of clusters, which indirectly requires a lower number of rounds to achieve the minimum consensus threshold in the feedback mechanism process (Chapter 6).

The global cluster consensus degree of a group of experts E, $\delta_{LC}\left(\hat{l}\right)$ is then defined as:

Definition 5.2.7. The global cluster consensus degree of a group of experts E is $\delta_{LC}\left(\hat{l}\right)$, with $\alpha_{\hat{l}}$-level being the optimal clustering level.

Algorithm 3 summarizes the preference similarity network clustering based consensus process.

Data: *Dendogram:*
A set of experts: $E = \{e^1, e^2, \dots, e^n\}$;
Set of all different α-levels: $\mathbb{L} = \{\alpha_l \; ; \; l = 2, \dots, n-1\}$;
Set of clusters at each α_l-level: $K_l = \{K_{lr} \; ; \; r = 1, \dots, l\}$;
Consensus threshold value: $= \rho$

begin
- Identify experts in each cluster of α_l-level;
- Compute $\delta_{int}\left(K_{lr}\right)$ and $\delta_{ext}\left(K_{lr}\right)$ for each cluster in K_l;
- Obtain $\delta_{CC}\left(K_{lr}\right)$ for each cluster in K_l;
- Calculate $\delta_{LC}\left(l\right)$ for all α_l-level in $\mathbb{L}$;
- Identify optimum agglomerative hierarchical clustering level: $\alpha_{\hat{l}}$-level;
- **if** $\delta_{LC}\left(\hat{l}\right) \geq \rho$ **then**
  - end consensus procedure and apply resolution process (Chapter 7);
- **else**
  - apply feedback mechanism and advice generation phase (Chapter 6);
- **end**

end

Algorithm 3: Preference similarity network clustering based consensus process

Example 5.2 (Continuation of Examples 5.1)**.** *We execute the agglomerative hierarchical clustering procedure (Algorithm 2 on page 59) and the clustering solution represented by a dendogram is acquired, as depicted in Figure 5.4.*

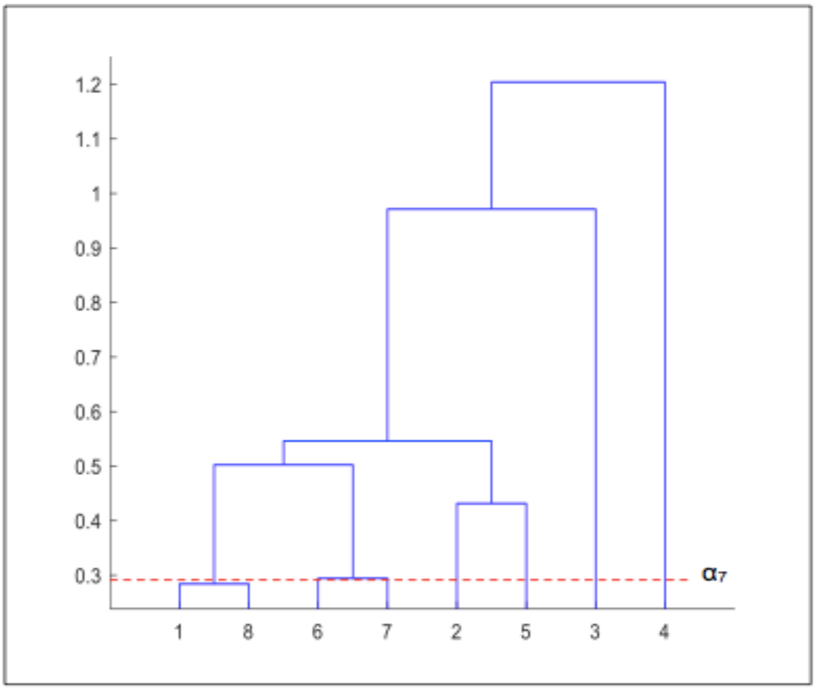

FIGURE 5.4: A dendogram consisting 8 experts, generated using Algorithm 2

By referring to Figure 5.4, each cluster at each α_l-level is identified. For instance, at α_2, there exist two clusters. K_{21} has $\{e^1, e^2, e^3, e^5, e^6, e^7, e^8, \}$ and K_{22} has only one expert, which is e^4.

Example of the calculations of internal cohesion degree, $\delta_{int}(K_{21})$, external cohesion degree, $\delta_{ext}(K_{21})$, cluster consensus degree, $\delta_{CC}(K_{21})$, and level consensus degree, $\delta_{LC}(2)$ are shown as follows:

$$\delta_{int}(K_{21})$$

$$= \frac{\begin{array}{r} S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18}+S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18} \\ +S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18}+S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18} \\ +S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18}+S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18} \\ +S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18}+S^{11}+S^{12}+S^{13}+S^{15}+S^{16}+S^{17}+S^{18} \end{array}}{7^2}$$

$$= \frac{\begin{array}{r} 1+0.896+0.769+0.909+0.872+0.905+0.945+0.896+1+0.829+0.878+ \\ 0.850+0.909+0.926+0.769+0.829+1+0.906+0.817+1+0.855+0.814+ \\ 0.909+0.878+0.906+1+0.879+0.859+0.868+0.872+0.850+0.817+0.879 \\ +1+0.938+0.843+0.905+0.909+0.855+0.859+0.938+1+0.929+0.945 \\ +0.926+0.814+0.868+0.843+0.929+1 \end{array}}{49}$$

$$= 0.894.$$

$$\begin{aligned} \delta_{ext}(K_{21}) &= \frac{S^{14}+S^{24}+S^{34}+S^{54}+S^{64}+S^{74}+S^{84}}{7(8-7)} \\ &= \frac{0.929+0.711+0.610+0.777+0.812+0.828+0.839}{7} \\ &= 0.786. \end{aligned}$$

$$\delta_{CC}(K_{21}) = \frac{7(0.894-0.786)}{8} + 0.786 = 0.880.$$

As calculated above, $\delta_{CC}(K_{21}) = 0.880$ *and in Table 5.1,* $\delta_{CC}(K_{22}) = 0.813$*. Thus,* $\delta_{LC}(2)$ *can be obtained by:*

$$\delta_{LC}(2) = \frac{0.880+0.813}{2} = 0.847.$$

The similar computations are executed and all values obtained are presented in Table 5.1. Based on the Definition 5.2.7 on page 62, the global cluster consensus degree is 0.868, with α_7 *as the optimal agglomerative hierarchical clustering level (Definition 5.2.5 on page 61).*

Table 5.1: The cluster internal and external cohesions, cluster consensus and level cluster consensus degrees at all α-levels

α	K	E	δ_{int}	δ_{ext}	δ_{CC}	δ_{LC}
2	1	$e^1, e^2, e^3, e^5, e^6, e^7, e^8$	0.894	0.786	0.880	0.847
	2	e^4	1	0.786	0.813	
3	1	$e^1, e^2, e^5, e^6, e^7, e^8$	0.911	0.824	0.890	0.843
	2	e^3	1	0.8	0.825	
	3	e^4	1	0.786	0.813	
4	1	e^1, e^6, e^7, e^8	0.929	0.860	0.895	0.853
	2	e^2, e^5	0.939	0.860	0.880	
	3	e^3	1	0.8	0.825	
	4	e^4	1	0.786	0.813	
5	1	e^1, e^8	0.972	0.875	0.899	0.861
	2	e^6, e^7	0.969	0.863	0.890	
	3	e^2, e^5	0.939	0.860	0.880	
	4	e^3	1	0.8	0.825	
	5	e^4	1	0.786	0.813	
6	1	e^1, e^8	0.972	0.875	0.899	0.864
	2	e^6, e^7	0.969	0.863	0.890	
	3	e^2	1	0.857	0.875	
	4	e^5	1	0.868	0.885	
	5	e^3	1	0.8	0.825	
	6	e^4	1	0.786	0.813	
7	1	e^1, e^8	0.972	0.875	0.899	**0.868**
	2	e^6	1	0.859	0.876	
	3	e^7	1	0.889	0.903	
	4	e^2	1	0.857	0.875	
	5	e^5	1	0.868	0.885	
	6	e^3	1	0.8	0.825	
	7	e^4	1	0.786	0.813	

Let the consensus threshold, $\rho = 0.90$*. It is obvious that consensus level is insufficient. From the clustering result (Figure 5.4 on page 64), the optimum cluster level is not led to a better solution in grouping the experts. This is because only 2 experts are grouped in a cluster (*$K_{71} = \{\mathrm{e}^1, \mathrm{e}^8\}$*), while the rest of the experts are in single member clusters. For this case, a feedback mechanism need to be carried out and this topic will be discussed further in the next chapter.*

Chapter 6

Influence-based Feedback Mechanism

The influence-based feedback mechanism [3] comprises three main phases: (1) identification of expert(s) with low contribution to consensus; (2) identification of a network influencer; and (3) generation of advice. The purpose of implementing this feedback system is to improve insufficient consensus state at the first consensus round and make use of this group agreement in finalizing experts' decision.

6.1 Identification of Experts With Low Contribution to Consensus

As defined in Definition 5.2.7 on page 62, the global cluster consensus degree of the group of experts, $\delta_{LC}\left(\hat{l}\right)$, is the average cluster consensus degree of the experts at the optimal consensus agglomerative hierarchical clustering $\alpha_{\hat{l}}$-level. At this level, those experts in a cluster $K_{\hat{l}r}$ with $\alpha_{\hat{l}}$-level cluster consensus below the global cluster consensus degree will be identified as contributing low to consensus (below the average). This is mathematically formulated as follows:

$$e_{low} = \{e^o \in E \mid e^o \in K_{\hat{l}r} \wedge K_{\hat{l}r} \in K_{low}\}, \tag{6.1}$$

[3]The content of this chapter has been presented in [138].

$$K_{low} = \left\{ K_{\hat{l}r} \mid \delta_{CC}\left(K_{\hat{l}r}\right) < \delta_{LC}\left(\hat{l}\right) \wedge \ r = 1, \ldots, \hat{l} \right\}; \tag{6.2}$$

Example 6.1 (Continuation of Examples 5.2)**.** *By referring to Table 5.1 (page 66), the global consensus degree of the group of experts,* $\delta_{LC}\left(\hat{l}\right)$*, is 0.868. Clusters having less cluster consensus degrees,* $\delta_{CC}\left(K_{\hat{l}r}\right)$*, than 0.868 are* K_{76} *and* K_{77}*. Thus, experts belong to these clusters* (e^3, e^4) *are experts who give insufficient contribution towards achieving consensus and need to receive recommendations on how to move their preferences closer to the group consensus.*

6.2 Identification of a Network Influencer

After the procedure which identifies experts with low contribution to the consensus, the experts identified need to be advised on how to change their preferences in order to increase the consensus level.

We utilize the preference similarity matrix, $\mathbb{S}$ contained in the Definition 5.1.1 (Page 55) as historical information used to generate advice. This idea is mostly used in *collaborative filtering* (CF), where the known (historical/initial) preferences of a group of users are used to provide 'valuable recommendations coming from someone who has shared similar history with other people in a group' [139].

According to this concept, we utilize the experts' initial evaluations as historical data to obtain the experts' preference similarity matrix, $\mathbb{S}$, as a criterion to be composed in our proposed Social Influence Network (SIN), which is constructed by a digraph that links the set of experts E (nodes), in such a way that every edge connects Expert e^i and e^j, (e^i, e^j), with the influence strength weight of the j^{th} expert over the i^{th} expert.

Our SIN comprises a set of experts, $E = \{e^1, e^2, \ldots, e^n\}$ and a row normalized preference similarity matrix, $\mathbb{S}_\eta = \left(S_\eta^{ij}\right)_{n \times n}$, where S_η^{ij} is the proportion of overall group influence on i that comes from j. Notice that $\mathbb{S}_\eta$ is obtained by taking a row normalization step on the preference similarity matrix, so that the following property $\sum_{j=1}^{n} S_\eta^{ij} = 1$ for all $i \in (1, \ldots, n)$ [104] is verified. This property ensures the influence of each expert towards all of his/her peers is 1 in total.

Formally, we name our SIN as *similarity social influence network (SSIN)*, which is visualised for a simple case of 3 expert nodes in Fig. 6.1 and defined below:

Definition 6.2.1. A similarity social influence network (SSIN) is an ordered triple, $G = \langle E, T, \mathbb{S}_\eta \rangle$ comprising a set of nodes E, a set of edges T, which are ordered pairs of experts in E, and a set of row normalized preference similarity weights, $\mathbb{S}_\eta = (S_\eta)_{n \times n}$ attached to T.

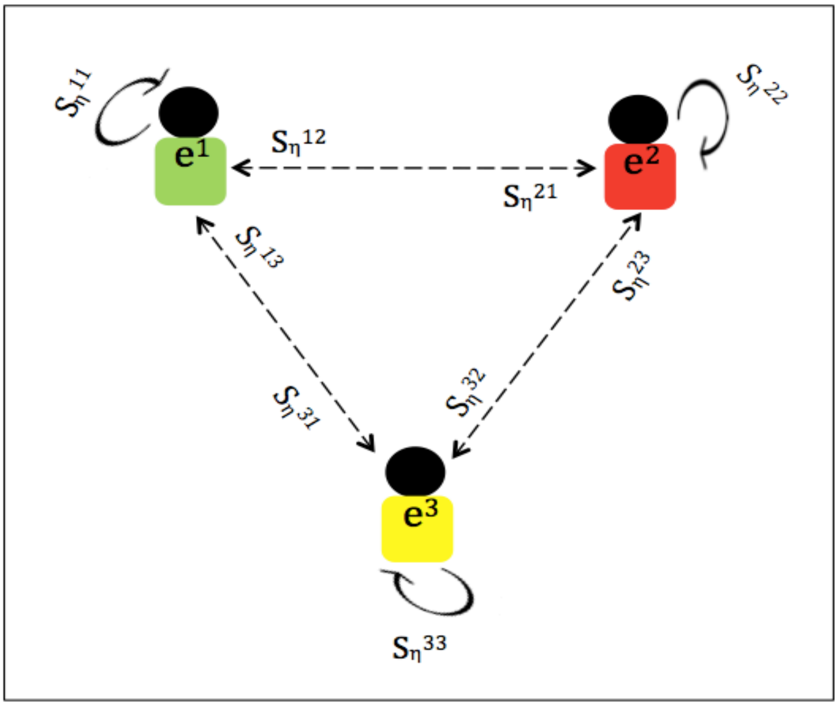

FIGURE 6.1: The general representation of SSIN consisting 3-expert nodes

We make use of the previously described influence measure (Page 30) by Bonacich and Llyod [124] . It is adapted to our preference similarity network CGDM model in order to identify the most influential expert of the network. The identified network influencer will act as the 'leader' in designing feedback rules with the aim to increase the group consensus level when this is below a satisfactory threshold value. We name our proposed influence measure the σ-centrality, which is formally defined as follows:

Definition 6.2.2. Let $\mathbb{S}_\eta$ be a set of row normalized preference similarity weights in G, the scalar σ be the relative importance of endogenous (internal network connections) versus exogenous (external) effects, and $Z = (z)_{m \times 1}$ be a set of individual expert exogenous effect values. Then, the influence score or σ-centrality of experts E, $Y = (y^1, \ldots, y^m)$, is:

$$Y = \left(I - \sigma\, \mathbb{S}_\eta^T\right)^{-1} Z.$$

In the absence of exogenous effect, Z is set as the unity vector, i.e. the vector with all components equal to 1.

For example, selection of the best employer will involve peer reviews, which can be presented as a relationship network in a workplace and this element is considered as an endogenous effect. In some situations, the selection process might involve evaluation from a company's top management, which is an exogenous factor to be included in the nomination of the best employer.

The combination of these two factors, endogenous and exogenous, produce stability in generating experts' influence scores because it is dependent on internal contributions to consensus from the generated preference similarity network and third-party importance evaluations over experts when available. Indeed, the value of Y represents the influence score of each expert, in such a way that both endogenous (internal) and exogenous (external) factors are considered. Experts who fall in clusters having higher cluster consensus degrees than or equal to the global cluster consensus degree of the group can be classified as belonging to the group of most influential experts in the network.

Let $K^*_{\hat{l}r}$ be set of clusters at the optimum clustering $\alpha_{\hat{l}}$-level with cluster consensus degrees, $\delta_{CC}(r)$, above the global cluster consensus degree of the group of experts, $\delta_{LC}\left(\hat{l}\right)$, and $e^{y^*}_{\hat{l}r}$ be the experts belonging to the clusters in $K^*_{\hat{l}r}$. Identification of possible network influencers can be formally written as:

$$e^{y^*}_{\hat{l}r} = \left\{ e^y \mid e^y \in K_{\hat{l}r} \wedge K_{\hat{l}r} \in K^*_{\hat{l}r} \right\}. \tag{6.3}$$

$$K^*_{\hat{l}r} = \left\{ K_{\hat{l}r} \mid \delta_{CC}(r) \geq \delta_{LC}\left(\hat{l}\right) \wedge \ r = 1, \ldots, \hat{l} \right\} \tag{6.4}$$

Thus, the network influencer, e^*, is the expert with highest influence score among those in $e^{y^*}_{\hat{l}r}$:

$$e^* = \max_{e^y \in K^*_{\hat{l}r}} Y(e^y). \tag{6.5}$$

The identified network influencer, e^*, will act as the leader for the experts who contribute low to consensus on how to change their opinions with the aim to increase and, subsequently, to reach the group consensus threshold level.

Example 6.2 (Continuation of Example 6.1). *As presented in Example 5.1 (Page 57), the matrix* $\mathbb{S}$ *is:*

$$\mathbb{S} = \begin{bmatrix} 1 & 0.896 & 0.769 & 0.929 & 0.909 & 0.872 & 0.905 & 0.945 \\ 0.896 & 1 & 0.829 & 0.711 & 0.878 & 0.850 & 0.909 & 0.926 \\ 0.769 & 0.829 & 1 & 0.610 & 0.906 & 0.817 & 0.855 & 0.814 \\ 0.929 & 0.711 & 0.610 & 1 & 0.777 & 0.812 & 0.828 & 0.839 \\ 0.909 & 0.878 & 0.906 & 0.777 & 1 & 0.879 & 0.859 & 0.868 \\ 0.872 & 0.850 & 0.817 & 0.812 & 0.879 & 1 & 0.938 & 0.843 \\ 0.905 & 0.909 & 0.855 & 0.828 & 0.859 & 0.938 & 1 & 0.929 \\ 0.945 & 0.926 & 0.814 & 0.839 & 0.868 & 0.843 & 0.929 & 1 \end{bmatrix}.$$

As an example, the $\mathbb{S}_\eta(1,1)$ *is calculated as follows:*

$$\begin{aligned} \mathbb{S}_\eta(1,1) &= \frac{S^{11}}{S^{11}+S^{12}+S^{13}+S^{14}+S^{15}+S^{16}+S^{17}+S^{18}} \\ &= \frac{1}{1+0.896+0.769+0.929+0.909+0.872+0.905+0.945} \\ &= 0.138. \end{aligned}$$

The similar calculation process is carried out and a complete matrix of a row normalised preference similarity weights, $\mathbb{S}_\eta$*, is constructed as:*

$$\mathbb{S}_\eta = \begin{bmatrix} 0.138 & 0.124 & 0.106 & 0.129 & 0.126 & 0.121 & 0.125 & 0.131 \\ 0.128 & 0.143 & 0.118 & 0.102 & 0.126 & 0.121 & 0.130 & 0.132 \\ 0.117 & 0.126 & 0.152 & 0.092 & 0.137 & 0.124 & 0.129 & 0.123 \\ 0.143 & 0.109 & 0.094 & 0.154 & 0.119 & 0.125 & 0.127 & 0.129 \\ 0.128 & 0.124 & 0.128 & 0.110 & 0.141 & 0.124 & 0.121 & 0.123 \\ 0.124 & 0.121 & 0.117 & 0.116 & 0.125 & 0.143 & 0.134 & 0.120 \\ 0.125 & 0.126 & 0.118 & 0.115 & 0.119 & 0.130 & 0.138 & 0.129 \\ 0.132 & 0.129 & 0.114 & 0.117 & 0.121 & 0.118 & 0.130 & 0.140 \end{bmatrix}.$$

Based on $\mathbb{S}_\eta$*, the similarity social influence network (SSIN) (Definition 6.2.1 on page 70) is then visualised as in Figure 6.2. Notice that only several row normalized cosine-similarity weights are displayed for the sake of simplicity.*

The main advantage of our influence score measure is that comprises both endogenous (network connections) and exogenous (external) effects. In this study, SSIN expresses the centrality degree with respect to the most influential expert status and

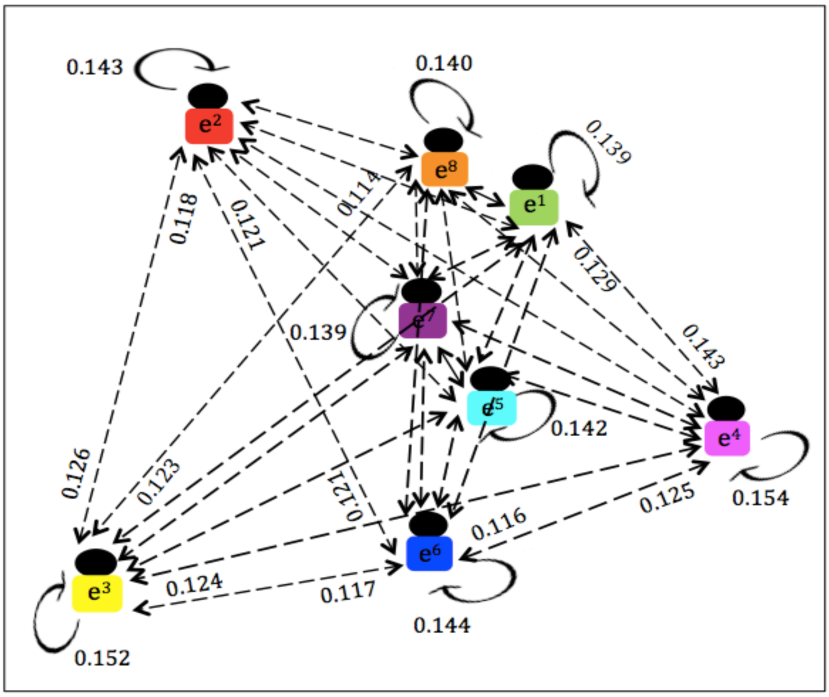

FIGURE 6.2: The similarity social influence network (SSIN) consisting 8 experts' nodes

exogenous factor involves third party opinion contributions, which is company's top management evaluations on experts' status (importance).

We implement Z for both cases ('no exogenous' and 'with exogenous' effects) in order to compare the impact of the influence factor in decision making process. For the 'no exogenous' effect case, we set Z as a matrix of ones and $Z = [0.8, 0.5, 0.5, 0.1, 0.9, 0.5, 1, 0.7]$ for the case 'with exogenous' effect. We set the scalar $\sigma = 0.5$, to represent equal (fair) status weights of relative importance of endogenous (network connections) with respect to exogenous (external) effects.

Related values are substituted into Definition 6.2.2 (Page 70) and the computation of the influence scores for 'no exogenous effect' case is stated in the following:

$$Y = \left(I - \sigma\ \mathbb{S}_\eta^{T}\right)^{-1} Z$$

$$= \left(I - (0.5) \begin{bmatrix} 0.138 & 0.124 & 0.106 & 0.129 & 0.126 & 0.121 & 0.125 & 0.131 \\ 0.128 & 0.143 & 0.118 & 0.102 & 0.126 & 0.121 & 0.130 & 0.132 \\ 0.117 & 0.126 & 0.152 & 0.092 & 0.137 & 0.124 & 0.129 & 0.123 \\ 0.143 & 0.109 & 0.094 & 0.154 & 0.119 & 0.125 & 0.127 & 0.129 \\ 0.128 & 0.124 & 0.128 & 0.110 & 0.141 & 0.124 & 0.121 & 0.123 \\ 0.124 & 0.121 & 0.117 & 0.116 & 0.125 & 0.143 & 0.134 & 0.120 \\ 0.125 & 0.126 & 0.118 & 0.115 & 0.119 & 0.130 & 0.138 & 0.129 \\ 0.132 & 0.129 & 0.114 & 0.117 & 0.121 & 0.118 & 0.130 & 0.140 \end{bmatrix}^{T} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= [2.035, 2.003, 1.947, 1.935, 2.014, 2.006, 2.034, 2.027]\,.$$

The same procedure is carried out for 'with exogenous' case, where

$$Z = [0.8, 0.5, 0.5, 0.1, 0.9, 0.5, 1, 0.7]$$

and the influence scores are shown below:

$$Y = [1.444, 1.130, 1.096, 0.676, 1.536, 1.128, 1.647, 1.343]\,.$$

Thus, the network influencer for 'no exogenous' case is e^1 and is e^7 for 'with exogenous' effect because both of experts have highest influence scores (2.035) and (1.647) respectively among others. This result proves that the exogoneous factor gives an impact on the appointment of a network influencer.

6.3 Generation of Advice

This section focuses on the contribution of a network influencer in controlling advice generation. As mentioned Section 6.1 (Page 68), in order to increase the consensus level of the group, an expert e^o in e_{low} is feedback with the following updated IPV, $\widetilde{V}^o$:

$$\widetilde{V}^o = (1-\beta)\cdot V^o + \beta\cdot V^* \tag{6.6}$$

where V^o is the IPV of e^o, V^* is the IPV of the network influencer, e^*, and $\beta \in [0, 1]$ is a control parameter that can be used to adjust the extent of the change of preferences feedback to the experts in e_{low}. Obviously, if $\beta = 0$ no changes are recommended and the original preferences of the expert, V^o, remain unchanged, while if $\beta = 1$ then the expert's preferences are completely substituted by those of the network influencer, V^*.

It is worth mentioning that this model deals with willingness of experts to change their preferences and accept those advises for the purpose of achieving consensus. If the low contributed to consensus expert(s) are against these recommendations, a sufficient consensus level will never be achieved.

In order to prove the validity of the proposed procedure, the following result proves that when implemented it will lead to an increase in group consensus:

Proposition 6.1. *Let V^o be the initial IPV of the Expert e^o, V^* be the IPV of the network influencer and $\widetilde{V}^o = (1 - \Omega) \cdot V^o + \Omega \cdot V^*$ be the updated IPV of e^o after the recommendation rule is implemented, where $\Omega \in [0, 1]$. Then, we have:*

$$S\left(\widetilde{V}^o, V^*\right) \geq S\left(V^o, V^*\right) \tag{6.7}$$

with equality holding if and only if $\Omega = 0$.

Proof. There exist points $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n(n-1)/2}$ such that $V^o = \overrightarrow{\mathbf{OA}}$, $V^* = \overrightarrow{\mathbf{OB}}$ and $\widetilde{V}^o = \overrightarrow{\mathbf{OC}}$, which can be represented as in Figure 6.3 below:

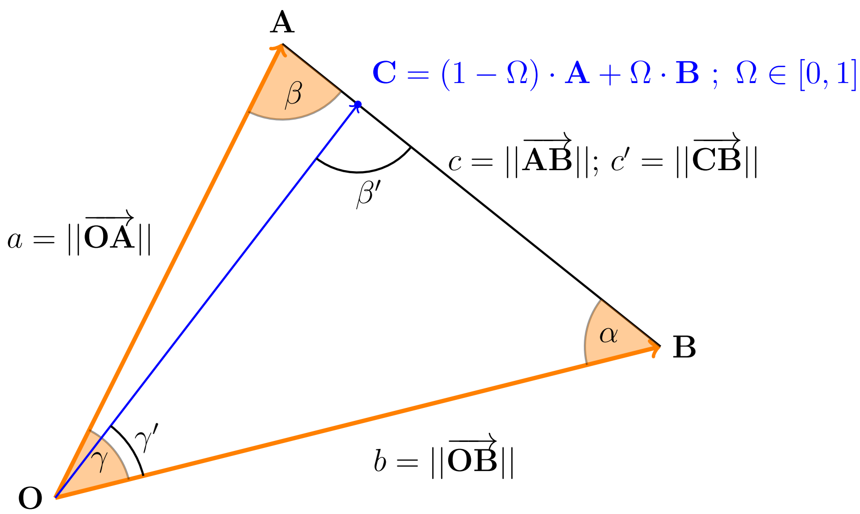

FIGURE 6.3: Spatial representation of intensity preference vectors

We have the following:

1. Because coordinates of points **A** and **B** are greater than or equal to zero it is $\gamma \in \left[0, \frac{\pi}{2}\right]$.

2. Clearly, $||\overrightarrow{\mathbf{AB}}|| = c \geq c' = ||\overrightarrow{\mathbf{CB}}||$, with equality holding if and only if $\Omega = 0$.

3. Because $\alpha + \beta + \gamma = \pi$, we have: $\sin\beta = \sin\alpha \cdot \cos\gamma + \cos\alpha \cdot \sin\gamma$.

The law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

implies that

$$c = b \cdot \frac{\sin\gamma}{\sin\beta}.$$

From Item 3, we have

$$c = b \cdot \frac{\sin\gamma}{\sin\alpha \cdot \cos\gamma + \cos\alpha \cdot \sin\gamma} = b \cdot \frac{1}{\sin\alpha \cdot \cot\gamma + \cos\alpha}.$$

From Item 1, we have

$$0 \leq \cot\gamma \leq \cot\gamma' \Leftrightarrow \gamma \geq \gamma'.$$

From Item 2, we have that $\gamma \geq \gamma'$, with equality holding if and only if $\Omega = 0$. Therefore, we have that

$$\cos\gamma \leq \cos\gamma'$$

with equality holding if and only if $\Omega = 0$. Finally, by Equation 5.1 (Page 54) we conclude that

$$S\left(\widetilde{V}^o, V^*\right) \geq S\left(V^o, V^*\right)$$

with equality holding if and only if $\Omega = 0$. $\square$

This section concludes with the below algorithmic representation of the proposed influence-based feedback mechanism procedure.

begin

At $\alpha_{\hat{l}}$-level, identify K_{low}, a set of clusters having less δ_{CC} than $\delta_{LC}\left(\hat{l}\right)$;

List all experts belong to K_{low}, denoted as e_{low};

Construct row normalized cosine-similarity matrix, $\mathbb{S}_\eta$;

Decide the individual expert exogenous effect, Z: (Assume Z as a vector of ones if no exogenous effect involved) ;

Determine σ-centrality using Definition (6.2.2) ;

Rank experts according to Y scores in descending order: $R(Y)$;

Identify $K^*_{\hat{l}r}$ as in Equation (6.4) ;

List all experts belonging to $K^*_{\hat{l}r}$ (Equation (6.3)) ;

Identify the network influencer, e^* by Equation (6.5) ;

Generate updated preferences, $\widetilde{V}^o$ using Equation (6.6).

end

Algorithm 4: Influence-based Feedback Mechanism

Example 6.3 (Continuation of Examples 6.2)**.** *From the Example 6.1 (Page 69), the experts with low consensus contributions are* e^3 *and* e^4*. In Example 6.2 (Page 74), the network influencer for 'no exogenous' case is* e^1 *and for 'with exogenous' is* e^7*.*

The example of calculation of generating advice for Expert e^3 *with 'no exogenous' case is presented as follows:*

Let $\beta = 0.1$*.* e^3 *is adviced to change his/her preferences,* V^o *closer to the network influencer (*e^1*) preferences,* V^**.*

$$V^o = (0.69, 0.12, 0.2, 0.36, 0.9, 0.06, 0.1, 0.2, 0.8, 0.64, 0.8, 0.98, 0.69, 0.97, 0.94);$$
$$V^* = (0.55, 0.45, 0.25, 0.7, 0.3, 0.7, 0.85, 0.4, 0.8, 0.65, 0.7, 0.6, 0.95, 0.6, 0.85)$$

$$\widetilde{V}^o = ((1 - 0.1) \cdot V^o) + (0.1 \cdot V^*) = (0.9 \cdot V^o) + (0.1 \cdot V^*)$$

$$\widetilde{V}^o_{1,1} = (0.9 \cdot 0.69) + (0.1 \cdot 0.55) = 0.676.$$

Therefore, the updated preferences for Expert e^3 *at* $\beta = 0.1$ *are:*

$$\widetilde{V}^o = (0.676, 0.153, 0.205, 0.394, 0.840, 0.124, 0.175, 0.22, 0.8, 0.641, 0.79, 0.942, 0.716, 0.933, 0.931).$$

6.4 Second Consensus Round

The generated updated preferences collected in the previous section will be utilized in implementing the second round of consensus, for the purpose of achieving a sufficient consensus state, if it is unsuccessful in the first round. For the influence-based feedback mechanism to be effective, a control parameter $\beta \in [0, 1]$ in Equation 6.6 (Page 74) should be selected to guarantee that the following two condition are verified:

- *Condition 1*: The global cluster consensus degree of the group of experts of the second consensus round , $\delta^2_{LC}\left(\hat{l}\right)$ must be greater than or equal the consensus threshold, ρ and;
- *Condition 2*: The optimal agglomerative hierarchical clustering level in second round of consensus process, $\alpha^2_{\hat{l}}$ must be less than the optimal agglomerative hierarchical clustering level of first round, $\alpha_{\hat{l}}$.

For simplicity, we use discrete values of β from the set $\{0.1, 0.2, \ldots, 0.9, 1\}$. Notice that the advice generation with $\beta = 0$ produces the first round of consensus solution. The first condition above states that sufficient consensus level will be achieved, while the second one is purposely introduced to achieve a better clustered solution (lower number of clusters) after the implementation of the feedback process.

Without imposing restrictions to the parameter of control, the above two conditions will be achieve at some extent because when the feedback advices are implemented, the experts will be more similar because the preferences will be closer to the network influencer's preferences, which also will have a positive effect on the cohesiveness within clusters. These two conditions are formally presented in the following definition.

Definition 6.4.1. The revised global cluster consensus degree, $\delta^2\left(\hat{l}\right)$, of the group of experts, E, for the second round of the consensus reaching process satisfies:

$$\left(\delta^2_{LC}\left(\hat{l}\right) \geq \rho\right) \wedge \left(\alpha^2_{\hat{l}} < \alpha_{\hat{l}}\right).$$

As mentioned before, the advice control paremeter, β, in recommendation rule of change is setting up as a discrete set $\{0.1, 0.2, \ldots, 0.9, 1\}$. For the second consensus round, the updated preferences at $\beta = 0.1$ are used in Algorithm 2 (Page 59) and 3 (Page 63) in order to obtain the global cluster consensus degree of the group of experts, $\delta^2_{LC}\left(\hat{l}\right)$, and the optimal agglomerative hierarchical clustering level, $\alpha^2_{\hat{l}}$.

If the above two conditions are not satisfied, the updated preferences at $\beta = 0.2$ are generated and will be used in the consensus procedure. This step will continue until the revised global cluster consensus degree of the group of experts, $\delta^2\left(\hat{l}\right)$, is obtained. There is a possibility when none of β-levels provide $\delta^2\left(\hat{l}\right)$, thus the process is repeated for the third round and so forth.

The algorithm to find the optimal parameter of control, within the discrete set of values $\{0.1, 0.2, \ldots, 0.9, 1\}$, with respect to Definition 6.4.1 is presented as:

begin
- At $\beta = 0.1$, replace V^o with $\widetilde{V}^o$ for all e_{low} in K_{low} ;
- Run Algorithm 2 and 3 ;
- Identify $\alpha^2_{\hat{l}}$-level and its corresponding $\delta^2_{LC}\left(\hat{l}\right)$;
- **if** $\left(\delta^2_{LC}\left(\hat{l}\right) \geq \rho\right) \wedge \left(\alpha^2_{\hat{l}} < \alpha_{\hat{l}}\right)$ **then**
  - end second round consensus procedure ;
- **else**
  - repeat this algorithm with next discrete β-level;
- **end**
- Run next round of consensus procedure;

end

Algorithm 5: The second round of consensus reaching procedure

Example 6.4 (Continuation of Examples 6.3)**.** *At each β-level, the initial preferences from the experts with low contribution to consensus will be replaced with the updated preferences and the second round of consensus process is executed. The optimal global consensus degrees and their corresponding optimal clustering solutions at all discrete β-levels are obtained. The optimal revised global consensus for the second consensus round is identified from the list of optimal global consensus degrees based on the two conditions stated on Page 78.*

Table 6.1 exhibits the optimal global consensus degrees and their corresponding optimal cluster solutions at each β-level for the second round of consensus reaching process for the cases 'no' and 'with exogenous' effects.

TABLE 6.1: The optimal global consensus degrees and their corresponding optimal cluster solutions at each β-level for the second round of consensus for 'no exogenous' and 'with exogenous' cases

β-levels	No Exogenous Effect		With Exogenous Effect	
	Global Consensus	Cluster Solution (α-level)	Global Consensus	Cluster Solution (α-level)
0	0.865	Level 7	0.865	Level 7
0.1	0.880	Level 2	0.878	Level 7
0.2	0.891	Level 7	0.892	Level 3
0.3	**0.900**	**Level 2**	**0.901**	**Level 2**
0.4	0.907	Level 2	0.907	Level 2
0.5	0.911	Level 7	0.910	Level 7
0.6	0.914	Level 7	**0.916**	**Level 2**
0.7	0.916	Level 7	0.919	Level 2
0.8	0.917	Level 7	0.920	Level 2
0.9	0.917	Level 7	0.920	Level 2
1	**0.911**	**Level 6**	0.919	Level 2

As mentioned in page 67, the consensus threshold, ρ, is 0.9. It is shown in Table 6.1 that the optimal global consensus degrees for the second consensus round are 0.9 and 0.901, respectively, with updated preferences at $\beta_{0.3}$ and the clustering solution falls at α_2 for both cases ('no exogenous' and 'with exogenous').

If the threshold is set at 0.910, different results are obtained. The optimal global consensus degree for 'with exogenous' case is 0.916 at α_2 and $\beta_{0.6}$. But, for the case of 'no exogenous', it falls at α_6 and β_1 with the degree of 0.911. At this β-level (β_1), the experts with low contribution to consensus have to replaced their preferences with the network influencer' preferences. Obviously, this is irrelevant to be considered in real decision making because those expert preferences are totally ignored.

However, the 'with exogenous' case provides more reasonable results when the optimal global consensus degree is 0.916 at α_2 and $\beta_{0.6}$. The experts with low consensus contribution are only required to modify their initial preferences with minimal amount of changes, while the clustering solution is better than the 'no exogenous' case. Notice that the global consensus degree at $\beta_{0.5}$ achieved consensus threshold (0.910), however the clustering solution is the same as the first consensus round, in which Condition 2 (Page 78) is not satisfied.

In addition, the optimal global consensus degrees in the case of 'with exogenous' effect are mostly greater than the optimal global consensus degree for the case of 'no exogenous'. We can conclude that our proposed influence-based feedback mechanism enables us to improve consensus to a sufficient state, by means of the σ-centrality influence measure.

The influence scores are used to select the most influential expert in the network. The chosen network influencer acts as a leader in giving recommendations to the experts with low contribution to consensus. This nomination not only relies on the network connections (consensus contribution) contructed in SSIN, but the external evaluations from the third parties are also taken into consideration. It is obviously shown that the utilization of exogenous effect in influence score measure positively effects the consensus reaching process.

Chapter 7

Influence-driven Resolution Process

This section describes two necessary phases involved in an influence-driven resolution process: (1) fusion phase; and (2) exploitation phase. These phases are described below.

7.1 Influence-based Preference Fusion

A brief review on the fusion of preferences is previously presented in Section 2.5.1 (Page 22). We focus on the development of new fusion operator based on the proposal of the IOWA operator by Yager and Filev [1].

The influence score of each expert, Y, obtained using Definition 6.2.2 (Page 70) is introduced in this context as the order inducing variable of the experts' preference evaluations to fuse, $\{p^1_{ij}, \ldots, p^m_{ij}\}$, which leads to the following *σ-IOWA operator*:

Definition 7.1.1. The σ-IOWA operator of dimension n, Φ^{σ}_{W}, is an IOWA operator with the set of influence score of experts in the network, $Y = (y^1, \ldots, y^n)$, as the order inducing variable.

Thus, denoting by W the weighting vector calculated using Equation 2.4 (Page 24) with Q a fuzzy linguistic quantifier (Page 23) representing the the concept of soft majority desired to implement, *collective preference relation* that derived using the σ-IOWA operator, Φ_W^σ, will be

$$p_{ij}^c = \Phi_W^\sigma \left(\langle y^1, p_{ij}^1\rangle, \ldots, \langle y^n, p_{ij}^n\rangle\right). \tag{7.1}$$

Clearly, the higher the influence score, the more influence an expert has in the network, and consequently the higher the contribution (weighting value) of that expert in the fusion process. Indirectly, the implication of less influential experts can be mitigated.

This can be achieved by implementing the concept of fuzzy majority via an increasing concave linguistic quantifier Q [6]. Yager [76] proposed the following parameterized family of RIM quantifiers $Q(r) = r^a,\ a \geq 0$ (Page 23) to model the majority concept *'most of'*, which is concave when $a \in [0, 1]$. For illustrative purpose, the value $a = 1/2$ will be used, and the collective preferences represent the degree of "preference of one alternative over another for '*most of*' the influential experts" in the network.

Example 7.1 (Continuation of Examples 6.4)**.** *As shown in Example 6.2 on page 73, the influence score of experts for 'no exogenous' case is,*

$$\begin{aligned} Y = [&e^1 = 2.035, e^2 = 2.003, e^3 = 1.947, e^4 = 1.935, e^5 = 2.014, e^6 = 2.006, \\ &e^7 = 2.034, e^8 = 2.027]. \end{aligned}$$

In order to obtain the weighting vector, W we have to compute the value of $D(1), \ldots, D(8)$ and substitute those values into Equation 2.4 on page 24 as follows:

Rank Y in descending order,

$$\begin{aligned} Rank(Y) = [&e^1 = 2.035, e^7 = 2.034, e^8 = 2.027, e^5 = 2.014, e^6 = 2.006, e^2 = 2.003, \\ &e^3 = 1.947, e^4 = 1.935]. \end{aligned}$$

From Rank Y,

$$D(1) = 2.035$$

$$\vdots$$

$$D(8) = 2.035 + 2.034 + 2.027 + 2.014 + 2.006 + 2.003 + 1.947 + 1.935 = 16.001.$$

$$
\begin{aligned}
W_1 &= Q\left(\frac{D(1)}{D(8)}\right) - Q\left(\frac{D(0)}{D(8)}\right) = Q\left(\frac{2.035}{16.001}\right) - Q\left(\frac{0}{16.001}\right) \\
&= Q\,(0.127) - Q\,(0) = (0.127)^{\frac{1}{2}} - 0 = 0.357 \\
&\vdots \\
W_8 &= Q\left(\frac{D(8)}{D(8)}\right) - Q\left(\frac{D(7)}{D(8)}\right) = Q\left(\frac{16.001}{16.001}\right) - Q\left(\frac{14.066}{16.001}\right) \\
&= Q\,(1) - Q\,(0.879) = (1)^{\frac{1}{2}} - (0.879)^{\frac{1}{2}} = 0.062.
\end{aligned}
$$

From above computations, the weighting vector, W is

$$(0.357, 0.147, 0.113, 0.095, 0.083, 0.075, 0.068, 0.062)\,.$$

Thus, the collective preference relations, derived using the σ-IOWA operator (Equation 7.1 on page 83) for p^c_{12} can be computed as:

$$
\begin{aligned}
p^c_{12} = \Phi^{\sigma}_{W}\ &(\langle 2.035, p^1_{12}\rangle, \langle 2.003, p^2_{12}\rangle, \langle 1.947, p^3_{12}\rangle, \langle 1.935, p^4_{12}\rangle, \\
&\langle 2.014, p^5_{12}\rangle, \langle 2.006, p^6_{12}\rangle, \langle 2.034, p^7_{12}\rangle, \langle 2.027, p^8_{12}\rangle) \\
&= \left(0.357 \times p^1_{12}\right) + \left(0.147 \times p^7_{12}\right) + \left(0.113 \times p^8_{12}\right) + \left(0.095 \times p^5_{12}\right) \\
&\left(0.083 \times p^6_{12}\right) + \left(0.075 \times p^2_{12}\right) + \left(0.068 \times p^3_{12}\right) + \left(0.062 \times p^4_{12}\right) \\
&= (0.357 \times 0.4) + (0.147 \times 0.55) + (0.113 \times 0.7) + (0.095 \times 0.34) \\
&(0.083 \times 0.13) + (0.075 \times 0.7) + (0.068 \times 0.69) + (0.062 \times 0.1) \\
&= 0.451.
\end{aligned}
$$

The complete computation of the collective preference relations, P^C is executed and presented as follows:

$$
P^C = \begin{bmatrix}
1 & 0.451 & 0.500 & 0.681 & 0.661 & 0.678 \\
0.549 & 1 & 0.595 & 0.749 & 0.658 & 0.783 \\
0.500 & 0.405 & 1 & 0.727 & 0.666 & 0.692 \\
0.319 & 0.251 & 0.273 & 1 & 0.557 & 0.562 \\
0.339 & 0.342 & 0.334 & 0.443 & 1 & 0.717 \\
0.322 & 0.217 & 0.308 & 0.438 & 0.283 & 1.
\end{bmatrix}
$$

7.2 Ranking of Alternatives

The second procedure involves in the influence-driven resolution process is the exploitation phase. This is necessary to be carried out in order to rank the alternatives so that the best one can be identified. A review on the ranking of alternatives is previously given in Section 2.5.2 (Page 25).

After the collective preference relations, P^C, are computed, the *Quantifier Guided Dominance Degree* (QGDD) [68] (Definition 2.5.3 on page 25) based on the utilization of the OWA operator (Definition 2.5.1 on page 23 and Equation 2.3 on page 24) guided by the linguistic quantifier Q is applied.

We make use of the maximal dominance set concept (Equation 2.5 on page 26) to choose the best alternative, meaning that *'most of'* the influential experts in SSIN have high contribution towards consensus, their individual preferences are appropriately considered and the decision made are accepted by the whole group of experts.

Algorithm 6 shows the consecutive steps of the proposed influence-driven resolution process.

begin
- Find IOWA weighting vector using Equation (2.4) ;
- Rank $Y = (y^1, \dots, y^n)$ in descending order, $R(Y)$;
- Determine p^c_{ij} (Equation (7.1)) by making use of σ-IOWA operator;
- Construct the collective preference matrix, $P^C = \left(p^c_{ij}\right)$;
- Compute $QGDD(A_i)$ based on Definition 2.5.3;
- Rank the alternatives and choose the best one using Equation (2.5).

end

Algorithm 6: Influence-driven Resolution Process

Example 7.2 (Continuation of Examples 7.1)**.** *The computational example to rank the alternatives begins with finding the OWA weights (Equation 2.3 on page 24).*

$$w_1 = Q\left(\frac{1}{5}\right) - Q\left(\frac{0}{5}\right) = Q(0.2) - Q(0) = (0.2)^{\frac{1}{2}} - (0) = 0.447$$
$$\vdots$$
$$w_5 = Q\left(\frac{5}{5}\right) - Q\left(\frac{4}{5}\right) = Q(1) - Q(0.8) = (1)^{\frac{1}{2}} - (0.8)^{\frac{1}{2}} = 0.106.$$

Thus, the OWA weighting vector, w, to rank the alternatives is

$$(0.447, 0.185, 0.142, 0.120, 0.106).$$

From first row of P^C, the collective elements for alternative A_1 (excluding p^c_{11}) are 0.451, 0.500, 0.681, 0.661, 0.678. These values are then ranked in descending order.

$$Rank\ (P^c_1) = [0.681, 0.678, 0.661, 0.500, 0.451].$$

Next, the QGDD (Definition 2.5.3 on page 25) for each alternative can be calculated and the example of $QGDD\,(A_1)$ is calculated as follows:

$$\begin{aligned} QGDD\,(A_1) = {} & \Phi_Q\ \left(\left\langle 0.447, p^c_{1,4}\right\rangle, \left\langle 0.185, p^c_{1,6}\right\rangle, \left\langle 0.142, p^c_{1,5}\right\rangle,\right. \\ & \left.\left\langle 0.120, p^c_{1,3}\right\rangle, \left\langle 0.106, p^c_{1,2}\right\rangle\right) \\ & = (0.447 \times 0.681) + (0.185 \times 0.678) + (0.142 \times 0.661) + (0.120 \times 0.500) + \\ & (0.106 \times 0.451) \\ & = 0.632. \end{aligned}$$

The same computations are carried out for the rest of alternatives and presented as:

$$\mathbb{A}^{QGDD} = \{0.632, 0.712, 0.651, 0.459, 0.527, 0.356\}.$$

Thus, the alternatives are ranked as the following:

$$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_6.$$

Based on the Equation 2.5 (page 26), the final ranking solution is A_2.

From Example 4.1 (Page 46), we concern that a committee of eight (8) heads of department in ABC company gives their evaluations over six (6) potential candidates for a nomination of the best employer of the year 20XX. After considering all individuals heads of departments' preferences, the best employer of the year 20XX is A_2. This final decision is accepted and agreed by the whole group of experts because they already achieved a sufficient consensus level.

The results of influence scores, IOWA weighting vectors, collective preferences, maximal dominance degrees and final ranking of alternatives for both cases ('no exogenous' and 'with exogenous') are presented in Table 7.1.

Referring to Table 7.1, all values obtained in the case of 'with exogenous' effect are slightly different with the case of having 'no exogenous' factor. When external (exogenous) factor is involved, the best employer of the year 20XX is remained A_2*. But the overall ranking of alternatives is* $A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_6$*. This means that the the exogenous factor affects the final ranking of alternatives.*

As in Perez et al. [95], the ranking of alternatives obtained by their work is $A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_6$*. This different result is expected because of the application of social influence in Perez et al. [95], which was implemented in a different context and no external factor was involved in their study.*

TABLE 7.1: The influence scores, IOWA weighting vectors, collective preferences, maximal dominance degrees and final ranking of alternatives for both cases ('no exogenous' and 'with exogenous')

	Influence scores	IOWA Weighting Vectors	Collective Preferences	Maximal Dominance Degrees	Ranking of Alternatives
No exogenous	$Y = (2.035, 2.003, 1.947, 1.935, 2.014, 2.006, 2.034, 2.027)$	$w = (0.357, 0.147, 0.113, 0.095, 0.083, 0.075, 0.068, 0.062)$	$\begin{bmatrix} 1 & 0.451 & 0.500 & 0.681 & 0.661 & 0.678 \\ 0.549 & 1 & 0.595 & 0.749 & 0.658 & 0.783 \\ 0.500 & 0.405 & 1 & 0.727 & 0.666 & 0.692 \\ 0.319 & 0.251 & 0.273 & 1 & 0.557 & 0.562 \\ 0.339 & 0.342 & 0.334 & 0.443 & 1 & 0.717 \\ 0.322 & 0.217 & 0.308 & 0.438 & 0.283 & 1 \end{bmatrix}$	$QGDD = (0.632, 0.712, 0.651, 0.459, 0.527, 0.356)$	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_6$
With exogenous	$Y = (1.444, 1.130, 1.096, 0.676, 1.536, 1.128, 1.647, 1.343)$	$w = (0.361, 0.147, 0.113, 0.096, 0.084, 0.074, 0.065, 0.060)$	$\begin{bmatrix} 1 & 0.463 & 0.496 & 0.671 & 0.651 & 0.687 \\ 0.537 & 1 & 0.584 & 0.738 & 0.641 & 0.788 \\ 0.504 & 0.416 & 1 & 0.722 & 0.664 & 0.694 \\ 0.329 & 0.262 & 0.278 & 1 & 0.559 & 0.565 \\ 0.349 & 0.359 & 0.336 & 0.441 & 1 & 0.718 \\ 0.313 & 0.212 & 0.306 & 0.435 & 0.282 & 1 \end{bmatrix}$	$QGDD = (0.707, 0.650, 0.632, 0.531, 0.464, 0.352)$	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_6$

Part IV

COMPLEXITY, APPLICATION AND ANALYSIS

Chapter 8

Complexity and Application of the Proposed Model

This chapter focuses on the computational complexity and the relevant applications of the proposed model.

8.1 Computational Complexity

The computational complexity of our proposed model is determined based on each algorithm involved as listed in the following:

- *Algorithm 1 (Page 50)*:
  The construction of the geo-uninorm consistent RPR (Step 1) involved a loop for $j > (i + 1)$ and $j < i$, thus the computation time is $O(n)$. Step 2 requires 1 multiplication and 1 division, thus the operation order is $O(1)$. Step 3 until 7 requires $O(n^2)$, which is the number of nesting in the loops of the personalized consistency control module. Thus, the total operation involved in Algorithm 1 is $O(n) + O(1) + O(n^2) = O(n^2)$.
- *Algorithm 2 (Page 59)*:
  The agglomerative hierarchical clustering with complete linkage function has the worst case time complexity at most $O(n^2 \ \log \ n)$ [140, 141, 142]. One $O(n^2 \ \log \ n)$ algorithm is to calculate the n^2 distance metric and then to sort each data point distance (overall time: $O(n^2 \ \log \ n)$). This distance

metric can be updated in $O(n)$ after each merging iteration. The next pair is picked and merged based on its smallest distance. This traversing step for the n sorted lists of distances produces n^2 steps at the end of the clustering procedure. By adding these orders, the complexity computation of this algorithm is $O(n^2 \ \log \ n)$.

- *Algorithm 3 (Page 63)*:
  In this algorithm, Step 1 until 4 has 1 addition and 1 division operation, and one nesting in the loop in Step 5 and 6. Therefore, the complexity time is $O(n)$.

- *Algorithm 4 (Page 77)*:
  There exist 1 addition and 1 division operation in Step 3, a matrix multiplication in Step 5 and 1 *maximum* operation in Step 9. Thus, the total operation time for this algorithm is $O(n^2)$.

- *Algorithm 5 (Page 79)*:
  There exist a loop in the procedure of second consensus round with a combination of two previous algorithms (Algorithm 2 and 3). Thus the total computation order is $O(n) + O(n^2 \ \log \ n) + O(n) = O(n^2 \ \log \ n)$.

- *Algorithm 6 (Page 85)*:
  This algorithm involved n multiplications and 1 *maximum* operation, produces $O(n+1)$ computation order.

According to the above analysis, the complexity computation of our proposed framework is:

$$O\left(n^2\right)+O\left(n^2 \ \log \ n\right)+O(n)+O\left(n^2\right)+O\left(n^2 \ \log \ n\right)+O(n+1) = O\left(n^2 \log \ n\right).$$

8.2 Relevant Applications

Generally, our proposed model is applicable in solving decision making problems, which requires a sufficient agreement level of a group of experts before the final decision is made. This decision making model allows bi-level experts' evaluation processes, where the internal and external (third party) opinions will also be taken into consideration.

Let assume that a government wants to implement new laws for the country. At the first place, the government needs to know about the public opinion regarding the proposed laws before it will be discussed further in the parliament. The simplest way to examine public opinion is via social media platforms, such as voting or polling using Facebook and Twitter. Other than 'Yes' or 'No' or a *Likert-scale*, various preference representation formats can be used for the purpose of expressing public opinions, such as *ordering, preference relations and utility functions.*

Thousands or millions of public opinion can be collected and the agreement (consensus) of their opinion can be measured using our preference similarity network clustering based consensus procedure. It is well-known that the clustering methodologies have capabilities in handling large user-base data, so as our proposed model. The implementation of the clustering algorithm can help the government to visualise public opinion similarities, whether between cities or states.

If the public consensus level is insufficient, the low contribution to consensus people, which disagree about the implementation of new laws is identified using our feedback mechanism process. The identified people will be asked to change their opinion towards the direction of a group consensus. This can be done by generating advises and guiding them on how to change their opinion, according to their leader's opinion. The *leader* is automatically appointed by the feedback system, considering public opinion (internal factor) and the members of parliament (external factor). He/she must be the most trusted or influenced person in the public network and also trusted by the parliament members.

After modifications of public opinion are considered, a sufficient consensus level will be achieved. The decision on which laws need to be firstly implemented can be obtained after considering both public opinion and members of parliament preferences.

Other than this application, our proposed framework can be utilized in judging a prestige entertainment award. The nominated actors or actresses will be voted by their public fans and judging by the professional judges/panels. The final decision made portrays both parties agreement, meaning that all individual preferences are appropriately taken into considerations.

In addition, our proposed model also can be applied for marketing purposes. For instance, a company wants to introduce new products. The company can identify certain customers, who can act as *influencers* based on their history of similarity preferences on social media networks. The company can give those products for free to the identified network influencers as trial packs. It is expected to have positive recommendations from the influencers and the company will gain benefit on it.

It is not limited to the above applications, but it is suitable to be implemented for the purpose of handling selection problems, investment, policy making, marketing and many more.

Chapter 9

Comparative Evaluations

This chapter focuses on the comparative analysis of our proposed model with previous studies in the literature.

9.1 Impact of Consistency Control Module Towards Consensus

From Kamis et al. [125], the comparison of results has been done between our proposed consistency control module with Chu et al.'s [98] work. As depicted in Table 9.1, five main elements are identified and the respective analyses are discussed in the following points:

1. Both techniques provided different consistency degrees because:
    - Chu focused on the reciprocity and additive consistency properties for a collective preference relation. However, we construct the consistent RPR based on the geo-uninorm operator, which is related to multiplicative consistency, rather than the additive consistency.
    - In obtaining consistency degree, we measure the similarity between the initial experts' preferences with the generated geo-uninorm consistent preferences. Chu measured consistency by comparing the similarity of individual expert's preferences with collective ones.

TABLE 9.1: Comparative results of consistency control module

Elements	**Chu et. al**[98]	**Proposed model**
Consistency degrees	$CCD\left(e^1\right) = 0.9400$, $CCD\left(e^2\right) = 0.8533$, $CCD\left(e^3\right) = 0.8311$, $CCD\left(e^4\right) = \mathbf{0.7089}$, $CCD\left(e^5\right) = 0.8289$, $CCD\left(e^6\right) = 0.8400$, $CCD\left(e^7\right) = 0.8422$, $CCD\left(e^8\right) = \mathbf{0.7511}$.	$CCD\left(e^1\right) = 0.8307$, $CCD\left(e^2\right) = 0.9513$, $CCD\left(e^3\right) = 0.9182$, $CCD\left(e^4\right) = \mathbf{0.7462}$, $CCD\left(e^5\right) = 0.8788$, $CCD\left(e^6\right) = 0.9435$, $CCD\left(e^7\right) = 0.9109$, $CCD\left(e^8\right) = \mathbf{0.7212}$.
Inconsistent expert(s) (Threshold 0.8)	e^4, e^8	e^4, e^8
Controlled parameter	$\gamma = 0.6$ for both e_4, e_8	Personalized: $\gamma(e^4) = 0.2,\ \gamma(e^8) = 0.2$
Feedback preferences	$CP^4 = \begin{bmatrix} 1 & \mathbf{0.3133} & 0.24 & 0.4877 & 0.6733 & 0.6867 \\ 0.6867 & 1 & \mathbf{0.3067} & 0.7333 & 0.32 & 0.4533 \\ 0.76 & 0.6933 & 1 & \mathbf{0.6867} & 0.27 & 0.2867 \\ 0.5133 & 0.2667 & 0.3133 & 1 & \mathbf{0.8067} & 0.7 \\ 0.3267 & 0.68 & 0.7267 & 0.1933 & 1 & \mathbf{0.5733} \\ 0.3133 & 0.5467 & 0.7133 & 0.3 & 0.4267 & 1 \end{bmatrix}$ $CP^8 = \begin{bmatrix} 1 & \mathbf{0.44} & 0.3667 & 0.3467 & 0.5333 & 0.6133 \\ 0.56 & 1 & \mathbf{0.3067} & 0.7067 & 0.2933 & 0.4333 \\ 0.6333 & 0.6933 & 1 & \mathbf{0.66} & 0.2467 & 0.2667 \\ 0.6533 & 0.2933 & 0.34 & 1 & \mathbf{0.8067} & 0.7067 \\ 0.4667 & 0.7067 & 0.7533 & 0.1933 & 1 & \mathbf{0.58} \\ 0.3867 & 0.5667 & 0.7333 & 0.2933 & 0.42 & 1 \end{bmatrix}$	$CP^4 = \begin{bmatrix} 1 & \mathbf{0.2} & 0.12 & 0.4773 & 0.84 & 0.84 \\ 0.8 & 1 & \mathbf{0.2} & 0.82 & 0.36 & 0.52 \\ 0.88 & 0.8 & 1 & \mathbf{0.8} & 0.28 & 0.28 \\ 0.5227 & 0.18 & 0.2 & 1 & \mathbf{1} & 0.84 \\ 0.16 & 0.64 & 0.72 & 0 & 1 & \mathbf{0.6} \\ 0.16 & 0.48 & 0.72 & 0.16 & 0.4 & 1 \end{bmatrix}$ $CP^8 = \begin{bmatrix} 1 & \mathbf{0.4} & 0.298 & 0.3333 & 0.68 & 0.76 \\ 0.6 & 1 & \mathbf{0.2} & 0.82 & 0.36 & 0.52 \\ 0.702 & 0.8 & 1 & \mathbf{0.8} & 0.28 & 0.28 \\ 0.6667 & 0.18 & 0.2 & 1 & \mathbf{1} & 0.84 \\ 0.32 & 0.64 & 0.72 & 0 & 1 & \mathbf{0.6} \\ 0.24 & 0.48 & 0.72 & 0.16 & 0.4 & 1 \end{bmatrix}$
Revised consistency indexes	$CCD_{\gamma=0.6}\left(e^4\right) = 0.8253$, $CCD_{\gamma=0.6}\left(e^8\right) = 0.8507$	$CCD_{\gamma=0.2}\left(e^1\right) = 0.8363$, $CCD_{\gamma=0.2}\left(e^2\right) = 0.8207$

2. Identification of inconsistent experts with respect to the consistency threshold:

    - Both approaches had the same number and the same inconsistent experts (e^4 and e^8) at consistency threshold of 0.8. However, if the threshold value had been set as 0.83, for example, our proposed model still has the same inconsistent expert, but Chu added expert e^5 in the inconsistent experts list because the consistency degree of expert e^5 (0.8289) was less than 0.83. We pointed out that when the threshold value is increased, more notable differences between the approaches are apparent.

3. Consistency control parameter in feedback mechanism:

    - Chu's work required the same parameter value ($\gamma = 0.6$) in controlling the preference change recommendations for all inconsistent experts.
    - In contrast, we provided personalized recommendations of change, depending on the experts' personal level of inconsistency. We restricted ourselves to the set of discrete values of γ: $\{0.1, 0.2, \ldots, 0.9, 1\}$ for illustrative purposes, so that the proposed model returns both experts with the same value of $\gamma = 0.2$.
    - Since $\gamma = 0.2$ is lower than $\gamma = 0.6$, we can say that our proposed model guaranteed consistency by advising inconsistent expert(s) to modify their preferences only with minimum changes, compared to the recommendations from Chu's proposal.

4. Different revised preferences between the two models:

    - We proposed a uninorm-based construction of consistent preference relations that makes use of $(m-1)$ original preference relation values, which remain unchanged in the next stage of the consistency process. This is not the case in the method implemented by Chu, since inconsistent experts are advised to change all of their preferences.
    - Therefore, the proposed approach is shown to be less expensive, not only in terms of the magnitude of the change recommended, but also computationally because a lower number of changes is required for an inconsistent expert to achieve the consistency threshold.

5. Both methods achieved a sufficient consistency level after the feedback mechanism is implemented:

- Both methods successfully improved the consistency level of the inconsistent experts to be above the threshold.
- However, our model is more efficient than Chu's work because consistency is achieved with only minimum changes required, in such a way that the inconsistent experts willingly accept those recommendations by ensuring that their initial preferences are still taken into consideration.

Another comparative result was presented in Kamis et al. [125] for the purpose of validating the effectiveness of the proposed consistency control module towards a sufficient consensus state. Our preference similarity network clustering based consensus model is implemented with and without the consistency control module and the following results were obtained (see Table 9.2).

TABLE 9.2: Comparison of results in analyzing the impact of consistency control module towards consensus

<table>
<tr><th>Elements</th><th>With consistency module</th><th>Without consistency module [8]</th></tr>
<tr><td>Consensus - 1st round (Threshold = 0.9)</td><td>0.901</td><td>0.893</td></tr>
<tr><td>Feedback consensus (2nd round)</td><td>Not required</td><td>
<table>
<tr><th>β</th><th>Consensus Index</th><th>Cluster Level</th></tr>
<tr><td>0</td><td>0.893</td><td>4</td></tr>
<tr><td>0.1</td><td>0.907</td><td>4</td></tr>
<tr><td>0.2</td><td>0.921</td><td>4</td></tr>
<tr><td>0.3</td><td>0.934</td><td>4</td></tr>
<tr><td>0.4</td><td>0.946</td><td>4</td></tr>
<tr><td>0.5</td><td>0.956</td><td>6</td></tr>
<tr><td>0.6</td><td>0.965</td><td>4</td></tr>
<tr><td>0.7</td><td>0.972</td><td>4</td></tr>
<tr><td>0.8</td><td>0.976</td><td>7</td></tr>
<tr><td>0.9</td><td>0.978</td><td>7</td></tr>
<tr><td>1</td><td>0.973</td><td>2</td></tr>
</table>
</td></tr>
</table>

Summarizing, we show that:

- First consensus round – The proposed consistency control module positively contributes to achieving a sufficient consensus level. It is obvious in Table 9.2, the consensus degree is above the threshold value when consistency is checked and improved, meaning that experts' consistency also contributed to expert's agreement. Without the consistency module, the consensus level is insufficient (lower than the threshold).
- Second consensus round – The activation of a consensus feedback mechanism process was not required when the consistency control module was implemented. Otherwise, a second round of consensus was needed, where it is expected to be more tedious.

9.2 Impact of Social Network Connections in GDM

As discussed in our published paper [8], the comparative results on the impact of social network connections in group decision making with Chu et al.'s work [98] were presented, consisting of 3 main elements: (1) Aggregation weighting vector; (2) collective preferences; and (3) ranking of alternatives. Referring to Table 9.3, the following conclusions are drawn:

1. A social network provides an impact on the generation of experts' associated weights:
    - The no network connection weighting vector is obviously different from the directed network and both of the undirected weighted preference similarity networks. This is because the weights are directly assumed by the authors in [98] since the experts are considered completely independent from each other.
    - Otherwise, weights for experts linked by networks are generally obtained from experts' centrality measures.
    - For the directed network in [98], the I-IOWA operator based on in-degree and out-degree centrality indices was developed with experts' weights being higher the higher their associated centrality indices.

TABLE 9.3: Comparative results on the social network connections in group decision making

	Weights vector	**Collective preferences**	**Ranking of alternatives**
No network connection [98]	(0.124, 0.123, 0.126, 0.125, 0.127, 0.122, 0.125, 0.128)	$\begin{bmatrix} 1 & 0.4579 & 0.3145 & 0.5289 & 0.5122 & 0.6248 \\ 0.5421 & 1 & 0.2760 & 0.6800 & 0.4150 & 0.5110 \\ 0.6982 & 0.6986 & 1 & 0.5187 & 0.2393 & 0.3074 \\ 0.4711 & 0.3000 & 0.4813 & 1 & 0.6523 & 0.5503 \\ 0.4878 & 0.5850 & 0.7607 & 0.3477 & 1 & 0.4698 \\ 0.3502 & 0.5040 & 0.6926 & 0.4497 & 0.5302 & 1 \end{bmatrix}$	$x_5 \succ x_6 \succ x_4 \succ x_3 \succ x_1 \succ x_2$
Directed network [98]	(0.408, 0.150, 0.103, 0.089, 0.079, 0.061, 0.061, 0.048)	$\begin{bmatrix} 1 & 0.3702 & 0.2540 & 0.4998 & 0.6169 & 0.6570 \\ 0.6298 & 1 & 0.3006 & 0.7056 & 0.3579 & 0.4706 \\ 0.7460 & 0.6988 & 1 & 0.5878 & 0.2725 & 0.2720 \\ 0.5002 & 0.3132 & 0.4122 & 1 & 0.7144 & 0.6297 \\ 0.3831 & 0.6421 & 0.7275 & 0.2856 & 1 & 0.5223 \\ 0.3336 & 0.5388 & 0.7280 & 0.3703 & 0.4777 & 1 \end{bmatrix}$	$x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_6 \succ x_1$
Undirected weighted preference similarity network (Not enough consensus)	(0.360, 0.149, 0.113, 0.095, 0.082, 0.073, 0.067, 0.061)	$\begin{bmatrix} 1 & 0.3958 & 0.2821 & 0.4985 & 0.5704 & 0.6421 \\ 0.6042 & 1 & 0.2940 & 0.6872 & 0.3523 & 0.4892 \\ 0.7179 & 0.6914 & 1 & 0.5899 & 0.2639 & 0.3044 \\ 0.5015 & 0.3456 & 0.4101 & 1 & 0.7033 & 0.6181 \\ 0.4296 & 0.6477 & 0.7361 & 0.2967 & 1 & 0.5122 \\ 0.3415 & 0.5272 & 0.6956 & 0.3819 & 0.4878 & 1 \end{bmatrix}$	$x_5 \succ x_3 \succ x_4 \succ x_2 \succ x_6 \succ x_1$
Undirected weighted preference similarity network (Enough consensus)	(0.356, 0.146, 0.112, 0.094,0.084, 0.075, 0.069, 0.064)	$\begin{bmatrix} 1 & 0.3675 & 0.2600 & 0.4960 & 0.6028 & 0.6558 \\ 0.6325 & 1 & 0.3024 & 0.6926 & 0.3216 & 0.4919 \\ 0.7400 & 0.6976 & 1 & 0.6281 & 0.2757 & 0.3080 \\ 0.5040 & 0.3074 & 0.3719 & 1 & 0.7350 & 0.6538 \\ 0.3972 & 0.6784 & 0.7243 & 0.2650 & 1 & 0.5283 \\ 0.3442 & 0.5081 & 0.6920 & 0.3462 & 0.4717 & 1 \end{bmatrix}$	$x_3 \succ x_5 \succ x_4 \succ x_2 \succ x_6 \succ x_1$

- The formation of the proposed preference similarity network implies the contribution of experts to consensus. Thus, when measuring consensus as well as in the proposed feedback mechanism, the similarity of opinion between experts are increased accordingly because some experts need to change their preferences closer to the network leader in order to achieve sufficient agreement. This might increase the centrality indices, which seems to be the explanation behind the slight difference between the weighting vectors of the undirected weighted preference similarity network without consensus and with sufficient consensus.
- In summary, the proposed novel preference network structure based on similarities between nodes provides similar weighting vectors in the different rounds of consensus, with slight differences reflecting the increase in experts centrality indices.

2. Social network effects the aggregation of preferences from an individual expert to a collective one, leading to an improved level of group consensus:

   - The collective preference relations are all different due to the difference in the aggregation operator used, and also due to the difference in the weighting vectors implemented.

3. Social network connections influence the ranking of alternatives:

   - Alternatives are ranked differently from the others when no network connections are involved, implying that the social network connections truly impact the decision making process. This is because the weightage from the experts' networks are collected, propagated to the others, and carried along the decision making procedure as additional input of the consensus contributions. The higher the expert's network weight, the higher the consensus contribution. Thus, experts opinion are appropriately taken into consideration with respect to their consensus contributions, which indirectly practise a *fair* decision making process.
   - The directed network in [98] and both the proposed undirected weighted preference similarity network without consensus and with sufficient consensus rank the same last three alternatives ($x_2 \succ x_6 \succ x_1$) and slightly differ in the ordering of the first three alternatives. The best alternative solution for the directed and undirected weighted preference similarity network (not enough consensus) are x_4 and x_5, respectively, with the

difference due to the different ranking approaches they use. The derivation of the priority weighting vector by Fedrizzi and Brunelli [143] was applied in the directed network connection in [98], while the dominance guided choice degree with fuzzy linguistic quantifier *'most of'* was applied in the proposed undirected weighted preference similarity network (not enough consensus).

- The ranking of alternatives for the proposed undirected weighted preference similarity networks with enough consensus is slightly different to the undirected weighted preference similarity networks (not enough consensus) because the consensus feedback process introduced changes in the individual preferences of half of the experts leading to the acceptance of the decision by the group as a whole.
- We can say that the proposed cluster based consensus measure gives some flexibility to the experts to revise their opinion for the sake of achieving sufficient group agreement and obtaining a good solution to satisfy them all.

9.3 Non-clustering Versus Clustering Based Consensus

For the purpose of validating our proposed clustering based consensus model in this thesis, we provide the second round of consensus solution at optimal advice control paremeter, $\beta_{0.3}$ (refer Table 6.1 on page 80) for both non-clustering (α_8) and clustering based method ('with exogenous') at optimal clustering level, α_2. At α_8, experts are not belong to any clusters or connected to other group members. Meaning that no clustering procedure is involved at this α-level.

By referring to the Table 9.4, several conclusions can be drawn.

TABLE 9.4: Comparison results of non-clustering and clustering based consensus models

Elements	Non-clustering based model	Clustering based model
Internal cohesions	$\delta_{int}\left(e^1\right) = 1$, $\delta_{int}\left(e^2\right) = 1$, $\delta_{int}\left(e^3\right) = 1$, $\delta_{int}\left(e^4\right) = 1$, $\delta_{int}\left(e^5\right) = 1$, $\delta_{int}\left(e^6\right) = 1$, $\delta_{int}\left(e^7\right) = 1$, $\delta_{int}\left(e^8\right) = 1$	$\delta_{int}\left(C_1\right) = 0.958$, $\delta_{int}\left(C_2\right) = 0.910$
External cohesions	$\delta_{ext}\left(e^1\right) = 0.901$, $\delta_{ext}\left(e^2\right) = 0.877$, $\delta_{ext}\left(e^3\right) = 0.874$, $\delta_{ext}\left(e^4\right) = 0.875$, $\delta_{ext}\left(e^5\right) = 0.877$, $\delta_{ext}\left(e^6\right) = 0.868$, $\delta_{ext}\left(e^7\right) = 0.912$, $\delta_{ext}\left(e^8\right) = 0.897$	$\delta_{ext}\left(C_1\right) = 0.873$, $\delta_{ext}\left(C_2\right) = 0.873$
Cluster(s)/expert(s) consensus	$\delta_{CC}\left(e^1\right) = 0.914$, $\delta_{CC}\left(e^2\right) = 0.892$, $\delta_{CC}\left(e^3\right) = 0.890$, $\delta_{CC}\left(e^4\right) = 0.890$, $\delta_{CC}\left(e^5\right) = 0.892$, $\delta_{CC}\left(e^6\right) = 0.885$, $\delta_{CC}\left(e^7\right) = 0.923$, $\delta_{CC}\left(e^8\right) = 0.910$	$\delta_{CC}\left(C_1\right) = 0.905$, $\delta_{CC}\left(C_2\right) = 0.896$,
Consensus degree - Consensus threshold = 0.9	0.899	0.901
Next consensus round	Required	Not required
Experts with low consensus contribution	e^2, e^3, e^4, e^5, e^6	None

- The clustering-based consensus model provides practical methodology compared to the non-clustering approach.
  The tediousness of computations (the internal, external, cluster consensus etc.) can be reduced when the clustering-consensus model is implemented. It is because the clustering-consensus technique presents group-based computation, while non-clustering focuses on the individual context. When large number of experts or alternatives are involved in the decision making process, the clustering based model is believed to be manageable, easy to be implemented and inaccuracy (error) can be minimised.

- Clustering-based consensus approach produces better consensus solution than the non-clustering consensus measure.
  The consensus degree obtained by the clustering-based model is higher than the non-clustering. It is shown that the consensus level for the clustering consensus procedure is sufficient (above the consensus threshold), thus no further consensus round is required. In the case of non-clustering consensus model, the experts with low contribution to consensus (e^2, e^3, e^4, e^5, e^6) need to be advised in the feedback mechanism. In addition, the third consensus round need to be activated in order to improve the group consensus level.

9.4 Proposed Model With Respect to the Existing Literature

In general, the main advantages of the proposed model and its differences with respect to previous studies in the literature are presented as follows:

(i) Our propose geo-uninorm consistency operator is introduced for the purpose of modelling transitivity, under the uninorm-based concepts. We prove that the geo-uninorm operator captures properties from both the geometric mean operator and the cross ratio uninorm operator. This work technically distincts from other existing uninorm-based operators, such as in [34, 144, 145].

(ii) Our preference network is constructed by incorporating generated weights from the similarity of experts' preferences based on the structural equivalence concept. This measure conceptually differs from previous work done in most similarity-based consensus models [24, 26] because they do not consider any network criterion, such as the connected ties and structural classes.

(iii) Our proposal provides an alternative solution to expert weight derivation, which overcomes the assumption that the weights of experts are known beforehand [20]. Meanwhile, the absence of a weighting vector does not necessarily mean that all of the experts are equally important. Indeed, once experts provide opinions, their preference similarities can be used to derive weighting values. This is an advantage in measuring consensus, as more importance will be given to experts with higher centrality indices/influence scores.

(iv) The cluster-based consensus model based on the proposed defined internal and external cohesions, cluster consensus and level consensus is one of the first efforts in deriving cluster-based group consensus model in decision making. Previous works done by Garcia-Lapresta and Perez-Roman [114, 115, 116], Abel et al. [117] and Li eta al. [118] focused on different contexts of clustering-based consensus.

(v) The use of the influence score to determine the network influencer is defined differently from Kamis et al. [8]. In this thesis, the identification procedure of a leader (network influencer) based on σ-centrality is introduced as an additional step in the feedback mechanism. The group consensus level is guaranteed to increase when experts are advised to get closer to the network influencer's preferences.

(vi) The integration of SIN in this proposed model is carried out from different perspectives from those in references [95, 123, 35, 105]. We reformulate the SIN concept and utilize it in the consensus feedback mechanism, instead of focusing on the evolution of preferences and estimation of missing information.

(vii) A new σ-IOWA aggregation operator is introduced, which produces ordering of the argument values based upon the influence score associated with each expert. This provides an alternative information fusion approach in the resolution process.

Part V

SUMMARY

Chapter 10

Conclusion and Future Work

10.1 Summary of Contributions

In this last chapter, we summarise the results obtained in this thesis with some conclusions derived from them and suggests directions for future works.

We have developed a novel consistent preference similarity network clustering and influence based group decision making model. From the development of this process, we extract the following conclusions:

- *Conclusion 1 – Through the introduction of geo-uninorm consistency operator, the correctness of expert preferences is secured and with less expense.*

  The transitivity property has been suggested in order to model consistency because of its hierarchy status with other basic properties of pairwise comparisons, asymmetry and indifference. Our proposed geo-uninorm consistency operator is a hybrid operator that is obtained by combining the best of the geometric average, a mean operator that assures that moderate stochastic transitivity is satisfied, and of the cross-ratio uninorm, which allows for the mean reinforcement property. In the RFPR context, these properties secured the consistency of experts' preferences and misleading solution in decision making can be avoided. In addition, validation via comparison with the existing study in [98] (Section 9.1 on page 96) proved that the proposed consistency module is less expensive, not only for the magnitude of the change recommended, but also computationally because a smaller number of changes is required for an inconsistent expert to achieve the consistency threshold.

- *Conclusion 2 – The consistency control module guarantees consistency only with minimum changes and provides fair individual recommendations.*

  The proposed geo-uninorm consistency measure allowed the building of a consistency control module based on a personalized feedback mechanism to be implemented when the consistency level is insufficient. The recommendation is personally generated for the specific identified inconsistent experts depending on their current individual level of inconsistency. Only minimum changes are considered for the purpose of improving consistency. This consistency control module effectively handled the feedback and advice generation process, in such a way that experts willingly changed their preferences without any pressure because the recommendation is considered as *fair* and appropriate to be accepted.

- *Conclusion 3 – Reformulations of SNA and SIN concepts in CGDM context have been successful.*

  We bridge a gap between SNA and SIN with CGDM frameworks by defining new terminologies and proposing algorithms related to experts preferences: similarity measure, structural equivalence, network structure, cohesion subgroups, influence and consensus measure.

- *Conclusion 4 – The clustering methodology helps in visualisation of the preference similarity network structure.*

  A group of experts is partitioned into subgroups, by means of structural equivalence relations using an agglomerative hierarchical clustering algorithm. A clustering solution, named as dendogram is used as a visualization tool to envision current similarity network pattern.

- *Conclusion 5 – The proposed clustering based consensus measure assures experts' homogeneity, leading to sufficient group agreement.*

  The use of the structural equivalence concept in representing the clustering solution guaranteed the homogeneity of experts preferences because experts are strongly connected with each other within the same cluster, rather than outsider experts. This means that experts are strongly connected and have very similar preferences among the clusters' members. It is logical that consensus (agreement) can easily be achieved when we are sorrounded by people who have similar opinions, instead of different ones.

- *Conclusion 6 – The proposed advice generation provides a solution on how to modify expert preferences towards sufficient consensus.*

  It is rational to focus on experts within cluster(s) with low consensus contribution and activate an appropriate feedback mechanism by providing them with appropriate recommendations on how to change their preferences to increase consensus. The role of a network influencer as a leader urges the experts with low contribution to consensus to modify their preferences closer to each other. The recommendation rule of change with a minimal control parameter generates advice to the identified experts, thus they are directed to a sufficient consensus level.

- *Conclusion 7 – The constructed preference network provides positive contribution to the network influencer.*

  The experts' preference similarity network, which is constructed based on SNA concept generates more weightage on the network influencer preferences than the others. This situation makes advises or recommendations propagated from the direction of the network influencer to the other experts in the network.

- *Conclusion 8 – The influence-driven feedback system improves consensus state and provides better clustering solution.*

  The influence-driven feedback mechanism positively contribute in achieving sufficient consensus state, when experts with low contribution to consensus are succesfully moved closer to each other, following recommendations from a network influencer. The updated preferences are then utilized in the second consensus round and the revised global cluster consensus degree of the group of experts, satisfying two important conditions is determined. These conditions are needed to ensure that the revised consensus degree is above the consensus threshold and the clustering solution is improved.

- *Conclusion 9 – The influence model allows external opinions as decision making contribution.*

  The σ-centrality allows both endogenous (internal) and exogenous (external) factors in expressing experts' importance weightage, where the endogenous element is based on the internal SSIN connections and the exogenous contribution from the third parties evaluations. Incorporation of both factors provides *fair* nomination of a network influencer or a leader.

- *Conclusion 10 – The influence factor gives positive impact to the fusion of preferences and alternative ranking.*

  The fusion of preferences is carried out by means of the σ-IOWA operator, where the experts' influence scores act as a set of order inducing fused vectors, associating with *fuzzy majority* environment. This makes the aggregation stage perform well in a *natural* sense. Use of the maximal dominance set concept helps to provide better ranking, ensuring that the decisions made are accepted by the whole group of experts.

10.2 Future Directions

Out of the research presented in this thesis, certain issues have emerged that would be interesting to be explored further, such as:

- It is well-known that knowledge contributions on decision making, SNA and SIN provide huge areas of studies to be explored, therefore continuous works, especially on consensual reaching process with application of SNA and SIN concepts need to be considered.

- Clustering techniques as used in this thesis have the potential to benefit decision making processes with big data arising from a large number of experts, criteria and/or alternatives. Thus, construction of a user-friendly interface is interesting to be explored.

- In this work, we utilized a symmetric similarity function to measure experts' preference similarities. Because of this, the construction of the preference similarity network produced undirected connections. In the future, asymmetric similarity functions, such as the Tversky index can be used to handle directed cases. Trust and influence networks, which are not symmetric, can also be considered.

- *Dynamic* consensus decision making is a relevant topic to be explored due to the demand of current web technologies that require real-time communications or time-varying individual relationships.

- It is worth mentioning that in any decision making model/process, no one can guarantee whether the decision is correct until after it is made, as agreed by Marakas [146] and Wu and Chiclana [35]. This is because decisions are theoretically based but might not be the best in practice as it relies on information/opinion provided/collected by experts and this might not be correct. Hence, post-decision evaluations are suggested to carry out in the future to preserve the quality of the decision making output.

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