Representations of Crossed Squares and Cat^2-Groups

Thesis submitted for the degree of

Doctor of Philosophy

at the

University of Leicester

by

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October 2018

Do not let your difficulties fill you with anxiety after all,

it is only in the darkest nights that the stars shine more brilliantly

Ali ibn Abi Talib (As)

Abstract

The concept of crossed modules was introduced by J.H.C. Whitehead in the late 1940s and then Loday [27] reformulated it as cat^1-groups. Crossed modules and cat^1-groups are two-dimensional generalisations of a group. Loday showed in [9] that crossed modules can be understood also as 2-groups. In much the same way, a higher dimensional analogue of crossed modules, the concept of crossed squares was introduced by Loday and Guin-Valery [27] and then Arvasi [2] linked it to the concept of higher categorical groups, namely cat^2-groups. From the same point of view, crossed squares and cat^2-groups are analogues of a three-dimensional generalisation of a group namely 3-groups. A group can be seen as a category with one object and morphisms given by the elements and with composition being the group multiplications. In classical representation theory the elements of a group can be realised as automorphisms of some object in some category, particularly in the category of vector spaces over a field K (see [13]). A 2-categorical analogue of the category of vector spaces over a field K has been described by Forrester-Barker [17] as the concept of a 2-category of length 1 chain complexes. Here, we describe a 3-groupoid of length 2 chain complexes as a 3-categorical analogue of the category of vector spaces over a field K. In this thesis, we first construct a 3-groupoid of length 2 chain complexes and describe it in a matrix language respecting the chain complex conditions. Also, imitating representations of a group G and homomorphisms of the group G into the general linear group of a vector space, we discuss representations of a category, which is a functor into a category of vector spaces over a field K. Here we develop a notion of representation of cat^2-groups and crossed squares, which will be defined as 3-functors. This extends the previous work by Forrester-Barker [17] where he defined the representation theory of cat^1-groups and crossed modules, which are given by 2-functors from the categorical dimension two to the categorical dimension three. The main objective in this thesis is to construct the general form of the automorphism $Aut(\gamma)$ after we introduce the path between matrices, which represents length 2 chain complexes γ and automorphisms of them.

Acknowledgements

It is a great pleasure to thank all the people who have given me support and encouragement and all who I have had a conversation with about my search.
First of all, my utmost gratitude to My supervisor Dr. Frank Neumann, who had the main role in helping me to complete my research, thanks for his guidance, advice and continued scientific support by encouraging me to take part in conferences, courses, and seminars to develop and support my scientific level without his continuous assist this study would not have been possible and I am very proud to be one of his students.
I am very grateful to my sponsor that has funded my study in Leicester university which is Kufa university and Ministry of Higher Education and Scientific Research in Iraq.
Furthermore, my sincere appreciation to all staff in College House for helping and making all the procedure more easier to me throughout the study.
Personally, the warmest of thanks to my husband Dr. Ammar for his understanding and patience while I was always busy during the period of my research in the university and the library.
As long as I am writing names down, I cannot neglect that of my parents. They provided me a lot of support and encouragement over the years.
Finally, I would like to thank the proofreader and all my colleagues in Michael Atiyah building for helping and they have given the strength to finish my work in the best way .

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Abbreviations

Cat = Category of categories and functors.

Grp = Category of groups.

$Grpd$ = Category of groupoids.

$2Cat$ = 2-Category of categories.

$GL(V)$ = Group of general linear transformations on a vector space V.

$Cat^1 - group$ = Category of cat^1-groups.

$Cat^2 - group$ = Category of cat^2-groups.

$Cat^n - group$ = Category of cat^n-groups.

$Vect_K$ = Category of K-vector spaces and linear transformations over a field K.

Rep_G^K = Category of K-linear representations of a group G.

$Ch_K^{(2)}$ = 3-category of length 2 chain complexes over $Vect_K$.

$Aut(\gamma)$ = Automorphism cat^2-group of a linear transformation γ.

K^n = $n \times n$ matrices with coefficients in a field K.

$K^{n,m}$ = $n \times m$ matrices with coefficients in a field K.

Dedicated to the soul of my mother and my father

Chapter 1

Introduction

Representation theory is an area that was started in 1896 by F. G. Frobenius. The theory investigates any abstract algebraic structure such as groups, Lie groups or modules, by representing their elements as linear transformations of vector spaces over a field. One of the reasons that representation theory is considered a very important subject is that it has many applications in different fields of mathematics such as algebra, number theory, probability theory, mathematical physics and many others. Representation theory comes in many ways depending on what the problem addresses; for instance, the representations of groups present groups in terms of linear transformations of vector spaces, so every element of the group is mapped to an invertible linear transformation and also permutation representations of groups are the same thing as group actions (see [10]). If a group G acts on a set S, then the action gives a homomorphism from G to the group of permutations on S,

$$G \times S \to S; (a, x) \to a.x$$

Linear representations are defined as group homomorphisms from the group G to the general linear group $GL(K)$ for a field K. As a basic example, we can see a linear representation as a functor from G (seen as a category with only one object and invertible morphisms) to the category of vector spaces over a field k. We are thus lead to Rep_G^K, which is the category of K-linear representation of a group G, whose objects are the functors $G \to Vect_K$ from the category of the group

G to a category of vector spaces over a field K and morphisms are the natural transformations between such functors (see [5] and [6]). In general, representation theory provides calculational tools using matrices, and it is a useful tool for connecting group theory with other abstract algebraic structures. While it is well known that the theory of representations is easily extended to group theory, Forrester-Barker [17] has also shown that it can be successfully applied to higher dimensional representation theory of 2-groups which are 2-dimensional categorical analogs of groups. The aim of this thesis is to extend Forrester-Barker's generalisation to the categorical dimension three. As such, this thesis focuses on the representation theory of higher dimensional groups referred to as 3-groups.

Let us put this in a general perspective. In homotopy theory, we define a homotopy n-type as a space X with trivial homotopy groups π_i for $i > n$ i.e. $\pi_i X = 0$ for $i > n$. In the case $n = 1$, the homotopy type of a space X is determined by its fundamental group $\pi_1(X, x)$, therefore a homotopy 1-type is modelled by the fundamental group $\pi_1(X, x)$. When $n = 2$ the homotopy type is determined by the action of $\pi_1(X, x)$ on $\pi_2(X, x)$ (see [31]). Furthermore, cat$^1-$groups or categorical groups as they are known in some other sources, are group objects in the category of small categories. They are a convenient algebraic model for homotopy 2-types. Cat$^1-$groups are given as a triple $\mathfrak{C} = (e; t, h, : G \to R)$, consisting of groups G and R, two surjections $t, h : G \to R$ and $e : R \to G$ satisfying two conditions:

$(CAT1) : te = he = id_R$.

$(CAT2) : [Ker\ t, Ker\ h] = 1_G$.

$$\begin{tikzcd} G \arrow[r, shift left, "t"] \arrow[r, shift right, "h"'] & R \arrow[l, bend left=60, "e"] \end{tikzcd}$$

Moving up to the higher dimensional cat^2-groups, we describe an algebraic model for homotopy 3-types which are analogous to Gray 3-groupoids in higher category theory (see [32]).

A number of phenomena in group theory are better seen from a crossed module perspective. A crossed module is a triple (G_1, G_2, σ) consisting of a group homomorphism $\sigma : G_1 \to G_2$ between two groups G_1 and G_2, together with an action

$(g_2, g_1) \to {}^{g_2}g_1$ of G_2 on G_1 satisfying:

$$\sigma({}^{g_2}g_1) = g_2\sigma(g_1){g_2}^{-1},$$

$$^{\sigma(s)}g_1 = sg_1s^{-1},$$

for all $g_2 \in G_2$ and $s, g_1 \in G_1$. In the 2-group case, an important example of crossed modules emerges which is equivalent to a cat^1-group (see [2] and [30]). Here, we study the higher dimensional versions such as 2-crossed modules and crossed squares. This model generalises a higher-dimensional categorical concept from dimension two to three namely 2-crossed modules of groups and cat^2-groups and morphisms between each of them. These concepts will be described explicitly. Furthermore, the collection of all cat^2-groups with their morphisms, crossed squares with their morphisms and 2-crossed modules with their morphisms form categories.

As the present study focuses on three-dimensional groups, our analysis will concentrate on the higher dimensional versions of crossed modules, which are 2-crossed modules or crossed squares, and cat^2-groups. Furthermore, these two concepts are equivalent and related to 3-groups.

To achieve this, an equivalence between cat^2-groups and 2-crossed modules and 3-groupoids will be described. 3-Categories, presented here as 3-groups with one object, in which all 1-cells, 2-cells and 3-cells are invertible, are the higher dimensional analogues of 2-categories, which can be described as strict 2-categories or bicategories. They are referred to as strict 3-categories, tricategories and Gray categories.

In the classical representation theory, the simplest algebraic structure of a representation of category $\mathcal{C}$ is a functor from $\mathcal{C}$ to the category of vector spaces $Vect(K)$ over a field K. In order to describe the representations of cat^2-groups $\mathfrak{G}$, we have to replace $Vect(K)$ by the Gray category of length 2 chain complexes of vector spaces.

$$\gamma : \; C_2 \xrightarrow{\gamma_2} C_1 \xrightarrow{\gamma_1} C_0$$

This study generalises the 2-category of length 1 chain complexes $Ch^{(1)}$ to a Gray category of length 2 chain complexes $Ch^{(2)}$. The latter consists of the following:

1. 0-morphisms, which are chain complexes of length 2,
2. 1-morphisms, which are chain maps between chain complexes of length 2,
3. 2-morphisms, which are homotopies between chain maps,
4. 3-morphisms, which are 2-homotopies between homotopies.

For ease of calculation, Gray category of length 2 chain complexes $Ch^{(2)}$ is converted into a matrix form, which respects the conditions and properties of chain complexes. One of the most important result in this thesis is the construction of the matrix form for 3-groupoids of length 2 chain complexes consists of two matrices: each one represents the morphism or differential or boundary map between two objects or chains depending on the algebraic structure under study, whether categories or a singular chain complex. This matrix form is far more informative than the chain complexes themselves; for instance, the forms for chain maps are constructed between chain complexes and homotopies between chain maps and also 2-homotopies between homotopies. In the cat^2-group and the crossed square representation, automorphisms of length 2 chain complexes should also be considered as a 3-group of chain automorphism on a length 2 chain complex and 1-homotopies and 2-homotopies between them. The notion of a group automorphism, which is an isomorphism from a group to itself, corresponds to the notion of a chain complex automorphism, which is a morphism from a length 2 chain complex to itself. If

$$\gamma : \ C_2 \xrightarrow{\gamma_2} C_1 \xrightarrow{\gamma_1} C_0$$

is a 2-length chain complex, a 3-groupoid of length 2 chain complexes $Aut(\gamma)$ can be described using the following diagram

$$\begin{tikzcd}
C_2 \arrow[r, "F_2"] \arrow[d, "\gamma_2"'] & C_2 \arrow[d, "\gamma_2"] \\
C_1 \arrow[r, "F_1"] \arrow[d, "\gamma_1"'] & C_1 \arrow[d, "\gamma_1"] \\
C_0 \arrow[r, "F_0"] & C_0
\end{tikzcd}$$

with different morphisms, starting with 0-morphisms and finishing with 3-morphisms to construct the group of automorphisms of cat^2-groups. The general form of automorphism of cat^2-groups $Aut(\gamma)$ are represented as matrices in different examples. For a representation of a category $\mathcal{C}$ by objects in another category $\mathcal{D}$ is nothing but a 1-functor $\mathcal{C} \to \mathcal{D}$ and Forrester-Barker [17] defined a representation of cat^1-groups, which are 2-groups $\mathfrak{C}$ as a 2-functor $\mathfrak{C} \to Ch_K^{(1)}$. Following this lead, the new notion of a cat^2-group representation will be defined as a 3-functor $\mathfrak{G} \to Ch_K^{(2)}$, generalising the notion of a cat^1-group representation.

Thesis outline

This thesis has six main chapters presenting the main subject coherently as follows:

The introduction in Chapter 1 reviews some basic background of representation theory and homotopy n-types and also describes algebraic models for homotopy types. Then, we introduce crossed modules and cat^1-groups and the relation between them. Then we introduce our primary idea about higher dimensional analogues of both cat^1-groups and crossed modules and discuss the relationship between them. At the end of this chapter, we give a basic description of how to convert the representation theory from the group theoretical language to the 3-group language that we will use throughout the next chapters.

Chapter 2 is first concerned with recalling the notions of categories and morphisms between them, namely functors and natural transformations. We give several examples for illustration. We also introduce 2-categories and morphisms between them and lay out some examples. The third section of this chapter presents the most important part which is the theory of 3-categories. As well as defining the morphisms between them, we give an explicit construction for 3-functors and 3-natural transformations by presenting some diagrams to explain them.

Chapter 3 introduces categorical representation theory. We describe how representation theory deals with different algebraic structures such as group representations and representations of a category and a 2-category with some examples.

Chapter 4 provides the definitions of crossed modules and cat^1-groups and the categorical equivalence between them. Then we discuss the higher dimensional analogues of crossed modules which are given by 2-crossed modules and crossed squares and the higher analogues of cat^1-groups which are cat^2-groups and the categorical equivalence between them.

Chapter 5 introduces the particular Gray categories of chain complexes Ch used here. We begin by constructing a Gray category of length 2 chain complexes $Ch_K^{(2)}$ as a 3-categorical analogue of vector spaces. The second section provides some important basic results which relate the length 2 chain complexes to particular matrices for easy calculations.

In chapter 6 a representation of cat^2-groups and crossed squares will be given. Then, we provide the definition of automorphisms of cat^2-groups and introduce a free path between matrices, which represent length 2 chain complexes γ and automorphisms of them. Then, we construct the general form of the automorphism $Aut(\gamma)$, and give examples of such automorphisms to remove any ambiguity in constructing Aut for cat^2-groups. Finally, .

Chapter 2

Classic Categorical Concepts

Category theory is a relatively new branch of abstract algebra which aims to describe general characteristics of structures in mathematics and the relationships between them. One of the most important reasons why categories are so interesting is that many similarities have been identified across very different areas of mathematics, thus providing a common unifying language. Category theory, like set theory, is now considered fundamental in the mathematical discourse. In set theory, the most basic concept is an element. In category theory, the basic structures are the objects and the maps between any two objects are called morphisms; whatever internal structure they may possess is ignored. The morphisms of category theory are often said to represent a process connecting two objects, or in many cases a structure-preserving transformation connecting two objects. This definition results in almost any structure either being its own category or the collection of all such structures with their obvious structure-preserving mappings forming a category. Gradually, the 2-categorical concepts are shown in this chapter as well for instance 2-categories, 2-functors and 2-natural transformation are explained with some examples. The main concepts in this chapter are Gray categories and 3-groupoids that are assumed to be a classification of the concept of 3-categories, using the definition of Gray-categories outlined in the paper by Crans in [11].

2.1 Categories, functors and natural transformations

2.1.1 Categories

Basic description will be given about our main concept in this section which is the notion of a category. The definition and some examples of categories will be considered. The main books will be used in this section [4] p.4-10 and [7] p.4-10.

Definition 2.1.1. A **category** C is given by a collection C_0 of objects and a collection C_1 of morphisms which have the following structure.

- Each morphism has two objects which are called domain (**dom**) and codomain (**cod**); one writes $f : X \to Y$, if X is the domain of the morphism f, and Y its codomain. One also writes $X = dom(f)$ and $Y = cod(f)$.

- Given two morphisms f and g such that $cod(f) = dom(g)$, the composition of f and g, written $g \circ f$, is defined and has $dom(f)$ and $cod(g)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \;\to\; X \xrightarrow{g \circ f} Z$$

- Composition is associative, that is: given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.

- For every object X there is an identity morphism $id_X : X \to X$, satisfying $id_X g = g$ for every $g : Y \to X$ and $f \circ id_X = f$ for every $f : X \to Y$.

Examples of Categories

Categories List		
Category	Objects	morphisms
Set	Sets	functions
Grp	Groups	group homomorphisms
Top	Topological Spaces	continuous functions
Vect	Vector Spaces	linear transformations
Pos	posets	monotone functions
CAT	Categories	functors
Funct	Functors	natural transformations

Table 2.1: Table of some examples of categories

On the first level, categories consist of a number of algebraic properties of transformations between mathematical objects, known as morphisms (such as binary relations, groups, sets, topological spaces, etc.) of the same type; conditions vary depending on the kind of collections. Moreover, these categories contain a unit (identity) morphism and a composition of morphisms. On the second level, 2- categories, which generalise categories consist of morphisms and 2-morphisms, which are morphisms between morphisms. There are, however, many applications where far more abstract concepts are represented by objects and morphisms.
The most important property of the morphisms is that they can be "composed" in other words, arranged in a sequence to form a new morphism.
In sets, we often consider elements and functions, but in category theory, we will consider objects and the morphisms between them in categories and in 2-categories. As well as, objects and morphisms, 2-morphisms will be considered.

2.1.2 Functors

The morphisms between categories, which are called functors, consist of a pair of functions sending objects and morphisms of the first category to items of the same types in the second category in order to preserve all of the categorical structures, as the following definition explains (see section 1 in [28] and also section 1 in [33]).

Definition 2.1.2. A **functor** is a morphism between categories. Given two categories C and D, a functor F from C to D consist of an object $F(A)$ in D for all A in C

- $F_{1_A} = 1_{F(A)}$
- whenever $f : A \to B$ and $g : B \to C$ is a pair of composable morphisms in C
  $F(g \circ f) = F(g) \circ F(f)$

Functors can be composed, and there is an identity functor for each category. In the category theory, there are lots of functors between the categories and every functor has a unique source category and a unique target category. Here we define covariant and contravariant functors.

Definition 2.1.3. A (**covariant**) functor F from one category C to another category D assigns

- to each object X in C an object $F(X)$ in D,
- to each morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$

such that the following two properties hold:

- $F(1_X) = 1_{F(X)}$, for every object X in C ,
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

Definition 2.1.4. A (**contravariant**) functor F from one category C to another category D assigns

- to each object X in C an object $F(X)$ in D,
- to each morphism $f : X \to Y$ a morphism $F(f) : F(Y) \to F(X)$

such that the following two properties hold:

- $F(1_X) = 1_{F(X)}$, for every object X in C ,
- $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

Examples of functors

- Forgetful functor($Grp \to Set$): This functor takes a group to its underlying set and homomorphism to its underlying function.
- The inclusion functor ($Ab \to Grp$).
- Free functor ($Set \to Vect_k$): This takes a set X to the space of formal k-linear combination of elements of X which is a vector space with X as basis.
- Order-Preserving functor: ($Pos \to Pos$) This takes a poset P to a poset Q.
- Diagram : A **diagram** Δ_I in a category C of shape I is a functor $\Delta_I : I \to C$ where I is a category (Index category). For example

$$\begin{tikzcd} I : \bullet \arrow[r] \arrow[d] & \bullet & & A_\bullet \arrow[r, "g"] \arrow[d, "f"'] & B_\bullet \\ \bullet & & \Longrightarrow & C_\bullet & \end{tikzcd}$$

$$I : \bullet \qquad \bullet \qquad \Longrightarrow \qquad A_\bullet \qquad B_\bullet$$

$$I : \bullet \rightrightarrows \bullet \qquad \Longrightarrow \qquad \begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

2.1.3 Natural transformations

Lifting up the dimension of morphisms, natural transformations are 2-morphisms, which are the morphisms between two functors, which are 1-morphisms. In this section we will show the definition and some examples of natural transformations.

Definition 2.1.5. : A **natural transformation** is a morphism between functors. Given functors F, G and categories C, D. Then $\eta : F \to G$ is a natural transformation as follows:

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D \end{tikzcd}$$

(η) is given by:

There is $\eta_x : F_x \to G_x$ for all objects x in C called a component of η at x satisfying

naturality which is for all f in C such that $f : x \to y$ the following diagram is commute:

$$\begin{tikzcd} F_x \arrow[r, "\eta_x"] \arrow[d, "F_f"'] & G_x \arrow[d, "G_f"] \\ F_y \arrow[r, "\eta_y"'] & G_y \end{tikzcd}$$

Naturality Square

This means

$$G_f \circ \eta_x = \eta_y \circ F_f$$

A cone over the diagram with vertex u is precisely one of the most important examples of natural transformation. So a cone $\triangle_u \to D(I)$ such that $\triangle_u : I \to C$, which maps $i : i \to u$ and $f : f \to 1_u$, thus for all I' and f in I there exists morphism $u \to D(f)$.

$$\begin{tikzcd} u \arrow[r, "\eta(I')"] \arrow[d, "i_u"'] & D(I') \arrow[d, "D(f)"] \\ u \arrow[r, "\eta(I'')"'] & D(I'') \end{tikzcd}$$

Composition of natural transformations

The important property of morphisms is the composition between them which satisfies the composition conditions. As the natural transformations are morphisms so they can be composed as the following definition explains.

Definition 2.1.6. Given two natural transformations η and α as in the following diagrams

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$

The composition of η and α is denoted as $\alpha \circ \eta : F \to H$ as in this diagram

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D \end{tikzcd} + \begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd} \Longrightarrow \begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha\circ\eta"] & D \end{tikzcd}$$

The composite $\alpha \circ \eta$ has components as follows

$$(\alpha \circ \eta)_x : \quad F_x \xrightarrow{\eta_x} G_x \xrightarrow{\alpha_x} H_x$$

$$\begin{tikzcd}
F_x \arrow[r, "\eta_x"] \arrow[d, "F_f"'] & G_x \arrow[r, "\alpha_x"] \arrow[d, "G_f"'] & H_x \arrow[d, "H_f"'] \\
F_y \arrow[r, "\eta_y"] & G_y \arrow[r, "\alpha_y"] & H_y
\end{tikzcd}$$

Definition 2.1.7. Two categories C and D are **isomorphic** in (Cat) if and only if there are functors

$$\begin{tikzcd}
C \arrow[r, shift left, "F"] & D \arrow[l, shift left, "G"]
\end{tikzcd}$$

such that $F \circ G = 1_D$ and $G \circ F = 1_C$.

Definition 2.2.3: A **natural isomorphism** is an isomorphism in the functor category. That means a natural transformation η can be described as a diagram

$$\begin{tikzcd}
C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D
\end{tikzcd}$$

such that for each c in C, ηc is an isomorphism in D. Moreover, the most interesting example of a natural isomorphism is a universal cone over the diagram with vertex u.

2.2 2-Categories, 2-functors and 2-natural transformations

Building on the previous discussion of categories, functors and natural transformations, the corresponding concepts will be presented in this section which are 2-categories, 2-functors and 2-natural transformations, referring to [2].

2.2.1 2-Categories

The notion of a 2-category generalises that of a category where besides of the objects and morphisms, there are also 2-morphisms, as in the following definition from (section 1.2 and 1.3 in [25]) and (section 7 in [8]).

Definition 2.2.1. A **2-category** consists of the following data, consisting of two kinds of things:

- Objects 0-morphisms : $A, B, C, ...$
- A small category $C(A,B)$ for each pair of objects. Objects of $C(A,B)$ are called 1-morphisms $\{f, g, h, ...\}$ while morphisms in $C(A,B)$ are called 2-morphisms $\{\alpha, \beta, \gamma, ...\}$.

There are the following operations:

1. A **unit functor**
$$u_A : 1 \to C(A,A)$$
for each object A and 1 as a terminal category (with one object(id_A) and one morphism (id_{id_A})).

2. An **associative composition functor** (compositions of 1-morphisms)
$$C_{ABC} : C(A,B) \times C(B,C) \to C(A,C)$$
for every triple of objects.

3. There are two kinds of composition in a 2-category (composition of 2-morphisms)
   - **vertical composition**: Given two functors (2-morphisms) α and β in $C(A,B)$ such that
   $$\alpha : f \to g \quad and \quad \beta : g \to h$$
   so
   $$\beta \circ \alpha : f \to h$$
   is a vertical composition.
   - **horizontal composition**: Given two functors (2-morphisms) $\alpha \in C(A,B)$ and $\beta \in C(B,C)$ such that
   $$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & B \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, bend left, "f'\circ f"] \arrow[r, bend right, "g'\circ g"'] \arrow[r, phantom, "\Downarrow\gamma*\beta"] & C \end{tikzcd}$$

so

$$\gamma * \beta : f' \circ f \to g' \circ g$$

is a horizontal composition.

There are objects and morphisms and 2-morphisms as follows:

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[r, bend left=60, "f'"] \arrow[r, "g'" description] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & C \end{tikzcd}$$

The interchange law satisfies the following

$$(\beta' * \beta) \circ (\alpha' * \alpha) = C_{ABC}(\beta, \beta') \circ C_{ABC}(\alpha, \alpha') \tag{1}$$

$$= C_{ABC}((\beta, \beta') \circ (\alpha, \alpha')) \tag{2}$$

$$= C_{ABC}((\beta \circ \alpha, \beta' \circ \alpha')) \tag{3}$$

$$= ((\beta \circ \alpha, \beta' \circ \alpha')) \tag{4}$$

$$= (\beta' \circ \alpha') * (\beta \circ \alpha) \tag{5}$$

Examples of 2-categories

List of 2-categories			
2-Category	**0-morphisms**	**1-morphisms**	**2-morphisms**
Cat	categories	functors	natural transformations
Mon cat.	monodial categories	monodial functors	monodial natural transformations
Enriched cat.	enriched categories	enriched functors	enriched natural transformations

Table 2.2: Table of some examples of 2-categories

Weak 2-categories or bicategories is a generalization of 2- categories, which consists of the data of a 2-category except that the associativity and unital axioms for horizontal composition are replaced by extra data of invertible natural transformations of 2-morphisms as the following points explain. We use the source [9]

- An associator is a natural family of isomorphisms of horizontal composition from

  $$(A \to B \to C) \to D$$

  to horizontal composition

  $$A \to (B \to C \to D)$$

  that means

  - for any 1-morphisms f, g, h such that

  $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

  there is an invertible 2-morphisms α such that

  $$\begin{tikzcd} A \arrow[r, bend left=60, "h\circ(g\circ f)"] \arrow[r, bend right=60, "(h\circ g)\circ f"'] \arrow[r, phantom, "\Downarrow \alpha_(f,g,h)"] & D \end{tikzcd}$$

  - for any 2-morphisms F, G, H

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow F"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow G"] & C \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow H"] & D \end{tikzcd}$$

the following diagram commutes.

$$\begin{tikzcd} h \circ (g \circ f) \arrow[r, Rightarrow, "\alpha(f,g,h)"] \arrow[d, Rightarrow, "H\circ(G\circ F)"'] & (h \circ g) \circ f \arrow[d, Rightarrow, "(H\circ G)\circ F"] \\ h' \circ (g' \circ f') \arrow[r, Rightarrow, "\alpha(f',g',h')"'] & (h' \circ g') \circ f' \end{tikzcd}$$

- A left identifier is a natural family of isomorphisms from horizontal composition with identity on the left to the identity functor on $A \to B$, that means:

  - for any 1-morphisms f such that

  $$A \xrightarrow{f} B$$

  an invertible 2-morphisms $\lambda(f)$ such that

  $$1_B \circ \ f \xrightarrow{\lambda(f)} f$$

  - for any 2-morphisms F

  $$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow F"] & B \end{tikzcd}$$

  the following diagram commutes.

  $$\begin{tikzcd} 1_B \circ f \arrow[r, Rightarrow, "\lambda(f)"] \arrow[d, Rightarrow, "1_{1_B}\circ F"'] & f \arrow[d, Rightarrow, "F"] \\ 1_B \circ f' \arrow[r, Rightarrow, "\lambda(f')"'] & f' \end{tikzcd}$$

- A right identifier is a natural family of isomorphisms from horizontal composition with identity on the right to the identity functor on $A \to B$, that means.

- for any 1-morphisms f such that

$$A \xrightarrow{f} B$$

an invertible 2-morphisms $\rho(f)$ such that

$$f \circ 1_A \xrightarrow{\rho(f)} f$$

- for any 2-morphisms F

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow F"] & B \end{tikzcd}$$

the following diagram commutes.

$$\begin{tikzcd} f \circ 1_A \arrow[r, Rightarrow, "\rho(f)"] \arrow[d, Rightarrow, "F \circ 1_{1_A}"'] & f \arrow[d, Rightarrow, "F"] \\ f' \circ 1_A \arrow[r, Rightarrow, "\rho(f')"'] & f' \end{tikzcd}$$

These are called associativity and unital constraints, as associativity pentagons and unit triangles are imposed, as the following diagrams explain.

$$\begin{tikzcd} & i \circ ((h \circ g) \circ f)) \arrow[dr, "\alpha(i, h \circ g, f)"] & \\ i \circ (h \circ (g \circ f)) \arrow[ur, "1_i \circ \alpha(h, g, f)"] \arrow[d, "\alpha(i, h, g \circ f)"'] & & (i \circ (h \circ g)) \circ f \\ (i \circ h) \circ (g \circ f) \arrow[rr, "\alpha(i \circ h, g, f)"] & & ((i \circ h) \circ g) \circ f \arrow[u, "\alpha(i, h, g) \circ 1_f"'] \end{tikzcd}$$

(Pentagon Diagram)

$$\begin{tikzcd}
g \circ (1_B \circ f) \arrow[rr, "\alpha(g, 1_B, f)"] \arrow[dr, "1_g \circ \lambda(f)"'] & & (g \circ 1_B) \circ f \arrow[dl, "\rho(g) \circ 1_f"] \\
& g \circ f &
\end{tikzcd}$$

(Unit Triangle Diagram)

2.2.2 2-Functors

In this section, the morphisms between 2-categories will be defined. This concept is referred to as a 2-functor, which consists of a triple of functions sending objects, morphisms and 2-morphisms of the first 2-category to items of the same types in the second 2-category in order to preserve all of the 2-categorical structures, as the following definition shows.

Definition 2.2.2. Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$, a **2-functor** F:$\mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to $\mathcal{B}$ consists of giving:

1. for each object A in $\mathcal{A}$, an object FA in $\mathcal{B}$.

2. for each pair of objects A and A' in $\mathcal{A}$, a functor $F_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$

This data is required to satisfy the following axioms:

- Compatibility with composition: given three objects A, A' and A'' in $\mathcal{A}$, the following equality holds:

$$F_{A,A''} \circ \mathcal{A}_{A,A',A''} = \mathcal{B}_{F_A,F_{A'},F_{A''}} \circ F_{A,A'} \times F_{A',A''}$$

$$\begin{tikzcd}
\mathcal{A}(A, A') \times \mathcal{A}(A', A'') \arrow[r, "\mathcal{A}_{A,A',A''}"] \arrow[d, "F_{A,A'} \times F_{A',A''}"'] & \mathcal{A}(A, A'') \arrow[d, "F_{A,A''}"] \\
\mathcal{B}(FA, FA') \times \mathcal{B}(FA', FA'') \arrow[r, "\mathcal{B}_{FA,FA',FA''}"'] & \mathcal{B}(FA, FA'')
\end{tikzcd}$$

- Unit: for every object A in $\mathcal{A}$, the following quality holds:

$$F_{A,A} \circ u_A = u_{F_A}$$

$$\begin{tikzcd} 1 \arrow[r, "u_A"] \arrow[dr, "u_{FA}"'] & \mathcal{A}(A,A) \arrow[d, "F_{A,A}"] \\ & \mathcal{B}(FA,FA) \end{tikzcd}$$

It is evident that a 2-functor includes an ordinary functor between underlying categories of objects and morphisms.

Examples of 2-functors

- A **constant 2-functor** $\Theta_A : D \to K$ for all A in K which is sending
  - every object D in D to the object A,
  - all 1-morphisms in D to the identity 1_A,
  - all 2-morphisms in D to the identity 2-morphism id_{1_A}.
- A **lax functor** is a 2-functor between bicategories (including 2-categories). Given bicategories $\mathcal{C}$ and $\mathcal{D}$, a lax functor $P : \mathcal{C} \to \mathcal{D}$ consist of :
  1. for each object x of $\mathcal{C}$, an object P_x of $\mathcal{D}$.
  2. for each $C(x,y)$ in $\mathcal{C}$, a functor $P_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(P_x, P_y)$.
  3. for each object x of $\mathcal{C}$, a 2-morphism $P_{idx} : id_{P_x} \Rightarrow P_{x,x}(id_x)$.
  4. for each triple x, y, z of $\mathcal{C}$, a 2-morphism

     $$P_{x,y,z}(f,g) : P_{x,y}(f) \circ P_{y,z}(g) \Rightarrow P_{x,z}(f \circ g)$$

     such that ($f : x \to y$ and $g : y \to z$).
  5. for each $\mathcal{C}(x,y)$ the following diagrams commute where

     $$\lambda f = id_x \circ f = f \circ id_y$$

     .

$$
\begin{tikzcd}
 & id_{Px} \circ P_{x,y}(f) \arrow[dl, "P_{id_x} \circ id_{P_{x,y}}(f)"'] \arrow[dr, "\lambda_{P_{x,y}(f)}"] & \\
P_{x,x}(id_x) \circ P_{x,y}(f) \arrow[dr, "P_{x,x,y}(id_x \circ f)"'] & & P_{x,y}(f) \\
 & P_{x,y}(id_x \circ f) \arrow[ur, "P_{x,y}(\lambda_f)"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & P_{x,y}(f) \circ id_{Py} \arrow[dl, "id_{P_{x,y}(f)} \circ P_{id_y}"'] \arrow[dr, "\lambda_{P_{x,y}(f)}"] & \\
P_{x,y}(f) \circ P_{y,y}(id_y) \arrow[dr, "P_{x,y,y}(f \circ id_y)"'] & & P_{x,y}(f) \\
 & P_{x,y}(f \circ id_y) \arrow[ur, "P_{x,y}(\lambda_f)"'] &
\end{tikzcd}
$$

6. for each quadruple w, x, y, z in $\mathcal{C}$ the following diagram commutes, where $\alpha_{f,g,h} : (f \circ g) \circ h \to f \circ (g \circ h)$.

$$
\begin{tikzcd}
(P_{w,x}(f) \circ P_{x,y}(g)) \circ P_{y,z}(h) \arrow[r, "\alpha_{P_{w,x}(f), P_{x,y}(g), P_{y,z}(h)}"] \arrow[d, "P_{w,x,y}(f,g) \circ id_{P_{y,z}(h)}"'] & P_{w,x}(f) \circ (P_{x,y}(g) \circ P_{y,z}(h)) \arrow[d, "id_{P_{w,x}(f)} \circ P_{x,y,z}(g,h)"] \\
P_{w,y}(f \circ g) \circ P_{y,z}(h) \arrow[d, "P_{w,y,z}(f \circ g, h)"'] & P_{w,x}(f) \circ P_{x,z}(g \circ h) \arrow[d, "P_{w,x,z}(f, g \circ h)"] \\
P_{w,z}((f \circ g) \circ h) \arrow[r, "P_{w,z}(\alpha_{f,g,h})"] & P_{w,z}(f \circ (g \circ h))
\end{tikzcd}
$$

- An **oplax functor** is a 2-functor between two bicategories (including 2-categories). If all morphisms in a lax functor are reversed, the notion of an oplax functor is obtained.

- A **pseudo functor** is a 2-functor between two bicategories (including 2-categories). The definition of a pseudo functor is the same as the definition of a lax functor, with the additional requirement that P_{idx} and $P_{x,y,z}(f,g)$ be invertible as follows.

Given bicategories $\mathcal{C}$ and $\mathcal{D}$, a pseudo functor $P : \mathcal{C} \to \mathcal{D}$ consists of:

1. for each object x of $\mathcal{C}$, an object P_x in $\mathcal{D}$.
2. for each $C(x,y)$ in $\mathcal{C}$, a functor $P_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(P_x, P_y)$.
3. for each object x of $\mathcal{C}$, an invertible 2-morphism $P_{idx} : id_{P_x} \Rightarrow P_{x,x}(id_x)$.
4. for each triple x, y, z of $\mathcal{C}$, an isomorphism (natural in $f : x \to y$ and $g : y \to z$)

$$P_{x,y,z}(f,g) : P_{x,y}(f) \circ P_{y,z}(g) \Rightarrow P_{x,z}(f \circ g)$$

5. for each $\mathcal{C}(x,y)$ the following diagrams commute where

$$\lambda f = id_x \circ f = f \circ id_y$$

$$\begin{tikzcd}
& id_{Px} \circ P_{x,y}(f) \arrow[dl, "P_{id_x} \circ id_{P_{x,y}(f)}"'] \arrow[dr, "\lambda_{P_{x,y}}(f)"] & \\
P_{x,x}(id_x) \circ P_{x,y}(f) \arrow[dr, "P_{x,x,y}(id_x \circ f)"'] & & P_{x,y}(f) \\
& P_{x,y}(id_x \circ f) \arrow[ur, "P_{x,y}(\lambda_f)"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& P_{x,y}(f) \circ id_{Py} \arrow[dl, "id_{P_{x,y}(f)} \circ P_{id_y}"'] \arrow[dr, "\lambda_{P_{x,y}(f)}"] & \\
P_{x,y}(f) \circ P_{y,y}(id_y) \arrow[dr, "P_{x,y,y}(f \circ id_y)"'] & & P_{x,y}(f) \\
& P_{x,y}(f \circ id_y) \arrow[ur, "P_{x,y}(\lambda_f)"'] &
\end{tikzcd}$$

6. for each quadruple w, x, y, z of $\mathcal{C}$ the following diagram commutes, where $\alpha_{f,g,h} : (f \circ g) \circ h \to f \circ (g \circ h)$

$$\begin{tikzcd}
(P_{w,x}(f) \circ P_{x,y}(g)) \circ P_{y,z}(h) \arrow[rr, "\alpha_{P_{w,x}(f),P_{x,y}(g),P_{y,z}(h)}"] \arrow[d, "P_{w,x,y}(f,g) \circ id_{P_{y,z}}(h)"'] & & P_{w,x}(f) \circ (P_{x,y}(g) \circ P_{y,z}(h)) \arrow[d, "id_{P_{w,x}(f)} \circ P_{x,y,z}(g,h)"] \\
P_{w,y}(f \circ g) \circ P_{y,z}(h) \arrow[d, "P_{w,y,z}(f \circ g, h)"'] & & P_{w,x}(f) \circ P_{x,z}(g \circ h) \arrow[d, "P_{w,x,z}(f, g \circ h)"] \\
P_{w,z}((f \circ g) \circ h) \arrow[rr, "P_{w,z}(\alpha_{f,g,h})"] & & P_{w,z}(f \circ (g \circ h))
\end{tikzcd}$$

2.2.3 2-Natural transformations

The notion of a 2-natural transformation is a generalisation of the notion of a natural transformation from category theory to 2-category theory. As a natural transformation is a morphism between two functors between categories, a 2-natural transformation is a morphism between two 2-functors between 2-categories; where a natural transformation has a commuting naturality square, a 2-natural transformation has a 2-morphism filling that square, as the following definition explains.

Definition 2.2.3. Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$ and two 2-functors F and G between them

$$F, G : \mathcal{A} \rightrightarrows \mathcal{B}.$$

A **2-natural transformation**

$$\theta : F \Rightarrow G$$

consisting in giving, for every object A in $\mathcal{A}$, a morphism $\theta_A : FA \Rightarrow GA$ such that the equality

$$\mathcal{B}(1_{FA}, \theta_{A'}) \circ F_{A,A'} = \mathcal{B}(\theta_A, 1_{GA'}) \circ G_{A,A'}.$$

holds for each pair of objects $A, A' \in \mathcal{A}$, as in the following diagram:

$$\begin{tikzcd}
\mathcal{A}(A,A') \arrow[r, "F_{A,A'}"] \arrow[d, "G_{A,A'}"'] & \mathcal{B}(FA,FA') \arrow[d, "\mathcal{B}(1_{FA},\theta_{A'})"] \\
\mathcal{B}(GA,GA') \arrow[r, "\mathcal{B}(\theta_A,1_{GA'})"'] & \mathcal{B}(FA,GA')
\end{tikzcd}$$

It can therefore be said that a 2-natural transformation includes an ordinary natural transformation between underlying functors.

Examples of 2-natural transformations

- A **2-cone** is one of the most important examples of a 2-natural transformation.
  A 2-cone (A,F):$\Theta_A \Rightarrow F$ where $A \in \mathcal{K}$ such that a 2-functor Θ_A is a constant 2-functor and a 2-functor between 2-categories $F : \mathcal{D} \to \mathcal{K}$.

- A **2-cocone** (F,A):$F \Rightarrow \Theta_A$ where $A \in \mathcal{K}$ such that a 2-functor Θ_A is a constant 2-functor and a 2-functor $F : \mathcal{D} \to \mathcal{K}$ between 2-categories .

- A **lax natural transformation** is a morphism between 2-functors between 2-categories.
  Given 2-categories $\mathcal{C}$ and $\mathcal{D}$ and (lax and oplax) 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, a lax natural transformation $\alpha : F \Rightarrow G$ is given by:

  1. for each A in $\mathcal{C}$ a 1-morphism $\alpha_A : F(A) \to G(A)$ in $\mathcal{D}$.
  2. for each $f : A \to B$ in $\mathcal{C}$ a 2-morphism $\alpha_f : G(f) \circ \alpha_A \Rightarrow \alpha_B \circ F(f)$.

- An **oplax natural transformation** is as above, only with 2-morphisms α_f reversed.

- A **pseudo natural transformation** is a lax natural transformation if each α_f is invertible.

2.3 3-Categories, 3-functors and 3-natural transformations

The notion of a 3-category theory is the part of category theory dealing with the higher dimensional categories. It generalizes that of 2-category, which consists objects, morphisms, 2-morphisms. Moreover, in 3-categories an additional kind of morphisms is added which are 3-morphisms. There are three kinds of 3-category which are strict 3-categories, the Gray categories which are semi strict 3-categories; and tricategories, which are weak 3-categories. We also have relations between 3-categories C and D which we could represent by 3-functors and 3- natural transformations just as 2-categories and categories. The main sources which will be used in this section are [11], [29] and [23]. To be more precise, see (section 2 in [11]), (section 1.2.4 in [29]) and (section 4.3 in [23]). [You could find more information about this section in the following sources [24], [22], [29], [32] and [35]].

2.3.1 Gray categories

Here we define the notion of Gray category. This notion explains the semi-strict kind of 3-categories, in which composition is strictly associative and unital, but the interchange law holds only up to isomorphism.

Definition 2.3.1. A **Gray category** $\mathcal{C}$ consists of collections C_0 of objects, C_1 of morphisms, C_2 of 2-morphisms and C_3 of 3-morphisms, together with:

- maps $s_n, t_n : C_i \to C_n$ for all $0 \leq n < i \leq 3$, also denoted d_n^- and d_n^+ and called n-source and n-target,
- maps $\#_n : C_{n+1} \;{}_{s_n}\times_{t_n} C_{n+1} \to C_{n+1}$ for all $0 \leq n < i \leq 3$, called vertical composition,
- maps $\#_n : C_i \;{}_{s_n}\times_{t_n} C_{n+1} \to C_i$ and $\#_n : C_{n+1} \;{}_{s_n}\times_{t_n} C_i \to C_i$ for all $0 \leq n < i \leq 3$, called whiskering,
- a map $\#_0 : C_2 \;{}_{s_0}\times_{t_0} C_2 \to C_3$, called horizontal composition, and
- maps $id_{\cdot} C_i \to C_{i+1}$ for all $0 \leq i \leq 2$, called identity,

  such that:

  i. $\mathbb{C}$ is a 3-skeletal reflexive globular set [19].

  ii. for every $C, C' \in C_0$, the collection of elements of $\mathbb{C}$ with 0-source C and 0-target C' form a 2-category $\mathbb{C}(C, C')$, with n-composition in $\mathbb{C}(C, C')$ given by $\#_{n+1}$ and identities given by $id-$,

  iii. for every $g : C' \to C''$in C_1 and every C and $C''' \in C_0, -\#_0 g$ is a 2-functor $\mathbb{C}(C'', C''') \to \mathbb{C}(C', C''')$ and $g\#_0-$ is 2-functor $\mathbb{C}(C, C') \to \mathbb{C}(C, C'')$.

  iv. for every $C' \in C_0$ and every C and $C'' \in C_0$, $-\#_0 id_{C'}$ is equal to the identity functor $\mathbb{C}(C', C'') \to \mathbb{C}(C', C'')$ and $id_{C'}\#_0-$ is equal to the identity functor $\mathbb{C}(C, C') \to \mathbb{C}(C, C')$,

  v. for every $\gamma :$
  $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in C_2 and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$s_1(\delta\#_0\gamma) = (g'\#_0\gamma)\#_1(\delta\#_0 f)$$

$$t_1(\delta\#_0\gamma) = (\delta\#_0 f')\#_1(g\#_0\gamma)$$

and $\delta\#_0\gamma$ is an iso 3-morphism,

vi. for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=70, "f"] \arrow[r, bend right=70, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ in C_3 and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$((\delta\#_0 f')\#_1(g\#_0\varphi))\#_2(\delta\#_0\gamma) = (\delta\#_0\gamma')\#_2((g'\#_0\varphi)\#_1(\delta\#_0 f))$$

and for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in C_2,$\psi :$ $\begin{tikzcd} C' \arrow[r, bend left=70, "g"] \arrow[r, bend right=70, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & C'' \end{tikzcd}$ in C_3,

$$(\delta'\#_0\gamma)\#_2((g'\#_0\gamma)\#_1(\psi\#_0 f)) = ((\psi\#_0 f')\#_1(g\#_0\gamma))\#_2(\delta\#_0\gamma)$$

vii. for every $\begin{tikzcd} C \arrow[r, bend left=70, "f"] \arrow[r, dashed, "f'" description] \arrow[r, bend right=70, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma'"] & C' \end{tikzcd}$ and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2

$$\delta\#_0(\gamma'\#_1\gamma) = ((\delta\#_0\gamma')\#_1(g\#_0\gamma))\#_2((g'\#_0\gamma')\#_1(\delta\#_0\gamma)),$$

and for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ and $\begin{tikzcd} C' \arrow[r, bend left=70, "g"] \arrow[r, dashed, "g'" description] \arrow[r, bend right=70, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\delta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta'"] & C'' \end{tikzcd}$ in C_2,

$$(\delta'\#_1\delta)\#_0\gamma = ((\delta'\#_0 f')\#_1(\delta\#_0\gamma))\#_2((\delta'\#_0\gamma)\#_1(\delta\#_0 f)),$$

viii. for every $f : C \to C'$ in C_1 and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$\delta\#_0 id_f = id_{\delta\#_0 f},$$

and for every $\gamma :$ \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd} in C_2 and $g : C' \to C''$ in C_1,

$$id_g \#_0 \gamma = id_{g\#_0\gamma},$$

ix. for every $c \in (C, C')_p$, $c' \in (C', C'')_q$ and $c'' \in (C'', C''')_r$ with $p+q+r \leq 2$,

$$(c'' \#_0 c')\#_0 c = c''\#_0(c'\#_0 c).$$

Example of Gray categories

There are many examples of Gray categories which are weak 3-categories. One of the important examples is a 3-groupoid, which is the higher generalisation of a groupoid or a 2-groupoid.

Definition 2.3.2. A **Gray 3-groupoid C** is given by a set C_0 of objects, a set C_1 of morphisms, a set C_2 of 2-morphisms and a set C_3 of 3- morphisms together with maps $s_i, t_i : C_k \to C_{i-1}$, where $i = 1, ..., k$ (and $k = 1, 2, 3$) such that:

1. $s_2t_2 \circ s_3t_3 = s_2t_2$, as maps $C_3 \to C_1$.

2. $s_1t_1 = s_1t_1 \circ s_2t_2 = s_1t_1 \circ s_3t_3$, as maps $C_3 \to C_0$.

3. $s_1t_1 = s_1t_1 \circ s_2t_2$, as maps $C_2 \to C_0$.

4. An horizontal multiplication $J\natural_3 J'$ of 3-morphisms if $s_3(J) = t_3(J')$, making C_3 into a groupoid whose set of objects is C_2.

5. A vertical composition $\Gamma\natural_2\Gamma' = \begin{bmatrix} \Gamma' \\ \Gamma \end{bmatrix}$ of 2-morphisms if $s_2(\Gamma) = t_2(\Gamma')$, making C_2 into a groupoid whose set of objects is C_1.

6. A vertical composition $J\natural_2 J' = \begin{bmatrix} J' \\ J \end{bmatrix}$ of 3-morphisms whenever $s_2(J) = t_2(J')$, making the set of 3-morphisms into a groupoid with set of objects C_1 and such that the boundaries $s_3, t_3 : C_3 \to C_2$ are functors.

7. The vertical and horizontal compositions of 3-morphisms satisfy the interchange law

$$(J\natural_3 J')\natural_2(J_1\natural_3 J_1') = (J\natural_2 J_1)\natural_3(J'\natural_2 J_1'),$$

whenever the compositions are well defined. This means that the vertical and horizontal compositions of 3-morphisms and vertical composition of 2-morphisms give C_3 the structure of a 2-groupoids, with set of objects being C_1, set of morphisms C_2 and set of 2-morphisms C_3 which is exactly a 2-groupoid.

8. (Existence of whiskering by 1-morphisms): For each x, y in C_0, we can therefore define a 2-groupoid $(C)(x,y)$ of all 1-, 2-, 3-morphisms a such that $t_1(a) = x$ and $s_1(a) = y$. Given a 1-morphism γ with $t_1(\gamma) = y$ and $s_1(\gamma) = z$ there exists a 2-groupoid map $\natural_1(\gamma) : (C)(x,y) \to (C)(y,z)$, called right whiskering. Similarly if $s(\gamma') = x$ and $t(\gamma') = w$ there exists a 2-groupoid map $\gamma'\natural_1 : (C)(x,y) \to (C)(w,y)$ called left whiskering.

9. The horizontal composition $\gamma\natural_1\gamma'$ of 1-morphism γ if $s_1(\gamma) = t_1(\gamma')$, which is to be associatove and to define a groupoid with set of objects C_0 and set of morphisms C_1.

10. Given $\gamma, \gamma' \in C_1$ we must have :

$$\natural_1\gamma \circ \natural_1\gamma' = \natural_1(\gamma\gamma')$$

$$\gamma\natural_1 \circ \gamma'\natural_1 = (\gamma\gamma')\natural_1$$

$$\gamma\natural_1 \circ \natural_1\gamma' = \natural_1\gamma' \circ \gamma\natural_1,$$

11. Now we define two horizontal composition of 2-morphisms

$$\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = \begin{bmatrix} s_2(\Gamma)\natural_1 & \Gamma' \\ \Gamma & \natural_1 t_2(\Gamma') \end{bmatrix} = (\Gamma\natural_1 t_2(\Gamma'))\natural_2(s_2(\Gamma)\natural_1\Gamma')$$

and

$$\begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix} = \begin{bmatrix} \Gamma & \natural_1 s_2(\Gamma') \\ t_2(\Gamma)\natural_1 & \Gamma' \end{bmatrix} = (t_2(\Gamma)\natural_1\Gamma')\natural_2(\Gamma\natural_1 s_2(\Gamma'))$$

and of 3-morphisms:

$$\begin{bmatrix} & J' \\ J & \end{bmatrix} = \begin{bmatrix} s_2(J)\natural_1 & J' \\ J & \natural_1 t_2(J') \end{bmatrix} = (J\natural_1 t_2(J'))\natural_2(s_2(J)\natural_1 J')$$

and

$$\begin{bmatrix} J & \\ & J' \end{bmatrix} = \begin{bmatrix} J & \natural_1 s_2(J') \\ t_2(J)\natural_1 & J' \end{bmatrix} = (t_2(J)\natural_1 J')\natural_2(J\natural_1 s_2(J'))$$

It follows from the previous axioms that they are associative. They also define the functor $C_3(x,y) \times C_3(y,z) \to C_3(x,z)$, where $C_3(x,y)$is the category with objects 2-morphisms Γ with $t_2(\Gamma) = x$ and $s_2(\Gamma) = y$ and the 3-morphisms J with $t_2(J) = x$ and $s_2(J) = y$, and the horizontal multiplication as composition.

12. (Interchange 3-morphisms): For any two 2-morphisms Γ and Γ' with $s_1(\Gamma) = t_1(\Gamma)$ a 3-morphism

$$\begin{bmatrix} & \Gamma' \\ \Gamma & \end{bmatrix} = t_3(\Gamma\#\Gamma') \xrightarrow{\Gamma\#\Gamma'} s_3(\Gamma\#\Gamma') = \begin{bmatrix} \Gamma & \\ & \Gamma' \end{bmatrix}$$

13. For any 3-morphisms

$$\Gamma_1 = t_3(J) \xrightarrow{J} s_3(J) = \Gamma_2 \quad \text{and} \quad \Gamma'_1 = t_3(J') \xrightarrow{J'} s_3(J') = \Gamma'_2$$

with $s_1(J) = t_1(J')$ the following horizontal compositions of 3-morphisms coincide

$$\begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} \xrightarrow{\Gamma_1\#\Gamma'_1} \begin{bmatrix} \Gamma_1 & \\ & \Gamma'_1 \end{bmatrix} \xrightarrow{\begin{bmatrix} J & \\ & J' \end{bmatrix}} \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix}$$

and

$$\begin{bmatrix} & \Gamma'_1 \\ \Gamma_1 & \end{bmatrix} \xrightarrow{\begin{bmatrix} & J' \\ J & \end{bmatrix}} \begin{bmatrix} & \Gamma'_2 \\ \Gamma_2 & \end{bmatrix} \xrightarrow{\Gamma_2\#\Gamma'_2} \begin{bmatrix} \Gamma_2 & \\ & \Gamma'_2 \end{bmatrix}$$

14. For any three 2-morphisms $\gamma \xrightarrow{\Gamma} \phi \xrightarrow{\Gamma'} \psi$ and $\gamma'' \xrightarrow{\Gamma''} \phi''$ with $s_2(\Gamma) =$

$t_2(\Gamma')$ and $s_1(\Gamma) = s_1(\Gamma') = t_1(\Gamma'')$, the following compositions of 3-morphisms coincide

$$\begin{bmatrix} \psi\natural_1 & \Gamma'' \\ \Gamma' & \natural_1\gamma'' \\ \Gamma & \natural_1\gamma'' \end{bmatrix} \xrightarrow{\begin{bmatrix} \Gamma'\#\Gamma'' \\ \Gamma\natural_1\gamma'' \end{bmatrix}} \begin{bmatrix} \Gamma' & \natural_1\phi'' \\ \phi\natural_1 & \Gamma'' \\ \Gamma & \natural_1\gamma'' \end{bmatrix} \xrightarrow{\begin{bmatrix} \Gamma'\natural_1\phi'' \\ \Gamma\#\Gamma'' \end{bmatrix}} \begin{bmatrix} \Gamma' & \natural_1\phi'' \\ \Gamma & \natural_1\phi'' \\ \gamma\natural_1 & \Gamma'' \end{bmatrix}$$

where the 2-morphism components of the 3-morphisms stand for the corresponding identity 3-morphism, and

$$\begin{bmatrix} \psi\natural_1 & \Gamma'' \\ \Gamma' & \natural_1\gamma'' \\ \Gamma & \natural_1\gamma'' \end{bmatrix} \xrightarrow{\begin{bmatrix} \Gamma' \\ \Gamma \end{bmatrix}\#\Gamma''} \begin{bmatrix} \Gamma' & \natural_1\phi'' \\ \Gamma & \natural_1\phi'' \\ \gamma\natural_1 & \Gamma'' \end{bmatrix}$$

2.3.2 3-Functors

There are now also higher morphism between 3-categories, which we explain now. As we know a **functor** F from a category C to a another one D is a map sending each object x in C to an object $F(x)$ in D. For higher dimensions, we can characterize a **3-functor** as a morphism between 3-categories in the following way.[The following definitions (2.3.3) and (2.3.4) has been taken from [35]].

Definition 2.3.3. We can describe 3-categories as special categories internal to $2Cat$. Based on this description, a 3-category has a 2-category of objects Q as in the following diagram

$$\begin{tikzcd}
 & x \arrow[dl, dash] \arrow[dr, dash] & \\
\gamma_1 \arrow[rr, "s"] \arrow[dr] & & \gamma_2 \\
 & y \arrow[ur] &
\end{tikzcd}$$

We have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$

$$\begin{tikzcd}
 & x \arrow[ddl, dash] \arrow[ddr] & \\
 & S_1 \arrow[dl, dash] \arrow[dr] \arrow[dd, Rightarrow, "V"] & \\
\gamma_1 \arrow[ddr] & & \gamma_2 \arrow[ddl] \\
 & S_2 \arrow[ul, dash] \arrow[ur] & \\
 & y &
\end{tikzcd}$$

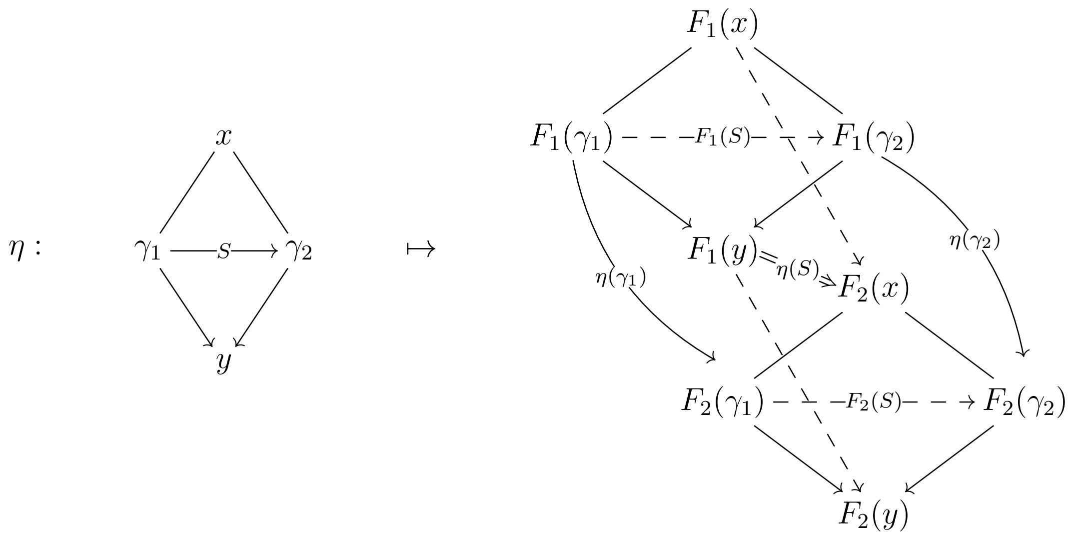
Instead of saying V is a morphism internal $2Cat$, we have to say that V is a 3-morphism, S_1 and S_2 are 2-morphisms, γ_1 and γ_2 are 1-morphisms and x and y are objects. A **3-functor** $F : S \to T$ between 3-categories S and T is a functor internal to $2Cat$, giving by the following morphism

$$F: \quad \begin{tikzcd}
 & x \arrow[ddl, dash] \arrow[ddr] & \\
 & S_1 \arrow[dl, dash] \arrow[dr] \arrow[dd, Rightarrow, "V"] & \\
\gamma_1 \arrow[ddr] & & \gamma_2 \arrow[ddl] \\
 & S_2 \arrow[ul, dash] \arrow[ur] & \\
 & y &
\end{tikzcd} \quad \mapsto \quad \begin{tikzcd}
 & F(x) \arrow[ddl, dash] \arrow[ddr] & \\
 & F(S_1) \arrow[dl, dash] \arrow[dr] \arrow[dd, Rightarrow, "F(V)"] & \\
F(\gamma_1) \arrow[ddr] & & F(\gamma_2) \arrow[ddl] \\
 & F(S_2) \arrow[ul, dash] \arrow[ur] & \\
 & F(y) &
\end{tikzcd}$$

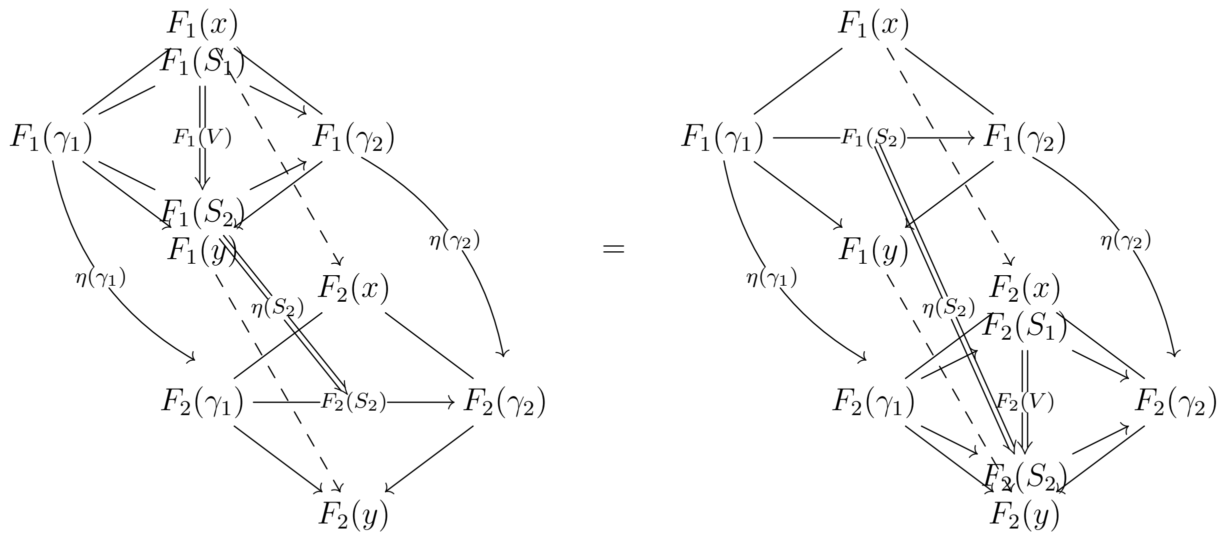
2.3.3 3-Natural transformations

A morphism between two 3-functors is a 3-natural transformation, which generalises the concept of a 2-natural transformation. The following definition will be a brief explanation of it.

Definition 2.3.4. A 1-morphism $\eta : F_1 \to F_2$ between two such 3-functors is a **natural transformation** internal to 2Cat, hence a 2-functor from the object 2-category to the morphism 2-category, so we can illustrate the natural transformations as in the following diagram:

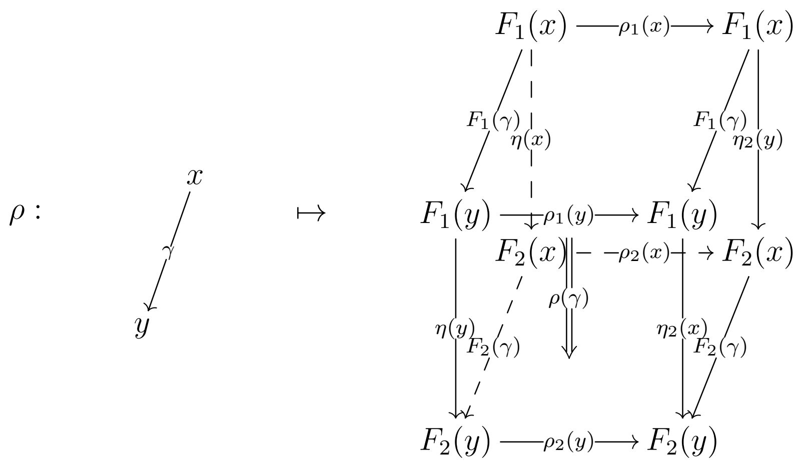


that satisfies the naturality condition

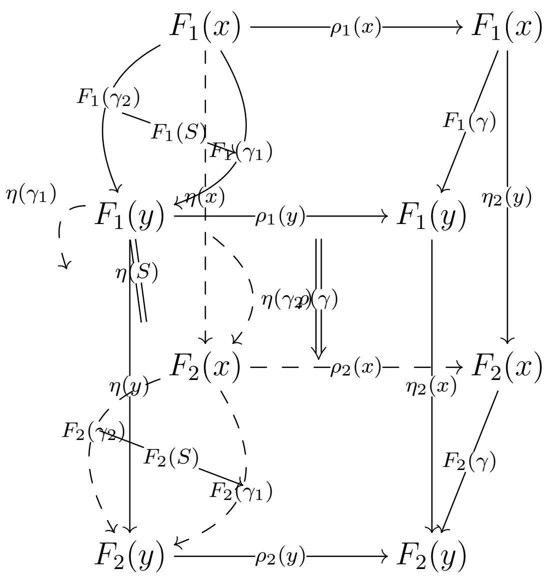


Systematically, 2-morphisms and 3-morphisms of 3-functors are 1-morphisms and 2-morphisms of these 2-functors η. For this reason, a 2-morphism $\eta \xrightarrow{\rho} \eta'$

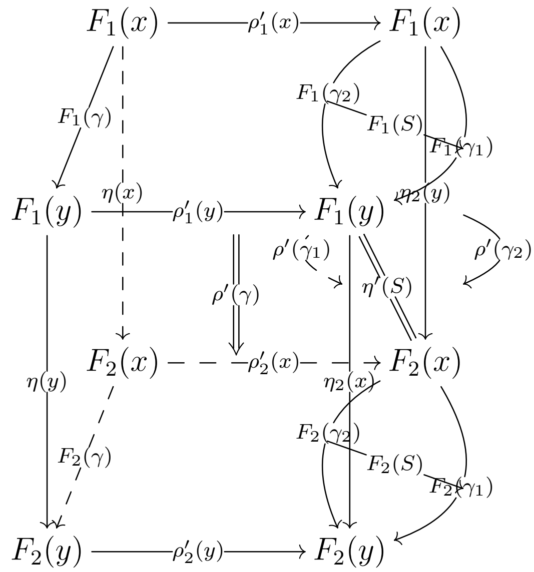
of our 3-functors is a 1-functor assignment.



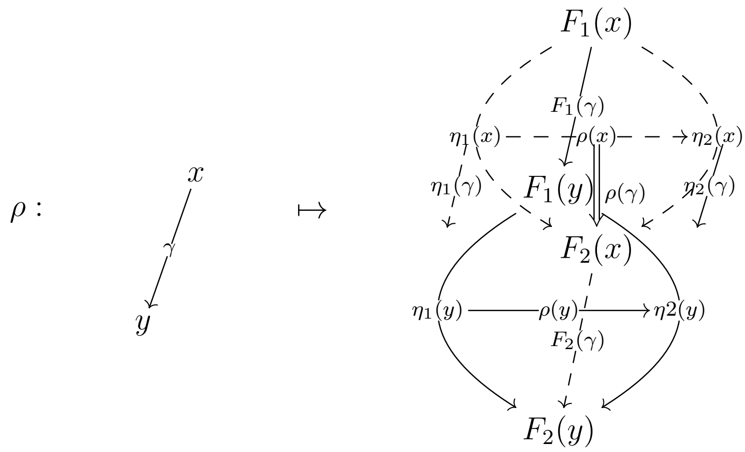
It also satisfies the naturality conditions as the following diagram illustrates



$$=$$

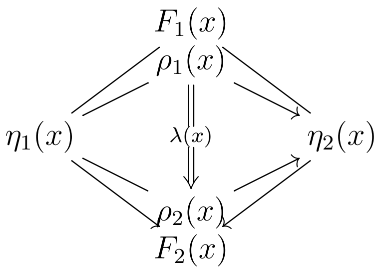


If we restrict those ρ for which the horizontal 1-morphisms $\rho_1(x)$ and $\rho_2(x)$ are identities only, we can illustrate this in the following diagram.

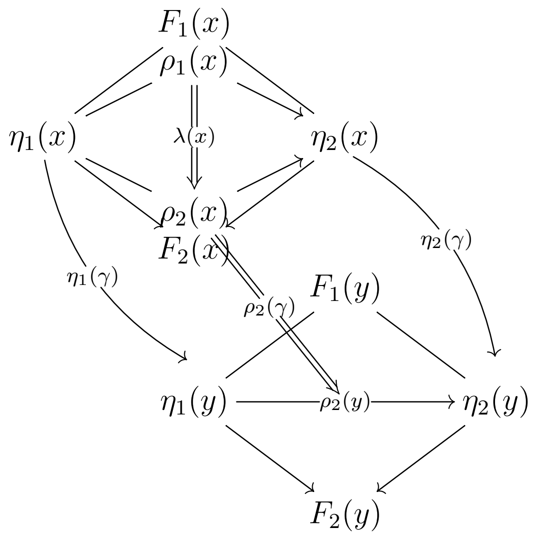


Moving to the higher dimension, we consider the morphisms between those natural transformations ρ which are called modifications $\lambda : \rho_1 \to \rho_2$. It is just a brief description of 3-morphisms of 3-functors λ and we can state the technical definition as follows

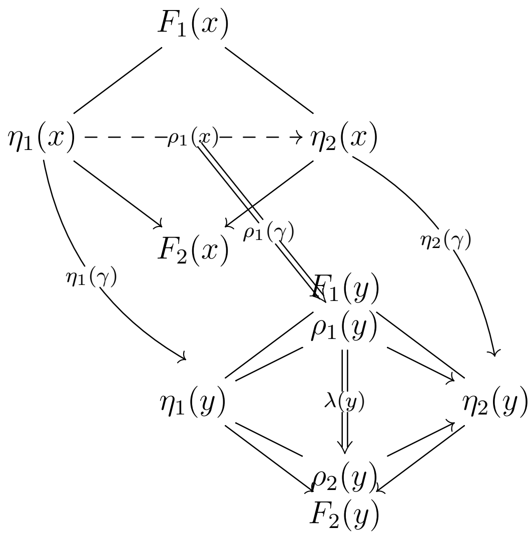
$\rho : \qquad x \qquad \mapsto$



The naturality conditions are also satisfied as the next picture illustrates.



$=$



Chapter 3

Linear Representations

To consider representations of cat^2-groups which are the aim of this thesis, various aspects of representation theory will be illustrated, along with examples. Representation theory is a very active subject which has many applications, created more than 100 years ago by Frobenius. The concept studies different kinds of algebraic structures by representing their elements as linear transformations of a vector space giving by matrices incorporating numerous algebraic operations. The categories and their higher dimensional versions are specific examples of an algebraic structure presented in this chapter, representations of them are shown as well. Here we use [15] and [40].

3.1 Representations of groups

Let V be a vector space over a field K and let $GL(V)$ be the group of all automorphisms $V \to V$. An element $a \in GL(V)$ is a linear map from V into V, which has an inverse a^{-1}; the latter is also linear. Suppose $\{e_1, e_2, ..., e_n\}$ is a basis for V. Then each linear map is defined by a $n \times n$ martix $T : V \to V$, as the following equation explains:

$$T(e_j) = \sum_i t_{ij} e_i$$

Definition 3.1.1. A **representation** α of a group G in a vector space V over

K is defined by a homomorphism

$$\alpha : G \to GL(V).$$

where $GL(V)$ - the general linear group on V- is the group of invertible linear maps, $t : V \to V$. **The degree** of the representation is the dimension of the vector space:

$$deg(\alpha) = dim_K(V).$$

Note that the representation α is finite if we require the vector space V to be a finite dimensional. In this case we have $GL(V) = GL(dim(V), K)$.

Definition 3.1.2. A representation of a group G on a vector space V

$$\alpha : G \to GL(V)$$

is a **faithful** representation if $ker(\alpha) = id_G$.

3.1.1 Group representation examples

Some examples will be given here for further clarity.

Example 3.1.1. Trivial representation
Given any K-vector space and any group G, we can define a representation $\alpha : G \to GL(V)$ by $\alpha(g) = id_V$ for any $g \in G$ which it is called the **trivial** representation.

Example 3.1.2. Regular representation
For any group G, we can associate the K-vector space G_k which has a basis $\{g : g \in G\}$, then G acts on G_k by multiplication on the left. The induced representation is called the **regular** representation.

Example 3.1.3. The symmetric group

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

has a representation on $\mathbb{R}$ by

$$\alpha(\sigma)v = sgn(\sigma)v$$

where sgn is the permutation symbol of the permutation σ.

Example 3.1.4. The circle group

$$G = C_6 =< c : c^6 = e > .$$

Let $GL(\mathbb{C})$ be a group of non-zero complex numbers when $n = 1$. Define

$$\alpha : G \to GL_1(\mathbb{C})$$

$$\alpha : c \mapsto e^{\frac{2\Pi i}{3}}$$

so $\alpha(c^k) = e^{\frac{2\Pi i}{3}}$. We check $\alpha(c)^6 = 1$, so this is a representation of C_6. But if $\alpha(c^3) = 1$, the kernel of α is $\{e, c^3\}$ and the image of α is $\{1, e^{\frac{2\Pi i}{3}}, e^{\frac{4\Pi i}{3}}\}$, which is isomorphic to C_3.

3.2 Matrix representations

The following section defines matrix representations, which are of great importance for this study. As such, it is useful to review the foundations of this particular kind of representation and understand its applicability to more general cases. [36] will be used in this section. Suppose $\beta = \{u_1, u_2, ..., u_n\}$ be an ordered basis for a finite-dimensional vector space V, so for any vector v in V write it as a linear combination of the vectors in the basis

$$v = \sum_{i=1}^{n} a_i u_i.$$

and the coordinate vector of x relative to β, denoted $[x]_\beta$ is

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Definition 3.2.1. Let $L : V \to W$ be a linear transformation and $\beta = \{v_1, v_2, ..., v_n\}$ and $\gamma = \{w_1, w_2, ..., w_n\}$ be a basis for V and W respectively.

A **matrix** representation A of L in the ordered bases β and γ is defined as a $m \times n$ matrix by the scalars a_{ij} as follows

$$L(v_i) = \sum_{i=1}^{n} a_{ij} w_i \qquad for \qquad 1 \leq j \leq n.$$

We write $A = [L]_\beta{}^\gamma$.

3.3 Representations of categories

The representation of a group G can be generalised to any category as the following definition indicates (see section 4 in [6]).

Definition 3.3.1. Suppose C is a category and G is a group. A **representation** of G in C is a pair (X, ρ), where X is an object in C and $\rho : G \to Aut(X)$ is a group homomorphism.
All the representations of G in C form a category of representations which has objects like (X, ρ) and (X', ρ'). As well as the objects, it has

$$mor((X, \rho), (X', \rho'))$$

which are the set of all morphisms $\psi : X \to X'$ between objects (X, ρ) and (X', ρ') such that for each g in G, the diagram

$$\begin{tikzcd} X \arrow[r, "\rho_g"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X' \arrow[r, "\rho'_g"'] & X' \end{tikzcd}$$

commute.

Every group can be represented by a category with one object (see [26]). Functors from this category into any other category are representations of that group in terms of the objects in the category, and choosing a functor into the category of vector spaces recovers classical representation theory. In this case, a representation of C on objects in another category D is a functor $C \to D$. Therefore, the representation category $Rep(C)$ is the functor category $Func(C, D)$ which has all the functors between C and D as objects and the natural transformations between these functors as morphisms of this category.

3.4 Representations of 2-categories

Let G be a group. G may be viewed as a 2-category with one object g, whose 1-morphism $Hom_G(g, g)$ is equal to G and whose 2-morphisms are all identities (see section 3 in [13]).

Definition 3.4.1. Let C be a 2-category. A **representation** of G in C is a lax-2-functor from G to C, providing the following:

1. An object c of C.

2. For each element g in G, a 1-morphism $\alpha(g) : c \to c$.

3. A 2-isomorphism, $\phi_1 : \alpha(1) \to id_c$.

4. For any pair g and h in G, a 2-isomorphism $\phi_{g,h} : (\alpha(g) \circ \alpha(h)) \Rightarrow \alpha(gh)$.

5. For any g, h and k in G, we have $\phi_{gh,k}(\phi_{g,h} \circ \alpha(k)) = \phi_{g,hk}(\alpha(g) \circ \phi_{h,k})$.

6. We have $\phi_{1,g} = \phi_1 \circ \alpha(g)$ and $\phi_{g,1} = \alpha(g) \circ \phi_1$.

Definition 3.4.2. Two 2-representations of G in C, α and α' are equivalent whenever there is a lax-2-natural transformation $\eta : \alpha \Rightarrow \alpha'$ such that each component of η is an equivalence in C.

Chapter 4

Higher Dimensional Groups

There are different ways to generalise the abstract concepts of group to higher dimensional categorical concepts. In this section we will focus on the concept of 3-groups. It is known that cat^1-groups and crossed modules are equivalent formulations of the categorical concepts of 2-groups. Here we will show similarly that cat^2-groups and crossed squares are equivalent. Crossed squares are equivalent to 2-crossed modules which can also be seen as Gray 3-grouoids. In this section we will assume that all 2-crossed modules and crossed squares are given as 2-crossed modules of groups and crossed squares of groups.

4.1 cat^1-groups and crossed modules

In this section, we introduce the concepts of both cat^1-groups and crossed modules with their morphisms. We also state the equivalence between them. (Section 2 in [1]) will be followed in this section. [Also, [16] and [38] could give quiet enough information about both cat^1-groups and crossed modules].

4.1.1 cat^1-groups

We consider one of the most important kinds of 2-categories namely cat^1-groups which are high dimensional analogues of categories and groups.

Definition 4.1.1. Suppose G is a group with a pair of endomorphisms

$$t, h : G \to G$$

having a common image R (i.e. $Im(t) = Im(h) = R$) and satisfying the following: A **cat^1-group $\mathfrak{C}$**, where $\mathfrak{C} = (e; t, h : G \to R)$ has domain G and codomain R with three homomorphisms; two surjections $t, h : G \to R$ and an embedding $e : R \to G$ as shown in the following diagram:

$$\begin{tikzcd} G \arrow[r, shift left, "t{,}h"] \arrow[r, shift left=2] & R \arrow[l, shift left, "e"] \end{tikzcd}$$

These homomorphisms are required to satisfy the following axioms:

$$C_1 : te(r) = he(r) = r \ \ for \ \ all \ \ r \in R$$

$$C_2 : [Ker(t), Ker(h)] = \{1_G\}.$$

Morphisms of cat^1-groups

The most important maps between two cat^1-groups are those that preserve the cat^1-group structure, and they are called morphisms of cat^1-groups which the following definition explains.

Definition 4.1.2. A **morphism of cat^1-groups** $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$t_2\gamma = \rho t_1, h_2\gamma = \rho h_1, e_2\rho = \gamma e_1.$$

4.1.2 Crossed modules

The concept of crossed modules originated in 1946 when J.H.C. Whitehead introduced it in his paper [39], arguing that the algebraic models for homotopy 2-types are the same as 2-groups. The equivalence between them was later established.

Definition 4.1.3. A **crossed module** (G_1, G_2, ϑ) consist of a group homomorphism $\vartheta : G_1 \to G_2$ called the boundary map, together with an action $(g_2, g_1) \to^{g_2} g_1$ of G_2 on G_1 satisfying

1. $\vartheta({}^{g_2}g_1) = g_2\vartheta(g_1){g_2}^{-1}$,

2. ${}^{\vartheta(s)}g_1 = sg_1s^{-1}$, for all g_2 in G_2 and g_1, s in G_1.

Suppose N is a subgroup of G. The inclusion homomorphism $N \to G$ together with the action ${}^g n = gng^{-1}$ of G on N is an inclusion crossed module which is the simplest example of a crossed module.

4.1.3 Morphisms of crossed modules

We consider a crossed module as an important fundamental algebraic structure. The following definition gives the notion of the morphism between crossed modules.

Definition 4.1.4. Suppose $\vartheta : G_1 \to G_2$ and $\vartheta' : G'_1 \to G'_2$ are two crossed modules, then a pair of homomorphisms φ, ϕ, where $\varphi : G_1 \to G'_1$, $\phi : G_2 \to G'_2$ is a **morphism of crossed modules** if

$$\phi(\vartheta(g_1)) = \vartheta'(\varphi(g_1))$$

and

$$\varphi({}^{g_2}g_1) =^{\phi(g_2)} \varphi(g_1)$$

for all g_1 in G_1 and g_2 in G_2.

Theorem 4.1.1. *The following data are equivalent:*

1. *a cat^1-group $\mathfrak{C}$.*
2. *a crossed module.*
3. *a 2-groupoid with a single object.*

Proof: See [17] p. 18-21.

4.2 cat^2-groups and 2-crossed modules

Some light on the background of cat^2-groups and crossed modules will be shed in this section and a neat description is given of the passage from higher dimensions of cat^1-groups to higher dimensions of crossed modules (see section 3 in [3]).

4.2.1 cat^2-groups

cat^2-groups are higher dimensional cat^1-groups. It is a cat^1-object in the category of cat^1-groups, which can also be described as double groupoids internal to the category of groupoids Grpd. In the order of cat^n-groups, we can extract the definition of cat^2-groups.

Definition 4.2.1. A **cat^n-group** is a group together with n categorical structures which commute pairwise, with n subgroups $N_1, N_2, ..., N_n$ of G and $2n$ group homomorphisms $s_i, t_i : G \to N_i$, $i = 1, 2, ..., n$, such that for $1 \leq i \leq n$, $1 \leq j \leq n$, satisfying the following conditions:

- $s_i/_{N_i} = t_i/_{N_i} = id_{N_i}$, ($s_i t_i = t_i$ and $t_i s_i = s_i$ for all i).
- $[Ker(s_i), Ker(t_i)] = 1$,
- $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$, and $t_i s_j = s_j t_i$, $i \neq j$, such that all morphisms s_i and t_i are endomorphisms for all $1 \leq i \leq n$.

For a cat^2-group $\mathfrak{G}$, we have a group, G, but this time with two independent cat^1-group $\mathfrak{C}$ structures on it. Therefore,
A cat^2-group $\mathfrak{G}$ is a 5-tuple (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

1. $s_i s_j = s_j s_i, t_i t_j = t_j t_i, s_i t_j = t_j s_i$ for $i, j = 1, 2$, $i \neq j$.
2. $[Kers_i, Kert_i] = 1$, for $i = 1, 2$.

Morphisms of cat^2-groups

The process, which connects two objects and preserves structures of them, is usually called a morphism. Here, morphisms between cat^2-groups will be shown.

Definition 4.2.2. Suppose we denote such cat^2-group $\mathfrak{G}$ by

$$\mathfrak{G} = (G, s_1, t_1, s_2, t_2)$$

and other one by

$$\mathfrak{G}' = (G', s_1', t_1', s_2', t_2').$$

A **morphism of cat^2-groups** $f : \mathfrak{G} \to \mathfrak{G}'$ is a group homomorphism $f : G \to G'$ such that

$$s'_1 f = f s_1 \quad \textit{and} \quad s'_2 f = f s_2$$

$$t'_1 f = f t_1 \quad \textit{and} \quad t'_2 f = f t_2.$$

cat^2-groups and their morphisms, along with their composition, can now be considered as a category of cat^2-groups.

4.2.2 Higher dimensional crossed modules

The dimension of crossed modules can then be elevated to the second dimension to produce two concepts: one is a 2-crossed module and the other is a crossed square. This section aims to define these two concepts and investigate if any relationship exists between them (see section 3 in [3] and section 4 in [30].

1. **Crossed squares**

   Crossed squares were defined by D. Guin-Walery and J.-L. Loday in [21] to study homotopy 3-types.

   Definition 4.2.3. Let P, L, M and N be groups. A **crossed square** of groups is a commutative square of groups together with actions of P on L, M and N.
   Moreover, there are actions of N on L and M via μ' and M acts on L and N via μ, and **h-map** $h : M \times N \to L$, such that

$$\begin{tikzcd} M \times N \arrow[dr, "h"] & & \\ & L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ & N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

   (a) the homomorphisms λ, λ', μ, μ' and $K = \mu\lambda = \mu'\lambda'$ are crossed modules for corresponding actions and the morphisms of maps $\lambda \to K$; $K \to \mu$; $\lambda' \to K'$; $K' \to \mu'$, are morphisms of crossed modules.

   (b) $\lambda h(m, n) = m^{\mu'(n)} m$.

(c) $\lambda' h(m,n) =^{\mu(m)} n(n)^{-1}$.

(d) $h(\lambda(l), n) = lnl^{-1}$.

(e) $h(m, \lambda'(l)) =^{m} ll^{-1}$.

(f) $h(mm', n) =^{m} h(m', n)h(m, n)$.

(g) $h(m, nn') = h(m, n)^{n} h(m, n)$.

(h) $h(^{P}m, ^{P}n) =^{P} h(m, n)$.

Examples of crossed squares

(a) Any pullback of a crossed module along a crossed module is an example of crossed square as the following explains:
Suppose N and M be two normal subgroups of P where $N \cap M$ acts by conjugation and $h : M \times N \to N \cap M$ is given by $h(m, n) = [m, n]$.

(b) From a simplicial group, which is a simplicial object in the category of groups, and two simplicial normal groups M and N, a crossed square can construct.

Morphisms of crossed squares

Morphisms refer to a structure-preserving morphism from one crossed square to another.

Definition 4.2.4. A **morphism of crossed squares** is a commutative diagram

$$\phi : (\phi_1, \phi_2, \phi_3, \phi_4) : (L_1, M_1, N_1, P_1) \to (L_2, M_2, N_2, P_2)$$

consisting of homomorphisms

$$\phi_1 : L_1 \to L_2$$

$$\phi_2 : M_1 \to M_2$$

$$\phi_3 : N_1 \to N_2$$

$$\phi_4 : P_1 \to P_2$$

such that the cube of homomorphisms is commutative

$$\phi_1 h(m_1, n_1) = h(\phi_2(m_1), \phi_3(n_1))$$

with $m_1 \in M_1$, $n_1 \in N_1$ and each of the homomorphisms ϕ_1, ϕ_2, ϕ_3 are ϕ_4-equivariant (see the definition of equivariant in [37] p.128) as the following diagram .

$$\begin{tikzcd}
& L_1 \arrow[rr, "\lambda_1"] \arrow[dl, "\phi_1"'] \arrow[dd, dotted, "\lambda'_1"] & & M_1 \arrow[dl, "\phi_3"'] \arrow[dd, "\mu_1"] \\
L_2 \arrow[rr, "\lambda_2"] \arrow[dd, "\lambda'_2"'] & & M_2 \arrow[dd, "u_2"] & \\
& N_1 \arrow[rr, "\mu'_1"] \arrow[dl, dotted, "\phi_2"'] & & P_1 \arrow[dl, "\phi_4"] \\
N_2 \arrow[rr, "\mu'_2"'] & & P_2 &
\end{tikzcd}$$

Crossed squares and their morphisms form a category which is equivalent to the category of internal crossed modules in the category of crossed modules.

2. **2-Crossed modules**

   A crossed module is an efficient algebraic tool. This section introduces the notion of a 2-crossed module which extends the concept of a crossed module (see [24] p.385).

Definition 4.2.5. A **2-crossed module** $\mathfrak{X}$ consists of a complex of groups

$$L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N$$

together with an action of N on L and M so that σ_1, σ_2 are morphisms of N-groups, where the group N acts on itself by conjugation, and an N-equivariant function

$\{,\} : M \times M \to L$ called a **Peiffer lifting**, which satisfies the following axioms:

(a) $\sigma_2\{m, m'\} = (mm'm^{-1})(\sigma_1(m) \rhd m'^{-1})$,

(b) $\{\sigma_2(l), \sigma_2(l')\} = [l, l']$, here $[l, l'] = ll'l^{-1}l'^{-1}$

(c) • $\{mm', m''\} = \{m, m'm''m'^{-1}\}\sigma_1(m) \rhd \ \{m', m''\}$, for each m, m' and $m'' \in M$

• $\{m, m'm''\} = \{m, m'\}(\sigma_1 m \rhd m') \rhd' \{m, m''\}$,

(d) $\{\sigma_2 l, m\}\{m, \sigma_2 l\} = l(\sigma_1(m) \rhd l^{-1})$,

(e) $\{m, m'\} \rhd n = \{m \rhd n, m' \rhd n\}$,

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$. We can define

$$m \rhd' l = l\{\sigma_2(l)^{-1}, m\} \tag{1}$$

From condition (5) yields:

$(m, m'm'') = (m, m')(\sigma_2 \rhd m') \rhd' \{m, m''\}$

$$= (\sigma_1(\{m, m'\}) \rhd' (\sigma_2(m) \rhd m') \rhd' \{m, m''\})\{m, m'\}$$

$$= ((m, m', m^{-1}) \rhd' \{m, m''\})\{m, m'\}, \tag{2}$$

However, the above definition of 2-crossed modules has many axioms, we need some more properties of them to make the path between 2-crossed modules and crossed squares easy to find. the following lamma has such important properties to use.(see [20] p.996).

Lemma 4.2.1. *In a 2-crossed module* $\mathfrak{T} = (\ L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N\ , \rhd, \{,\})$ *and for each* $m, m' \in M$*, we have*

$$\{m, m'\}^{-1} = \sigma_1(m) \rhd \{m^{-1}, mm'e^{-1}\}, \tag{3}$$

$$\{m, m'\}^{-1} = (mm'm^{-1}) \rhd' \{m, m^{-1}\}, \tag{4}$$

$$\{m, m'\}^{-1} = (\sigma_1(m) \rhd m') \rhd' \{m, m'^{-1}\}, \tag{5}$$

by the condition (d) of definition (4.2.5) it follows that

$$\sigma_1(m) \rhd l = (m \rhd' l)\{m, \sigma_2(l)^{-1}\}. \tag{6}$$

also,

$(\sigma_1(m) \rhd m') \rhd' (\sigma_1(m) \rhd l)$

$$= (\sigma_1(m) \rhd l)\{(\sigma_1(m) \rhd \sigma_2(l^{-1}), \sigma_1(m) \rhd m'\}$$

$$= (\sigma_1(m) \rhd l)(\sigma_1(m) \rhd \{\sigma_2(l^{-1}), m'\}$$

$$= \sigma_1(m) \rhd (m' \rhd' l)$$

Proof: Using the property $\{1_M, m\} = \{m, 1_M\} = 1_L$ for all $m \in M$ which apply directly axioms (4) and (5) of the definition of 2-crossed modules (4.2.5) as well as equation (2).

where we have used the fact that the Peiffer lifting $\{,\}$ is G-equivariant and that G acts on L by automorphisms. We thus have the following identity for each $e, f \in E$ and $l \in L$:

$$(\sigma_2(m) \rhd m') \rhd' (\sigma_2(m) \rhd l) = \sigma_2(m) \rhd (m' \rhd' l)$$

We have also:

$(m \rhd' \{m', m''\})\{m, \sigma_2(m') \rhd m''\}$

$$= \sigma_2(m) \rhd \{m', m''\}\{m, (\sigma_2(m') \rhd m'')m'm''^{-1}m'^{-1}\}^{-1}\{m, \sigma_2(m') \rhd m''\}$$

$$= \sigma_2(m) \rhd \{m', m''\}(\sigma_2(m) \rhd \sigma_2(m') \rhd m'') \rhd' \{m, m'm''^{-1}m'^{-1}\}^{-1}$$

$$= \sigma_2(m) \rhd \{m', m''\}(\sigma_2(m) \rhd \sigma_2(m') \rhd m'') \rhd' (\sigma_2(m) \rhd (m'm''^{-1}m'^{-1}))$$

$\rhd' \{m, m'm''m'^{-1}\}$

$$= \sigma_2(m) \rhd \{m', m''\}(\sigma_2(m) \rhd \sigma_1\{m', m''\}^{-1} \rhd' \{m, m'm''m'^{-1}\}$$

$$= \{m, m'm''m'^{-1}\}\sigma_2(m) \rhd \{m', m''\} \ = \{mm', m''\},$$

by the equation (6) in the lemma (4.2.1) and the condition (e) of the definition (4.2.5) and also the equation (5). The next step follows from the fact that $(\sigma_1 : L \to E, \rhd')$ is a crossed module. From all the above, the following equations have already proved (see [29] p.8):

Lemma 4.2.2. *In each 2-crossed module we have, for each $m, m', m'' \in M$:*

$$\{mm', m''\} = (m \rhd' \{m', m''\})\{e, \sigma(m') \rhd m''\}, \tag{7}$$

$$\{mm', m''\} = (m, m'm''m'^{-1}\}\sigma(m) \rhd \{m', m''\}, \tag{8}$$

and

$$\{m, m'm''\} = \{m, m'\}(\sigma(e) \rhd' m' \rhd' \{m, m''\}, \tag{9}$$

$$\{m, m'm''\} = ((mm'm^{-1}) \rhd' \{m, m''\})\{m, m'\}, \tag{10}$$

and also,

$$\{m, m'\}^{-1} = m \rhd' \{m^{-1}, \sigma(m) \rhd m'\}, \tag{11}$$

Examples of 2-crossed modules

(a) The simplest example of 2-crossed modules are crossed modules themselves. So, any crossed module $\chi = [\varphi : G_2 \to G_1]$ gives a 2-crossed module, $L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N$, by setting $L = 1$, the trivial group and $M = G_2, N = G_1$.

(b) Crossed complexes are another example of 2-crossed modules. So, any crossed complex,

$$\cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow C_3 \xrightarrow{\sigma_3} C_2 \xrightarrow{\sigma_2} C_1 \ ,$$

in which all higher dimensional terms are trivial , by supposing $L = C_3$, $M = C_2$ and $N = C_1$, with trivial Peiffer lifting.

Morphisms of 2-crossed modules

As a 2-crossed module is one of the most important algebraic models, there is a relationship between them that preserves the entire structure of 2-crossed modules known as a morphism (see section 2 in [34]).

Definition 4.2.6. A **morphism** between 2-crossed modules: If we denote such a 2-crossed module by $\{L, M, N, \sigma_2, \sigma_1\}$ and the other by $\{L', M', N', \sigma_2', \sigma_1'\}$. A morphism between them is given by the diagram:

$$\begin{tikzcd} L \arrow[r, "\sigma_2"] \arrow[d, "f_2"'] & M \arrow[r, "\sigma_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"] \\ L' \arrow[r, "\sigma_2'"'] & M' \arrow[r, "\sigma_1'"'] & N' \end{tikzcd}$$

where $f_0\sigma_1 = \sigma_1' f_1,\ f_1\sigma_2 = \sigma_2' f_2$

$$f_1({}^n m_1) =^{f_0(n)} f_1(m_1), f_2({}^n l) =^{f_0(n)} f_2(l)$$

and

$$\{,\} f_1 \times f_1 = f_2\{,\}$$

for all $l \in L$, $m_1 \in M$, $n \in N$.

4.2.3 Equivalence between cat^2-groups and higher dimensional crossed modules

After we have now described cat^2-groups and 2-crossed modules, we can know state the following theorem which gives the categorical equivalences of the different concepts we are using here. The content of this theorem can be extracted from various sources (see [2], [29] and [34]), but we state it in a comprehensive form collecting all the different categorical equivalences. We also give more explicit details in the proofs which again follow the material from (section 3 in [2]), (section 1 in [29]) and (section 2 in [34]).

Theorem 4.2.3. *The following data are equivalent:*

1. *a cat^2-group $\mathfrak{G}$.*

2. *a crossed square.*

3. *a 2-crossed module.*

4. *a Gray 3-groupoid with single object.*

Proof: (1)$\Leftrightarrow$ (2).

Starting with the cat^2-group $(\mathfrak{G}, s_1, s_2, t_1, t_2)$. Here, the cat^1-group $(\mathfrak{G}, s_1, t_1)$ will give us a crossed module with $M = ker(s_1)$, $N = Im(s_1)$ and $\sigma = t_1/_M$ and the other cat^1-group $(\mathfrak{G}, s_2, t_2)$ will give us another crossed module with $M' = ker(s_2)$, $N' = Im(t_2)$ and $\sigma' = t_2/_{M'}$ morphism of cat^1-group. We thus get a morphism of crossed modules.

$$\begin{array}{ccc} ker(s_1) \cap ker(s_2) & \longrightarrow & Im(s_1) \cap ker(s_2) \\ \downarrow & & \downarrow \\ ker(s_1) \cap Im(s_2) & \longrightarrow & Im(s_1) \cap Im(s_2) \end{array}$$

It remains to produce the h-map (see definition (4.2.3)) but this is given by commutators within G.
If $x = ker(s_1) \cap Im(s_2)$ and $y = ker(s_2) \cap Im(s_1)$ then,

$$[x, y] \in ker(s_1) \cap ker(s_2) = L,$$

and the h-map is given by

$$h : x \times y \to L$$

such that $h(m, m') = [m, m']$.
On the other hand, let us suppose we have a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\mu']{} & P \end{array}$$

this gives a morphism

$$\sigma : (L \rtimes N, s, t) \to (M \rtimes P, s', t')$$

of cat^1-groups. There is an action of $(m, p) \in M \rtimes P$ on $(l, n) \in L \rtimes N$,

$$^{(m,p)}(l, n) =^m (^pl,^p n) = (^{\mu(m)p}h(m,^p n),^p n)$$

using this action, we thus form the associated cat^1-group with $(L \rtimes N) \rtimes (M \rtimes P)$ and induced endomorphism s_1,s_2, t_1 and t_2.

(2)$\Leftrightarrow$ (3).

Suppose we have a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\mu']{} & P \end{array}$$

now we can derive

$$L \xrightarrow{\sigma_2} M \rtimes N \xrightarrow{\sigma_1} P$$

a 2- crossed module, where $\sigma_2(z) = (\lambda^{-1}(z), \lambda'(z))$ for $z \in L$,

$$\sigma_1(xy) = \mu(x)\mu'(y), \text{for} x \in M, y \in N,$$

Moreover the Peiffer lifting is given by

$$(x, y), (x', y') = h(x, yy'y^{-1}).$$

On the other hand, let us suppose $L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N$ be a 2-crossed module and let G^* be the corresponding simplicial group. The derived crossed square associated to G^*.

$$\begin{array}{ccc} N(G)_2 & \xrightarrow{\sigma_2} & ker(d_0) \\ {\scriptstyle\sigma_2}\downarrow & & \downarrow{\scriptstyle i} \\ ker(d_1) & \xrightarrow[i]{} & G_1 \end{array}$$

with

$$N(G)_2 = ker(d_0) \bigcap ker(d_1)$$

because

$$N(G)_n = \cap ker(d_i)$$

So we have shown that cat^2-groups, crossed squares and 2-crossed modules are equivalent. Now, it will be demonstrated that these are equivalent to 3-categories.

(3)$\Leftrightarrow$ (4).

Starting with a 2-crossed module

$$\mathfrak{T} = \ L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N$$

we can construct a Gray 3-groupoid C with a single object. Let us suppose $C_0 = \{*\}$, $C_1 = N$, $C_2 = N \times M$ and $C_3 = N \times M \times L$. Now as boundaries $s_1, t_1 : C_k \to C_0 = \{*\}$, where $k = 1, 2, 3$ (see [24]). Furthermore, $t_2(X, e) = X$ and $s_2(X, e) = \sigma_1(e)^{-1}X$. Let us put a vertical composition as the following

$$(X, e)\natural_2(\sigma_1(e)^{-1}X, f) = (X, ef),$$

and also

$$(Y, e)\natural_1 X = (Y X, e)$$

and

$$X\natural_1(Y, e) = (XY, X \rhd e)$$

In the same way,

$$X\natural_1(Y, e, l) = (XY, X \rhd e, X \rhd l)$$

and

$$(Y, e, l)\natural_1 X = (Y X, e, l).$$

Moving to 3-morphisms, put

$$t_3(X, e, l) = (X, e)$$

and

$$s_3(X, e, l) = (X, \sigma_2(l)^{-1}e)$$

and also

$$t_2(X, e, l) = X$$

and

$$s_2(X, e, l) = \sigma_1(e)^{-1}X.$$

We note that $s_2s_3(X, e, l) = s_2(X, \sigma_2(l)^{-1}e) = \sigma_1(e)^{-1}X = s_2(X, e, l)$, because

$\sigma_1\sigma_2 = 1$. As vertical composition of 3-morphisms we put:

$$(X, e, l)\natural_2(\sigma_1(e)^{-1}X, f, k) = \begin{bmatrix} (\sigma_1(e)^{-1}X, f, k) \\ (X, e, l) \end{bmatrix} = (X, ef, (e \rhd' k)l)$$

and as the horizontal composition of 3-morphisms, we put

$$(X, e, l)\natural_2(X, \sigma_2(l)^{-1}e, k) = (X, e, lk)$$

The vertical and horizontal compositions of 2-morphisms define a 2-groupoid that's because $(\sigma_2 : N \times L \to N \times M, \rhd')$ is a crossed module of groupoids. Let us now define the interchange 3-morphisms. We can see that:

$$\begin{bmatrix} & (Y, f) \\ (X, e) & \end{bmatrix} = (XY, e(\sigma_1(e)^{-1}X) \rhd f)$$

and

$$\begin{bmatrix} (X, e) & \\ & (Y, f) \end{bmatrix} = (XY, (X \rhd f)e).$$

We therefore take:

$$(X, e)\#(Y, f) = (XY, e(\sigma_1(e)^{-1}X) \rhd f, e \rhd' \{e^{-1}, X \rhd f\}^{-1}).$$

We have to note that

$\sigma_2(e \rhd' \{e^{-1}, X \rhd f\}^{-1})^{-1}e(\sigma_1(e)^{-1}X) \rhd f$

$$= ee^{-1}(X \rhd f)e(\sigma_1(e)^{-1}X) \rhd f^{-1}e^{-1}e(\sigma_1(e)^{-1}X) \rhd f = (X \rhd f)e.$$

It is easy to see that:

$$\begin{bmatrix} & (Y, f, l) \\ (X, e, k) & \end{bmatrix} = \begin{bmatrix} (\sigma_1(e)^{-1}XY, \sigma_1(e)^{-1}X \rhd f, \sigma_1(e)^{-1}X \rhd l) \\ (XY, e, k) \end{bmatrix}$$

$$= (XY, e(\sigma_1(e)^{-1}X) \rhd f, (e \rhd' \sigma_1(e)^{-1}X \rhd l)k)$$

and

$$\begin{bmatrix} (X,e,k) & \\ & (Y,f,l) \end{bmatrix} = \begin{bmatrix} (X\sigma_1(f)^{-1}Y,e,k) \\ (XY, X \rhd f, X \rhd l) \end{bmatrix}$$

$$= (XY, (X \rhd f)e, ((X \rhd f) \rhd' k)X \rhd l).$$

Now, to prove the condition 13 in the definition (2.3.2), we must prove that for each $X \in N$, $e, f \in M$ and $k, l \in L$.

$e \rhd' \{e^{-1}, X \rhd f\}^{-1}((X \rhd f) \rhd' k)X \rhd l$

$= (e \rhd' \sigma_1(e)^{-1}X \rhd l)k(\sigma_2(k)^{-1}e) \rhd' \{e^{-1}\sigma_2(k), X \rhd (\sigma_2(l)^{-1}f)\}^{-1}$. By using $(\sigma_2 : L \to M, \rhd')$ is a crossed module

$e \rhd' \{e^{-1}, X \rhd f\}^{-1}((X \rhd f) \rhd' k)X \rhd l$

$= (e \rhd' \sigma_1(e)^{-1}X \rhd l)e \rhd' \{e^{-1}\sigma_2(k), X \rhd (\sigma_2(l)^{-1}f)\}^{-1}k.(a)$ (12) For $l = 1$ this is equivalent to:

$$e \rhd' \{e^{-1}, X \rhd f\}^{-1}((X \rhd f) \rhd' k) = e \rhd' \{(e)^{-1}\sigma_2(k), X \rhd f\}^{-1}k.$$

or

$$((X \rhd f) \rhd' k^{-1})e \rhd' \{e^{-1}, X \rhd f\} = k^{-1}e \rhd' \{(e)^{-1}\sigma_2(k), X \rhd f\}$$

from the equation (7) of the lemma(4.2.2)and the definition of

$$e \rhd' l = l\{\sigma_2(l^{-1}), e\}.$$

Note that $\sigma_1\sigma_2 = 1_L$. For $k = 1$ the equation (12) is the same as:

$$e \rhd' \{e^{-1}, X \rhd f\}^{-1}X \rhd l = (e \rhd' \sigma_1(e)^{-1}X \rhd l)e \rhd' \{e^{-1}, X \rhd (\sigma_2(l)^{-1}f)\}^{-1},$$

or

$$(X \rhd l^{-1})e \rhd' \{e^{-1}, X \rhd f\} = e \rhd' \{(e)^{-1}, X \rhd (\sigma_2(l)^{-1}f)\}(e \rhd' \sigma_1(e)^{-1}X \rhd l^{-1}).$$

This can be proved as follows, by using the equation(11) of the lemma(4.2.2)

$e \rhd' \{e^{-1}, X \rhd (\sigma_2(l)^{-1}f)\}(e \rhd' \sigma_1(e)^{-1}X \rhd l^{-1})$

$$= (\sigma_2(X \rhd l^{-1})e \rhd' \{e^{-1}, X \rhd f\})(e \rhd' \{e^{-1}, X \rhd \sigma_2(l)^{-1}\})(e \rhd' \sigma_1(e)^{-1}X \rhd l^{-1})$$

$$= (X \rhd l^{-1}e \rhd' \{e^{-1}, X \rhd f\})(X \rhd l)(e \rhd' \{e^{-1}, X \rhd \sigma_2(l)^{-1}\})(e \rhd' \sigma_1(e)^{-1}X \rhd l^{-1})$$

where we have used that $(\sigma_2 : L \to M, \rhd')$ is a crossed module.
Now , by using the condition(d) of the definition(4.2.5):
$(e \rhd' \{e^{-1}, X \rhd \sigma_2(l)^{-1}\})(e \rhd' \sigma_1(e)^{-1}X \rhd l^{-1})$

$$= e \rhd' (\{X \rhd \sigma_2(l)^{-1}, e^{-1}\}^{-1}(X \rhd l^{-1}))$$

$$= (e\sigma_2(X \rhd l^{-1})) \rhd' (\{X \rhd \sigma_2(l), e^{-1}\})e \rhd' (X \rhd l^{-1})$$

by equation (11)

$$= (e \rhd' X \rhd l^{-1})e \rhd' (\{X \rhd \sigma_2(l), e^{-1}\})$$

since $(\sigma_2 : L \to M, \rhd')$ is a crossed module.

$$= e \rhd' ((X \rhd l^{-1})\{X \rhd \sigma_2(l), e^{-1}\})$$

$$= X \rhd l^{-1}$$

the general case of equation(12) follows from $k = 1$ and $l = 1$ cases by the interchange law for the horizontal and vertical compositions.
Let us now prove the condition (13) of the definition (2.3.2) , the first condition is equivalent to:

$$(ef) \rhd' \{f^{-1}e^{-1}, X \rhd g\}^{-1} = (ef) \rhd' \{f^{-1}, \sigma_1(e)^{-1}X \rhd g\}^{-1}e \rhd' \{e^{-1}, X \rhd g\}^{-1}$$

this follows from the equation (7). The second condition is equivalent to:

$$e \rhd' \{e^{-1}, X \rhd fX \rhd g\}^{-1} = e \rhd' \{e^{-1}, X \rhd f\}^{-1}((X \rhd f)e) \rhd' \{e^{-1}, X \rhd g\}^{-1}$$

this follows from the equation (10). We have proved that any 2-crossed module $\mathfrak{T}$ defines a Gray 3-groupoid $C(\mathfrak{T})$, with a single object. Conversely, with revising the process: a Gray 3-groupoid C together with an object $x \in C_0$ of it defines a

2-crossed module.

□

Chapter 5

A Gray Category of Chain Complexes

As Forrester-Barker in [17] concluded, a 2-category of length 1 chain complexes $ch_k^{(1)}$ is equivalent to k-vector spaces in groups. In this chapter, we discuss the 3-categories of length 2 chain complexes which are analogues of K-vector spaces in groups as Gray categories (see [24]).

Definition 5.0.1. We work in Ch, the category of chain complexes over a field K. Suppose C is a chain complex, then we set

$$(I \otimes C)_n = C_n \oplus C_n \oplus C_{n-1},$$

with differential

$$\delta^{I\otimes C}(x, y, z) = (\delta_{x-z}, \delta_{y+z}, -\delta_z),$$

to get a chain complex $I \otimes C$. This is a cylinder structure given by:

$$e_0 : C \to I \otimes C$$

such that

$$e_0(c)(x) = (x, 0, 0)$$

$$e_1 : C \to I \otimes C$$

such that

$$e_1(c)(y) = (0, y, 0)$$

$$\sigma : I \otimes C \to C$$

such that

$$\sigma(x, y, z) = x + y$$

Suppose $f_0, f_1 : C \to D$ are two chain maps and $h : I \otimes C \to D$ is a homotopy such that $h : f_0 \simeq f_1$.
Then, there is a degree 1 map h' where $h'_n : C_n \to D_{n+1}$ such that

$$h(x, y, z) = f_0(x) + f_1(y) + h'(z).$$

We can recover f_1 from f_0 and h' by the chain homotopy formula

$$f_1 = f_0 + \delta h'.$$

So, we will use (f_0, h') as an alternative form for h,

$$h : h \to (f_0, h') \quad or \quad h = (f_0, h').$$

The cylinder construction $I \otimes I \otimes C$ is given by

$$(I \otimes I \otimes C)_n = (C_n \oplus C_n \oplus C_{n-1}) \oplus (C_n \oplus C_n \oplus C_{n-1}) \oplus (C_{n-1} \oplus C_{n-1} \oplus C_{n-2})$$

Suppose $X = (x, y, z) \in I \otimes C$. The differential and the face operator:

$$e_i(I \otimes C) = I \otimes e_i(C) : I \otimes C \to I \otimes I \otimes C,$$

are given by the formula
$\delta(X_0, X_1, X_2) =$
$(\delta_{x_0 - z_0 - x_2}, \delta_{y_0 - z_0 - y_2} \delta_{z_0 - z_2}; \delta_{x_1 - z_1 + x_2}, \delta_{y_1 + z_1 + y_2}, -\delta_{z_1 + z_2}; \delta_{x_2 - z_2}, \delta_{y_2 + z_2}, -\delta_{Z_1})$
$e_0(I \otimes C)(X) = (X; 0; 0)$
so $I \otimes e_0(C) = (x, 0, 0; y, 0, 0; z, 0, 0)$ $e_1(I \otimes C)(X) = (0; X; 0)$
so $I \otimes e_1(C) = (0, x, 0; 0, y, 0; 0, z, 0)$.

A 2-homotopy α is a homotopy between homotopies having the same *dom* and *codom*:

$$\alpha : I \otimes I \otimes C \to D; \alpha : h_0 \simeq h_1$$

$$h_0 e_0(c) = h_1 e_0(c) = f_0 \quad and \quad h_0 e_1(c) = h_1 e_1(c) = f_1$$

$$\alpha e_0(I \otimes C)(X) = h_0 \quad or \quad \alpha(I \otimes e_0(C)) = f_0 \delta_c$$

$$\alpha e_1(I \otimes C)(X) = h_1 \quad or \quad \alpha(I \otimes e_1(C)) = f_1 \delta_c.$$

Now, α is a chain map $\alpha \to (f_0, h'_0, \alpha')$ where $\alpha' \in Hom(C, D)_2$, that is $(\alpha')_n : C_n \to D_{n+2}$, is a degree 2 map.
In fact, writing $X = (x, y, z) \in I \otimes C$, we get:

$$\alpha(X_0, X_1, X_2) = f_0(x_0 + x_1) + f_1(y_0 + y_1) + h'_0(z_0) + h'_1(z_1) + \alpha'(z_2).$$

But as $\alpha : h_0 \simeq h_1$ it can be recovered from α', h'_0 and f_1, that means

$$f_1 = f_0 + \delta.h'_0 \quad and \quad h'_1 = h'_0 + \delta.\alpha'.$$

Two 2-homotopies α_0, α_1 determine the same 2-track, which is an equivalence class of relative homotopy classes of 2-homotopies (see [18]).
If

$$\alpha_0 \to (f_0, h'_0, \alpha'_0)$$

$$\alpha_1 \to (f_1, h'_1, \alpha'_1)$$

We write $[\alpha_0] = [\alpha_1]$ if there is $A \in Hom(C, D)_3$ such that $\alpha'_1 = \alpha'_0 + \delta.A$.

From the definition above, we can define a length 2 chain complex over a field K simply as the following:
Suppose C_0, C_1 and C_2 be vector spaces over K and if $\sigma_2 : C_2 \to C_1$ and $\sigma_1 : C_1 \to C_0$ are linear transformations. Then $C_2 \xrightarrow[\sigma^1]{} C_1 \xrightarrow{\sigma_1} C_0$ is a length 2 chain complex.
As we are dealing with a 3-categorical generalisation of group representations, we need to find a 3-categorical analogue of $Vect_K$.
Now, we include an algebraic structure of Gray categories on Ch which is Gray-

category of arbitrary length chain complexes over $Vect_K$.

Definition 5.0.2. A structure of a Gray category of length 2 chain complexes (ch_k^2) consists of:

Ch_0: collection of chain complexes (objects);

Ch_1: collection of chain maps (1-morphisms);

Ch_2: collection of homotopies (2-morphisms);

Ch_3: collection of 2-homotopies (3-morphisms);

Source and target maps as follows:

$$s_n^i, t_n^i : Ch_i \to Ch_n; 0 \leq n \leq i \leq 3$$

1. $i = 1, n = 0$

   - $s_0^1 = dom(f_0)$
   - $t_0^1 = codom(f_0)$

2. $i = 2, n = 1$

   - $s_1^2(f_0, h_0') = f_0$
   - $t_1^2(f_0, h_0') = f_0 + \sigma h'$

3. $i = 3, n = 2$

   - $s_2^3(f_0, h_0, [\alpha']) = (f_0, h_0')$
   - $t_2^3(f_0, h_0', [\alpha']) = (f_0, h_0' + \sigma\alpha')$

$$\begin{tikzcd}
C_2 \arrow[r] \arrow[d, "\sigma_2^C"'] & D_2 \arrow[r] \arrow[d, "\sigma_2^D"'] & E_2 \arrow[d, "\sigma_2^E"] \\
C_1 \arrow[r] \arrow[d, "\sigma_1^C"'] \arrow[ur, "h_2'"] & D_1 \arrow[r] \arrow[d, "\sigma_1^D"'] \arrow[ur, "k_2'"] & E_1 \arrow[d, "\sigma_1^E"] \\
C_0 \arrow[r] \arrow[ur, "h_1'"] \arrow[uur, dotted, "\alpha'"] & D_0 \arrow[r] \arrow[ur, "k_1'"] \arrow[uur, dotted, "\alpha'"] & E_0
\end{tikzcd}$$

1- vertical composition

$$\#_n : Ch_{n+1 s_n} \times_{t_n} Ch_{n+1} \to Ch_{n+1}$$

Here we have different levels of vertical compositions starting with 3-morphisms and ending with 1-morphisms.

- n=2 suppose $\alpha \to (f_0, h_0', [\alpha'])$
  $\beta \to (g_0, k_0', [\beta']$, with $t_2(\alpha) = s_2(\beta)$, $(f_0, h_0' + \sigma\alpha') = (g, k_0')$

$$\beta \#_2 \alpha = (f_0, h_0', [\alpha' + \beta'])$$

- n=1
  $\#_1 : Ch_{2s_1} \times_{t_1} Ch_2 \to Ch_2$
  suppose $h \to (f_0, h')$ and $k \to (g_0, k')$
  with $t_1(h) = s_1(k)$, $(f_0 + \sigma h') = g_0$

$$k \#_1 h = (f_0, h' + k')$$

- n=0
  $\#_0 : Ch_{1s_0} \times_{t_0} Ch_1 \to Ch_1$
  suppose $f \to f_0$ and $g \to g_0$
  with $t_0(f) = s_0(g)$, $codom(f) = dom(g)$

As a Gray category has various forms of whiskering (see definitions (2.3.1) and (2.3.2)), we do that as in the following:

$$\#_n : Ch_{is_n} \times_{t_n} Ch_{n+1} \to Ch_i$$

and

$$\#_n : Ch_{n+1s_n} \times_{t_n} Ch_i \to Ch_i,$$

where $n + 1 < i \leq 3$.
Here, we apply the above two forms of whisking on different levels.

1. n=1 , i=3

$$\#_1 : Ch_{3s_1} \times_{t_1} Ch_2 \to Ch_3$$

If

$$h = (f_0, h') \in Ch_2 \qquad \textit{and} \qquad \beta = (g_0, k', [\beta'])$$

with $t_1(h) = s_1(\beta)$, $(f_0 + \sigma h') = g_0$, it follows

$$\beta \#_1 h = (f_0, k' + h', [\beta'])$$

The whiskering from the other hand gives:

$$\#_1 : Ch_{2s_1} \times_{t_1} Ch_3 \to Ch_3$$

we have

$$\alpha = (f_0, h'_0, [\alpha']) \in Ch_3 \quad and \quad g = (g_0, k') \in Ch_2$$

with $t_1(\alpha) = s_1(g)$, $(f_0 + \sigma h') = g_0$, it follows

$$g \#_1 \alpha = (f_0, h' + k', [\alpha'])$$

2. n=0 , i=2

$$\#_0 : Ch_{2s_0} \times_{t_0} Ch_1 \to Ch_2$$

If

$$k = (g_0, k') \in Ch_2 \quad where \quad f \in Ch_1$$

with $t_0(f) = s_0(k)$, it follows

$$k \#_0 f = (g.f, k'f)$$

Again for n=0 , i=2

$$\#_0 : Ch_{1s_0} \times_{t_0} Ch_2 \to Ch_2$$

If

$$h = (f_0, h') \in Ch_2 \quad where \quad g \in Ch_1$$

with $t_0(h) = s_0(g)$, it follows

$$g \#_0 h = (g.f_0, g.h')$$

3. n=0 , i=2

$$\#_0 : Ch_{3s_0} \times_{t_0} Ch_1 \to Ch_3$$

If

$$\alpha = (g_0, k'_0, [\alpha']) \in Ch_3 \quad where \quad f \in Ch_1$$

with $t_0(f) = s_0(\alpha)$, it follows

$$\alpha\#_0 f = (g.f_0, k'_0.f, [\alpha'.f])$$

Again for n=0 , i=3

$$\#_0 : Ch_{1s_0} \times_{t_0} Ch_3 \to Ch_3$$

If

$$g \in Ch_1, \alpha = (f_0, h'_0, [\alpha']) \;\; where \;\; g \in Ch_3$$

with $t_0(g) = s_0(\alpha)$, it follows

$$g\#_0\alpha = (g.f_0, g.h'_0, [g.\alpha'])$$

Now, we determine another kind of composition as follows:

2- horizontal composition

$$\#_0 : Ch_{2s_0} \times_{t_0} Ch_2 \to Ch_3$$

If

$$h = (f_0, h') \in Ch_2 \quad where \quad k = (g_0, k') \in Ch_2$$

with $t_0(f) = s_0(g)$, it follows

$$k\#_0 h = (g_0.f_0, g_0.h' + k'.f_1, [k'.h']).$$

Now, we note that $Ch(C, D)$ is the collection of elements (1-,2-,3- morphisms) with a source C and a target D. There are n-compositions on $Ch(C, D)$ given by $\#_{n+1}$, $n = 0, 1$ and identities.

- Here we describe the identity of 1-morphisms
  $id- : C_i \to C_{i+1}$, $0 \leq i \leq 2$.
  Then,

$$idf = (f, 0) \quad and \quad idh = (f_0, h', 0)$$

- If $h \in Ch_2$ is a 2-morphism, then $h = (f_0, h')$, where

$$dom(f_0) = C, codom(f_0) = D, h \in Hom(C, D)_1 \quad and \quad k : f_1 \to f_1 + \sigma k'$$

  since, $k \#_1 h = (f_0, k' + h')$ for all $h \in Ch(C, D)$.
  There is

$$h^{-1} \in Ch(C, D), h = (f_0, h') \quad and \quad h^{-1} = (f_0, \sigma.h', -h')$$

  In 3-morphism of $Ch(C, D)$ act in a similar way under $\#_2$ making $Ch(C, D)$ in to 2-groupoid.

- Let $g : D \to E$, where

$$g \#_0 - : Ch(C, D) \to Ch(D, E)$$

  so, $g \#_0 (k \#_1 h) = (g.f_0, g.(k' + h'))$, where

$$h = (f_0, \sigma.h') \quad and \quad k = (g_0, \sigma.k')$$

  Then

$$(g \#_0 k) \#_1 (g \#_0 h) = (g.g_0, g.k') \#_1 (g.f_0, g.h') = (g.f_0, g.(k' + h'))$$

  Similarly, if α and β are 3-morphisms, so that $\beta \#_2 \alpha$ is defined where $\alpha = (f_0, h'_0, [\alpha'])$ and $\beta = (g_0, k'_0, [\beta'])$
  Then

$$g \#_0 (\beta \#_2 \alpha) = g \#_0 (f_0, h'_0, [\alpha' + \beta']) = (g.f_0, g.h'_0, [g\alpha' + g\beta'])$$

and

$$(g\#_0\beta)\#_2(g\#_0\alpha) = (gg_0, g.k'_0, [g\beta'])\#_2(gf_0, gh'_0, [g\alpha'])$$

$$= (g.f_0, g.h'_0, [g\alpha' + g\beta'])$$

- If g is an identity, the $g\#_0-$ and $-\#_0 g$ likewise the relevant identity 2-functor.

- Let $h = (f_0, h')$ and $k = (g_0, k')$, where $codom(f) = dom(g)$ and $f_1 = f_0 + \sigma h'$ and $g_1 = g_0 + \sigma k'$, $k\#_0 h$ is defined
  so,

  $$s_2(k\#_0 h) = (g_0.f_0, g_0.h' + k'.f_1)$$

  Then

  $$(k\#_0 f_1)\#_1(g_0\#_0 h) = (g_0.f_1, k'.f_1)\#_1(g_0.f_0, g_0.h')$$

  $$= (g_0 f_0, g_0 h' + k'.f_1)$$

  to know $t_2(k\#_0 h)$, we will need to know $\sigma(k'h')$

  $$(k'h')_n : C_n \to D_{n+1} \to E_{n+2}$$

  If $\alpha' \in Hom(C, E)_2$, then the usual differential formula gives

  $$(\sigma\alpha')_n(c) = \sigma^E(\alpha'_n(c)) - \alpha'_{n-1}(\sigma^C(c)) \quad for \quad c \in C$$

  For $\alpha' = k'h'$ and $c \in C$
  $\sigma(k'h')(c) = \sigma^E(k'_{n+1}h'_n(c) - k'_n h'_{n-1}(\sigma^C(c)) = (\sigma k')_{n+1}h'_n(c) - k'_n((\sigma h')_{n-1}(c))$
  Then

  $$t_2(k\#_0 h) = (g_0 f_0, g_0 h' + f_1 k' + \sigma k'h' - k'\sigma h')$$

  $$= (g_0 f_0, g_1 h' + k' f_0)$$

  $$= (g\#_0 h)\#_1(k\#_0 f_0)$$

- suppose we have $\alpha = (f_0, h'_0, [\alpha'])$ such that $dom(f_0) = C$ and $codom(f_0) = D$ where $f_1 = f_0 + \sigma h'_0$ and $h'_1 = h_0 + \sigma\alpha'$
  When

$$s_2(\alpha) = h_0 = (f_0, h'_0) \quad and \quad t_2 = h_1 = (f_0, h'_1)$$

Let $k = (g_0, k') : g_0 \simeq g_1$ so $g_1 = g_0 + \sigma k' : D \to E$.

Then,

$$((g_1\#_0\alpha)\#_1(k\#_0 f_0))\#_2(k\#_0 h_0) \quad and \quad (k\#_0 h')\#_2((k\#_0 f_1)\#_1(g_0\#_0\alpha))$$

that means

$$[g_0\alpha' + k'h'_1] = [k'h_0 + g_1\alpha']$$

consider the composite $k'\alpha' \in Hom(C, E)_3$ this is given by

$$(k'\alpha')_n = k_{n+1}\alpha'_n.$$

So,

$$\begin{aligned}
\sigma(k'\alpha')(c) &= \sigma^E(k'\alpha')(c) + k'\alpha'(\sigma^c c)\\
&= \sigma^E k'(\alpha'(c)) + k'(\sigma^D\alpha'(c)) - k'(\sigma^D\alpha'(c)) + k'\alpha'\sigma^c(c)\\
&= \sigma k'(\alpha'(c)) - k'(\sigma\alpha'(c))\\
&= g_1\alpha'(c) - g_0\alpha'(c) - k'h'(c) - k'h'_0(c)\\
&= (g_1\alpha'(c) - k'h'_0(c)) - (g_0\alpha'(c) + k'h'_1(c)).
\end{aligned}$$

So, the two classes are the same.

The dual rule takes $\beta \in Ch_3$ and $h = (f_0, h') \in Ch_2$,

$\beta = (g_0, k'_0, [\beta'])$ with $g_0 \simeq g_1 = g_0 + \sigma k'_0$ and $k'_1 = k'_0 + \sigma\beta'$ with $g_0 : D \to E$, where $f_0 : C \to D$, $h = f_0 \simeq f_1 = f_0 + \sigma h'$.

The formula that needs verifying is the equality of

$$(k_1\#_0 h)\#_2((\beta\#_0 f_1)\#_1(g_0\#_0 h))$$

and

$$((g_1\#_0 h)\#_1(\beta\#_0 f_0))\#_2(k_0\#_0 h)$$

That means $\beta' h' : C \to E$ of degree 3.

Suppose $\beta = (g_0, k'_0, [\beta'])$ with $dom(g_0) = D$, $codom(g_0) = E$ and $h =$

(f_0, h'_0) where $f_0 : C \to D$

$$[k'_0 h' + f_0 \beta'] = [f_1 \beta' + k'_1 h'].$$

So

$$\begin{aligned}
\sigma(\beta' h')(c) &= \sigma^E(\beta' h'(c)) \\
&= \sigma^E(\beta' h'(c)) + \beta'(\sigma^c(c))h' \\
&= \sigma^E(\beta'(c))h' + (\sigma^D \beta'(c))h' - (\sigma^D(\beta')h' + \sigma\beta'\sigma^C(c)h' \\
&= \beta'(c)\sigma h' - \sigma(\beta'(c))h'
\end{aligned}$$

- This axiom describes the interaction of interchange with composition.
  If we have
  $h = (f_0, h')f_0 \simeq f_1 = f_0 + \sigma h', k = (f_1, k')$, such that $h, k \in Ch(C, D)$.
  Again $\iota = (g_0, \iota') : g_0 \simeq g_1 = g_0 + \sigma\iota'$, such that $\iota \in Ch(D, E)$.
  The axioms states the equality of

  $$\iota \#_0 (k \#_1 h) \quad \textit{and} \quad (g_1 \#_0 k)\#_1(\iota \#_0 h))\#_2((\iota \#_0 k)\#_1(g_0 \#_0 h))$$

  as $k\#_1 h = (f_0, k' + h')$
  We have

  $$\iota \#_0 (k \#_1 h) = (g_0 f_0, g_0(k' + h') + \iota' f_1, [\iota'(k' + h')])$$

  The dual equality is the same
  $(k\#_1 h)\#_0 \iota$ such that $k\#_1 h = (f_0, k' + h')$,
  so

  $$(k\#_1 h)\#_0 \iota = (f_0 g_0, (k' + h')g_0 + f_1 \iota', [(k' + h')\iota'])$$

- This axiom describes the interaction of interchange with identities
  Given $f : C \to D, f \in Ch_1$
  and

  $$k = (g, k') = g \simeq g = g + \sigma k' : D \to E$$

  so

  $$k\#_0 id_f = (g, k')\#_0(f, 0) = (gf, k'f, 0) = id_{k\#_0 f}$$

similarly for the dual.

- This last axiom gives that $\#_0$ is associative.
  Given $c \in Ch(C,D)_p$, $c' \in Ch(D,E)_q$ and $c'' \in Ch(E,F)_r$.
  Then,
  $$(c''\#_0c')\#_0c = c''\#_0(c'\#_0c).$$

From here we suppose (ch_k^2) as a sub3-groupoid by taking in consideration just the invertible maps.

5.1 A matrix formulation for calculations

Proceeding from the fact, in linear algebra, is that the linear transformations between vector spaces are equivalent to matrices over a field K. $Ch_K^{(2)}$ have been considered in this chapter earlier as linear transformation, we will work in this section to represent $Ch_K^{(2)}$ as the matrices for making the calculation easy to performed. The category of length 2 chain complexes of K-vector spaces is denoted by $Ch_K^{(2)}$. As it has been shown in the last section that $Ch_K^{(2)}$ is an 3-groupoid with 0,1,2,3-morphisms such that 0-morphisms are length 2 chain complexes, 1-morphisms are chain maps between chain complexes, 2-morphisms are homotopies between chain maps and 3-morphisms are 2-homotopies between homotopies with all kinds of compositions and whiskers between morphisms. To make calculation simpler, we will describe $Ch_K^{(2)}$ by matrices.
In 2003, Forrester-Barker in [17] showed that for a category of length 1 chain complexes $\mathfrak{X} = Ch_K^{(1)}$ with differential $\delta : C_1 \to C_0$ can be represented by an $n_0 \times n_1$ a single matrix Δ^C for each one object in this category where n_i is the dimension of C_i.
Here a certain matrix formulation will be considered for a length 2 chain complex. It will be shown that this can be extended depending on the length of the chain complex. An example using $Ch_K^{(2)}$ will also be given.
Objects in $Ch_K^{(2)}$ are chain complexes of length 2, denoted by (γ) with two differentials $\gamma_2^C : C_2 \to C_1$ and $\gamma_1^C : C_1 \to C_0$. These differentials can be represented by $n_1 \times n_2$ and $n_0 \times n_1$ matrices Δ_2^C and Δ_1^C with $\Delta_C^1.\Delta_C^2 = [0]$.
Suppose that γ' is another chain complex of length 2, with two differentials:

$\gamma_2'^d : d_2 \to d_1$ and $\gamma_1'^d : d_1 \to d_0$, where the dimension of D_i is m_i and $\Delta_D^1.\Delta_D^2 = [0]$. Moving on to the chain map between γ and γ' which is $f : \gamma \to \gamma'$ is given by the triple of matrices $(F_2(m_2 \times n_2), F_1(m_1 \times n_1), F_0(m_0 \times n_0))$ with the following conditions and (further details are shown in the diagram):

1. $F_0.\Delta_C^1 = \Delta_D^1.F_1$ which is an $m_0 \times n_1$ matrix.

2. $F_1.\Delta_C^2 = \Delta_D^2.F_2$ which is an $m_1 \times n_2$ matrix.

$$\begin{tikzcd}
C_2 \arrow[r, "F_2"] \arrow[d, "\gamma_2^C"'] & D_2 \arrow[d, "\gamma_2'^d"] \\
C_1 \arrow[r, "F_1"] \arrow[d, "\gamma_1^C"'] & D_1 \arrow[d, "\gamma_1'^d"] \\
C_0 \arrow[r, "F_0"] & D_0
\end{tikzcd}$$

Moving up to dimension two, we define a homotopy which is a map between two chain maps. So let us suppose we have another chain map between γ and γ' such that $f' : \gamma \to \gamma'$ and a homotopy $h : f \simeq f'$. As we work on $Ch_K^{(2)}$ so there are two chain homotopies

1. $h_1' : C_0 \to D_1$ with a corresponding $m_1 \times n_0$ matrix H_1 such that $H_1.\Delta_1^C = F_1' - F_1$ and $\Delta_1^D.H_1 = F_0' - F_0$

2. $h_2' : C_1 \to D_2$ with a corresponding $m_2 \times n_1$ matrix H_2 such that $H_2.\Delta_2^C = F_2' - F_2$ and $\Delta_2^D.H_2 = F_1' - F_1$ we can describe it just like in the following diagram:

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "F_2"] \arrow[r, shift right, "F_2'"'] \arrow[d, "\gamma_2^C"'] & D_2 \arrow[d, "\gamma_2'^d"] \\
C_1 \arrow[ur, dotted, "h_2'"] \arrow[r, shift left, "F_1"] \arrow[r, shift right, "F_1'"'] \arrow[d, "\gamma_1^C"'] & D_1 \arrow[d, "\gamma_1'^d"] \\
C_0 \arrow[ur, dotted, "h_1'"] \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_0'"'] & D_0
\end{tikzcd}$$

Again raising the dimension to three, a 2-homotopy is a map between two homotopies. Suppose we have another homotopy between f and f' which is $h^* : f \simeq f'$ such that there are another two chain homotopies

1. $h_1^{*\prime} : C_0 \to D_1$ with a corresponding $m_1 \times n_0$ matrix H_1^* such that $H_1^*.\Delta_1^C = F_1' - F_1$ and $\Delta_1^D.H_1^* = F_0' - F_0$

2. $h_2^{*\prime} : C_1 \to D_2$ with a corresponding $m_2 \times n_1$ matrix H_2^* such that $H_2^*.\Delta_2^C = F_2' - F_2$ and $\Delta_2^D.H_2^* = F_1' - F_1$

A 2-homotopy between homotopies $\alpha : H \simeq H^*$ is given by $\alpha : C_0 \to D_2$, where α is a matrix with dimension $m_2 \times n_0$ which must satisfy the following conditions:

1. $\Delta_D^2.\alpha = H_1^* - H_1$

2. $\alpha.\Delta_C^1 = H_2^* - H_2$

$$\begin{tikzcd}
C_2 \arrow[r] \arrow[d, "\gamma_2^C"'] & D_2 \arrow[d, "\gamma_2^d"] \\
C_1 \arrow[ur, Rightarrow, shift left, "h_2'"] \arrow[ur, Rightarrow, shift right, "h_2^*"'] \arrow[d, "\gamma_1^C"'] & D_1 \arrow[d, "\gamma_1'^d"] \\
C_0 \arrow[r] \arrow[ur, Rightarrow, shift left, "h_1'"] \arrow[ur, Rightarrow, shift right, "h_1^*"'] \arrow[uur, dotted, "\alpha'"] & D_0
\end{tikzcd}$$

To summarize, we have obtain the following theorem. Here we will use the same notations in this section:

Theorem 5.1.1. *The morphisms of 3-groupoid length 2 chain complexes $Ch_K^{(2)}$ can be described as matrices as the following:*

1. *Chain complexes of length 2 (0-morphisms), denoted by (γ^C) with two differentials $\gamma_2^C : C_2 \to C_1$ and $\gamma_1^C : C_1 \to C_0$. These differentials can be represented by $n_1 \times n_2$ and $n_0 \times n_1$ matrices Δ_2^C and Δ_1^C with $\Delta_C^1.\Delta_C^2 = [0]$*

2. *Suppose we have another chain complexes of length 2, denoted by (γ^D) with two differentials Δ_2^D and Δ_1^D with $\Delta_D^1.\Delta_D^2 = [0]$.*
   A chain map between γ^C and γ^D (1-morphisms) which is $f : \gamma^C \to \gamma^D$ is given by the triple of matrices $(F_2(m_2 \times n_2), F_1(m_1 \times n_1), F_0(m_0 \times n_0))$ with the following conditions:

   (a) $F_0.\Delta_C^1 = \Delta_D^1.F_1$ *which is an $m_0 \times n_1$ matrix.*

   (b) $F_1.\Delta_C^2 = \Delta_D^2.F_2$ *which is an $m_1 \times n_2$ matrix.*

3. *A homotopy (2-morphisms) which is a map between two chain maps. So let us suppose we have another chain map between γ^C and γ^D such that $f' : \gamma \to \gamma'$ and a homotopy $h : f \simeq f'$. As we work on $Ch_K^{(2)}$ so there are two chain homotopies*

(a) $h_1' : C_0 \to D_1$ *with a corresponding* $m_1 \times n_0$ *matrix* H_1 *such that* $H_1.\Delta_1^C = F_1' - F_1$ *and* $\Delta_1^D.H_1 = F_0' - F_0$

(b) $h_2' : C_1 \to D_2$ *with a corresponding* $m_2 \times n_1$ *matrix* H_2 *such that* $H_2.\Delta_2^C = F_2' - F_2$ *and* $\Delta_2^D.H_2 = F_1' - F_1$.

4. *A 2-homotopy (3-morphisms) is a map between two homotopies. Suppose we have another homotopy between* f *and* f' *which is* $h^* : f \simeq f'$ *such that there are another two chain homotopies A 2-homotopy between homotopies* $\alpha : H \simeq H^*$ *is given by* $\alpha : C_0 \to D_2$, *where* α *is a matrix with dimension* $m_2 \times n_0$ *which must satisfy the following conditions:*

   (a) $\Delta_D^2.\alpha = H_1^* - H_1$

   (b) $\alpha.\Delta_C^1 = H_2^* - H_2$.

Proof: This follows from the section 5.1 which explains explicitly the fact that morphisms of 3-groupoids can be described as matrices with some conditions.

5.2 Examples of a matrix formulation

For ease of computation, we work over $\mathbb{R}$ with the standard basis for I !R^n. A chain complexes of length 2, $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^2$, is given
$C = (\Delta_C^1, \Delta_C^2; \Delta_C^1.\Delta_C^2 = [0])$ where

$$\Delta_C^1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad and \quad \Delta_C^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

with $\Delta_C^1.\Delta_C^2 = [0]$
$D = (\Delta_D^1, \Delta_D^2; \Delta_D^1.\Delta_D^2 = [0])$ such that

$$\Delta_D^1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad and \quad \Delta_D^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

with $\Delta_C^1.\Delta_C^2 = [0]$.
Chain maps between C and D as the following:

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad and \;\; F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & o \\ 0 & 0 & 2/3 \end{bmatrix},$$

such that $F_0.\Delta^1_C = \Delta^1_D.F_1$ and $F_1.\Delta^2_C = \Delta^2_D.F_2$.

Suppose we have another chain map, as follows:

$$F'_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F'_1 = \begin{bmatrix} 2 & 1/2 \\ 0 & 0 \end{bmatrix} \quad and \;\; F'_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & o \\ 0 & 0 & 2/3 \end{bmatrix},$$

such that $F'_0.\Delta^1_C = \Delta^1_D.F'_1$ and $F'_1.\Delta^2_C = \Delta^2_D.F'_2$.

Moving up to the homotopy which are (2×2) two matrices H_1 *and* $H_2 = F \simeq F'$ such that

1. $H_1.\Delta^1_C = F'_1 - F_1$ and $\Delta^1_D.H_1 = F'_0 - F_0$

2. $H_2.\Delta^2_C = F'_2 - F_2$ and $\Delta^2_D.H_2 = F'_1 - F_1$

where

$$H_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \quad and \;\; H_2 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \\ 0 & 1/6 \end{bmatrix}$$

Again suppose there are another chain maps as the following:

$$F''_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F''_1 = \begin{bmatrix} 2 & 1/2 \\ 0 & 0 \end{bmatrix} \quad and \;\; F''_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & o \\ 0 & 0 & 2/3 \end{bmatrix},$$

such that $F''_0.\Delta^1_C = \Delta^1_D.F''_1$ and $F''_1.\Delta^2_C = \Delta^2_D.F''_2$.

It follows that, there are other homotopies which are given by two (2×2) matrices H'_1 and $H'_2 = F' \simeq F''$ such that

1. $H'_1.\Delta^1_C = F''_1 - F'_1$ and $\Delta^1_D.H'_1 = F''_0 - F'_0$

2. $H'_2.\Delta^2_C = F''_2 - F'_2$ and $\Delta^2_D.H'_2 = F''_1 - F'_1$

where

$$H_1' \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad and \;\; H_2' = \begin{bmatrix} 0 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Progressing to a 2-homotopy which is a (3×2) matrix $\alpha = H \simeq H'$ such that $\Delta_D^2.\alpha = H_1' - H_1$ and $\alpha.\Delta_c^1 = H_2' - H_2$ where

$$\alpha = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 2/3 & -1/6 \end{bmatrix}$$

Hence, we actually represent length 2 chain complexes $C = \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^2$ and $D = \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^2$ in thr matrix form with Δ_C^1 and Δ_C^2 as objects of C and Δ_D^1 and Δ_D^2 as objects of D. Also, chain maps between them F_0, F_1 and F_2 are matrices and so on with chain homotopies and 2-homotopies.

Chapter 6

Automorphisms of Linear Transformations

Suppose that $\gamma : C_2 \to C_1 \to C_0$ is a linear transformation of vector spaces. As explained previously it is an object in $ch_k^{(2)}$. Objects of $ch_k^{(2)}$, chain automorphisms $\gamma \to \gamma$, homotopies between them and 2-homotopies, form a subcategory of $ch_k^{(2)}$. As is known, automorphisms indicate an isomorphism of an object to itself. Our aim is to develop a representation of cat^2-groups and crossed squares. In this chapter, we will introduce some examples of Automorphisms and generalise the form of automorphisms $Aut(\gamma)$.

Definition 6.0.1. Let $\gamma : C_2 \to C_1 \to C_0$ be a length 2 chain complex of k-vector spaces. The **automorphism** cat^2-group of γ, $Aut(\gamma)$, consists of:

1. the group $Aut(\gamma)_1$ of all chain automorphisms $\gamma \to \gamma$.

2. the group $Aut(\gamma)_2$ of all homotopies on $Aut(\gamma)_1$.

3. the group $Aut(\gamma)_3$ of all 2-homotopies on $Aut(\gamma)_2$.

4. morphisms

$$s_0, t_0 : Aut(\gamma)_1 \to Aut(\gamma)_0$$

$$s_1, t_1 : Aut(\gamma)_2 \to Aut(\gamma)_1$$

$$s_2, t_2 : Aut(\gamma)_3 \to Aut(\gamma)_2$$

5. morphisms

$$i_1 : Aut(\gamma)_1 \to Aut(\gamma)_2$$

$$i_2 : Aut(\gamma)_2 \to Aut(\gamma)_3$$

Here, after we show that the morphisms in $Aut(\gamma)$ form a Gray category, we need to show that there is a 0-morphism as $Aut(\gamma)_0$ which is a length 2 chain complex, which can be unpacked as a pair (γ_1, γ_2).
$Aut(\gamma_1)$ consists of 1-morphisms $\gamma \xrightarrow{f} \gamma$,(f_0, f_1, f_2) and $\gamma = (\gamma_1, \gamma_2)$,
$Aut(\gamma_2)$ consists of 2-morphisms

$$\begin{tikzcd} \gamma \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \gamma \end{tikzcd}$$

between chain maps $f = (f_0, f_1, f_2)$ and $f' = (f'_0, f'_1, f'_2)$, which can be unpacked as a triple (h, f, γ) such that $h = (h'_1, h'_2)$, $f = (f_0, f_1, f_2)$ and $\gamma = (\gamma_1, \gamma_2)$, while $Aut(\gamma_3)$ consists of 3-morphisms

$$\begin{tikzcd} f \arrow[r, bend left, "h"] \arrow[r, bend right, "\hat{h}"'] \arrow[r, phantom, "\Downarrow \alpha"] & f' \end{tikzcd}$$

between homotopies $h = (h'_1, h'_2)$ and $\hat{h} = (\hat{h}'_1, \hat{h}'_2)$, which can be unpacked as a quadruple $(\tilde{\alpha}, h, f, \gamma)$
The following diagram explains all the morphisms:

$$\begin{tikzcd} C_2 \arrow[r, shift left, dotted, "f'_2"] \arrow[r, shift right, dotted, "f_2"'] \arrow[d, "\gamma_2"'] & C_2 \arrow[d, "\gamma_2"] \\ C_1 \arrow[ur, shift left, "h'_2"] \arrow[ur, shift right, dotted, "\hat{h}'_2"'] \arrow[r, shift left, dotted, "f_1"] \arrow[r, shift right, dotted, "f_1"'] \arrow[d, "\gamma_1"'] & C_1 \arrow[d, "\gamma_1"] \\ C_0 \arrow[ur, shift left, "h'_1"] \arrow[ur, shift right, dotted, "\hat{h}'_1"'] \arrow[uur, dotted, "\tilde{\alpha}"] \arrow[r, shift left, dotted, "f'_0"] \arrow[r, shift right, dotted, "f_0"'] & C_0 \end{tikzcd}$$

It is easy to compose between the elements of $Aut(\gamma_1)$. Now we are moving to $Aut(\gamma_2)$ which are homotopies as follows:
A homotopy $h : f \to f'$ is a triple

$$((h'_1, h'_2), (f_0, f_1, f_2), (f'_0, f'_1, f'_2)),$$

with h'_1, h'_2 chain homotopies and f the source of h, along with f' the target. These satisfy the chain homotopy conditions

$$f'_0 - f_0 = \gamma_1 h'_1, f'_1 - f_1 = h'_1\gamma_1$$

and

$$f'_1 - f_1 = \gamma_2 h'_2, f'_2 - f_2 = h'_2\gamma_2,$$

we can make an equivalent condition for all these which is given by

$$f' - f = \gamma_1 h'_1 + h'_1\gamma_1 + \gamma_2 h'_2 + h'_2\gamma_2.$$

$Aut(\gamma_3)$ is a 2-homotopy between $Aut(\gamma_2)$ homotopies; $\alpha : h \simeq k$ where $\alpha \in Aut(\gamma_3)$. We can describe them as a triple $(\tilde{\alpha}, h, f)$, where $h = (h'_1, h'_2)$ and $f = (f_0, f_1, f_2)$, with $\tilde{\alpha}$ is a 2-chain homotopy such that h is a source and k as a target, satisfying the following conditions:

$$\tilde{\alpha}\gamma_1 = k'_2 - h'_2$$

and

$$\gamma_2\tilde{\alpha} = k'_1 - h'_1.$$

There is no doubt that the condition for the structure homomorphism in $Aut(\gamma)$ are satisfied. The maps s_0, t_0 give respectively the source $f = (f_0, f_1, f_2)$ and the target $f' = (f'_0, f'_1, f'_2)$, while i_0 is an identity map of homotopy $(h', f) = ((h'_1, h'_2), (f_0, f_1, f_2))$ which comes from a chain map $f = (f_0, f_1, f_2)$ together with the identity homotopy

$$1_{f_0} : f_0 \to f_0, 1_{f_1} : f_1 \to f_1, 1_{f_2} : f_2 \to f_2$$

such that

$$i_0 : f_0 \to 1_{f_0}, i_1 : f_1 \to 1_{f_1}, i_2 : f_2 \to 1_{f_2}$$

Moving up to $Aut(\gamma)_3$ with two of 2-chain homotopies α and $\tilde{\alpha}$. The source of chain homotopy $h = (h', f) = ((h'_1, h'_2), (f_0, f_1, f_2))$ while i_1 maps each chain homotopy $h = (h', f)$ to the identity 2-chain homotopy such that

$$1_{h'_1} : h'_1 \to h'_1, 1_{h'_2} : h'_2 \to h'_2$$

and

$$i_1 : h'_1 \to 1_{h'_1}, i_2 : h'_2 \to 1_{h'_2}$$

The group operation in $Aut(\gamma)_1$ is the composition of chain automorphism. The identity is id_γ, the chain map consisting of the identity linear transformation at both levels.
Since, every $f \in Aut(\gamma)_1$ is a chain automorphism, it has an inverse f^{-1}, which is also a chain automorphism on $\gamma \in Ch_k^2$ and $f^{-1} \in Aut(\gamma)_1$.
Vertical composition provides the group operation $Aut(\gamma)_2$. For example Ch_k^2, in this case if

$$h = ((h'_1, h'_2), (f_0, f_1, f_2) \quad \text{and} \quad k = ((k'_1, k'_2), (g_0, g_1, g_2))$$

are homotopies, the composition $k\#h$ is the homotopy specified by the source chain map $g\#f$ and the chain homotopy

$$(g_1 h'_1 + k'_1 f_0, g_2 h'_2 + k'_2 f_1)$$

and the inverse of

$$((h'_1, h'_2), (f_0, f_1, f_2))$$

is the element

$$((-{f_1}^{-1} h'_1 (f'_0)^{-1}, -{f_2}^{-1} h'_2 (f'_1)^{-1}), f^{-1})$$

where $f = (f_0, f_1, f_2)$. The element in $Aut(\gamma)_2$ can be also joined by vertical composition $\#_1$, which is defined for pairs of 2-morphisms for which the target 1-morphism of the first is the source of the second, that is, if

$$((h'_1, h'_2), (f_0, f_1, f_2))$$

and

$$\hat{h} = (\hat{h}_1, f + \gamma_1 h'_1 + h'_1\gamma_1 + \gamma_2 h'_2 + h'_2\gamma_2) \quad \text{where} \quad h, \hat{h} \in Aut(\gamma)_2$$

The vertical composite is

$$\hat{h}\#_1 h = (\hat{h}_1 + h', f) = ((\hat{h}'_1, \hat{h}'_2) + (h'_1, h'_2), (f_0, f_1, f_2))$$

in $Aut(\gamma)_2$. Turning to the composition in $Aut(\gamma)_3$, in this case, if

$$\alpha = (\tilde{\alpha}, h', f) = (\tilde{\alpha}, (h'_1, h'_2), (f_0, f_1, f_2))$$

and

$$\beta = (\tilde{\beta}, k', g) = (\tilde{\beta}, (k'_1, k'_2), (g_0, g_1, g_2))$$

with target of α is the source of β. The composition is given by

$$\beta\#_2\alpha = (\tilde{\alpha} + \tilde{\beta}, (h'_1, h'_2), (f_0, f_1, f_2))$$

and the identity of this composition as $(0, 1_h, 1_f)$. The inverse of

$$(\tilde{\alpha}, (h'_1, h'_2), (f_0, f_1, f_2))$$

is the element

$$(-f_2^{-1}\tilde{\alpha}(f'_0)^{-1}, h^{-1})$$

An element of $Aut(\gamma)_3$ can also be joined by vertical composition $\#_2$, which is defined for a pair of 3-morphisms for which the target 2-morphism of the first is the same as the source of the second. That is,

$$\alpha = (\tilde{\alpha}, h', f) \quad \text{where} \quad h' = (h'_1, h'_2)$$

and

$$f = (f_0, f_1, f_2)$$

and

$$\beta = (\tilde{\beta}, h' + \gamma\tilde{\alpha} + \tilde{\alpha}\gamma)$$

are in $Aut(\gamma)_3$, the vertical composite is

$$\beta \#_2 \alpha = ([\tilde{\alpha} + \tilde{\beta}], h', f),$$

in the same groupoid operation with each 2-morphism

$$h = (h', f)$$

having an identity

$$1_h = (0, h', f)$$

for vertical composition and every 3-morphism

$$(\tilde{\alpha}, h', f)$$

having the inverse

$$(-\tilde{\alpha}, h' + \gamma\tilde{\alpha} + \tilde{\alpha}\gamma).$$

6.1 Representations of cat^2-groups and crossed squares

The idea of representations of cat^1-groups will be extended to the representation of cat^2-groups in this section as we lift the dimension up. The representation theory of cat^1-groups was defined by Forrester-Barker in [17] as follows:
Suppose $\mathfrak{C}$ is a cat^1-group and $ch_k^{(1)}$ is a length 1 chain complex so the representation ϕ is as follows:

$$\phi : \mathfrak{C} \to ch_k^{(1)}.$$

However, the dimension of cat^1-groups have been lifted in this thesis to cat^2-groups and we have shown that it is the same thing as 3-groups. After that the definition of cat^2-groups representations can be established as follows:

$$\phi^* : \mathfrak{G} \to ch_k^{(2)},$$

where $\mathfrak{G}$ is a cat^2-group and ϕ^* is a 3-functor which takes every single element of $\mathfrak{G}$ to the length 2 chain complexes.

Given $\mathfrak{G}$, to define the representation of $\mathfrak{G}$, which is $\phi^*(\mathfrak{G})$ we must find a chain complex γ to represent as a target object, $\phi^*(\star) = \gamma$. All the elements of cat^2-group $\mathfrak{G}$ must be mapped to elements of $ch_k^{(2)}$, with 0-morphism (a lengths 2 chain complex), 1-morphisms (chain maps), 2-morphisms (homotopies) and 3-morphisms (2-homotopies) and to ϕ^* be a functor, all this mapping must preserve identities and compositions. $Aut(\gamma)$, which is a main concept that was considered as an automorphism in this chapter earlier, is the image of $\mathfrak{G}$.

Considering $Aut(\gamma)$ as a cat^2-group give us another way to define the representation of cat^2-group as follows:

$$\phi^* : \mathfrak{G} \to Aut(\gamma).$$

We explained earlier in section (4.2.3) that cat^2-groups are equivalent to crossed squares. Therefore, we conclude a representation of crossed squares as itself a representation of cat^2-groups.

6.1.1 Faithful representations of cat^2-groups

In group terms, the faithful representation is defined in (3.1.2), to use this concept from a categorical viewpoint, we can define it as follows:

Suppose G is a category and $\phi : G \Longrightarrow Vect_K$ is a faithful functor i.e. if for every $g, h \in G$ and $\phi(g) = \phi(h)$ then $g = h$. As a representation of a cat^2-group is defined as a 3-functor, the faithful representations of cat^2-groups are the faithful 3-functors. With an accurate search to define a faithful 3-functor we can rely on the analysis, mentioned in [11] that a faithful representation of a cat^1-group is a faithful 2-functor, and develop this idea to higher dimensions as we study in this thesis so the faithful representation of cat^2-groups are exactly the faithful 3-functors.

6.1.2 The category of representations of cat^2-groups

Here, we take a categorical view of cat ^{2} - groups representations and determine the morphisms between them to discover a new category.

In sections (3.3) and (3.4) it was introduced the definitions of representations of categories and 2-categories respectively. Here, we define the 3-category of representation of cat^2-groups.
Since the category of (K-linear) representation of a group G is a functor category

$$Rep_k^G = (Vect_k)^G$$

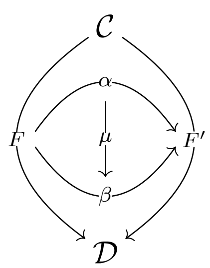
whose objects are functors $G \to Vect_K$ and whose morphisms are natural transformations between such functors.
Lifting up the dimension of categories to 2-categories, cat^1-groups have been described as 2-categories in [14]. So, a 2-category of representations of cat^1-group $\mathfrak{C}$ is a 2-functor 2-category

$$Rep_k^{\mathfrak{C}} = (Ch_k^{(1)})^{\mathfrak{C}}$$

whose objects are 2-functors $\mathfrak{C} \to Ch_k^{(1)}$, whose 1-morphisms are 2-natural transformations between such 2-functors and whose 2-morphisms are called **modification** which are as follows:
For any two 2-natural transformations α and β between two 2-functors F and F' between two 2-categories $\mathcal{C}$ and $\mathcal{D}$ as follows

$$\alpha, \beta : F \to F' : \mathcal{C} \to \mathcal{D}$$

would consist of a function $\mu : \alpha \to \beta$ such that for each 0-morphism $C \in \mathcal{C}$ there is a 2-morphism $\mu_C : \alpha_C \to \beta_C$ in $\mathcal{D}$, we can describe it as follows:



Now moving to the higher dimension, we can define the 3-category of representations of cat^2-group $\mathfrak{G}$ as follows, using the concepts and notations from the previous chapters.

Definition 6.1.1. We consider cat^2-groups as 3-groupoids which are kind of 3-categories in the theorem (4.2.3). A **3-category of representations of cat^2-group $\mathfrak{G}$** is a 3-functor 3-category

$$Rep_k^{\mathfrak{G}} = (Ch_k^{(2)})^{\mathfrak{G}},$$

whose objects are 3-functors $\mathfrak{G} \to Ch_k^{(2)}$, whose 1-morphisms are 3-natural transformations between such 3-functors, whose 2-morphisms are modifications between such 3-natural transformations and whose 3-morphisms are called **perturbation** (see section 2.3 [12] p.15) which are as follows:
For any two modifications μ and μ' between two 3-natural transformations η and η' between two 3-functors K and K' between two 3-categories $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\mu, \mu' : \eta \to \eta' : K \to K' : \mathcal{A} \to \mathcal{B}$$

would consist of a function $\nu : \mu \to \mu'$ such that for each 0-morphism $A \in \mathcal{A}$ there is a 3-morphism $\nu_A : \mu_A \to \mu'_A$ in $\mathcal{B}$, we can describe it simply as follows:

$$\begin{tikzcd}[column sep=large]
& \mathcal{A} \arrow[dd, bend right=80, "K"'] \arrow[dd, bend left=80, "K'"] & \\
K \arrow[rr, bend left=50, "\eta"] \arrow[rr, bend right=50, "\eta'"'] & \mu \xrightarrow{\nu} \mu' & K' \\
& \mathcal{B} &
\end{tikzcd}$$

If $K_1, K_2 : \mathfrak{G} \to Ch_k^{(2)}$ are two representations of $\mathfrak{G}$ with representation complexes γ and γ' respectively. A 3-natural transformation is a morphism $\eta : \gamma \to \gamma'$ such that the following diagram commutes

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\gamma \arrow[d, bend right, "K_1"'] \arrow[d, bend left, "K'_1"] \arrow[r, "\eta_1"] & \gamma' \arrow[d, bend right, "K_2"'] \arrow[d, bend left, "K'_2"] \\
\gamma \arrow[r, "\eta_2"] & \gamma'
\end{tikzcd}$$

i.e. $\eta_2 K_1 = K_2 \eta_1$.

The following section establish to find the path between matrices and automorphisms.

6.2 Connection between matrices and automorphisms

Suppose that $\gamma : C_2 \xrightarrow{\Delta_2} C_1 \xrightarrow{\Delta_1} C_0$ is a linear transformation of vector spaces which consists of two matrices: Δ_2 which is an $n_1 \times n_2$-matrix, where n_1 and n_2 are the dimensions of C_1 and C_2 respectively; and the other matrix Δ_1 which is an $n_0 \times n_1$-matrix, where n_0 is the dimension of C_0. In order to connect the matrices and automorphisms of linear transformation, an element of $Aut(\gamma)_1$ is used. This is a triple

$$F = (F_2, F_1, F_0)$$

of matrices such that

$$\gamma_1 F_2 = F_1 \gamma_1 \quad \text{and} \quad \gamma_2 F_1 = F_0 \gamma_2.$$

As it is known that F_2, F_1 and F_0 are invertible matrices, so the above equation can be rewritten as the following:

$$F_2 = \gamma_1^{-1} F_1 \gamma_1 \ \text{ or } \ F_1 = \gamma_1 F_2 \gamma_1^{-1}$$

and

$$F_1 = \gamma_2^{-1} F_0 \gamma_2 \ \text{ or } \ F_0 = \gamma_2 F_1 \gamma_2^{-1}$$

The elements of $Aut(\gamma)_2$ are homotopies

$$h : f \to f',$$

where f and f' are two linear transformations

$$f = (f_2, f_1, f_0) \text{ and } f' = (f_2', f_1', f_0')$$

respectively. This we can now analyse further, the homotopy element can be expressed as a pair (h', f) where

$$f = (f_2, f_1, f_0) \text{ and } h' = (h'_1, h'_2)$$

with h'_1 as a $n_1 \times n_0$ matrix H_1 and h'_2 as a $n_2 \times n_1$ matrix H_2.
As with the elements in $Aut(\gamma)_1$, the compatibility condition in $Aut(\gamma)_2$ gives the following

$$H_1\gamma_2 = F'_1 - F_1 \text{ and } \gamma_2 H_1 = F'_0 - F_0$$

and

$$H_2\gamma_1 = F'_2 - F_2 \text{ and } \gamma_1 H_2 = F'_1 - F_1.$$

When this information is converted into the matrix language, it can be assumed that

$$H_1 \in K^{n_1,n_0} \text{ while } \gamma_1 \in K^{n_0,n_1}$$

so both

$$\gamma_2 H_1 \text{ and } H_1\gamma_2$$

are defined and

$$H_2 \in K^{n_2,n_1} \text{ while } \gamma_2 \in K^{n_1,n_2}$$

as well as both

$$H_2\gamma_1 \text{ and } \gamma_1 H_2$$

are defined. At this point, the s, t and i maps in the matrix formulation can be checked.
Suppose that

$$F = (f_2, f_1, f_0) \text{ and } F' = (f'_2, f'_1, f'_0)$$

are elements in $Aut(\gamma)_1$ and

$$(H, F) : F \Rightarrow F'$$

is an element in $Aut(\gamma)_2$ such that

$$s(H,F) = F \text{ and } t(H,F) = F + \gamma H + H\gamma.$$

This means that

$$t((H_1', H_2'), (f_2, f_1, f_0)) = (f_2, f_1, f_0) + \gamma_2 H_1 + \gamma_1 H_2 + H_1\gamma_2 + H_2\gamma_1$$

and

$$i(F) = (0, F)$$

Moving on to the elements of $Aut(\gamma)_3$, a 2-homotopy

$$\alpha : h \longrightarrow \hat{h},$$

where h and $\hat{h}$ are two homotopies

$$h = (h_1', h_2') \text{ and } \hat{h} = (\hat{h}'_1, \hat{h}'_2),$$

the 2-homotopy element can be expressed as a triple

$$\alpha = (\tilde{\alpha}, h, F)$$

where

$$h' = (h_1', h_2') \text{ and } F = (f_2, f_1, f_0)$$

with $\tilde{\alpha}$ is an $n_2 \times n_0$ α. To check the compatibility condition in $Aut(\gamma)_3$, the following criteria should be satisfied:

$$\gamma_1\alpha = \hat{H}_1 - H_1 \quad \text{and} \quad \alpha\gamma_2 = \hat{H}_2 - h_2$$

As we know that

$$\gamma_1 \in K^{n_0,n_1}, \alpha \in K^{n_2,n_0} \text{ and } \gamma_2 \in K^{n_1,n_2}$$

both

$$\gamma_1\alpha \text{ and } \alpha\gamma_2$$

can be defined.

Furthermore, the s, t and i maps in the matrix formulation must be checked.

Suppose that

$$H = (h', F) \text{ and } \hat{H} = (\hat{h'}, F)$$

where

$$H, \hat{H} \in Aut(\gamma)_2$$

and

$$(\alpha, H, F) : H \longrightarrow \hat{H}$$

which is an element in $Aut(\gamma)_3$, then

$$s(\alpha, H, F) = H, t(\alpha, H, F) = H + \gamma_1\alpha + \alpha\gamma_2$$

and

$$i(\alpha, H, F) = (0, H, F).$$

In vector space language, suppose

$$V =< v_1, ..., v_n > \text{ and } W =< w_1, ..., w_n >$$

as bases for K^n. So, there is a unique non-singular matrix

$$P \in GL_n(k)$$

sometimes known as a change matrix from V to W such that if

$$x \in K^n$$

is a vector space expressed in terms of coefficients with respect to the basis V then Px is the same vector expressed with respect to the basis W.

Each vector can be expressed depending on the basis that comes from it: for example X_v, X_w. A matrix from V to W is P, so P^{-1} is a matrix from W to V. Suppose that

$$F : K^m \to K^n$$

is a linear transformation such that v, w and s are bases of K^m and v', w' and s' are bases of K^n, with matrices P_1, P_2, P_1' and P_2' such that P_1 from V to W, P_2 from W to S, P_1' from V' to W' and P_2' is a matrix from W' to S'.

Now, reverting to the linear transformation F, assume that there are three matrices that can be extracted depending on the bases. For instance, F_v is the matrix which can obtained from F by using V, V' as bases, F_w is the matrix which can obtained from F using W, W' and the last one, F_s, can be extracted using S, S' as bases. A a result all of the above matrices can be collected in the following formulae:

$$F_w = P_1' F_v P_1^{-1}$$

$$F_s = P_2' F_w P_2^{-1}$$

Depending on the linear transformation between the vector spaces, all can be defined as $Aut(\gamma)$

6.3 Examples of $Aut(\gamma)$

For more in-depth understanding of automorphisms of linear transformation and their appearance in higher categories, some simple examples will be considered, as well as the matrix formulation of these examples. Here, we will introduce the examples in order of the difficulty beginning with the simplest.

6.3.1 Simple example

The cyclic group is a suitable example to describe its $Aut(\gamma)$. Suppose

$$\gamma : C^2 \to C \to C$$

to be a linear transformation between cyclic groups C^2 and C

$$\gamma : C^2 \to C^2 \to C^1$$

$$\gamma : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \end{bmatrix}$$

We can convert these in matrix language,

$$\gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in C^{2\times 2} \quad \text{and} \quad \gamma_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \in C^{1\times 2}$$

Now, we describe the elements of $Aut(\gamma)_1$ which are chain automorphisms. They consist of a triple of non-singular matrices (F_2, F_1, F_0) such that

$$F_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, F_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, F_0 = \begin{bmatrix} a_0 \end{bmatrix}$$

and they should satisfy the following conditions:

$$F_0\gamma_1 = \gamma_1 F_1 \quad \text{and} \quad F_1\gamma_2 = \gamma_2 F_2.$$

To apply these conditions on what we get from matrices, for the first condition $F_0\gamma_1 = \gamma_1 F_1$ we will get

$$F_0\gamma_1 = \begin{bmatrix} a_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a_0 \end{bmatrix}$$

$$\gamma_1 F_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} c_1 & d_1 \end{bmatrix},$$

so

$$\begin{bmatrix} 0 & a_0 \end{bmatrix} = \begin{bmatrix} c_1 & d_1 \end{bmatrix},$$

that means $c_1 = 0$ and $a_0 = d_1$.

We will keep going to check the second condition $F_1\gamma_2 = \gamma_2 F_2$

$$F_1\gamma_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix}$$

and

$$\gamma_2 F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$$

As the condition is

$$F_1 \gamma_2 = \gamma_2 F_2,$$

so

$$a_1 = a_2, b_2 = 0 \ \ and \ \ c_1 = 0.$$

After that F will be given as follows

$$F = (F_2, F_1, F_0) = (\begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} d_1 \end{bmatrix})$$

where $a_2, c_2, d_2, b_1, d_1 \in C$ and $a_2 \neq 0$.

The simplest example for matrices of the above automorphism on (γ_1, γ_2) is

$$id_F = (F_2, F_1, F_0) = (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix})$$

It definitely satisfies the compability conditions above.

Suppose we have another automorphism on

$$\gamma = (\gamma_1, \gamma_2),$$

for instance

$$G = (G_1, G_2, G_0)$$

with

$$G_2 = \begin{bmatrix} a_2' & 0 \\ c_2' & d_2' \end{bmatrix}, G_1 = \begin{bmatrix} a_2' & b_1' \\ 0 & d_1' \end{bmatrix}, G_0 = \begin{bmatrix} d_1' \end{bmatrix}$$

with the same conditions

$$G_0 \gamma_1 = \gamma_1 G_1 \ \ and \ \ G_1 \gamma_2 = \gamma_2 G_2.$$

Now, to prove that $Aut(\gamma)_1$ is a group, we have to check the group operations. Assuming that we have two automorphisms F and G as follows:

$$F = (F_2, F_1, F_0) = (\begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} d_1 \end{bmatrix})$$

$$G = (G_2, G_1, G_0) = (\begin{bmatrix} a_2' & 0 \\ c_2' & d_2' \end{bmatrix}, \begin{bmatrix} a_2' & b_1' \\ 0 & d_1' \end{bmatrix}, \begin{bmatrix} d_1' \end{bmatrix})$$

Then, $F\#_0 G$ is a chain automorphism with

$$(F\#_0 G)_0 = F_0 G_0,$$

$$(F\#_0 G)_1 = \begin{bmatrix} a_2 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} a_2' & b_1' \\ 0 & d_1' \end{bmatrix}$$

$$= \begin{bmatrix} a_2 a_2' & a_2 b_1' + b_1 d_1' \\ 0 & d_1 d_1' \end{bmatrix},$$

$$(F\#_0 G)_2 = \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_2' & 0 \\ c_2' & d_2' \end{bmatrix}$$

$$= \begin{bmatrix} a_2 a_2' & 0 \\ c_2 a_2' + d_2 c_2' & d_2 d_2' \end{bmatrix}$$

Moving up to the homotopies ($Aut(\gamma)_2$) which are the morphisms between chain automorphisms F and G: Suppose given a homotopy

$$H : F \to G$$

between two chain automorphisms F and G. This is a pair of matrices

$$H = (H_1, H_2),$$

where H_1 is a (2×1) matrix and H_2 is a (2×2) matrix and they must satisfy the following condition:

$$H_1\gamma_1 = G_1 - F_1 \text{ and } \gamma_1 H_1 = G_0 - F_0$$

Also

$$H_2\gamma_2 = G_2 - F_2 \text{ and } \gamma_2 H_2 = G_1 - F_1,$$

Suppose $H = (H_1, H_2)$ such that

$$H_1 = \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} e_2 & m_2 \\ f_2 & n_2 \end{bmatrix}$$

Then, we show how it satisfies the above conditions

$$H_1\gamma_1 = \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & e_1 \\ 0 & f_1 \end{bmatrix}$$

and

$$G_1 - F_1 = \begin{bmatrix} a_2' & b_1' \\ 0 & d_1' \end{bmatrix} - \begin{bmatrix} a_2 & b_1 \\ 0 & d_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_2' - a_2 & b_1' - b_1 \\ 0 & d_1' - d_1 \end{bmatrix}$$

So, by the condition

$$H_1\gamma_1 = G_1 - F_1$$

this implies

$$a_2' - a_2 = 0, e_1 = b_1' - b_1 \text{ and } f_1 = d_1' - d_1$$

Again, with the next condition

$$\gamma_1 H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} f_1 \end{bmatrix}$$

and

$$G_0 - F_0 = \begin{bmatrix} d_1' \end{bmatrix} - \begin{bmatrix} d_1 \end{bmatrix} = \begin{bmatrix} d_1' - d_1. \end{bmatrix}$$

So, by the condition

$$\gamma_1 H_1 = G_0 - F_0$$

this implies

$$f_1 = d_1' - d_1.$$

The process continues with the next couple of conditions

$$H_2\gamma = \begin{bmatrix} e_2 & m_2 \\ f_2 & n_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e_2 & 0 \\ f_2 & 0 \end{bmatrix}$$

and

$$G_2 - F_2 = \begin{bmatrix} a_2' & 0 \\ c_2' & d_2' \end{bmatrix} - \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_2' - a_2 & 0 \\ c_2' - c_2 & d_2' - d_2 \end{bmatrix}.$$

So, by the condition

$$H_2\gamma_2 = G_2 - F_2$$

this implies

$$e_2 = a_2' - a_2, f_2 = c_2' - c_2 \text{ and } d_2' - d_2 = 0.$$

Also with the last section of conditions

$$\gamma_2 H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_2 & m_2 \\ f_2 & n_2 \end{bmatrix}$$

$$= \begin{bmatrix} e_2 & m_2 \\ 0 & 0 \end{bmatrix}$$

By the condition

$$\gamma_2 H_2 = G_1 - F_1$$

this implies

$$e_2 = a_2' - a_2, m_2 = b_1' - b_1 \text{ and } d_1' - d_1 = 0.$$

So,

$$H = (H_1, H_2) = (\begin{bmatrix} b'_1 - b_1 \\ d'_1 - d_1 \end{bmatrix}, \begin{bmatrix} a'_2 - a_2 & b'_1 - b_1 \\ c'_2 - c_2 & n_2 \end{bmatrix})$$

with some processes, we have to prove that $Aut(\gamma)_2$ is a group. The elements of $Aut(\gamma)_2$ are the homotopies

$$(H, F)$$

where

$$H = (H_1, H_2) \text{ and } F = (F_2, F_1, F_0).$$

By this point, we have two kinds of compositions, the first one is a horizontal composition.

If we have two homotopies

$$(H, F) \text{ and } (H', F')$$

such that

$$H = (H_1, H_2) : F \to G$$

where

$$F = (F_2, F_1, F_0) \text{ and } G = (G_2, G_1, G_0)$$

and the other homotopy

$$(H', F')$$

such that

$$H' = (H'_1, H'_2) : F' \to G'$$

where

$$F' = (F'_2, F'_1, F'_0) \text{ and } G' = (G'_2, G'_1, G'_0).$$

So, the horizontal composite

$$(H', F')\#_0(H, F)$$

is defined to be the homotopy with source $F'\#_0F$ and the chain homotopy is $g_1H + H'f_0$. The second one is a vertical composition.

If

$$(H, F) : F \to F' \text{ and } (H', F') : F' \to F'',$$

then, the vertical composition

$$(H', F')\#_1(H, F)$$

is also defined to be a homotopy with source F and its chain homotopy is the sum of both chain homotopies $H + H'$. Lift up the dimension to get 2- homotopies $Aut(\gamma)_3$ which are the morphisms between homotopies and we can denote them as a triple

$$(\alpha, H, F), \text{ where } H = (H_1, H_2) \text{ and } F = (F_2, F_1, F_0).$$

In this case, there are more complicated compositions; as well as all of the kinds of compositions which have already been shown, there are many whiskers:

- The vertical composition: The vertical composition of the element in $Aut(\gamma)_3$. Suppose we have

  $$\alpha, \beta \in Aut(\gamma)_3$$

  such that

  $$\alpha = (\alpha, H, F) \text{ where } H = (H_1, H_2) \text{ and } F = (F_2, F_1, F_0)$$

  and the other element which is

  $$\beta = (\beta, K, G), \text{ where } K = (K_1, K_2) \text{ and } G = (G_2, G_1, G_0)$$

  So

  $$\beta\#_2\alpha$$

  defines a 2-homotopy with source F_0 and 2-homotopy $[\alpha + \beta]$.
  But regarding the horizontal compositions and whiskers will be more complicated than the elements of $Aut(\gamma)_2$.

- The horizontal composition: If

$$(\alpha, H, F) : H \to K$$

and

$$(\beta, H', F') : H' \to K'$$

are 2-homotopies, the horizontal composite

$$(\beta, H', F')\#_0(\alpha, H, F)$$

is a 2-homotopy with source

$$F'\#_0 F \in Aut(\gamma)_1$$

and 2-chain homotopy

$$g_2\alpha + \beta f_0.$$

There are many whiskers between the elements in different groups.

1. suppose we have $G \in Aut(\gamma)_1$ and $k, h \in Aut(\gamma)_2$ such that

$$G = (g_2, g_1, g_0),$$

$$k = (G, k) \text{ where } k = (k_1, k_2)$$

and

$$h = (F, h) \text{ where } h = (h_1, h_2).$$

So

$$g\#_0(k\#_1 h)$$

is an element in $Aut(\gamma)_2$ with source gf and its chain homotopy $g(k' + h')$.

2. suppose we have $G \in Aut(\gamma)_1$, there are whiskers with two elements in $Aut(\gamma)_3$

$$\alpha \text{ and } \beta$$

such that

$$\alpha = (F, h, \alpha) \text{ and } \beta = (G, K, \beta)$$

so

$$g\#_0(\beta\#_2\alpha)$$

is an element in $Aut(\gamma)_3$ with source $g.f$ and its 2-homotopy is $g\alpha' = g\beta'$.

6.3.2 Inclusion example

As is well known, an inclusion map is a function that sends each element of the domain which is a subset of the co-domain to the co-domain. Here the same idea applies in our constructions.

Suppose $\gamma : K^m \hookrightarrow K^n \hookrightarrow K^r$ is an inclusion linear transformation, such that $m \leq n \leq r$, where $n = m + p$ and $r = n + s = m + p + s$ therefore

$$K^n = K^m \oplus K^p$$

and

$$K^r = K^n \oplus K^s = K^m \oplus K^p \oplus K^s.$$

The matrix formulation which corresponds to γ is a pair

$$\gamma = (\gamma_1, \gamma_2) \text{ and } \gamma_1 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \text{ and } \gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix},$$

seen as elements in $Aut(\gamma)_0$. Gradually, we can describe the connection between them as a chain automorphism between inclusion linear transformations. They are invertible matrices $F = (F_2, F_1, F_0)$ which must satisfy the chain automorphism conditions

$$F_0.\gamma_1 = \gamma_1.F_1 \text{ and } F_1.\gamma_2 = \gamma_2.F_2$$

To describe them let us suppose that

$$F = (F_2, F_1, F_0) = (F_2, \begin{bmatrix} F_2 & a \\ b & c \end{bmatrix}, \begin{bmatrix} F_2 & a & d \\ b & c & e \\ f & g & h \end{bmatrix}),$$

where $a \in GL_m(K)$, $b \in K^{m,p}$, $c \in K^{m,s}$, $d \in K^{p,m}$, $e \in GL_p(k)$, $f \in K^{p,s}$, $g \in K^{s,m}$, $h \in K^{s,p}$, $i \in GL_s(k)$ and they must satisfy the commutativity condition $F_0.\gamma_2 = \gamma_2.F_1$.

$$F_0.\gamma_2 = \begin{bmatrix} F_2 & a & d \\ b & c & e \\ f & g & h \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \\ 0 & g \end{bmatrix}$$

and

$$\gamma_2.F_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_2 & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b & c \\ 0 & 0 \end{bmatrix}$$

So

$$\begin{bmatrix} 0 & a \\ 0 & c \\ 0 & g \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b & c \\ 0 & 0 \end{bmatrix}$$

that means, $b = 0$, $a = 0$ and $g = 0$.

Also

$$F_1.\gamma_1 = \begin{bmatrix} F_2 & a \\ b & c \end{bmatrix} \cdot \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} F_2 \\ b \end{bmatrix}$$

$$\gamma_1 F_2 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \cdot \begin{bmatrix} F_2 \end{bmatrix} = \begin{bmatrix} F_2 \\ 0 \end{bmatrix}$$

So

$$\begin{bmatrix} F_2 \\ b \end{bmatrix} = \begin{bmatrix} F_2 \\ 0 \end{bmatrix}$$

that means, $b = 0$ and we can describe

$$F = (F_2, F_1, F_0) = (\begin{bmatrix} F_2 \end{bmatrix}, \begin{bmatrix} F_2 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} F_2 & 0 & d \\ 0 & c & 0 \\ f & 0 & h \end{bmatrix})$$

as an element of $Aut(\gamma)_1$. Now to consider the elements of $Aut(\gamma)_2$ we have to use another element of $Aut(\gamma)_1$ which is

$$G = (G_2, G_1, G_0) = (\begin{bmatrix} G_2 \end{bmatrix}, \begin{bmatrix} G_2 & 0 \\ 0 & c' \end{bmatrix}, \begin{bmatrix} G_2 & 0 & d' \\ 0 & c' & 0 \\ f' & 0 & h' \end{bmatrix})$$

the product of G and F is $F\#_0 G$, it is also a chain automorphism with

$$(F\#_0 G)_2 = \begin{bmatrix} F_2.G_2 \end{bmatrix},$$

$$(F\#_0 G)_1 = \begin{bmatrix} F_2 & 0 \\ 0 & c \end{bmatrix} . \begin{bmatrix} G_2 & 0 \\ 0 & c' \end{bmatrix} = \begin{bmatrix} F_2 G_2 & 0 \\ 0 & cc' \end{bmatrix},$$

$$(F\#_0 G)_0 = \begin{bmatrix} F_2 & 0 & d \\ 0 & c & 0 \\ f & 0 & h \end{bmatrix} . \begin{bmatrix} G_2 & 0 & d' \\ 0 & c' & 0 \\ f' & 0 & h' \end{bmatrix} = \begin{bmatrix} F_2 G_2 + df' & 0 & F_2 d' + dh' \\ 0 & cc' & 0 \\ fG_2 + hf' & 0 & fd' + hh' \end{bmatrix}.$$

Here, we describe the homotopy $H = F \simeq G$ which is an element of $Aut(\gamma)_2$ and it consists of a pair of matrices (h_1, h_2), where

$$h_1 : K^n \to K^m \text{ a m} \times \text{n matrix}$$

and

$$h_2 : K^r \to K^n \text{ a n} \times \text{r matrix}$$

as the following diagram explains:

$$\begin{tikzcd}
K_m \arrow[r, shift left, "F_2"] \arrow[r, shift right, "G_2"'] \arrow[d, "\gamma_1"'] & K_m \arrow[d, "\gamma_1"] \\
K_n \arrow[ur, dotted, "h_2"] \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] \arrow[d, "\gamma_2"'] & K_n \arrow[d, "\gamma_2"] \\
K_r \arrow[ur, dotted, "h_1"] \arrow[r, shift left, "F_0"] \arrow[r, shift right, "G_0"'] & K_r
\end{tikzcd}$$

The homotopy should satisfy the homotopy conditions which are

$$\gamma_2.h_1 = G_0 - F_0 \text{ and } h_1.\gamma_2 = G_1 - F_1.$$

Also,

$$\gamma_1.h_2 = G_1 - F_1 \text{ and } h_2.\gamma_1 = G_2 - F_2.$$

Let us now suppose that

$$H = (h_1, h_2) = (\begin{bmatrix} x & y & z \\ l & q & w \end{bmatrix}, \begin{bmatrix} x & y \end{bmatrix}),$$

where $x \in GL_m(K)$, $y \in K^{m,p}$, $z \in K^{m,s}$, $l \in K^{p,m}$, $q \in GL_p(K)$ and $w \in K^{p,s}$.

Therefore,

$$\gamma_2.h_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x & y & z \\ l & q & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ l & q & w \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$G_0 - F_0 = \begin{bmatrix} G_2 - F_2 & 0 & d' - d \\ 0 & c' - c & 0 \\ f' - f & 0 & h' - h \end{bmatrix}$$

So, $G_2 - F_2 = 0$, $d' - d = 0$, $f' - f = 0$, $h' - h = 0$, $l = 0$ and $w = 0$ that means, $G_2 = F_2$, $f' = f$, $h' = h$ and $c' = c$.

Turning to check the second part of the first condition

$$h_1.\gamma_2 = \begin{bmatrix} x & y & z \\ l & q & w \end{bmatrix} . \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & y \\ 0 & q \end{bmatrix}$$

and

$$G_1 - F_1 = \begin{bmatrix} G_2 - F_2 & 0 \\ 0 & c' - c \end{bmatrix}.$$

So, $G_2 - F_2 = 0$, $y = 0$ and $c' - c = q$, then $G_2 = F_2$.

Checking the other condition as the following:

$$\gamma_1.h_2 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} . \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$

and

$$G_1 - F_1 = \begin{bmatrix} G_2 - F_2 & 0 \\ 0 & c' - c \end{bmatrix}$$

so, $G_2 - F_2 = x$, $c' - c = 0$ and $y = 0$.

The other part to check is

$$h_2.\gamma_1 = \begin{bmatrix} x & y \end{bmatrix} . \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} x \end{bmatrix}$$

and

$$G_2 - F_2 = \begin{bmatrix} G_2 - F_2 \end{bmatrix}.$$

So, $G_2 - F_2 = x$.

We can describe it as follows:

$$H = (h_1, h_2) = (\begin{bmatrix} 0 & 0 & z \\ 0 & 0 & e' - e \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix}).$$

Moving up to higher dimensions to describe 2-homotopies which are chain automorphisms between homotopies, let us suppose that we have another chain automorphism

$$J = (J_2, J_1, J_0) = (\begin{bmatrix} J_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & c'' \end{bmatrix}, \begin{bmatrix} J_2 & 0 & d'' \\ 0 & c'' & 0 \\ f'' & 0 & h'' \end{bmatrix})$$

and there is another homotopy $K = G \simeq J$ between G and J therefore we can describe K as a pair of matrices

$$(k_1, k_2) = (\begin{bmatrix} 0 & 0 & z' \\ 0 & 0 & e'' - e' \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix}),$$

at this point we can consider a 2-homotopy $\alpha : K^r \to K^m$ as a $r \times m$ matrix

$$\begin{bmatrix} \lambda & \beta & \sigma \end{bmatrix}$$

between two homotopies H and K with a vertical composition such that $\alpha = H \simeq K$ such as $\alpha\gamma_2 = k_2 - h_2$ and $\gamma_1\alpha = k_1 - h_1$.
Checking the above condition, we have to work out the following:

$$\alpha.\gamma_2 = \begin{bmatrix} \lambda & \beta & \sigma \end{bmatrix} . \begin{bmatrix} 0 & 0 \\ 0 & I_p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta \end{bmatrix}$$

and

$$k_2 - h_2 = \begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

so $\beta = 0$.
Also

$$\gamma_1.\alpha = \begin{bmatrix} I_m \\ 0 \end{bmatrix} . \begin{bmatrix} \lambda & \beta & \sigma \end{bmatrix} = \begin{bmatrix} \lambda & \beta & \sigma \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$k_1 - h_1 = \begin{bmatrix} 0 & 0 & z' \\ 0 & 0 & e'' - e' \end{bmatrix} - \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & e' - e \end{bmatrix} = \begin{bmatrix} 0 & 0 & z' - z \\ 0 & 0 & e'' + e \end{bmatrix}$$

so $\lambda = 0$, $\sigma = z' - z$ and $e'' + e = 0$.

After that we can consider the form of 2-homotopy as a matrix

$$\alpha = \begin{bmatrix} 0 & 0 & z' - z \end{bmatrix}$$

6.3.3 Projection example

Suppose

$$\gamma : K^n \oplus K^m \oplus K^s \to K^n \oplus K^m \to K^n$$

be a linear transformation such that

$$\gamma_1 : K^n \oplus K^m \oplus K^s \to K^n \oplus K^m$$

is the projection of $K^n \oplus K^m$ on its direct summands, $K^n \oplus K^m$ is a quotient space of $K^n \oplus K^m \oplus K^s$ and also

$$\gamma_2 : K^n \oplus K^m \to K^n$$

is the projection of K^n on its direct summands, K^n is a quotient space of $K^n \oplus K^m$. So, we can choose a basis

$$V = \{v_1, v_2, v_3, ..., v_n, v_{n+1}, ..., v_{n+m+s}\}$$

for $K^n \oplus K^m \oplus K^s$ and a basis

$$V^* = \{v_1^*, v_2^*, v_3^*, ..., v_{n+m}^*\}$$

for $K^n \oplus K^m$ and a basis

$$\bar{V} = \{\bar{v_1}, \bar{v_2}, \bar{v_3}, ..., \bar{v_n}\}$$

for K^n, where $v_i^* = v_i + K^s$, where $i \leq n$ and $\bar{v} = v_i^* + K^m = v_i + K^m + K^s$, where $i \leq n$.

We can describe how γ works:

$$\gamma(\sum_{i=1}^{n+m+s} \alpha_i v_i) = \sum_{i=1}^{n+m} \alpha_i v_i^* = \sum_{i=1}^{n} \alpha_i \bar{v}_i$$

Turning to the matrix formulation, we can describe γ as a pair of matrices $\gamma = (\gamma_1, \gamma_2)$, where

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \in K^{n+m+s,n+m} \quad \text{and} \quad \gamma_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} \in K^{n,n+m}$$

Moving up the dimension, the maps between the above matrices can be considered, which are chain automorphisms $F = (F_2, F_1, F_0)$. These automorphisms must satisfy the commutativity conditions of chain automorphisms.
Let us suppose that $F = (F_2, F_1, F_0)$ such that $F_0 \in GL_n(K)$, $F_1 \in K^{n+m,n+m}$ and $F_2 \in K^{n+m+s,n+m+s}$ has the following form:

$$F = (F_2, F_1, F_0) = F = (\begin{bmatrix} F_0 & L & A \\ C & D & B \\ E & P & J \end{bmatrix}, \begin{bmatrix} F_0 & L \\ C & D \end{bmatrix}, F_0),$$

where $A \in K^{n,s}$,$C \in K^{m,n}$, $D \in GL_m(K)$, $B \in K^{m,s}$, $E \in K^{s,n}$, $S \in K^{s,m}$ and $J \in GL_s(K)$.
The commutativity conditions should be satisfied according to the following:

- $F_0.\gamma_2 = \gamma_2.F_1$ $F_0.\gamma_2 = F_0. \begin{bmatrix} I_n & 0 \end{bmatrix} = \begin{bmatrix} F_0 & L \end{bmatrix}$,

  $$\gamma_2.F_1 = \begin{bmatrix} I_n & 0 \end{bmatrix} . \begin{bmatrix} F_0 & L \\ C & D \end{bmatrix} = \begin{bmatrix} F_0 & L \end{bmatrix} \text{ so, } L = 0$$

- $F_1.\gamma_1 = \gamma_1.F_2$ $F_1.\gamma_1 = \begin{bmatrix} F_0 & L \\ C & D \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & L & 0 \\ 0 & D & 0 \end{bmatrix}$,

  $$\gamma_1.F_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} . \begin{bmatrix} F_0 & L & A \\ C & D & B \\ E & P & J \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ C & D & B \end{bmatrix}$$

So $L = 0$, $B = 0$ and $C = 0$.

Now, we can rewrite

$$F = (F_2, F_1, F_0) = \begin{bmatrix} F_0 & 0 & A \\ 0 & D & 0 \\ E & 0 & J \end{bmatrix}, \begin{bmatrix} F_0 & 0 \\ 0 & D \end{bmatrix}, F_0).$$

All the elements of $Aut(\gamma)_1$ have the same form, so suppose

$$G = (G_2, G_1, G_0) = (\begin{bmatrix} G_0 & 0 & A' \\ 0 & D' & 0 \\ E' & 0 & J' \end{bmatrix}, \begin{bmatrix} G_0 & 0 \\ 0 & D' \end{bmatrix}, G_0).$$

Then the composition of the elements G and F in $Aut(\gamma)_1$ is $F\#_0 G$ which consists of

$$(F\#_0 G)_0 = F_0 G_0$$

$$(F\#_0 G)_1 = \begin{bmatrix} F_0 G_0 & 0 \\ 0 & DD' \end{bmatrix}$$

$$(F\#_0 G)_2 = \begin{bmatrix} F_0 G_0 + AE' & 0 & F_0 A' + AJ' \\ 0 & DD' & 0 \\ EG_0 + JE' & 0 & EA' + JJ' \end{bmatrix}.$$

The homotopy consists of a pair (H, F) where $H = (h_1, h_2)$ such that $h_1 : K^n \to K^{n+m}$ and $h_2 : K^{n+m} \to K^{n+m+s}$.

Assuming that

$$H = (\begin{bmatrix} X \\ Y \end{bmatrix}, \begin{bmatrix} X & M \\ Y & N \\ Z & S \end{bmatrix}),$$

as the following diagram explains

$$\begin{tikzcd}
K_{m+n+s} \arrow[r, shift left, "F_2"] \arrow[r, shift right, "G_2"'] \arrow[d, "\gamma_1"'] & K_{m+n+s} \arrow[d, "\gamma_1"] \\
K_{m+n} \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] \arrow[d, "\gamma_2"'] \arrow[ur, dotted, "h_2"] & K_{m+n} \arrow[d, "\gamma_2"] \\
K_n \arrow[r, shift left, "F_0"] \arrow[r, shift right, "G_0"'] \arrow[ur, dotted, "h_1"] & K_n
\end{tikzcd}$$

Homotopy should satisfy the homotopy conditions, which are

$$\gamma_2 h_1 = G_0 - F_0 \quad \textit{and} \quad h_1\gamma_2 = G_1 - F_1.$$

Also,

$$\gamma_1 h_2 = G_1 - F_1 \quad \textit{and} \quad h_2\gamma_1 = G_2 - F_2.$$

This can be proven as follows:

$$\gamma_2 h_1 = \begin{bmatrix} I_n & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \end{bmatrix} = G_0 - F_0$$

and

$$h_1\gamma_2 = \begin{bmatrix} X \\ Y \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix} = G_1 - F_1 = \begin{bmatrix} G_0 - F_0 & 0 \\ 0 & D' - D \end{bmatrix}.$$

Also

$$\gamma_1 h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \cdot \begin{bmatrix} X & M \\ Y & N \\ Z & S \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Y & N \end{bmatrix}$$

$$= G_1 - F_1 = \begin{bmatrix} G_0 - F_0 & 0 \\ 0 & D' - D \end{bmatrix}$$

and

$$h_2.\gamma_1 = \begin{bmatrix} X & M \\ Y & N \\ Z & S \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & M & 0 \\ 0 & N & 0 \\ 0 & S & 0 \end{bmatrix}$$

$$= G_2 - F_2 = \begin{bmatrix} G_0 - F_0 & 0 & A' - A \\ 0 & D' - D & 0 \\ E' - E & 0 & J' - J \end{bmatrix}.$$

This means, $X = G_0 - F_0 = 0$, $Y = 0$, $N = D' - D$ and $M = S = 0$.

So

$$H = (h_1, h_2) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z & 0 \end{bmatrix})$$

Moving on to 2-homotopy, another homotopy is needed. To obtain this, it is necessary to find another chain automorphism. Suppose S is another chain automorphism such that

$$Q = (Q_2, Q_1, Q_0) = (\begin{bmatrix} Q_0 & 0 & A'' \\ 0 & D'' & 0 \\ E'' & 0 & J'' \end{bmatrix}, \begin{bmatrix} Q_0 & 0 \\ C'' & D'' \end{bmatrix}, Q_0),$$

and assume another homotopy $\hat{H} = (\hat{h_1}, \hat{h_2})$ between two chain automorphisms G and Q such that

$$\hat{H} = (\hat{h_1}, \hat{h_2}) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z' & 0 \end{bmatrix}).$$

Here, the 2- homotopy α between H and $\hat{H}$ is given as:

$$(\alpha, H, F) : K^n \to K^{n+m+s}$$

where

$$\alpha = \begin{bmatrix} \lambda \\ \beta \\ \sigma \end{bmatrix}$$

and it must satisfy the 2-homotopy conditions which are

$$\gamma_1 \alpha = \hat{h_1} - h_1 \quad \text{and} \quad \alpha \gamma_2 = \hat{h_2} - h_2.$$

So

$$\gamma_1\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ \beta \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

and since

$$\hat{h_1} - h_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To apply the second part of the condition:

$$\alpha\gamma_2 = \begin{bmatrix} \lambda \\ \beta \\ \sigma \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \beta & 0 \\ \sigma & 0 \end{bmatrix}.$$

Also

$$\hat{h_2} - h_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z' - Z & 0 \end{bmatrix}.$$

By the above condition that means

$$\begin{bmatrix} \lambda & 0 \\ \beta & 0 \\ \sigma & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z' - Z & 0 \end{bmatrix}.$$

So the 2-homotopy will be

$$\alpha = \begin{bmatrix} 0 \\ 0 \\ Z' - Z \end{bmatrix}.$$

6.4 General form of $Aut(\gamma)$

The automorphism $Aut(\gamma)$ of a linear transformation of a vector space has been defined in section 5.0.1. In this section, we aim to describe a general form of the automorphism $Aut(\gamma)$, starting with arbitrary linear transformation γ, chain automorphisms F, then homotopies H and 2-homotopies α as matrices with a specific form. It is necessary to generalise all the aforementioned examples and apply a specific form to the 3-categories of automorphisms $Aut(\gamma)$ over a vector space.

We can now assume that we have a linear transformation

$$\gamma : K^a \to K^b \to K^c$$

and it can be described as

$$K^a \xrightarrow{\gamma_1} K^b \xrightarrow{\gamma_2} K^c$$

with $ker(\gamma_1) = K^m$ and $ker(\gamma_2) = K^s$ so that

$$K^n \oplus K^m \oplus K^s \xrightarrow{\gamma_1} K^n \oplus K^p \oplus K^s \ ,$$

where $n + s = a - m$ and $p = b - (n + s)$ such that $\gamma_1(n, m, s) = (n', 0, s')$, where $n' \in K^n$, $s' \in K^s$ and

$$K^n \oplus K^p \oplus K^s \xrightarrow{\gamma_2} K^n \oplus K^p \oplus K^{p'}$$

where $n + p = b - s$ and $p' = c - (n + p)$ such that $\gamma_2(n', 0, s') = (n'', 0, 0)$, where $n'' \in K^n$.

Now turning to matrix language, we can convert γ_1 to γ_1 and γ_2 to γ_2, such that

$$\gamma_1 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in K^n \oplus K^p \oplus K^s, K^n \oplus K^m \oplus K^s$$

and

$$\gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \in K^n \oplus K^p \oplus K^{p'}, K^n \oplus K^p \oplus K^s.$$

Suppose F is a chain map between linear transformations, it consists of three matrices F_2, F_1 and F_0 where

$$F_2 \in GL_{n+m+s}, F_1 \in GL_{n+p+s} \ \ and \ \ F_0 \in GL_{n+p+p'}.$$

Assume

$$F_2 = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix}, F_1 = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix}, F_0 = \begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix}$$

satisfying the commutativity conditions which are

$$\gamma_2 F_1 = F_0.\gamma_2 \quad \text{and} \quad \gamma_1.F_2 = F_1.\gamma_1.$$

We prove that

$$\gamma_2 F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ d_1 & e_1 & f_1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$F_0 \gamma_2 = \begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_0 & 0 \\ 0 & e_0 & 0 \\ 0 & h_0 & 0 \end{bmatrix}.$$

Here we get $b_0 = 0, h_0 = 0, d_1 = 0, f_1 = 0$ and $e_1 = e_0$. Also

$$\gamma_1 F_2 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$F_1 \gamma_1 = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ d_1 & 0 & 0 \\ g_1 & 0 & 0 \end{bmatrix}.$$

For both of these we get $a_1 = a_2, b_2 = 0, c_2 = c_1, d_0 = 0, h_2 = 0$ and $g_2 = g_1$, therefore the last form to the chain automorphism F will be as follows:

$$F = (F_2, F_1, F_0) = (\begin{bmatrix} a_2 & 0 & 0 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_1 & c_1 \\ 0 & e_1 & 0 \\ 0 & h_1 & i_1 \end{bmatrix}, \begin{bmatrix} a_0 & 0 & c_0 \\ d_0 & e_1 & f_0 \\ g_0 & 0 & i_0 \end{bmatrix}),$$

where F is an element in $Aut(\gamma)_1$.

To moving up to the elements of $Aut(\gamma)_2$, we must have another element of G in $Aut(\gamma)_1$ such that

$$G = (G_2, G_1, G_0) = (\begin{bmatrix} a'_2 & 0 & 0 \\ d'_2 & e'_2 & f'_2 \\ g'_2 & h'_2 & i'_2 \end{bmatrix}, \begin{bmatrix} a'_2 & b'_1 & c'_1 \\ 0 & e'_1 & 0 \\ 0 & h'_1 & i'_1 \end{bmatrix}, \begin{bmatrix} a'_0 & 0 & c'_0 \\ d'_0 & e'_1 & f'_0 \\ g'_0 & 0 & i'_0 \end{bmatrix}).$$

The elements of chain automorphisms can be composed as follows:

$$(F\#_0G)_2 = \begin{bmatrix} a_2 & 0 & 0 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix} \cdot \begin{bmatrix} a'_2 & 0 & 0 \\ d'_2 & e'_2 & f'_2 \\ g'_2 & h'_2 & i'_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_2a'_2 & 0 & 0 \\ d_2a'_2 + e_2d'_2 + f_2g'_2 & e_2e'_2 + f_2h'_2 & e_2f'_2 + f_2i'_2 \\ g_2a'_2 + h_2d'_2 + i_2g'_2 & h_2e'_2 + i_2h'_2 & h_2f'_2 + i_2i'_2 \end{bmatrix}$$

$$(F\#_0G)_1 = \begin{bmatrix} a_2 & b_1 & c_1 \\ 0 & e_1 & 0 \\ 0 & h_1 & i_1 \end{bmatrix} \begin{bmatrix} a'_2 & b'_1 & c'_1 \\ 0 & e'_1 & 0 \\ 0 & h'_1 & i'_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_2a_2' & a_2b_1' + b_1e_1' + c_1h_1' & a_2c_1' + c_1i_1' \\ 0 & e_1e_1' & 0 \\ 0 & h_1e_1' + i_1h_1' & i_1i_1' \end{bmatrix}$$

$$(F\#_0G)_0 = \begin{bmatrix} a_0 & 0 & c_0 \\ d_0 & e_1 & f_0 \\ g_0 & 0 & i_0 \end{bmatrix} \begin{bmatrix} a_0' & 0 & c_0' \\ d_0' & e_1' & f_0' \\ g_0' & 0 & i_0' \end{bmatrix}$$

$$= \begin{bmatrix} a_0a_0' + c_0g_0' & 0 & a_0c_0' + c_0i_0' \\ d_0a_0' + e_1d_0' + f_0g_0' & e_1e_1' & d_0c_0' + e_1f_0' + f_0i_0' \\ g_0a_0' + i_0g_0' & 0 & g_0c_0' + i_0i_0' \end{bmatrix}.$$

Now consider H, the homotopy between two chain automorphisms F and G, $H = F \simeq G$, which consists of a pair of matrices, h_1 and h_2, $H = (h_1, h_2)$, where $h_1 \in K^{n+p+p',n+p+s}$ and $h_2 \in K^{n+p+s,n+m+s}$.

Let us assume now the form of the homotopy as $H = (h_1, h_2)$

$$h_1 = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ W_1 & Q_1 & R_1 \\ S_1 & M_1 & N_1 \end{bmatrix}, h_2 = \begin{bmatrix} X_2 & Y_2 & Z_2 \\ W_2 & Q_2 & R_2 \\ S_2 & M_2 & N_2 \end{bmatrix}.$$

At this point, the chain homotopy must satisfy the following conditions:

1. $\gamma_2.h_1 = G_0 - F_0$ and $h_1.\gamma_2 = G_1 - F_1$

$$\gamma_2.h_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 & Z_1 \\ W_1 & Q_1 & R_1 \\ S_1 & M_1 & N_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ W_1 & Q_1 & R_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_0 - F_0 = \begin{bmatrix} a_0' - a_0 & 0 & c_0' - c_0 \\ d_0' - d_0 & e_1' - e_1 & f_0' - f_0 \\ g_0' - g_0 & 0 & i_0' - i_0 \end{bmatrix},$$

which gives us $W_1 = d_0' - d_0, Q_1 = e_1' - e_1, R_1 = f_0' - f_0$.

Also,

$$h_1.\gamma_2 = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ W_1 & Q_1 & R_1 \\ S_1 & M_1 & N_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Y_1 & 0 \\ 0 & Q_1 & 0 \\ 0 & M_1 & 0 \end{bmatrix}$$

$$G_1 - F_1 = \begin{bmatrix} a_2' & b_1' & c_1' \\ 0 & e_1' & 0 \\ 0 & h_1' & i_1' \end{bmatrix} - \begin{bmatrix} a_2 & b_1 & c_1 \\ 0 & e_1 & 0 \\ 0 & h_1 & i_1 \end{bmatrix} = \begin{bmatrix} a_2' - a_2 & b_1' - b_1 & c_1' - c_1 \\ 0 & e_1' - e_1 & 0 \\ 0 & h_1' - h_1 & i_1' - i_1 \end{bmatrix},$$

which means $Y_1 = b_1' - b_1$ and $M_1 = h_1' - h_1$

2. $\gamma_1.h_2 = G_1 - F_1$ and $h_2.\gamma_1 = G_2 - F_2$

$$\gamma_1 h_2 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_2 & Y_2 & Z_2 \\ W_2 & Q_2 & R_2 \\ S_2 & M_2 & N_2 \end{bmatrix} = \begin{bmatrix} X_2 & Y_2 & Z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since

$$G_1 - F_1 = \begin{bmatrix} a_2' - a_2 & b_1' - b_1 & c_1' - c_1 \\ 0 & e_1' - e_1 & 0 \\ 0 & h_1' - h_1 & i_1' - i_1 \end{bmatrix}.$$

For the above we get

$$\begin{bmatrix} X_2 & Y_2 & Z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_2' - a_2 & b_1' - b_1 & c_1' - c_1 \\ 0 & e_1' - e_1 & 0 \\ 0 & h_1' - h_1 & i_1' - i_1 \end{bmatrix}.$$

Therefore $X_2 = a_2' - a_2$, $Y_2 = b_1' - b_1$ and $Z_2 = c_2' - c_2$

Also $h_2.\gamma_1 = G_2 - F_2$

$$\begin{bmatrix} X_2 & Y_2 & Z_2 \\ W_2 & Q_2 & R_2 \\ S_2 & M_2 & N_2 \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} X_2 & 0 & 0 \\ W_2 & 0 & 0 \\ S_2 & 0 & 0 \end{bmatrix}.$$

Since

$$G_2 - F_2 = \begin{bmatrix} a'_2 & 0 & 0 \\ d'_2 & e'_2 & f'_2 \\ g'_2 & h'_2 & i'_2 \end{bmatrix} - \begin{bmatrix} a_2 & 0 & 0 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix}$$

$$= \begin{bmatrix} a'_2 - a_2 & 0 & 0 \\ d'_2 - d_2 & e'_2 - e_2 & f'_2 - f_2 \\ g'_2 - g_2 & h'_2 - h_2 & i'_2 - i_2 \end{bmatrix},$$

which gives us

$$\begin{bmatrix} X_2 & 0 & 0 \\ W_2 & 0 & 0 \\ S_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a'_2 - a_2 & 0 & 0 \\ d'_2 - d_2 & e'_2 - e_2 & f'_2 - f_2 \\ g'_2 - g_2 & h'_2 - h_2 & i'_2 - i_2 \end{bmatrix}.$$

Therefore $W_2 = d'_2 - d_2$ and $S_2 = g'_2 - g_2$.

Form the above information, we get:

$$F \simeq G \text{ or } (F_2, F_1, F_0) \simeq (G_2, G_1, G_0).$$

The chain homotopy can therefore be reformulated as follows:

$$H = (h_1, h_2) = (\begin{bmatrix} X_1 & b'_1 - b_1 & Z_1 \\ d'_0 - d_0 & 0 & f'_0 - f_0 \\ S_1 & 0 & N_1 \end{bmatrix}, \begin{bmatrix} 0 & b'_1 - b_1 & 0 \\ d'_2 - d_2 & Q_2 & R_2 \\ 0 & M_2 & N_2 \end{bmatrix}).$$

Sequentially, moving to a higher dimension to describe the elements of $Aut(\gamma)_3$, another homotopy is needed which can compose with them either horizontally or vertically depending on the source and target of the chain homotopy.

Suppose that T is another chain automorphism such that $\hat{H} : G \simeq T$ where $F \simeq G \simeq T$. Consequently, H and $\hat{H}$ can be joined to a vertical composition because, the target of H is itself a source of $\hat{H}$ as follows

$$S = (T_2, T_1, T_0) = (\begin{bmatrix} a''_2 & 0 & 0 \\ d''_2 & e''_2 & f''_2 \\ g''_2 & h''_2 & i''_2 \end{bmatrix}, \begin{bmatrix} a''_2 & b''_1 & c''_1 \\ 0 & e''_1 & 0 \\ 0 & h''_1 & i''_1 \end{bmatrix}, \begin{bmatrix} a''_0 & 0 & c''_0 \\ d''_0 & e''_1 & f''_0 \\ g''_0 & 0 & i''_0 \end{bmatrix}),$$

and the homotopy $\hat{H}$ as follows

$$\hat{H} = (\hat{h_1}, \hat{h_2}) = (\begin{bmatrix} X_1' & b_1'' - b_1' & Z_1' \\ d_0'' - d_0' & 0 & f_0'' - f_0' \\ S_1' & 0 & N_1' \end{bmatrix}, \begin{bmatrix} 0 & b_1'' - b_1' & 0 \\ d_2'' - d_2' & Q_2' & R_2' \\ 0 & M_2' & N_2' \end{bmatrix}).$$

Here, we consider another homotopy $\bar{H} : \bar{F} \simeq \bar{G}$. The horizontal composition between H and $\bar{H}$ is $(H, F)\#_0(\bar{H}, \bar{F})$ having $F\#_0\bar{F}$ as a source and the chain homotopy are $G_1.\bar{h_1} + h_1.F_0$ and $G_2.\bar{h_2} + h_2.F_1$.

Assume that α is a 2-homotopy $\alpha : H \simeq \hat{H}$, which is an element in $Aut(\gamma)_3$, we get:

$$\alpha = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix},$$

the 2-homotopy condition should satisfy the following:

$$\gamma_1.\alpha = \hat{h_1} - h_1 \text{ and } \alpha.\gamma_2 = \hat{h_2} - h_2.$$

Hence, this gives:

$$\gamma_1.\alpha = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} A & B & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{h_1} - h_1 = \begin{bmatrix} X_1' & b_1'' - b_1' & Z_1' \\ d_0'' - d_0' & 0 & f_0'' - f_0' \\ S_1' & 0 & N_1' \end{bmatrix} - \begin{bmatrix} X_1 & b_1' - b_1 & Z_1 \\ d_0' - d_0 & 0 & f_0' - f_0 \\ S_1 & 0 & N_1 \end{bmatrix}$$

$$= \begin{bmatrix} X_1' - X_1 & b_1'' + b_1 & Z_1' - Z_1 \\ d_0'' + d_0 & 0 & f_0'' + f_0 \\ S_1' - S_1 & 0 & N_1' - N_1 \end{bmatrix}.$$

Therefore $A = X_1' - X_1, B = b_1'' + b_1$ and $C = Z_1' - Z_1$.

Also

$$\alpha\gamma = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & B & 0 \\ 0 & E & 0 \\ 0 & H & 0 \end{bmatrix}$$

$$\hat{h_2} - h_2 = \begin{bmatrix} 0 & b_1'' - b_1' & 0 \\ d_2'' - d_2' & Q_2' & R_2' \\ 0 & M_2' & N_2' \end{bmatrix} - \begin{bmatrix} 0 & b_1' - b_1 & 0 \\ d_2' - d_2 & Q_2 & R_2 \\ 0 & M_2 & N_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b_1'' + b_1 & 0 \\ d_2'' + d_2 & Q_2' - Q_2 & R_2' - R_2 \\ 0 & M_2' - M_2 & N_2' - N_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} 0 & B & 0 \\ 0 & E & 0 \\ 0 & H & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_1'' + b_1 & 0 \\ d_2'' + d_2 & Q_2' - Q_2 & R_2' - R_2 \\ 0 & M_2' - M_2 & N_2' - N_2 \end{bmatrix}.$$

Therefore $B = b_1'' + b_1, E = Q_2' - Q_2$ and $H = M_2' - M_2$.

The final form of the 2-homotopy is

$$\alpha = \begin{bmatrix} X_1' - X_1 & b_1'' + b_1 & Z_1' - Z_1 \\ D & Q_2' - Q_2 & F \\ G & M_2' - M_2 & I \end{bmatrix}.$$

To summarize, we have obtain the following theorem. Here we will use the same notations in this section:

Theorem 6.4.1. *The general form of the automorphism $Aut(\gamma)$ of a linear transformation of a vector space as a matrices with the following forms:*
Suppose that γ_1 and γ_2 are the differentials of linear transformation γ.

$$\gamma_1 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and } \gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A chain map F between linear transformations is as follows:

$$F = (F_2, F_1, F_0) = (\begin{bmatrix} a_2 & 0 & 0 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_1 & c_1 \\ 0 & e_1 & 0 \\ 0 & h_1 & i_1 \end{bmatrix}, \begin{bmatrix} a_0 & 0 & c_0 \\ d_0 & e_1 & f_0 \\ g_0 & 0 & i_0 \end{bmatrix}),$$

where F is an element in $Aut(\gamma)_1$.

The chain homotopy H between chain maps formulates as the following:

$$H = (h_1, h_2) = (\begin{bmatrix} X_1 & b_1' - b_1 & Z_1 \\ d_0' - d_0 & 0 & f_0' - f_0 \\ S_1 & 0 & N_1 \end{bmatrix}, \begin{bmatrix} 0 & b_1' - b_1 & 0 \\ d_2' - d_2 & Q_2 & R_2 \\ 0 & M_2 & N_2 \end{bmatrix}).$$

where H is an element in $Aut(\gamma)_2$.

The form of 2-homotopy between chain homotopies is

$$\alpha = \begin{bmatrix} X_1' - X_1 & b_1'' + b_1 & Z_1' - Z_1 \\ D & Q_2' - Q_2 & F \\ G & M_2' - M_2 & I \end{bmatrix}.$$

where α is an element in $Aut(\gamma)_3$.

Proof: This follows from the explicit construction of a general form of $Aut(\gamma)$ in section 6.4 starting with a linear transformation γ, then chain automorphisms, then homotopies and finally 2-homotopies.

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