Quantum mechanics approach to option pricing

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John von Neumann

"The only way to do great work is to love what you do. If you haven't found it yet, keep looking. Don't settle."

Steve Jobs

Abstract

Options are financial derivatives on an underlying security. The Schrodinger and Heisenberg approach to the quantum mechanics together with the Dirac matrix approaches are applied to derive the Black-Scholes formula and the quantum Cox-Rubinstein formula.

The quantum mechanics approach to option pricing is based on the interpretation of the option price as the Schrodinger wave function of a certain quantum mechanics model determined by Hamiltonian H. We apply this approach to continuous time market models generated by Levy processes.

In the discrete time formulization, we construct both self-adjoint and non self-adjoint quantum market. Moreover, we apply the discrete time formulization and analyse the quantum version of the Cox-Ross-Rubinstein Binomial Model. We find the limit of the N-period bond market, which convergences to planar Brownian motion and then we made an application to option pricing in planar Brownian motion compared with Levy models by Fourier techniques and Monte Carlo method.

Furthermore, we analyse the quantum conditional option price and compare for the conditional option pricing in the quantum formulization. Additionally, we establish the limit of the spectral measures proving the convergence to the geometric Brownian motion model. Finally, we found Binomial Model formula and Path integral formulization gave are close to the Black-Scholes formula.

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Finally, I would like to give some special thanks to my family for their love, faith, and support on me, without which the completion of this thesis is impossible.

Contents

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Symbols

S	Set
$\mathcal{B}$	Borel set
Ω	Sample space
$\mathcal{F}$	Filtration
μ	Measure function
$(S, \mathcal{F}, \mu)$	Measurable space
$(\Omega, \mathcal{F}, P)$	Probability space
$P_x = P(\xi = x)$	Probability of a random variable ξ taking value x
$\mathcal{H}$	Hilbert space
(x, y)	Inner product
$\mathcal{A}$	Von Neumann algebra
$P(\mathcal{H})$	Set of orthogonal projection
$L(\mathcal{H})$	Banach algebra
X	Random variable / Quantum observable
ν	Quantum state space
ρ	Interest rate / Quantum state
ψ	State vector in ν
ν_{dual}	Dual space
χ	State vector in ν_{dual}
$\|\psi>$	Ket vector in ν
$<\chi\|$	Bra vector in ν_{dual}
$<\chi\|\psi>$	Scalar product
$\|\psi><\chi\|$	Operator (dyad)
A	Semigroup
U	Unitary operator
I	Identity operator
J	Jordan matrix
tr	Trace

H	Hamiltonian operator
P	Projection operator
$\mathcal{P}$	Set of all quantum states in $\mathcal{H}$
$A \otimes B$	Kronecker product (a special case of tensor product) of A and B
S_t	Share price / Quantum share price
W	Capital
C	Payoff / Option claim
OP	Option price
θ	Number of shares
δ	Number of bonds

To my parents

Chapter 1

Introduction

Options are financial derivatives on an underlying security [31]. Black-Scholes model of continuous time market is based on the Ito calculus [32] [34]. Together with the Stochastic Modelling it gives the comprehensive analysis of the option pricing. On the other hand, Cox-Ross-Rubinstein Binomial Model of the discrete time market provides a powerful alternative for the overall justification of the no-arbitrage market assumption, hedging strategies and option pricing derivations [33].

Because of the shortfalls and holes within the Black-Scholes model based on its restrictive assumptions, establishing some models can make a reasonable explanation for financial market, which is why so many researchers are looking for new ways to price derivatives. The beginnings of this approach can be traced back to several papers [54] and [55] where the interest rates and option prices are treated as quantum field. Segal and Segal in [16] on the Black-Scholes pricing formula in the quantum context [24]. Chen shows a quantization of the classical Black-Scholes-Merton based binomial option pricing model developed by Cox-Ross-Rubinstein [11]. Quantum model based on quantum probability, instead of classical probability, which is generalisation for classical probability. It is not clear how existing methods in classical probability can cover the quantum models. On the other hand, we have proved that quantum models do cover the classical non-quantum models. In this thesis, we start to extend Chen's work and analyse the quantum conditional option price. Also, we establish several quantum market and the related quantum model in the discrete time version, which is does not considered by Baaquie too much.

The Schrodinger and Heisenberg approach to the quantum mechanics together with the Dirac matrix approaches are applied to derive the Black-Scholes formula and the quantum Cox-Rubinstein formula.

The quantum mechanics approach to option pricing, as stated in Belal E. Baaquie [1],[2], is based on the interpretation of the option price as the Schrodinger wave function of a certain quantum mechanics model determined by Hamiltonian H. Considering H, the Hamiltonian for the Black-Scholes, we derive the Black-Scholes formula for the option price. The quantum mechanics formulization is based on the identity decomposition. We apply this approach to continuous time market models generated by Levy processes.

In the discrete time formulization, following CHEN Zeqian (2002) [4] and (2004) [5], we construct self-adjoint and non self-adjoint quantum markets. Instead of considering the eigenvalues, we consider the diagonal elements, apply the discrete time formalizm and analyse the quantum version of the Cox-Ross-Rubinstein Binomial Model. We find the limit of the N-period bond market, which convergences to planar Brownian motion and we compare the option pricing for planar Brownian motion with the option pricing for Levy models by Fourier techniques and Monte Carlo method.

Furthermore, we analyse the quantum conditional option price via the quantum conditional expectation [9], [10], [12], [29] and we give a proof for the conditional option pricing in the quantum formulization. Besides, we establish the limit of the spectral measures proving the convergence to the geometric Brownian motion model.

An efficient computational algorithm to price financial derivatives needs to be considered. The path integral method is a very famous and powerful method in Physics. This method is nowadays widely employed in physics [23], and very recently in finance too [19], [20], [21] because it gives the possibility of applying powerful analytical and numerical techniques.

1.1 Results

Results have been presented at the conferences in Samos (Greece) in May, 2016 and Beijing (China) in June, 2016 (both were oral presentations).

The joint paper with S. Utev is under preparation, which is based on [35].

1.1.1 Theoretical Results

New findings are presented in Chapter 3 to Chapter 6.

• Several quantum markets have been established in the discrete time formulization in the section 4.2.

• Several relative examples on quantum markets are provided in the sections 5.1 and 5.2.

• The limit of N-period bond market, it connected with planar Brownian motion in the section 4.4.

• The non self-adjoint quantum market, it made a connection with Jordan matrices in the section 4.3.

• In the continuous time quantum formulization, lengthy calculations for hamiltonians and OP have been presented in the sections 3.4, 3.5 and 3.6.

• The analysis and proofs of the conditional option pricing for generalized quantum N period Binomial model have been presented in the section 4.5.

• Establishing the limit of the spectral measures proving the convergence to the geometric Brownian motion model has been presented in the section 4.6.

1.1.2 Numerical Results

• The results of Binomial Model formula and Path integral are close to the Black-Scholes formula even for relatively small $n \sim 40$ in the section 6.1.

• Calibration for European option by Fourier techniques and Monte Carlo compared with planar Brownian motion model and Levy model have been presented in the section 6.2.

• Calibration for the coefficient of interest rate is given in the section 6.3.

1.2 Structure of the Dissertation

Chapter 2. It covers preliminary knowledge based on probability theory from 2.1.1 up to 2.1.16, including definitions, examples, and selected proofs in the chapter 2. Moreover, it covers random walks, stochastic processes on finite dimensional and infinite dimensional linear algebra from 2.2.1 to 2.2.6, continuous semigroups and generators between 2.2.7 and 2.2.8, quantum mechanics, Hamiltonian, and Dirac notaion from 2.2.9 to 2.2.11, quantum probability notations from 2.3.1 to 2.3.6, Tensor product and diagonal decomposion on 2.3.7 and 2.3.8, and financial markets on 2.4.

Chapter 3. It covers the main structure in terms of quantum formalism on 3.2. Moreover, it introduces the calculation for different resolvents for different stochastic processes from 3.3.1 to 3.3.3. Also, it covers several examples for Hamiltonian and pricing kernels from 3.4.1 to 3.4.10. Furthermore, it covers Transformed Hamiltoinian from 3.5.1 to 3.5.3. Besides, it covers resolvent method for Hamiltonian operators from 3.6.1 to 3.6.3.

Chapter 4. Considering the quantum observable [10], we apply the discrete time quantum formalizm and construct Option pricing of Tensor product of non- commutative Market on 4.2. Also, we analyse the quantum version of the Cox-Ross-Rubinstein Binomial Model. Furthermore, on 4.3, we analyse the quantum conditional option price via the quantum conditional expectation [9], [10], [12], [29]. Besides, we establish the limit of the spectral measures proving the convergence to the geometric Brownian motion model (GBM model) on 4.4.

Chapter 5. It covers some examples for quantum markets. On the one hand, starting from the one period quantum market, we introduce one step quantum market, including the one step bond market. In this part, we consider the non-commutative case and non-self adjoint case on 5.1.1 and 5.1.2. Moreover, we consider the 5×5 non-Diagonalizable case on 5.1.3. Furthermore, we consider the commuitative market on 5.1.4. On the other hand, considering the tensor product of market, we introduce two period quantum market, including self-adjoint and non self-adjoint case.

Chapter 6. It covers some numerical implementations. Firstly, we introduce the Path integral for Black-Scholes Lagrangian on 6.1. Secondly, we introduce the Calibration for European option by Fourier techniques and Monte Carlo compared with Black-Scholes model and Levy model. Finally, we introduce Calibration for the coefficient of interest rate.

Chapter 2

Preliminary knowledge

2.1 Introduction

Let us give some preliminary knowledge for this chapter. We introduce some basic definition in terms of probability theory and some stochastic processes, such as Brownian motion, Poisson process, Levy process, and Markov process. And we give the definition and theorem for Itô lemma and resolvent for the mainly derivations in this chapter. Moreover, we start from vector space, and introduce some spaces we used. Furthermore, we give a review about quantum probability, quantum mechanics, such as Hamiltonian operator, dirac notation, Schördinger equation, and so forth. Besides, we introduce some basic concept for finance, such as Black-Scholes, Binomial model, and so on.

2.2 Preliminary knowledge for Classical market and Markov market

2.2.1 σ-field

The Definition 2.1 follows [36] on the page 15.

Definition 2.1. Let S be a non-empty set. A collection of subsets of S is called a σ-field $\mathcal{F}$ on S if it satisfies

i) $S \in \mathcal{F}$;
ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
iii) If A_i, where $i = 1, 2, \ldots$, is a sequence of sets in $\mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

2.2.2 Measurable space and Measurable sets

The Definition 2.2 follows [36] on the page 16.

Definition 2.2. Let S be a set and $\mathcal{F}$ is a σ-field on S. A pair $(S, \mathcal{F})$ is called a measurable space. An element of $\mathcal{F}$ is called a $\mathcal{F}$-measurable subset of S.

Example Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F}_1 = \{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4, 6\}\}$. $\mathcal{F}_1$ is a σ-field. Then, we obtain

i) $\Omega \in \mathcal{F}_1$;
ii) $\Omega^c = \emptyset \in \mathcal{F}_1$, $\emptyset^c = \Omega \in \mathcal{F}_1$, $\{1, 3, 5\}^c = \{2, 4, 6\} \in \mathcal{F}_1$, and $\{2, 4, 6\}^c = \{1, 3, 5\} \in \mathcal{F}_1$;
iii) $\emptyset \cup \{1, 3, 5\} = \{1, 3, 5\} \in \mathcal{F}_1$, $\emptyset \cup \{2, 4, 6\} = \{2, 4, 6\} \in \mathcal{F}_1$, $\Omega \cup \emptyset = \Omega \in \mathcal{F}_1$, $\{1, 3, 5\} \cup \{2, 4, 6\} = \Omega \in \mathcal{F}_1$, $\Omega \cup \{1, 3, 5\} = \Omega \in \mathcal{F}_1$, and $\Omega \cup \{2, 4, 6\} = \Omega \in \mathcal{F}_1$.

Example

$\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the smallest σ-field.

A family of all subsets $\mathcal{F}_2 = P(\Omega) = \{A : A \subseteq \Omega\}$ is the largest σ-field of subsets of Ω.

2.2.3 Borel set

The Definition 2.3 follows [36] on the page 8.

Definition 2.3. Let S be a set, a topology on S is a collection $\mathcal{X}$ of subsets of S if it satisfies:
i) $\emptyset \in \mathcal{X}$ and $S \in \mathcal{X}$;
ii) For any intersection $U \cap V \in \mathcal{X}$, where $U \in \mathcal{X}$ and $V \in \mathcal{X}$;
iii) For any union $\cup\, \mathcal{U} \in \mathcal{X}$, where $\mathcal{U} \subset \mathcal{X}$.

If $\mathcal{X}$ is a topology on S, then the pair $(S, \mathcal{X})$ is called a topology space. The members of $\mathcal{X}$ are called open sets in S. A subset of S is called closed if its complement is in $\mathcal{X}$. A subset of S may be open, closed, clopen set, or neither.

The most familiar topological spaces are the metric spaces. The Definition 2.4 follows [36] on the page 9.

Definition 2.4. A metric space is a set S in which a distance function (or metric) ρ is defined, with the following properties:
i) $0 \le \rho(x, y) < \infty$ for all x and $y \in S$;
ii) $\rho(x, y) = 0$ if and only if $x = y$;
iii) $\rho(x, y) = \rho(y, x)$ for all x and $y \in S$;
iv) $\rho(x, y) \le \rho(x, z) + \rho(y, z)$ for all x, y, and $z \in S$.

The Definition 2.5 follows [40] on the page 4.

Definition 2.5. In any metric space, the open ball with center at x and radius r is the set

$$B_r(x) = \{y : d(x, y) < r\}.$$

Definition 2.6. Let S be a set and $\mathcal{B}(\mathbb{R})$ be the σ-field generated by the family of open subsets of S. One can say that the Borel σ-field on S. Subsets of $\mathcal{B}(\mathbb{R})$ belong to $\mathcal{B}(\mathbb{R})$ are called Borel sets.

Example Let us fix Ω be $\{1, 2, 3, 4, 5, 6\}$.

a) Suppose we have the pseudo distance

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

If the distance is more than 1, we have all set Ω. And if the distance is less than or equal to 1, we only have one point x.

$$\mathcal{B}_r(x) = \begin{cases} \Omega, \text{ if } r > 1, \\ x, \text{ if } r \le 1. \end{cases} \quad .$$

For this situation, the topology here is that any set is open so that Borel σ-algebra $\mathcal{B}(\Omega)$ is a class of all subsets in Ω.

b) Suppose we have the pseudo distance $d(x, y) = 0$ for all x, y, then

$$\mathcal{B}_r(x) = \Omega\,, \ \ r > 0.$$

This topology is for $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

2.2.4 Measure space

The Definition 2.7 follows [36] on the page 18 in the book.

Definition 2.7. Let $(S, \mathcal{F})$ be a measurable space, $\mathcal{F}$ is a σ-field on S. A map

$$\mu : S \to [0, \infty]$$

is called a measure on $(S, \mathcal{F})$ if μ is countablely additive. The triple $(S, \mathcal{F}, \mu)$ is called a measure space.

2.2.5 Measurable function

The Definition 2.8 follows [36] on the page 18.

Definition 2.8. Suppose that $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ are measurable spaces and h is a map

$$h : S_1 \to S_2.$$

Then, h is measurable if

$$h^{-1} : \mathcal{F}_2 \to \mathcal{F}_1;$$

if the inverse image

$$h^{-1}(A) := \{s \in S : h(s) \in A\}$$

of every set $A \in \mathcal{F}_2$ is in $\mathcal{F}_1$.

2.2.6 Measure and Probability measure

The Definition 2.9 follows [36] on the page 18.

Definition 2.9. A measure is a countably additive set function such that a function $\mu : \mathcal{F} \to R$ with

i) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$;
ii) If $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets, then

$$\mu(\cup_i A_i) = \sum_i \mu(A_i).$$

If $\mu(S) = 1$, then we call μ a probability measure.

2.2.7 Probability space

Definition 2.10. A probability space is a triple $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω and P is the probability measure on $\mathcal{F}$.

Example Suppse $\mathcal{F}_1 = \{\{1,3,5\},\{2,4,6\}\}$. Let $\{1,3,5\}$, $\{2,4,6\}$ be A_1, A_2, respectively. Clearly, $A_i \in \mathcal{F}_1$, where $i = 1, 2$. Then,

$$\mu_1(\cup_{i=1}^2 A_i) = \sum_{i=1}^2 \mu_1(A_i) = 1.$$

Therefore, μ_1 is a probability measure on $\mathcal{F}_1$.

2.2.8 Event

The Definition 2.11 follows [36] on the page 23.

Definition 2.11. The σ-field $\mathcal{F}$ on Ω is called the family of events, then an event is an element of $\mathcal{F}$, which is an $\mathcal{F}$-measurable subset of Ω.

2.2.9 Random variable

The Definition 2.12 follows [36] on the page 31.

Definition 2.12. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(\Omega, \mathcal{F})$ and $(\mathcal{R}, \mathcal{B})$ be measurable spaces. If an element $X : (\Omega, \mathcal{F}) \to (\mathcal{R}, \mathcal{B})$, for any $A \in \mathcal{B}$, $X^{-1}(A) \in \mathcal{F}$, then X is randorm variable.

2.2.10 Filtration

The Definition 2.13 follows [36] on the page 93.

Definition 2.13. For any time t, we define the σ-field $\mathcal{F}_t$ generated by the random variables $\{B_s, s \leq t\}$ and the events in $\mathcal{F}$ of probability zero. That is, $\mathcal{F}_t$ is the smallest σ-field that contains the sets of the form

$$\{B_s \in A\} \quad \cup \quad N,$$

where $0 \leq s \leq t$, A is a Borel subset of $\mathbb{R}$, and $N \in \mathcal{F}$ is such that $P(N) = 0$. Notice that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$, that is , $\{\mathcal{F}_t, t \geq 0\}$ is a non-decreasing family of σ-fields. We say that $\{\mathcal{F}_t, t \geq 0\}$ is a *filtration* in the probability space $(\Omega, \mathcal{F}, P)$.

Example $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\{1, 3, 5\}, \{2, 4, 6\}\}$ are defined. And $\mathcal{F}_2$ are all subsets in Ω. Then, we have

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2,$$

which is a collection $\{\mathcal{F}_t, t = 0, 1, 2\}$ of σ-fields is a filtration.

Example Let Ω be a set $\{1, 2, 3, 4, 5, 6\}$. $\mathcal{F}_t$ is Filtration of σ-algebra if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. X_t is $\mathcal{F}_t$ adapted if $X_t \in \mathcal{F}_t$. As we know,

$$X(i) = i, \text{ where } i \in \Omega.$$

However, we have

$$X^{-1}(\{1\}) = \{1\} \notin \mathcal{F}_1.$$

Therefore, X is not $\mathcal{F}_1$-measurable.

2.2.11 Distribution

The Definition 2.14 follows [36] on the page 32.

Definition 2.14. A random variable defines a probability measure on the Borel σ-field $\mathcal{B}_R$, that is

$$P_X(\mathcal{B}) = P(X^{-1}(\mathcal{B})) = P(\{\omega : X(\omega) \in \mathcal{B}\}).$$

The probability measure P_X is called the law or the distribution of X.

Example A random variable has the Uniform distribution $U(0,1)$ if

$$P(x) = \frac{x-a}{b-a},$$

for $a \leq x \leq b$, $P(x) = 0$ for $x < a$, and $P(x) = 1$ for $x > b$.

Example A random variable has the Binomial distribution $B(n,p)$ if

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

for $k = 0, \ldots, n$.

Example A random variable has the normal distribution $N(\mu, \sigma^2)$ if

$$P(a < X < b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx,$$

for all $a < b$.

2.2.12 Expectation

The Definition 2.15 follows [36] on the page 58.

Definition 2.15. Let $(\Omega, \mathcal{F}, P)$ be the probability space. If a random variable X on $(\Omega, \mathcal{F}, P)$, then we define the expectation $E(X)$ of X as the integral of X with respect to the probability measure P as follows

$$E(X) = \int_\Omega X dP.$$

If X is a discrete variable that takes the values $v_1, \ldots,$ on the sets $A_1, \ldots,$ then its expectation will be

$$E(X) = P(A_1)v_1 + \ldots.$$

Example If X is a random variable with normal distribution $N(0, \sigma^2)$ and x is a constant, then we obtain

$$\begin{aligned} E(e^{cX}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\infty}^{\infty} e^{cx} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\infty}^{\infty} e^{\frac{\sigma^2 c^2}{2} - \frac{(x-\sigma^2 c)^2}{2\sigma^2}} dx \\ &= e^{\frac{c^2\sigma^2}{2}}. \end{aligned}$$

2.2.13 Independence

The Definition 2.16 follows [36] on the page 38.

Definition 2.16. The event A and B are said to be independent if $P(A \cap B) = P(A)P(B)$.

Theorem 2.17. *Random variables $X_1, ..., X_n$ are independent if*

$$E[f_(X_1)...f_n(X_n)] = E[f_1(X_1)]...E[f_n(X_n)]$$

for all measurable functions f_j when all expectations are defined.

2.2.14 Conditional Probability

Definition 2.18. The conditional probability of an event A given event B such that $P(B) > 0$ is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

2.2.15 Conditional Expectation

The Definition 2.19 follows [36] on the page 84.

Definition 2.19. The expectation of an integrable random variable X with respect to this new probability will be the conditional expectation of X given B and it can be computed as follows:

$$E(X|B) = \frac{1}{P(B)} E(X1_B),$$

where 1_B is indicator function.

2.2.16 Characteristic function

The Definition 2.20 follows [36] on the page 172.

Definition 2.20. The characteristic function $\psi = \psi_X$ of a random variable X is defined to be the map

$$\psi : \mathbb{R} \to \mathbb{C},$$

which is defined by

$$\psi(\xi) := E(e^{i\xi X}).$$

Let $F := F_X$ be the distribution function of X, and let $\mu = \mu_X$ be the distribution of X. Then,

$$\psi(\xi) = \int_R e^{i\xi x} dF(x) := \int_R e^{i\xi x} \mu(dx),$$

where ψ is the Fourier transform of μ.

2.2.17 Random Walk

Definition 2.21. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent, identically distributed discrete random variables. For each positive integer n, we let S_n denote the sum $X_1 + \ldots + X_n$. The sequence $\{S_n\}_{n=1}^{\infty}$ is called a random walk. If the common range of the X_k's is R^m, then we say that $\{S_n\}$ is a random walk in R^m.

2.2.18 Brownian motion

These definitions and theorems given bellow follows [39].

Definition 2.22. A stochastic process $B(t)$, $t \geq 0$ is called a Brownian motion, if it satisfies the following conditions:
(i) $B(0) = 0$ a.s..;
(ii) For all positive integer $n \geq 2$ and $0 \leq t_1 < \cdots < t_n$, the increments $B(t_n) -$

$B(t_{n-1}), ..., B(t_2) - B(t_1)$, are independent random variables;
(iii) If $0 \leq s < t$, the increment $B(t) - B(s)$ has the normal distribution $N(0, t-s)$;
(iv) The process $\{B(t)\}$ has continuous trajectories.

Lemma 2.23. *Let $B(t)$ be a Brownian motion, then we have*

$$E[B(a)B(b)] = min(a, b)$$

Proof: let $0 < a < b$, then we have

$$\begin{aligned} E[B(a)B(b)] &= E[(B(b) - B(a))B(a) + (B(a))^2] \\ &= E[(B(b) - B(a))(B(a) - B(0)) + (B(a) - (B(0))^2] \\ &= E[(B(b) - B(a))]E[(B(a) - B(0)) + E(B(a) - B(0))^2 \\ &= 0 + a = a. \end{aligned}$$

Notice that a Brownian motion is standard if it satisfies

$$B(0) = 0 \text{ a.s., } E[B(t)] = 0 \text{ and } E[(B(t))^2] = t.$$

Example Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. By the definition, $B(1)$ and $B(3) - B(1)$ are independent cause

$$E[B(1)(B(3) - B(1))] = E[B(1)]E[B(3) - B(1)] = 0.$$

But, $B(3)$ and $B(1)$ are not independent since

$$E[B(1)B(3)] = 1.$$

Since as it follows from the lemma below.

The Definition 2.24 follows [9] on the page 89.

Definition 2.24. Suppose that $\mathcal{D}$ and $\mathcal{R}$ are sets that possess an addition operation as well as a scalar multiplication operation. A function f that maps points in $\mathcal{D}$ to points in $\mathcal{R}$ is said to be a linear function whenever f satisfies the conditions that

$$f(x + y) = f(x) + f(y)$$

and

$$f(\alpha x) = \alpha f(x)$$

for every x and y in $\mathcal{D}$ and for all scalars α.

Definition 2.25. If B_1, B_2 are independent linear Brownian motions started in x_1, x_2, then the stochastic process $\{B(t) : t \geq 0\}$ given by

$$B(t) = (B_1(t), B_2(t))^T$$

is called a 2-dimensional Brownian motion started in $(x_1, x_2)^T$. The d-dimensional Brownian motion started in the origin is also called started Brownian motion. One-dimensional Brownian is also called linear, two-dimensional Brownian motion, planar Brownian motion.

2.2.19 Poisson process

These definitions and theorems given bellow follows [39].

Definition 2.26. A stochastic process $N_t \geq 0$ defined on a probability space (Ω, F, P) is said to be a Poisson process of rate λ if it verifies these properties:
i) $N_t = 0$ a.s..;
ii) for any $n \geq 1$ and for any $0 \leq t_1 < ... < t_n$, the increments $N_{t_n} - N_{t_{n-1}}, ..., N_{t_2} - N_{t_1}$, are independent random variables;
iii) for any $0 \leq s < t$, the increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t-s)$, that is,

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

$k = 0, 1, 2, \ldots,$ where $\lambda > 0$ is a fixed constant;
iv) The sample paths $N_t, t \geq 0$ are increasing functions of t changing only by the jump of size 1.

Example Let $\{N_t, t \geq 0\}$ be a Poisson process with parameter $\lambda = 10$. Assume that $k = 5$, then we obtain

$$P(N_2 - N_1 \leq 5) = \sum_{n=0}^{5} e^{-10} \frac{10^n}{n!}.$$

The variance of Possion variable is calculated by

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= E((X)(X-1)+X) - (E(X))^2 \\ &= E((X)(X-1)) + E(X) - (E(X))^2 \\ &= E((X)(X-1)) + \lambda - \lambda^2 \\ &= \lambda. \end{aligned}$$

Example If X is a random variable with Poisson distribution with parameter λ, then we obtain

$$E(X) = \sum_{j=0}^{\infty} j\frac{e^{-j}\lambda^j}{j!} = \lambda e^{-\lambda}\sum_{j=1}^{\infty}\frac{e^{-\lambda}\lambda^{j-1}}{(j-1)!} = \lambda.$$

Definition 2.27. Let X and Y be two random variables with means $E(X)$ and $E(Y)$, respectively. Covariance is the expected value of the products of deviations from the means:

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

Example Given a Poisson process $\{N_t, t \geq 0\}$ with parameter λ, calculate the covariance of $N(2)$ and $N(3)$.

From the definition (2.24), according to the independent increments, we derive

$$\begin{aligned} Cov(N(2), N(3)) &= Cov(N(2), N(3) - N(2)) + Cov(N(2), N(2)) \\ &= Var(N(2)) \\ &= 2\lambda. \end{aligned}$$

2.2.20 Levy Process

These definitions and theorems given bellow follows [30].

Definition 2.28. A 1-dimensional Levy process on a probability space $(\Omega, \mathcal{F}, P)$ is a collection $X = \{X_t\}_{t\geq 0}$ of R-valued random variables on Ω satisfying the following properties:
i) Given an integer $n \geq 1$ and a collection of times $0 \leq t_0 < t_1 < ... < t_n$, the

random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.
ii) $X_0 = 0$ almost surely.
iii) For any $t \geq 0$, the distribution of $X_{s+t} - X_s$ is independent of $s \geq 0$.
iv) Stochastic continuity: given $t \geq 0$ and $\epsilon > 0$, we have $\lim_{s \to t} P[|X_s - X_t| > \epsilon] = 0$.
v) There exists a subset $\Omega_0 \in \Omega$ such that $P[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$, the trajectory $t \to X_t(\omega)$ is right continuous in $t \geq 0$, and has left limits for all $t > 0$.

Remark For the property v), a function $f : R^+ \to R^d$ is càdlàg if it is right continuous with left limits, such a function has jump discontinuities. Here is the definition about right continuous with left limit.

Definition 2.29. A function f on R is said to be right continuous with left limit at a point $a = b$ if it satisfies the following properties:
i) $f(b)$ is defined, where a point b is in the domain of a function f;
ii) Right limit of the function as a approaches b from the right hand side exists, i.e. $\lim_{a \to b+} f(a) = f(b+)$. Left limit of the function as a approaches b from the left hand side exists, i.e. $\lim_{a \to b-} f(a) = f(b-)$.
iii) $f(b+) = f(b)$.

Lemma 2.30. *Consider $X+Y$, where $P(Y = 0) = 1$. Then, E[f(X+Y)]=E[f(X)]. We write $X + Y = X$ a.s..*

Example Consider the process Z_t, we have

$$Z_t = X_t + I(t = U),$$

where X_t is a Levy process and I is an indicator function and U follows the uniform distribution $U[0, 1]$.

According to the lemma 2.30, for the variable Y, we have $P(Y = 0) = 1$ and $P(1_{t=a} = 1) = 0$. If we take supermum of this point, then $P(\sup_{0 \leq t \leq 1} 1_{t=a} = 1) = 1$. Therefore, X_t is càdlàg, but Z_t is not.

Before the discussion of characteristic funtion for Levy process, we need to give the definition of infinite divisibility. Motivated by Andreas E. Kyprianou, the finite divisible is given in his book, named Introductory Lectures on Fluctuations of Levy Processes with Applications (2006).

Definition 2.31. Let X be a random variable taking values in R with the law μ_X. We say that X is infinitely divisible if, for all $n \in N$, there exist independent and identically distributed random variables $Y_1, \ldots, Y_n$ such that

$$X \overset{d}{=} Y_1 + \ldots + Y_n.$$

The characteristic function of the distribution of X_t admits the representation

$$E[e^{i\xi X_t}] = e^{-t\psi(\xi)}, \xi \in R, t \geq 0.$$

The function ψ is called the characteristic exponent of X. The Levy-Khintchine formula describes all possible characteristic exponents. Motivated by Andreas E. Kyprianou, the explaination of the characteristic function for Levy process is given in his book, named Introductory Lectures on Fluctuations of Levy Processes with Applications (2006).

Proof of characteristic function for Levy process: From the definition of a Levy process, we know X_t, which is a random variable belonging to the class of infinitely divisible distributions for any $t > 0$,. It follows

$$X_t = X + t/n + (X_{2t/n} - X_{t/n}) + \ldots + (X_t - X_{(n-1)t/n}). \tag{2.1}$$

Assume that

$$\psi_t(\theta) = -logE(e^{i\theta X_t}).$$

By using (2.1) twice, we have two integers m, n such that

$$m\psi_t(\theta) = \psi_m(\theta) = n\psi_{m/n}(\theta).$$

Therefore, for any Levy process, it has the property

$$E(e^{i\theta X_t}) = e^{-t\psi(\theta)}.$$

The proof of the theorem is motivated by Levy Processes and Stochastic Calculus (2004) written by David Applebaum.

Theorem 2.32. *Let X be a Levy process on R. Then its characteristic exponent admits the representation*

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_{R\setminus 0}(1 + i\xi x 1_{[-1,1]} - e^{ix\xi})F(dx),$$

where $\sigma^2 \geq 0, b \in R$, and F is a measure on $R \setminus 0$ satisfying

$$\int_{R\setminus 0} min\{|x|^2, 1\}F(dx) \leq \infty.$$

Example The Brownian motion with drift μ and valatility σ is a Levy process. Let $P(x)$ be the probability density function for X_t. Note that the characteristic function of the Brownian motion can be obtained by using the definition of a characteristic function.

$$\begin{aligned}
E[e^{i\xi X_t}] &= \int_{-\infty}^{\infty} e^{i\xi x} P(x)dx \\
&= \int_{-\infty}^{\infty} e^{i\xi x}\frac{1}{\sqrt{2\pi\sigma^2 t}} exp(-\frac{(x-\mu t)^2}{2\sigma^2 t})dx \\
&= e^{(i\mu t\xi - \sigma^2 t\xi^2/2)} \\
&= e^{-t(-i\mu\xi + \sigma^2\xi^2/2)}.
\end{aligned}$$

Example The Possion process with parameter λ is a Levy process. Let $P(x)$ be the probability density function for X_t. The characteristic function of Possion process is

$$\begin{aligned}
E[e^{i\xi X_t}] &= \sum_{k=0}^{\infty} P(X_t = k)e^{i\xi k} \\
&= \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} e^{i\xi k} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t e^{i\xi})^k}{k!} \\
&= e^{\lambda t(e^{i\xi}-1)}.
\end{aligned}$$

Example If $F(dx) = 0$, then this Levy process is a Brownian notion with variance σ^2 and drift μ. Note that the Brownian motion is the only (subclass of) Levy process(es) with continuous sample paths. Sample paths of any other Levy process exhibit jumps.

Example Assume that $\sigma = 0$ and $F(dx) = c\lambda e^{-\lambda x} 1_{(0,1)}(x)dx$, where $\lambda > 0$ and c is a non-negative constant. For this Levy measure, the characteristic exponent is

$$\begin{aligned}\psi(\xi) &= \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_0^1 (1 - e^{ix\xi}) c\lambda e^{-\lambda x} dx \\ &= \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_0^1 c\lambda e^{-\lambda x} dx - \int_0^1 c\lambda e^{(-\lambda + i\xi)x} dx \\ &= \frac{\sigma^2}{2}\xi^2 - ib\xi + c(1 - e^{-1}) - \frac{c\lambda}{\lambda - i\xi}(1 - e^{-(\lambda - i\xi)}).\end{aligned}$$

Definition 2.33. Suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}$ are given. A geometric Levy process is given by

$$S_t = S_0 e^{X_t},$$

where X_t is a Levy process with the generating triplet $(\sigma^2, F(dx), b)$.

2.2.21 Markov Process

Notice that the Definition 2.34 follows [38] on the page 227.

Definition 2.34. A Markov process X with state space $(S, \mathcal{F})$ is defined by Markov kernel as follows

$$X = (\Omega, \{\mathcal{F}_t\}, \{X_t\}, \{P_t : t \geq 0\}, \{P_X : x \in S\}),$$

which is a S-valued stochastic process adapted to $\{\mathcal{F}_t\}$, such that, for $0 \leq s \leq t$, and $f \in \mathcal{B}(S)$ and $x \in S$,

$$E^x[f(X_{s+t})|\mathcal{F}_s] = (P_t f)(X_s), \ \ P_X \ \texttt{a.s.},$$

where $\{P_t\}$ is a transition function on $(S, \mathcal{F})$, a family of kernels $P_t : S \times \mathcal{F} \to [0, 1]$ such that

(i) for $t \geq 0$ and $x \in S, P_t(x, \cdot)$ is a measure on $\mathcal{F}$ with $P_t(x, S) = 1$;
(ii) for $t \geq 0$ and $\Gamma \in \mathcal{F}, P_t(\cdot, \Gamma)$ is $\mathcal{F}$-measurable;
(iii) for $s, t \geq 0$, $x \in S$ and $\Gamma \in \mathcal{F}$,

$$P_{t+s}(x, \Gamma) = \int_E P_s(x, dy) P_t(y, \Gamma).$$

Important example is the class of Levy and geometric Levy processes.

2.2.22 Itô's formula

These definitions and theorems follows [34].

Definition 2.35. Let $\mathcal{L}$ be the class of funtions

$$f : [0, \infty) \times \Omega \to R$$

such that
i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$.
ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted.
iii) $E[\int_S^T f(t, \omega)^2 dt] < \infty$.

The Itô integral $\int f dB$ can be defined for a larger class of integrals f than $\mathcal{L}$. First, the measurability condition ii) of the definition can be changed as follows, we say a):
a) There exists an increasing family of σ-algebras $\mathcal{H}_t, t \geq 0$ such that B_t is a martingale with respect to $\mathcal{H}_t$ and f_t is $\mathcal{H}_t$-adapted.
b) The extension of the Itô integral consists of weakening condition iii) of the definition to $P[\int_S^T f(s, \omega)^2 ds < \infty] = 1$.
Let $L_{a,T}$ be the space of processes satisfied with a) and b). And let $L^1_{a,T}$ be the space of processes ν satisfied with a) and the following condition c) instead of b)

$$c) \;\; P[\int_0^T |\nu_t| dt < \infty] = 1.$$

.

Definition 2.36. A continuous and adapted stochastic process $\{X_t, 0 \leq t \leq T\}$ is called an Itô process as follows

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t \nu_s ds,$$

where $u \in L_{a,T}$ and $\nu \in L^1_{a,T}$.

Theorem 2.37. *Let X_t be an Itô process given by*

$$dX_t = u_t dt + \nu_t dB_t. \tag{2.2}$$

Let $f(t,x) \in C^2([0,\infty) \times R)$. *Then,*

$$Y_t = f(t, X_t)$$

is an Ito process and

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) \cdot (dX_t)^2,$$

which follows the product rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0 \text{ and } dB_t \cdot dB_t = dt.$$

Theorem 2.38. *Suppose that X is an Itô process of the form (2.2). Let $f(t,x)$ be a function twice differentiable with respect to the variable x and once differentiable with respect to t, with continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial f}{\partial t}$ (we say that f is of class $C^{1,2}$). Then, the process $Y_t = f(t, X_t)$ is again an Itô process with the representation*

$$\begin{aligned} Y_t = \quad & f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(x, X_s)u_s dB_s \\ & + \int_0^t \frac{\partial f}{\partial x}(s, X_s)v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)u_s^2 ds. \end{aligned}$$

2.3 Preliminary knowledge for Hamiltonian and Markov kernel

2.3.1 Vector space

The Definition 2.39 follows [9] on the page 159.

Definition 2.39. A set $\mathcal{V}$ is called a vector space over $\mathcal{F}$, where $\mathcal{F}$ is a scalar field (the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers), when the vector addition and scalar multiplication operations satisfy the following properties:

i) $x + y \in \mathcal{V}$ for all $x, y \in \mathcal{V}$. This is called the closure property for vector addition;
ii) $(x + y) + z = x + (y + z)$ for every $x, y, z \in \mathcal{V}$;
iii) $x + y = y + x$ for every $x, y \in \mathcal{V}$;
iv) There is an element $0 \in \mathcal{V}$ such that $x + 0 = x$ for every $x \in \mathcal{V}$;

v) For each $x \in \mathcal{V}$, there is an element $(-x) \in \mathcal{V}$ such that $x + (-x) = 0$;
vi) $\alpha x \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $x \in \mathcal{V}$. This is the closure property for scalar multiplication;
vii) $(\alpha\beta)x = \alpha(\beta x)$ for all $\alpha, \beta \in \mathcal{F}$ and every $x \in \mathcal{V}$;
viii) $\alpha(x + y) = \alpha x + \alpha y$ for every $\alpha \in \mathcal{F}$ and all $x, y \in \mathcal{V}$;
iv) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in \mathcal{F}$ and every $x \in \mathcal{V}$;
x) $1x = x$ for every $x \in \mathcal{V}$.

Definition 2.40. Let E be a vector space over $\mathbb{C}$. A subset M of E is called a subspace of E if and only if it satisfies the following properties:

i) If u and v are in M, then $u + v$ is in M.
ii) If u is in M and α is in $\mathbb{C}$, then αu is in M.

2.3.2 Banach space

The following definition follows [41] on the page 194.

Definition 2.41. Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A norm $|| \cdot ||$ on $\mathcal{V}$ is a real valued function on $\mathcal{V}$, which has the following properties:

i) $||x|| \geq 0$, $||x|| = 0$ implies $x = 0$.
ii) $||ax|| = |a|||x||, a \in \mathbb{C}$.
iii) $||x + y|| \leq ||x|| + ||y||$.
Then, $\mathcal{V}$ is called a normed linear space.

A sequence $\{x_n\}$ in a normed linear space $\mathcal{V}$ is called a Cauchy sequence if $||x_n - x_m|| \to 0$ as $n, m \to \infty$. $\mathcal{V}$ is complete if every Cauchy sequence in $\mathcal{V}$ converges. A complete normed linear space is called a Banach space.

2.3.3 Dual space

The Definition 2.42 follows [42] on the page 169 in the book.

Definition 2.42. A linear functional on $\mathcal{V}$ is a function $f : \mathcal{V} \to \mathcal{F}$ such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(x_2)$$

for all $\alpha, \beta \in \mathcal{F}$ and all $x, y \in \mathcal{V}$.

Definition 2.43. Let $\mathcal{V}$ be a linear vector space and $\mathbb{C}$ be the field of complex numbers. The dual space $\mathcal{V}_{dual}$ of $\mathcal{V}$ is defined as

$$\mathcal{V}_{dual} = \{f : \mathcal{V} \to \mathbb{C}, \text{ where } f \text{ is linear functional}\}.$$

The dual space is also a linear vector space.

Let us give a simple proof for that the dual space is a linear verctor space.

Lemma 2.44. *Let f_1, f_2 be linear functionals and k_1, k_2 are in $\mathbb{C}$. $f = k_1 f_1 + k_2 f_2$, then f is linear. We need to show that $f(\alpha x+\beta y) = k_1 f_1(\alpha x+\beta y)+k_2 f_1(\alpha x+\beta y)$.*

Proof:

$$\begin{aligned}
f(\alpha x + \beta y) &= k_1 f_1(\alpha x + \beta y) + k_2 f_1(\alpha x + \beta y) \\
\text{since} &\quad f_1, f_2 \texttt{ are linear.} \\
\text{then} &\quad k_1(\alpha f_1(x) + \beta f_2(y)) + k_2(\alpha f_2(x) + \beta f_2(y)) \\
&= k_1 \alpha f_1(x) + k_1 \beta f_2(y) + k_2 \alpha f_2(x) + k_2 \beta f_2(y) \\
&= \alpha(k_1 f_1(x) + k_2 f_2(x)) + \beta(k_1 f_1(y) + k_2 f_2(y)) \\
&= \alpha f(x) + \beta f(y).
\end{aligned}$$

2.3.4 Hilbert space

The Definition 2.45 follows [42] on the page 105.

Definition 2.45. A complex vector space $\mathcal{H}$ is called an inner product space if to each ordered pair of vectors x and y in $\mathcal{H}$ is associated a complex number (x, y), called the inner product or scalar product of x and y, such that the following rules hold:

(1) $(y, x) = \overline{(x, y)}$. (The bar denotes complex conjugation.)
(2) $(x + y, z) = (x, z) + (y, z)$.
(3) $(\alpha x, y) = \alpha(x, y)$ if $x \in \mathcal{H}, y \in \mathcal{H}, \alpha \in \mathbb{C}$.
(4) $(x, x) \geq 0$ for all $x \in \mathcal{H}$.
(5) $(x, x) = 0$ only if $x = 0$.

Every inner product space can be normed by defining

$$||x|| = (x, x)^{1/2}. \tag{2.3}$$

If the normed space is complete, it is called a Hilbert space.

Definition 2.46. Given two vector spaces E and F over the complex field $\mathbb{C}$, a function $f : E \to F$ is semilinear if

$$f(u + \nu) = f(u) + f(\nu) \text{ and } f(\lambda u) = \bar{\lambda} f(u),$$

for all $u, \nu \in E$ and all $\lambda \in \mathbb{C}$. The set of all semilinear maps $f : E \to \mathbb{C}$ is denoted by $\bar{E}^*$.

Notice that the inner product is not linear, but it is semi-linear with respect to second coordinate by the following proof.

Proof.

$$\begin{aligned} (x, iy) &= \overline{(iy, x)} \\ &= \overline{i(y, x)} \\ &= \bar{i}\overline{(y, x)} \\ &= -i\overline{(y, x)} \\ &= -i(x, y). \end{aligned}$$

□

2.3.5 Bounded linear operator

These definitions follows [41] on the page 51 and 52.

Definition 2.47. A function A which maps $\mathcal{H}_1$ to $\mathcal{H}_2$ is called a linear operator if for all x, y in $\mathcal{H}_1$ and $\alpha \in \mathbb{C}$ satisfied the following properties:

i) $A(x + y) = A(x) + A(y)$;
ii) $A(\alpha x) = \alpha A(x)$.

Notice that $\mathcal{H}_1, \mathcal{H}_2$ are two separate Hilbert spaces.

Definition 2.48. The linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is called bounded if

$$\sup_{||x|| \leq 1} ||Ax|| < \infty.$$

The norm of A, written $||A||$, is given by

$$||A|| = \sup_{||x|| \leq 1} ||Ax||.$$

Lemma 2.49. *For any $\Gamma > 0$, A is linear and $K = \sup_{||x|| \leq 1} ||Ax|| < \infty$, then we have*

$$\sup_{||x|| \leq \Gamma} ||Ax|| < \infty.$$

Proof. Let $y = \frac{x}{\Gamma}$. Then, we obtain

$$\begin{array}{rcl} ||Ax|| & = & ||A\Gamma \dfrac{x}{\Gamma}|| \\ & = & ||\Gamma(A\dfrac{x}{\Gamma})|| \\ & = & \Gamma ||A\dfrac{x}{\Gamma}|| \\ & \text{since} & ||\dfrac{x}{\Gamma}|| = \dfrac{||x||}{\Gamma} \leq 1 \\ & \leq & \Gamma K < \infty. \end{array}$$

□

2.3.6 Eigenfunction and Eigenvalue

Definition 2.50. Let $A \in \mathbb{C}^{n \times n}$ be a n by n matrix. The eigenvalues λ of A are the zeros of the characteristic polynomial

$$p(\lambda) = det(A - \lambda I),$$

where I is an identity martix. If the eigenvalues of A are real, then we index them from largest to smallest

$$\lambda_n(A) \leq \ldots \leq \lambda_1(A).$$

Let $\lambda(A)$ be a set of all possible eigenvalues of A is called its spectrum. If $\lambda \in \lambda(A)$, then there exists a nonzero vector x so that

$$Ax = \lambda x.$$

Such a vector is called an eigenvector for A associated with λ.

Notice that the Definition 2.50 follows [43].

2.3.7 Strongly continuous semigroup

Definition 2.51. A family $T(t)$ of bounded linear operators on a Banach space X is a strongly continuous semigroup such that
i) $T(0) = I$, (identity property);
ii) $\forall t, s > 0 : T(t + s) = T(t)T(s)$;
iii) $\forall x_0 \in X : \|T(t)x_0 - x_0\| \to 0$, as $t \to 0$.

Notice that the Definition 2.51 follows [38] on the page 234.

For the Definition of Chaos follows [53] on the page 50.

Here is an example about non-continuous semigroup.
Example Let $\{X_t : t > 0\}$ be a set of i.i.d. Chaos and $X_0 = 0$ if it satisfies with
i) $T_0 = I$.
ii) Let $P = F_{X_t} = T_t$, where P is the distribution of X_t and is a projection. Then, we have $T_{t+s} = P = T_tT_s = P^2$.
iii) $||T_tx_0 - x_0|| = ||Px_0 - x_0|| \neq 0$.

Then, $T(s) = P$ is a projection, for $s > 0$. And $\{X_t : t > 0\}$ is a non-continuous semigroup.

Notice that x_0 is a function, say f, and $Px_0 = E[f(X)]$, where $X = X_t$ is a number. Then, $E[f(X)]$ is not equal to f.

2.3.8 Infinitesimal Generator

Definition 2.52. Let $T(t)$ be a strongly continuous semigroup. The Infinitesimal Generator of $T(t)$ is defined by

$$A\nu = \lim_{t\to 0} \frac{T(t)\nu - \nu}{t} \text{ for } \nu \in D(A),$$

where $D(A)$ is the domain of A, which defined as

$$D(A) = \{\nu \in X : \lim_{t\to 0} \frac{T(t)\nu - \nu}{t} \text{ exists.}\}$$

The Definition 2.52 follows [34].

2.3.9 Quantum Mechanics Approach

"Quantum mechanics is the description of the behavior of matter and light in all its details and, in particular, of the happenings on an atomic scale." This is given by Feymann in 1964. Quantum mechanics is a branch of physics which is the fundamental theory of nature in terms of the small scales and energy levels of atoms. In quantum mechanics, the particle's evolution is random. The quantum particle's position at each instant t is a degree of freedom in physics, which is a random variable in probability theory. If one collects all the degrees of freedom over time, one would obtain a collection of random variables.

In quantum mechanics, the Hamiltonian is the operator corresponding to the total energy of the system in most of the cases. It is the sum of the kinetic energies of all the particles, plus the potential energy of the particles associated with the system. For different situations or number of particles, the Hamiltonian is different since it includes the sum of kinetic energies of the particles, and the potential energy function corresponding to the situation.

We introduce special Hamiltonian and apply Dirac continuous formalism for continuous markets. For discrete markets, we apply the traditional trace technique.

2.3.10 Dirac notation

In quantum mechanics, a state space is a complex Hilbert space, which is defined as the Definition 2.45. Introduce $\mathcal{V}$ is a quantum state space, which is a linear vector space defined as the Definition 2.39. ψ is a state vector in $\mathcal{V}$. $\mathcal{V}_{dual}$ is a dual space, which is also a linear vector space. χ is a state vector in $\mathcal{V}_{dual}$. In Dirac's bracket notation:

- $|\psi>$: a ket vector in $\mathcal{V}$.
- $<\chi|$: a bra vector in $\mathcal{V}_{dual}$.
- $<\chi|\psi>$: a scalar product.
- $|\psi><\chi|$: an operator(or called a dyad).

Example For ket vectors a and b, we write

$$|a>=\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ and } |b>=\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Consider one of them as a bra vector

$$<a| = \begin{pmatrix} a_1^*, \ldots, a_n^* \end{pmatrix} \text{ and } |b>=\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

then, we have the inner product

$$<a|b>=\begin{pmatrix} a_1^*, \ldots, a_n^* \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + \ldots + a_n^* b_n.$$

We can write operators in terms of bras and kets by the outer product.

$$|b><a| = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1^*, \ldots, a_n^* \end{pmatrix} = \begin{pmatrix} b_1 a_1^* & \ldots & b_1 a_n^* \\ \vdots & \ddots & \vdots \\ b_n a_1^* & \ldots & b_n a_n^* \end{pmatrix}.$$

In the Dirac formalism, the Hamiltonian is typically implemented as an operator on a Hilbert space. That is the eigenvectors of H, denoted $|x>$, provide an orthonormal basis for the Hilbert space. E_x is the spectrum of allowed energy levels of the system is given by the set of eigenvalues,

$$H|x>= E_x|x> .$$

2.3.11 Schördinger equation

We are motivated by Belal E. Baaquie for Schordinger equation, which is given at [3] on the page 20.

For describing a quantum system, one of the fundamental goals of physics is to obtain the dynamical equations that predict the future state of a system. Let ψ be a state funtion. This requirement in quantum mechanics is met by the Schrödinger partial differential equation that determines the future time evolution of the state function $\psi(t, \mathcal{F})$, where t parameterizes time. And the Schrödinger equation is time reversible.

The Schrödinger equation is expressed by the state space and operators and determines the time evolution of the state function $|\psi(t)>$, with t being the time parameter. One needs to specify the degrees of freedom of the system in question, that in turn specifies the nature of the state space $\mathcal{V}$; one also needs to specify the Hamiltonian H.

Then, the Schrödinger equation is given by

$$\frac{\partial|\psi(t)>}{\partial t} = -iH|\psi(t)> . \tag{2.4}$$

Let $|\psi>$ be the initial value of the state vector at $t = 0$ with $<\psi|\psi>= 1$. From Stone Theorem, the Schrödinger equation can be integrated to yield the following formal solution

$$|\psi(t)>= e^{-itH}|\psi>= U(t)|\psi> . \tag{2.5}$$

the Hamiltonian H is an operator that translates the initial state vector in time, as in the Schördinger equation. The evolution operator $U(t)$ is defined by

$$U(t) = e^{-itH}, U^{\dagger}(t) = e^{itH} \tag{2.6}$$

and is unitary since H is Hermitian.

The operator $U(t)$ is the exponential of the Hamiltonian H that in many cases, as is the case given in the Schrödinger equation, is a differential operator. The Feynman path integral is a mathematical tool for analyzing $U(t)$.

2.3.12 Fourier transform

Let f is an integrable function $f : \mathbb{R} \to \mathbb{C}$. The Fourier transform of the function f is $\hat{f}$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx, \text{ where } \xi \in R.$$

If $\hat{f}$ is an integrable function, then f is continuous and is determined by $\hat{f}$ via the inverse transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi x}d\xi, \text{ where } x \in R.$$

2.3.13 Fourier transform of tempered distributions

We begin by introducing the Schwart space which is the basic space to define the Fourier transform of tempered distributions.

Definition 2.53. The Schwartz space or space of rapidly decreasing functions on R^n is the function space

$$S(R^n) = \{f \in C^\infty(R^n) : ||f||_{\alpha,\beta} \leq \infty, \forall \alpha, \beta \in Z_+^n\},$$

where α, β are multi-indices, $C^\infty(R^n)$ is the set of smooth functions from R^n to C, and

$$||f||_{\alpha,\beta} = \sup_{x \in R^n} |x^\alpha D^\beta f(x)|.$$

Notice that if f is infinitely differentiable, all the $D^\beta f$ are infinitely differentiable functions.

The Definition 2.54 and 2.55 follows [42] on the page 220.

Definition 2.54. Let S be a Schwartz space. The set $\mathcal{T}$ consists of all the linear continuous mappings

$$T : S \to C,$$

i.e., we have $T \in \mathcal{T}$ if and only if

$$T(\alpha x + \beta y) = \alpha T x + \beta T y \text{ for all } \alpha, \beta \in C, x, y \in S,$$

and, as $n \to \infty$,

$$x_n \xrightarrow{S} x \text{ implies } T x_n \to T x.$$

The elements T of $\mathcal{T}$ are called tempered distribution.

Definition 2.55. Let $T \in \mathcal{T}$. The Fourier transform FT of T is defined by

$$(FT)(x) := T(Fx) \text{ for all } x \in S.$$

2.3.14 Completeness equation

We follows [2] on the page 48.

Definition 2.56. A projection on a linear space X is a linear map $P : X \to X$ such that

$$P^2 = P.$$

Definition 2.57. An orthogonal projection on a Hilbert space $\mathcal{H}$ is a linear map $P : \mathcal{H} \to \mathcal{H}$ that satisfies

$$P^2 = P, \quad < Px, y >=< x, Py > \text{ for all } x, y \in \mathcal{H}.$$

The completeness equation for the degree of freedom is given by

$$\int_{-\infty}^{\infty} dx |x >< x| = I : \text{ Completeness Equation} \tag{2.7}$$

where I is the identity operator on state space.
Let $\{\varphi_i\}$ be an orthonormal system, φ_i is an orthonormal basis. If $|a>$ is a vector,

then

$$|\varphi_i >< \varphi_i|(|a>) = |\varphi_i >< \varphi_i|a> = < \varphi_i|a> |\varphi_i> = P_{\varphi_i}a, \tag{2.8}$$

Let M be the subspace of H consisting of all linear combinations of kets belonging to an orthonormal collection $\{|\varphi_i>\}$,

$$\sum P_{\varphi_i}a = |a> . \tag{2.9}$$

Here, the projector R onto M is the sum of the dyad projectors:

$$R = \sum |\varphi_i >< \varphi_i| \tag{2.10}$$

If every element of R can be in the equation (2.10), the orthonormal collection form an orthonormal basis of H and the decomposition of the identity is

$$\sum_{i=-\infty}^{\infty} |\varphi_i >< \varphi_i| = I. \tag{2.11}$$

For the continuous time version, it justified via Fourier of tempered distributions.

$$\int_{-\infty}^{\infty} dx|x >< x| = I. \tag{2.12}$$

2.3.15 Hamiltonian for a periodic completeness equation

We follows [3] on the page 39. If the Hamiltonian operator of particle freely moving on a circle S^1 with radius L, we are going to apply it with $L = 1$ and $m = 1$. And the free particle Hamiltonian operator H, which is given by

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2}, \tag{2.13}$$

where $x \in S^1$ is defined on a periodic domain $[0, 2\pi]$, the variable x has the periodicity

$$x = x + 2\pi. \tag{2.14}$$

Consider the normalized eigenfunctions of H given by

$$H\psi_n(x) = E_n\psi_E(x), \tag{2.15}$$

$$\int_0^{2\pi} dx|\psi_n(x)|^2 = 1. \tag{2.16}$$

2.3.16 Resolvent

Notice that the Definition 3.4 follows [38] on the page 233.

Definition 2.58. Suppose that $\{P_t\}$ is the transition function of joint measurability with respect to time and space on the measurable space $(S, \mathcal{F})$, so that we have

$\forall \Gamma \in \mathcal{F}$, the map $(x, t) \to P_t(x, \Gamma)$ is $(\mathcal{F} \times \mathcal{B}[0, \infty))$-measurable from $E \times [0, \infty)$ to R.

For $\lambda > 0$, we define a map $R_\lambda : m\mathcal{B}_X \to m\mathcal{B}_X$: for $x \in S$, we have

$$R_\lambda f(x) := \int_{[0,\infty)} e^{-\lambda t} P_t f(x) dt = \int_S R_\lambda(x, dy) f(y),$$

where

$$R_\lambda(x, \Gamma) := \int_{[0,\infty)} e^{-\lambda t} P_t(x, \Gamma) dt.$$

Thus, for each Markov process, we can write down its resolvent.

2.4 Preliminary knowledge for Quantum Market

Quantum probability is the generalisation of the classical theory of probability made necessary by the noncommutative multiplication of quantum observables, which are usually represented by self-adjoint operators in a Hilbert space.

2.4.1 Projection

Definition 2.59. A projection on a linear space X is a linear map $P : X \to X$ such that

$$P^2 = P.$$

Definition 2.60. An orthogonal projection on a Hilbert space $\mathcal{H}$ is a linear map $P : \mathcal{H} \to \mathcal{H}$ that satisfies

$$P^2 = P, \quad < Px, y > = < x, Py > \text{ for all } x, y \in \mathcal{H}.$$

Example The orthographic projection which maps the point (a, b, c) in three-dimensional space R^3 to the point $(a, b, 0)$ is an orthogonal projection onto the $x - y$ plane. The orthographic projection is give by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The action of the matrix on the arbitrary vector is

$$P \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

Then, we obtain

$$P^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = P \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = P \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Definition 2.61. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is self-adjoint if $A^* = A$.

If A is a self-adjoint operator, it has the following fact.

If $A^* = A$ and A admits a representation $A = UDU^*$. Then, $f(A) = Uf(D)U^*$ for any function f.

2.4.2 Quantum state

The Definition 2.62, 2.63, 2.64 follows [44] on the page 28.

Definition 2.62. An involution on an algebra A is a map $a \to a^*$ of A onto itself such that
i) $(a^*)^* = a$;
ii) $(ab)^* = b^* a^*$;
iii) $(a + \gamma b)^* = a^* + \bar{\gamma} b^*$;
for all $a, b \in A$ and $\gamma \in \mathbb{C}$.

Definition 2.63. A *-algebra is an algebra A together with an involution. A Banach *-algebra is a Banach algebra A together with an isometric involution. An algebra homomorphism $\psi : A \to B$ between *-algebras is called a *-homomorphism if $\psi(a^*) = \psi(a)^*$ for all $a \in A$.

Definition 2.64. A Banach *-algebra is called a C *-algebra if $||a^*a|| = ||a||^2$ for all $a \in A$.

The Definition 2.65 and the Theorem 2.66 follow [45] on the page 63.

Definition 2.65. A quantum state on a given C^*-algebra $\mathcal{A}$ is a positive self-adjoint operator with unit-trace, i.e., $tr(\rho) = 1$.

The class of all quantum states on the C^*-algebra $\mathcal{A}$ shall be denoted by $\mathcal{S}(\mathcal{A})$, where

$$\mathcal{S}(\mathcal{A}) = \{\rho \in \mathcal{A} | \rho \geq 0, tr(\rho) = 1\}.$$

The theorem 2.66 follows [45].

Theorem 2.66. *A quantum state $\rho \in \mathcal{S}(\mathcal{A})$ has a canonial convex decomposition of the form*

$$\rho = \sum_{j=1}^{\infty} \lambda_j P_j, \tag{2.17}$$

where $\{\lambda_j\}$ is a sequence of non-negative numbers with $\sum_{j=1}^{\infty} \lambda_j = 1$ summing to one and $(P_j)_{j=1}^{\infty}$ is an orthonormal sequence of one-dimensional projections. If there are infinitely many nonzero terms, then the sum convergences with respect to the trace norm $|| \cdot ||_1$.

2.4.3 Quantum probability space

A positive operator ρ of unit trace is called a state. The set of all states in $\mathcal{H}$ is denoted by $\mathcal{P}$. For any fixed state ρ, the triple $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \rho)$ is called a finite dimensional quantum probability space.

Definition 2.67. A quantum probability space is a pair $(\mathcal{A}, \rho)$, where $\mathcal{A}$ is a von Neumann algebra and ρ is a normal (i.e., σ-weakly continous) state. The events in $(\mathcal{A}, \rho)$ are the orthogonal projections $p \in \mathcal{A}$. The probability that p occurs is $\rho(p)$.

2.4.4 Quantum observable

This section follows [48] on the page 9.

Let $\mathcal{H}$ be a Hilbert space of dimension $n \leq \infty$. Elements of $\mathcal{O}(\mathcal{H})$, i.e., Hermitian operators in $\mathcal{H}$, are called observables. An observable in quantum probability is what a random variable is in classical probability. Any observable X, being a self-adjoint operator, has the spectral resolution $X = \sum_j x_j E_j^X$, where $x_1, x_2, \ldots$ are its distinct eigenvalues and E_i^X is the event that X takes the value x_i.

Definition 2.68. A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ between real or complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is said to be orthogonal or unitary, respectively, if it is invertible and if

$$< Ux, Uy >_{\mathcal{H}_2} = < x, y >_{\mathcal{H}_1} \text{ for all } x, y \in \mathcal{H}_1.$$

Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic as Hilbert spaces if there is a unitary linear map between them.

A unitary operator is one-to-one and onto, and preserves the inner product. A map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary if and only if $U^*U = UU^* = I$.

Example Let $\mathcal{H} = C^n$ with its canonical orthonormal basis $\{e_j, 0 \leq j \leq n-1\}$. For any set $\{x_0, x_1, \ldots, x_{n-1}\}$ of n distinct real numbers consider the pure state e_0 and the observable $X = \sum_j x_j |e_j >< e_j|$. In the canonical basis X has the diagonal matrix representation

$$X = \begin{pmatrix} x_0 & 0 & 0 & \ldots & 0 \\ 0 & x_1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x_{n-1} \end{pmatrix}.$$

For any unitary operator $U = ((u_{ij}))$, the observable U^*XU takes the values $x_0, x_1, \ldots, x + n - 1$ and

$$\begin{aligned} < e_0, (U^*XU)^k e_0 > &= < Ue_0, X^k U e_0 > \\ &= \sum_{j=0}^{n-1} x_j^k |u_{j0}|^2. \end{aligned}$$

2.4.5 Quantum expectation

Let ρ be a state and the probability of event E_j^X, X takes the value x_j in the state ρ, is equal to $tr\rho E_j^X$. X has expectation $E_\rho(X)$ in the state ρ,

$$\begin{aligned} E_\rho(X) &= \sum_j x_j tr\rho E_j^X \\ &= tr\rho \sum_j x_j E_j^X \\ &= tr\rho X. \end{aligned}$$

Definition 2.69. (Projection-valued measure) Let $\mathcal{H}$ be a Hilbert space and let $P(\mathcal{H})$ be the set of orthogonal projections in the Banach algebra $L(\mathcal{H})$. A (finite) projection valued measure a on $\mathcal{H}$ is a map $\mathcal{B}(R) \to P(\mathcal{H})$ and $A \to a_A$ from the σ-algebra of Borel subsets of R to the set of projections, such that the following conditions hold:
(i) $a = 0, a_R = Id$;
(ii) For some constant $R > 0$, we have $a_{[-R,R]} = Id$;
(iii) If $A_n, n \geq 1$, is an arbitrary sequence of pairwise disjoint Borel subsets of R, let

$$A = \bigcup_{n \geq 1} A_n \in \mathcal{B}(R),$$

and then we have

$$a_A = \sum_{n \geq 1} a_{A_n}$$

where the series converges in the strong operator topology of $\mathcal{H}$.

Let A be a positive self-adjoint operator with the eigenvalues $\{x_1, x_2, \ldots, x_n\}$ and associated projections $P_1, P_2, \ldots, P_n$ so that the projection-valued measure $a(E)$ on the σ-field of Borel subsets of the real line R can be defined by $\sum_{x_i \in E} P_i$ for all $E \subseteq R$. For any eigenvalue x_i, assume that a measurement which gives the values x_i transforms the state ρ into the new state $\hat{\rho_x}$.

2.4.6 Quantum conditional expectation

First, we consider a set of points such that the probability of points is equal to $tr(\rho P_x)$, where x takes the value of x_i in the state ρ, then

$$\begin{aligned} tr(\rho P_x) &= tr(\rho P_x^2) \\ &= tr(P_x \rho P_x) \\ &= tr(\rho_x). \end{aligned}$$

If we have a random variable ξ such that $P(\xi = x) = P_x$, then we can derive the new density

$$\begin{aligned} tr(\rho P_i) &= \frac{tr(\rho_x P_i)}{tr(\rho_x)} \\ &= \frac{\rho_x}{tr(\rho_x)} \\ &= \hat{\rho}_x. \end{aligned}$$

For the quantum condition probability $P[Y|A = x]$,

$$\begin{aligned} \text{quantum } \frac{tr(\rho_x Y)}{tr(\rho_x)} &= \frac{tr(P_x \rho P_x Y)}{tr(P_x \rho P_x)} \\ &= \frac{tr(P_x \rho P_x S_N)}{tr(P_x \rho P_x)}, \text{ where } Y = S_N. \end{aligned}$$

The quantum conditional expectation $E[Y = x'|A = x]$,

$$\text{quantum } \sum x' \frac{tr(P_x \rho P_x Y)}{tr(P_x \rho P_x)}. \tag{2.18}$$

2.4.7 Tensor product

Definition 2.70. Let F be a field. The kronecker product of $A = [a_{ij}] \in M_{m,n}(F)$ and $B = [b_{ij}] \in M_{p,q}(F)$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B \equiv \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ . & \ldots & . \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix} \in M_{mp,nq}(F).$$

Theorem 2.71. *Let $A \in M_{m,n}$, $B \in M_{p,q}$, $C \in M_{n,k}$, and $D \in M_{q,r}$. Then,*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{2.19}$$

Proof. Let $A = [a_{ih}]$ and $C = [c_{hj}]$. By the definition of Kronecker product, $A \otimes B = [a_{ih}B]$ and $(C \otimes D) = C = [c_{hj}D]$. The i, j^{th} block of $(A \otimes B)(C \otimes D)$ is $\sum_{n=1}^{n} a_{ih}c_{hj}$ which implies that the i, j^{th} block of $(AC) \otimes (BD)$ is $[\sum_{n=1}^{n} a_{ih}c_{hj}](BD)$. Thus, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. □

Theorem 2.72. *Let $A \in M_{m,n}(F)$. Then,*

$$(A \otimes B)^* = A^* \otimes B^* \ \texttt{for B} \ \in M_{p,q}(F). \tag{2.20}$$

Proof.

$$\begin{aligned}
(A \otimes B)^* &= \begin{pmatrix} a_{11}B & ... & a_{1n}B \\ . & ... & . \\ a_{m1}B & ... & a_{mn}B \end{pmatrix}^* \\
&= \begin{pmatrix} \bar{a_{11}}B^* & ... & \bar{a_{1n}}B^* \\ . & ... & . \\ \bar{a_{m1}}B^* & ... & \bar{a_{mn}}B^* \end{pmatrix} \\
&= A^* \otimes B^*.
\end{aligned}$$

□

Corollary 2.73. *The above Theorem can be generalized in the following way:*

$$(A_1 \otimes A_2 \otimes ... \otimes A_k)(B_1 \otimes B_2 \otimes ... \otimes B_k) = A_1B_1 \otimes A_2B_2 \otimes ... \otimes A_kB_k. \tag{2.21}$$

Proof. By the above Theorem, since $(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1B_1 \otimes A_2B_2$, which is given by

$$\begin{aligned}
\text{the L.H.S of (2.21)} &= [(A_1 \otimes A_2 \otimes ... \otimes A_n) \otimes A_{n+1}][(B_1 \otimes B_2 \otimes ... \otimes B_n) \otimes B_{n+1}] \\
&= [(A_1 \otimes A_2 \otimes ... \otimes A_n)][(B_1 \otimes B_2 \otimes ... \otimes B_n)] \otimes [A_{n+1}B_{n+1}] \\
&= [A_1B_1 \otimes A_2B_2 \otimes ... \otimes A_nB_n] \otimes [A_{n+1}B_{n+1}] \\
&= A_1B_1 \otimes A_2B_2 \otimes ... \otimes A_nB_n \otimes A_{n+1}B_{n+1}.
\end{aligned}$$

□

2.4.8 Diagonal decomposition

Theorem 2.74. *(Symmetric Schur Dcomposition) If $A \in R^{n\times n}$ is symmetric, then there exists a real orthogonal Q such that*

$$Q^* AQ = \Lambda = diag(\lambda_1, \dots, \lambda_n).$$

Moreover, for $k = 1 : n, AQ(:, k) = \lambda_k Q(:, k)$.

Theorem 2.75. *(Schur Decomposition) If $A \in C^{n\times n}$, then there exists a unitary $Q \in C^{n\times n}$ such that*

$$Q^{-1}AQ = D + N,$$

where $D = diag(\lambda_1, \dots, \lambda_n)$ and $N \in C^{n\times n}$ is strictly upper triangular. Furthermore, Q can be chosen so that the eigenvalues λ_i appear in any order along the diagonal.

2.4.9 Jordan Matrix

Motivated by Darl D. Meyer, the definition 2.76 follows [9] on the page 590.

First, we present some general results about Jordan decomposition.

Definition 2.76. Function of a Jordan Blocks For a $k \times k$ Jordan block $J_{k\times k}$ with eigenvalue u and for a function f such that $f(u), f'(u), \dots, f^{(k-1)}(u)$ exist, $f(J_{k,k})$ is defined to be

$$\begin{aligned} f(J_{k,k}) &= f\begin{pmatrix} u & 1 & \dots & \dots & 0 \\ 0 & u & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u & 1 \\ 0 & \dots & \dots & \dots & u \end{pmatrix} \\ &= \begin{pmatrix} f(u) & \frac{f'(u)}{1!} & \frac{f''(u)}{2!} & \dots & \frac{f^{(k-1)}(u)}{(k-1)!} \\ 0 & f(u) & \frac{f'(u)}{1!} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(u) & \frac{f'(u)}{1!} \\ 0 & \dots & \dots & \dots & f(u) \end{pmatrix} \end{aligned}$$

where

$$J_{k,k} = \begin{pmatrix} u & 1 & \dots & \dots & 0 \\ 0 & u & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u & 1 \\ 0 & \dots & \dots & \dots & u \end{pmatrix}.$$

Functions of Matrix For A with $\sigma(A) = \{u, u, \dots, u\}$, let $k_k = index(u)$.
(i) A funtion f is said to be defined (or to exist) at A when $f(u), f'(u), \dots, f^{(k-1)}(u)$ exist for each $u \in \sigma(A)$.
(ii) Suppose $A = PJP^{-1}$, where J is in Jordan form with representing the various Jordan blocks. If f exists at A, then the value of f at A is defined to

$$f(A) = P \begin{pmatrix} f(J_{k_1,k_1}) & 0 & \dots & \dots & 0 \\ 0 & f(J_{k_2,k_2}) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(J_{k_{k-1},k_{k-1}}) & 0 \\ 0 & \dots & \dots & \dots & f(J_{k_k,k_k}) \end{pmatrix} P^{-1},$$

where

$$A = P \begin{pmatrix} J_{k_1,k_1} & 0 & \dots & \dots & 0 \\ 0 & J_{k_2,k_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{k_{k-1},k_{k-1}} & 0 \\ 0 & \dots & \dots & \dots & J_{k_k,k_k} \end{pmatrix} P^{-1}.$$

2.4.10 Monte Carlo

Monte Carlo simulation is a method by using random numbers for iteratively evaluating a deterministic model. For derivative pricing, it simulates a large number of price paths of the underlying assets with probability corresponding to the underlying stochastic process, calculates the discounted payoff of the derivative for each path, and averages the discounted payoffs to yield the derivative price. The validity of Monte Carlo simulation relies on the law of large numbers.

2.5 Geometric Levy processes market

2.5.1 Binomial market (Cox-Rubinstein model) by Hedging and No Arbitrage

Consider a discrete market with one non-risky asset (bond) and one risky asset (share). One can take $t = 0, 1, \ldots, N$. Assume that one non-risky asset is a riskless bond or bank account B, which yields

$$B(t+1) = (1+\rho)B(t), B(0) = 1,$$

where $\rho > 0$ is a riskless rate of return in each time interval $[t, t+1]$. Thus, its price process is $B(t) = (1+\rho)^t, t = 0, 1, \ldots, N$.

Suppose that the interest rate r is fixed and the share price S, where $S_0, S_1, \ldots$, are defined by the N-step binomial model

$$S_0, S_t = S_{t-1}Y_t,$$

where $Y, Y_1, Y_2, \ldots$ are iid random variables with the probability $P(Y = u) + P(Y = d) = 1$. Here, each share price can rise to a value Su or fall to a value Sd go down. The no-arbitrage condition $d < 1 + \rho < u$ holds.

Motivated by the lecture notes, I state the following derivation of hedging for both Binomial market and Black-Scholes market.

The option game Suppose we want to find the initial portfolio (x_0, x_1), where x_1 invested in shares and x_0 invested in bonds, such that the final capital $W_1 = C$ is payoff or the option claim. Let ρ be the risk-free interest rate. Let us remind that

$$W_1 = \text{ capital at time } 1 = x_0(1+r) + x_1 S_1.$$

Notice that the C payoff is the function of S_1. Therefore, the equation $W_1 = C$ consists of two equations with two unknowns

$$[W_1 | S_1 = S_0 u] = x_0(1+r) + x_1 S_0 u = C_u,$$

and

$$[W_1|S_1 = S_0 d] = x_0(1+r) + x_1 S_0 d = C_d.$$

The solution are

$$x_1 = \frac{C_u - C_d}{S_0(u-d)},$$

which means that the number of units of stock you must be held in a portfolio that replicates the payoff to the option and

$$x_0 = \frac{uC_u - dC_d}{(1+\rho)(u-d)},$$

which it means that the value of the borrowing (or a short position in bonds) required in a portfolio that replicates the payoff to the option. Then the option price

$$\begin{aligned}
OP &= OP(C) = \text{ how much to invest to get the option claim } C \\
&= W_0 = x_0 + x_1 S_0 \\
&= \frac{uC_u - dC_d}{(1+\rho)(u-d)} + S_0 \frac{C_u - C_d}{S_0(u-d)} \\
&= C_u \frac{-d+1+\rho}{(1+\rho)(u-d)} + C_d \frac{u-(1+\rho)}{(1+\rho)(u-d)} \\
&= \frac{1}{1+\rho}[C_u \frac{1+\rho-d}{u-d} + C_d \frac{u-(1+\rho)}{u-d}] \\
&= \frac{1}{1+\rho}[C_u q_u + C_d q_d],
\end{aligned}$$

where

$$q_u = \frac{1+\rho-d}{u-d} \text{ and } q_d = \frac{u-(1+\rho)}{u-d}.$$

The definition 2.77 and the theorem 2.78 follows [46].

Definition 2.77. Consider the binomial asset-pricing model. Let $W_0, W_1, \ldots, W_T$ be a sequence of random variables, with each W_j depending only on the first j paths. Such a sequence of random variables is called an adapted stochastic process. If

$$W_j = E_j[W_{j+1}], \ j = 0, 1, \ldots, T,$$

we say this process is a martingale.

Theorem 2.78. *Suppose that the general binomial model with* $0 < d < 1 + \rho < u$ *satisfy the risk-neutral probabilities be given by*

$$q_u = \frac{1+\rho-d}{u-d} \text{ and } q_d = \frac{u-(1+\rho)}{u-d}.$$

Then, under the risk-neutral measure, the discounted stock price is a martingale.

Then, the arbitrage free option price formula of the option claim $C = f(S_T)$ is defined by

$$\begin{aligned} W_0 = OP(f(S_T)) &= (1+\rho)^{-T} E_Q[f(S_T)] \\ &= (1+\rho)^{-T} \sum_{j=0}^{T} f(S_0 u^j d^{T-j}) \frac{T!}{j!(T-j)!} q_u^j q_d^{T-j}. \end{aligned}$$

Notice that risk-neutral measure allows the option to be priced as the discounted value of its expected payoff with the risk-free interest rate.

2.5.2 Black-Scholes market by Hedging and No Arbitrage

Consider a continuous time market with one non-risky bond with the fixed interest rate ρ, no transaction costs, and an risky asset. Assume that the share price process $\{S_t; t \geq 0\}$ is modelled by the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.22}$$

Then, $\{S_t; t \geq 0\}$ is a Geometric Brownian Motion (GBM) with mean parameter μ and variance parameter σ^2, i.e. S_t follows the following lemma:

Lemma 2.79. *The process* S *given by the formula*

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}, \forall t \in [0, T],$$

is the unique solution of the stochastic differential equation (2.22).

The lemma 2.79 follows [47].

Then, we know

(i) $\{S_t; t \geq 0\}$ is a GBM with mean $a = \mu - \frac{\sigma^2}{2}$ and variance σ^2;

(ii) The discounted share process $e^{-\rho t} S_t$ is a martingale and the no-arbitrage condition holds if and only if $a = \rho - \frac{\sigma^2}{2}$;

(iii) The Black-Scholes formula holds

$$\begin{aligned} OP(f(S_T|t)) &= E_Q[e^{-\rho(T-t)} f(S_T)|\mathcal{F}_t] \\ &= e^{-\rho(T-t)} E[f(S_t e^{N(a(T-t),\sigma^2(T-t))})|S_t]. \end{aligned}$$

To derive the Black-Scholes market via hedging, we recall some terminology for the option game:
(θ_t, δ_t) is the portfolio at time t, where θ is the number of shares to hold at time t (invest in shares) and δ is the number of bonds to hold at time t (invest in bonds);
ρ is the risk-free interest rate;
X_t is the capital at time t;
$f(S_T)$ is option claim;
S_t is the risky asset (share price) at time t.
1 is the price of the bond at time 0.

Note we have the following connection

$$X_t = \theta_t S_t + \delta_t.$$

The target is to get option claim $f(S_T)$ at time T to hedge

$$X_T = f(S_T).$$

Then, X_t denotes how much to invest at time t to get the option claim and the option price of the option claim $f(S_T)$ at time t is $OP(f(S_T)|t)$.

Again, we assume that the share price follows the GBM and is defined via the stochastic differential equation

$$dS_t = S_t \mu dt + S_t \sigma dS_t,$$

where $\{B_t; t \geq 0\}$ is a standard Brownian motion.

Under the risk-neutral measure, the drift of a stock is changed to the risk-free rate of return so that

$$dS_t = S_t\rho dt + S_t\sigma dS_t.$$

It reflects the fact that the hedging strategy ensures that the underlying drift of the stock is balanced against the drift of the option. The drifts are balanced since drift reflects the risk premium demanded by investors to account for uncertainty and that uncertainty has been hedged away.

As before, let us follow the capital change process and use its differential $t \to t+dt$. At time t, we have the capital

$$X_t = \theta_t S_t + 1\delta_t.$$

And at time $t + dt$, we have the capital

$$X_t + dX_t = \theta_t S_t + 1\delta_t + \theta_t dS_t + \delta_t \rho dt.$$

Then, we obtain

$$\begin{aligned} dX_t &= \theta_t dS_t + \delta_t \rho dt \\ &= \theta_t (S_t \rho dt + S_t \sigma dB_t) + \delta_t \rho dt \\ &= \theta_t S_t \sigma dB_t + (\delta_t \rho + \theta_t S_t \rho) dt. \end{aligned}$$

Notice that it changes in share price: $S_t \to S_t + dS_t$; it gains from shares: $\theta_t S_t \to \theta(S_t + dS_t)$; it gains from interest rate: $\delta \to \delta e^{dt} = \delta(1 + \rho dt)$.

On the other hand, by BS formula

$$\begin{aligned} X_t &= OP(f(S_T)|t) = \text{ BS formula } = E_Q[e^{\rho(T-t)} f(S_T)|S_t] \\ &= g(t, S_t), \end{aligned}$$

which is an Ito process, and hence by the Ito formula

$$\begin{aligned} dX_t &= dg(t, S_t) g'_{S_t} dS_t + g'_t dt + \frac{1}{2} g''_{S_t,S_t} (dS_t)^2 \\ &= g'_{S_t} S_t \sigma dB_t + (g'_{S_t} S_t \rho + g'_t + \frac{1}{2} g''_{S_t,S_t} S_t^2 \sigma^2) dt. \end{aligned}$$

Recall that

$$dX_t = \theta_t S_t \sigma dB_t + (\delta_t \rho + \theta_t S_t \rho) dt.$$

Then, we derive

$$g'_{S_t} S_t \sigma = \theta_t S_t \sigma \text{ and } \theta_t = g'_{S_t} = \frac{\partial g(t, S_t)}{\partial S_t},$$

where $\partial g(t, x)/\partial x$ is called delta. So, the portfolio policy corresponding to this option game is referred to as the data hedging and it means we need to hold delta $(\partial g(t, S_t)/\partial S_t)$ shares.

Finally, to derive the Blach-Scholes equation, consider $f(t, X_t) = e^{-\rho t} X_t$, which is an Ito martingale. By the Ito formula,

$$\begin{aligned} de^{-\rho t} X_t &= df(t, X_t) = f'_{X_t} dX_t + f'_t dt + \frac{1}{2} f''_{X_t,X_t} (dX_t)^2 \\ &= e^{-\rho t} dX_t - \rho e^{-\rho t} X_t dt. \end{aligned}$$

Now, applying from above

$$dX_t = g'_{S_t} S_t \sigma dB_t + (g'_{S_t} S_t \rho + g'_t + \frac{1}{2} g''_{S_t,S_t} S_t^2 \sigma^2) dt,$$

we derive

$$\begin{aligned} de^{-\rho t} X_t &= e^{-\rho t} g'_{S_t} S_t \sigma dB_t + e^{-\rho t} (-\rho g + g'_{S_t} S_t \rho + g'_t + \frac{1}{2} g''_{S_t,S_t} S_t^2 \sigma^2) dt \\ &= \sigma_t dB_t \text{ since Ito martingale,} \end{aligned}$$

It means that our option price is expected to grow at the same rate as the bank account and hence the growth of each cancels out in the given process. That is what it means to be a martingale. We do not expect change over time so we have zero expected growth. Hence, the dt term is zero implying

$$0 = e^{-\rho t} (-\rho g + g'_{S_t} S_t \rho + g'_t + \frac{1}{2} g''_{S_t,S_t} S_t^2 \sigma^2)$$

which is with $x = S_t$ after rearrangement gives Black-scholes equation for the option price:

$$\rho g = g'_t + g'_x x\rho + \frac{1}{2} g''_{x,x} x^2 \sigma^2.$$

The equation is derived for the European call option

$$\begin{aligned} C(t,x) &= E_Q[e^{-\rho(T-t)(S_T-K)_+|S_t=x}] \\ &= OP((S_T-K)_+|t,x). \end{aligned}$$

The Black-Scholes equation is then

$$\rho C(t,x) = \frac{\partial C(t,x)}{\partial t} + \frac{\partial C(t,x)}{\partial x}\mu x + \frac{1}{2}\frac{\partial^2 C(t,x)}{\partial x^2} x^2\sigma^2.$$

If you sell an option, the BS model says that you can completely remove the risk of the call by continuously rebalancing your stock holding to neutralize the delta of the option.

$$C(t) = S(t)e^{-\rho(T-t)}N(d_1) - e^{-r(T-t)}KN(d_2),$$

where

$$d_1 = \frac{\frac{ln(S(t))}{K} + (r - \rho + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Here, $N(x)$ is the cumulative density function of the standard normal distribution and $N'(x)$ is the probability density function of the standard normal distribution:

$$N(x) = \int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,$$

and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

The option's delta is given by $\frac{\partial c(t)}{\partial S(t)} = e^{-\rho(T-t)}N(d_1)$. It measures how the call price changes per unit change in the price of the underlying and is the number of units of stock you must hold in a continuously rebalanced portfolio that replicates the payoff to the call. The term $S(t)N(d_1)$ is the discounted value of the expected benefit of owning the option expectations taken under risk-neutral probability measure. The term $e^{-r(T-t)}KN(d_2)$ is the discounted value of the expected cost of owning the option with expectations taken under risk-neutral probability measure. $N(d_2)$ is the risk-neutral probability that the call option finishes in the money.

2.5.3 Geometric Levy Market

For the content in this section, it follows [49].

Consider a continuous time market with one non-risky bond with the fixed interest rate ρ and on risky asset. The share price process $\{S_t = S_0 e^{X_t}; t \geq 0\}$ is defined by the Geometric Levy process since $\{X_t; t \geq 0\}$ is a Levy process. There are several properties as follows:

(i) Markov property

$$E[g(S_T)|\mathcal{F}_t] = E[g(S_T)|S_t];$$

(ii) GL-conditioning $[S_T|S_t = u] = ue^{X_{T_t}}$.
(iv) $M_t = e^{-\rho t} S_t$ is a martingale if and only if

$$\rho = H(1)$$

where the function $H(u)$ is defined by

$$E[e^{zX_t}] = e^{tH(z)}.$$

(v) The transformed Levy process $\tilde{X}_t$ is the Levy process with

$$E_Q[f(X_T)] = E[f(X_T)e^{uX_T}]/E[e^{uX_T}]$$

with the transformed function $\tilde{H}(z) = H(z+u) - H(z)$ which can be chosen to satisfy $\tilde{H}(1) = \rho$. Then the new measure is called martingale measure.
(vi) Option pricing holds for the Geometric Levy process with the martingale measure Q in the following form

$$\begin{aligned} OP(f(S_T)|t,x) &= E_Q[e^{-\rho(T-t)} f(S_T)|S_t = x] \\ &= E_Q[e^{-\rho(T-t)} f(xe^{X_{T-t}})] \end{aligned}$$

2.5.4 Markov Property

Filtration $(\mathcal{F}_t)_t$ is a non-decreasing set of σ-fields. i.e.

$$\mathcal{F}_s \subset \mathcal{F}_t, \text{ for } t \geq s.$$

Let X_t be a stochastic process which is adapted to the filtration, i.e. $X_t \in \mathcal{F}_t$ or X_t is $\mathcal{F}_t$-measurable. Then, X_t satisfies the Markov Property if for all $t \leq T$ and f (for which the Lebesgue integral is finite $E|f(X_T)| \leq \infty$)

$$E[f(X_T)|\mathcal{F}_t] = E[f(X_T)|X_t].$$

The Markov Property shows how the general conditional expectation is reduced to a usual conditional expectation of one variable on another.

In other words, $OP(S_T|t) = S_t$, works in any market (hedging via buying one share).

$$OP(f(S_T)|t) \quad = \quad e^{-\rho(T-t)}E[f(S_T)|\mathcal{F}_t].$$

Then, by Markov property, we obtain

$$OP(f(S_T)|t) \quad = \quad e^{-\rho(T-t)}E[f(S_T)|S_t].$$

Chapter 3

Hamiltonians and Markov kernels

3.1 Introduction

In this chapter, the objective is to derive option pricing via quantum formalism. Firstly, we apply the quantum formalism to derive several Hamiltonian operators, such as Hamiltonian for standard Brownian motion, Hamiltonian for Brownian motion, Hamiltonian for geometric Brownian motion, Hamiltonian for possion process, Hamiltonian for compound poisson process, Hamiltonian for possion process with shift, and even Hamiltonian for Levy process. Then, we use these Hamiltonian operators to derive pricing kernel and the relative option price defined via Feynman-Kac formula.

3.2 Main Structure

In my thesis, the quantum model starts from the derivation of eigenvalues of the Hamiltonian operator according to different stochastic processes.

Let us outline the structure of the next subsection.

Lemma 3.1. *Let H be a Hamiltonian operator, f be an eigenvector of H, and K be the related eigenvalue. Then,*

$$H^s f = K^s f.$$

If we consider the exponential form, it will become

$$e^{tH} f = e^{tK} f. \tag{3.1}$$

Proof: For the simple connection, it based on the following expression

$$e^{tH} f = \sum_{s=0}^{\infty} \frac{(tH)^s f}{s!} = \frac{\sum_{s=0}^{\infty} K^s t^s f}{s!} = f e^{Kt}.$$

3.2.1 Quantum formalism

In this section, the objective is to derive Option Pricing via Quantum formalism. The formalism is based on the Fourier transform of tempered distributions on the $|p>$ basis to momentum space. Let us write

$$< x|p >= e^{ipx} \ , \ < p|x >= e^{-ipx}. \tag{3.2}$$

Then,

$$< x|x' >= \delta(x - x') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|p >< p|x' > .$$

This is equivalent to the the completeness equation for momentum space basis $|p>$

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p >< p| = I$$

We first apply the formalism to compute the pricing kernel

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \end{aligned}$$

The option pricing is based on the following assumptions
(1) All financial instruments, including the price of the option, are elements of a state space. The stock price is given by

$$S(x) =< x|S >= e^x.$$

The option price is given by a state vector. For the call option price, the payoff function is given by

$$C(t,x) =< x|C,t > \ , \ g(x) =< x|g >,$$

and similarly for the put option.
(2) The option price is evolved by a Hamiltonian operator H, that, due to put-call parity, evolves both the call and put options.
(3) The price of the option satisfies the Schrödinger equation

$$H|C,t >= \frac{\partial}{\partial t}|C,t >$$

Notice that solution to the Schrödinger equation is derived by the Feynman-Kac formula (form of the completeness equation)

$$C(t,x) = \int_{-\infty}^{\infty} dx' < x|e^{-(T-t)H}|x' > g(x').$$

Before the final step, we need to introduce the Feynman-Kac formula. The Feynman-Kac formula estabilishes a connection between parabolic partial differential equations and stochastic processes. The Definition 3.2 follows [34] on the page 143.

Definition 3.2. Let $f \in C_0^2(R)$ and $g \in C(R)$. Suppose that g is a nonegative, continuous function and f is a bounded and continuous funtion. Assume that $u(t,x)$ is a bounded funtion that satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - g(x)u.$$

and the initial condition

$$u(0,x) = f(x).$$

Then, the Feynman-Kac formula is

$$u(t,x) = E^x[e^{-\int_0^t g(X_s)ds} f(X_t)],$$

where, under the probability measure P^x, the process $\{X_s\}_{t\geq 0}$ is the Brownian motion started at x.

Finally, to find the option price of the option claim $Q(S(T))$ for the risky price $S(t)$, we have the following lemma.

Lemma 3.3. *Let $S(t)$ be the risky price and $Q(S(T))$ detemrined via Hamiltonian H be the option claim. Then, Option Price is defined via Feynman-Kac formula.*

$$OP(Q(S(T))|S(t) = x) \;=\; \int_{-\infty}^{\infty} < x|e^{-(T-t)H}|x' > Q(x')dx'.$$

3.3 Resolvents

We begin with calculating resolvents for several Levy processes.

3.3.1 The Resolvent for Brownian motion

Let $f(y) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ be the Brownian motion transition function. Then, we motivated by and derive the resolvent for the Brownian motion. We start from the the definition of resolvent 3.4, and we plug in the Brownian motion transition funtion as follows

$$(R_\alpha f)(x) = \frac{1}{\sqrt{2\alpha}} \int_{\infty}^{\infty} e^{-\sqrt{2\alpha}|y-x|} f(y)dy,$$

Then, we derive the resolvent for Brownian motion, which is

$$\begin{aligned}
& (\int_{-\infty}^{\infty} e^{-\alpha t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dt = \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|x|}) \\
\overset{t=s^2}{\Longrightarrow} \quad & 2\int_0^{\infty} e^{-\alpha s^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2s^2}} ds \\
\overset{s=\sqrt{c}u}{\Longrightarrow} \quad & \frac{2\sqrt{c}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\alpha c u^2 - \frac{x^2}{2cu^2}} du \\
\overset{c=\frac{|x|}{\sqrt{2\alpha}}, \beta=\sqrt{\frac{\alpha}{2}}|x|}{\Longrightarrow} \quad & \frac{2\sqrt{c}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\beta(u^2+\frac{1}{u^2})} du \\
= \quad & \frac{2\sqrt{c}}{\sqrt{2\pi}} e^{-\beta} \int_0^{\infty} exp(-\beta(u-\frac{1}{u})^2) du.
\end{aligned}$$

The map $u \mapsto s(u) := u - \frac{1}{u}$ maps $(0, \infty)$ one-to-one onto $(-\infty, \infty)$ and the inverse map $s \mapsto u(s)$ satifies

$$u(s) = s + u(-s) \Rightarrow u'(s) + u'(-s) = 1.$$

Proof:

$$
\begin{array}{ll}
 & s(u) = u - \dfrac{1}{u} \\
 & s(\dfrac{1}{u}) = \dfrac{1}{u} - u = -s(u) \\
\Rightarrow & s(\dfrac{1}{u}) = -s(u) \\
Because & \text{the map } s(u) := u - \dfrac{1}{u} \text{ maps } (0,\infty) \text{ one-to-one onto } (-\infty,\infty) \\
 & \text{ and the inverse map } s \mapsto u(s). \\
Therefore & \text{given } u, \text{ we can get } u - \dfrac{1}{u} \\
 & \text{given } \dfrac{1}{u} \text{ , we can get } -s \\
\Rightarrow & s \mapsto u(s) = s - \dfrac{1}{s} \\
= & s + u(-s).
\end{array}
$$

3.3.2 The Resolvent for Poisson process

Let $P_t = e^{-\lambda t}(\lambda t)^{n-1}/\Gamma(n)$ be Poisson transition function. Then, we derive the resolvent for Poisson process, which is

$$R_\lambda := \int_0^\infty e^{-zt} P_t dt = \frac{\lambda^{n-1}}{(z+\lambda)^n},$$

We start from the the definition of resolvent (Definition 3.4), then we plug in the Poisson transition funtion as follows

$$
\begin{aligned}
R_\lambda &= \int_0^\infty e^{-zt} P_t dt \\
&= \int_0^\infty e^{-zt} \frac{e^{-\lambda t}(\lambda t)^{n-1}}{\Gamma(n)} dt \\
&= \frac{1}{\Gamma(n)} \int_0^\infty e^{-t(z+\lambda)} \lambda^{n-1} t^{n-1} dt.
\end{aligned}
$$

Let $(t(z+\lambda))^n = a$. Then we obtain $t(z+\lambda) = a^{\frac{1}{n}}, x \in (0,\infty), a \in (0,\infty)$. Also, we have $da = n(t(z+\lambda))^{n-1}(z+\lambda) = nt^{n-1}(z+\lambda) = nt^{n-1}(z+\lambda)^n dt$. Then, we get

$$\frac{\lambda^{n-1}}{\Gamma(n)n(z+\lambda)^n} \int_0^\infty e^{-a^{\frac{1}{n}}} da$$

Let $a^{\frac{1}{n}} = x$. Then $x^n = a, da = nx^{n-1}dx$. Therefore, we get

$$\begin{aligned}\frac{\lambda^{n-1}}{\Gamma(n)n(z+\lambda)^n}\int_0^\infty e^{-x}nx^{n-1}dx &= \frac{\lambda^{n-1}}{\Gamma(n)(z+\lambda)^n}\int_0^\infty e^{-x}x^{n-1}dx \\ &= \frac{\lambda^{n-1}}{(z+\lambda)^n}.\end{aligned}$$

3.3.3 The Resolvent for Compound Poisson process

Let P_t be the transition function for the Compound Poisson process as follows

$$P(X_t \leq X) = \sum_{n=0}^{\infty} P(\sum_{k=1}^{m_t} Y_k \leq x | m_t = n)\frac{(\lambda t)^n e^{-\lambda t}}{n!} = \sum_{n=0}^{\infty}\frac{(\lambda t)^n e^{-\lambda t}}{n!}G^{(n)}(x).$$

As before, we derive the resolvent for Poisson process, which is

$$R_\lambda := \int_0^\infty e^{-zt}P_t dt = \frac{\lambda^{n-1}}{(z+\lambda)^n}.$$

Similarly, we start from the the definition of resolvent (Definition 3.4), then we plug in the Compound Poisson transition funtion as follows

$$\begin{aligned}\int_0^\infty E[f(X_t)|X_0 = a]dt &= \sum_{n=0}^{\infty} E[f(a + \sum_{k=1}^{n} Y_k)]\int_0^\infty e^{-zt}P_t dt \\ &= \sum_{n=0}^{\infty} E[f(a + \sum_{k=1}^{n} Y_k)]\frac{\lambda^{n-1}}{(\lambda+z)^n} \\ &= \sum_{n=0}^{\infty} E[f(a + X_n)]\frac{\lambda^{n-1}}{(\lambda+z)^n}.\end{aligned}$$

3.4 Examples for Hamiltonian and pricing kernel

Here are several examples for the Option Pricing Calculation.

3.4.1 Hamiltonian and Markov kernel for the Standard Brownian Motion

The objective in this part is to compute the Option Pricing $\mathrm{OP}(f(B_T)|t)$ for the Standard Brownian Motion (SBM) case via the Quantum Mechanics Formalism.

Assume that the interest rate $\rho = 0$. Firstly, we show that Hamiltonian H for SBM is defined by

$$H = -\frac{1\partial^2 f}{2\partial x^2}. \tag{3.3}$$

Then, we derive the Markov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time and continuous space quantum mechanics formalism

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \quad . \end{aligned}$$

Firstly, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(B_t)|B_0 = x]}{t} \\ &= \frac{f(x) - E[f(x + B_t)]}{t} \\ &= -E[\frac{f(x + B_t) - f(x)}{t}]. \end{aligned}$$

Then, via the Ito formula,

$$f(x + B_t) = f(x) + \int_0^t f'(x + B_s)dB_s + \frac{1}{2}\int_0^t f''(x + B_s)ds$$

implying

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''(x + B_s)ds]] \\ &= -\frac{1}{2t}\int_0^t E[f''(x + B_s)ds] \\ &\to -\frac{1}{2}f''(x) \end{aligned}$$

since $f''(B_s) = f''(x + B_s) \to f''(x)$ as $s \to 0$ a.s..

Then, we apply QMF approach to obtain the SBM Markov kernel. It can be seen that

$$\begin{aligned}
p(x,\tau;x') &= <x|e^{-\tau H}|x'> \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} <x|e^{-\tau H}|p><p|x'> \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\tau(-\frac{1}{2}(ip)^2)} e^{ip(x-x')}
\end{aligned}$$

by applying that $e^{-\tau H}|p>= e^{-\tau(-\frac{1}{2}(ip)^2)}|p>$, $<x|p>= e^{ipx}$, and $<p|x>= e^{-ipx}$ (See before (3.2)).

From the Gaussian kernel,

$$\begin{aligned}
p(x,\tau;x') &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{\frac{1}{2}\tau(ip)^2} e^{ip(x-x')} \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f(ip)^2} e^{ipA}, \text{ where } f = \frac{1}{2}\tau \text{ and } A = x - x' \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f((ip)^2 + \frac{ip}{f}A + \frac{A^2}{4f^2} - \frac{A^2}{4f^2})} \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f(ip+\frac{A}{2f})^2 - \frac{A^2}{4f})}.
\end{aligned}$$

Let $y = p + \frac{A}{2f}$ and $z = \sqrt{f}y$, and then $dy = \frac{1}{\sqrt{f}}dz$. Thus, by simple algebra

$$\begin{aligned}
p(x,\tau;x') &= \frac{\sqrt{\pi}}{2\pi\sqrt{f}} e^{-\frac{A^2}{4f}} \\
&= \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}A^2} \\
&= \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(x-x')^2} \\
&= pdf_{N(0,\tau)}(x'-x).
\end{aligned}$$

Notice that according to the main Markov argument (see main structure (3.1)). As we know B_t is a martingale and F is a natural filtration. Then, we define

$$\begin{aligned}
OP(f(B_T)|t) &= E[f(B_T)|F_t] \\
&= E[f(B_T)|B_t] \text{ by Markov Property} \\
&= A_{T-t}f(B_t) \text{ by Semigroup Property} \\
&= e^{-(T-t)H}f(B_t).
\end{aligned}$$

Then, using the pricing kernel we derive

$$\begin{aligned}
OP(f(B_T)|t,x)) &= \int_{-\infty}^{\infty} p(x,T-t;x')f(x')dx' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2(T-t)}}(x-x')^2 f(x')dx' \\
&= \int_{-\infty}^{\infty} pdf_{N(0,T-t)}(x'-x)f(x')dx' \\
&= E[f(x+N(0,T-t))].
\end{aligned}$$

Example 1: Notice that $f(x) = e^{hx}$ is eigenvector of H with eigenvalue $K_h = -h^2/2$. Then, for the particular option claim $f(x)$,

$$\begin{aligned}
OP(f(B_T)|B_t = x) &= A_{T-t}f(x) \\
&= e^{-(T-t)H}f(x) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_h)^j}{j!} \\
&= e^{-(T-t)K_h}f(x), \text{ where } K_h = -\frac{h^2}{2} \\
&\textit{via} \quad x \to B_t \\
OP(e^{hB_T}|t) &= OP(f(B_T)|t) = OP(f(B_T)|B_t) \\
&= e^{(T-t)h^2/2}f(B_t) = e^{(T-t)h^2/2}e^{hB_t}.
\end{aligned}$$

3.4.2 Hamiltonian and Markov kernel for the Geometric Standard Brownian motion

Now, the goal in this part is to compute the similar Option Pricing OP$(f(S_T)|t)$ for the Geometric Standard Brownian Motion (GSBM) case via the Quantum Mechanics Formalism. As before, we assume that the interest rate $\rho = 0$. Acutally, the option price is not justified. In the geometric standard brownian motion model $S_t = S_0 e^{B_t}$, where $a = 0, \sigma = 1$. Under the no arbitrage condition $a = \mu - \frac{\sigma^2}{2} = 0$, we obtain $\mu = \frac{\sigma^2}{2}$, which means that $\rho = \mu = \frac{1}{2}$.

Then, the option pricing formula becomes

$$\begin{aligned} OP(f(S_T)|t) &= e^{-\rho(T-t)}E(f(S_T)|S_t) \\ &= e^{-\frac{1}{2}(T-t)}E(f(S_T)|S_t). \end{aligned}$$

We define that Hamiltonian H for GSBM as

$$H = -\frac{1}{2}(x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x}). \tag{3.4}$$

Similarly, we derive the Markov kernel via the continuous time continuous space quantum mechanics formalism, which is

$$\begin{aligned} p(x,\tau;x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \quad . \end{aligned}$$

Now, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(S_t)|S_0 = x]}{t} \\ &= \frac{f(x) - E[f(xe^{B_t})]}{t} \\ &= -E[\frac{f(xe^{B_t}) - f(x)}{t}] \end{aligned}$$

and , via the Ito formula,

$$f(S_t) = f(x) + \int_0^t f'(xe^{B_t})dB_s + \frac{1}{2}\int_0^t f''(xe^{B_t})ds).$$

This yields

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''(xe^{B_t})ds]] \\ &= -\frac{1}{2t}\int_0^t E[f''(xe^{B_t})ds] \\ &\to -\frac{1}{2}(x^2 f''(x) + xf'(x)) \end{aligned}$$

since $f'(xe^{B_t}) = xe^y{*}f'(xe^y)$ and $f''(xe^{B_t}) = (xe^y{*}f')' = f''(xe^y) = (xe^y)^2 f''(xe^y) + xe^y f'(xe^y)$.

Next step is to apply the QMF approach implying

$$\begin{aligned} p(x,\tau;x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > . \end{aligned}$$

However, $|p >$ is no more eigenvector and the approach can not be applied.

Example 2: Now, we choose a particular class of claims $f(x) = x^m = f_m$. Notice that f_m is an eigenvector of H with the eigenvalue K_m i.e. $Hf_m = K_m f_m$ where $K_m = -m^2/2$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned} OP(f(X_T)|X_t = x) &= A_{T-t} f(x) \\ &= e^{-H(T-t)} f(x) \\ &= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\ &= e^{-\frac{1}{2}(T-t)} f(x) \sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_m)^j}{j!} \\ &= e^{-\frac{1}{2}(T-t)} e^{-(T-t)K_m} f(x), \text{ where } K_m = -\frac{m^2}{2} \\ via &\quad x \to X_t \\ OP(X_T^m|t) &= OP(f(X_T)|t) = OP(f(X_T)|X_t) \\ &= e^{-\frac{1}{2}(T-t)} e^{(T-t)m^2/2} f(X_t) = e^{-\frac{1}{2}(T-t)} e^{(T-t)m^2/2} e^{mB_t}. \end{aligned}$$

Notice that $X_T^m = e^{mB_T}$ and not surprisingly the answers in Examples 1 and 2 are the same.

3.4.3 Hamiltonian and Markov kernel for the Brownian Motion

The objective in this part is to compute the Option Pricing OP$(f(B_T)|t)$ for the Brownian Motion (BM) case via the Quantum Mechanics Formalism. Again, we assume that the interest rate $\rho = 0$.

Firstly, we show that Hamiltonian H for BM is defined by

$$H = -\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} - a\frac{\partial f}{\partial x}. \tag{3.5}$$

Then, we derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time continuous space quantum mechanics formalism

$$\begin{aligned} p(x,\tau;x') &= <x|e^{-\tau H}|x'> \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} <x|e^{-\tau H}|p><p|x'> \ . \end{aligned}$$

First, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(B_t)|\sigma B_0 = x]}{t} \\ &= \frac{f(x) - E[f(x + at + \sigma B_t)]}{t} \\ &= -E[\frac{f(x + at + \sigma B_t) - f(x)}{t}]. \end{aligned}$$

Then, via the Ito formula,

$$f(t, B_t) = f(0, B_0) + \int_0^t f'(s, B_s)_{B_s} dB_s + \frac{1}{2}\int_0^t f''_{B_s,B_s}(s, B_s)ds + \int_0^t f''_{u,B_s}(B_u)ds$$

implying

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''(x + at + \sigma B_t)ds] + \int_0^t E[f''(x + at + \sigma B_t)ds]] \\ &= -\frac{1}{2t}\int_0^t E[f'(x + B_s)ds] \\ &\to -\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} - a\frac{\partial f}{\partial x} \end{aligned}$$

since $f'_B(0) = \sigma f'(x), f''(0)_{BB} = \sigma^2 f''(x), f'_t(0) = af'(x)$.

Then, we apply QMF approach to obtain the BM Markov kernel.

$$\begin{aligned} p(x,\tau;x') &= <x|e^{-\tau H}|x'> \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} <x|e^{-\tau H}|p><p|x'> \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\tau(-\frac{1}{2}\sigma^2(ip)^2 - ap)} e^{ip(x-x')} \end{aligned}$$

applying that $e^{-\tau H}|p> = e^{-\tau(-\frac{1}{2}(ip)^2)}|p>$, $<x|p> = e^{ipx}$, and $<p|x> = e^{-ipx}$ (See before (3.2)).

From the Gaussian kernel,

$$\begin{aligned}
p(x,\tau;x') &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{\frac{1}{2}\tau\sigma^2(ip)^2} e^{ip(x-x'+\tau a)} \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f(ip)^2} e^{ipA}, \text{ where } f = \frac{1}{2}\tau\sigma^2 \text{ and } A = x - x' + \tau a \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f((ip)^2 + \frac{ip}{f}A + \frac{A^2}{4f^2} - \frac{A^2}{4f^2})} \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{f(ip + \frac{A}{2f})^2 - \frac{A^2}{4f})}
\end{aligned}$$

Let $y = p + \frac{A}{2f}$ and $z = \sqrt{f}y$, then $dy = \frac{1}{\sqrt{f}}dz$

$$\begin{aligned}
p(x,\tau;x') &= \frac{\sqrt{\pi}}{2\pi\sqrt{f}} e^{-\frac{A^2}{4f}} \\
&= \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}A^2} \\
&= \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2}(x-x'+\tau a)^2}. \\
&= pdf_{N(0,\tau)}(x' - x + \tau a).
\end{aligned}$$

Notice that according to the main Markov argument (see main structure (3.1)). As we know B_t is a martingale and F is a natural filtration. Then, we define

$$\begin{aligned}
OP(f(B_T)|t) &= E[f(B_T)|F_t] \\
&= E[f(B_T)|B_t] \text{ by Markov Property} \\
&= A_{T-t}f(B_t) \\
&= e^{-(T-t)H} f(B_t).
\end{aligned}$$

Then, using the pricing kernel we derive

$$\begin{aligned}
OP(f(B_T)|t,x)) &= \int_{-\infty}^{\infty} p(x,T-t;x')f(x')dx' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)\sigma^2}} e^{-\frac{1}{2(T-t)\sigma^2}(x-x'+\tau a)^2} f(x')dx' \\
&= \int_{-\infty}^{\infty} pdf_{N(0,T-t)}(x'-x+\tau a)f(x')dx' \\
&= E[f(x + N(0,T-t))].
\end{aligned}$$

Example 3: Notice that $f(x) = e^{\widehat{h}x}$ is the eigenvector of H with eigenvalue $K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h}$. Then, for the particular option claim $f(x)$,

$$
\begin{aligned}
OP(f(B_T)|t) &= OP(f(B_T)|B_t = x) \\
&= A_{T-t}f(x) \\
&= e^{-(T-t)H}f(x) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_{\widehat{h}})^j}{j!} \\
&= e^{-(T-t)K_{\widehat{h}}}f(x),\ K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h} \\
via &\quad x \to B_t \\
&= e^{-(T-t)K_{\widehat{h}}}f(B_t),\ K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h}.
\end{aligned}
$$

3.4.4 Hamiltonian and Markov kernel for the Brownian Motion with constant interest rate

The objective in this part is to compute the Option Pricing OP$(f(B_T)|t)$ for the Brownian Motion with constant interest rate (BMR) case via the Quantum Mechanics Formalism.

Firstly, we show that Hamiltonian H for BMR is defined by

$$
H = -\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} - a\frac{\partial f}{\partial x} + \rho. \tag{3.6}
$$

Then, we derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time continuous space quantum mechanics formalism

$$
\begin{aligned}
p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \ .
\end{aligned}
$$

First, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(B_t)|\sigma B_0 = x]}{t} \\ &= \frac{f(x) - E[f(x + at + \sigma B_t)]}{t} \\ &= -E[\frac{f(x + at + \sigma B_t) - f(x)}{t}] \end{aligned}$$

Then, via the Ito formula,

$$f(t, B_t) = f(0, B_0) + \int_0^t f'(s, B_s)_{B_s} dB_s + \frac{1}{2}\int_0^t f''_{B_s, B_s}(s, B_s)ds + \int_0^t f''_{u, B_s}(B_u)ds$$

implying

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''(x + at + \sigma B_t)ds] + \int_0^t E[f''(x + at + \sigma B_t)ds]] \\ &= -\frac{1}{2t}\int_0^t E[f'(x + B_s)ds] \\ &\to -\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} - a\frac{\partial f}{\partial x} + \rho \end{aligned}$$

since $f'_B(0) = \sigma f'(x), f''(0)_{BB} = \sigma^2 f''(x), f'_t(0) = af'(x) - \rho$.

Then, we apply QMF approach to obtain the BMR Markov kernel.

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\tau(-\frac{1}{2}\sigma^2(ip)^2 - aip + \rho)} e^{ip(x - x')} \end{aligned}$$

applying that $e^{-\tau H}|p >= e^{-\tau(-\frac{1}{2}\sigma^2(ip)^2 - aip + \rho)}|p >$, $< x|p >= e^{ipx}$, and $< p|x >= e^{-ipx}$ (See before (3.2)).

From the Gaussian kernel,

$$
\begin{aligned}
p(x,\tau;x') &= e^{-(T-t)\rho}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{\frac{1}{2}\tau\sigma^2(ip)^2}e^{ip(x-x'+\tau a)} \\
&= e^{-(T-t)\rho}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{f(ip)^2}e^{ipA}, \text{ where } f=\frac{1}{2}\tau\sigma^2 \text{ and } A=x-x'+\tau a \\
&= e^{-(T-t)\rho}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{f((ip)^2+\frac{ip}{f}A+\frac{A^2}{4f^2}-\frac{A^2}{4f^2})} \\
&= e^{-(T-t)\rho}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{f(ip+\frac{A}{2f})^2-\frac{A^2}{4f})}.
\end{aligned}
$$

Let $y=p+\frac{A}{2f}$ and $z=\sqrt{f}y$, then $dy=\frac{1}{\sqrt{f}}dz$

$$
\begin{aligned}
p(x,\tau;x') &= e^{-(T-t)\rho}\frac{\sqrt{\pi}}{2\pi\sqrt{f}}e^{-\frac{A^2}{4f}} \\
&= e^{-(T-t)\rho}\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-\frac{1}{2\tau\sigma^2}A^2} \\
&= e^{-(T-t)\rho}\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-\frac{1}{2\tau\sigma^2}(x-x'+\tau a)^2} \\
&= e^{-(T-t)\rho}pdf_{N(0,\tau)}(x'-x).
\end{aligned}
$$

Notice that according to the main Markov argument (see main structure (3.1)). As we know B_t is a martingale, and F is a natural filtration. Then, we define

$$
\begin{aligned}
OP(f(B_T)|t) &= E[f(B_T)|F_t] \\
&= E[f(B_T)|B_t] \text{ by Markov Property} \\
&= A_{T-t}f(B_t) \\
&= e^{-(T-t)H}f(B_t)
\end{aligned}
$$

Then, using the pricing kernel we derive

$$
\begin{aligned}
OP(f(B_T)|t,x)) &= \int_{-\infty}^{\infty}p(x,T-t;x')f(x')dx' \\
&= e^{-(T-t)\rho}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{1}{2(T-t)}(x-x'+\tau a)^2 f(x')dx'} \\
&= e^{-(T-t)\rho}\int_{-\infty}^{\infty}pdf_{N(0,T-t)}(x'-x+\tau a)f(x')dx' \\
&= e^{-(T-t)\rho}E[f(x+N(0,T-t))].
\end{aligned}
$$

Example 4: Notice that $f(x) = e^{\widehat{h}x}$ is eigenvector of H with eigenvalue $K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h}$. Then, for the particular option claim $f(x)$,

$$\begin{aligned}
OP(f(B_T)|t) &= OP(f(B_T)|B_t = x) \\
&= A_{T-t}f(x) \\
&= e^{-(T-t)H}f(x) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_{\widehat{h}})^j}{j!} \\
&= e^{-(T-t)\rho}e^{-(T-t)K_{\widehat{h}}}f(x), \text{ where } K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h} \\
via \quad & x \to B_t \\
&= e^{-(T-t)\rho}e^{-(T-t)K_{\widehat{h}}}f(B_t), \text{ where } K_{\widehat{h}} = -\frac{\sigma^2\widehat{h}^2}{2} - a\widehat{h}.
\end{aligned}$$

3.4.5 Hamiltonian and Markov kernel for the Geometric Brownian Motion

The objective in this part is to compute the Option Pricing OP$(f(S_T)|t)$ for the Geometric Brownian Motion (GBM) case via the Quantum Mechanics Formalism. Assume that the intereat rate $\rho = 0$.

Firstly, we show that Hamiltonian H for GBM is defined by

$$H = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\sigma^2 x \frac{\partial f}{\partial x} - ax\frac{\partial f}{\partial x}. \tag{3.7}$$

Then, we derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time continuous space quantum mechanics formalism

$$\begin{aligned}
p(x,\tau;x') &= < x|e^{-\tau H}|x' > \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \quad .
\end{aligned}$$

First, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(S_t)|\sigma S_0 = x]}{t} \\ &= \frac{f(x) - E[f(xe^{at+\sigma S_t})]}{t} \\ &= -E[\frac{f(xe^{at+\sigma S_t}) - f(x)}{t}] \end{aligned}$$

Then, via the Ito formula,

$$f(t, B_t) = f(0, B_0) + \int_0^t f'_{B_s}(s, B_s)dB_s + \frac{1}{2}\int_0^t f''_{B_s,B_s}(s, B_s)ds + \int_0^t f''_{u,B_s}(B_u)ds$$

implying

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''_{B_s,B_s}(s, B_s)dt] + \int_0^t E[f'_s(u, B_u)dt]] \\ &\to -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\sigma^2 x \frac{\partial f}{\partial x} - ax\frac{\partial f}{\partial x} \end{aligned}$$

since $f'_B(0) = \sigma x e^{at+\sigma B_t} f'(x)$, $f'_t(0) = axe^{at+\sigma B_t} f'(x)$, $f''_{BB}(0) = \sigma^2 x^2 e^{at+\sigma B_t} f''(x) +$ $\sigma^2 x e^{at+\sigma B_t} f'(x)$.

Then, we apply QMF approach to obtain the GBM Markov kernel.

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \end{aligned}$$

However, $|p >$ is no more eigenvector and the approach does not apply.

Example 5: Now, we choose a particular class of claims $f(x) = x^{\widehat{m}} = f_{\widehat{m}}$. Assume that $f_{\widehat{m}}$ is an eigenvector of H with the eigenvalue $K_{\widehat{m}}$ i.e. $Hf_{\widehat{m}} = K_{\widehat{m}} f_{\widehat{m}}$ where

$K_{\widehat{m}} = -\frac{\sigma^2 \widehat{m}^2}{2} - a\widehat{m}$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned}
OP(f(X_T)|t) &= OP(f(X_T)|X_t = x) \\
&= A_{T-t} f(x) \\
&= e^{-H(T-t)} f(x) \\
&= \sum_{j=0}^{\infty} \frac{-(T-t)^j H^j f(x)}{j!} \\
&= f(x) \sum_{j=0}^{\infty} \frac{-(T-t)^j (K_{\widehat{m}})^j}{j!} \\
&= e^{-(T-t)K_{\widehat{m}}} f(x), \text{ where } K_{\widehat{m}} = -\frac{\sigma^2 \widehat{m}^2}{2} - a\widehat{m} \\
via \quad & x \to X_t \\
&= e^{-(T-t)K_{\widehat{m}}} f(X_t), \text{ where } K_{\widehat{m}} = -\frac{\sigma^2 \widehat{m}^2}{2} - a\widehat{m}.
\end{aligned}$$

3.4.6 Hamiltonian and Markov kernel for the Geometric Brownian Motion with constant interest rate

The objective in this part is to compute the Option Pricing OP$(f(S_T)|t)$ for the Geometric Brownian Motion with constant interest rate (GBMR) case via the Quantum Mechanics Formalism.

Firstly, we show that Hamiltonian H for GBMR is defined by

$$H = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\sigma^2 x \frac{\partial f}{\partial x} - ax\frac{\partial f}{\partial x} + \rho. \tag{3.8}$$

Then, we derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time continuous space quantum mechanics formalism

$$\begin{aligned}
p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \quad .
\end{aligned}$$

First, we notice that

$$\begin{aligned} H &= \frac{If - A^t f}{t}(x) \\ &= \frac{f(x) - E[f(S_t)|\sigma S_0 = x]}{t} \\ &= \frac{f(x) - E[f(xe^{at+\sigma S_t})]}{t} \\ &= -E[\frac{f(xe^{at+\sigma S_t}) - f(x)}{t}]. \end{aligned}$$

Then, via the Ito formula,

$$f(t, B_t) = f(0, B_0) + \int_0^t f'_{B_s}(s, B_s) dB_s + \frac{1}{2}\int_0^t f''_{B_s,B_s}(s, B_s) ds + \int_0^t f''_{u,B_s}(B_u) ds$$

implying

$$\begin{aligned} H &= -\frac{1}{t}[E(Ito) + \frac{1}{2}\int_0^t E[f''_{B_s,B_s}(s, B_s)dt] + \int_0^t E[f'_s(u, B_u)dt]] \\ &\to -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\sigma^2 x \frac{\partial f}{\partial x} - ax\frac{\partial f}{\partial x} + \rho \end{aligned}$$

since $f'_B(0) = \sigma x e^{at+\sigma B_t} f'(x)$, $f'_t(0) = axe^{at+\sigma B_t} f'(x) - \rho$, $f''_{BB}(0) = \sigma^2 x^2 e^{at+\sigma B_t} f''(x) + \sigma^2 x e^{at+\sigma B_t} f'(x)$.

Then, we apply QMF approach to obtain the GBMR Markov kernel.

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \end{aligned}$$

And the option claim as before is the example 5.

3.4.7 Hamiltonian and Markov kernel for the Poisson process

The objective in this part is to compute the Option Pricing $\mathrm{OP}(f(N_t)|t)$ for the Poisson process case via the Quantum Mechanics Formalism. As before, we assume that the interest rate $\rho = 0$.

Firstly, we show that Hamiltonian H for Poisson process is defined by

$$H = -\lambda(f(x+1) - f(x)). \tag{3.9}$$

Then, to derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ we apply the continuous time discrete space (Z_+) quantum mechanics formalism

$$\begin{aligned} p(x, \tau; x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \end{aligned}$$

where

$$< x|p >= e^{ixp}, < p|x >= e^{-ixp}$$

and x and x' are both treated as positive integers and Dirac functions are concentrated at x and x'.

Firstly, we notice that

$$\begin{aligned} H &= \frac{f(x) - E[f(x+N_t)]}{t} \\ &= \frac{f(x) - \sum_{k=0}^{\infty} f(x+k)P(N_t = k)}{t}. \end{aligned}$$

Assume that $f(x)$ is a real-valued function that is infinitely differentiable at real number a. Then, via the Taylor expansion, $f(x)$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

which yields

$$\begin{aligned} H &= \frac{f(x) - (f(x)e^{-\lambda t} + f(x+1)e^{-\lambda t}\lambda t + o(t^2))}{t} \\ &= \frac{f(x)(1-e^{-\lambda t}) - f(x+1)e^{-\lambda t}\lambda t - o(t^2)}{t} \\ &\to f(x)\lambda - \lambda f(x+1) \\ &= \lambda(f(x) - f(x+1)). \end{aligned}$$

Finally, we obtain

$$H = -\lambda(f(x+1) - f(x)).$$

Notice that the space of eigenvalues BZ_+ is now the class of functions

$$< k|p >= e^{ikp}, < p|j >=^{defn} \overline{< j|p >} = e^{-ijp}.$$

Then, we apply QMF approach to obtain the Poisson process Markov kernel $p_P(\cdot)$.

The formalism now yields

$$\begin{aligned} p_P(x,\tau;x') &= < x|e^{-\tau H}|p > \\ &= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \\ &= \frac{1}{2\pi}\int_0^{2\pi} < x|p >< p|x' > e^{\lambda\tau(e^{ip}-1)}dp. \end{aligned}$$

Notice that x and x' are both treated as positive integers and Dirac functions are concentrated at x and x'. More exactly, for $k \in Z_+$, we set up $k = \delta_k$ (the Dirac function $\delta_k(y)$, equal to 1 at point k and 0 otherwise, i.e

$$\delta_k(k) = 1,\ \delta_k(y) = 0, y \neq k.$$

Overall, we get

$$p_P(x,\tau;x') = \frac{1}{2\pi}\int_0^{2\pi} e^{ip(x-x')}e^{\lambda\tau(e^{ip}-1)}dp.$$

Then by changing the variable $z = e^{ip}$

$$p_P(x,\tau;x') = \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda\tau(z-1)}}{z^{j-k+1}}dz.$$

Applying the Cauchy formula,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z^{a+1}}dz = \frac{f^{(a)}(0)}{a!}$$

with $f(z) = e^{\lambda\tau(z-1)}$ and $a = j - k$ we then derive

$$\begin{aligned} p_P(x,\tau;x') &= \frac{f^{(j-k)}(0)}{(j-k)!} \\ &= \frac{e^{-\lambda\tau}(\lambda\tau)^{j-k}}{(j-k)!}. \end{aligned}$$

Notice that according to the main Markov argument (See before (3.1)). As we know N_t is a martingale, and F is a natural filtration. Then, we define

$$\begin{aligned} OP(f(N_T)|t) &= E[G(T,N_T)|F_t] \\ &= E[G(T,N_T)|N_t] \text{ by Markov Property} \\ &= A_{T-t}f(N_t) \\ &= e^{-(T-t)H}f(N_t). \end{aligned}$$

Then, using the pricing kernel derived in above we get

$$\begin{aligned} OP(f(N_T)|t,x) &= \sum_{j=x}^{\infty} p_P(x,T-t,j)f(j) \\ &= \sum_{j=x}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^{j-x}}{(j-x)!}f(j). \end{aligned}$$

By changing j to $j - x$, we have

$$OP(f(N_T)|t) = \sum_{j=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^{j}}{(j)!}f(j+x).$$

Now, writing it as an expectation of the Poisson variable, we eventually derive the analogue of the Black-Schole formula when share price is modelled by the Geometric Poisson process.

$$OP(f(N_T)|t,x) = E[f(x+Po(\lambda(T-t))].$$

Example 6: Now, we choose a particular class of claims $f(x) = e^{hx} = f_h$. Notice that f_h is an eigenvector of H with the eigenvalue K_h i.e. $Hf_h = K_h f_h$ where $K_h = -\lambda(e^{hx} - 1)$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned}
OP(f(N_T)|t) &= OP(f(N_T)|N_t = x) \\
&= A_{T-t}f(x) \\
&= e^{-(T-t)H}f(x) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_h)^j}{j!} \\
&= e^{-(T-t)K_h}f(x), \text{ where } K_h = -\lambda(e^h - 1) \\
via &\quad x \to N_t \\
&= e^{-(T-t)K_h}f(N_t), \text{ where } K_h = -\lambda(e^h - 1).
\end{aligned}$$

3.4.8 Hamiltonian and Markov kernel for the Geometric Poisson process

The objective in this part is to compute the Option Pricing OP$(f(S_t)|t)$ for the Geometric Poisson process case via the Quantum Mechanics Formalism. Again, we assume that the interest rate $\rho = 0$.

Firstly, we show that Hamiltonian H for Geometric Poisson process is defined by

$$H = -\lambda(f(xe) - f(x)). \tag{3.10}$$

Then, we derive the Makrov kernel (pricing kernel) $p(x, \tau; x')$ via the continuous time discrete space (e^{Z_+}) quantum mechanics formalism

$$\begin{aligned}
p(x,\tau;x') &= < x|e^{-\tau H}|x' > \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \quad .
\end{aligned}$$

where

$$< x|p >= x^{ip}, < p|x >= x^{-ip}$$

and x and x' are both treated as elements in e^{Z_+} and Z_+ positive integers and Dirac functions are concentrated at x and x'.

Firstly, we notice that

$$\begin{aligned} H &= \frac{f(x) - E[f(xZ^{N_t})]}{t} \\ &= \frac{f(x) - \sum_{k=0}^{\infty} f(xZ^k) P(N_t = k)}{t}. \end{aligned}$$

Then, via the taylor expansion, here, $f(x)$ is a real-valued function that is infinitely differentiable at real number a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

implying

$$\begin{aligned} H &= \frac{f(x) - (f(x)e^{-\lambda t} + f(xz)e^{-\lambda t}\lambda t + o(t^2))}{t} \\ &= \frac{f(x)(1 - e^{-\lambda t}) - f(xz)e^{-\lambda t}\lambda t - o(t^2)}{t} \\ &\to f(x)\lambda - \lambda f(xz) \\ &= -\lambda(f(xz) - f(x)). \end{aligned}$$

Finally, we obtain

$$H = -\lambda(f(xe) - f(x)).$$

Then, we apply QMF approach to obtain the Geometric Poisson process Markov kernel $p_{GP}(\cdot)$.

$$\begin{aligned} p_{GP}(x, \tau; x') &= < x|e^{-\tau H}|p > \\ &= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \\ &= \frac{1}{2\pi} \int_0^{2\pi} < x|p >< p|x' > e^{-\lambda\tau(z^{im}-1)} dp. \end{aligned}$$

Notice that x and x' are both treated as e^k and e^j where k, j are positive integers and also as a functions concentrated at x and x'. Remind that now

$$< x|p >= x^{ip}, < p|x >= x^{-ip}.$$

Overall, we get

$$\begin{aligned} p_{GP}(x,\tau;x') &= \frac{1}{2\pi}\int_0^{2\pi} x^{ip}(x')^{-ip}e^{\lambda\tau(e^{ip}-1)}dp \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\lambda\tau(e^{ip}-1)}e^{ip(k-j)}dp. \end{aligned}$$

Notice that this is exactly the same formula as we derived for the Poisson case with $\ln x = k$ and $\ln x' = j$ that is

$$p_{GP}(x,\tau;x') = p_P(\ln x,\tau;\ln x')$$

Notice that according to the main Markov argument (See before (3.1)). As we know N_t is a martingale and F is a natural filtration. Then, we define

$$\begin{aligned} OP(f(N_T)|t) &= E[G(T,N_T)|F_t] \\ &= E[G(T,N_T)|N_t] \text{ by Markov Property} \\ &= A_{T-t}f(Z^{N_t}) \\ &= e^{-(T-t)H}f(Z^{N_t}). \end{aligned}$$

Then, using the pricing kernel we derive

$$\begin{aligned} OP(f(N_T)|t,x)) &= \sum_{j=x}^{\infty} p_{GP}(x,T-t;x')f(\ln x') \\ &= \frac{e^{-\lambda\tau}(\lambda\tau)^{\ln x'-\ln x}}{(\ln x'-\ln x)!}f(\ln x'). \end{aligned}$$

By changing $\ln x'$ to $\ln x$,

$$OP(f(Z^{N_T})|t) = \sum_{\ln x'=0}^{\infty}\frac{e^{\lambda\tau}(\lambda\tau)^{\ln x'}}{(\ln x')!}f(\ln x'+\ln x).$$

As before, we write it as an expectation form to derive the relative Black-Scholes formula as follows

$$OP(f(Z^{N_T})|t,\ln x) = E[f(\ln x + Po(\lambda(T-t)))].$$

Example 7: Now, we choose as a particular class of claims $f(x) = z^m = f_m$. Notice that f_m is an eigenvector of H with the eigenvalue K_m i.e. $Hf_m = K_m f_m$

where $K_m = -\lambda(z^m - 1)$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned}
OP(f(N_T)|t) &= OP(Z^{mN_t}|t), \text{ where } e^{hx} = Z^{mx} = e^{(mlnZ)x} \\
&= OP(f(X_T)|t) \\
&= A_{T-t}f(X_t) \\
&= e^{-H(T-t)}f(x) \\
&= \sum_{j=0}^{\infty} \frac{-(T-t)^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{-(T-t)^j (K_m)^j}{j!} \\
&= e^{-(T-t)K_m}f(x), \text{ where } K_m = -\lambda(z^m - 1) = -\lambda(e^h - 1) \\
via &\quad x \to N_t \\
&= e^{-(T-t)K_m}f(N_t), \text{ where } K_m = -\lambda(e^h - 1).
\end{aligned}$$

3.4.9 Hamiltonian and Markov kernel for the Poisson process with shift

The objective in this part is to compute the Option Pricing OP$(f(N_t+ct)|t)$ for the Poisson process with shift case via the Quantum Mechanics Formalism. Assume that the interest rate $\rho = 0$.

Firstly, we show that Hamiltonian H for Poisson process with shift is defined by

$$H = -\lambda(f(\omega + 1) - f(\omega)) - cf'(\omega), \text{ where } \omega = x + y. \tag{3.11}$$

Then, to derive the Makrov kernel (pricing kernel) $p_S(x, \tau; x')$ we apply the continuous time discrete space (Z_+) quantum mechanics formalism

$$\begin{aligned}
p_S(x, \tau; x') &= < x|e^{-\tau H}|x' > \\
&= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' >
\end{aligned}$$

where

$$< x|p >= e^{ixp}, < p|x >= e^{-ixp}$$

and x and x' are both treated as positive integers and Dirac functions are concentrated at x and x'.

Derive hamiltonian via probability method

First, we notice that

$$\begin{aligned} H &= \frac{f(\omega) - E[f(X_t)|X_0 = \omega]}{t}, \text{ where } X_t = N_t + ct \\ &= \frac{f(\omega) - \sum_{k=0}^{\infty} f(N_t + ct + \omega)P(N_t = k)}{t}. \end{aligned}$$

Then, via the taylor expansion, here, $f(x)$ is a real-valued function that is infinitely differentiable at real number a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

which yields

$$\begin{aligned} H &= \frac{f(\omega) - (f(ct+\omega)P(N_t = 0) + f(ct+\omega)P(N_t = 1))}{t} + o(1) \\ &= \frac{f(\omega) - ((f(\omega) + f'(\omega))(1 - \lambda t) + f(\omega+1)\lambda t)}{t} + o(1) \\ &\to (\lambda f(\omega) - f'(\omega)c - f(\omega+1)\lambda) + o(1) \\ &= -\lambda(f(\omega+1) - f(\omega)) - cf'(\omega). \end{aligned}$$

Finally, we obtain

$$H = -\lambda(f(\omega+1) - f(\omega)) - cf'(\omega), \text{ where } \omega = x + y.$$

Notice that the space of eigenvalues BZ_+ is now the class of functions

$$< k|p >= e^{ikp}, < p|j >=^{defn} \overline{< j|p >} = e^{-ijp}.$$

Then, we apply QMF approach to obtain the Poisson process Markov kernel $p_P(\cdot)$.

The formalism now yields

$$\begin{aligned} p_S(x,\tau;x') &= < x|e^{-\tau H}|p > \\ &= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > \\ &= \frac{1}{2\pi} \int_0^{2\pi} < x|p >< p|x' > e^{\lambda\tau(e^{ip}-1) - ice^{ip}} dp. \end{aligned}$$

Notice that x and x' are both treated as positive integers and Dirac functions are concentrated at x and x'. More exactly, for $k \in Z_+$, we set up $k = \delta_k$ (the Dirac function $\delta_k(y)$), is equal 1 at point k and 0 otherwise, i.e.

$$\delta_k(k) = 1,\ \delta_k(y) = 0, y \neq k.$$

Overall, we get

$$p_S(x,\tau;x') = \frac{1}{2\pi}\int_0^{2\pi} e^{ip(x-x')}e^{\lambda\tau(e^{ip}-1)-ice^{ip}}dp.$$

And then, we derive the Hamiltonian for poisson process with drift via pair and transform as follows.

Derive Hamiltonian via pair

Suppose we have a pair $\xi_t = (N_t, Y_t)$ and

$$A_t f(x,y) = E[f(N_t,Y_t)|N_0 = x, Y_0 = y] = E[f(x+N_t, y+ct)].$$

Recall that

$$[N_T|N_t = u] = u + P_0(t-t) = u + N_{T-t} \text{ and } [Y_T|Y_t = \omega] = \omega + c(T-t).$$

$$\begin{aligned} H &= \frac{f(x,y) - Ef(x+N_t, y+ct)}{t} \\ &= \frac{1}{t}(f(x,y) - (f(x,y+ct)P(N_t=0) + f(x+1,y+ct)P(N_t=0))) + o(1) \\ &= \frac{1}{t}(f(x,y) - (f(x,y) + ctf'(x,y))(1-\lambda t) + f(x+1,y)\lambda t) + o(1) \\ &\to -\lambda(f(x+1,y) - f(x,y)) - xf'_y(x,y). \end{aligned}$$

Then by changing the variable $z = e^{ip}$

$$p_S(x,\tau;x') = \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda\tau(z-1)-ice^z}}{z^{j-k+1}}dz.$$

Applying the Cauchy formula,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{a+1}}dz = \frac{f^{(a)}(0)}{a!}$$

with $f(z) = e^{\lambda\tau(z-1)-ice^z}$ and $a = j - k$ we then derive

$$\begin{aligned} p_S(x, \tau; x') &= \frac{f^{(j-k)}(0)}{(j-k)!} \\ &= \frac{e^{-\lambda\tau}(\lambda\tau)^{j-k}}{(j-k)!}. \end{aligned}$$

Notice that according to the main Markov argument (See before (3.1)). As we know N_t is a martingale and F is a natural filtration. Then, we define

$$\begin{aligned} OP(f(N_T + cT)|t) &= E[G(T, N_T)|F_t] \\ &= E[G(T, N_T)|N_t] \text{ by Markov Property} \\ &= A_{T-t} f(N_t + cT) \\ &= e^{-(T-t)H} f(N_t + cT). \end{aligned}$$

Then, using the pricing kernel derived in above we get

$$\begin{aligned} OP(f(N_T + cT)|t, x) &= \sum_{j=x}^{\infty} p_S(x, T-t, j) f(j + ct) \\ &= \sum_{j=x}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^{j-x}}{(j-x)!} f(j + ct) \end{aligned}$$

by changing j to $j - x$

$$OP(f(N_T + cT)|t) = \sum_{j=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^{j}}{(j)!} f(j + x + ct)$$

and now writing it as an expectation of the Poisson variable, we eventually derive the analogue of the Black-Scholl formula when share price is modelled by the Geometric Poisson process.

$$OP(f(N_T + cT)|t, x) = E[f(x + ct + Po(\lambda(T-t))].$$

Example 8: Now, we choose a particular class of claims $f(x) = e^{hx} = f_h$. Notice that f_h is an eigenvector of H with the eigenvalue K_h i.e. $Hf_h = K_h f_h$ where

$K_h = -\lambda(e^{hx} - 1)$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned}
OP(f(N_T + cT)|t) &= OP(f(N_T + cT)|N_t = x) \\
&= A_{T-t} f(x + ct) \\
&= e^{-(T-t)H} f(x + ct) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x + ct)}{j!} \\
&= f(x + ct) \sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_h)^j}{j!} \\
&= e^{-(T-t)K_h} f(x + ct), \text{ where } K_h = -\lambda(e^{hx} - 1) \\
via \quad & x \to N_t \\
&= e^{-(T-t)K_h} f(N_t + ct), \text{ where } K_h = -\lambda(e^{hx} - 1).
\end{aligned}$$

3.4.10 Hamiltonian and Markov kernel for Levy process via resolvet method and Ito formula

The objective in this part is to compute the Option Pricing OP$(f(X_T)|t) = E[f(X_T)|X_t]$ for the Levy process via the Quantum Mechanics Formalism. Assume that the interest rate $\rho = 0$.

The target is to compute the Hamiltonian on a dense set $D = \{|p>: e^{ipx}\}$. Consider a characteristic function for Levy process as follows

$$\begin{aligned}
A_t|p> &= E_x e^{ipX_t} \\
&= E_0 e^{ip(x+X_t)} \\
&= |p> (Ee^{ipX_t}) \\
\text{via} \quad & Ee^{ipX_t} = e^{-t\psi(p)} \\
&= |p> e^{-t\psi(p)}.
\end{aligned}$$

Then, we derive the Hamiltonian for Levy process by

$$\begin{aligned}\frac{I - A_t}{t}|p> &= \frac{I|p> - A_t|p>}{t} \\ &= |p> (\frac{1 - e^{-t\psi(p)}}{t}) \\ &\to |p> \psi(p) =: H|p> .\end{aligned}$$

For more specific situation, we derive the Hamiltonian operator for Levy process. Let $\mathcal{G}$ be the space of infinitely differentiable functions on the real line. $D_{\mathcal{G}}$ is dense in C_0. Let X_t be a Levy process in R with generator $\mathcal{A}$. There are drift $a \in R$, volatility $\sigma \geq 0$, and μ is a measure on R satisfying $\int_{R\backslash 0}(1 \wedge |x|^2)\mu(dx) \leq \infty$. $\forall f \in \mathcal{G}$, we obtain Hamiltonian for Levy process as follows:

$$\mathcal{A}f(x) = af'(x) - \frac{1}{2}\sigma^2 f''(x) + \int_{R\backslash 0} (f(x) - f(x+y) + \frac{y}{1+y^2})\mu(dy).$$

Proof. Recall the Levy-Knintchine formula:

$$\varphi(\xi) = ia\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{R\backslash 0} (1 - e^{i\xi x} + \frac{y}{1+y^2})\mu(dy).$$

We need to calculation

$$\mathcal{A}_t f(x) = \lim_{t\to 0} \frac{1}{t}(f(x) - E_t f(x)).$$

We know $\forall f$, $E_t f(x) = E(f(X_{t+s})|X_s = x) = E(f(X_t)|X_0 = x) = \int f(y)\pi(t, x \to dy)$, where π is a probability density function.

Also, $exp(-\varphi_t(\xi)) = E(exp(i\xi(X_{t+s} - X_s))) = E(exp(i\xi(X_t - X_0))) = \int exp(i\xi y)\pi(t, 0 \to dy)$.

Besides, $\forall f \in \mathcal{A}$, let g be a function s.t.

$$f(x) = \int_\xi e^{ix\xi} g(\xi) d\xi.$$

Then, we obtain

$$\begin{aligned} E_t f(x) &= \int_y \pi(t,dy) \int_\xi e^{i(x+y)\xi} g(\xi) d\xi \\ &= \int_\xi e^{ix\xi} g(\xi) d\xi \int_y e^{iy\xi} \pi(t,dy) \\ &= \int_\xi e^{ix\xi} g(\xi) d\xi e^{-\varphi_t(\xi)}. \end{aligned}$$

$$\begin{aligned} f(x) - E_t f(x) &= \int_\xi e^{ix\xi} g(\xi) d\xi - \int_\xi e^{ix\xi} g(\xi) d\xi e^{-\varphi_t(\xi)} \\ &= \int_\xi e^{ix\xi} g(\xi)(1 - e^{-\varphi_t(\xi)}) d\xi \\ &= \int_\xi e^{ix\xi} g(\xi)(1 - e^{-\varphi_1(\xi)}) d\xi \\ &= \int_\xi e^{ix\xi} g(\xi)(t\varphi_1(\xi) + o(t)) d\xi \end{aligned}$$

$$\begin{aligned} \mathcal{A}_t f(x) &= \lim_{t\to 0} \frac{1}{t}(f(x) - E_t f(x)) \\ &= \int_\xi e^{ix\xi} g(\xi)(\varphi_1(\xi)) d\xi \\ &= \int_\xi e^{ix\xi} g(\xi)(ia\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{R\backslash 0}(1 - e^{i\xi y} + \frac{y}{1+y^2})\mu(dy)) d\xi \\ &= ia \int_\xi e^{ix\xi} g(\xi)(\xi) d\xi + \frac{1}{2}\sigma^2 \int_\xi e^{ix\xi} g(\xi)\xi^2 d\xi \\ &\quad + \int_y (\int_{R\backslash 0} (e^{i\xi x}\xi g(\xi) d\xi - \int_{R\backslash 0} (e^{i\xi(x+y)} g(\xi) d\xi + \frac{iy}{1+y^2} \int_{R\backslash 0} e^{i\xi(x)}\xi g(\xi) d\xi)\mu(dy). \end{aligned}$$

Therefore, we obtain

$$\mathcal{A} f(x) = af'(x) - \frac{1}{2}\sigma^2 f''(x) + \int_{R\backslash 0} (f(x) - f(x+y) + \frac{y}{1+y^2})\mu(dy).$$

$\square$

Then, we derive the Makrov kernel (pricing kernel) $p(x,\tau;x')$ via the continuous time continuous space quantum mechanics formalism

$$\begin{aligned} p(x,\tau;x') &= < x|e^{-\tau H}|x' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > . \end{aligned}$$

However, $|p>$ is no more eigenvector and the approach does not apply.

Example 10: Now, we choose a particular class of claims $f(x) = e^h = f_h$. Notice that f_h is an eigenvector of H with the eigenvalue K_h i.e. $Hf_h = K_h f_h$ where $K_h = -\frac{\sigma^2 h^2}{2} - ah - E(f(x+X_1) - f(x))\lambda$. Then, for the particular class of option claims $f(x)$,

$$\begin{aligned}
OP(f(X_T)|X_t = x) &= A_{T-t}f(x) \\
&= e^{-H(T-t)}f(x) \\
&= \sum_{j=0}^{\infty} \frac{[-(T-t)]^j H^j f(x)}{j!} \\
&= f(x)\sum_{j=0}^{\infty} \frac{[-(T-t)]^j (K_h)^j}{j!} \\
&= e^{-(T-t)K_h} f(x), \text{ where} \\
& \quad\ K_h = -\frac{\sigma^2 h^2}{2} - ah - E(f(x+X_1) - f(x))\lambda. \\
& via \quad x \to X_t, \\
&= e^{-(T-t)K_h} f(X_t).
\end{aligned}$$

3.5 Transformed Hamiltonian

Motivated by the discrete time version Hamiltonian for changing of basis. We introduce a simple example first.

Example 1 Suppose we have

$$|1>= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \texttt{and} \quad |2>= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S.$$

Also, we have

$$|1>_*= |1^*>= \begin{pmatrix} A \\ 0 \end{pmatrix} \quad \texttt{and} \quad |2>_*= |2^*>= \begin{pmatrix} 0 \\ B \end{pmatrix} \in S.$$

Then, we derive

$$|1>= \frac{1}{A}|1^*> \quad \texttt{and} \quad |2>= \frac{1}{B}|2^*>.$$

Obviously, we have

$$< 1|2 >= 0, < 1|1 >=< 2|2 >= 1 \texttt{ and } < 1^*|2^* >_*= 0, < 1^*|1^* >_*=< 2^*|2^* >_*= 1.$$

Also, we have

$$< a|b >=< a, b >=< wa^*, wb^* >=< a^*, w^*wb^* >=< a^*, \sum b^* >, \texttt{ where } \sum = w^*w.$$

Finally, we obtain

$$\begin{aligned} < 1|1 > &= \frac{1}{A^2} < 1^*|1^* >_* \\ &= \frac{1}{A^2} < 1^*| \sum 1^* > \\ &= \frac{1}{A^2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 1. \end{aligned}$$

Example 2 Assume that

$$|1 >= \frac{1}{A}|1^* >, |2 >= \frac{1}{B}|2^* > \texttt{ and } \sum = w^*w, p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Also, we have

$$< wa^*, wb^* >=< a^*, w^*wb^* >=< a^*, \sum b^* >, \texttt{ where } \sum = w^*w.$$

Then, we derive

$$
\begin{aligned}
<1|p|2> &= <1, p2> \\
&= <\frac{1}{A}1^*, p^*\frac{2^*}{B}>_* \\
&= <\frac{1}{A}1^* w^*|p^*|w\frac{1}{B}2^*> \\
&= \frac{1}{A}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\frac{1}{B}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{AB}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} Ap_{11} & Ap_{12} \\ Bp_{21} & Bp_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} A^2p_{11} & ABp_{12} \\ BAp_{21} & B^2p_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{AB}{AB}p_{12} = p_{12}.
\end{aligned}
$$

Then, we derive different Hamiltonian opertator and make some applications in option pricing via Transform method.

3.5.1 Option pricing based on Black-scholes via Hamiltonian quantum technique for BM (Transform Hamiltonian)

Introduce a function $f : X \to R$ on a Banach space $B_{X\to R}$, and a transform $T : X \to Y$. If we have an operator $U : B_{X\to R} \to B_{Y\to R}$ ($B_{Y\to R}$ is another Banach space), then

$$Uf(x) = f(Tx)\text{, } x \in X \text{ and } Tx \in Y, \tag{3.12}$$

Introduce a Hamiltonian $H : B_{X\to R} \to B_{X\to R}$, such as $H : D_1 \to D_2$, where $D_1 \in B_{X\to R}$.

Theorem (Fugle-Putnam-Rosenblum) Assume that M, N, $T \in B(H)$, M and N are normal, and

$$MT = TN. \tag{3.13}$$

Then $M^*T = TN^*$ (See pp 315 in Functional analysis by Walter Rudin, 2003).

If (6.2.2) holds, then $M^kT = TN^k$ for k=1,2,3...by induction. Hence,

$$exp(M)T = Texp(N). \tag{3.14}$$

Thus, if we have

$$VH := HU, \tag{3.15}$$

then

$$VHf(x) = H(Uf)(x) = Hf(Tx). \tag{3.16}$$

The most important thing is

$$H_T = UH \quad \texttt{, defined by } H_Tf(x) = H(f(Tx)). \tag{3.17}$$

Derive pricing kernel via transformed Hamiltonian method Assume that we have a transformation $T : x \to T(x) = \sigma x$, where σ is a parameter for the transformation T, then we derive

$$Uf(x) = f(Tx) = f(\tilde{x}) = \tilde{f}$$

and

$$f(x) = U^{-1}f(\tilde{x}) = U^{-1}\tilde{f}.$$

Then, we obtain

$$U^{-1}f(x) = f(T^{-1}x) = f(\frac{x}{\sigma}).$$

Implying

$$
\begin{aligned}
< x|e^{-\tau H}|x' > &= \int_{-\infty}^{\infty} < x|e^{-\tau H}|p >< p|x' > \frac{dp}{2\pi} \\
&= \int^{\infty} < x|A|p >< p|x' > d\mu(p) \\
&= \int^{\infty} < f|A|p >< p|g > d\mu(p) \\
&= \int_{\infty}^{\infty} < U^{-1}\tilde{f}|\hat{A}|U^{-1}\tilde{p} >< U^{-1}\tilde{p}|\tilde{g} > d\tilde{\mu}(\tilde{p}), \texttt{ where } p = U^{-1}\tilde{p} \\
&= \int_{\infty}^{\infty} < \frac{x}{\sigma}|e^{-\tau\hat{H}}|\frac{p}{\sigma} >< \frac{p}{\sigma}|\frac{x'}{\sigma} > d\tilde{\mu}(p) \\
&= pdf_{B_\tau}(x'-x).
\end{aligned}
$$

EXAMPLE 1

Given a Hamiltonian $H_{T_1} = Hf(x) = -\frac{1}{2}f''(x)$ for SBM, B_t, a transform $T_\sigma = T_x = \sigma x$, try to find H_{T_σ}.

$$
\begin{aligned}
H_{T_\sigma}f(x) \quad &= \quad H_{T_1}(Uf(x)) \\
&= \quad H_{T_1}(f(T_\sigma x)) \\
&= \quad Hg \\
&= \quad -\frac{1}{2}g''(x) \\
By \quad & \quad g' = (f(T(x)))' = T'(x)f'(T(x)) \\
& \quad g'' = T''(x)f'(T(x)) + (T'(x))^2 f''(T(x)) \\
&= \quad -\frac{1}{2}[T''(x)f'(T(x)) + (T'(x))^2 f''(T(x))] \\
(T_\sigma = \sigma x) \quad & \quad -\frac{1}{2}\sigma^2 f''(\sigma x) \\
(\sigma x \to y) \quad & \quad -\frac{1}{2}\sigma^2 f''(y)
\end{aligned}
$$

Derive pricing kernel in Brownian motion without drift case via transformed Hamiltonian method Suppose we have $Hf(x) = -\frac{f''(x)}{2}$, we apply the transformation $T : x \to T(x) = \sigma x$ to obtain $H_\sigma f(x) = -\frac{\sigma^2 f''(x)}{2}$ as follows.

$$
Hf(\tilde{x}) := U\tilde{H}f(x) := \tilde{H}(Uf)(\tilde{x}) = \tilde{H}(f(T\tilde{x})) = -\frac{\sigma^2 f''(\sigma\tilde{x})}{2}.
$$

Notice that $Uf(\tilde{B}_t) = f(T\tilde{B}_t) = f(B_t)$, where $T\tilde{B}_t = B_t$. Then, we derive the pricing kernel for Brownian motion without drift case like before.

$$< x|e^{-\tau H}|x' > \; = \; \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H_\sigma}|p >< p|x' >$$

Changing some variables by $U^{-1}U = I$, we have

$$Ux = \frac{x}{\sigma}, \ U^{-1}p = p\sigma, \ \texttt{and} \ Ux' = \frac{x'}{\sigma}.$$

Implying

$$\begin{aligned}
< x|e^{-\tau H}|x' > &= \int_{-\infty}^{\infty} d\mu(p) < Ux|e^{-\tau H}|U^{-1}p >< U^{-1}p|Ux' > \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{p^2\sigma^2}{2}\tau + ip(x-x')} \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{p^2\sigma^2}{2}\tau + ip\sigma(y-y')}, \ \texttt{where} \ \frac{x}{\sigma} = y \ \texttt{and} \ \frac{x'}{\sigma} = y' \\
&= \frac{1}{\sigma} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{q^2\tau}{2} + iq(y-y')}, \ \texttt{where} \ p\sigma = q \\
&= \frac{1}{\sigma} \Gamma(\tilde{x}, \tilde{x}'|\tau), \ \texttt{where} \ y = \tilde{x} \ \texttt{and} \ y' = \tilde{x}' \\
&= pdf_{B_\tau(x'|\sigma B_0 = x)}
\end{aligned}$$

As before, according to the main Markov argument (see main structure (3.1)). Then, using the pricing kernel we derive

$$\begin{aligned}
OP(f(B_T)|t, x)) &= \int_{-\infty}^{\infty} p(x, T-t; x') f(x') dx' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2(T-t)}} (x - x')^2 f(x') dx' \\
&= \int_{-\infty}^{\infty} pdf_{N(0,T-t)}(x' - x) f(x') dx' \\
&= E[f(x + N(0, T-t))].
\end{aligned}$$

EXAMPLE 2

Given a Hamiltonian $H_{T_1} = Hf(x) = -\frac{1}{2}f''(x)$ for SBM, B_t, a transform $T_\sigma = T_x = e^x$, try to find H_{T_σ}.

$$
\begin{array}{rcl}
H_{T_\sigma} f(x) & = & H_{T_1}(Uf(x)) \\
& = & H_{T_1}(f(T_x x)) \\
& = & Hg \\
& = & -\frac{1}{2} g''(x) \\
& By & g' = (f(T(x)))' = T'(x) f'(T(x)) \\
& & g'' = T''(x) f'(T(x)) + (T'(x))^2 f''(T(x)) \\
& = & -\frac{1}{2}[T''(x) f'(T(x)) + (T'(x))^2 f''(T(x))] \\
& (T_x = e^x) & -\frac{1}{2}(e^x f'(e^x) + e^{2x} f''(e^x)) \\
& (e^x \to y) & -\frac{1}{2}(y f'(y) + y^2 f''(y)).
\end{array}
$$

Although we obtain the Hamiltonian from the geometrc standard brownian motion case to the standard brownian motion case, we cannot find the related transformation of the Hamiltonian for the pricing kernel.

EXAMPLE 3

Given a Hamiltonian $H_{T_2} = Hf(t) = -\frac{\partial f}{\partial t}$ for t, a transform $T_x = \mu_x$, try to find H_{T_μ}.

$$
\begin{array}{rcl}
H_{T_\mu} f(x) & = & H_{T_2}(Uf(x)) \\
& = & H_{T_2}(f(T_\mu x)) \\
& = & Hg \\
& = & -g'(x) \\
& By & g' = (f(T(x)))' = T'(x) f'(T(x)) \\
& = & -T'(x) f'(T(x)) \\
& (T_\mu = \mu x) & -\mu f'(\mu x) \\
& (\mu x \to y) & -\mu f'(y).
\end{array}
$$

Notice that for the transformation of the volatility, please see the example 1.

Then, we need to consider the pair, if $T(x, y) = x + y$, $f(x, y) = f(x + y)$, then

$$
H_{\mu t + \sigma B_t} f = H_{A_t \oplus B_t} f(T(x, y)) = -\frac{\sigma^2}{2} f'' - \mu f' \tag{3.18}
$$

$$\begin{aligned} H_{A_t \oplus B_t} f(x,y) &= H_B f'(\cdot, y) + H_A f(x, \cdot) \\ &= \frac{1}{2}\sigma^2 f''_{yy} - \mu f'_x, \texttt{ since } A_t \to \mu t, B_t \to \sigma B_t. \end{aligned}$$

Here, we obtain Brownian motion Hamiltonian by the transformation.

Derive pricing kernel in Brownian motion case via transformed Hamiltonian method Suppose we have $Hf(x) = -\frac{f''(x)}{2}$, we apply the transformation $T_A : x \to T_A(x) = \sigma x$ and $T_B : x \to T_B(x) = \mu$ to obtain $H_A f(x) = -\frac{\sigma^2 f''(x)}{2}$ and $H_B f(x) = -\mu f'(x)$ as follows.

$$H_A f(\tilde{x}) := U_A \tilde{H} f(x) := \tilde{H}(U_A f)(\tilde{x}) = \tilde{H}(f(T_A \tilde{x})) = -\frac{\sigma^2 f''(\sigma \tilde{x})}{2}.$$

$$H_B f(\tilde{x}) := U_B \tilde{H} f(x) := \tilde{H}(U_B f)(\tilde{x}) = \tilde{H}(f(T_B \tilde{x})) = -\mu f'(\mu \tilde{x}).$$

Notice that $U_A f(\tilde{B}_t) = f(T_A \tilde{B}_t) = f(B_t)$, where $T_A \tilde{B}_t = B_t$. Changing some variables by $U_A^{-1} U_A = I$ and $U_B^{-1} U_B = I$, we have

$$U_A x = \frac{x}{\sigma}, \; U_A^{-1} p = p\sigma, \texttt{ and } U_A x' = \frac{x'}{\sigma}.$$

$$U_B x = \mu, \; U_B^{-1} p = p\mu, \texttt{ and } U_B x' = 0?.$$

Implying

$$\begin{aligned} < x|e^{-\tau H}|x' > &= \int_{-\infty}^{\infty} d\mu(p) < Ux|e^{-\tau H}|U^{-1}p >< U^{-1}p|Ux' > \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{p^2\sigma^2}{2}\tau + ip(x-x')} \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{p^2\sigma^2}{2}\tau + ip\sigma(y-y')}, \texttt{ where } \frac{x}{\sigma} = y \texttt{ and } \frac{x'}{\sigma} = y' \\ &= \frac{1}{\sigma} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{q^2\tau}{2} + iq(y-y')}, \texttt{ where } p\sigma = q \\ &= \frac{1}{\sigma} \Gamma(\tilde{x}, \tilde{x}'|\tau), \texttt{ where } y = \tilde{x} \texttt{ and } y' = \tilde{x}' \\ &= pdf_{B_\tau (x'|\sigma B_0 = x)} \end{aligned}$$

As before, according to the main Markov argument (see main structure (3.1)). Then, using the pricing kernel we derive

$$\begin{aligned}
OP(f(B_T)|t,x)) &= \int_{-\infty}^{\infty} p(x,T-t;x')f(x')dx' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2(T-t)}}(x-x')^2 f(x')dx' \\
&= \int_{-\infty}^{\infty} pdf_{N(0,T-t)}(x'-x)f(x')dx' \\
&= E[f(x+N(0,T-t))].
\end{aligned}$$

EXAMPLE 4

Given a Hamiltonian $H_{T_\sigma} = Hf(x) = -\frac{\sigma^2}{2}f''(x) - af''$ for SBM, B_t, a transform $T_\sigma = T_x = e^x$, try to find H_{T_σ}.

$$\begin{aligned}
H_{T_\sigma}f(x) &= H_{T_1}(Uf(x)) \\
&= H_{T_1}(f(T_x x)) \\
&= Hg \\
&= -\frac{1}{2}\sigma^2 g''(x) - ag'(x) \\
By \quad & g' = (f(T(x)))' = T'(x)f'(T(x)) = yf'(y) \\
& g'' = T''(x)f'(T(x)) + (T'(x))^2 f''(T(x)) = yf'(y) + y^2 f''(y) \\
&= -\frac{\sigma^2}{2}(yf'(y) + y^2 f''(y)) - ayf'(y).
\end{aligned}$$

Although we obtain the Hamiltonian from the geometrc brownian motion case to the brownian motion case, we cannot find the related transformation of the Hamiltonian for the pricing kernel.

EXAMPLE 5

Given a Hamiltonian $H_{T_1} = Hf(x) = -\frac{1}{2}(xf'(x) + x^2 f''(x))$ for GSBM, e^{B_t}, a transform $T_\sigma = T_x = \ln(x)$, try to find H_{T_σ}.

$$
\begin{aligned}
H_{T_\sigma} f(x) &= H_{T_1}(Uf(x)) \\
&= H_{T_1}(f(T_x x)) \\
&= Hg \\
&= -\frac{1}{2}(xg'(x) + x^2 g''(x)) \\
By \quad & g' = (f(T(x)))' = T'(x)f'(T(x)) = (ln(x))'f'(ln(x)) = \frac{1}{x}f'(ln(x)) \\
& g'' = (\frac{1}{x}f'(ln(x)))' = -\frac{1}{x^2}f'(ln(x)) + \frac{1}{x^2}f''(ln(x)) \\
&= -\frac{1}{2}(x\frac{1}{x}f'(ln(x)) + x^2(-\frac{1}{x^2}f'(ln(x)) + \frac{1}{x^2}f''(ln(x)))) \\
&= -\frac{1}{2}(f'(ln(x)) - f'(ln(x)) + f''(ln(x)))) \\
&= -\frac{1}{2}(f''(ln(x))).
\end{aligned}
$$

Although we obtain the Hamiltonian from the brownian motion case to the geometrc brownian motion case, we cannot find the related transformation of the Hamiltonian for the pricing kernel.

3.5.2 Derive a Hamiltonian via Probabolity method

First, we derive a Hamiltonian for standard Brownian motion.

$$
\begin{aligned}
Atf &= \frac{f(x) - E[f(x + B_t)]}{t} \\
By \quad & (f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + C||f'''||h^3, \\
& \text{ where the space } ||f''' < \infty||) \\
&= \frac{1}{t}(f(x) - E[f(x) + f'(x)B_t + \frac{1}{2}f''(x)B_t^2 + C||f'''||(B_t)^3]) \\
By \quad & (E(B_t)^3 = t^{\frac{3}{2}}C \texttt{ and } E(B_t) = 0) \\
&= -\frac{1}{2}f''(x)\frac{E(B_t)^2}{t} + o(t^{\frac{1}{2}}) \\
&\to -\frac{1}{2}f''(x)
\end{aligned}
$$

3.5.3 Derive a Hamiltonian via limit method

Given $X_t = \frac{N_t - \lambda t}{\sqrt{\lambda}}, X_t = \epsilon, N_t = at$, we derive a Hamiltonian for standard Brownian motion.

$$
\begin{aligned}
& E[f(X_t)|X_0 = x] \\
= \quad & Atf(x) = \frac{I - At}{t} f \\
= \quad & \frac{f(x) - E[f(X_t)|X_0 = x]}{t} \\
By \quad & ([X_t|X_0 = x] = x + \epsilon N_t + at) \\
= \quad & \frac{f(x) - E[f(x + \epsilon N_t + at)]}{t} \\
= \quad & \frac{f(x) - f(x + at)P(N_t = 0) + f(x + \epsilon N_t + at)P(N_t = 1) + others = 0}{t} \\
= \quad & \frac{f(x) - f(x + at)e^{-\lambda t} + f(x + \epsilon N_t + at)e^{-\lambda t}\lambda t}{t} + o(t) \\
= \quad & \frac{1}{t}(f(x) - (f(x) + f'(x)at + f(x)o(t^2))(1 - \lambda t + o(t^2)) + \\
& (f(x + \epsilon) + f'(x + \epsilon)at)\lambda t) + o(t) \\
By \quad & e^{-\lambda t} = 1 - \lambda t + o(t^2)), e^y = 1 + y + \frac{y^2}{2} + ... \\
& f(x + at) = (t \to 0) = f(x) + f'(x)at + o(t^2) \\
& f(x + \delta) = f(x) + f'(x)\delta \\
& f(x + \epsilon + at) = (t \to 0) = f(x + \epsilon) + f'(x + \epsilon)at + o(t^2) \\
= \quad & \frac{1}{t}(f(x) - (f(x) + f'(x)at + f(x)o(t^2))(1 - \lambda t) + (f(x + \epsilon) \\
& + o(t)(1 + o(t)))\lambda t) + o(t) \\
= \quad & \frac{1}{t}(f(x)\lambda t - f'(x)at - f(x + \epsilon)\lambda t) + o(t) \\
= \quad & \frac{1}{t}(-\lambda(f(x + \epsilon) - f(x))t - f'(x)at) + o(t) \\
= \quad & \frac{1}{t}(-\lambda \delta_\epsilon f(x) - f'(x)at) + o(t) =: H(x)
\end{aligned}
$$

Here, we need to change some variables: $\epsilon = \frac{1}{\sqrt{\lambda}}, a = -\sqrt{\lambda} = -\frac{1}{\epsilon}, \lambda = \frac{1}{\epsilon^2}$.

$$
\begin{aligned}
H_\epsilon f \quad &= \quad -\frac{1}{\epsilon^2}(f(x + \epsilon) - f(x) - f'(x)\epsilon) \\
&= \quad -\frac{1}{\epsilon}(\frac{f(x + \epsilon) - f(x)}{\epsilon} - f'(x)\epsilon) \\
&\to \quad -\frac{1}{2}f''(x).
\end{aligned}
$$

Thus, we obtain $H^\epsilon_{X_t} \to H_{\omega_t} = -\frac{1}{2}f''$ for each f pointwise, where $X_t = \frac{N_t - \lambda t}{\sqrt{\lambda}} = \omega_t$.

3.6 Resolvent method for Hamiltonians

In the classical probability language this material is the connection between generators of markov processes and Resolvents. Resolvents are roughly Laplace transform of the markov semigroup.

The general idea is as follows:

$$\begin{aligned} R_z &= \int_0^\infty e^{-zt} A_t dt, \texttt{ where } A_t \texttt{ is a semigroup,} \\ &= \frac{1}{z}\int_0^\infty e^{-zt} A_{t_z/z} dt_z \end{aligned}$$

and then, we have $zR_z = \int_0^\infty e^{-u} A_u du$.

The Definition 3.4 follows [50] on the page 67.

Let A be a closed linear operator on the real Banach space X, with domain $D(A)$.

Definition 3.4. i) Let the real number $\lambda \in \rho(A)$, the resolvent set of A, provided the operator

$$\lambda I - A : D(A) \to X$$

is one-to-one and onto.

ii) If $\lambda \in \rho(A)$, the resolvent operator $R_\lambda : X \to X$ is defined by

$$R_\lambda u := (\lambda I - A)^{-1} u.$$

Now, we have

$$\begin{aligned} z(I - zR_z) &= \int_0^\infty e^{-u}\left(\frac{I - A_{\frac{u}{z}}}{\frac{u}{z}}\right) u du \\ \texttt{Let } \tfrac{u}{z} = \epsilon & \int_0^\infty e^{-u}\left(\frac{I - A_\epsilon}{\epsilon}\right) u du \\ \texttt{As } \tfrac{u}{z} = \epsilon \to 0 & \int_0^\infty e^{-u} H du = H. \end{aligned}$$

Applying it for any fixed function f and point x, we obtain

$$\int_0^\infty e^{-u}\left(\frac{I - A_\epsilon}{\epsilon} f(x)\right) u du.$$

In details, we derive

$$\begin{aligned}
z(I - zR_z) &= zI|p> - z^2 R_z|p> \\
&= z|p> - \frac{z^2}{z+K_p}|p> \\
&= (z - \frac{z^2}{z+K_p})|p> \\
&= (\frac{z(z+K_p) - z^2}{z+K_p})|p> \\
&= \frac{zK_p}{z+K_p}|p> = \frac{K_p}{1+\frac{K_p}{z}}|p> \\
&= K_p|p> \texttt{, as } \to \infty, \\
&= H|p> .
\end{aligned}$$

3.6.1 Hamiltonian for Poisson process via Resolvent method

$$R_z I_{n-1}(0) = \frac{\lambda^n}{(\lambda+z)^n}, \quad R_z|p> = \frac{1}{z+K_p}|p>,$$

where $\varphi(p) = K_p = \lambda(e^{ip} - 1)$ and

$$R_z I_n(0) = \frac{\lambda^n}{(\lambda+z)^{n+1}} = \int_0^\infty e^{-zt} P_{0,n}^{(t)} dt,$$

where $P_0(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$.

Recall that $A_t f(x) = E_{N_0=x} f(N_t) = Ef(x+N_t) = \sum_{j=0}^\infty f(x+j) P_{0,j}^{(t)}$. And

$$\begin{aligned}
R_z f(x) &= \int_0^\infty e^{-zt} A_t f(x) dt \\
&= \sum_0^\infty f(x+j) \int_0^\infty e^{-zt} P_{0,j}^{(t)} dt \\
&= \sum_{j=0}^\infty f(x+j) \frac{\lambda^j}{(\lambda+z)^{j+1}}
\end{aligned}$$

Also, we have

$$(z(I - zR_z))f(x) = zf(x) - z^2 R_z f(x) = zf(x) - z^2 \sum_{j=0}^\infty f(x+j) \frac{\lambda^j}{(\lambda+z)^{j+1}}.$$

Then, we obtain

$$z(I - zR_z)f \quad \rightarrow \quad Hf = -\lambda(f(x+1) - f(x)),$$

for $f \in \mathcal{B}_\infty : ||f|| = sup|f(u)| \leq \infty$.

3.6.2 Hamiltonian for Compound Poisson process via Resolvent method

$$A_t f(x) = E_{N_0=x} f(N_t) = Ef(x+N_t) = \sum_{j=0}^{\infty} f(x+j) P_{0,j}^{(t)}.$$

$$A_t f(x) = \sum_{j=0}^{\infty} Ef(x + \sum_{k=1}^{j} Y_k) P_{0,j}^{(t)}$$

We have

$$\begin{aligned}
R_z f(x) &= \int_0^\infty e^{-zt} A_t f(x) dt \\
&= \sum_{j=0}^{\infty} E(f(x + \sum_{k=1}^{j} Y_k)) \int_0^\infty e^{-zt} P_{0,j}^{(t)} dt \\
&= \sum_{j=0}^{\infty} E(f(x + X_j)) \frac{\lambda^j}{(\lambda+z)^{j+1}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
z(I - zR_z)f(x) &= zf(x) - z^2 R_z f(x) \\
&= zf(x) - z^2 \sum_{j=0}^{\infty} E(f(x+X_j)) \frac{\lambda^j}{(\lambda+z)^{j+1}} \\
&= z^2 \sum_{j=0}^{\infty} \frac{\lambda^j}{(\lambda+z)^{j+1}} f(x) - z^2 \sum_{j=0}^{\infty} E(f(x+X_j)) \frac{\lambda^j}{(\lambda+z)^{j+1}} \\
&= -\sum_{j=1}^{\infty} (E(f(x+X_j)) - f(x)) \frac{\lambda^j z^2}{(\lambda+z)^{j+1}} \\
&= o(\frac{1}{z}) - E(f(x+X_1)) - f(x)) \frac{\lambda^j z^2}{(\lambda+z)^{j+1}} \\
&\rightarrow 0 - E(f(x+X_1)) - f(x))\lambda, \texttt{ as } z \rightarrow \infty, \\
&= Hf(x).
\end{aligned}$$

3.6.3 Hamiltonian for Levy process via Resolvent method

Hamiltonian on a dense set $\mathcal{D} = \{|p>: e^{ipx}\}$.

$$\begin{aligned} A_t|p> &= E_x e^{ipX_t} \\ &= E_0 E^{ip(x+X_t)} \\ &= |p> (Ee^{ipX_t}) \\ &= |p> e^{-t\psi(p)}. \end{aligned}$$

Notice that: The characteristic function of Levy process is defined by

$$Ee^{ipX_t} = e^{-t\psi(p)}.$$

Then, one can derive Hamiltonian for Levy process via Resolvent method as follows:

$$R_z = \int_0^\infty e^{-zt} dt,$$

and

$$\begin{aligned} R_z|p> &= \int_0^\infty e^{-zt} A_t|p> dt \\ &= \int_0^\infty e^{-zt}|p> e^{-t\psi(p)} dt \\ &= |p> \int_0^\infty e^{-t(z+\psi(p))} dt \\ &= |p> \frac{1}{z+\psi(p)}. \end{aligned}$$

Finally, one can obtain

$$\begin{aligned} z(I - zR_z)|p> &= z|p> - \frac{z^2}{z+\psi(p)}|p> \\ &= (z - \frac{z^2}{z+\psi(p)})|p> \\ &= (\frac{z(z+\psi(p)) - z^2}{z+\psi(p)})|p> \\ &= \frac{z\psi(p)}{z+\psi(p)}|p> \\ &\to \psi(p)|p>. \end{aligned}$$

3.7 Conclusion

In this chapter, It covers the main structure in terms of quantum formalism. Moreover, it introduces the calculation for different resolvents for different stochastic processes. Also, it covers several examples for Hamiltonian and pricing kernels. Furthermore, it covers Transformed Hamiltonian. Besides, it covers resolvent method for Hamiltonian operators.

The objective is to derive option pricing via quantum formalism in this chapter. Firstly, we apply several method, such as generator approach, probability method to derive several Hamiltonian operators for different stochastic processes, such as standard Brownian motion, Brownian motion, geometric Brownian motion, Poisson process, geometric Poisson process, Poisson process with shift, and even Levy process. Also, we apply transformed Hamiltonian method and resolvent method for Hamiltonians to justify our results. Surprisingly, we obtain the same result by different method. Then, we use these Hamiltonian operators to derive pricing kernel and the relative option price defined via Feynman-Kac formula. Here is the starting point that we make a connection between classical model and non-classical model.

Chapter 4

Quantum markets

4.1 Introduction

In this chapter, we analyse the quantum version of Binomial model, including both self-adjoint market and non self-adjoint market. And we ananlyse the quantum bond markets. Moreover, we analyse the quantum conditional option price via the quantum conditional expectation. Besides, we establish the limit of the spectral measures proving the convergence to the Geometric Brownian motion model (GBM model).

4.2 N period quantum Binomial market

In this part, we define the generalised N-period quantum binomial model and develop a relevant option pricing.

Quantum share price S_N. Quantum share price for the quantum binomial market is defined by

$$S_N = S_0 H_1 \otimes \ldots \otimes H_N, \tag{4.1}$$

where H_i is self-adjoint 2×2 matrices, which represents the changing of share price with jumps u_i (jump up) or d_i (jump down), $i = 1, \ldots, N$. For the quantum model, u_i and d_i are diagonal elements.

Observe that since $H_i^* = H_i$

$$\begin{aligned} S_N^* &= (S_0 H_1 \otimes \ldots \otimes H_N)^* \\ &= S_0 H_1^* \otimes \ldots \otimes H_N^* \\ &= S_N. \end{aligned}$$

So, S_N is a self-adjoint operator with non-negative diagonal elements, $z_1 \ldots z_N$, where z_j is u_j or d_j. Hence, quantum share price S_N is self-adjoint non-negative operator.

Quantum claim $C = f(S_N)$. In particular, by the general fact 4.2.1, we can introduce the quantum claim $C = f(S_N)$ for any function f via the following general formula.

Fact 4.2.1. Let $A^* = A$, and assume that A admits a representation $A = UDU^*$. Then, $f(A) = Uf(D)U^*$.

In our case,

$$\begin{aligned} S_N &= S_0 H_1 \otimes \ldots \otimes H_N && (4.2) \\ &= US_0 D_1 \otimes \ldots \otimes D_N U^* \\ \Longrightarrow & \; f(S_N) = Uf(S_0 D_1 \otimes \ldots \otimes D_N)U^*. && (4.3) \end{aligned}$$

Quantum state ρ. The generalised N-period binomial model is then introduced a quantum state ρ in the Euclidean space $\mathcal{E} = R^{2^N}$ of dimension 2^N as the tensor product

$$\rho = \rho_1 \otimes \ldots \otimes \rho_N.$$

Note that from the standard definition of the quantum trace, $\rho_i = \rho_i^*$ are self-adjoint non-negative 2×2 matrices such that

$$tr(\rho_i) = 1, i = 1, \ldots, N.$$

Then, by the property of self-adjoint operator, we obtain

$$\begin{aligned} \rho^* &= (\rho_1 \otimes \ldots \otimes \rho_N)^* \\ &= \rho_1^* \otimes \ldots \otimes \rho_N^* \\ &= \rho. \end{aligned}$$

Moreover, ρ is a non-negative self-adjoint operator because ρ has diagonal elements $y_1 \dots y_N$, where y_j is q_u^j or q_d^j. Also, note that

$$\begin{aligned} tr(\rho) &= tr(\rho_1 \otimes \dots \otimes \rho_N) \\ &= tr(\rho_1) \cdots tr(\rho_N) \\ &= 1. \end{aligned}$$

Notice that the second equation is derived from the property of trace. Hence, ρ is a proper quantum state.

Quantum state risk-neutral world. The risk-neutral world of the quantum model (B,S) consists of self-adjoint non-negative 2×2 matrices ρ (referred to as states) satisfying

$$tr(\rho_1 H_1) = 1 + r_1,\ tr(\rho_2 H_2) = 1 + r_2, \dots,\ \texttt{and}\ tr(\rho_N H_N) = 1 + r_N. \tag{4.4}$$

Transformed quantum state. Furthermore, we define

$$H_i = U_i^* D_i U_i \text{ and } \rho_i \Longrightarrow \tilde{\rho}_i = U_i \rho_i U_i^*, \text{ where i=1,2,...N.} \tag{4.5}$$

Note that

$$\begin{aligned} \tilde{\rho_i}^* &= (U_i \rho U_i^*)^* \\ &= (U_i^*)^* \rho_i^* U_i^* = U_i \rho_i^* U_i^* \\ &= \tilde{\rho}_i, \text{ where i=1,2,..,N.} \end{aligned}$$

We obtain the first equation by the property of unitary operator and trace.

And then,

$$\begin{aligned} tr(\tilde{\rho}_i) &= tr(U_i \rho_i U_i^*) \\ &= tr(U_i^* U_i \rho_i) \\ &= tr(\rho_i) = 1, \text{ where i=1,2,...,N.} \end{aligned}$$

Under the general fact 4.2.1, we obtain the first equation. Then, we apply the trace rule for the second equation. And we obtain the third equation because of the definition of transformed quantum state. The final equation derived by the definition of quantum state.

Finally, we obtain

$$\begin{aligned} < \tilde{\rho}_i x, x > &= < U \rho_i U^* x, x > \\ &= < \rho_i U^* x, U^* x > \\ &= < \rho_i y, y > \\ &\geq 0, \end{aligned}$$

where $U^* x = y$ and since $\rho_i \geq 0$. We obtain the first and second equation because of the property of unitary operator. And $\tilde{\rho}_i$ have non-negative diagonal elements $q_u^{(i)}, q_d^{(i)}$ and have a representation

$$\tilde{\rho}_i = \begin{pmatrix} q_u^{(i)} & \bar{x} \\ x & q_d^{(i)} \end{pmatrix}$$

Transformed quantum state is then defined by

$$\begin{aligned} \widetilde{\rho} &= U^*(\rho_1 \otimes ... \otimes \rho_N) U \\ &= (U_1 \otimes U_2 \otimes \ldots \otimes U_N)(\rho_1 \otimes ... \otimes \rho_N)(U_1^* \otimes U_2^* \otimes \ldots \otimes U_N^*) \\ &= (U_1 \rho_1 U_1^*) \otimes (U_2 \rho_2 U_2^*) \ldots (U_N \rho_N U_N^*) \end{aligned} \tag{4.6}$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Finally, we derive the last equation by the property of tensor product.

Notice that tensor product of positive self-adjoint operators is positive self-adjoint. Hence, $\widetilde{\rho}$ is positive self-adjoint operator with $tr(\rho) = 1$. Moreover, the quantum no arbitrage condition is satisfied for the transformed density:

$$tr(\tilde{\rho_1} D_1) = 1 + r_1,\ tr(\tilde{\rho_2} D_2) = 1 + r_2, ..., \text{ and } tr(\tilde{\rho_N} D_N) = 1 + r_N. \tag{4.7}$$

Let

$$D = D_1 \otimes D_2 ... \otimes D_N.$$

In addition, we have

$$\begin{aligned} tr(\widetilde{\rho}H) &= tr(\widetilde{\rho}_1 \cdots \otimes \widetilde{\rho}_N H_1 \otimes \cdots \otimes \widetilde{\rho}_N) && (4.8)\\ &= tr(\widetilde{\rho}_1 H_1) \cdots tr(\widetilde{\rho}_N H_N) \\ &= tr(\widetilde{\rho}_1 D_1) \cdots tr(\widetilde{\rho}_N D_N) && (4.9)\\ &= (1+r_1)\cdots(1+r_N). && (4.10) \end{aligned}$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Also, we derive the last equation via the property of unitary operator. Finally, we obtain the final equation via the no-arbitrary condition.

The arbitrage free time 0, the general option price of quantum option claim $C = f(S_N)$ for generalised N period quantum binomial model is defined by

$$\begin{aligned} OP(f(S_N)) &= \frac{tr(\rho f(S_N))}{tr\rho H} && (4.11)\\ &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N))}{(1+r_1)(1+r_2)...(1+r_N)}. && (4.12) \end{aligned}$$

Applying (4.3) and (4.12), we obtain

$$\begin{aligned} OP(f(S_N)) &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N))}{(1+r_1)(1+r_2)...(1+r_N)} \\ &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N U f(S_0 D_1 \otimes ... \otimes D_N) U^*)}{(1+r_1)(1+r_2)...(1+r_N)} \\ &= \frac{tr([U^*(\rho_1 \otimes ... \otimes \rho_N) U] f(S_0 D_1 \otimes ... \otimes D_N))}{(1+r_1)(1+r_2)...(1+r_N)} \\ &= \frac{tr(\widetilde{\rho} f(S_0 D_1 \otimes ... \otimes D_N))}{(1+r_1)(1+r_2)...(1+r_N)} \\ &= \frac{tr(\widetilde{\rho_1} \otimes \ldots \otimes \widetilde{\rho_N} f(S_0 D_1 \otimes ... \otimes D_N))}{(1+r_1)(1+r_2)...(1+r_N)}. \end{aligned}$$

We obtain the second equation via the general fact 4.2.1. And the third equation has been obtain by the trace property. And then, we obtain the fourth equation by the definition of transformed quantum state.

Since $\widetilde{\rho}$ is a tensor product and $f(S_0 D_1 \otimes ... \otimes D_N)$ is a diagonal matrix, we derive the following option pricing formulas.

Theorem 4.1. *(*Option pricing for generalized N-period quantum binomial market*). Under notation in above,*

$$OP(f(S_N)) = \frac{\sum_\sigma f(S_0 y_\sigma) q_\sigma}{(1+r_1)(1+r_2)...(1+r_N)}$$

Corollary 4.2. *In particular, for the N-period quantum binomial model (the quantum analogue of Cox-Ross-Rubinstein Binomial market), $q_u^i = q_u, r_i = r, u_i = u, d_i = d, i = 1, 2, ..., N$, we derive*

$$OP(f(S_N)) = \frac{1}{(1+r)^N} \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} f(S_0 u^n d^{N-n}) q_d^n q_u^{N-n}.$$

4.3 N period quantum Binomial model in the non Self-adjoint market

In this part, we define the generalised N-period quantum binomial model and develop a relevant option pricing.

Quantum share price S_N for Non self-adjoint case. Quantum share price for the quantum binomial market is defined by

$$S_N = S_0 H_1 \otimes \ldots \otimes H_N, \tag{4.13}$$

where H_i is (non) self-adjoint 2×2 matrices, which represents the changing of share price with jumps u_i (jump up) or d_i (jump down), $i = 1, \ldots, N$. For the quantum model, u_i and d_i are diagonal elements. The only difference is that H_i are Non self-adjoint in general.

Quantum claim $C = f(S_N)$. In particular, we can introduce the quantum claim $C = f(S_N)$ via the following general formula.

Fact 4.3.1. (a) Assume an operator H admits a representation

$$H = uP_u + dP_d(6)$$

where P_u and P_d are generalized orthogonal projections such that

$$P_u^2 = P_u, P_d^2 = P_d, P_u P_d = P_d P_u = 0, P_u + P_d = I.$$

Then,

$$f(H) = f(u)P_u + f(d)P_d.$$

(b) (Case 2×2 matrices) Assume that H admits a representation

$$H = P \begin{pmatrix} u & 0 \\ 0 & d \end{pmatrix} P^{-1}.$$

Then,

$$f(H) = P \begin{pmatrix} f(u) & 0 \\ 0 & f(d) \end{pmatrix} P^{-1}.$$

In general, the tensor product of Jordan matrices is not a Jordan matrix. However, the argument of (4.13) works as well for the diagonizable case. In our case,

$$\begin{aligned} S_N &= S_0 H_1 \otimes \ldots \otimes H_N && (4.14) \\ &= P S_0 D_1 \otimes \ldots \otimes D_N P^{-1} \\ \Longrightarrow & \; f(S_N) = P f(S_0 D_1 \otimes \ldots \otimes D_N) P^{-1}. && (4.15) \end{aligned}$$

Quantum state ρ. As in the self-adjoint case, the quantum state ρ in the Euclidean space $\mathcal{H} = R^{2^N}$ of dimension 2^N is defined as the tensor product

$$\rho = \rho_1 \otimes \ldots \otimes \rho_N.$$

which is again is a non-negative self-adjoint operator with trace 1.

Note that the density is self-adjoint by default.

Quantum state risk-neutral world. The risk-neutral condition is defined again as before via (4.4).

Transformed quantum state. Notice now that the similar definition , yields

$$H_i = P_i D_i P_i^{-1} \text{ and } \rho_i \Longrightarrow \tilde{\rho}_i = P_i \rho_i P_i^{-1}, \text{ where i=1,2,...N.} \tag{4.16}$$

And the transformed quantum state is then defined by

$$\begin{aligned}
\widetilde{\rho} &= P(\rho_1 \otimes ... \otimes \rho_N)P^{-1} \\
&= (P_1 \otimes P_2 \otimes \ldots \otimes P_N)(\rho_1 \otimes ... \otimes \rho_N)(P_1^{-1} \otimes P_2^{-1} \otimes \ldots \otimes P_N^{-1}) \\
&= (P_1\rho_1 P_1^{-1}) \otimes (P_2\rho_2 P_2^{-1}) \ldots (P_N\rho_N P_N^{-1}) \qquad (4.17)
\end{aligned}$$

The last two equation derived by the property of tensor product.

Notice that again $\tilde{\rho}_i$ have trace 1

$$\begin{aligned}
tr(\tilde{\rho}_i) &= tr(P_i\rho_i P_i^{-1}) \\
&= tr(P_i^{-1}P_i\rho_i) \\
&= tr(\rho_i) = 1, \text{ where i=1,2,...,N.}
\end{aligned}$$

Under the definition of transformed state, we obtain the first equation. Then, we apply the trace rule for the second equation. And we obtain the third equation because of the property of the projection. The final equation derived by the definition of quantum state.

Remark 4.3. However, in general, neither $\tilde{\rho}_i$ nor $\tilde{\rho}$ are self adjoint operators, that is in general they are not proper quantum states!

The risk neutral condition is transformed to the same form as before 4.7.

The arbitrage free time 0, the general option price of quantum option claim $C = f(S_N)$, for generalised N period quantum binomial model is defined by

$$\begin{aligned}
OP(f(S_N)) &= \frac{tr(\rho f(S_N))}{tr\rho H} \\
&= \frac{tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N))}{(1+r_1)(1+r_2)...(1+r_N)}.
\end{aligned}$$

Applying (4.3) and (4.12), we obtain

$$\begin{aligned}
OP(f(S_N)) &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N))}{(1+r_1)(1+r_2)...(1+r_N)} \\
&= \frac{tr(\rho_1 \otimes ... \otimes \rho_N P f(S_0 D_1 \otimes ... \otimes D_N)P^{-1})}{(1+r_1)(1+r_2)...(1+r_N)} \\
&= \frac{tr([P^{-1}(\rho_1 \otimes ... \otimes \rho_N)P]f(S_0 D_1 \otimes ... \otimes D_N))}{(1+r_1)(1+r_2)...(1+r_N)}
\end{aligned}$$

We obtain the second equation via the general fact 4.2.1. And the third equation has been obtain by the trace property. Finally, we yield the same answer as in the self-adjont case.

Theorem 4.4. *(*Option pricing for generalized N-period quantum binomial market*). Under notation in above,*

$$OP(f(S_N)) = \frac{\sum_\sigma f(S_0 y_\sigma) q_\sigma}{(1+r_1)(1+r_2)...(1+r_N)}.$$

4.4 N step bond market

We carry on the N step bond market. Now, our share price is also a bond matrix that is with all diagonal elements u.

Case of a self adjoint bond. The share(bond) price matrix is defined by

$$H^{\otimes N} = U(D_1 \otimes \ldots \otimes D_N)U^{*.}$$

The state ρ satisfies usual conditions $tr(\rho_1) = 1, \ldots, tr(\rho_N) = 1$. Hence, for the transformed density $\tilde{\rho} = U\rho U^*$, the no-arbitrage condition becomes

$$\begin{aligned}
tr(\rho^{\otimes N} H^{\otimes N}) &= tr(\rho_1 \otimes \ldots \otimes \rho_N U(D_1 \otimes \ldots \otimes D_N)U^*) \\
&= tr(U(\rho_1 \otimes \ldots \otimes \rho_N)U^* D_1 \otimes \ldots \otimes D_N) \\
&= tr(\tilde{\rho_1} \otimes \ldots \otimes \tilde{\rho_N} D_1 \otimes \ldots \otimes D_N) \\
&= (\tilde{\rho_{11}} + \ldots + \tilde{\rho_{NN}})u^N \\
&= tr(\tilde{\rho})u^N \\
&= u^N \\
&= (1+r)^N.
\end{aligned}$$

Notice that in one step bond market, we have $H = U^* \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} U = uI_2$. Then, we introduce a general lemma for the risk neutral condition.

Lemma 4.4.1. (Option price for self-adjoint bond market) A local no arbitrage condition is equivalent to the global no arbitrage condition as follows

$$tr(\rho^{\otimes N} H^{\otimes N}) = tr(\rho H)^N = (1+r)^N \text{ if and only if } tr(\rho H) = 1+r.$$

Besides, $C = f(S_N)$ is the quantum claim for any function f via the general fact 4.3.1 in one step model. Then, we obtain

$$\begin{aligned} OP(f(S_0 H^{\otimes N})) &= \frac{tr(\rho^{\otimes N} f(S_0 H^{\otimes N}))}{1+r} \\ &= \frac{tr(\tilde{\rho}^{\otimes N}) f(S_0 u^N)}{(1+r)^N} \\ &= \frac{\tilde{\rho}_{11} f(S_0 u^N) + \ldots + \tilde{\rho}_{NN} f(S_0 u^N)}{(1+r)^N} \\ &= \frac{f(S_0 u^N)}{(1+r)^N}. \end{aligned}$$

Case of non self-adjoint bond. Now, we assume that

$$\begin{aligned} H = H_u^n &= P^{\otimes n} \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}^{\otimes n} P^{-1 \otimes n} \\ &= P^{\otimes n} J_{2,u}^{\otimes n} P^{-1 \otimes n} \end{aligned}$$

Define the transformed state $\tilde{\rho} = P^{-1} \rho P$. For the no-arbitrage condition, we derive

$$tr(\rho H_u^i) = tr\left(\tilde{\rho} H_u^i\right) = 1 + r_i, \text{ where } i = 1, \ldots, n.$$

Besides, $C = f(S_n)$ is the quantum claim for any function f via the general fact 4.3.1. Then, we obtain

$$\begin{aligned} OP(f(S_0 H_u)) &= \frac{tr(\tilde{\rho} f(S_0 J_{n,u}))}{1+r} \\ &= \frac{tr\left(\tilde{\rho}\begin{pmatrix} f(S_0 u_1 u_2) & S_0 \frac{f'(S_0 u_1)}{1!} & \ldots & S_0^3 \frac{f^{(n-1)}(S_0)}{(n-1)!} \\ 0 & f(S_0 u_1 d_2) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & f(S_0 d_1 d_2) \end{pmatrix}\right)}{(1+r_1)(1+r_2)\ldots(1+r_n)}. \end{aligned}$$

Limit of N step bond market

Lemma 4.4.1. As $k = n$, we have $H = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}$, and $\rho = \begin{pmatrix} \rho_{11} & y \\ y & \rho_{22} \end{pmatrix}$, where $y = 1+r-u$ and $r = \frac{\lambda}{n}$. Then the option price tends to $E(\xi^N) = Ee^{(N(iB_{2\sigma}+\sigma+\lambda))} = e^{(N(\sigma+\lambda)-N^2\sigma)}$ as $n \to \infty$.

Proof.

$$\begin{aligned} OP((S_0 H^{\otimes n})^N) &= \frac{(tr(\rho H^N))^n}{(1+r)^n} \\ &= \frac{(\rho_{11}u^N + \rho_{22}u^N + yu^{N-1}N)^N}{(1+r)^n} \\ &= (\frac{u^N + yu^{N-1}N}{1+r})^n, \text{where } e^{\frac{\sigma}{n}} = u \\ &\to e^{(N\sigma+\lambda N-N^2\sigma-\lambda)} \text{ as } n \to \infty \\ &= e^{-\lambda}E(\xi^N). \end{aligned}$$

□

Connection with the planar process, it is a representation and play with planar Brownian Motion by Fourier techniques. For the distribution of this process, we interpret R^2 as the complex plane. Hence a planar Brownian motion becomes a complex Brownian motion. A complex-valued stochastic process called a martingale. if its real and imaginary parts are martingales. Let $\{B(t) : t \geq 0\}$ be a complex Brownian motion started in i, the imaginary unit.

Remark the definition of planer Brownian Motion is given in the subsection about Brownian Motion.

Lemma 4.5. *Let $\tilde{B}_t$ be a planar Brownian motion, $e^{z\tilde{B}_t}$ is a martingale with respect to its natural filtration.*

Proof.

$$\begin{aligned} E[e^{z\tilde{B}_T}|\mathcal{F}_t] &= E[e^{zX_T}|\mathcal{F}_t]E[e^{zY_T}|\mathcal{F}_t], \texttt{ where } z = x+iy \\ &= e^{zX_t+tz^2/2}e^{izY_t+t(iz)^2/2} \\ &= e^{zX_t+izY_t} \\ &= e^{z(X_t+iY_t)} = e^{z\tilde{B}_t}. \end{aligned}$$

□

Then, the option price under the risk neutral condition with the interest rate $\rho = 0$ is defined by

$$\begin{aligned} OP(f(\tilde{B_T})|t) &= E[f(\tilde{B_T})|\mathcal{F}_t] \\ &= e^{(N(\lambda+\sigma)-N^2\sigma^2)}. \end{aligned}$$

Notice that when $\sigma = 0$, the result is the same with the bound; when $\sigma \leq 0$, it tends to Black-scholes formula; And if $\sigma \leq 0$, then it seems to be a kind of complex Brownian motion.

4.5 Conditional option pricing for generalized quantum N period Binomial model

In this part, we derive the Quantum Cox-Ross-Rubinstein Binomial Model by considering the distribution of eigenvalues. Let $\sigma = y_1 \dots y_N$ means the sum over all paths with $y_i \in \{u_i, d_i\}$ and the probability of the path being $q_\sigma = q_{\sigma_1}^{(1)} \dots q_{\sigma_N}^{(N)}$. Let

$$x = S_t = S_0 y_1 \dots y_t$$

be the initial position. The corresponding projection P_x is a projection operator on S_t and ρ is a state, which can transform a new state $\hat{\rho} = P_x \rho P_x$.

Finally, notice that (refer to the general fact 4.3.1)

$$f(S_0 H_1 \otimes ... \otimes H_N) = U^* f(S_0 D_1 \otimes ... \otimes D_N) U,$$

where $U = U_1 \otimes \dots \otimes U_N$ and $U^* = U_1^* \otimes \dots \otimes U_N^*$. Notice that for generalised N period quantum binomial model, the arbitrage free time t, option price of quantum option claim $C = f(S_N)$ is defined by

$$OP(f(S_N)|t, x) = \frac{tr(P_x \rho P_x f(S_N))}{tr(P_x \rho P_x)}$$

The main results in this section is the following theorem.

Theorem 4.5.1. (Conditional option pricing for generalized N-period quantum binomial market). Under notation in above,

$$OP(f(S_N)|t,x) = \sum_{y_{t+1},\ldots,y_N} f(S_0 y_1 \ldots y_N) \prod_{k=t+1}^{N} q_{y_k}^{(k)} \Big/ [(1+r_{t+1})\ldots(1+r_N)]$$

To prove th Theorem 4.5.1, we first establish several lemmas. The following fact appears to be useful.

Fact 4.5.1. Let $\{p_\alpha\}$ be $\alpha \in \omega$ or set of orthogonal projection. Let $A_i \in \omega$ be disjoint subsets. Then, $\sum_{\alpha \in A_i} P_\alpha$ are orthogonal projections. Moreover,

$$\sum_{\alpha \in A} P_\alpha \sum_{\beta \in B} P_\beta = \sum_{j \in A \bigcap B} P_j.$$

In the N step quantum binomial model, we consider the observable as a tensor product. Let

$$\begin{aligned}
\rho &= \otimes_{k=1}^{N} \rho_k \ , \ S_Y = \sum_{j \in Y} \tilde{U}^* D_j^{(s)} \tilde{U}, \\
U &= U_1 \otimes \ldots \otimes U_n \ , \ \tilde{U} = \tilde{U}_1 \otimes \ldots \otimes \tilde{U}_n, \\
D_j^{(s)} &= D_{j_1}^{(s)} \otimes \ldots \otimes D_{j_N}^{(s)}
\end{aligned}$$

where $j = j_1, \ldots, j_n$, $j = 1$ (up) and $j = 0$ (down).

Lemma 4.5.2. Under notation in above,

$$tr(\rho S_Y) = \sum_{j \in Y} \prod_{k=1}^{N} (q_{y_{j_k}}^{(k)} y_{j_k})$$

Proof of Lemma 4.5.2. Observe that by the linearity of the trace and properties of the tensor product

$$\begin{aligned}
tr(\rho S_Y) &= \sum_{j \in Y} tr(\rho \tilde{U}^* D_j^{(s)} \tilde{U}) \\
&= \sum_{j \in Y} tr\big((\otimes_{k=1}^{N} \rho_k)(\otimes_{k=1}^{N} \tilde{U}_k^*)(\otimes_{k=1}^{N} D_{j_k}^{(s)})(\otimes_{k=1}^{N} \tilde{U}_k)\big) \\
&= \sum_{j \in Y} tr(\otimes_{k=1}^{N} \rho_k \tilde{U}_k^* D_{j_k}^{(s)} \tilde{U}_k) \\
&= \sum_{j \in Y} \prod_{k=1}^{N} tr(\rho_k \tilde{U}_k^* D_{j_k}^{(s)} \tilde{U}_k)
\end{aligned}$$

Notice that by the cycling property of the trace $tr(ABC) = tr(BCA)$.

$$
\begin{aligned}
tr(\rho_k \tilde{U}_k^* D_{j_k}^{(s)} \tilde{U}_k) &= tr((\tilde{U}_k \rho_k \tilde{U}_k^*) D_{j_k}^{(s)}) \\
&= tr(\hat{\rho}_k D_{j_k}^{(s)}) \quad \text{where } \hat{\rho}_k = \tilde{U}_k \rho_k \tilde{U}_k^* ,
\end{aligned}
$$

implying that

$$
\begin{aligned}
tr(\rho S_Y) &= \sum_{j \in Y} \prod_{k=1}^{N} tr(\hat{\rho}_k D_{j_k}^{(s)}) \\
&= \sum_{j \in Y} \prod_{k=1}^{N} (q_{y_{j_k}}^{(k)} y_{j_k})
\end{aligned}
$$

The proof is complete $\diamond$.

Lemma 4.5.3. Let P_x be a projection operator on $S_0, \dots, S_t = S_0 y_1 \cdots y_t = x$. Then

$$
tr(\rho P_x) = q_{y_1}^{(1)} \dots q_{y_t}^{(t)}.
$$

Proof of Lemma 4.5.3. We apply the previous Lemma 4.5.2 to the set

$$
Y = \{1 : \text{ at place of eigenvalue } \prod_{t=1}^{N} y_{j_t} \text{ such that } y_1, \dots, y_t \text{ are fixed}\}
$$

The statistics $S_Y = P_x$ that is all eigenvalues $y_i = 1$. Notice that here a tensor product is treated both as a number and a vector $(y_1, \dots, y_n)$. Remind that in this case $q_{y_1}^{(1)}, \dots, q_{y_t}^{(t)}$ are fixed and so

$$
\begin{aligned}
tr(\rho P_x) &= tr(\rho S_Y) \\
&= \sum_{j \in Y} \prod_{t=1}^{N} q_{y_t}^{(t)} \\
&= q_{y_1}^{(1)} \cdots q_{y_t}^{(t)} \sum_{y_{t+1}, \dots, y_N} \prod_{k=t+1}^{N} q_{y_k}^{(k)} \\
&= q_{y_1}^{(1)} \dots q_{y_t}^{(t)} \prod_{k=t+1}^{N} \left(\sum_{y_k} q_{y_k}^{(k)} \right) \\
&= q_{y_1}^{(1)} \dots q_{y_t}^{(t)}.
\end{aligned}
$$

The proof is complete $\diamond$.

Lemma 4.5.4. Let P_x be the projection operator defined in Lemma 4.5.3. Then

$$\begin{aligned} tr(\rho P_x S_N P_x) &= q^{(1)}_{y_1}\dots q^{(t)}_{y_t} x' P(\ln Y_{m+1} + \dots + \ln Y_N \\ &= \ln(x'/x)) \end{aligned}$$

Proof. We apply Lemma 4.5.2 to the set

$$Y(x,x') = \{j : x' = S_0 y_1 \dots y_N, x = S_t = S_0 y_1 \dots y_t, \text{ and } y_1,\dots,y_t \text{ are fixed.}\}.$$

we observe that

$$P_x S_N P_x = S_{Y(x,x')}.$$

and therefore

$$tr(\rho P_x S_N P_x) = tr(\rho S_{Y(x,x')}).$$

Notice, $j \in Y$ means that $y_{t+1}\cdots y_N = x'/x$ and the path $y_1,\dots,y_t$ is fixed. Overall by the similar argument as in Lemma 4.5.3, we get

$$\begin{aligned} tr(\rho P_x S_N P_x) &= tr(\rho S_{Y(x,x')}) \\ &= \sum_{j\in Y(x.x')} \prod_{k=1}^{N} (q^{(k)}_{y_{j_k}} y_{j_k}) \\ &= q^{(1)}_{y_1}\dots q^{(t)}_{y_t} x' \sum_{j\in Y(x')} \prod_{m=t+1}^{N} q_{y_m} \\ &= q^{(1)}_{y_1}\dots q^{(t)}_{y_t} x' P(\ln Y_{m+1} + \dots + \ln Y_N = \ln(x'/x)). \end{aligned}$$

The proof is complete $\diamond$.

If $q^{(1)}_{y_1},\dots,q^{(t)}_{y_t}$ are not fixed, then we obtain the following lemma.

Lemma 4.5.5. Let P_x be a projection operator on $Y(x) = \{1 : x = \{y_0,\dots,y_t\}\}$. Then

$$\begin{aligned} tr(\rho P_x) &= \sum_{y_t : S_0 y_0 \dots y_t = x} q^{(1)}_{y_1}\dots q^{(t)}_{y_t} \\ &= P(S_t = x). \end{aligned}$$

Lemma 4.5.6. Let P_x be the projection operator defined in Lemma 4.5.5. Then

$$\begin{aligned} tr(\rho P_x S_N P_x) &= P(S_t = x) P(\ln Y_{m+1} + \ldots + \ln Y_N \\ &= \ln(x'/x)) \end{aligned}$$

Now, we are ready to finish the proof of the Theorem.

Proof of Theorem 4.5.1. Applying the quantum conditional expectation with the general fact 4.3.1, we derive by using the cycle trace property and that $P_x^2 = P_x$ is a projection.

$$\begin{aligned} OP(f(S_N)|t,x)) &= OP(f(S_N)|S_t = x)) \\ &= \frac{tr(\hat{\rho} f(S_N))}{tr(\hat{\rho})} \text{ where } \hat{\rho} \text{ is a new state and } S_N \text{ is an observable,} \\ &= \frac{tr(P_x \rho P_x f(S_N))}{tr(P_x \rho P_x)(1 + r_{t+1}) \ldots (1 + r_N)} \\ &= \frac{tr(\rho P_x f(S_N) P_x)}{tr(P_x \rho P_x)(1 + r_{t+1}) \ldots (1 + r_N)} \\ &= \frac{tr(\rho P_x f(S_N) P_x))}{q_{y_1}^{(1)} \ldots q_{y_t}^{(t)} (1 + r_{t+1}) \ldots (1 + r_N)} \end{aligned}$$

by applying Lemmas 4.5.3 at the last step.

Proceeding as in Lemma 4.5.4, we derive

$$\begin{aligned} & tr(\rho P_x S_N P_x)) \\ = \; & q_{y_1}^{(1)} \ldots q_{y_t}^{(t)} \sum_{x'} S_0 y_1 \ldots y_N P(\ln Y_{m+1} + \ldots + \ln Y_N = \ln(x'/x)) \\ = \; & q_{y_1}^{(1)} \ldots q_{y_t}^{(t)} \sum_{y_{t+1}, \ldots, y_N} S_0 y_1 \ldots y_N P(\ln Y_{m+1} + \ldots + \ln Y_N = \ln(y_{t+1} \ldots y_N)) \\ = \; & q_{y_1}^{(1)} \ldots q_{y_t}^{(t)} \sum_{y_{t+1}, \ldots, y_N} S_0 y_1 \ldots y_N \prod_{k=t+1}^{N} q_{y_k}^{(k)} \end{aligned}$$

Plugging the last expression into the representation for $OP(f(S_N)|t,x)$ with the general fact 4.3.1, we derive

$$OP(f(S_N)|t,x) = \sum_{y_{t+1}, \ldots, y_N} f(S_0 y_1 \ldots y_N) \prod_{k=t+1}^{N} q_{y_k}^{(k)} \Big/ [(1 + r_{t+1}) \ldots (1 + r_N)]$$

The proof of Theorem 4.5.1 is complete $\diamond$.

4.6 Convergence to the Black-Scholes model

Let d be a positive integer, let $M_d(\mathbb{C})$ be the algebra of $d \times d$ complex matrices with usual matrix multiplication, and let $A_n \in M_d(\mathbb{C})$.

Lemma 4.6.1. Let μ_n be the measure of Eigenvalues of A_n with respect to $tr(\rho^{\otimes n} H^{\otimes n})$, where ρ is a state, which is a positive operator with unit trace and H is an observable with the spectral resolution. Then, there is a particular μ satisfied $\int f(x) d\mu_n(x) \to \int f(x) d\mu(x)$ for μ_n as $n \to \infty$.

Proof of Lemma 4.6.1. We begin with the **Representation via spectral measure** μ_n. Observe that

$$
\begin{aligned}
Ef(H) &= tr(\rho^{\otimes n} f(S_0 H^{\otimes n})) && (4.18)\\
&= \int f(S_0 x) \mu_n(dx), && (4.19)
\end{aligned}
$$

where μ_n is the measure of Eigenvalues of A_n with respect to the quantum probability, $tr(\rho^{\otimes n} H^{\otimes n})$, defined by

$$
\begin{aligned}
\mu_n(x) &= \sum_{\sigma : \lambda_\sigma \leq x} q_\sigma && (4.20)\\
&= \sum_{\sigma_i : y_1^{\sigma 1} \ldots y_n^{\sigma n}} q_{\sigma_1} \ldots q_{\sigma_n} \leq x, && (4.21)
\end{aligned}
$$

where the sum is over all paths $\sigma = \sigma_1 \ldots \sigma_N$ with $\sigma_i \in \{u, d\}$, the probability of the path being $q_\sigma = q_{\sigma_1} \ldots q_{\sigma_N}$ with $q_{\sigma_i} \in \{q_u, q_d\}$. And, the corresponding eigenvalues are

$$
\lambda_\sigma = y_1^{\sigma^{i_1}} \ldots y_n^{\sigma_{i_n}}, \ \texttt{where} \ y_i^{\sigma_i} = \begin{cases} u_i & , \sigma_i = u, \\ d_i & , \sigma_i = d. \end{cases} \qquad (4.22)
$$

Reduction to the weak convergence. Hence, we need to find the limiting distribution μ, which satisfied

$$
\int f(x) d\mu_n(x) \to \int f(x) d\mu(x)
$$

that is to establish the weak limit $\mu_n \to \mu$.

Probability modelling Notice that

$$\begin{aligned} q_{\sigma_1} \dots q_{\sigma_n} &= P(Y_1 = y_1^{\sigma_1})P(Y_2 = y_2^{\sigma_2}) \dots P(Y_n = y_n^{\sigma_n}) && (4.23) \\ &= P(Y_1 = y_1^{\sigma_1}, \dots, Y_n^{\sigma_n}), && (4.24) \end{aligned}$$

where Y_i are iid with

$$P(Y_i = u) = q_u, P(Y_i = d) = q_d. \quad (4.25)$$

And, also

$$P(\sigma : \lambda_\sigma^n \leq x) = Q(Y_1 \dots Y_n \leq x). \quad (4.26)$$

Overall, from (4.18) - (4.26)

$$\begin{aligned} \mu_n(x) &= P(Y_1 \times \dots \times Y_n \leq x) \\ &= P(\ln Y_1 + \dots + \ln Y_n \leq \ln x) \\ &= P(T_n \leq \ln x) \end{aligned}$$

where $T_n = \ln Y_1 + \dots + \ln Y_n$.

Observe that

$$E[T_n] = E[\ln(\frac{u}{d})Bin(n, q_u) + n\ln(d)] \to 2\sigma a,$$

and,

$$Var(T_n) = Var(n\ln(d) + \ln(\frac{u}{d})Bin(n, q_u)) \to \sigma^2.$$

Hence, by the central limit theorem,

$$\begin{aligned} \mu_n(x) &= P(T_n \leq \ln x) \to P(e^{\sigma(2a+N(0,1))} \leq x) \\ &= \mu(x). \end{aligned}$$

The proof is complete. $\diamond$.

Comment 4.6.1. In the above equation, if $t = 1$, $\rho = \lambda$, $a = \rho - \frac{\sigma^2}{2}$. Then, we derive that the limiting measure corresponds to the Geometric Brownian Motion model

$$P(e^{\sigma(2a+N(0,1))} \leq x) = P(S_0 e^{at+\sigma B_t} \leq x).$$

where $\{B_t : t \geq 0\}$ is a Standard Brownian Motion (Wiener process).

Comment 4.6.2. The traditional way to prove the convergence is to establish the convergence of the Cauchy transforms. More exactly, the Cauchy transform for the measure μ is defined by

$$S_\mu(z) = \int_R \frac{\mu(dt)}{t-z}, \quad z \in C \backslash R.$$

So the measure is then determined by

$$\mu((a,b)) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_a^b \Im S_\mu(x + i\epsilon),$$

for all open interval (a,b) with a, b where μ does not have an atom at a and b.

One needs to show that $S_{\mu_n}(z) \to S_\mu(z)$. In addition, it is common to apply

$$\int \frac{d\mu(x)}{z-x} \quad = \quad \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int x^k d\mu(x)$$

However, for the limiting measure μ defined in above the last sum is equal to

$$\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} S_0^k e^{ak+\sigma^2 k^2/2} \quad = \quad \infty.$$

showing that, the traditional approach is not appropriate in this case.

4.7 Conclusion

Quantum model based on quantum probability, instead of classical probability, which is generalisation for classical probability. In this chapter, we have proved that quantum models do cover the classical non-quantum models.

Firstly, we analyse the quantum version of Binomial model, including both self-adjoint market and non self-adjoint market. Considering the quantum observable, we apply the discrete time quantum formulization and construct option pricing of tensor product of non-commutative market.

Moreover, we analyse the quantum version of the Cox-Ross-Rubinstein Binomial Model. And we analyse the quantum bond markets and found that N step bond market can connect to planar Brownian motion.

Furthermore, we analyse the quantum conditional option price via the quantum conditional expectation. Besides, we establish the limit of the spectral measures proving the convergence to the geometric Brownian motion model. Here is the other point that we make a connection between classical model and non-classical model.

In the next chapter, we will give several examples to demonstrate our results.

Chapter 5

Examples for Quantum Market

5.1 One period Quantum Market

5.1.1 Option pricing for generalized one period quantum Binomial market

We start from the one step quantum binomial model. Let ρ be a quantum state and H be an observable with spectral resolution, which is a self-adjoint operator in the probability theory, $H = \sum_j h_j E_j^H$. Also,

$$H = U \begin{pmatrix} u & 0 \\ 0 & d \end{pmatrix} U^{*}, \text{ where U is Unitary operator.}$$

It represents the changing of share price with jumps u (jump up) or d (jump down). The probability of Event E_j^H, H takes the value h_j in the state ρ, is equal to $tr\rho E_j^H$. H has expectation $E_\rho(H)$ in the state ρ,

$$\begin{aligned} E_\rho(H) &= \sum_j h_j tr\rho E_j^H \\ &= tr\rho \sum_j h_j E_j^H \\ &= tr\rho H. \end{aligned}$$

Thus, the risk-neutral world of the quantum model (B,S) consists of states ρ on the space $\mathcal{H} = R^2$ satisfying

$$\begin{aligned} tr\rho H &= uq_u + dq_d \\ &= 1 + r, \end{aligned}$$

where $tr\rho H$ is the probability of the event in the state ρ and q_u, q_d are diagonal elements of ρ.

Quantum share price for the quantum binomial market is defined by

$$S_1 = S_0 H.$$

In this case, a quantum model for the binmial market (B,S) is presented.

We remind that if $A^* = A$ then A admits a representation $A = UDU^*$. Then, $f(A) = Uf(D)U^*$.

By using the general fact 4.2.1 and the definition of self-adjoint operator $H = U^*DU$, we obtain

$$\begin{aligned} f(S_1) &= Uf(S_0 D_1)U^* \\ &= U \begin{pmatrix} f(S_0 u) & 0 \\ 0 & f(S_0 d) \end{pmatrix} U^*. \end{aligned}$$

And then, we define transformed quantum state

$$\rho \Longrightarrow \tilde{\rho} = U\rho U^* = \begin{pmatrix} \tilde{\rho_{11}} & \tilde{\rho_{12}} \\ \tilde{\rho_{21}} & \tilde{\rho_{22}} \end{pmatrix}, \tag{5.1}$$

where U is a Unitary operator.

In particular, we know $\tilde{\rho} = \rho$, $tr(\tilde{\rho}) = tr(\rho) = 1$ since $U^*U = UU^* = I$, ρ is a quantum state, and $< \rho_1 x, x > = < \rho y, y > \geq 0$, where $U^* x = y$ because of $\rho \geq 0$.

Finally, we define $f(H) = U^* f(D) U$.

$$\begin{aligned} tr(\rho f(S_0 H)) &= tr(\rho U^* f(S_0 D) U) \\ &= tr((U \rho U^*) f(S_0 D)) \\ &= tr(\tilde{\rho} f(S_0 D)) \\ &= tr\left(\begin{pmatrix} \tilde{\rho_{11}} & \tilde{\rho_{12}} \\ \tilde{\rho_{21}} & \tilde{\rho_{22}} \end{pmatrix} \begin{pmatrix} f(S_0 u) & 0 \\ 0 & f(S_0 d) \end{pmatrix}\right). \end{aligned}$$

Under the general fact, we obtain the first equation. Then, we apply the trace rule for the second equation. And we obtain the final equation because of the definition of transformed quantum state.

Under the no-arbitrage condition, the option price of quantum option claim $C = f(S_1)$ for one step quantum binomial model is defined by

$$\begin{aligned} OP(f(S_1)) &= \frac{tr(\rho f(S_1))}{1+r} \\ &= \frac{tr\left(\rho U \begin{pmatrix} f(S_0 u) & 0 \\ 0 & f(S_0 d) \end{pmatrix} U^*\right)}{1+r} \\ &= \frac{\tilde{q_u} f(S_0 u) + \tilde{q_d} f(S_0 d)}{1+r}. \end{aligned}$$

Moreover, the quantum binomial model is arbitrage-free if and only if $-1 \leq a < r < b$. For the European call option in the quantum binomial market with the exercise price K. Its payoff is of the form

$$H = (S_1 - K)^+, \tag{5.2}$$

which takes two values

$$h_a = max(0, S_0 d - K), h_b = max(0, S_0 u - K). \tag{5.3}$$

Thus, the option value C is

$$\begin{aligned} C &= \frac{1}{1+r} tr(\rho H) \\ &= \frac{1}{1+r}\left(\frac{1+r-d}{u-d} h_a + \frac{u-(1+r)}{u-d} h_b\right) \end{aligned}$$

for all states ρ in the risk-neutral world.

5.1.2 One step quantum market

Let us introduce several cases for One step quantum market.

Case 1. In general, this is 2x2 non self adjoint matrix case.

Example 1 Market: (2x2 case). Discrete time market. Two quantum states on the Euclidean space $E = M_{2\times 2}$ are the bond. And the share price modelled by the dynamics

$$S_1 = S_0 H$$

where 2×2 matrix $H = PDP^{-1} = P \begin{pmatrix} u & 0 \\ 0 & d \end{pmatrix} P^{-1}$ has two diagonal elements u and d. They represent the change of share price with jumps u (jump up) or d (jump down).

The risk-neutral world of the quantum model (B, S) consists of states ρ on H which are self adjoint non-negative matrices with density $tr(\rho) = 1$, and satisfy the risk neutral condition

$$\begin{aligned} tr(\rho H) &= utr(\rho P_u) + dtr(\rho P_d) \\ &= 1 + r, \end{aligned}$$

where $tr(\rho P_u)$ and $tr(\rho P_d)$ are the probability of the event in the state ρ. Notice that, with the density assumption $tr(\rho) = 1$, the orthogonal projections P_u, P_d implies that

$$\begin{aligned} tr(\rho P_u) &= q_u \\ &= \frac{1 + r - d}{u - d}, \end{aligned}$$

$$tr(\rho P_d) = 1 - q_u,$$

where ρ has positive diagonals q_u or q_d. However, ρ is not self-adjoint in general.

The general quantum claim is $C = f(S_1)$, i.e. a function of operator H. Notice the quantum claim

$$\begin{aligned} f(S_1) &= f(S_0 H) \\ &= Pf(S_0 D)P^{-1} \\ &= P \begin{pmatrix} f(S_0 u) & 0 \\ 0 & f(S_0 d) \end{pmatrix} P^{-1} \\ &= f(S_0 u)P_u + f(S_0 d)P_d \end{aligned}$$

is then defined via the general fact 4.3.1.

We also can find the transformed quantum density

$$\rho \Longrightarrow \tilde{\rho} = P\rho P^{-1},$$

The matrix $\tilde{\rho}$ has positive diagonals q_u or q_d. However, $\tilde{\rho}$ is in general not self-adjoint (that is not a proper density).

The option price formula in the model is defined as follows:

$$\begin{aligned} OP(C) &= \frac{tr(\rho f(S_0 H))}{1+r} \\ &= \frac{tr(\rho P_u)f(S_0 u) + tr(\rho P_d)f(S_0 d)}{1+r} \quad \text{(via partition)} \\ &= \frac{tr(\tilde{\rho} f(S_0 D_H))}{1+r} \quad \text{(via transform)} \\ &= \frac{q_u f(S_0 u) + q_d f(S_0 d)}{1+r}. \end{aligned}$$

◇

Here, we introduce more examples about 2×2 matrix case.

Generalized 2×2 bond market We carry on the 2×2 market. Now, our share price is also a bond matrix that is with equal diagonal elements $u = d$.

Case of a self-adjoint bond. The share(bond) price matrix is defined by

$$H = U \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} U^{*.},$$

where U is unitary operator. The state ρ satisfies usual conditions

$$tr(\rho H) = 1 + r, tr(\rho) = 1.$$

Hence, for the transformed density $\tilde{\rho} = U\rho U^*$, the no-arbitrage condition becomes

$$\begin{aligned} tr(\rho H) &= tr\left(\begin{pmatrix} \tilde{\rho_{11}} & \tilde{\rho_{12}} \\ \tilde{\rho_{21}} & \tilde{\rho_{22}} \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\right) \\ &= u(\tilde{\rho_{11}} + \tilde{\rho_{22}}) \\ &= utr(\tilde{\rho}) \\ &= u \\ &= 1 + r. \end{aligned}$$

Besides, $C = f(S_1)$ is the quantum claim for any function f via the general fact 4.3.1 in one step model. Then, we obtain

$$\begin{aligned} OP(f(S_0H)) &= \sum_{j=0}^{0} \frac{f^{(j)}(S_0u)}{j!}\frac{tr(\rho H_u^j G_u)}{1+r} \\ &= \frac{tr(\rho f(S_0H))}{1+r} \\ &= \frac{\rho_{11}f(S_0u) + \rho_{22}f(S_0u)}{1+r} \\ &= \frac{f(S_0u)}{1+r} \\ &= \frac{f(S_0(1+r))}{1+r}. \end{aligned}$$

Case of non self-adjoint bond. Now, we assume that

$$\begin{aligned} H_u &= P\begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}P^{-1} \\ &= PJ_{2,u}P^{-1} \end{aligned}$$

Define the transformed state $\tilde{\rho} = T^{-1}\rho T$. For the no-arbitrage condition, we derive

$$\begin{aligned} tr(\rho H) &= tr\left(\begin{pmatrix} \tilde{\rho_{11}} & \tilde{\rho_{12}} \\ \tilde{\rho_{21}} & \tilde{\rho_{22}} \end{pmatrix}\begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}\right) \\ &= u(\tilde{\rho_{11}} + \tilde{\rho_{22}}) + \tilde{\rho_{21}} \\ &= u + \tilde{\rho_{21}} \\ &= 1 + r. \end{aligned}$$

Besides, $C = f(S_1)$ is the quantum claim for any function f via the general fact 4.3.1 in one step model. Then, we obtain

$$\begin{aligned} OP(f(S_0H)) &= \frac{tr(\tilde{\rho} f(S_0 J_{2,u}))}{1+r} \\ &= \frac{tr\left(\begin{pmatrix} \tilde{\rho_{11}} & \tilde{\rho_{12}} \\ \tilde{\rho_{21}} & \tilde{\rho_{22}} \end{pmatrix}\begin{pmatrix} f(S_0u) & S_0 f'(S_0u) \\ 0 & f(S_0u) \end{pmatrix}\right)}{1+r} \\ &= \frac{\tilde{\rho_{11}} f(S_0u) + \tilde{\rho_{22}} f(S_0u) + \tilde{\rho_{21}} S_0 f'(S_0u)}{1+r}. \end{aligned}$$

Case 2. Diagonalizable market with two eigenvalues.

Consider a quantum market, two quantum states on the Euclidean space $E = M_{k\times k}$, the share price is modelled by the dynamics

$$S_1 = S_0 H$$

where operator H has two quantum observable with eigenvalues u and d. They represent the change of share price with jumps u (jump up) or d (jump down). We assume that they have a general arbitrary multiplicities say, k_u and k_d, respectively, such that $k_u + k_d = k$.

The risk-neutral world of the quantum model (B, S) consists of states ρ on H satisfying

$$\begin{aligned} tr(\rho H) &= utr(\rho P_u) + dtr(\rho P_d) && (5.4) \\ &= 1 + r, && (5.5) \end{aligned}$$

Notice that (5.4) together with the density assumption $tr(\rho) = 1$ implies that

$$tr(\rho P_u) = q_u = \frac{1 + r - d}{u - d}, tr(\rho P_d) = 1 - q_u. \qquad (5.6)$$

The general quantum claim is $C = f(S_1)$, i.e. a function of operator H. Notice the quantum claim

$$\begin{aligned} f(S_1) &= f(S_0H) && (5.7) \\ &= f(S_0u)P_u + f(S_0d)P_d && (5.8) \end{aligned}$$

is then defined via the general fact 4.3.1.

We refer to this model as a one step non-self adjoint binomial market.

The option price formula in the model is defined as follows:

$$\begin{aligned}
OP(C) &= \frac{tr(\rho f(S_0H))}{1+r} \\
&= \frac{tr(\rho P_u)f(S_0u) + tr(\rho P_d)f(S_0d)}{1+r} \\
&= \frac{q_u f(S_0u) + q_d f(S_0d)}{1+r}.
\end{aligned}$$

Case 3. Non-Diagonalizable market with two eigenvalues.

The difference is now that the share price admits only a Jordan decomposition in general.

If there are several Jordan blocks, then $H = \sum H_i$, and $f(H) = \sum f(H_i)$. For each H, we have $H = H_u + H_d$, where

$$H_u = PJ'_{k,k}P^{-1} = PJ'P^{-1},$$

and

$$H_u = PJ''_{k,k}P^{-1} = PJ''P^{-1},$$

with the properties

$$H_uH_d = PJ'P^{-1}PJ''P^{-1} = PJ'J''P^{-1} \text{ and } J'J'' = J''J' = 0.$$

From the general definition, the option claim is defined by

$$f(S_1) = \sum_{i=1}^{s_u}\sum_{j=0}^{k_i-1} \frac{f^{(j)}(S_0u)}{j!}H_u^j + \sum_{i=1}^{s_d}\sum_{j=0}^{m_i-1} \frac{f^{(j)}(S_0d)}{j!}H_d^j.$$

This implies the following option price formula

$$OP(f(S_1)) = \sum_{i=1}^{s_u}\sum_{j=0}^{k_i-1} \frac{f^{(j)}(S_0u)}{j!}\frac{tr(\rho H_u^j)}{1+r} + \sum_{i=1}^{s_d}\sum_{j=0}^{m_i-1} \frac{f^{(j)}(S_0d)}{j!}\frac{tr(\rho H_d^j)}{1+r}.$$

And the risk neutral condition is define by OP$(S_1) = S_0$.

5.1.3 5×5 non diagonalizable quantum Binomial market

Consider one step market, it consists of a risky asset (share) and non-risky asset (bond). In this example, the return are eigenvalues of 5x5 matrix. ρ is the quantum state, which is a non-negative self-adjoint operator with trace 1. The share price consist of two Jordan Blocks

$$J_{3\times 3} = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 1 \\ 0 & 0 & u \end{pmatrix},$$

and

$$J_{2\times 2} = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}.$$

Then, the share price is defined by

$$H = P \begin{pmatrix} u & 1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 & d \end{pmatrix} P^{-1}.$$

If $C = S_1^2$ is the quantum claim in this situation, we obtain

$$C = PS_0^2 \begin{pmatrix} u & 1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}^2 P^{-1}.$$

The risk-neutral world of the quantum model consists of states ρ on the Euclidean space E satisfying

$$tr(\rho H) = 1 + r.$$

We define the transformed quantum state $\tilde{\rho} = P\rho P^{-1}$, which has trace 1 but ρ is not self adjoint in general.

Then, we obtain

$$
\begin{aligned}
& OP(S_1^2) \\
= \; & \frac{tr(\rho S_1^2)}{(1+r)^2} \\
= \; & \frac{tr(\rho(S_0^2 H^2))}{(1+r)^2} \\
= \; & \frac{tr\left(\begin{pmatrix} q_1 & \tilde{\rho}_{12} & \tilde{\rho}_{13} & \tilde{\rho}_{14} & \tilde{\rho}_{15} \\ \tilde{\rho}_{21} & q_2 & \tilde{\rho}_{23} & \tilde{\rho}_{24} & \tilde{\rho}_{25} \\ \tilde{\rho}_{31} & \tilde{\rho}_{32} & q_3 & \tilde{\rho}_{34} & \tilde{\rho}_{35} \\ \tilde{\rho}_{41} & \tilde{\rho}_{42} & \tilde{\rho}_{43} & q_4 & \tilde{\rho}_{45} \\ \tilde{\rho}_{51} & \tilde{\rho}_{52} & \tilde{\rho}_{53} & \tilde{\rho}_{54} & q_5 \end{pmatrix} \begin{pmatrix} S_0^2 u^2 & S_0^2 2u & 1 & 0 & 0 \\ 0 & S_0^2 u^2 & S_0^2 2u & 0 & 0 \\ 0 & 0 & S_0^2 u^2 & 0 & 0 \\ 0 & 0 & 0 & S_0^2 d^2 & S_0^2 2d \\ 0 & 0 & 0 & 0 & S_0^2 d^2 \end{pmatrix}\right)}{(1+r)} \\
= \; & \frac{1}{(1+r)}(q_1 S_0^2 u^2 + \tilde{\rho}_{21} S_0^2 2u + q_2 S_0^2 u^2 + \tilde{\rho}_{31} + \tilde{\rho}_{32} S_0^2 2u + q_3 S_0^2 u^2 \\
& + q_4 S_0^2 d^2 + \tilde{\rho}_{54} S_0^2 2d + q_5 S_0^2 d^2).
\end{aligned}
$$

5.1.4 Commutative market

Consider a quantum Market, it consists of a risky asset (share) and a non-risky asset (bond). They have been presented as two self-adjoint operators defined on the Euclidean space E, where H is quantum share price and ρ is quantum state. The interest rate r is fixed. $1+r$ is a return on the non-risky asset. We apply physical symmetry to compute the one step option pricing. In our market definition, all three components ρ (state operator) and H (share price operator) are interchangeable. And r is fixed.

Notice that $\rho^* = \rho$, $H^* = H$ and $\rho, H \geq 0$. Then, we have a global risk neutral condition $tr(\rho H) = 1 + r$.

The general quantum claim is $C = f(S_0 H)$, i.e. a function of H.

Consider a quantum Market, assume that ρ and H are commutative. And r is fixed. Then,

$$
\begin{aligned}
\rho &= U^* D_\rho U \texttt{ and } H = U^* D_H U, \\
f(S_0 H_1) &= U^* f(S_0 D_H) U = f(S_1).
\end{aligned}
$$

Besides, $C = f(S_1)$ is the quantum claim for any function f via the general fact 4.3.1 in one step model.

Under the no arbitrage condition, we have $tr(\rho H) = 1 + r$. Then, we obtain

$$\begin{aligned} OP(f(S_1)) &= tr(\rho f(S_0 H)\frac{1}{1+r}) \\ &= tr(U^* D_\rho U U^* f(S_1) U \frac{1}{1+r}) \\ &= tr(D_\rho f(S_0 D_H)\frac{1}{1+r}) \\ &= \sum d^\rho f(S_0 d^{(H)})\frac{1}{1+r}. \end{aligned}$$

Example based on data analysis (estimate α_r) In this section, we introduce how to mimics the interest rate from the market, which means that we need to estimate the coefficient of interest rate. It is $\alpha_r H = 1+r$, where α_r is a proportional share price as an operator. We calculate the average share price of component A_v in terms of a certain index. Then, we have $Ie^{\rho T} = \alpha_r(A_v)I$ where I is the identity operator. Finally, we obtain

$$\alpha_r = \frac{1+r}{e^{\widehat{\ln S_T}}},$$

where

$$\widehat{\ln S_T} = \frac{1}{M}\sum_{k=1}^{M}\frac{1}{N}\sum_{j=1}^{N}\ln(\frac{S_k^{(j)}}{S_{k-1}^{(j)}}).$$

Notice that N is the number of stocks, M is the number of days, and S is stock price for a certain stock.

We take the stock price of component of FTSE 100 for last month and DAX (30 shares) for last year from the market. Applying the bank interest rate from Bank of England (0.0025), we obtain the coefficients of interest are 1.0022 and 1.0020, respectively.

Remark 5.1. For more details of the numerical implementation, it has been represented in the section 6.3.

5.2 Two period Quantum Market

In the following content, we use a quantum model of mutil-period binomial markets to re-deduce the Cox-Ross-Rubinstein binomial option pricing formula. We start from the two step model as follows.

5.2.1 Two period Quantum Market

There are several quantum states, which are on the Euclidean space $E = M$ are the bond $B = (B_0, B_1, B_2)$ and the share price modelled by the dynamics $S = (S_0, S_1, S_2)$:

$$\begin{aligned} S_2 &= S_0 H_1 \otimes H_2 \\ &= S_0 \begin{pmatrix} u_1 & 0 \\ 0 & d_1 \end{pmatrix} \otimes \begin{pmatrix} u_2 & 0 \\ 0 & d_2 \end{pmatrix} \\ &= S_0 \begin{pmatrix} u_1 u_2 & 0 & 0 & 0 \\ 0 & u_1 d_2 & 0 & 0 \\ 0 & 0 & d_1 u_2 & 0 \\ 0 & 0 & 0 & d_1 d_2 \end{pmatrix}. \end{aligned}$$

where H_i is self-adjoint 2×2 matrices, which represents the changing of share price with jumps u_i (jump up) or d_i (jump down), $i = 1, 2$. For the quantum model, u_i and d_i are diagonal elements.

Observe that since $H_i^* = H_i$

$$\begin{aligned} S_2^* &= (S_0 H_1 \otimes H_2)^* \\ &= S_0 H_1^* \otimes H_2^* \\ &= S_2. \end{aligned}$$

So, S_2 is a self-adjoint operator with non-negative diagonal elements, z_j, where $j = 1, 2$. For each z_j, it is u_j or d_j. Hence, quantum share price S_2 is self-adjoint non-negative operator.

By the general fact 4.2.1, we can introduce the quantum claim $C = f(S_2)$ for any function f via the following general formula. In our case,

$$\begin{aligned} S_2 &= S_0 H_1 \otimes H_2 &(5.9)\\ &= U S_0 D_1 \otimes D_2 U^* \\ &\Longrightarrow f(S_2) = U f(S_0 D_1 \otimes D_2) U^*. &(5.10) \end{aligned}$$

The two-period binomial model is then introduced a quantum state ρ in the Euclidean space $E = R^{2^2}$ of dimension 2^2 as the tensor product

$$\begin{aligned} \rho &= \rho_1 \otimes \rho_2 \\ &= \left(\begin{array}{cc} q_u^1 & 0 \\ 0 & q_d^1 \end{array} \right) \otimes \left(\begin{array}{cc} q_u^2 & 0 \\ 0 & q_d^2 \end{array} \right) \\ &= \begin{pmatrix} q_u^1 q_u^2 & 0 & 0 & 0 \\ 0 & q_u^1 q_d^2 & 0 & 0 \\ 0 & 0 & q_d^1 q_u^2 & 0 \\ 0 & 0 & 0 & q_d^1 q_d^2 \end{pmatrix}. \end{aligned}$$

Note that from standard definition, $\rho_i = \rho_i^*$ are self-adjoint non-negative 2×2 matrices such that

$$tr(\rho_1) = tr(\rho_2) = 1.$$

Note that from standard definition, $\rho_i = \rho_i^*$ are self-adjoint non-negative 2×2 matrices such that

$$tr(\rho_i) = 1, i = 1, 2.$$

Then, by the property of self-adjoint operator, we obtain

$$\begin{aligned} \rho^* &= (\rho_1 \otimes \rho_2)^* \\ &= \rho_1^* \otimes \rho_2^* \\ &= \rho. \end{aligned}$$

Moreover, ρ is a non-negative self-adjoint operator because ρ has diagonal elements y_j, where $j = 1, 2$. And it is q_u^j or q_d^j. Also, note that

$$\begin{aligned} tr(\rho) &= tr(\rho_1 \otimes \rho_2) \\ &= tr(\rho_1)tr(\rho_2) \\ &= 1. \end{aligned}$$

Hence, ρ is a proper quantum state.

The risk-neutral world of the quantum model (B, S) consists of self-adjoint non-negative 2×2 matrices ρ (referred to as states) satisfying

$$tr(\rho_1 H_1) = 1 + r_1, \text{ and } tr(\rho_2 H_2) = 1 + r_2. \tag{5.11}$$

Furthermore, we define the Transformed quantum state.

$$H_i = U_i^* D_i U_i \text{ and } \rho_i \Longrightarrow U_i \rho_i U_i^*, \text{ where i=1,2.} \tag{5.12}$$

Note that

$$\begin{aligned} \tilde{\rho_i}^* &= (U_i \rho U_i^*)^* \\ &= (U_i^*)^* \rho_i^* U_i^* = U_i \rho_i^* U_i^* \\ &= \tilde{\rho_i}, \text{ where i=1,2.} \end{aligned}$$

We obtain the first equation by the property of unitary operator and trace.

And then,

$$\begin{aligned} tr(\tilde{\rho_i}) &= tr(U_i \rho_i U_i^*) \\ &= tr(U_i^* U_i \rho_i) \\ &= tr(\rho_i) = 1, \text{ where i=1,2.} \end{aligned}$$

Under the property of trace, we obtain the first equation. Then, we apply the trace rule for the second equation. And we obtain the third equation because of the definition of transformed quantum state. The final equation derived by the definition of quantum state.

Finally, we obtain

$$\begin{aligned} < \tilde{\rho}_i x, x > &= < U\rho_i U^* x, x > \\ &= < \rho_i U^* x, U^* x > \\ &= < \rho_i y, y > \\ &\geq 0, \end{aligned}$$

where $U^*x = y$ and since $\rho_i \geq 0$. We obtain the first and second equation because of the property of unitary operator. And $\tilde{\rho}_i$ have non-negative diagonal elements $q_u^{(i)}, q_d^{(i)}$ and have a representation

$$\tilde{\rho}_i = \begin{pmatrix} q_u^{(i)} & \bar{x} \\ x & q_d^{(i)} \end{pmatrix}, \text{ where } i = 1, 2.$$

Transformed quantum state is then defined by

$$\begin{aligned} \widetilde{\rho} &= U^*(\rho_1 \otimes \rho_2)U \\ &= (U_1 \otimes U_2)(\rho_1 \otimes \rho_2)(U_1^* \otimes U_2^*) \\ &= (U_1\rho_1U_1^*) \otimes (U_2\rho_2U_2^*) \qquad (5.13) \\ &= \begin{pmatrix} \tilde{q_u}^1\tilde{q_u}^2 & \tilde{x} & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{q_u}^1\tilde{q_d}^2 & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{x} & \tilde{q_d}^1\tilde{q_u}^2 & \tilde{x} \\ \tilde{x} & \tilde{x} & 0 & \tilde{q_d}^1\tilde{q_d}^2 \end{pmatrix}. \qquad (5.14) \end{aligned}$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Finally, we derive the last equation the property of tensor product.

Notice that tensor product of positive self-adjoint operators is positive self-adjoint. Hence, $\widetilde{\rho}$ is positive self-adjoint operator with $tr(\rho) = 1$. Moreover, the quantum no arbitrage condition (5.21) is satisfied for the transformed density:

$$tr(\tilde{\rho}_1 D_1) = 1 + r_1, \text{ and } tr(\tilde{\rho}_2 D_2) = 1 + r_2. \qquad (5.15)$$

Let

$$D = D_1 \otimes D_2.$$

In addition, we have

$$
\begin{aligned}
tr(\widetilde{\rho}H) &= tr(\widetilde{\rho}_1 \otimes \widetilde{\rho}_2 H_1 \otimes H_2) && (5.16)\\
&= tr(\widetilde{\rho}_1 H_1)tr(\widetilde{\rho}_2 H_2) \\
&= tr(\widetilde{\rho}_1 D_1)tr(\widetilde{\rho}_2 D_2) && (5.17)\\
&= (1+r_1)(1+r_2). && (5.18)
\end{aligned}
$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Also, we derive the last equation the property of unitary operator. Finally, we obtain the final equation via the no-arbitrary condition.

The arbitrage free time 0, the general option price of quantum option claim $C = f(S_2)$ for generalised two-period quantum binomial model is defined by

$$
\begin{aligned}
OP(f(S_2)) &= \frac{tr(\rho f(S_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\rho_1 \otimes \rho_2 f(S_0 H_1 \otimes H_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\rho_1 \otimes \rho_2 U^* f(S_0 D_1 \otimes D_2) U)}{(1+r_1)(1+r_2)} \\
&= \frac{tr(U^*(\rho_1 \otimes \rho_2) U f(S_0 D_1 \otimes D_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\widetilde{\rho} U f(S_0 D_1 \otimes D_2))}{(1+r_1)(1+r_2)} \\
&= \frac{1}{(1+r_1)(1+r_2)} tr(\begin{pmatrix} \tilde{q_u}^1 \tilde{q_u}^2 & \tilde{x} & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{q_u}^1 \tilde{q_d}^2 & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{x} & \tilde{q_d}^1 \tilde{q_u}^2 & \tilde{x} \\ \tilde{x} & \tilde{x} & 0 & \tilde{q_d}^1 \tilde{q_d}^2 \end{pmatrix} \\
&\quad \begin{pmatrix} f(S_0 u_1 u_2) & 0 & 0 & 0 \\ 0 & f(S_0 u_1 d_2) & 0 & 0 \\ 0 & 0 & f(S_0 d_1 u_2) & 0 \\ 0 & 0 & 0 & f(S_0 d_1 d_2) \end{pmatrix}) \\
&= \frac{f(S_0 u_1 u_2) q_u^1 q_u^2 + f(S_0 u_1 d_2) q_u^1 q_d^2 + f(S_0 d_1 u_2) q_u^1 q_d^2 + f(S_0 d_1 d_2) q_d^1 q_d^2}{(1+r_1)(1+r_2)}.
\end{aligned}
$$

We obtain the second equation via the general 4.2.1. And the third equation has been obtain by the trace property. And then, we obtain the fourth equation by the definition of transformed quantum state.

5.2.2 Two period Quantum Market for non self-adjoint case

There are several quantum states, which are on the Euclidean space $E = M$ are the bond $B = (B_0, B_1, B_2)$ and the share price modelled by the dynamics $S = (S_0, S_1, S_2)$:

$$\begin{aligned} S_2 &= S_0 H_1 \otimes H_2 \\ &= S_0 \left(\begin{array}{cc} u_1 & 1 \\ 0 & d_1 \end{array} \right) \otimes \left(\begin{array}{cc} u_2 & 1 \\ 0 & d_2 \end{array} \right) \\ &= S_0 \begin{pmatrix} u_1u_2 & u_1 & u_2 & 1 \\ 0 & u_1d_2 & 0 & d_2 \\ 0 & 0 & d_1u_2 & d_1 \\ 0 & 0 & 0 & d_1d_2 \end{pmatrix}. \end{aligned}$$

where H_i is non self-adjoint 2×2 matrices, which represents the changing of share price with jumps u_i (jump up) or d_i (jump down), $i = 1, 2$. For the quantum model, u_i and d_i are eigenvalues.

Observe that since $H_i^* = H_i$

$$\begin{aligned} S_2^* &= (S_0 H_1 \otimes H_2)^* \\ &= S_0 H_1^* \otimes H_2^* \\ &= S_2. \end{aligned}$$

So, S_2 is a self-adjoint operator with non-negative diagonal elements, z_j, where $j = 1, 2$. For each z_j, it is u_j or d_j. Hence, quantum share price S_2 is self-adjoint non-negative operator.

By the general fact 4.2.1, we can introduce the quantum claim $C = f(S_2)$ for any function f via the following general formula. In our case,

$$\begin{aligned} S_2 &= S_0 H_1 \otimes H_2 && (5.19) \\ &= U S_0 D_1 \otimes D_2 U^* \\ &\Longrightarrow f(S_2) = U f(S_0 D_1 \otimes D_2) U^*. && (5.20) \end{aligned}$$

The two-period binomial model is then introduced a quantum state ρ in the Euclidean space $E = R^{2^2}$ of dimension 2^2 as the tensor product

$$\begin{aligned}
\rho &= \rho_1 \otimes \rho_2 \\
&= \left(\begin{array}{cc} q_u^1 & 0 \\ 0 & q_d^1 \end{array} \right) \otimes \left(\begin{array}{cc} q_u^2 & 0 \\ 0 & q_d^2 \end{array} \right) \\
&= \begin{pmatrix} q_u^1 q_u^2 & 0 & 0 & 0 \\ 0 & q_u^1 q_d^2 & 0 & 0 \\ 0 & 0 & q_d^1 q_u^2 & 0 \\ 0 & 0 & 0 & q_d^1 q_d^2 \end{pmatrix}.
\end{aligned}$$

Note that from standard definition, $\rho_i = \rho_i^*$ are self-adjoint non-negative 2×2 matrices such that

$$tr(\rho_1) = tr(\rho_2) = 1.$$

Note that from standard definition, $\rho_i = \rho_i^*$ are self-adjoint non-negative 2×2 matrices such that

$$tr(\rho_i) = 1, i = 1, 2.$$

Then, by the property of self-adjoint operator, we obtain

$$\begin{aligned}
\rho^* &= (\rho_1 \otimes \rho_2)^* \\
&= \rho_1^* \otimes \rho_2^* \\
&= \rho.
\end{aligned}$$

Moreover, ρ is a non-negative self-adjoint operator because ρ has diagonal elements y_j, where $j = 1, 2$. And it is q_u^j or q_d^j. Also, note that

$$\begin{aligned}
tr(\rho) &= tr(\rho_1 \otimes \rho_2) \\
&= tr(\rho_1) tr(\rho_2) \\
&= 1.
\end{aligned}$$

Hence, ρ is a proper quantum state.

The risk-neutral world of the quantum model (B, S) consists of self-adjoint non-negative 2×2 matrices ρ (referred to as states) satisfying

$$tr(\rho_1 H_1) = 1 + r_1, \text{ and } tr(\rho_2 H_2) = 1 + r_2. \tag{5.21}$$

Furthermore, we define the Transformed quantum state.

$$H_i = U_i^* D_i U_i \text{ and } \rho_i \Longrightarrow U_i \rho_i U_i^*, \text{ where i=1,2.} \tag{5.22}$$

Note that

$$\begin{aligned} \tilde{\rho_i}^* &= (U_i \rho U_i^*)^* \\ &= (U_i^*)^* \rho_i^* U_i^* = U_i \rho_i^* U_i^* \\ &= \tilde{\rho}_i, \text{ where i=1,2.} \end{aligned}$$

We obtain the first equation by the property of unitary operator and trace.

And then,

$$\begin{aligned} tr(\tilde{\rho}_i) &= tr(U_i \rho_i U_i^*) \\ &= tr(U_i^* U_i \rho_i) \\ &= tr(\rho_i) = 1, \text{ where i=1,2.} \end{aligned}$$

Under the property of trace, we obtain the first equation. Then, we apply the trace rule for the second equation. And we obtain the third equation because of the definition of transformed quantum state. The final equation derived by the definition of quantum state.

Finally, we obtain

$$\begin{aligned} < \tilde{\rho}_i x, x > &= < U \rho_i U^* x, x > \\ &= < \rho_i U^* x, U^* x > \\ &= < \rho_i y, y > \\ &\geq 0, \end{aligned}$$

where $U^* x = y$ and since $\rho_i \geq 0$. We obtain the first and second equation because of the property of unitary operator. And $\tilde{\rho}_i$ have non-negative diagonal elements

$q_u^{(i)}, q_d^{(i)}$ and have a representation

$$\tilde{\rho}_i = \begin{pmatrix} q_u^{(i)} & \bar{x} \\ x & q_d^{(i)} \end{pmatrix}, \text{ where } i = 1, 2.$$

Transformed quantum state is then defined by

$$\begin{aligned} \widetilde{\rho} &= U^*(\rho_1 \otimes \rho_2)U \\ &= (U_1 \otimes U_2)(\rho_1 \otimes \rho_2)(U_1^* \otimes U_2^*) \\ &= (U_1\rho_1U_1^*) \otimes (U_2\rho_2U_2^*) & (5.23) \\ &= \begin{pmatrix} \tilde{q_u}^1\tilde{q_u}^2 & \tilde{x} & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{q_u}^1\tilde{q_d}^2 & \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{x} & \tilde{q_d}^1\tilde{q_u}^2 & \tilde{x} \\ \tilde{x} & \tilde{x} & 0 & \tilde{q_d}^1\tilde{q_d}^2 \end{pmatrix}. & (5.24) \end{aligned}$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Finally, we derive the last equation the property of tensor product.

Notice that tensor product of positive self-adjoint operators is positive self-adjoint. Hence, $\widetilde{\rho}$ is positive self-adjoint operator with $tr(\rho) = 1$. Moreover, the quantum no arbitrage condition (5.21) is satisfied for the transformed density:

$$tr(\tilde{\rho}_1 D_1) = 1 + r_1, \ \texttt{and} \ tr(\tilde{\rho}_2 D_2) = 1 + r_2. \tag{5.25}$$

Let

$$D = D_1 \otimes D_2.$$

In addition, we have

$$\begin{aligned} tr(\widetilde{\rho}H) &= tr(\widetilde{\rho}_1 \otimes \widetilde{\rho}_2 H_1 \otimes H_2) & (5.26) \\ &= tr(\widetilde{\rho}_1 H_1)tr(\widetilde{\rho}_2 H_2) \\ &= tr(\widetilde{\rho}_1 D_1)tr(\widetilde{\rho}_2 D_2) & (5.27) \\ &= (1 + r_1)(1 + r_2). & (5.28) \end{aligned}$$

We obtain the first equation by the definition of tensor product. And we apply the property of tensor product to obtain the second equation. Also, we derive the last equation the property of unitary operator. Finally, we obtain the final equation via the no-arbitrary condition.

The arbitrage free time 0, the general option price of quantum option claim $C = f(S_2)$ for generalised two-period quantum binomial model is defined by

$$\begin{aligned}
OP(f(S_2)) &= \frac{tr(\rho f(S_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\rho_1 \otimes \rho_2 f(S_0 H_1 \otimes H_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\rho_1 \otimes \rho_2 U^* f(S_0 D_1 \otimes D_2)U)}{(1+r_1)(1+r_2)} \\
&= \frac{tr(U^*(\rho_1 \otimes \rho_2)U f(S_0 D_1 \otimes D_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\widetilde{\rho} U f(S_0 D_1 \otimes D_2))}{(1+r_1)(1+r_2)} \\
&= \frac{tr(\tilde{\rho} \begin{pmatrix} f(S_0u_1u_2) & S_0\frac{f'(S_0u_1)}{1!} & S_0^2\frac{f''(S_0u_2)}{2!} & S_0^3\frac{f'''(S_0)}{3!} \\ 0 & f(S_0u_1d_2) & S_0\frac{f'(S_0u_1)}{1!} & S_0^2\frac{f''(S_0u_2)}{2!} \\ 0 & 0 & f(S_0d_1u_2) & S_0\frac{f'(S_0u_1)}{1!} \\ 0 & 0 & 0 & f(S_0d_1d_2) \end{pmatrix})}{(1+r_1)(1+r_2)} \\
&= \frac{1}{(1+r_1)(1+r_2)}(f(S_0u_1u_2)\tilde{q_u}^1\tilde{q_u}^2 + S_0\frac{f'(S_0u_1)}{1!}\tilde{q_u}^1\tilde{q_u}^2 \\
&\quad + S_0^2\frac{f''(S_0u_2)}{2!}\tilde{q_u}^1\tilde{q_u}^2 + S_0^3\frac{f'''(S_0)}{3!}\tilde{q_u}^1\tilde{q_u}^2 + f(S_0u_1d_2)\tilde{q_u}^1\tilde{q_d}^2 \\
&\quad + S_0\frac{f'(S_0u_1)}{1!}\tilde{q_u}^1\tilde{q_d}^2 + S_0^2\frac{f''(S_0u_2)}{2!}\tilde{q_u}^1\tilde{q_d}^2 + f(S_0d_1u_2)\tilde{q_u}^2\tilde{q_d}^1 \\
&\quad + S_0\frac{f'(S_0u_1)}{1!}\tilde{q_u}^2\tilde{q_d}^1 + f(S_0d_1d_2)\tilde{q_d}^1\tilde{q_d}^2).
\end{aligned}$$

We obtain the second equation via the general 4.2.1. And the third equation has been obtain by the trace property. And then, we obtain the fourth equation by the definition of transformed quantum state.

5.3 Conculsion

In this chapter, we give several examples for quantum model. We start from one-step quantum model to two-step quantum model and consider both self-adjoint and non self-adjoint matrix case. Also, we consider the 5×5 non-diagonalizable matrix case in a certain quantum Binomial market. In this case, we obtain a slightly different answer from the self-adjoint matrix case because we got extra elements from off diagonal elements.

Chapter 6

Numerical implementation

6.1 Introduction

Alternative approach to derive the Option price for the Geometric Brownian motion model is provided by the Feynman Path integral [22].

The Feymann Path Integral has been proven as a powerful tool for analytical and computational studies of random system [23] and [22].This chapter is an introduction to the application of path integrals to option pricing. Instead of studying Hamiltonian operator, we shift it to the study of Lagrangian, which is a fundamental mathematical structure in the path-integral formulation of quantum mechanics [51] and [52].

6.2 Path integral for Black-Scholes Lagrangian

To find the pricing kernel

$$p(x, \tau; x') =< x|e^{-\tau H}|x' > . \tag{6.1}$$

for $\tau = T - t$, we discretize the time by choosing the N time steps and letting $x_i = x(t_i)$, where $t_i = i\epsilon$ and $0 \leq i \leq N$.

Then, (6.1) reduces to

$$p(x; \tau|x') = (\prod_{i=1}^{N-1} \int dx_i) \prod_{i=1}^{N} < x_i|e^{-\epsilon H}|x_{i-1} >$$

with boundary conditions

$$x_N = x, x_0 = x'.$$

Applying the Feynman formula for the Hamiltonian

$$< x_i|e^{-\epsilon H}|x_{i-1} >= \mathcal{N}_i(\epsilon)e^{\epsilon L(x_i;x_{i-1};\epsilon)}$$

where $\mathcal{N}_i(\epsilon) = \frac{1}{\sqrt{2\pi\sigma^2\epsilon}}$ is normalization constant and $L(x_i; x_{i-1}; \epsilon) = -\frac{1}{2\epsilon\sigma^2}((x(i) - x(i-1) - \epsilon(r - \frac{\sigma^2}{2}))^2)$ is the corresponding Lagrangian for the system [3].

Overall, the pricing kernal as follows:

$$p(x;\tau;x') = \int DXe^S, \texttt{ where the action is given by } S = \epsilon \sum_{i=1}^{N} L(x_i; x_{i-1}; \epsilon), \tag{6.2}$$

and the path-integration measure on R^{N-1} is given by

$$\int_A DX = \int_A \mathcal{N}_N(\epsilon) \prod_{i=1}^{N-1} \mathcal{N}_i(\epsilon)dx_i.$$

Finally, we provide the comparison of numerical calculations for the European call option with strike price $K = 100$, interest rate $r = 0.006$, volatility $\sigma = 0.2$, and expiry date $T = 1$ via Black-Scholes formula, Binomial model formula and the path integral (6.2).

Numerical calculations show that Binomial Model formula and Path integral give are close to the Black-Scholes formula even for relatively small $n \sim 40$.

6.3 Calibration for European option by Fourier techniques and Monte Carlo compared with Black-Scholes model and Levy model

In this section, it follows the book by Svetlana Boyarchenko and Sergei Levendorskii (2002).

Assume that under a risk-neutral measure Q chosen for pricing of options on the underlying stock, $\tilde{B_T}$ is a planar Brownian motion, and consider a contingent claim with the deterministic life span $[0, T]$ and terminal payoff $G(\tilde{B_T})$. Assume further that for some $\omega \in (\lambda_-, \lambda_+)$, function $G_\omega(x) := e^{\omega x}G(x)$ belongs to $L_1(R)$. Then we can decompose G into the Fourier integral

$$G(x) = \frac{1}{2\pi}\int_{Im\xi=\omega} e^{ix\xi}\hat{G}(\xi)d\xi, \texttt{where } \hat{G}(\xi) = \int_R = e^{-ix\xi}G(x)d\xi$$

is the Fourier image of G. Substituting it into the pricing formula

$$V(t,x) = E^Q[e^{-r\tau}G(\tilde{B_T})|\tilde{B}_t = x]$$

and changing the order of taking expectation and integration, we obtain

$$\begin{aligned} V(z_s) &= E^Q[e^{-r\tau}G(\tilde{B}_T)|\tilde{B}_t = x] \\ &= E^Q[\frac{e^{-r(T-t)}}{2\pi}\int_{Im\xi=\omega} e^{i\tilde{B}_T\xi}\hat{G}(\xi)d\xi|\tilde{B}_t = x] \\ &= \frac{1}{2\pi}\int_{Im\xi=\omega} e^{ix\xi}e^{-r(T-t)}E^Q[exp(-i\xi e^{i\tilde{B}_\alpha+\sigma})]\hat{G}(\xi)d\xi. \end{aligned}$$

For simplicity, let $\lambda = 0$, we consider the limit of $tr(\rho H^{\otimes n})$ by moments problem. Given a_n, we need to find whether there exists μ so that $a_n = \int x^n d\mu_n$. We know $a_{2N} = EJ^{2N} \geq (EJ^N)^2$, $a_{2N} \geq a_N^2$. It is not true if $\sigma > 0$. Thus, $e^{2N\sigma-(2N)^2\sigma} \geq (e^{N\sigma-N^2\sigma})^2 = e^{2N\sigma-2N^2\sigma} \to e^{2N^2\sigma} \leq 1 \to \sigma \leq 0$, which means that there is no real probability measure and no distribution.

We try to use Cauchy transform, but, it fails to get a explicit result, that's why we calculate

$$E^Q[exp(-i\xi e^{i\tilde{B}_\alpha+\sigma})] = [\frac{1}{Q}\sum_{j=1}^{Q} exp(-i\xi e^{i\xi z_j\sqrt{\alpha}\sigma})]$$

by Monte Carlo, where $z_j \sim (0,1)$, $Q = 2^6$, and $\sigma = 0.35$.

$$\begin{aligned} &= \frac{1}{2\pi}\int_{Im\xi=\omega} e^{ix\xi}e^{-r(T-t)}[\frac{1}{Q}\sum_{j=1}^{Q} exp(-i\xi e^{i\xi z_j\sqrt{\alpha}\sigma})]\hat{G}(\xi)d\xi \\ &= \frac{1}{M\Delta}\sum_{k=1}^{M} e^{ix_s\xi_k}e^{-r(T-t)}[\frac{1}{Q}\sum_{j=1}^{Q} exp(-i\xi_k e^{i\xi z_j\sqrt{\alpha}\sigma})]\hat{G}(\xi)d\xi. \end{aligned}$$

Notice that $\hat{G}(\xi)$ is the Fourier image of G, we calculate as follows:

$$\begin{aligned}
\hat{G}(\xi) &= \int_R e^{-ix\xi}G(x)dx \\
&= \int_R e^{-ix\xi}(x-K)_+dx \\
&= \int_{\ln K}^{+\infty}(e^{-ix\xi}x - e^{-ix\xi}K)dx \\
&= \frac{1}{i\xi}xe^{-i\xi\ln K} - \frac{1}{\xi^2}e^{-i\xi\ln K} + C \\
&\to (\frac{\ln K}{i\xi} - \frac{1}{\xi^2})e^{-i\xi\ln K}.
\end{aligned}$$

Then, we obtain

$$\to \frac{1}{M\Delta}\sum_{k=1}^{M} e^{ix_s\xi_k}e^{-r(T-t)}[\frac{1}{Q}\sum_{j=1}^{Q} exp(-i\xi_k e^{i\xi z_j\sqrt{\alpha}\sigma})][(\frac{lnK}{i\xi_k} - \frac{1}{\xi_k^2}) * e^{-i\xi_k lnK}],$$

where $\hat{G}(\xi)$ is the Fourier image of G.

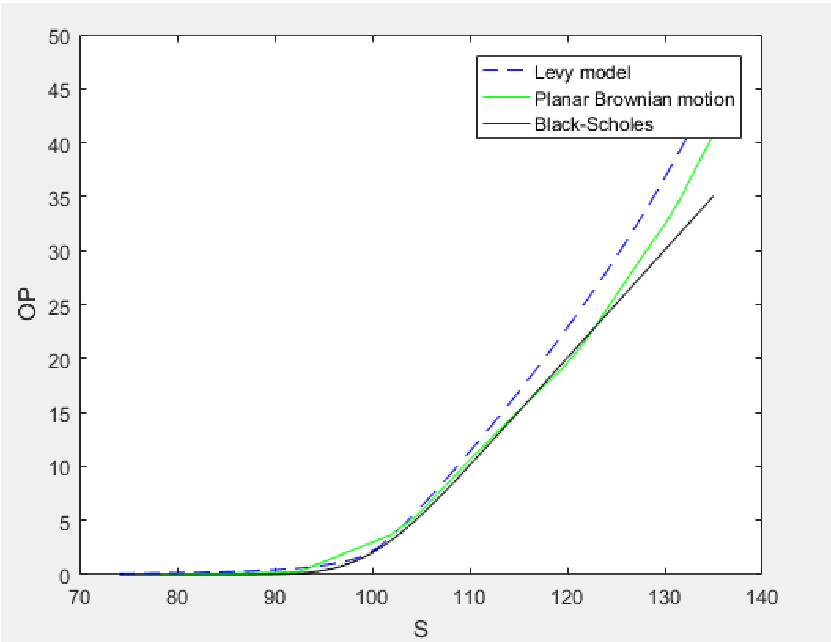

FIGURE 6.1: Calibration for European option by Fourier techniques and Monte Carlo compared with Black-Scholes model and Levy model. Strike price K = 100, riskless rate r = 0.04, volatility σ = 0.35, Maturity date T = 1.

6.4 Calibration for the coefficient of interest rate

In this section, we introduce how to mimics the interest rate from the market. Under the market $(\rho, H, 1+r)$, we need to estimate the coefficient of interest rate, which is $\alpha_r H = 1+r$, where α_r is a proportional share price as an operator. We calculate the average share price of component A_v in terms of a certain index. Then, we have $Ie^{\rho T} = \alpha_r(A_v)I$ with the identity operator. Finally, we obtain

$$\alpha_r = \frac{1+r}{e^{\widehat{\ln S_T}}},$$

where

$$\widehat{\ln S_T} = \frac{1}{M}\sum_{k=1}^{M}\frac{1}{N}\sum_{j=1}^{N}\ln(\frac{S_k^{(j)}}{S_{k-1}^{(j)}}).$$

Notice that N is the number of stocks, M is the number of days, and S is stock price for a certain stock.

We take the stock price of component of FTSE 100 for last month and DAX (30 shares) for last year from the market. Applying the bank interest rate from Bank of England (0.0025), we obtain the coefficients of interest are 1.0022 and 1.0020, respectively.

6.5 Conclusion

In this Chapter, it shows several numerical results for quantum model.

Firstly, we did a simulation among of classcical Binomial model, quantum Binomial model, and Black-Scholes model. We found that the results of Binomial model formula and Path integral are close to the Black-Scholes formula even for relatively small $n \sim 40$.

Moreover, we did a calibration for European option by Fourier techniques and Monte Carlo method. In this part, we apply Monte Carlo method for planar Brownian motion model and Levy model. And both of them compared with Black-Scholes model. It shows that under some certain parameters, the result of planar Brownian motion model, which is obtained by taking a limit in the N-period bond market, is better than the result of Levy model for pricing European option.

Chapter 7

Conclusions and Further Research

7.1 Conclusions

Quantum model based on quantum probability, instead of classical probability, which is generalisation for classical probability. In this thesis, we have proved that quantum models do cover the classical non-quantum models. Specifically, we start to extend Chen's work and analyse the quantum conditional option price. We establish the limit of the spectral measures proving the convergence to the geometric Brownian motion model. Moreover, we establish several quantum markets and the related quantum models in the discrete time version. We found that N step bond market can connect to planar Brownian motion model. Furthermore, we found Binomial Model formula and Path integral formulization gave are close to the Black-Scholes formula. And under some certain parameters, the result of planar Brownian motion model is better than the result of Levy model for pricing European option. Besides, we consider the 5×5 non-diagonalizable matrix case in a certain quantum Binomial market and we obtain a slightly different answer from the self-adjoint matrix case because we got extra elements from off diagonal elements.

There are some futher research in the future. For example, we consider the different type of markets. Also, we apply Heisenberg group to derive a new Black-Scholes formula.

7.2 Further Research

7.2.1 General quantum market

Consider a quantum Market as a triple (ρ, H, r), there are three self-adjoint operators defined on the Euclidean space $\mathcal{E}$, where H is quantum share price with the diagonal decomposition

$$H = UDU^*, \text{ where U is an unitary operator,}$$

which has two observable eigenvalues u and d for each step. It represents the changing of share price with jumps u (jump up) or d (jump down). And ρ is quantum state, and the interest rate is r.

Notice that $\rho^* = \rho$, $H^* = H$, $r^* = r$, and $\rho, H, r \geq 0$, r^{-1} exists.

By applying the general fact 4.3.1, we obtain the quantum claim for any function f

$$S_N = S_0H = US_0DU^*$$

$$\rightarrow f(S_N) = Uf(S_0D)U^*.$$

Now, by the similar definition, yields

$$H = UDU^*, \;\; \rho = U\rho U^*, \text{ and } r = UDU^*.$$

And the transformed quantum state is then defined by

$$\tilde{\rho} = U\rho U^*.$$

Notice that again $\tilde{\rho}$ has trace 1 and we have a risk neutral condition $tr(\rho H r^{-1}) = 1$.

In this case, a quantum model for the binomial market B, S is presented. We obtain the generalised one period binomial model.

$$OP(S_N) \;\; = \;\; \sum \tilde{\rho_{ii}} f(S_0 A_{ii}),$$

where $\tilde{\rho}_{ii}$ is the transformed quantum state and A_{ii} is a diagonal matrix of the share price.

7.2.2 Symmetric market

Consider a quantum Market as a triple (ρ, H, r), there are three self-adjoint operators defined on the Euclidean space E, where H is quantum share price and ρ is quantum state. We apply physical symmetry to compute the one step option pricing. In our market definition, all three components ρ (state operator), H (share price operator) and r (interest rate operator) are interchangeable.

7.2.3 OP for Heisenberg product dynamic of Discrete Quantum markets

Motivated by Segal [24], we apply Heisenberg group to establish a 'nova' Black-Scholes formula and found that the interest rate depends on the share price in quantum way.

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