# RUIN PROBABILITY VIA SEVERAL NUMERICAL METHODS

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*Dedicated to*

All good people, regardless of their race, nation or religion.

# RUIN PROBABILITY VIA SEVERAL NUMERICAL METHODS


## Abstract

In this thesis, ruin probabilities of insurance companies are studied. Ruin probability in finite time is considered because it is more realistic compared with infinite time ruin probabilities. However, infinite time methods are also mentioned in order to compare them with the finite time methods.

The thesis will initially provide some information about ruin probability of a risk process in finite and infinite time, and then the Markov chain and quantum mechanics approaches will be shown in order to compute the ruin probability.

Using a reinsurance agreement, which is a risk sharing tool in actuarial science, the ruin probability of a modified surplus process in finite time via the quantum mechanics approach is studied. Furthermore, some optimization problems about capital injections, withdrawals and reinsurance premiums are taken into consideration in order to minimise the ruin probability.

Finally, the thesis compares the finite time method under the reinsurance agreement in terms of the ruin probability and total injections amount with an infinite time counterpart method.

# Papers

- Tamturk, M. and Utev, S., 2017. Ruin probability via Quantum Mechanics Approach. *Insurance: Mathematics and Economics.*
- Tamturk, M. and Utev, S., 2017. Optimum reinsurance via Dirac-Feynman Approach submitted to *Methodology and Computing in Applied Probability*(under review).

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# List of Abbreviations and Symbols

| | |
|---|---|
| $A$ | Transition matrix or operator |
| $\|\alpha\rangle$ | *Ket* representing column vector in quantum mechanics |
| $\langle\alpha\|$ | *Bra* is transpose of *Ket* which representing row vector in quantum mechanics |
| $\langle\alpha, \beta\rangle$ | Inner product in quantum mechanics |
| $c$ | Premium rate |
| $H$ | Hamiltonian operator |
| $I$ | Identity operator |
| IID | Independent and identically distributed |
| $K$ | Shift operator |
| $K_p$ | Eigenvalue of Hamiltonian operator |
| $N(t)$ | Claim number up to time $t$ |
| $p$ | Eigenvector of Hamiltonian operator |
| $Q$ | Generator matrix |
| $R(t)$ | Capital at time $t$ in classical Surplus process |
| $R^*(t)$ | Capital at time $t$ in a modified surplus process by reinsurance |
| $\overline{R(t)}$ | Capital at time $t$ in a modified surplus process by capital injection |
| $S(t)$ | Total claim amount at time $t$ |
| $u$ | Initial capital of an insurance company |

| | |
|---|---|
| $X_j$ | $j-th$ claim amount |
| $Y$ | Expected total injection amount |
| $z$ | Reinsurance premium amount |
| $\lambda$ | Claim frequency rate |

# Chapter 1

# INTRODUCTION

## 1.1 Introduction and Literature Review

All over the world, people are attempting to reduce the probability of any risk in order to improve their life and safety [75]. Simply, insurance is a form of protection for people against something going wrong, but actuarially speaking it is a contract between an insurance company and a policyholder. According to the insurance agreement, insurance companies are responsible for covering policyholders' losses. In other words, the policyholder transfers the risk of financial loss to the insurance company by paying a premium. In actuarial sciences, an insurance company's probability of ruin is an important risk measure.

The following three research areas in the analysis of ruin are important [52]:

(i) ruin time and ruin probability,

(ii) the deficit at ruin,

(iii) the reserve immediately before ruin.

In this thesis, the main focus is on ruin probability.

**Definition 1 (Risk Process)**

*The classical ruin probability is dealing with the classical surplus (or risk) process of* *an insurance company $R(t)$ (or $R_t$), which is defined by [3, 18, 24, 38, 55]*

$$R(t) = u + ct - \sum_{j=1}^{N(t)} X_j$$

*where $u$ is the initial capital of an insurance company, $c$ is a premium rate, $t$ is time, $N(t)$ is the number of claims up to time $t$, $X_j$ is the $j$-th claim amount, and $X$ and $N(t)$ are mutually independent processes. The process is also refereed to as a compound Poisson process when $N(t)$ has a Poisson process and $X$ is i.i.d.*

**Definition 2 (Ruin Probability)**
*The ruin probability is defined via the ruin time by*

$$T = \begin{cases} \min\{t \geq 0 | R(t) < 0\} & \textit{for discrete time}, \\ \inf\{t \geq 0 | R(t) < 0\} & \textit{for continuous time}. \end{cases}$$

*Ruin will occur as soon as the capital of the insurance company becomes negative. In particular, the infinite time ruin probability (also called ultimate ruin) is defined by*

$$P(T < \infty | R(0) = u).$$

*The ruin probability in finite time horizon is defined by*

$$P(T \leq t | R(0) = u).$$

**Definition 3 (Non Ruin Probability)** *The finite time non ruin probability, known as survival probability, is defined by*

$$P(T > t | R(0) = u).$$

The survival probability means that a ruin does not occur until a certain time. It plays a crucial role in this thesis.
Numerical analysis plays an important role in actuarial sciences. There are a number of numerical methods developed to estimate the ruin probability [4, 27, 37], which deal with the infinite time ruin probability even though finite time methods are more

realistic.

In actuarial applications, it is important to tackle modified surplus processes that incorporate financial interferences, such as Capital Allocations, Capital Injections, Withdrawals and Reinsurance. Those financial instruments are widely studied in [28], [57], [80] and [58]. In particular, the finite time ruin probability techniques appear to us to be more powerful in dealing with these characteristics in terms of reality.

The traditional techniques of finite and infinite time ruin probabilities are based on classical probability analysis such as the Markov Chain argument. The quantum mechanics approach provides an alternative powerful tool. Although the method became more popular in financial mathematics [5,6], there are only scattered applications in actuarial sciences [44,45].

This thesis suggests two numerical approaches in order to compute the finite time ruin probability. The first method is based on a modification of the traditional Markov Chain approach [10,11,72]. The second method is based on the Dirac Matrix and Feynman Path calculations method [5,6,60]. These two approaches are successfully applied to compute the finite time ruin probability with and without capital injections and withdrawals.

For the classical surplus process, Picard and Lefevre [61] suggested a powerful approach to computing finite time ruin probability for integer claim sizes. This approach is based on Appell polynomial expansions and so is referred to as the Appell polynomial approach. The method was modified by Ignatov et al. [37].

The numerical results derived from the quantum mechanics approach for finite time ruin and non ruin probabilities are compared with the Appell polynomials approach [37,61,63,69], and a modification of the traditional Markov chain approach [10,11,72] in this thesis.

Many optimization problems have been studied in actuarial science [8,25,30,33]. Similarly, it is dealt with in this thesis by applying the quantum mechanics approach in order to solve numerical capital allocation type problems in actuarial science, such as

- How to maximize the proportion of the total claim amount paid with the prescribed ruin level,

- How to minimize the ruin probability via the optimization of the time and amounts of capital allocation of investments and withdrawals,

- How to minimize the ruin probability via optimization of allocation of initial capitals.

In this thesis, reinsurance agreements are also taken into consideration. Therefore, the computation of ruin probability of the modified surplus process with reinsurance, and the optimal reinsurance via the Dirac-Feynman approach will also be examined. Reinsurance is a risk-sharing arrangement between a primary insurer and a reinsurer. There are different types of reinsurance agreements and various optimality approaches to reinsurance, including those of Castaner, Claramunt and Lefevre [12], Denuit and Vermandele [20], Dickson and Waters [22], Ignatov, Kaishev and Krachunov [39], Kaishev and Dimitrova [40], Schmidli [28], and Zhou and Yuen [80].

The ruin probability of the modified surplus process with reinsurance by capital injections attracted the interest of several academics, such as Nie et al. [57,58]. In this thesis, the following reinsurance agreement motivated by Nie et al. is considered: the insured companies pay reinsurance premiums in advance in order to get capital injections at times when the capital goes below a given retention level. Capital injection is an important topic in risk management, especially during unpredictable economic crises or some natural disasters.

Several optimal strategies are discussed and numerically illustrated for the reinsurance agreement. All the methods have the main objective to decrease the finite time ruin probability on the one hand, and on the other hand, to guarantee that reinsurance premium covers an average of overall capital injections. In addition, the first type of optimality is to find the optimal reinsurance premium and retention level to obtain the smallest ruin probability. In the second type, the upper level for compensation of claims and the reinsurance premium are investigated. The third type is to find the largest paid proportion of claims against the retention level, and the final type is to find the smallest premium rate against the retention level.

In all our calculations, we apply the Dirac matrix approach (motivated by Baaquie [5] [6]). More exactly, all computations are based on the Dirac-Feynman path calculation approach applied to the Dirac-Feynman operator (convolution type operator, defined in Theorem 6.1.3) and perturbed by the Injection operator (shift type operator, introduced in equation (6.3.7)).

We analyse the difference between finite time reinsurance contracts and their infinite time counterparts as suggested in Nie et al. [57, 58]. In particular, the finite and infinite time ruin probabilities and the expected injection amounts in modified surplus processes by reinsurance are compared. In addition, a peculiar connection between the capital injection operator and the convolution operator is established and the effect of the injection operator is analysed.

There are also curious applications such as a Fuzzy sets technique [36, 50, 73] and Game theory [51] to ruin probability. However, they are beyond the scope of this thesis.

## 1.2 Structure and Results

We present the structure of the thesis and highlight the main results.

- Chapter 2 begins with an introduction to the risk process. Then it states the stochastic process and distribution of the sum of random variables. Secondly, known finite and infinite time methods that compute ruin probability of an insurance company are considered. These methods are compared with modified Markov chain and quantum mechanics approaches mentioned in the following chapters.

- In Chapter 3, we modify a Markov chain approach to compute the finite time ruin probability. Firstly, for a small grid size $\varepsilon > 0$, a particular $d \times d$ transition matrix $A = A_\varepsilon$ with 0 absorption level is introduced. The generator matrix $Q$ for the corresponding continuous time Markov chain version of $A$ is defined.

  The finite time ruin probabilities are then computed via matrix $A$ in chosen grid level $\varepsilon$ by

$$P_u(T > t) = (1 + o(\varepsilon)) \sum_{j=1}^{d-1} A(t)_{u,j\varepsilon} \ .$$

Furthermore, the surplus process with capital injections and reduction is introduced by adding a shift type operator matrix $K$ (see definition of $K$ in Section 3.3). With the operator $K$, the finite time non ruin probability of the modified surplus process with capital injections and reductions is computed by

$$\begin{aligned} P_u(T > t) &= (1 + o(\varepsilon)) \sum_{j=1}^{d-1} \Big( A^{[t_1/\varepsilon]} K(a_1) A^{[(t_2-t_1)/\varepsilon]} K(a_2) \dots \\ & \qquad \dots A^{[(t_k-t_{k-1})/\varepsilon]} K(a_k) P^{[(t-t_k)/\varepsilon]} \Big)_{u,j\varepsilon} . \end{aligned}$$

Lastly, some results in case the claim size has a discretized exponential distribution are shown.

- The fourth chapter is about the quantum mechanics approach, and Dirac matrix approach and relevant terminology are defined. Then, computation of transition probability via various Hamiltonian operators in terms of claim size distributions are derived via the so-called discrete time formalism

$$\begin{aligned} P(x \xrightarrow{t} x') &= < x|e^{-tH}|x' > \\ &= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-tH}|p >< p|x' > \end{aligned}$$

where

  - $|x\rangle$ is the column vector and $\langle x|$ is its row vector (transposed vector),
  - $\langle x|x'\rangle$ is the inner product,
  - $|p\rangle \langle p|$ is the projection operator,
  - $H$ is the Hamiltonian operator.

After this, the Feynman's Path integral method and the Dirac matrix are applied to compute ruin probabilities, such as

$$P_u(T > t) = (1 + o(1)) \sum_{x_1=1} < u|e^{-t_1 H}|x_1 > \sum_{x_2=1} < x_1|e^{-(t_2-t_1)H}|x_2 >$$
$$\cdots \sum_{x_n=1} < x_{n-1}|e^{-(t-t_{n-1})H}|x_n > .$$

The chapter continues by representing the numerical results for discretized exponential distribution and Gaussian distributions. As in Chapter 3, the modified surplus process with capital injections and withdrawals is treated. Finally, we compare the quantum mechanics approach with the Appell polynomial approach and Markov chain approach.

- Chapter 5 is devoted to optimization problems. Three different actuarial examples are considered. Firstly, optimization of the initial capitals of two different surplus processes is shown by giving the results and graphs. In the second example, the optimum proportion of total claim compensation is computed with respect to a given specific ruin level. Lastly, optimization of the capital allocation of investment and withdrawals is considered.

- Then, in Chapter 6, we analyse the modified surplus process with reinsurance and capital injections. The modified surplus process is defined by

$$\begin{aligned} R^*(t) &= u + ct - z - H(S(t)) + Y(t) \\ &= w + ct - H(S(t)) + Y(t) \end{aligned}$$

where
$$H(S(t)) = \sum_{i=1}^{N(t)} X_i I(X_i \leq h) + hI(X_i > h).$$

and then, ruin probability under reinsurance contract is computed via the quantum mechanics approach. Furthermore, the effect of the injection operator $K$ and expected total capital injections amount $E[Y(t)]$ are also shown. In this chapter, numerical results are given in order to find optimum reinsurance cost $z$ and proportional claim payment $h$.

- In Chapter 7, the finite time method suggested in previous parts of the thesis is compared with the infinite time method stated by Nie at al. [57]. The comparison is made with respect to the ruin probability and expectations of injection amounts in terms of retention levels and reinsurance premiums.
- In the last chapter, future works are outlined.

Some parts of this thesis have been submitted as papers. Part of the content of Chapters 3-5 is included in a published paper entitled “Ruin Probability via Quantum Mechanics Approach” [76]. Furthermore, several parts of Chapter 5-7 are used in a submitted paper entitled “Optimum reinsurance via Dirac-Feynman Approach” [77].

# Chapter 2

# RISK PROCESS AND KNOWN METHODS

In this chapter, classical risk process is defined, and then stochastic processes and the distribution of the sum of random variables are mentioned. Additionally, known finite and infinite time methods are given in order to compute ruin probability of an insurance company.

We start by defining the risk process that is also called the surplus process.

## 2.1 Risk Process

The classical risk process at time $t$ consists of four components: premium rate ($c$), initial capital ($u$), claim amounts ($X_i$), number of claims $N(t)$ up to time $t$. Let $R(u,t)$ or $R(t)$ be the capital of insurance company at time $t$ with initial reserve $u$. In this case, the process with respect to time can be basically formalized [3,18,24,38,55] by

$$R(t) = u + ct - S(t)$$

where $S(t)$ is the total claim amount up to time $t$. It may be modelled by approaches as the individual and collective risk models [78]. In the individual risk model, the claim number is fixed.

Let $X_i$ be iid (independent and identically distributed) random sequence of positive claim sizes. In this circumstance,

$$S_n = X_1 + X_2 + \cdots + X_n.$$

In the collective risk model, the aggregate loss amount has compound distribution, so

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)} = \sum_{i=1}^{N(t)} X_i.$$

In the classical model, $X_i$ and $N(t)$ are independent processes from each other. However, models with various dependence structures, such as dependent claims or dependence between claim size and claim intervals become more popular [1, 2].
$X$ may have different distributions, such as exponential, normal, gamma, weibull, pareto and so on [9].
$N(t)$ is an integer value representing the claim number up to time $t$. It may have a different distribution such as Geometric, Negative binomial, Poisson distributions. Throughout this thesis, claim number $N(t)$ is assumed to be a Poisson process with intensity $\lambda > 0$. Therefore, $S(t)$ is a compound Poisson process.
The claim number process has the following property

$$N(t + \triangle t) - N(t) \sim Poisson(\lambda \triangle t) \text{ for all t and } \triangle > 0.$$

The probability that number of claims is equal to k in the interval $(t, t + \triangle t)$, can be found by

$$P(N(t + \triangle t) - N(t) = k) = \frac{e^{-\lambda \triangle t}(\lambda \triangle t)^k}{k!} \qquad k = 0, 1, 2, \ldots.$$

**Convolution**

Let $X_1$ and $X_2$ be random variables representing claim amounts with probability density (or mass in discrete time) functions $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$.
The probability mass function of $Y = X_1 + X_2$ is found by following the convolution

formula for discrete claim size

$$f_Y(y) = \sum_{x_1} P(X_1 = x_1)P(X_2 = y - x_1)$$
$$= \sum_{x_1} f_{X_1}(x_1)f_{X_2}(y - x_1).$$

Similarly, the probability density function in the continuous claim case is written as

$$f_Y(y) = \int_0^y f_{X_1}(x_1)f_{X_2}(y - x_1)dx_1.$$

Here, $f_Y$ is a two-fold convolution. A convolution can be recursively evaluated. For example, with fixed claim number $N(t) = 3$, three-fold convolution can be shown as

$$f_{X_1+X_2+X_3}(y) = (f_{X_1+X_2} * f_{X_3})(y) = (f_{X_1} * f_{X_2} * f_{X_3})(y).$$

Similarly,

$$f_{X_1+X_2+\ldots+X_{N(t)}}(y) = \sum_{n=0}^{\infty} f^{*n}_{X_1+X_2+\ldots+X_n}(y)P(N(t) = n),$$

where $f^{*n}_{X_1+X_2+\ldots+X_n}(y)$ is n-fold convolution for the continuous value, which can be written as

$$f^{*n}_{X_1+X_2+\ldots+X_n}(y) = \int_0^y f^{*n-1}_{X_1+X_2+\ldots+X_{n-1}}(y - x)f_{X_n}(x)dx.$$

If $X_i$ has exponential distributions with mean $1/\lambda$, then $S_n = X_1 + X_2 + \cdots + X_n$ has a gamma distribution and its pdf is

$$f^{*n}_{S_n} = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

with parameter $n$ and $\lambda$.

Similarly,

if $X_i$ has a normal (Gaussian) distribution with mean $\mu$ and variance $\sigma^2$, then $S_n$ has a normal distribution with mean $n\mu$ and $n\sigma^2$.

An example of an insurer cash flow can be seen in figure 2.1.

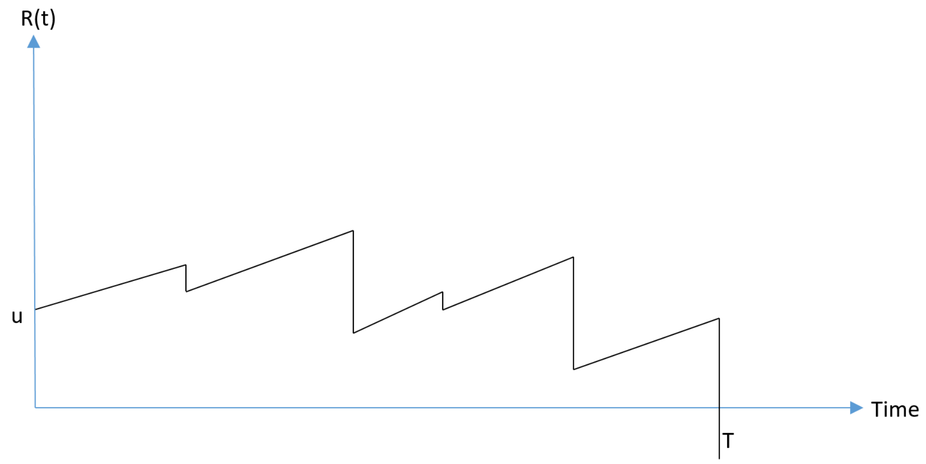



Figure 2.1: The cash flow of an insurer.

In this thesis, other expenses for insurance companies are not taken into consideration, such as operation cost. However, in the real sector, operational cost should be taken into account and the surplus process should be exposed to shifting. In the subsequent chapters, a reinsurance agreement with capital injections and withdrawals will be added into the surplus process.

The ruin time $T$ is the minimum non negative time when the capital of an insurance company is below zero. However, it is convenient in our research to add zero to ruin as an absorption level, so ruin will occur as soon as the capital of the insurance company becomes negative or null.

$$T = \begin{cases} \min\{t \geq 0 | R(t) \leq 0\} & \text{for discrete time,} \\ \inf\{t \geq 0 | R(t) \leq 0\} & \text{for continuous time.} \end{cases}$$

Finite time ruin probability at time $t$ with initial capital $u$ is denoted by

$$P_u(T \leq t).$$

Ultimate ruin probability for infinite time with initial capital $u$ is denoted by

$$P_u(T < \infty).$$

It is obvious that longer time gives rise to an increase in ruin probability, which means

$$P_u(T \leq t_1) \leq P_u(T \leq t_2) \leq P_u(T < \infty)$$

for every $t_1 < t_2$.

On the other hand, more initial capital leads to a decrease in ruin probability.

$$P_{u_1}(T \leq t) \leq P_{u_2}(T \leq t)$$

for all $u_1 > u_2$.

In this step, it is convenient to define the non-ruin probability, which is known as survival probability in the present context [67].

$$\varphi(u,t) = P_u(T > t) = 1 - P_u(T \leq t).$$

**Definition 4 (Net Profit Condition and Loading factor)**
*In the real insurance system, the premium rate in the unit time should be bigger than the expected aggregate claim, which is called the net profit condition:*

$$c > m\lambda.$$

*where $c$ is premium rate, $m$ is claim mean, and $\lambda$ is claim frequency.*
*Recall that*

$$P(T < \infty) = 1 \qquad \textit{when} \qquad c < m\lambda.$$

*Notice that infinite time methods are not applied without this condition in general because the ruin will happen eventually. However, finite time methods work without this condition.*
*Now, let $\theta$ be the loading factor. $\theta > 0$ satisfies the net profit condition.*

$$c = (1+\theta)m\lambda \qquad \textit{gives} \qquad \theta = \frac{c - m\lambda}{m\lambda}.$$

*The loading factor is used to determine the premium rate by insurance companies.*

**Definition 5 (Lundberg's inequality and adjustment coefficient)**
*Ultimate ruin probability satisfies the following inequality*

$$P_u(T < \infty) \leq e^{-Ru}.$$

This inequality is called Lundberg's inequality [24], and it gives an upper barrier for ultimate ruin probability.
Since an ultimate ruin probability is bigger than ruin probability in finite time, Lundberg's inequality can also be applied as an upper barrier in finite time methods.

$$P_u(T \leq t) \leq P_u(T < \infty) \leq e^{-Ru}.$$

In the inequality, $R$ is known as the adjustment coefficient, which is a parameter related to the surplus process. $R$ depends on premium income and distribution of aggregate claims.
$R$ can be found as solution of

$$\lambda M_X(R) = \lambda + cR \tag{2.1.1}$$

where $M_X(R) = E[e^{RX}]$ is the moment generating function of claim size.
Assume that claim sizes have exponential distribution with claim mean $m$ and the net profit condition holds ($c > \lambda m$), then the moment generating function is

$$M_X(R) = \frac{\frac{1}{m}}{\frac{1}{m} - R} \qquad \text{for} \qquad R < \frac{1}{m}.$$

When putting $M_X(R)$ into equation (2.1.1), we have

$$cmR^2 + R(m\lambda - c) = 0.$$

When the equation is solved, we get

$$R = \frac{-\lambda}{c} + \frac{1}{m}.$$

## 2.2 Stochastic Processes and Distributions

In this section, several basic definitions from probability theory are given [56,67,74].

**Definition 6 (Measurable space)**

*Let $\mathcal{F}$ be a nonempty family of subsets of $\Omega$ such that:*

- $A \in \mathcal{F}$ *implies* $A^c \in \mathcal{F}$,
- $\{A_n : n \in N\}$ *a sequence of sets in* $\mathcal{F}$ *implies* $\bigcup\limits_{n \in N} A_n \in \mathcal{F}$.

$(\Omega, \mathcal{F})$ *is called a measurable space, where* $\mathcal{F}$ *is a* $\sigma$*-algebra of subsets of* $\Omega$.

**Definition 7 (Probability Measure)**

*A probability measure or probability distribution is a real valued function*

$$P : \mathcal{F} \rightarrow [0, 1]$$

*where* $\mathcal{F}$ *is* $\sigma$ *field on* $\Omega$*, which satisfies the following conditions:*

- $P(\Omega) = 1$,
- *For any subset of* $A \in \Omega$, $0 \leq P(A) \leq 1$,
- *If* $A_i, i \in I$ *are disjoint collection of events, then*

$$P(\bigcup_{i \in I} A_i) = \sum_{i \in I} P(A_i).$$

**Definition 8 (Measurable)**

*Let* $(\Omega, \mathcal{F})$ *be a measurable space. A function* $X : \Omega \rightarrow \mathbb{R}$ *is* $\mathcal{F}$ *measurable if* $X^{-1}(A) \in \mathcal{F}$ *for any Borel subset* $A \subset \mathbb{R}$.

**Definition 9 (Random variables)**

*Let* $X : \Omega \rightarrow \mathbb{R}$ *be* $\mathcal{F}$ *measurable in a probability space* $(\Omega, \mathcal{F}, P)$*, then* $X$ *is a random variable on the probability space.*

**Definition 10 (Independence)**

*Let $A$ and $B$ be subsets of $\Omega$, then $A$ and $B$ events are independent if*

$$P(A \cap B) = P(A)P(B).$$

*Two independent events are written as $A \perp B$.*

**Definition 11 (Stochastic Process)**

*A Stochastic process (or random process) is a collection of random variables on the probability space [56, 67]. Let $\{X_t,\ t \in \tau\}$ be a stochastic process. If $\tau$ is countable, the process is a discrete time process. If $\tau$ is not countable, the process is a continuous process.*

**Definition 12 (Levy Process)**

*A stochastic process $\{X_t; t \geq 0\}$ is a Levy process if*
*(i)Disjoint increments are independent,*
*(ii)$X_{t+\triangle t} - X_{\triangle t} \sim X_t$.*

A Brownian motion and a Poisson process are also Levy processes.

**Definition 13 (Brownian Motion)**

*A stochastic process $S_t,\ t \geq 0$ is called a Brownian motion with drift $\mu$ and diffusion coefficient $\sigma^2$ if*

- *$S_{t+y} - S_y \backsim N(\mu t, \sigma^2 t)$ for all $t, y \geq 0$,*
- *Disjoint increments $S_{t_n} - S_{t_{n-1}}, ..., S_{t_2} - S_{t_1}$ are independent for all $0 \leq t_1 < ... < t_n$.*

The Brownian motion is applied for approximation of random walks.

$$S_t = \mu t + \sigma B_t \qquad t \geq 0 \qquad for \qquad \sigma > 0 \qquad and \qquad \mu \in \mathbb{R}$$

where $B_t$ is the Standard Brownian Motion that has $\mu = 0$ and $\sigma^2 = 1$.
A Brownian motion is a Gaussian process. Let $Z$ be

$$Z = \frac{S_t - E[S_t]}{\sqrt{var(S_t)}} = \frac{S_t - \mu t}{\sigma \sqrt{t}}.$$

Notice that $Z \sim N(0,1)$, so

$$P(Z < x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

Also

$$P(Z \in [x, x+\varepsilon]) = \varepsilon \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

is useful in tackling Gaussian claims.

### 2.2.1 Distribution of the sum of random variables

The sum of independent and identically distributed positive random variables is an important topic in insurance applications. The main question is to find the distribution of

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)}.$$

Let's observe the distribution of $S_n$ for fix $N(t) = n$ with respect to the moment generating function and convolution of distributions with a distribution of $X$.

1)Say $X_1$ has an exponential distribution with mean $m$, then observe the $S(t)$ for both approaches.

$$\begin{aligned} M_S(t) &= E[e^{tS_n}] = E[e^{t(X_1+X_2+\cdots+X_n)}] \\ &= E[e^{tX_1}]E[e^{tX_2}]\cdots E[e^{tX_n}]. \end{aligned}$$

Therefore,

$$M_S(t) = M_X(t)^n \tag{2.2.2}$$

because $X_i$'s are independent and identically distributed.

The moment generating function can be written in the following form when it has exponential distribution.

$$M_X(t) = \frac{\frac{1}{m}}{\frac{1}{m} - t} \qquad \text{provided} \qquad t < \frac{1}{m}.$$

From 2.2.2,

$$M_S(t) = \Big(\frac{\frac{1}{m}}{\frac{1}{m} - t}\Big)^n = \Big(1 - \frac{t}{\frac{1}{m}}\Big)^{-n}.$$

This means $S_n$ has a gamma distribution with $\gamma(n, \frac{1}{m})$.

2)Now let's do it by using a convolution of distributions.

For n=2;

$$\begin{aligned} f^{*2} &= \int_0^y f(y-x)f(x)dx \\ &= \int_0^y \frac{1}{m}e^{-\frac{1}{m}(y-x)}\frac{1}{m}e^{-\frac{1}{m}x}dx \\ &= (\frac{1}{m})^2 ye^{-\frac{1}{m}y}. \end{aligned}$$

This means $S_2$ has a $\gamma(2, \frac{1}{m})$ distribution for n=2.

For n=3;

$$\begin{aligned} f^{*3} &= \int_0^y f^{*2}(y-x)f(x)dx \\ &= \int_0^y f^{*2}(x)f(y-x)dx \\ &= \int_0^y \Big(\frac{1}{m}\Big)^2 xe^{-\frac{1}{m}x}\frac{1}{m}e^{-\frac{1}{m}(y-x)}dx \\ &= \frac{1}{2}\Big(\frac{1}{m}\Big)^3 y^2e^{-\frac{1}{m}y}. \end{aligned}$$

This means $S_3$ has a $\gamma(3, \frac{1}{m})$ distribution. Similarly, $S_n$ has a gamma distribution with $\gamma(n, \frac{1}{m})$.

Let's look at the moment generating function of the compound Poisson distribution

$S(t)$ when $N(t)$ has a Poisson distribution.

$$
\begin{aligned}
M_S(t) &= E[M_X(t)^N] \\
&= E[e^{log(M_X(t)^N)}] \\
&= E[e^{NlogM_X(t)}] \\
&= M_N[logM_X(t)]. \qquad (2.2.3)
\end{aligned}
$$

The moment generating function of $S(t)$ is shown in terms of the moment generating functions of $N$ and $X$.

Equation (2.2.3) can be written in the following form because $N(t)$ has a Poisson distribution with $\lambda$ claim frequency.

$$
\begin{aligned}
M_S(t) &= e^{\lambda(e^{logM_X(t)}-1)} \\
&= e^{\lambda(M_X(t)-1)}.
\end{aligned}
$$

### 2.2.2 Gambler's ruin problem

Let's consider a game between two players with fair coin flipping. Let £$z_1$ and £$z_2$ be the initial fortune of the players. In the game, £1 will be transferred from loser to winner in each event. The game will continue until one of the players has all money or the other loses his or her own money. The main objective of the game is to reach the total possible fortune of £$z_1 + z_2$ without ruining. Let $R_t$ denote the fortune after the t-th flip. For the first player, $R_0 = z_1$ and $R_t = z_1 + \delta_1 + ... + \delta_t$ where $\delta_i$ are IID and

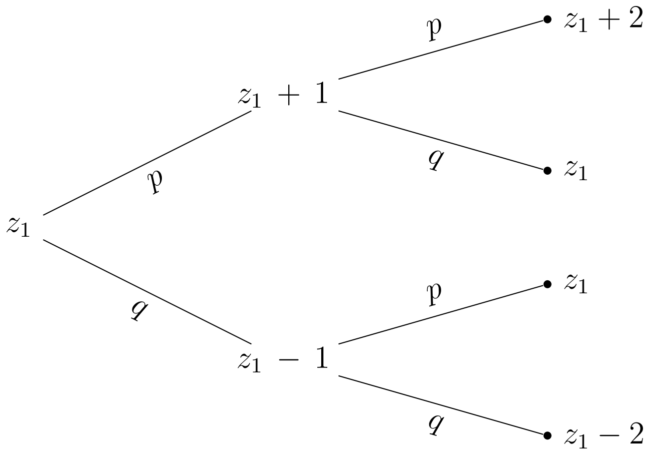
$$
\delta_i = \begin{cases} 1 & \text{if win} \\ -1 & \text{if lose} \end{cases}.
$$

The random walk will stop when it hits 0 or $z_1 + z_2$.

Let $T$ be the stopping time, defined by

$$
T = min\{t \geq 0 : R_t \in \{0, z_1 + z_2\} | R_0 = z_1\}.
$$

When the capital of the first player is equal to 0 or $z_1 + z_2$, the game will stop. In the game, the first player wins 1 with probability $p$ or loses 1 with probability $q = 1 - p$.




Let $P_1(z_1)$ denote the chance of winning the game for the first player with initial fortune $z_1$. We assume that

$$P_1(0) = 0 \text{ and } P_1(z_1 + z_2) = 1.$$

Here, the key idea is that we derive an equation by conditioning on the first step

$$P_1(z_1) = P_1(z_1 + 1)p + P_1(z_1 - 1)q.$$

In this circumstance,

$$P_1(z_1) = \begin{cases} \frac{1-(\frac{q}{p})^{z_1}}{1-(\frac{q}{p})^{z_1+z_2}} & \text{if} \quad p \neq q \\ \frac{z_1}{z_1+z_2} & \text{if} \quad p = q \end{cases}. \tag{2.2.4}$$

**Proof.** we start with

$$P_1(z_1) = P_1(z_1 + 1)p + P_1(z_1 - 1)q.$$

The equation can be written by little algebra as

$$P_1(z_1)(p+q) = P_1(z_1+1)p + P_1(z_1-1)q \qquad \text{because} \qquad p+q=1,$$

$$\begin{aligned}
\text{so} \qquad & P_1(z_1+1) - P_1(z_1) = \frac{q}{p}(P_1(z_1) - P_1(z_1-1) \\
& P_1(z_1+1) - P_1(z_1) = \frac{q}{p}(\frac{q}{p}(P_1(z_1-1) - P_1(z_1-2)) \qquad \text{by iterating.} \\
\text{We have} \qquad & P_1(z_1+1) - P_1(z_1) = \big(\frac{q}{p}\big)^{z_1}(P_1(1)).
\end{aligned}$$

$P_1(z_1)$ can be written as $\sum_{k=1}^{z_1} P_1(k) - P_1(k-1)$, then

$$\begin{aligned}
P_1(z_1) &= \sum_{k=0}^{z_1-1} \big(\frac{q}{p}\big)^k P_1(1) \\
&= \begin{cases} \frac{1-\left(\frac{q}{p}\right)^{z_1}}{1-\frac{q}{p}} P_1(1) & \text{if} \quad p \neq q \\ z_1 P_1(1) & \text{if} \quad p = q. \end{cases}
\end{aligned} \tag{2.2.5}$$

When $P_1(z_1+z_2) = 1$ is taken into account,

$$P_1(1) = \begin{cases} \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^{z_1+z_2}} & \text{if} \quad p \neq q \\ \frac{1}{z_1+z_2} & \text{if} \quad p = q. \end{cases} \tag{2.2.6}$$

From equations (2.2.5) and (2.2.6), we obtain equation (2.2.4). ■

At time $t$, expectation of the capital of the first player $E[R_t]$ is defined by

$$E[R_t | R_{t-1} = x] = E[R_{t-1} + \delta_{t-1} | R_{t-1} = x] = x + E[\delta]$$

and a Markov chain can be defined in terms of the capital of the player with transition matrix $P$ as

$$P(R_{n+1} = x_{n+1} | R_n = x_n, \cdots, R_0 = z_1) = P(R_{n+1} = x_{n+1} | R_n = x_n).$$

For the total fortune $z_1 + z_2 = £5$, the transition matrix over one step probability is defined by

$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} & p_{05} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \\ p_{50} & p_{51} & p_{52} & p_{53} & p_{54} & p_{55} \end{pmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

At time $t$, expected capital of the player is found via $P^t$. Note that

$$P^t = P^{t-1}P,$$

which is dealt with in the next chapter.

## 2.3 Infinite time (ultimate) ruin probability

The probability of ruin in infinite time is known as the ultimate ruin probability, and different approaches can be taken to computing this.

Ultimate ruin probability for the surplus process, where claim size has an exponential distribution, is computed by [24, 68]

$$P_u(T < \infty) = \frac{\lambda m}{c} e^{-(\frac{1}{m} - \frac{\lambda}{c})u}. \tag{2.3.7}$$

This formula is obtained by using survival probability as below.

Let $\varphi(u) = 1 - P_u(T < \infty)$ be the survival probability that ruin never occurs.

The survival probability can be shown by considering the first claim time and amount.

$$\varphi(u) = \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} f(x_1)\varphi(u + ct - x_1)dx_1 dt. \tag{2.3.8}$$

Note that $u + ct - x_1$ is the capital of an insurance company after the first claim occurs.

When substituting $y = u + ct$ in the previous equation and taking derivative with respect to $u$, we have.

$$\frac{d}{du}\varphi(u) = \frac{\lambda^2}{c^2}e^{\frac{\lambda u}{c}} \int\limits_u^\infty e^{-\lambda\frac{y}{c}} \int\limits_0^y f(x_1)\varphi(y - x_1)dx_1dy - \frac{\lambda}{c}\int\limits_0^u f(x_1)\varphi(u - x_1)dx_1$$
$$= \frac{\lambda}{c}\varphi(u) - \frac{\lambda}{c}\int\limits_0^u f(x_1)\varphi(u - x_1)dx_1. \tag{2.3.9}$$

We need to eliminate the integral part in the equation in order to get a differential equation to solve easily.

Let's consider 2.3.9 in case that claim sizes have exponential distribution with parameter $\alpha$.

In this circumstance, $F(x) = 1 - e^{-\alpha x}$ for $x \geq 0$. Then,

$$\frac{d}{du}\varphi(u) = \frac{\lambda}{c}\varphi(u) - \frac{\lambda}{c}\int\limits_0^u \alpha e^{-\alpha x}\varphi(u - x_1)dx_1$$
$$= \frac{\lambda}{c}\varphi(u) - \frac{\alpha\lambda}{c}e^{-\alpha u}\int\limits_0^u e^{\alpha x_1}\varphi(x_1)dx_1. \tag{2.3.10}$$

Differentiating of equation (2.3.10) gives the following equation.

$$\frac{d^2}{du^2}\varphi(u) = \frac{\lambda}{c}\frac{d}{du}\varphi(u) - \frac{\alpha^2\lambda}{c}e^{-\alpha u}\int\limits_0^u e^{\alpha X_1}\varphi(x_1)dx_1 - \frac{\alpha\lambda}{c}\varphi(u). \tag{2.3.11}$$

If the equation (2.3.10) is added to equation (2.3.11) by multiplying by $\alpha$, then the following equation is obtained.

$$\frac{d^2}{du^2}\varphi(u) + \alpha\frac{d}{du}\varphi(u) = \frac{\lambda}{c}\frac{d}{du}\varphi(u), \tag{2.3.12}$$

The general solution to a second order differential equation above is in the form below

$$\varphi(u) = \sigma_0 + \sigma_1 e^{-(\alpha - \frac{\lambda}{c})u} \tag{2.3.13}$$

where $\sigma_0$ and $\sigma_1$ are constant.

$\sigma_0 = 1$ because $\lim\limits_{u\to\infty} \varphi(u) = 1$. For u=0 and $\sigma_0 = 1$, equation (2.3.13) is

$$\varphi(0) = 1 + \sigma_1.$$

Therefore, $\sigma_1 = \varphi(0) - 1 = -P_0(T < \infty)$.

When putting $\sigma_0$ and $\sigma_1$ into equation (2.3.13),

$$\varphi(u) = 1 - P_0(T < \infty)e^{-(\alpha - \frac{\lambda}{c})u}.$$

Now, $P_0(T < \infty)$ needs to be solved.

If we get $1 - P_u(T < \infty)$ instead of $\varphi(u)$ in equation (2.3.9), and integrate the equation over $(0, \infty)$, the following equation is obtained,

$$\begin{aligned} -P_0(T < \infty) = {} & \frac{\lambda}{c}\int_0^\infty P_u(T < \infty)du - \frac{\lambda}{c}\int_0^\infty\int_0^u f(x_1)P_{u-x_1}(T < \infty)dx_1du \\ & - \frac{\lambda}{c}\int_0^\infty (1 - F(u))du. \end{aligned} \tag{2.3.14}$$

When the double integral term in equation (2.3.14) is taken into consideration, this term can be written in a different way by changing the order of integration.

$$\begin{aligned} \int_0^\infty\int_0^u f(x_1)P_{u-x_1}(T < \infty)dx_1du &= \int_0^\infty\int_x^\infty P_{u-X_1}(T < \infty)du\ f(x_1)\ dx_1 \\ &= \int_0^\infty P_y(T < \infty)dy. \end{aligned}$$

In this circumstance, in the right hand side of equation (2.3.14), the sum of the first two terms is zero. Therefore, the equation can be written as follows:

$$P_0(T < \infty) = \frac{\lambda}{c}\int_0^\infty (1 - F(u))du = \frac{\lambda m_1}{c} \tag{2.3.15}$$

where $m_1 = \frac{1}{\alpha}$.

Now, $\varphi(u)$ can be written in terms of $P_0(T < \infty)$ in the following equation when $F(x_1) = 1 - e^{-\alpha x_1}$, $x_1 \geq 0$ :

$$\varphi(u) = 1 - \frac{\lambda m_1}{c} e^{-(\alpha - \frac{\lambda}{c})u}. \tag{2.3.16}$$

As mentioned in the previous sections, the adjustment coefficient for exponential claim distribution was $R = \frac{-\lambda}{c} + \frac{1}{m}$.

Let's give $P_u(T < \infty)$ in terms of the adjustment coefficient.

$$P_u(T < \infty) = P_0(T < \infty)e^{-Ru}.$$

This equation shows that Lundberg's inequality gives an upper bound for ruin probability because $P_0(T < \infty) < 1$ under the net profit condition $(c > \lambda m)$.

In case of $c = 5, \lambda = 1$, and $\mu = 4$, the way in which the ultimate ruin probability and upper level with Lundberg's inequality change by initial capital can be seen in the following graph.

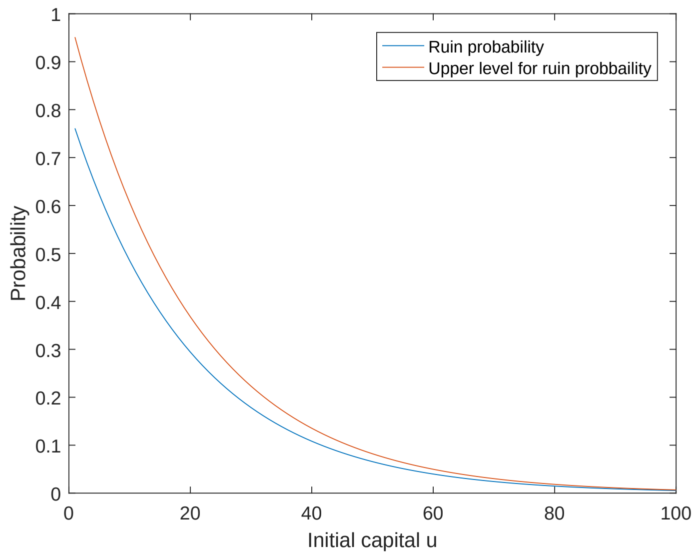



Figure 2.2: Ultimate ruin probability and upper bound with respect to initial capital

## 2.4 Finite time ruin probability

We deal with finite time (non) ruin probability via the Picard-Lefevre approach, which was introduced in 1997 by Philippe Picard and Claude Lefevre. This approach is compared with our results in next chapters. The approach is referred to Picard-Lefevre or Appell polynomial approach in the forward parts of this thesis.

### 2.4.1 Expansion of functions

Let $f(x)$ be a real or complex valued differentiable function at $\zeta$, then the function's power series is defined by

$$f(x) = \sum_{k=0}^{\infty} c_k (x-\zeta)^k. \tag{2.4.17}$$

Taylor series expansion can be defined as a sum of terms of a function's infinite derivatives at point $\zeta$ by

$$\begin{aligned} f(x) &= f(\zeta) + \frac{f'(\zeta)}{1!}(x-\zeta) + \frac{f''(\zeta)}{2!}(x-\zeta)^2 + \frac{f'''(\zeta)}{3!}(x-\zeta)^3 + ... \\ &= \sum_{k=0}^{\infty} \frac{f^k(\zeta)}{k!}(x-\zeta)^k \end{aligned}$$

which shows equation (2.4.17) with

$$c_k = \frac{f^{(k)}(\zeta)}{k!}.$$

For $\zeta = 0$, the Taylor series is referred to as the Maclaurin series.
For example, the Maclaurin series of $e^x$ and $sin(x)$ at $\zeta = 0$ are defined by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

For $z = e^{i\theta}$, the Fourier series of the function $f(e^{i\theta})$ as a function of the polar angle $\theta$ is defined by

$$g(\theta) = \sum_{k=0}^{\infty} c_k e^{ki\theta}$$

where

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-ki\theta} d\theta.$$

As seen, a Fourier series is a particular example of a complex Taylor series because $f(e^{i\theta}) := g(\theta)$ .

Therefore, $c_k$ in the analytic expansion and Fourier should be equal to each other.

$$\begin{aligned}
\frac{f^{(k)}(0)}{k!} &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-ki\theta} d\theta \\
f^{(k)}(0) &= \frac{k!}{2\pi} \int_0^{2\pi} g(\theta) e^{-ki\theta} d\theta \\
&= \frac{k!}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ki\theta} d\theta \\
&= \frac{k!}{2\pi i} \oint_0^{2\pi} \frac{f(z)}{z^{k+1}} dz.
\end{aligned}$$

We consider $z$ on a unit circle where $\{z \in \mathbb{C} : |z| = 1\}$ with $z = e^{i\theta}$ and $dz = ie^{i\theta} d\theta$. The equation is referred to as Cauchy's differentiation formula.

### 2.4.2 Appell polynomial approach

In this method [46, 47, 61, 62], it is assumed that the claim amounts are positive integer values. Let $R(t)$ be a surplus process for an insurance company with initial capital $u$.

$$R(t) = u + ct - S(t)$$

where $c$ is the premium income per unit time, and $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregate claim amount.

$S(t)$ has a discrete compound process. Let $p_n(t)$ be the probability mass function of $S(t)$.

$$p_n(t) = P(S(t) = n) \qquad n = 0, 1, ...$$

$$p_0(t) = e^{-\lambda t} \quad \text{for n=0 and} \quad p_n(t) = e^{-\lambda t} \sum_{j=1}^{n} \frac{(\lambda t)^j}{j!} p_n^{*j} \qquad n = 1, 2...$$

where $p_n^{*j}$ is the j-th convolution of $X$.

According to Panjer's [59] recursion formula,

$$p_0(t) = e^{-\lambda t} \quad \text{and} \quad p_n(t) = \lambda t \sum_{j=1}^{n} \frac{j}{n} q_j p_{n-j}(t), \qquad n = 1, 2, ...$$

where $q_j = P(X = j)$.

In the surplus process, let's define the income function by

$$h(t) = u + ct.$$

If $h(t)$ is not continuous, then $h^{-1}(x) = inf\{y;\ h(y) \geq x\}$

$$v_n = h^{-1}(n) = max\{0, \frac{n-u}{c}\}, n = 0, 1, 2, ...$$

Therefore,

$$v_0 = v_1 = ... = v_u, \qquad v_n = \frac{n-u}{c} \qquad for \qquad n \geq u + 1.$$

Let $T$ be the ruin time, then

$$P_n(x) = P(S(x) = n \quad and \quad T > x).$$

$S_x$ is the outcome function representing total claim amount at time $x$.

Non ruin probability is defined by

$$P(T > x) = \sum_{n=0}^{[u+cx]} P_n(x).$$

It is obvious that

$$P_0(x) = P(S(x) = 0 \quad and \quad T > x) = e^{-\lambda x}$$

because the claim number is zero, so $\frac{e^{-\lambda x}(\lambda x)^k}{k!} = e^{-\lambda x}$ for $k = 0$.

When $x < v_n$,

$$P_n(x) = 0.$$

Notice that $x < v_n$ means that outcome is bigger than the income at time $x$.

In the other cases, $P_n(x)$ can be written with respect to last claim $J$ before the ruin

$$P_n(x) = \int_{v_n}^{x} \sum_{j=1}^{n} q_j P_{n-j}(t) \lambda e^{-\lambda(x-t)} dt \tag{2.4.18}$$

where $q_j = P(J = j)$ is probability of last claim amount $J$ before ruin.

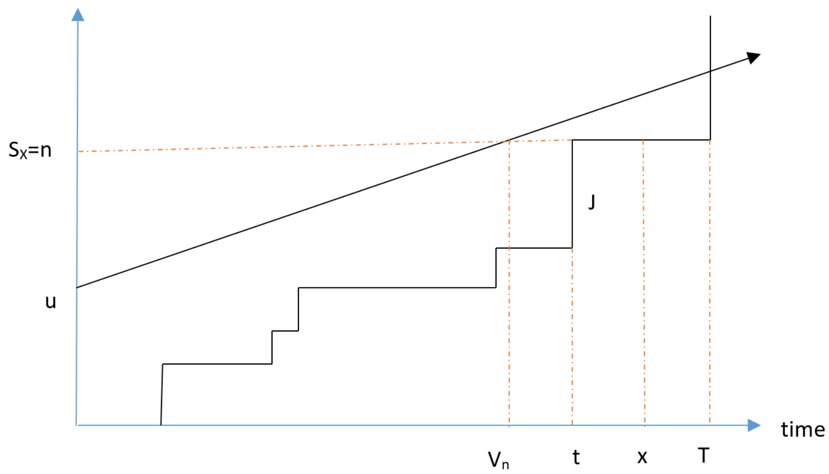



Figure 2.3: Income and outcome in the surplus process

The approach is based on the following fundamental assumption:

$$P_n(x) = E[P_{n-J}(t)].$$

Picard and Lefevre pointed out that $P_n(x)$ has a polynomial structure, so it can be written as

$$P_n(x) = e^{-\lambda x} B_n(x) \tag{2.4.19}$$

where $B_n, n = 0, 1, 2...$ is a sequence of generalized Appell polynomials of degree $n$ in $x$ with

$$B_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ \int\limits_{v_n}^{x} \sum\limits_{j=1}^{n} \lambda q_j B_{n-j}(t)dt & \text{if } n > 0 \end{cases}.$$

We derive

$$P_n(x) = P(S(x) = n \quad and \quad T > x) = e^{-\lambda x} B_n(x).$$

COROLLARY:

$$P(T > x) = e^{-\lambda x} \sum_{n=0}^{\infty} B_n(x).$$

When $v_j < x \leq v_{j+1}$,

$$P(T > x) = e^{-\lambda x} \sum_{n=0}^{j} B_n(x). \tag{2.4.20}$$

In the family of generalized Appell polynomials, each polynomial $B_n(x)$ can be written in the expansion form [47] as

$$B_n(x) = \sum_{k=0}^{n} B_k(0) e_{n-k}(x), \qquad n = 0, 1, ...$$

**Definition 14 (Generalized Appell polynomials)** *[61]*
*$e_n(x)$ is a family of generalized Appell polynomials if its generating function is written in following form.*

$$\sum_{n=0}^{\infty} e_n(x) z^n = e^{xG(z)}$$

*where*

$$G(z) = \sum_{j=1}^{\infty} \lambda q_j z^j$$

The equations below are equivalent to each other for generalized Appell polynomial families.

- $B'_n = \sum\limits_{j=1}^{n} \lambda q_j B_{n-j}, \quad n > 0.$
- $\Delta B_n = B_{n-1}, \quad n > 0$ where $\Delta$ is operator that $\Delta^{k+1} = \Delta(\Delta^k)$ with $\Delta^0$ the identity operator .
- $B_n = \sum\limits_{i=0}^{n} b_i e_{n-i}, \quad n \geq 0$

where $b_i = B_i(0)$ is a family of numbers.

As mentioned before,

$$p_n(x) = \sum_{k=0}^{n} e^{-\lambda x} \frac{(\lambda x)^k}{k!} q_n^{*k}, n \geq 0,$$

where

$$q_j^{*k} = P(X_1 + X_2 + ... + X_k = j), k > 0.$$

We write $p_n(x)$ in terms of a polynomial of degree $n$ in time $t$ as

$$p_n(x) = e^{-\lambda x} e_n(x), \quad n \geq 0,$$

where $e_0(x) = 1$ and $e_n(0) = 0$.

$e_n(x)$ is written as

$$e_n(x) = \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} q_n^{*k}.$$

Picard and Lefevre suggested that $B_n$ is expressed in the theorem below.

**Theorem 15** *For the linear case of $h = u + ct$,*

$$B_n(x) = \begin{cases} e_n(x) & \text{when } 0 \leq n \leq u \\ \sum_{j=0}^{u} e_j(\frac{j-u}{c}) f_{n-j}(x + \frac{u-n}{c}) & \text{when } n > u \\ = \sum_{j=0}^{u} e_j(\frac{j-u}{c}) \frac{cx-n+u}{cx-j+u} e_{n-j}(x + \frac{u-j}{c}) & \end{cases}$$

where $f_n(x) = \frac{cx}{cx+n} e_n(x + \frac{n}{c})$, which has an Appell structure.

From equation (2.4.20) and Theorem 15, the next theorem is deduced.

**Theorem 16 (Picard-Lefevre polynomial approach)**

*For the linear case,*

$$P(T > x)|R_0 = u) = e^{-\lambda x} \sum_{j=0}^{u} \left\{ e_j(x) + \sum_{n=u+1}^{[cx+u]} e_j(\frac{j-u}{c}) \frac{cx-n+u}{cx-j+u} e_{n-j}(x + \frac{u-j}{c}) \right\} \tag{2.4.21}$$

*where*

$$e_n(x) = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} q_n^{*k}$$

*and*

$$q_j^{*k} = P(X_1 + X_2 + ... + X_k = j).$$

According to the formula in Theorem 16, non ruin probability with respect to time is displayed for $u = 20$, $c = 1$, $\lambda = 0.1$, and the claims have an exponential distribution with claim mean $m = 9$.

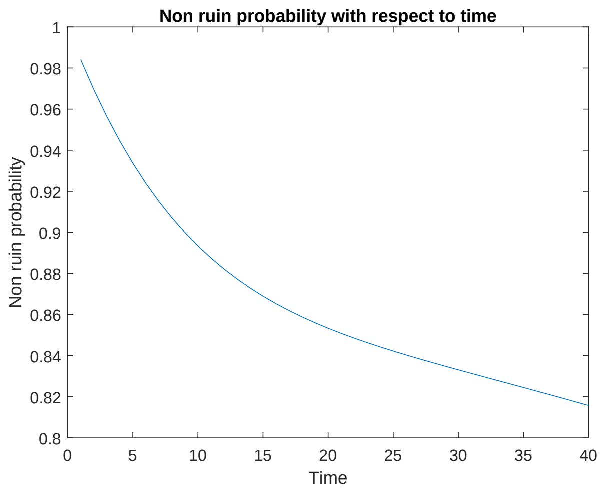



Figure 2.4: Non ruin probability via the Picard-Lefevre approach

The Picard-Lefevre approach also provides a formula for computation of non ruin probability in infinite time for the linear case.

$$P(T < \infty | R_0 = u) = 1 - (1 - \frac{\lambda m}{c}) \sum_{j=0}^{u} e^{\lambda(u-j)/c} e_j(\frac{j-u}{c}). \tag{2.4.22}$$

In order to analyse the results obtained from the Appell polynomial approach in infinite time and the formula defined in equation (2.3.7), let's consider the next graph for $c = 1$, $\lambda = 0.1$, $m = 90$ and $u = [1, 100]$.

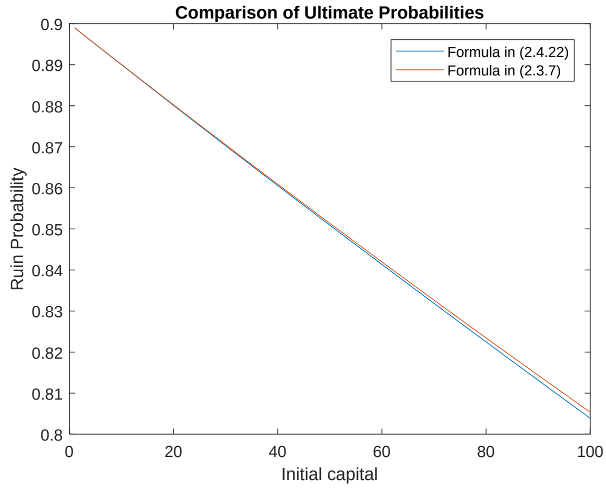



Figure 2.5: Ultimate ruin probability via the Picard-Lefevre approach and classical approach

In the graph, the red line gives the results of the Picard- Lefevre method while the blue one is for the formula defined in equation (2.3.7).

As seen from the graph, both methods give close results in small initial capitals. However, an increase in the initial capital causes a slight difference.

# Chapter 3

# MARKOV CHAIN APPROACH

In this chapter, the Markov chain model is observed in the classic and modified surplus processes by capital injections. The application of this model in the computation of ruin probability is the subject of various papers [16, 21, 49, 53].

Predicting what will happen at time $n+1$ in a stochastic process is complicated. In general, it depends on all the previous history up to time $n$. However, this prediction can basically be done by adjusting the information at time $n$ in some approaches without further information before time $n$ [64]. Under this condition, let's look at the probability of $X_{i+1}$ at time $n+1$.

$$P(X_{n+1} = x_{n+1} | X_n = x_n, \cdots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n).$$

This equation is known as the Markov property. If a discrete time stochastic process with discrete variables satisfies this property, then this process is called a discrete time Markov chain [64]. This process was named by Andrey Markov.

Let $X_0$,$X_1$,$\cdots$ be a sequence of random variables on the $V$ state space with transition probabilities $p_{i,j} = P(X_{n+1} = j | X_n = i)$, $i, j \in V$. This process is called a homogeneous Markov chain if there is a time independent transition matrix of $X$ [68].

In other words, if $P(X_{n+m} = j | X_m = i) = P(X_n = j | X_0 = i)$ for all $n, m \in N$ and all $i, j \in V$, then $X_n$ are homogeneous Markov chain.

In the transition matrix $P$,

$$\sum_{j\in V} p_{i,j} = 1 \qquad \text{and} \qquad p_{i,j} \geq 0.$$

Let $P^{(m)}$ be the matrix with $m$ step transition probability, then

$$p_{i,j}^{(m)} = P(X_{t+m} = j | X_t = i).$$

For homogeneous and discrete Markov chain, the Chapman-Kolmogorov equation gives [54]

$$p_{i,j}(t_1 + t_2) = \sum_k p_{i,k}(t_1) p_{k,j}(t_2)$$

and

$$P^{t_1+t_2} = P^{t_1} P^{t_2}.$$

For example,

$$\begin{aligned} p_{i,j}^{(2)} &= \sum_k p_{i,k} p_{k,j} \\ &= \sum_k P(X_{t+1} = k | X_t = i) P(X_{t+2} = j | X_{t+1} = k) \\ &= P(X_{t+2} = j | X_t = i). \end{aligned}$$

$\{X_t\}$,$t \geq 0$ is a continuous time Markov chain if

$$P(X_{t+\tau} = j | X_\tau = i, X_\eta = x_\eta, 0 \leq \eta < \tau) = P(X_{t+\tau} = j | X_\tau = i).$$

The Chapman-Kolmogorov equation for continuous time is defined by

$$p_{i,j}(t_1 + t_2) = \int_k p_{i,k}(t_1) p_{k,j}(t_2) dk.$$

## 3.1 Ruin Probability via the Markov chain approach

Modification of the traditional Markov chain approach [10, 11, 72] is taken into consideration in order to compute the ruin probability.

As mentioned in the first chapter, the risk process $R(t)$ of an insurance company is formalized by

$$R(t) = u + ct - S(t) \qquad \text{with} \qquad S(t) = \sum_{i=1}^{N(t)} X_i$$

where $u$ is the initial capital, $c$ is the premium amount at a unit time, $S(t)$ is the compound Poisson process representing the total claim amount up to time $t$, $X_i$ is the i-th claim size, and $N(t)$ is Poisson process representing the number of claims up to time $t$.

An example of the movement of the surplus process is shown in Figure 3.1.

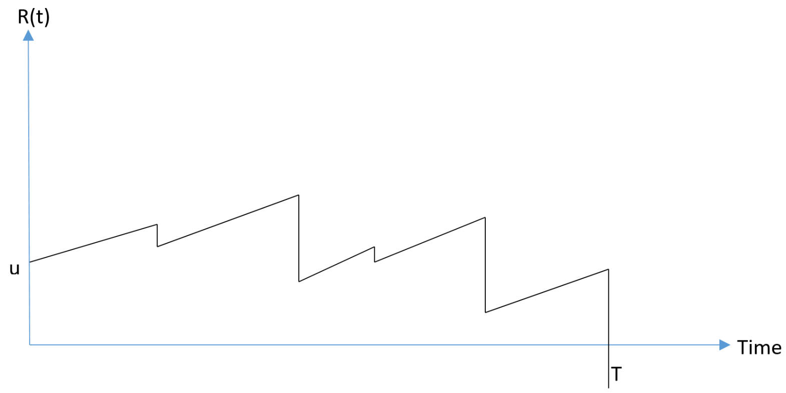



Figure 3.1: Surplus process

After a small time interval $\varepsilon$ is taken into consideration, the movement of the capital can be shown as in Figure 3.2.

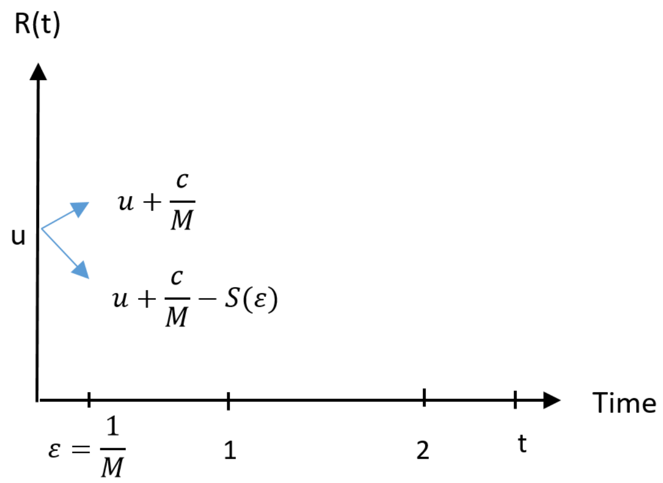



Figure 3.2: Movement of the capital in small time interval $\varepsilon$

Let $P_u(T > t)$ be the probability of non ruin at time $t$ with initial capital $u$.
If $S(\varepsilon) = w$ takes integer values between 0 and $n$, then the non ruin probability can be written in the following form when $u + \frac{c}{M} - w > 0$

$$P_u(T > t) = \sum_{w=0}^{n} P(u \to u + \frac{c}{M} - w) P_{u+\frac{c}{M}-w}(T > t - \epsilon) \qquad \text{for} \quad \varepsilon = \frac{1}{M} \quad (3.1.1)$$

where

$$\begin{aligned} P(u \to u + \frac{c}{M} - w) &= P(R(\varepsilon) = u + \frac{c}{M} - w | R(0) = u) \\ &= \frac{e^{-\lambda\varepsilon}\lambda\varepsilon}{1!} P(X_1 = w) + \frac{e^{-\lambda\varepsilon}(\lambda\varepsilon)^2}{2!} P(X_1 + X_2 = w) \\ &+ \frac{e^{-\lambda\varepsilon}(\lambda\varepsilon)^3}{3!} P(X_1 + X_2 + X_3 = w) + \ldots \qquad \text{for} \quad w \geq 1 \\ P(u \to u + \frac{c}{M}) &= e^{-\lambda\varepsilon} \qquad \text{for} \quad w = 0. \end{aligned}$$

The equation above (3.1.1) can be defined in matrix form for $M = 1$ with respect to different initial capitals.

$$\underbrace{\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ a_{u,0} & a_{u,1} & a_{u,2} & a_{u,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} P_0(T > t - \epsilon) \\ P_1(T > t - \epsilon) \\ P_2(T > t - \epsilon) \\ P_3(T > t - \epsilon) \\ \vdots \\ P_u(T > t - \epsilon) \\ \vdots \end{pmatrix}}_{f(t-\epsilon)} = \underbrace{\begin{pmatrix} P_0(T > t) \\ P_1(T > t) \\ P_2(T > t) \\ P_3(T > t) \\ \vdots \\ P_u(T > t) \\ \vdots \end{pmatrix}}_{f(t)} \tag{3.1.2}$$

where the first matrix $A$ is a our transition matrix consisting of $a_{i,j} = P(i \to j)$.
In the transition matrix $A$, we consider that 0 is the absorption state.
Elements of transition matrix $A$ in d dimensional is defined by

$$A_{i,j} = a_{i,j} = \begin{cases} 1, & \text{for } i = j = 0; \\ 0, & \text{for } i = 0, j \neq 0; \\ 1 - \sum_{j=1}^{d-1} a_{i,j}, & \text{for } j = 0, i \neq 0; \\ P(R_{k+1} = j | R_k = i), & \text{for the other cases} \end{cases}.$$

Note that

$$P_0(T > t) = P_0(T > t - \varepsilon) = 0$$

because 0 is the absorption state.
In this circumstance, it can be written as

$$A(x)f(t-x) = f(t)$$

where $f$ is the column vector function representing non ruin probabilities.
Similarly,

$$A(x+y)f(t) = A(x)A(y)f(t) = f(t+x+y).$$

The capital of an insurance company at time $t$ can be found with the help of $A(x) =$

$A^x$ in the case where the grid size is equal to 1.

If the grid size is equal to $\varepsilon = \frac{1}{M}$, $M \in \mathbf{N}^+$, then

$$A(x) = A^{\frac{x}{\varepsilon}} = A^{xM}.$$

In continuous time, the transition matrix can be found via the generator matrix.

$$A(0) = \lim_{t \to 0} A(t) = I$$

$$A'(0) = \lim_{\varepsilon \to 0} \frac{A(\varepsilon) - I}{\varepsilon} = Q$$

where $Q$ is called the generator of Markov process

$$Q = \begin{pmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & q_{0,5} & \cdots \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} & \cdots \\ q_{2,0} & q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & q_{2,5} & \cdots \\ q_{3,0} & q_{3,1} & q_{3,2} & q_{3,3} & q_{3,4} & q_{3,5} & \cdots \\ q_{4,0} & q_{4,1} & q_{4,2} & q_{4,3} & q_{4,4} & q_{4,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.1.3}$$

The sum of the elements in each row of $Q$ is zero because

$$\sum_{j=0, j \neq i}^{D-1} q_{i,j} = -q_{i,i} \qquad (D \text{ is dimention of the matrix})$$

$$q_{i,j} = \lim_{\varepsilon \to 0} \frac{A_{i,j}(\varepsilon)}{\varepsilon} \geq 0 \qquad \text{and} \qquad q_{i,i} \leq 0.$$

For the small $\varepsilon$,

$$A_{i,j} = q_{i,j}\varepsilon + O(\varepsilon) \qquad \text{for} \qquad i \neq j$$

$$A_{i,i} = 1 + q_{i,i}\varepsilon + O(\varepsilon).$$

Let $N(t)$ be a Poisson process with frequency $\lambda$. Therefore,

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0, 1, 2, \ldots.$$

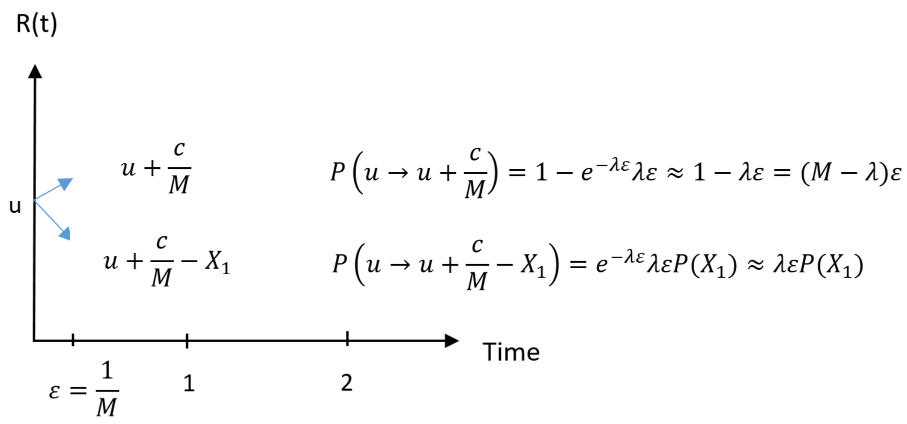
For very small grid size $\varepsilon$ and small claim frequency $\lambda$ in numerical computation, the following computation can be taken into consideration.

$$P(N(\varepsilon)=1)=e^{-\lambda\varepsilon}\lambda\varepsilon \approx \lambda\varepsilon \text{ because } \lambda\varepsilon e^{\lambda\varepsilon}=\lambda\varepsilon(1+\lambda\varepsilon+\frac{(\lambda\varepsilon)^2}{2!}+...)\approx\lambda\varepsilon.$$

If we just consider a case in which the number of claims is equal to zero or one by ignoring more than one because the grid size is very small, then

$$P(N(\varepsilon)=0)=e^{-\lambda\varepsilon}\approx 1-\lambda\varepsilon.$$

For $S(\varepsilon)=X_1$, claim and non claim cases are shown by




For example, if $X_1$ takes integer values between 1 and k with $P(X_1=w)=\frac{1}{k}$ for all $1\le w\le k$, then

$P(u\to u+\frac{c}{M}-1)=\frac{\lambda\varepsilon}{k}$,

$P(u\to u+\frac{c}{M}-2)=\frac{\lambda\varepsilon}{k}$,

$\vdots$

$P(u\to u+\frac{c}{M}-k)=\frac{\lambda\varepsilon}{k}$.

Therefore,

$A_{ij}(\varepsilon)=P(R(\varepsilon)=u+\frac{c}{M}|R(0)=u)=(M-\lambda)\varepsilon$ and $q_{i,j}=(M-\lambda)$ for $j>i$

$A_{ij}(\varepsilon)=P(R(\varepsilon)=u+\frac{c}{M}-w|R(0)=u)=P(X_1=w)\lambda\varepsilon$ and $q_{i,j}=P(X_1=w)\lambda$ for $i<j$.

## 3.2 Discretization of the semigroup

The matrix $A(t)$ is differentiable for all $t \geq 0$ with

$$A'(t) = A(t)Q = QA(t).$$

The solution of the equation with A(0)=I [7,65] is

$$A(t) = e^{Qt}$$

where Q is generator operator of Markovian process.

Rather than analytic formula, the discretization method will be applied to find $A(t)$ by

$$\begin{aligned}
\lim_{\triangle t \to 0} \frac{A(t + \triangle t) - A(t)}{\triangle t} &= A'(t), \\
\text{so} \qquad A(t + \triangle t) &= A(t) + A'(t) \triangle t + O((\triangle t)^2) \\
A(t + \triangle t) &= A(t) + A'(t) \triangle t + \frac{A''(t)(\triangle t)^2}{2!} + O((\triangle t)^3)
\end{aligned} \tag{3.2.4}$$

where $A''(t) = Q^2 A(t)$ because

$$\begin{aligned}
A''(t) = (e^{Qt})'' &= \sum_{k=0}^{\infty} (\frac{Q^k t^k}{k!})'' = \sum_{k=0}^{\infty} Q^k (\frac{(t^k)''}{k!}) = \sum_{k=0}^{\infty} Q^k t^{k-2} \frac{k(k-1)}{k!} = Q^2 \sum_{k=2}^{\infty} Q^{k-2} \frac{t^{k-2}}{(k-2)!} \\
&= Q^2 \sum_{j=0}^{\infty} Q^j \frac{t^j}{j!} = Q^2 e^{Qt} = Q^2 A(t).
\end{aligned}$$

When $A'$ and $A''$ are put into equation (4.3.6), the equation can be written in the following form.

$$A(t + \triangle t) = A(t) + A(t)Q \triangle t + \frac{A(t)Q^2(\triangle t)^2}{2!} + O((\triangle t)^3). \tag{3.2.5}$$

With equation (3.2.5), better approximation in order to find $A(t)$ is obtained.

**Example 3.2.1** Let's consider a case where the premium rate $c = 1$, the grid size

$\epsilon = 0.01, (M = 100)$ and $X_i = \{1, 2, 3\}$ with probability $p_1$, $p_2$ and $p_3$ defined for very small claim frequency $\lambda$ by

$$\begin{aligned} p_1 &= P(u \to u + c\epsilon - 1) = \frac{e^{-\lambda\varepsilon}\lambda\varepsilon}{3} \approx \frac{\lambda\varepsilon}{3}, \\ p_2 &= P(u \to u + c\epsilon - 2) = \frac{e^{-\lambda\varepsilon}\lambda\varepsilon}{3} \approx \frac{\lambda\varepsilon}{3}, \\ p_3 &= P(u \to u + c\epsilon - 3) = \frac{e^{-\lambda\varepsilon}\lambda\varepsilon}{3} \approx \frac{\lambda\varepsilon}{3} \end{aligned}$$

and no claim probability is

$$p = 1 - p_1 - p_2 - p_3 = P(u \to u + c\epsilon) = 1 - e^{-\lambda\varepsilon}\lambda\varepsilon \approx (M - \lambda)\epsilon.$$

Let $A$ be the transition matrix over time unit $\varepsilon$, and its elements be consist of $0, p, p_1, p_2$, and $p_3$.

Notice that

$$p + p_1 + p_2 + p_3 = 1.$$

In this circumstance, the matrix form can be shown as below:

$$Af(t - 0.01) = A^{t_1 M} f(t - t_1) = A^{t_2 M} f(t - t_2) = f(t) \text{ for } t > t_1, t_2.$$

$$
\begin{array}{cc}
u & \begin{array}{cccccccccccccccccccc} 0 & 0.01 & 0.02 & 0.03 & \cdots & \cdots & \cdots & 1 & 1.01 & 1.02 & \cdots & \cdots & 2 & 2.01 & 2.02 & \cdots & \cdots & 3 & 3.01 & 3.02 \end{array} \\
\begin{array}{c} 0 \\ 0.01 \\ 0.02 \\ \\ 0.99 \\ 1 \\ 1.01 \\ \\ 1.99 \\ 2 \\ 2.01 \\ \\ 2.99 \\ 3 \\ 3.01 \\ \\ \end{array} &
\left(\begin{array}{cccccccccccccccccccc}
1 & & & & \cdots & \cdots & \cdots & & & & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
1-p & & p & & \cdots & \cdots & \cdots & & & & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
1-p & & & p & \cdots & \cdots & \cdots & & & & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
1-p & & & & \cdots & \cdots & \cdots & p & & & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
p_2+p_3 & p_1 & & & \cdots & \cdots & \cdots & & p & & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
p_2+p_3 & & p_1 & & \cdots & \cdots & \cdots & & & p & \cdots & \cdots & & & & \cdots & \cdots & & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
p_2+p_3 & & & & \cdots & \cdots & \cdots & p_1 & & & \cdots & \cdots & p & & & \cdots & \cdots & & & \cdots \\
p_3 & p_2 & & & \cdots & \cdots & \cdots & & p_1 & & \cdots & \cdots & & p & & \cdots & \cdots & & & \cdots \\
p_3 & & p_2 & & \cdots & \cdots & \cdots & & & p_1 & \cdots & \cdots & & & p & \cdots & \cdots & & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
p_3 & & & & \cdots & \cdots & \cdots & p_2 & & & \cdots & \cdots & p_1 & & & \cdots & \cdots & p & & \cdots \\
 & p_3 & & & \cdots & \cdots & \cdots & & p_2 & & \cdots & \cdots & & p_1 & & \cdots & \cdots & & p & \cdots \\
 & & p_3 & & \cdots & \cdots & \cdots & & & p_2 & \cdots & \cdots & & & p_1 & \cdots & \cdots & & & p\cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\end{array}
\underbrace{\begin{pmatrix} P_0(T>t-0.01) \\ P_{0.01}(T>t-0.01) \\ P_{0.02}(T>t-0.01) \\ \vdots \\ P_{0.99}(T>t-0.01) \\ P_1(T>t-0.01) \\ P_{1.01}(T>t-0.01) \\ \vdots \\ P_{1.99}(T>t-0.01) \\ P_2(T>t-0.01) \\ P_{2.01}(T>t-0.01) \\ \vdots \\ P_{2.99}(T>t-0.01) \\ P_3(T>t-0.01) \\ P_{3.01}(T>t-0.01) \\ \vdots \end{pmatrix}}_{f(t-0.01)}
=
\underbrace{\begin{pmatrix} P_0(T>t) \\ P_{0.01}(T>t) \\ P_{0.02}(T>t) \\ \vdots \\ P_{0.99}(T>t) \\ P_1(T>t) \\ P_{1.01}(T>t) \\ \vdots \\ P_{1.99}(T>t) \\ P_2(T>t) \\ P_{2.01}(T>t) \\ \vdots \\ P_{2.99}(T>t) \\ P_3(T>t) \\ P_{3.01}(T>t) \\ \vdots \end{pmatrix}}_{f(t)}
$$

Now, let's look at the generator matrix of $A$

$$
Q=\begin{array}{c|ccccccccccccccccccccc}
u & 0 & 0.01 & 0.02 & 0.03 & \cdots & \cdots & 0.99 & 1 & 1.01 & 1.02 & \cdots & 1.99 & 2 & 2.01 & 2.02 & \cdots & 2.99 & 3 & 3.01 & 3.02 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0.01 & \lambda & -M & (M-\lambda) & & \cdots & \cdots & & & & & \cdots & & & & & \cdots & & & & & \cdots \\
0.02 & \lambda & & -M & (M-\lambda) & \cdots & \cdots & & & & & \cdots & & & & & \cdots & & & & & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0.99 & \lambda & & & & \cdots & \cdots & -M & (M-\lambda) & & & \cdots & & & & & \cdots & & & & & \cdots \\
1 & \frac{2\lambda}{3} & \frac{\lambda}{3} & & & \cdots & \cdots & & -M & (M-\lambda) & & \cdots & & & & & \cdots & & & & & \cdots \\
1.01 & \frac{2\lambda}{3} & & \frac{\lambda}{3} & & \cdots & \cdots & & & -M & (M-\lambda) & \cdots & & & & & \cdots & & & & & \cdots \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
1.99 & \frac{2\lambda}{3} & & & & \cdots & \cdots & & \frac{\lambda}{3} & & & \cdots & -M & (M-\lambda) & & & \cdots & & & & & \cdots \\
2 & \frac{\lambda}{3} & \frac{\lambda}{3} & & & \cdots & \cdots & & & \frac{\lambda}{3} & & \cdots & & -M & (M-\lambda) & & \cdots & & & & & \cdots \\
2.01 & \frac{\lambda}{3} & & \frac{\lambda}{3} & & \cdots & \cdots & & & & \frac{\lambda}{3} & \cdots & & & -M & (M-\lambda) & \cdots & & & & & \cdots \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\
2.99 & \frac{\lambda}{3} & & & & \cdots & \cdots & & \frac{\lambda}{3} & & & \cdots & & \frac{\lambda}{3} & & & \cdots & -M & (M-\lambda) & & & \cdots \\
3 & & \frac{\lambda}{3} & & & \cdots & \cdots & & & \frac{\lambda}{3} & & \cdots & & & \frac{\lambda}{3} & & & & -M & (M-\lambda) & & \cdots \\
3.01 & & & \frac{\lambda}{3} & & \cdots & \cdots & & & & \frac{\lambda}{3} & \cdots & & & & \frac{\lambda}{3} & \cdots & & & -M & (M-\lambda) & \cdots \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{array}
$$

The transition matrix can be found via the discretization method by using $Q$ matrix.

$$A(t+\epsilon) = A(t) + A(t)Q\epsilon + \frac{A(t)Q^2\epsilon^2}{2!} + O(\epsilon^3). \tag{3.2.6}$$

Let us define the transition matrix $A$ as transition probabilities over a single time period for grid size $\varepsilon = 1$.

$$A_{i,j} = P(R_{k+1} = j | R_k = i).$$

Let $A^n$ denote the matrix with $A(n)_{i,j} = P(R_n = j|R_0 = i)$ where $A(n)_{i,j}$ is an element of $A^n$.

**Proposition 17** *Assuming that* $0$ *is an absorption state in the* $d \times d$ *transition matrix, the ruin and non ruin probability are defined by*

$$\begin{aligned} P_u(T \leq t) &= (1+o(1))A(t)_{u,0}, \\ P_u(T > t) &= 1 - P(T \leq t|R(0) = u) \\ &= (1+o(1))\sum_{j=1}^{d-1} A(t)_{u,j} \end{aligned} \tag{3.2.7}$$

*where the error terms depend on the grid size.*
*When the grid size is equal to* $\varepsilon$*, ruin and non ruin probability are defined by*

$$\begin{aligned} P_u(T \leq t) &= (1+o(\varepsilon))A(t)_{u,0} \quad (3.2.8) \\ &= (1+o(\varepsilon))A^{[\frac{t}{\varepsilon}]}_{u,0}, \\ P_u(T > t) &= (1+o(\varepsilon))\sum_{j=1}^{d-1} A(t)_{u,j\varepsilon} \\ &= (1+o(\varepsilon))\sum_{j=1}^{d-1} A^{[\frac{t}{\varepsilon}]}_{u,j\varepsilon} \quad (3.2.9) \end{aligned}$$

*where* $[\frac{t}{\varepsilon}]$ *is an integer part of* $\frac{t}{\varepsilon}$*.*

**Gambler's Ruin problem via the Markov chain approach**

Let's consider the game mentioned in Section 2.2.2. In the game, $\pounds z_1$ and $\pounds z_2$ are the initial fortunes of the two players, and $p$ and $q = 1-p$ are the winning and losing probabilities of the first player. In each gamble of the game, £1 will be transferred from loser to winner in each event. The game will end when one player has all the money or the other has lost all of his or her own money. The main objective of the game is to reach the total fortune of £$z_1 + z_2$ without ruining.

$X_n$ is denoted as the fortune of the first player after the nth gamble. In this circumstance, $X_n$ is a Markov process with $a_{00} = a_{z_1+z_2\ z_1+z_2} = 1$ because 0 and $z_1 + z_2$ are up and down barriers. The corresponding transition matrix is defined by

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ z_1+z_2 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & z_1+z_2 \end{array} \\
\begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots & a_{0\ z_1+z_2} \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots & a_{1\ z_1+z_2} \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots & a_{2\ z_1+z_2} \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots & a_{3\ z_1+z_2} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
a_{z_1+z_2\ 0} & a_{z_1+z_2\ 1} & a_{z_1+z_2\ 2} & a_{z_1+z_2\ 3} & \cdots & a_{z_1+z_2\ z_1+z_2}
\end{pmatrix}
\end{array}
$$

$$
= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ z_1+z_2 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & z_1+z_2 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
q & 0 & p & 0 & 0 & \cdots & 0 \\
0 & q & 0 & p & 0 & \cdots & 0 \\
0 & 0 & q & 0 & p & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{array}.
$$

The non ruin probability of the first player until the nth gamble is found by

$$P_{z_1}(T > n) = (1 + o(1)) \sum_{j=1}^{z_1+z_2} A^n_{z_1,j}.$$

## 3.3 Capital injection and reduction

Insurance companies may be exposed to capital injections or withdrawal because of unpredictable economic and natural processes such as:

(i) risk, investment, unpredictable financial crisis, payment to shareholders, tax, and charges in the tax system

(ii) volcanic eruptions, earthquakes, landslides, mudflows etc.

Capital injection allows insurance companies to keep their surplus process above a certain fixed level in order to decrease the ruin probability. Therefore, capital injection plays an important role in the insurance sector.

The modified surplus process with the injections at time $t_i$ with amounts $a_i$ for $i = 1, 2, ...k$ can be formalized as

$$\overline{R(t)} = u + ct - S(t) + \sum_{j=1}^{k} a_i I_{(t \geq t_j)}$$

where

$\overline{R(t_1)} = R(t_1) + a_1,$

$\overline{R(t_2)} = R(t_2) + a_1 + a_2,$

$\vdots$

$\overline{R(t_k)} = R(t_k) + a_1 + a_2 + \cdots + a_k.$

An example of one capital injection at time $t_1$ with amount $a$ can be seen in Figure 3.3.

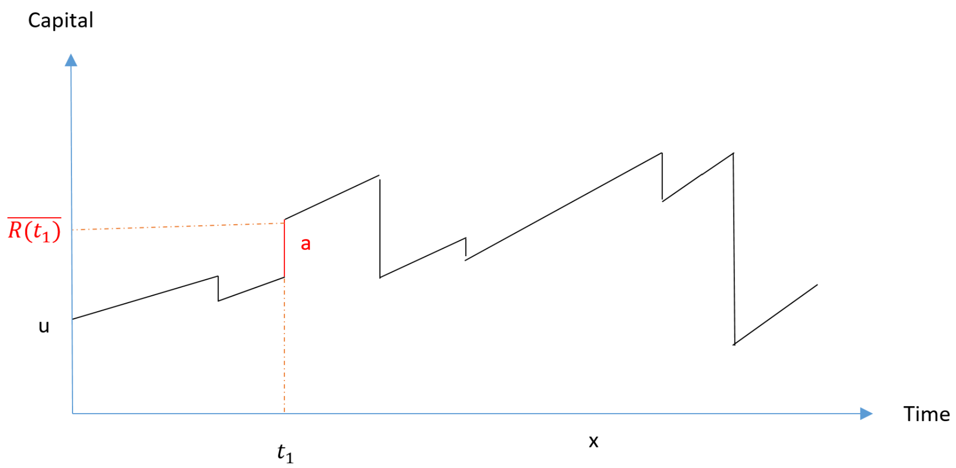



Figure 3.3: Surplus process with a capital injection

An example of movement of the surplus process exposed to two capital injections with the amount of $a_1$ and $a_2$ at time $t_1$ and $t_2$, respectively is as

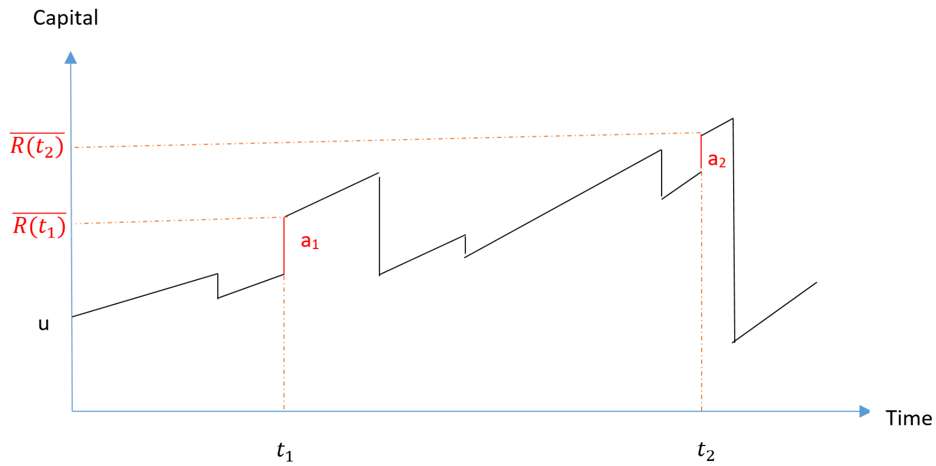



Figure 3.4: Surplus process with capital injections

Now, let's introduce a shift operator under the assumption that zero is the absorption state. The shift operator is necessary to change the capital in injection or withdraw times.

$$K(a)\overline{R_t} = \begin{cases} \overline{R_t} + a, & \text{if } \overline{R_t} + a > 0 \\ 0, & \text{if } \overline{R_t} + a \leq 0 \end{cases}.$$

$K(a)$ means it will give rise to a change in the reserve at amount $a$. From equation (3.2.9) and the shift operator, we derive the following result.

**Proposition 18** *With the small and fixed positive grid size $\varepsilon > 0$, consider the surplus process exposed to $k$ times capital injections or reductions at time $t_i$ with amount $a_i$ ,$i = 1, \ldots, k$, respectively. Then, ruin and non ruin probability can be found by*

$$P_u(T \leq t) = (1 + o(\varepsilon))\Big(A^{[t_1/\varepsilon]}K(a_1)A^{[(t_2-t_1)/\varepsilon]}K(a_2)\ldots A^{[(t_k-t_{k-1})/\varepsilon]}K(a_k)A^{[(t-t_k)/\varepsilon]}\Big)_{u,0},$$
$$P_u(T > t) = (1 + o(\varepsilon))\sum_{j=1}^{d-1}\Big(A^{[t_1/\varepsilon]}K(a_1)A^{[(t_2-t_1)/\varepsilon]}K(a_2)\ldots A^{[(t_k-t_{k-1})/\varepsilon]}K(a_k)A^{[(t-t_k)/\varepsilon]}\Big)_{u,j\varepsilon} \tag{3.3.10}$$

*where the error term depends on the grid size $\varepsilon$.*

Let the grid size be equal to $\varepsilon = \frac{1}{M}$, then the matrix form of the $K$ shift matrix for $a > 0$ is generated by

$$K(a) = \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & aM & aM+1 & aM+2 & aM+3 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \end{array}.$$

In the reduction case ($a < 0$), the matrix form of $K$ is defined as below.

$$K(a) = \begin{array}{c} 0 \\ \varepsilon \\ 2\varepsilon \\ 3\varepsilon \\ \\ -a-\varepsilon \\ -a \\ -a+\varepsilon \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

## 3.4 Discretization strategy on claim distributions

Discretization strategy [13] in applied mathematics is a method to transform variables from continuous values into discrete counterparts.

In our numerical computations, we generally study discrete claim sizes by using exponential distribution and Gaussian distribution even though they are continuous functions.

The probability density function of an exponential distribution with mean $m$ is defined by

$$f(x) = \begin{cases} \frac{1}{m}e^{-\frac{1}{m}x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

The probability density function of a Gaussian distribution with mean $m$ and variance $\sigma^2$ is defined by

$$f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(x-m)^2}{2\sigma^2}}.$$

$\int\limits_{-\infty}^{\infty} f(x)dx = 1$ for both distributions.

According to the discretization strategy in this thesis, a probability mass function is the discrete version of density probability function that is defined for an exponential distribution by

$$P(X = x) = \begin{cases} \frac{1}{m}e^{-\frac{1}{m}x}\varepsilon b & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\varepsilon$ is the grid size, and $b$ is the normalizing constant that is defined by

$$b = \frac{1}{\sum\limits_{k=1}^{\infty} \frac{1}{m}e^{-\frac{1}{m}k\varepsilon}\varepsilon}.$$

Similarly, the probability mass function for discretized of Gaussian distribution is defined by

$$P(X = x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(x-m)^2}{2\sigma^2}} \varepsilon b \tag{3.4.11}$$

with the normalizing constant

$$b = \frac{1}{\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(k\varepsilon-m)^2}{2\sigma^2}} \varepsilon}.$$

Notice that we consider a sum over positive values, since claims are positive values.

With the normalizing constant,

$$\sum_x P(X = x) = 1.$$

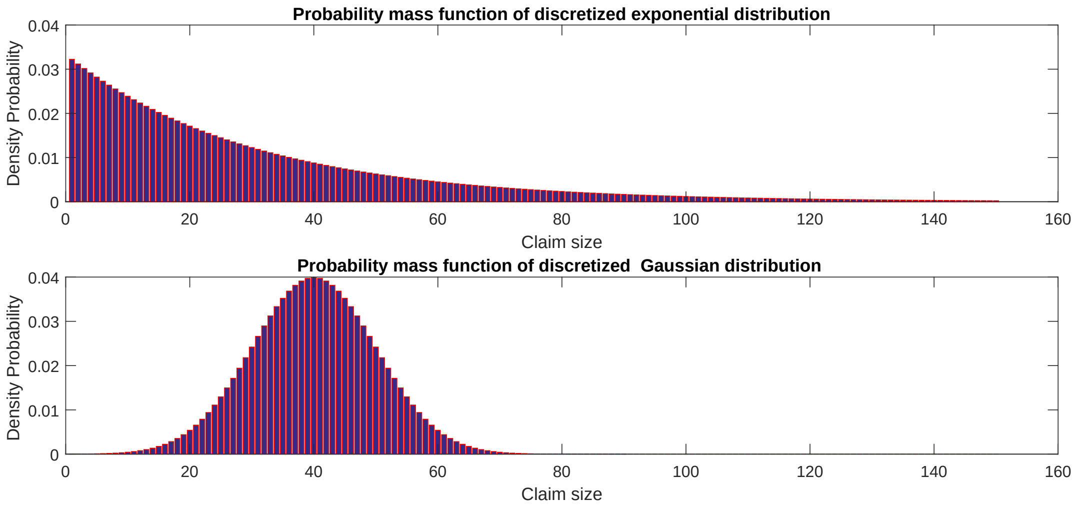



Figure 3.5: Discretization of the distributions

## 3.5 Results

1) We observe ruin probability via the Markov chain approach with respect to different initial capitals and times assuming that claim sizes are integer values with discretized exponential distribution with mean $m$

$$P(X = x) = \frac{\frac{1}{m}e^{-\frac{1}{m}x}}{\sum\limits_{k=1}^{\infty} \frac{1}{m}e^{-\frac{1}{m}k}}. \tag{3.5.12}$$

Ruin probabilities created by using a transition matrix or generator matrix are listed in Table 3.1 for claim premium=1, claim frequency =0.02, claim mean=45, and grid size =1.

Table 3.1: Ruin probability via Markov chain approach

| Initial capital | Time | Ruin probability | Ruin probability via generator matrix |
|---|---|---|---|
| 5 | 50 | 0.4356 | 0.4334 |
| 5 | 100 | 0.5652 | 0.5641 |
| 5 | 150 | 0.6296 | 0.6289 |
| 5 | 200 | 0.6696 | 0.6691 |
| 10 | 50 | 0.4048 | 0.4028 |
| 10 | 100 | 0.5346 | 0.5334 |
| 10 | 150 | 0.601 | 0.6003 |
| 10 | 200 | 0.6429 | 0.6424 |
| 20 | 50 | 0.3493 | 0.3476 |
| 20 | 100 | 0.4775 | 0.4764 |
| 20 | 150 | 0.5469 | 0.5462 |
| 20 | 200 | 0.5919 | 0.5914 |
| 30 | 50 | 0.301 | 0.2996 |
| 30 | 100 | 0.4257 | 0.4247 |
| 30 | 150 | 0.4969 | 0.4962 |
| 30 | 200 | 0.5441 | 0.5436 |

It is obvious from Table 3.1, an increase in the initial capital gives rise to a decrease in the ruin probability, while time increase causes bigger ruin probability as expected. According to the table, it seems that the ruin probabilities via the transition matrix and generator matrix give very close results.

2) Now, let's assume the claim sizes are integers and their distribution look like a discretized normal distribution as below.

$$P(X = x) = \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(x-m)^2}{2\sigma^2}}}{\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(k-m)^2}{2\sigma^2}}}. \tag{3.5.13}$$

For the same values, and the standard derivation is 10, the results are produced in the following table.

Table 3.2: Ruin probability via Markov chain approach

| Initial capital | Time | Ruin probability | Ruin probability via generator matrix |
|---|---|---|---|
| 5 | 50 | 0.554 | 0.5503 |
| 5 | 100 | 0.6574 | 0.6562 |
| 5 | 150 | 0.7067 | 0.7061 |
| 5 | 200 | 0.7367 | 0.7364 |
| 10 | 50 | 0.5214 | 0.5198 |
| 10 | 100 | 0.6282 | 0.6274 |
| 10 | 150 | 0.6801 | 0.6796 |
| 10 | 200 | 0.712 | 0.7116 |
| 20 | 50 | 0.4481 | 0.4483 |
| 20 | 100 | 0.5618 | 0.5615 |
| 20 | 150 | 0.6191 | 0.6187 |
| 20 | 200 | 0.6551 | 0.6547 |
| 30 | 50 | 0.3688 | 0.3676 |
| 30 | 100 | 0.4852 | 0.4843 |
| 30 | 150 | 0.5475 | 0.5468 |
| 30 | 200 | 0.5877 | 0.5873 |

3)Now let us state the ruin probability of surplus process exposed to one capital injection with respect to different injection times and injection amounts for both claim size distributions mentioned above.

The ruin probabilities at time 200 with the initial capital is 5, the premium rate is 1, the claim frequency is 0.03, the claim size mean is 30 and the standard deviation is 10, is displayed in the following table.

Table 3.3: Ruin probability via Markov chain approach

| Capital injection time | Capital injection amount | Ruin probability (Discretized exponential distribution) | Ruin probability (Discretized Gaussian distribution) |
|---|---|---|---|
| 10 | 5 | 0.6832 | 0.7251 |
| 10 | 10 | 0.6578 | 0.6899 |
| 10 | 15 | 0.6335 | 0.6548 |
| 50 | 5 | 0.6967 | 0.7434 |
| 50 | 10 | 0.6845 | 0.7291 |
| 50 | 15 | 0.6728 | 0.7158 |
| 100 | 5 | 0.7027 | 0.7504 |
| 100 | 10 | 0.6962 | 0.7431 |
| 100 | 15 | 0.6902 | 0.7364 |
| 150 | 5 | 0.7062 | 0.7544 |
| 150 | 10 | 0.7031 | 0.7508 |
| 150 | 15 | 0.7003 | 0.7477 |

According to Table 3.3, early injection time and increased injection amount result in less ruin probability. It can also be seen from the table exponential and Gaussian claim size distributions give close but different results.
To minimise the computational error amount in the examples,

- a small claim frequency in the small time grid size is chosen
- or the probability of a large claim number $N(\varepsilon)$ in the grid size is taken into account.

## 3.6 Appell Polynomial Approach in modified surplus processes

As mentioned in Section 2.4,

$$P_n(x) = P(S_x = n \quad and \quad T > x) = e^{-\lambda x} B_n(x)$$

where

$$B_n(x) = \begin{cases} e_n(x) & \text{when } 0 \le n \le u \\ \sum\limits_{m=0}^{u} e_m(\frac{m-u}{c}) \frac{cx-n+u}{cx-m+u} e_{n-m}(x + \frac{u-m}{c}) & \text{when } n > u \end{cases}$$

with

$$e_n(x) = \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} q_n^{*k} \qquad \text{and} \qquad q_n^{*k} = X_1 + X_2 + ... + X_k = n.$$

Now, elements of transition matrix $A$ can be computed by applying the Appell polynomial approach

$$\begin{aligned} A(\varepsilon)_{i,j} = a_{i,j} = P(i \to j) &= P(S_\varepsilon = n | R_0 = i) \\ &= P(S_\varepsilon = n \quad and \quad T > \varepsilon | R_0 = i) \qquad \text{because } j > 0 \\ &= e^{-\lambda\varepsilon} B_n(\varepsilon) \end{aligned}$$

where

$$\begin{aligned} n = S_\varepsilon &= R_0 + c\varepsilon - R_\varepsilon \\ &= i + c - j \qquad \text{for } \varepsilon = 1. \end{aligned}$$

Therefore,

$$A(\varepsilon)_{i,j} = \begin{cases} e^{-\lambda} e_{i+c-j}(1) & \text{when } 0 \leq i + c - j \leq i \\ e^{-\lambda} \sum\limits_{m=0}^{i} e_m(\frac{m-i}{c}) \frac{c-(i+c-j)+i}{c-m+i} e_{(i+c-j)-m}(1 + \frac{i-m}{c}) & \text{when } i + c - j > i \end{cases}.$$

After obtaining the transition matrix by using the Appell polynomial approach, the ruin and non ruin probability of the modified surplus process is found by

$$\begin{aligned} P_u(T \leq t) &= (1 + o(\varepsilon))\Big(A^{[t_1/\varepsilon]} K(a_1) A^{[(t_2-t_1)/\varepsilon]} K(a_2) \ldots A^{[(t_k-t_{k-1})/\varepsilon]} K(a_k) A^{[(t-t_k)/\varepsilon]}\Big)_{u,0}, \\ P_u(T > t) &= (1 + o(\varepsilon)) \sum_{j=1}^{d-1} \Big(A^{[t_1/\varepsilon]} K(a_1) A^{[(t_2-t_1)/\varepsilon]} K(a_2) \ldots A^{[(t_k-t_{k-1})/\varepsilon]} K(a_k) A^{[(t-t_k)/\varepsilon]}\Big)_{u,j\varepsilon} \end{aligned} \tag{3.6.14}$$

where the error term depends on the grid size $\varepsilon$.

# Chapter 4

# QUANTUM MECHANICS APPROACH

In this chapter, the ruin probability of an insurance company in classical and modified surplus processes is computed via the quantum mechanics approach. Some parts of this chapter can also be found in a paper entitled "Ruin Probability via Quantum Mechanics Approach" [76].

## 4.1 Introduction to Quantum Mechanics

Quantum mechanics consists of laws that provide us a mode of description for microscopic systems. Since the beginning of the twentieth century, scientists have used quantum mechanics to explain the structure of atoms and molecules, and some of the properties of electromagnetic radiation [19] [66]. The quantum theory is a general framework, and it is about what is possible or impossible rather than what is in reality [34].

The universe is governed by amplitudes. Dirac showed a special way for amplitudes. According to Dirac notation $\langle\alpha|$ and $|\beta\rangle$ are called bra and ket, respectively. They represent a state vector or wavefunction.

The sum of two bras or kets gives another bra or ket.

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

or

$$\langle\alpha| + \langle\beta| = \langle\gamma| .$$

For scalar $v$ , then

$$v\,|\alpha\rangle = |\alpha\rangle\, v.$$

$\langle\alpha|\beta\rangle$ represent an amplitude for an event [15]. For example, if $|x\rangle$ is a state on state space $V$ and function of f, then $\langle f|x\rangle$ is also event amplitude.
Bra and ket form a scalar product together:

$$\langle\alpha|\beta\rangle = \int_{-\infty}^{\infty} dx \alpha^*(x)\beta(x) = \langle\beta|\alpha\rangle^* .$$

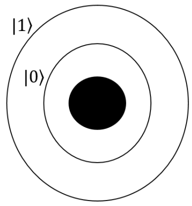
Each wave function can be written as the sum of basis state vectors:

$$|\beta\rangle = \lambda_1 |\beta_1\rangle + \lambda_2 |\beta_2\rangle + ....$$

Now we consider the discrete basis. Let $|0\rangle , |1\rangle , |2\rangle , ..., |k-1\rangle$ be the basis states. The superposition is denoted as a linear combination of basis states

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle + ... + \alpha_{k-1} |k-1\rangle$$

where $\alpha_i \in \mathbb{C}$ and $\sum_i |\alpha_i| = 1$.
If the system level is two, they are called qubits.
Let's consider the two-system level as Hydrogen atom, and define $|0\rangle$ and $|1\rangle$ as the ground energy state of the electron and the first energy state of the electron, respectively. The electron can be found in some linear superposition of any of the two energy levels.



In quantum mechanics, qubits 0 and 1 are represented as $|0\rangle$ and $|1\rangle$ that form a two-dimensional basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An arbitrary qubit $\alpha$ is a linear superposition of the basis states.

$$\alpha = \alpha_1 |0\rangle + \alpha_2 |1\rangle \quad \text{where} \quad \alpha_1^2 + \alpha_2^2 = 1.$$

The combination of two qubits can be done with the help of a tensor product.

**Definition 19 (Tensor Product)** *Let $V_1$ and $V_2$ be two vector spaces. Then the tensor product operator is defined by*

$$\otimes : V_1 \times V_2 \to V_1 \otimes V_2.$$

*For*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix},$$

*the tensor product of the two matrices is*

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{1,1} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{1,2} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \\ a_{2,1} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{2,2} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}. \end{aligned}$$

Similarly, a tensor product of two kets is found by

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Before giving the Fourier transform in quantum mechanics, it will be shown in classical probability in order to demonstrate the differences between the two.

## 4.2 Fourier Transform

The Fourier transform is the decomposition of a time function. The classical Fourier transform of a function f on $\mathbb{R}$ is defined by

$$F(p) = \int_{-\infty}^{\infty} f(t)e^{-ipt}dt,$$

where $F$ is function of real variable $p$ while $F(p)$ is a complex number.
The inverse of the Fourier transform is then defined by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)e^{ipt}dp.$$

If $F(p)$ is put into the equation above, $f(t)$ is obtained.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)e^{ipt}dp &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(t_2)e^{-ipt_2}dt_2\Big)e^{ipt}dp \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t_2)\Big(\int_{-\infty}^{\infty} e^{-ip(t_2-t)}dp\Big)dt_2 \\
&= \int_{-\infty}^{\infty} f(t_2)\delta(t_2-t)dt_2 \\
&= f(t).
\end{aligned}$$

**Example 4.2.1** Let's look at the Fourier transform of $f(t) = e^{-2|t|}$.

$$\begin{aligned} F(p) = \int_{-\infty}^{\infty} e^{-2|t|} e^{-ipt} dt &= \int_{-\infty}^{0} e^{2t} e^{-ipt} dt + \int_{0}^{\infty} e^{-2t} e^{-ipt} dt \\ &= \frac{1}{2 - ip} + \frac{1}{2 + ip} \\ &= \frac{4}{4 + p^2}. \end{aligned}$$

**Definition 20 (Parseval's theorem)**

*Let $f(x)$ and $g(x)$ be integrable functions of the Fourier transform $F(\xi)$ and $G(\xi)$, respectively, then*

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx &= \int_{-\infty}^{\infty} F(\xi)\overline{G(\xi)}d\xi \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt \int_{-\infty}^{\infty} \overline{g(t_2)}e^{-i\xi t_2}dt_2 d\xi. \end{aligned} \tag{4.2.1}$$

*where $\overline{g(x)}$ and $\overline{G(\xi)}$ are complex conjugates of $g(x)$ and $G(\xi)$, respectively.*

**Definition 21 (Plancherel theorem)**

*For $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, the norm of a function's squared is equal to the norm of its Fourier transform's squared*

$$\int_{-\infty}^{\infty} ||f(x)||^2 dx = \int_{-\infty}^{\infty} ||F(\xi)||^2 d\xi \tag{4.2.2}$$

where

$$F(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi\xi x}dx.$$

The Plancherel theorem is related to the Parseval theorem. When we get $f = g$ in the Parsevel theorem, equation (4.2.2) is obtained.

As seen from equation (4.2.1), there are three integrals on the right side of the equation. The Parseval's theorem will be shown in the next section by getting rid of the integrals with the help of the Dirac notation.

## 4.3 Quantum Mechanics

In quantum mechanics [5, 14, 26, 32, 42, 70], $< x|A|x' >$ is called a propagator, where $\langle x|$ and $|x'\rangle$, respectively called *bra* and *ket*, are used to define quantum states. The propagator gives the probability (amplitude) for the particle to travel in a given space time from point $(x, t_1)$ to point $(x', t_2)$.

Let's start with

$$|x\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \text{ x-th position and } \langle x| = \begin{pmatrix} 0 & \cdots & 1 & 0 & \cdots \end{pmatrix}$$

then

$$\langle x|y\rangle = \delta_{x-y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Let $A$ and $|x\rangle$ be the $n+1 \times n+1$ dimension transition matrix and the $n \times 1$ vector as follows:

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad |x\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \tag{4.3.3}$$

In this circumstance, the propagator can be showed by

$$\langle x| A |x'\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

$$= a_{22}.$$

$\langle x| A |x'\rangle$ is a bilinear form on x and x'.

**Example 4.3.1** Let's compute $\langle 1| A |2\rangle$ for $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ by applying the completeness equation $I = \sum_p \langle p|p\rangle$.
Without the resolution of the identity, the result is

$$\langle 1| A |2\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = a_{12}.$$

Now, it is done via the resolution of the identity

$$\begin{aligned} \langle x| A |x'\rangle &= \langle x| AI |x'\rangle \\ &= \sum_p \langle x| A |p\rangle \langle p|x'\rangle . \end{aligned}$$

Therefore,

$$\begin{aligned} \langle 1| A |2\rangle &= \sum_p \langle 1| A |p\rangle \langle p|2\rangle \\ &= \langle 1| A |1\rangle \langle 1|2\rangle + \langle 1| A |2\rangle \langle 2|2\rangle + \langle 1| A |3\rangle \langle 3|2\rangle \\ &= a_{12} \end{aligned}$$

because of $\langle 1|2\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$ and $\langle 3|2\rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.$

**The Fourier transform** of $|x\rangle$ to the momentum space is

$$\begin{aligned} \langle x|x'\rangle = \delta(x - x') &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|p\rangle \langle p|x'\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')}. \end{aligned} \tag{4.3.4}$$

Therefore, $\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p >< p|$ is the resolution of the identity with respect to the momentum basis $|p\rangle$ with the scalar product $\langle x|p\rangle = e^{ixp}$.

Let's write **Parseval's theorem** in 4.2.1 with the Dirac notation,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \langle f|g\rangle = \langle f|\, I\, |g\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle f|p\rangle \langle p|g\rangle$$

where $I = \int_{-\infty}^{\infty} \frac{dp}{2\pi} |p >< p|$ is the completeness equation for the momentum basis $|p\rangle$ with the scalar product $\langle f|p\rangle = f(p) = e^{ixp}$.

The advantage of this equation is that we have just one integral in 4.3, in contrast with 4.2.1.

The starting point for $A = e^{-tH}$ is the following formalism for the homogeneous continuous time and continuous space random walk from point $(x, T)$ to $(x', T + t)$ [5,6].

$$P(x \xrightarrow{t} x') =< x|e^{-tH}|x' >= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-tH}|p >< p|x' > \tag{4.3.5}$$

with the identity resolution $\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p >< p'| = I$.

Similar formalism for the homogeneous continuous time discrete space random walk is defined with the completeness equation now being $\int_{0}^{2\pi} \frac{dp}{2\pi} |p >< p'| = I$ by

$$P(x \xrightarrow{t} x') =< x|e^{-tH}|x' >= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-tH}|p >< p|x' > \qquad (4.3.6)$$

where $< x|p >= e^{ixp}$ and $< p|x >= e^{-ixp}$.

In this quantum mechanics formalism, $A = e^{-tH}$ is a transition operator, and $H$ is the Hamiltonian operator. In most of our cases, $H = -Q$ where $Q$ is the generator matrix.

The Hamiltonian operator represents the total energy of a system, so it is equal to the sum of kinetic energy and potential energy.

For example, the Schrodinger Hamiltonian is defined by

$$H = \bar{T} + V$$

where $\bar{T}$ and $V$ are kinetic and potential energy operators, respectively.

The kinetic energy operator is defined by

$$\bar{T} = \frac{p^2}{2m} = \frac{-h}{2m} \bigtriangledown^2$$

where $m$ is the mass of the particle, and the momentum operator is

$$p = -ih \bigtriangledown .$$

In this circumstance, the Hamiltonian is defined by

$$\begin{aligned} H &= \bar{T} + V \\ &= \frac{-h}{2m} \bigtriangledown^2 + V. \end{aligned}$$

The link between the quantum formalism and the classical notions from the Markov process theory is established via the Hamiltonian operator.

The Markovian stochastic process $X_t$ is characterized by a generator $Q$ and the continuous semigroup $A_t = e^{tQ}$. As such, when $H = -Q$, we have the explicit equivalence.

In general, the Hamiltonian operator $H$ does not need to be a generator.

In Dirac formalism, the Hamiltonian operator is defined in Hilbert space.

**Definition 22 (Hilbert Space)**

*A Hilbert space is a real (or complex) complete inner product space. Let $V$ be an inner product space where $\langle x, y\rangle : V \times V \to \mathbb{C}$.*
*For $x, y, z \in V$ and $\alpha$ is a scalar,*

- $\langle x, x\rangle = 0$ *if and only if* $x = 0$,
- $\langle x, x\rangle \geq 0$,
- $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$,
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

*In Hilbert space, every inner product produces a norm.*

$$|x| = \sqrt{\langle x, x\rangle}.$$

## 4.4 Deriving of the Hamiltonian operators and computation of transition probability for different Hamiltonian operators

### 4.4.1 Case 1: Fixed claim sizes and shifted Poisson Hamiltonian

Let $A$ be the transition probability matrix. The Hamiltonian is found by the traditional probability analysis.

**Case 1 :** Let $X_j$ be the fixed claim size $X_j = m$ for $j = 1, 2, ...$ with claim frequency $\lambda$. Firstly, we find the Hamiltonian by

$$
\begin{aligned}
Hf(x) &= lim_{t\to 0}\frac{I - A(t)}{t}f(x) \\
&= lim_{t\to 0}\frac{f(x) - E[f(x + ct - S(t)]}{t} \\
&= lim_{t\to 0}\frac{1}{t}[f(x) - \sum_{j=0}^{\infty} f(x + ct - jm)\frac{e^{-\lambda t}(\lambda t)^j}{j!}]
\end{aligned}
$$

$E[f(x + ct - S(t)]$ depends on $m, \lambda$ and $j$ because of $S(t)$.

we consider for j=0 and j=1 because $j > 1$, $\frac{(\lambda t)^j}{t}$ goes to zero.

$$
\begin{aligned}
&= lim_{t\to 0}\frac{1}{t}[f(x) - f(x + ct)e^{-\lambda t} - f(x + ct - m)e^{-\lambda t}\lambda t] \\
&= lim_{t\to 0}\frac{f(x) - f(x + ct)e^{-\lambda t}}{t} - f(x + ct - m)e^{-\lambda t}\lambda \\
&= -cf'(x) + \lambda(f(x) - f(x - m)). \qquad (4.4.7)
\end{aligned}
$$

Secondly, we analyse the spectral decomposition for $H$. Notice that $H$ is not self adjoint. However, the technique works.

In Dirac formalism, the Hamiltonian operator is defined in Hilbert space, and Eigenvectors $|p>$ provide an orthonormal basis for Hilbert space.

$$H|p >= K_p|p >$$

where the eigenvector of the Hamiltonian operator is $|p>$, and $f(p) = e^{ixp}$, $x$ is an integer value, $i$ is a complex imaginary unit.

In order to obtain the eigenvalue of Hamiltonian $K_p$, we put them into the following equation as

$$
\begin{aligned}
H|p> &= -cipe^{ixp} + \lambda(e^{ixp} - e^{ip(x-m)}) \\
&= (-cip + \lambda - \lambda e^{-imp})e^{ixp} \\
&= K_p|p> .
\end{aligned}
$$

So, the eigenvalue of the Hamiltonian operator is

$$K_p = -cip + \lambda - \lambda e^{-imp}.$$

Furthermore,

$$\begin{aligned}
H|p> &= K_p|p> \\
H^2|p> &= K_p^2|p> \\
&\vdots \\
H^n|p> &= K_p^n|p> .
\end{aligned}$$

Therefore,

$$e^{-tH}|p>=\sum_{j=0}^{\infty}\frac{(-tH)^j}{j!}|p>= e^{-tK_p}|p> .$$

Finally, we compute the transition probabilities by the formula 4.3.6.

Note that

$$\begin{aligned}
|x_i> &= R(t) \text{ and } |x_{i+1}>= R(t+\triangle t) \\
<x_i|p> &= e^{ix_ip} \text{ and } <p|x_{i+1}>= e^{-ix_{i+1}p}.
\end{aligned}$$

From equation (4.3.6) in the above calculations,

$$\begin{aligned}
P(x_i \to x_{i+1}) =< x_i|e^{-\triangle tH}|x_{i+1}> &= \int_0^{2\pi}\frac{dp}{2\pi} < x_i|e^{-\triangle tH}|p>< p|x_{i+1}> \\
&= \int_0^{2\pi}\frac{dp}{2\pi} < x_i|p>< p|x_{i+1}> e^{-tK_p} \\
&= \frac{1}{2\pi}\int_0^{2\pi}(e^{ix_ip}e^{-ix_{i+1}p})e^{-\triangle tK_p}dp \\
&= \frac{e^{-\lambda\triangle t}}{2\pi}\int_0^{2\pi} e^{ip(x_i-x_{i+1})+\triangle ticp+\triangle t\lambda e^{-imp}}dp.
\end{aligned}$$

The main result is then stated in the following lemma.

**Lemma 23** *Assume $x_{i+1} - x_i$ is an integer. Then,*

$$P(x_i \to x_{i+1}) = < x_i | e^{-\triangle t H} | x_{i+1} >$$
$$= \frac{e^{-\lambda \triangle t}}{2\pi} \int_0^{2\pi} e^{ip(x_i - x_{i+1}) + \triangle ticp + \triangle t \lambda e^{-imp}} dp. \tag{4.4.8}$$

When the integral in equation (4.4.8) is solved by the trapezoidal rule for $h = \frac{2\pi}{N}$ numerically, the results in case $u = 30, t = 50, c = 1, \lambda = 0.5$, and $m = 3$, for N=5000 and N=200, are displayed in Figures 4.1 and 4.2.

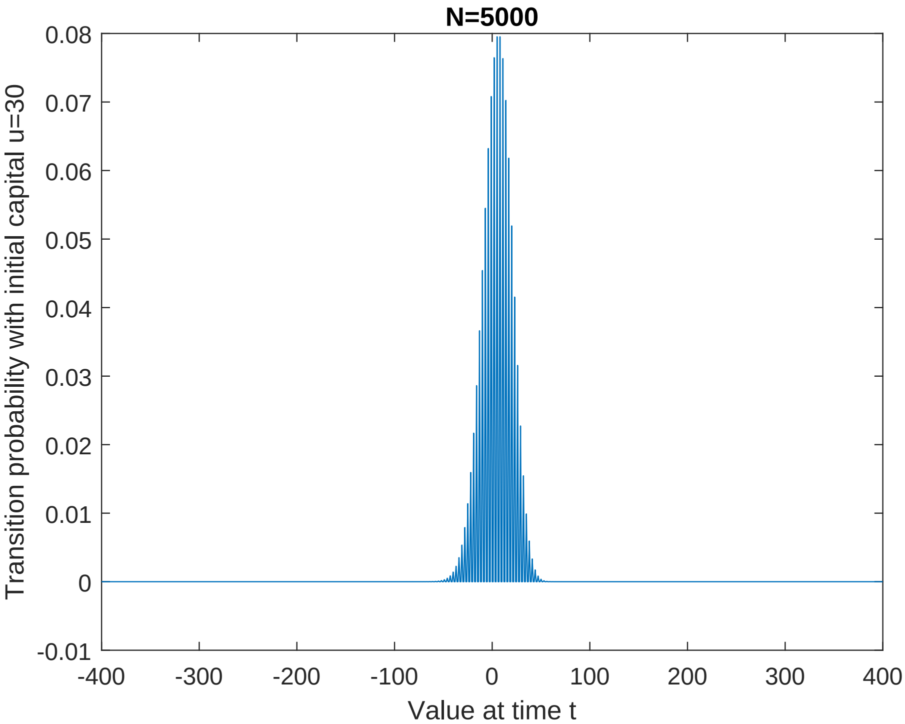



Figure 4.1: $P(30 \to \text{value at time } 50)$ for N=5000 (the iteration number)

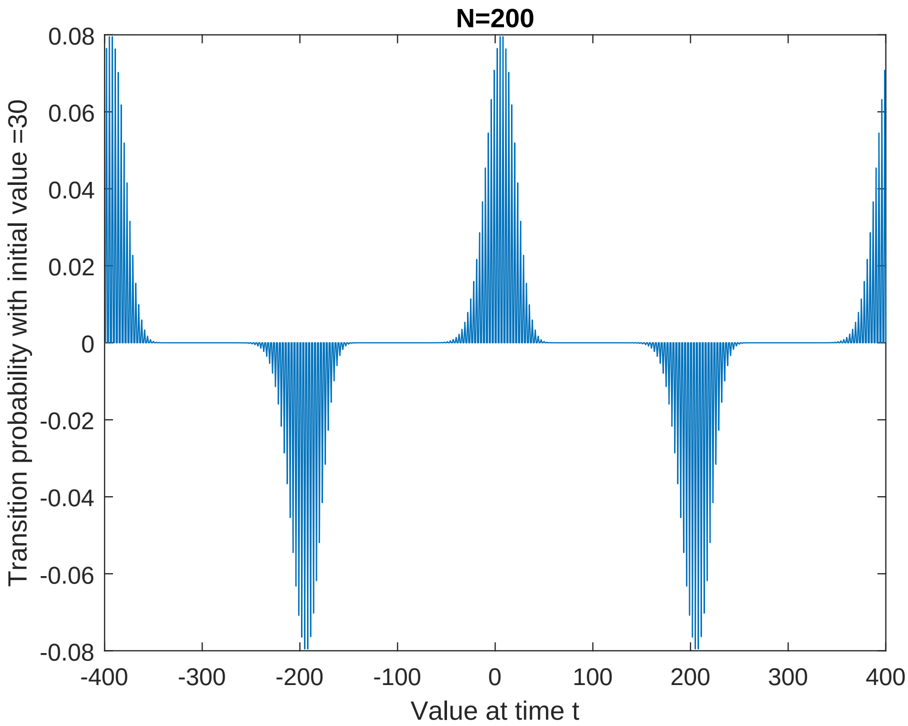



Figure 4.2: $P(30 \rightarrow \text{value at time } 50)$ for N=200

### 4.4.2 Case 2: Random integer valued claim sizes and shifted Compound Poisson Hamiltonian

We start by computing the Hamiltonian for this case. A splitting strategy is applied for the Poisson process as a sum of the independent Poisson processes.

According to the splitting strategy, let $N(t)$ be a Poisson process with rate $\lambda$. When $N(t)$ is divided into $Z$ independent processes, then [29]

- $N_j(t)$ is a Poisson process with rate $\lambda_j = \lambda P(X = j)$ for integer valued claim size $j = 1, 2, ....$
- $N(t) \sim N_1(t) + N_2(t) + ... + N_Z(t)$.
- $\lambda \sim \lambda_1 + \lambda_2 + ... + \lambda_Z$ because $\sum_{j=1}^{Z} P(X = j) \sim 1$ for large Z in numerical calculations.

$$
\begin{aligned}
Hf(x) &= lim_{t\to 0}\frac{I-A(t)}{t}f(x)\\
&= lim_{t\to 0}\frac{f(x)-E[f(x+ct-S(t)]}{t}\\
&= lim_{t\to 0}\frac{1}{t}\Big[f(x)-\sum_{j_1,j_2,j_3,\cdots=0}^{\infty}[f(x+ct-j_1-2j_2-3j_3-\cdots)\frac{e^{-\lambda_1 t}(\lambda_1 t)^{j_1}}{j_1!}\\
&\qquad \frac{e^{-\lambda_2 t}(\lambda_2 t)^{j_2}}{j_2!}\frac{e^{-\lambda_3 t}(\lambda_3 t)^{j_3}}{j_3!}\cdots]\Big]
\end{aligned}
$$

In expectation of the compound Poisson process, we consider for $i_1+i_2+i_3+\ldots=0$

and $i_1+i_2+i_3+\ldots=1$ because $\dfrac{t^{i_1+i_2+i_3+\ldots}}{t}$ goes to zero for $i_1+i_2+i_3+\ldots>1$

$$
\begin{aligned}
&= lim_{t\to 0}\frac{1}{t}\Big[f(x)-f(x+ct)e^{-(\lambda_1+\lambda_2+\lambda_3+\cdots)t}-f(x+ct-1)e^{-\lambda_1 t}\lambda_1 t\\
&\qquad -f(x+ct-2)e^{-\lambda_2 t}\lambda_2 t-f(x+ct-3)e^{-\lambda_3 t}\lambda_3 t-\cdots\Big]\\
&= lim_{t\to 0}\Big[\frac{f(x)-f(x+ct)e^{-(\lambda_1+\lambda_2+\lambda_3+\cdots)t}}{t}-f(x+ct-1)e^{-\lambda_1 t}\lambda_1\\
&\qquad -f(x+ct-2)e^{-\lambda_2 t}\lambda_2-f(x+ct-3)e^{-\lambda_3 t}\lambda_3-\cdots\Big]\\
&= -cf'(x)+\lambda f(x)-\sum_{j=1}^{\infty}f(x-j)\lambda_j.
\end{aligned}
\tag{4.4.9}
$$

Now, we compute the eigenvalue by

$$
\begin{aligned}
H|p> &= -cipe^{ixp}+(\lambda_1+\lambda_2+\lambda_3+\cdots)e^{ixp}-e^{ip(x-1)}\lambda_1-e^{ip(x-2)}\lambda_2-e^{ip(x-3)}\lambda_3-\cdots\\
&= (-cip+\lambda-\sum_{j=1}^{\infty}\lambda_j e^{-jip})e^{ixp}\\
&= K_p|p>.
\end{aligned}
$$

Therefore, the eigenvalue of the Hamiltonian is

$$K_p=-cip+\lambda-\sum_{j=1}^{\infty}\lambda_j e^{-jip}.$$

Finally, equation (4.3.6) with the above calculations is written as

$$
\begin{aligned}
P(x_i \to x_{i+1}) &=< x_i|e^{-\triangle tH}|x_{i+1} > \\
&= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|e^{-\triangle tH}|p >< p|x_{i+1} > \\
&= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|p >< p|x_{i+1} > e^{-\triangle tK_p} \\
&= \frac{1}{2\pi}\int_0^{2\pi} (e^{ix_ip}e^{-ix_{i+1}p})e^{-\triangle tK_p}dp \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ip(x_i-x_{i-1})}e^{-\triangle t(-cip+\lambda_1+\lambda_2+..+\lambda_k+\ldots-e^{-ip}\lambda_1-e^{-2ip}\lambda_2-\ldots-e^{-kip}\lambda_k-\ldots)}dp \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ip(x_i-x_{i+1})+\triangle ticp-\triangle t\sum\limits_{j=1}^{\infty}\lambda_j(1-e^{-jip})}dp.
\end{aligned}
$$

Now, the main result is stated in the lemma.

**Lemma 24** *Assume $x_{i+1} - x_i$ is an integer. Then,*

$$
\begin{aligned}
P(x_i \to x_{i+1}) &=< x_i|e^{-\triangle tH}|x_{i+1} > \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ip(x_i-x_{i+1})+\triangle ticp-\triangle t\sum\limits_{j=1}^{\infty}\lambda_j(1-e^{-jip})}dp. \qquad (4.4.10)
\end{aligned}
$$

Note that $\lambda_j$ is found by splitting the Poisson process with respect to the probability mass function as

$$\lambda_j = \lambda P(X = j).$$

- If the claim sizes are the integer values and have discretized exponential distribution with claim mean $m$, then

$$\lambda_j = \lambda P(X = j) = \lambda \frac{\frac{1}{m}e^{-\frac{1}{m}j}}{\sum\limits_{k=1}^{\infty}\frac{1}{m}e^{-\frac{1}{m}k}}.$$

- If the claim sizes are the integer values and have discretized Gaussian distribution with claim mean $m$ and variance $\sigma^2$, then

$$\lambda_j = \lambda P(X = j) = \lambda \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(j-m)^2}{2\sigma^2}}}{\sum_{k=1}^{\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\dfrac{(k-m)^2}{2\sigma^2}}}.$$

- If the claim sizes are the integer values and have discrete uniform distribution with claim mean $m$, and $m = 1w_1 + 2w_2 + 3w_3 + ... + Lw_L$ where $w_k$ are weights for $k = 1, 2, ..., L$ then

$$\lambda_j = \lambda P(X = j) = \lambda \frac{w_j}{w_1 + w_2 + ... + w_L} = \lambda w_j \quad \text{because of} \quad w_1 + w_2 + ... + w_L = 1.$$

**Example 4.4.1** Let the claim amounts consist of $\{1, 2, 3\}$ with claim frequency $\lambda_1$, $\lambda_2$, and $\lambda_3$, respectively. In this circumstance, the transition probability is computed as below.

$$\begin{aligned} P(x \to x') = < x|e^{-tH}|x' > &= \int_0^{2\pi} \frac{dp}{2\pi} < x|e^{-tH}|p >< p|x' > \\ &= \int_0^{2\pi} \frac{dp}{2\pi} < x|p >< p|x' > e^{-tK_p} \\ &= \frac{1}{2\pi} \int_0^{2\pi} (e^{ixp} - e^{-ix'p}) e^{-tK_p} dp \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ip(x-x')} e^{-t(-cip+\lambda_1+\lambda_2+\lambda_3-e^{-ip}\lambda_1-e^{-2ip}\lambda_2-e^{-3ip}\lambda_3)} dp \\ &= \frac{e^{-(\lambda_1+\lambda_2+\lambda_3)t}}{2\pi} \int_0^{2\pi} e^{ip(x-x')+t(cip+e^{-ip}\lambda_1+e^{-2ip}\lambda_2+e^{-3ip}\lambda_3)} dp. \end{aligned} \tag{4.4.11}$$

When the integral in 4.4.11 is solved by the trapezoidal rule for $h = \frac{2\pi}{N}$ numerically, the results for $u = 20$, $t = 40$, $c = 2$, $\lambda = 0.9$, $X_i = \{1, 2, 3\}$ with $\lambda_i = \frac{\lambda}{3}$, are displayed in Figures 4.3 and 4.4 for $N = 5000$ and $N = 200$.

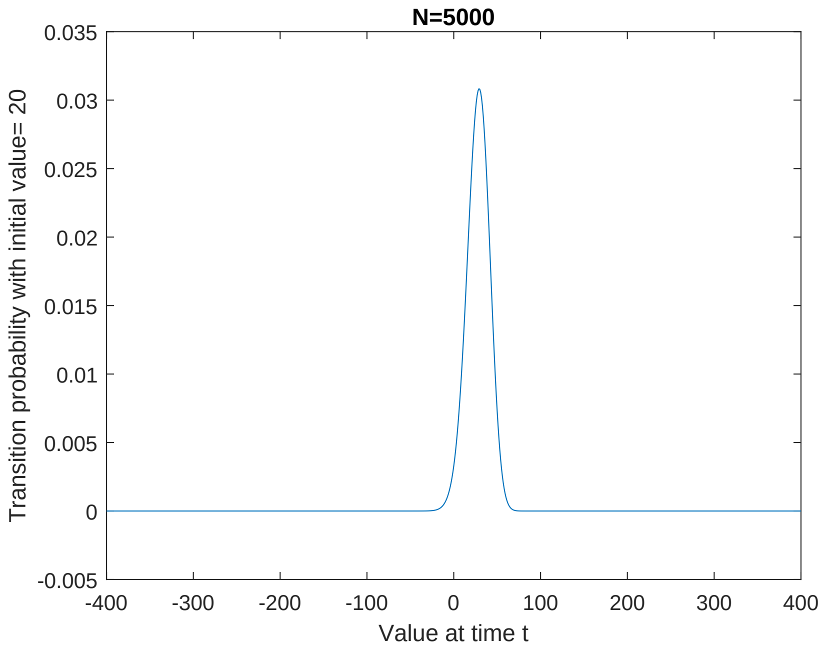



Figure 4.3: $P(20 \to \text{value at time } 40)$ for N=5000

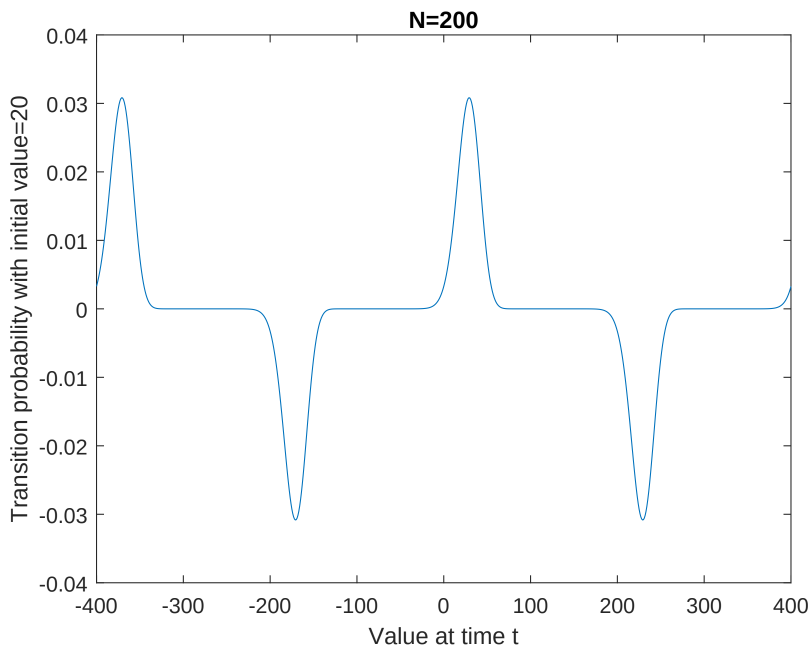



Figure 4.4: $P(20 \to \text{value at time } 40)$ for N=200

### 4.4.3 Case 3: Gaussian claim sizes and Gaussian Hamiltonian

Now, we replace the surplus process $R_t$ by the Brownian motion $\{B_t \,;\, t \geq 0\}$ with mean parameter $\mu$ and variance parameter $b^2$. The Traditional Hamiltonian is found via

$$\begin{aligned} Hf(x) &= \lim_{t\to 0} \frac{I - P(t)}{t} f(x) \\ &= \lim_{t\to 0} \frac{f(x) - E[f(x+B_t)]}{t} \\ &= -\frac{b^2}{2} f''(x) - \mu f'(x). \end{aligned} \tag{4.4.12}$$

Proof of this equation is achieved via Ito's Lemma.

**Definition 25 (Ito's Lemma)**
*$\{S_t : t \geq 0\}$ is an Ito process if it satisfies the following stochastic differential equation.*

$$dS_t = \mu_t dt + \sigma_t dB_t$$

*where $B_t$ is a Brownian process (also called a Wiener process), $\mu_t$ is a drift, and $\sigma_t$ is volatility.*

$f(t, S_t)$ is also an Ito process with

$$\begin{aligned} df(t, S_t) &= f'_{S_t} dS_t + f'_t dt + \frac{1}{2} f''_{S_t,S_t} (dS_t)^2 \\ &= f'_{S_t}[\mu_t dt + \sigma_t dB_t] + f'_t dt + \frac{1}{2} f''_{S_t,S_t} \sigma_t^2 dt \end{aligned}$$

because $(dS_t)^2 = \sigma_t^2 dt$ in Ito's lemma.

**Proof of 4.4.12.**

$$\begin{aligned} HQ(x) &= \lim_{t\to 0} \frac{I - P(t)}{t} Q(x) \\ &= \lim_{t\to 0} \frac{Q(x) - E[Q(x+B_t)]}{t}. \end{aligned}$$

Let's get $f(B_t) = Q(x + B_t)$.

$$f(B_t) = f(0) + \int_0^t f'_B(B_u)dBu + \int_0^t \frac{1}{2} f''_{BB}(B_u)du$$
$$E[f(B_t)] = E[f(0)] + E[ito] + E[\int_0^t \frac{1}{2} f''_{BB}(B_u)du]$$

where $E[ito] = 0$ and $E[f(0)] = Q(x + B_0) = Q(x)$.

Therefore,

$$\begin{aligned} HQ(x) &= \lim_{t\to 0} \frac{Q(x) - E[Q(x + B_t)]}{t} \\ &= \lim_{t\to 0} \frac{Q(x) - (Q(x) - \frac{1}{2}\int_0^t E[Q''(x + B_u)]du)}{t} \\ &= -\frac{1}{2} f''(x). \end{aligned}$$

Here we get a standard Brownian motion where $\sigma^2 = 1$ and $\mu = 0$. Similarly, when we consider $f(B_t) = Q(x+\sigma B_t+\mu t)$ for different values of $\sigma$ and $\mu$, equation (4.4.12) is obtained. ■

However, instead of using this Hamiltonian in equation (4.4.12) directly, it is more convenient to apply the slightly modified Dirac-Feynman formula [5] stated in the lemma below.

**Lemma 26**

$$\begin{aligned} P(x_i \to x_{i+1}) &= \; < x_i | e^{-\Delta t H - V} | x_{i+1} > \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{\Delta t}}} e^{\frac{-(x_{i+1} - (x_i + c\Delta t - m\lambda\Delta t))^2}{2\sigma^2_{\Delta t}}} e^{-V(x_{i+1})} \end{aligned} \tag{4.4.13}$$

*where*

$$V(x_{i+1}) = \begin{cases} 0, & \text{if } x_{i+1} > 0, \\ \infty, & \text{if } x_{i+1} < 0. \end{cases}.$$

Notice that in this formula, the mean and variance parameters are found by matching the corresponding parameters of the surplus process $R_t$, which is the Levy process. Moreover, in this case we assume that the claim sizes have a Gaussian distribution with mean $m$ and variance $\sigma^2$. Then,

$$\begin{aligned} E[R_t] &= R_0 + t(c - m\lambda) \ , \\ \sigma^2_{\triangle t} &= \mathrm{Var}(R_{\Delta t}) = var(S(\triangle\, t)) \\ &= E[N(\triangle\, t)]var(X) + var(N(\triangle\, t))E[X]^2 \\ &= \lambda \triangle t\sigma^2 + \lambda \triangle tm^2 \ . \end{aligned}$$

In Figure 4.5, the way that equation (4.4.13) is changing with respect to for different variances ($var = 10, 50, 100, 200$) can be seen for $u = 20,\ t = 30,\ c = 6,\ \lambda = 0.5,\ m = 10$.

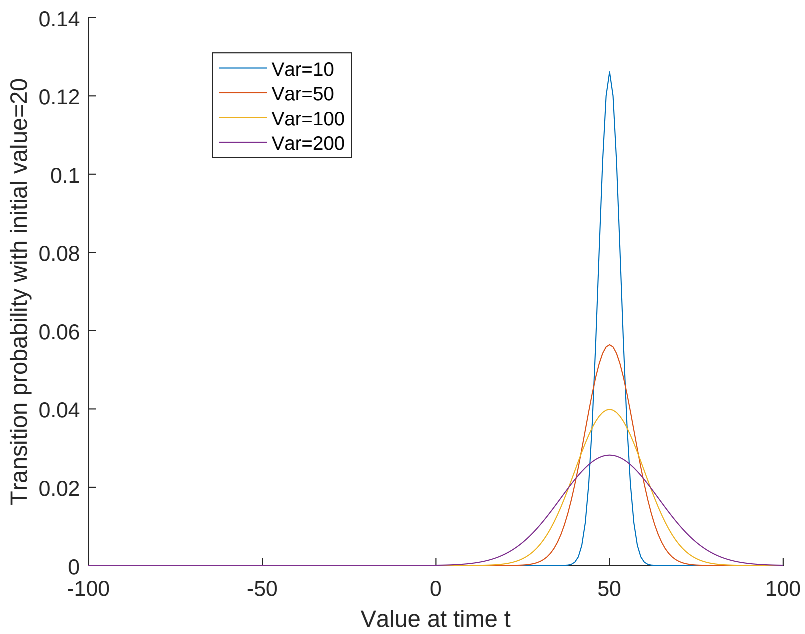



Figure 4.5: $P(20 \to \text{value at time } 30)$ for different variance

## 4.5 Ruin Probability via Quantum Mechanics

### 4.5.1 Path integral, Path calculations

$\langle x_0| e^{-tH} |x_n\rangle$ gives the probability for the particle to travel in a given space time $t$ from point $(x_0, 0)$ to point $(x_n, 0)$. When all the possible paths are taken into consideration,

$$\int_{-\infty}^{\infty} dx_n \langle x_0| e^{-tH} |x_n\rangle = 1.$$

For $t_1 < t$, when the particle goes to $(x_n, t)$ from $(x_0, 0)$ providing it is $x_1$ at time $t_1$,

$$\langle x_0| e^{-tH} |x_n\rangle = \int_{-\infty}^{\infty} dx_1 \langle x_0| e^{-t_1 H} |x_1\rangle \langle x_1| e^{-(t-t_1)H} |x_n\rangle .$$

Similarly, let $x_i$ be the position of the particle at times $t_i < t$ , $i = 0, 1, ..., n$.

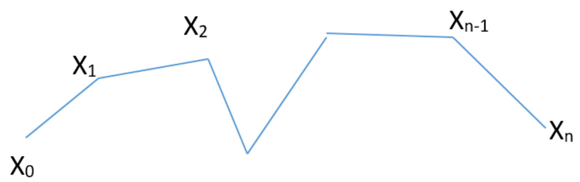



Figure 4.6: Path of the capital

In this circumstance, $\langle x_0| e^{-tH} |x_n\rangle$ can be formalized [5, 6, 31] by

$$\langle x_0| e^{-tH} |x_n\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1 dx_2 ... dx_{n-1} \langle x_0| e^{-t_1 H} |x_1\rangle \langle x_1| e^{-(t_2-t_1)H} |x_2\rangle$$
$$... \langle x_{n-1}| e^{-(t-t_{n-1})H} |x_n\rangle .$$

Non ruin probability via the quantum mechanics approach can be computed by the path integral method. Clearly, we can compute the non ruin probability for the continuous process by restricting the integral over region $0 < x_1 < \infty, \dots, 0 < x_{n-1} < \infty$.

Here, the initial capital is $x_0 = u$ and capital at time $t_n$ (or t) is $x_n$.

Note that $t_1 + (t_2 - t_1) + t_3 - t_2)... + (t - t_{n-1}) = t$.

In case of integer claim size and grid size $\varepsilon = 1$, the non ruin probability at time $t$ is computed by all possible determinate paths

$$
\begin{aligned}
P_u(T > t) = (1 + o(1)) \sum_{x_1=1} \langle u| \, e^{-t_1 H} \, |x_1\rangle \sum_{x_2=1} \langle x_1| \, e^{-(t_2-t_1)H} \, |x_2\rangle \sum_{x_3=1} \langle x_2| \, e^{-(t_3-t_2)H} \, |x_3\rangle \\
\cdots \sum_{x_n=1} \langle x_{n-1}| \, e^{-(t-t_{n-1})H} \, |x_n\rangle \, . \qquad (4.5.14)
\end{aligned}
$$

For case 3, the error $o(1)$ depends on the grid size. For other cases, it depends on the grid and the numerical approximation of the integral in 4.4.8 and 4.4.10.

To overpass the computational complexity in 4.6.21 due to the large number of paths, we apply the Markov chain approach (see e.g. [11], [64]) mentioned in the previous chapter. More exactly, let us define $d \times d$ transition matrix $A$ as transition probabilities over a single time period for grid size 1.

$$A_{i,j} = P(R_{k+1} = j | R_k = i).$$

Let $A^n$ denote the matrix with $A_{i,j}^{(n)} = P(R_n = j | R_0 = i)$ where $A_{i,j}^{(n)}$ is an element of $A^n$.

From the Chapman-Kolmogorov equation for the discrete and homogeneous Markov chain,

$$A(t_1 + t_2) = A(t_1) A(t_2).$$

Notice that it is convenient in our approach to define 0 as the absorption state for the ruin probability. Then,

$$
\begin{aligned}
P_u(T \leq t) &= (1 + o(1)) A_{u,0}^{(t)} \, , \\
P_u(T > t) &= 1 - P(T < t | R(0) = u) = (1 + o(1)) \sum_{j=1}^{d-1} A_{u,j}^{(t)}
\end{aligned}
$$

where the error terms depend on the grid.

**Quantum path approach**. Now, similarly let $A$ denote the transition matrix via quantum mechanics characteristics. The modified matrix is stated in the following lemma.

**Lemma 27**

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 1 - \sum\limits_{i=1} < 1|e^{-H}|i > & < 1|e^{-H}|1 > & < 1|e^{-H}|2 > & < 1|e^{-H}|3 > & \cdots & < 1|e^{-H}|u > & \cdots \\ 1 - \sum\limits_{i=1} < 2|e^{-H}|i > & < 2|e^{-H}|1 > & < 2|e^{-H}|2 > & < 2|e^{-H}|3 > & \cdots & < 2|e^{-H}|u > & \cdots \\ 1 - \sum\limits_{i=1} < 3|e^{-H}|i > & < 3|e^{-H}|1 > & < 3|e^{-H}|2 > & < 3|e^{-H}|3 > & \cdots & < 3|e^{-tH}|u > & \cdots \\ 1 - \sum\limits_{i=1} < 4|e^{-H}|i > & < 4|e^{-H}|1 > & < 4|e^{-H}|2 > & < 4|e^{-H}|3 > & \cdots & < 4|e^{-H}|u > & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ 1 - \sum\limits_{i=1} < u|e^{-H}|i > & < u|e^{-H}|1 > & < u|e^{-H}|2 > & < u|e^{-H}|3 > & \cdots & < u|e^{-H}|u > & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \tag{4.5.15}$$

*where transition probabilities* $\langle i|\, e^{-H} \,|j\rangle$ *are computed via* (4.4.8), (4.4.10), *or* (4.4.13) *according to the case considered.*

The next theorem states our main numerical approach, which will be applied in all relevant numerical results further on.

**Theorem 28** *Assuming the above, for function* $f : Z_+ \to R$ *with* $f(0) = 0$

$$E[f(R_t)I(T > t)|R_0 = u] = (1 + o(1))A^t f(u). \tag{4.5.16}$$

*where* $A_t$ *is a semi group and* $A_t = A^t$*.*

***Proof.*** *Let us consider a family of operators*

$$A_t f(u) = E[I(T > t)f(R_t)|R_0 = u].$$

*Firstly, we show that* $A_t$ *is a semigroup. Let* $\mathcal{F}_t$ *be the* $\sigma$*-algebra generated by* $R_s, s \leq$

*$t$. We apply the Chapman-Kolmogorov argument and write for $t > s > 0$,*

$$\begin{aligned} A_t f(R_0)) &= E\{E[I(T > t)f(R_t)|\mathcal{F}_s\}|R_0] \\ &= EI(T > s)\{E[I(T > t)f(R_t)|\mathcal{F}_s\}|R_0 = u] \\ &\quad \textit{using the Markov property and the time shift} \\ &= E[I(T > s)A_{t-s}f(R_s)|R_0] \\ &= A_s(A_{t-s}f)(R_0). \end{aligned}$$

*Finally, by discretization and approximation of $A_1$, we prove the theorem.*

■

*In particular, the non-ruin and ruin probabilities are found by*

$$\begin{aligned} P_u(T > t) &= (1 + o(1)) \sum_{j=1}^{\infty} A^t_{u,j} \ , \qquad (4.5.17) \\ P_u(T \le t) &= (1 + o(1))A^t_{u,0}. \end{aligned}$$

## 4.6 Comparison with the other methods

In this section, the quantum mechanics approach will be compared with the Markov Chain and the Picard-Lefevre methods.
According to the **Picard-Lefevre** approach mentioned in Section 2.4, the finite time non ruin probability is found by

**Lemma 29**

$$P(T > t|R_0 = u) = e^{-\lambda t} \sum_{j=0}^{u} \{e_j(t) + \sum_{n=u+1}^{[ct+u]} e_j(\frac{j-u}{c}) \frac{ct-n+u}{ct-j+u} e_{n-j}(t + \frac{u-j}{c})\} \qquad (4.6.18)$$

*where*

$$e_n(t) = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} q_n^{*k} \quad \textit{and} \quad q_n^{*k} = P(X_1 + X_2 + ... + X_k = n).$$

In order to compare the quantum mechanics approach with a second method, the Markov Chain approach mentioned in Chapter 2 will be used. According to that approach, ruin and non ruin probability are found by

$$
\begin{aligned}
P_u(T \leq t) &= (1 + o(\varepsilon))A(t)_{u,0} \\
P_u(T > t) &= 1 - P(T \leq t | R(0) = u) \\
&= (1 + o(\varepsilon)) \sum_{j=1} A(t)_{u,j\varepsilon}
\end{aligned}
$$

where A(t) is found by

$$A(t) = A^{[\frac{t}{\varepsilon}]}$$

or

$$A(t + \varepsilon) = A(t) + A(t)Q\varepsilon + \frac{A(t)Q^2(\varepsilon)^2}{2!} + O((\varepsilon)^3). \tag{4.6.19}$$

### 4.6.1 Numerical results for the comparison

In this part, numerical results on non-ruin probability are compared via the following approaches:

(i) Quantum Mechanics Approach with Poisson Hamiltonian operator;

(ii) Quantum Mechanics Approach with Compound Poisson Hamiltonian operator;

(iii) Quantum Mechanics Approach with the Gaussian Hamiltonian operator;

(iv) Appell Polynomials approach as introduced by Picard and Lefevre;

(v) Classical Markov approach

### 4.6.2 Fixed claim sizes

In the case we assume that an insurance company covers all claims with the same amount, the claim premium $c = 1$, the claim frequency $\lambda = 0.4$, and the claim mean $m = 2$.

In Table 4.1, the numerical results for grid size $\varepsilon = 1$ via the quantum approach with the Poisson Hamiltonian, the Appell Polynomials approach, traditional Markov chains approach are summarized.

Table 4.1: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Approach |
|---|---|---|---|---|
| 2 | 5 | 0.7042 | 0.7041 | 0.7041 |
| 5 | 5 | 0.9331 | 0.9331 | 0.9331 |
| 10 | 5 | 0.9982 | 0.9981 | 0.9981 |
| 20 | 5 | 1.0001 | 1 | 1 |
| 2 | 20 | 0.5308 | 0.5306 | 0.5306 |
| 5 | 20 | 0.8126 | 0.8124 | 0.8124 |
| 10 | 20 | 0.9683 | 0.9681 | 0.9681 |
| 20 | 20 | 0.9998 | 0.9996 | 0.9996 |
| 2 | 40 | 0.4835 | 0.4833 | 0.4833 |
| 5 | 40 | 0.7568 | 0.7564 | 0.7564 |
| 10 | 40 | 0.9397 | 0.9393 | 0.9393 |
| 20 | 40 | 0.9981 | 0.9977 | 0.9977 |

Similarly, for $c = 1,\ \lambda = 0.3,\ m = 3$, the results are listed in Table 4.2.

Table 4.2: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Approach |
|---|---|---|---|---|
| 2 | 5 | 0.5612 | 0.5612 | 0.5612 |
| 5 | 5 | 0.8571 | 0.857 | 0.857 |
| 10 | 5 | 0.9802 | 0.9802 | 0.9802 |
| 20 | 5 | 1 | 0.9999 | 0.9999 |
| 2 | 20 | 0.3614 | 0.3614 | 0.3614 |
| 5 | 20 | 0.6338 | 0.6338 | 0.6338 |
| 10 | 20 | 0.8708 | 0.8708 | 0.8708 |
| 20 | 20 | 0.992 | 0.9919 | 0.9919 |
| 2 | 40 | 0.2916 | 0.2916 | 0.2916 |
| 5 | 40 | 0.5281 | 0.528 | 0.528 |
| 10 | 40 | 0.7802 | 0.7801 | 0.7801 |
| 20 | 40 | 0.965 | 0.9649 | 0.9649 |

As seen from the tables above,

- the non ruin probabilities via the three methods are very close.
- the computation process in quantum approach takes more time compared to the others. Note that the computation time in Markov Approach depends on dimension of transition matrix.

### 4.6.3 Random integer valued claim sizes with discretized exponential distribution

In this case, we assume that all claims are integer valued and in addition the probability mass functions are discretized exponential distribution. The results for the quantum approach with the compound Poisson Hamiltonian, the Appell Polynomial, the Markov approaches and Monte Carlo Approach are summarized in Tables 4.3 and 4.4, where claim premium $c = 1$, claim frequency $\lambda = 0.04$, claim mean $m = 20$ and the iteration number is 200 for Monte Carlo Approach.

Difference Monte Carlo and Markov chain Approaches is random claim samplings. Monte Carlo Approach, Given the grid size $\varepsilon = 1$ and assuming the claim size has integer values with discrete exponential distribution and the claim mean $m$,

$$P(X) = \frac{\frac{1}{m} e^{-\frac{1}{m}x}}{\sum\limits_{k=1}^{\infty} \frac{1}{m} e^{-\frac{1}{m}k}}. \tag{4.6.20}$$

Table 4.3: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Appraoch | Markov Monte Carlo Approach |
|---|---|---|---|---|---|
| 2 | 5 | 0.8484 | 0.8478 | 0.8484 | 0.8554 |
| 5 | 5 | 0.8676 | 0.8667 | 0.8677 | 0.8716 |
| 10 | 5 | 0.8944 | 0.893 | 0.8945 | 0.8965 |
| 20 | 5 | 0.9329 | 0.9307 | 0.933 | 0.9338 |
| 2 | 20 | 0.6174 | 0.616 | 0.6176 | 0.6318 |
| 5 | 20 | 0.6558 | 0.6543 | 0.656 | 0.6626 |
| 10 | 20 | 0.7116 | 0.71 | 0.712 | 0.7212 |
| 20 | 20 | 0.7985 | 0.7965 | 0.7988 | 0.7998 |
| 2 | 40 | 0.4919 | 0.4933 | 0.4923 | 0.5093 |
| 5 | 40 | 0.5328 | 0.5347 | 0.5332 | 0.5449 |
| 10 | 40 | 0.5943 | 0.5972 | 0.5948 | 0.6108 |
| 20 | 40 | 0.6958 | 0.7006 | 0.6964 | 0.7144 |
| 2 | 60 | 0.4287 | 0.4338 | 0.4293 | 0.4415 |
| 5 | 60 | 0.4682 | 0.4743 | 0.4688 | 0.4835 |
| 10 | 60 | 0.5287 | 0.5366 | 0.5294 | 0.5523 |
| 20 | 60 | 0.6319 | 0.6439 | 0.6328 | 0.6518 |

The results of the four methods with claims distributed exponentially (claim frequency $\lambda = 0.03$ and mean of claims $m = 30$) are listed in Table 4.4.

Table 4.4: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Appraoch | Markov Monte Carlo Approach |
|---|---|---|---|---|---|
| 2 | 5 | 0.8766 | 0.8766 | 0.8768 | 0.8768 |
| 5 | 5 | 0.8875 | 0.8873 | 0.8877 | 0.8901 |
| 10 | 5 | 0.9037 | 0.9032 | 0.9038 | 0.9026 |
| 20 | 5 | 0.9293 | 0.9284 | 0.9294 | 0.9319 |
| 2 | 20 | 0.6557 | 0.6554 | 0.6563 | 0.6619 |
| 5 | 20 | 0.6808 | 0.6804 | 0.6814 | 0.6868 |
| 10 | 20 | 0.7187 | 0.7181 | 0.7193 | 0.7297 |
| 20 | 20 | 0.7819 | 0.7808 | 0.7824 | 0.7948 |
| 2 | 40 | 0.515 | 0.5157 | 0.516 | 0.5314 |
| 5 | 40 | 0.5437 | 0.5446 | 0.5447 | 0.5531 |
| 10 | 40 | 0.5881 | 0.5892 | 0.5891 | 0.5947 |
| 20 | 40 | 0.6651 | 0.6666 | 0.6662 | 0.6726 |
| 2 | 60 | 0.4391 | 0.4418 | 0.4405 | 0.4508 |
| 5 | 60 | 0.4674 | 0.4705 | 0.4688 | 0.4764 |
| 10 | 60 | 0.5118 | 0.5156 | 0.5133 | 0.5272 |
| 20 | 60 | 0.591 | 0.596 | 0.5925 | 0.6074 |

As we see from Tables 4.3 and 4.4,

- the non ruin probabilities in the quantum, the polynomial, and the Markov approaches are very close again. Therefore, it is not possible to determine which methods give better results.
- Computation takes more time in quantum approach than the others. However, it can take more time in Monte Carlo because it depends on the iteration number.

### 4.6.4 Discretized Gaussian Distributions

The quantum mechanics approach with the Gaussian Hamiltonian is compared with the Picard-Lefevre and Markov approaches when claim amounts have a discretized Gaussian distribution.

The non ruin probabilities are summarized in Table 4.5, where claim premium $c = 1$, claim frequency $\lambda = 0.04$, claim mean $m = 20$, and claim variance $\sigma^2 = 100$.

Table 4.5: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Approach |
|---|---|---|---|---|
| 10 | 5 | 0.6903 | 0.9051 | 0.9051 |
| 20 | 5 | 0.9242 | 0.9086 | 0.9086 |
| 40 | 5 | 0.999 | 0.9589 | 0.9589 |
| 60 | 5 | 1 | 0.9947 | 0.9947 |
| 10 | 10 | 0.5474 | 0.82 | 0.82 |
| 20 | 10 | 0.8085 | 0.831 | 0.831 |
| 40 | 10 | 0.9833 | 0.931 | 0.931 |
| 60 | 10 | 0.9995 | 0.984 | 0.984 |
| 10 | 20 | 0.4248 | 0.6814 | 0.6814 |
| 20 | 20 | 0.6675 | 0.7227 | 0.7227 |
| 40 | 20 | 0.9214 | 0.8881 | 0.8881 |
| 60 | 20 | 0.9887 | 0.9549 | 0.9549 |
| 10 | 30 | 0.3659 | 0.5926 | 0.5925 |
| 20 | 30 | 0.588 | 0.6672 | 0.6672 |
| 40 | 30 | 0.861 | 0.8444 | 0.8444 |
| 60 | 30 | 0.966 | 0.9273 | 0.9273 |

For $\lambda = 0.03$ and $m = 30$, the non ruin probabilities are shown in Table 4.6.

Table 4.6: Comparison of the methods

| Initial capital (u) | Time (t) | Quantum Approach | Appell Polynomial Approach | Markov Approach |
|---|---|---|---|---|
| 10 | 5 | 0.701 | 0.8659 | 0.8659 |
| 20 | 5 | 0.9343 | 0.8901 | 0.8901 |
| 40 | 5 | 0.9994 | 0.9769 | 0.977 |
| 60 | 5 | 1 | 0.9953 | 0.9955 |
| 10 | 10 | 0.5508 | 0.7571 | 0.7571 |
| 20 | 10 | 0.8189 | 0.8109 | 0.811 |
| 40 | 10 | 0.987 | 0.952 | 0.9522 |
| 60 | 10 | 0.9997 | 0.9857 | 0.9859 |
| 10 | 20 | 0.4194 | 0.6131 | 0.6131 |
| 20 | 20 | 0.6696 | 0.7164 | 0.7164 |
| 40 | 20 | 0.928 | 0.8969 | 0.897 |
| 60 | 20 | 0.9911 | 0.9623 | 0.9623 |
| 10 | 30 | 0.3555 | 0.5406 | 0.5405 |
| 20 | 30 | 0.5828 | 0.6563 | 0.6562 |
| 40 | 30 | 0.8648 | 0.8477 | 0.8476 |
| 60 | 30 | 0.9698 | 0.9372 | 0.9369 |

As seen from the tables, while the Appell polynomial and Markov methods produce close results, the quantum approach with the Gaussian Hamiltonian gives slightly different results at relatively small initial capital $u$, when the Gaussian approximation is not good enough, as expected.

### 4.6.5 Advantages and Disadvantages

The disadvantage of the method is that the computation process can take more time for the Levy process in comparison with the Appell and Markov Approaches.
The advantage of quantum mechanics appraoch is that we do not need to choose particular Hamiltonian operator or eigenvalue $K_p$ of the Hamiltonian operator corresponding to the Levy process. Therefore, it makes the method more flexible.

$$
\begin{aligned}
P(x_i \to x_{i+1}) =< x_i|e^{-\Delta tH}|x_{i+1} > &= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|e^{-\Delta tH}|p >< p|x_{i+1} > \\
&= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|p >< p|x_{i+1} > e^{-\Delta tK_p}.
\end{aligned}
$$

In computing the propagator above, let's consider Gambler's ruin problem.

**Example 4.6.1 (Gambler's Ruin Problem)** Here, we consider a random walk that cannot be embedded in a Levy process.
As mentioned in Section 2.2.2, according to the game, the $x_i$ goes to $x_i + 1$ with probability $\alpha$ or goes to $x_i - 1$ with probability $\beta = 1 - \alpha$.

$$
\begin{aligned}
A\,|p\rangle\,(x_i) &= E[e^{ipx_{i+1}}] \\
&= (e^{ip}\alpha + e^{-ip}\beta)e^{ipx_i} \\
e^{-H}\,|p\rangle &= e^{-K_p}\,|p\rangle
\end{aligned}
$$

where $K_p$ and $|p\rangle$ are eigenvalue and eigenvector of Hamiltonian operator $H$.
In this circumstance,

$$K_p = -ln(e^{ip}\alpha + e^{-ip}\beta).$$

According to $K_p$, the propagator for $t = 1$ is defined by

$$
\begin{aligned}
P(x_i \to x_{i+1}) &= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|p >< p|x_{i+1} > e^{-tK_p} \\
&= \int_0^{2\pi} \frac{dp}{2\pi} e^{ip(x_i - x_{i+1})}(e^{ip}\alpha + e^{-ip}\beta). \\
&= \alpha \int_0^{2\pi} \frac{dp}{2\pi} e^{ip(x_i - x_{i+1}+1)} + \beta \int_0^{2\pi} \frac{dp}{2\pi} e^{ip(x_i - x_{i+1}-1)} \\
&= \begin{cases} \alpha & \text{for} \quad x_i - x_{i+1} + 1 = 0 \\ \beta & \text{for} \quad x_i - x_{i+1} - 1 = 0 \end{cases}
\end{aligned}
$$

Similarly, it can be shown for $t = 2$ by

$$
\begin{aligned}
P(x_i \to x_{i+1}) &= \int_0^{2\pi} \frac{dp}{2\pi} < x_i|p >< p|x_{i+1} > e^{-2K_p} \\
&= \int_0^{2\pi} \frac{dp}{2\pi} e^{ip(x_i - x_{i+1})}(e^{2ip}\alpha^2 + 2\alpha\beta + e^{-2ip}\beta^2). \\
&= \begin{cases} \alpha^2 & \text{for} \quad x_i - x_{i+1} + 2 = 0 \\ 2\alpha\beta & \text{for} \quad x_i - x_{i+1} = 0 \\ \beta^2 & \text{for} \quad x_i - x_{i+1} - 2 = 0 \end{cases}
\end{aligned}
$$

As seen above, it is just quantum version of binomial model.

Numerical calculations give

$$
\begin{aligned}
P_{20}(T > 100) &= 0.3972, \quad \text{for u=20, t=100, p=0.4,} \\
P_{40}(T > 200) &= 0.9994, \quad \text{for u=40, t=200, p=0.6.}
\end{aligned}
$$

**Example 4.6.2 (Non Levy process)** Let's consider a non Levy process. Let us choose the operator as $P(i \to j) = P(X = j)$ with $P(X = 0) = p$ and $P(X = j) =$

$q/M$, $j = 1, \ldots, M$.

$$A = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \ldots & M \end{array} \\ \begin{pmatrix} p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \\ p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \end{pmatrix} \end{array}$$

where $p = 0.3$, $q = 0.7$, and $M = 100$.

In this circumstance, non ruin probability via the following formula gives the same answer regardless of the initial capital.

$$P_u(T > t) = (1 + o(1)) \sum_{x_1=1} < u|A|x_1 > \sum_{x_2=1} < x_1|A|x_2 > \sum_{x_3=1} < x_2|A|x_3 > \\ \cdots \sum_{x_n=1} < x_{n-1}|A|x_n > . \tag{4.6.21}$$

The numerical results agree with the simple theoretical answer

$$P_u(T > t) = (1 - p)^t.$$

**Example 4.6.3 (Non iid chain)** Now, each odd row is replaced by $P(2j + 1 \to 0) = 1$, so the matrix operator is defined when $M$ is an even number by

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \ldots & M \end{array} \\ \begin{pmatrix} p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \\ 1 & 0 & 0 & \ldots & 0 \\ p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p & \frac{q}{M} & \frac{q}{M} & \ldots & \frac{q}{M} \end{pmatrix} \end{array}$$

for u=40, t=5, p=0.3, $P_{40}(T > 5) = 0.0105$,

for u=20, t=5, p=0.3, $P_{42}(T > 5) = 0.0105$.

Again, the results agree with the theoretical answer

$$P_u(T > t) = 2(q/2)^t \qquad \text{when } u \text{ is even.}$$

# Chapter 5

# OPTIMIZATION

This Chapter is based on a paper entitled "Ruin Probability via Quantum Mechanics Approach" [76].

There are many insurance companies in the world. The sector is competitive because customers compare various companies and make a selection based on the premium rate, the money the insurer has to cover.

In general, low premium rates and a high percentage of covered claims attract the interest of customers. However, this potentially increases the ruin probability and thus affects the amount of profit. Therefore, there should be a balance between the interest of customers and the profit.

Insurance companies arrange claim payment, premium rate, and initial capital to keep the number of policies and expected profit amount at a high level and the risk probability at a low level. Optimization plays an important role in the competitive market for these reasons.

Optimization is the selection of the best available parameters in all possible alternatives. Many optimization problems like optimal investment policies [8], optimal dividends problems [23, 79], optimal reinsurance [20], and optimal insurance [30] have been studied in actuarial science.

In this chapter, optimization is taken into consideration with regard to non ruin probability. In solving the subsequent optimization problems in this chapter, equations (4.4.8), (4.4.10), and path integral formula from (4.6.21) in Chapter 4 will be used.

In the following optimization problems, the quantum mechanics techniques as a numerical approach has been applied. This is one of the novelties in this thesis.

## 5.1 Optimization of allocation of initial capitals

Let us consider two surplus processes with different claim frequencies and different claim means. This case can be seen that an insurance company invests in two different insurance sectors or two insurance companies have a partnership. They are common business practices in insurance sector. However, here novelty is that all computations have been done via quantum mechanics approach.
Let $T_1$ and $T_2$ be the ruin times for first and second processes, respectively.

$$\begin{aligned} \tau &= \min(T_1, T_2) \text{ and} \\ P(\tau > t | u_1, u_2) &= P(T_1 > t | R_0 = u_1\ ) P(T_2 > t | R_0 = u_2\ ) \end{aligned}$$

where $u = u_1 + u_2$ will be referred to as the allocation of initial capitals.
Optimization of allocation of initial capitals is to find the allocation $(u_1, u_2)$ that results in the largest non ruin probability.
Non ruin probability that depends on $u_1$, $u_2$ and time $t$ in case of the claim frequencies $\lambda_1 = 0.4,\ \lambda_2 = 0.3$, the claim means $m = 2,\ m = 3$, the premium rate $c = 1$, and the total initial capital $u_1 + u_2 = 20$ is displayed in Figure 5.1.

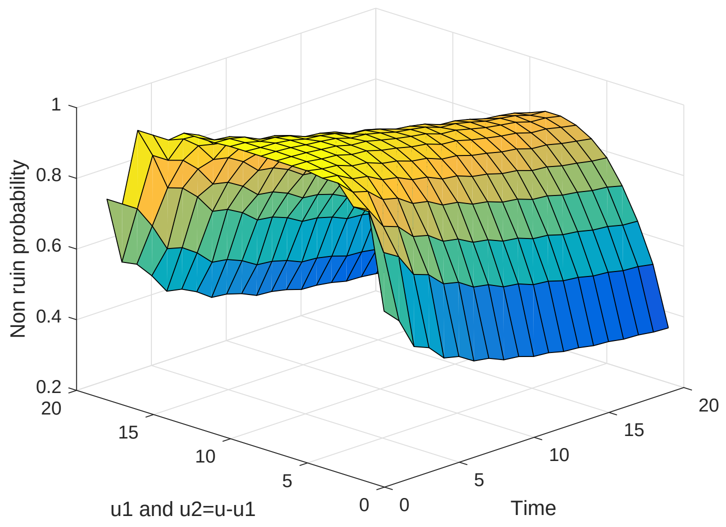



Figure 5.1: Optimization of allocation of initial capitals

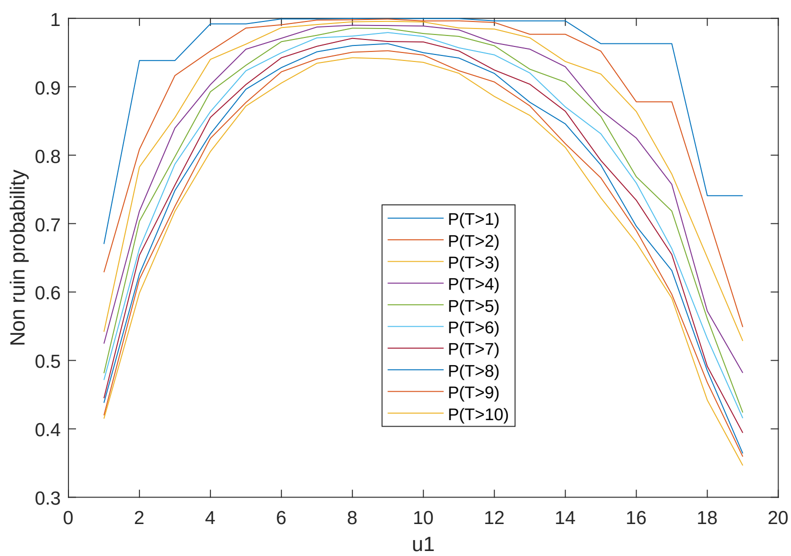



Figure 5.2: Optimization of allocation of initial capitals

From Figure 5.1, for the largest non ruin probability, the optimum initial capitals $u_1, u_2$ are summarized in the following table.

| $u_1$ | $u_2$ | P(T>1) | P(T>2) | P(T>3) | P(T>4) | P(T>5) | P(T>6) | P(T>7) | P(T>8) | P(T>9) | P(T>10) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 19 | 0.670233 | 0.628882 | 0.541918 | 0.524605 | 0.481359 | 0.471456 | 0.444809 | 0.438071 | 0.419787 | 0.414925 |
| 2 | 18 | 0.938306 | 0.808554 | 0.782717 | 0.718272 | 0.70344 | 0.663557 | 0.653764 | 0.626283 | 0.618708 | 0.598601 |
| 3 | 17 | 0.938306 | 0.916337 | 0.854954 | 0.8397 | 0.797787 | 0.786954 | 0.7563 | 0.748351 | 0.725232 | 0.71801 |
| 4 | 16 | 0.991901 | 0.952229 | 0.940021 | 0.903154 | 0.8927 | 0.863654 | 0.855259 | 0.831076 | 0.824605 | 0.805282 |
| 5 | 15 | 0.991901 | 0.985788 | 0.962249 | 0.954835 | 0.931342 | 0.923335 | 0.903174 | 0.89632 | 0.876976 | 0.871751 |
| 6 | 14 | 0.9991 | 0.990613 | 0.986368 | 0.970863 | 0.965909 | 0.949707 | 0.942344 | 0.928022 | 0.921827 | 0.905448 |
| 7 | 13 | 0.9991 | 0.997461 | 0.990934 | 0.987268 | 0.975196 | 0.971471 | 0.959107 | 0.951067 | 0.940569 | 0.934347 |
| 8 | 12 | 0.9998 | 0.99793 | 0.994942 | 0.989917 | 0.98564 | 0.974041 | 0.970804 | 0.960135 | 0.950481 | 0.942441 |
| 9 | 11 | 0.9995 | 0.998922 | 0.99559 | 0.989452 | 0.985105 | 0.979249 | 0.966158 | 0.962953 | 0.952636 | 0.940805 |
| 10 | 10 | 0.9996 | 0.996192 | 0.994727 | 0.988737 | 0.977778 | 0.973495 | 0.965468 | 0.949793 | 0.946367 | 0.935656 |
| 11 | 9 | 0.9996 | 0.996371 | 0.985937 | 0.983162 | 0.973594 | 0.956962 | 0.95239 | 0.942027 | 0.92344 | 0.919696 |
| 12 | 8 | 0.9963 | 0.993927 | 0.984337 | 0.964205 | 0.959955 | 0.946603 | 0.924466 | 0.91949 | 0.907028 | 0.885839 |
| 13 | 7 | 0.9963 | 0.976704 | 0.971577 | 0.954947 | 0.925648 | 0.920137 | 0.903688 | 0.877312 | 0.872023 | 0.858089 |
| 14 | 6 | 0.9963 | 0.976704 | 0.936903 | 0.929259 | 0.906776 | 0.870561 | 0.864181 | 0.845715 | 0.816778 | 0.811358 |
| 15 | 5 | 0.963 | 0.952038 | 0.91863 | 0.865671 | 0.856661 | 0.831326 | 0.792324 | 0.785693 | 0.766834 | 0.737692 |
| 16 | 4 | 0.963 | 0.877996 | 0.863784 | 0.825041 | 0.768421 | 0.75927 | 0.734073 | 0.696164 | 0.689841 | 0.672035 |
| 17 | 3 | 0.963 | 0.877996 | 0.772359 | 0.757319 | 0.718252 | 0.663426 | 0.654814 | 0.631386 | 0.596624 | 0.590896 |
| 18 | 2 | 0.7408 | 0.71339 | 0.65042 | 0.572163 | 0.561022 | 0.532081 | 0.491466 | 0.485089 | 0.467736 | 0.441991 |
| 19 | 1 | 0.7408 | 0.548785 | 0.52848 | 0.481831 | 0.423858 | 0.415605 | 0.394166 | 0.364079 | 0.359356 | 0.346502 |

According to Figures 5.1,5.2 and Table 5.1, the non ruin probability is higher when the difference between $u_1$ and $u_2$ is smaller. However, an increase in time causes less non ruin probability.

## 5.2 Optimization of proportion of the total claim amount paid with the prescribed ruin level

In insurance contracts, the companies can either refuse to cover all claims or they can just give a proportion of claim. The second situation can affect satisfaction of the insured and number of customers.

Optimization of proportional factor is studied in proportional reinsurance models [35].

With the proportion of the total claim amount, the surplus process is defined by

$$\begin{aligned} R_t &= u + ct - kS(t) \\ &= u + ct - \sum_{i=1}^{N(t)} kX_i, \end{aligned}$$

where $k$ is the proportionality factor determining the total claim amount the insurer covers. The proportionality factor is used in proportional reinsurance agreements as well.

The optimization problem is to maximize the covered level $k$ with respect to the prescribed ruin level $\ell$,

$$\max\{k: \text{ such that } P_u(T > t) \geq \ell\}.$$

Table 5.2 shows non ruin probabilities at time 8, 9 and 10 for $u = 5,\ c = 1,\ \lambda = 0.1,\ m = 10$ with respect to different proportionality factors $k = [0.6, 0.7, 0.8, 0.9, 1]$.

| | $P(T>8)$ | $P(T>9)$ | $P(T>10)$ |
|---|---|---|---|
| k=0.6 | 0.7931 | 0.7887 | 0.7817 |
| k=0.7 | 0.7189 | 0.6912 | 0.6898 |
| k=0.8 | 0.6740 | 0.6505 | 0.6254 |
| k=0.9 | 0.6291 | 0.6099 | 0.5886 |
| k=1 | 0.5841 | 0.5692 | 0.5518 |

For $\ell = 0.6$,

$$\max\{k: \text{ such that } P_u(T>t) \geq \ell\} = \begin{cases} 0.9, & \text{for } t=8, \\ 0.9, & \text{for } t=9 \\ 0.8, & \text{for } t=10 \end{cases}.$$

## 5.3 Optimization of allocation of investments and withdrawals

In this section, we consider the market consisting of two cooperative insurance companies with the overall capital investment 0.

This case like section 5.1 can be seen that an insurance company invests in two different insurance sectors or two insurance companies have a partnership. However, additionally, there is capital swap transaction in this case.

Let $R_t^{(1)}$ and $R_t^{(2)}$ be surplus processes of two insurance companies.

$$R_t^{(1)} = u_1 + c_1 t - S_1(t) + C_1(t)$$
$$R_t^{(2)} = u_2 + c_2 t - S_2(t) + C_2(t)$$

where $C_1(t) = \sum_{j=1}^{k} a_j \varepsilon_1(t_j) I_{(t>t_j)}$ and $C_2(t) = \sum_{j=1}^{k} a_j \varepsilon_2(t_j) I_{(t>t_j)}$ are swapped capitals between two companies by injection (money coming in) or reduction (money coming out).

**Swap strategy:** Let us consider two different actuarial companies that can swap the money between them.

**Assumption 1** We assume that claim processes are independent.

**Assumption 2** Total capital allocation is $C_1(t)+C_2(t) = 0$ and only a finite number of the capital allocation (C-Allocation) occurs.

This means that when one of the insurance companies is exposed to the capital injection, the other gets the capital withdrawal.

More exactly, at specific times $t_i,\ i = 1, \ldots, k$, the companies swap the capital by amount $a_i > 0$, so that one of them gets a positive amount $a$ (i.e. an injection occurs $R_{t_i} \to R_{t_i+0} = R_{t_i} + a_i$) and the other gets the withdrawal of the capital by $a_i$ (i.e. reduction occurs $R_{t_i} \to R_{t_i+0} = R_{t_i} - a_i$).

Observing of the process can be done by daily, monthly or annually. It just depends on the grid time size.

Let

$$\varepsilon_i(t_1) = \begin{cases} 1, & \text{when injection} \\ -1, & \text{when withdrawal.} \end{cases}$$

We say that the ruin occurs if one of the companies is ruined. Let $T_i$ be the ruin time of the $i-th$ company and let $\tau$ be the ruin time of the market. Then,

$$\begin{aligned} \tau &= \min(T_1, T_2) \ \text{ and} \\ P(\tau > t|u_1, u_2) &= P(T > t|R_0 = u_1 \text{ C-Allocation})P(T > t|R_0 = u_2 \text{ and reverse C-Allocation}). \end{aligned}$$

The goal of optimizing the allocation of injections and withdrawals is to find an optimal injection (or reduction) amount $a_i$ and time allocations $t_i$ to get the largest non-ruin probability.

To compute the non ruin probability, the numerical approach of the quantum mechanics techniques mentioned in Chapter 4 has been applied.

Figure 5.3 shows the non ruin probability as a function of one time allocation and the capital allocation for $u_1 = 5,\ u_2 = 5,\ t = 10,\ c_1 = 1,\ c_2 = 1,\ \lambda_1 = 0.4,\ \lambda_2 = 0.3,\ m_1 = 2$, and $m_2 = 3$.

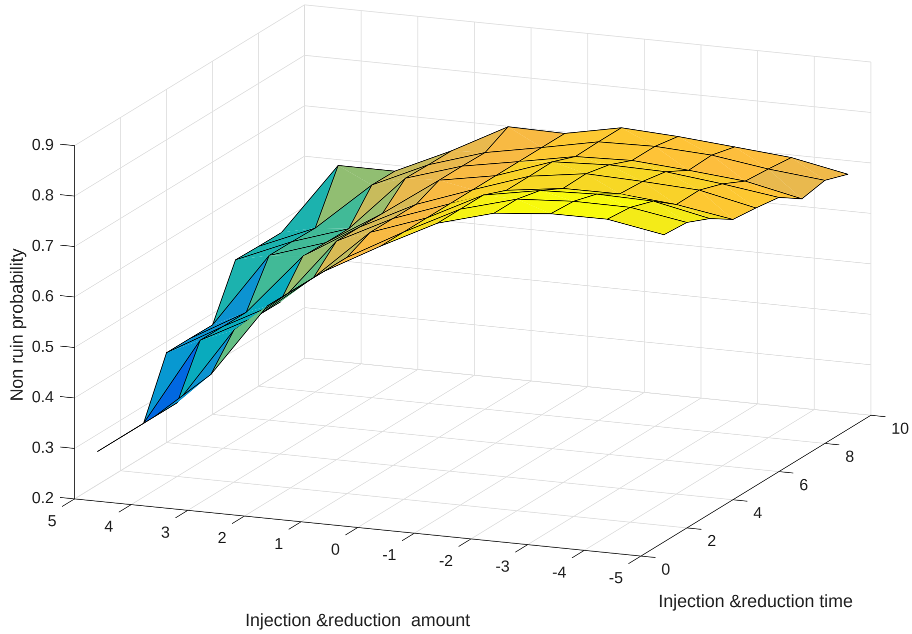



Figure 5.3: Optimization of injections and withdrawals

Figure 5.3 and Table 5.3 shows that the capital swap time and amount affect the non ruin probability on a large scale. According to the figure and table, the largest non-ruin probability is 0.656699, which is attained on the capital transfer of amount 1 from Company 2 to Company 1 at time 3.

| $C_1(t)$ | $C_2(t)$ | P(T>10) $t_1 = 1$ | P(T>10) $t_1 = 2$ | P(T>10) $t_1 = 3$ | P(T>10) $t_1 = 4$ | P(T>10) $t_1 = 5$ | P(T>10) $t_1 = 6$ | P(T>10) $t_1 = 7$ | P(T>10) $t_1 = 8$ | P(T>10) $t_1 = 9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| -5 | 5 | 0.259183 | 0.257431 | 0.346468 | 0.330291 | 0.398342 | 0.389073 | 0.432634 | 0.429795 | 0.480147 |
| -4 | 4 | 0.374788 | 0.443478 | 0.434007 | 0.473003 | 0.470459 | 0.512538 | 0.492741 | 0.539394 | 0.528556 |
| -3 | 3 | 0.52213 | 0.519978 | 0.547878 | 0.528229 | 0.561836 | 0.553138 | 0.568125 | 0.565763 | 0.590597 |
| -2 | 2 | 0.609322 | 0.62263 | 0.614133 | 0.617418 | 0.617418 | 0.63098 | 0.613585 | 0.635218 | 0.625311 |
| -1 | 1 | 0.64868 | 0.64868 | 0.656699 | 0.641004 | 0.650293 | 0.650293 | 0.644434 | 0.644435 | 0.653876 |
| 0 | 0 | 0.649378 | 0.649378 | 0.649378 | 0.649378 | 0.649378 | 0.649378 | 0.649378 | 0.649378 | 0.649378 |
| 1 | -1 | 0.615974 | 0.615974 | 0.629726 | 0.629726 | 0.623291 | 0.641943 | 0.635493 | 0.635493 | 0.649378 |
| 2 | -2 | 0.560112 | 0.558744 | 0.551159 | 0.591263 | 0.582644 | 0.580757 | 0.610694 | 0.608645 | 0.599689 |
| 3 | -3 | 0.425655 | 0.481704 | 0.473545 | 0.471437 | 0.521797 | 0.519499 | 0.5106 | 0.562701 | 0.553459 |
| 4 | -4 | 0.338234 | 0.336766 | 0.421842 | 0.419473 | 0.410869 | 0.487955 | 0.47854 | 0.476013 | 0.553778 |
| 5 | -5 | 0.261353 | 0.260074 | 0.254665 | 0.360136 | 0.352305 | 0.350177 | 0.435406 | 0.432973 | 0.425124 |

Note that here the surplus process is just observed for one capital swap. Similarly, it can be observed for several capital swaps.

# Chapter 6

# REINSURANCE

This chapter is based on a published paper entitled “Ruin Probability via Quantum Mechanics Approach” [76] and submitted paper entitled “Optimal reinsurance via Dirac-Feynman Approach” [77].

This chapter examines the numerical computation of the (non) ruin probability of a modified surplus process with reinsurance and the optimal reinsurance via the Dirac-Feymnan approach.

Reinsurance is a risk sharing arrangement between a primary insurer and a reinsurer, and can also be used to refer to risk managing and transferring in the insurance industry [17].

There are a number of different types of reinsurance agreements, including

- Proportional reinsurance,
- Non-proportional reinsurance,
- Excess-of-loss reinsurance,
- Facultative coverage.

With the different types of reinsurance, there are various optimality approaches such as in Castaner, Claramunt and Lefevre [12], Denuit and Vermandele [20], Dickson and Waters [22], Ignatov, Kaishev and Krachunov [39], Kaishev and Dimitrova [40], Schmidli [28], Zhou and Yuen [80], Schmidli [71].

We consider following non-proportional reinsurance agreement in this chapter.

**Reinsurance Agreement**: Our insurance agreement is motivated by Nie et al [58] [57]. According to the non proportional reinsurance agreement, the insured company pays a reinsurance premium in advance in order to get capital injections at times when the capital goes below a fixed given retention level. At the end of each time interval, we observe the capital and if it is found to be below the retention level, then the reinsurance company is supposed to raise the capital to the retention level.

## 6.1 Preliminary

Recall that the ruin process is defined by the following equation (see Chapter 1)

$$R(t) = u + ct - S(t)$$

where $u$ is the initial capital, $c$ is the premium rate per unit time, $t$ is time, $S(t) = \sum_{i=1}^{N(t)} X_i$ is the total claim amount. $N(t)$ is the claim number up to time $t$, and $X_i$ is the i-th claim amount.

Given the surplus process $R(t)$, $t \geq 0$, let the ruin time be

$$T = \begin{cases} \min\{t \geq 0 | R(t) \leq 0\} & \text{for discrete time,} \\ \inf\{t \geq 0 | R(t) \leq 0\} & \text{for continuous time.} \end{cases}$$

We apply the following non ruin probability formula via the path integral approach stated in equation (4.6.21) for $\varepsilon = 1$. It has been derived as

$$P_u(T > t) = (1 + o(1)) \sum_{x_1=1} < u|A(t_1)|x_1 > \sum_{x_2=1} < x_1|A(t_2 - t_1)|x_2 > \sum_{x_3=1} < x_2|A(t_3 - t_2)|x_3 > \cdots \sum_{x_n=1} < x_{n-1}|A(t - t_{n-1})|x_n > . \qquad (6.1.1)$$

where $A$ is an operator.

For $A$ is transition operator over a single time period $\varepsilon$ ,

$$A(t) = A^{[\frac{t}{\varepsilon}]}.$$

Then, the powers of the transition matrix are in such a form that

$$\left[\frac{t_1}{\varepsilon}\right], \left[\frac{t_2 - t_1}{\varepsilon}\right], \cdots, \left[\frac{t - t_n}{\varepsilon}\right]$$

which are integer parts of the values.

For example, for the grid size $\varepsilon = 1$ , transition matrix in d dimensional is defined by

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,u} & \cdots \\ 1 - \sum_{i=1}^{d-1} a_{1,i} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,u} & \cdots \\ 1 - \sum_{i=1}^{d-1} a_{2,i} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,u} & \cdots \\ 1 - \sum_{i=1}^{d-1} a_{3,i} & a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,u} & \cdots \\ 1 - \sum_{i=1}^{d-1} a_{4,i} & a_{4,1} & a_{4,2} & a_{4,3} & \cdots & a_{4,u} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ 1 - \sum_{i=1}^{d-1} a_{u,i} & a_{u,1} & a_{u,2} & a_{u,3} & \cdots & a_{u,u} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \tag{6.1.2}$$

where

$$A_{i,j} = a_{i,j} = \begin{cases} 1, & \text{if } i = j = 0; \\ 0, & \text{if } i = 0, j \neq 0; \\ 1 - \sum_{j=1}^{d-1} a_{i,j}, & \text{if } j = 0, i \neq 0; \\ P(R_{k+1} = j | R_k = i), & \text{for the other cases} \end{cases} .$$

$< u|A(t_1)|x_1 >=< u|A^{[\frac{t_1}{\varepsilon}]}|x_1 >$ is equal to the element in $u+1$ th row and $x_1+1$ th column of the matrix $A^{[t_1]}$ for $\varepsilon = 1$ under assumption that zero is the absorption state in our transition matrix. The assumption means that when the capital becomes negative or null, ruin occurs.

**Theorem 30** *Under the assumption that* 0 *is an absorption state representing ruin probability, and the observing unit time* $\varepsilon$ *for bounded continuous function with*

$f(0) = 0$

$$E[f(R_t)I(T > t)|R_0 = u] = (1 + o(\varepsilon))A^{[\frac{t}{\varepsilon}]}f(u). \tag{6.1.3}$$

*with equation (4.6.21), ruin and non-ruin probability can be computed by*

$$P_u(T > t) = (1 + o(\varepsilon)) \sum_{j=1}^{\infty} A^{[\frac{t}{\varepsilon}]}_{u,j\varepsilon} , \tag{6.1.4}$$

$$P_u(T \leq t) = (1 + o(\varepsilon))A^{[\frac{t}{\varepsilon}]}_{u,0}, \tag{6.1.5}$$

*where the error terms depend on the grid time size.*

Note that in our method, the interval [0,t] is observed by $[\frac{t}{\varepsilon} - 1]$ times.
For example, when the time is 20 with grid size =0.01, the capital $[\frac{t}{\varepsilon} - 1] = 1999$ times will be analysed in order to check that it is below the retention level or not.

## 6.2 Modified ruin model

In this section, we introduce the modified surplus process that incorporates the reinsurance by capital injections.

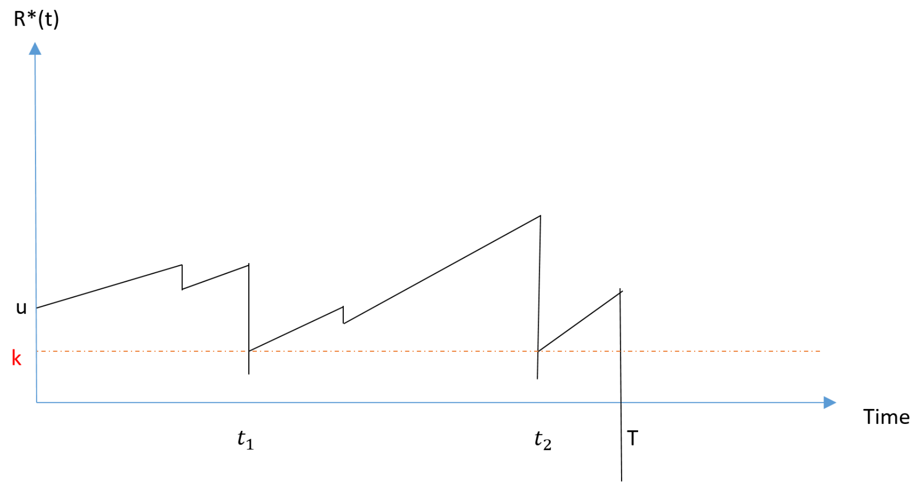
## 6.3 Ruin probabilities for the modified ruin model

As mentioned above, there are various types of reinsurance arrangements provided by reinsurers. For example, the modified risk process under a reinsurance agreement where a primary insurance company pays the reinsurance premium regularly to keep its capital above the retention level by getting capital injections, is defined by

$$R^*(t) = u + (c - z)t - S(t) + Y(t)$$

where $z$ is the reinsurance premium that the insured insurance company has to pay at every time unit, and $Y(t)$ is the expected total injection amount.
However, in this thesis we consider the reinsurance contract as discussed in Nie et al.

(2011). For this contract, the first insurance company has to pay the initial premium amount $z$ in advance to the second insurance company (referred to as the reinsurer), which restores the surplus of the first insurance company to a fixed retention level ($k$) when the surplus process is below this retention level.




**Example 6.3.1** In Figure 6.1, we consider a discrete version of a risk process with three moves. After each time interval, the capital moves up with probability $p_1$, remains constant with probability $p_2$, or goes down with probability $p_3$.
In addition, at the end of each interval, the movement is observed. The position is moved up to the retention level if it was below the retention level.

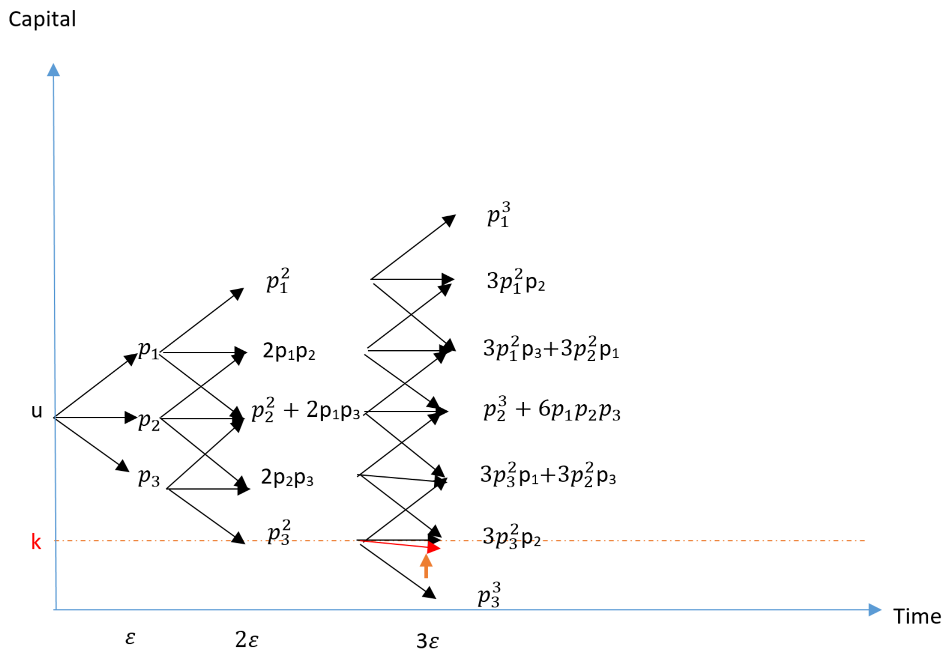



Figure 6.1: Random walk of the capital

As seen from Figure 6.1, there is no need for the capital injection at time $\varepsilon$ and $2\varepsilon$ because it is above the retention level, but the injection is necessary at time $3\varepsilon$ with probability $p_3^3$.

To make it more realistic, we additionally assume that the primary insurance company set an upper level for the compensation of claims. Then, the aggregating claim amount with $h$ upper bound is defined by

$$H(S(t)) = \sum_{i=1}^{N(t)} \big[X_i I(X_i \leq h) + hI(X_i > h)\big]. \tag{6.3.6}$$

The modified surplus process is then defined by

$$\begin{aligned} R^*(t) &= u + ct - z - H(S(t)) + Y(t) \\ &= w + ct - H(S(t)) + Y(t) \end{aligned}$$

where $w = u - z$ is the new initial capital after buying reinsurance and

$$Y(t) = Y(w,k,t) = \sum_{i=1}^{[\frac{t}{\varepsilon}-1]} y_i$$

is the total injection amount up to time $t$, defined by the retention level $k$, grid time size $\varepsilon$, and exact initial capital $w$. Notice that under this reinsurance agreement the capital injections may happen at each time $j\varepsilon$, $j = 1, 2, \ldots$.

Now, let us introduce an **Injection operator** (shift type operator) with 0 absorption level and $k$ retention level

$$(Kf)(x) = \begin{cases} f(x), & \text{if } x \geq k \\ f(k), & \text{if } 0 < x < k \\ f(0), & \text{if } x \leq 0. \end{cases} \tag{6.3.7}$$

The matrix form of $K$ with respect to barrier $k$ is defined by

$$K = \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & & k & k+1 & k+2 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \end{pmatrix} \end{array}. \tag{6.3.8}$$

Let $\overline{P}_w^k(T > t)$ and $\overline{P}_w^k(T \leq t)$ be non ruin and ruin probabilities of the modified surplus process, respectively. From equation (6.1.3) of Theorem 30 and equation

(4.6.21), we derive

$$\begin{aligned}
\overline{P}^k_w(T>t) &= (1+o(1))\sum_{x_1=1}^{d-1} < u|AK|x_1 > \sum_{x_2=1}^{d-1} < x_1|AK|x_2 > \sum_{x_3=1}^{d-1} < x_2|AK|x_3 > \\
&\quad \cdots \sum_{x_n=1}^{d-1} < x_{n-1}|A|x_n > \\
&= (1+o(1))\sum_{j=1}^{d-1}\Big(\underbrace{AKAK\ldots K}_{t-1 \text{ times}} A\Big)_{w,j} \\
&= (1+o(1))\sum_{j=1}^{d-1}\Big((AK)^{t-1}A\Big)_{w,j}.
\end{aligned}$$

$$\overline{P}^k_w(T\le t) = (1+o(1))\Big(\underbrace{AKAK\ldots K}_{t-1 \text{ times}} A\Big)_{w,0} = (1+o(1))\Big((AK)^{t-1}A\Big)_{w,0}.$$

In particular, we derive the following proposition for grid size $\varepsilon$.

**Proposition 31** *Under notation in above,*

$$\begin{aligned}
\overline{P}^k_w(T>t) &= (1+o(\varepsilon))\sum_{j=1}^{d-1}\Big((AK)^{[\frac{t}{\varepsilon}-1]}A\Big)_{w,j\varepsilon}, \\
\overline{P}^k_w(T\le t) &= (1+o(\varepsilon))\Big((AK)^{[\frac{t}{\varepsilon}-1]}A\Big)_{w,0}
\end{aligned}$$

*where the error term depends on the grid time size $\varepsilon$.*

## 6.4 Effect of the injection operator

Notice that $K^n = K$ because $K^2 = K$. This can be seen in the matrix form of the injection operator in 6.3.8.

Therefore, it is easier to work with

$$A^n K^n A = A^n K A.$$

Operators $(AK)^n A$ and $A^n K A$ are types of transition matrices. Note that in $(AK)^n A$ we apply the injection operator $n$ times whereas in $A^n K A$ it is applied only once.

The operators $K$ and $A$ are non-commutative in general, which clearly poses numerical complications.

**Example 6.4.1** Let's define matrix A and K by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0 & 0.2 & 0.2 \\ 0.5 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.1 & 0.1 & 0.2 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

According to the matrix forms of operators $A$ and $K$ above, $AK \neq KA$.

Although $AK \neq KA$ and more generally $(AK)^n A \neq A^n KA$, the ruin probability computations via $A^n KA$ give close results for some values of claim frequency, claim mean, and premium rate, as seen in Table 6.1.

### 6.4.1 Stochastic comparison of $(AK)^n A$ and $A^n KA$

We first model the movement of the capital with initial capital $w$ via the operators $(AK)^n A$ and $A^n KA$ by a coupling construction. By abuse of notation, here by $K$ we also denote a function

$$K(x) = \begin{cases} x & \text{if } x \geq k \\ k, & \text{if } 0 < x < k \\ 0, & \text{if } x \leq 0. \end{cases}$$

Notice that

$$K(x) \geq x. \tag{6.4.9}$$

Clearly, if $x > k$, then $K(x) = x$, and if not, then $K(x) = k$.

Let

$$a_{i,j} = P(R_0^* + \xi = j | R_0 = i) = P(\xi = j - i | R_0^* = i) \text{ where } \xi = c - X_1.$$

For $n = 1$, the initial capital $w$ goes via $(AK)^n A$ to

$$\begin{aligned} w \overset{A}{\overbrace{\longrightarrow}} w + \xi_1 \overset{K}{\overbrace{\longrightarrow}} & K(w + \xi_1) \\ \overset{A}{\overbrace{\longrightarrow}} & K(w + \xi_1) + \xi_2. \end{aligned}$$

For $n = 2$, $w$ goes to

$$\begin{aligned} w \overset{A}{\overbrace{\longrightarrow}} w + \xi_1 \overset{K}{\overbrace{\longrightarrow}} & K(u - z + \xi_1) \\ \overset{A}{\overbrace{\longrightarrow}} & K(w + \xi_1) + \xi_2 \\ \overset{K}{\overbrace{\longrightarrow}} & K(K(w + \xi_1) + \xi_2) \\ \overset{A}{\overbrace{\longrightarrow}} & K(K(w + \xi_1) + \xi_2) + \xi_3. \end{aligned}$$

In general, for $n$ steps and $w > 0$,

$$[(AK)^n A]_{w,R^*(n+1)} = w \overset{(AK)^n A}{\overbrace{\longrightarrow}} R^*(n+1) = K(K(\cdots K(w + \xi_1) + \xi_2) + \cdots) + \xi_{n+1}. \tag{6.4.10}$$

A similar pattern for $A^n KA$ is given by

$$\begin{aligned} w \overset{A^n}{\overbrace{\longrightarrow}} & w + \xi_1 + \xi_2 + \cdots + \xi_n \\ \overset{K}{\overbrace{\longrightarrow}} & K(w + \xi_1 + \xi_2 + \cdots + \xi_n) \\ \overset{A}{\overbrace{\longrightarrow}} & K(w + \xi_1 + \xi_2 + \cdots + \xi_n) + \xi_{n+1} \end{aligned}$$

which gives the capital at time $n + 1$ when the initial capital is $w$ and the grid size is 1. Therefore,

$$[A^n KA]_{wR^{**}(n+1)} = w \overset{A^n KA}{\overbrace{\longrightarrow}} R^{**}(n+1) = K(w + \xi_1 + \xi_2 + \cdots + \xi_n) + \xi_{n+1}. \tag{6.4.11}$$

From equations (6.4.10), (6.4.11) and (6.4.9), we derive the following coupling inequality

$$\begin{aligned} R^*(n+1) &= K(K(\cdots K(w+\xi_1)+\xi_2)+\cdots)+\xi_{m+1} \\ &\geq K(w+\xi_1+\xi_2+\cdots+\xi_m)+\xi_{m+1} \\ &\geq R^{**}(n+1) \end{aligned}$$

which implies the following result.

**Proposition 32** *Under the notation above, for any integers $x \geq 1$ and $n \geq 0$,*

$$\sum_{j=x}((AK)^n A)_{w,j} \geq \sum_{j=x}(A^n KA)_{w,j}$$

*implying that the ruin probabilities computed via $(AK)^n A$ are approximately smaller than the corresponding ruin probabilities computed via $A^n KA$.*

Due to the stochastic comparison, the Wasserstein distance [41,48] can be computed via the Monte Carlo approach simultaneously

$$d_w(R^*(t), R^{**}(t)) = E[R(t) - R^*(t)] \approx \frac{1}{N}\sum_{j=1}^{N}[R^{j*}(t) - R^{j**}(t)].$$

In Table 6.1, we compare ruin probabilities computed via $(AK)^n A$ and $A^n KA$ for various times $t$ and various retention levels. Notice that

$$t = (n+1)\varepsilon.$$

The results are listed for the initial capital is 50, the premium rate is 20, the claim sizes have a discretized exponential distribution with claim mean 18, claim frequency 1, and for no upper barrier $h = \infty$.

Table 6.1: Ruin probability via $(AK)^n A$ and $A^n KA$

| Time | k | via $(AK)^n A$ | via $A^n KA$ | $d_w(R(t), \overline{R}(t))$ |
|---|---|---|---|---|
| 30 | 5 | 0.4513 | 0.4527 | 0.1588 |
| 30 | 10 | 0.4463 | 0.4526 | 0.7519 |
| 30 | 15 | 0.4084 | 0.4524 | 5.6104 |
| 45 | 5 | 0.4837 | 0.4851 | 0.1745 |
| 45 | 15 | 0.4700 | 0.4850 | 2.0119 |
| 45 | 30 | 0.4206 | 0.4849 | 9.1205 |
| 60 | 5 | 0.5005 | 0.5018 | 0.1777 |
| 60 | 30 | 0.4376 | 0.5017 | 9.2851 |
| 60 | 45 | 0.3602 | 0.5015 | 21.6717 |

As seen from the table above, the ruin probabilities via $(AK)^n A$ are always smaller than $A^n KA$, as supported by the further discussion. The Wassertstein difference is relatively small when the retention level $k$ is small, but it increases significantly with the retention level $k$ and time $t$.

## 6.5 Expectation of the total capital injections amount

For a reasonable reinsurance contract in terms of reinsurance company, reinsurance cost $z$ is required to cover the average of the total injection amount, that is

$$E[Y(u-z,k,t)] < z.$$

We begin by stating a numerical formula for the expected total injection amount $E[Y(w,k,t)]$.

**Proposition 33** *Let* $0$ *be the absorption level and* $\varepsilon$ *be the grid time size. We emphasize that* $Y(w,k,t)$ *is treated for the discretized version as below.*

$$E[Y(w,k,t)] = \sum_{j=1}^{[\frac{t}{\varepsilon}-1]} \sum_{i=1}^{[\frac{k}{\varepsilon}-1]} (k - i\varepsilon)\Big((AK)^{j-1}A\Big)_{w,i\varepsilon}.$$

**Proof.** For simplicity, let's consider the case of $\varepsilon = 1$.

Let $y_i$ and $v_i$ be the $i$-th injection amount and injection time,respectively, then

$$y_i = \begin{cases} k - R^*(v_i), & \text{if } 0 < R^*(v_i) < k \\ 0, & \text{if } R^*(v_i) \geq k. \end{cases}$$

Clearly, the expectation of capital injections paid by reinsurer at time $t$ is defined by

$$\begin{aligned} E[y_t] &= \sum_{i=1}^{k-1}(k-i)\Big(AKAK...KA\Big)_{w,i} \\ &= \sum_{i=1}^{k-1}(k-i)\Big((AK)^{t-1}A\Big)_{w,i}. \end{aligned}$$

Therefore, the total injection amount is computed as follows.

$$\begin{aligned} E[Y(w,k,t)] &= \sum_{j=1}^{t-1} E[y_j] \\ &= \sum_{i=1}^{k-1}(k-i)A_{w,i} + \sum_{i=1}^{k-1}(k-i)\Big(AKA\Big)_{w,i} + ... \quad + \sum_{i=1}^{k-1}(k-i)\Big(AKAK...KA\Big)_{w,i} \\ &= \sum_{j=1}^{t-1}\sum_{i=1}^{k-1}(k-i)\Big((AK)^{j-1}A\Big)_{w,i}. \end{aligned}$$

■

## 6.6 Numerical Results

In this section, three optimization examples are discussed and numerically illustrated by applying the Dirac-Feynman approach.

1. **Gaussian Claim size**

   In the Dirac-Feynman notation stated in Chapter 4 , we get $A = e^{-\triangle tH - V}$ and, for Gaussian claim distribution, we derive

$$\begin{aligned} P(x_i \to x_{i+1}) &= \; < x_i|e^{-\triangle tH-V}|x_{i+1} > \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{\triangle t}}} e^{\frac{-(x_{i+1}-(x_i+c\triangle t - m\lambda\triangle t))^2}{2\sigma^2_{\triangle t}}} e^{-V(x_{i+1})} \end{aligned} \quad (6.6.12)$$

where the potential function $V(x_{i+1})$ is defined by

$$V(x_{i+1}) = \begin{cases} 0, & \text{if } x_{i+1} > 0, \\ \infty, & \text{if } x_{i+1} \leq 0. \end{cases} \tag{6.6.13}$$

In the first type of optimality, the goal is to find the optimal reinsurance premium and retention level to obtain the smallest ruin probability. In the second type, the upper level for compensation of claims and the reinsurance premium are investigated. The aim of the third type is to find the largest paid proportion of claims against the retention level.

### 6.6.1 Optimization of reinsurance cost $z$

In this part, the finite time ruin probability of the modified surplus process and expected total injection amount are numerically computed using the methods outlined above. The results are stated in Tables 6.2 and 6.3, respectively.

More exactly, we fix the time $t = 20$, initial capital $u = 20$, premium rate $c = 14$, claim frequency $\lambda = 1$, claim mean $m = 12$, $var(X) = 144$, and $h = \infty$.

From Theorem 30, we find that the finite time ruin without reinsurance is equal to

$$P_{15}(T \leq 20) = 0.6110.$$

In addition, for simplicity we choose reinsurance costs $z = \{1, 2, ..., 10\}$ and retention levels $k = \{5, 6, ..., 10\}$.

Table 6.2: Ruin probability of the modified surplus process with respect to $z$ and $k$

$\overline{P}_w^k(T \leq 20)$

| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
|---|---|---|---|---|---|---|
| z=1 | 0.616 | 0.612 | 0.6071 | 0.6011 | 0.594 | 0.5859 |
| z=2 | 0.6285 | 0.6244 | 0.6195 | 0.6135 | 0.6064 | 0.5983 |
| z=3 | 0.6409 | 0.6369 | 0.6319 | 0.6259 | 0.6189 | 0.6107 |
| z=4 | 0.6535 | 0.6495 | 0.6445 | 0.6385 | 0.6315 | 0.6233 |
| z=5 | 0.666 | 0.662 | 0.6571 | 0.6511 | 0.6441 | 0.636 |
| z=6 | 0.6786 | 0.6746 | 0.6696 | 0.6637 | 0.6568 | 0.6487 |
| z=7 | 0.6911 | 0.6871 | 0.6822 | 0.6764 | 0.6695 | 0.6615 |
| z=8 | 0.7036 | 0.6997 | 0.6948 | 0.689 | 0.6822 | 0.6743 |
| z=9 | 0.716 | 0.7122 | 0.7074 | 0.7016 | 0.6949 | 0.6871 |
| z=10 | 0.7284 | 0.7246 | 0.7198 | 0.7141 | 0.7075 | 0.6999 |

Table 6.3: Expected total injection amount $E(Y)$ with respect to $z$ and $k$

$E[Y]$

| z \ k | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|
| 1 | 0.5675 | 0.8684 | 1.2401 | 1.6861 | 2.2096 | 2.8134 |
| 2 | 0.5709 | 0.8732 | 1.2463 | 1.6936 | 2.2183 | 2.8231 |
| 3 | 0.5734 | 0.8765 | 1.2504 | 1.6983 | 2.2232 | 2.8278 |
| 4 | 0.5749 | 0.8783 | 1.2523 | 1.6999 | 2.2241 | 2.8274 |
| 5 | 0.5754 | 0.8786 | 1.2519 | 1.6984 | 2.2209 | 2.8217 |
| 6 | 0.5748 | 0.8771 | 1.2491 | 1.6937 | 2.2134 | 2.8105 |
| 7 | 0.5731 | 0.874 | 1.244 | 1.6856 | 2.2015 | 2.7937 |
| 8 | 0.5703 | 0.8692 | 1.2363 | 1.6742 | 2.1852 | 2.7712 |
| 9 | 0.5663 | 0.8626 | 1.2261 | 1.6593 | 2.1643 | 2.7431 |
| 10 | 0.5612 | 0.8542 | 1.2133 | 1.6409 | 2.139 | 2.7092 |

Our aim is to minimise the finite time ruin probability and corresponding reinsurance premium $z$ with

$$\min\{\overline{P}_{u-z}^k(T \leq t) : \ z > E[Y(u-z,k,t)] \ \text{ and } \ \overline{P}_{u-z}^k(T \leq t) < P_u(T \leq t)\}.$$

From Tables 6.2 and 6.3, it is clear that the reinsurance is appropriate for several values of $k$ and $z$. However, we choose the value of reinsurance cost $z = 3$ and the retention level $k = 10$ because this gives the smallest ruin probability (0.6107) under the conditions that $z > E[Y(u-z,k,t)]$ and $\overline{P}_{u-z}^k(T \leq t) < P_u(T \leq t)\}$.

### 6.6.2 Optimization of proportional payment $h$

In this part, instead of the compensation of claim in 6.3.6 a slightly different claim process $H(S(t) = hS(t)$ is considered, which is that the insurance company covers only proportion $h$ of the claim. We analyse a numerical example to find the maximum proportional payment $h$ under the reinsurance strategy. In this example,

$$u = 15,\ t = 30,\ c = 14,\ \lambda = 1,\ m = 12 \text{ and } Var(X) = \sigma_X^2 = 144.$$

In addition, for the reinsurance contract, we take

$$z = 2,\ k = \{5, 6, \ldots, 10\},\ \text{ and } h = \{0.5, 0.6, \ldots, 1\}.$$

Notice that the ruin probabilities now depend on the level $h$ written as $P_u(T \leq t|h)$ and $\overline{P}_w^k(T \leq t|h)$.

Our aim is to find maximum $h$ so that there exists $k = k_h$, satisfying

$$L \leq \overline{P}_{13}^k(T \leq 30|h) \leq P_{15}(T \leq 30|h) \ \text{ and } \ z > E(Y).$$

Table 6.4: Ruin probability of the normal and modified process with respect to $h$ and $k$

| | $P_{15}(T \leq 30\|h)$ (No reinsurance) | $\overline{P}_{13}^k(T \leq 30\|h)$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| h=0.5 | 0.2537 | 0.2753 | 0.2724 | 0.2689 | 0.2649 | 0.2603 | 0.2552 |
| h=0.6 | 0.3109 | 0.3329 | 0.3296 | 0.3256 | 0.3209 | 0.3155 | 0.3095 |
| h=0.7 | 0.3794 | 0.4011 | 0.3974 | 0.3929 | 0.3876 | 0.3815 | 0.3746 |
| h=0.8 | 0.4592 | 0.4799 | 0.4759 | 0.471 | 0.4653 | 0.4585 | 0.4509 |
| h=0.9 | 0.5485 | 0.5672 | 0.5632 | 0.5582 | 0.5522 | 0.5452 | 0.5372 |
| h=1 | 0.6428 | 0.6588 | 0.655 | 0.6502 | 0.6444 | 0.6376 | 0.6297 |

Table 6.5: Expected total injection amount $E(Y)$ with respect to $h$ and $k$

| | $E[Y]$ | | | | | |
|---|---|---|---|---|---|---|
| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| h=0.5 | 0.3082 | 0.4719 | 0.6736 | 0.9146 | 1.1956 | 1.5174 |
| h=0.6 | 0.3585 | 0.5488 | 0.7831 | 1.0631 | 1.3897 | 1.7638 |
| h=0.7 | 0.4152 | 0.6354 | 0.9068 | 1.2312 | 1.6101 | 2.0445 |
| h=0.8 | 0.4769 | 0.7299 | 1.0418 | 1.4153 | 1.8523 | 2.3543 |
| h=0.9 | 0.5405 | 0.8273 | 1.1815 | 1.6063 | 2.1044 | 2.6783 |
| h=1 | 0.6009 | 0.9199 | 1.3144 | 1.7885 | 2.346 | 2.9905 |

According to Tables 6.4 and 6.5, which are derived from Theorem 30, the clear increase in $h$ makes the ruin probability and the expected total injection amount bigger while an increase in the retention level causes less ruin probability and more injection amount. Optimals $k$ and $h$ depend on $L$.
For example, for $L = 0.3$ and $h = 0.6$, the optimum reinsurance agreement is obtained by the retention level $k = 10$. For $k = 9$ and $L = 0.4$, the highest proportional payment $h$ is 0.8.

### 6.6.3 Optimization of the premium rate $c$

Given the aggregate claim process in Section 4.3, a numerical example to find the lowest premium $c$ is considered. As before, we find it via optimization of the retention level $k$. In this case,

$$\begin{aligned} u &= 20,\ \lambda = 1,\ m = 12,\ t = 40,\ Var(X) = \sigma_X^2 = 144 \\ z &= 5,\ k = \{5, 6, \ldots, 10\},\ h = \infty \text{ and } c = \{10, 11, 12, 13, 14, 15\}. \end{aligned}$$

The ruin probabilities now depend on the premium rate $c$ written as $P_u(T \leq t|c)$ and $\overline{P}_w^k(T \leq t|c)$. The goal is to find minimal $c$ so that there exists $k$, satisfying

$$L \leq \overline{P}_{15}^k(T \leq 40|c) \leq P_{20}(T \leq 40|c) \ \text{ and } \ z > E(Y).$$

Table 6.6: Ruin probability with respect to $k$ and $c$

| | $P_{20}(T \leq 40)$ (No reinsurance) | $\overline{P}^{k}_{15}(T \leq 40)$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| c=10 | 0.9298 | 0.9274 | 0.9261 | 0.9243 | 0.9221 | 0.9194 | 0.916 |
| c=11 | 0.8824 | 0.8789 | 0.8769 | 0.8743 | 0.8711 | 0.8672 | 0.8625 |
| c=12 | 0.8195 | 0.8147 | 0.812 | 0.8086 | 0.8045 | 0.7994 | 0.7934 |
| c=13 | 0.7439 | 0.738 | 0.7347 | 0.7306 | 0.7256 | 0.7195 | 0.7124 |
| c=14 | 0.6608 | 0.654 | 0.6503 | 0.6457 | 0.6401 | 0.6335 | 0.6257 |
| c=15 | 0.5762 | 0.569 | 0.5651 | 0.5602 | 0.5544 | 0.5476 | 0.5396 |

Table 6.7: Expected total injection amount $E(Y)$ with respect to $k$ and $c$

| | $E[Y]$ | | | | | |
|---|---|---|---|---|---|---|
| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| c=10 | 0.7401 | 1.1342 | 1.6243 | 2.218 | 2.9239 | 3.7514 |
| c=11 | 0.7318 | 1.122 | 1.6073 | 2.1948 | 2.8925 | 3.709 |
| c=12 | 0.7068 | 1.0839 | 1.5526 | 2.1193 | 2.7911 | 3.5752 |
| c=13 | 0.6661 | 1.0214 | 1.4624 | 1.995 | 2.6247 | 3.3575 |
| c=14 | 0.6131 | 0.9401 | 1.3454 | 1.8337 | 2.4097 | 3.0776 |
| c=15 | 0.5534 | 0.8482 | 1.2132 | 1.6521 | 2.1684 | 2.7652 |

Again, the optimal $c$ depends on the level $L$ and $k$. For $L = 0.8$ and $k = 8$, the lowest premium rate is attained for $c = 12$.

2. **Exponential Claim size**

In Chapter 4, the transition probability for the compound Poisson process with discretized exponential claim size was defined by

$$
\begin{aligned}
P(x_i \to x_{i+1}) &= < x_i | e^{-\triangle t H} | x_{i+1} > \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{ip(x_i - x_{i+1}) + \triangle t i c p - \triangle t \sum\limits_{j=1}^{\infty} \lambda_j (1 - e^{-jip})} dp
\end{aligned}
\tag{6.6.14}
$$

where

$$
\lambda_j = \lambda P(X = j) = \lambda \frac{\frac{1}{m} e^{-\frac{1}{m} j}}{\sum\limits_{k=1}^{\infty} \frac{1}{m} e^{-\frac{1}{m} k}}.
$$

### 6.6.4 Optimization of reinsurance cost $z$ for discretized exponential claim size

Similar to Section 6.6.1, the ruin probability and total injection amount are displayed in Tables 6.8 and 6.9 for discretized exponential claim distribution and initial capital $u = 20$, premium rate $c = 14$, claim frequency $\lambda = 1$, claim mean $m = 12$, time $t = 20$, and $h = \infty$.

Table 6.8: Ruin probability of the modified surplus process with respect to $z$ and $k$

| | $\overline{P}^k_w(T \leq 20)$ | | | | | |
|---|---|---|---|---|---|---|
| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| z=1 | 0.5491 | 0.5471 | 0.5447 | 0.5418 | 0.5385 | 0.5348 |
| z=2 | 0.5585 | 0.5564 | 0.5539 | 0.551 | 0.5477 | 0.5439 |
| z=3 | 0.5679 | 0.5658 | 0.5633 | 0.5604 | 0.557 | 0.5532 |
| z=4 | 0.5775 | 0.5754 | 0.5728 | 0.5698 | 0.5664 | 0.5625 |
| z=5 | 0.5872 | 0.585 | 0.5824 | 0.5794 | 0.5759 | 0.572 |
| z=6 | 0.5969 | 0.5948 | 0.5922 | 0.5891 | 0.5856 | 0.5816 |
| z=7 | 0.6068 | 0.6047 | 0.602 | 0.5989 | 0.5953 | 0.5913 |
| z=8 | 0.6169 | 0.6146 | 0.6119 | 0.6088 | 0.6052 | 0.6011 |
| z=9 | 0.627 | 0.6247 | 0.622 | 0.6188 | 0.6151 | 0.611 |
| z=10 | 0.6372 | 0.6349 | 0.6322 | 0.6289 | 0.6252 | 0.621 |

Table 6.9: Expected total injection amount $E(Y)$ with respect to $z$ and $k$

| | $E[Y]$ | | | | | |
|---|---|---|---|---|---|---|
| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| z=1 | 0.3304 | 0.5032 | 0.7149 | 0.9668 | 1.26 | 1.5955 |
| z=2 | 0.3356 | 0.5111 | 0.726 | 0.9818 | 1.2795 | 1.6202 |
| z=3 | 0.3408 | 0.5189 | 0.7372 | 0.9969 | 1.2991 | 1.6449 |
| z=4 | 0.346 | 0.5268 | 0.7484 | 1.012 | 1.3187 | 1.6698 |
| z=5 | 0.3512 | 0.5348 | 0.7597 | 1.0271 | 1.3384 | 1.6946 |
| z=6 | 0.3564 | 0.5427 | 0.7709 | 1.0423 | 1.3582 | 1.7195 |
| z=7 | 0.3617 | 0.5507 | 0.7822 | 1.0575 | 1.3779 | 1.7444 |
| z=8 | 0.3669 | 0.5586 | 0.7934 | 1.0727 | 1.3976 | 1.7693 |
| z=9 | 0.3721 | 0.5666 | 0.8047 | 1.0878 | 1.4173 | 1.7941 |
| z=10 | 0.3774 | 0.5745 | 0.8159 | 1.103 | 1.4369 | 1.8188 |

In case of no reinsurance, the ruin probability is

$$P_{20}(T \leq 20) = 0.5438.$$

According to Tables 6.8 and 6.9, the optimum reinsurance is attained by $k = 8$ and $z = 1$ because

$$\min\{\overline{P}^k_{u-z}(T \leq t)\} = \overline{P}^8_{19}(T \leq 20)$$

providing to

$$z > E[Y(u-z,k,t)] \text{ and } \overline{P}^k_{u-z}(T \leq t) < P_u(T \leq t).$$

### 6.6.5 Optimization of the premium rate $c$ for exponential distribution

As in Section 6.6.3, the ruin probability and the expected total injection amount for exponential claim distribution are listed in Tables 6.10 and 6.11 when $u = 20,\ z = 5,\ t = 40,\ \lambda = 1$, and $m = 12$.

Table 6.10: Ruin probability with respect to $k$ and $c$

| | $P_{20}(T \leq 40)$ (No reinsurance) | $\overline{P}^k_{15}(T \leq 40)$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| c=10 | 0.9129 | 0.9256 | 0.9245 | 0.9231 | 0.9214 | 0.9194 | 0.9171 |
| c=11 | 0.8537 | 0.8733 | 0.8718 | 0.87 | 0.8678 | 0.8652 | 0.8622 |
| c=12 | 0.781 | 0.8068 | 0.8051 | 0.8029 | 0.8003 | 0.7973 | 0.7938 |
| c=13 | 0.6988 | 0.7313 | 0.7294 | 0.727 | 0.7242 | 0.7209 | 0.7171 |
| c=14 | 0.6148 | 0.6525 | 0.6505 | 0.648 | 0.6451 | 0.6418 | 0.638 |
| c=15 | 0.5344 | 0.5767 | 0.5748 | 0.5724 | 0.5696 | 0.5664 | 0.5628 |

Table 6.11: Expected total injection amount $E(Y)$ with respect to $k$ and $c$

| | $E[Y]$ | | | | | |
|---|---|---|---|---|---|---|
| | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
| c=10 | 0.5429 | 0.8285 | 1.1801 | 1.6008 | 2.0938 | 2.6622 |
| c=11 | 0.5155 | 0.7864 | 1.1195 | 1.5177 | 1.9836 | 2.52 |
| c=12 | 0.4791 | 0.7306 | 1.0395 | 1.4084 | 1.8394 | 2.3347 |
| c=13 | 0.4365 | 0.6654 | 0.9463 | 1.2813 | 1.6723 | 2.121 |
| c=14 | 0.3913 | 0.5962 | 0.8476 | 1.147 | 1.4961 | 1.8963 |
| c=15 | 0.3466 | 0.528 | 0.7504 | 1.0151 | 1.3234 | 1.6765 |

As seen in the listed results, ruin probabilities under the reinsurance agreement are higher than the case without reinsurance. Therefore, the reinsurance agreement is not reasonable for the values. Different retention level and reinsurance premium should be determined.

According to the optimization examples, it is obvious that bigger retention level causes smaller ruin probability because of more capital injections.

# Chapter 7

# COMPARISON OF FINITE AND INFINITE TIME METHODS UNDER REINSURANCE AGREEMENT

In this chapter, we numerically compare our finite time method suggested in previous chapters with the infinite time method stated by Nie et al. [57]. The relationship between the finite and infinite time methods are analysed with respect to ruin probabilities and the expected injection amounts. Moreover, some optimum values of retention level and reinsurance premium are determined in order to obtain optimum reinsurance contract.

Some parts of this chapter have been submitted under the title "Optimal reinsurance via Dirac-Feynman Approach" [77].

Novelty and originality in the comparison include

- the application of the Dirac matrix with Feynman path calculation,
- differences in behaviour of the finite and the infinite time methods,
- computation of total capital injection amount, besides the ruin probability.

## 7.1 Finite and infinite time models for comparison

The modified surplus process is defined as in Chapter 6,

$$\begin{aligned} R^*(t) &= u + ct - z - H(S(t)) + Y(t) \\ &= w + ct - H(S(t)) + Y(t) \end{aligned}$$

where

$$Y(t) = Y(w,k,t) = \sum_{i=1}^{[\frac{t}{\varepsilon}-1]} y_i$$

is the total injection amount up to time $t$, defined by the retention level $k$, grid time size $\varepsilon$, and exact initial capital after reinsurance premium payment $w = u - z$.
In the modified surplus process, the aggregating claim amount with upper limit $h$ is defined as in Chapter 6 by

$$H(S(t)) = \sum_{i=1}^{N(t)} X_i I(X_i \le h) + hI(X_i > h). \tag{7.1.1}$$

Remember that $P_u(T < \infty)$ and $\overline{P}_u^k(T < \infty)$ denote the ultimate ruin probabilities for the classical and modified surplus processes with retention level $k$, respectively. To compare the finite time method with the infinite time counterpart, the following approach is considered.

$$\frac{P(T \le M)}{P(T \le L)} \to 1 \text{ (as } L, M \to \infty), \quad \text{which implies} \quad \frac{P(T < \infty)}{P(T \le L)} \to 1, (\text{as } L \to \infty).$$

For $L \le M$,

$$P(T \le L) \le P(T \le M)$$

because of

$$\begin{aligned} P(T > M|R(0) = u) &= P(T > L + (M - L)|R(0) = u) \\ &= P(T > L|R(0) = u) \int_0^\infty P(T > M - L|R(L) = x)dx. \end{aligned} \tag{7.1.2}$$

Notice that the integral part in equation (7.1.2) is less than or equal to 1, so this implies $P(T \leq L) \leq P(T \leq M)$.

With these expressions above, we analyse that

$$P(T < \infty) \geq P(T < t) \quad \text{and} \quad E[Y(w,k,t)] \leq E[Y(w,k,\infty)].$$

We apply the following exact expressions for the infinite time ruin probabilities of the modified surplus processes derived in [57]:

$$\overline{P}_w^k(T < \infty) = P_{w-k}(T < \infty) - G(w-k,k)\frac{1 - P_0(T < \infty)}{1 - G(0,k)} \tag{7.1.3}$$

where $u - z > k$, $G(x,k) = P_x(T < \infty)(1 - e^{-\alpha k})$ and the claim size has an exponential distribution with parameter $\alpha$.

Moreover, the expectation of the total injection amount is defined in [57] by

$$E[Y(w,k)] = \int_0^k yg(w-k,y)dy + E[Y(k,k)]G(w-k,k) \tag{7.1.4}$$

where $g(w-k,y) = P_{w-k}(T < \infty)\alpha e^{-\alpha y}$.

In comparison with the infinite time formula above, our finite time method introduced in Proposition 31 yields

$$\overline{P}_w^k(T > t) = (1 + o(\varepsilon)) \sum_{j=1}^{d-1} \left( (AK)^{[\frac{t}{\varepsilon} - 1]} A \right)_{w,j\varepsilon}$$

where we consider the discretized exponential claim sizes defined by

$$P(X) = \frac{\frac{1}{m}e^{-\frac{1}{m}x}}{\sum_{k=1}^{\infty}\frac{1}{m}e^{-\frac{1}{m}k}}. \tag{7.1.5}$$

In addition, recall that the ultimate ruin probability of the classical surplus process is defined by (e.g. [57])

$$P_u(T < \infty) = \frac{\lambda m}{c}e^{-(\frac{1}{m}-\frac{\lambda}{c})u}.$$

## 7.2 Numerical Results

In all our computations, Matlab software was used. In Matlab, the dimension of the matrix introduced in 4.5.15 for $\varepsilon = 0.01$ is usually taken at $20000 \times 20000$ because the dimension must be taken at least $(w + ct + E[Y])\frac{1}{\varepsilon}$.
In the following computations, we consider grid size $\varepsilon = 1$, and the transition matrix dimension is $1500 \times 1500$. The time of each computation of ruin probability and total injection amount is roughly 5 minutes on the HPC (High-performance computing) computer of the University of Leicester. In normal computers, it takes more time. HPC should be preferred in computations to avoid "out of memory" errors. The Matlab codes can be found in Appendix A.4.

### 7.2.1 Comparison of the ruin probability and the expected total injection amount

In Tables 7.1 and 7.2, the ruin probabilities of modified surplus processes under capital injections are compared for our finite approach with the infinite time approach as defined in [57].
In both tables, we let $R_0 = u - z = w$ in case of reinsurance and $R_0 = u$ in case of no reinsurance.

- Ruin probability and expected total injection amount in the finite and infinite time methods for the initial capital $u = 20$, the insurance premium $c = 1$, the claim frequency $\lambda = 0.03$, the claim mean $m = 30$, and the retention level

$k = 5$ are listed with respect to various reinsurance premiums $z = 1, 2, ..., 10$ in Table 7.1.

Table 7.1: Ruin probabilities and total injection amounts

| | | Infinite time method | | Finite time method | |
|---|---|---|---|---|---|
| Reinsurance premium | w=u-z | $\overline{P}_w^k(T < \infty)$ | $E[Y(w,k)]$ | $\overline{P}_w^k(T \leq 1400)$ | $E[Y(w,k,1400)]$ |
| z=1 | 19 | 0.8437 | 0.3719 | 0.8374 | 0.2746 |
| z=2 | 18 | 0.8465 | 0.3731 | 0.8404 | 0.2758 |
| z=3 | 17 | 0.8493 | 0.3744 | 0.8435 | 0.2771 |
| z=4 | 16 | 0.8521 | 0.3756 | 0.8466 | 0.2784 |
| z=5 | 15 | 0.855 | 0.3769 | 0.8496 | 0.2797 |
| z=6 | 14 | 0.8578 | 0.3781 | 0.8527 | 0.281 |
| z=7 | 13 | 0.8607 | 0.3794 | 0.8558 | 0.2823 |
| z=8 | 12 | 0.8636 | 0.3807 | 0.8589 | 0.2836 |
| z=9 | 11 | 0.8665 | 0.3819 | 0.862 | 0.2849 |
| z=10 | 10 | 0.8694 | 0.3832 | 0.8652 | 0.2862 |

- For the same values but different retention level (k=10), the results are listed in Table 7.2.

Table 7.2: Ruin probabilities and total injection amounts

| | | Infinite time method | | Finite time method | |
|---|---|---|---|---|---|
| Reinsurance premium | w=u-z | $\overline{P}_w^k(T < \infty)$ | $E[Y(w,k)]$ | $\overline{P}_w^k(T \leq 1400)$ | $E[Y(w,k,1400)]$ |
| z=1 | 19 | 0.8402 | 1.5697 | 0.8339 | 1.301 |
| z=2 | 18 | 0.843 | 1.575 | 0.8369 | 1.307 |
| z=3 | 17 | 0.8458 | 1.5802 | 0.84 | 1.3131 |
| z=4 | 16 | 0.8486 | 1.5855 | 0.843 | 1.3192 |
| z=5 | 15 | 0.8514 | 1.5908 | 0.8461 | 1.3253 |
| z=6 | 14 | 0.8543 | 1.5961 | 0.8492 | 1.3315 |
| z=7 | 13 | 0.8571 | 1.6014 | 0.8522 | 1.3377 |
| z=8 | 12 | 0.86 | 1.6068 | 0.8553 | 1.3439 |
| z=9 | 11 | 0.8629 | 1.6122 | 0.8584 | 1.3501 |
| z=10 | 10 | 0.8657 | 1.6175 | 0.8616 | 1.3563 |

As seen from Tables 7.1 and 7.2, the infinite time method gives larger ruin probability and expected injection amount compared with the finite time method. As expected, an increase in the retention level $k$ causes a decrease in ruin probability with larger expected injection amount.

- For the initial capital $u = 100$, reinsurance premiums $z = 5, 10, 15...70$, insurance premium $c = 1$, the claim frequency $\lambda = 0.02$, the claim mean $m = 45$, and the retention level $k = 30$, the ruin probabilities and expected total capital injection amount in finite and infinite time are shown in Table 7.3.

Table 7.3: Ruin probabilities and total injection amounts

| | | Infinite time method | | Finite time method | |
|---|---|---|---|---|---|
| Reinsurance premium | w=u-z | $\overline{P}^k_w(T < \infty)$ | $E[Y(w,k)]$ | $\overline{P}^k_w(T \leq 1400)$ | $E[Y(w,k,1400)]$ |
| z=70 | 30 | 0.8221 | 10.3978 | 0.7736 | 8.8315 |
| z=65 | 35 | 0.813 | 10.2829 | 0.7617 | 8.669 |
| z=60 | 40 | 0.804 | 10.1693 | 0.75 | 8.5087 |
| z=55 | 45 | 0.7951 | 10.0569 | 0.7384 | 8.3504 |
| z=50 | 50 | 0.7864 | 9.9458 | 0.727 | 8.1942 |
| z=45 | 55 | 0.7777 | 9.8359 | 0.7223 | 8.0308 |
| z=40 | 60 | 0.7691 | 9.7272 | 0.7112 | 7.8785 |
| z=35 | 65 | 0.7606 | 9.6197 | 0.7003 | 7.7284 |
| z=30 | 70 | 0.7522 | 9.5134 | 0.6895 | 7.5804 |
| z=25 | 75 | 0.7439 | 9.4083 | 0.6788 | 7.4345 |
| z=20 | 80 | 0.7356 | 9.3044 | 0.6682 | 7.2906 |
| z=15 | 85 | 0.7275 | 9.2015 | 0.6578 | 7.1487 |
| z=10 | 90 | 0.7195 | 9.0999 | 0.6474 | 7.0088 |
| z=5 | 95 | 0.7115 | 8.9993 | 0.6372 | 6.871 |

The ruin probabilities without reinsurance for both methods are

$$P_{100}(T < \infty) = 0.7207 \quad \text{and} \quad P_{100}(T < 1400) = 0.6389.$$

For the logical reinsurance agreement, the following conditions are necessary

- $\overline{P}^k_{u-z}(T < \infty) < P_u(T < \infty)$,
- $E[Y(w,k)] < z$.

In this circumstance, optimum values for the reinsurance agreement can be seen in Figure 7.1.

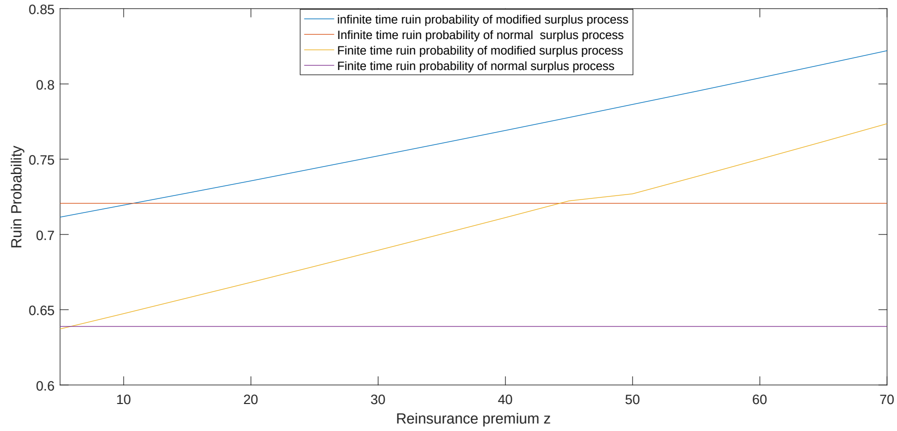



Figure 7.1: Ruin probabilities with respect to various $z$

- Now, let's observe the ruin probabilities of the surplus process with and without reinsurance for both methods by keeping the initial capital $w = u - z$ being fixed.

  The ruin probabilities of the modified surplus process and the expected total injection amount for $w = 10,\ c = 1,\ \lambda = 0.01$, and $m = 90$ are listed in Table 7.4.

Table 7.4: Ruin probabilities and total injection amounts

| | Infinite time method | | Finite time method | |
|---|---|---|---|---|
| Retention level | $\overline{P}_w^k(T < \infty)$ | $E[Y(w,k)]$ | $\overline{P}_w^k(T \leq 1400)$ | $E[Y(w,k,1400)]$ |
| k=1 | 0.8901 | 0.005 | 0.8073 | 0 |
| k=2 | 0.89 | 0.0199 | 0.8073 | 0.009 |
| k=3 | 0.89 | 0.045 | 0.8073 | 0.027 |
| k=4 | 0.89 | 0.0803 | 0.8073 | 0.0542 |
| k=5 | 0.8899 | 0.1259 | 0.8072 | 0.0906 |
| k=6 | 0.8899 | 0.182 | 0.8071 | 0.1364 |
| k=7 | 0.8898 | 0.2486 | 0.807 | 0.1916 |
| k=8 | 0.8897 | 0.3259 | 0.8069 | 0.2564 |
| k=9 | 0.8896 | 0.414 | 0.8067 | 0.3309 |
| k=10 | 0.8895 | 0.513 | 0.8066 | 0.4151 |

Similarly, for $w = 40,\ c = 1,\ \lambda = 0.01$, and $m = 90$, the results correspond to various retention levels, as shown in Table 7.5.

Table 7.5: Ruin probabilities and total injection amounts

| | Infinite time method | | Finite time method | |
|---|---|---|---|---|
| Retention level | $\overline{P}_w^k(T<\infty)$ | $E[Y(w,k)]$ | $\overline{P}_w^k(T\leq 1400)$ | $E[Y(w,k,1400)]$ |
| k=5 | 0.8512 | 0.1218 | 0.7519 | 0.0843 |
| k=10 | 0.8509 | 0.4962 | 0.7513 | 0.3861 |
| k=15 | 0.8503 | 1.137 | 0.7503 | 0.917 |
| k=20 | 0.8494 | 2.0581 | 0.7488 | 1.6881 |
| k=25 | 0.8482 | 3.274 | 0.7468 | 2.7104 |
| k=30 | 0.8467 | 4.799 | 0.7444 | 3.9944 |
| k=35 | 0.845 | 6.6478 | 0.7415 | 5.5502 |
| k=40 | 0.8523 | 8.8351 | 0.738 | 7.3873 |

The tables above show that the reinsurance is not always appropriate (as also discussed in [57]).

Notice that we chose small claim frequencies in the examples above in order to avoid high error rate.

The ultimate ruin probability does not work without the net profit condition ($c \not> \lambda m$) while the finite time methods work.

# Chapter 8

# FUTURE WORK

In this thesis, several approaches to computing the finite time ruin probabilities for the classical and modified surplus processes are analysed and compared with infinite time methods.

Although the quantum method gives good numerical results of the ruin probabilities via finite time characteristics, the infinite time counterparts of the finite time methods are interesting and present a challenging open question.

There are many realistic modifications to the surplus process and we only consider some of them. For example, the following modification appears to be popular in car insurance practice. It is referred to as the voluntary excess that affects premium rate in unit time and the capital of an insurance company. Some insurance companies put compulsory excess as well. More exactly, the total claim amount with voluntary excess (let $VE$ denote as the amount of voluntary excess) is defined by

$$S(t) = \sum_{i=0}^{N(t)} (X(i) - VE).$$

As seen, *VE* is a deductible amount from claim amount. The company gives the customer the option to choose the *VE*. Choosing a higher level voluntary excess decreases the insurance premium [43]. Therefore, it is a kind of bet. Optimization problems on the choice of the *VE*, and the relationship between the *VE* and premium rate together with an analysis of the finite time ruin probabilities, prompt interesting questions.

Similar questions (e.g., legal costs or late payment penalties) may also lead to interesting optimization questions.

Of course, adding interest rates is another important problem, as they play vital role in the computation of the current value of future claims. Therefore, interest rates should also be taken into account.

Adjusting the quantum approach and developing new methods to compute the finite and infinite ruin probabilities for dependent claims and claim occurrences should also be a future focus.

Lastly, in this thesis, heavy tailed distributions are not considered. The methods can be taken into consideration with heavy tailed distributions.

# Appendix A

# CODES

## A.1 Codes for comparison of ultimate ruin probabilities

```
function a=createplotforultimateruin(c,lambda,claimmean)
for u=1:100
x(u)=Appell_ultimateruin(u,c,lambda,claimmean);
y(u)=ultimateruin(u,c,lambda,claimmean);
end
plot(1:100,x)
hold on
plot(1:100,y)
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function ruinprobability=Appell_ultimateruin(u,c,lambda,
    claimmean)
%computation of ruin probability via Appell polynomial
    Approach
sum=0;
 for j=0:u
sum=sum+exp(lambda*(u-j)/c)*ee(j,(j-u)/c,lambda,claimmean);
```

```
 end
    ruinprobability=1-((c-lambda*claimmean)/c)*sum;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function result=ee(n,time,lambda,claimmean)
sum2=0;
for k=0:n
    sum2=sum2+((lambda*time)^k)*convolution(n,k,claimmean)/
        factorial(k);
end
result=sum2;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function probabilitymassfunction=convolution(n,k,claimmean)
if n==0;
    probabilitymassfunction=1;
elseif k==0;
    probabilitymassfunction=0;
else
   probabilitymassfunction=gamma3(n,k,claimmean);
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function ad=gamma3(sum,numberofconvolution,claimmean)
x=gampdf(sum,numberofconvolution,claimmean);
ad=x;
end
%%%%%%%%%%%%%%%%%%%%%%%%
function d=ultimateruin(u,c,lambda,mean)
a=1/mean;
d=(lambda*exp(-u*(a-(lambda/c))))/(a*c);
```

```
end
```

## A.2 Codes to compute the non ruin probability via Markov approach

```
function nonruin=MARKOV_1_CLAIM_EXPONENTIAL(u,t,c,lambda,m)
%Grid size =1
%Claims have exponential distributions
% Here, we consider N(1)=1,2,3,4,5,6
n=1200; %dimension of matrix, it may change with respect to
    premium rate, claim frequency and claim mean
A=single(zeros(n+c)); % A is the transition matrix
X=exppdf(1:1200,m);%probability mass function for discreted
    exponential distribution
X=X*exp(-lambda)*(lambda)/sum(X);
X2=gampdf(1:1200,2,m);
X2=X2*[(exp(-lambda)*(lambda)^2)/2]/sum(X2);
X3=gampdf(1:1200,3,m);
X3=X3*[(exp(-lambda)*(lambda)^3)/6]/sum(X3);
X4=gampdf(1:1200,4,m);
X4=X4*[(exp(-lambda)*(lambda)^4)/24]/sum(X4);
X5=gampdf(1:1200,5,m);
X5=X5*[(exp(-lambda)*(lambda)^5)/120]/sum(X5);
X6=gampdf(1:1200,5,m);
X6=X6*[(exp(-lambda)*(lambda)^6)/720]/sum(X6);
no_claim_probability=1-exp(-lambda)*(lambda)-[(exp(-lambda)
    *(lambda)^2)/2]-[(exp(-lambda)*(lambda)^3)/6]-[(exp(-
    lambda)*(lambda)^4)/24]-[(exp(-lambda)*(lambda)^5)
    /120]-[(exp(-lambda)*(lambda)^6)/720];
A(1,1)=1;
```

```
for i=2:n-c;
     A(i,i+c)=no_claim_probability;
end
 for ss=n-c+1:n
      A(ss,ss)=A(ss,ss)+no_claim_probability;
 end
for j=2:n;
     for kk=1:1200;
     if j-1+c-kk>0;
         A(j,j+c-kk)=A(j,j+c-kk)+X(kk)+X2(kk)+X3(kk)+X4(kk)+
            X5(kk)+X6(kk);
     else
     A(j,1)=A(j,1)+X(kk)+X2(kk)+X3(kk)+X4(kk)+X5(kk)+X6(kk);
     end
     end
end
D=A^t;   %D=A(t)
nonruin=sum(D(u+1,2:n+c));
end
```

## A.3 Codes to compute the ruin probability and the total injection amount for Poisson process

```
function [ruin,injectionamount]=MODIFIEDCASE11(w,t,c,
   claimfrequency,claimsize,retentionlevel)
%We compute ruin probability of modified surplus process and
    total injection amount
%Gridsize=1
```

```
tic
n=1500; %dimension of matrix, it may change with respect to
    premium rate, claim frequency and claim mean
N=3000; %splitting number to solve the integral numerically.
h=mtimes(2*pi,1/N);
p=single(0:h:2*pi);
A=single(zeros(n,n));
B=(2:n)'-(2:n);
C=single(zeros(n));
%The transition matrix
C(2:n,2:n)=0.5*(exp(i*B*p(1)+i*c*p(1)+claimfrequency*exp(-i*
    claimsize*p(1)))+exp(i*B*p(length(p))+i*c*p(length(p))+
    claimfrequency*exp(-i*claimsize*p(length(p)))));
for jjj=2:(length(p)-1)
A(2:n,2:n)=exp(i*(B*p(jjj))+i*(c*p(jjj))+claimfrequency*exp
    (-i*(claimsize*p(jjj))));
C=C+A;
end
C=h*C;
C=mtimes(exp(-claimfrequency)/(2*pi),C);
C(:,1)=1-sum(C(:,2:n),2);
C(1,1)=1;
AA=C^t;
Shiftmatrixoperator=single(zeros(n));% We create the shift
    matrix here
Shiftmatrixoperator(1,1)=1;
pppp=retentionlevel+1;
for ii=2:pppp;
Shiftmatrixoperator(ii,pppp)=1;
end
for j=(pppp+1):n;
```

```
    Shiftmatrixoperator(j,j)=1;
end
pp=mpower(C*Shiftmatrixoperator,t-1)*C;

Totalinjectionamount=0;
for ii=1:retentionlevel;
    Totalinjectionamount=Totalinjectionamount+(
        retentionlevel-ii)*C(u+1,ii+1);
end
 G=C;
 x=1:1:retentionlevel;
    y=retentionlevel-x;
    up=retentionlevel+1;
    uu=u+1;
for j=2:t;
    G=mtimes(mtimes(G,Shiftmatrixoperator),C);
    Totalinjectionamount=Totalinjectionamount+sum(y.*G(uu,2:
        up));
end
ruin=pp(u+1,1);  % Ruin probability
injectionamount=Totalinjectionamount; % Total injection
    maount
toc
end
```

## A.4 Codes to compute the ruin probability and total injection amount for Compound Poisson process

```
function [ruin,injectionamount]= Ruinprobabiliry__and__injectionamount(w,t,c,claimfrequency,claimmean,retentionlevel)
%we compute ruin probability of modified surplus process and total injection amount
%Gridsize=1
%Claim size have exponential distribution
tic
n=1500;
X=exppdf(1:400,claimmean);%probability mass function
i_th_claimfrequency=claimfrequency.*(X./sum(X));
N=3000; %splitting number to solve the integral numerically
h=mtimes(2*pi,1/N);
p=single(0:h:2*pi);
A=single(zeros(n));
B=(2:n)'-(2:n);
C=single(zeros(n));
% C is transition matrix
C(2:n,2:n)=0.5*(exp(i*B*p(1)+i*c*p(1)-sum(i_th_claimfrequency.*(1-exp(-i*(1:200)*p(1)))))+exp(i*B*p(length(p))+i*c*p(length(p))-sum(i_th_claimfrequency.*(1-exp(-i*(1:200)*p(length(p)))))));
for jjj=2:(length(p)-1)
A(2:n,2:n)=exp(i*B*p(jjj)+i*c*p(jjj)-sum(i_th_claimfrequency.*(1-exp(-i*(1:200).*p(jjj)))));
C=C+A;
end
C=h*C;
C=mtimes(1/(2*pi),C);
 C(:,1)=1-sum(C(:,2:n),2);
 C(1,1)=1;
```

```
AA=C^t;
Shiftmatrixoperator=single(zeros(n));
Shiftmatrixoperator(1,1)=1;
pppp=retentionlevel+1;
for  ii=2:pppp;
Shiftmatrixoperator(ii ,pppp)=1;
end
for  j=(pppp+1):n;
     Shiftmatrixoperator(j,j)=1;
end
pp=mpower(C*Shiftmatrixoperator ,t-1)*C;
Totalinjectionamount=0;
for  ii=1:retentionlevel;
     Totalinjectionamount=Totalinjectionamount+(
         retentionlevel-ii)*C(u+1,ii+1);
end
 G=C;
 x=1:1:retentionlevel;
     y=retentionlevel-x;
     up=retentionlevel+1;
     uu=u+1;
 count=1;
for  j=2:t;
     G=mtimes(mtimes(G,Shiftmatrixoperator),C);
     Totalinjectionamount=Totalinjectionamount+sum(y.*G(uu,2:
         up));
     end
ruin=pp(u+1,1);  % Ruin probability
injectionamount=Totalinjectionamount; % Total injection
   amount
toc
```

```
end
```

## A.5 Codes to compute the ruin probability for Appell Polynomial Approach

```
 function NONRUIN=APPELL_EXPONENTIAL(u,time,c,claimfrequency,claimmean)
tic
  %Here X__i have exponential distributions.
  % Computation of nonruin via Appell Approach.
  %u is initial  capital.
  %c is premium.
if u<0
     NONRUIN=0;
else
zz=0;
for n=0:u;
    zz=zz+ee(n,time,claimfrequency,claimmean);
end
zp=0;
for n=(u+1):round(c*time+u);
for j=0:u;
     qq=(j-u)/c;
     qqq=(time*c+u-j)/c;
     zp=zp+(ee(j,qq,claimfrequency,claimmean)*(c*time-n+u)*ee(n-j,qqq,claimfrequency,claimmean))/(c*time-j+u);
     end
end
NONRUIN=exp(-claimfrequency*time)*(zz+zp);
end
```

```
toc
end
%%%%%%%%%%
function ennnxxx=ee(n,time,claimrequency,claimmean)
summ=0;
if time==0
    ennnxxx=0;
else
for k=0:n;
    ttttt=(app222(n,k,claimmean)*(claimrequency*time)^(k))/
        factorial(k);
    summ=summ+ttttt;
end
ennnxxx=summ;
end
end
function appellprobability=app222(n,k,claimmean)
%here X1+X2+...+Xk=n
if n==0;
    appellprobability=1;
elseif k==0;
    appellprobability=0;
else
    appellprobability=gamma3(n,k,claimmean);
end
end
function ad=gamma3(valueofsum,numberofconvolution,mu)
% mu = E[X_1]
x=gampdf(valueofsum,numberofconvolution,mu);
ad=x;
end
```

## A.6 Codes to compute the (non)ruin probability for Monte Carlo Approach

```
function nonruin=MONTECARLO(u,t,c,lambda,m,M)
%M is iteration number
%u is initial capital
%t is time
%c is premium rate
%lambda is claim frequancy
%m is claim mean
%grid time size =1
%Claims are random samplings  distributed exponentially
%we use monte carlo approach as well.
% here, we consider N(1)=1,2,3,4,5,6
n=200;
for kk=1:M
B=round(exprnd(m,n,6)); %random sampling distributed
    exponentially
A=single(zeros(n+c)); % A will be the transition matrix
N(1)=exp(-lambda)*(lambda);
N(2)=[(exp(-lambda)*(lambda)^2)/2];
N(3)=[(exp(-lambda)*(lambda)^3)/6];
N(4)=[(exp(-lambda)*(lambda)^4)/24];
N(5)=[(exp(-lambda)*(lambda)^5)/120];
N(6)=[(exp(-lambda)*(lambda)^6)/720];
no_claim_probability=1-sum(N);

A(1,1)=1;
for i=2:n-c;
    A(i,i+c)=no_claim_probability;
end
```

```
%%%%%%%%%%%%%%%%
 for ss=n-c+1:n
      A(ss,ss)=A(ss,ss)+no_claim_probability;
 end
%%%%%%%%%%%%%%%%
for j=2:n;
     for k=1:6
     if j-1+c-sum(B(j,1:k))>0
          A(j,j+c-sum(B(j,1:k)))=A(j,j+c-sum(B(j,1:k)))+N(k);
     else
          A(j,1)=A(j,1)+N(k);
     end
     end
end
D=A^t;
result(kk)=sum(D(u+1,2:n+c)); %non ruin
end
nonruin=sum(result)/M;
end
```

## A.7 Codes of optimization problems in Chapter 5

```
 function d=optimization_of_initial_capitals(u,c,t,lambda,
    lambda2,mean,mean2)
%optimization of initial capitals with respect to time
for k=1:u-1; % initial capital
     for kk=1:t; %time
    nonruin(k,kk)=quantumandmarkov(k,kk,c,lambda,mean)*
        quantumandmarkov(u-k,kk,c,lambda2,mean2);
```

```
        end
end
surf(nonruin) %plot
d=nonruin
end
%%%%%%%%%%%%%%%
function as=quantumandmarkov(u,t,c,lambda,mean)
%Non ruin probability
%here the grid time is 1
n=100;
A=single(zeros(n));% A is  transition amtrix
A(1,1)=1;
for ii=2:n;
  for   jj=2:n;
         A(ii,jj)=quanintg7modify(ii-1,jj-1,1,c,lambda,mean);
  end
  A(ii,1)=1-sum(A(ii,2:n));
end
tt=t/1;
D=A^tt;
as=sum(D(u+1,2:n));
end
%%%%%%%%%%%%%%%%
function aa=quanintg7modify(u,newu,t,c,lambda,mean)
% Computation of elements of the transition matrix
for k=0:100;
        j(k+1)=(2*pi*k)/100;
end
sum=0;
for ii=1:100;
        p=(j(ii+1)+j(ii))/2;
```

```
        ttt=(j(ii+1)-j(ii))*Integralsolving(p,u,newu,t,c,lambda,mean);
        sum=sum+ttt;
end
aa=real(sum/(2*pi));
end
%%%%%%%%%%%%%%%
function dd=Integralsolving(p,u,newu,t,c,lambda,mean)
dd=exp(i*p*(u-newu)-t*(-c*i*p+lambda- lambda*exp(-i*mean*p)));
end
%%%%%%%%%%%%%%%%%%%
function asd=optimizationofproportionofclaims(u,c,t,lambda,mean)
a=1;
for k=1:5
    for j=1:t
   Y(a,j)= quantumandmarkov(u,j,c,lambda,k*mean);
    end
end
surf(Y)
asd=Y;
end
 %%%%%%%%%%%%%%%%%%%%%555
function as=optimizationinjection3(u1,u2,time,c1,c2,lambda1,lambda2,mean1,mean2)
% we need to find optimum injection(or reduction) time and amount
%here we used quantum method
kk=1;
for a=-5:5; %a is amount of injection or reduction
```

```
    for injectiantime=1:time-1;
        if a<0
            C(kk,injectiantime)=quantumandmarkovreduction(u1
                ,injectiantime,time,c1,lambda1,mean1,-a)*
                quantumandmarkovinjection(u2,injectiantime,
                time,c2,lambda2,mean2,-a);
        elseif a>0
          C(kk,injectiantime)=quantumandmarkovinjection(u1,
              injectiantime,time,c1,lambda1,mean1,a)*
              quantumandmarkovreduction(u2,injectiantime,time
              ,c2,lambda2,mean2,a);
        else
          C(kk,injectiantime)=quantumandmarkov(u1,time,c1,
              lambda1,mean1)*quantumandmarkov(u2,time,c2,
              lambda2,mean2);
        end
      end
    kk=kk+1;
end
as=C;
end
%%%%%%%%%%%%%%%%%%
function as=quantumandmarkovinjection(u,t1,t,c,lambda,mean,a
   )
%Non ruin probability
%here the grid time size is 1
n=1000; % dimension of transition matrix. if you change
   here, you need to change quanintg7modify
A=single(zeros(n));% transition matrix
KKK=single(zeros(n)); %shift matrix
A(1,1)=1;
```

```
for ii=2:n;
 for   jj=2:n;
      A(ii,jj)=quanintg7modify(ii-1,jj-1,1,c,lambda,mean);
 end
 A(ii,1)=1-sum(A(ii,2:n));
end
%tt=t/0.01;
tt=t/1;
for i=2:n;
    if  a+i <n+1
     KKK(i,a+i)=1; %  capital shifter
    end
end
KKK(1,1)=1;
D=(A^(t1))*KKK*A^(t-t1);
as=sum(D(u+1,2:n)); %nonruin
end
%%%%%%%%%%%%%%%%%%%%
function as=quantumandmarkovreduction(u,t1,t,c,lambda,mean,a
    )
%Non ruin probability
%here the grid time size is 1
n=100; %dimension of transition matrix
A=single(zeros(n));%transition matrix
BBB=single(zeros(n));
A(1,1)=1;
for ii=2:n;
 for   jj=2:n;
      A(ii,jj)=quanintg7modify(ii-1,jj-1,1,c,lambda,mean);
 end
 A(ii,1)=1-sum(A(ii,2:n));
```

```
end
%tt=t/0.01;
 tt=t/1;
 for  i=1:a+1;
     BBB(i,1)=1; % this is capital shifter
 end
ww=2;
 for  j=a+2:n
       BBB(j,ww)=1;
       ww=ww+1;
 end
D=(A^(t1))*BBB*A^(t-t1);
 as=sum(D(u+1,2:n));%non ruin probability
 end
```

## A.8 Codes of optimization problems in Chapter 6

```
 function  aa=OPTIMIZATIONOFGAUSSIAN2(u,t,c,lambda,mean,varx)
%optimization of reinsurance premium z and retention level k
%computation of expectation of injection
%gridsize=1
%distribution is Gaussian
  n=1000; %dimension of transition matrix
A=single(zeros(n,n));%transition matrix
B=(2:n)-(2:n)';
 var=lambda*varx+lambda*(mean)^2;
A(2:n,2:n)= exp(((B-c+mean*lambda).^2)/(-2*var))/sqrt(2*pi*
    var);
  A(:,1)=1-sum(A(:,2:n),2);
```

```
 A(1,1)=1;
step=1;
for  retentionlevel=5:10;
     shiftmatrix=single(zeros(n));
shiftmatrix(1,1)=1;
pppp=retentionlevel+1;
for  ii=2:pppp;
shiftmatrix(ii,pppp)=1;
end
for  j=(pppp+1):n;
     shiftmatrix(j,j)=1;
end
pp=mpower(A*shiftmatrix,t-1)*A;
topp=0;
for  ii=1:retentionlevel;
     topp=topp+(retentionlevel-ii)*A(u-[1:10]+1,ii+1);
end
 G=A;
 x=1:1:retentionlevel;
     y=retentionlevel-x;
     up=retentionlevel+1;
     uu=u-[1:10]+1;
 count=1;
for  j=2:t;
     G=mtimes(mtimes(G,shiftmatrix),A);
     topp=topp+sum(y.*G(uu,2:up),2);
end
 aa([1:10],step)=topp;
 step=step+1
end
end
```

```
%%%%%%%%%%%%%%%%%%
function aa=OPTIMIZATIONOFGAUSSIAN3(u,t,c,lambda,mean,varx,z)
%optimization of reinsurance premium h and retention level k
%with respect to ruin probability
%grid time size=1
%distribute is Gaussian
n=1000; %dimension of transition matrix
A=single(zeros(n,n)); % transition matrix
B=(2:n)-(2:n)';
var=lambda*varx+lambda*(mean)^2; % computation of variance in grid time.
meannn=mean;
stepp2=1
for h=0.5:0.1:1
    mean=meannn*h;
A(2:n,2:n)= exp(((B-c+mean*lambda).^2)/(-2*var))/sqrt(2*pi*var);
 A(:,1)=1-sum(A(:,2:n),2);
 A(1,1)=1;
step=1;
for retentionlevel=5:10;
    shiftmatrix=single(zeros(n));
shiftmatrix(1,1)=1;
pppp=retentionlevel+1;
for ii=2:pppp;
shiftmatrix(ii,pppp)=1;
end
for j=(pppp+1):n;
    shiftmatrix(j,j)=1;
end
```

```
  pp=mpower(A*shiftmatrix ,t-1)*A;
 aa(stepp2 ,step)=pp(u-z+1,1);

 step=step+1;
end
stepp2=stepp2+1
end
end
%%%%%%%%%%%%%%%%
function aa=OPTIMIZATIONOFGAUSSIAN5(u,t,lambda,mean,varx,z)
%optimization of reinsurance premium c and retention level k
%with respect to ruin probability
%grid time size=1
%distribute is Gaussian

n=1000; % dimension of transition matrix
A=single(zeros(n,n)); % transition matrix
B=(2:n)-(2:n)';
var=lambda*varx+lambda*(mean)^2;% computation of variance in
    the grid time.
steppp=1;
for c=10:1:15
A(2:n,2:n)= exp(((B-c+mean*lambdalambda).^2)/(-2*var))/sqrt
    (2*pi*var);
 A(:,1)=1-sum(A(:,2:n),2);
 A(1,1)=1;
step=1;
for retentionlevel=5:10;
     shiftmatrix=single(zeros(n));
shiftmatrix(1,1)=1;
pppp=retentionlevel+1;
```

```
for ii=2:pppp;
shiftmatrix(ii,pppp)=1;
end
for j=(pppp+1):n;
    shiftmatrix(j,j)=1;
end
  pp=mpower(A*shiftmatrix,t-1)*A;
%to find expected capital injection amount, following codes
   can be used if necessary.
% topp=0;
% for ii=1:retentionlevel;
%       topp=topp+(retentionlevel-ii)*A(u+1,ii+1);
% end
%  G=A;
%  x=1:1:retentionlevel;
%       y=retentionlevel-x;
%       up=retentionlevel+1;
%       uu=u+1;
%  count=1;
% for j=2:t;
%       G=mtimes(mtimes(G,shiftmatrix),A);
%
%
% topp=topp+sum(y.*G(uu,2:up));

%end
 aa(steppp,step)=pp(u-z+1,1);


 step=step+1;
%aa(1,2)=topp;
end
```

```
steppp=steppp+1
end
end
```

## A.9 Codes of computation of ultimate ruin under reinsurance in (7.1.3)

```
  function  ff=ultimateruinwithreinsurance(w,k,c,frequency,mean)
%This function gives ultimate ruin of modified surplus process.
%Equation in (7.1.3)
if w==k
    result=(ultimateruin(0,c,frequency,mean)-GGG(0,k,c,frequency,mean))/(1-GGG(0,k,c,frequency,mean));
else
    result=ultimateruin(w-k,c,frequency,mean)-GGG(w-k,k,c,frequency,mean)*((1-ultimateruin(0,c,frequency,mean))/(1-GGG(0,k,c,frequency,mean)));
end
ff=result;
end
%%%%%%%%%%%%%%%%%%%%%
function
%ultimate ruin without reinsurance d=ultimateruin(w,c,frequency,mean)
a=1/mean; % a=1/mean
d=(frequency*exp(-w*(a-(frequency/c))))/(a*c);
end
%%%%%%%%%%%%%%%%
```

```
function dddd=GGG(w,y,c,frequency,mean)
a=1/mean;
dddd=ultimateruin(w,c,frequency,mean)*(1-exp(-a*y));
end
%%%%%%%%%%%%%%%
function dddd=gggg(w,y,c,frequency,mean)
a=1/mean;
dddd=ultimateruin(w,c,frequency,mean)*a*exp(-a*y)
end
%%%%%%%%%%%%%%%
```

## A.10 Codes of computation of total injection amount in (7.1.4)

```
  function sad=injectionamount1(w,c,frequency,mean,k)
%This is for w>k
a=1/mean;
sum=0;
h=k/1000;
for n=1:999; % the number should be big enough, so it
    depends on variables
sum=sum+n*h*gggg(w-k,n*h,c,frequency,mean);
end
ttt=h*(sum+(k*gggg(w-k,k,c,frequency,mean)/2));
sad=ttt+injectionamount2(k,c,frequency,mean,k)*GGG(w-k,k,c,
    frequency,mean);
end
%%%%%%%%%%%%%%%
function saddas=injectionamount2(w,c,frequency,mean,k)
%This is for w=k
```

```
a=1/mean;
sum=0;
h=k/1000;
for n=1:999;
sum=sum+n*h*gggg(0,n*h,c,frequency,mean);
end
ttt=h*(sum+(k*gggg(0,k,c,frequency,mean)/2));
saddas=ttt/(1-GGG(0,k,c,frequency,mean));
end
```

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