Cohomology and finiteness conditions for generalisations of Koszul algebras

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Abstract. We study finite dimensional Koszul algebras and their generalisations including d-Koszul algebras and (D, A)-stacked algebras, together with their projective resolutions and Hochschild cohomology. Then we introduce the stretched algebra $\tilde{\Lambda}$ and give a functorial construction of the projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ and the projective bimodule resolution of $\tilde{\Lambda}$. Following this, we show that if $E(\Lambda)$ is finitely generated then so is $E(\tilde{\Lambda})$. We investigate the connection between $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$ and the finiteness condition **(Fg)** using the theory of stratifying ideals. We give sufficient conditions for a finite dimensional Koszul monomial algebra to have **(Fg)** and generalize this result to finite dimensional d-Koszul monomial algebras. It is known that if Λ is a d-Koszul algebra then $\tilde{\Lambda}$ is a (D, A)-stacked algebra, where $D = dA$. We investigate the converse. We give the construction of the algebra $\mathcal{B}$ from a (D, A)-stacked algebra $\mathcal{A}$ and show that if $\mathcal{A}$ is a (D, A)-stacked monomial algebra, then $\mathcal{B}$ is d-Koszul with $D = dA$.

This thesis is dedicated to the memory of my father,

Yousuf Jawad,

who always believed in my ability to be successful in the academic arena. You are gone but your belief in me has made this journey possible

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Contents

1. Introduction

This thesis studies cohomology of finite dimensional algebras, particularly Koszul algebras and their generalisations. Koszul algebras have been shown to be important in algebra and algebraic topology [38], see also [21]. The class of Koszul algebras was introduced by Priddy in 1970, [38]. There are several generalisations of Koszul algebras; the d-Koszul algebras were introduced by Berger [5] and (D, A)-stacked monomial algebras were introduced by Green and Snashall [26]. Leader [36] extended this theory to the class of non-monomial (D, A)-stacked algebras. The Ext algebra of a Koszul algebra, a d-Koszul algebra and a (D, A)-stacked algebra are all finitely generated.

We are interested in the cohomology and finiteness conditions for generalizations of Koszul algebras where $\Lambda = K\mathcal{Q}/I$ is a finite dimensional algebra for K a field and $\mathcal{Q}$ a finite quiver and I an admissible ideal. Homological algebra has been used to study the representations of finite dimensional algebras (see [2]) and the study of cohomology theories (including the Ext algebra and the Hochschild cohomology ring) has proved extremely useful. The Hochschild cohomology of finite dimensional algebras was introduced by Hochschild [30]. We study projective resolutions, the Ext algebra, Hochschild cohomology and the **(Fg)** condition for generalisations of Koszul algebras.

In this thesis, we begin with some background information on finite dimensional algebras Λ given by quiver and relations. We also remind the reader of the construction of the minimal projective resolution of $\Lambda/\mathfrak{r}$ of Green, Solberg and Zacharia, given in [28], where $\mathfrak{r}$ is the Jacobson radical of Λ. This is followed by the construction of the beginning of a minimal projective bimodule resolution of Λ of Green and Snashall given in [24].

In Chapter 3 we introduce Koszul algebras and their generalisations. This is followed by giving a brief introduction to Gröbner bases following [13], [15] and [17]. The main result in this chapter is:

Theorem 3.28 Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra with $\operatorname{gldim}\Lambda \geq 4$ and with a reduced Gröbner basis $\mathcal{G}$ of elements of length D. Then $A|D$.

In Chapter 4 we explicitly give the construction of a new algebra $\tilde{\Lambda}$ from a finite dimensional algebra Λ. We call $\tilde{\Lambda}$ a stretched algebra. This generalises work by

Leader [36], where she takes a d-Koszul algebra Λ and gives a construction of a new algebra, $\tilde{\Lambda}$. We prove in Theorem 4.8 that the algebras Λ and $\varepsilon\tilde{\Lambda}\varepsilon$ are isomorphic where $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ (as an element of $\tilde{\Lambda}$). After that we describe the relationship between the projective resolutions of $\Lambda/\mathfrak{r}$ and $\tilde{\Lambda}/\tilde{\mathfrak{r}}$, and between the projective bimodule resolutions of Λ and $\tilde{\Lambda}$. We show that if $E(\Lambda)$ is finitely generated then so is $E(\tilde{\Lambda})$:

Theorem 4.47 Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Suppose there is some $m \geq 1$ such that the Ext algebra $E(\Lambda)$ is generated in degree at most m. Then the Ext algebra $E(\tilde{\Lambda})$ is also finitely generated, and has generators in degree at most $m+2$.

Chapter 5 calculates the Hochschild cohomology groups of algebras Λ and $\tilde{\Lambda}$ in Examples 5.1 and 5.2, where we find $\mathrm{HH}^3(\Lambda) \cong \mathrm{HH}^3(\tilde{\Lambda})$. Moreover, it is shown in Chapter 6 that the results of these examples can be extended to the general case (see Theorem 6.24).

In Chapter 6, we study the Hochschild cohomology rings of Λ and its stretched algebra $\tilde{\Lambda}$ and the finiteness condition **(Fg)** using the theory of stratifying ideals. We give some results on the stretched algebra $\tilde{\Lambda}$ showing in Theorem 6.9 that $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal:

Theorem 6.9 Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters with $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ and $B = \varepsilon\tilde{\Lambda}\varepsilon$. Then $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal of $\tilde{\Lambda}$.

We then show in Corollary 6.13 that $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ has finite projective dimension:

Corollary 6.13 Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. With the above notation, $\mathrm{pdim}_{\tilde{\Lambda}^e} \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = 2$.

After that, we build on the work of Koenig and Nagase [34], Nagase [37] and Psaroudakis, Skartsæterhagen and Solberg [39]. We prove that $\tilde{\Lambda}$ satisfies **(Fg)** if and only if Λ satisfies **(Fg)**:

Theorem 6.35 Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the previous notation. Then $\tilde{\Lambda}$ satisfies **(Fg)** if and only if Λ satisfies **(Fg)**.

We also find a relationship between the injective dimension of $\tilde{\Lambda}$ and the injective dimension of Λ:

Theorem 6.37 Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra and let $\tilde{\Lambda}$ be the stretched algebra. Then $\mathrm{idim}_{\tilde{\Lambda}} \tilde{\Lambda} \leq \sup\{\mathrm{idim}_{\Lambda} \Lambda, 2\}$.

Chapter 7 presents some other Koszul algebras which have **(Fg)**. We begin with existing work and results by Erdmann and Solberg [12]. We then give some properties for a finite dimensional monomial d-Koszul algebra $\Lambda = K\mathcal{Q}/I$; let $\mathcal{R}^n$ be the set of overlaps R^n and let ρ be a minimal set of monomials in the generating set of I. In Proposition 7.13, in which we show that all subpaths of $R \in \mathcal{R}^n$ $(n \geq 2)$ of length d are in ρ and in Proposition 7.14 we show that all paths of length d which lie on the closed trail T are in ρ:

Proposition 7.14 Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional monomial d-Koszul algebra, where $d \geq 2$, and let ρ be a minimal generating set for I consisting of monomials of length d. Suppose that $T = a_1 a_2 \cdots a_n$ is a closed trail in $\mathcal{Q}$ so that $a_1, a_2, \ldots, a_n$ are distinct arrows. Suppose also that $d \geq n+1$. Then all paths of length d which lie on the closed trail $T = \alpha_1 \cdots \alpha_m$ are in ρ.

After that we give sufficient conditions for a finite dimensional d-Koszul monomial algebra to have **(Fg)**, and we prove this in Theorem 7.15:

Theorem 7.15 Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional monomial d-Koszul algebra, where $d \geq 2$, and let ρ be a minimal generating set for I consisting of monomials of length d. Suppose $\operatorname{char} K \neq 2$ and $\operatorname{gldim} \Lambda \geq 4$. Suppose that Λ satisfies the following conditions:

(1) If α is a loop in $\mathcal{Q}$, then $\alpha^d \in \rho$ but there are no elements in ρ of the form $\alpha^{d-1}\beta$ or $\beta\alpha^{d-1}$ with $\beta \neq \alpha$.

(2) If $T = \alpha_1 \cdots \alpha_m$ is a closed trail in $\mathcal{Q}$ with $m > 1$ such that the set $\rho_T = \{\alpha_1 \cdots \alpha_d, \alpha_2 \cdots \alpha_{d+1}, \ldots, \alpha_m \alpha_1 \cdots \alpha_{d-1}\}$ is contained in ρ, then there are no elements in $\rho \setminus \rho_T$ which begin or end with the arrow α_i, for all $i = 1, \ldots, m$.

Then Λ has **(Fg)**.

In Chapter 8, we give a construction of an algebra $\mathcal{B}$ from a (D, A)-stacked algebra $\mathcal{A}$, where $D = dA$, $A \geq 1$ and $d \geq 2$. One of the main results of this chapter is Theorem 8.4, where we prove that the algebra $\mathcal{B}$ we have constructed from a (D, A)-stacked monomial algebra $\mathcal{A}$ is a d-Koszul monomial algebra. Also we give conditions in Theorem 8.5 under which $\mathcal{A}$ and $\tilde{\mathcal{B}}$ are isomorphic:

Theorem 8.5 Let $\mathcal{A}$ be a (D, A)-stacked monomial connected algebra with $\operatorname{gldim} \mathcal{A} \geq$

4, so $D = dA$, for some $d \geq 2$. Let $\mathcal{B}$ be the algebra constructed from Definition 8.2. Assume that the following conditions hold:

(1) Each arrow occurs in precisely one A-subpath;

(2) If v is properly internal to some $x \in \mathcal{S}_A$, then

  (a) v is not properly internal to $y \in \mathcal{S}_A$ for $y \neq x$.

  (b) $v \neq \mathfrak{o}(z)$ and $v \neq \mathfrak{t}(z)$, for all $z \in \mathcal{S}_A$.

Then $\tilde{\mathcal{B}} \cong \mathcal{A}$.

This key result naturally leads to the intriguing question of how the cohomology of $\tilde{\mathcal{B}}$ is related to that of $\mathcal{A}$, which could prove to be an interesting topic for future research.

2. Finite dimensional algebras and modules

In this chapter, we review the concepts of quiver, path algebra, and admissible ideal and we discuss the result that every finite dimensional basic K-algebra, where K is an algebraically closed field, is isomorphic to $K\mathcal{Q}/I$ for a quiver $\mathcal{Q}$ and an admissible ideal I. This material is covered in many books on representation theory, for example [2], [3], [9], [10], and [28]. We follow the approach of [2].

Definition 2.1. A quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{o}, \mathfrak{t})$, consists of two sets, $\mathcal{Q}_0$ which is the set of vertices and $\mathcal{Q}_1$ which is the set of arrows, together with two maps $\mathfrak{o}, \mathfrak{t} : \mathcal{Q}_1 \to \mathcal{Q}_0$, which associate to each arrow $\alpha \in \mathcal{Q}_1$ its origin $\mathfrak{o}(\alpha)$ and its tail $\mathfrak{t}(\alpha)$.

Note that the quiver is finite if both of $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are finite. We assume that all quivers are finite.

Definition 2.2. The quiver $\mathcal{Q}$ is connected if the underlying graph $\overline{\mathcal{Q}}$ which is obtained from $\mathcal{Q}$ with no orientation on the arrows, is a connected graph.

We assume throughout this thesis that K is field.

Now we want to define the path algebra of a quiver. Before that we define what is meant by a path. A path is a sequence $\alpha_1\alpha_2\cdots\alpha_n$ of arrows $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{Q}_1$ with $\mathfrak{t}(\alpha_l) = \mathfrak{o}(\alpha_{l+1})$ for $l = 1, \ldots, n-1$. We say that the length of the path $p = \alpha_1\alpha_2\cdots\alpha_n$ is n and is denoted by $\ell(p)$. For each vertex i, we let e_i denote the trivial path at i of length 0, so that $e_i^2 = e_i$. We also use v, v_i, w, w_i to denote vertices of a quiver. Then, to avoid too many subscripts, we use the same letters v, v_i, w, w_i to denote the trivial path at that vertex. It is clear from the context as to whether we mean the vertex or the trivial path at the vertex.

We write our arrows in a path from left to right.

Definition 2.3. Let $\mathcal{Q}$ be a quiver. The path algebra $K\mathcal{Q}$ of $\mathcal{Q}$ is the K-algebra whose underlying K-vector space has basis the set of all paths $\alpha_1\cdots\alpha_n$ of length $n \geq 0$ and such that the product of two paths $\alpha_1\cdots\alpha_n$ and $\beta_1\cdots\beta_m$ is equal to zero if $\mathfrak{t}(\alpha_n) \neq \mathfrak{o}(\beta_1)$ and is equal to the composed path $\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m$ if $\mathfrak{t}(\alpha_n) = \mathfrak{o}(\beta_1)$.

There is a direct sum decomposition of $K\mathcal{Q}$ into vector spaces,

$$K\mathcal{Q} = K\mathcal{Q}_0 \oplus K\mathcal{Q}_1 \oplus \cdots \oplus K\mathcal{Q}_n \oplus \cdots$$

where $K\mathcal{Q}_n$ is the subspace of $K\mathcal{Q}$ generated by the set $\mathcal{Q}_n$ of all paths of length n, for all $n \geq 0$.

In addition, this decomposition defines a grading on $K\mathcal{Q}$, since $(K\mathcal{Q}_n)(K\mathcal{Q}_m) \subseteq K\mathcal{Q}_{n+m}$ for all $n, m \geq 0$. This will subsequently be called the length grading.

Let Λ be an K-algebra throughout this chapter.

An element $e \in \Lambda$ is called an idempotent if $e^2 = e$. The idempotents e_1, e_2 are called orthogonal if $e_1e_2 = e_2e_1 = 0$, and the idempotent e is said to be primitive if e cannot be written as a sum of two nonzero orthogonal idempotents of Λ.

Proposition 2.4. *Let $\mathcal{Q}$ be a quiver. The element $1 = \sum_{a \in \mathcal{Q}_0} e_a$ is the identity of $K\mathcal{Q}$ and the set $\{e_a | a \text{ in } \mathcal{Q}_0\}$ of all trivial paths is a complete set of primitive orthogonal idempotents for $K\mathcal{Q}$.*

Definition 2.5. Let $\mathcal{Q}$ be a finite connected quiver. The two-sided ideal of the path algebra $K\mathcal{Q}$ generated (as an ideal) by the arrows of $\mathcal{Q}$ is called the arrow ideal of $K\mathcal{Q}$ and is denoted by $R_{\mathcal{Q}}$.

Definition 2.6. Let $\mathcal{Q}$ be a quiver and $R_{\mathcal{Q}}$ be the arrow ideal of the path algebra $K\mathcal{Q}$. A two-sided ideal I of $K\mathcal{Q}$ is said to be admissible if there exists $m \geq 2$ such that $R_{\mathcal{Q}}^m \subseteq I \subseteq R_{\mathcal{Q}}^2$.

The following result is Gabriel's Theorem (see [10]). This theorem is the reason why it is useful to study algebras of the form $K\mathcal{Q}/I$.

Theorem 2.7. *Let K be an algebraically closed field and let Λ be a basic and connected finite dimensional K-algebra. Then there exists a unique quiver $\mathcal{Q}$ and an admissible ideal I of $K\mathcal{Q}$ such that $\Lambda \cong K\mathcal{Q}/I$.*

Next we look at projective modules and projective resolutions.

Definition 2.8. A right Λ-module P is projective if for any epimorphism $f : M \to N$ and any homomorphism $g : P \to N$, there is an homomorphism $h : P \to M$ such that the diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "f"'] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 2.9. *Let e be an idempotent element in Λ. Then $e\Lambda$ is a projective right Λ-module.*

Definition 2.10. A chain complex is a sequence

$$\cdots \longrightarrow X^{n+1} \xrightarrow{d^{n+1}} X^n \xrightarrow{d^n} X^{n-1} \longrightarrow \cdots$$

of right Λ-modules with Λ-homomorphisms such that $d^n d^{n+1} = 0$ for all $n \geq 0$.

Definition 2.11. A sequence

$$\cdots \longrightarrow X^{n+1} \xrightarrow{d^{n+1}} X^n \xrightarrow{d^n} X^{n-1} \longrightarrow \cdots$$

of right Λ-modules connected by Λ-homomorphisms is called exact if $\operatorname{Ker} d^n = \operatorname{Im} d^{n+1}$ for all n. In particular

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a short exact sequence if f is a monomorphism and g is an epimorphism and $\operatorname{Ker} g = \operatorname{Im} f$.

Definition 2.12. Let M be a right Λ-module. A projective resolution of M is an exact sequence

$$\cdots \to P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

where P^n is a projective Λ-module for all $n \geq 0$.

It is called a minimal projective resolution of M if $\operatorname{Im} d^n \subseteq \operatorname{rad} P^{n-1}$ for all $n \geq 0$.

Proposition 2.13. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra. Then every finitely generated module has a minimal projective resolution.*

Definition 2.14. A right Λ-module N is injective if, for any monomorphism $f : L \longrightarrow M$ and any homomorphism $g : L \longrightarrow N$, there is an homomorphism $h : M \to N$ such that the diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "h"'] \\ & N & \end{tikzcd}$$

Definition 2.15. Let N be a right Λ-module. An injective resolution of N is an exact sequence

$$0 \to N \xrightarrow{h^0} I^0 \longrightarrow \cdots \to I^n \overset{h^{n+1}}{\to} I^{n+1} \longrightarrow \cdots$$

where I^n is an injective Λ-module for all $n \geq 0$.

2.1. **The Ext algebra.** Now we look at cohomology theory. This material can be found in many books on representation theory, including [32], [33], [40] and [28].

Let K be a field, Λ be a K-algebra with Jacobson radical $\mathfrak{r}$. We denote by $\operatorname{Mod}\Lambda$ the abelian category of all right Λ-modules, that is, the category whose objects are right Λ-modules, the morphisms are Λ-module homomorphisms, and the composition of morphisms is the usual composition of maps and we denote by $\operatorname{mod}\Lambda$ the subcategory of $\operatorname{Mod}\Lambda$, where objects are finitely generated modules.

We assume that all our modules are in $\operatorname{mod}\Lambda$, that is, they are finitely generated.

Definition 2.16. Let M, N be right Λ-modules, and let (P^n, d^n) be a minimal projective resolution of M,

$$\cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0.$$

We apply $\operatorname{Hom}_\Lambda(-, N)$ to give the complex

$$0 \longrightarrow \operatorname{Hom}_\Lambda(P^0, N) \xrightarrow{\delta^0} \operatorname{Hom}_\Lambda(P^1, N) \xrightarrow{\delta^1} \operatorname{Hom}_\Lambda(P^2, N) \xrightarrow{\delta^2} \cdots$$

where δ^n is the map induced by d^{n+1} such that $\delta^n : \operatorname{Hom}_\Lambda(P^n, N) \longrightarrow \operatorname{Hom}_\Lambda(P^{n+1}, N)$ with $f \longmapsto f \circ d^{n+1}$. The nth cohomology group is denoted by $\operatorname{Ext}^n_\Lambda(M, N)$, and is defined by $\operatorname{Ext}^n_\Lambda(M, N) = \operatorname{Ker}\delta^n / \operatorname{Im}\delta^{n-1}$ for all $n \geq 0$.

In particular, $\operatorname{Ker}\delta^0 = \operatorname{Ext}^0_\Lambda(M, N) = \operatorname{Hom}_\Lambda(M, N)$.

The following theorem says that $\operatorname{Ext}^n_\Lambda(M, N)$ is independent of the choice of projective resolution.

Theorem 2.17. *If $\{P_n\}$ and $\{Q_n\}$ are two projective resolutions of M, with $\operatorname{Ext}^n_\Lambda(M, N)$ and $\overline{\operatorname{Ext}^n_\Lambda}\,(M, N)$ computed with these resolutions, then*

$$\operatorname{Ext}^n_\Lambda(M, N) \cong \overline{\operatorname{Ext}^n_\Lambda}(M, N).$$

Definition 2.18. The Ext algebra of Λ is

$$E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \oplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}).$$

We need to describe the product structure of $\operatorname{Ext}^*_\Lambda(M, M) = \oplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(M, M)$ (see [26]). So we start with a minimal projective resolution (P^n, d^n) of M as a right Λ-module. We apply $\operatorname{Hom}_\Lambda(-, M)$ to get the chain complex

$$0 \to \operatorname{Hom}_\Lambda(P^0, M) \xrightarrow{\delta^0} \cdots \to \operatorname{Hom}_\Lambda(P^n, M) \xrightarrow{\delta^n} \operatorname{Hom}_\Lambda(P^{n+1}, M) \cdots$$

Let $\eta \in \operatorname{Ext}^n_\Lambda(M, M)$. Then η is represented by an element of $\operatorname{Ker} \delta^n \subseteq \operatorname{Hom}_\Lambda(P^n, M)$ and we also denote this element by η.

Let $\eta \in \operatorname{Ext}^n_\Lambda(M, M)$ and $\theta \in \operatorname{Ext}^m_\Lambda(M, M)$ be represented by $\eta \in \operatorname{Hom}_\Lambda(P^n, M)$ and $\theta \in \operatorname{Hom}_\Lambda(P^m, M)$ respectively. We have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+m+1} \arrow[r, "d^{n+m+1}"] \arrow[d, "\mathcal{L}^{n+1}\theta"'] & P^{n+m} \arrow[r, "d^{n+m}"] \arrow[d, "\mathcal{L}^{n}\theta"'] & \cdots \arrow[r, "d^{m+2}"] & P^{m+1} \arrow[r, "d^{m+1}"] \arrow[d, "\mathcal{L}^{1}\theta"'] & P^{m} \arrow[d, "\mathcal{L}^{0}\theta"'] \arrow[dr, "\theta"] & & \\
\cdots \arrow[r] & P^{n+1} \arrow[r, "d^{n+1}"] & P^{n} \arrow[r] & \cdots \arrow[r, "d^{2}"] & P^{1} \arrow[r, "d^{1}"] & P^{0} \arrow[r, "d^{0}"] & M \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{L}^n\theta$ is a lifting of θ, so that the diagram commutes.

Now we can represent the element $\eta\theta$ by the map $\eta \circ \mathcal{L}^n\theta : P^{n+m} \longrightarrow M$, where $\mathcal{L}^n\theta$ is the nth lifting of θ. So we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+m+1} \arrow[r, "d^{n+m+1}"] \arrow[d, "\mathcal{L}^{n+1}\theta"'] & P^{n+m} \arrow[r, "d^{n+m}"] \arrow[d, "\mathcal{L}^{n}\theta"'] & \cdots \arrow[r, "d^{m+2}"] & P^{m+1} \arrow[r, "d^{m+1}"] \arrow[d, "\mathcal{L}^{1}\theta"'] & P^{m} \arrow[d, "\mathcal{L}^{0}\theta"'] \arrow[dr, "\theta"] & & \\
\cdots \arrow[r] & P^{n+1} \arrow[r, "d^{n+1}"] & P^{n} \arrow[r] \arrow[dr, "\eta"] & \cdots \arrow[r, "d^{2}"] & P^{1} \arrow[r, "d^{1}"] & P^{0} \arrow[r, "d^{0}"] & M \arrow[r] & 0 \\
& & & M & & & &
\end{tikzcd}$$

Remark 2.19. The liftings $\mathcal{L}^0\theta, \mathcal{L}^1\theta, \ldots, \mathcal{L}^{n+1}\theta, \ldots$ are not unique; however the element $\eta\theta \in \operatorname{Ext}^{n+m}_\Lambda(M, M)$ is independent of the choice of liftings.

Proposition 2.20. *[5, Corollary 2.5.4]* $\operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \cong \operatorname{Hom}_\Lambda(P^n, \Lambda/\mathfrak{r})$*, where* (P^n, d^n) *is a minimal projective resolution of* $\Lambda/\mathfrak{r}$*.*

Definition 2.21. Let M be a right Λ-module. Then $\operatorname{pdim}_\Lambda M \leq n$, (writing $\operatorname{pdim}_\Lambda M$ to denote the projective dimension of M) if there is a finite projective resolution of M

$$0 \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{d^{n-1}} \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0.$$

If no such finite resolution exists, then $\operatorname{pdim}_\Lambda M = \infty$. Note that, $\operatorname{pdim}_\Lambda M = n$ if M has no projective resolution of length $n-1$.

Definition 2.22. [41, Section 8.1] Let Λ be a finite dimensional K-algebra. Then the global dimension of Λ is defined to be

$$\operatorname{gldim} \Lambda = \sup\{\operatorname{pdim}_\Lambda M, M \text{ is a right } \Lambda\text{-module}\}.$$

Theorem 2.23. *Let M be a right Λ-module. Then the following statements are equivalent:*

(1) $\operatorname{pdim}(M) \leq n$*;*
(2) There exists a projective resolution of M of length n such that

$$0 \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

(3) $\operatorname{Ext}^{n+1}_\Lambda(M, N) = 0$ *for all right modules N.*

Definition 2.24. Let N be a right Λ-module. Then $\operatorname{idim}_\Lambda N \leq n$, (writing $\operatorname{idim}_\Lambda N$ to denote the injective dimension of N) if there is a finite injective resolution of N

$$0 \longrightarrow N \xrightarrow{h^0} I^0 \xrightarrow{h^1} \cdots \longrightarrow I^{n-2} \xrightarrow{h^{n-1}} I^{n-1} \xrightarrow{h^n} I^n \longrightarrow 0.$$

If no such finite resolution exists, then $\operatorname{idim}_\Lambda N = \infty$. Note that, $\operatorname{idim}_\Lambda N = n$ if N has no injective resolution of length $n-1$.

Proposition 2.25. *Let N be a right Λ-module. Then* $\operatorname{idim}_\Lambda N = n$ *if and only if* $\operatorname{Ext}^{n+1}_\Lambda(-, N) = 0$ *and* $\operatorname{Ext}^{n}_\Lambda(-, N) \neq 0$.

2.2. **The construction of a minimal projective resolution.** Now we study the construction of a minimal projective resolution for the module $\Lambda/\mathfrak{r}$ from [28] and the construction of a minimal projective bimodule resolution for Λ from [24]. Let $R = K\mathcal{Q}$, where $\mathcal{Q}$ is a finite quiver, and let $\Lambda = K\mathcal{Q}/I$, where I is an admissible ideal. All modules are finitely generated modules. We start with the construction of a minimal projective resolution from [28]. The following definition is well-known.

Definition 2.26. An algebra Λ is called hereditary if any submodule of a projective module is projective.

Proposition 2.27. *[9, Section 1] Let $K\mathcal{Q}$ be a path algebra. Then $K\mathcal{Q}$ is hereditary.*

Definition 2.28. [28] An element $x \in K\mathcal{Q}$ is called uniform if there exist two vertices v, w such that $x = vx = xw$. We define $\mathfrak{o}(x) = v, \mathfrak{t}(x) = w$.

Now we look at the minimal projective resolution (P^n, d^n) for $\Lambda/\mathfrak{r}$ as described by Green, Solberg and Zacharia in [28].

Let $\Lambda = K\mathcal{Q}/I$ and let $\Lambda/\mathfrak{r} = \oplus_i S_i$, where S_i are right simple Λ-modules. They define sets g^n which we will use to describe the resolution as follows:

$g^0 =$ set of vertices of $\mathcal{Q}$,

$g^1 =$ set of arrows of $\mathcal{Q}$,

$g^2 =$ a minimal set of uniform relations which generate I.

In [28] Green, Solberg and Zacharia define $\oplus_i g_i^{n*} R = (\oplus_i g_i^{n-1} R) \cap (\oplus_i g_i^{n-2} I)$, for all $n \geq 3$, where $R = K\mathcal{Q}$. We discard all elements g_i^{n*} which are in $\oplus_i g_i^{n-1} I$ and we denote the remaining elements by g_i^n. We may choose all the g_i^n to be uniform elements; we will assume that they are all uniform. We let g^n be the set of all elements g_i^n. The set g^n can be chosen in such a way that there exists a minimal projective resolution of $\Lambda/\mathfrak{r}$ as follows:

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where $P^n = \oplus_i \mathfrak{t}(g_i^n)\Lambda$. The P^n are projective right Λ-modules. For each $x \in g^n$ there are unique elements $r_j \in R$ such that $x = \sum_{j=1}^m g_j^{n-1} r_j$, where $|g^{n-1}| = m$. For each $n \geq 1$, they define $d^n : P^n \longrightarrow P^{n-1}$ to be the Λ-homomorphism given by: $\mathfrak{t}(x)\lambda \longmapsto \sum_j \mathfrak{t}(g_j^{n-1}) r_j \mathfrak{t}(x)\lambda$, and $\mathfrak{t}(g_j^{n-1}) r_j \mathfrak{t}(x)\lambda$ is in the component of P^{n-1} corresponding to $\mathfrak{t}(g_j^{n-1})$.

We summarize this in the following theorem.

Theorem 2.29. *[28, Theorem 2.4] With the above notation, we can choose the sets g^n in such way that for each n, no proper linear combination of a subset of g^n lies in $(\oplus g^{n-1} I + \oplus g^{n*} R_{\mathcal{Q}})$. Then (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$.*

Remark 2.30. We note that each g_i^n is in $(\oplus_i g_i^{n-1}R) \cap (\oplus_i g_i^{n-2}I)$, and I has minimal generating set g^2. In particular, for $n = 3$, we can write

$$g_i^3 = \sum_i g_i^2 p_i = \sum_i q_i g_i^2 r_i \tag{1}$$

with $p_i, q_i, r_i \in K\mathcal{Q}$ and $q_i \in R_{\mathcal{Q}}$.

We will use Theorem 2.29 in the next example.

Example 2.31. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} & 1 \arrow[dr, "\alpha"] & \\ 3 \arrow[ur, "\gamma"] & & 2 \arrow[ll, "\beta"] \end{tikzcd}$$

and let $\Lambda = K\mathcal{Q}/I$, where $I = \langle \alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta \rangle$. We begin by finding the resolution of each simple module.
For $S_1 = e_1\Lambda/e_1\mathfrak{r}$, the minimal projective resolution of S_1 is given by

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} S_1 \longrightarrow 0$$

where we define the modules P^n and the maps d^n as follows:

- Let $P^0 = e_1\Lambda$. We define the map $d^0 : e_1\Lambda \mapsto e_1\Lambda/e_1\mathfrak{r}$ by $d^0(e_1\lambda) = e_1\lambda + e_1\mathfrak{r}$ so we have $\operatorname{Ker} d^0 = \{e_1\lambda : d^0(e_1\lambda) = 0\} = \alpha e_2\Lambda$.
- Let $P^1 = e_2\Lambda$ and define $d^1 : P^1 \longrightarrow P^0$ by $e_2\lambda \mapsto \alpha e_2\lambda$ where $\lambda \in \Lambda$. Then $\operatorname{Im} d^1 = \operatorname{Ker} d^0$. We have $\operatorname{Ker} d^1 = \{e_2\lambda : d^1(e_2\lambda) = 0\}$. Since $e_2\Lambda = \operatorname{sp}\{e_2, \beta, \beta\gamma\}$, then we can write $e_2\lambda = c_1e_2 + c_2\beta + c_3\beta\gamma \in \operatorname{Ker} d^1$ and $\alpha e_2\lambda = c_1\alpha + c_2\alpha\beta = 0$. Hence $c_1 = c_2 = 0$ and $e_2\lambda = c_3\beta\gamma$. Thus, $\operatorname{Ker} d^1 = \beta\gamma e_1\Lambda$.
- Let $P^2 = e_1\Lambda$ and define $d^2 : P^2 \longrightarrow P^1$ by $d^2(e_1\lambda) = \beta\gamma e_1\lambda$, where $\lambda \in \Lambda$. Then $\operatorname{Im} d^2 = \operatorname{Ker} d^1$. Here $\operatorname{Ker} d^2 = \{e_1\lambda : d^2(e_1\lambda) = 0\} = \{e_1\lambda : \beta\gamma e_1\lambda = 0\}$. Since $e_1\Lambda = \operatorname{sp}\{e_1, \alpha, \alpha\beta\}$, we have $\operatorname{Ker} d^2 = \alpha e_2\Lambda$.
- Let $P^3 = e_2\Lambda$ and we define $d^3 : P^3 \longrightarrow P^2$ by $d^3(e_2\lambda) = \alpha e_2\lambda$ where $\lambda \in \Lambda$. Then $\operatorname{Im} d^3 = \operatorname{Ker} d^2$ and $\operatorname{Ker} d^3 = \beta\gamma e_1\Lambda$.
- For all $n \geq 4$, n even we have $P^n = e_1\Lambda$ and define $d^n : P^n \mapsto P^{n-1}$ by $d^n(e_1\lambda) = \beta\gamma e_1\lambda$ so $\operatorname{Ker} d^n = \alpha e_2\Lambda$.

- For all $n \geq 5$, n odd we have $P^n = e_2\Lambda$ and define the map $d^n : P^n \longrightarrow P^{n-1}$ by $d^n(e_2\lambda) = \alpha e_2\lambda$. So we have $\operatorname{Ker} d^n = \beta\gamma e_1\Lambda$.

We can find the projective resolutions of S_2 and S_3 in a similar way.

Now we give the projective resolution (P^n, d^n) for $\Lambda/\mathfrak{r}$. To be able to construct this resolution we need the sets g^n. They are

- $g^0 = \{e_1, e_2, e_3\}$ and we label the elements of the set g^0 by $g^0_1 = e_1, g^0_2 = e_2$, and $g^0_3 = e_3$.
- $g^1 = \{\alpha, \beta, \gamma\}$ and we label the elements of the set g^1 by $g^1_1 = \alpha, g^1_2 = \beta$, and $g^1_3 = \gamma$.
- $g^2 = \{\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\}$ and we label the elements of the set g^2 by $g^2_1 = \alpha\beta\gamma, g^2_2 = \beta\gamma\alpha$, and $g^2_3 = \gamma\alpha\beta$.
- For all $n \geq 3$, n odd, we have $g^n_1 = g^{n-1}_1\alpha$, $g^n_2 = g^{n-1}_2\beta$, and $g^n_3 = g^{n-1}_3\gamma$.
- For all $n \geq 3$, n even, we have $g^n_1 = g^{n-1}_1\beta\gamma$, $g^n_2 = g^{n-1}_2\gamma\alpha$, and $g^n_3 = g^{n-1}_3\alpha\beta$.

The minimal projective resolution of $\Lambda/\mathfrak{r}$ is given by

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where we define the modules P^n and the maps d^n as follows:

- Let $P^0 = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ and define $d^0 : P^0 \longrightarrow \Lambda/\mathfrak{r}$ to be the canonical surjection given by $d^0(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = \sum_{i=1}^3 e_i\lambda_i + \mathfrak{r}$, where $\lambda_i \in \Lambda$ for all $i = 1, 2, 3$ and hence $\operatorname{Ker} d^0 = \alpha e_2\Lambda + \beta e_3\Lambda + \gamma e_1\Lambda$.
- Let $P^1 = e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$ and we define $d^1 : P^1 \longrightarrow P^0$ by the following $\mathfrak{t}(g^1_1) \longmapsto (\alpha, 0, 0), \mathfrak{t}(g^1_2) \longmapsto (0, \beta, 0),$ and $\mathfrak{t}(g^1_3) \longmapsto (0, 0, \gamma)$. Then $\operatorname{Ker} d^1 = \beta\gamma e_1\Lambda + \gamma\alpha e_2\Lambda + \alpha\beta e_3\Lambda$.
- Let $P^2 = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ and we define $d^2 : P^2 \longrightarrow P^1$ via $\mathfrak{t}(g^2_1) \longmapsto \mathfrak{t}(g^1_1)\beta\gamma = (\beta\gamma, 0, 0)$, $\mathfrak{t}(g^2_2) \longmapsto \mathfrak{t}(g^1_2)\gamma\alpha = (0, \gamma\alpha, 0)$, and $\mathfrak{t}(g^2_3) \longmapsto \mathfrak{t}(g^1_3)\alpha\beta = (0, 0, \alpha\beta)$. Then $\operatorname{Ker} d^2 = \alpha e_2\Lambda + \beta e_3\Lambda + \gamma e_1\Lambda$.
- For all $n \geq 3$, n odd we have $P^n = e_2\Lambda \oplus e_3\Lambda \oplus e_1\Lambda$ and define the map $d^n : P^n \longrightarrow P^{n-1}$ via $d^n(e_2\lambda_1, e_3\lambda_2, e_1\lambda_3) = (\alpha e_2\lambda_1, \beta e_3\lambda_2, \gamma e_1\lambda_3)$. Then $\operatorname{Ker} d^n = \beta\gamma e_1\Lambda + \gamma\alpha e_2\Lambda + \alpha\beta e_3\Lambda$.
- For all $n \geq 3$, n even we have $P^n = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda$ and $d^n : P^n \longrightarrow P^{n-1}$ is given by $(e_1\lambda, e_2\lambda, e_3\lambda) \longmapsto (\beta\gamma\lambda_1, \gamma\alpha\lambda_2, \alpha\beta\lambda_3)$. So we have $\operatorname{Ker} d^n = \alpha e_2\Lambda + \beta e_3\Lambda + \gamma e_1\Lambda$.

We can also get the minimal projective resolution of a simple module S_i from the minimal projective resolution of $\Lambda/\mathfrak{r}$. Let Λ_0 be the semisimple algebra generated by $e_1, \ldots, e_n$ so that Λ_0 is isomorphic to $\Lambda/\mathfrak{r}$. Then we tensor the minimal projective resolution of $\Lambda/\mathfrak{r}$ by $S_i \otimes_{\Lambda_0}$. This gives an exact sequence which is a minimal projective resolution of S_i. Note that the n-th projective in the minimal projective resolution of S_i is $\oplus \mathfrak{t}(g_j^n)\Lambda$ where the sum is over all g_j^n with $\mathfrak{o}(g_j^n) = i$.

We introduce now the concept of the opposite algebra.

Definition 2.32. Let Λ be a K-algebra. Then we define the opposite algebra Λ^{op} of Λ, where Λ^{op} has the same K-module structure as Λ and the elements of Λ^{op} are the same as those of Λ. The multiplication in Λ^{op} is denoted $*$ and is defined by $a * b = ba$. where ba is the product of b and a in Λ.

Definition 2.33. Let Λ be a K-algebra. Then its enveloping algebra is $\Lambda^e = \Lambda^{op} \otimes_K \Lambda$ with the multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 * a_2 \otimes b_1 b_2 = a_2 a_1 \otimes b_1 b_2$. We write $\otimes$ instead of $\otimes_K$ when there is no confusion.

The categories of Λ-Λ-bimodules and right Λ^e-modules are isomorphic.

Now we can say that a Λ-Λ-bimodule P is a projective bimodule if P is projective as a right Λ^e-module. Also, if e_i, e_j are any two idempotents corresponding to vertices i, j in $\mathcal{Q}$, then $(e_i \otimes e_j)^2 = (e_i \otimes e_j)(e_i \otimes e_j) = e_i^2 \otimes e_j^2 = e_i \otimes e_j$. Thus $e_i \otimes e_j$ is an idempotent in Λ^e. Hence $(e_i \otimes e_j)\Lambda^e$ is a projective Λ^e-module. Now, $(e_i \otimes e_j)\Lambda^e$ corresponds to the Λ-Λ-bimodule $\Lambda e_i \otimes e_j \Lambda$. Thus $\Lambda e_i \otimes e_j \Lambda$ is a projective bimodule.

Let $\{Q^n, d^n\}$ be a projective bimodule resolution of Λ

$$\cdots \longrightarrow Q^n \xrightarrow{d^n} \cdots \xrightarrow{d^2} Q^1 \xrightarrow{d^1} Q^0 \xrightarrow{d^0} \Lambda \longrightarrow 0$$

where Q^i are projective bimodules and d^n are Λ-Λ-homomorphisms. We apply $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to this exact sequence to get the chain complex

$$0 \longrightarrow \mathrm{Hom}_{\Lambda^e}(Q^0, \Lambda) \xrightarrow{\delta^0} \mathrm{Hom}_{\Lambda^e}(Q^1, \Lambda) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n+1}} \mathrm{Hom}_{\Lambda^e}(Q^n, \Lambda) \xrightarrow{\delta^n} \cdots$$

Definition 2.34. The nth Hochschild cohomology group is the nth cohomology group of this complex and it is denoted by $\mathrm{HH}^n(\Lambda)$. So $\mathrm{HH}^n(\Lambda) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$. The Hochschild cohomology ring is defined to be $\mathrm{HH}^*(\Lambda) = \oplus_{n \geq 0} \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$.

Definition 2.35. Let Λ be a K-algebra. Then the centre of Λ is $Z(\Lambda) = \{z \in \Lambda : z\lambda = \lambda z \text{ for all } \lambda \in \Lambda\}$.

Theorem 2.36. *[31] Let Λ be a finite dimensional K-algebra. Then* $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$.

Definition 2.37. [21]

(1) Let Λ be a K-algebra. Then Λ is a graded algebra if $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, with $\Lambda_m \cdot \Lambda_n \subseteq \Lambda_{m+n}$ for all $m, n \geq 0$.

(2) Let Λ be a graded algebra. Then Λ is graded commutative if $xy = (-1)^{|x||y|}yx$, for all homogeneous elements x and y in Λ where $|z|$ denotes the degree of z.

The ring $\mathrm{HH}^*(\Lambda)$ is graded commutative (see [44]).

Theorem 2.38. *[44, Corollary 1.2(a)] Let Λ be an algebra over a commutative ring K, where Λ is a flat as a module over K. Then* $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ *is graded commutative.*

Now we look at the construction of the projective bimodule resolution (Q^n, δ^n) for Λ from [24].

In [24] Green and Snashall construct the first four projective Λ-Λ-bimodules Q^i for $i = 0, 1, 2, 3$ and maps δ^i in a minimal projective bimodule resolution of Λ, namely

$$\cdots \longrightarrow Q^3 \xrightarrow{\delta^3} Q^2 \xrightarrow{\delta^2} Q^1 \xrightarrow{\delta^1} Q^0 \xrightarrow{\delta^0} \Lambda \longrightarrow 0.$$

They define $Q^0 = \oplus_i \Lambda \mathfrak{o}(g_i^0) \otimes \mathfrak{t}(g_i^0)\Lambda = \oplus_i \Lambda e_i \otimes e_i \Lambda$ and they define the map $\delta^0 : Q^0 \longrightarrow \Lambda$ by $\delta^0(\lambda e_i \otimes e_i \mu) = \lambda e_i \mu$. They set $Q^1 = \oplus_i \Lambda \mathfrak{o}(g_i^1) \otimes \mathfrak{t}(g_i^1)\Lambda = \oplus_\alpha \Lambda e_{\mathfrak{o}(\alpha)} \otimes e_{\mathfrak{t}(\alpha)}\Lambda$ and they define this map $\delta^1 : Q^1 \longrightarrow Q^0$ by the matrix A_1 where the rows of A_1 are indexed by the vertices of $\mathcal{Q}$, the columns by the arrows of $\mathcal{Q}$ and the entry in the (g_i^0, g_j^1)-place is given by

$$\begin{cases} \mathfrak{o}(g_j^1) \otimes g_j^1 & \text{if } \mathfrak{o}(g_j^1) = g_i^0 \text{ and } \mathfrak{t}(g_j^1) \neq g_i^0 \\ -g_j^1 \otimes \mathfrak{t}(g_j^1) & \text{if } \mathfrak{t}(g_j^1) = g_i^0 \text{ and } \mathfrak{o}(g_j^1) \neq g_i^0 \\ \mathfrak{o}(g_j^1) \otimes g_j^1 - g_j^1 \otimes \mathfrak{t}(g_j^1) & \text{if } \mathfrak{o}(g_j^1) = g_i^0 = \mathfrak{t}(g_j^1) \\ 0 & \text{otherwise.} \end{cases}$$

Also, they define $Q^2 = \oplus_i \Lambda \mathfrak{o}(g_i^2) \otimes \mathfrak{t}(g_i^2)\Lambda$, where g_i^2 are uniform elements in the set g^2 and we define $\delta^2 : Q^2 \longrightarrow Q^1$ by the matrix A_2 where the rows of A_2 are indexed by the arrows, and the columns of A_2 by the set of the minimal generators for the ideal I and the entry in the (g_i^1, g_l^2)-place is given by $\sum_{j=1}^r c_j \sum_{k=1}^{s_j} \varepsilon_{kj} a_{1j} a_{2j} \dots a_{k-1j} \otimes a_{k+1j} \dots a_{s_j j}$, where $g_l^2 = \sum_{j=1}^r c_j a_{1j} a_{2j} \dots a_{kj} \dots a_{s_j j}$ such that $c_j \in K$, the a_{kj} are arrows in $\mathcal{Q}$ and

$$\varepsilon_{kj} = \begin{cases} 1 & a_{kj} = g_i^1 \\ 0 & \text{otherwise.} \end{cases}$$

They set $Q^3 = \oplus_i \Lambda \mathfrak{o}(g_i^3) \otimes \mathfrak{t}(g_i^3)\Lambda$. We know that by [28, Section 1] and (1) (see Remark 2.30), each element of g^3 is in $(\oplus_i g_i^2 R) \cap (\oplus_i g_i^1 I)$, so

$$g_j^3 = \sum_{j=1}^{m_2} g_j^2 p_j = \sum_{j=1}^{m_2} q_j g_j^2 r_j$$

where $p_j, q_j, r_j \in K\mathcal{Q}$ with $q_j \in R_{\mathcal{Q}}$. So they define the map $\delta^3 : Q^3 \longrightarrow Q^2$ by the matrix A_3 where the rows are indexed by the elements of g^2 and the columns are indexed by the elements of g^3. The (g_i^2, g_j^3) entry of the matrix A_3 is given by $\mathfrak{o}(g_i^2) \otimes p_i - q_i \otimes r_i$.

We summarize this in the following theorem.

Theorem 2.39. *[24, Theorem 2.9] With the above notation, the following sequence is part of a minimal projective resolution of Λ as a bimodule.*

$$Q^3 \longrightarrow Q^2 \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow \Lambda \longrightarrow 0$$

with maps $A_i : Q^i \longrightarrow Q^{i-1}$ for $i = 1, 2, 3$.

We now illustrate this construction with an example.

Example 2.40. Let $\Lambda = K\mathcal{Q}/I$ be the algebra which is given by the quiver $\mathcal{Q}$

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha_2"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend left, "\alpha_3"] & 3 \arrow[l, bend left, "\alpha_4"] \end{tikzcd}$$

and let $I = \langle \alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2 \rangle$. The sets g^0, g^1, g^2 are given as follows:

- $g^0 = \{e_1, e_2, e_3\}$ and we label the elements of the set g^0 by $g_1^0 = e_1, g_2^0 = e_2$, and $g_3^0 = e_3$.

- $g^1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and we label the elements of the set g^1 by $g^1_1 = \alpha_1, g^1_2 = \alpha_2, g^1_3 = \alpha_3,$ and $g^1_4 = \alpha_4$.
- $g^2 = \{\alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2\}$ and we label the elements of the set g^2 by $g^2_1 = \alpha_1\alpha_2\alpha_1\alpha_2, g^2_2 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2,$ and $g^2_3 = \alpha_3\alpha_4\alpha_3\alpha_4$.

So we can construct the minimal projective resolution as follows:

- Let $P^0 = \oplus_{i=1}^3 e_i\Lambda$ and let d^0 be the natural epimorphism $d^0 : \Lambda \longrightarrow \Lambda/\mathfrak{r}$ given by $d^0(e_1\lambda_1, e_2\lambda_2, e_3\lambda_3) = \sum_{i=1}^3 e_i\lambda_i + \mathfrak{r}$.
- Let $P^1 = \oplus_{i=1}^4 \mathfrak{t}(g^1_i)\Lambda = e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda \oplus e_2\Lambda$ and $d^1 : P^1 \longrightarrow P^0$ be given by

  $\mathfrak{t}(g^1_1) \mapsto (0, \alpha_1, 0)$,
  $\mathfrak{t}(g^1_2) \mapsto (\alpha_2, 0, 0)$,
  $\mathfrak{t}(g^1_3) \mapsto (0, \alpha_3, 0)$ and
  $\mathfrak{t}(g^1_4) \mapsto (0, 0, \alpha_4)$.
- Let $P^2 = \oplus_{i=1}^3 \mathfrak{t}(g^2_i)\Lambda = e_2\Lambda \oplus e_2\Lambda \oplus e_2\Lambda$ and $d^2 : P^2 \longrightarrow P^1$ be given by

  $\mathfrak{t}(g^2_1) \mapsto t(g^1_1)\alpha_2\alpha_1\alpha_2 = (\alpha_2\alpha_1\alpha_2, 0, 0, 0)$,
  $\mathfrak{t}(g^2_2) \mapsto t(g^1_1)\alpha_2\alpha_3\alpha_4 - \mathfrak{t}(g^1_3)\alpha_4\alpha_1\alpha_2 = (\alpha_2\alpha_3\alpha_4, 0, -\alpha_4\alpha_1\alpha_2, 0)$, and
  $\mathfrak{t}(g^2_3) \mapsto t(g^1_3)\alpha_4\alpha_3\alpha_4 = (0, 0, \alpha_4\alpha_3\alpha_4, 0)$.

We can see that $\operatorname{Ker} d^2 = (\alpha_1\alpha_2, 0, 0)e_2\Lambda + (\alpha_3\alpha_4, -\alpha_1\alpha_2, 0)e_2\Lambda$
$$+ (0, \alpha_3\alpha_4, \alpha_1\alpha_2)e_2\Lambda + (0, 0, \alpha_3\alpha_4)e_2\Lambda.$$

By induction for $n \geq 3$, we have

- $g^n_1 = g^{n-1}_1\alpha_1\alpha_2$;
- For $2 \leq r \leq n$, we have $g^n_r = g^{n-1}_{r-1}\alpha_3\alpha_4 + (-1)^{r-1}g^{n-1}_r\alpha_1\alpha_2$;
- $g^n_{n+1} = g^{n-1}_n\alpha_3\alpha_4$.

Continuing in this way, we have the projective resolution for $\Lambda/\mathfrak{r}$ as

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where $P^n = \oplus_{i=1}^{n+1}\mathfrak{t}(g^n_i)\Lambda$ and we define the map $d^n : P^n \longrightarrow P^{n-1}$ as follows:

$\mathfrak{t}(g^n_1) \mapsto \mathfrak{t}(g^{n-1}_1)\alpha_1\alpha_2$
$\mathfrak{t}(g^n_r) \mapsto \mathfrak{t}(g^{n-1}_{r-1})\alpha_3\alpha_4 + (-1)^{r-1}\mathfrak{t}(g^{n-1}_r)\alpha_1\alpha_2$ where $2 \leq r \leq n$,
$\mathfrak{t}(g^n_{n+1}) \mapsto \mathfrak{t}(g^{n-1}_n)\alpha_3\alpha_4$.

For this algebra and for $n \geq 3$, we have

- $g_1^n = \alpha_1\alpha_2 g_1^{n-1}$;
- For $2 \leq r \leq n$, we have $g_r^n = \alpha_1\alpha_2 g_r^{n-1} + (-1)^{n+r-1}\alpha_3\alpha_4 g_{r-1}^{n-1}$;
- $g_{n+1}^n = \alpha_3\alpha_4 g_n^{n-1}$.

Now we construct the minimal projective bimodule resolution for Λ. In particular we construct the first four projective bimodules in a projective resolution of Λ from [24] and then we generalize this construction.

We have the following:

- $Q^0 = \oplus_{i=1}^3 \Lambda e_i \otimes e_i\Lambda = \Lambda e_1 \otimes e_1\Lambda \oplus \Lambda e_2 \otimes e_2\Lambda \oplus \Lambda e_3 \otimes e_3\Lambda$ and define $\delta^0 : Q^0 \longrightarrow \Lambda$ by $\delta^0(\lambda_1 e_1 \otimes e_1\mu_1, \lambda_2 e_2 \otimes e_2\mu_2, \lambda_3 e_3 \otimes e_3\mu_3) = \sum_{i=1}^3 \lambda_i e_i \mu_i$.
- $Q^1 = \oplus_{i=1}^4 \Lambda\mathfrak{o}(g_i^1) \otimes \mathfrak{t}(g_i^1)\Lambda = \Lambda e_2 \otimes e_1\Lambda \oplus \Lambda e_1 \otimes e_2\Lambda \oplus \Lambda e_2 \otimes e_3\Lambda \oplus \Lambda e_3 \otimes e_2\Lambda$, and define $\delta^1 : Q^1 \longrightarrow Q^0$ as follows:

  $\mathfrak{o}(g_1^1) \otimes \mathfrak{t}(g_1^1) \mapsto (-\alpha_1 \otimes \mathfrak{t}(g_1^1), \mathfrak{o}(g_1^1) \otimes \alpha_1, 0)$
  $\mathfrak{o}(g_2^1) \otimes \mathfrak{t}(g_2^1) \mapsto (\mathfrak{o}(g_2^1) \otimes \alpha_2, -\alpha_2 \otimes \mathfrak{t}(g_2^1), 0)$
  $\mathfrak{o}(g_3^1) \otimes \mathfrak{t}(g_3^1) \mapsto (0, \mathfrak{o}(g_2^1) \otimes \alpha_3, -\alpha_3 \otimes \mathfrak{t}(g_3^1))$
  $\mathfrak{o}(g_4^1) \otimes \mathfrak{t}(g_4^1) \mapsto (0, -\alpha_4 \otimes \mathfrak{t}(g_2^1), \mathfrak{o}(g_3^1) \otimes \alpha_4)$.

  We can also write A_1 as the matrix:

$$\begin{pmatrix} -\alpha_1 \otimes e_1 & e_1 \otimes \alpha_2 & 0 & 0 \\ e_2 \otimes \alpha_1 & -\alpha_2 \otimes e_2 & e_2 \otimes \alpha_3 & -\alpha_4 \otimes e_2 \\ 0 & 0 & -\alpha_3 \otimes e_3 & e_3 \otimes \alpha_4 \end{pmatrix}.$$

- $Q^2 = \oplus_{i=1}^3 \Lambda\mathfrak{o}(g_i^2) \otimes \mathfrak{t}(g_i^2)\Lambda = \Lambda e_2 \otimes e_2\Lambda \oplus \Lambda e_2 \otimes e_2\Lambda \oplus \Lambda e_2 \otimes e_2\Lambda$ and define the map $\delta^2 : Q^2 \longrightarrow Q^1$ as follows:

$$\mathfrak{o}(g_1^2) \otimes \mathfrak{t}(g_1^2) \mapsto (\mathfrak{o}(g_1^1) \otimes \alpha_2\alpha_1\alpha_2 + \alpha_1\alpha_2 \otimes \alpha_2, \alpha_1 \otimes \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_1 \otimes \mathfrak{t}(g_2^1), 0, 0)$$

$$\mathfrak{o}(g_2^2) \otimes \mathfrak{t}(g_i^2) \mapsto (\mathfrak{o}(g_1^1) \otimes \alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes \alpha_2, \alpha_1 \otimes \alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1 \otimes \mathfrak{t}(g_2^1), \alpha_1\alpha_2 \otimes \alpha_4 - \mathfrak{o}(g_3^1) \otimes \alpha_4\alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3 \otimes \mathfrak{t}(g_4^1) - \alpha_3 \otimes \alpha_1\alpha_2)$$

$$\mathfrak{o}(g_3^2) \otimes \mathfrak{t}(g_3^2) \mapsto (0, 0, \mathfrak{o}(g_3^1) \otimes \alpha_4\alpha_3\alpha_4 + \alpha_3\alpha_4 \otimes \alpha_4, \alpha_3 \otimes \alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_3 \otimes \mathfrak{t}(g_4^1)).$$

Moreover, we can write the matrix A_2 as follows:

$$\begin{bmatrix} \mathfrak{o}(\alpha_1)\otimes\alpha_2\alpha_1\alpha_2+\alpha_1\alpha_2\otimes\alpha_2 & \mathfrak{o}(\alpha_1)\otimes\alpha_2\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\alpha_2 & 0 \\ \alpha_1\otimes\alpha_1\alpha_2+\alpha_1\alpha_2\alpha_1\otimes\mathfrak{t}(\alpha_2) & \alpha_1\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\alpha_1\otimes\mathfrak{t}(\alpha_2) & 0 \\ 0 & \alpha_1\alpha_2\otimes\alpha_4-\mathfrak{o}(\alpha_3)\otimes\alpha_4\alpha_1\alpha_2 & \mathfrak{o}(\alpha_3)\otimes\alpha_4\alpha_3\alpha_4+\alpha_3\alpha_4\otimes\alpha_4 \\ 0 & \alpha_1\alpha_2\alpha_3\otimes\mathfrak{t}(\alpha_4)-\alpha_3\otimes\alpha_1\alpha_2 & \alpha_3\otimes\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_3\otimes\mathfrak{t}(\alpha_4) \end{bmatrix}$$

- $Q^3=\oplus_{i=1}^4\Lambda\mathfrak{o}(g_i^3)\otimes\mathfrak{t}(g_i^3)\Lambda=\Lambda e_2\otimes e_2\Lambda\oplus\Lambda e_2\otimes e_2\Lambda\oplus\Lambda e_2\otimes e_2\Lambda\oplus\Lambda e_2\otimes e_2\Lambda$ and define $\delta^3: Q^3\longrightarrow Q^2$ as follows:

  $\mathfrak{o}(g_1^3)\otimes\mathfrak{t}(g_1^3)\mapsto(\mathfrak{o}(g_1^2)\otimes\alpha_1\alpha_2-\alpha_1\alpha_2\otimes\mathfrak{t}(g_1^2),0,0)$

  $\mathfrak{o}(g_2^3)\otimes\mathfrak{t}(g_2^3)\mapsto(\mathfrak{o}(g_1^2)\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\mathfrak{t}(g_1^2),\mathfrak{o}(g_2^2)\otimes(-\alpha_1\alpha_2)-$
  $\alpha_1\alpha_2\otimes\mathfrak{t}(g_2^2),0)$

  $\mathfrak{o}(g_3^3)\otimes\mathfrak{t}(g_3^3)\mapsto(0,\mathfrak{o}(g_2^2)\otimes\alpha_3\alpha_4+\alpha_3\alpha_4\otimes\mathfrak{t}(g_2^2),\mathfrak{o}(g_3^2)\otimes\alpha_1\alpha_2-\alpha_1\alpha_2\otimes\mathfrak{t}(g_3^2))$

  $\mathfrak{o}(g_4^3)\otimes\mathfrak{t}(g_4^3)\mapsto(0,0,\mathfrak{o}(g_3^2)\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\mathfrak{t}(g_3^2))$.

  Note that we used the general formula to construct Q^3 and δ^3, since

  $g_1^3=g_1^2(\alpha_1\alpha_2)=(\alpha_1\alpha_2)g_1^2\mathfrak{t}(g_1^2)$,
  $g_2^3=g_1^2(\alpha_3\alpha_4)+g_2^2(-\alpha_1\alpha_2)=(\alpha_1\alpha_2)g_2^2\mathfrak{t}(g_2^2)+(\alpha_3\alpha_4)g_1^2\mathfrak{t}(g_1^2)$,
  $g_3^3=g_2^2(\alpha_3\alpha_4)+g_3^2(\alpha_1\alpha_2)=(\alpha_1\alpha_2)g_3^2\mathfrak{t}(g_3^2)+(-\alpha_3\alpha_4)g_2^2\mathfrak{t}(g_2^2)$,
  $g_4^3=g_3^2(\alpha_3\alpha_4)=(\alpha_3\alpha_4)g_1^2\mathfrak{t}(g_3^2)$.

Furthermore, we can write the matrix A_3, where the first column is
$\begin{pmatrix}\mathfrak{o}(g_1^2)\otimes\alpha_1\alpha_2-\alpha_1\alpha_2\otimes\mathfrak{t}(g_1^2) & 0 & 0\end{pmatrix}$,
the second column is
$\begin{pmatrix}\mathfrak{o}(g_1^2)\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\mathfrak{t}(g_1^2) & (-1)[\mathfrak{o}(g_2^2)\otimes\alpha_1\alpha_2+\alpha_1\alpha_2\otimes\mathfrak{t}(g_2^2)] & 0\end{pmatrix}$,
the third column is
$\begin{pmatrix}0 & \mathfrak{o}(g_2^2)\otimes\alpha_3\alpha_4+\alpha_3\alpha_4\otimes\mathfrak{t}(g_2^2) & \mathfrak{o}(g_3^2)\otimes\alpha_1\alpha_2-\alpha_1\alpha_2\otimes\mathfrak{t}(g_3^2)\end{pmatrix}$,
and the fourth column is $\begin{pmatrix}0 & 0 & \mathfrak{o}(g_3^2)\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\mathfrak{t}(g_3^2)\end{pmatrix}$.

In this example we constructed the projective bimodules Q^i for all $i=0,1,2,3$ and maps in the start of a minimal projective bimodule resolution of Λ using [24].

We can generalize the construction and find the minimal projective bimodule resolution of Λ using the minimal projective resolution of $\Lambda/\mathfrak{r}$ and the method of [18] and [45]. However, this algebra Λ will be a stretched algebra which we introduce in Chapter 4. So we can use Theorem 4.43 to say that the minimal projective bimodule resolution for Λ is

$$\cdots\longrightarrow Q^n\xrightarrow{\delta^n}Q^{n-1}\longrightarrow\cdots\longrightarrow Q^3\xrightarrow{\delta^3}Q^2\xrightarrow{\delta^2}Q^1\longrightarrow Q^0\xrightarrow{\delta^0}\Lambda\longrightarrow 0$$

where $Q^n = \oplus_i \Lambda\mathfrak{o}(g_i^n) \otimes \mathfrak{t}(g_i^n)\Lambda$ and the map $\delta^n : Q^n \longrightarrow Q^{n-1}$ is given by

$$\mathfrak{o}(g_1^n) \otimes \mathfrak{t}(g_1^n) \mapsto \mathfrak{o}((g_1^{n-1}) \otimes \alpha_1\alpha_2 + (-1)^n \alpha_1\alpha_2 \otimes \mathfrak{t}(g_1^{n-1});$$

$$\mathfrak{o}(g_r^n) \otimes \mathfrak{t}(g_r^n) \mapsto \mathfrak{o}(g_{r-1}^{n-1}) \otimes \alpha_3\alpha_4 + (-1)^{r-1}\alpha_3\alpha_4 \otimes \mathfrak{t}(g_{r-1}^{n-1}) + (-1)^{r-1}\mathfrak{o}(g_r^{n-1}) \otimes \alpha_1\alpha_2 + (-1)^n \alpha_1\alpha_2 \otimes \mathfrak{t}(g_r^{n-1}) \text{ where } 2 \leq r \leq n;$$

$$\mathfrak{o}(g_{n+1}^n) \otimes \mathfrak{t}(g_{n+1}^n) \mapsto \mathfrak{o}(g_n^{n-1}) \otimes \alpha_3\alpha_4 + (-1)^n \alpha_3\alpha_4 \otimes \mathfrak{t}(g_n^{n-1}).$$

3. Koszul algebras and generalisations

Koszul algebras were introduced by Priddy [38] to study algebraic topology. They also occur in many places in representation theory of algebras. It is known that the Ext algebra of a Koszul algebra is finitely generated in degrees $0, 1$. In this chapter we look at Koszul algebras and some generalisations. We also consider Gröbner bases; our main result here is Theorem 3.28 which concerns (D, A)-stacked algebras.

We assume throughout this thesis that K is a field, $\Lambda = K\mathcal{Q}/I$ for some quiver $\mathcal{Q}$ and admissible ideal I, so that Λ is a finite dimensional K-algebra. All modules are finitely generated right Λ-modules.

Lemma 3.1. *[21, Lemma 2.1] Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a graded algebra. Then the following statements are equivalent:*

- *Λ is generated in degrees 0 and 1;*
- *For all $i, j \geq 0$, $\Lambda_i\Lambda_j = \Lambda_{i+j}$;*
- *For all $k \geq 1$, Λ_k is the product of k copies of Λ_1.*

Definition 3.2. [21] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra. Then Λ is a Koszul algebra if Λ is a graded algebra with the length grading and if $\Lambda/\mathfrak{r}$ (considered as a graded Λ-module in degree 0) has a graded projective resolution

$$\cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

such that P^i is generated in degree i. In this case, we say that $\Lambda/\mathfrak{r}$ has a linear resolution.

Green and Martínez-Villa [21] prove that if $\Lambda = K\mathcal{Q}/I$ is a Koszul algebra, then I is quadratic. The converse holds when the admissible ideal I is quadratic and monomial [29, Proposition 2.2].

Theorem 3.3. *[21, Theorem 6.1] Let $\Lambda = K\mathcal{Q}/I$ be a Koszul algebra. Then $E(\Lambda)$ is a Koszul algebra.*

Moreover, Green and Martínez-Villa [22] show that $E(\Lambda) \cong K\mathcal{Q}^{\mathrm{op}}/I^{\perp}$, where the description of $I^{\perp}$ is as follows. Let V_2 be the vector space with basis all paths of length 2 in $K\mathcal{Q}$. Then V_2^{op} is the vector space with basis all paths of length 2 in $K\mathcal{Q}^{\mathrm{op}}$. We know that I is quadratic so let W be the subset of V_2 which consists of a minimal set of generators of I. They define a bilinear form on $V_2 \times V_2^{\text{op}}$ by $\langle \alpha\beta, a^{\text{op}}b^{\text{op}} \rangle = \delta_{\alpha\beta}(ab)$ where α, β, a, b are arrows in $\mathcal{Q}$. Then $W^{\perp}$ is the orthogonal complement in V_2^{op} of W with respect to this bilinear form. The ideal $I^{\perp}$ is the ideal of $K\mathcal{Q}^{\text{op}}$ which is generated by $W^{\perp}$.

Example 3.4. Let $\mathcal{Q}$ be the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and let $I = \langle \alpha\beta, \beta\alpha \rangle$. For the algebra $\Lambda = K\mathcal{Q}/I$ the sets g^n are given as follows:

- $g^0 = \{e_1, e_2\}$;
- $g^1 = \{\alpha, \beta\}$ with $g_1^1 = \alpha$ and $g_2^1 = \beta$;
- $g^2 = \{\alpha\beta, \beta\alpha\}$ with $g_1^2 = \alpha\beta$ and $g_2^2 = \beta\alpha$;
- For $n \geq 3$
  - $g_1^n = g_1^{n-1}\alpha$ and $g_2^n = g_2^{n-1}\beta$, where n odd
  - $g_1^n = g_1^{n-1}\beta$ and $g_2^n = g_2^{n-1}\alpha$, where n even.

So the minimal projective resolution for $\Lambda/\mathfrak{r}$ is

$$\cdots \longrightarrow P^2 \xrightarrow{d^2} P_1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda/\mathfrak{r} \longrightarrow 0$$

where

- $P^0 = e_1\Lambda \oplus e_2\Lambda$ and $d^0(e_1\lambda, e_2\mu) = (e_1\lambda + e_2\mu) + \mathfrak{r}$, where $\lambda, \mu \in \Lambda$.
- $P^1 = e_2\Lambda \oplus e_1\Lambda$ and the map d^1 is given by $\mathfrak{t}(g_1^1) \mapsto (\alpha, 0)$, $\mathfrak{t}(g_2^1) \mapsto (0, \beta)$.
- $P^2 = e_1\Lambda \oplus e_2\Lambda$ and the map d^2 is given by $\mathfrak{t}(g_1^2) \mapsto (\beta, 0)$, $\mathfrak{t}(g_2^2) \mapsto (0, \alpha)$.
- For $n \geq 3$ we have
  - If n odd, then $P^n = e_2\Lambda \oplus e_1\Lambda$ and the map $P^n \xrightarrow{d^n} P^{n-1}$ is given by

    $\mathfrak{t}(g_1^n) \mapsto \mathfrak{t}(g_1^{n-1})\alpha$,

    $\mathfrak{t}(g_2^n) \mapsto \mathfrak{t}(g_2^{n-1})\beta$.
  - If n even, then $P^n = e_1\Lambda \oplus e_2\Lambda$ and the map $P^n \xrightarrow{d^n} P^{n-1}$ is given by

    $\mathfrak{t}(g_1^n) \mapsto \mathfrak{t}(g_1^{n-1})\beta$,

    $\mathfrak{t}(g_2^n) \mapsto \mathfrak{t}(g_2^{n-1})\alpha$.

We can now see that the elements $g_i^n \in g^n$ have length n,

- $\ell(g_i^0) = 0$, for all $i = 1, 2$.
- $\ell(g_i^1) = 1$, for all $i = 1, 2$.

- $\ell(g_i^2) = 2$, for all $i = 1, 2$.
- For $n \geq 2$, $\ell(g_i^n) = n$, where $i = 1, 2$.

Hence each projective P^n is generated in degree n. So, we have shown that $\Lambda/\mathfrak{r}$ has a linear resolution. Hence Λ is a Koszul algebra, and thus the Ext algebra $E(\Lambda)$ is generated in degrees 0 and 1. In addition it can be shown that $E(\Lambda) = K\mathcal{Q}^{\text{op}}$.

There are several generalisations of Koszul algebras. The d-Koszul algebras were introduced by Berger [6] to study Artin-Schelter regular algebras.

Definition 3.5. [6] Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be a graded K-algebra generated in degrees 0 and 1. Assume that $\Lambda_0 = \Lambda/\mathfrak{r}$ is a finitely generated semisimple K-algebra, Λ_1 is a finitely generated K-module and that (P^n, d^n) is a minimal graded Λ-projective resolution of $\Lambda/\mathfrak{r}$. Let $d \geq 2$. We say that Λ is a d-Koszul algebra if, for each $n \geq 0$, P^n can be generated in exactly one degree, $\delta(n)$, and

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ even,} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ odd.} \end{cases}$$

We can see from the definition that every Koszul algebra is a 2-Koszul algebra.

Theorem 3.6. *[20, Theorem 4.1] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra, where I is generated by homogeneous elements of length d for some $d \geq 2$. Then Λ is d-Koszul if and only if the* Ext *algebra $E(\Lambda)$ can be generated in degrees $0, 1$ and 2.*

We give an example of a d-Koszul algebra which is not a Koszul algebra.

Example 3.7. We take the algebra as in Example 2.31. Recall that the sets g^n are

- $g^0 = \{e_1, e_2, e_3\}$;
- $g^1 = \{\alpha, \beta, \gamma\}$;
- $g^2 = \{\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\}$;
- For all $n \geq 3$, n odd, we have $g_1^n = g_1^{n-1}\alpha$, $g_2^n = g_2^{n-1}\beta$, and $g_3^n = g_3^{n-1}\gamma$
- For all $n \geq 3$, n even, we have $g_1^n = g_1^{n-1}\beta\gamma$, $g_2^n = g_2^{n-1}\gamma\alpha$, and $$g_3^n = g_3^{n-1}\alpha\beta$$

We can now see that the elements $g_i^n \in g^n$ have length $\delta(n)$ for $d = 3$, since

- $\ell(g_i^0) = 0$, for all $i = 1, 2, 3$.

- $\ell(g_i^1) = 1$, for all $i = 1, 2, 3$.
- $\ell(g_i^2) = 3$, for all $i = 1, 2, 3$.
- $\ell(g_i^3) = 4$, for all $i = 1, 2, 3$.
- For $n \geq 2$, $\ell(g_i^{2n}) = nd$, where $i = 1, 2, 3$.
- For $n \geq 2$, $\ell(g_i^{2n+1}) = nd + 1$, where $i = 1, 2, 3$.

Hence each projective P^n is generated in degree $\delta(n)$, so Λ is a 3-Koszul algebra.

Definition 3.8. [25] Let I be an ideal generated by a set of paths (or monomials) in $K\mathcal{Q}$. Then we say $\Lambda = K\mathcal{Q}/I$ is a monomial algebra.

The (D, A)-stacked monomial algebras were introduced by Green and Snashall in [26, Definition 3.1].

Definition 3.9. [26, Definition 3.1] Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra, where I is an admissible ideal with minimal set of generators ρ. Then Λ is a (D, A)-stacked monomial algebra if there are natural numbers $D \geq 2$ and $A \geq 1$ such that, for all $n \geq 0$ and $g_i^n \in g^n$,

$$\ell(g_i^n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n}{2}D & \text{if } n \text{ is even, } n \geq 2 \\ \frac{n-1}{2}D + A & \text{if } n \text{ is odd, } n \geq 3. \end{cases}$$

In particular all relations in ρ are of length D.

Note that the length of each path in g^2 is D and the length of each path in g^3 is $D + A$. Then $\ell(g_i^3) - \ell(g_i^2) = A$, for all $g_i^2 \in g^2$ and $g_i^3 \in g^3$.

Green and Snashall showed in [25] that they are precisely the finite dimensional monomial algebras for which every projective module in the minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in a single degree and for which the Ext algebra of Λ is finitely generated as a K-algebra [25]. Furthermore, $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3 (see [25, Theorem 3.6]).

Leader and Snashall introduced (D, A)-stacked algebras in [35].

Definition 3.10. [35, Definition 1.1] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra. Then Λ is a (D, A)-stacked algebra if there are natural numbers $D \geq 2$, $A \geq 1$ such that, for all $0 \leq n \leq \operatorname{gldim}\Lambda$, the projective module P^n in a minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in degree $\delta(n)$, where

$$\delta(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \frac{n}{2}D & \text{if } n \text{ even, } n \geq 2 \\ \frac{n-1}{2}D + A & \text{if } n \text{ odd, } n \geq 3. \end{cases}$$

The (D, A)-stacked algebras with $A = 1$ are precisely the finite dimensional D-Koszul algebras of Berger.

Theorem 3.11. *[35, Theorem 2.4] Let $\Lambda = KQ/I$ be a (D, A)-stacked algebra with $D \geq 2$ and $A \geq 1$. Then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3.*

Proposition 3.12. *The algebra of Example 2.40 is a $(4, 2)$-stacked algebra.*

Proof. We take the algebra as in Example 2.40, so we have the following,

- P^0 is generated in degree 0; since $\ell(g_i^0) = 0$, where $i = 1, 2, 3$.
- P^1 is generated in degree 1; since $\ell(g_i^1) = 1$, where $i = 1, 2, 3, 4$.
- P^2 is generated in degree 4; since $\ell(g_i^2) = 4$, where $i = 1, 2, 3$.
- P^3 is generated in degree 6; since $\ell(g_i^3) = 6$, where $i = 1, 2, 3, 4$.
- For $n \geq 1$, we have P^{2n} is generated in degree $4n$; since $\ell(g_i^{2n}) = 4n$, where $i = 1, 2, \ldots, 2n + 1$.
- For $n \geq 1$, we have P^{2n+1} is generated in degree $4n+2$; since $\ell(g_i^{2n+1}) = 4n+2$, where $i = 1, 2, \ldots, 2n + 2$.

Hence, from Definition 3.10 we have $D = 4$ and $A = 2$ and Λ is a $(4, 2)$-stacked algebra. □

Example 3.13. Let Λ be the algebra of Example 2.40.

Since Λ is a $(4, 2)$-stacked algebra, then $E(\Lambda)$ is generated in degrees $0, 1, 2$ and 3. For each n and each $g_i^n \in g^n$, we let $f_i^n \in \operatorname{Hom}(P^n, \Lambda/\mathfrak{r})$ be the map given by

$$\mathfrak{t}(g_j^n) \mapsto \begin{cases} \mathfrak{t}(g_i^n) + \mathfrak{r} & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

where $f^n = \{f_i^n\}$ and $|f^n| = |g^n|$. So we have the following

- The basis of $\operatorname{Ext}^0(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is $\{f_1^0, f_2^0, f_3^0\}$.

- The basis of $\mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is $\{f_1^1, f_2^1, f_3^1, f_4^1\}$, where $f_i^1 : P^1 \longrightarrow \Lambda/\mathfrak{r}$ is given by

  $f_1^1 : \mathfrak{t}(g_1^1) = e_1 \mapsto e_1 + \mathfrak{r}$, else $\mapsto 0$;

  $f_2^1 : \mathfrak{t}(g_2^1) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_3^1 : \mathfrak{t}(g_3^1) = e_3 \mapsto e_3 + \mathfrak{r}$, else $\mapsto 0$;

  $f_4^1 : \mathfrak{t}(g_4^1) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$.
- The basis of $\mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is $\{f_1^2, f_2^2, f_3^2\}$, where $f_i^2 : P^2 \longrightarrow \Lambda/\mathfrak{r}$ is given by

  $f_1^2 : \mathfrak{t}(g_1^2) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_2^2 : \mathfrak{t}(g_2^2) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_3^2 : \mathfrak{t}(g_3^2) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$.
- The basis of $\mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is $\{f_1^3, f_2^3, f_3^3, f_4^3\}$, where $f_i^3 : P^3 \longrightarrow \Lambda/\mathfrak{r}$ is given by

  $f_1^3 : \mathfrak{t}(g_1^3) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_2^3 : \mathfrak{t}(g_2^3) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_3^3 : \mathfrak{t}(g_3^3) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$;

  $f_4^3 : \mathfrak{t}(g_4^3) = e_2 \mapsto e_2 + \mathfrak{r}$, else $\mapsto 0$.

More generally, $f^n = \{f_i^n\}$ is a basis of $\mathrm{Ext}^n_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$.

Now we need to find the products in the Ext algebra. Since the algebra is a (D, A)-stacked algebra, then by [35, Proposition 3.1] we have $\mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) = 0$. Hence $f_i^1 f_j^1 = 0$ for all $i, j = 1, 2, 3, 4$. Moreover, we see in [35, Proposition 3.2] $\mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) = 0 = \mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$. Also $\mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) = 0 = \mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^1(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$. We will find the products in $\mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$,
$\mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$, $\mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ and
$\mathrm{Ext}^3(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$.

For the elements in $\mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r}) \times \mathrm{Ext}^2(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$, we want to find $f_i^2 f_j^2 = f_i^2 \circ \mathcal{L}^2 f_j^2 : P^4 \longrightarrow P^2 \longrightarrow \Lambda/\mathfrak{r}$, where $i, j = 1, 2, 3$ and $\mathcal{L}^2 f_j^2$ denotes a lifting of f_j^2. This product is in $\mathrm{Ext}^4(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$. Consider $f_1^2 : P^2 \longrightarrow \Lambda/\mathfrak{r}$. Then we have the following diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & P^5 \arrow[r, "d^5"] \arrow[d, "\mathcal{L}^3 f_1^2"'] & P^4 \arrow[r, "d^4"] \arrow[d, "\mathcal{L}^2 f_1^2"'] & P^3 \arrow[r, "d^3"] \arrow[d, "\mathcal{L}^1 f_1^2"'] & P^2 \arrow[d, "\mathcal{L}^0 f_1^2"'] \arrow[dr, "f_1^2"] & & \\
\cdots \arrow[r] & P^3 \arrow[r, "d^3"] & P^2 \arrow[r, "d^2"] \arrow[dr, "f_i^2"'] & P^1 \arrow[r, "d^1"] & P^0 \arrow[r, "d^0"] & \Lambda/\mathfrak{r} \arrow[r] & 0 \\
 & & & \Lambda/\mathfrak{r} & & &
\end{tikzcd}
$$

We define $\mathcal{L}^0 f_1^2 : P^2 \longrightarrow P^0$ by $\mathfrak{t}(g_1^2) = e_2 \mapsto e_2$, else $\mapsto 0$. So we have $d^0 \circ \mathcal{L}^0 f_1^2(e_2) = d^0(e_2) = e_2 + \mathfrak{r}$ and thus the diagram commutes. Now we want to define $\mathcal{L}^1 f_1^2 : P^3 \longrightarrow P^1$ such that the following diagram commutes.

$$
\begin{tikzcd}
P^3 \arrow[r, "d^3"] \arrow[d, "\mathcal{L}^1 f_1^2"'] & P^2 \arrow[d, "\mathcal{L}^0 f_1^2"] \\
P^1 \arrow[r, "d^1"] & P^0
\end{tikzcd}
$$

We define $\mathcal{L}^1 f_1^2$ by

$\mathfrak{t}(g_1^3) \mapsto \mathfrak{t}(g_1^1)\alpha_2$, since $d^3(\mathfrak{t}(g_1^3)) = \mathfrak{t}(g_1^2)\alpha_1\alpha_2$ and $\mathcal{L}^0 f_1^2(d^3(\mathfrak{t}(g_1^3))) = \alpha_1\alpha_2$,

$\mathfrak{t}(g_2^3) \mapsto \mathfrak{t}(g_3^1)\alpha_4$, since $d^3(\mathfrak{t}(g_2^3)) = \mathfrak{t}(g_1^2)\alpha_3\alpha_4 + \mathfrak{t}(g_2^2)(-\alpha_1\alpha_2)$ and
$\mathcal{L}^0 f_1^2(d^3(\mathfrak{t}(g_2^3))) = \alpha_3\alpha_4$

$\mathfrak{t}(g_3^3) \mapsto 0$, since $d^3(\mathfrak{t}(g_3^3)) = \mathfrak{t}(g_2^2)\alpha_3\alpha_4 + \mathfrak{t}(g_3^2)\alpha_1\alpha_2$ and $\mathcal{L}^0 f_1^2(d^3(\mathfrak{t}(g_2^3))) = 0$

$\mathfrak{t}(g_4^3) \mapsto 0$, since $d^3(\mathfrak{t}(g_4^3)) = \mathfrak{t}(g_3^2)\alpha_3\alpha_4$ and $\mathcal{L}^0 f_1^2(d^3(\mathfrak{t}(g_4^3))) = 0$.

Now we define $\mathcal{L}^2 f_1^2 : P^4 \longrightarrow P^2$ such that the diagram commutes

$$
\begin{tikzcd}
P^4 \arrow[r, "d^4"] \arrow[d, "\mathcal{L}^2 f_1^2"'] & P^3 \arrow[d, "\mathcal{L}^1 f_1^2"] \\
P^2 \arrow[r, "d^2"] & P^1
\end{tikzcd}
$$

by

$\mathfrak{t}(g_1^4) \mapsto \mathfrak{t}(g_1^2)$, since $d^4(\mathfrak{t}(g_1^4)) = \mathfrak{t}(g_1^3)\alpha_1\alpha_2$ and $\mathcal{L}^1 f_1^2(d^4(\mathfrak{t}(g_1^4))) = \mathfrak{t}(g_1^1)\alpha_2\alpha_1\alpha_2$.

$\mathfrak{t}(g_2^4) \mapsto \mathfrak{t}(g_2^2)$, since $d^4(\mathfrak{t}(g_2^4)) = \mathfrak{t}(g_1^3)\alpha_3\alpha_4 + \mathfrak{t}(g_2^3)(-\alpha_1\alpha_2)$
and $\mathcal{L}^1 f_1^2(d^4(\mathfrak{t}(g_2^4))) = \mathfrak{t}(g_1^1)\alpha_2\alpha_3\alpha_4 - \mathfrak{t}(g_3^1)\alpha_4\alpha_1\alpha_2$.

$\mathfrak{t}(g_3^4) \mapsto \mathfrak{t}(g_3^2)$, since $d^4(\mathfrak{t}(g_3^4)) = \mathfrak{t}(g_2^3)\alpha_3\alpha_4 + \mathfrak{t}(g_3^3)\alpha_1\alpha_2$
and $\mathcal{L}^1 f_1^2(d^4(\mathfrak{t}(g_3^4))) = \mathfrak{t}(g_3^1)\alpha_4\alpha_3\alpha_4$.

$\mathfrak{t}(g_4^4) \mapsto 0$, since $d^4(\mathfrak{t}(g_4^4)) = \mathfrak{t}(g_3^3)\alpha_3\alpha_4 + \mathfrak{t}(g_4^3)(-\alpha_1\alpha_2)$

and $\mathcal{L}^1 f_1^2(d^4(\mathfrak{t}(g_4^4))) = 0$.

$\mathfrak{t}(g_5^4) \mapsto 0$, since $d^4(\mathfrak{t}(g_5^4)) = \mathfrak{t}(g_4^3)\alpha_3\alpha_4$ and $\mathcal{L}^1 f_1^2(d^4(\mathfrak{t}(g_5^4))) = 0$.

We also need to define $\mathcal{L}^3 f_1^2 : P^5 \longrightarrow P^3$ such that the diagram commutes

$$\begin{tikzcd} P^5 \arrow[r, "d^5"] \arrow[d, "\mathcal{L}^3 f_1^2"'] & P^4 \arrow[d, "\mathcal{L}^2 f_1^2"] \\ P^3 \arrow[r, "d^3"] & P^2 \end{tikzcd}$$

So we define $\mathcal{L}^3 f_1^2$ by

$\mathfrak{t}(g_1^5) \mapsto \mathfrak{t}(g_1^3)$, since $d^5(\mathfrak{t}(g_1^5)) = \mathfrak{t}(g_1^4)\alpha_1\alpha_2$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_1^5))) = \mathfrak{t}(g_1^2)\alpha_1\alpha_2$.

$\mathfrak{t}(g_2^5) \mapsto \mathfrak{t}(g_2^3)$, since $d^5(\mathfrak{t}(g_2^5)) = \mathfrak{t}(g_1^4)\alpha_3\alpha_4 + \mathfrak{t}(g_2^4)(-\alpha_1\alpha_2)$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_2^5))) = \mathfrak{t}(g_1^2)\alpha_3\alpha_4 + \mathfrak{t}(g_2^2)(-\alpha_1\alpha_2)$.

$\mathfrak{t}(g_3^5) \mapsto \mathfrak{t}(g_3^3)$, since $d^5(\mathfrak{t}(g_3^5)) = \mathfrak{t}(g_2^4)\alpha_3\alpha_4 + \mathfrak{t}(g_3^4)\alpha_1\alpha_2$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_3^5))) = \mathfrak{t}(g_2^2)\alpha_3\alpha_4 + \mathfrak{t}(g_3^2)\alpha_1\alpha_2$.

$\mathfrak{t}(g_4^5) \mapsto \mathfrak{t}(g_4^3)$, since $d^5(\mathfrak{t}(g_4^5)) = \mathfrak{t}(g_3^4)\alpha_3\alpha_4 + \mathfrak{t}(g_4^4)(-\alpha_1\alpha_2)$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_4^5))) = \mathfrak{t}(g_3^2)\alpha_3\alpha_4$.

$\mathfrak{t}(g_5^5) \mapsto 0$, since $d^5(\mathfrak{t}(g_5^5)) = \mathfrak{t}(g_4^4)\alpha_3\alpha_4 + \mathfrak{t}(g_5^4)\alpha_1\alpha_2$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_5^5))) = 0$.

$\mathfrak{t}(g_6^5) \mapsto 0$, since $d^5(\mathfrak{t}(g_6^5)) = \mathfrak{t}(g_5^4)\alpha_3\alpha_4$ and $\mathcal{L}^2 f_1^2(d^5(\mathfrak{t}(g_5^5))) = 0$.

Now that we have the liftings, we can compute the product $f_i^2 f_1^2$ in $\mathrm{Ext}^4(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. So the products $f_i^2 f_1^2 = f_i^2 \circ \mathcal{L}^2 f_1^2 : P^4 \longrightarrow P^2 \longrightarrow \Lambda/\mathfrak{r}$ are

- $f_1^4 = f_1^2 f_1^2 = f_1^2 \circ \mathcal{L}^2 f_1^2$.
- $f_2^4 = f_2^2 f_1^2 = f_2^2 \circ \mathcal{L}^2 f_1^2$.
- $f_3^4 = f_3^2 f_1^2 = f_3^2 \circ \mathcal{L}^2 f_1^2$.

In a similar way, we compute the liftings for f_2^2, f_3^2 and f_4^2. The results of the products $f_j^2 f_i^2$ are as follows:

- For $i = 2$, we have
  - $f_2^4 = f_1^2 f_2^2 = f_1^2 \circ \mathcal{L}^2 f_2^2$.
  - $-f_3^4 = f_2^2 f_2^2 = f_2^2 \circ \mathcal{L}^2 f_2^2$, since $\mathfrak{t}(g_3^4) \xrightarrow{\mathcal{L}^2 f_2^2} -\mathfrak{t}(g_2^2) \xrightarrow{f_2^2} -e_2 + \mathfrak{r}$, else $\mapsto 0$.
  - $f_4^4 = f_3^2 f_2^2 = f_3^2 \circ \mathcal{L}^2 f_2^2$, since $\mathfrak{t}(g_4^4) \xrightarrow{\mathcal{L}^2 f_2^2} \mathfrak{t}(g_3^2) \xrightarrow{f_3^2} e_2 + \mathfrak{r}$, else $\mapsto 0$.
- For $i = 3$, we have
  - $f_3^4 = f_1^2 f_3^2 = f_1^2 \circ \mathcal{L}^2 f_3^2$.

- $f_4^4 = f_2^2 f_3^2 = f_2^2 \circ \mathcal{L}^2 f_3^2$.
- $f_5^4 = f_3^2 f_3^2 = f_3^2 \circ \mathcal{L}^2 f_3^2$.

Hence, $f_2^2 f_1^2 = f_1^2 f_2^2$, $f_3^2 f_1^2 = -f_2^2 f_2^2 = f_1^2 f_3^2$ and $f_3^2 f_2^2 = f_2^2 f_3^2$. By a similar argument we can find the elements in $\operatorname{Ext}^3(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \times \operatorname{Ext}^3(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$.

So we have $f_2^3 f_1^3 = -f_1^3 f_2^3$, $f_3^3 f_1^3 = f_2^3 f_2^3 = f_1^3 f_3^3$, $f_4^3 f_1^3 = -f_3^3 f_2^3 = f_2^3 f_3^3 = -f_1^3 f_4^3$, $f_4^3 f_2^3 = f_3^3 f_3^3 = f_2^3 f_4^3$, and $f_4^3 f_3^3 = -f_3^3 f_4^3$.

For the elements in $\operatorname{Ext}^2(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \times \operatorname{Ext}^3(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, we have the products $f_2^2 f_1^3 = f_1^2 f_2^3$, $f_3^2 f_1^3 = -f_2^2 f_2^3 = f_1^2 f_3^3$, $f_3^2 f_2^3 = f_2^2 f_3^3 = f_1^2 f_4^3$, $f_3^2 f_3^3 = -f_2^2 f_4^3$.

For the elements in $\operatorname{Ext}^3(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \times \operatorname{Ext}^2(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, we have the products $f_2^3 f_1^2 = -f_1^3 f_2^2$, $f_3^3 f_1^2 = f_2^3 f_2^2$, $f_2^3 f_2^2 = f_1^3 f_3^2$, $f_3^3 f_2^2 = f_2^3 f_3^2$, $f_3^3 f_1^2 = f_2^3 f_2^2$, $f_2^3 f_2^2 = -f_1^3 f_3^2$, and $f_4^3 f_2^2 = f_3^3 f_3^2$.

In the same way we can find products $f_i^3 f_j^3 = f_r^2 f_s^2 f_t^2$ in $\operatorname{Ext}^6(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, where $i, j = 1, \ldots, 4$ and $r, s, t = 1, \ldots, 3$. So we have,

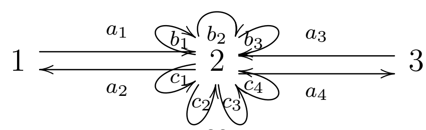
$f_1^3 f_1^3 = f_1^2 f_1^2 f_1^2$, $f_2^3 f_1^3 = -f_1^3 f_2^3 = f_2^2 f_1^2 f_1^2 = f_1^2 f_2^2 f_1^2$,
$f_3^3 f_1^3 = f_2^3 f_2^3 = f_1^3 f_3^3 = f_3^2 f_1^2 f_1^2 = f_2^2 f_2^2 f_1^2$,
$f_4^3 f_1^3 = f_3^3 f_2^3 = f_2^3 f_3^3 = -f_1^3 f_4^3 = f_3^2 f_2^2 f_1^2 = f_2^2 f_3^2 f_1^2 = f_1^2 f_3^2 f_2^2$,
$f_4^3 f_2^3 = f_3^3 f_3^3 = f_2^3 f_4^3 = f_3^2 f_3^2 f_1^2 = -f_2^2 f_3^2 f_2^2 = f_1^2 f_3^2 f_3^2$,
$f_4^3 f_3^3 = -f_3^3 f_4^3 = f_3^2 f_2^2 f_3^2$, and
$f_4^3 f_4^3 = f_3^2 f_3^2 f_3^2$.

Recall that we write paths in a quiver from left to right. So, if f_i^n corresponds to the path $g_i^n \in g^n$ and $g_i^n = \mathfrak{o}(g_i^n) g_i^n \mathfrak{t}(g_i^n)$, then $f_i^n = f^0_{\mathfrak{t}(g_i^n)} f_i^n f^0_{\mathfrak{o}(g_i^n)}$ where $f^0_{\mathfrak{t}(g_i^n)}$ (respectively, $f^0_{\mathfrak{o}(g_i^n)}$) denotes the element of f^0 that corresponds to $\mathfrak{t}(g_i^n)$ (respectively, $\mathfrak{o}(g_i^n)$).

Notation: we set $f_i^1 = a_i$, $f_j^2 = b_j$, and $f_k^3 = c_k$ where $i = 1, \ldots, 4$, $j = 1, \ldots, 3$, and $k = 1, \ldots, 4$.

For the algebra of Example 2.40 we can now describe the Ext algebra by quiver and relations.

Theorem 3.14. *Let Λ be the algebra of Example 2.40, and keep the above notation. The* Ext *algebra $E(\Lambda)$ is $K\Delta/\mathcal{I}$, where Δ is the quiver with vertex set $\Delta_0 = f^0$ and arrow set $\Delta_1 = f^1 \cup f^2 \cup f^3$, so that Δ is*



.

and $\mathcal{I}$ is generated by

- a_ia_j, *for all* i, j.
- a_ib_j, b_jb_i *for all* i, j.
- a_ic_k, c_ka_i *for all* i, k.
- $b_1b_2 - b_2b_1$, $b_3b_1 + b_2b_2$, $b_1b_3 + b_2b_2$, $b_2b_3 - b_3b_2$.
- $c_2c_1 + c_1c_2$, $c_3c_1 - c_2c_2$, $c_3c_1 - c_1c_3$, $c_4c_1 + c_3c_2$, $c_4c_1 - c_2c_3$, $c_2c_3 + c_1c_4$, $c_4c_2 - c_3c_3$, $c_3c_3 - c_2c_4$, $c_4c_3 + c_3c_4$.
- $b_2c_1 - b_1c_2$, $b_3c_1 + b_2c_2$, $b_2c_2 - b_1c_3$, $b_3c_2 - b_2c_3$, $b_2c_3 - b_1c_4$, $b_3c_3 + b_2c_4$.
- $c_2b_1 + c_1b_2$, $c_3b_1 - c_2b_2$, $c_2b_2 + c_1b_3$, $c_2b_2 + c_1b_3$, $c_3b_2 - c_2b_3$, $c_4b_2 - c_3b_3$.
- $c_1c_1 - b_1b_1b_1$, $c_2c_1 - b_2b_1b_1$, $c_3c_1 - b_3b_1b_1$, $c_4c_1 - b_3b_2b_1$, $c_4c_2 - b_3b_3b_1$, $c_4c_3 - b_3b_3b_2$, $c_4c_4 - b_3b_3b_3$.

Note that, since $c_1^2 - b_1^3$ is in the minimal generating set for $\mathcal{I}$, so $\mathcal{I}$ is not length homogeneous. Hence $\mathcal{I}$ is not generated by linear combinations of paths of the same length. We can see also that

- $\mathrm{Ext}^4_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ has basis $\{b_1^2, b_1b_2, b_2^2, b_2b_3, b_3^2\}$
- $\mathrm{Ext}^5_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ has basis $\{b_1c_1, b_2c_1, b_3c_1, b_3c_2, b_3c_3, b_3c_4\}$.
- $\mathrm{Ext}^6_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ has basis $\{c_1^2, c_2c_1, c_3c_1, c_4c_1, c_4c_2, c_4c_3, c_4^2\}$.

We now introduce some Gröbner basis theory; we follow the approach of [13], [15] and [17], see also the discussion in [36]. Let $\mathcal{Q}$ be a finite quiver, and let $\mathfrak{B}$ be the basis of all paths in $K\mathcal{Q}$. We note that $\mathfrak{B}$ is a multiplicative basis of $K\mathcal{Q}$, so that if $p, q \in \mathfrak{B}$ then either $pq \in \mathfrak{B}$ or $pq = 0$. Our main result is Theorem 3.28.

Definition 3.15. Let $\mathfrak{B}$ be the basis of all paths in $K\mathcal{Q}$. Then we say $>$ is a well-order on $\mathfrak{B}$ if $>$ is a total order on $\mathfrak{B}$ and every non empty subset of $\mathfrak{B}$ has a minimal element.

Definition 3.16. An admissible order on $\mathfrak{B}$ is a well-order $>$ on $\mathfrak{B}$ which has the following properties where p, q, r in $\mathfrak{B}$:

(1) if $p > q$, then $pr > qr$ if both $pr \neq 0$ and $qr \neq 0$.
(2) if $p > q$, then $rp > rq$ if both $rp \neq 0$ and $rq \neq 0$.
(3) if $p = qr$, then $p \geq q$ and $p \geq r$.

We have from [15], the left length lexicographic order is an admissible order and is given as follows: The vertices and arrows have arbitrary order such that every vertex is smaller than every arrow. So if $p, q \in \mathfrak{B}$ are paths of length more than 1 where $p = \alpha_1 \cdots \alpha_n$ and $q = \beta_1 \cdots \beta_m$ with $\alpha_i, \beta_i \in \mathcal{Q}_1$, then $p > q$ if $n > m$ or, if $n = m$, then there is some $1 \leq i \leq n$ such that $\alpha_j = \beta_j$ for $j < i$ and $\alpha_i > \beta_i$.

We now introduce the Gröbner basis of an ideal. Let $K\mathcal{Q}$ be a path algebra and let $\mathfrak{B}$ be the basis of all paths, with admissible order $>$.

Definition 3.17. Let x be an element of $K\mathcal{Q}$, so $x = \sum_{j=1}^{n} c_j p_j$ with $0 \neq c_j \in K$ and $p_j \in \mathfrak{B}$. The $\mathrm{Tip}(x)$ is the largest p_i, with respect to the ordering $>$, occurring in x. We denote the coefficient of $\mathrm{Tip}(x)$ by $\mathrm{CTip}(x)$. The paths $p_1, \ldots, p_n$ of $\mathfrak{B}$ which occur in x are called the support of x, denoted by $\mathrm{Supp}(x)$. If I is an ideal in $K\mathcal{Q}$, then $\mathrm{Tip}(I) = \{\mathrm{Tip}(y) : y \in I \setminus \{0\}\}$. The set of finite paths in $K\mathcal{Q}$ which are not in $\mathrm{Tip}(I)$ is called $\mathrm{Nontip}(I)$.

Definition 3.18. [13] Let $K\mathcal{Q}$ be a path algebra and let $\mathfrak{B}$ be the basis of all paths. Then a non-empty subset $\mathcal{G}$ is a Gröbner basis for an ideal I if for each $0 \neq x \in I$, there exists $r, s \in \mathfrak{B}$ such that $\mathrm{Tip}(x) = r\,\mathrm{Tip}(g)s$, for some $g \in \mathcal{G}$.

Definition 3.19. Let $0 \neq a \in K\mathcal{Q}$. A simple (algebra) reduction for a by f is determined by a 4-tuple (c, r, f, s) where $c \in K \setminus \{0\}$; $f \in K\mathcal{Q} \setminus \{0\}$ and $r, s \in \mathfrak{B}$, satisfying the following properties:

(1) $r\,\mathrm{Tip}(f)s \in \mathrm{Supp}(a)$,

(2) $r\,\mathrm{Tip}(f)s \notin \mathrm{Supp}(a - crfs)$.

Moreover we say that a reduces over f to $a - crfs$ and write $a \Rightarrow_f a - crfs$. In general, a reduces to a' over a set $X = \{f_1, \ldots, f_n\}$, denoted by $a \Rightarrow_X a'$, if there is a finite sequence of reductions such that a reduces to a_1 over f_1, a_i reduces to a_{i+1} over f_{i+1} for $i = 1, \ldots, n-2$, and a_{n-1} reduces to a' over f_n.

Let $a, b \in \mathfrak{B}$. Then we say that $a | b$, if there exist $r, s \in \mathfrak{B}$ such that $b = ras$.

Definition 3.20. Let $h_1, h_2 \in K\mathcal{Q}$ and suppose there are elements $p, q \in \mathfrak{B}$ such that:

(1) $\mathrm{Tip}(h_1)p = q\,\mathrm{Tip}(h_2)$;

(2) $\mathrm{Tip}(h_1)$ does not divide q and $\mathrm{Tip}(h_2)$ does not divide p.

Then the overlap difference of h_1 and h_2 by p, q is defined as

$$\mathfrak{o}(h_1, h_2, p, q) = (1/\operatorname{CTip}(h_1))h_1 p - (1/\operatorname{CTip}(h_2))q h_2.$$

We now define a reduced Gröbner basis. We take the definition from [17], but see also [13]; note that a reduced Gröbner basis of I is called MINSHARP(I) in [13].

Definition 3.21. [17, Section 1] Let $K\mathcal{Q}$ be a path algebra and let $\mathfrak{B}$ be the basis of all paths, with admissible order $>$. Let I be a ideal in $K\mathcal{Q}$. An element $x \in I$ is sharp if $x = p + \sum_i \alpha_i q_i$ where $\operatorname{Tip}(x) = p$, $\alpha_i \in K$ and $q_i \in \operatorname{Nontip}(I)$ for all i. A set $\mathcal{G}$ is a reduced Gröbner basis for I if the following conditions hold:

(1) each $g \in \mathcal{G}$ is sharp;
(2) if $x \in I \setminus \{0\}$ then there is some $g \in \mathcal{G}$ such that $\operatorname{Tip}(g)$ is a subpath of $\operatorname{Tip}(x)$;
(3) if g, g' are distinct elements in $\mathcal{G}$, then $\operatorname{Tip}(g)$ is not a subpath of $\operatorname{Tip}(g')$.

Theorem 3.22. *[13, Theorem 13] Let $K\mathcal{Q}$ be a path algebra and let $\mathcal{H} = \{h_j : j \in J\}$ be a subset of non-zero uniform elements in $K\mathcal{Q}$, which generates the ideal I. Assume that the following conditions hold;*

(1) $\operatorname{CTip}(h_j) = 1$*, for all $j \in J$,*
(2) h_i does not reduce over h_j for all $i \neq j$,
(3) every overlap difference for two (not necessarily distinct) members of $\mathcal{H}$ always reduces to zero over $\mathcal{H}$.

Then $\mathcal{H}$ is a reduced Gröbner basis of I.

As a consequence we get the following result.

Theorem 3.23. *Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked monomial algebra, and let $\mathcal{G}$ be a minimal set of homogeneous elements which generate I. Then $\mathcal{G}$ is a reduced Gröbner basis of I consisting of elements of length D.*

Proof. We show that $\mathcal{G}$ is a reduced Gröbner basis of I by satisfying the conditions of Theorem 3.22. Since Λ is a monomial algebra, $\mathcal{G}$ is a set of monomials. Also, all monomials in any minimal generating set of I will have length D since Λ is a (D, A)-stacked monomial algebra. We can see that the condition (1) holds by inspection.

Now, consider two arbitrary elements h_i and h_j in $\mathcal{G}$ with $h_i \neq h_j$ and we assume h_i reduces over h_j. Then there are $u, v \in \mathfrak{B}$ such that $u \operatorname{Tip}(h_j)v \in \operatorname{Supp}(h_i)$. Since $\mathcal{G}$ is a minimal generating set of monomials of length D, then $uh_jv = h_i$ and hence $u, v \in \mathcal{Q}_0$. So $h_i = h_j$ which is a contradiction. So h_i does not reduce over h_j, for all $i \neq j$.

It remains to show that the overlap difference for two elements of $\mathcal{G}$ reduces to zero over $\mathcal{G}$. So let consider two arbitrary elements h_i and h_j in $\mathcal{G}$ and assume there are elements $p, q \in \mathfrak{B}$ such that $h_ip = qh_j$, where h_i does not divide q and h_j does not divide p. Then we have

$$\mathfrak{o}(h_i, h_j, p, q) = (1/\operatorname{CTip}(h_i))h_ip - (1/\operatorname{CTip}(h_j))qh_j = 0.$$

Thus $\mathcal{G}$ is a reduced Gröbner basis consisting of elements of length D. □

We have the following result from [26] concerning (D, A)-stacked monomial algebras. Our aim is to generalise this to other (D, A)-stacked algebras, and we give a partial generalisation in Theorem 3.28.

Proposition 3.24. *[26, Proposition 3.3(3)] Let Λ be a (D, A)-stacked monomial algebra with* $\operatorname{gldim}\Lambda \geq 4$. *Then $D = dA$ for some $d \geq 2$.*

Definition 3.25. [19, Section 3] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra. We say Λ is δ-resolution determined if there is a map $\delta : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $n > 0$ with $n \leq \operatorname{gldim}\Lambda$, the projective module P^n in a minimal projective resolution of $\Lambda/\mathfrak{r}$ is generated in degree $\delta(n)$.

It is clear from Definition 3.10 that every (D, A)-stacked algebra is a δ-resolution determined algebra.

Definition 3.26. [25] Let $\Lambda = K\mathcal{Q}/I$ and let I_M be the ideal generated by $\operatorname{Tip}(I)$ in $K\mathcal{Q}$. So I_M is a monomial ideal. Set $\Lambda_M = K\mathcal{Q}/I_M$.

The following result of Green and Snashall considers δ-resolution determined algebras.

Theorem 3.27. *[25, Corollary 3.4] Let $\Lambda = K\mathcal{Q}/I$. Suppose that I is generated by length homogeneous elements. Let $\mathcal{G}$ be the reduced Gröbner basis for I with respect to the length-lexicographic order. Then Λ is δ-resolution determined and $\mathcal{G}$ consists* *of length homogeneous elements of one degree D if and only if Λ_M is δ-resolution determined. Furthermore, in this case, there exist D, A and B with $D > A > B \geq 0$ so that*

$$\delta(n) = \begin{cases} 0 & n = 0; \\ 1 & n = 1; \\ \frac{n}{2}D + \frac{(n-2)}{2}B & \textit{if } n \textit{ even}, \ 2 \leq n \leq \operatorname{gldim}\Lambda; \\ \frac{n-1}{2}D + A + \frac{(n-3)}{2}B & \textit{if } n \textit{ odd}, \ 3 \leq n \leq \operatorname{gldim}\Lambda. \end{cases}$$

We now give a generalisation of Proposition 3.24 to some (D, A)-stacked algebras.

Theorem 3.28. *Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra with $\operatorname{gldim}\Lambda \geq 4$ and with a reduced Gröbner basis $\mathcal{G}$ of elements of length D. Then $A|D$.*

Proof. Let $\Lambda = K\mathcal{Q}/I$ be a (D, A)-stacked algebra and assume $\mathcal{G}$ consists of length homogeneous elements of one degree D. Then Λ is δ-resolution determined. So using Theorem 3.27 we have Λ_M is δ-resolution determined. Hence Λ_M is a (D, A)-stacked monomial algebra (with the same δ and hence the same values of D and A as in Λ). Hence, by Proposition 3.24 we have that A divides D. $\square$

4. The stretched algebra and projective resolutions

In this chapter we will give the construction of a new algebra $\tilde{\Lambda}$ from a finite dimensional algebra Λ. This generalises work by Leader [36], where she takes a d-Koszul algebra Λ and a natural number A to create a new algebra, $\tilde{\Lambda}$, and she shows that the new algebra is a (D, A)-stacked algebra. Leader begins by using the quiver $\mathcal{Q}$ and ideal I of $\Lambda = K\mathcal{Q}/I$ to define a new quiver $\tilde{\mathcal{Q}}_A$ and ideal $\tilde{I}_A$ of $K\tilde{\mathcal{Q}}_A$, where $D = dA$, $A \geq 1$ and $d \geq 2$. She then defines $\tilde{\Lambda}_A = K\tilde{\mathcal{Q}}_A/\tilde{I}_A$. Furthermore, she described the construction of the minimal projective resolution of $\tilde{\Lambda}/\mathfrak{r}$ as a right $\tilde{\Lambda}$-module and the construction of the minimal projective bimodule resolution of $\tilde{\Lambda}$ in [36].

This construction can be generalised by taking any finite dimensional algebra Λ. We describe this construction and the generalisation here.

We assume throughout this section that $\Lambda = K\mathcal{Q}/I$ is a finite dimensional algebra and I is an admissible ideal. We set $\mathfrak{r}$ to be the Jacobson radical of Λ.

Definition 4.1. (see [36, Definition 8.1]) Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra where I is generated by a minimal set g^2 of uniform elements in $K\mathcal{Q}$. Let $A \geq 1$. We construct the new quiver $\tilde{\mathcal{Q}}_A$ as follows:

- All vertices of $\mathcal{Q}$ are also vertices in $\tilde{\mathcal{Q}}_A$.
- For each arrow α in $\mathcal{Q}$ we have A arrows $\alpha_1, \dots, \alpha_A$ in $\tilde{\mathcal{Q}}_A$ and additional vertices $w_1, w_2, \dots, w_{A-1}$ in $\tilde{\mathcal{Q}}_A$ such that
$$\begin{array}{lllll} \mathfrak{o}(\alpha) & = & \mathfrak{o}(\alpha_1) & & \\ \mathfrak{t}(\alpha_1) & = & \mathfrak{o}(\alpha_2) & = & w_1 \\ \mathfrak{t}(\alpha_2) & = & \mathfrak{o}(\alpha_3) & = & w_2 \\ & & \vdots & & \vdots \\ \mathfrak{t}(\alpha_{A-1}) & = & \mathfrak{o}(\alpha_A) & = & w_{A-1} \\ \mathfrak{t}(\alpha_A) & = & \mathfrak{t}(\alpha) & & \end{array}$$
and the only arrows incident with the vertex w_j are α_j and α_{j+1}.
- We construct the ideal $\tilde{I}_A$ of $K\tilde{\mathcal{Q}}_A$ as follows. Let $g^2 = \{g_1^2, g_2^2, \dots, g_m^2\}$ be the minimal generating set of uniform elements of I. Since each g_i^2 can be written as linear combination of paths, then $g_i^2 = \sum_j c_j \alpha_{j_1} \dots \alpha_{j_{d(j)}}$, for $i = 1, \dots, m$, where $c_i \in K$ and α_{j_k} is an arrow in $\mathcal{Q}$ for each k. We know that every arrow α_{j_k} in $\mathcal{Q}$ corresponds to the path $\alpha_{j_k,1} \cdots \alpha_{j_k,A}$ in $\tilde{\mathcal{Q}}_A$. We define $\tilde{g}_i^2 =$

$\sum_j c_j(\alpha_{j_1,1}\cdots\alpha_{j_1,A})\cdots(\alpha_{j_{d(j)},1}\cdots\alpha_{j_{d(j)},A})$, and define $\tilde{g}^2 = \{\tilde{g}^2_1,\ldots,\tilde{g}^2_m\}$ to be the minimal generating set of $\tilde{I}_A$. We define $\tilde{\Lambda}_A = K\tilde{\mathcal{Q}}_A/\tilde{I}_A$.

We now illustrate this construction.

Example 4.2. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \bullet \arrow[loop left, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

and let $I = \langle x^2, y^2, xy - yx\rangle$. Let $\Lambda = K\mathcal{Q}/I$. So we have the following

- $g^0 = \{v\}$
- $g^1 = \{x, y\}$
- $g^2 = \{x^2, y^2, xy - yx\}$
- For all $n \geq 3$ we have

$$\begin{aligned}
&g^n_1 = g^{n-1}_1 x;\\
&g^n_r = g^{n-1}_{r-1} y + (-1)^{r-1} g^{n-1}_r x, \text{ where } 2 \leq r \leq n;\\
&g^n_{n+1} = g^{n-1}_n y.
\end{aligned}$$

It can be seen that the sets g^n have length n. So $\Lambda/\mathfrak{r}$ has a linear resolution. Hence Λ is a Koszul algebra.

We want to construct the new quiver $\tilde{\mathcal{Q}}_A$ and ideal $\tilde{I}_A$ of $K\tilde{\mathcal{Q}}_A$. Let $A = 2$. Each arrow in $\mathcal{Q}$ corresponds to path of length 2 in $\tilde{\mathcal{Q}}_A$ in such a way that

$$\begin{array}{rclcl}
\mathfrak{o}(x) & = & \mathfrak{o}(\alpha_1) & & \\
\mathfrak{t}(\alpha_1) & = & \mathfrak{o}(\alpha_2) & = & e_1 \\
\mathfrak{t}(\alpha_2) & = & \mathfrak{t}(x) & = & e_2
\end{array}$$

and

$$\begin{array}{rclcl}
\mathfrak{o}(y) & = & \mathfrak{o}(\alpha_3) & & \\
\mathfrak{t}(\alpha_3) & = & \mathfrak{o}(\alpha_4) & = & e_3 \\
\mathfrak{t}(\alpha_4) & = & \mathfrak{t}(y) & = & e_2
\end{array}$$

Hence, x, y correspond to $\alpha_1\alpha_2$ and $\alpha_3\alpha_4$ respectively. Thus, the following diagram illustrates this process of defining $\tilde{\mathcal{Q}}_A$ from $\mathcal{Q}$:

$$\begin{tikzcd}
1 \arrow[r, bend right, "\alpha_2"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend left, "\alpha_3"] & 3 \arrow[l, bend left, "\alpha_4"]
\end{tikzcd}$$

Now we want to find $\tilde{I}_A$. We have $g^2_1 = x^2$, $g^2_2 = xy - yx$, and $g^2_3 = y^2$ so by using the above construction, we have $\tilde{g}^2_1 = \alpha_1\alpha_2\alpha_1\alpha_2$, $\tilde{g}^2_2 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2$, and

$\tilde{g}_3^2 = \alpha_3\alpha_4\alpha_3\alpha_4$. We note that $\tilde{\Lambda}_A = K\tilde{\mathcal{Q}}_A/\tilde{I}_A$ is the algebra of Example 2.40; see also Example 3.13.

For the above algebra $\tilde{\Lambda}_A$ we have the following sets:

- $\tilde{g}^0 = \{e_1, e_2, e_3\}$;
- $\tilde{g}^1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$;
- $\tilde{g}^2 = \{\alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2\}$.

We have the following properties of this construction, some of which can be found in [36, Chapter 8].

Proposition 4.3. *Let m_0 be the number of vertices of $\mathcal{Q}$ and m_1 be the number of arrows of $\mathcal{Q}$. With the above notation, we have:*

- *$\tilde{\Lambda}_A$ is a finite dimensional algebra;*
- *The quiver $\tilde{\mathcal{Q}}_A$ has $m_0 + m_1(A-1)$ vertices;*
- *The quiver $\tilde{\mathcal{Q}}_A$ has m_1A arrows;*
- *The set $\tilde{g}^2 = \{\tilde{g}_1^2, \ldots, \tilde{g}_m^2\}$ is a minimal generating set of uniform elements for $\tilde{I}_A$.*
- *If I is generated by length homogeneous elements, then $\tilde{I}_A$ is generated by length homogeneous elements.*
- *If Λ is a monomial algebra, then $\tilde{\Lambda}_A$ is a monomial algebra.*

Leader [36] shows that the new algebra $\tilde{\Lambda}_A$ is a (D, A)-stacked algebra when Λ is a d-Koszul algebra.

Theorem 4.4. *[36, Theorem 8.15] Let $\Lambda = K\mathcal{Q}/I$ be a d-Koszul algebra. Let $A \geq 1$ and set $D = dA$. With the above construction, the algebra $\tilde{\Lambda}_A = K\tilde{\mathcal{Q}}_A/\tilde{I}_A$ is a (D, A)-stacked algebra.*

We now write $\tilde{\Lambda}$ instead of $\tilde{\Lambda}_A$ to avoid excessive subscripts. We call $\tilde{\Lambda}$ a stretched algebra.

Example 4.5. The algebra $\tilde{\Lambda}$ of Example 4.2 is a $(4, 2)$-stacked algebra using Theorem 4.4, since Λ is a Koszul algebra and hence a 2-Koszul algebra.

Definition 4.6. We keep the above notation. Let $\theta^* : K\mathcal{Q} \longrightarrow K\tilde{\mathcal{Q}}_A$ be the K-algebra homomorphism which is given by

$$\begin{cases} v \mapsto v & \text{for each vertex } v \text{ in } \mathcal{Q}, \\ \alpha \mapsto \alpha_1\alpha_2\cdots\alpha_A & \text{for each arrow } \alpha \text{ in } \mathcal{Q}. \end{cases}$$

Then θ^* is also a monomorphism.

We define $\theta : K\mathcal{Q}/I \longrightarrow K\tilde{\mathcal{Q}}/\tilde{I}$ by $\theta(x+I) = \theta^*(x) + \tilde{I}$ for all $x \in K\mathcal{Q}$. Then the map θ is also a K-algebra monomorphism.

Definition 4.7. Let $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ (as an element of $\tilde{\Lambda}$). Note that ε is an idempotent element of $\tilde{\Lambda}$.

The following result shows that the algebras Λ and $\varepsilon\tilde{\Lambda}\varepsilon$ are isomorphic.

Theorem 4.8. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra. Then $\Lambda \cong \varepsilon\tilde{\Lambda}\varepsilon$ where $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ (as an element of $\tilde{\Lambda}$).*

Proof. By using the first isomorphism theorem we have $\Lambda/\operatorname{Ker}\theta \cong \operatorname{Im}\theta$. Since $\operatorname{Ker}\theta = 0$, then $\Lambda \cong \operatorname{Im}\theta$. Now, we want to prove that $\operatorname{Im}\theta = \varepsilon\tilde{\Lambda}\varepsilon$. We note that $\theta(v)$ and $\theta(\alpha)$ in $\varepsilon\tilde{\Lambda}\varepsilon$, for all $v \in \mathcal{Q}_0$ and $\alpha \in \mathcal{Q}_1$. Since θ is an algebra homomorphism, it follows that $\operatorname{Im}\theta \subseteq \varepsilon\tilde{\Lambda}\varepsilon$. Conversely, let $z \in \varepsilon\tilde{\Lambda}\varepsilon$, so we have $z = \varepsilon\tilde{y}\varepsilon$ where $\tilde{y} \in \tilde{\Lambda}$. From the construction of $K\tilde{\mathcal{Q}}$ and the map θ, if $\theta(\alpha) = \alpha_1\alpha_2\cdots\alpha_A$, then the only arrow which starts at $\mathfrak{t}(\alpha_i)$ is α_{i+1} and the only arrow which ends at $\mathfrak{t}(\alpha_i)$ is α_i, for all $i = 1, \ldots, A-1$. So, if an element $\tilde{p} \in \tilde{\Lambda}$ has $\mathfrak{o}(\tilde{p}) \in \mathcal{Q}_0$ and $\mathfrak{t}(\tilde{p}) \in \mathcal{Q}_0$, then $\tilde{p} = \theta(p)$, for some $p \in \Lambda$. Hence, $z = \varepsilon\theta(y)\varepsilon = \theta(\varepsilon y\varepsilon)$, for some $y \in \Lambda$. Thus $\varepsilon\tilde{\Lambda}\varepsilon \subseteq \operatorname{Im}\theta$. So $\operatorname{Im}\theta \cong \varepsilon\tilde{\Lambda}\varepsilon$ and $\Lambda \cong \varepsilon\tilde{\Lambda}\varepsilon$. □

Let $\tilde{\mathfrak{r}}$ denote the Jacobson radical of $\tilde{\Lambda}$.

Proposition 4.9. *[1, Corollary 17.13] Let R be a ring with radical* $\operatorname{rad}(R)$ *and let e be an idempotent in R. Then* $\operatorname{rad}(eRe) = e\operatorname{rad}(R)e$.

Applying this to our construction gives the following result.

Proposition 4.10. *Let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ be constructed as above. Then $\varepsilon\tilde{\mathfrak{r}}\varepsilon = \operatorname{rad}\varepsilon\tilde{\Lambda}\varepsilon$. Moreover,* $\operatorname{rad}\varepsilon\tilde{\Lambda}\varepsilon \cong \mathfrak{r}$.

Our aim is to describe the relationship between the projective resolutions of $\Lambda/\mathfrak{r}$ and $\tilde{\Lambda}/\tilde{\mathfrak{r}}$, and between the projective bimodule resolutions of Λ and $\tilde{\Lambda}$. To do this, we first introduce some more notation.

Definition 4.11. (See [36, Theorem 8.14]) Suppose $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Let $\tilde{p}_w$ be the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at a vertex in $\mathcal{Q}_0$ and ends at w. Let $\tilde{q}_w$ be the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at the vertex w and ends at a vertex in $\mathcal{Q}_0$. We illustrate these in the following diagram

$$v \overset{\tilde{p}_w}{\rightsquigarrow} w \overset{\tilde{q}_w}{\rightsquigarrow} v'$$

where v, v' are vertices in $\mathcal{Q}_0$.

Definition 4.12. [23, Definition 3.1] Let $v \in \mathcal{Q}_0$. We say v is properly internal to the path p if $p = p_1 v p_2$, where $\ell(p_1), \ell(p_2) \geq 1$ and $\mathfrak{o}(p) \neq v \neq \mathfrak{t}(p)$.

Remark 4.13. Let $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. So w is properly internal to $\theta(\alpha)$ for some arrow $\alpha \in \mathcal{Q}_1$. Let $v = \mathfrak{o}(\alpha)$ and let $v' = \mathfrak{t}(\alpha)$. Keeping the notation of Definition 4.1, the quiver $\tilde{\mathcal{Q}}$ contains the subquiver

$$v \xrightarrow{\alpha_1} w_1 \xrightarrow{\alpha_2} w_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{A-1}} w_{A-1} \xrightarrow{\alpha_A} v'$$

For each $i = 1, \ldots, A-1$, we have $\tilde{p}_{w_i} = \alpha_1 \cdots \alpha_i$ and $\tilde{q}_{w_i} = \alpha_{i+1} \cdots \alpha_A$; moreover $\tilde{p}_{w_i}\tilde{q}_{w_i} = \alpha_1 \cdots \alpha_A$.

Proposition 4.14. *Let $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$, and let $v' = \mathfrak{t}(\tilde{q}_w)$ and $v = \mathfrak{o}(\tilde{p}_w)$.*

(1) If $0 \neq v'\tilde{\lambda} \in v'\tilde{\Lambda}$, then $\tilde{q}_w\tilde{\lambda} \neq 0$ in $\tilde{\Lambda}$.
(2) If $0 \neq \tilde{\lambda}v \in \tilde{\Lambda}v$, then $\tilde{\lambda}\tilde{p}_w \neq 0$ in $\tilde{\Lambda}$.
(3) If $\lambda = v'\lambda v \in v'\Lambda v$, so that $\theta(\lambda) \in v'\tilde{\Lambda}v$, then the following are equivalent:
- *(i)* $\tilde{q}_w\theta(\lambda)\tilde{p}_w = 0$.
- *(ii)* $\tilde{q}_w\theta(\lambda) = 0$.
- *(iii)* $\theta(\lambda)\tilde{p}_w = 0$.
- *(iv)* $\theta(\lambda) = 0$.
- *(v)* $\lambda = 0$.

Proof. We show (3)$(i) \Rightarrow (v)$. Suppose $\tilde{q}_w\theta(\lambda)\tilde{p}_w = 0$, and $\lambda = v'\lambda v$. Then considering $\tilde{q}_w\theta(\lambda)\tilde{p}_w$ as an element of $K\tilde{\mathcal{Q}}$, we have that $\tilde{q}_w\theta(\lambda)\tilde{p}_w \in \tilde{I}$, and $\tilde{I}$ is generated by a set of uniform elements $\tilde{g}_i^2$ which all start and end at a vertex in $\mathcal{Q}_0$. So by construction of $\tilde{\Lambda}$, recall that if an element $\tilde{p} \in \tilde{\Lambda}$ has $\mathfrak{o}(\tilde{p}) \in \mathcal{Q}_0$ and $\mathfrak{t}(\tilde{p}) \in \mathcal{Q}_0$ then $\tilde{p} = \theta(p)$ for some $p \in K\mathcal{Q}$. However, no vertices of $\mathcal{Q}_0$ are properly internal to either $\tilde{q}_w$ or $\tilde{p}_w$. So it follows that $\theta(\lambda) = 0$. Since θ is one-to-one, we have $\lambda = 0$.

The rest of the proof is similar, and we leave it to the reader. □

Proposition 4.15. *Let $w \in \tilde{\mathcal{Q}}_0 \setminus Q_0$ and let $v' = \mathfrak{t}(\tilde{q}_w)$ and $v = \mathfrak{o}(\tilde{p}_w)$. Let $B = \varepsilon\tilde{\Lambda}\varepsilon$. Then we have the following properties:*

(1) $v'B \cong \tilde{q}_w B$ as right B-modules.
(2) $v'\tilde{\Lambda} \cong \tilde{q}_w\tilde{\Lambda}$ as right $\tilde{\Lambda}$-modules.
(3) $Bv \cong B\tilde{p}_w$ as left B-modules.
(4) $\tilde{\Lambda}v \cong \tilde{\Lambda}\tilde{p}_w$ as left $\tilde{\Lambda}$-modules.

Proof. We prove (1) only. We define a map $\varphi_w : v'B \longrightarrow \tilde{q}_w B$ by $\varphi_w(v'x) = \tilde{q}_w v'x$, where $x \in B$. It is straightforward to show that φ_w is a right B-module homomorphism and is onto. The fact that φ_w is one-to-one follows from Proposition 4.14. □

Proposition 4.16. *With the notation of Remark 4.13, $w_i \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$ for all $i = 1, \ldots, A-1$. Then we have the following properties:*

(1) An element of $\tilde{\Lambda}w_i$ is of the form

$$\tilde{\lambda}w_i = \sum_{1 \le j \le i} c_j w_j \alpha_{j+1} \cdots \alpha_i w_i + \tilde{\mu}\tilde{p}_{w_i}$$

where $c_j \in K$, $\tilde{\mu} \in \tilde{\Lambda}$.

(2) An element of $w_i\tilde{\Lambda}$ is of the form

$$w_i\tilde{\lambda} = \sum_{i \le j \le A-1} c_i w_i \alpha_{i+1} \cdots \alpha_j w_j + \tilde{q}_{w_i}\tilde{\mu}.$$

where $c_i \in K$, $\tilde{\mu} \in \tilde{\Lambda}$.

(3) $\dim \tilde{\Lambda}w_i = i + \dim \tilde{\Lambda}v$.
(4) $\dim w_i\tilde{\Lambda} = (A - i) + \dim v'\tilde{\Lambda}$.

Proof. We prove (1). It is clear that $\tilde{\Lambda}w_i$ has a basis which consists of all paths $w_j\alpha_{j+1} \cdots \alpha_i w_i$, where $1 \le j \le i$ together with paths of the form $\tilde{\gamma}\tilde{p}_{w_i}$ since

$\tilde{p}_{w_i} = \alpha_1 \cdots \alpha_i$. So we can write

$$\tilde{\lambda} w_i = \sum_{1 \leq j \leq i} c_j w_j \alpha_{j+1} \cdots \alpha_i w_i + \tilde{\mu} \tilde{p}_{w_i}$$

where $c_j \in K$, $\tilde{\mu} \in \tilde{\Lambda}$. Then (3) follows from (1) and Proposition 4.15. We leave the rest of the proof to the reader. $\square$

We next study the properties of the idempotent embedding functor T_e. This material is covered in many books on category theory, and we follow the approach of [2].

Let $e \in A$ be an idempotent in a finite dimensional K-algebra A and consider the algebra $B = eAe \cong \operatorname{End} eA$ with the identity element $e \in B$. In [2, Chapter 1] the authors give three additive K-linear covariant functors,

$$\operatorname{mod} B \underset{\operatorname{res}_e}{\overset{T_e, L_e}{\rightleftarrows}} \operatorname{mod} A$$

which are defined by $\operatorname{res}_e(-) = (-)e$, $T_e(-) = (-) \otimes_B eA$ and $L_e(-) = \operatorname{Hom}_B(Ae, -)$. More specifically, for the functor $T_e : \operatorname{mod} B \longrightarrow \operatorname{mod} A$ we have:

- For $X \in \operatorname{mod} B$, then $T_e(X) = X \otimes_B eA$.
- For each B-module homomorphism $f : X \longrightarrow Y$ where $X, Y \in \operatorname{mod} B$, the A-module homomorphism $T_e(f) : T_e(X) \longrightarrow T_e(Y)$ is given by $T_e(f) : x \otimes_B ea \mapsto f(x) \otimes_B ea$ for all $x \in X$ and all $a \in A$.

We remind the reader of some category theory concepts.

Definition 4.17. [2, A2, Definition 2.2]

(1) The functor $T : C \longrightarrow C'$ is additive if T preserves direct sums and, for all $X, Y \in \operatorname{Ob} C$, the map $T_{XY} : \operatorname{Hom}_C(X, Y) \longrightarrow \operatorname{Hom}_{C'}(T(X), T(Y))$, given by $h \mapsto T(h)$, satisfies $T(f + g) = T(f) + T(g)$, for all $f, g \in \operatorname{Hom}_C(X, Y)$.

(2) Let C and C' be abelian categories. A covariant additive functor $T : C \longrightarrow C'$ is right exact if, for any exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in C, then the induced sequence $T(X) \xrightarrow{T(f)} T(Y) \xrightarrow{T(g)} T(Z) \longrightarrow 0$ is exact in C'.

(3) A functor $T : C \longrightarrow C'$ is faithful, if the map $T_{XY} : \operatorname{Hom}_C(X, Y) \longrightarrow \operatorname{Hom}_{C'}(T(X), T(Y))$ given by $f \mapsto T(f)$ is an injective map, for all $X, Y \in \operatorname{Ob} C$.

(4) The functor $T : C \longrightarrow C'$ is full, if the map $T_{XY} : \mathrm{Hom}_C(X,Y) \longrightarrow \mathrm{Hom}_{C'}(T(X),T(Y))$ given by $f \mapsto T(f)$ is surjective map, for all $X, Y \in \mathrm{Ob}\, C$.

Theorem 4.18. *[2, Chapter 1.6, Theorem 6.8] Suppose that A is a finite dimensional K-algebra and that $e \in A$ is an idempotent, and let $B = eAe$. The functors T_e, L_e associated to $e \in A$ satisfy the following conditions*

(1) T_e and L_e are full and faithful K-linear functors such that $res_e T_e \cong I_{\mathrm{mod}_B} \cong \mathrm{res}_e\, L_e$, the functor L_e is right adjoint to res_e and T_e is left adjoint to res_e, that is, there are functorial isomorphisms

$$\mathrm{Hom}_A(X_A, L_e(Y_B)) \cong \mathrm{Hom}_B(\mathrm{res}_e(X_A), Y_B)$$

$$\mathrm{Hom}_A(T_e(Y_B), X_A) \cong \mathrm{Hom}_B(Y_B, \mathrm{res}_e(X_A))$$

for every A-module X_A and every B-module Y_B.

(2) The restriction functor res_e is exact, T_e is right exact, and L_e is left exact.

(3) The functor T_e and L_e preserve indecomposability, T_e carries projectives to projectives, and L_e carries injectives to injectives.

(4) A module X_A is in the category $\mathrm{Im}\, T_e$ if and only if there is an exact sequence $P^1 \longrightarrow P^0 \longrightarrow X_A \longrightarrow 0$, where P^1 and P^0 are direct sums of summands of eA.

Proposition 4.19. *[2, Chapter 1.6, p36] Let A be a finite dimensional algebra, let $e \in A$ be an idempotent, and let $B = eAe$. Suppose that $e = \sum_{j=1}^{s} e_j$, with e_j primitive orthogonal idempotents, for all $j = 1, \ldots, s$. Then $\mathfrak{m}_j : e_j B \otimes_B eA \longrightarrow e_j A$, where $e_j b \otimes_B ea \mapsto e_j bea$, for all $a \in A$ and $b \in B$, is a right A-module isomorphism for $i = 1, \ldots, s$.*

We now relate this to our algebras Λ and $\tilde{\Lambda}$ and use the idempotent ε. We set $B = \varepsilon\tilde{\Lambda}\varepsilon$, so $B \cong \Lambda$.

Proposition 4.20. *Let Λ, $\tilde{\Lambda}$ be finite dimensional algebras as above. Then $\mathfrak{m}_v : vB \otimes_B \varepsilon\tilde{\Lambda} \longrightarrow v\tilde{\Lambda}$, where $vb \otimes_B ea \mapsto vbea$, for all $a \in \tilde{\Lambda}$ and $b \in B$, is a right $\tilde{\Lambda}$-module isomorphism for all $v \in \mathcal{Q}_0$.*

Our aim in this chapter is give a new and functorial approach to the projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ and the projective bimodule resolution of $\tilde{\Lambda}$ and results of [36]. We keep the notation of previous chapters.

4.1. **Functorial approach to the projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$.** Now we take a minimal projective resolution (P^n, d^n) of $\Lambda/\mathfrak{r}$ as given by [28], so we have the sets g^n. We use these sets to give sets $\tilde{g}^n$ in order to describe the minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$.

Definition 4.21. (see [36, Definition 8.7]) Define the sets $\tilde{g}^n \in K\tilde{\mathcal{Q}}$, for $n \geq 0$ as follows

- $\tilde{g}^0$ is the set of vertices of $\tilde{\mathcal{Q}}$
- $\tilde{g}^1$ is the set of arrows of $\tilde{\mathcal{Q}}$
- $\tilde{g}^2$ is a minimal generating set of $\tilde{I}$ as given in Definition 4.1
- For $n \geq 3$, let $\tilde{g}_i^n = \theta^*(g_i^n)$ for each $g_i^n \in g^n$, and set $\tilde{g}^n = \{\tilde{g}_i^n\}$.

For $n \geq 2$, it can be seen that each $\tilde{g}_i^n$ is a uniform element which starts and ends at the vertex corresponding to the vertex $\mathfrak{o}(g_i^n)$ and $\mathfrak{t}(g_i^n)$ respectively in $\mathcal{Q}_0$ and so $\tilde{g}_i^n = \varepsilon \tilde{g}_i^n \varepsilon$.

Definition 4.22. (see [36, Chapter 8]) We define $\tilde{P}^n$ to be the projective $\tilde{\Lambda}$-module $\tilde{P}^n = \oplus_i \mathfrak{t}(\tilde{g}_i^n)\tilde{\Lambda}$, for all n and define $\tilde{\Lambda}$-module homomorphisms by

- $\tilde{d}^0 : \tilde{P}^0 \longrightarrow \tilde{\Lambda}/\tilde{\mathfrak{r}}$, where $\tilde{d}^0$ is the canonical surjection.
- $\tilde{d}^1 : \tilde{P}^1 \longrightarrow \tilde{P}^0$ is given by $\mathfrak{t}(\tilde{\alpha})\tilde{\lambda} \mapsto \tilde{\alpha}\tilde{\lambda}$, where $\tilde{\alpha}\tilde{\lambda}$ is in the component of $\tilde{P}^0$ which corresponds to $\mathfrak{o}(\tilde{\alpha})$, for all $\tilde{\lambda} \in \tilde{\Lambda}$.
- Write $\tilde{g}_i^2 = \sum_j \tilde{\alpha}_j \tilde{\eta}_j$, where $\tilde{\alpha}_j$ is an arrow in $\tilde{\mathcal{Q}}$ and $\tilde{\eta}_j \in K\tilde{\mathcal{Q}}$. Then $\tilde{d}^2 : \tilde{P}^2 \longrightarrow \tilde{P}^1$ is given by $\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}$ has entry $\tilde{\eta}_j\tilde{\lambda}$ in the summand of $\tilde{P}^1$ which corresponds to $\mathfrak{t}(\tilde{\alpha}_j)$, for $\tilde{\lambda} \in \tilde{\Lambda}$.
- For $n \geq 3$ we have $g_i^n = \sum_j g_j^{n-1} q_j$ for some $q_j \in K\mathcal{Q}$. Then $\tilde{g}_i^n = \theta^*(g_i^n)$ and so we have $\tilde{g}_i^n = \sum_j \tilde{g}_j^{n-1}\theta^*(q_j)$. Thus $\tilde{d}^n : \tilde{P}^n \longrightarrow \tilde{P}^{n-1}$ is given by $\tilde{d}^n(\mathfrak{t}(\tilde{g}_i^n)\tilde{\lambda})$ has entry $\mathfrak{t}(\tilde{g}_j^{n-1})\theta(q_j)\tilde{\lambda}$ in the summand of $\tilde{P}^{n-1}$ which corresponds to $\mathfrak{t}(\tilde{g}_j^{n-1})$, for $\tilde{\lambda} \in \tilde{\Lambda}$.

Hence, we have a sequence of $\tilde{\Lambda}$-modules and homomorphisms

$$\cdots \longrightarrow \tilde{P}^n \xrightarrow{\tilde{d}^n} \tilde{P}^{n-1} \longrightarrow \cdots \longrightarrow \tilde{P}^1 \xrightarrow{\tilde{d}^1} \tilde{P}^0 \xrightarrow{\tilde{d}^0} \tilde{\Lambda}/\tilde{\mathfrak{r}} \longrightarrow 0 \qquad (2)$$

It remains to consider whether this sequence is exact. In the case where Λ is a d-Koszul algebra, we have the following theorem.

Theorem 4.23. *[36, Theorem 8.14] With the above notation and for a d-Koszul algebra Λ, then $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ as a right $\tilde{\Lambda}$-module.*

We now generalise this result. We start by considering the part of the sequence (1) with $n \leq 2$. We observe that

$$0 \longrightarrow \operatorname{Ker} \tilde{d}^2 \xrightarrow{i} \tilde{P}^2 \xrightarrow{\tilde{d}^2} \tilde{P}^1 \xrightarrow{\tilde{d}^1} \tilde{P}^0 \xrightarrow{\tilde{d}^0} \tilde{\Lambda}/\tilde{\mathfrak{r}} \longrightarrow 0$$

where $i : \operatorname{Ker} \tilde{d}^2 \longrightarrow \tilde{P}^2$ is the inclusion map, is the start of a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$ from [28]. Hence this sequence

$$0 \longrightarrow \operatorname{Ker} \tilde{d}^2 \xrightarrow{i} \tilde{P}^2 \xrightarrow{\tilde{d}^2} \tilde{P}^1 \xrightarrow{\tilde{d}^1} \tilde{P}^0 \xrightarrow{\tilde{d}^0} \tilde{\Lambda}/\tilde{\mathfrak{r}} \longrightarrow 0$$

is exact. Now consider $n \geq 2$.

Proposition 4.24. *Let P^n and $\tilde{P}^n$ be as above. Then $T_\varepsilon P^n \cong \tilde{P}^n$ for all $n \geq 2$.*

Proof. We have $P^n = \oplus_i \mathfrak{t}(g_i^n)\Lambda$ which by Theorem 4.8 we identify with $\oplus_i \mathfrak{t}(\tilde{g}_i^n)\varepsilon\tilde{\Lambda}\varepsilon$ via $(\mathfrak{t}(g_1^n)\lambda_1, \ldots, \mathfrak{t}(g_m^n)\lambda_m) = (\mathfrak{t}(\tilde{g}_1^n)\varepsilon\theta(\lambda_1)\varepsilon, \ldots, \mathfrak{t}(\tilde{g}_m^n)\varepsilon\theta(\lambda_m)\varepsilon) = (\mathfrak{t}(\tilde{g}_1^n)\theta(\lambda_1), \ldots, \mathfrak{t}(\tilde{g}_m^n)\theta(\lambda_m))$. Now we use the functor $T_\varepsilon : \operatorname{mod} B \longrightarrow \operatorname{mod} \tilde{\Lambda}$ which is given by $T_\varepsilon(-) = (-) \otimes_B \varepsilon\tilde{\Lambda}$. So, we have

$$T_\varepsilon P^n = T_\varepsilon(\oplus_i \mathfrak{t}(\tilde{g}_i^n)\varepsilon\tilde{\Lambda}\varepsilon) = \oplus_i T_\varepsilon(\mathfrak{t}(\tilde{g}_i^n))\varepsilon\tilde{\Lambda}\varepsilon = \oplus_i \mathfrak{t}(\tilde{g}_i^n)\varepsilon\tilde{\Lambda}\varepsilon \otimes_B \varepsilon\tilde{\Lambda}.$$

By Proposition 4.20 we have $\oplus_i \mathfrak{t}(\tilde{g}_i^n)\varepsilon\tilde{\Lambda}\varepsilon \otimes_B \varepsilon\tilde{\Lambda} \cong \oplus_i \mathfrak{t}(\tilde{g}_i^n)\tilde{\Lambda} = \tilde{P}^n$. Thus $T_\varepsilon P^n \cong \tilde{P}^n$ for all $n \geq 2$. $\square$

We identify the elements of $T_\varepsilon P^n$ with those of $\tilde{P}^n$ as follows. For $(0, \ldots, 0, \mathfrak{t}(g_i^n)\lambda, 0, \ldots, 0)$ in P^n and $\tilde{\mu}$ in $\tilde{\Lambda}$ we have

$$(0, \ldots, 0, \mathfrak{t}(g_i^n)\lambda, 0, \ldots, 0)) \otimes_B \varepsilon\tilde{\mu} = (0, \ldots, 0, \mathfrak{t}(\tilde{g}_i^n)\theta(\lambda)\tilde{\mu}, 0, \ldots, 0)$$

for $\lambda \in \Lambda, \tilde{\mu} \in \tilde{\Lambda}$.

Proposition 4.25. *With the above notation and identifications, then $T_\varepsilon d^n = \tilde{d}^n$, for all $n \geq 3$.*

Proof. As above, we write $g_i^n = \sum_{j=1}^m g_j^{n-1} q_j$, where m is the number of elements in the set g^{n-1} and let $n \geq 3$. Then $d^n(0, \ldots, \mathfrak{t}(g_i^n)\lambda, \ldots, 0)$ has entry $\mathfrak{t}(g_j^{n-1})q_j\lambda$ in the summand of P^{n-1} which corresponds to $\mathfrak{t}(g_j^{n-1})$, for all $\lambda \in \Lambda$. Using the identification above, for $\tilde{\mu} \in \tilde{\Lambda}$,

$$\begin{aligned}
T_\varepsilon d^n(0, \ldots, 0, \mathfrak{t}(\tilde{g}_i^n)\tilde{\mu}, 0, \ldots, 0) &= T_\varepsilon d^n(0, \ldots, 0, \mathfrak{t}(g_i^n), 0, \ldots, 0) \otimes_B \varepsilon\tilde{\mu} \\
&= d^n(0, \ldots, 0, \mathfrak{t}(g_i^n), 0, \ldots, 0) \otimes_B \varepsilon\tilde{\mu} \\
&= (\mathfrak{t}(g_1^{n-1})q_1, \ldots, \mathfrak{t}(g_m^{n-1})q_m) \otimes_B \varepsilon\tilde{\mu} \\
&= (\mathfrak{t}(\tilde{g}_1^{n-1})\theta(q_1), \ldots, \mathfrak{t}(\tilde{g}_m^{n-1})\theta(q_m))\varepsilon\tilde{\mu} \\
&= \tilde{d}^n(0, \ldots, 0, \mathfrak{t}(\tilde{g}_i^n)\tilde{\mu}, 0, \ldots, 0)
\end{aligned}$$

as required. $\square$

Using Propositions 4.24 and 4.25, we have the following identification and commutative diagram for $n \geq 3$, which we use without further comment.

$$\begin{tikzcd}
T_\varepsilon P^n \arrow[r, "T_\varepsilon(d^n)"] \arrow[d, leftrightarrow, "\cong"'] & T_\varepsilon P^{n-1} \arrow[d, leftrightarrow, "\cong"] \\
\tilde{P}^n \arrow[r, "\tilde{d}^n"] & \tilde{P}^{n-1}
\end{tikzcd}$$

So we can identify

$$\cdots \longrightarrow \tilde{P}^n \xrightarrow{\tilde{d}^n} \tilde{P}^{n-1} \cdots \longrightarrow \tilde{P}^3 \xrightarrow{\tilde{d}^3} \tilde{P}^2$$

with

$$\cdots \longrightarrow T_\varepsilon P^n \xrightarrow{T_\varepsilon d^n} T_\varepsilon P^{n-1} \cdots \longrightarrow T_\varepsilon P^3 \xrightarrow{T_\varepsilon d^3} T_\varepsilon P^2.$$

Proposition 4.26. *Let $\tilde{P}^n$ and $\tilde{d}^n$ be as given above, for $n \geq 3$. Then*

$$\cdots \longrightarrow T_\varepsilon P^n \xrightarrow{T_\varepsilon d^n} T_\varepsilon P^{n-1} \cdots \longrightarrow T_\varepsilon P^3 \xrightarrow{T_\varepsilon d^3} T_\varepsilon P^2$$

is a complex of right $\tilde{\Lambda}$-modules.

Proof. We want to show that $T_\varepsilon(d^n) \circ T_\varepsilon(d^{n+1}) = 0$, for all $n \geq 3$. From the definition of a functor, we have $T_\varepsilon(d^n) \circ T_\varepsilon(d^{n+1}) = T_\varepsilon(d^n \circ d^{n+1})$. But (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$, so $d^n \circ d^{n+1} = 0$ and hence $T_\varepsilon(d^n \circ d^{n+1}) = 0$. Thus $T_\varepsilon(d^n) \circ T_\varepsilon(d^{n+1}) = 0$, for all $n \geq 3$ and the result follows. $\square$

Lemma 4.27. *With the above notation, an element of $T_\varepsilon P^n$ is of the form*

$$x \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} x_w \otimes_B \varepsilon \tilde{p}_w,$$

where $x, x_w \in P^n$.

Proof. Let $z \in T_\varepsilon P^n$. Then we can write z as $z = z\mathbf{1}_{\tilde{\Lambda}} = z\varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} zw$, where $\varepsilon = \sum_{v \in \mathcal{Q}_0} v$. We have $z = \sum_i x_i \otimes_B \varepsilon a_i$, where $a_i \in \tilde{\Lambda}$ and $x_i \in P^n$. Now $z\varepsilon = (\sum_i x_i \otimes_B \varepsilon a_i)\varepsilon = \sum_i x_i \otimes_B \varepsilon a_i \varepsilon = \sum_i x_i \varepsilon a_i \varepsilon \otimes_B \varepsilon = x \otimes_B \varepsilon$, where $x = \sum_i x_i \varepsilon a_i \varepsilon$. Now consider zw, where $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. We have $zw = (\sum_i x_i \otimes_B \varepsilon a_i)w = \sum_i x_i \otimes_B \varepsilon a_i w$. By the construction of the quiver $\varepsilon a_i w = \varepsilon a_i{}' \tilde{p}_w$, for some $a_i{}' \in \varepsilon\tilde{\Lambda}\varepsilon$. So $zw = \sum_i x_i \varepsilon a_i{}' \varepsilon \otimes_B \varepsilon \tilde{p}_w = x_w \otimes_B \varepsilon \tilde{p}_w$, where $x_w = \sum_i x_i \varepsilon a_i{}' \varepsilon$. So $z = x \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} x_w \otimes_B \varepsilon \tilde{p}_w$ as required. $\square$

Proposition 4.28. *For $n \geq 3$, $\operatorname{Ker}(T_\varepsilon d^n) \subseteq \operatorname{Im}(T_\varepsilon d^{n+1})$.*

Proof. Let $z \in \operatorname{Ker}(T_\varepsilon d^n)$ and $n \geq 3$. Then $z \in T_\varepsilon P^n$ so using Lemma 4.27, we write

$$z = x \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} x_w \otimes_B \varepsilon \tilde{p}_w,$$

where $x, x_w \in P^n$. Since $z \in \operatorname{Ker}(T_\varepsilon d^n)$, then $T_\varepsilon d^n(z) = 0$, so

$$d^n(x) \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} d^n(x_w) \otimes_B \tilde{p}_w = 0.$$

We may write $d^n(x) = (\mathfrak{t}(\tilde{g}_1^{n-1})\mu_1, \ldots, \mathfrak{t}(\tilde{g}_m^{n-1})\mu_m)$ and $d^n(x_w) = (\mathfrak{t}(\tilde{g}_1^{n-1})\mu_{1,w}, \ldots, \mathfrak{t}(\tilde{g}_m^{n-1})\mu_{m,w})$ for some $\mu_j, \mu_{j,w} \in \Lambda$ and where m is the number of elements in the set $\tilde{g}^{n-1}$. Hence for each $1 \leq j \leq m$, we have

$$\mathfrak{t}(\tilde{g}_j^{n-1})\mu_j \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \mathfrak{t}(\tilde{g}_j^{n-1})\mu_{j,w} \otimes_B \tilde{p}_w = \mathfrak{t}(\tilde{g}_j^{n-1})(\mu_j \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \mu_{j,w} \otimes_B \tilde{p}_w) = 0.$$

Applying the isomorphism $\mathfrak{m}_{\mathfrak{t}(\tilde{g}_j^{n-1})}$ in Proposition 4.20, this gives that

$$\mathfrak{t}(\tilde{g}_j^{n-1})(\mu_j \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \mu_{j,w} \tilde{p}_w) = 0.$$

Hence $\mathfrak{t}(\tilde{g}_j^{n-1})\mu_j \varepsilon = 0$ and $\mathfrak{t}(\tilde{g}_j^{n-1})\mu_{j,w}\tilde{p}_w = 0$ for all $1 \leq j \leq m$ and all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. So, from Proposition 4.14, $\mathfrak{t}(\tilde{g}_j^{n-1})\mu_{j,w} = 0$ for all $1 \leq j \leq m$. Thus $d^n(x) = 0$ and $d^n(x_w) = 0$, so that $x, x_w \in \operatorname{Ker} d^n$ for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$.

Now, (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$, so $\operatorname{Im} d^{n+1} = \operatorname{Ker} d^n$. Thus there are elements $y, y_w \in P^{n+1}$ such that $x = d^{n+1}(y)$ and $x_w = d^{n+1}(y_w)$ for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. So we have

$$\begin{aligned} z &= d^{n+1}(y) \otimes_B \varepsilon + \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} d^{n+1}(y_w) \otimes_B \varepsilon\tilde{p}_w \\ &= T_\varepsilon d^{n+1}(y \otimes_B \varepsilon + \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} y_w \otimes_B \varepsilon\tilde{p}_w) \end{aligned}$$

and hence $z \in \operatorname{Im} T_\varepsilon d^{n+1}$. Thus $\operatorname{Ker}(T_\varepsilon d^n) \subseteq \operatorname{Im}(T_\varepsilon d^{n+1})$ as required. □

We summarize Proposition 4.26 and Proposition 4.28 as follows:

Theorem 4.29. *The sequence*

$$\cdots \longrightarrow T_\varepsilon P^n \xrightarrow{\tilde{d}^n} T_\varepsilon P^{n-1} \cdots \longrightarrow T_\varepsilon P^3 \xrightarrow{\tilde{d}^3} \operatorname{Im} T_\varepsilon d^3 \longrightarrow 0$$

is exact.

Proposition 4.30. *With the above notation, then* $\operatorname{Ker} \tilde{d}^2 = \operatorname{Im} T_\varepsilon d^3$.

Proof. First we show $\operatorname{Ker} \tilde{d}^2 \subseteq \operatorname{Im} T_\varepsilon d^3$. Let $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$, then $\tilde{x} = (\mathfrak{t}(\tilde{g}_1^2)\tilde{\lambda}_1, \ldots, \mathfrak{t}(\tilde{g}_m^2)\tilde{\lambda}_m)$ is an element of $\tilde{P}^2$, for some $\tilde{\lambda}_i \in \tilde{\Lambda}$ and where m is the number of elements in $\tilde{g}^2$. We can write $\tilde{x} = \tilde{x}\mathbf{1}_{\tilde{\Lambda}} = \tilde{x}\varepsilon + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}} \tilde{x}w$ where $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$. For each $i = 1, \ldots, m$,

$$\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i = \mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i\varepsilon + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i w.$$

We may write $\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i\varepsilon = \mathfrak{t}(\tilde{g}_i^2)\theta(\lambda_i)$ for some $\lambda_i \in \Lambda$ and $\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i w = \mathfrak{t}(\tilde{g}_i^2)\theta(\lambda_{i,w})\tilde{p}_w$, for some $\lambda_{i,w} \in \Lambda$ where $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Hence

$$\tilde{x}\varepsilon = (\mathfrak{t}(\tilde{g}_1^2)\theta(\lambda_1)\varepsilon, \ldots \mathfrak{t}(\tilde{g}_m^2)\theta(\lambda_m)\varepsilon)$$

and

$$\begin{aligned} \tilde{x}w &= (\mathfrak{t}(\tilde{g}_1^2)\theta(\lambda_{1,w})\tilde{p}_w, \ldots, \mathfrak{t}(\tilde{g}_m^2)\theta(\lambda_{m,w})\tilde{p}_w) \\ &= (\mathfrak{t}(\tilde{g}_1^2)\theta(\lambda_{1,w})\varepsilon, \ldots, \mathfrak{t}(\tilde{g}_m^2)\theta(\lambda_{m,w})\varepsilon)\tilde{p}_w. \end{aligned}$$

So, $\tilde{x}\varepsilon = T_\varepsilon(x_\varepsilon)$ where $x_\varepsilon = (\mathfrak{t}(g_1^2)\lambda_1\varepsilon, \ldots, \mathfrak{t}(g_m^2)\lambda_m\varepsilon)$, and $\tilde{x}w = T_\varepsilon(x_w)\tilde{p}_w$ where $x_w = (\mathfrak{t}(g_1^2)\lambda_{1,w}\varepsilon, \ldots, \mathfrak{t}(g_m^2)\lambda_{m,w}\varepsilon)$, with $x_\varepsilon, x_w \in P^2$ and for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Hence

$$\tilde{x} = T_\varepsilon(x_\varepsilon) + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} T_\varepsilon(x_w)\tilde{p}_w.$$

Now, $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$. Then $\tilde{d}^2(\tilde{x}) = 0$, and so

$$\tilde{d}^2(\tilde{x}\varepsilon) + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \tilde{d}^2(\tilde{x}w) = 0.$$

We may write $g_i^2 = \sum_{j=1}^r \alpha_j \beta_{j,i}$, where α_j is an arrow in $\mathcal{Q}$ and r is the number of arrows in $\mathcal{Q}_1$. Then $\tilde{g}_i^2 = \sum_{j=1}^r \theta^*(\alpha_j)\theta^*(\beta_{j,i}) = \sum_{j=1}^r \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,A}\theta^*(\beta_{j,i})$. So, $\tilde{d}^2(\tilde{x})$ has entry $\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\tilde{\lambda}_i$ in the summand of $\tilde{P}^1$ corresponding to $\mathfrak{t}(\alpha_{j,1})$, and 0 otherwise. Thus, for all $j = 1, \ldots, r$,

$$\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\tilde{\lambda}_i = 0.$$

So $\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\tilde{\lambda}_i\varepsilon = 0$ and $\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\tilde{\lambda}_i w = 0$, for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$ and all $j = 1, \ldots, r$. Firstly,

$$\begin{aligned} 0 &= \textstyle\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i\varepsilon \\ &= \textstyle\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\theta(\lambda_i)\varepsilon \\ &= \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}(\textstyle\sum_{i=1}^m \theta(\beta_{j,i}\lambda_i)). \end{aligned}$$

Since $\alpha_{j,2}\cdots\alpha_{j,A} = \tilde{q}_{\mathfrak{t}(\alpha_{j,1})}$, Proposition 4.14 gives $\sum_{i=1}^m \theta(\beta_{j,i}\lambda_i) = 0$. Since θ is one-to-one, then $\sum_{i=1}^m \beta_{j,i}\lambda_i = 0$ for all $j = 1, \ldots, r$. Hence $d^2(x_\varepsilon) = 0$. Also,

$$\begin{aligned} 0 &= \textstyle\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})\mathfrak{t}(\tilde{g}_i^2)\tilde{\lambda}_i w \\ &= \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}(\textstyle\sum_{i=1}^m \theta(\beta_{j,i})\theta(\lambda_{i,w})\tilde{p}_w) \\ &= \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}(\textstyle\sum_{i=1}^m \theta(\beta_{j,i}\lambda_{i,w}))\tilde{p}_w. \end{aligned}$$

A similar argument shows that $\sum_{i=1}^m \theta(\beta_{j,i}\lambda_{i,w}) = 0$ for all $j = 1, \ldots, r$ and hence $d^2(x_w) = 0$. Thus x_ε and x_w are in $\operatorname{Ker} d^2$, for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$.

But, (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$, so $\operatorname{Im} d^3 = \operatorname{Ker} d^2$. Thus there are elements $y_\varepsilon, y_w \in P^3$ such that $x_\varepsilon = d^3(y_\varepsilon)$ and $x_w = d^3(y_w)$ for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. So we have

$$\begin{aligned} \tilde{x} &= T_\varepsilon(x_\varepsilon) + \textstyle\sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} T_\varepsilon(x_w)\tilde{p}_w \\ &= T_\varepsilon\left(d^3(y_\varepsilon)\right) + \textstyle\sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} T_\varepsilon\left(d^3(y_w)\right)\tilde{p}_w \\ &= T_\varepsilon d^3\left(y_\varepsilon \otimes_B \varepsilon + \textstyle\sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} y_w \otimes_B \tilde{p}_w\right) \end{aligned}$$

and hence $\tilde{x} \in \operatorname{Im} T_\varepsilon d^3$. Thus $\operatorname{Ker} \tilde{d}^2 \subseteq \operatorname{Im}(T_\varepsilon d^3)$.

Conversely, we now prove $\operatorname{Im} T_\varepsilon d^3 \subseteq \operatorname{Ker} \tilde{d}^2$. As above we write $g_i^2 = \sum_{j=1}^r \alpha_j \beta_{j,i}$, so $\tilde{g}_i^2 = \sum_{j=1}^r \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,A}\theta^*(\beta_{j,i})$. Let $\tilde{x} \in \operatorname{Im} T_\varepsilon d^3$, so $\tilde{x} = T_\varepsilon d^3(\tilde{y})$, for some $\tilde{y} \in T_\varepsilon P^3 \cong \tilde{P}^3$. So using Lemma 4.27, we have $\tilde{y} = y_\varepsilon \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} y_w \otimes_B \varepsilon \tilde{p}_w$, for some $y_\varepsilon, y_w \in P^3$. Then

$$\tilde{x} = d^3(y_\varepsilon) \otimes_B \varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} d^3(y_w) \otimes_B \varepsilon \tilde{p}_w.$$

But, (P^n, d^n) is a minimal projective resolution of $\Lambda/\mathfrak{r}$, so $\operatorname{Im} d^3 = \operatorname{Ker} d^2$. Thus let $x_\varepsilon = d^3(y_\varepsilon) \in \operatorname{Ker} d^2$ and $x_w = d^3(y_w) \in \operatorname{Ker} d^2$ for all $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$, and write $x_\varepsilon = (\mathfrak{t}(g_1^2)\lambda_1, \ldots, \mathfrak{t}(g_m^2)\lambda_m)$, $x_w = (\mathfrak{t}(g_1^2)\lambda_{1,w}, \ldots, \mathfrak{t}(g_m^2)\lambda_{m,w})$ where $\lambda_i, \lambda_{i,w} \in \Lambda$. Then we identify $\tilde{x}$ with

$$\Big(\mathfrak{t}(\tilde{g}_1^2)\theta(\lambda_1)\varepsilon + \textstyle\sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \mathfrak{t}(\tilde{g}_1^2)\theta(\lambda_{1,w})\tilde{p}_w, \ldots, \mathfrak{t}(\tilde{g}_m^2)\theta(\lambda_m)\varepsilon \\ + \textstyle\sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \mathfrak{t}(\tilde{g}_m^2)\theta(\lambda_{m,w})\tilde{p}_w\Big).$$

Then $\tilde{d}^2(\tilde{x})$ has entry $\sum_{i=1}^m \mathfrak{t}(\alpha_{j,1})\alpha_{j,2}\cdots\alpha_{j,A}\theta(\beta_{j,i})(\theta(\lambda_i)\varepsilon + \sum_{w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \theta(\lambda_{i,w})\tilde{p}_w)$ in the summand of $\tilde{P}^1$ corresponding to $\mathfrak{t}(\alpha_{j,1})$ and 0 otherwise. But $x_\varepsilon, x_w \in \operatorname{Ker} d^2$ so $d^2(x_\varepsilon) = 0$ and $d^2(x_w) = 0$. Thus $\sum_{i=1}^m \beta_{j,i}\lambda_i = 0$ and $\sum_{i=1}^m \beta_{j,i}\lambda_{i,w} = 0$ for all $j = 1, \ldots, r$ and $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Hence $\tilde{d}^2(\tilde{x}) = 0$ and so $\tilde{x} \in \operatorname{Ker} \tilde{d}^2$. Thus $\operatorname{Im} T_\varepsilon d^3 \subseteq \operatorname{Ker} \tilde{d}^2$. Hence $\operatorname{Ker} \tilde{d}^2 = \operatorname{Im} T_\varepsilon d^3$. $\square$

Theorem 4.31. *With the above notation, the equation (2)*

$$\cdots \longrightarrow \tilde{P}^n \xrightarrow{\tilde{d}^n} \tilde{P}^{n-1} \longrightarrow \cdots \longrightarrow \tilde{P}^1 \xrightarrow{\tilde{d}^1} \tilde{P}^0 \xrightarrow{\tilde{d}^0} \tilde{\Lambda}/\tilde{\mathfrak{r}} \longrightarrow 0$$

is exact. Moreover, it is a minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$.

4.2. **Functorial approach to the projective bimodule resolution of $\tilde{\Lambda}$.** We now construct a minimal projective resolution of $\tilde{\Lambda}$ as a right $\tilde{\Lambda}^e$-module from a given minimal projective resolution for Λ as a right Λ^e-module. We begin with some background information and some more definitions from [28].

Definition 4.32. [39, Section 2] We define the functor

$$T_\xi : \operatorname{mod} \xi\tilde{\Lambda}^e\xi \longrightarrow \operatorname{mod} \tilde{\Lambda}^e$$

by $T_\xi(-) = (-) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi\tilde{\Lambda}^e$, where $\xi = \varepsilon \otimes \varepsilon$, an idempotent in $\tilde{\Lambda}^e$.

Proposition 4.33. *With the above notation we have $\Lambda^e \cong \xi\tilde{\Lambda}^e\xi$.*

Proof. We have $\Lambda \cong \varepsilon\tilde{\Lambda}\varepsilon$ by Theorem 4.8, and so $\Lambda^{\mathrm{op}} \cong \varepsilon\tilde{\Lambda}^{\mathrm{op}}\varepsilon$. Using [39, Section 2], we have $(\varepsilon\tilde{\Lambda}\varepsilon)^e = \xi\tilde{\Lambda}^e\xi$. Hence $\Lambda^e \cong \xi\tilde{\Lambda}^e\xi$. □

Definition 4.34. Let $\Lambda^e, \tilde{\Lambda}^e$ be as above. Then, we define $\phi : \Lambda^e \longrightarrow \tilde{\Lambda}^e$ to be the composition of the isomorphism $\Lambda^e \cong \xi\tilde{\Lambda}^e\xi$ of Proposition 4.33 with the inclusion map $i : \xi\tilde{\Lambda}^e\xi \longrightarrow \tilde{\Lambda}^e$ which is given by

$$\lambda \otimes \mu \mapsto \xi(\theta(\lambda) \otimes \theta(\mu))\xi = \theta(\lambda) \otimes \theta(\mu)$$

Moreover ϕ is an algebra monomorphism.

The projective bimodules in a minimal projective bimodule resolution of an algebra are given by Happel in [30].

Proposition 4.35. *[30] Let Λ be a finite dimensional algebra and let*

$$\cdots \longrightarrow Q^n \xrightarrow{\delta^n} Q^{n-1} \longrightarrow \cdots \longrightarrow Q^1 \xrightarrow{\delta^1} Q^0 \xrightarrow{\delta^0} \Lambda \longrightarrow 0$$

be a minimal projective resolution of Λ as a Λ-Λ-bimodule. Then

$$Q^n = \oplus_{i,j} P(i,j)^{\dim \mathrm{Ext}^n_\Lambda(S_i,S_j)}$$

where $P(i,j)$ is the projective Λ-Λ-bimodule $\Lambda(e_i \otimes e_j)\Lambda$, and S_i, S_j are the simple modules corresponding to $e_i\Lambda$ and $e_j\Lambda$ respectively.

From [28], let (P^n, d^n) be a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a right Λ-module, so $P^n = \oplus_k \mathfrak{t}(g^n_k)\Lambda$. Then the n-th projective in a minimal projective resolution of the simple module S_i is $\oplus \mathfrak{t}(g^n_k)\Lambda$ where the sum is now over all g^n_k with $\mathfrak{o}(g^n_k) = i$. So, using the result by Benson ([5, Corollary 2.5.4]) (see Proposition 2.20) we can index $\mathrm{Ext}^n_\Lambda(S_i, S_j)$ by the elements of g^n which start at i and end at j. We now sum over all i and j and use the result by Happel (Proposition 4.35) to give the following description of the projective modules in a minimal projective bimodule resolution of Λ.

Definition 4.36. Let $n \geq 0$. Define $Q^n = \oplus_{g^n_i \in g^n} \Lambda\mathfrak{o}(g^n_i) \otimes \mathfrak{t}(g^n_i)\Lambda$ so that Q^n is the nth projective in a minimal projective bimodule resolution of Λ. Let (Q^n, δ^n) be a minimal projective bimodule resolution for Λ with the part up to Q^3 as given by [24].

Define $\tilde{Q}^n = \oplus_{\tilde{g}_i^n \in \tilde{g}^n} \tilde{\Lambda}\mathfrak{o}(\tilde{g}_i^n) \otimes \mathfrak{t}(\tilde{g}_i^n)\tilde{\Lambda}$ so that $\tilde{Q}^n$ is the nth projective in a minimal projective bimodule resolution of $\tilde{\Lambda}$. Define $\tilde{\delta}^n : \tilde{Q}^n \longrightarrow \tilde{Q}^{n-1}$ by $\tilde{\delta}^n = T_\xi \delta^n$ for all $n \geq 3$, and $\tilde{\delta}^0, \tilde{\delta}^1, \tilde{\delta}^2$ to be the maps given in [24].

Proposition 4.37. *Let Q^n and $\tilde{Q}^n$ be as above. Then $T_\xi Q^n \cong \tilde{Q}^n$, for all $n \geq 2$.*

Proof. We identify Q^n with $\oplus_i(\mathfrak{o}(\tilde{g}_i^n) \otimes \mathfrak{t}(\tilde{g}_i^n))\xi\tilde{\Lambda}^e\xi$ by Proposition 4.33. So

$$\begin{aligned} T_\xi Q^n &= \oplus_i(\mathfrak{o}(\tilde{g}_i^n) \otimes \mathfrak{t}(\tilde{g}_i^n))\xi\tilde{\Lambda}^e\xi \otimes_{\xi\tilde{\Lambda}^e\xi} \xi\tilde{\Lambda}^e \\ &\cong \oplus_i(\mathfrak{o}(\tilde{g}_i^n) \otimes \mathfrak{t}(\tilde{g}_i^n))\tilde{\Lambda}^e \\ &= \tilde{Q}^n \end{aligned}$$

as required. □

Proposition 4.38. *The definition of $\tilde{\delta}^3$ from Definition 4.36 coincides with that given by the construction of [24].*

Proof. From Remark 2.30 equation (1) we write $g_i^3 = \sum_j g_j^2 p_j = \sum_j q_j g_j^2 r_j$. Then

$$\tilde{g}_i^3 = \sum_j \tilde{g}_j^2 \theta^*(p_j) = \sum_j \theta^*(q_j)\tilde{g}_j^2\theta^*(r_j).$$

Since $\delta^3(\mathfrak{o}(g_i^3) \otimes \mathfrak{t}(g_i^3))$ is given by the matrix A_3 where the (g_j^2, g_i^3)-entry is $\mathfrak{o}(g_j^2) \otimes p_j - q_j \otimes r_j$, from Definition 4.36 and for all $\tilde{\eta} \in \tilde{\Lambda}^e$, we have

$$\begin{aligned} \tilde{\delta}^3(\mathfrak{o}(\tilde{g}_i^3) \otimes \mathfrak{t}(\tilde{g}_i^3))\tilde{\eta} &= T_\xi\delta^3(\mathfrak{o}(g_i^3) \otimes \mathfrak{t}(g_i^3)) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi\tilde{\eta} \\ &= \delta^3(\mathfrak{o}(g_i^3) \otimes \mathfrak{t}(g_i^3)) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi\tilde{\eta} \\ &= (\mathfrak{o}(\tilde{g}_j^2) \otimes \theta(p_j) - \theta(q_j) \otimes \theta(r_j))\xi\tilde{\eta}. \end{aligned}$$

Hence $\tilde{\delta}^3(\mathfrak{o}(\tilde{g}_i^3) \otimes \mathfrak{t}(\tilde{g}_i^3))$ is given by the matrix $\tilde{A}^3$ where the $(\tilde{g}_j^2, \tilde{g}_i^3)$-entry is $\mathfrak{o}(\tilde{g}_j^2) \otimes \theta(p_j) - \theta(q_j) \otimes \theta(r_j)$. Hence the result follows. □

Hence, we have a sequence of $\tilde{\Lambda}$-$\tilde{\Lambda}$ bimodules and homomorphisms

$$\cdots \longrightarrow \tilde{Q}^n \xrightarrow{\tilde{\delta}^n} \tilde{Q}^{n-1} \longrightarrow \cdots \longrightarrow \tilde{Q}^1 \xrightarrow{\tilde{\delta}^1} \tilde{Q}^0 \xrightarrow{\tilde{\delta}^0} \tilde{\Lambda} \longrightarrow 0 \tag{3}$$

We want to show the sequence is exact and we start by considering the part of the sequence:

$$\tilde{Q}^3 \xrightarrow{\tilde{\delta}^3} \tilde{Q}^2 \xrightarrow{\tilde{\delta}^2} \tilde{Q}^1 \xrightarrow{\tilde{\delta}^1} \tilde{Q}^0 \xrightarrow{\tilde{\delta}^0} \tilde{\Lambda} \longrightarrow 0.$$

This is the start of a minimal projective bimodule resolution of $\tilde{\Lambda}$ from [24]. Hence this sequence is exact.

Now consider the case $n \geq 2$. We want to show the following sequence is exact

$$\cdots \longrightarrow T_\xi Q^n \xrightarrow{T_\xi \delta^n} T_\xi Q^{n-1} \cdots \longrightarrow T_\xi Q^3 \xrightarrow{T_\xi \delta^3} T_\xi Q^2.$$

Proposition 4.39. *The sequence*

$$\cdots \longrightarrow T_\xi Q^n \xrightarrow{T_\xi \delta^n} T_\xi Q^{n-1} \cdots \longrightarrow \ T_\xi Q^3 \xrightarrow{T_\xi \delta^3} T_\xi Q^2$$

is a complex of right $\tilde{\Lambda}^e$-modules.

Proof. We want to show that $T_\xi(\delta^n) \circ T_\xi(\delta^{n+1}) = 0$, for all $n \geq 3$. From the definition of a functor, we have $T_\xi(\delta^n) \circ T_\xi(\delta^{n+1}) = T_\xi(\delta^n \circ \delta^{n+1})$. But (Q^n, δ^n) is a minimal projective resolution of Λ, so $\delta^n \circ \delta^{n+1} = 0$ and hence $T_\xi(\delta^n \circ \delta^{n+1}) = 0$. Thus $T_\xi(\delta^n) \circ T_\xi(\delta^{n+1}) = 0$, for all $n \geq 3$ and the result follows. $\square$

Lemma 4.40. *With the above notation, an element of $T_\xi Q^n$ is of the form*

$$x_{\varepsilon \otimes \varepsilon} \otimes_{\xi \tilde{\Lambda}^e \xi} \xi + \sum_{w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big(x_{\varepsilon \otimes w} \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\varepsilon \otimes \tilde{p}_w) +$$

$$x_{w \otimes \varepsilon} \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\tilde{q}_w \otimes \varepsilon) + x_{w \otimes w'} \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'}) \Big)$$

where $x_{\varepsilon \otimes \varepsilon}, x_{\varepsilon \otimes w}, x_{w \otimes \varepsilon}, x_{w \otimes w'}$ are in Q^n.

Proof. Let $z \in T_\xi Q^n$. Then we can write z as

$$z = z\mathbf{1}_{\tilde{\Lambda}^e} = z\xi + \sum_{w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big(z(\varepsilon \otimes w) + z(w \otimes \varepsilon) + z(w \otimes w') \Big)$$

where $\xi = \varepsilon \otimes \varepsilon$. We have $z = \sum_i x_i \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)$, where $x_i \in Q^n$ and $(\tilde{\lambda}_i \otimes \tilde{\mu}_i) \in \tilde{\Lambda}^e$. So

$$\begin{aligned} z\xi &= \big(\textstyle\sum_i x_i \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i) \big) \xi \\ &= \textstyle\sum_i x_i \otimes_{\xi \tilde{\Lambda}^e \xi} \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)\xi \\ &= \textstyle\sum_i x_i \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)\xi \otimes_{\xi \tilde{\Lambda}^e \xi} \xi \\ &= x_{\varepsilon \otimes \varepsilon} \otimes_{\xi \tilde{\Lambda}^e \xi} \xi \end{aligned}$$

where $x_{\varepsilon \otimes \varepsilon} = \sum_i x_i \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)\xi$.

Now,

$$\begin{aligned} z(\varepsilon \otimes w) &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)(\varepsilon \otimes w) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon \otimes \varepsilon)(\tilde{\lambda}_i \otimes \tilde{\mu}_i)(\varepsilon \otimes w) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon\tilde{\lambda}_i\varepsilon \otimes \varepsilon\tilde{\mu}_i w). \end{aligned}$$

From the construction of $K\tilde{\mathcal{Q}}$, we may write $\varepsilon\tilde{\lambda}_i\varepsilon = \varepsilon\theta(\lambda_i)\varepsilon$ and $\varepsilon\tilde{\mu}_i w = \varepsilon\theta(\mu_i)\tilde{p}_w$, for some $\lambda_i, \mu_i \in \Lambda$. Hence

$$\begin{aligned} z(\varepsilon \otimes w) &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon\theta(\lambda_i)\varepsilon \otimes \varepsilon\theta(\mu_i)\varepsilon\tilde{p}_w) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon \otimes \varepsilon)(\theta(\lambda_i) \otimes \theta(\mu_i))(\varepsilon \otimes \varepsilon)(\varepsilon \otimes \tilde{p}_w) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi(\varepsilon \otimes \tilde{p}_w) \\ &= \textstyle\sum_i x_i\xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) \\ &= x_{\varepsilon\otimes w} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) \end{aligned}$$

where $x_{\varepsilon\otimes w} = \sum_i x_i\xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi$.

Similarly

$$\begin{aligned} z(w \otimes \varepsilon) &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{\lambda}_i \otimes \tilde{\mu}_i)(w \otimes \varepsilon) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon \otimes \varepsilon)(\tilde{\lambda}_i \otimes \tilde{\mu}_i)(w \otimes \varepsilon) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (w\tilde{\lambda}_i\varepsilon \otimes \varepsilon\tilde{\mu}_i\varepsilon) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\tilde{q}_w\varepsilon\theta(\lambda_i)\varepsilon \otimes \varepsilon\theta(\mu_i)\varepsilon) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi(\tilde{q}_w \otimes \varepsilon) \\ &= \textstyle\sum_i x_i\xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) \\ &= x_{w\otimes\varepsilon} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) \end{aligned}$$

where $x_{w\otimes\varepsilon} = \sum_i x_i\xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi$.

Also,

$$\begin{aligned} z(w \otimes w') &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\varepsilon \otimes \varepsilon)(\tilde{\lambda}_i \otimes \tilde{\mu}_i)(w \otimes w') \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (w\tilde{\lambda}_i\varepsilon \otimes \varepsilon\tilde{\mu}_i w') \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} (\tilde{q}_w\varepsilon\theta(\lambda_i)\varepsilon \otimes \varepsilon\theta(\mu_i)\tilde{p}_{w'}) \\ &= \textstyle\sum_i x_i \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi(\tilde{q}_w \otimes \tilde{p}_{w'}) \end{aligned}$$

$$= \sum_i x_i \xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'})$$

$$= x_{w\otimes w'} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'})$$

where $x_{w\otimes w'} = \sum_i x_i \xi(\theta(\lambda_i) \otimes \theta(\mu_i))\xi$, for some $\lambda_i, \mu_i \in \Lambda$. $\square$

We now show that $\mathrm{Ker}(T_\xi \delta^n) \subseteq \mathrm{Im}(T_\xi \delta^{n+1})$, for $n \geq 3$.

Proposition 4.41. *For $n \geq 3$, $\mathrm{Ker}(T_\xi \delta^n) \subseteq \mathrm{Im}(T_\xi \delta^{n+1})$.*

Proof. Let $z \in \mathrm{Ker}(T_\xi \delta^n)$. Then $z \in T_\xi Q^n$ so using Lemma 4.40, we write

$$z = x_{\varepsilon\otimes\varepsilon} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi + \sum_{w,w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big(x_{\varepsilon\otimes w} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) +$$

$$x_{w\otimes\varepsilon} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) + x_{w\otimes w'} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'}) \Big)$$

where $x_{\varepsilon\otimes\varepsilon}, x_{\varepsilon\otimes w}, x_{w\otimes\varepsilon}, x_{w\otimes w'}$ are in Q^n. Since $z \in \mathrm{Ker}(T_\xi \delta^n)$, we have $T_\xi \delta^n(z) = 0$. So

$$\delta^n(x_{\varepsilon\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi + \sum_{w,w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big(\delta^n(x_{\varepsilon\otimes w}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) +$$

$$\delta^n(x_{w\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) + \delta^n(x_{w\otimes w'}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'}) \Big) = 0.$$

We want to show that $x_{\varepsilon\otimes\varepsilon}, x_{\varepsilon\otimes w}, x_{w\otimes\varepsilon}, x_{w\otimes w'} \in \mathrm{Ker}\, \delta^n$. For each $\tau \in \{\varepsilon\otimes\varepsilon, \varepsilon\otimes w, w\otimes\varepsilon, w\otimes w'\}$, x_τ is in Q^n so $\delta^n(x_\tau)$ is in Q^{n-1} and we may write

$$\delta^n(x_\tau) = \big(\mathfrak{o}(g_1^{n-1}) \otimes \mathfrak{t}(g_1^{n-1})(\lambda_{1,\tau} \otimes \mu_{1,\tau}), \ldots, \mathfrak{o}(g_m^{n-1}) \otimes \mathfrak{t}(g_m^{n-1})(\lambda_{m,\tau} \otimes \mu_{m,\tau}) \big)$$

with the ith component in the summand of Q^{n-1} corresponding to $\mathfrak{o}(g_i^{n-1}) \otimes \mathfrak{t}(g_i^{n-1})$, where m is the number of elements in the set g^{n-1} and $\lambda_{i,\tau}, \mu_{i,\tau} \in \Lambda$. Hence, for each $1 \leq j \leq m$, we have

$$\begin{aligned}
&(\mathfrak{o}(g_j^{n-1}) \otimes \mathfrak{t}(g_j^{n-1}))(\lambda_{j,\varepsilon\otimes\varepsilon} \otimes \mu_{j,\varepsilon\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi + \\
&\quad \textstyle\sum_{w,w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big((\mathfrak{o}(g_j^{n-1}) \otimes \mathfrak{t}(g_j^{n-1}))(\lambda_{j,\varepsilon\otimes w} \otimes \mu_{j,\varepsilon\otimes w}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) + \\
&\qquad (\mathfrak{o}(g_j^{n-1}) \otimes \mathfrak{t}(g_j^{n-1}))(\lambda_{j,w\otimes\varepsilon} \otimes \mu_{j,w\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) + \\
&\qquad\quad (\mathfrak{o}(g_j^{n-1}) \otimes \mathfrak{t}(g_j^{n-1}))(\lambda_{j,w\otimes w'} \otimes \mu_{j,w\otimes w'}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'}) \Big) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,\varepsilon\otimes\varepsilon} \otimes \mu_{j,\varepsilon\otimes\varepsilon})\xi + \\
&\quad \textstyle\sum_{w,w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0} \Big((\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,\varepsilon\otimes w} \otimes \mu_{j,\varepsilon\otimes w})\xi(\varepsilon \otimes \tilde{p}_w) +
\end{aligned}$$

$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes\varepsilon} \otimes \mu_{j,w\otimes\varepsilon})\xi(\tilde{q}_w \otimes \varepsilon) +$$
$$\left.(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes w'} \otimes \mu_{j,w\otimes w'})\xi(\tilde{q}_w \otimes \tilde{p}_{w'})\right) = 0.$$

Hence

$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1})(\lambda_{j,\varepsilon\otimes\varepsilon} \otimes \mu_{j,\varepsilon\otimes\varepsilon})\xi = 0,$$
$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,\varepsilon\otimes w} \otimes \mu_{j,\varepsilon\otimes w})\xi(\varepsilon \otimes \tilde{p}_w) = 0,$$
$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes\varepsilon} \otimes \mu_{j,w\otimes\varepsilon})\xi(\tilde{q}_w \otimes \varepsilon) = 0,$$
$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes w'} \otimes \mu_{j,w\otimes w'})\xi(\tilde{q}_w \otimes \tilde{p}_{w'}) = 0$$

for all $w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. Since the ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$ is generated by uniform elements $\tilde{g}_1^2, \ldots, \tilde{g}_m^2$ which all start and end at a vertex in $\mathcal{Q}_0$, a similar argument to Proposition 4.14 gives

$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,\varepsilon\otimes w} \otimes \mu_{j,\varepsilon\otimes w})\xi = 0,$$
$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes\varepsilon} \otimes \mu_{j,w\otimes\varepsilon})\xi = 0 \text{ and}$$
$$(\mathfrak{o}(\tilde{g}_j^{n-1}) \otimes \mathfrak{t}(\tilde{g}_j^{n-1}))(\lambda_{j,w\otimes w'} \otimes \mu_{j,w\otimes w'})\xi = 0.$$

Thus $\delta^n(x_{\varepsilon\otimes\varepsilon}) = 0$, $\delta^n(x_{\varepsilon\otimes w}) = 0$, $\delta^n(x_{w\otimes\varepsilon}) = 0$ and $\delta^n(x_{w\otimes w'}) = 0$, so that $x_{\varepsilon\otimes\varepsilon}, x_{\varepsilon\otimes w}, x_{w\otimes\varepsilon}, x_{w\otimes w'}$ are in $\operatorname{Ker}\delta^n$ for all $w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$.

Now (Q^n, δ^n) is a minimal projective resolution of Λ so $\operatorname{Ker}\delta^n = \operatorname{Im}\delta^{n+1}$. Thus there are elements $y_{\varepsilon\otimes\varepsilon}, y_{\varepsilon\otimes w}, y_{w\otimes\varepsilon}, y_{w\otimes w'}$ in Q^{n+1} such that $x_{\varepsilon\otimes\varepsilon} = \delta^{n+1}(y_{\varepsilon\otimes\varepsilon})$, $x_{\varepsilon\otimes w} = \delta^{n+1}(y_{\varepsilon\otimes w})$, $x_{w\otimes\varepsilon} = \delta^{n+1}(y_{w\otimes\varepsilon})$ and $x_{w\otimes w'} = \delta^{n+1}(y_{w\otimes w'})$ for all $w, w' \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$. So we have

$$\begin{aligned} z &= \delta^{n+1}(y_{\varepsilon\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi + \textstyle\sum_{w,w'\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\Big(\delta^{n+1}(y_{\varepsilon\otimes w}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) \\ &\quad +\delta^{n+1}(y_{w\otimes\varepsilon}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) + \delta^{n+1}(y_{w\otimes w'}) \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'})\Big) \\ &= T_\xi\delta^{n+1}\Big(y_{\varepsilon\otimes\varepsilon} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi + \textstyle\sum_{w,w'\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} y_{\varepsilon\otimes w} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\varepsilon \otimes \tilde{p}_w) \\ &\quad +y_{w\otimes\varepsilon} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \varepsilon) + y_{w\otimes w'} \otimes_{\xi\tilde{\Lambda}^e\xi} \xi(\tilde{q}_w \otimes \tilde{p}_{w'})\Big) \end{aligned}$$

and hence $z \in \operatorname{Im} T_\xi\delta^{n+1}$. Thus $\operatorname{Ker}(T_\xi\delta^n) \subseteq \operatorname{Im} T_\xi\delta^{n+1}$ as required. □

We summarize Proposition 4.39 and Proposition 4.41 as follows:

Theorem 4.42. *The sequence*

$$\cdots \longrightarrow T_\xi Q^n \xrightarrow{T_\xi\delta^n} T_\xi Q^{n-1} \cdots \longrightarrow T_\xi Q^3 \xrightarrow{T_\xi\delta^3} T_\xi Q^2$$

is exact.

Theorem 4.43. *With the above notation, the sequence (3)*

$$\cdots \longrightarrow \tilde{Q}^n \xrightarrow{\tilde{\delta}^n} \tilde{Q}^{n-1} \longrightarrow \cdots \longrightarrow \tilde{Q}^1 \xrightarrow{\tilde{\delta}^1} \tilde{Q}^0 \xrightarrow{\tilde{\delta}^0} \tilde{\Lambda} \longrightarrow 0$$

is exact. Moreover, it is a minimal projective bimodule resolution of $\tilde{\Lambda}$.

4.3. **The relation between the** Ext **algebras of** Λ **and** $\tilde{\Lambda}$**.** We look now at the relationship between $E(\Lambda)$ and $E(\tilde{\Lambda})$.

Definition 4.44. [36, Definition 9.1] Let $f_i^n : P^n \to \Lambda/\mathfrak{r}$ be the Λ-module homomorphism given by

$$\mathfrak{t}(g_j^n) \mapsto \begin{cases} \mathfrak{t}(g_i^n) + \mathfrak{r} & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

We set $f^n = \{f_i^n\}$ so that $|f^n| = |g^n|$.

Let $\tilde{f}_i^n : \tilde{P}^n \to \tilde{\Lambda}/\tilde{\mathfrak{r}}$ be the $\tilde{\Lambda}$-module homomorphism given by

$$\mathfrak{t}(\tilde{g}_j^n) \mapsto \begin{cases} \mathfrak{t}(\tilde{g}_i^n) + \tilde{\mathfrak{r}} & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

We set $\tilde{f}^n = \{\tilde{f}_i^n\}$ so that $|\tilde{f}^n| = |\tilde{g}^n|$.

The set f^n forms a basis for $\operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ and the set $\tilde{f}^n$ forms a basis for $\operatorname{Ext}^n_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ by Proposition 2.20. Moreover, for $n \geq 2$ we have $|g^n| = |\tilde{g}^n|$, so $|f^n| = |g^n| = |\tilde{g}^n| = |\tilde{f}^n|$

Definition 4.45. [36, Definition 9.2] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra and let $\tilde{\Lambda}$ be the stretched algebra. Let $\operatorname{Ext}^{\geq 2}_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) = \bigoplus_{n\geq 2} \operatorname{Ext}^n_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ and let $\operatorname{Ext}^{\geq 2}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}}) = \bigoplus_{n\geq 2} \operatorname{Ext}^n_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$. We define a K-module homomorphism $\Psi : \operatorname{Ext}^{\geq 2}_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \operatorname{Ext}^{\geq 2}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ by

$$\Psi(f_i^n) = \tilde{f}_i^n \text{ for } n \geq 2.$$

We note that Ψ is clearly 1-1 and onto so that it is a K-module isomorphism. In [36, Theorem 9.15], Leader showed that Ψ is an algebra homomorphism when Λ is a d-Koszul algebra. However, her arguments do not require that Λ is d-Koszul, and hold more generally. Hence we have the following result.

Theorem 4.46. *Let* $\Psi : \operatorname{Ext}^{\geq 2}_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \to \operatorname{Ext}^{\geq 2}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ *be the map given in Definition 4.45. Then* Ψ *is a* K*-algebra isomorphism.*

We end this chapter with the following result.

Theorem 4.47. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Suppose there is some $m \geq 1$ such that the Ext algebra $E(\Lambda)$ is generated in degree at most m. Then the Ext algebra $E(\tilde{\Lambda})$ is also finitely generated, and has generators in degree at most $m + 2$.*

Proof. Assume $m \geq 1$. Suppose that $E(\Lambda)$ is generated in degrees $0, 1, \ldots, m$. We use induction to show that $E(\tilde{\Lambda})$ is generated in degrees $0, 1, \ldots, m + 2$.

We start by considering the case $n = m + 3$. Let $x \in E(\tilde{\Lambda})$ and suppose that $x \in \mathrm{Ext}^{m+3}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$. From Theorem 4.46 there is some y in $E(\Lambda)$ with $\Psi(y) = x$, and $|y| = m + 3$. By hypothesis, y is a sum of products of elements of degree at most m. Without loss of generality, suppose that $y = y_1 y_2 \cdots y_r$ where $1 \leq |y_i| \leq m$ for each i.

If $|y_1| \geq 2$ so that necessarily we have $m \geq 2$, then $3 \leq |y_2 \cdots y_r| \leq m + 1$. Let $z = y_2 \cdots y_r$; then $y = y_1 z$ with $y_1, z \in \mathrm{Ext}^{\geq 2}_{\Lambda}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. So $x = \Psi(y) = \Psi(y_1)\Psi(z)$ and $\Psi(y_1), \Psi(z)$ both have degree at most $m + 1$. So x can be written as a product of elements of degree at most $m + 1$.

Otherwise $|y_1| = 1$. In this case, $y_2 \cdots y_r$ has degree $m + 2$. Then $y = (y_1 y_2)(y_3 \cdots y_r)$. Let $z_1 = y_1 y_2$ and $z_2 = y_3 \cdots y_r$. Since y_2 has degree at most m, it follows that $|z_2| \geq 2$. Then $y = z_1 z_2$ with $z_1, z_2 \in \mathrm{Ext}^{\geq 2}_{\Lambda}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. So $x = \Psi(y) = \Psi(z_1)\Psi(z_2)$ and $\Psi(z_1), \Psi(z_2)$ both have degree at most $m + 1$. So x can be written as a product of elements of degree at most $m + 1$.

Now we assume that elements of $\mathrm{Ext}^{n}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ can be written as sums of products of elements of degree at most $m + 2$, where $m + 3 \leq n \leq m + k$. We let $x \in E(\tilde{\Lambda})$ and now suppose that $x \in \mathrm{Ext}^{n}_{\tilde{\Lambda}}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$ with $n = m + (k + 1)$. From Theorem 4.46 there is some y in $E(\Lambda)$ with $\Psi(y) = x$, and $|y| = m + (k + 1)$. By hypothesis, y is a sum of products of elements of degree at most m. Without loss of generality, suppose that $y = y_1 y_2 \cdots y_r$ where $1 \leq |y_i| \leq m$ for each i.

If $|y_1| \geq 2$ so that necessarily we have $m \geq 2$, then $k + 1 \leq |y_2 \cdots y_r| \leq m + (k + 1) - 2$. Let $z = y_2 \cdots y_r$; then $y = y_1 z$ with $y_1, z \in \mathrm{Ext}^{\geq 2}_{\Lambda}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. So $x = \Psi(y) = \Psi(y_1)\Psi(z)$ where $\Psi(y_1)$ has degree at most m and $\Psi(z)$ has degree at most $m + k - 1$. By hypothesis $\Psi(z)$ can be written as a sum of products of elements of degree at most $m+2$. Hence x can be written as a sum of products of elements of degree at most $m+2$.

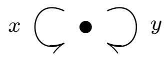
Otherwise $|y_1| = 1$. In this case, $y_2 \cdots y_r$ has degree $m+k$. Then $y = (y_1y_2)(y_3 \cdots y_r)$. Let $z_1 = y_1y_2$ and $z_2 = y_3 \cdots y_r$. Since y_2 has degree at most m, it follows that $k \leq |z_2| \leq m+k-1$, where $k \geq 3$. Then $y = z_1z_2$ with $z_1, z_2 \in \operatorname{Ext}_{\tilde{\Lambda}}^{\geq 2}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$. So $x = \Psi(y) = \Psi(z_1)\Psi(z_2)$ where $\Psi(z_1)$ has degree at most $m+1$ and $\Psi(z_2)$ has degree at most $m+k-1$. By hypothesis $\Psi(z_2)$ can be written as a sum of products of elements of degree at most $m+2$. Hence x can be written as a sum of products of elements of degree at most $m+2$.

Hence for each element x in $\operatorname{Ext}_{\tilde{\Lambda}}^{\geq 2}(\tilde{\Lambda}/\tilde{\mathfrak{r}}, \tilde{\Lambda}/\tilde{\mathfrak{r}})$, x can be written as a sum of products of elements of degree at most $m+2$ and thus $E(\tilde{\Lambda})$ is generated in degree at most $m+2$. $\square$

5. The Hochschild cohomology of two examples

In this Chapter we study the Hochschild cohomology groups of algebras Λ and $\tilde{\Lambda}$ in Examples 5.1 and 5.2. Throughout this chapter we will write $e_i \otimes_r e_j$ for the generator of the summand of Q^n corresponding to the relation g_r^n where $\mathfrak{o}(g_r^n) = e_i$ and $\mathfrak{t}(g_r^n) = e_j$.

Example 5.1. Let $\Lambda = K\mathcal{Q}/I$ be the algebra which is given by the quiver



and $I = \langle x^2, y^2, xy - yx \rangle$. We denote the vertex of the quiver $\mathcal{Q}$ by v. We have

- $g^0 = \{v\}$;
- $g^1 = \{x, y\}$;
- $g^2 = \{x^2, y^2, xy - yx\}$ with $g_1^2 = x^2$, $g_2^2 = xy - yx$, $g_3^2 = y^2$;
- For all $n \geq 3$, we have

$$g_1^n = g_1^{n-1}x = xg_1^{n-1};$$
$$g_r^n = g_{r-1}^{n-1}y + (-1)^{r-1}g_r^{n-1}x = (-1)^{n+r-1}yg_{r-1}^{n-1} + xg_r^{n-1},$$
$$\text{where } 2 \leq r \leq n;$$
$$g_{n+1}^n = g_n^{n-1}y = yg_n^{n-1}.$$

Now, for $n \geq 1$, keeping the above notation, we define the map $d^n : Q^n \longrightarrow Q^{n-1}$ for the algebra Λ, where $Q^n = \Lambda\mathfrak{o}(g_i^n) \otimes \mathfrak{t}(g_i^n)\Lambda$ are Λ-Λ-bimodules, by

$$\mathfrak{o}(g_1^n) \otimes \mathfrak{t}(g_1^n) \mapsto \mathfrak{o}(g_1^{n-1}) \otimes_1 x + (-1)^n x \otimes_1 \mathfrak{t}(g_1^{n-1})$$
$$\mathfrak{o}(g_r^n) \otimes \mathfrak{t}(g_r^n) \mapsto \mathfrak{o}(g_{r-1}^{n-1}) \otimes_{r-1} y + (-1)^{r-1}\mathfrak{o}(g_r^{n-1}) \otimes_r x$$
$$+(-1)^n\big((-1)^{n+r-1}y \otimes_{r-1} \mathfrak{t}(g_{r-1}^{n-1}) + x \otimes_r \mathfrak{t}(g_r^{n-1})\big)$$
$$\mathfrak{o}(g_{n+1}^n) \otimes \mathfrak{t}(g_{n+1}^n) \mapsto \mathfrak{o}(g_n^{n-1}) \otimes_n y + (-1)^n y \otimes_n \mathfrak{t}(g_n^{n-1}).$$

Note that $\mathfrak{o}(x) = \mathfrak{t}(g_1^{n-1})$ so $\mathfrak{o}(g_1^{n-1}) \otimes_1 \mathfrak{t}(g_1^{n-1})x = \mathfrak{o}(g_1^{n-1}) \otimes_1 x$.

This algebra has been well-studied and (Q^n, d^n) is a minimal projective bimodule resolution of Λ; see [18].

So we have, for each $f \in \mathrm{Hom}(Q^n, \Lambda)$, f is determined by the elements $f(v \otimes_r v)$ and they can be written as a linear combination of the basis elements in $v\Lambda v$, for $n \geq 0$. Hence $f(v \otimes_r v) = c_1 v + c_2 x + c_3 y + c_4 xy$, with $c_i \in K$.

The following example is the stretched algebra of Example 5.1 with $A = 2$. This is also the algebra of Examples 2.40, 3.13 and 4.2.

Example 5.2. Let $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ be the algebra which is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha_2"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend left, "\alpha_3"] & 3 \arrow[l, bend left, "\alpha_4"] \end{tikzcd}$$

and $\tilde{I} = \langle \alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2 \rangle$. We have

- $\tilde{g}^0 = \{e_1, e_2, e_3\}$
- $\tilde{g}^1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
- $\tilde{g}^2 = \{\alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2\}$
  with $\tilde{g}^2_1 = \alpha_1\alpha_2\alpha_1\alpha_2$, $\tilde{g}^2_2 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2$, $\tilde{g}^2_3 = \alpha_3\alpha_4\alpha_3\alpha_4$.
- For $n \geq 3$, we have
  - $\tilde{g}^n_1 = \tilde{g}^{n-1}_1\alpha_1\alpha_2 = \alpha_1\alpha_2\tilde{g}^{n-1}_1$;
  - For $2 \leq r \leq n$, we have $\tilde{g}^n_r = \tilde{g}^{n-1}_{r-1}\alpha_3\alpha_4 + (-1)^{r-1}\tilde{g}^{n-1}_r\alpha_1\alpha_2$
    $= \alpha_1\alpha_2\tilde{g}^{n-1}_r + (-1)^{n+r-1}\alpha_3\alpha_4\tilde{g}^{n-1}_{r-1}$;
  - $\tilde{g}^n_{n+1} = \tilde{g}^{n-1}_n\alpha_3\alpha_4 = \alpha_3\alpha_4\tilde{g}^{n-1}_n$.

Keeping the above notation and for $n \geq 3$, we define the map $\tilde{d^n} : \tilde{Q}^n \longrightarrow \tilde{Q}^{n-1}$ for the algebra $\tilde{\Lambda}$, where $\tilde{Q}^n = \oplus_{\tilde{g}^n_i \in \tilde{g}^n} \tilde{\Lambda}\mathfrak{o}(\tilde{g}^n_i) \otimes \mathfrak{t}(\tilde{g}^n_i)\tilde{\Lambda}$ are $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodules, as follows:

$$\begin{aligned}
\mathfrak{o}(\tilde{g}^n_1) \otimes \mathfrak{t}(\tilde{g}^n_1) &\mapsto \mathfrak{o}(\tilde{g}^{n-1}_1) \otimes_1 \alpha_1\alpha_2 + (-1)^n\alpha_1\alpha_2 \otimes_1 \mathfrak{t}(\tilde{g}^{n-1}_1)\\
\mathfrak{o}(\tilde{g}^n_r) \otimes \mathfrak{t}(\tilde{g}^n_r) &\mapsto \mathfrak{o}(\tilde{g}^{n-1}_{r-1}) \otimes_{r-1} \alpha_3\alpha_4 + (-1)^{r-1}\mathfrak{o}(\tilde{g}^{n-1}_r) \otimes_r \alpha_1\alpha_2\\
&\qquad + (-1)^n\big((-1)^{n+r-1}\alpha_3\alpha_4 \otimes_{r-1} \mathfrak{t}(\tilde{g}^{n-1}_{r-1}) + \alpha_1\alpha_2 \otimes_r \mathfrak{t}(\tilde{g}^{n-1}_r)\big)\\
\mathfrak{o}(\tilde{g}^n_{n+1}) \otimes \mathfrak{t}(\tilde{g}^n_{n+1}) &\mapsto \mathfrak{o}(\tilde{g}^{n-1}_n) \otimes_n \alpha_3\alpha_4 + (-1)^n\alpha_3\alpha_4 \otimes_n \mathfrak{t}(\tilde{g}^{n-1}_n).
\end{aligned}$$

Using Example 5.1 and Theorem 4.43, it can be shown that $(\tilde{Q}^n, \tilde{d^n})$ is a minimal projective resolution of $\tilde{\Lambda}$ as a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule. See also the comment at the end of Example 2.40. Alternatively, we can use the argument in [24, Proposition 2.8]; see also [45, Theorem 1.6]. In this case we need to note that $(\tilde{\Lambda}/\tilde{\mathfrak{r}} \otimes_{\tilde{\Lambda}} \tilde{Q}^n, \mathrm{id} \otimes_{\tilde{\Lambda}} \tilde{d^n})$ is precisely the minimal projective resolution of $\tilde{\Lambda}/\tilde{\mathfrak{r}}$; this was studied in Example 2.40.

5.1. **The centre of the algebras.** We now look at $\mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^0(\tilde{\Lambda})$ in Examples 5.1 and 5.2.

Example 5.3. Let Λ be the algebra in Example 5.1. It can be seen that each element in Λ is in $Z(\Lambda)$. This is because Λ is a commutative algebra. In particular $\Lambda = K[x,y]/(x^2,y^2)$. So $Z(\Lambda) = \Lambda$.

Example 5.4. Let $\tilde{\Lambda}$ be the algebra from Example 5.2. Let $z \in Z(\tilde{\Lambda})$. Firstly, we will show that z can be written as $z = e_1ze_1 + e_2ze_2 + e_3ze_3$. Since $z \in Z(\tilde{\Lambda})$, then $z\tilde{\lambda} = \tilde{\lambda}z$, for all $\tilde{\lambda} \in \tilde{\Lambda}$. In particular, $ze_i = e_iz$, for all $i = 1,2,3$, so we have $e_i(ze_i) = e_i(e_iz) = (e_i^2)z = e_iz$. Now, $z = 1.z = (e_1+e_2+e_3)z = e_1z + e_2z + e_3z = e_1ze_1 + e_2ze_2 + e_3ze_3$.

Now, z can be written as follows $z = d_1e_1 + d_2\alpha_2\alpha_1 + d_3\alpha_2\alpha_3\alpha_4\alpha_1 + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1 + d_6e_2 + d_7\alpha_1\alpha_2 + d_8\alpha_3\alpha_4 + d_9\alpha_1\alpha_2\alpha_3\alpha_4 + d_{10}e_3 + d_{11}\alpha_4\alpha_3 + d_{12}\alpha_4\alpha_3\alpha_4\alpha_3 + d_{13}\alpha_4\alpha_1\alpha_2\alpha_3 + d_{14}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$, where $d_i \in K$.

- We have $\alpha_1z = z\alpha_1$, then $\alpha_1z = d_1\alpha_1 + d_2\alpha_1\alpha_2\alpha_1 + d_3\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$, since $\alpha_1\alpha_2\alpha_1\alpha_2 = 0$ and $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_3\alpha_4\alpha_1\alpha_2$. Also, $z\alpha_1 = d_6\alpha_1 + d_7\alpha_1\alpha_2\alpha_1 + d_8\alpha_3\alpha_4\alpha_1 + d_9\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$. Hence, $d_1 = d_6, d_2 = d_7, d_3 = d_9, d_8 = 0$.
- Similarly, we find $d_1 = d_6, d_2 = d_7, d_3 = d_9, d_8 = 0$ when we consider $\alpha_2z = z\alpha_2$.
- We have $\alpha_3z = z\alpha_3$, then $\alpha_3z = d_{10}\alpha_3 + d_{11}\alpha_3\alpha_4\alpha_3 + d_{13}\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3$. Also, $z\alpha_3 = d_6\alpha_3 + d_7\alpha_1\alpha_2\alpha_3 + d_8\alpha_3\alpha_4\alpha_3 + d_9\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$. Hence $d_{10} = d_6, d_{11} = d_8, d_{13} = d_9, d_7 = 0$.
- Similarly, we find $d_{10} = d_6, d_{11} = d_8, d_{13} = d_9, d_7 = 0$ when we consider $\alpha_4z = z\alpha_4$.

Thus $z = d_1(e_1+e_2+e_3) + d_3(\alpha_2\alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_4\alpha_1\alpha_2\alpha_3) + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1 + d_{12}\alpha_4\alpha_3\alpha_4\alpha_3 + d_{14}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$, where $d_i \in K$.

Now, we want to show if $z = d_1\mathbf{1} + d_3(\alpha_2\alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_4\alpha_1\alpha_2\alpha_3) + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1 + d_{12}\alpha_4\alpha_3\alpha_4\alpha_3 + d_{14}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$, where $d_i \in K$, then $z \in Z(\tilde{\Lambda})$. We have $ze_1 = d_1e_1 + d_3\alpha_2\alpha_3\alpha_4\alpha_1 + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1$ and $e_1z = d_1e_1 + d_3\alpha_2\alpha_3\alpha_4\alpha_1 + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1$. Thus, $e_1z = ze_1$. Similarly, $e_iz = ze_i$, where $i = 2,3$.

Next we show that $z\alpha_i = \alpha_iz$, where $i = 1,\ldots,4$.

- For $i = 1$, we have $\alpha_1z = d_1\alpha_1 + d_3\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$ and $z\alpha_1 = d_1\alpha_1 + d_3\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$ and hence $z\alpha_1 = \alpha_1z$.

- For $i = 2$, we have $\alpha_2 z = d_1\alpha_2 + d_3\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4$ and $z\alpha_2 = d_1\alpha_1 + d_3\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2$ and hence $z\alpha_2 = \alpha_2 z$.
- For $i = 3$, we have $\alpha_3 z = d_1\alpha_3 + d_3\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3$ and $z\alpha_3 = d_1\alpha_3 + d_3\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$ and hence $z\alpha_3 = \alpha_3 z$, since $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_3\alpha_4\alpha_1\alpha_2$.
- For $i = 4$, we have $\alpha_4 z = d_1\alpha_4 + d_3\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4$ and $z\alpha_4 = d_1\alpha_4 + d_3\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4$ and hence $z\alpha_4 = \alpha_4 z$.

So $z \in Z(\tilde{\Lambda})$. Hence for each $z \in Z(\tilde{\Lambda})$, z can be written as $z = d_1\mathbf{1} + d_3(\alpha_2\alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_4\alpha_1\alpha_2\alpha_3) + d_4\alpha_2\alpha_1\alpha_2\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1 + d_{12}\alpha_4\alpha_3\alpha_4\alpha_3 + d_{14}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$, where $d_i \in K$ and $\dim Z(\tilde{\Lambda}) = 6$.

5.2. HH^1. We compute HH^1 explicitly for Example 5.1 and Example 5.2.

We denote the map $Q^n \longrightarrow Q^{n-1}$ by d^n and we denote the induced map $\mathrm{Hom}(Q^n, \Lambda) \longrightarrow \mathrm{Hom}(Q^{n+1}, \Lambda)$ by δ^n. So δ^n is induced from d^{n+1}.

5.2.1. $\mathrm{HH}^1(\Lambda)$. In order to find $\mathrm{HH}^1(\Lambda)$ For Example 5.1, we need to find $\mathrm{Ker}\,\delta^1$ and $\mathrm{Im}\,\delta^0$, where $\delta^1 : \mathrm{Hom}(Q^1, \Lambda) \longrightarrow \mathrm{Hom}(Q^2, \Lambda)$ and $\delta^0 : \mathrm{Hom}(Q^0, \Lambda) \longrightarrow \mathrm{Hom}(Q^1, \Lambda)$. We have $\mathrm{Ker}\,\delta^1 = \{f \in \mathrm{Hom}(Q^1, \Lambda) : \delta^1(f) = 0\}$. Let f in $\mathrm{Ker}\,\delta^1$, then $f \in \mathrm{Hom}(Q^1, \Lambda)$. So $f : Q^1 \longrightarrow \Lambda$ is given by

$$v \otimes_x v \mapsto c_1 v + c_2 x + c_3 y + c_4 xy$$
$$v \otimes_y v \mapsto c_5 v + c_6 x + c_7 y + c_8 xy$$

where c_i in K. Since $f \circ d^2 = 0$, then we have

$$\begin{aligned}
\mathfrak{o}(g_1^2) \otimes \mathfrak{t}(g_1^2) &\overset{d^2}{\mapsto} (v \otimes_x x + x \otimes_x v, 0) \\
&\overset{f}{\mapsto} f(v \otimes_x v)x + xf(v \otimes_x v) \\
&= 2c_1 x + 2c_3 xy \\
&= 0.
\end{aligned}$$

We consider two cases. If $\mathrm{char}\, K = 2$, then there is no condition on constants. If $\mathrm{char}\, K \neq 2$, then $c_1 = c_3 = 0$.

$$\begin{aligned}
\mathfrak{o}(g_2^2) \otimes \mathfrak{t}(g_2^2) &\overset{d^2}{\mapsto} (v \otimes y - y \otimes v, x \otimes v - v \otimes x) \\
&\overset{f}{\mapsto} f(v \otimes_x v)y - yf(v \otimes_x v) + xf(v \otimes_y v) - f(v \otimes_y v)x \\
&= c_1 y + c_2 xy - c_1 y - c_2 yx + c_5 x + c_7 xy - c_5 x - c_7 yx \\
&= 0.
\end{aligned}$$

$$\begin{array}{rcl}\mathfrak{o}(g_3^2)\otimes\mathfrak{t}(g_3^2) & \stackrel{d^2}{\mapsto} & (0, v\otimes y + y\otimes v)\\ & \stackrel{f}{\mapsto} & f(v\otimes_y v)y + yf(v\otimes_y v)\\ & = & 2c_5y + 2c_6xy\\ & = & 0.\end{array}$$

Now we consider two cases. If char $K = 2$, then there is no condition on constants. If char $K \neq 2$, then $c_5 = c_6 = 0$.

Hence we have two cases. If char $K = 2$, then $\operatorname{Ker}\delta^1 = \operatorname{Hom}(Q^1, \Lambda)$. If char $K \neq 2$, then $\operatorname{Ker}\delta^1 = \{f \in \operatorname{Hom}(Q^1, \Lambda) :$

$$f(v\otimes_x v) = c_2x + c_4xy,$$
$$f(v\otimes_y v) = c_7y + c_8xy\}.$$

So we have two cases. If char $K = 2$, then $\dim\operatorname{Ker}\delta^1 = 8$, and if char $K \neq 2$, then $\dim\operatorname{Ker}\delta^1 = 4$.

Next we find $\operatorname{Im}\delta^0$. We know $\operatorname{Im}\delta^0 = \{\delta^0(f) : f \in \operatorname{Hom}(Q^0, \Lambda)\}$. Let $f \in \operatorname{Hom}(Q^0, \Lambda)$, then $f : Q^0 \longrightarrow \Lambda$ is given by $f(v\otimes_v v) = c'_1v + c'_2x + c'_3y + c'_4xy$. So we have

$$\begin{array}{rcl}\mathfrak{o}(g_1^1)\otimes\mathfrak{t}(g_1^1) & \stackrel{d^1}{\mapsto} & v\otimes x - x\otimes v\\ & \stackrel{f}{\mapsto} & f(v\otimes_v v)x - xf(v\otimes_v v)\\ & = & c'_1x + c'_3yx - c'_1x - c'_3xy\\ & = & 0.\\ \mathfrak{o}(g_2^1)\otimes\mathfrak{t}(g_2^1) & \stackrel{d^1}{\mapsto} & v\otimes y - y\otimes v\\ & \stackrel{f}{\mapsto} & f(v\otimes_v v)y - yf(v\otimes_v v)\\ & = & c'_1x + c'_2yx - c'_1x - c'_2xy\\ & = & 0.\end{array}$$

Hence $\operatorname{Im}\delta^0 = 0$ and $\dim\operatorname{Im}\delta^0 = 0$. Thus $\operatorname{HH}^1(\Lambda) = \operatorname{Hom}(Q^1, \Lambda)$, if char $K = 2$, and if char $K \neq 2$, then $\operatorname{HH}^1(\Lambda) = \{f \in \operatorname{Hom}(Q^1, \Lambda) :$

$$f(v\otimes_x v) = c_2x + c_4xy,$$
$$f(v\otimes_y v) = c_7y + c_8xy\}.$$

Hence $\dim\operatorname{HH}^1(\Lambda) = 8$, if char $K = 2$, and $\dim\operatorname{HH}^1(\Lambda) = 4$, if char $K \neq 2$.

5.2.2. $\operatorname{HH}^1(\tilde{\Lambda})$. To find $\operatorname{HH}^1(\tilde{\Lambda})$, we need to find $\operatorname{Ker}\tilde{\delta}^1$ and $\operatorname{Im}\tilde{\delta}^0$ where $\tilde{\delta}^1 : \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^2, \Lambda)$ and $\tilde{\delta}^0 : \operatorname{Hom}(\tilde{Q}^0, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda})$. We have $\operatorname{Ker}\tilde{\delta}^1 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda}) : \tilde{\delta}^1(\tilde{f}) = 0\}$ and let $\tilde{f}$ in $\operatorname{Ker}\tilde{\delta}^1$, then $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda})$.

So $\tilde{f} : \tilde{Q}^1 \longrightarrow \tilde{\Lambda}$ is given by

$$\tilde{f}(e_2 \otimes_{\alpha_1} e_1) \mapsto r_1\alpha_1 + r_2\alpha_1\alpha_2\alpha_1 + r_3\alpha_3\alpha_4\alpha_1 + r_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$$
$$\tilde{f}(e_1 \otimes_{\alpha_2} e_2) \mapsto r_5\alpha_2 + r_6\alpha_2\alpha_1\alpha_2 + r_7\alpha_2\alpha_3\alpha_4 + r_8\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2$$
$$\tilde{f}(e_2 \otimes_{\alpha_3} e_3) \mapsto r_9\alpha_3 + r_{10}\alpha_3\alpha_4\alpha_3 + r_{11}\alpha_1\alpha_2\alpha_3 + r_{12}\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3$$
$$\tilde{f}(e_3 \otimes_{\alpha_4} e_2) \mapsto r_{13}\alpha_4 + r_{14}\alpha_4\alpha_3\alpha_4 + r_{15}\alpha_4\alpha_1\alpha_2 + r_{16}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4,$$

where r_i in K. Since $\tilde{f} \circ \tilde{d}^2 = 0$, then we have

$$\begin{aligned}
\mathfrak{o}(\tilde{g}_1^2) \otimes \mathfrak{t}(\tilde{g}_1^2) &\overset{\tilde{d}^2}{\mapsto} (e_2 \otimes \alpha_2\alpha_1\alpha_2 + \alpha_1\alpha_2 \otimes \alpha_2, \alpha_1 \otimes \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_1 \otimes e_2, 0, 0) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2\alpha_1\alpha_2 + \alpha_1\alpha_2\tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2 + \alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2)\alpha_1\alpha_2 \\
&\quad +\alpha_1\alpha_2\alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2) \\
&= 0. \\
\mathfrak{o}(\tilde{g}_2^2) \otimes \mathfrak{t}(\tilde{g}_2^2) &\overset{\tilde{d}^2}{\mapsto} (e_2 \otimes \alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes \alpha_2, \alpha_1 \otimes \alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1 \otimes e_2, \alpha_1\alpha_2 \otimes \alpha_4 \\
&\quad -e_2 \otimes \alpha_4\alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3 \otimes e_2 - \alpha_3 \otimes \alpha_1\alpha_2) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2 + \alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2)\alpha_3\alpha_4 \\
&\quad -\alpha_3\alpha_4\alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2) + \alpha_1\alpha_2\tilde{f}(e_2 \otimes_{\alpha_3} e_3)\alpha_4 - \tilde{f}(e_2 \otimes_{\alpha_3} e_3) \\
&\quad \alpha_4\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3\tilde{f}(e_3 \otimes_{\alpha_4} e_2) - \alpha_3\tilde{f}(e_3 \otimes_{\alpha_4} e_2)\alpha_1\alpha_2 \\
&= 0. \\
\mathfrak{o}(\tilde{g}_3^2) \otimes \mathfrak{t}(\tilde{g}_3^2) &\overset{\tilde{d}^2}{\mapsto} (0, 0, e_2 \otimes \alpha_4\alpha_3\alpha_4 + \alpha_3\alpha_4 \otimes \alpha_4, \alpha_3 \otimes \alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_3 \otimes e_2) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_{\alpha_3} e_3)\alpha_4\alpha_3\alpha_4 + \alpha_3\alpha_4\tilde{f}(e_2 \otimes_{\alpha_3} e_3)\alpha_4 \\
&\quad +\alpha_3\tilde{f}(e_3 \otimes_{\alpha_4} e_2)\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_3\tilde{f}(e_3 \otimes_{\alpha_4} e_2) \\
&= 0.
\end{aligned}$$

Hence $\operatorname{Ker}\tilde{\delta}^1 = \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda})$ and $\dim \operatorname{Ker}\tilde{\delta}^1 = 16$.

Now we want to find $\operatorname{Im}\tilde{\delta}^0$. Since $\operatorname{Im}\tilde{\delta}^0 = \{\tilde{\delta}^0(\tilde{f}) : \tilde{f} \in \operatorname{Hom}(\tilde{Q}^0, \tilde{\Lambda})\}$. Let $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^0, \tilde{\Lambda})$, then $\tilde{f} : \tilde{Q}^0 \longrightarrow \tilde{\Lambda}$ is given by

$$\tilde{f}(e_1 \otimes_{e_1} e_1) = d_1e_1 + d_2\alpha_2\alpha_1 + d_3\alpha_2\alpha_1\alpha_2\alpha_1 + d_4\alpha_2\alpha_3\alpha_4\alpha_1 + d_5\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\alpha_1$$
$$\tilde{f}(e_2 \otimes_{e_2} e_2) = d_6e_2 + d_7\alpha_1\alpha_2 + d_8\alpha_3\alpha_4 + d_9\alpha_1\alpha_2\alpha_3\alpha_4$$
$$\tilde{f}(e_3 \otimes_{e_3} e_3) = d_{10}e_3 + d_{11}\alpha_4\alpha_3 + d_{12}\alpha_4\alpha_3\alpha_4\alpha_3 + d_{13}\alpha_4\alpha_1\alpha_2\alpha_3 + d_{14}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3$$

where $d_i \in K$. So we have

$$\begin{aligned}
\mathfrak{o}(\tilde{g}_1^1) \otimes \mathfrak{t}(\tilde{g}_1^1) &\overset{\tilde{d}^1}{\mapsto} (-\alpha_1 \otimes e_1, e_2 \otimes \alpha_1, 0) \\
&\overset{\tilde{f}}{\mapsto} -\alpha_1\tilde{f}(e_1 \otimes_{e_1} e_1) + \tilde{f}(e_2 \otimes_{e_2} e_2)\alpha_1 \\
&= -d_1\alpha_1 - d_2\alpha_1\alpha_2\alpha_1 - d_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1 + d_6\alpha_1 + d_7\alpha_1\alpha_2\alpha_1 \\
&\quad +d_8\alpha_3\alpha_4\alpha_1 + d_9\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1
\end{aligned}$$

$$
\begin{aligned}
&= (-d_1+d_6)\alpha_1+(-d_2+d_7)\alpha_1\alpha_2\alpha_1+(-d_4+d_9)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\\
&\quad +d_8\alpha_3\alpha_4\alpha_1.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{o}(\tilde g_2^1)\otimes\mathfrak{t}(\tilde g_2^1) &\overset{\tilde d^1}{\mapsto} (e_1\otimes\alpha_2,-\alpha_2\otimes e_2,0)\\
&\overset{\tilde f}{\mapsto} \tilde f(e_1\otimes_{e_1}e_1)\alpha_2-\alpha_2\tilde f(e_2\otimes_{e_2}e_2)\\
&= d_1\alpha_2+d_2\alpha_2\alpha_1\alpha_2+d_4\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2-(d_6\alpha_2+d_7\alpha_2\alpha_1\alpha_2+d_8\alpha_2\alpha_3\alpha_4\\
&\quad +d_9\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4)\\
&= (d_1-d_6)\alpha_2+(d_2-d_7)\alpha_2\alpha_1\alpha_2+(d_4-d_9)\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2\\
&\quad +d_8\alpha_2\alpha_3\alpha_4.\\
\mathfrak{o}(\tilde g_3^1)\otimes\mathfrak{t}(\tilde g_3^1) &\overset{\tilde d^1}{\mapsto} (0,e_2\otimes\alpha_3,-\alpha_3\otimes e_3)\\
&\overset{\tilde f}{\mapsto} \tilde f(e_2\otimes_{e_2}e_2)\alpha_3-\alpha_3\tilde f(e_3\otimes_{e_3}e_3)\\
&= d_6\alpha_3+d_7\alpha_1\alpha_2\alpha_3+d_8\alpha_3\alpha_4\alpha_3+d_9\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3-(d_{10}\alpha_3\\
&\quad +d_{11}\alpha_3\alpha_4\alpha_3+d_{13}\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3)\\
&= (d_6-d_{10})\alpha_3+d_7\alpha_1\alpha_2\alpha_3+(d_8-d_{11})\alpha_3\alpha_4\alpha_3+(d_9-d_{13})\alpha_1\alpha_2\alpha_3\\
&\quad \alpha_4\alpha_3.\\
\mathfrak{o}(\tilde g_4^1)\otimes\mathfrak{t}(\tilde g_4^1) &\overset{\tilde d^1}{\mapsto} (0,-\alpha_4\otimes e_2,e_3\otimes\alpha_4)\\
&\overset{\tilde f}{\mapsto} -\alpha_4\tilde f(e_2\otimes_{e_2}e_2)+\tilde f(e_3\otimes_{e_3}e_3)\alpha_4\\
&= -(d_6\alpha_4+d_7\alpha_4\alpha_1\alpha_2+d_8\alpha_4\alpha_3\alpha_4+d_9\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4)+d_{10}\alpha_4\\
&\quad +d_{11}\alpha_4\alpha_3\alpha_4+d_{13}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\\
&= (-d_6+d_{10})\alpha_4-d_7\alpha_4\alpha_1\alpha_2+(-d_8+d_{11})\alpha_4\alpha_3\alpha_4+(-d_9+d_{13})\\
&\quad \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\operatorname{Im}\tilde\delta^0 = \{\tilde\delta^0(\tilde f)\in \operatorname{Hom}(\tilde Q^1,\tilde\Lambda):&\\
\tilde\delta^0(\tilde f)(\mathfrak{o}(\tilde g_1^1)\otimes_{\alpha_1}\mathfrak{t}(\tilde g_1^1)) &= (-d_1+d_6)\alpha_1+(-d_2+d_7)\alpha_1\alpha_2\alpha_1\\
&\quad +(-d_4+d_9)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1+d_8\alpha_3\alpha_4\alpha_1,\\
\tilde\delta^0(\tilde f)(\mathfrak{o}(\tilde g_2^1)\otimes_{\alpha_2}\mathfrak{t}(\tilde g_2^1)) &= (d_1-d_6)\alpha_2+(d_2-d_7)\alpha_2\alpha_1\alpha_2\\
&\quad +(d_4-d_9)\alpha_2\alpha_3\alpha_4\alpha_1\alpha_1+d_8\alpha_2\alpha_3\alpha_4,\\
\tilde\delta^0(\tilde f)(\mathfrak{o}(\tilde g_3^1)\otimes_{\alpha_3}\mathfrak{t}(\tilde g_3^1)) &= (d_6-d_{10})\alpha_3+d_7\alpha_1\alpha_2\alpha_3+(d_8-d_{11})\alpha_3\alpha_4\alpha_3\\
&\quad +(d_9-d_{13})\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3,\\
\tilde\delta^0(\tilde f)(\mathfrak{o}(\tilde g_4^1)\otimes_{\alpha_4}\mathfrak{t}(\tilde g_4^1)) &= (-d_6+d_{10})\alpha_4-d_7\alpha_4\alpha_1\alpha_2+(-d_8+d_{11})\alpha_4\alpha_3\alpha_4\\
&\quad +(-d_9+d_{13})\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\}.
\end{aligned}
$$

So $\dim\operatorname{Im}\tilde\delta^0=8$ and hence $\dim \mathrm{HH}^1(\tilde\Lambda)=8$.

5.3. HH^2. We now compute HH^2 explicitly for Example 5.1 and Example 5.2.

5.3.1. $\mathrm{HH}^2(\Lambda)$. To find $\mathrm{HH}^2(\Lambda)$ for Example 5.1, we compute $\operatorname{Ker}\delta^2$, then $\operatorname{Im}\delta^1$, where $\delta^2 : \operatorname{Hom}(Q^2,\Lambda) \longrightarrow \operatorname{Hom}(Q^3,\Lambda)$ and $\delta^1 : \operatorname{Hom}(Q^1,\Lambda) \longrightarrow \operatorname{Hom}(Q^2,\Lambda)$. We know $\operatorname{Ker}\delta^2 = \{f \in \operatorname{Hom}(Q^2,\Lambda), \delta^2(f) = 0\}$. Let f in $\operatorname{Ker}\delta^2$, then $f \in \operatorname{Hom}(Q^2,\Lambda)$. So $f : Q^2 \longrightarrow \Lambda$ is given by

$$v \otimes_1 v \mapsto c_1 v + c_2 x + c_3 y + c_4 xy$$

$$v \otimes_2 v \mapsto c_5 v + c_6 x + c_7 y + c_8 xy$$

$$v \otimes_3 v \mapsto c_9 v + c_{10} x + c_{11} y + c_{12} xy$$

where $c_i \in K$. Since $\delta^2(f) = 0$, then

$$\begin{aligned}
\mathfrak{o}(g_1^3) \otimes \mathfrak{t}(g_1^3) &\overset{d^3}{\mapsto} (\mathfrak{o}(g_1^2) \otimes_1 x - x \otimes_1 \mathfrak{t}(g_1^2), 0, 0)\\
&\overset{f}{\mapsto} f(v \otimes_1 v)x - xf(v \otimes_1 v)\\
&= c_1 x + c_3 yx - c_1 x - c_3 xy\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{o}(g_2^3) \otimes \mathfrak{t}(g_2^3) &\overset{d^3}{\mapsto} (\mathfrak{o}(g_1^2) \otimes_1 y - y \otimes_1 \mathfrak{t}(g_1^2), -\mathfrak{o}(g_2^2) \otimes_2 x - x \otimes_2 \mathfrak{t}(g_2^2), 0)\\
&\overset{f}{\mapsto} f(v \otimes_1 v)y - yf(v \otimes_1 v) - f(v \otimes_2 v)x - xf(v \otimes_2 v)\\
&= c_1 y + c_2 xy - c_1 x - c_2 yx - c_5 x - c_7 yx - c_5 x - c_7 xy\\
&= -2c_5 x - 2c_7 xy\\
&= 0.
\end{aligned}$$

Now we need to consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $c_5 = c_7 = 0$.

$$\begin{aligned}
\mathfrak{o}(g_3^3) \otimes \mathfrak{t}(g_3^3) &\overset{d^3}{\mapsto} (0, \mathfrak{o}(g_2^2) \otimes_2 y + y \otimes_2 \mathfrak{t}(g_2^2), \mathfrak{o}(g_3^2) \otimes_3 x - x \otimes_3 \mathfrak{t}(g_3^2))\\
&\overset{f}{\mapsto} f(v \otimes_2 v)y + yf(v \otimes_2 v) + f(v \otimes_3 v)x - xf(v \otimes_3 v)\\
&= 2c_5 x + 2c_6 xy\\
&= 0.
\end{aligned}$$

Now we need to consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $c_5 = c_6 = 0$.

$$\begin{aligned}
\mathfrak{o}(g_4^3) \otimes \mathfrak{t}(g_4^3) &\overset{d^3}{\mapsto} (0, 0, \mathfrak{o}(g_3^2) \otimes_3 y - y \otimes_3 \mathfrak{t}(g_3^2))\\
&\overset{f}{\mapsto} f(v \otimes_3 v)y - yf(v \otimes_3 v)\\
&= c_9 y + c_{10} xy - c_9 y - c_{10} xy\\
&= 0.
\end{aligned}$$

Hence we have two cases. If $\operatorname{char} K = 2$, then $\operatorname{Ker}\delta^2 = \operatorname{Hom}(Q^2,\Lambda)$.

If $\operatorname{char} K \neq 2$, then $\operatorname{Ker} \delta^2 = \{f \in \operatorname{Hom}(Q^2, \Lambda) :$

$$\begin{aligned}
&f(v \otimes_1 v) = c_1 v + c_2 x + c_3 y + c_4 xy,\\
&f(v \otimes_2 v) = c_8 xy,\\
&f(v \otimes_3 v) = c_9 v + c_{10} x + c_{11} y + c_{12} xy\}.
\end{aligned}$$

We have two cases. If $\operatorname{char} K = 2$, then $\dim \operatorname{Ker} \delta^2 = 12$ and if $\operatorname{char} K \neq 2$, then $\dim \operatorname{Ker} \delta^2 = 9$.

Now we need to find $\operatorname{Im} \delta^1$. We know $\operatorname{Im} \delta^1 = \{\delta^1(f), f \in \operatorname{Hom}(Q^1, \Lambda)\}$. So for each $f \in \operatorname{Hom}(Q^1, \Lambda)$, then we have $f : Q^1 \longrightarrow \Lambda$ is given by

$$\begin{aligned}
&v \otimes_x v \mapsto c'_1 v + c'_2 x + c'_3 y + c'_4 xy\\
&v \otimes_y v \mapsto c'_5 v + c'_6 x + c'_7 y + c'_8 xy.
\end{aligned}$$

where $c'_i \in K$.

Now we will find $\delta^1(f)$,

$$\begin{aligned}
\mathfrak{o}(g_1^2) \otimes \mathfrak{t}(g_1^2) &\overset{d^2}{\mapsto} (v \otimes x + x \otimes v, 0)\\
&\overset{f}{\mapsto} f(v \otimes_x v)x + xf(v \otimes_x v)\\
&= 2c'_1 x + 2c'_3 xy.\\
\mathfrak{o}(g_2^2) \otimes \mathfrak{t}(g_2^2) &\overset{d^2}{\mapsto} (v \otimes y - y \otimes v, x \otimes v - v \otimes x)\\
&\overset{f}{\mapsto} f(v \otimes_x v)y - yf(v \otimes_x v) + xf(v \otimes_y v) - f(v \otimes_y v)x\\
&= c'_1 y + c'_2 xy - c'_1 y - c'_2 xy + c'_5 x + c'_7 xy - c'_5 x - c'_7 yx\\
&= 0.\\
\mathfrak{o}(g_3^2) \otimes \mathfrak{t}(g_3^2) &\overset{d^2}{\mapsto} (0, v \otimes y + y \otimes v)\\
&\overset{f}{\mapsto} f(v \otimes_y v)y + yf(v \otimes_y v)\\
&= 2c'_5 x + 2c'_6 xy.
\end{aligned}$$

Now we need to consider two cases. If $\operatorname{char} K = 2$, then $\operatorname{Im} \delta^1 = 0$.

If $\operatorname{char} K \neq 2$, then $\operatorname{Im} \delta^1 = \{f \in \operatorname{Hom}(Q^2, \Lambda) :$

$$\begin{aligned}
&f(v \otimes_1 v) = 2c'_1 x + 2c'_3 xy,\\
&f(v \otimes_2 v) = 0,\\
&f(v \otimes_3 v) = 2c'_5 x + 2c'_6 xy\}.
\end{aligned}$$

Hence $\dim \operatorname{Im} \delta^1 = 0$, if $\operatorname{char} K = 2$ and $\operatorname{Im} \delta^1 = 4$, if $\operatorname{char} K \neq 2$.

Thus $\dim \operatorname{HH}^2(\Lambda) = 12$, if $\operatorname{char} K = 2$ and $\dim \operatorname{HH}^2(\Lambda) = 5$, if $\operatorname{char} K \neq 2$.

Hence we have two cases. If $\operatorname{char} K = 2$, then $\operatorname{HH}^2(\Lambda) = \operatorname{Hom}(Q^2, \Lambda)$.

If $\operatorname{char} K \neq 2$, then $\mathrm{HH}^2(\Lambda) = \{f \in \operatorname{Hom}(Q^2, \Lambda) :$

$$\begin{aligned} f(v \otimes_1 v) &= c_1 v + c_3 y, \\ f(v \otimes_2 v) &= c_8 xy, \\ f(v \otimes_3 v) &= c_9 v + c_{11} y\}. \end{aligned}$$

5.3.2. $\mathrm{HH}^2(\tilde{\Lambda})$. To find $\mathrm{HH}^2(\Lambda)$ for Example 5.2, we compute $\operatorname{Ker} \tilde{\delta}^2$, then $\operatorname{Im} \tilde{\delta}^1$ where $\tilde{\delta}^2 : \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda})$ and $\tilde{\delta}^1 : \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})$. We know $\operatorname{Ker} \tilde{\delta}^2 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) : \tilde{\delta}^2(\tilde{f}) = 0\}$. We have $\operatorname{Ker} \tilde{\delta}^2 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}), \tilde{\delta}^2(\tilde{f}) = 0\}$. Let $\tilde{f}$ in $\operatorname{Ker} \tilde{\delta}^2$, then $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})$. So $\tilde{f} : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ is given by

$$\begin{aligned} e_2 \otimes_1 e_2 &\mapsto k_1 e_2 + k_2 \alpha_1\alpha_2 + k_3 \alpha_3\alpha_4 + k_4 \alpha_1\alpha_2\alpha_3\alpha_4 \\ e_2 \otimes_2 e_2 &\mapsto k_5 e_2 + k_6 \alpha_1\alpha_2 + k_7 \alpha_3\alpha_4 + k_8 \alpha_1\alpha_2\alpha_3\alpha_4 \\ e_2 \otimes_3 e_2 &\mapsto k_9 e_2 + k_{10} \alpha_1\alpha_2 + k_{11} \alpha_3\alpha_4 + k_{12} \alpha_1\alpha_2\alpha_3\alpha_4 \end{aligned}$$

where $k_i \in K$. We have $\tilde{\delta}^2(\tilde{f}) = 0$, then

$$\begin{aligned} \mathfrak{o}(\tilde{g}_1^3) \otimes \mathfrak{t}\tilde{g}_1^3) &\overset{\tilde{d}^3}{\mapsto} (\mathfrak{o}(\tilde{g}_1^2) \otimes_1 \alpha_1\alpha_2 - \alpha_1\alpha_2 \otimes_1 \mathfrak{t}(\tilde{g}_1^2), 0, 0) \\ &\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_1\alpha_2 - \alpha_1\alpha_2 \tilde{f}(e_2 \otimes_1 e_2) \\ &= k_1\alpha_1\alpha_2 + k_3\alpha_3\alpha_4\alpha_1\alpha_2 - k_1\alpha_1\alpha_2 - k_3\alpha_1\alpha_2\alpha_3\alpha_4 \\ &= 0. \\ \mathfrak{o}(\tilde{g}_2^3) \otimes \mathfrak{t}(\tilde{g}_2^3) &\overset{\tilde{d}^3}{\mapsto} (\mathfrak{o}(\tilde{g}_1^2) \otimes_1 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_1 \mathfrak{t}(\tilde{g}_1^2), -\mathfrak{o}(\tilde{g}_2^2) \otimes_2 \alpha_1\alpha_2 \\ &\qquad -\alpha_1\alpha_2 \otimes_2 \mathfrak{t}(\tilde{g}_2^2), 0) \\ &\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_3\alpha_4 - \alpha_3\alpha_4 \tilde{f}(e_2 \otimes_1 e_2) - \tilde{f}(e_2 \otimes_2 e_2)\alpha_1\alpha_2 \\ &\quad -\alpha_1\alpha_2 \tilde{f}(e_2 \otimes_2 e_2) \\ &= k_1\alpha_3\alpha_4 + k_2\alpha_1\alpha_2\alpha_3\alpha_4 - k_1\alpha_3\alpha_4 - k_2\alpha_3\alpha_4\alpha_1\alpha_2 - k_5\alpha_1\alpha_2 \\ &\quad -k_7\alpha_3\alpha_4\alpha_1\alpha_2 - k_5\alpha_1\alpha_2 - k_7\alpha_1\alpha_2\alpha_3\alpha_4 \\ &= -2k_5 x - 2k_7 xy \\ &= 0. \end{aligned}$$

Now we need to consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $k_5 = k_7 = 0$.

$$\begin{aligned}
\mathfrak{o}(\tilde{g}_3^3) \otimes \mathfrak{t}(\tilde{g}_3^3) &\overset{\tilde{d}^3}{\mapsto} (0, \mathfrak{o}(\tilde{g}_2^2) \otimes_2 \alpha_3\alpha_4 + \alpha_3\alpha_4 \otimes_2 \mathfrak{t}(\tilde{g}_2^2), \mathfrak{o}(\tilde{g}_3^2) \otimes_3 \alpha_1\alpha_2 - \alpha_1\alpha_2 \otimes_3 \mathfrak{t}(\tilde{g}_3^2)) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_2 e_2)\alpha_3\alpha_4 + \alpha_3\alpha_4\tilde{f}(e_2 \otimes_2 e_2) + \tilde{f}(e_2 \otimes_3 e_2)\alpha_1\alpha_2 \\
&\quad -\alpha_1\alpha_2\tilde{f}(e_2 \otimes_3 e_2) \\
&= k_5\alpha_3\alpha_4 + k_6\alpha_1\alpha_2\alpha_3\alpha_4 + k_5\alpha_3\alpha_4 + k_6\alpha_3\alpha_4\alpha_1\alpha_2 \\
&\quad +k_9\alpha_1\alpha_2 + k_{11}\alpha_3\alpha_4\alpha_1\alpha_2 - k_9\alpha_1\alpha_2 - k_{11}\alpha_1\alpha_2\alpha_3\alpha_4 \\
&= 2k_5\alpha_3\alpha_4 + 2k_6\alpha_1\alpha_2\alpha_3\alpha_4 \\
&= 0.
\end{aligned}$$

Now we need to consider two case. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $k_5 = k_6 = 0$.

$$\begin{aligned}
\mathfrak{o}(\tilde{g}_4^3) \otimes \mathfrak{t}(\tilde{g}_4^3) &\overset{\tilde{d}^3}{\mapsto} (0, 0, \mathfrak{o}(\tilde{g}_3^2) \otimes_3 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_3 \mathfrak{t}(\tilde{g}_3^2)) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_3 e_2)\alpha_3\alpha_4 - \alpha_3\alpha\tilde{f}(e_2 \otimes_3 e_2) \\
&= k_9\alpha_3\alpha_4 + k_{10}\alpha_1\alpha_2\alpha_3\alpha_4 - k_9\alpha_3\alpha_4 - k_{10}\alpha_3\alpha_4\alpha_1\alpha_2 \\
&= 0.
\end{aligned}$$

Hence we have two cases. If $\operatorname{char} K = 2$, then $\operatorname{Ker} \tilde{\delta}^2 = \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})$.
If $\operatorname{char} K \neq 2$, then $\operatorname{Ker} \tilde{\delta}^2 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) :$

$$\begin{aligned}
&\tilde{f}(e_2 \otimes_1 e_2) = k_1e_2 + k_2\alpha_1\alpha_2 + k_3\alpha_3\alpha_4 + k_4\alpha_1\alpha_2\alpha_3\alpha_4, \\
&\tilde{f}(e_2 \otimes_{e_2} e_2) = k_8\alpha_1\alpha_2\alpha_3\alpha_4, \\
&\tilde{f}(e_2 \otimes_3 e_2) = k_9e_2 + k_{10}\alpha_1\alpha_2 + k_{11}\alpha_3\alpha_4 + k_{12}\alpha_1\alpha_2\alpha_3\alpha_4\}.
\end{aligned}$$

So we have if $\operatorname{char} K = 2$, then $\dim \operatorname{Ker} \tilde{\delta}^2 = 12$, and if $\operatorname{char} K \neq 2$, then $\dim \operatorname{Ker} \tilde{\delta}^2 = 9$.

We now find $\operatorname{Im} \tilde{\delta}^1$. We have $\operatorname{Im} \tilde{\delta}^1 = \{\tilde{\delta}^1(\tilde{f}), \tilde{f} \in \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda})\}$.
Let $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^1, \tilde{\Lambda})$, then $\tilde{f} : \tilde{Q}^1 \longrightarrow \tilde{\Lambda}$ is given by

$$\begin{aligned}
&e_2 \otimes_{\alpha_1} e_1 \mapsto r_1\alpha_1 + r_2\alpha_1\alpha_2\alpha_1 + r_3\alpha_3\alpha_4\alpha_1 + r_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1 \\
&e_1 \otimes_{\alpha_2} e_2 \mapsto r_5\alpha_2 + r_6\alpha_2\alpha_1\alpha_2 + r_7\alpha_2\alpha_3\alpha_4 + r_8\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4. \\
&e_2 \otimes_{\alpha_3} e_3 \mapsto r_9\alpha_3 + r_{10}\alpha_3\alpha_4\alpha_3 + r_{11}\alpha_1\alpha_2\alpha_3 + r_{12}\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3 \\
&e_3 \otimes_{\alpha_4} e_2 \mapsto r_{13}\alpha_4 + r_{14}\alpha_4\alpha_3\alpha_4 + r_{15}\alpha_4\alpha_1\alpha_2 + r_{16}\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4,
\end{aligned}$$

where $r_i \in K$.

Now we find $\tilde{\delta}^1(\tilde{f})$,

$$\begin{aligned}
\mathfrak{o}(\tilde{g}_1^2) \otimes \mathfrak{t}(\tilde{g}_1^2) &\overset{\tilde{d}^2}{\mapsto} (e_2 \otimes \alpha_2\alpha_1\alpha_2 + \alpha_1\alpha_2 \otimes \alpha_2, \alpha_1 \otimes \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_1 \otimes e_2, 0) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2\alpha_1\alpha_2 + \alpha_1\alpha_2\tilde{f}(e_2 \otimes_{\alpha_1} e_1)\alpha_2 + \alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2) \\
&\quad \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_1\tilde{f}(e_1 \otimes_{\alpha_2} e_2) \\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}_2^2)\otimes\mathfrak{t}(\tilde{g}_2^2) &\stackrel{\tilde{d}^2}{\mapsto} (e_2\otimes\alpha_2\alpha_3\alpha_4-\alpha_3\alpha_4\otimes\alpha_2, \alpha_1\otimes\alpha_3\alpha_4-\alpha_3\alpha_4\alpha_1\otimes e_2,\\
&\quad\ \alpha_1\alpha_2\otimes\alpha_4-e_2\otimes\alpha_4\alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3\otimes e_2-\alpha_3\otimes\alpha_1\alpha_2)\\
&\stackrel{\tilde{f}}{\mapsto} \tilde{f}(e_2\otimes_{\alpha_1}e_1)\alpha_2\alpha_3\alpha_4-\alpha_3\alpha_4\tilde{f}(e_2\otimes_{\alpha_1}e_1)\alpha_2+\alpha_1\tilde{f}(e_1\otimes_{\alpha_2}e_2)\alpha_3\alpha_4\\
&\quad\ -\alpha_3\alpha_4\alpha_1\tilde{f}(e_1\otimes_{\alpha_2}e_2)+\alpha_1\alpha_2\tilde{f}(e_2\otimes_{\alpha_3}e_3)\alpha_4-\tilde{f}(e_2\otimes_{\alpha_3}e_3)\alpha_4\alpha_1\alpha_2\\
&\quad\ +\alpha_1\alpha_2\alpha_3\tilde{f}(e_3\otimes_{\alpha_4}e_2)-\alpha_3\tilde{f}(e_3\otimes_{\alpha_4}e_2)\alpha_1\alpha_2\\
&= r_1\alpha_1\alpha_2\alpha_3\alpha_4-r_1\alpha_3\alpha_4\alpha_1\alpha_2+r_5\alpha_1\alpha_2\alpha_3\alpha_4-r_5\alpha_3\alpha_4\alpha_1\alpha_2\\
&\quad\ +r_9\alpha_1\alpha_2\alpha_3\alpha_4-r_9\alpha_3\alpha_4\alpha_1\alpha_2+r_{13}\alpha_1\alpha_2\alpha_3\alpha_4-r_{13}\alpha_3\alpha_4\alpha_1\alpha_2\\
&= 0.\\
\mathfrak{o}(\tilde{g}_3^2)\otimes\mathfrak{t}(\tilde{g}_3^2) &\stackrel{\tilde{d}^2}{\mapsto} (0,0,e_2\otimes\alpha_3\alpha_4\alpha_3+\alpha_3\alpha_4\otimes\alpha_4, \alpha_3\otimes\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_3\otimes e_2,\\
&\stackrel{\tilde{f}}{\mapsto} \tilde{f}(e_2\otimes_{\alpha_3}e_3)\alpha_4\alpha_3\alpha_4+\alpha_3\alpha_4\tilde{f}(e_2\otimes_{\alpha_3}e_3)\alpha_4+\alpha_3\tilde{f}(e_3\otimes_{\alpha_4}e_2)\alpha_3\alpha_4\\
&\quad\ +\alpha_3\alpha_4\alpha_3\tilde{f}(e_3\otimes_{\alpha_4}e_2)\\
&= 0.
\end{aligned}
$$

Hence $\operatorname{Im}\tilde{\delta}^1 = 0$ and $\dim\operatorname{Im}\tilde{\delta}^1 = 0$. Thus we have two cases. If $\operatorname{char} K = 2$, then $\dim \mathrm{HH}^2(\tilde{\Lambda}) = 12$, and if $\operatorname{char} K \neq 2$, then $\dim \mathrm{HH}^2(\tilde{\Lambda}) = 9$.

Hence we have two cases. If $\operatorname{char} K = 2$, then $\mathrm{HH}^2(\tilde{\Lambda}) = \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})$.

If $\operatorname{char} K \neq 2$, then $\mathrm{HH}^2(\tilde{\Lambda}) = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) :$

$$
\begin{aligned}
&\tilde{f}(e_2\otimes_1 e_2) = k_1e_2+k_2\alpha_1\alpha_2+k_3\alpha_3\alpha_4+k_4\alpha_1\alpha_2\alpha_3\alpha_4,\\
&\tilde{f}(e_2\otimes_2 e_2) = k_8\alpha_1\alpha_2\alpha_3\alpha_4,\\
&\tilde{f}(e_2\otimes_3 e_2) = k_9e_2+k_{10}\alpha_1\alpha_2+k_{11}\alpha_3\alpha_4+k_{12}\alpha_1\alpha_2\alpha_3\alpha_4\}.
\end{aligned}
$$

5.3.3. *The relation between* $\mathrm{HH}^2(\Lambda)$ *and* $\mathrm{HH}^2(\tilde{\Lambda})$ *of Examples 5.1 and 5.2.* We now find the connection between $\mathrm{HH}^2(\Lambda)$ and $\mathrm{HH}^2(\tilde{\Lambda})$ of Examples 5.1 and 5.2.

For Example 5.1 we have the basis of $\mathrm{HH}^2(\Lambda) = \mathrm{sp}\{z_1, z_2, z_3, z_4, z_5\}$, where

$z_1 : Q^2 \longrightarrow \Lambda$ which is given by $(v\otimes_1 v) \mapsto v$, else$\mapsto 0$.

$z_2 : Q^2 \longrightarrow \Lambda$ which is given by $(v\otimes_1 v) \mapsto c_3y$, else$\mapsto 0$.

$z_3 : Q^2 \longrightarrow \Lambda$ which is given by $(v\otimes_2 v) \mapsto c_8xy$, else$\mapsto 0$.

$z_4 : Q^2 \longrightarrow \Lambda$ which is given by $(v\otimes_3 v) \mapsto v$, else$\mapsto 0$.

$z_5 : Q^2 \longrightarrow \Lambda$ which is given by $(v\otimes_3 v) \mapsto c_{11}y$, else$\mapsto 0$.

For Example 5.2 we have the basis of $\mathrm{HH}^2(\tilde{\Lambda}) = \mathrm{sp}\{\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4, \tilde{z}_5, \tilde{z}_6, \tilde{z}_7, \tilde{z}_8, \tilde{z}_9\}$, where

$\tilde{z}_1 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2\otimes_1 e_2) \mapsto e_2$, else$\mapsto 0$.

$\tilde{z}_2 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2\otimes_1 e_2) \mapsto k_2\alpha_1\alpha_2$, else$\mapsto 0$.

$\tilde{z}_3 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2\otimes_1 e_2) \mapsto k_3\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_4 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_1 e_2) \mapsto k_4\alpha_1\alpha_2\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_5 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_2 e_2) \mapsto k_8\alpha_1\alpha_2\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_6 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_3 e_2) \mapsto e_2$, else$\mapsto 0$.

$\tilde{z}_7 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_3 e_2) \mapsto k_{10}\alpha_1\alpha_2$, else$\mapsto 0$.

$\tilde{z}_8 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_3 e_2) \mapsto k_{11}\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_9 : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_3 e_2) \mapsto k_{12}\alpha_1\alpha_2\alpha_3\alpha_4$, else$\mapsto 0$.

Now we define the group homomorphism $\phi^2 : \mathrm{HH}^2(\Lambda) \longrightarrow \mathrm{HH}^2(\tilde{\Lambda})$ via

$$z_1 \mapsto \tilde{z}_1$$
$$z_2 \mapsto \tilde{z}_3$$
$$z_3 \mapsto \tilde{z}_4$$
$$z_4 \mapsto \tilde{z}_6$$
$$z_5 \mapsto \tilde{z}_8.$$

5.4. HH^3. We find $\mathrm{HH}^3(\Lambda)$ for Example 5.1, and $\mathrm{HH}^3(\tilde{\Lambda})$ for Example 5.2.

5.4.1. $\mathrm{HH}^3(\Lambda)$. To find $\mathrm{HH}^3(\Lambda)$ for Example 5.1, we firstly find $\operatorname{Ker}\delta^3$, then $\operatorname{Im}\delta^2$, where $\delta^3 : \mathrm{Hom}(Q^3, \Lambda) \longrightarrow \mathrm{Hom}(Q^4, \Lambda)$ and $\delta^2 : \mathrm{Hom}(Q^2, \Lambda) \longrightarrow \mathrm{Hom}(Q^3, \Lambda)$. We have $\operatorname{Ker}\delta^3 = \{f \in \mathrm{Hom}(Q^3, \Lambda) : \delta^3(f) = 0\}$. Let f in $\operatorname{Ker}\delta^3$, then $f \in \mathrm{Hom}(Q^3, \Lambda)$. So $f : Q^3 \longrightarrow \Lambda$ is given by

$$v \otimes_1 v \mapsto c_1 v + c_2 x + c_3 y + c_4 xy$$
$$v \otimes_2 v \mapsto c_5 v + c_6 x + c_7 y + c_8 xy$$
$$v \otimes_3 v \mapsto c_9 v + c_{10} x + c_{11} y + c_{12} xy$$
$$v \otimes_4 v \mapsto c_{13} v + c_{14} x + c_{15} y + c_{16} xy$$

where $c_i \in K$. Since $\delta^3(f) = 0$, then

$$\begin{aligned} \mathfrak{o}(g_1^4) \otimes \mathfrak{t}(g_1^4) &\overset{d^4}{\mapsto} (\mathfrak{o}(g_1^3) \otimes_1 x + x \otimes_1 \mathfrak{t}(g_1^3), 0, 0, 0) \\ &\overset{f}{\mapsto} f(v \otimes_1 v)x + xf(v \otimes_1 v) \\ &= 2c_1 x + 2c_3 xy \\ &= 0. \end{aligned}$$

We need to consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $c_1 = c_3 = 0$.

$$\begin{aligned} \mathfrak{o}(g_2^4) \otimes \mathfrak{t}(g_2^4) &\overset{d^4}{\mapsto} (\mathfrak{o}(g_1^3) \otimes_1 y - y \otimes_1 \mathfrak{t}(g_1^3), -\mathfrak{o}(g_2^3) \otimes_2 x + x \otimes_2 \mathfrak{t}(g_2^3), 0, 0) \\ &\overset{f}{\mapsto} f(v \otimes_1 v)y - yf(v \otimes_1 v) - f(v \otimes_2 v)x + xf(v \otimes_2 v) \\ &= 0. \end{aligned}$$

$$\begin{array}{rcl}
\mathfrak{o}(g^4_3) \otimes \mathfrak{t}(g^4_3) & \overset{d^4}{\mapsto} & (0, \mathfrak{o}(g^3_2) \otimes_2 y + y \otimes_2 \mathfrak{t}(g^3_2), \mathfrak{o}(g^3_3) \otimes_3 x + x \otimes_3 \mathfrak{t}(g^3_3), 0) \\
& \overset{f}{\mapsto} & f(v \otimes_2 v)y + yf(v \otimes_2 v) + f(v \otimes_3 v)x + xf(v \otimes_3 v) \\
& = & 2c_5 y + 2c_9 x + 2(c_6 + c_{11})xy \\
& = & 0.
\end{array}$$

So we consider two case. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $c_5 = c_9 = 0$ and $c_6 = c_{11}$.

$$\begin{array}{rcl}
\mathfrak{o}(g^4_4) \otimes \mathfrak{t}(g^4_4) & \overset{d^4}{\mapsto} & (0, 0, \mathfrak{o}(g^3_3) \otimes_3 y - y \otimes_3 \mathfrak{t}(g^3_3), -\mathfrak{o}(g^3_4) \otimes_4 x + x \otimes_4 \mathfrak{t}(g^3_4), 0) \\
& \overset{f}{\mapsto} & f(v \otimes_3 v)y - yf(v \otimes_3 v) - f(v \otimes_4 v)x + xf(v \otimes_4 v) \\
& = & 0. \\
\mathfrak{o}(g^4_5) \otimes \mathfrak{t}(g^4_5) & \overset{d^4}{\mapsto} & (0, 0, 0, \mathfrak{o}(g^3_4) \otimes_4 y + y \otimes_4 \mathfrak{t}(g^3_4)) \\
& \overset{f}{\mapsto} & f(v \otimes_4 v)y + yf(v \otimes_4 v) \\
& = & 2c_{13} y + 2c_{14} xy \\
& = & 0.
\end{array}$$

Then we have two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $c_{13} = c_{14} = 0$.

Hence we have if $\operatorname{char} K = 2$, then $\operatorname{Ker} \delta^3 = \operatorname{Hom}(Q^3, \Lambda)$. If $\operatorname{char} K \neq 2$, then

$$\begin{aligned}
\operatorname{Ker} \delta^3 = \{ f \in \operatorname{Hom}(Q^3, \Lambda) : & \\
& f(v \otimes_1 v) = c_2 x + c_4 xy, \\
& f(v \otimes_2 v) = c_6 x + c_7 y + c_8 xy, \\
& f(v \otimes_3 v) = c_{10} x + c_6 y + c_{12} xy, \\
& f(v \otimes_4 v) = c_{15} x + c_{16} xy \}.
\end{aligned}$$

So $\dim \operatorname{Ker} \delta^3 = 16$, if $\operatorname{char} K = 2$ and $\dim \operatorname{Ker} \delta^3 = 9$, if $\operatorname{char} K \neq 2$.

We now find $\operatorname{Im} \delta^2$. Since $\operatorname{Im} \delta^2 = \{\delta^2(f), f \in \operatorname{Hom}(Q^2, \Lambda)\}$ and let $f \in \operatorname{Hom}(Q^2, \Lambda)$, then $f : Q^2 \longrightarrow \Lambda$ is given by

$$\begin{aligned}
& v \otimes_1 v \mapsto c'_1 v + c'_2 x + c'_3 y + c'_4 xy \\
& v \otimes_2 v \mapsto c'_5 v + c'_6 x + c'_7 y + c'_8 xy \\
& v \otimes_3 v \mapsto c'_9 v + c'_{10} x + c'_{11} y + c'_{12} xy,
\end{aligned}$$

where $c'_i \in K$. So we have,

$$\begin{array}{rcl}
\mathfrak{o}(g^3_1) \otimes \mathfrak{t}(g^3_1) & \overset{d^3}{\mapsto} & (v \otimes x - x \otimes v, 0, 0) \\
& \overset{f}{\mapsto} & f(v \otimes_1 v)x - xf(v \otimes_1 v) \\
& = & c'_1 x + c'_3 xy - c'_1 x - c'_3 xy \\
& = & 0.
\end{array}$$

$$\begin{array}{rcl}
\mathfrak{o}(g_2^3) \otimes \mathfrak{t}(g_2^3) & \stackrel{d^3}{\mapsto} & (v \otimes_1 y - y \otimes_1 v, -v \otimes_2 x - x \otimes_2 v, 0) \\
& \stackrel{f}{\mapsto} & f(v \otimes_1 v)y - yf(v \otimes_1 v) - f(v \otimes_2 v)x - xf(v \otimes_2 v) \\
& = & c'_1 y + c'_2 xy - c'_1 y - c'_2 xy - c'_5 x - c'_7 xy - c'_5 x - c'_7 yx \\
& = & -2c'_5 x - 2c'_7 xy. \\
\mathfrak{o}(g_3^3) \otimes \mathfrak{t}(g_3^3) & \stackrel{d^3}{\mapsto} & (0, v \otimes_2 y + y \otimes_2 v, v \otimes_3 x - x \otimes_3 v) \\
& \stackrel{f}{\mapsto} & f(v \otimes_2 v)y + yf(v \otimes_2 v) + f(v \otimes_3 v)x - xf(v \otimes_3 v) \\
& = & 2c'_5 x + 2c'_6 xy. \\
\mathfrak{o}(g_4^3) \otimes \mathfrak{t}(g_4^3) & \stackrel{d^3}{\mapsto} & (0, 0, v \otimes_3 y - y \otimes_3 v) \\
& \stackrel{f}{\mapsto} & f(v \otimes_3 v)y - yf(v \otimes_3 v) \\
& = & c'_9 x + c'_{10} xy - c'_9 x - c'_{10} xy \\
& = & 0.
\end{array}$$

Considering two cases, we have if $\operatorname{char} K = 2$, then $\operatorname{Im} \delta^2 = 0$.

If $\operatorname{char} K \neq 2$, then $\operatorname{Im} \delta^2 = \{f \in \operatorname{Hom}(Q^2, \Lambda) :$

$$\begin{array}{l}
f(v \otimes_1 v) = 0, \\
f(v \otimes_2 v) = -2c'_5 x - 2c'_7 xy, \\
f(v \otimes_3 v) = 2c'_5 y + 2c'_6 xy, \\
f(v \otimes_4 v) = 0\}.
\end{array}$$

So $\dim \operatorname{Im} \delta^2 = 0$, if $\operatorname{char} K = 2$ and $\dim \operatorname{Im} \delta^2 = 3$, if $\operatorname{char} K \neq 2$.

Thus $\dim \operatorname{HH}^3(\Lambda) = 16$, if $\operatorname{char} K = 2$ and $\dim \operatorname{HH}^3(\Lambda) = 6$, if $\operatorname{char} K \neq 2$.

So we have two cases. If $\operatorname{char} K = 2$, then $\operatorname{HH}^3(\Lambda) = \operatorname{Hom}(Q^3, \Lambda)$.

If $\operatorname{char} K \neq 2$, then $\operatorname{HH}^3(\Lambda) = \{f \in \operatorname{Hom}(Q^3, \Lambda) :$

$$\begin{array}{l}
f(v \otimes_1 v) = c_2 x + c_4 xy, \\
f(v \otimes_2 v) = c_7 y, \\
f(v \otimes_3 v) = c_{10} x, \\
f(v \otimes_4 v) = c_{15} x + c_{16} xy\}.
\end{array}$$

5.4.2. $\operatorname{HH}^3(\tilde{\Lambda})$. For Example 5.2, let $\tilde{\delta}^3 : \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^4, \tilde{\Lambda})$ and $\tilde{\delta}^2 : \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) \longrightarrow \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda})$. We have $\operatorname{Ker} \tilde{\delta}^3 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda}) : \tilde{\delta}^3(\tilde{f}) = 0\}$. Let $\tilde{f}$ in $\operatorname{Ker} \tilde{\delta}^3$, then $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda})$. So $\tilde{f} : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ is given by

$$\begin{array}{l}
e_2 \otimes_1 e_2 \mapsto k_1 e_2 + k_2 \alpha_1\alpha_2 + k_3 \alpha_3\alpha_4 + k_4 \alpha_1\alpha_2\alpha_3\alpha_4 \\
e_2 \otimes_2 e_2 \mapsto k_5 e_2 + k_6 \alpha_1\alpha_2 + k_7 \alpha_3\alpha_4 + k_8 \alpha_1\alpha_2\alpha_3\alpha_4 \\
e_2 \otimes_3 e_2 \mapsto k_9 e_2 + k_{10} \alpha_1\alpha_2 + k_{11} \alpha_3\alpha_4 + k_{12} \alpha_1\alpha_2\alpha_3\alpha_4 \\
e_2 \otimes_4 e_2 \mapsto k_{13} e_2 + k_{14} \alpha_1\alpha_2 + k_{15} \alpha_3\alpha_4 + k_{16} \alpha_1\alpha_2\alpha_3\alpha_4
\end{array}$$

where $k_i \in K$. Since $\tilde{\delta}^3(\tilde{f}) = 0$, then

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}^4_1) \otimes \mathfrak{t}(\tilde{g}^4_1) &\overset{\tilde{d}^4}{\mapsto} (\mathfrak{o}(\tilde{g}^3_1) \otimes_1 \alpha_1\alpha_2 + \alpha_1\alpha_2 \otimes_1 \mathfrak{t}(\tilde{g}^3_1), 0, 0, 0) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_1\alpha_2 + \alpha_1\alpha_2\tilde{f}(e_2 \otimes_1 e_2) \\
&= 2k_1\alpha_1\alpha_2 + 2k_3\alpha_1\alpha_2\alpha_3\alpha_4 \\
&= 0.
\end{aligned}
$$

We need to consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $k_1 = k_3 = 0$.

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}^4_2) \otimes \mathfrak{t}(\tilde{g}^4_2) &\overset{\tilde{d}^4}{\mapsto} (\mathfrak{o}(\tilde{g}^3_1) \otimes_1 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_1 \mathfrak{t}(\tilde{g}^3_1), -\mathfrak{o}(\tilde{g}^3_2) \otimes_2 \alpha_1\alpha_2 \\
&\qquad +\alpha_1\alpha_2 \otimes_2 \mathfrak{t}(\tilde{g}^3_2), 0) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_3\alpha_4 - \alpha_3\alpha_4\tilde{f}(e_2 \otimes_1 e_2) - \tilde{f}(e_2 \otimes_2 e_2)\alpha_1\alpha_2 \\
&\quad -\alpha_1\alpha_2\tilde{f}(e_2 \otimes_2 e_2) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}^4_3) \otimes \mathfrak{t}(\tilde{g}^4_3) &\overset{\tilde{d}^4}{\mapsto} (0, \mathfrak{o}(\tilde{g}^3_2) \otimes_2 \alpha_3\alpha_4 + \alpha_3\alpha_4 \otimes_2 \mathfrak{t}(\tilde{g}^3_2), \mathfrak{o}(\tilde{g}^3_3) \otimes_3 \alpha_1\alpha_2 \\
&\qquad +\alpha_1\alpha_2 \otimes_3 \mathfrak{t}(\tilde{g}^3_3), 0) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_2 e_2)\alpha_3\alpha_4 + \alpha_3\alpha_4\tilde{f}(e_2 \otimes_2 e_2) + \tilde{f}(e_2 \otimes_3 e_2)\alpha_1\alpha_2 + \alpha_1\alpha_2 \\
&\quad \tilde{f}(e_2 \otimes_3 e_2) \\
&= 2k_5\alpha_3\alpha_4 + 2k_6\alpha_1\alpha_2\alpha_3\alpha_4 + 2k_9\alpha_1\alpha_2 + 2k_{11}\alpha_1\alpha_2\alpha_3\alpha_4 \\
&= 2k_5\alpha_3\alpha_4 + 2k_9\alpha_1\alpha_2 + 2(k_6 + k_{11})\alpha_1\alpha_2\alpha_3\alpha_4.
\end{aligned}
$$

We consider two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $k_5 = k_9 = 0$ and $k_6 = k_{11}$.

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}^4_4) \otimes \mathfrak{t}(\tilde{g}^4_4) &\overset{\tilde{d}^4}{\mapsto} (0, 0, \mathfrak{o}(\tilde{g}^3_3) \otimes_3 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_3 \mathfrak{t}(\tilde{g}^3_3), -\mathfrak{o}(\tilde{g}^3_4) \otimes_4 \alpha_1\alpha_2 \\
&\qquad +\alpha_1\alpha_2 \otimes_4 \mathfrak{t}(\tilde{g}^3_4)) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_3 e_2)\alpha_3\alpha_4 - \alpha_3\alpha_4\tilde{f}(e_2 \otimes_3 e_2) - \tilde{f}(e_2 \otimes_4 e_2)\alpha_1\alpha_2 + \alpha_1\alpha_2 \\
&\quad \tilde{f}(e_2 \otimes_4 e_2) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{o}(\tilde{g}^4_5) \otimes \mathfrak{t}(\tilde{g}^4_5) &\overset{\tilde{d}^4}{\mapsto} (0, 0, 0, \mathfrak{o}(\tilde{g}^3_4) \otimes_4 y + y \otimes_4 \mathfrak{t}(\tilde{g}^3_4)) \\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_4 e_2)\alpha_3\alpha_4 + \alpha_3\alpha_4\tilde{f}(e_2 \otimes_4 e_2) \\
&= 2k_{13}\alpha_1\alpha_2 + 2k_{14}\alpha_1\alpha_2\alpha_3\alpha_4.
\end{aligned}
$$

So we have two cases. If $\operatorname{char} K = 2$, then there is no condition on constants. If $\operatorname{char} K \neq 2$, then $k_{13} = k_{14} = 0$.

Hence we have two cases. If $\operatorname{char} K = 2$, then $\operatorname{Ker} \tilde{\delta}^3 = \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda})$.

If $\operatorname{char} K \neq 2$, then $\operatorname{Ker} \tilde{\delta}^3 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda}) :$

$$\begin{aligned}
\tilde{f}(e_2 \otimes_1 e_2) &= k_2\alpha_1\alpha_2 + k_4\alpha_1\alpha_2\alpha_3\alpha_4,\\
\tilde{f}(e_2 \otimes_2 e_2) &= k_6\alpha_1\alpha_2 + k_7\alpha_3\alpha_4 + k_8\alpha_1\alpha_2\alpha_3\alpha_4,\\
\tilde{f}(e_2 \otimes_3 e_2) &= k_{10}\alpha_1\alpha_2 + k_6\alpha_3\alpha_4 + k_{12}\alpha_1\alpha_2\alpha_3\alpha_4,\\
\tilde{f}(e_2 \otimes_4 e_2) &= k_{15}\alpha_3\alpha_4 + k_{16}\alpha_1\alpha_2\alpha_3\alpha_4\}.
\end{aligned}$$

So we have, if $\operatorname{char} K = 2$, then $\dim \operatorname{Ker} \tilde{\delta}^3 = 16$. If $\operatorname{char} K \neq 2$, then $\dim \operatorname{Ker} \tilde{\delta}^3 = 9$.

We need to find $\operatorname{Im} \tilde{\delta}^2$. Since $\operatorname{Im} \tilde{\delta}^2 = \{\tilde{\delta}^2(\tilde{f}), \tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})\}$ and let $\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda})$, then $\tilde{f} : \tilde{Q}^2 \longrightarrow \tilde{\Lambda}$ is given by

$$\begin{aligned}
&e_2 \otimes_1 e_2 \mapsto k'_1e_2 + k'_2\alpha_1\alpha_2 + k'_3\alpha_3\alpha_4 + k'_4\alpha_1\alpha_2\alpha_3\alpha_4\\
&e_2 \otimes_2 e_2 \mapsto k'_5e_2 + k'_6\alpha_1\alpha_2 + k'_7\alpha_3\alpha_4 + k'_8\alpha_1\alpha_2\alpha_3\alpha_4\\
&e_2 \otimes_3 e_2 \mapsto k'_9e_2 + k'_{10}\alpha_1\alpha_2 + k'_{11}\alpha_3\alpha_4 + k'_{12}\alpha_1\alpha_2\alpha_3\alpha_4,
\end{aligned}$$

where $k'_i \in K$. We have,

$$\begin{aligned}
\mathfrak{o}(\tilde{g}^3_1) \otimes \mathfrak{t}(\tilde{g}^3_1) &\overset{\tilde{d}^3}{\mapsto} (e_2 \otimes_1 \alpha_1\alpha_2 - \alpha_1\alpha_2 \otimes_1 e_2, 0, 0)\\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_1\alpha_2 + \alpha_1\alpha_2\tilde{f}(e_2 \otimes_1 e_2)\\
&= k'_1\alpha_1\alpha_2 + k'_3\alpha_1\alpha_2\alpha_3\alpha_3 - k'_1\alpha_1\alpha_2 - k'_3\alpha_1\alpha_2\alpha_3\alpha_4\\
&= 0.\\
\mathfrak{o}(\tilde{g}^3_2) \otimes \mathfrak{t}(\tilde{g}^3_2) &\overset{\tilde{d}^3}{\mapsto} (e_2 \otimes_1 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_1 e_2, -e_2 \otimes_2 \alpha_1\alpha_2 - \alpha_1\alpha_2 \otimes_2 e_2, 0)\\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_1 e_2)\alpha_3\alpha_4 - \alpha_3\alpha_4\tilde{f}(e_2 \otimes_1 e_2) - \tilde{f}(e_2 \otimes_2 e_2)\alpha_1\alpha_2 - \alpha_1\alpha_2\\
&\quad \tilde{f}(e_2 \otimes_2 e_2)\\
&= k'_1\alpha_3\alpha_4 + k'_2\alpha_1\alpha_2\alpha_3\alpha_4 - k'_1\alpha_3\alpha_4 - k'_2\alpha_1\alpha_2\alpha_3\alpha_4 - k'_5\alpha_1\alpha_2\\
&\quad -k'_7\alpha_1\alpha_2\alpha_3\alpha_4 - k'_5\alpha_1\alpha_2 - k'_7\alpha_3\alpha_4\alpha_1\alpha_2\\
&= -2k'_5\alpha_1\alpha_2 - 2k'_7\alpha_1\alpha_2\alpha_3\alpha_4.\\
\mathfrak{o}(\tilde{g}^3_3) \otimes \mathfrak{t}(\tilde{g}^3_3) &\overset{\tilde{d}^3}{\mapsto} (0, e_2 \otimes_2 \alpha_3\alpha_4 + y \otimes_2 e_2, e_2 \otimes_3 \alpha_1\alpha_2 - \alpha_1\alpha_2 \otimes_3 e_2)\\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_2 e_2)\alpha_3\alpha_4 + \alpha_3\alpha_4\tilde{f}(e_2 \otimes_2 e_2) + \tilde{f}(e_2 \otimes_3 e_2)\alpha_1\alpha_2 - \alpha_1\alpha_2\\
&\quad \tilde{f}(e_2 \otimes_3 e_2)\\
&= 2k'_5\alpha_1\alpha_2 + 2k'_6\alpha_1\alpha_2\alpha_3\alpha_4.\\
\mathfrak{o}(\tilde{g}^3_4) \otimes \mathfrak{t}(\tilde{g}^3_4) &\overset{\tilde{d}^3}{\mapsto} (0, 0, e_2 \otimes_3 \alpha_3\alpha_4 - \alpha_3\alpha_4 \otimes_3 e_2)\\
&\overset{\tilde{f}}{\mapsto} \tilde{f}(e_2 \otimes_3 e_2)\alpha_3\alpha_4 - \alpha_3\alpha_4\tilde{f}(e_2 \otimes_3 e_2)\\
&= k'_9x + k'_{10}\alpha_1\alpha_2\alpha_3\alpha_4 - k'_9\alpha_1\alpha_2 - k'_{10}\alpha_1\alpha_2\alpha_3\alpha_4\\
&= 0.
\end{aligned}$$

Now we need to consider two cases. If $\operatorname{char} K = 2$, then $\operatorname{Im} \tilde{\delta}^2 = 0$, and if char $K \neq 2$, then $\operatorname{Im} \tilde{\delta}^2 = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^2, \tilde{\Lambda}) :$

$$\tilde{f}(e_2 \otimes_1 e_2) = 0,$$
$$\tilde{f}(e_2 \otimes_2 e_2) = -2k'_5\alpha_1\alpha_2 - 2k'_7\alpha_1\alpha_2\alpha_3\alpha_4,$$
$$\tilde{f}(e_2 \otimes_3 e_2) = 2k'_5\alpha_3\alpha_4 + 2k'_6\alpha_1\alpha_2\alpha_3\alpha_4,$$
$$\tilde{f}(e_2 \otimes_4 e_2) = 0\}.$$

So we have $\dim \operatorname{Im} \tilde{\delta}^2 = 0$, if char $K = 2$ and $\dim \operatorname{Im} \tilde{\delta}^2 = 3$, if char $K \neq 2$. Thus, we have two cases. If char $K = 2$, then $\dim \mathrm{HH}^3(\tilde{\Lambda}) = 16$. If char $K \neq 2$, then $\dim \mathrm{HH}^3(\tilde{\Lambda}) = 6$.

Hence we have two cases. If char $K = 2$, then $\mathrm{HH}^3(\tilde{\Lambda}) = \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda})$.
If char $K \neq 2$, then $\mathrm{HH}^3(\tilde{\Lambda}) = \{\tilde{f} \in \operatorname{Hom}(\tilde{Q}^3, \tilde{\Lambda}) :$

$$\tilde{f}(e_2 \otimes_1 e_2) = k_2\alpha_1\alpha_2 + k_4\alpha_1\alpha_2\alpha_3\alpha_4,$$
$$\tilde{f}(e_2 \otimes_2 e_2) = k_7\alpha_3\alpha_4,$$
$$\tilde{f}(e_2 \otimes_3 e_2) = k_{10}\alpha_1\alpha_2,$$
$$\tilde{f}(e_2 \otimes_4 e_2) = k_{15}\alpha_3\alpha_4 + k_{16}\alpha_1\alpha_2\alpha_3\alpha_4\}.$$

5.4.3. *The relation between* $\mathrm{HH}^3(\Lambda)$ *and* $\mathrm{HH}^3(\tilde{\Lambda})$ *of Examples 5.1 and 5.2.* We find the relationship between $\mathrm{HH}^3(\Lambda)$ and $\mathrm{HH}^3(\tilde{\Lambda})$ of Examples 5.1 and 5.2.

For Example 5.1 we have the basis of $\mathrm{HH}^3(\Lambda) = \operatorname{sp}\{z_1, z_2, z_3, z_4, z_5, z_6\}$, where

$z_1 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_1 v) \mapsto c_2 x$, else$\mapsto 0$.

$z_2 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_2 v) \mapsto c_4 xy$, else$\mapsto 0$.

$z_3 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_2 v) \mapsto c_7 y$, else$\mapsto 0$.

$z_4 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_4 v) \mapsto c_{10} x$, else$\mapsto 0$.

$z_5 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_4 v) \mapsto c_{15} y$, else$\mapsto 0$.

$z_6 : Q^3 \longrightarrow \Lambda$ which is given by $(v \otimes_4 v) \mapsto c_{16} xy$, else$\mapsto 0$.

For Example 5.2 we have the basis of $\mathrm{HH}^3(\tilde{\Lambda}) = \operatorname{sp}\{\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4, \tilde{z}_5, \tilde{z}_6\}$, where

$\tilde{z}_1 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_1^3} e_2) \mapsto k_2\alpha_1\alpha_2$, else$\mapsto 0$.

$\tilde{z}_2 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_1^3} e_2) \mapsto k_4\alpha_1\alpha_2\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_3 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_1^3} e_2) \mapsto k_7\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_4 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_2^3} e_2) \mapsto k_{10}\alpha_1\alpha_2$, else$\mapsto 0$.

$\tilde{z}_5 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_1^3} e_2) \mapsto k_{15}\alpha_3\alpha_4$, else$\mapsto 0$.

$\tilde{z}_6 : \tilde{Q}^3 \longrightarrow \tilde{\Lambda}$ which is given by $(e_2 \otimes_{\tilde{g}_1^3} e_2) \mapsto k_{16}\alpha_1\alpha_2\alpha_3\alpha_4$, else$\mapsto 0$.

So we define the group homomorphism

$\phi^3 : \mathrm{HH}^3(\Lambda) \longrightarrow \mathrm{HH}^3(\tilde{\Lambda})$ via $z_i \mapsto \tilde{z}_i$, for all $i = 1, \ldots, 6$.

By the same argument we can show that $\mathrm{HH}^4(\Lambda) \cong \mathrm{HH}^4(\tilde{\Lambda})$ and $\mathrm{HH}^5(\Lambda) \cong \mathrm{HH}^5(\tilde{\Lambda})$. We come back to this in the next chapter.

6. Stratifying ideals and Hochschild cohomology for $\tilde{\Lambda}$

Chapters 6 and 7 study the Hochschild cohomology rings of Λ and its stretched algebra $\tilde{\Lambda}$, and the finiteness condition **(Fg)**. In this chapter, we give some results on the stretched algebra $\tilde{\Lambda}$ showing in Theorem 6.9 that $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal. After that we investigate the connection between $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$ and finiteness conditions. Chapter 7 will study when a d-Koszul algebra has **(Fg)** in order to apply the results of this chapter. We begin Chapter 6 with some properties of stratifying ideals.

6.1. Stratifying ideals.

Definition 6.1. [8, Definition 2.1.1] Let A be an algebra and e an idempotent in A. The two sided ideal AeA generated by e is called a stratifying ideal if

- The multiplication map $Ae \otimes_{eAe} eA \longrightarrow AeA$ is an isomorphism, and
- For all $n > 0$, $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$.

In order to decide if an ideal is a stratifying ideal, we give some properties of Tor.

Definition 6.2. [40, Chapter 6] Let R be a ring. If M is a left R-module and

$$P_n : \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} N \longrightarrow 0$$

is a projective resolution of a right R-module N, then

$$\mathrm{Tor}_n^R(N, M) = \mathrm{Ker}(d_n \otimes \mathbf{1}_M)/\,\mathrm{Im}(d_{n+1} \otimes \mathbf{1}_M).$$

Theorem 6.3. *[40, Theorem 7.2] If a right R-module F is flat, then $\mathrm{Tor}_n^R(F, M) = 0$, for all $n > 0$ and for every left R-module M.*

Since every projective module is flat, then we have the following corollary.

Corollary 6.4. *If P is a projective right R-module, then $\mathrm{Tor}_n^R(P, M) = 0$, for all $n > 0$ and for every left R-module M.*

Remark 6.5. Let e be an idempotent element in A. If Ae is projective as a right eAe-module and the multiplication map $Ae \otimes_{eAe} eA \longrightarrow AeA$ is an isomorphism, then AeA is a stratifying ideal.

Now we use Remark 6.5 to prove $\tilde{\Lambda}e_2\tilde{\Lambda}$ is a stratifying ideal for the algebra $\tilde{\Lambda}$ in Example 5.2.

Proposition 6.6. *Let $\tilde{\Lambda}$ be the algebra in Example 5.2 and we keep the notation of the previous chapters. Then $\tilde{\Lambda}e_2\tilde{\Lambda}$ is a stratifying ideal.*

Proof. For ease of notation, we set $B = e_2\tilde{\Lambda}e_2$. We show first $\tilde{\Lambda}e_2$ is projective as a right B-module, then we prove the multiplication map $\tilde{\Lambda}e_2 \otimes_B e_2\tilde{\Lambda} \longrightarrow \tilde{\Lambda}e_2\tilde{\Lambda}$ is an isomorphism.

We have

$$\begin{aligned}\tilde{\Lambda}e_2 = \text{sp}\{&e_2, \alpha_2, \alpha_4, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_2\alpha_1\alpha_2, \alpha_2\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2, \alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4,\\ &\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\} \text{ and}\\ B = \text{sp}\{&e_2, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4\}.\end{aligned}$$

So we have $e_2B = \text{sp}\{e_2, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4\}$,

$$\alpha_2B = \text{sp}\{\alpha_2, \alpha_2\alpha_1\alpha_2, \alpha_2\alpha_3\alpha_4, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\}$$
$$\alpha_4B = \text{sp}\{\alpha_4, \alpha_4\alpha_1\alpha_2, \alpha_4\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\}.$$

Hence $\tilde{\Lambda}e_2 = e_2B \oplus \alpha_2B \oplus \alpha_4B$, where $e_2B, \alpha_2B, \alpha_4B$ are right B-modules. Since e_2 is an idempotent in B, then e_2B is projective as a right B-module. Now, using Proposition 4.15, we have $e_2B \cong \alpha_2B$ and $e_2B \cong \alpha_4B$. Hence $\tilde{\Lambda}e_2 \cong e_2B \oplus e_2B \oplus e_2B$, where e_2B is projective as a right B-module. Since the direct sum of projective modules is projective, then $\tilde{\Lambda}e_2$ is projective as a right B-module.

Next we show that the multiplication map $\tilde{\Lambda}e_2 \otimes_B e_2\tilde{\Lambda} \longrightarrow \tilde{\Lambda}e_2\tilde{\Lambda}$ is an isomorphism. We define the map $\psi : \tilde{\Lambda}e_2 \otimes_B e_2\tilde{\Lambda} \longrightarrow \tilde{\Lambda}e_2\tilde{\Lambda}$ via $\tilde{\lambda}e_2 \otimes_B e_2\tilde{\mu} \mapsto \tilde{\lambda}e_2\tilde{\mu}$. It is clear that ψ is a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule homomorphism and is onto. So we need to show that ψ is one-to-one. We refer the reader to the general result in Theorem 6.9, noting that we can write $\tilde{\lambda}e_2 \otimes_B e_2\tilde{\mu}$ as

$$e_2 \otimes_B e_2\nu_1 + \tilde{q}_{e_1} \otimes_B e_2\nu_2 + \tilde{q}_{e_3} \otimes_B e_2\nu_3$$

where ν_1, ν_2, ν_3 are elements of $\tilde{\Lambda}$ and $\tilde{q}_{e_1} = \alpha_2$, $\tilde{q}_{e_3} = \alpha_4$ (See also Lemma 6.7). Thus $\tilde{\Lambda}e_2\tilde{\Lambda}$ is a stratifying ideal.

□

Lemma 6.7. *Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters with $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ and $B = \varepsilon\tilde{\Lambda}\varepsilon$. An element of $\tilde{\Lambda}\varepsilon \otimes_B \varepsilon\tilde{\Lambda}$ is of the form*

$$\varepsilon \otimes_B \varepsilon\nu + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon \otimes_B \varepsilon\nu_w$$

where ν, ν_w are elements of $\tilde{\Lambda}$.

Proof. Let $\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu}$ in $\tilde{\Lambda}\varepsilon\otimes_B\varepsilon\tilde{\Lambda}$. Then $\mathbf{1}_{\tilde{\Lambda}}(\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu}) = \varepsilon\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu}+\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} w\tilde{\lambda}\varepsilon\otimes_B \varepsilon\tilde{\mu}$. So we have $\varepsilon\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu} = \varepsilon\varepsilon\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu} = \varepsilon\otimes_B\varepsilon\tilde{\lambda}\varepsilon\varepsilon\tilde{\mu} = \varepsilon\otimes_B\varepsilon\tilde{\lambda}\varepsilon\tilde{\mu} = \varepsilon\otimes_B\varepsilon\nu$, where $\nu = \tilde{\lambda}\varepsilon\tilde{\mu}$ which is an element in $\tilde{\Lambda}$.

Also, by the construction of the quiver, $w\tilde{\lambda}\varepsilon = \tilde{q}_w\theta(\lambda)\varepsilon$, for some $\lambda\in\Lambda$, so

$$\begin{aligned}
\textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} w\tilde{\lambda}\varepsilon\otimes_B\varepsilon\tilde{\mu} &= \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\theta(\lambda)\varepsilon\otimes_B\varepsilon\tilde{\mu}\\
&= \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon\otimes_B\varepsilon\theta(\lambda)\varepsilon\varepsilon\tilde{\mu}\\
&= \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon\otimes_B\varepsilon\theta(\lambda)\varepsilon\tilde{\mu}\\
&= \textstyle\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon\otimes_B\varepsilon\nu_w
\end{aligned}$$

where $\nu_w = \varepsilon\theta(\lambda)\varepsilon\tilde{\mu}$ which is an element in $\tilde{\Lambda}$, and the result follows. $\square$

Proposition 6.8. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters with $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ and $B = \varepsilon\tilde{\Lambda}\varepsilon$. Then $\tilde{\Lambda}\varepsilon$ is projective as a right B-module.*

Proof. We will show $\tilde{\Lambda}\varepsilon = \varepsilon B\oplus\big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} v'B\big)$. For $v\in\mathcal{Q}_0$, $\tilde{\Lambda}v$ has basis which consists of all paths which start from a vertex in $\mathcal{Q}_0$ and end at v, and all paths start from a vertex w and end at v, where $w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0$. Then using Definition 4.11, we can write $\tilde{\Lambda}v$ as $\tilde{\Lambda}v = \varepsilon\tilde{\Lambda}v + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w\tilde{\Lambda}v$, for all $v\in\mathcal{Q}_0$. Hence

$$\tilde{\Lambda}\varepsilon = \varepsilon B + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B.$$

Next we show that $\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B = \oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B$. Let $\tilde{q}_w b\in\tilde{q}_w B\cap\sum_{w'\neq w}\tilde{q}_{w'}B$. Then $\tilde{q}_w b = w\tilde{q}_w b = w\sum_{w'\neq w} w'\tilde{q}_{w'}b_{w'} = 0$, where $b, b_{w'}\in B$. So $\tilde{q}_w B\cap\sum_{w'\neq w}\tilde{q}_{w'}B = 0$ and hence $\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B = \oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B$. Then

$$\tilde{\Lambda}\varepsilon = \varepsilon B + (\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B).$$

Now we show that $\varepsilon B\cap\big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B\big) = 0$. Let $\varepsilon b\in\varepsilon B\cap\big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B\big)$. So $\varepsilon b = \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w b_w$, where b, b_w in B. Then $\varepsilon b = \varepsilon\varepsilon b = \varepsilon\sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w b_w = 0$. Hence $\varepsilon B\cap\big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B\big) = 0$ and thus

$$\tilde{\Lambda}\varepsilon = \varepsilon B\oplus\big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0}\tilde{q}_w B\big).$$

Since ε is an idempotent in B, then εB is projective as a right B-module. By using Proposition 4.15(1) we have $v'B \cong \tilde{q}_w B$ as right B-modules. Hence, $\tilde{\Lambda}\varepsilon \cong \varepsilon B \oplus \big(\oplus_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} v'B\big)$ where εB, $v'B$ are projective right B-modules. Since the direct sum of projective modules is projective, then $\tilde{\Lambda}\varepsilon$ is projective as a right B-module. $\square$

We can now generalize Proposition 6.6.

Theorem 6.9. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters with $\varepsilon = \sum_{v\in\mathcal{Q}_0} v$ and $B = \varepsilon\tilde{\Lambda}\varepsilon$. Then $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal of $\tilde{\Lambda}$.*

Proof. By using Proposition 6.8, we have $\tilde{\Lambda}\varepsilon$ is projective as a right B-module. From Remark 6.5, it remains to show that the multiplication map $\tilde{\Lambda}\varepsilon \otimes_B \varepsilon\tilde{\Lambda} \longrightarrow \tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is an isomorphism. We define the map $\psi : \tilde{\Lambda}\varepsilon \otimes_B \varepsilon\tilde{\Lambda} \longrightarrow \tilde{\Lambda}\varepsilon\tilde{\Lambda}$ via $\tilde{\lambda}\varepsilon \otimes_B \varepsilon\tilde{\mu} \mapsto \tilde{\lambda}\varepsilon\tilde{\mu}$. It is clear that ψ is a $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule homomorphism and is onto. So we need to show that ψ is one-to-one. Suppose that

$$\psi(\tilde{\lambda}_1\varepsilon \otimes_B \varepsilon\tilde{\mu}_1) = \psi(\tilde{\lambda}_2\varepsilon \otimes_B \varepsilon\tilde{\mu}_2).$$

Then by Lemma 6.7, we have

$$\tilde{\lambda}_1\varepsilon \otimes_B \varepsilon\tilde{\mu}_1 = \varepsilon \otimes_B \varepsilon\nu + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon \otimes \varepsilon\nu_w$$

and

$$\tilde{\lambda}_2\varepsilon \otimes_B \varepsilon\tilde{\mu}_2 = \varepsilon \otimes_B \varepsilon\nu' + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon \otimes \varepsilon\nu'_w$$

where ν, ν_w, ν', ν'_w are elements in $\tilde{\Lambda}$. Then

$$\psi(\varepsilon \otimes_B \varepsilon\nu + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon \otimes \varepsilon\nu_w) = \psi(\varepsilon \otimes_B \varepsilon\nu' + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\varepsilon \otimes \varepsilon\nu'_w)$$

So

$$\varepsilon\nu + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\nu_w = \varepsilon\nu' + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w\nu'_w.$$

and hence

$$\varepsilon\nu - \varepsilon\nu' + \sum_{w\in\tilde{\mathcal{Q}}_0\setminus\mathcal{Q}_0} \tilde{q}_w(\nu_w - \nu'_w) = 0. \tag{4}$$

First we multiply (3) by ε, then we get $\varepsilon\nu - \varepsilon\nu' = 0$. Second we multiply (3) by w, where $w \in \tilde{\mathcal{Q}}_0 \setminus \mathcal{Q}_0$, then we have $\tilde{q}_w(\nu_w - \nu'_w) = 0$. Since the ideal $\tilde{I}$ of $K\tilde{\mathcal{Q}}$ is generated by uniform elements $\tilde{g}_1^2, \ldots, \tilde{g}_m^2$ which all start and end at a vertex in $\mathcal{Q}_0$, a similar argument to Proposition 4.14 shows that $\nu_w = \nu'_w$. Thus $\tilde{\lambda}_1\varepsilon \otimes_B \varepsilon\tilde{\mu}_1 = \tilde{\lambda}_2\varepsilon \otimes_B \varepsilon\tilde{\mu}_2$ and hence ψ is one-to-one.

Therefore $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal. $\square$

6.2. **The projective dimension of** $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$**.** We start with the algebra $\tilde{\Lambda}$; in Examples 2.40, 3.13, 4.2 and see also 5.2; and show that $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}$ has projective dimension 2 in this case. We omit some of the details as they are in the general case, which is Theorem 6.12. We keep the notation from previous chapters.

Proposition 6.10. *Let $\tilde{\Lambda}$ be the algebra in Example 5.2. Then there exists a minimal projective $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule resolution of $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}$*

$$0 \longrightarrow \tilde{R}^2 \xrightarrow{\Delta^2} \tilde{R}^1 \xrightarrow{\Delta^1} \tilde{R}^0 \xrightarrow{\Delta^0} \tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda} \longrightarrow 0.$$

Proof. To construct this resolution, we need to find projective bimodules $\tilde{R}^n$ and maps Δ^n for all $n = 0, 1, 2$. Since $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda} = \mathrm{sp}\{e_1, e_3\}$, we define $\tilde{R}^0 = \tilde{\Lambda}e_1 \otimes e_1\tilde{\Lambda} \oplus \tilde{\Lambda}e_3 \otimes e_3\tilde{\Lambda}$ and we define the map

$\Delta^0 : \tilde{R}^0 \longrightarrow \tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}$ via

$(\tilde{\lambda}_1e_1 \otimes e_1\tilde{\mu}_1, \tilde{\lambda}_2e_3 \otimes e_3\tilde{\mu}_2) \mapsto (\tilde{\lambda}_1e_1\tilde{\mu}_1 + \tilde{\lambda}_2e_3\tilde{\mu}_2) + \tilde{\Lambda}e_2\tilde{\Lambda}$, where $\tilde{\lambda}_i, \tilde{\mu}_i \in \tilde{\Lambda}$.

We have $\operatorname{Ker}\Delta^0 = \{(\tilde{\lambda}_1e_1 \otimes e_1\tilde{\mu}_1, \tilde{\lambda}_2e_3 \otimes e_2\tilde{\mu}_2) : \tilde{\lambda}_1e_1\tilde{\mu}_1 + \tilde{\lambda}_2e_3\tilde{\mu}_3 \in \tilde{\Lambda}e_2\tilde{\Lambda}\}$ and we have

$\tilde{\Lambda}e_1 = \mathrm{sp}\{e_1, \alpha_1, \alpha_4\alpha_1, \alpha_2\alpha_1, \alpha_3\alpha_4\alpha_1, \alpha_1\alpha_2\alpha_1, \alpha_4\alpha_3\alpha_4\alpha_1, \alpha_2\alpha_3\alpha_4\alpha_1, \alpha_4\alpha_1\alpha_2\alpha_1,$
$\alpha_2\alpha_1\alpha_2\alpha_1, \alpha_1\alpha_2\alpha_3\alpha_4\alpha_1, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\}$,

$e_1\tilde{\Lambda} = \mathrm{sp}\{e_1, \alpha_2, \alpha_2\alpha_1, \alpha_2\alpha_3, \alpha_2\alpha_3\alpha_4, \alpha_2\alpha_1\alpha_2, \alpha_2\alpha_3\alpha_4\alpha_3, \alpha_2\alpha_3\alpha_4\alpha_1, \alpha_2\alpha_1\alpha_2\alpha_1,$
$\alpha_2\alpha_1\alpha_2\alpha_3, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\}$,

$\tilde{\Lambda}e_3 = \mathrm{sp}\{e_3, \alpha_3, \alpha_2\alpha_3, \alpha_4\alpha_3, \alpha_1\alpha_2\alpha_3, \alpha_3\alpha_4\alpha_3, \alpha_2\alpha_1\alpha_2\alpha_3, \alpha_4\alpha_1\alpha_2\alpha_3, \alpha_4\alpha_3\alpha_4\alpha_3,$
$\alpha_2\alpha_3\alpha_4\alpha_3, \alpha_1\alpha_2\alpha_3\alpha_4\alpha_3, \alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3\}$,

$e_3\tilde{\Lambda} = \mathrm{sp}\{e_3, \alpha_4, \alpha_4\alpha_3, \alpha_4\alpha_1, \alpha_4\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2, \alpha_4\alpha_3\alpha_4\alpha_1, \alpha_4\alpha_1\alpha_2\alpha_3, \alpha_4\alpha_1\alpha_2\alpha_1,$
$\alpha_4\alpha_3\alpha_4\alpha_3, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3\}$.

$\tilde{\Lambda}e_2 = \mathrm{sp}\{e_2, \alpha_2, \alpha_4, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_2\alpha_1\alpha_2, \alpha_2\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2, \alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4,$
$\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\}$ and

$e_2\tilde{\Lambda} = \mathrm{sp}\{e_2, \alpha_1, \alpha_3, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_1\alpha_2\alpha_1, \alpha_3\alpha_4\alpha_1, \alpha_1\alpha_2\alpha_3, \alpha_3\alpha_4\alpha_3, \alpha_1\alpha_2\alpha_3\alpha_4,$

$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1, \alpha_1\alpha_2\alpha_3\alpha_4\alpha_3\}.$

Also, we note that $\tilde{p}_{e_1} = \alpha_1$, $\tilde{q}_{e_1} = \alpha_2$, $\tilde{p}_{e_3} = \alpha_3$, $\tilde{q}_{e_3} = \alpha_4$. So $\dim \tilde{\Lambda}e_1 \otimes e_1\tilde{\Lambda} = 169$ and $\dim \tilde{\Lambda}e_3 \otimes e_3\tilde{\Lambda} = 169$. Then it can be shown that the generators of $\operatorname{Ker}\Delta^0$ are $e_1 \otimes \tilde{q}_{e_1}, \tilde{p}_{e_1} \otimes e_1, e_3 \otimes \tilde{q}_{e_3}, \tilde{p}_{e_3} \otimes e_3$.

So we let $\tilde{R}^1 = \tilde{\Lambda}e_1 \otimes e_2\tilde{\Lambda} \oplus \tilde{\Lambda}e_2 \otimes e_1\tilde{\Lambda} \oplus \tilde{\Lambda}e_3 \otimes e_2\tilde{\Lambda} \oplus \tilde{\Lambda}e_2 \otimes e_3\tilde{\Lambda}$ and we define the map $\Delta^1 : \tilde{R}^1 \longrightarrow \tilde{R}^0$ as follows

$e_1 \otimes e_2 \mapsto e_1 \otimes \tilde{q}_{e_1}$

$e_2 \otimes e_1 \mapsto \tilde{p}_{e_1} \otimes e_1$

$e_3 \otimes e_2 \mapsto e_3 \otimes \tilde{q}_{e_3}$

$e_2 \otimes e_3 \mapsto \tilde{p}_{e_3} \otimes e_3.$

Now we have

$$\begin{aligned}\dim \operatorname{Ker}\Delta^1 &= \dim \tilde{R}^1 - \dim \operatorname{Ker}\Delta^0\\ &= \dim \tilde{R}^1 - (\dim \tilde{R}^0 - \dim(\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}))\\ &= 624 - 336\\ &= 288.\end{aligned}$$

It can be shown that the generators of $\operatorname{Ker}\Delta^1$ are $(\tilde{p}_{e_1} \otimes e_2, -e_2 \otimes \tilde{q}_{e_1}, 0, 0)$ and $(0, 0, \tilde{p}_{e_3} \otimes e_2, -e_2 \otimes \tilde{q}_{e_3})$. Hence we define the bimodule $\tilde{R}^2 = \tilde{\Lambda}e_2 \otimes e_2\tilde{\Lambda} \oplus \tilde{\Lambda}e_2 \otimes e_2\tilde{\Lambda}$, and we define the map $\Delta^2 : \tilde{R}^2 \longrightarrow \tilde{R}^1$ via

$e_2 \otimes e_2 \mapsto (\tilde{p}_{e_1} \otimes e_2, -e_2 \otimes \tilde{q}_{e_1}, 0, 0)$

$e_2 \otimes e_2 \mapsto (0, 0, \tilde{p}_{e_3} \otimes e_2, -e_2 \otimes \tilde{q}_{e_3}).$

Now $\dim \tilde{R}^2 = 288$, so $\dim \operatorname{Ker}\Delta^2 = \dim \tilde{R}^2 - \dim \operatorname{Ker}\Delta^1 = 0$. Hence, we have the minimal projective resolution

$$0 \longrightarrow \tilde{R}^2 \xrightarrow{\Delta^2} \tilde{R}^1 \xrightarrow{\Delta^1} \tilde{R}^0 \xrightarrow{\Delta^0} \tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda} \longrightarrow 0.$$

as required. $\square$

We now consider the general case and recall some notation from Chapter 4. For each arrow α in $\mathcal{Q}_1$, we have $\theta(\alpha) = \alpha_1 \cdots \alpha_A$ and additional vertices $w_1, \ldots, w_{A-1}$ in $\tilde{\mathcal{Q}}_0$, where $w_i = \mathfrak{t}(\alpha_i)$ for $i = 1, \ldots, A-1$. So w_i is properly internal to $\theta(\alpha)$. Also we set $\dim \tilde{\Lambda}v = V$, and $\dim v'\tilde{\Lambda} = V'$, where $v = \mathfrak{o}(\alpha)$ and $v' = \mathfrak{t}(\alpha)$. We recall that $\tilde{p}_{w_i}$ is the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at a vertex in $\mathcal{Q}_0$ and ends at w_i and $\tilde{q}_{w_i}$ is the unique shortest path in $K\tilde{\mathcal{Q}}$ which starts at the vertex w_i and ends at a vertex in $\mathcal{Q}_0$. We note that $\tilde{p}_{w_i} = \alpha_1 \cdots \alpha_i$ and $\tilde{q}_{w_i} = \alpha_{i+1} \cdots \alpha_A$.

We define Γ_α to be the subquiver of $\tilde{\mathcal{Q}}$

$$\Gamma_\alpha := \overset{w_1}{\cdot} \xrightarrow{\alpha_2} \overset{w_2}{\cdot} \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{A-1}} \overset{w_{A-1}}{\cdot}$$

We have

$$\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} \cong \oplus_{\alpha\in\mathcal{Q}_1}(\tilde{\Lambda}w_1\tilde{\Lambda} + \tilde{\Lambda}w_2\tilde{\Lambda} + \cdots + \tilde{\Lambda}w_{A-1}\tilde{\Lambda} + \tilde{\Lambda}\varepsilon\tilde{\Lambda})/\tilde{\Lambda}\varepsilon\tilde{\Lambda}.$$

Then we define $X_\alpha = (\tilde{\Lambda}w_1\tilde{\Lambda} + \tilde{\Lambda}w_2\tilde{\Lambda} + \cdots + \tilde{\Lambda}w_{A-1}\tilde{\Lambda} + \tilde{\Lambda}\varepsilon\tilde{\Lambda})/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$, so

$$\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} \cong \oplus_{\alpha\in\mathcal{Q}_1} X_\alpha$$

and hence, $K\Gamma_\alpha \cong X_\alpha$ as algebras.

Proposition 6.11. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters. Then* $\dim \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = m_1((A-1)A/2)$, *where m_1 is the number of arrows of $\mathcal{Q}$.*

Proof. From the construction of $\tilde{\Lambda}$, and for each arrow α in $\mathcal{Q}_1$, we have the following basis elements in $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$:

$A-1$ vertices

$A-2$ arrows $\alpha_2, \dots, \alpha_{A-1}$

$A-3$ paths of length 2

$\vdots$

and 1 path of length $A-2$ (which is the path $\alpha_2 \cdots \alpha_{A-1}$). So

$$\dim X_\alpha = \sum_{i=1}^{A-1} i = \frac{(A-1)A}{2}.$$

Then

$$\dim \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = m_1 \dim X_\alpha = \frac{m_1(A-1)A}{2}$$

where m_1 is the number of arrows of $\mathcal{Q}$. □

We now ready to generalize Proposition 6.10.

Theorem 6.12. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Keeping the above notation, X_α has a minimal projective $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule resolution*

$$0 \longrightarrow \tilde{R}^2_\alpha \xrightarrow{\Delta^2_\alpha} \tilde{R}^1_\alpha \xrightarrow{\Delta^1_\alpha} \tilde{R}^0_\alpha \xrightarrow{\Delta^0_\alpha} X_\alpha \longrightarrow 0.$$

Moreover, we have a minimal projective $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule resolution of $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$

$$0 \longrightarrow \tilde{R}^2 \xrightarrow{\Delta^2} \tilde{R}^1 \xrightarrow{\Delta^1} \tilde{R}^0 \xrightarrow{\Delta^0} \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} \longrightarrow 0$$

where $\tilde{R}^i = \oplus \tilde{R}^i_\alpha$, for $i = 0, 1, 2$ and $\alpha \in \mathcal{Q}_1$.

Proof. Let $\alpha \in \mathcal{Q}_1$, and consider X_α, with the above notation. Now, we define the bimodule $\tilde{R}^0_\alpha = \oplus_{i=1}^{A-1} \tilde{\Lambda} w_i \otimes w_i \tilde{\Lambda}$ and we define the map $\Delta^0_\alpha : \tilde{R}^0_\alpha \longrightarrow X_\alpha$ via $w_i \otimes w_i \mapsto w_i + \tilde{\Lambda}\varepsilon\tilde{\Lambda}$ where $i = 1, \ldots, A-1$.

Using Proposition 4.16 with $\dim \tilde{\Lambda} v = V$, $\dim v'\tilde{\Lambda} = V'$, we have

$$\begin{aligned}
\dim \tilde{R}^0_\alpha &= \textstyle\sum_{i=1}^{A-1} \dim(\tilde{\Lambda} w_i) \dim(w_i \tilde{\Lambda}) \\
&= \textstyle\sum_{i=1}^{A-1} (i + V)((A-i) + V') \\
&= \textstyle\sum_{i=1}^{A-1} i(A-i) + \sum_{i=1}^{A-1} iV + \sum_{i=1}^{A-1} iV' + (A-1)VV' \\
&= ((A-1)A(A+1))/6 + ((A-1)AV)/2 + ((A-1)AV')/2 \\
&\qquad + (A-1)VV' \\
&= (A-1)(VV' + A(V+V')/2 + A(A+1)/6).
\end{aligned}$$

Hence

$$\begin{aligned}
\dim \operatorname{Ker} \Delta^0_\alpha &= \dim \tilde{R}^0_\alpha - \dim X_\alpha \\
&= (A-1)(VV' + A(V+V')/2 + A(A+1)/6) - (A(A-1))/2 \\
&= (A-1)\big(VV' + A(V+V')/2 + A(A-2)/6\big).
\end{aligned}$$

The next step is to find the generators of $\operatorname{Ker} \Delta^0_\alpha$. Let K be the $\tilde{\Lambda}$-$\tilde{\Lambda}$-bimodule generated by $\{\tilde{p}_{w_1} \otimes w_1, w_{A-1} \otimes \tilde{q}_{w_{A-1}}, w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1}, \text{where } i = 1, \ldots, A-2\}$. So we can see that $K \subseteq \operatorname{Ker} \Delta^0_\alpha$. Note that, for all $i = 1, \ldots, A-1$, $\tilde{p}_{w_i} \otimes w_i$ is in $\operatorname{Ker} \Delta^0_\alpha$. Indeed $\tilde{p}_{w_i} \otimes w_i \in K$, as we can write

$$\tilde{p}_{w_i} \otimes w_i = (\tilde{p}_{w_1} \otimes w_1)\alpha_2 \cdots \alpha_i - \sum_{j=1}^{i-1} \alpha_1 \cdots \alpha_j (w_j \otimes \alpha_{j+1} - \alpha_{j+1} \otimes w_{j+1})\alpha_{j+2} \cdots \alpha_i.$$

We claim that $\operatorname{Ker} \Delta^0_\alpha = K$. We have

$$K = \tilde{\Lambda}\tilde{p}_{w_1} \otimes \ w_1 \tilde{\Lambda} + \tilde{\Lambda} w_{A-1} \otimes \tilde{q}_{w_{A-1}} \tilde{\Lambda} + \sum_{i=1}^{A-2} \tilde{\Lambda}(w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1})\tilde{\Lambda}.$$

We note that

$$\tilde{p}_{w_1} \otimes \tilde{q}_{w_1} = \sum_{j=1}^{A-2} \tilde{p}_j (w_j \otimes \alpha_{j+1} - \alpha_{j+1} \otimes w_{j+1}) \tilde{q}_{j+1} + \tilde{p}_{w_{A-1}} \otimes \tilde{q}_{w_{A-1}}.$$

First suppose that $A = 2$. We set $U_1 = \tilde{\Lambda}\tilde{p}_{w_1} \otimes w_1\tilde{\Lambda}$ and $U_2 = \tilde{\Lambda}w_1 \otimes \tilde{q}_{w_1}\tilde{\Lambda}$. Then $K = U_1 + U_2$. So $\dim K = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. We can see that $U_1 \cap U_2 = \tilde{\Lambda}\tilde{p}_{w_1} \otimes \tilde{q}_{w_1}\tilde{\Lambda}$, since $\tilde{p}_{w_1}$ and $\tilde{q}_{w_1}$ are arrows in $\tilde{\mathcal{Q}}$. So, from Proposition 4.16, we have $\dim K = V(1+V') + (1+V)V' - VV' = V + V' + VV'$. So $\dim K = \dim \operatorname{Ker} \Delta^0_\alpha$. Therefore $K = \operatorname{Ker} \Delta^0_\alpha$.

Now suppose that $A \geq 3$. We set $U_1 = \tilde{\Lambda}\tilde{p}_{w_1} \otimes\ w_1\tilde{\Lambda}$, $U_2 = \tilde{\Lambda}w_{A-1} \otimes \tilde{q}_{w_{A-1}}\tilde{\Lambda}$, and $U_3 = \sum_{i=1}^{A-2} \tilde{\Lambda}(w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1})\tilde{\Lambda}$. We can see that $\tilde{\Lambda}(w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1})\tilde{\Lambda} \cong \tilde{\Lambda}(w_i \otimes w_{i+1})\tilde{\Lambda}$. Then $\dim K = \dim(U_1 + U_2) + \dim U_3 - \dim(U_1 + U_2) \cap U_3$.
Here $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. We want to show that $U_1 \cap U_2 = \{0\}$. We can see that $U_1 \subseteq \tilde{\Lambda}w_1 \otimes w_1\tilde{\Lambda}$ and $U_2 \subseteq \tilde{\Lambda}w_{A-1} \otimes w_{A-1}\tilde{\Lambda}$. Since $A \geq 3$ we have $U_1 \cap U_2 = \{0\}$. So using Proposition 4.15, we have

$$\begin{aligned}
\dim U_1 &= V((A-1) + V') \\
\dim U_2 &= ((A-1) + V)V' \\
\dim U_3 &= \textstyle\sum_{i=1}^{A-2}(i+V)((A-(i+1)) + V').
\end{aligned}$$

We can see that $\tilde{p}_{w_1} \otimes \tilde{q}_{w_1} - \tilde{p}_{w_{A-1}} \otimes \tilde{q}_{w_{A-1}}$ is in $U_1 + U_2$. Also

$$\begin{aligned}
\tilde{p}_{w_1} \otimes \tilde{q}_{w_1} - \tilde{p}_{w_{A-1}} \otimes \tilde{q}_{w_{A-1}} &= \tilde{p}_{w_1}(w_1 \otimes \alpha_2 - \alpha_2 \otimes w_2)\tilde{q}_{w_2} \\
&\quad + \tilde{p}_{w_2}(w_2 \otimes \alpha_3 - \alpha_3 \otimes w_3)\tilde{q}_{w_3} \\
&\quad + \cdots + \\
&\quad + \tilde{p}_{w_{A-2}}(w_{A-2} \otimes \alpha_{A-1} - \alpha_{A-1} \otimes w_{A-1})\tilde{q}_{w_{A-1}}.
\end{aligned}$$

so it is in U_3 and hence it is in $(U_1 + U_2) \cap U_3$. Moreover, $(U_1 + U_2) \cap U_3$ is generated by $\tilde{p}_{w_1} \otimes \tilde{q}_{w_1} - \tilde{p}_{w_{A-1}} \otimes \tilde{q}_{w_{A-1}}$. So

$$\dim((U_1 + U_2) \cap U_3) = \dim \tilde{\Lambda}(\tilde{p}_{w_1} \otimes \tilde{q}_{w_1} - \tilde{p}_{w_{A-1}} \otimes \tilde{q}_{w_{A-1}})\tilde{\Lambda} = VV'.$$

Hence, $\dim K$

$$\begin{aligned}
&= V((A-1) + V') + ((A-1) + V)V' + \textstyle\sum_{i=1}^{A-2}(i+V)((A-(i+1)) + V') - VV' \\
&= V(A-1) + VV' + V'(A-1) + \textstyle\sum_{i=1}^{A-2} i(A-(i+1)) + \sum_{i=1}^{A-2} iV' + \sum_{i=1}^{A-2} V(A-(i+1)) \\
&\qquad + \textstyle\sum_{i=1}^{A-2} VV' \\
&= (A-1)(V+V') + (A-1)VV' + (A(A-2)(A-1))/2 - ((A-2)(A-1)(2A-3))/6 \\
&\qquad - ((A-2)(A-1))/2 + (V'(A-1)(A-2))/2 + (V(A-1)(A-2))/2 \\
&= (A-1)\big(VV' + A(V+V')/2 + A(A-2)/6\big).
\end{aligned}$$

So $\dim K = \dim \operatorname{Ker} \Delta^0_\alpha$. Therefore $K = \operatorname{Ker} \Delta^0_\alpha$.

Now we define $\tilde{R}^1_\alpha = \tilde{\Lambda}v \otimes w_1\tilde{\Lambda} \oplus (\bigoplus_{i=1}^{A-2} \tilde{\Lambda}w_i \otimes w_{i+1}\tilde{\Lambda}) \oplus \tilde{\Lambda}w_{A-1} \otimes v'\tilde{\Lambda}$, and we define the map $\Delta^1_\alpha : \tilde{R}^1_\alpha \longrightarrow \tilde{R}^0_\alpha$ via

$$v \otimes w_1 \mapsto \tilde{p}_{w_1} \otimes w_1$$
$$w_i \otimes w_{i+1} \mapsto w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1}$$
$$w_{A-1} \otimes v' \mapsto w_{A-1} \otimes \tilde{q}_{w_{A-1}}$$

where $\tilde{p}_{w_1} \otimes w_1$ lies in the $\mathfrak{t}(\alpha_1) \otimes \mathfrak{t}(\alpha_1)$-component of $\tilde{R}^0_\alpha$, $w_{A-1} \otimes \tilde{q}_{w_{A-1}}$ lies in the $\mathfrak{o}(\alpha_A) \otimes \mathfrak{o}(\alpha_A)$-component of $\tilde{R}^0_\alpha$, $w_i \otimes \alpha_{i+1}$ lies in the $\mathfrak{o}(\alpha_{i+1}) \otimes \mathfrak{o}(\alpha_{i+1})$-component of $\tilde{R}^0_\alpha$ and $\alpha_{i+1} \otimes w_{i+1}$ lies in the $\mathfrak{t}(\alpha_{i+1}) \otimes \mathfrak{t}(\alpha_{i+1})$-component of $\tilde{R}^0_\alpha$. Then

$$\begin{aligned}
\dim \tilde{R}^1_\alpha &= \dim \tilde{\Lambda}v \dim w_1\tilde{\Lambda} + \textstyle\sum_{i=1}^{A-2} \dim \tilde{\Lambda}w_i \dim w_{i+1}\tilde{\Lambda} + \dim \tilde{\Lambda}w_{A-1} \dim v'\tilde{\Lambda} \\
&= V((A-1) + V') + \textstyle\sum_{i=1}^{A-2} (i+V)((A-(i+1)) + V') \\
&\qquad\qquad +((A-1)+V)V' \\
&= \textstyle\sum_{i=1}^{A-1} Vi + \sum_{i=1}^{A-1} V'i + AVV' + \sum_{i=1}^{A-2} i(A-(i+1)) \\
&= AVV' + (A(A-1)(V+V'))/2 + (A(A-1)(A-2))/6.
\end{aligned}$$

Then $\dim \operatorname{Ker} \Delta^1_\alpha = \dim \tilde{R}^1_\alpha - \dim \operatorname{Ker} \Delta^0_\alpha = VV'$.

Now we want to find the generators of $\operatorname{Ker} \Delta^1_\alpha$. We can see that

$$z = (v \otimes \tilde{q}_{w_1}, -\tilde{p}_{w_1} \otimes \tilde{q}_{w_2}, \ldots, -\tilde{p}_{w_i} \otimes \tilde{q}_{w_{i+1}}, \ldots, -\tilde{p}_{w_{A-2}} \otimes \tilde{q}_{w_{A-1}}, -\tilde{p}_{w_{A-1}} \otimes v')$$

is in $\operatorname{Ker} \Delta^1_\alpha$. Now z generates a sub-bimodule of $\operatorname{Ker} \Delta^1_\alpha$ of dimension VV'. Hence $\operatorname{Ker} \Delta^1_\alpha$ is generated by z. So

$$\operatorname{Ker} \Delta^1_\alpha = \tilde{\Lambda}(v \otimes \tilde{q}_{w_1}, -\tilde{p}_{w_1} \otimes \tilde{q}_{w_2}, \ldots, -\tilde{p}_{w_i} \otimes \tilde{q}_{w_{i+1}}, \ldots, -\tilde{p}_{w_{A-2}} \otimes \tilde{q}_{w_{A-1}}, -\tilde{p}_{w_{A-1}} \otimes v')\tilde{\Lambda}.$$

Then we define $\tilde{R}^2_\alpha = \tilde{\Lambda}v \otimes v'\tilde{\Lambda}$ and we define the map $\Delta^2_\alpha : \tilde{R}^2_\alpha \longrightarrow \tilde{R}^1_\alpha$ via

$$v \otimes v' \mapsto (v \otimes \tilde{q}_{w_1}, -\tilde{p}_{w_1} \otimes \tilde{q}_{w_2}, \ldots, -\tilde{p}_{w_i} \otimes \tilde{q}_{w_{i+1}}, \ldots, - \tilde{p}_{w_{A-2}} \otimes \tilde{q}_{w_{A-1}}, -\tilde{p}_{w_{A-1}} \otimes v').$$

Then $\dim \tilde{R}^2_\alpha = VV'$ and so $\dim \operatorname{Ker} \Delta^2_\alpha = \dim \tilde{R}^2_\alpha - \dim \operatorname{Ker} \Delta^1_\alpha = 0$. Thus, we have the minimal projective resolution of X_α

$$0 \longrightarrow \tilde{R}^2_\alpha \xrightarrow{\Delta^2_\alpha} \tilde{R}^1_\alpha \xrightarrow{\Delta^1_\alpha} \tilde{R}^0_\alpha \xrightarrow{\Delta^0_\alpha} X_\alpha \longrightarrow 0.$$

The result follows. □

Corollary 6.13. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. With the above notation,* $\operatorname{pdim}_{\tilde{\Lambda}^e} \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = 2$.

6.3. **Hochschild cohomology, finiteness conditions, and eventually homological isomorphisms.** We review some definitions and results that relate to Hochschild cohomology, finiteness conditions, and eventually homological isomorphisms which are used in the next section.

We start by introducing the ideal of $\mathrm{HH}^*(A)$ which is generated by nilpotent elements. First note that, if A is a finite dimensional algebra over a field K with $\operatorname{char} K \neq 2$, then any homogeneous element of $\mathrm{HH}^*(A)$ of odd degree is nilpotent. To see this, let $z \in \mathrm{HH}^*(A)$ be a homogeneous element of odd degree. By using $\mathrm{HH}^*(A)$ is a graded commutative ring (see Theorem 2.38), then $z^2 = -z^2$ and hence $2z^2 = 0$. So $z^2 = 0$, and thus z is nilpotent.

Definition 6.14. [44] Let $\mathcal{N}$ be the ideal in $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Note that $\mathcal{N}$ is a graded ideal and also that $\mathcal{N}$ is the set of all nilpotent elements in $\mathrm{HH}^*(A)$, since $\mathrm{HH}^*(A)$ is graded commutative.

Proposition 6.15. *[44] Let A be a finite dimensional algebra over a field K and let $M \in \operatorname{mod} A$. Then*

$$\varphi_M : \mathrm{HH}^*(A) \longrightarrow \operatorname{Ext}^*_A(M, M)$$

is a homomorphism of graded rings, which is given by $\varphi_M(-) = M \otimes_A -$.

Let M be a right A-module. The ring homomorphism φ_M in Proposition 6.15 gives $\operatorname{Ext}^*_A(M, M)$ a left and a right $\mathrm{HH}^*(A)$-module structure. In general, suppose we have graded rings R and S and a graded ring homomorphism $f : S \to R$. Then we have a right S-module structure on R which is given by $r \cdot s = rf(s)$ and a left S-module structure on R which is given by $s \cdot r = f(s)r$. Moreover, Snashall and Solberg show in [44, Theorem 1.1] that the left and right module structures are connected in the following way: let $\eta \in \mathrm{HH}^n(A)$ and $\theta \in \operatorname{Ext}^m_A(M, M)$, then

$$\varphi_M(\eta)\theta = (-1)^{mn}\theta\varphi_M(\eta).$$

We now introduce the assumption **(Fg)** which relates to finiteness conditions on Hochschild cohomology and was introduced by Erdmann, Holloway, Snashall, Solberg, and Taillefer in [11]. See also [46].

Definition 6.16. [11] Let A be an indecomposable finite dimensional algebra over an algebraically closed field K.

(Fg1). H is a graded subalgebra of $\mathrm{HH}^*(A)$ such that H is a commutative Noetherian ring and $H^0 = \mathrm{HH}^0(A)$.

(Fg2). $E(A)$ is a finitely generated H-module.

We say A satisfies **(Fg)** if **(Fg1)** and **(Fg2)** hold for some H.

Remark 6.17. Note that [11] shows that the two assumptions **(Fg1)** and **(Fg2)** imply that $\mathrm{HH}^*(A)$ is a finitely generated H-module, and consequently $\mathrm{HH}^*(A)$ itself is finitely generated as a K-algebra. Moreover, $E(A)$ is a finitely generated K-algebra.

This work is connected to the more general concept of an eventually homological isomorphism, which was introduced by Psaroudakis, Skartsæterhagen and Solberg in [39]. We start with some definitions.

Definition 6.18. [39] Given a functor $\mathfrak{f} : \mathcal{B} \longrightarrow \mathcal{C}$ between abelian categories and an integer t, the functor $\mathfrak{f}$ is called a t-homological isomorphism if there is a group isomorphism

$$\mathrm{Ext}^j_{\mathcal{B}}(B, B') \cong \mathrm{Ext}^j_{\mathcal{C}}(\mathfrak{f}(B), \mathfrak{f}(B'))$$

for every pair of objects B, B' in $\mathfrak{B}$, and every $j > t$. Moreover these isomorphisms are not required to be induced by the functor $\mathfrak{f}$.

If $\mathfrak{f}$ is a t-homological isomorphism for some t, then we say that $\mathfrak{f}$ is an eventually homological isomorphism.

We recall the restriction functor from [2] and Chapter 4. For an algebra A, set $B = eAe$, where e is an idempotent in A. Then res_e is the functor $res_e : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, B$ which is given by $res_e(N) = (N)e$, where N is in $\mathrm{mod}\, A$.

Theorem 6.19. *[39, Lemma 8.23] Let A be a finite dimensional algebra over an algebraically closed field K. Suppose that AeA is a stratifying ideal in A. Then the following are equivalent:*

(1) $\mathrm{pdim}_{A^e} A/AeA < \infty$.

(2) The functor $res_e : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, B$ is an eventually homological isomorphism.

We note that the proof of [39, Lemma 8.23] gives the following result.

Proposition 6.20. *[39, Proof of Lemma 8.23] Let A be a finite dimensional algebra over an algebraically closed field K. Suppose that AeA is a stratifying ideal in A. Suppose also that* $\operatorname{pdim}_{A^e} A/AeA < \infty$*. If* $\operatorname{pdim}_{A^e} A/AeA = t$ *then the functor res_e is a t-homological isomorphism.*

Definition 6.21. (See [39, Section 4, p63]) Let A be a finite dimensional algebra over a field. Then A is called Gorenstein if $\operatorname{idim} A_A < \infty$ and $\operatorname{idim}_A A < \infty$.

We have the following result from [11].

Theorem 6.22. *[11, Theorem 2.5 (a)] Let A be an indecomposable finite dimensional algebra over an algebraically closed field K. Suppose that A and H satisfy* **(Fg)**. Then the algebra A is Gorenstein.

6.4. **Hochschild cohomology of Λ and $\tilde{\Lambda}$.** We now use these results, especially Theorem 6.9 and Corollary 6.13 to investigate the relationship between $\mathrm{HH}^*(\Lambda)$ and $\mathrm{HH}^*(\tilde{\Lambda})$. We build on the work of Koenig and Nagase [34], Nagase [37] and Psaroudakis, Skartsæterhagen and Solberg [39]. Our main results in this section are Theorem 6.24, Theorem 6.27, Theorem 6.32, Theorem 6.35, and Theorem 6.37.

Proposition 6.23. *[37, Proposition 6(1)] Let A be an algebra with a stratifying ideal AeA. Suppose* $\operatorname{pdim}_{A^e} A/AeA < \infty$*. Then we have* $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(eAe)$ *as graded algebras, where* $n = \operatorname{pdim}_{A^e} A/AeA + 1$.

Combining this with Corollary 6.13 gives the following result.

Theorem 6.24. *We keep the above notation, so that $\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is a stratifying ideal of $\tilde{\Lambda}$. Then* $\mathrm{HH}^{\geq 3}(\Lambda) \cong \mathrm{HH}^{\geq 3}(\tilde{\Lambda})$ *as graded algebras.*

This generalises the result of Example 5.2 from Chapter 5 when we saw $\mathrm{HH}^3(\Lambda) \cong \mathrm{HH}^3(\tilde{\Lambda})$.

Koenig and Nagase [34] also look at the Hochschild cohomology groups of the algebra A/AeA.

Proposition 6.25. *[34, Proposition 3.3(3)] Let A be an algebra with a stratifying ideal AeA. For any $n \geq 0$, then* $\mathrm{HH}^n(A/AeA) \cong \mathrm{Ext}^n_{A^e}(A/AeA, A/AeA)$.

Using Corollary 6.13, we get the following result when we apply it to $\tilde{\Lambda}$.

Proposition 6.26. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Keeping the notation of the previous chapters, then we have*

$$\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \cong \mathrm{Ext}^n_{\tilde{\Lambda}^e}(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}).$$

Hence $\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$, *for all* $n \geq 3$.

However, the following result shows that we also have $\mathrm{HH}^2(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$.

Theorem 6.27. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters. Then*

$$\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0,$$

for all $n \geq 2$.

Proof. The case $n \geq 3$ is in Proposition 6.26, so we assume $n = 2$. By using the projective resolution of $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ and applying $\mathrm{Hom}_{\tilde{\Lambda}^e}(-, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})$ we get the following complex

$$0 \longrightarrow \mathrm{Hom}_{\tilde{\Lambda}^e}(\tilde{R}^0, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \xrightarrow{\delta^0} \mathrm{Hom}_{\tilde{\Lambda}^e}(\tilde{R}^1, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \xrightarrow{\delta^1} \mathrm{Hom}_{\tilde{\Lambda}^e}(\tilde{R}^2, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \xrightarrow{\delta^2} 0.$$

We have $\mathrm{Ext}^2_{\tilde{\Lambda}^e}(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = \mathrm{Ker}\,\delta^2/\operatorname{Im}\delta^1 = \mathrm{Hom}_{\tilde{\Lambda}^e}(\tilde{R}^2, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/\operatorname{Im}\delta^1$. Let $g \in \mathrm{Hom}(\tilde{R}^2, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})$. Recall from Theorem 6.12 that $\tilde{R}^2 = \oplus_{\alpha\in\mathcal{Q}_1}\tilde{\Lambda}\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha)\tilde{\Lambda}$. Then for each α in $\mathcal{Q}_1$, we have $g(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha)) = x + \tilde{\Lambda}\varepsilon\tilde{\Lambda}$ for some x in $\tilde{\Lambda}$. Since $\mathfrak{o}(\alpha)(g(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha)))\mathfrak{t}(\alpha) = g(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha))$, then $x + \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = 0 + \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ and so $g = 0$. So $\mathrm{Hom}_{\tilde{\Lambda}^e}(\tilde{R}^2, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$. Hence $\mathrm{Ext}^2_{\tilde{\Lambda}^e}(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}, \tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$. Thus $\mathrm{HH}^2(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$. □

Remark 6.28. We give now an alternative proof for Theorem 6.27. By using Theorem 2.27, we have that X_α is hereditary and then $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is hereditary. Thus $\operatorname{gldim}\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda} = 1$ and hence $\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$, for all $n \geq 2$.

Proposition 6.29. *Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the above notation. Then*

$$\mathrm{HH}^*(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/\mathcal{N} = \mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/(\mathcal{N}\cap\mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})).$$

Proof. By Theorem 6.27 we have $\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0$, for all $n \geq 2$. So

$$\mathrm{HH}^*(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = \mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \oplus \mathrm{HH}^1(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}).$$

Since $\mathrm{HH}^*(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})$ is a graded algebra, then again from Theorem 6.27 we have

$$\mathrm{HH}^1(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \times \mathrm{HH}^1(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) = 0.$$

So for each $\eta \in \mathrm{HH}^1(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})$, we have $\eta^2 = 0$ and hence η is nilpotent. Thus, $\mathrm{HH}^*(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/\mathcal{N} = \mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/(\mathcal{N} \cap \mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}))$. □

Koenig and Nagase show in [34] that there are three long exact sequences which connect $\mathrm{HH}^n(A)$, $\mathrm{HH}^n(eAe)$ and $\mathrm{HH}^n(A/AeA)$.

Theorem 6.30. *[34, Theorem 3.4] Let A be an algebra with a stratifying ideal AeA. Then there are long exact sequences as follows:*

(1) $\cdots \longrightarrow \mathrm{Ext}^n_{A^e}(A, AeA) \longrightarrow \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^n(A/AeA) \longrightarrow \ldots$

(2) $\cdots \longrightarrow \mathrm{Ext}^n_{A^e}(A/AeA, A) \longrightarrow \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^n(eAe) \longrightarrow \ldots$

(3) $\cdots \longrightarrow \mathrm{Ext}^n_{A^e}(A/AeA, AeA) \longrightarrow \mathrm{HH}^n(A) \xrightarrow{f_n}$
$\mathrm{HH}^n(A/AeA) \oplus \mathrm{HH}^n(eAe) \longrightarrow \ldots$

Corollary 6.31. *[34, Corollary 3.5] Let A be an algebra with a stratifying ideal AeA.*

(1) Let $f : \mathrm{HH}^*(A) \to \mathrm{HH}^*(A/AeA) \times \mathrm{HH}^*(eAe)$ be the graded algebra homomorphism from Theorem 6.30(3). Then $(\mathrm{Ker}\, f)^2$ vanishes.

(2) The induced homomorphism

$$\bar{f} : \mathrm{HH}^*(A)/\mathcal{N} \to \mathrm{HH}^*(A/AeA)/\mathcal{N} \times \mathrm{HH}^*(eAe)/\mathcal{N}$$

is injective.

Recalling that $\varepsilon\tilde{\Lambda}\varepsilon \cong \Lambda$, then we get the following result when we apply Theorem 6.30 and Corollary 6.31 to $\tilde{\Lambda}$.

Theorem 6.32. *Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the previous notation.*

(1) There is a long exact sequence

$\cdots \longrightarrow \mathrm{Ext}^n_{\tilde{\Lambda}^e}(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}, \tilde{\Lambda}\varepsilon\tilde{\Lambda}) \longrightarrow \mathrm{HH}^n(\tilde{\Lambda}) \xrightarrow{f_n}$
$\mathrm{HH}^n(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \oplus \mathrm{HH}^n(\varepsilon\tilde{\Lambda}\varepsilon) \longrightarrow \ldots$

(2) Let $f : \mathrm{HH}^*(\tilde{\Lambda}) \to \mathrm{HH}^*(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}) \times \mathrm{HH}^*(\Lambda)$ *be the graded algebra homomorphism from (1). Then* $(\operatorname{Ker} f)^2$ *vanishes.*

(3) The induced homomorphism $\bar{f} : \mathrm{HH}^*(\tilde{\Lambda})/\mathcal{N} \to \mathrm{HH}^0(\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda})/\mathcal{N} \times \mathrm{HH}^*(\Lambda)/\mathcal{N}$ *is injective.*

Note that the algebras Λ and $\tilde{\Lambda}$ may be connected but $\tilde{\Lambda}/\tilde{\Lambda}\varepsilon\tilde{\Lambda}$ is not necessarily connected.

Example 6.33. Let $\tilde{\Lambda}$ be the algebra of Examples 2.40, 3.13, 4.2 and 5.2. From Proposition 6.6 $\tilde{\Lambda}e_2\tilde{\Lambda}$ is a stratifying ideal. It can be seen that $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda} = S_1 \oplus S_3$. We may illustrate $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}$ as follows:

$$.e_1 \qquad\qquad\qquad .e_3$$

We can see that $\tilde{\Lambda}/\tilde{\Lambda}e_2\tilde{\Lambda}$ is disconnected.

We now consider the finiteness condition **(Fg)**.

Proposition 6.34. *[37, Proposition 6(2)] Let A be an algebra with a stratifying ideal AeA. Suppose* $\operatorname{pdim}_{A^e} A/AeA < \infty$. *Then A satisfies* **(Fg)** if and only if eAe satisfies **(Fg)**.

We recall in our construction of $\tilde{\Lambda}$ from Λ we have that $\Lambda \cong \varepsilon\tilde{\Lambda}\varepsilon$. So using Corollary 6.13 we have the following result.

Theorem 6.35. *Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the previous notation. Then $\tilde{\Lambda}$ satisfies* **(Fg)** if and only if Λ satisfies **(Fg)**.

For our construction of $\tilde{\Lambda}$, we use Corollary 6.13, Theorem 6.19 and Proposition 6.20 to get the following result.

Corollary 6.36. *Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. We keep the notation of the previous chapters. Let K be an algebraically closed field. Then the functor res_ε : $\operatorname{mod}\tilde{\Lambda} \longrightarrow \operatorname{mod}\varepsilon\tilde{\Lambda}\varepsilon$ is an eventually homological isomorphism. Indeed, $res_\varepsilon : \operatorname{mod}\tilde{\Lambda} \longrightarrow \operatorname{mod}\varepsilon\tilde{\Lambda}\varepsilon$ is a 2-homological isomorphism.*

Recall that Erdmann, Holloway, Snashall, Solberg, and Taillefer showed in Theorem 6.22, that when the algebra A has **(Fg)**, then A is Gorenstein. We now use the method of [39, Theorem 4.3] to prove the following result.

Theorem 6.37. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra and let $\tilde{\Lambda}$ be the stretched algebra. Then* $\operatorname{idim}_{\tilde{\Lambda}} \tilde{\Lambda} \leq \sup\{\operatorname{idim}_{\Lambda} \Lambda, 2\}$.

Proof. The inequality holds if Λ has infinite injective dimension. We assume $\operatorname{idim}_{\Lambda} \Lambda = n$. Then $\sup\{\operatorname{idim}_{\Lambda} \Lambda, 2\} = \max\{\operatorname{idim}_{\Lambda} \Lambda, 2\}$. Let $m = \max\{\operatorname{idim}_{\Lambda} \Lambda, 2\} + 1$. By Corollary 6.36, res_{ε} is a 2-homological isomorphism, so we have

$$\operatorname{Ext}^m_{\tilde{\Lambda}}(X, Y) \cong \operatorname{Ext}^m_{\Lambda}(res_{\varepsilon}(X), res_{\varepsilon}(Y))$$

for all $X, Y \in \operatorname{mod} \tilde{\Lambda}$. Now, set $Y = \tilde{\Lambda}$. Then

$$\operatorname{Ext}^m_{\tilde{\Lambda}}(X, \tilde{\Lambda}) \cong \operatorname{Ext}^m_{\Lambda}(res_{\varepsilon}(X), res_{\varepsilon}(\tilde{\Lambda})) \cong \operatorname{Ext}^m_{\Lambda}(res_{\varepsilon}(X), \tilde{\Lambda}\varepsilon).$$

From Proposition 6.8 we have $\tilde{\Lambda}\varepsilon$ is projective as a right $\varepsilon\tilde{\Lambda}\varepsilon$-module. So $\operatorname{idim}_{\Lambda} \tilde{\Lambda}\varepsilon \leq n$, since Λ has injective dimension n and $\Lambda \cong \varepsilon\tilde{\Lambda}\varepsilon$. It follows that $\operatorname{Ext}^{n+1}_{\Lambda}(res_{\varepsilon}(X), \tilde{\Lambda}\varepsilon) = 0$. Hence $\operatorname{Ext}^m_{\tilde{\Lambda}}(X, \tilde{\Lambda}) = 0$ and so $\operatorname{idim}_{\tilde{\Lambda}} \tilde{\Lambda} \leq m - 1 = \max\{\operatorname{idim}_{\Lambda} \Lambda, 2\}$, using Proposition 2.25. $\square$

We end this chapter with some examples to illustrate the result above.

Example 6.38. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha_1"'] & 2 \arrow[l, bend right, "\alpha_2"'] \end{tikzcd}$$

and $I = \langle \alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2 \rangle$. The sets g^n are

- $g^0 = \{e_1, e_2\}$, with $g^0_1 = e_1$ and $g^0_2 = e_2$;
- $g^1 = \{\alpha_1, \alpha_2\}$, with $g^1_1 = \alpha_1$ and $g^1_2 = \alpha_2$;
- $g^2 = \{\alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2\}$, with $g^2_1 = \alpha_1\alpha_2\alpha_1$ and $g^2_2 = \alpha_2\alpha_1\alpha_2$;
- For all $n \geq 3$, n odd, we have $g^n_1 = g^{n-1}_1\alpha_2$, $g^n_2 = g^{n-1}_2\alpha_1$;
- For all $n \geq 3$, n even, we have $g^n_1 = g^{n-1}_1\alpha_1\alpha_2$, $g^n_2 = g^{n-1}_2\alpha_2\alpha_1$

We can now see that the elements $g^n_i \in g^n$ have length $\delta(n)$ for $d = 3$, since

- $\ell(g^0_i) = 0$, for $i = 1, 2$.
- $\ell(g^1_i) = 1$, for $i = 1, 2$.
- $\ell(g^2_i) = 3$, for $i = 1, 2, 3$.
- For $n \geq 2$, $\ell(g^{2n}_i) = nd$, where $i = 1, 2$.
- For $n \geq 1$, $\ell(g^{2n+1}_i) = nd + 1$, where $i = 1, 2$.

Hence each projective P^n is generated in degree $\delta(n)$. So Λ is a 3-Koszul monomial algebra. We can see that the algebra Λ has two indecomposable projective modules $P(1)$ and $P(2)$ and we may illustrate $P(i)$, where $i = 1, 2$, with the following diagrams:

$$\begin{array}{cc} P(1) = e_1\Lambda & P(2) = e_2\Lambda \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{array}$$

Moreover the indecomposable injective modules $I(i)$, where $i = 1, 2$, are given as follows:

$$\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ 1 & 2 \\ I(1) & I(2) \end{array}$$

Hence $I(1) = P(1) = e_1\Lambda$ and $I(2) = P(2) = e_2\Lambda$. So $\mathrm{idim}_\Lambda\, e_1\Lambda = \mathrm{idim}_\Lambda\, e_2\Lambda = 0$.

Then the injective dimension of Λ is 0, so Λ is self-injective.

Now with $A = 2$ the algebra $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ has the quiver

$$\begin{tikzcd} 1 \arrow[r, "\gamma_1"] & v_1 \arrow[d, "\gamma_2"] \\ v_2 \arrow[u, "\gamma_4"] & 2 \arrow[l, "\gamma_3"] \end{tikzcd}$$

and $\tilde{I} = \langle \gamma_1\gamma_2\gamma_3\gamma_4\gamma_1\gamma_2, \gamma_3\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4 \rangle$. Then $\tilde{\Lambda}$ is a $(6, 2)$-stacked monomial algebra. This is the algebra of [14, Example 3.2]. The four indecomposable projective modules are:

$$\begin{array}{cccc} \tilde{P}(1) = e_1\tilde{\Lambda} & \tilde{P}(v_1) = v_1\tilde{\Lambda} & \tilde{P}(2) = e_2\tilde{\Lambda} & \tilde{P}(v_2) = v_2\tilde{\Lambda} \\ 1 & v_1 & 2 & v_2 \\ v_1 & 2 & v_2 & 1 \\ 2 & v_2 & 1 & v_1 \\ v_2 & 1 & v_1 & 2 \\ 1 & v_1 & 2 & v_2 \\ v_1 & 2 & v_2 & 1 \\ & v_2 & & v_1 \end{array}$$

Moreover the indecomposable injective modules are:

$$\begin{array}{cccc}
 & v_2 & & v_1 \\
v_2 & 1 & v_1 & 2 \\
1 & v_1 & 2 & v_2 \\
v_1 & 2 & v_2 & 1 \\
2 & v_2 & 1 & v_1 \\
v_2 & 1 & v_1 & 2 \\
1 & v_1 & 2 & v_2 \\
\tilde{I}(1) & \tilde{I}(v_1) & \tilde{I}(2) & \tilde{I}(v_2)
\end{array}$$

Hence $\tilde{I}(v_2) = \tilde{P}(v_1) = v_1\tilde{\Lambda}$ and $\tilde{I}(v_1) = \tilde{P}(v_2) = v_2\tilde{\Lambda}$. So $\operatorname{idim}_{\tilde{\Lambda}} v_1\tilde{\Lambda} = \operatorname{idim}_{\tilde{\Lambda}} v_2\tilde{\Lambda} = 0$. In addition we have the following injective resolutions of $e_1\tilde{\Lambda}$ and $e_2\tilde{\Lambda}$:

$$\begin{tikzcd}
0 \arrow[r] & e_1\tilde{\Lambda} \arrow[r] & \tilde{I}(v_1) \arrow[rr] \arrow[dr] & & \tilde{I}(v_2) \arrow[r] & \tilde{I}(2) \arrow[r] & 0 \\
 & & & S(v_2) \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & e_2\tilde{\Lambda} \arrow[r] & \tilde{I}(v_2) \arrow[rr] \arrow[dr] & & \tilde{I}(v_1) \arrow[r] & \tilde{I}(1) \arrow[r] & 0 \\
 & & & S(v_1) \arrow[ur] & & &
\end{tikzcd}$$

where $S(v_1), S(v_2)$ are the simple modules corresponding to the vertices v_1, v_2. It is clear that the injective dimensions of $e_1\tilde{\Lambda}$ and $e_2\tilde{\Lambda}$ are 2. Hence the injective dimension of $\tilde{\Lambda}$ is 2. So in this case, $\operatorname{idim}_{\Lambda} \Lambda = 0$ and $\operatorname{idim}_{\tilde{\Lambda}} \tilde{\Lambda} = 2$. We note from [14] that $\tilde{\Lambda}$ has **(Fg)**. It follows from Theorem 6.35 that Λ has **(Fg)**.

Example 6.39. Let $\Lambda = K\mathcal{Q}/I$ be the algebra which is given by the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$$

and $I = \langle ab, bc \rangle$. The indecomposable projective modules are:

$$\begin{array}{cccc}
P(1) & P(2) & P(3) & P(4) \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 &
\end{array}$$

and the indecomposable injective modules are:

$$\begin{array}{cccc}
 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
I(1) & I(2) & I(3) & I(4)
\end{array}$$

It is clear that $P(i) = I(i+1)$, where $i = 1, 2, 3$. It remains to find the injective dimension of $P(4)$. We have:

$$\begin{tikzcd}
0 \arrow[r] & P(4) \arrow[r] & I(4) \arrow[rr] \arrow[dr] & & I(3) \arrow[rr] \arrow[dr] & & I(2) \arrow[r] & I(1) \arrow[r] & 0 \\
 & & & S(3) \arrow[ur] & & S(2) \arrow[ur] & & &
\end{tikzcd}$$

So $\operatorname{idim}_\Lambda P(4) = 3$ and hence $\operatorname{idim}_\Lambda \Lambda = 3$.

Now with $A = 2$ the algebra $\tilde{\Lambda} = K\tilde{\mathcal{Q}}/\tilde{I}$ has the quiver

$$1 \xrightarrow{a_1} v_1 \xrightarrow{a_2} 2 \xrightarrow{b_1} v_2 \xrightarrow{b_2} 3 \xrightarrow{c_1} v_3 \xrightarrow{c_2} 4$$

and $\tilde{I} = \langle a_1a_2b_1b_2, b_1b_2c_1c_2 \rangle$. The indecomposable projective modules are:

$$\begin{array}{ccccccc}
\tilde{P}(1) & \tilde{P}(v_1) & \tilde{P}(2) & \tilde{P}(v_2) & \tilde{P}(3) & \tilde{P}(v_3) & \tilde{P}(4) \\
1 & v_1 & 2 & v_2 & 3 & v_3 & 4 \\
v_1 & 2 & v_2 & 3 & v_3 & 4 & \\
2 & v_2 & 3 & v_3 & 4 & & \\
v_2 & 3 & v_3 & 4 & & & \\
 & v_3 & & & & &
\end{array}$$

and the indecomposable injective modules are:

$$\begin{array}{ccccccc}
 & & & & & v_1 & \\
 & & & 1 & v_1 & 2 & v_2 \\
 & & 1 & v_1 & 2 & v_2 & 3 \\
 & 1 & v_1 & 2 & v_2 & 3 & v_3 \\
1 & v_1 & 2 & v_2 & 3 & v_3 & 4 \\
\tilde{I}(1) & \tilde{I}(v_1) & \tilde{I}(2) & \tilde{I}(v_2) & \tilde{I}(3) & \tilde{I}(v_3) & \tilde{I}(4)
\end{array}$$

So we have $\tilde{P}(1) = \tilde{I}(v_2)$, $\tilde{P}(v_1) = \tilde{I}(v_3)$, and $\tilde{P}(v_2) = \tilde{I}(4)$. It can be seen that $\operatorname{idim}_{\tilde{\Lambda}} \tilde{P}(3) = 2$, $\operatorname{idim}_{\tilde{\Lambda}} \tilde{P}(2) = 2$, $\operatorname{idim}_{\tilde{\Lambda}} \tilde{P}(v_3) = 3$, and $\operatorname{idim}_{\tilde{\Lambda}} \tilde{P}(4) = 3$. Thus the injective dimension of $\tilde{\Lambda}$ is 3. So here we have $\operatorname{idim}_\Lambda \Lambda = 3 = \operatorname{idim}_{\tilde{\Lambda}} \tilde{\Lambda}$.

We now want to show that Λ has **(Fg)**. We know $Z(\Lambda) \cong K$, so take $H = \mathrm{HH}^0(\Lambda) = K$. We have from Green and Martínez-Villa [22], $E(\Lambda) \cong K\mathcal{Q}^{\mathrm{op}}/I^{\perp}$. Here $I^{\perp} = 0$. So $E(\Lambda) \cong K\mathcal{Q}^{\mathrm{op}}$ which is finite dimensional, since $\mathcal{Q}$ is finite and acyclic. It follows $E(\Lambda)$ is a finitely generated K-module and hence a finitely generated H-module. So Λ has **(Fg)**. Therefore $\tilde{\Lambda}$ has **(Fg)**, by using Theorem 6.6 and Theorem 6.35.

7. **(Fg)** for d-Koszul algebras

The aim of this chapter is to give sufficient conditions for a finite dimensional d-Koszul monomial algebra to have **(Fg)**, and we do this in Theorem 7.15.

We start this chapter by introducing some other Koszul algebras which have **(Fg)**; these are mostly from the work of Erdmann and Solberg in [12]. Then we introduce overlaps; these were used by Green and Zacharia to describe a basis for the Ext algebra of a monomial algebra in [29], and by Bardzell to describe the minimal projective bimodule resolution of a monomial algebra in [4]. We need this concept of overlaps to use the work of Green and Snashall in [26] where they find the Hochschild cohomology ring of a stacked monomial algebra. We can then give the commutative ring H for the **(Fg)** condition. In Theorem 7.11 we give sufficient conditions for a finite dimensional Koszul monomial algebra to have **(Fg)**, and in Theorem 7.15 we give sufficient conditions for a finite dimensional d-Koszul monomial algebra to have **(Fg)**.

We begin with results from [12] on **(Fg)** and the graded centre of the Ext algebra of a Koszul algebra.

Definition 7.1. [7] Let Λ be a finite dimensional algebra. The graded centre of the Ext algebra $Z_{\text{gr}}(E(\Lambda))$ is the subring of $E(\Lambda)$ generated by all homogeneous elements z such that $zy = (-1)^{|z||y|}yz$ for each homogeneous element y in $E(\Lambda)$, where $|x|$ denotes the degree of a homogeneous element x.

Theorem 7.2. *[12, Theorem 1.3] Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra over an algebraically closed field K.*

(a) *If Λ satisfies* **(Fg)**, then $Z_{\text{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\text{gr}}(E(\Lambda))$-module.

(b) When Λ is Koszul, then the converse implication also holds, that is, if $Z_{\text{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\text{gr}}(E(\Lambda))$-module, then Λ satisfies **(Fg)**.

We now illustrate this with an example.

Example 7.3. Let Λ be the algebra in Example 4.2 with char $K \neq 2$ and we keep the notation of the previous chapters. Since the algebra Λ is Koszul, then by Theorem 3.3, $E(\Lambda) = K\mathcal{Q}^{\text{op}}/I^{\perp}$ is Koszul and is generated by f_1^0, f_1^1, f_2^1 where f_1^0 corresponds to $e_1 \in g^0$, f_1^1 corresponds to $x \in g^1$, f_2^1 corresponds to $y \in g^1$. Moreover, the quiver of $E(\Lambda)$ is the same as the quiver of Λ. It can be seen that there is only one relation for the Ext algebra, which is $f_1^1 f_2^1 = -f_2^1 f_1^1$ and this corresponds to $xy + yx = 0$. So we have $E(\Lambda) = K\mathcal{Q}/(xy + yx)$.

By induction, it can be shown that $\eta \in Z_{\mathrm{gr}}(E(\Lambda))$ with η homogeneous of degree $2n$ if and only if $\eta = \sum_{i=0}^{n} c_i x^{2n-2i} y^{2i}$, where $c_i \in K$. We do not give details here but note that this example has been well studied (see [12], [45]) and is contained in Proposition 7.6 below with $n = 2$ and $q_{12} = -1$. In fact $Z_{gr}(E(\Lambda)) = K[x^2, y^2]$ and thus $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian. Moreover $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module, where the generators are $\{1, x, y, xy\}$. Thus Λ satisfies **(Fg)**, by using Theorem 7.2.

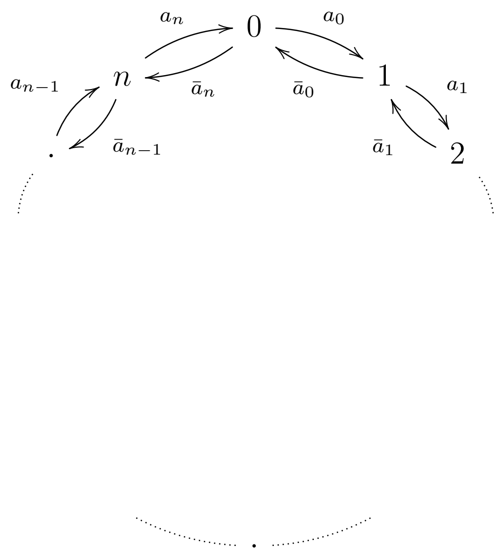
Definition 7.4. [43, Chapter 1] A finite dimensional algebra A over a field K is said to be of finite representation type if the number of isomorphism classes of indecomposable modules in $\operatorname{mod} A$ is finite.

Now we present some work of Erdmann and Solberg ([12]) on symmetric Koszul algebras which have **(Fg)**.

Theorem 7.5. *[12, Theorem] Let Λ be a finite dimensional symmetric algebra over an algebraically closed field with radical cube zero and radical square non-zero. Then Λ satisfies* **(Fg)** if and only if Λ is of finite representation type, Λ is of type $\tilde{\mathbb{D}}_n$ for $n \geq 4$, $\tilde{\mathbb{Z}}_n$ for $n > 0$, $\widetilde{D\mathbb{Z}_n}$ for $n \geq 2$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, $\tilde{\mathbb{E}}_8$, or Λ is of type $\tilde{\mathbb{Z}}_0$ or $\tilde{\mathbb{A}}_n$ for $n \geq 1$ when q is a root of unity. The algebras are described as follows:

(1) The case $\tilde{\mathbb{A}}_n$.

Let $\mathcal{Q}$ be the quiver given by



and $I = \langle \{a_i a_{i+1}\}_{i=0}^{n}, \{\bar{a}_{i+1}\bar{a}_i\}_{i=0}^{n}, \{a_i\bar{a}_i + \bar{a}_{i-1}a_{i-1}\}_{i=1}^{n} \cup \{a_0\bar{a}_0 + q\bar{a}_n a_n\}\rangle$, for some nonzero element $q \in K$.

(2) The case $\tilde{\mathbb{Z}}_n$ where $n > 0$:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd} 0 \arrow[loop left, "b"] \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\bar{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & \cdots\cdots \arrow[l, shift left, "\bar{a}_2"] \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\bar{a}_{n-2}"] \arrow[r, shift left, "a_{n-1}"] & n \arrow[l, shift left, "\bar{a}_{n-1}"] \arrow[loop right, "c"] \end{tikzcd}$$

and $I = \langle b^2 + a_0\bar{a}_0, ba_0, \bar{a}_0 b, \{a_i a_{i+1}\}_{i=0}^{n-2}, \{\bar{a}_i\bar{a}_{i-1}\}_{i=1}^{n-1}, \{a_i\bar{a}_i + \bar{a}_{i-1}a_{i-1}\}_{i=1}^{n-1}, a_{n-1}c, c\bar{a}_{n-1}, c^2 + q\bar{a}_{n-1}a_{n-1}\rangle$.

(3) The case $\tilde{\mathbb{Z}}_0$:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd} \cdot \arrow[loop left, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

and $I = \langle b^2, c^2, bc + qcb\rangle$.

(4) The $\widetilde{\mathbb{DZ}}_n$-case:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
0 \arrow[dr, shift right, "a_0"'] & & & & & & \\
& 2 \arrow[ul, shift right, "\bar{a}_0"'] \arrow[dl, shift right, "\bar{a}_1"'] \arrow[r, shift right, "a_2"'] & 3 \arrow[l, shift right, "\bar{a}_2"'] \arrow[r, shift right, "a_3"'] & \cdots\cdots \arrow[l, shift right, "\bar{a}_3"'] \arrow[r, shift right, "a_{n-3}"'] & n-2 \arrow[l, shift right, "\bar{a}_{n-3}"'] \arrow[r, shift right, "a_{n-2}"'] & n-1 \arrow[l, shift right, "\bar{a}_{n-2}"'] \arrow[r, shift right, "a_{n-1}"'] & n \arrow[l, shift right, "\bar{a}_{n-1}"'] \arrow[loop right, "b"] \\
1 \arrow[ur, shift right, "a_1"'] & & & & & &
\end{tikzcd}$$

where $n > 2$ and $I = \langle a_0\bar{a}_1, a_0a_2, a_1\bar{a}_0, \bar{a}_2\bar{a}_1, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_2\bar{a}_0, \bar{a}_1a_1 - a_2\bar{a}_2,$ $\{a_ia_{i+1}\}_{i=1}^{n-2}, \{\bar{a}\bar{a}_{i-1}\}_{i=2}^{n-1}, \{\bar{a}_{i-1}a_{i-1} + a_i\bar{a}_i\}_{i=3}^{n-1}, a_{n-1}b, \bar{a}_{n-1}a_{n-1} + qb^2, b\bar{a}_{n-1}\rangle$, for some $q \in K \setminus \{0\}$. If $n = 2$, then the relations are $a_0b, a_1b, b\bar{a}_0, b\bar{a}_1, a_0\bar{a}_1,$ $a_1\bar{a}_0, \bar{a}_0a_0 - b^2, \bar{a}_1a_1 - b^2$.

(5) The $\tilde{\mathbb{D}}_n$-case:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
0 \arrow[dr, shift right, "a_0"'] & & & & & n-1 \arrow[dl, shift right, "\bar{a}_{n-2}"'] \\
& 2 \arrow[ul, shift right, "\bar{a}_0"'] \arrow[dl, shift right, "\bar{a}_1"'] \arrow[r, shift right, "a_2"'] & 3 \arrow[l, shift right, "\bar{a}_2"'] \arrow[r, shift right, "a_3"'] & \cdots\cdots \arrow[l, shift right, "\bar{a}_3"'] \arrow[r, shift right, "a_{n-3}"'] & n-2 \arrow[l, shift right, "\bar{a}_{n-3}"'] \arrow[ur, shift right, "a_{n-2}"'] \arrow[dr, shift right, "b"'] & \\
1 \arrow[ur, shift right, "a_1"'] & & & & & n \arrow[ul, shift right, "\bar{b}"']
\end{tikzcd}$$

Assume that $n > 4$, then $I = \langle a_0\bar{a}_1, a_0a_2, a_1\bar{a}_0, \bar{a}_2\bar{a}_1, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_2\bar{a}_0, \bar{a}_1a_1 -$ $a_2\bar{a}_2,\ \{a_ia_{i+1}\}_{i=1}^{n-3}, \{\bar{a}\bar{a}_{i-1}\}_{i=2}^{n-2}, \{\bar{a}_{i-1}a_{i-1} + a_i\bar{a}_i\}_{i=3}^{n-3}, a_{n-2}b, \bar{b}a_{n-2}, a_{n-3}b,$ $a_{n-2}\bar{a}_{n-2} - b\bar{b}, b\bar{b} - \bar{a}_{n-3}a_{n-3}\rangle$.

For $n = 4$, then I is generated by $\{a_0\bar{a}_1, a_0a_2, a_0b, a_1\bar{a}_0, a_1a_2, a_1b, \bar{a}_2\bar{a}_0,$ $\bar{a}_2\bar{a}_1, \bar{a}_2b, \bar{b}\bar{a}_0, \bar{b}\bar{a}_1, \bar{b}a_2, \bar{a}_0a_0 - \bar{a}_1a_1, \bar{a}_1a_1 - b\bar{b}, b\bar{b} - a_2\bar{a}_2\}$.

(6) The $\tilde{\mathbb{E}}_6$-case:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
& & 4 \arrow[d, shift right, "\bar{a}_3"'] & & \\
& & 3 \arrow[u, shift right, "a_3"'] \arrow[d, shift right, "\bar{a}_2"'] & & \\
0 \arrow[r, shift right, "a_0"'] & 1 \arrow[l, shift right, "\bar{a}_0"'] \arrow[r, shift right, "a_1"'] & 2 \arrow[l, shift right, "\bar{a}_1"'] \arrow[u, shift right, "a_2"'] \arrow[r, shift right, "a_4"'] & 5 \arrow[l, shift right, "\bar{a}_4"'] \arrow[r, shift right, "a_5"'] & 6 \arrow[l, shift right, "\bar{a}_5"']
\end{tikzcd}$$

with $I = \langle \{a_i a_{i+1}\}_{i=0}^{4}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=1}^{5}, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=1,3,5}, a_1 a_4, \bar{a}_2 a_4, \bar{a}_4 a_2, \bar{a}_1 a_1 - a_2 \bar{a}_2, a_2 \bar{a}_2 - a_4 \bar{a}_4 \rangle$.

(7) The $\tilde{\mathbb{E}}_7$-case:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
 & & & 4 \arrow[d, shift right, "\bar{a}_3"'] & & & \\
0 \arrow[r, shift right, "a_0"'] & 1 \arrow[l, shift right, "\bar{a}_0"'] \arrow[r, shift right, "a_1"'] & 2 \arrow[l, shift right, "\bar{a}_1"'] \arrow[r, shift right, "a_2"'] & 3 \arrow[l, shift right, "\bar{a}_2"'] \arrow[u, shift right, "a_3"'] \arrow[r, shift right, "a_4"'] & 5 \arrow[l, shift right, "\bar{a}_4"'] \arrow[r, shift right, "a_5"'] & 6 \arrow[l, shift right, "\bar{a}_5"'] \arrow[r, shift right, "a_6"'] & 7 \arrow[l, shift right, "\bar{a}_6"']
\end{tikzcd}$$

with $I = \langle \{a_i a_{i+1}\}_{i=0}^{5}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=1}^{6}, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=1,2,5,6}, a_2 a_4, \bar{a}_3 a_4, \bar{a}_4 a_3, \bar{a}_2 a_2 - a_3 \bar{a}_3, a_3 \bar{a}_3 - a_4 \bar{a}_4 \rangle$.

(8) The $\tilde{\mathbb{E}}_8$-case:

Let $\mathcal{Q}$ be the quiver given by

$$\begin{tikzcd}
 & & 3 \arrow[d, shift right, "\bar{a}_2"'] & & & & & \\
0 \arrow[r, shift right, "a_0"'] & 1 \arrow[l, shift right, "\bar{a}_0"'] \arrow[r, shift right, "a_1"'] & 2 \arrow[l, shift right, "\bar{a}_1"'] \arrow[u, shift right, "a_2"'] \arrow[r, shift right, "a_3"'] & 4 \arrow[l, shift right, "\bar{a}_3"'] \arrow[r, shift right, "a_4"'] & 5 \arrow[l, shift right, "\bar{a}_4"'] \arrow[r, shift right, "a_5"'] & 6 \arrow[l, shift right, "\bar{a}_5"'] \arrow[r, shift right, "a_6"'] & 7 \arrow[l, shift right, "\bar{a}_6"'] \arrow[r, shift right, "a_7"'] & 8 \arrow[l, shift right, "\bar{a}_7"']
\end{tikzcd}$$

with $I = \langle \{a_i a_{i+1}\}_{i=0}^{6}, \{\bar{a}_i \bar{a}_{i-1}\}_{i=1}^{7}, \{\bar{a}_{i-1} a_{i-1} + a_i \bar{a}_i\}_{i=1,4,5,6,7}, a_1 a_3, \bar{a}_2 a_3, \bar{a}_3 a_2, \bar{a}_3 \bar{a}_1, \bar{a}_1 a_1 - a_2 \bar{a}_2, a_2 \bar{a}_2 - a_3 \bar{a}_3 \rangle$.

Proposition 7.6. *[12, Proposition 9.1] Let* $\Lambda = K\langle x_1, x_2, \ldots, x_n \rangle / (\{x_i x_j + q_{ij} x_j x_i\}_{i<j}, \{x_i^2\}_{i=1}^{n}), q_{ij} \in K^*$. *Then* Λ *satisfies* **(Fg)** if and only if all elements $\{q_{ij}\}_{i<j}$ are roots of unity.

Remark 7.7. There are other examples in the literature of Koszul algebras which have **(Fg)**.

(1) The algebra Λ_1 of [45] is the algebra of case $\tilde{\mathbb{A}}_n$ in [12].
(2) The algebra of [44] is the algebra of case $\tilde{\mathbb{Z}}_1$ in [12].
(3) The algebra which is given by the following quiver

$$\begin{tikzcd} \cdot \arrow[loop left, "\alpha"] \end{tikzcd}$$

and relation α^2 is of finite representation type and is therefore **(Fg)** by [12].
(4) The algebra of Example 6.38 is the algebra in [14, Example 3.2] which satisfies the condition **(Fg)**.

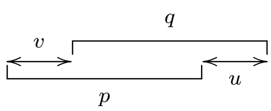
(5) The algebra of Example 6.39 satisfies the condition **(Fg)**.

The algebras we are interested in are monomial algebras; these are very rarely Koszul or d-Koszul algebras. So Theorem 7.11 and Theorem 7.15 give new results for finding algebras with **(Fg)**. We now introduce the concept of overlaps. Green and Zacharia use this in [29] to describe a basis of the Ext algebra of a monomial algebra.

We assume that $\Lambda = K\mathcal{Q}/I$ is a monomial algebra unless otherwise stated.

Definition 7.8. [25, Definition 1.1]

(1) A path q overlaps a path p with overlap pu if there are paths u and v such that $pu = vq$ and $1 \leq \ell(u) \leq \ell(q)$. We may illustrate the definition with the following diagram:



Note that we allow $\ell(v) = 0$ here.

(2) A path q properly overlaps a path p with overlap pu if q overlaps p and $\ell(v) \geq 1$.

(3) A path p has no overlaps with a path q if p does not properly overlap q and q does not properly overlap p.

Definition 7.9. [25] A path p is a prefix of a path q if there is some path p' such that $q = pp'$.

We refer the reader to look at Example 7.12.

We now describe the minimal projective resolution of Λ over Λ^e, by using the concept of overlaps from [29] and [16], see also [26]. We keep the notation of [26].

The sets $\mathcal{R}^n$ are defined as follows:

$\mathcal{R}^0$ = set of vertices of $\mathcal{Q}$,

$\mathcal{R}^1$ = set of arrows of $\mathcal{Q}$,

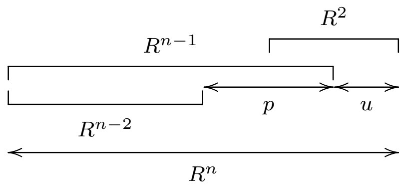
$\mathcal{R}^2$ = the minimal set of monomials in the generating set of I.

Then for all $n \geq 3$, $R^2 \in \mathcal{R}^2$ maximally overlaps $R^{n-1} \in \mathcal{R}^{n-1}$ with overlap $R^n = R^{n-1}u$ for some $u \in K\mathcal{Q}$, if it satisfies the following conditions:

- $R^{n-1} = R^{n-2}p$, for some path p in $K\mathcal{Q}$;

- R^2 overlaps p with overlap pu;
- There is no element in $\mathcal{R}^2$ which overlaps p with overlap being a proper prefix of pu.

The set $\mathcal{R}^n$ is defined to be the set of all overlaps R^n. We may illustrate R^n with the following diagram:



Green, Happel and Zacharia construct the minimal projective resolution (P^n, d^n) of $\Lambda/\mathfrak{r}$ (see [16]), also the construction can be found in [29] and [25]. For all $n \geq 0$, let $P^n = \oplus_{R^n \in \mathcal{R}^n} \mathfrak{t}(R^n)\Lambda$. The sets $\mathcal{R}^n$ are precisely the sets g^n of [28] which we used in Chapter 4. Define, for $n \geq 1$ and $R^n \in \mathcal{R}^n$, the map $d^n : P^n \longrightarrow P^{n-1}$ via $\mathfrak{t}(R^n) \mapsto (0, \dots, 0, p, 0, \dots)$, where $R^n = R^{n-1}p$ and p occurs in the component of P^{n-1} corresponding to R^{n-1}.

Now let (Q^*, δ^*) be the minimal projective Λ^e-resolution of Λ of [4]. We use the notation of Green and Snashall in [26]. Then $Q^n = \oplus_{R^n \in \mathcal{R}^n} \Lambda\mathfrak{o}(R^n) \otimes \mathfrak{t}(R^n)\Lambda$. From [4, Lemma 3.3], each element $R^n \in \mathcal{R}^n$, can be expressed uniquely as $R_j^{n-1}a_j$ and as $b_k R_k^{n-1}$ for some $R_j^{n-1}, R_k^{n-1} \in \mathcal{R}^{n-1}$ and paths a_j, b_k. The map $\delta^{2n+1} : Q^{2n+1} \to Q^{2n}$ is given via:

$$\mathfrak{o}(R^{2n+1}) \otimes \mathfrak{t}(R^{2n+1}) \mapsto \mathfrak{o}(R_j^{2n}) \otimes a_j - b_k \otimes \mathfrak{t}(R_k^{2n}),$$

where $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n} \in \mathcal{R}^{2n+1}$. Note that the first tensor lies in the summand corresponding to R_j^{2n} and the second tensor lies in the summand corresponding to R_k^{2n}.

For even degree the elements R^{2n} can be expressed as follows: $R^{2n} = p_j R_j^{2n-1} q_j$, for some $R_j^{2n-1} \in \mathcal{R}^{2n-1}$ and paths p_j, q_j with $n \geq 1$ and $j = 1, \dots, r$. So they define the map $\delta^{2n} : Q^{2n} \to Q^{2n-1}$ by $\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \mapsto \sum_{j=1}^{r} p_j \otimes q_j$, where the tensor $p_j \otimes q_j$ lies in the summand of Q^{2n-1} corresponding to R_j^{2n-1}.

We define f_i^n to be the Λ-homomorphism $P^n \longrightarrow \Lambda/\mathfrak{r}$ given by

$$\mathfrak{t}(R_j^n) \mapsto \begin{cases} \mathfrak{t}(R_i^n) + \mathfrak{r} & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

We set $f^n = \{f_i^n\}$ so that $|f^n| = |\mathcal{R}^n|$; see also Definition 4.44. We list the elements of $\mathcal{R}^n$ as $R_1^n, \ldots, R_s^n$ for some s. We compose module homomorphisms from right to left. So the composition $f \circ g$ means we apply g first then f. We recall that we write paths in a quiver from left to right. So if f_i^n corresponds to the path $R_i^n \in \mathcal{R}^n$ and if $R_i^n = eR_i^n e'$, where $e = \mathfrak{o}(R_i^n)$ and $e' = \mathfrak{t}(R_i^n)$ are in $\mathcal{R}^0$, then $f_i^n = f_{e'}^0 f_i^n f_e^0$ where f_e^0 (respectively, $f_{e'}^0$) denotes the element of f^0 that corresponds to e (respectively, e'). With this notation, we have from [29] that $f_j^m f_i^n \neq 0$ in $E(\Lambda)$ if and only if $R_i^n R_j^m = R_k^{n+m} \in \mathcal{R}^{n+m}$ for some k and $f_j^m f_i^n = f_k^{n+m}$. So we identify the set $\mathcal{R}^n$ with a basis of $\operatorname{Ext}_\Lambda^n(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$; see also [29]. We use this identification without further comment.

Definition 7.10. [26, Section 2]

(1) A closed path C in $\mathcal{Q}$ is a non-trivial path C in $K\mathcal{Q}$ such that $C = vCv$ for some vertex v. We may say that C is a closed path at vertex v.

(2) A closed trail T in $\mathcal{Q}$ is a non-trivial closed path $T = \alpha_1 \cdots \alpha_m$ in $K\mathcal{Q}$ such that $\alpha_1, \ldots, \alpha_m$ are all distinct arrows.

(3) Let p be any path and let q be a closed path in $\mathcal{Q}$. Then p lies on q if p is a subpath of q^s for some $s \geq 1$.

(4) We say that two trails are distinct if neither lies on the other.

We now give sufficient conditions for a finite dimensional Koszul monomial algebra to have **(Fg)**.

Theorem 7.11. *Let $\Lambda = KQ/I$ be a finite dimensional monomial Koszul algebra and let ρ be a minimal generating set for I consisting of quadratic monomials. Suppose* $\operatorname{char} K \neq 2$ *and* $\operatorname{gldim} \Lambda \geq 4$. *Suppose that the following two conditions hold:*

(1) If α is a loop in $\mathcal{Q}$, then $\alpha^2 \in \rho$ but there are no elements in ρ of the form $\alpha\beta$ or $\beta\alpha$ with $\beta \neq \alpha$.

(2) If $T = \alpha_1 \cdots \alpha_m$ is a closed trail in $\mathcal{Q}$ with $m > 1$ such that the set $\rho_T = \{\alpha_1\alpha_2, \ldots, \alpha_{m-1}\alpha_m, \alpha_m\alpha_1\}$ is contained in ρ, then there are no elements in $\rho \setminus \rho_T$ which begin or end with any of the arrows α_j for $j = 1, \ldots, m$.

Then Λ has **(Fg)**.

Proof. Let $\alpha_1, \ldots, \alpha_u$ be all the loops in the quiver $\mathcal{Q}$, and suppose that α_i is a loop at the vertex v_i. Since Λ is a finite dimensional quadratic monomial algebra, α_i^2 is necessarily in the minimal generating set ρ. By hypothesis, we have that there are no elements in ρ of the form $\alpha_i\beta$ or $\beta\alpha_i$ with $\beta \neq \alpha_i$. So there are no overlaps of α_i^2 with any element of $\rho \setminus \{\alpha_i^2\}$. Again, using that Λ is a finite dimensional monomial algebra, it follows that the vertices $v_1, \ldots, v_u$ are distinct.

Let $T_{u+1}, \ldots, T_r$ be all the distinct closed trails in $\mathcal{Q}$ such that for each $i = u+1, \ldots, r$, we have $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$, where $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$ are arrows, and the set $\rho_{T_i} = \{\alpha_{i,1}\alpha_{i,2}, \alpha_{i,2}\alpha_{i,3}, \ldots, \alpha_{i,m_i}\alpha_{i,1}\}$ is contained in ρ. By hypothesis, there are no elements in $\rho \setminus \rho_{T_i}$ which begin or end with any of the arrows $\alpha_{i,j}$ for $j = 1, \ldots, m$. So, for $j = 1, \ldots, m$, no arrow $\alpha_{i,j}$ has overlaps with any element of $\rho \setminus \rho_{T_i}$. Let $T_{i,1}, \ldots, T_{i,m_i}$ be defined by

$$\begin{aligned}
T_{i,1} &= T_i = \alpha_{i,1}\alpha_{i,2} \cdots \alpha_{i,m_i};\\
T_{i,2} &= \alpha_{i,2}\alpha_{i,3} \cdots \alpha_{i,m_i}\alpha_{i,1};\\
&\vdots\\
T_{i,m_i} &= \alpha_{i,m_i}\alpha_{i,1} \cdots \alpha_{i,m_i-1}.
\end{aligned}$$

Then the paths $T_{i,1}, \ldots, T_{i,m_i}$ are all of length m_i and lie on the closed path T_i.

Now we show that Λ satisfies **(Fg1)**.

Since Λ is a Koszul monomial algebra, then Λ is a $(2,1)$-stacked monomial algebra. Using [26, Theorem 3.4], we have $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$, where

- for $i = 1, \ldots, u$, the vertices $v_1, \ldots, v_u$ are distinct and the element x_i corresponding to the loop α_i is in degree 2 and is represented by the map $Q^2 \longrightarrow \Lambda$ where for $R^2 \in \mathcal{R}^2$,

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v_i & \text{if } R^2 = \alpha_i^2 \\ 0 & \text{otherwise.} \end{cases}$$

- and for $i = u+1, \ldots, r$, the element x_i corresponding to the closed path $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$ is in degree $2\mu_i$ such that $\mu_i = m_i / \gcd(2, m_i)$ and is represented by the map $Q^{2\mu_i} \longrightarrow \Lambda$, where for $R^{2\mu_i} \in \mathcal{R}^{2\mu_i}$,

$$\mathfrak{o}(R^{2\mu_i}) \otimes \mathfrak{t}(R^{2\mu_i}) \mapsto \begin{cases} \mathfrak{o}(T_{i,k}) & \text{if } R^{2\mu_i} = T_{i,k}^{2/\gcd(2,m_i)} \text{ for all } k = 1, \ldots, m_i \\ 0 & \text{otherwise.} \end{cases}$$

The action of the homogeneous element $x \in \mathrm{HH}^n(\Lambda)$ on $E(\Lambda)$ is given by left multiplication by $\sum_j R_j^n$ where the sum is over all j such that $x(\mathfrak{o}(R_j^n) \otimes \mathfrak{t}(R_j^n)) \neq 0$ and $n = |x|$. Thus if $x_i \in \mathrm{HH}^2(\Lambda)$ corresponds to the loop α_i, then the action of x_i on $E(\Lambda)$ is given by left multiplication by α_i^2. And if x_i in degree $2\mu_i$ corresponds to the closed path T_i then the action of x_i on $E(\Lambda)$ is given by left multiplication by $\sum_{k=1}^{m_i} T_{i,k}^{2/\gcd(2,m_i)}$.

Let H be the subring of $\mathrm{HH}^*(\Lambda)$ generated by $Z(\Lambda)$ and $\{x_1, \ldots, x_r\}$. We want to show that H is a commutative Noetherian ring. Since $Z(\Lambda) = \mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^*(\Lambda)$ is graded commutative, we know that $zx_i = x_i z$ for all $z \in Z(\Lambda)$ and $i = 1, \ldots, r$. So, using [26, Theorem 3.4] we have that $H = Z(\Lambda)[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$. Hence H is a commutative ring. Moreover, $Z(\Lambda)$ is finite dimensional so is a commutative Noetherian ring. Thus H is a Noetherian ring (see [42, Corollary 8.11]). Therefore Λ satisfies **(Fg1)**.

We claim $E(\Lambda)$ is finitely generated as a left H-module with generating set consisting of all f_i^n with $n \leq \max\{|x_1|, \ldots, |x_r|, |\mathcal{Q}_1|\}$. Let $N = \max\{|x_1|, \ldots, |x_r|, |\mathcal{Q}_1|\}$.

Let $0 \neq y \in E(\Lambda)$. Then y is a linear combination of f_i^n. We consider $y \in f^n$, with $n > N$. So y is a homogeneous element of $E(\Lambda)$ of degree n. Consider the element $R \in \mathcal{R}^n$ which corresponds to $y \in f^n$, where $n > N$. We know R is a maximal overlap sequence. Since Λ is Koszul, then $\ell(R) = n$. So we can write $R = a_1 a_2 \cdots a_n$, such that $a_i a_{i+1} \in \rho = \mathcal{R}^2$ for $i = 1, \ldots, n-1$. We may illustrate R with the following diagram:

$$a_1\, a_2\, a_3\, a_4 \cdots a_{n-1}\, a_n$$

Since $R \in \mathcal{R}^n$ with $n > N$, then there is some repeated arrow. So we choose j, k with k minimal and $k \geq 1$ such that a_j is a repeated arrow, $a_j, \ldots, a_{j+k-1}$ are all distinct arrows and $a_{j+k} = a_j$. Write

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_n).$$

There are two cases to consider.

Case (1): $k = 1$. Then $a_j = a_{j+1}$ and so a_j is a loop. It follows that

$$R = (a_1 \cdots a_{j-1})(a_j a_j)(a_{j+2} \cdots a_n).$$

By hypothesis $a_j^2 \in \rho$ and there is no relation of the form $a_j\beta$ or βa_j with $\beta \neq a_j$. But R is a maximal overlap sequence, so $a_{j-1}a_j \in \rho$ and $a_j a_{j+2} \in \rho$. Hence $a_{j-1} = a_j$ and $a_{j+2} = a_j$. Inductively, we see that $R = a_j^n$.

From above, let x_i be the generator in H corresponding to the loop a_j, so $1 \leq i \leq u$ and $|x_i| = 2$. Then x_i acts on $E(\Lambda)$ as left multiplication by the central element a_j^2. Hence

$$R = \begin{cases} (a_j^2)^{(n/2)} & \text{if } n \text{ even;} \\ (a_j^2)^{(n-1)/2} a_j & \text{if } n \text{ odd.} \end{cases}$$

So

$$R = \begin{cases} (x_i)^{(n/2)} \mathfrak{o}(a_j) & \text{if } n \text{ even;} \\ (x_i)^{(n-1)/2} a_j & \text{if } n \text{ odd} \end{cases}$$

where $\ell(\mathfrak{o}(a_j)) = 0$ and $\ell(a_j) = 1$ and so $|\mathfrak{o}(a_j)| \leq N$ and $|a_j| \leq N$.

Case (2): $k > 1$. We note by our choice of j, k that $a_j \cdots a_{j+k-1}$ is a closed trail of length k. Let $T = a_j \cdots a_{j+k-1}$. We have

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_n)$$

and since R is a maximal overlap sequence we may illustrate R with the following diagram:

$$a_1\ a_2\ a_3\ a_4\ \cdots\ a_{j-1}\ a_j\ a_{j+1}\ a_{j+2}\ \cdots\ a_{j+k-1}\ a_j\ a_{j+k+1}\ a_{j+k+2}\ \cdots\ a_{n-1}\ a_n$$

Then $a_j a_{j+1}, a_{j+1}a_{j+2}, \ldots, a_{j+k-1}a_j$ are all in ρ, so the set ρ_T is contained in ρ. By hypothesis, there are no elements in $\rho \setminus \rho_T$ which begin or end with any of the arrows $a_j, \ldots, a_{j+k-1}$.

Now $a_j a_{j+k+1}$ is also in ρ since all subpaths of R of length 2 are in ρ. So, by hypothesis $a_{j+k+1} = a_{j+1}$. We also have that $a_{j+k+1}a_{j+k+2}$ is in ρ, so $a_{j+1}a_{j+k+2}$ is in ρ. Again, by hypothesis, $a_{j+k+2} = a_{j+2}$. Inductively, we see that R lies on the closed trail T. So we may relabel the trail T so that $R = T^q p$, where $T = a_1 \cdots a_k$, p is a prefix of T with $1 \leq \ell(p) \leq k$, and $n = kq + \ell(p)$.

Thus there is a generator in H which corresponds to this closed trail T. Without loss of generality, suppose that x_r is the generator in H corresponding to T. Let

$$\begin{aligned} T_{r,1} &= T = a_1 a_2 \cdots a_k; \\ T_{r,2} &= a_2 a_3 \cdots a_k a_1; \\ &\vdots \\ T_{r,k} &= a_k a_1 \cdots a_{k-1}. \end{aligned}$$

The action of x_r on $E(\Lambda)$ is left multiplication by

$$T_{r,1}^{2/\gcd(2,k)} + T_{r,2}^{2/\gcd(2,k)} + \cdots + T_{r,k}^{2/\gcd(2,k)}.$$

Suppose first that k is odd. Then $\gcd(2,k) = 1$. So the action of x_r on $E(\Lambda)$ is left multiplication by $T_{r,1}^2 + T_{r,2}^2 + \cdots + T_{r,k}^2$. We have $|x_r| = 2k$. Since $n > N \geq 2k$, it follows that $q \geq 2$. Now $R = T^q p$ with $1 \leq \ell(p) \leq k$. Hence

$$R = \begin{cases} (T^2)^{(q/2)} p & \text{if } q \text{ even;} \\ (T^2)^{(q-1)/2} T p & \text{if } q \text{ odd.} \end{cases}$$

We note that $\ell(p) \leq k \leq N$ and $\ell(Tp) = k + \ell(p) \leq k + k = 2k \leq N$. We now show that $T_{r,l} p = 0$, for all $2 \leq l \leq k$. The element $T_{r,l} p$ in $E(\Lambda)$ can be written as

$$a_l\, a_{l+1}\, a_{l+2}\, a_{l+3} \cdots a_{l-2} a_{l-1} \cdot a_1\, a_2\, a_3 \cdots a_{\ell(p)-1} a_{\ell(p)}$$

where $T_{r,l} = a_l a_{l+1} \cdots a_k a_1 \cdots a_{l-1}$, and $p = a_1 a_2 \cdots a_{\ell(p)}$. If $a_{l-1} \cdot a_1$ is in ρ, then we have a closed trail $a_1 a_2 \cdots a_{l-1}$ of length $l-1$. But $l - 1 < l \leq k$. So this contradicts the minimality of k. Hence $a_{l-1} \cdot a_1$ is not in ρ. So $T_{r,l} p$ does not represent a maximal overlap sequence and hence $T_{r,l} p = 0$ for $2 \leq l \leq k$. Similarly $T_{r,l}^2 p = 0$, for $2 \leq l \leq k$. Hence

$$R = \begin{cases} (x_r)^{(q/2)} p & \text{if } q \text{ even;} \\ (x_r)^{(q-1)/2} T p & \text{if } q \text{ odd.} \end{cases}$$

Suppose now k is even. Then $\gcd(2,k) = 2$. So the action of x_r on $E(\Lambda)$ is left multiplication by $T_{r,1} + T_{r,2} + \cdots + T_{r,k}$, and $|x_r| = k$. Here we have $q \geq 1$. Now $R = T^q p$ with $1 \leq \ell(p) \leq k$ (and $T = T_{r,1}$). We now show that $T_{r,l} p = 0$, for all

$2 \leq l \leq k$. Again, the element $T_{r,l}p$ in $E(\Lambda)$ can be written as

$$a_l\, a_{l+1}\, a_{l+2}\, a_{l+3} \cdots a_{l-2} a_{l-1} \cdot a_1\, a_2\, a_3 \cdots a_{\ell(p)-1} a_{\ell(p)}$$

where $T_{r,l} = a_l a_{l+1} \cdots a_k a_1 \cdots a_{l-1}$, and $p = a_1 a_2 \cdots a_{\ell(p)}$. If $a_{l-1} \cdot a_1$ is in ρ, then we have a closed trail $a_1 a_2 \cdots a_{l-1}$ of length $l-1$, which contradicts the minimality of k. Hence $a_{l-1} \cdot a_1$ is not in ρ. So $T_{r,l}p$ does not represent a maximal overlap sequence and hence $T_{r,l}p = 0$ for $2 \leq l \leq k$. Hence $R = x_r^q p$.

This shows that $E(\Lambda)$ is generated by $f^0, \ldots, f^N$ as a left H-module. Hence Λ satisfies **(Fg2)** and thus Λ has **(Fg)**. $\square$

The next example illustrates the above theorem; however $\{f^0, f^1, \ldots, f^N\}$ is not a minimal generating set of $E(\Lambda)$ as a left H-module in this example.

Example 7.12. Let $\Lambda = K\mathcal{Q}/I$ be the algebra which is given by the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[d, "\alpha_3"] & & & & & \\
5 \arrow[u, "\alpha_5"] & & 4 \arrow[ll, "\alpha_4"] \arrow[r, "\gamma_1"] & 6 \arrow[loop above, "\beta"] \arrow[r, "\gamma_2"] & 7 \arrow[r, "\gamma_3"] & 8 \arrow[r, "\gamma_4"] & 9 \arrow[r, "\gamma_5"] & 10
\end{tikzcd}$$

and $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_4\alpha_5, \alpha_5\alpha_1, \beta^2, \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_3\gamma_4, \gamma_4\gamma_5 \rangle$. Then Λ is a Koszul monomial algebra. The algebra Λ satisfies the conditions of Theorem 7.11 with the loop β and closed trail $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$. From Theorem 7.11, let x_1, x_2 be the generators in H, where x_1 corresponds to the loop β with $|x_1| = 2$ and x_2 corresponds to the closed trail $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ with $|x_2| = 10$. So the action of x_1 on $E(\Lambda)$ is left multiplication given by $x_1 \mapsto \beta^2$, and the action of x_2 on $E(\Lambda)$ is left multiplication given by

$$\begin{aligned} x_2 \quad \mapsto \quad & (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2 + (\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1)^2 + (\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2)^2 + (\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3)^2 \\ & + (\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4)^2. \end{aligned}$$

Moreover $Z(\Lambda) = K$. Here $N = 11$ and so $E(\Lambda)$ is generated by $f^0, \ldots, f^{11}$ as a left H-module.

We identify f^n with $\mathcal{R}^n$ and list the elements of the set $\mathcal{R}^n$, for $n = 0, \ldots, 11$:

$\mathcal{R}^0 = \{e_1, \ldots, e_{10}\}$

$\mathcal{R}^1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$.

$\mathcal{R}^2 = \{\alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_4\alpha_5, \alpha_5\alpha_1, \beta^2, \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_3\gamma_4, \gamma_4\gamma_5\}$

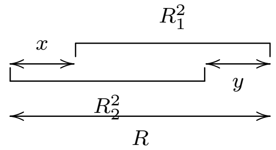
$$\begin{aligned}
\mathcal{R}^3 &= \{\alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5, \alpha_4\alpha_5\alpha_1, \alpha_5\alpha_1\alpha_2, \beta^3, \gamma_1\gamma_2\gamma_3, \gamma_2\gamma_3\gamma_4, \gamma_3\gamma_4\gamma_5\}\\
\mathcal{R}^4 &= \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2\alpha_3\alpha_4\alpha_5, \alpha_3\alpha_4\alpha_5\alpha_1, \alpha_4\alpha_5\alpha_1\alpha_2, \alpha_5\alpha_1\alpha_2\alpha_3, \beta^4, \gamma_1\gamma_2\gamma_3\gamma_4,\\
&\quad \gamma_2\gamma_3\gamma_4\gamma_5\}\\
\mathcal{R}^5 &= \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5, \alpha_2\alpha_3\alpha_4\alpha_5\alpha_1, \alpha_3\alpha_4\alpha_5\alpha_1\alpha_2, \alpha_4\alpha_5\alpha_1\alpha_2\alpha_3, \alpha_5\alpha_1\alpha_2\alpha_3\alpha_4, \beta^5,\\
&\quad \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\}\\
\mathcal{R}^6 &= \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1, \alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3, \alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4,\\
&\quad \alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5, \beta^6\}\\
\mathcal{R}^7 &= \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2, \alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3, \alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5,\\
&\quad \alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1, \beta^7\}\\
\mathcal{R}^8 &= \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5, ,\\
&\quad \alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2, \beta^8\}\\
\mathcal{R}^9 &= \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5, \alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1,\\
&\quad \alpha_4\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2, \alpha_5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3, \beta^9\}\\
\mathcal{R}^{10} &= \{(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2, (\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1)^2, (\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2)^2, (\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3)^2,\\
&\quad (\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4)^2, \beta^{10}\}\\
\mathcal{R}^{11} &= \{(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2\alpha_1, (\alpha_2\alpha_3\alpha_4\alpha_5\alpha_1)^2\alpha_2, (\alpha_3\alpha_4\alpha_5\alpha_1\alpha_2)^2\alpha_3,\\
&\quad (\alpha_4\alpha_5\alpha_1\alpha_2\alpha_3)^2\alpha_4, (\alpha_5\alpha_1\alpha_2\alpha_3\alpha_4)^2\alpha_5, \beta^{11}\}.
\end{aligned}$$

Since $\beta^{11} \in \mathcal{R}^{11}$, then from above β^{11} corresponds to $x_1^5\beta$. Also, we have $(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2\alpha_1$ is in $\mathcal{R}^{11}$. So $(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)^2\alpha_1$ corresponds to $x_2\alpha_1$. Similarly, each element in $\mathcal{R}^{11}$ is in the left H-module generated by $\mathcal{R}^0, \ldots, \mathcal{R}^{10}$. Thus $E(\Lambda)$ is generated by $\mathcal{R}^0, \ldots, \mathcal{R}^{10}$ as a left H-module.

Our final result of this chapter is Theorem 7.15 where we give sufficient conditions for a finite dimensional d-Koszul monomial algebra to have **(Fg)**. First we need two propositions.

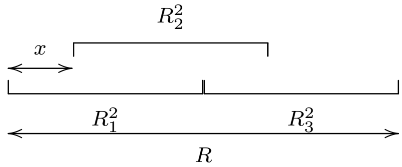
Proposition 7.13. *Let Λ be a finite dimensional monomial d-Koszul algebra, where $d \geq 2$. Suppose that $R \in \mathcal{R}^n$ for some $n \geq 2$. Then all subpaths of R of length d are in ρ.*

Proof. This is clearly true for $n = 2$, so we start by considering $\mathcal{R}^3$. An element $R \in \mathcal{R}^3$ is constructed from R_1^2 which maximally overlaps R_2^2. So that $R = R_2^2 y$ as follows:

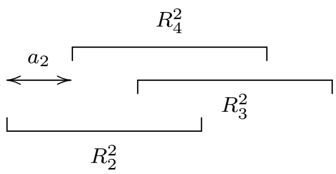


Since Λ is d-Koszul, then $\delta(3) = d + 1$ and so $\ell(R) = d + 1$. So $\ell(x) = 1 = \ell(y)$. Write $R = a_1 a_2 \cdots a_d a_{d+1}$. So there are only two subpaths of length d in R which are $R_2^2 = a_1 a_2 \cdots a_d$ and $R_1^2 = a_2 a_3 \cdots a_{d+1}$. Then R_1^2 and R_2^2 are both in ρ.

Now let $n = 4$. An element $R \in \mathcal{R}^4$ is constructed from a sequence of overlaps. Since Λ is d-Koszul, then $\delta(4) = 2d$ and so $\ell(R) = 2d$. We may illustrate R with the following diagram:



and $R = R_1^2 R_3^2$. Write $R = a_1 \cdots a_d a_{d+1} \cdots a_{2d}$. By hypothesis ($R$ is a maximal overlap sequence so the overlap of R_2^2 with R_1^2 gives an element in $\mathcal{R}^3$ of length $\delta(3) = d+1$), $\ell(x) = 1$ so $x = a_1$ and $R_2^2 = a_2 \cdots a_{d+1} \in \mathcal{R}^2 = \rho$ which maximally overlaps R_1^2. We know R_3^2 overlaps R_2^2. So there is a relation R_4^2 such that R_4^2 maximally overlaps R_2^2 with maximal overlap $R_2^2 a_{d+2} = a_2 R_4^2 \in \mathcal{R}^3$ of length $d + 1$:



So $R_4^2 = a_3 \cdots a_{d+2} \in \rho$. Continuing inductively, we see that all subpaths of R of length d are in ρ:

$$
\begin{aligned}
&a_1 \cdots a_d \\
&a_2 \cdots a_{d+1}, \\
&a_3 \cdots a_{d+2}, \\
&\ \vdots \\
&a_d \cdots a_{2d-1}, \\
&a_{d+1} \cdots a_{2d}.
\end{aligned}
$$

The general case uses a similar argument to that for $n = 4$. For $n \geq 4$, an element $R \in \mathcal{R}^n$ is constructed from a sequence of overlaps as follows:

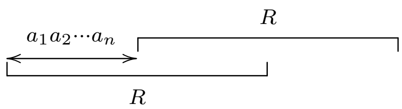
Moreover, since Λ is d-Koszul, then $\ell(R) = \delta(n)$ with δ as in Definition 3.5). So

$$\ell(x) = 1 \text{ and } \ell(y) = \begin{cases} d-1 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

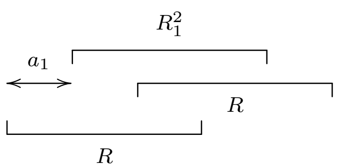
Using the above argument we see inductively that all $n - d + 1$ subpaths of R of length d are in ρ. $\square$

Proposition 7.14. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional monomial d-Koszul algebra, where $d \geq 2$, and let ρ be a minimal generating set for I consisting of monomials of length d. Suppose that $T = a_1 a_2 \cdots a_n$ is a closed trail in $\mathcal{Q}$ so that $a_1, a_2, \ldots, a_n$ are distinct arrows. Suppose also that $d \geq n + 1$. Then all paths of length d which lie on the closed trail T are in ρ.*

Proof. Since Λ is finite dimensional, we know that some subpath R of T is in ρ, and that $\ell(R) = d$. Without loss of generality, we may suppose that $R = (a_1 a_2 \cdots a_n)^m a_1 a_2 \cdots a_s$ for some $1 \leq s \leq n - 1$ with $d = nm + s$. We know that $d \geq n + 1$ so the path R has prefix $a_1 a_2 \cdots a_n a_1$. Thus there is an overlap of R with itself as follows:



So there is a relation R_1^2 such that R_1^2 maximally overlaps R with maximal overlap $Ra_{s+1} = a_1 R_1^2 \in \mathcal{R}^3$ of length $d + 1$:



So $R_1^2 = (a_2 \cdots a_n a_1)^m a_2 \cdots a_s a_{s+1} \in \rho$. Continuing inductively (in the same way as in Proposition 7.13), we see that ρ contains all paths of length d which lie on the closed trail T. $\square$

Note that the set of all paths of length d which lie on the closed trail T in Proposition 7.14 is the set ρ_T of [26] with $A = 1$.

Theorem 7.15. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional monomial d-Koszul algebra, where $d \geq 2$, and let ρ be a minimal generating set for I consisting of monomials of length d. Suppose* $\operatorname{char} K \neq 2$ *and* $\operatorname{gldim} \Lambda \geq 4$. *Suppose that Λ satisfies the following conditions:*

(1) If α is a loop in $\mathcal{Q}$, then $\alpha^d \in \rho$ but there are no elements in ρ of the form $\alpha^{d-1}\beta$ or $\beta\alpha^{d-1}$ with $\beta \neq \alpha$.

(2) If $T = \alpha_1 \cdots \alpha_m$ is a closed trail in $\mathcal{Q}$ with $m > 1$ such that the set $\rho_T = \{\alpha_1 \cdots \alpha_d, \alpha_2 \cdots \alpha_{d+1}, \ldots, \alpha_m\alpha_1 \cdots \alpha_{d-1}\}$ is contained in ρ, then there are no elements in $\rho \setminus \rho_T$ which begin or end with the arrow α_i, for all $i = 1, \ldots, m$.

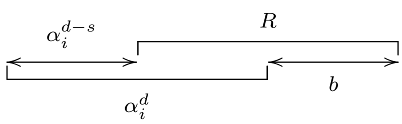
Then Λ has **(Fg)**.

Note: If $T = \alpha_1 \cdots \alpha_m$ is a closed trail then the subscript i of α_i is taken modulo m within the range $1 \leq i \leq m$. Thus ρ_T is the set of all paths of length d which lie on the closed trail T.

Proof. The case where $d = 2$ is the case where Λ is Koszul, and is proved in Theorem 7.11. So we assume here that $d \geq 3$.

Let $\alpha_1, \ldots, \alpha_u$ be all the loops in the quiver $\mathcal{Q}$, and suppose that α_i is a loop at the vertex v_i. Since Λ is a finite dimensional monomial d-Koszul algebra, α_i^d is necessarily in the minimal generating set ρ. By hypothesis, we have that there are no elements in ρ of the form $\alpha_i^{d-1}\beta$ or $\beta\alpha_i^{d-1}$ with $\beta \neq \alpha_i$, for $i = 1, \ldots, u$.

We show that there are no overlaps of α_i^d with any element of $\rho \setminus \{\alpha_i^d\}$. Suppose for contradiction, that $R \in \rho \setminus \{\alpha_i^d\}$ and that R overlaps α_i^d. Then either $R = \alpha_i^s b$ or $R = b\alpha_i^s$ where $1 \leq s \leq d-1$ and b is a path of length $d-s$ which does not begin (respectively, end) with the arrow α_i. Suppose first that $R = \alpha_i^s b$ where α_i is not the first arrow of b. Then R overlaps α_i^d with overlap of length $2d - s$ as follows:



Now, this is a maximal overlap since α_i is not the first arrow of b and thus gives an element $R_1^3 \in \mathcal{R}^3$. However, $\ell(R_1^3) = d + 1$ since Λ is d-Koszul. Thus $2d - s = d + 1$

and so $s = d - 1$. But then $R = \alpha_i^{d-1} b$ and b is an arrow. This is a contradiction to the hypothesis. The second case where $R = b\alpha_i^s$ is similar. So there are no overlaps of α_i^d with any element of $\rho \setminus \{\alpha_i^d\}$.

Again, using that Λ is a finite dimensional monomial algebra, it follows that the vertices $v_1, \ldots, v_u$ are distinct.

Let $T_{u+1}, \ldots, T_r$ be all the distinct closed trails in $\mathcal{Q}$ such that for each $i = u+1, \ldots, r$, we have $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$, where $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$ are arrows, and the set

$$\rho_{T_i} = \{\alpha_{i,1} \cdots \alpha_{i,d}, \alpha_{i,2} \cdots \alpha_{i,d+1}, \ldots, \alpha_{i,m_i}\alpha_{i,1} \cdots \alpha_{i,d-1}\}$$

is contained in ρ.

By hypothesis, for each closed trail T_i $(u+1 \leq i \leq r)$, there are no elements in $\rho \setminus \rho_{T_i}$ which begin or end with the arrow $\alpha_{i,j}$, for all $j = 1, \ldots, m$. So no arrow $\alpha_{i,j}$ has overlaps with any element in $\rho \setminus \rho_{T_i}$.

Let $i \in \{u+1, \ldots, r\}$ and let $T_{i,1}, \ldots, T_{i,m_i}$ be defined by

$$\begin{aligned}
T_{i,1} &= T_i = \alpha_{i,1}\alpha_{i,2} \cdots \alpha_{i,m_i};\\
T_{i,2} &= \alpha_{i,2}\alpha_{i,3} \cdots \alpha_{i,m_i}\alpha_{i,1};\\
&\vdots\\
T_{i,m_i} &= \alpha_{i,m_i}\alpha_{i,1} \cdots \alpha_{i,m_{i-1}}.
\end{aligned}$$

Then the paths $T_{i,1}, \ldots, T_{i,m_i}$ are all of length m_i and lie on the closed path T_i.

Now we show that Λ satisfies **(Fg1)**.

Since Λ is a d-Koszul monomial algebra, then Λ is a $(d,1)$-stacked monomial algebra. By using [26, Theorem 3.4], we have $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$, where

- for $i = 1, \ldots, u$, the vertices $v_1, \ldots, v_u$ are distinct and the element x_i corresponding to the loop α_i is in degree 2 and is represented by the map $Q^2 \longrightarrow \Lambda$ where for $R^2 \in \mathcal{R}^2$,

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \mapsto \begin{cases} v_i & \text{if } R^2 = \alpha_i^d \\ 0 & \text{otherwise} \end{cases}$$

- and for $i = u+1, \ldots, r$, the element x_i corresponding to the closed path $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$ is in degree $2\mu_i$ such that $\mu_i = m_i/\gcd(d, m_i)$ and is represented by the map $Q^{2\mu_i} \longrightarrow \Lambda$, where for $R^{2\mu_i} \in \mathcal{R}^{2\mu_i}$,

$$\mathfrak{o}(R^{2\mu_i}) \otimes \mathfrak{t}(R^{2\mu_i}) \mapsto \begin{cases} \mathfrak{o}(T_{i,k}) & \text{if } R^{2\mu_i} = T_{i,k}^{d/\gcd(d,m_i)} \text{ for all } k = 1, \dots, m_i \\ 0 & \text{otherwise.} \end{cases}$$

Let H be the subring of $\mathrm{HH}^*(\Lambda)$ generated by $Z(\Lambda)$ and $\{x_1, \dots, x_r\}$. We show that H is a commutative Noetherian ring. Since $Z(\Lambda) = \mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^*(\Lambda)$ is graded commutative, we know that $zx_i = x_i z$ for all $z \in Z(\Lambda)$ and $i = 1, \dots, r$. So, using [26, Theorem 3.4] we have that $H = Z(\Lambda)[x_1, \dots, x_r]/\langle x_a x_b \text{ for } a \neq b \rangle$. Hence H is a commutative ring. Moreover, $Z(\Lambda)$ is finite dimensional so is a commutative Noetherian ring. Thus H is a Noetherian ring (see [42, Corollary 8.11]). Therefore Λ satisfies **(Fg1)**.

The rest of this proof shows that Λ satisfies **(Fg2)**. We will show that $E(\Lambda)$ is finitely generated as a left H-module with generating set consisting of all f_i^n with $n \leq \max\{3, |x_1|, \dots, |x_r|, |\mathcal{Q}_1|\}$. Let $N = \max\{3, |x_1|, \dots, |x_r|, |\mathcal{Q}_1|\}$.

Let $0 \neq y \in E(\Lambda)$. Then y is a linear combination of f_i^n. We consider $y \in f^n$, with $n > N$. So y is a homogeneous element of $E(\Lambda)$ of degree n. Consider the element $R \in \mathcal{R}^n$ which corresponds to $y \in f^n$, where $n > N$. Since $n > 2$, we know R is a maximal overlap sequence of length $\delta(n)$ where (from Definition 3.5)

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ even,} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ odd.} \end{cases}$$

So we can write $R = a_1 a_2 \cdots a_{\delta(n)}$. From Proposition 7.13, we know that all subpaths of R of length d are in ρ, that is, $a_1 \cdots a_d, a_2 \cdots a_{d+1}, \dots, a_{\delta(n)-d+1} \cdots a_{\delta(n)}$ all are in ρ. We may illustrate R with the following diagram:

$$a_1\, a_2 \, \cdots \, a_d \, a_{d+1} \cdots \, a_{\delta(n)-d+1} \, \cdots \, a_{\delta(n)}$$

Since $R \in \mathcal{R}^n$ with $n > N$, then there is some repeated arrow. So we choose j, k with k minimal and $k \geq 1$ such that a_j is a repeated arrow, $a_j, \dots, a_{j+k-1}$ are all distinct arrows and $a_{j+k} = a_j$. Write

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_{\delta(n)}).$$

There are two cases to consider.

Case (1): $k = 1$. Then $a_j = a_{j+1}$ and so a_j is a loop. It follows that

$$R = (a_1 \cdots a_{j-1})(a_j a_j)(a_{j+2} \cdots a_{\delta(n)}).$$

Note that $n > N \geq 3$ so that $n \geq 4$. Hence $\delta(n) \geq 2d$.

Suppose first that $j \leq d-1$. Then from Proposition 7.13, $a_j^2 a_{j+2} \cdots a_{j+d-1}$ is in ρ since $j+d-1 \leq \delta(n)$. But $a_j^d \in \rho$ and we have seen earlier in the proof that there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$. Thus $a_j = a_{j+2} = \cdots = a_{j+d-1}$. Inductively we see that $R = (a_1 \cdots a_{j-1})a_j^{\delta(n)-j+1}$. Using Proposition 7.13 again, we know that $a_1 \cdots a_{j-1}a_j^{d-j+1}$ is in ρ and $d-j+1 \geq 2$. And there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$. So $a_j = a_1 = \cdots = a_{j-1}$. Thus we have $R = a_j^{\delta(n)}$.

Now suppose that $j \geq d$. Then $j-d+1 \geq 1$, so by Proposition 7.13, we have that $a_{j-d+1} \cdots a_{j-1}a_j$ is in ρ. Since there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$, it follows that $a_{j-d+1} = \cdots = a_{j-1} = a_j$, so inductively $R = a_j^{j+1}(a_{j+2} \cdots a_{\delta(n)})$. Using Proposition 7.13 again, we know that $a_j^{d-1}a_{j+2}$ is in ρ as $j+1 \geq d-1$. There are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$. So $a_j = a_{j+2}$. Thus inductively, we see that $R = a_j^{\delta(n)}$.

So, for all j, we have that $R = a_j^{\delta(n)}$.

From above, let x_i be the generator in H corresponding to the loop a_j, so $1 \leq i \leq u$ and $|x_i| = 2$. Then x_i acts on $E(\Lambda)$ as left multiplication by the central element a_j^d. Hence

$$R = \begin{cases} (a_j^d)^{(n/2)} & \text{if } n \text{ even;} \\ (a_j^d)^{((n-1)/2)}a_j & \text{if } n \text{ odd.} \end{cases}$$

So

$$R = \begin{cases} (x_i)^{(n/2)}\mathfrak{o}(a_j) & \text{if } n \text{ even;} \\ (x_i)^{((n-1)/2)}a_j & \text{if } n \text{ odd} \end{cases}$$

with $|\mathfrak{o}(a_j)| \leq N$ and $|a_j| \leq N$.

Case (2): $k > 1$. We note by our choice of j, k that $a_j \cdots a_{j+k-1}$ is a closed trail of length k. We denote this closed trail by T.

The first step is to show that ρ_T is contained in ρ, where ρ_T is the set of all paths of length d which lie on the closed trail T. If $d \geq k+1$, then we can use Proposition 7.14 to see that ρ_T is contained in ρ.

Otherwise, suppose that $d \leq k$. Recall that

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_{\delta(n)}).$$

Then:

$$\begin{aligned}
&a_j a_{j+1} \cdots a_{j+d-1}, \\
&a_{j+1} a_{j+2} \cdots a_{j+d}, \\
&\quad \vdots \\
&a_{j+k-d} a_{j+k-d+1} \cdots a_{j+k-1}, \\
&a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j
\end{aligned}$$

are all paths of length d which are subpaths of R. From Proposition 7.13, these paths are all in ρ.

Now $a_j a_{j+1} \cdots a_{j+d-1}$ overlaps $a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j$. So there is a relation $R_1^2 \in \rho$ such that R_1^2 maximally overlaps $a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j$ with maximal overlap of length $d+1$. Then we have that

$$R_1^2 = a_{j+k-d+2} a_{j+k-d+3} \cdots a_{j+k-1} a_j a_{j+1}$$

and this maximal overlap is $(a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j)\, a_{j+1} = a_{j+k-d+1} R_1^2$. Continuing in this way, $a_{j+1} a_{j+2} \cdots a_{j+d}$ overlaps R_1^2. So there is a relation $R_2^2 \in \rho$ such that R_2^2 maximally overlaps R_1^2 with maximal overlap of length $d+1$. So

$$R_2^2 = a_{j+k-d+3} a_{j+k-d+4} \cdots a_{j+k-1} a_j a_{j+1} a_{j+2}$$

and this maximal overlap is $R_1^2 a_{j+2} = a_{j+k-d+2} R_2^2$. Inductively, we see that every path of length d on the closed trail T is in ρ. So ρ_T is contained in ρ.

So for all $k > 1$ we have that ρ_T is contained in ρ. Thus, by hypothesis, there are no paths in $\rho \setminus \rho_T$ which begin or end with any of the arrows $a_j, a_{j+1}, \ldots, a_{j+k-1}$.

The next step is to show that we can write R in the form $R = T^q p$ where p is a prefix of T. We recall that all subpaths of

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_{\delta(n)})$$

of length d are in ρ (Proposition 7.13).

Suppose first that $d \leq k$. Then $a_{j+k-d+2} \cdots a_{j+k-1} a_j a_{j+k+1}$ is a subpath of R of length d which begins with the arrow $a_{j+k-d+2} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So, by hypothesis, this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}$. Inductively, we have $a_{j+k+2} = a_{j+2},\ a_{j+k+3} = a_{j+3}, \dots$. Also, $a_{j-1}a_j \cdots a_{j+d-2}$ is a subpath of R of length d which ends with the arrow $a_{j+d-2} \in \{a_j, a_{j+1}, \dots, a_{j+k-1}\}$. So, by hypothesis, this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}$. Inductively, we have $a_{j-2} = a_{j+k-2},\ a_{j-3} = a_{j+k-3}, \dots$. So we may write $R = p_1 T^q p_2$, where $T = a_j \cdots a_{j+k-1}$, p_1 is a suffix of T and p_2 is a prefix of T.

Now suppose that $d \geq k+1$. We consider $j \leq d-1$ and $j \geq d$ separately. Let $j \leq d-1$. Then, we know that $a_{j+1}a_{j+2} \cdots a_{j+k-1}a_j a_{j+k+1} \cdots a_{j+d}$ is a subpath of R (since $j + d < 2d \leq \delta(n)$) and is of length d and starts with the arrow $a_{j+1} \in \{a_j, a_{j+1}, \dots, a_{j+k-1}\}$. So by hypothesis, this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}, a_{j+k+2} = a_{j+2}, \dots$. Also $a_1 a_2 \cdots a_{j-1} \cdots a_d$ is a subpath of R of length d and starts with the arrow $a_1 \in \{a_j, a_{j+1}, \dots, a_{j+k-1}\}$. So by hypothesis, this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}, a_{j-2} = a_{j+k-2} \dots$. So we may write $R = p_1 T^q p_2$, where $T = a_j \cdots a_{j+k-1}$, p_1 is a suffix of T and p_2 is a prefix of T.

Otherwise $j \geq d$. Then, we know that $a_{j-d+k} \cdots a_{j-1}a_j \cdots a_{j+k-1}$ is a subpath of R of length d and ends with the arrow $a_{j+k-1} \in \{a_j, a_{j+1}, \dots, a_{j+k-1}\}$. So by hypothesis, this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}, a_{j-2} = a_{j+k-2}, \dots$. Also $a_{j+k-d+2} \cdots a_{j-1}a_j \cdots a_{j+k+1}$ is a subpath of R of length d and starts with the arrow $a_{j+k-d+2}$. But we have just shown that $a_{j+k-d+2} \in \{a_j, a_{j+1}, \dots, a_{j+k-1}\}$. So by hypothesis, this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}$. Inductively, $a_{j+k+2} = a_{j+2}, \dots$. So again we may write $R = p_1 T^q p_2$, where $T = a_j \cdots a_{j+k-1}$, p_1 is a suffix of T and p_2 is a prefix of T.

In all cases we have written R as $R = p_1 T^q p_2$, where $T = a_j \cdots a_{j+k-1}$, p_1 is a suffix of T and p_2 is a prefix of T. Without loss of generality, we can relabel the trail T so that $R = T^q p$, where $T = a_1 \cdots a_k$, p is a prefix of T with $1 \leq \ell(p) \leq k$, and $\delta(n) = kq + \ell(p)$.

Thus there is a generator in H which corresponds to this closed trail T. Let x_r be the generator in H corresponding to T. Let

$$\begin{aligned} T_{r,1} &= T = a_1 a_2 \cdots a_k; \\ T_{r,2} &= a_2 a_3 \cdots a_k a_1; \\ &\vdots \\ T_{r,k} &= a_k a_1 \cdots a_{k-1}. \end{aligned}$$

The action of x_r on $E(\Lambda)$ is left multiplication by

$$T_{r,1}^{d/\gcd(d,k)} + T_{r,2}^{d/\gcd(d,k)} + \cdots + T_{r,k}^{d/\gcd(d,k)}.$$

Suppose first that $\gcd(d, k) = 1$. So the action of x_r on $E(\Lambda)$ is left multiplication by $T_{r,1}^d + T_{r,2}^d + \cdots + T_{r,k}^d$, and $|x_r| = 2k$. So $N \geq 2k$. Now $R = T^q p$ with $1 \leq \ell(p) \leq k$. Write $q = cd + w$ with $0 \leq w \leq d - 1$. Then $R = (T^d)^c(T^w p)$. Moreover, $T^w p$ corresponds to an element in $E(\Lambda)$ of length $kw + \ell(p)$. Now $kw + \ell(p) \leq k(d-1) + k = kd$, so $\ell(T^w p) \leq kd$. So $T^w p$ corresponds to an element in $E(\Lambda)$ of degree at most $2k$ since $\delta(2k) = kd$. Thus $T^w p$ corresponds to an element in $E(\Lambda)$ of degree at most N.

More generally, the action of x_r on $E(\Lambda)$ is left multiplication by

$$T_{r,1}^{d/\gcd(d,k)} + T_{r,2}^{d/\gcd(d,k)} + \cdots + T_{r,k}^{d/\gcd(d,k)}$$

and $|x_r| = 2k/\gcd(d, k)$. So $N \geq 2k/\gcd(d, k)$. Now $R = T^q p$ with $1 \leq \ell(p) \leq k$. Write $q = \frac{d}{\gcd(d,k)}c + w$ with $0 \leq w \leq \frac{d}{\gcd(d,k)} - 1$. Then

$$R = \left(T^{d/\gcd(d,k)}\right)^c (T^w p).$$

Moreover, $T^w p$ corresponds to an element in $E(\Lambda)$ of length $kw + \ell(p)$. Now $kw + \ell(p) \leq k\left(\frac{d}{\gcd(d,k)} - 1\right) + k = kd/\gcd(d, k)$, so $\ell(T^w p) \leq kd/\gcd(d, k)$. So $T^w p$ corresponds to an element in $E(\Lambda)$ of degree at most $2k/\gcd(d, k)$ since $\delta(2k/\gcd(d, k)) = kd/\gcd(d, k)$. Thus $T^w p$ corresponds to an element in $E(\Lambda)$ of degree at most N.

Now we show that $T_{r,l}^{d/\gcd(d,k)} p = 0$, for all $2 \leq l \leq k$. Let $2 \leq l \leq k$. We have $T_{r,l} = a_l a_{l+1} \cdots a_k a_1 \cdots a_{l-1}$ and $p = a_1 a_2 \cdots a_{\ell(p)}$. So the element $T_{r,l}^{d/\gcd(d,k)} p$ in $E(\Lambda)$ can be written as

$$(a_l a_{l+1} \cdots a_k a_1 \cdots a_{l-1})^{d/\gcd(d,k)} \cdot a_1 a_2 \cdots a_{\ell(p)}$$

and is a path of length $\frac{kd}{\gcd(d,k)} + \ell(p)$ where $1 \leq \ell(p) \leq k$. If this represents a non-zero element in $E(\Lambda)$, then $\mathfrak{t}(a_{l-1}) = \mathfrak{o}(a_1)$ so that $a_1 \cdots a_{l-1}$ is a closed trail. But $l - 1 < k$, so this contradicts the minimality of k. Hence $T_{r,l}^{d/\gcd(d,k)} p = 0$ in $E(\Lambda)$ for all $2 \leq l \leq k$.

Finally, we recall that we have $R = \left(T^{d/\gcd(d,k)}\right)^c (T^w p)$. Hence $R = x_r^c T^w p$ and $T^w p$ is in the set $\{f^0, f^1, \ldots, f^N\}$.

This shows that $E(\Lambda)$ is generated by $f^0, \ldots, f^N$ as a left H-module. Hence Λ satisfies **(Fg2)** and thus Λ has **(Fg)**. $\square$

We now give two examples to illustrate these results. The first example is of a Koszul monomial algebra and this is the algebra of [14, Example 3.1], where Furuya and Snashall show that Λ has **(Fg)**. The second is a d-Koszul algebra, where we show that the algebra Λ has **(Fg)**.

Example 7.16. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd}
2 \arrow[dd, "\beta"'] & & 4 \arrow[dd, "\eta"] \\
& 1 \arrow[ul, "\alpha"'] \arrow[ur, "\zeta"] & \\
3 \arrow[ur, "\gamma"'] & & 5 \arrow[ul, "\theta"]
\end{tikzcd}$$

and $I = \langle \alpha\beta, \beta\gamma, \gamma\alpha, \zeta\eta, \eta\theta, \theta\zeta \rangle$. Then Λ is a Koszul monomial algebra. The algebra Λ satisfies the conditions of Theorem 7.11 with two closed trails $\alpha\beta\gamma$ and $\eta\zeta\theta$. We have $H = K[x, y]/(xy)$, where $Z(\Lambda) = K$, and x (respectively y) corresponds to the closed trail $\alpha\beta\gamma$ (respectively $\eta\zeta\theta$). The algebra Λ has **(Fg)**. Here $N = 6$ and so $E(\Lambda)$ is generated by $f^0, \ldots, f^6$ as a left H-module.

The action of x on $E(\Lambda)$ is left multiplication given by

$$x \mapsto (\alpha\beta\gamma)^2 + (\beta\gamma\alpha)^2 + (\gamma\alpha\beta)^2,$$

and the action of y on $E(\Lambda)$ is left multiplication given by

$$y \mapsto (\eta\theta\zeta)^2 + (\zeta\eta\theta)^2 + (\theta\zeta\eta)^2.$$

Example 7.17. Let $\Lambda = K\mathcal{Q}/I$ be the 5-Koszul monomial algebra which is given by the quiver

$$\begin{tikzcd}
1 \arrow[r, "a_1"] & 2 \arrow[r, "a_2"] & 3 \arrow[loop above, "b"] \arrow[r, "a_3"] & 4 \arrow[r, "a_4"] & 5
\end{tikzcd}$$

with $I = \langle a_1a_2b^2a_3, b^5, a_2b^2a_3a_4\rangle$. The algebra Λ satisfies the conditions of Theorem 7.15 with the loop b. Moreover, $Z(\Lambda) = K$ and $H = K[x]$, where the element x corresponds to the loop b in degree 2. We have $N = 5$ and so $E(\Lambda)$ is generated by $f^0, \ldots, f^5$ as a left H-module. The algebra Λ has **(Fg)**.

We list the elements of the set $\mathcal{R}^n$, for $n = 0, \ldots, 5$:

$\mathcal{R}^0 = \{e_1, \ldots, e_5\}$

$\mathcal{R}^1 = \{a_1, a_2, b, a_3, a_4\}$.

$\mathcal{R}^2 = \{a_1a_2b^2a_3, b^5, a_2b^2a_3a_4\}$

$\mathcal{R}^3 = \{a_1a_2b^2a_3a_4, b^6\}$

$\mathcal{R}^4 = \{b^{10}\}$

$\mathcal{R}^5 = \{b^{11}\}$.

The action of x on $E(\Lambda)$ is left multiplication given by $x \mapsto b^5$.

It is an open question as to whether the conditions of Theorem 7.11 and of Theorem 7.15 give necessary as well as sufficient conditions for a Koszul monomial algebra (respectively d-Koszul monomial algebra) to have **(Fg)**.

8. Constructing d-Koszul algebras

In this final chapter we give a construction of an algebra $\mathcal{B}$ from a (D, A)-stacked algebra $\mathcal{A}$, where $D = dA$, $A \geq 1$ and $d \geq 2$. One of the main results of this chapter is Theorem 8.4, where we show that the algebra $\mathcal{B}$ we have constructed from a (D, A)-stacked monomial algebra $\mathcal{A}$ is a d-Koszul monomial algebra and we give conditions in Theorem 8.5 under which $\mathcal{A}$ and the stretched algebra $\tilde{\mathcal{B}}$ are isomorphic.

We start with the construction of an algebra $\mathcal{B}$ from a (D, A)-stacked algebra $\mathcal{A}$. Let $\mathcal{A} = K\Gamma/\mathcal{I}$ be a (D, A)-stacked algebra and assume that $D = dA$ for some $d \geq 2$, where $\mathcal{I}$ is generated by a minimal set $\tilde{\rho}$ of homogeneous uniform relations of length D. We fix the set $\tilde{\rho}$ and label the elements of $\tilde{\rho}$ as $\tilde{\rho}_1, \ldots, \tilde{\rho}_m$. We construct a new algebra $\mathcal{B}$ using the quiver Γ and ideal $\mathcal{I}$, and relate $\tilde{\mathcal{B}}$ to the algebra $\mathcal{A}$.

Definition 8.1. We keep the above notation.

(1) Let x in $K\Gamma$ be a linear combination of paths of length D where $D = dA$. We write $x = \sum_k c_k \alpha_{k,1} \cdots \alpha_{k,D}$ where c_k are non-zero elements of K and the $\alpha_{k,j}$ are arrows in Γ. We define the A-subpaths of x to be the paths $\alpha_{k,rA+1} \cdots \alpha_{k,(r+1)A}$ for some k, and $0 \leq r \leq d-1$.

(2) Fix a minimal generating set $\tilde{\rho}$ for $\mathcal{I}$. We define the A-subpaths of $\mathcal{A}$ to be the set of A-subpaths of x where $x \in \tilde{\rho}$. We denote the set of A-subpaths of $\mathcal{A}$ by $\mathcal{S}_A$. Note that we consider $\mathcal{S}_A$ a set with no repeats.

We start by defining a new quiver $\mathfrak{Q}$ and ideal $\mathfrak{I}$ of $K\mathfrak{Q}$ and let $\mathcal{B} = K\mathfrak{Q}/\mathfrak{I}$.

Definition 8.2. Let $\mathcal{A} = K\Gamma/\mathcal{I}$ be a (D, A)-stacked algebra and assume that $D = dA$ for some $d \geq 2$, where $\mathcal{I}$ is generated by a minimal set $\tilde{\rho} = \{\tilde{\rho}_1, \ldots, \tilde{\rho}_m\}$ of uniform relations of length D. For each $i = 1, \ldots, m$, write $\tilde{\rho}_i = \sum_k c_k \alpha_{i,k,1} \cdots \alpha_{i,k,D}$ where c_k are non-zero scalars in K and $\alpha_{i,k,j}$ arrows in Γ, for all $j = 1, \ldots, D$. Then

- The vertex set of $\mathfrak{Q}$ is the set $\{\mathfrak{o}(\tilde{\rho}_i), \mathfrak{t}(y)$ for all $y \in \mathcal{S}_A$ and all $i = 1, \ldots, m\}$. Note that $\mathfrak{t}(\alpha_{i,k,dA}) = \mathfrak{t}(\alpha_{i,k,D}) = \mathfrak{t}(\tilde{\rho}_i)$. This set does not include any repeats, so if $\mathfrak{t}(\alpha_{i,k,rA}) = \mathfrak{t}(\alpha_{j,l,sA})$ as vertices of Γ for some i, j, k, l, r, s then we identify $\mathfrak{t}(\alpha_{j,k,rA})$ and $\mathfrak{t}(\alpha_{i,l,sA})$ as the same vertex in $\mathfrak{Q}$.
- The arrows of $\mathfrak{Q}$ are constructed as follows. Each $y \in \mathcal{S}_A$ corresponds to an arrow β_y in $\mathfrak{Q}$. Recall that we do not include any repeats in $\mathcal{S}_A$.

We illustrate this construction in the following diagram. Consider the path $\alpha_{i,k,1}\cdots\alpha_{i,k,D}$. Then $e_0 = o(\tilde{\rho}_i), e_1 = t(\alpha_{i,k,A}), \ldots, e_d = t(\alpha_{i,k,dA})$ are vertices in $\mathfrak{Q}$ and $\beta_1, \ldots, \beta_d$ are arrows in $\mathfrak{Q}$ corresponding to the A-subpaths $\alpha_{i,k,1}\ldots\alpha_{i,k,A}, \ldots, \alpha_{i,k,(d-1)A+1}\ldots\alpha_{i,k,dA}$ respectively. Then the $\alpha_{i,k,1}\cdots\alpha_{i,k,D}$ may be consider as the path of length D in $K\Gamma$

$$e_0 \xrightarrow{\alpha_{i,k,1}\cdots\alpha_{i,k,A}} e_1 \xrightarrow{\alpha_{i,k,A+1}\cdots\alpha_{i,k,2A}} \ldots \xrightarrow{\alpha_{i,k,(d-1)A+1}\cdots\alpha_{i,k,dA}} e_d$$

which corresponds to the path $\beta_1\cdots\beta_d$ of length d in $K\mathfrak{Q}$

$$e_0 \xrightarrow{\beta_1} e_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_d} e_d$$

- We now define the ideal $\mathfrak{I}$ of $K\mathfrak{Q}$. With the above notation, for each $i = 1, ..., m$, we define $\rho_i = \sum_k c_k \beta_{k1}\cdots\beta_{kd}$ in $K\mathfrak{Q}$ where β_j is the arrow in $K\mathfrak{Q}$ corresponds to $\alpha_{j,k,rA+1}\cdots\alpha_{j,k,(r+1)A}$ for all $j = 1, \ldots, d$ and $r = 0, \ldots, d-1$. We let $\mathfrak{I}$ be the ideal of $K\mathfrak{Q}$ which is generated by the set $\rho = \{\rho_1, \ldots, \rho_m\}$. Note that ρ is necessarily a minimal generating set for $\mathfrak{I}$, since $\tilde{\rho}$ is a minimal generating set for $\mathcal{I}$.
- Now we define $B = K\mathfrak{Q}/\mathfrak{I}$.

We give examples of algebras $\mathcal{A}$ and $\mathcal{B}$ to illustrate different cases where $\tilde{\mathcal{B}}$ is and is not isomorphic to $\mathcal{A}$. This motivates Theorem 8.5. We start by restricting ourselves to the monomial case.

Example 8.3. (1) Let $\mathcal{A}_1 = K\Gamma/\mathcal{I}$ be the $(4, 2)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
 & & & 4 \arrow[r, "\alpha_4"] & 5 & & & & \\
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[ur, "\alpha_3"] \arrow[dr, "\alpha_5"'] & & & & & & \\
 & & & 6 \arrow[r, "\alpha_6"] & 7 \arrow[r, "\alpha_7"] & 8 \arrow[r, "\alpha_8"] & 9 \arrow[r, "\alpha_9"] & 10 \arrow[r, "\alpha_{10}"] & 11
\end{tikzcd}$$

and with $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_5\alpha_6, \alpha_5\alpha_6\alpha_7\alpha_8, \alpha_7\alpha_8\alpha_9\alpha_{10} \rangle$.

The sets g^n are given as follows:

- $g^0 = \{e_1, \ldots, e_{11}\}$, with $g^0_i = e_i$, where $i = 1, \ldots, 11$.
- $g^1 = \{\alpha_1, \ldots, \alpha_{10}\}$, with $g^1_i = \alpha_i$, where $i = 1, \ldots, 10$.

- $g^2 = \tilde{\rho} = \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_5\alpha_6, \alpha_5\alpha_6\alpha_7\alpha_8, \alpha_7\alpha_8\alpha_9\alpha_{10}\}$, with $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $\tilde{\rho}_2 = \alpha_1\alpha_2\alpha_5\alpha_6$, $\tilde{\rho}_3 = \alpha_5\alpha_6\alpha_7\alpha_8$, and $\tilde{\rho}_4 = \alpha_7\alpha_8\alpha_9\alpha_{10}$.
- $g^3 = \{\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8, \alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}\}$, with $g^3_1 = \alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8$ and $g^3_2 = \alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}$.
- $g^4 = \{\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}\}$, with $g^4_1 = \alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}$.

We can now see that the elements $g^n_i \in g^n$ have length $\delta(n)$ for $D = 4$ and $A = 2$, since

- $\ell(g^0_i) = 0$, where $i = 1, \dots, 11$.
- $\ell(g^1_i) = 1$, where $i = 1, \dots, 10$.
- $\ell(g^2_i) = 4$, where $i = 1, 2, 3, 4$.
- $\ell(g^3_i) = 6$, where $i = 1, 2$.
- $\ell(g^4_1) = 8$.

Hence each projective P^n is generated in degree $\delta(n)$. So $\mathcal{A}_1$ is a (D, A)-stacked monomial algebra with $D = 4$, $A = 2$ and $d = 2$. We have $S_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \alpha_7\alpha_8, \alpha_9\alpha_{10}\}$. Then by using the construction above and we have $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $\tilde{\rho}_2 = \alpha_1\alpha_2\alpha_5\alpha_6$, $\tilde{\rho}_3 = \alpha_5\alpha_6\alpha_7\alpha_8$, and $\tilde{\rho}_4 = \alpha_7\alpha_8\alpha_9\alpha_{10}$. So $\mathfrak{Q}$ has six vertices $\{\mathfrak{o}(\alpha_1\alpha_2), \mathfrak{t}(\alpha_1\alpha_2), \mathfrak{t}(\alpha_3\alpha_4), \mathfrak{t}(\alpha_5\alpha_6), \mathfrak{t}(\alpha_7\alpha_8), \mathfrak{t}(\alpha_9\alpha_{10})\}$ and five arrows $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$, where

β_1 corresponds to $\alpha_1\alpha_2$,

β_2 corresponds to $\alpha_3\alpha_4$,

β_3 corresponds to $\alpha_5\alpha_6$,

β_4 corresponds to $\alpha_7\alpha_8$,

β_5 corresponds to $\alpha_9\alpha_{10}$,

Thus $\mathcal{B}_1 = K\mathfrak{Q}/\mathfrak{I}$ is given by the quiver:

$$\begin{tikzcd}
 & & 5 & & \\
1 \arrow[r, "\beta_1"] & 3 \arrow[ur, "\beta_2"] \arrow[dr, "\beta_3"] & & & \\
 & & 7 \arrow[r, "\beta_4"] & 9 \arrow[r, "\beta_5"] & 11
\end{tikzcd}$$

and $\mathfrak{I} = \langle \beta_1\beta_2, \beta_1\beta_3, \beta_3\beta_4, \beta_4\beta_5 \rangle$. Noting that every quadratic monomial algebra is Koszul (see Chapter 3 p24, which references [29]), then $\mathcal{B}_1$ is Koszul. Moreover, $\mathcal{A}_1 \cong \tilde{\mathcal{B}}_1$.

(2) Let $\mathcal{A}_2 = K\Gamma/\mathcal{I}$ be the $(4,2)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & 3 \arrow[r, "\alpha_3"] & 4 \arrow[r, "\alpha_4"] & 5 \arrow[r, "\alpha_5"] & 6 \arrow[r, "\alpha_6"] & 7 \\
& 8 \arrow[r, "\beta_2"] & 9 \arrow[r, "\beta_3"] & 10 \arrow[r, "\beta_4"] & 11 \arrow[r, "\beta_5"] & 12 \arrow[r, "\beta_6"] & 13
\end{tikzcd}$$

and $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6 \rangle$.

The sets g^n are given as follows:

- $g^0 = \{e_1, \ldots, e_{13}\}$, with $g_i^0 = e_i$, for $i = 1, \ldots, 13$.
- $g^1 = \{\alpha_i, \beta_j\}$, for $i = 1, \ldots, 6$ and $j = 1, \ldots, 6$, we label the elements of the set g^1 by $g_1^1, g_2^1, \ldots, g_{12}^1$ in the order they are given here.
- $g^2 = \tilde{\rho} = \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6\}$, with
$$\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4,\ \tilde{\rho}_2 = \alpha_3\alpha_4\alpha_5\alpha_6,\ \tilde{\rho}_3 = \beta_1\beta_2\beta_3\beta_4,\ \tilde{\rho}_4 = \beta_3\beta_4\beta_5\beta_6.$$
- $g^3 = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\}$, with $g_1^3 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ and $g_2^3 = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$.

We can now see that the elements $g_i^n \in g^n$ have length $\delta(n)$ for $D = 4$ and $A = 2$, since

- $\ell(g_i^0) = 0$, where $i = 1, \ldots, 13$.
- $\ell(g_i^1) = 1$, where $i = 1, \ldots, 12$.
- $\ell(g_i^2) = 4$, where $i = 1, 2, 3, 4$.
- $\ell(g_i^3) = 6$, where $i = 1, 2$.

Hence each projective P^n is generated in degree $\delta(n)$. So $\mathcal{A}_2$ is a (D, A)-stacked monomial algebra with $D = 4$, $A = 2$ and $d = 2$. We have $S_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \beta_1\beta_2, \beta_3\beta_4, \beta_5\beta_6\}$. Then by using the construction above, $\mathfrak{Q}$ has eight vertices and six arrows and $\mathcal{B}_2 = K\mathfrak{Q}/\mathfrak{I}$ where $\mathfrak{Q}$ is the quiver:

$$\begin{tikzcd}
1 \arrow[r, "\gamma_1"] & 3 \arrow[r, "\gamma_2"] & 5 \arrow[r, "\gamma_3"] & 7 \\
2 \arrow[r, "\gamma_4"] & 9 \arrow[r, "\gamma_5"] & 11 \arrow[r, "\gamma_6"] & 13
\end{tikzcd}$$

and $\mathfrak{I} = \langle \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_4\gamma_5, \gamma_5\gamma_6 \rangle$. We can see that the algebra $\mathcal{B}_2$ is disconnected.

(3) Let $\mathcal{A}_3 = K\Gamma/\mathcal{I}$ be the $(6,3)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
 & & & & & 6 \arrow[r, "\alpha_6"] & 7 & & & \\
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[r, "\alpha_3"] & 4 \arrow[r, "\alpha_4"] & 5 \arrow[ur, "\alpha_5"] \arrow[dr, "\alpha_7"] & & & & & \\
 & & & & & 8 \arrow[r, "\alpha_8"] & 9 \arrow[r, "\alpha_9"] & 10 \arrow[r, "\alpha_{10}"] & 11 \arrow[r, "\alpha_{11}"] & 12
\end{tikzcd}$$

and with $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_1\alpha_2\alpha_3\alpha_4\alpha_7\alpha_8, \alpha_4\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11} \rangle$.

The sets g^n are given as follows:

- $g^0 = \{e_1, \ldots, e_{12}\}$, with $g_i^0 = e_i$, where $i = 1, \ldots, 12$.
- $g^1 = \{\alpha_1, \ldots, \alpha_{11}\}$, with $g_i^1 = \alpha_i$, where $i = 1, \ldots, 11$.
- $g^2 = \tilde{\rho} = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_1\alpha_2\alpha_3\alpha_4\alpha_7\alpha_8, \alpha_4\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\}$, with $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$, $\tilde{\rho}_2 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_7\alpha_8$, and $\tilde{\rho}_3 = \alpha_4\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}$.
- $g^3 = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}\}$, with $g_1^3 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_7\alpha_8\alpha_9\alpha_{10}\alpha_{11}$.

We can now see that the elements $g_i^n \in g^n$ have length $\delta(n)$ for $D = 6$ and $A = 3$, since

- $\ell(g_i^0) = 0$, for $i = 1, \ldots, 12$.
- $\ell(g_i^1) = 1$, for $i = 1, \ldots, 11$.
- $\ell(g_i^2) = 6$, for $i = 1, 2, 3$.
- $\ell(g_1^3) = 9$.

Hence each projective P^n is generated in degree $\delta(n)$. So $\mathcal{A}_3$ is a (D, A)-stacked monomial algebra with $D = 6$, $A = 3$ and $d = 2$. We have $S_A = \{\alpha_1\alpha_2\alpha_3, \alpha_4\alpha_5\alpha_6, \alpha_4\alpha_7\alpha_8, \alpha_9\alpha_{10}\alpha_{11}\}$. Then by using the construction above, $\mathfrak{Q}$ has five vertices and four arrows. Thus $\mathcal{B}_3 = K\mathfrak{Q}/\mathfrak{I}$ is given by the quiver:

$$\begin{tikzcd}
 & & 7 & \\
1 \arrow[r, "\beta_1"] & 4 \arrow[ur, "\beta_2"] \arrow[dr, "\beta_3"] & & \\
 & & 9 \arrow[r, "\beta_4"] & 12
\end{tikzcd}$$

and $\mathfrak{I} = \langle \beta_1\beta_2, \beta_1\beta_3, \beta_3\beta_4 \rangle$. Noting that every quadratic monomial algebra is Koszul (see [29]), then $\mathcal{B}_3$ is Koszul. However, with $A = 3$ the algebra $\tilde{\mathcal{B}}_3 \ncong \mathcal{A}_3$.

(4) Let $\mathcal{A}_4 = K\Gamma/\mathcal{I}$ be the $(4,2)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[dr, "\alpha_3"] & & 4 \arrow[r, "\beta_5"] & 5 \arrow[r, "\beta_6"] & 6 \\
& & & 7 \arrow[ur, "\beta_4"] \arrow[dr, "\alpha_4"] & & & \\
8 \arrow[r, "\beta_1"] & 9 \arrow[r, "\beta_2"] & 10 \arrow[ur, "\beta_3"] & & 11 \arrow[r, "\alpha_5"] & 12 \arrow[r, "\alpha_6"] & 13
\end{tikzcd}$$

with $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6 \rangle$.

The sets g^n are given as follows:

- $g^0 = \{e_1, \dots, e_{13}\}$, with $g_i^0 = e_i$, where $i = 1, \dots, 13$.
- $g^1 = \{\alpha_i, \beta_j\}$, where $i = 1, \dots, 6$ and $j = 1, \dots, 6$, we label the elements of the set g^1 by $g_1^1, g_2^1, \dots, g_{12}^1$ in the order they are given here.
- $g^2 = \tilde{\rho} = \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6\}$, with $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $\tilde{\rho}_2 = \alpha_3\alpha_4\alpha_5\alpha_6$, $\tilde{\rho}_3 = \beta_1\beta_2\beta_3\beta_4$, and $\tilde{\rho}_4 = \beta_3\beta_4\beta_5\beta_6$.
- $g^3 = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\}$, with $g_1^3 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$, and $g_2^3 = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$.

We can now see that the elements $g_i^n \in g^n$ have length $\delta(n)$ for $D = 4$ and $A = 2$, since

- $\ell(g_i^0) = 0$, where $i = 1, \dots, 13$.
- $\ell(g_i^1) = 1$, where $i = 1, \dots, 12$.
- $\ell(g_i^2) = 4$, where $i = 1, 2, 3, 4$.
- $\ell(g_i^3) = 6$, where $i = 1, 2$.

Hence each projective P^n is generated in degree $\delta(n)$. So $\mathcal{A}_4$ is a (D,A)-stacked monomial algebra with $D = 4$, $A = 2$ and $d = 2$. We have $\mathcal{S}_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \beta_1\beta_2, \beta_3\beta_4, \beta_5\beta_6\}$. Then by using the construction above, we have eight vertices and six arrows. Hence $\mathcal{B}_4 = K\mathfrak{Q}/\mathfrak{I}$ is given by the quiver:

$$\begin{tikzcd}
1 \arrow[r, "\gamma_1"] & 3 \arrow[r, "\gamma_2"] & 11 \arrow[r, "\gamma_3"] & 13 \\
8 \arrow[r, "\gamma_4"] & 10 \arrow[r, "\gamma_5"] & 4 \arrow[r, "\gamma_6"] & 6
\end{tikzcd}$$

and $\mathfrak{I} = \langle \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_4\gamma_5, \gamma_5\gamma_6 \rangle$. We can see that the algebra $\mathcal{B}_4$ is disconnected.

(5) Let $\mathcal{A}_5 = K\Gamma/\mathcal{I}$ be the $(4,2)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
 & & 3 \arrow[r, "\beta_3"] & 4 \arrow[r, "\beta_4"] & 5 \arrow[r, "\beta_5"] & 6 \arrow[r, "\beta_6"] & 7 \arrow[r, "\beta_7"] & 8 \arrow[r, "\beta_8"] & 9 \\
1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\beta_1"'] & 2 \arrow[ur, "\beta_2"] \arrow[dr, "\alpha_2"'] & & & & & & & \\
 & & 10 \arrow[r, "\alpha_3"] & 11 \arrow[r, "\alpha_4"] & 12 \arrow[r, "\alpha_5"] & 13 \arrow[r, "\alpha_6"] & 14 & &
\end{tikzcd}$$

and with $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6, \beta_5\beta_6\beta_7\beta_8 \rangle$.

The sets g^n are given as follows:

- $g^0 = \{e_1, \ldots, e_{14}\}$, with $g_i^0 = e_i$, where $i = 1, \ldots, 14$.
- $g^1 = \{\alpha_i, \beta_j\}$, where $i = 1, \ldots, 6$ and $j = 1, \ldots, 8$, we label the elements of the set g^1 by $g_1^1, g_2^1, \ldots, g_{14}^1$ in the order they are given here.
- $g^2 = \tilde{\rho} = \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4, \beta_3\beta_4\beta_5\beta_6, \beta_5\beta_6\beta_7\beta_8\}$, with $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $\tilde{\rho}_2 = \alpha_3\alpha_4\alpha_5\alpha_6$, $\tilde{\rho}_3 = \beta_1\beta_2\beta_3\beta_4$, $\tilde{\rho}_4 = \beta_3\beta_4\beta_5\beta_6$, and $\tilde{\rho}_5 = \beta_5\beta_6\beta_7\beta_8$.
- $g^3 = \{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6, \beta_3\beta_4\beta_5\beta_6\beta_7\beta_8\}$, with $g_1^3 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$, $g_2^3 = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$ and $g_3^3 = \beta_3\beta_4\beta_5\beta_6\beta_7\beta_8$.
- $g^4 = \{\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7\beta_8\}$, with $g_1^4 = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7\beta_8$.

We can now see that the elements $g_i^n \in g^n$ have length $\delta(n)$ for $D = 4$ and $A = 2$, since

- $\ell(g_i^0) = 0$, where $i = 1, \ldots, 14$.
- $\ell(g_i^1) = 1$, where $i = 1, \ldots, 14$.
- $\ell(g_i^2) = 4$, where $i = 1, \ldots, 5$.
- $\ell(g_i^3) = 6$, where $i = 1, 2, 3$.
- $\ell(g_1^4) = 8$.

Hence each projective P^n is generated in degree $\delta(n)$. So $\mathcal{A}_5$ is a (D,A)-stacked monomial algebra with $D = 4$, $A = 2$ and $d = 2$. We have $\mathcal{S}_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \beta_1\beta_2, \beta_3\beta_4, \beta_5\beta_6, \beta_7\beta_8\}$. Then by using the construction above, we have eight vertices and seven arrows. Hence $\mathcal{B}_5 = K\mathfrak{Q}/\mathfrak{I}$ is given by the quiver:

$$\begin{tikzcd}
 & 3 \arrow[r, "\gamma_2"] & 5 \arrow[r, "\gamma_3"] & 7 \arrow[r, "\gamma_4"] & 9 \\
1 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_5"'] & & & & \\
 & 10 \arrow[r, "\gamma_6"] & 12 \arrow[r, "\gamma_7"] & 14 &
\end{tikzcd}$$

and $\mathfrak{I} = \langle \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_3\gamma_4, \gamma_5\gamma_6, \gamma_6\gamma_7 \rangle$. We note that every quadratic monomial algebra is Koszul (see [29]). Then $\mathcal{B}_5$ is Koszul. However, with $A = 2$ the algebra $\tilde{\mathcal{B}}_5 \ncong \mathcal{A}_5$.

(6) Let $\mathcal{A}_6 = K\Gamma/\mathcal{I}$ be the $(4, 2)$-stacked algebra which is given by the quiver

$$\begin{tikzcd}
 & 2 \arrow[r, "\beta_2"] & 3 \arrow[r, "\beta_3"] & 4 \arrow[r, "\beta_4"] & 5 & & & & \\
1 \arrow[ur, "\beta_1"] \arrow[dr, "\alpha_1"] & & & & & & & & \\
 & 6 \arrow[r, "\alpha_2"] & 7 \arrow[r, "\alpha_3"] & 8 \arrow[r, "\alpha_4"] & 9 \arrow[r, "\alpha_5"] & 10 \arrow[r, "\alpha_6"] & 11 \arrow[r, "\alpha_7"] & 12 \arrow[r, "\alpha_8"] & 13
\end{tikzcd}$$

and with $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7\alpha_8, \beta_1\beta_2\beta_3\beta_4 \rangle$. So we have $\mathcal{S}_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \alpha_7\alpha_8, \beta_1\beta_2, \beta_3\beta_4\}$. Then by using the construction above, we have seven vertices and six arrows. Hence $\mathcal{B}_6 = K\mathfrak{Q}/\mathfrak{I}$ is given by the quiver:

$$\begin{tikzcd}
 & 3 \arrow[r, "\gamma_2"] & 5 & & \\
1 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_3"'] & & & & \\
 & 7 \arrow[r, "\gamma_4"] & 9 \arrow[r, "\gamma_5"] & 11 \arrow[r, "\gamma_6"] & 13
\end{tikzcd}$$

and $\mathfrak{I} = \langle \gamma_3\gamma_4, \gamma_5\gamma_6, \gamma_1\gamma_2 \rangle$. It is clear that $\mathcal{B}_6$ is Koszul (see Theorem 3.3). Moreover, with $A = 2$ the algebra $\tilde{\mathcal{B}}_6 \cong \mathcal{A}_6$.

We keep the notation of Chapter 7 and now prove one of our main results.

Theorem 8.4. *Let $\mathcal{A} = K\Gamma/\mathcal{I}$ be a (D, A)-stacked monomial algebra with* $\operatorname{gldim} \mathcal{A} \geq 4$, *so $D = dA$, for some $d \geq 2$. Let $\mathcal{B}$ be the algebra constructed from $\mathcal{A}$ using Definition 8.2. Then $\mathcal{B}$ is a d-Koszul monomial algebra.*

Proof. Let $\mathcal{B} = K\mathfrak{Q}/\mathfrak{I}$ be the algebra constructed from $\mathcal{A} = K\Gamma/\mathcal{I}$. Then $\mathfrak{I}$ is monomial. Using the notation of overlaps, set

$\tilde{\mathcal{R}}^0 =$ set of vertices of Γ,

$\tilde{\mathcal{R}}^1 =$ set of arrows of Γ,

$\tilde{\mathcal{R}}^2 =$ the minimal set of monomials in the generating set of $\mathcal{I}$ (denoted $\tilde{\rho}$ in Definition 8.2).

and

$\mathcal{R}^0 =$ set of vertices of $\mathfrak{Q}$,

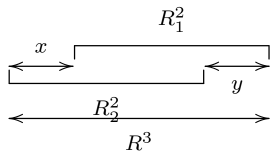
$\mathcal{R}^1 =$ set of arrows of $\mathfrak{Q}$,

$\mathcal{R}^2 =$ the minimal set of monomials in the generating set of $\mathfrak{I}$ (denoted ρ in Definition 8.2).

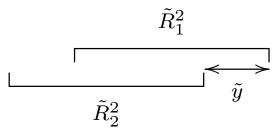
So it is clear that $\ell(R^0) = 0$, $\ell(R^1) = 1$, and $\ell(R^2) = d$, for all $R^0 \in \mathcal{R}^0$, $R^1 \in \mathcal{R}^1$ and $R^2 \in \mathcal{R}^2$. For $n \geq 3$, let $\tilde{\mathcal{R}}^n$ (respectively $\mathcal{R}^n$) denote the set of overlaps in $\mathcal{A}$ (respectively $\mathcal{B}$).

Since $\mathcal{A}$ is a (D, A)-stacked monomial algebra, then the nth projective module in a minimal resolution of $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ is $\tilde{P}^n = \oplus_{\tilde{R}^n \in \tilde{\mathcal{R}}^n} \mathfrak{t}(\tilde{R}^n)\mathcal{A}$ and is generated in degree $\delta(n)$ (see Definition 3.10) for all $n \geq 0$.

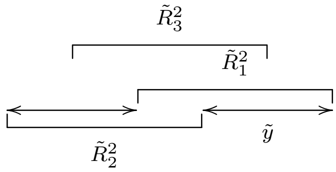
We start by considering $\mathcal{R}^3$. An element $R^3 \in \mathcal{R}^3$ is constructed from R_1^2 which maximally overlap R_2^2 of the form $R^3 = R_2^2 y$ as follows:



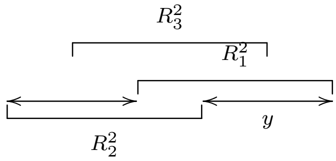
By Definition 8.2, R_1^2 (respectively R_2^2) corresponds to $\tilde{R}_1^2$ (respectively $\tilde{R}_2^2$) in the minimal generating set $\tilde{\mathcal{R}}^2$ for $\mathcal{I}$. So this element gives an overlap of $\tilde{R}_1^2$ with $\tilde{R}_2^2$:



We wish to show that $\tilde{R}_2^2\tilde{y}$ is in $\tilde{\mathcal{R}}^3$. If $\tilde{R}_1^2$ does not maximally overlap $\tilde{R}_2^2$, then we have $\tilde{R}_3^2 \in \tilde{\mathcal{R}}^2$ which maximally overlaps $\tilde{R}_2^2$:

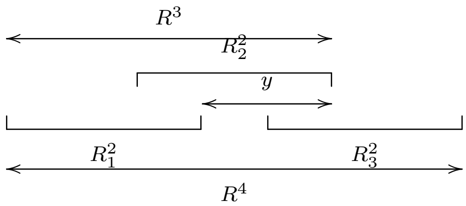


But $\tilde{R}_3^2$ corresponds to some $R_3^2 \in \mathcal{R}^3$ and so R_3^2 overlaps R_2^2:

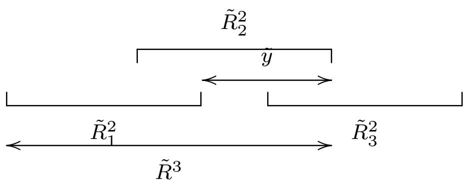


which is a contradiction, since R_1^2 maximally overlaps R_2^2. Hence $\tilde{R}_1^2$ does maximally overlap $\tilde{R}_2^2$ and so $\tilde{R}_2^2\tilde{y} \in \tilde{\mathcal{R}}^3$. Now write $\tilde{R}_2^2\tilde{y} = \tilde{R}^3 \in \tilde{\mathcal{R}}^3$. Since $\ell(\tilde{R}^3) = D + A$ and $\ell(\tilde{R}_2^2) = D$, then $\ell(\tilde{y}) = A$. However, $\tilde{y}$ is a suffix of $\tilde{R}_1^2$ and so $\tilde{y} \in \mathcal{S}_A$. So $\tilde{y}$ corresponds to an arrow in $\mathcal{B}$. Hence $\ell(y) = 1$ and $\ell(R^3) = d + 1$.

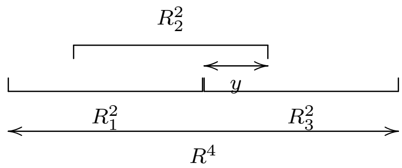
We use a similar argument for the elements of $\mathcal{R}^4$. An element $R^4 \in \mathcal{R}^4$ is constructed from a sequence of overlaps as follows:



From Definition 8.2, R_1^2 (respectively R_2^2, R_3^2) corresponds to $\tilde{R}_1^2$ (respectively $\tilde{R}_2^2, \tilde{R}_3^2$) in the minimal generating set $\tilde{\mathcal{R}}^2$ for I. So using the above argument regarding $\tilde{R}^3$, we have:



and $\ell(\tilde{y}) = A$. So in our construction of R^4 we have $\ell(y) = 1$. But R_3^2 overlaps y, so y must be a prefix of R_3^2. Hence $\mathfrak{t}(R_1^2) = \mathfrak{o}(R_3^2)$ and we have



Then $R^4 = R_1^2 R_3^2$ and $\ell(R^4) = 2d$.

So continuing in this way, by induction for all $n \geq 0$, and all $R^n \in \mathcal{R}^n$ we have

$$\ell(R^n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ even,} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ odd.} \end{cases}$$

Thus $\mathcal{B}$ is d-Koszul, where $\mathcal{B}$ is constructed from $\mathcal{A}$ using Definition 8.2. $\square$

We now give conditions under which $\mathcal{A}$ and $\tilde{\mathcal{B}}$ are isomorphic.

Theorem 8.5. *Let $\mathcal{A}$ be a (D, A)-stacked monomial connected algebra with* gldim $\mathcal{A} \geq 4$, *so $D = dA$, for some $d \geq 2$. Let $\mathcal{B}$ be the algebra constructed from Definition 8.2. Assume that the following conditions hold:*

(1) Each arrow occurs in precisely one A-subpath;

(2) If v is properly internal to some $x \in \mathcal{S}_A$, then

(a) v is not properly internal to $y \in \mathcal{S}_A$ for $y \neq x$.

(b) $v \neq \mathfrak{o}(z)$ and $v \neq \mathfrak{t}(z)$, for all $z \in \mathcal{S}_A$.

Then $\tilde{\mathcal{B}} \cong \mathcal{A}$.

Proof. We define a map $\mathcal{F} : K\Gamma \longrightarrow \tilde{\mathcal{B}}$ and will show that $\mathcal{F}$ is surjective. First, let $v \in K\Gamma$. Suppose first that $v = \mathfrak{o}(\alpha)$ or $\mathfrak{t}(\alpha)$ for some arrow α. Then by (1) α occurs in some subpath $x \in \mathcal{S}_A$. So either $v = \mathfrak{o}(x)$ or $\mathfrak{t}(x)$, for some $x \in \mathcal{S}_A$. Then v corresponds to a vertex in $\mathcal{B}$ and hence to a vertex in $\tilde{\mathcal{B}}$ which we also denote by v. Then we define

$$\mathcal{F}(v) = v.$$

Otherwise, v is properly internal to a unique $x \in \mathcal{S}_A$. Note that by (2) we cannot have $v = \mathfrak{o}(x)$ or $\mathfrak{t}(x)$ for some $x \in \mathcal{S}_A$ and v being properly internal to any element of $\mathcal{S}_A$. So we write $x = \alpha_1 \cdots \alpha_r v \alpha_{r+1} \cdots \alpha_A$ and from Definition 8.2, x corresponds to a unique arrow β in $\mathcal{B}$ and then, by construction of the stretched algebra, we have $\theta(\beta) = \beta_1 \cdots \beta_A$. Hence we define

$$\mathcal{F}(v) = \mathfrak{t}(\beta_r).$$

Let α be an arrow in $\mathcal{A}$, so by hypothesis (1) α occurs in precisely one A-subpath x. So $x = \alpha_1 \cdots \alpha_A$ with $\alpha = \alpha_j$ for some $1 \leq j \leq A$. Then there is an arrow $\beta \in \mathcal{B}$ corresponding to x and we can write $\theta(\beta) = \beta_1 \cdots \beta_A$ in $\tilde{\mathcal{B}}$. So we define

$$\mathcal{F}(\alpha) = \beta_j.$$

It is straightforward to verify that the map $\mathcal{F}$ is well defined.

We extend this to $K\Gamma$ by defining $\mathcal{F}$ to be the linear map with

$$\mathcal{F}(\alpha_1 \cdots \alpha_r) = \mathcal{F}(\alpha_1) \cdots \mathcal{F}(\alpha_r).$$

It is clear from the construction that $\mathcal{F}$ gives a surjective map $K\Gamma \longrightarrow \tilde{\mathcal{B}}$.

Next we show that $\operatorname{Ker} \mathcal{F} = \mathcal{I}$. By our constructions, Definition 8.2 and Definition 4.1, there is a 1-1 correspondence between the elements in the minimal generating set $\tilde{\rho}$ of $\mathcal{I}$, the elements in the minimal generating set ρ of $\mathfrak{I}$ and the elements in the minimal generating set $\tilde{\mathfrak{I}}$ of $\tilde{\mathcal{B}}$ such that $\mathcal{F}(\mathcal{I}) = \tilde{\mathfrak{I}}$ which is zero in $\tilde{\mathcal{B}}$. Thus $\operatorname{Ker} \mathcal{F} = \mathcal{I}$. From the first isomorphism theorem we have $\mathcal{A} = K\Gamma/\mathcal{I} \cong \tilde{\mathcal{B}}$. $\square$

This result is illustrated in Example 8.3 (2) and (8).

Example 8.6. Let $\mathcal{A} = K\Gamma/\mathcal{I}$ be a algebra which is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend right, "\alpha_2"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend left, "\alpha_3"] & 3 \arrow[l, bend left, "\alpha_4"] \end{tikzcd}$$

and $\mathcal{I} = \langle \alpha_1\alpha_2\alpha_1\alpha_2, \alpha_3\alpha_4\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2 \rangle$. This algebra was studied in Example 2.40. Then $\mathcal{A}$ is a $(4, 2)$-stacked algebra. We want to construct the quiver $\mathfrak{Q}$ and ideal $\mathfrak{I}$ of $K\mathfrak{Q}$. We use the construction above, with $\tilde{\rho}_1 = \alpha_1\alpha_2\alpha_1\alpha_2$, $\tilde{\rho}_2 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_1\alpha_2$, $\tilde{\rho}_3 = \alpha_3\alpha_4\alpha_3\alpha_4$. Then we have one vertex $\{\mathfrak{o}(\alpha_1\alpha_2)\}$ and we have two arrows $\{\beta_1, \beta_2\}$, where β_1 corresponds to $\alpha_1\alpha_2$, and β_2 corresponds to $\alpha_3\alpha_4$. So we have the following quiver $\mathfrak{Q}$:

$$\begin{tikzcd} 2 \arrow[loop left, "\beta_1"] \arrow[loop right, "\beta_2"] \end{tikzcd}$$

Again we use the construction above to get $\rho_1 = \beta_1^2$, $\rho_2 = \beta_1\beta_2 - \beta_2\beta_1$, $\rho_3 = \beta_2^2$. We can see that $\mathcal{B}$ is Koszul and $\mathcal{A} \cong \tilde{\mathcal{B}}$.

9. Conclusion

In this thesis we studied Koszul algebras and generalisations of these algebras. We have given the construction of a stretched algebra $\tilde{\Lambda}$ from a finite dimensional algebra Λ. We used a functorial approach to determine the projective resolutions and the projective bimodule resolution of a stretched algebra. We used stratifying ideals to give information on finite generation of the Hochschild cohomology ring for a stretched algebra. We have shown that a d-Koszul algebra satisfies the **(Fg)** finiteness condition if and only if its stretched algebra (which is a (D, A)-stacked algebra) also satisfies the **(Fg)** finiteness condition. We have also given sufficient conditions for a finite dimensional d-Koszul monomial algebra to have **(Fg)**. Furthermore, we have given a construction of an algebra $\mathcal{B}$ from a (D, A)-stacked algebra $\mathcal{A}$, where $D = dA$, $A \geq 1$ and $d \geq 2$.

A further study for research would be to focus on how the cohomology of $\tilde{\mathcal{B}}$ relates to that of $\mathcal{A}$, and investigate whether every d-Koszul non-monomial algebra arises from a (D, A)-stacked non-monomial algebra with $D = dA$ via our construction.

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