Towards a classification of constant length substitution tiling spaces

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Statement

The accompanying thesis submitted for the degree of PhD entitled "Towards a classification of constant length substitution tiling spaces" is based upon work conducted by the author in the department of Mathematics at the University of Leicester during the period between January 2014 and December 2018

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Abstract

We focus on families of hyperbolic attractors for diffeomorphisms of the solid surface of genus two which admit the structure of a substitution tiling space for a substitution of constant length. These spaces are similar to solenoids in that each one admits an almost one to one map onto a solenoid. In particular, we investigate a special class we call difference d substitutions. The simplest families, where $d = 1$, are very close to solenoids in that they can be considered as a solenoid with one path component replaced with a pair of asymptotic path components. Despite this close link to the underlying solenoid, their topological classification is considerably more complicated than those for solenoids, and there is more than one homeomorphism class of difference 1 substitution tiling space corresponding to the same n-adic solenoid. In particular, we show that two difference one constant length substitution tiling spaces arising from the substitutions θ and θ' are homeomorphic if and only if the shift on the subshift Ω_θ is topologically conjugate to the shift on $\Omega_{\theta'}$ or the inverse of the shift on $\Omega_{\theta'}$. Using the known classification of these shifts based on [15], this allows a complete classification of these families.

We then carry out analysis with the general and more complex case where $d > 1$.

The three key elements in our analysis are a detailed analysis of a factor map onto a solenoid based on a construction of Coven and Keane and the classification at which they arrive [15], the structure of affine maps of solenoids and the rigidity of substitution tiling spaces determined by Barge, Swanson [7] and Kwapisz [20].

To the memory of my dear parents and my first teacher Salim Alkutaibi.

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LIST OF ABBREVIATIONS AND SYMBOLS

card	cardinality
clopen	closed and open
det	determinant
dim	dimension
int	interior
cl	closure
$cl\mathcal{O}$	the orbit closure operator
$supp(\mathcal{T})$	support of a tiling $\mathcal{T}$
$susp(f)$	suspension of a function f
mod	modulo
PF	Perron–Frobenius
$\mathcal{A}$	alphabet
ϕ	the empty set
$\mathbb{N}$	the set of non-negative integers
$\mathbb{Z}^+$	the set of positive integers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\approx$	equivalent
$\sim_w$	weak equivalent
$\sim_p$	proximal relation
$:=$	definition
$\simeq$	homeomorphic; isomorphic

CONTENTS

1. INTRODUCTION

We start by defining a dynamical system as a continuous action of an abelian topological group on a compact metric space and define some basic related concepts such as transitivity, minimality and recurrence and investigate some of their relations. A minimal set is a set for which the orbits of its points are dense, and the construction of a minimal sub–dynamical system from the orbit of a point will play a role in later sections. Our main interest is in the dynamical systems arising from actions of $\mathbb{R}^n$ and $\mathbb{Z}^n$. Of particular importance will be the two of minimal systems given by the adding machines and the solenoids we obtain from their suspensions. The adding machines are Cantor sets obtained from the countable product of a finite set of integers with an algebraic structure and endowed with the product topology. The solenoids are then obtained as suspensions of adding machines, and these solenoids are factors of the attractors that form our primary consideration. To put our topological considerations of classification in context, we also consider the measure theoretic classification, which turns out to be significantly simpler.

Smale in [29] introduced the hyperbolic attractor of an expanding smooth map on 3-manifold which is homeomorphic to the dyadic solenoid. Williams ([31], [32], [33]) generalized the solenoid attractor of Smale and characterized the general expanding attractor as an inverse limit of branched manifolds. More specifically, Williams showed that if f is a diffeomorphism of a compact manifold and Λ is an expanding attractor for f, then Λ is the inverse limit of a system:

$$\Lambda = \varprojlim\{M \stackrel{g}{\leftarrow} M \stackrel{g}{\leftarrow} M \stackrel{g}{\leftarrow} ...\}$$

where M is a branched manifold and g is an immersion. We use the notation $\Lambda = \Sigma_g$ for inverse limit where the inverse sequence of spaces and maps are all the same. For such an inverse system, there is a shift map $\sigma : \Sigma_g \to \Sigma_g$ defined by $((x_0, x_1, x_2, ...)) = (g(x_0), x_0, x_1, ...)$. Williams also showed that $f\mid_\Lambda$ is conjugate to this shift map. Williams' characterization of expanding attractors has provided very useful information about the topology of these attractors and the dynamics of $f\mid_\Lambda$. For instance, Λ is locally homeomorphic to the product of a Cantor set and a Euclidean open disk of the dimension of the attractor, the periodic points of $f\mid_\Lambda$ are dense in Λ, and the additional properties which can be deduced from the properties of the conjugate shift map.

A very similar attractor to Smale's original dyadic solenoid example can be constructed on the solid torus corresponding to multiplication by any given integer $n > 1$ on an $n-$adic solenoid. In [14], Coelho, Parry and Williams, examine in detail an attractor of a diffeomorphism of solid surface of genus two which has a one–dimensional, orientable hyperbolic attractor that is not homeomorphic to any $n-$adic solenoid. This attractor has the structure of a substitution tiling space for a substitution tiling over a finite alphabet as detailed in [5].
In this particular case, the substitution is on two letters and is of constant length. This construction will provide the framework for the construction of the attractors that we consider. In particular, by varying the number of times that the handles wind around inside each other, we obtain variations on the original construction of Coelho, Parry and Williams [14] in much the same way that the n–adic solenoid is a generalization of Smale's dyadic solenoid.
The topological classification of one–dimensional solenoids is discussed in section 2.2.

The spaces we consider admit the structure of a substitution tiling space. In fact, Anderson and Putnam [2] have shown that all substitution tiling spaces occur as hyperbolic attractors of diffeomorphisms; that is, Smale–William solenoids.

Barge and Diamond [5] proved that all one-dimensional substitution tiling spaces have a finite and non-empty collection of asymptotic composants (path connected components) and that every orientable hyperbolic one-dimensional attractor is either homogeneous, and hence a solenoid or non-homogeneous, and then a one-dimensional substitution tiling space where the inhomogeneity is clear by the existence of asymptotic composants.

We can associate a symbolic dynamical system in a natural way with any primitive substitution. If the substitution θ is primitive then the symbolic dynamical system associated with θ is that denoted by (X_θ, σ) formed by taking the closure of the shift orbit of any $\theta-$periodic point. The space X_θ is finite if and only if there is a $\theta-$periodic point which is also a shift-periodic. In this case the substitution is also called shift-periodic. We are mainly interested in primitive and shift aperiodic substitutions.

The dynamical system associated with a primitive substitution can be endowed with a Borel probability measure μ. Furthermore, this measure is invariant under the action of the shift σ; that is, $\mu(\sigma^{-1}(B)) = \mu(B)$, for every Borel set B. Indeed this measure is uniquely defined by its values on the cylinder. The measure of the cylinder $[w]$ is defined as the frequency of the finite word w in any element of X which does exist. Let us note that the system (X_θ, σ) is uniquely ergodic: there exists a unique shift invariant measure.

In section 3.5 we will follow the procedures of Coven and Keane [15] to define a factor map between the substitution minimal set of a discrete substitution of constant length n and the adding machine on the $n-$adic integers and show that this factor map is a measure theoretic isomorphism. This result allows us to use the classification of n-adic spaces to measure theoretically classify the discrete substitutions of constant length n :

Theorem[15][17] *There is an almost one to one map from a discrete substitution of length n onto an adding machine on the n-adic integers (Corollary 3.49). Two such adding machines are measure theoretically isomorphic if and only if their lengths have the same prime factors (Theorem 3.50).*

This generalizes directly to the suspensions of the corresponding spaces, to show that the suspensions of discrete constant length substitutions are measure theoretically isomorphic if and only if their lengths have the same prime factors. This factor result is extremely useful when we introduce the special type of discrete substitutions of constant length which we will focus on in chapter 5.

We define a difference one substitution to be primitive, of constant length $n \geqslant 2$ and its difference set J_1 (Definition 3.32) to consist of precisely one element. The more general case, difference $d-$substitution is defined to be primitive, of constant length $n \geqslant 2$ and its difference set J_1 to consist of precisely d elements with an extra condition that, $\theta(0)$ and $\theta(1)$ agree on the first symbol, the final symbol or on both.

For the simplest families (difference one substitutions of constant prime length) the classification proves to be more involved than that of solenoids. In particular, we obtain the following Theorem:

Theorem. (Theorem 5.8) *For any difference one substitutions θ and θ' of prime length p, the suspensions $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic if and only if either the shift of Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ or the shift of Ω_θ is conjugate to the inverse of the shift on $\Omega_{\theta'}$.*

Based on the algorithmic results found in [15] for the classification of such substitutions up to topological conjugacy, this allows us to decide by a definite procedure whether any given two such $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic. The key elements in our classification are a detailed analysis of a factor map onto a solenoid [15], the structure of affine maps of solenoids and the rigidity of substitution tiling spaces determined by Barge, Swanson [7] and Kwapisz [22].

We discuss the more general case (difference d substitutions of constant length and non–prime lengths), but we are not able to obtain corresponding results.

The Structure of this Thesis

This thesis is organized into five chapters. The first chapter is the introduction and in chapters 2,3 and 4, we summaries the basic notions and elementary results that we will be using throughout the thesis specifically in chapter 5.

In Chapter 2, we define basic notions on dynamical systems that arise from an abelian topological group G acting on a compact Hausdorff space X by an action T and provide some general characteristics to be used in the next chapters.

In Chapter 3, we will introduce the symbolic dynamical system and provide some related notions and results. We will focus on the notion of primitive substitutions in more details as the main base for the next chapter. The last section will focus on a special type of substitutions which are the constant length substitutions defined on two letter alphabets.

In Chapter 4, we introduce the general notion of tilings, tiling spaces and related dynamical system. We investigate the topological structure of one-dimensional tiling spaces that arise from substitutions which are compact and connected, everywhere the product of a Cantor set with an arc and each of their arc components is dense.

In Chapter 5, we introduce our main type of constant length substitution tiling spaces which we call difference 1 substitutions and present some notes related to the classification of such spaces.

2. DYNAMICAL SYSTEMS

A dynamical system will be a pair (X, T) where X is a compact metric space (the phase space) and T is a continuous action of a group. In what follows we will restrict our attention to actions of $\mathbb{R}^d$ and $\mathbb{Z}^d$. Also, some abstract characteristics will be given to be used in the next chapters. Many notions and results on dynamical systems work more naturally if G is assumed to be an abelian group, not just a group. An important class of examples are given by the minimal systems of the adding machines and solenoids and their relationship with attractors. The study of the topological properties of dynamical systems is called topological dynamics and is our primary concern here. The study of the "statistical properties" of dynamical systems is called ergodic theory, and the latter notion will be introduced briefly at the end of this chapter.

2.1 Topological dynamical systems

In this section we will only define basic notions on dynamical systems arise from an abelian group acting on a compact Hausdorff space.

Definition 2.1. *Assume that X is an arbitrary set and G is an abelian group. An* action *of G on X is a function $T : G \times X \to X$, $T(g, x) = gx$, such that:*

(1) $T(e, x) = ex = x, \forall x \in X$,

(2) $T(g_1, T(g_2, x)) = g_1(g_2 x) = (g_1 g_2)x = T(T(g_1, g_2), x) \, \forall x \in X$ *and* $g_1, g_2 \in G$.

We will need to add the topological structure to define the required system:

Definition 2.2. *A* dynamical system over an abelian topological group G *or* G*-system is a pair* (X, T) *where:*

(1) X *is non-empty compact Hausdorff space,*

(2) T *is a continuous action of* G *on* X*,*

(3) for every $g \in G$ *the function* $T_g : X \to X, T_g(x) = gx$ *is a homeomorphism.*
X *is said to be the phase space.*

Remark 2.3. *The main types of group action systems we study are when* $G = \mathbb{R}$ *(or in general* $\mathbb{R}^d$*), which is called a flow (or in general a continuous system) and when* $G = \mathbb{Z}$ *(or in general* $\mathbb{Z}^d$*) which is called a discrete system.*

Definition 2.4. *Let* (X, T) *be a dynamical system. The* orbit *of a point* $x \in X$ *is the set* $\mathcal{O}(x) := \{T_g(x) | \, g \in G\}$*. A subset* $Z \subseteq X$ *is said to be* invariant *(under* T*) if the orbit of any point in* Z *still in* Z*, that is, if* $T(Z) \subseteq Z$*, where:*

$$T(Z) = \cup_{z \in Z} \mathcal{O}(z) = \cup_{z \in Z} \{T_g(z) | \, g \in G\}.$$

For an invariant $Z \subseteq X$ the set $cl(Z)$ is also invariant, since T is continuous and hence $T(cl(Z)) \subseteq cl(T(Z)) \subseteq cl(Z)$. It is clear that, for any $x \in X$, $\mathcal{O}(x)$ is an invariant set and hence $(cl(\mathcal{O}(x), T)$ is a sub system of (X, T).

The equivalence classes of the quotient space X/G is said to be the orbit space of X under the action of G. For instance, the action of the integer lattice $\mathbb{Z}^d$ by translations in $\mathbb{R}^d$, the orbit of a point $p \in \mathbb{R}^d$ is the copy of $\mathbb{Z}^d$ translated by the vector p and the quotient space $\mathbb{R}^d/\mathbb{Z}^d$ is homeomorphic to the torus $\mathbb{T}^d$.

Definition 2.5. *[20] A* morphism $\pi : (X, T_1) \to (Y, T_2)$, *where* (X, T_1) *and* (Y, T_2) *are dynamical systems, is a continuous and equivariant function. In other words,* π *is a continuous function and satisfies* $\pi \circ T_1 = T_2 \circ \pi$, *that is the diagram:*

$$\begin{tikzcd} X \arrow[r, "T_1"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "T_2"] & Y \end{tikzcd}$$

is commutative.

If π *is one to one, then* (X, T_1) *is a* subsystem *of* (Y, T_2),
If π *is onto, then* π *is said to be a* factor map *(or semi topological conjugacy),* (Y, T_2) *is said to be a* factor *of* (X, T_1), (X, T_1) *is said to be an* extension *of* (Y, T_2) *and* (X, T_1), (Y, T_2) *are said to be semi conjugate,*
If π *is bijective with continuous inverse (In particular* π *is a homeomorphism), then* π *is said to be a* topological conjugacy,
Two dynamical systems (X, T_1) *and* (Y, T_2) *with topological conjugacy map* π *are said to be topologically conjugate.*

In [15] factor map and topological conjugacy are called a homomorphism and an isomorphism respectively.

Definition 2.6. *[20] Let (X, T), a point $x \in X$ is said to be :*

(1) periodic *if there exists a positive integer n such that $T^n(x) = x$.*

(2) repetitive *if for any neighborhood $U_x \ni x$ there exists $p > 0$, for any $n \geqslant 0$ there exist $k < p$ such that, $T^{n+k}(x) \in U_x$.*

(3) recurrent *if for any neighborhood $U_x \ni x$ there exists $n > 0$ such that $T^n(x) \in U_x$.*

Every dynamical system has a repetitive point.

Remark 2.7. *We have the following relations:*

$$\textit{periodic} \Rightarrow \textit{repetitive} \Rightarrow \textit{recurrent.}$$

2.1.1 Minimality and transitivity

Definition 2.8. *A dynamical system (X, T) is said to be* minimal *if it does not have a proper sub system, that is, if $Y \subseteq X$ is closed and $T-$invariant, then $Y = X$ or $Y = \phi$.*

Every dynamical system (X, T) has a minimal T-invariant subset $Y \subseteq X$.

Proposition 2.9. *[20, Theorem 2.19] A dynamical system $(cl\mathcal{O}(x), T)$ is minimal if and only if x is repetitive.*

From the minimality of the dynamical system $(cl\mathcal{O}(x), T)$ that $cl\mathcal{O}(y) = cl\mathcal{O}(x)$ for all $y \in cl\mathcal{O}(x)$. So, y is repetitive too.

Definition 2.10. *[20] A dynamical system (X, T) is* transitive *if it admits a dense orbit for at least one point $x \in X$. A point $x \in X$ is transitive if its orbit is dense. The set of all transitive points in X is $\{x \in X \mid cl\mathcal{O}(x) = X\}$.*

It is clear that every minimal dynamical system is transitive.

Definition 2.11. *A factor mapping $\pi : X \to Y$ is almost one to one if there is a point $y_0 \in Y$ so that $card\, \pi^{-1}(y_0) = 1$.*

The name of this kind of factor comes from the fact that if (Y, T) is transitive, then $\{y \in Y \mid card(\pi^{-1}(y)) = 1\}$ is a dense G_δ set.

Example 2.12. (Rotations of the torus) Let $\mathbb{T}^n$ be the n-dimensional torus and $a = (a_1, \ldots, a_n) \in \mathbb{T}^n$, define the rotation function on $\mathbb{T}^n$ by $r_a(x)_i = (x_i + a_i)$ *mod* 1. Then $(\mathbb{T}^n, r_a)$ is a dynamical system. In dimension one, let $(\mathbb{T}, r_a)$ be the rotation of the circle then :

(1) If a is rational then every point is periodic and the system does not contain any dense orbit.

(2) If a is irrational then no point is periodic and every orbit is dense hence, the system is minimal.

Proposition 2.13. *[11, Proposition 2.2.1] A dynamical system (X, T) is transitive if and only if for any nonempty open sets U, V in X, there exists a point $g \in G$ such that $T_g(U) \cap V \neq \phi$.*

Proposition 2.14. *[20, Proposition 2.10] Let $(X, T_1), (Y, T_2)$ be dynamical systems and let $\pi : (X, T_1) \to (Y, T_2)$ ba a factor map, if (X, T_1) is minimal (re-*

spectively, transitive, has dense orbit) then so is (Y, T_2).

Definition 2.15. *Let* (X, T) *be a dynamical system, a point* $x \in X$ *is said to be an* equicontinuous point *of* (X, T) *if:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall g \in G, d(T_g(y), T_g(x)) < \varepsilon.$$

The system (X, T) *is* equicontinuous *if each point of* X *is equicontinuous.*

Remark 2.16. *[20, p. 60] If* (X, T) *is a dynamical system with* T *is an isometry (that is,* $\forall g \in G$, $d(x, y) = d(T_g(x), T_g(y))$ *for all* $x, y \in X$*) then* (X, T) *is equicontinuous.*

Proposition 2.17. *[20, Corollary 2.34] A transitive equicontinuous dynamical system is minimal.*

Proposition 2.18. *[20, Proposition 2.43]*

(1) A subsystem of an equicontinuous system is equicontinuous.

(2) The product of equicontinuous systems is equicontinuous.

Definition 2.19. *A dynamical system* (X, T) *is said to be* expansive *if:*

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists g \in G, d(T_g(x), T_g(y)) \geqslant \varepsilon.$$

Definition 2.20. *Let* $(X, T), (Y, T')$ *be dynamical systems. A factor map* $\pi : (X, T) \to (Y, T')$ *is said to be* equicontinuous factor map *if* (Y, T') *is equicontinuous. (that is the image* $\pi(X, T)$ *is equicontinuous).*

And π is said to be maximal equicontinuous factor map *if for any equicontinuous factor $\tilde{\pi} : (X,T) \to (Z,T'')$ there exists a unique factor map $\psi : (Y,T') \to (Z,T'')$ such that $\psi \circ \pi = \tilde{\pi}$.*

$$\begin{tikzcd}
(\mathbf{X},\mathbf{T}) \arrow[r, "\pi"] \arrow[dr, "\tilde{\pi}"'] & (\mathbf{Y},\mathbf{T}') \arrow[d, "\psi"] \\
 & (\mathbf{Z},\mathbf{T}'')
\end{tikzcd}$$

The image of the maximal equicontinuous factor map is said to be the structure system *([15]) of the original system.*

Proposition 2.21. *[20, Theorem 2.44] Any dynamical system (X,T) possesses a maximal equicontinuous factor (Y,T') which is unique up to topological conjugacy.*

Definition 2.22. *[30] Let (X,T) be a dynamical system where G acting on the compact metric space (X,d), two points $x,y \in X$ are said to be* proximal, *denoted $x \sim_p y$ if:*

$$\inf_{g\in G} d(T_g(x),T_g(y)) = 0$$

Otherwise, x,y are said to be distal.

Proposition 2.23. *[20, Proposition 2.47] Let (X,T) be a minimal system and let $\pi : (X,T) \to (Y,T')$ be the maximal equicontinuous factor map. If $x,y \in X$ are proximal then $\pi(x) = \pi(y)$.*

In general, $\sim_p$ is not an equivalence relation on X.

Proposition 2.24. *[20, Theorem 2.49] Let (X, T) be minimal and let $\pi : (X, T) \to (Y, T')$ be an almost one to one extension of an abelian compact group Y, where, $T'(y) = a + y$ for some $a \in Y$. Then (Y, g) is a maximal equicontinuous factor of (X, T).*

Proposition 2.25. *[30, p. 400] If (X, T) is an almost one to one extension of (Y, T') the proximal relation is an equivalence relation.*

Proposition 2.26. *[20, Theorem 2.56] Two minimal equicontinuous systems are topologically conjugate if and only if their maximal equicontinuous factors are topologically conjugate.*

2.2 Adding machines and solenoids

In this section we will give a brief description to two examples of dynamical systems which will play a role in the rest of this work, the adding machines and solenoids which form special classes of minimal sets. This section will provide the general definitions and the relation between these two notions with some properties.

2.2.1 Adding machines

Definition 2.27. *[9] Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a sequence of integers such that each $\alpha_i \geqslant 2$. Let $\Delta_\alpha = \{(x_1, x_2, \dots) \mid$ for each $i, x_i \in \{0, 1, \dots, \alpha_i - 1\}\}$ then*

Δ_α *(endowed with the product topology) is called an* adding machine space*.*

Define the metric d_α on Δ_α such that for $x = x_1x_2\ldots$, $y = y_1y_2\ldots$ in Δ_α:

$$d_\alpha(x, y) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}, \textit{where } d(x_i, y_i) = \begin{cases} 0 & \text{if } x_i = y_i, \\ 1 & \text{if } x_i \neq y_i. \end{cases}$$

The addition on Δ_α is defined by :
$x + y = (x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots)$, where:
$z_1 = (x_1 + y_1) \bmod \alpha_1$
$z_2 = (x_2 + y_2 + r_1) \bmod \alpha_2$ and r_1(the remainder) is 0 if $x_1 + y_1 < \alpha_1$ and 1 if $x_1 + y_1 \geqslant \alpha_1$, and so on.

Define the adding machine map $\tau : \Delta_\alpha \to \Delta_\alpha$ (or add one map) as:
$\tau(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) + (1, 0, 0, \ldots)$ or simply $\tau(x) = x + \mathbf{1}$. This map is a homeomorphism.

The system (Δ_α, τ) is called the adding machine dynamical system or simply the adding machine.

If we consider each α_i set to be the set $\mathbb{Z}_{\alpha_i}$, the set of integers modulo α_i, then:

$$\Delta_\alpha = \prod_{i=0}^{\infty} \{0, 1, \ldots, \alpha_i - 1\} = \prod_{i=0}^{\infty} \mathbb{Z}_{\alpha_i}.$$

When each α_i is a constant n then Δ_α is called the $n-$adic integers, denoted by $Z(n)$ and when $n = 2$ then Δ_α is called the dyadic or the $2-$adic adding machine that is:

$$Z(n) := \prod_{i=0}^{\infty} \mathbb{Z}_n = \left\{ \sum_{i=0}^{\infty} z_i \, n^i \mid \text{ for all } i, \; z_i \in \{0, 1, \ldots, n-1\} \right\}.$$

The adding machines are infinite minimal sets where each point is recurrent:

Proposition 2.28. *[9, Corollary 2.5] Let $f : X \to X$ be a continuous map of a compact Hausdorff space to itself. There is a sequence α of prime numbers such that f is topologically conjugate to the adding machine map τ if and only if X is an infinite minimal set for f and each point of X is recurrent.*

2.2.2 Solenoids

Solenoids can be defined in a few different ways. We will introduce some of these definitions but we are primarily interested in the definition of a solenoid as the suspension of an adding machine.

Definition 2.29. *[12] An $n-$*dimensional solenoid *is an inverse limit space:*

$$\Sigma := \varprojlim \{ f_i : M_i \to M_{i-1} \}$$

where for each $i \geqslant 1$, M_i is a compact, connected (i.e. continuum), $n-$dimensional manifold without boundary, and $f_i : M_i \to M_{i-1}$ is a $k-$fold covering, $k \geqslant 2$.

$$M_0 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \ldots$$

M_0 is called the base space, for each i, f_i is called a bonding map. The set $F = \{ f_i \mid f_i : M_i \to M_{i-1}, i \geqslant 1 \}$ is said to be a presentation of the solenoid.

If each covering map $\pi_n : M_n \to M_1, n \geqslant 2$ *defined as* $\pi_n = f_1 \circ f_2 \circ \cdots \circ f_n$ *is a proper (regular in [26]) covering map (the inverse image* $\pi_n^{-1}(A) \in M_n$ *is compact for each compact subset* $A \in M_1$ *) then, the solenoid is said to be a McCord solenoid.*

We exclude the case when $k = 1$ since the $1-$fold covering map is a homeomorphism. We are interested in $1-$dimensional solenoids and in the following we will illustrate some famous examples.

Example 2.30.

(1) Vietoris solenoid : Is a $1-$dimensional solenoid where the base space is the circle, that is, $M_0 = S^1$.

A special type of Vietoris solenoids is the dyadic solenoid (or $2-$adic solenoid or just $2-$solenoid) : Let $P = (p_1, p_2, \dots)$ be a sequence of positive integers, let $M_0 = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and for each $i \geqslant 1$, $f_i : M_i \to M_{i-1}$ defined as $f_i(z) = z^{p_i}$. Since any positive integer can be written as a product of prime factors, the sequence P can be considered as a sequence of prime numbers.

$$\Sigma_P = \varprojlim\{S^1 \xleftarrow{f_1(z)=z^{p_1}} S^1 \xleftarrow{f_2(z)=z^{p_2}} S^1 \xleftarrow{f_3(z)=z^{p_3}} \dots\}$$

When $p_i = 2$ for each i, that is $P = (2, 2, 2, \dots)$, then the inverse limit space :

$$\Sigma(2) := \varprojlim\{S^1 \xleftarrow{f(z)=z^2} S^1 \xleftarrow{f(z)=z^2} S^1 \xleftarrow{f(z)=z^2} \dots\}$$

The $n-$addic solenoid (or $n-$solenoid) $\Sigma(n)$ can be defined similarly.

(2) Geometrical approach: Let $\mathbb{T}_{\mathbb{S}}^2$ denote the solid torus and let $\mathbb{T}_{\mathbb{S}_1}^2, \mathbb{T}_{\mathbb{S}_2}^2, \dots$ be a sequence of solid tori such that, the function f_i wrapping $\mathbb{T}_{\mathbb{S}_{i+1}}^2$ around inside $\mathbb{T}_{\mathbb{S}_i}^2$ longitudinally p_i times without folding back where each p_i is a prime

number. Then the inverse limit:

$$\Sigma_P = \varprojlim\{f_i : \mathbb{T}_{\mathbb{S}_i}^2 \to \mathbb{T}_{\mathbb{S}_{i+1}}^2\}$$

is a one dimensional solenoid (figure 2.1).

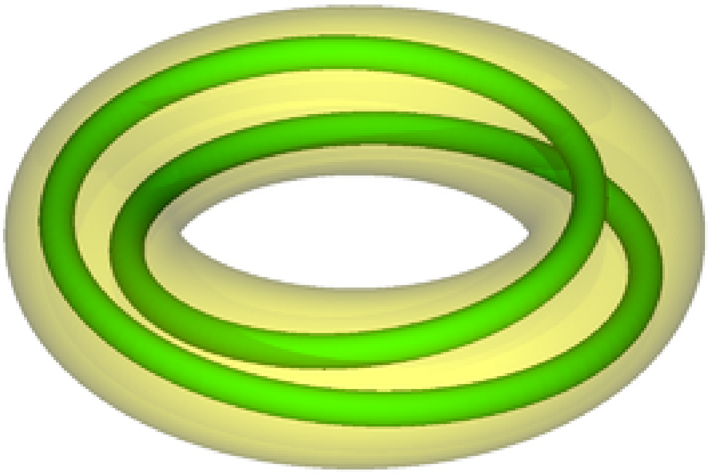


Fig. 2.1: Construction of dyadic solenoid (f_1)

The cross (or transverse) section of the torus ${\mathbb{T}_\mathbb{S}}^2 = D \times S^1$ is the disk D but the cross section of the solenoid is an infinite, totally disconnected and perfect set of points, that is, a Cantor set.

Definition 2.31. *The* suspension *of the homeomorphism $h : X \to X$ is the quotient space*

$$susp(h) := X \times [0,1]/\approx$$

where $\approx$ is the equivalence relation generated by the relation:

$$(x,0) \approx (h(x),1).$$

Remark 2.32. *The cross section of a solenoid is a Cantor set (respectively, adding machine). By following the arc components in one of the two possible* *directions, we can define a return map (Poincaré return map) from this section to itself which corresponds to the adding machine map. Thus, for a given* $n \geq 2$*, the* $n-$*solenoid can be defined as* $\Sigma(n) := susp(\tau)$*, the suspension of the adding machine map* $\tau : Z(n) \to Z(n)$*.*

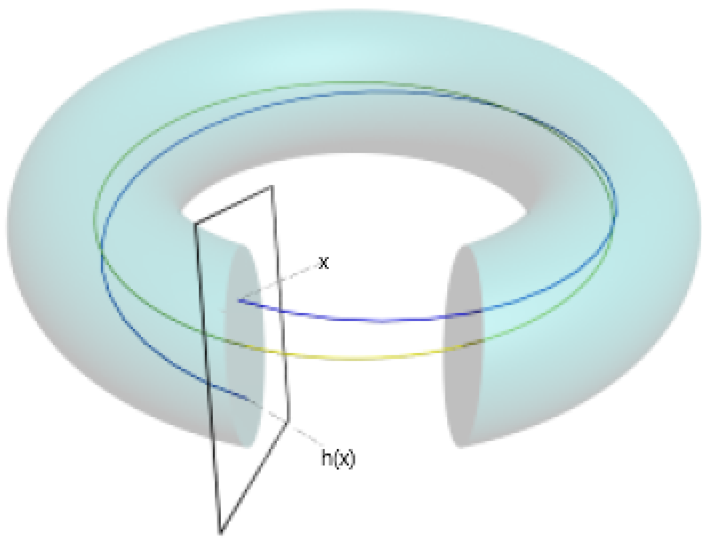


Fig. 2.2: Cross section and first return map

In the following, the definitions of the first return equivalent, the equivalence of sequences and the theorem of classification of solenoids as mentioned in [1].

Definition 2.33. *[1] Let* $p = (p_1, p_2, ...)$ *and* $q = (q_1, q_2, ...)$ *be sequences of primes. The adding machines* (Δ_p, τ) *and* (Δ_q, τ) *are* first return equivalent *if there exist clopen(closed and open) subsets* U *and* V *of* Δ_p *and* Δ_q *, respectively, such that* $(U, \tau\mid_U)$ *and* $(V, \tau\mid_V)$ *are conjugate, that is, there is a homeomorphism* $h : U \longrightarrow V$ *such that* $h \circ \tau \mid_U = \tau \mid_V \circ h$ *.*

We say two sequences p and q of primes are equivalent, and write $p \sim q$, if a finite number of primes can be deleted from each sequence so that every prime number occurs the same number of times in the deleted sequences.

Theorem 2.34. *[1, p.1162] Let* $p = (p_1, p_2, ...)$ *and* $q = (q_1, q_2, ...)$ *be sequences of primes. The following conditions are equivalent:*
(a) The spaces Σ_p *and* Σ_q *are homeomorphic.*
(b) The adding machines (Δ_p, τ) *and* (Δ_q, τ) *are first return equivalent.*
(c) $p \sim q$.

We are interest in the system $(Z(n), \tau)$ and the solenoid obtained from suspension of this system, i.e. $\Sigma(n) = \mathrm{susp}(Z(n), \tau)$, after that we will use the notation "adding machines" and "solenoid" restrictively to denote such systems.

2.3 Attractors

Definition 2.35. *Let* M *be a compact topological space, and* $f : M \to M$ *be a continuous map. A closed region* $N \subset M$ *is called a* trapping region *for* f *provided* $f(N) \subset int(N)$. *A set* Λ *is called an* attracting set *provided there is a trapping region* N *such that* $\Lambda = \bigcap_{k>0} f^k(N)$.

Definition 2.36. *Let* X *be a metric space with metric function* d. *A map* $f : X \longrightarrow X$ *is called a* contracting map *if for some* $c < 1$ *we have* $d(f(x), f(y)) \leqslant c\, d(x, y)$ *for all* x, y *in* X.

Proposition 2.37. *Contraction principle [10, p. 244]*
Let X be a complete metric space. If $f : X \longrightarrow X$ is a contracting map then there is a unique point $x \in X$ which is fixed under f, that is, $f(x) = x$.

Definition 2.38. *[31] Let M be a compact manifold and $f \in Diff^r(M), r \geqslant 1$. Let $x \in M$, define the* stable *(respectively* unstable*)* manifold *at f relative to x, $W^s(x, f)$ (respectively $W^u(x, f)$) such that:*

$$W^s(x, f) = \{y \in M \mid lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$$

$$W^u(x, f) = W^s(x, f^{-1}).$$

A closed invariant subset $\Lambda \subseteq M$ has a hyperbolic structure *$E^u + E^s$ provided the tangent bundle $\mathsf{T}M$ restricted to Λ splits as a direct sum, $\mathsf{T}M \mid_\Lambda = E^u + E^s$, which is invariant under the derivative $\mathsf{T}f$ of f and such that $\mathsf{T}f \mid_{E^u}$ is an expansion and $\mathsf{T}f \mid_{E^s}$ is contraction.*

Definition 2.39. *[31] Let M be a compact manifold and $f \in Diff^r(M), r \geqslant 1$. A subset $\Lambda \subset M$ is said to be an attractor if there is a closed neighborhood N such that:*

(1) $f(N) \subset int(N)$, that is, N is a trapping region for f,

(2) $\Lambda = \bigcap_{k\geqslant 0} f^k(N)$, that is, Λ is an attracting set,

(3) $f \mid_\Lambda$ is transitive.

Remark 2.40. *[20, p. 77] Let $\Lambda = \bigcap_{k>0} f^k(N)$ be an attractor then:*

(1) Λ is invariant under f, that is, $f(\Lambda) = \Lambda$.

(2) The orbit of any point $x \in N$ converges to Λ.

Definition 2.41.

(1) An attractor Λ with hyperbolic structure is said to be a hyperbolic attractor*.*

(2) A hyperbolic attractor Λ with hyperbolic structure $E^u + E^s$ is said to be expanding *if $dim\Lambda = u$.*

Smale in [29] introduced the hyperbolic attractor of an expanding smooth map on 3-manifold which is homeomorphic to the dyadic solenoid. Williams ([31], [32], [33]) generalized the solenoid attractor of Smale to the higher genus cases by replacing the compact manifold of Smale (the circle S^1 in the dyadic solenoid) with a branched 1-manifold (the wedge of two circles in [14]).

Williams represented his expanding attractor as a special type of solenoids that is, as the inverse limit:

$$\Lambda = \Sigma_g = \varprojlim\{M \xleftarrow{g} M \xleftarrow{g} M \xleftarrow{g} \dots\}$$

where M is a branched manifold and g is an immersion defined with the shift map $\sigma_g : \Sigma_g \to \Sigma_g$ defined by:

$$\sigma_g((x_0, x_1, x_2, \dots)) = (g(x_0), g(x_1), g(x_2), \dots)) = (g(x_0), x_0, x_1, \dots).$$

The new solenoid is said to be Smale–Williams solenoid.

Remark 2.42. *[31, p.171] Let Λ be a Smale–Williams solenoid, which is an expanding attractor for $f \in$ Diff (M), then:*

(1) $f \mid_\Lambda$ is topologically conjugate to the shift map σ_g.

(2) Λ is locally homeomorphic to the product of a Cantor set and $\mathbb{R}^d$, where $d =$ dim(Λ).

(3) The periodic points of $f\mid_\Lambda$ is dense in Λ.

In [14], Coelho, Parry and Williams, examine in detail an attractor of a diffeomorphism of solid surface of genus two that admits the structure of a substitution tiling space for a substitution of constant length. which represents our main interest (Chapter 4).

Example 2.43. [14] Consider the (oriented) 1−dimensional branched manifold M as the wedge of two circles:

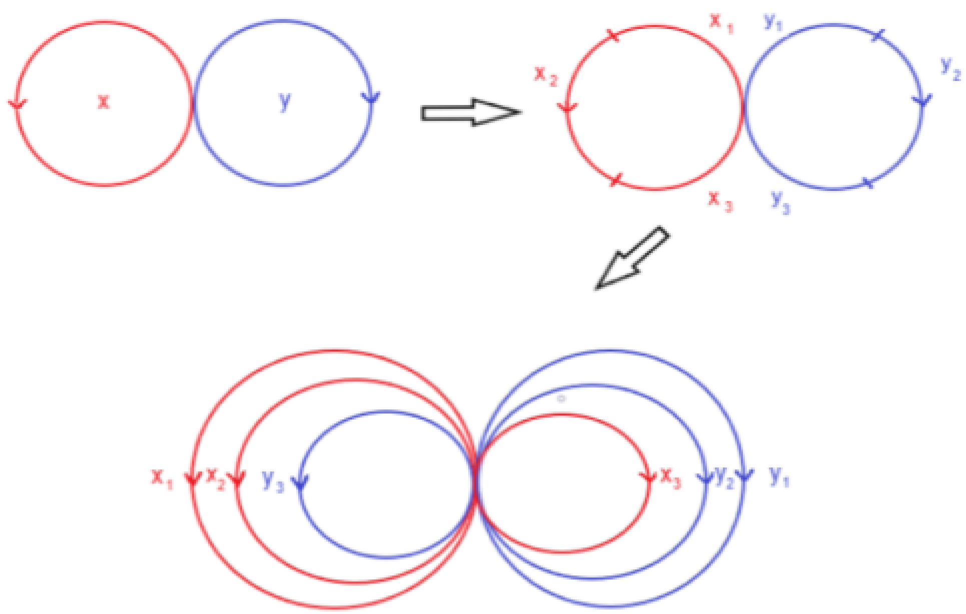


Fig. 2.3: The map g

and the map g (figure 2.3) which expands the first circle (by a factor of 3) and wraps each of x_1, x_2 successively around $x\,(x = x_1x_2x_3)$ and x_3 around $y\,(y = y_1y_2y_3)$ preserving orientation. In short $g : x \rightarrow x^2y$. Similarly $g : y \rightarrow xy^2$. We can simulate this behaviour by constructing a diffeomorphism f of the solid of genus 2:

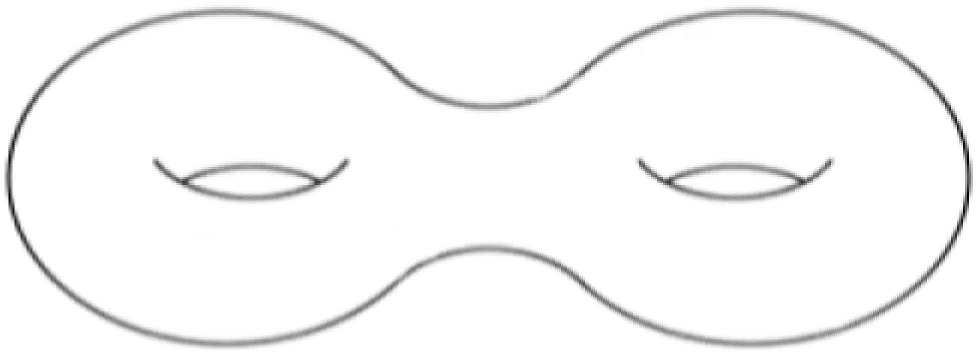


Fig. 2.4: the solid torus of genus 2

obtained by thickening the one dimensional branched manifold. The diffeomorphism f expands (like g) in one direction and contracts the transverse discs. Mimicking g we have the following picture of the manifold and its image under f:

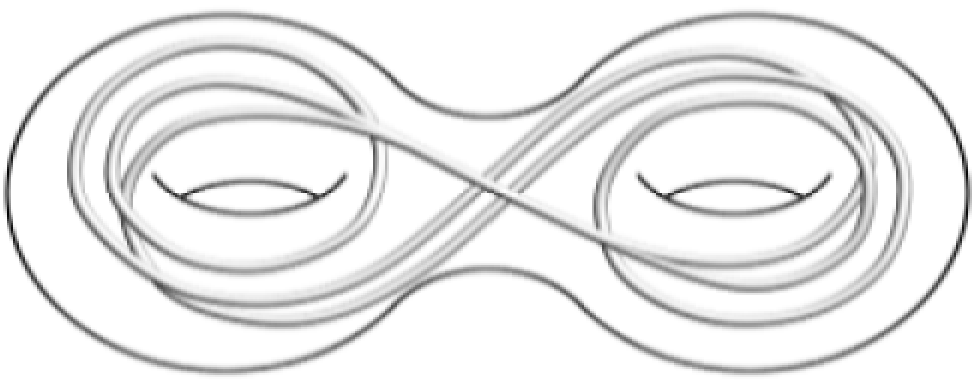


Fig. 2.5: The diffeomorphism f

Williams ([31],[33]) show that the attractor $\Lambda = \bigcap_{k>0} f^k(M)$ is topologically conjugate to the shift $\sigma_g : \Sigma_g \to \Sigma_g$ on the inverse limit solenoid defined by $g : M \to M$ which is locally the direct product of a Cantor set and an arc. The branch point presents no problem here since the image under g of a (branched) neighbourhood is an arc and f is hyperbolic ([33]). Clearly there is a projection (contracting transverse discs) which maps M onto Σ_g which in turn projects to M. Hence any Hölder continuous function defined on M can be lifted to Σ_g or M.

2.4 Measure theoretic dynamical systems

As we will use the measure theoretic isomorphism as a comparison tool we will introduce here some basic notions which support this purpose.

Definition 2.44. *[7] A non-empty collection* A *of subsets of a set* X *is called a* σ-algebra *if* A *is closed under complements and countable unions (and hence countable intersections). A* measure μ *on* A *is a non-negative function on* A *that is* σ*-additive, i.e.,* $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ *for any countable collection of disjoint sets* $A_i \in \mathsf{A}$*. A set of measure 0 is called a* null set*. A set whose complement is a null set is said to have* full measure*.*

Definition 2.45. *A* measure space *is a triple* (X, A, μ)*, where* X *is a set,* A *is a* $\sigma-$*algebra of subsets of* X*, and* μ *is a* $\sigma-$*additive measure. The elements of* A *are called* measurable sets*. If* $\mu(X) = 1$*, then* (X, A, μ) *is called a* probability space *and* μ *is a* probability measure*.*

The smallest $\sigma-$algebra containing all the open subsets of a topological space X is called the Borel $\sigma-$algebra of X. If A is the Borel $\sigma-$algebra, then a measure μ on A is a Borel measure if the measure of any compact set is finite.

Definition 2.46. *Let* (X, A, μ) *and* (Y, B, ν) *be measure spaces. A map* $T : X \to Y$ *is called* measurable *if the preimage of any measurable set is measurable. A measurable map* T *is* non-singular *if the preimage of measure 0 set has measure 0, and is* measure-preserving *if* $\mu(T^{-1}(B)) = \nu(B)$ *for every* $B \in \mathsf{B}$*. A non-singular map from a measure space into itself is called a* non-singular transformation *(or simply a transformation). If a transformation* T *preserves a* *measure μ, then μ is called $T-$*invariant. *Two measurable functions are* equivalent *if they coincide on a set of full measure.*

Proposition 2.47. *[30, p.98] Every dynamical system on a compact space has an invariant probability measure.*

Definition 2.48. *A* measure-theoretic dynamical system *is defined as a system (X, T, μ) where μ is a probability measure defined on the $\sigma-$algebra A of subsets of X and $T : X \to X$ is a measurable map which preserves the measure.*

Definition 2.49. *A measure-theoretic dynamical system (X, T, μ) is said to be* ergodic *if every Borel subset B of X such that $T^{-1}(B) = B$ has zero measure or full measure. (X, T, μ) is said to be* uniquely ergodic *if there is a unique probability measure which is $T-$invariant.*

Definition 2.50. *Measure spaces (X, A, μ) and (Y, B, ν) are* measure theoretic isomorphic *if there is a subset X' of full measure in X, a subset Y' of full measure in Y, and an invertible bijection $T : X' \to Y'$ such that T and T^{-1} are measurable and measure-preserving with respect to (A, μ) and (B, ν). An isomorphism from a measure space into itself is an automorphism.*

3. SYMBOLIC DYNAMICAL SYSTEMS

Symbolic dynamics studies a special kind of dynamical system called a symbolic dynamical system. The classical set-up is $1-$dimensional, but we describe here the general $d-$dimensional case. For the group we take $\mathbb{Z}^d$ and we let $X_n = \{1, \ldots, n\}^{\mathbb{Z}^d}, n > 1$, with the product topology. Letting T be the shift action of $\mathbb{Z}^d$ on X_n, we obtain a dynamical system (X_n, T) called the $d-$dimensional full shift on n symbols. It has very complicated subsets. A $\mathbb{Z}^d$-symbolic dynamical system is defined to be a pair (X, T) where X is a closed T-invariant subset $X \subseteq X_n$ called a shift space.

3.1 Alphabet, words and language

An alphabet $\mathcal{A}$ is a finite set of letters (or symbols) of cardinality $card(\mathcal{A}) \geqslant 2$. A word or block is a finite sequence (string) of letters in $\mathcal{A}$. The empty word (contains no letters) will denoted by ϵ. The set of all finite nonempty words in $\mathcal{A}$ will denoted by $\mathcal{A}^*$.

For a word $u = u_1 \ldots u_n \in \mathcal{A}$, $|u| = n$ is the length of the word u and $|u|_a$ is the number of occurrences of a letter $a \in \mathcal{A}$ appearing in the word $u \in \mathcal{A}^*$. The concatenation of two words $v = v_1 \ldots v_m$ and $w = w_1 \ldots w_s$ is the word $u = vw = v_1 \ldots v_m w_1 \ldots w_s$, for such $u \in \mathcal{A}^*$, i.e., $u = vw$ and $v, w \in \mathcal{A}^*$, the word v is said to be a prefix of u and w is a suffix of u.

Remark 3.1. *The set $\mathcal{A}^* \cup \{\epsilon\}$ with the operation of concatenation form a free monoid with identity element ϵ. The set $\mathcal{A}^*$ with the operation of concatenation endowed with the structure of a free semi group.*

A bisequence (or biinfinite word) is a word $v = \{v_i, i \in \mathbb{Z}\}$ of elements $v_i \in \mathcal{A}$. The set of all such v is denoted by $\mathcal{A}^{\mathbb{Z}}$. That is:

$$v = \dots v_{-2}v_{-1}v_0v_1v_2 \dots \quad \text{and} \quad \mathcal{A}^{\mathbb{Z}} = \dots \times \mathcal{A} \times \mathcal{A} \times \dots$$

A sequence (or a right infinite word) is the word $u = \{u_i, i \in \mathbb{N}\}$ of elements $u_i \in \mathcal{A}$. The set of all such u is denoted $\mathcal{A}^{\mathbb{N}}$. That is:

$$u = u_0u_1u_2 \dots \quad \text{and} \quad \mathcal{A}^{\mathbb{N}} = \mathcal{A} \times \mathcal{A} \times \dots$$

We identify the 0th coordinate in a bi-infinite word u by either an indexing, as above, or by use of a decimal point{.}.

A word $v = v_1 \dots v_r$ is said to occur at position m in a word u (infinite or finite) if there exists an integer m such that $u_m = v_1, \dots, u_{m+r-1} = v_r$ then v is denoted by $u[m, r]$ and v is said to be a factor of u.

The language (respectively the language of length n) of the sequence u, denoted by $\mathcal{L}(u)$ (respectively, $\mathcal{L}_n(u)$) is the set of all words (respectively, of length n) in $\mathcal{A}^*$ which occur in u.

A sequence u is recurrent if every factor occurs infinitely often.

Definition 3.2. *A sequence u is periodic (respectively, ultimately periodic) if there exists a positive integer q such that $\forall n \in \mathbb{N}, u_n = u_{n+q}$ (respectively, $\exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0, u_n = u_{n+q}$).*

Notice that, every periodic sequence is ultimately periodic but the converse is not always true.

Definition 3.3. *Let u be a sequence, the* complexity function *of u is denoted by p_u which associates to each positive integer n, the number $p_u(n)$ of different words (factors) of length n occurring in u. That is, $p_u(n) = card\mathcal{L}_n(u)$.*

Remark 3.4. *[18, p. 3] The complexity function is non-decreasing and for any positive integer n we have, $1 \leqslant p(n) \leqslant (card(A))^n$.*

Proposition 3.5. *[18, Proposition 1.1.1] A sequence u is ultimately periodic if and only if $p_u(n)$ is bounded function.*

Definition 3.6. *Let $v = v_1 v_2 \dots v_m$ be a factor of the sequence $u \in \mathcal{A}^{\mathbb{N}}$ then a letter $a \in \mathcal{A}$ is said to be an extension of v if $va = v_1 v_2 \dots v_m a$ (or $av = av_1 v_2 \dots v_m$) is also a factor of u.*

Definition 3.7. *[18] A* Sturmian sequence *(respectively,* Arnoux-Rauzy *sequence) is a one-sided sequence u with complexity function p_u such that, $\forall n \in \mathbb{N}$, $p_u(n) = n + 1$ (respectively, $p_u(n) = 2n + 1$). In particular it is defined over a two-letter (respectively, three-letter) alphabet.*

3.2 Symbolic systems

Consider the set $\mathcal{A}^{\mathbb{Z}}$, define a topology on $\mathcal{A}^{\mathbb{Z}}$ which is the product topology of the discrete topologies of each copy of $\mathcal{A}$ and hence this space will be a compact space (Tychonoff's theorem). A second way to define this topology is by the following metric function:
for each $u, v \in \mathcal{A}^{\mathbb{Z}}$,

$$d(u, v) = 2^{-min\{n \in \mathbb{N} | u_{|n|} \neq v_{|n|}\}}.$$

Thus, u is said to be close to v if the first terms on both sides around u_0 and v_0 are equal.

$\mathcal{A}^{\mathbb{Z}}$ with this metric d is a complete metric space and $\mathcal{A}^{\mathbb{Z}}$ is a Cantor set.

Define the map:

$$\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \text{ such that } \sigma((u_n)_{n \in \mathbb{Z}}) = (u_{n+1})_{n \in \mathbb{Z}}.$$

which is called the shift map. Observe that σ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$.
On $\mathcal{A}^{\mathbb{N}}$, σ is a uniformly continuous, onto, but not one to one.

For any $u \in \mathcal{A}^{\mathbb{Z}}$, we will use the notation $\Omega_u := cl\mathcal{O}(u)$, where the closure is in $\mathcal{A}^{\mathbb{Z}}$.

Definition 3.8. *[18] The symbolic dynamical system associated with $u \in \mathcal{A}^{\mathbb{Z}}$ (respectively, $u \in \mathcal{A}^{\mathbb{N}}$) is the pair (Ω_u, σ).*

Definition 3.9. *Let $w = w_0 w_1 ... w_n$ be a word,* the cylinder set*:*

$$[w] := \{x \in \Omega_u \mid x_0 = w_0, ... x_n = w_n\}$$

Remark 3.10. *[18, p. 4,5]*

(1) When $u \in \mathcal{A}^{\mathbb{Z}}$, the map $\sigma : \Omega_u \to \Omega_u$ is a homeomorphism.

(2) The cylinder sets are clopen sets and form a base for the topology of Ω_u.

Proposition 3.11. *[18, p. 5]*

(1) $x \in \Omega_u$ if and only if $\mathcal{L}_n(x) \subset \mathcal{L}_n(u)$ for all n.

(2) u is recurrent if and only if σ is onto on $\mathcal{L}_n(u)$.

A bisequence $u = (u_n)$ is *minimal* (or *recurrent*) if every word occurring in u occurs in an infinite number of positions with bounded gaps; that is, if for every factor w there exists s such that for every n, w is a factor of $u_n \dots u_{n+s-1}$.

Proposition 3.12. *[18, p. 6] A bisequence u is minimal if and only if the dynamical system (Ω_u, σ) is minimal.*

Definition 3.13. *[15] A bisequence u is said to be a* Toeplitz bisequence *if there is a collection of pairwise disjoint arithmetic progressions $\{Q_i\}$ such that $\bigcup_i Q_i = \mathbb{Z}$ and $u_n = u_m$ if n and m belong to the same Q_i. A Toeplitz bisequence is said to be* regular *if $\sum_i 1/q_i = 1$ where $Q_i = \{p_i + kq_i \mid k = 0, \pm 1, \pm 2, ...\}$ and $q_i > 0$*

Proposition 3.14. *[15, p. 92] If u is a Toeplitz bisequence, then (Ω_u, σ) is minimal. If in addition, u is regular, then (Ω_u, σ) is strictly ergodic.*

3.3 Substitutions

Definition 3.15. *A* substitution *is a function θ from $\mathcal{A}$ into $\mathcal{A}^* \setminus \epsilon$. θ can be extended naturally to $\mathcal{A}^{\mathbb{Z}}$ such that:*

$$\theta : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}, \theta(u) = \ldots \theta(u_{-2})\theta(u_{-1})\theta(u_0)\theta(u_1)\theta(u_2)\ldots$$

(similarly, for $\mathcal{A}^{\mathbb{N}}$).

A fixed point of the substitution θ (θ-fixed point) is an infinite word u with $\theta(u) = u$. A periodic point of θ (θ-periodic point) is an infinite word u with $\theta^n(u) = u$ for some $n > 0$.

By repeating the substitution $\theta(a)$, which is start with a, infinitely many we will get a $\theta-$fixed sequence.

Example 3.16. The Morse sequence u is the fixed point beginning with a of the Morse substitution θ_1 defined over the alphabet $\{a, b\}$ by $\theta_1(a) = ab$ and $\theta_1(b) = ba$. The Fibonacci sequence is the fixed point v beginning with a of the Fibonacci substitution θ_2 defined over the two-letter alphabet $\{a, b\}$ by $\theta_2(a) = ab$ and $\theta_2(b) = a$.

The Fibonacci sequence is an example of Sturmian sequences.

Definition 3.17. *Let $\mathcal{A} = \{a_1 \ldots a_n\}$ an alphabet and $\theta : \mathcal{A} \to \mathcal{A}^* \setminus \epsilon$ be a substitution then the $n \times n$ matrix M_θ is said to be the* incidence matrix *(or* substitution matrix*) of the substitution θ if each entry $M_{(i,j)}$ denotes the number of occurrences of a_i in $\theta(a_j)$.*

Definition 3.18. *A substitution θ is said to be* unimodular *if the determinant of its incidence matrix is* $+1$ *or* -1 *(i.e.* $det(M_\theta) = \pm 1$*).*

Example 3.19.

(1) The Morse substitution θ_1 is not unimodular since the incidence matrix $M_{\theta_1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has a determinant $det(M_{\theta_1}) = 0$.

(2) Fibonacci substitution θ_2 with incidence matrix $M_{\theta_2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is unimodular.

3.4 Primitivity

Definition 3.20. *A substitution θ over the alphabet $\mathcal{A}$ is* primitive *if there exists a positive integer n such that for all a and b in $\mathcal{A}$ the letter a occurs in $\theta^n(b)$.*

The primitivity guarantees that each word $\theta^n(a)$, after a finite number n of applications to the substitution, will contain all the letters of the alphabet.

Definition 3.21. *For a primitive substitution θ, the* symbolic dynamical system associated with θ *is the orbit closure of any θ-periodic point $u \in \mathcal{A}^{\mathbb{Z}}$ and is denoted by (Ω_θ, σ). That is, $(\Omega_\theta, \sigma) = cl\mathcal{O}(u)$.*

Proposition 3.22. *[18, Proposition 1.2.3] If θ is primitive then any $\theta-$periodic point u is minimal, that is (Ω_u, σ) is minimal.*

Proposition 3.23. *[18, p. 12] For a primative substitution, all the symbolic dynamical systems associated with periodic points are the same.*

Let θ be a substitution such that:

(1) There exists a letter $a \in \mathcal{A}$ such that $\theta(a)$ begins with a, that is, there is a fixed point starting with a,

(2) All the sequences are infinite, that is, $\forall\, b \in \mathcal{A}$, $lim_{n\to\infty}|\theta^n(b)| = \infty$,

(3) There is an infinite fixed point (obtained from (1) and (2)) which contains all the letters of the alphabet $\mathcal{A}$.

When θ is primitive then there exists $k \geqslant 1$ such that, θ^k satisfies conditions 1,2 and 3 above. Then we have the following proposition:

Proposition 3.24. *[18, Proposition 1.2.4] The substitution θ (which satisfies 1,2 and 3 above) is primitive if and only if the fixed point (which starts with a) is minimal.*

A matrix M is called non-negative when all its entries are non-negative numbers. A non-negative matrix is called positive when at least one entry is > 0, and it is called strictly positive when all its entries are > 0.

Definition 3.25. *[3] A non-negative square matrix $M = (M_{ij})$ is said to be* irreducible *if, for each index pair (i, j), there is an integer $k \in N$ with $(M^k)_{ij}$ is positive, otherwise M is said to be reducible.*

Definition 3.26. *[3] A non-negative square matrix M is called* primitive *if there exists an integer $k \in \mathbb{N}$ such that M^k strictly positive.*

It is clear that any primitive matrix is irreducible.

Proposition 3.27. *[3, Lemma 4.1] A substitution θ is primative (respectively, irreducible) if and only if its incidence matrix M_θ is primitive (respectively, irreducible).*

For an $n \times n$ matrix M, scalars λ and $n \times 1$ vectors $v \neq 0$ satisfying $Mv = \lambda v$ are called eigenvalues and eigenvectors of M, respectively. The characteristic polynomial of M is $p(\lambda) = det(M - \lambda I)$ (where I is the $n \times n$ unit matrix) which is of degree n. The characteristic equation for M is $p(\lambda) = 0$. The eigenvalues of M are solutions of the characteristic equation or, equivalently, the roots of the characteristic polynomial. The algebraic multiplicity of an eignvalue λ is the number of times it is repeated as a root of the characteristic polynomial.
When the algebraic multiplicity of $\lambda = 1$ then λ is said to be a simple eigenvalue.

Theorem 3.28. *(Perron-Frobenius theorem) [18, Theorem 1.2.6] Let M be a non-negative irreducible matrix. Then M has a positive eigenvalue λ which is greater than or equal the modulus of the other eigenvalues $\alpha : \lambda \geqslant\mid \alpha \mid$. The eigenvalue λ is a simple eigenvalue and there exists an eigenvector with positive entries associated with λ.*
Furthermore, if M is primitive, then the eigenvalue λ is strictly greater than the modulus of the other eigenvalues $\alpha : \lambda >\mid \alpha \mid$.

The Perron-Frobenius theorem, eigenvalue and eigenvector of primitive substitutions will denoted by PF-theorem, PF-eigenvalue and PF-eigenvector respectively. This property of primitive matrices, which provided by PF-theorem, implies the existence of “frequencies” for every factor of a fixed point of a primitive substitution.

Definition 3.29. *[18] Let u be a sequence. The frequency f_w of a factor w of u is defined as the limit (when n tends towards infinity), if it exists, of the number of occurrences of the factor w in $u_0 u_1 \dots u_{n-1}$ divided by n.*

Proposition 3.30. *[18, Theorem 1.2.7] Let θ be a primitive substitution. Let u be a fixed point of θ. Then every factor of u has a frequency. Furthermore, all the frequencies are positive. The frequencies of the letters are given by the coordinates of the PF-eigenvector renormalized in such a way that the sum of its coordinates equals 1.*

The symbolic system (Ω_θ, σ) is finite if there is a θ-periodic point which is also shift periodic (σ-periodic), then the substitution θ is said to be shift periodic (σ-periodic) or just periodic.

We will consider only the substitutions which are not periodic (details in the next chapter). Such substitutions are said to be aperiodic.

3.5 Constant length 2-letter substitutions

In this section we will follow the procedures of Coven and Keane [15] to define a factor map between the substitution minimal set of a discrete substitution of constant length n and the set of $n-$adic integers and show that, this factor map is a measure theoretic isomorphism. This result allows one to classify the discrete substitutions of constant length n through their conjugated n-adic spaces. Although there are more general results for substitutions on larger alphabets (see e.g. [3], [17]), we focus on 2-letters alphabets and hence, we will focus on the detailed results for those.

We will use the notation:

$$\theta(0) = a_0 \dots a_{n-1}$$

$$\theta(1) = b_0 \dots b_{n-1}$$

for constant length substitution θ of length n on the alphabet $\{0,1\}$, that is, for each i, $a_i, b_i \in \{0,1\}$ and the notation $\tilde{a}$ for the reverse of the word a, that is, $\tilde{a} = a_{n-1} \dots a_0$.

Definition 3.31. *A substitution θ of constant length n defined on the alphabet $\mathcal{A} = \{0,1\}$, where $\theta(0) = a = a_1a_2 \dots a_{n-1}$ and $\theta(1) = b = b_1b_2 \dots b_{n-1}$, have been divided into three classes such that:*

(1) θ is said to be finite *if any of the following holds:*

(i) $a_i = b_i$ for all i, i.e. $\theta(0) = \theta(1)$.

(ii) $a_i = 0$ for all i and $b_i = 1$ for all i, i.e. $\theta(0) = 000 \dots 0$ and $\theta(1) = 111 \dots 1$.

(iii) $a_i = 1$ for all i and $b_i = 0$ for all i, i.e. $\theta(0) = 111 \dots 1$ and $\theta(1) = 000 \dots 0$.

(iv) n is odd and $a = \tilde{b} = 0101 \dots 010$ or $a = \tilde{b} = 1010 \dots 101$, i.e. $\theta(0) = 0101 \dots 010$ and $\theta(1) = 1010 \dots 101$, or $\theta(0) = 1010 \dots 101$ and $\theta(1) = 0101 \dots 010$.

(2) θ is said to be continuous *if it is not finite and $a = \tilde{b}$.*

(3) θ is said to be discrete *if it is neither finite nor continuous.*

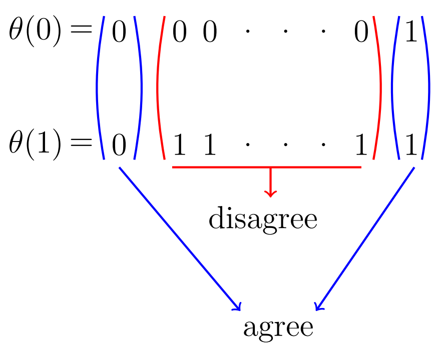
Definition 3.32. *For a substitution θ of constant length n defined on the alphabet $\mathcal{A} = \{0,1\}$, where $\theta(0) = a = a_1a_2 \dots a_{n-1}$ and $\theta(1) = b = b_1b_2 \dots b_{n-1}$,. The set $I_1 := \{i : a_i = b_i\}$ is said to be the* agreement set, *and $J_1 := \{i : a_i \neq b_i\}$, is said to be the* difference set.

Remark 3.33. *[15, p. 92] A substitution θ is said to be* discrete *if and only if both the agreement set I_1 and the difference set J_1 are nonempty.*

In other words; a substitution is discrete if and only if $\theta(0)$ and $\theta(1)$ agree in some places and disagree in some other places.

From now on we will focus only on discrete substitutions.

Example 3.34. Let θ be defined as follows:
$\theta(0) = 000\dots01$ and $\theta(1) = 011\dots11$, then it is clear that θ is discrete.



Remark 3.35. *[15, p. 92] θ and θ^k are in the same class (finite, continuous or discrete) for all $k \in \mathbb{Z}^+$.*

A substitution θ of length n, where $\theta(0) = a = a_1a_2\dots a_{n-1}$ and $\theta(1) = b = b_1b_2\dots b_{n-1}$, defines a map β on the set of 2-blocks $\{00, 01, 10, 11\}$ by:

$$\beta(00) = a_{n-1}a_0,$$

$$\beta(01) = a_{n-1}b_0,$$

$$\beta(10) = b_{n-1}a_0,$$

$$\beta(11) = b_{n-1}b_0.$$

β^2 can be defined exactly like β but with respect to θ^2, which is of length n^2, instead of θ.

Remark 3.36. *[15, p. 93] The map β^2 has one, two or four fixed points whenever $\theta(0)$ and $\theta(1)$ agree at both ends, one end or neither end respectively.*

Example 3.37.

(1) Let $\theta(0) = 001$, $\theta(1) = 100$, which do not agree in both ends.
Hence, $\beta(00) = 10$, $\beta(01) = 11$, $\beta(10) = 00$, $\beta(11) = 01$
Then: $\theta^2(0) = 001001100$, $\theta^2(1) = 100001001$
And: $\beta^2(00) = 00$, $\beta^2(01) = 01$, $\beta^2(10) = 10$, $\beta^2(11) = 11$
Thus, β^2 possess four fixed points.

(2) Let $\theta(0) = 001$, $\theta(1) = 010$, which agree in one end.
Hence, $\beta(00) = 10$, $\beta(01) = 10$, $\beta(10) = 00$, $\beta(11) = 00$
Then: $\theta^2(0) = 001001010$, $\theta^2(1) = 001010001$
And: $\beta^2(00) = 00$, $\beta^2(01) = 00$, $\beta^2(10) = 10$, $\beta^2(11) = 10$
Thus, β^2 possess two fixed points.

(3) Let $\theta(0) = 001$, $\theta(1) = 011$, which agree in both ends.
Hence, $\beta(00) = 10$, $\beta(01) = 10$, $\beta(10) = 10$, $\beta(11) = 10$
Then: $\theta^2(0) = 001001011$, $\theta^2(1) = 001011011$
And: $\beta^2(00) = 10$, $\beta^2(01) = 10$, $\beta^2(10) = 10$, $\beta^2(11) = 10$
Thus, β^2 possess one fixed point.

We will get the same result for any $n \geqslant 2$.

Let Ω be the $2-$symbolic space, that is, $\Omega := \{0,1\}^{\mathbb{Z}}$.

Remark 3.38. *[15, p. 93] Let pq be the fixed point of β^2 mentioned above. Then θ^2 is a contraction on the closed set $\{u \in \Omega \mid u_{-1}u_0 = pq\}$.*
Therefore, by the contraction principle, there is a unique point $w \in \Omega$ which is fixed under θ^2 (i.e. $\theta^2(w) = w$) and $w_{-1}w_0 = pq$. This fixed point will be denoted by w^{pq}.

Proposition 3.39. *[15, p. 93] Let θ be a discrete substitution then there is a strictly ergodic minimal subsystem (Ω_θ, σ) of (Ω, σ) such that for any pq invariant under β^2, $\Omega_\theta = cl\mathcal{O}(w^{pq})$, the orbit closure of the fixed point w^{pq}.*

Remark 3.40. *[15, p. 93] $\Omega_\theta = \Omega_{\theta^k}$ for all $k \in \mathbb{N}$.*

Remark 3.41. *[15, p. 93] In a finite substitution, the only minimal sets contained in $\Omega_\theta = cl\mathcal{O}(w^{pq})$ are the periodic orbits, which are considered as trivial minimal sets. Also, since for each continuous substitution θ there is an associated discrete substitution $\tilde{\theta}$ of the same length such that (Ω_θ, σ) is a distal extension of $(\Omega_{\tilde{\theta}}, \sigma)$. Therefore, we restrict our work to discrete substitutions.*

In the following results we can see that the classification of two substitutions should be for substitutions of same class(discrete or continuous).

Proposition 3.42. *[15, p. 93] Let θ be a substitution. If θ is continuous then Ω_θ is the orbit closure of an extended Morse sequence. If θ is discrete then Ω_θ is the orbit closure of a regular Toeplitz bisequence.*

For a discrete substitution θ, Ω_θ is minimal (Proposition 3.42 and Proposition 3.14). Ω_θ is said to be a substitution minimal set.

Corollary 3.43. *[15, p. 94] Every substitution minimal set is strictly ergodic.*

Corollary 3.44. *[15, p. 94] If θ_1 is continuous and θ_2 is discrete, then the systems $(\Omega_{\theta_1}, \sigma)$ and $(\Omega_{\theta_2}, \sigma)$ are not isomorphic.*

Since the structure system of (Ω_θ, σ) is the $n-$adic system $(Z(n), \tau)$ and since the structure system is a maximal equicontinuous factor map image of (Ω_θ, σ) then we have a factor map $\pi : (\Omega_\theta, \sigma) \to (Z(n), \tau)$ which defined in [15] by:

$$\pi(w) = lim_{i\to\infty}\sigma^{k_i}(w^{pq}) = w$$

Where k_i is chosen such that $\sigma^{k_i}(w^{pq}) \to w$ for some pq invariant under β^2.

Lemma 3.45. *[15, Lemma 3] Let pq be invariant under β^2. Let $w \in \Omega$ and $\{k_i\}$ be a sequence of integers such that, $lim_{i\to\infty}\sigma^{k_i}(w^{pq}) = w$, then for each positive integer j there is an integer N_j such that, $i, \tilde{i} \geqslant N_j$ implies $k_i \equiv k_{\tilde{i}}$ mod n^j.*

The existence of the factor map π and some of its characteristics are given in the following result:

Proposition 3.46. *[15, Theorem 1] There is a factor map $\pi : (\Omega_\theta, \sigma) \to (Z(n), \tau)$ such that $\pi(w^{pq}) = 0$ for each θ^2-invariant point w^{pq}. When $\pi(w) = \sum_{i=1}^{\infty} z_i n^i$ then the word $w\left[-\sum_{i=0}^{k-1} z_i n^i, n^k\right]$ is either $\theta^k(0)$ or $\theta^k(1)$, for each $k \geqslant 1$.*

Let $z^{(k)}$ denotes the kth partial sum of $z = \sum_{i=o}^{\infty} z_i n^i$ that is, $z^{(k)} = \sum_{i=o}^{k-1} z_i n^i$. From the last proposition we can say that, $w\left[-z^{(k)}, n^k\right]$ is either $\theta^k(0)$ or $\theta^k(1)$, for each $k \geqslant 1$.

Now we need to determine where π is one-to-one, for each positive integer k, we have:

$$I_k = \{m \mid 0 \leqslant m \leqslant n^k - 1,\ \theta^k(0)\,\text{and}\,\theta^k(1)\ \text{agree at place}\,m\}$$

$$J_k = \{m \mid 0 \leqslant m \leqslant n^k - 1,\ \theta^k(0)\,\text{and}\,\theta^k(1)\ \text{disagree at place}\,m\}$$

and then,

$$I_k = \{\sum_{i=0}^{k-1} z_i n^i \mid z_i \in I_1\ \text{for some}\,i\}$$

,

$$J_k = \{\sum_{i=0}^{k-1} z_i n^i \mid z_i \in J_1\ \text{for all}\,i\}$$

.

And when $k \to \infty$,

$$I_\infty = \{\sum_{i=0}^{\infty} z_i n^i \mid z_i \in I_1\ \text{for some}\,i\}$$

,

$$J_\infty = \{\sum_{i=0}^{\infty} z_i n^i \mid z_i \in J_1\ \text{for all}\,i\}$$

.

It is clear that, J_∞ is a subset of $Z(n)$.
Let, $E = \bigcup\{\tau^k J_\infty \mid k = 0, \pm1, \pm2, ...\}$, and let, $Z^* := Z(n) - E$.
Then J_∞ is closed and nowhere dense. Hence Z^* is a G_δ set which, by the Baire category theorem, is dense in $Z(n)$.

Proposition 3.47. *[15, Theorem 2] Let μ denote normalized Haar measure on $Z(n)$. Then $\mu(Z^*) = 1$.*

The following proposition determines where π is one-to-one.

Proposition 3.48. *[15, Theorem 3] If $z \in Z^*$ then $\pi^{-1}(z)$ is a singleton. If $z \in E$ then $\pi^{-1}(z)$ consist of two points unless $z \in \mathcal{O}(0)$, a and b disagree at both endpoints then $\pi^{-1}(z)$ consists of four points.*

Proof. For each $z \in Z(n)$, if $w \in \pi^{-1}(z)$ then $w[-z^{(k)}, n^k]$ is either $\theta^k(0)$ or $\theta^k(1)$ (proposition(3.46)) that is, w_0 is at the z^kth place of either $\theta^k(0)$ or $\theta^k(1)$. Now, if $z \in Z^*$ then, $z^k \in I_k$ for all large k and hence $z \in I_\infty$ and w_0 is uniquely determined. Since Z^* is $\tau-$invariant then, for any m, $\tau^m(z) \in Z^*$ and since π is a factor map then $\pi(\sigma^m(w)) = \tau^m(z)$, that is, $\sigma^m(w) \in \pi^{-1}(\tau^m(z))$ and $w_m = (\sigma^m(w))_0$ is uniquely determined for all m. Thus, $\pi^{-1}(z)$ is a singleton.
Next, when $z \in E = \bigcup\{\tau^k J_\infty \mid k = 0, \pm 1, \pm 2, ...\}$ and $z \notin \mathcal{O}(0)$ hence, $z \notin Z^*$ and since E and $\mathcal{O}(0)$ are $\tau-$invariant, we can say $z \in J_\infty$. Both $z^{(k)}$ and $n^k - z^{(k)}$ are increase with k, thus, there are points $w, \acute{w} \in \Omega_\theta$ such that for each $k \geqslant 1$, $\{w[-z^k, n^k], \acute{w}[-z^k, n^k]\} = \{\theta^k(0), \theta^k(1)\}$.
Since $z \in J_\infty$, $w_0 \neq \acute{w}_0$ and hence, $w \neq \acute{w}$, proposition(3.46) shows that $\pi(w) = \pi(\acute{w}) = z$ that is, $\pi^{-1}(z) = \{w, \acute{w}\}$, exactly two points.
Finally, let $z \in E \cap \mathcal{O}(0)$, we can assume that, $z = 0$. But, $\pi^{-1}(0) =\{$ $w^{pq} \mid pq$ invariant under $\beta^2\}$ and when a and b disagree at both end points then, π^{-1} consists of four points.

□

Corollary 3.49. *[15, p. 96] The map $\pi : (\Omega_\theta, \sigma) \to (Z(n), \tau)$ is a measure-theoretic isomorphism.*

The following theorem is a direct result of the previous one and the classification of adding machines, it will be useful in the later sections.

Theorem 3.50. *[15, p. 96] Two discrete substitutions have measure-theoretically isomorphic minimal sets if and only if their lengths have the same prime factors.*

Definition 3.51. *Let θ be a discrete substitution such that $\theta(0) = a$ and $\theta(1) = b$, then θ is said to be in* normal form *if:*

(1) $a_0 \neq b_0$,

(2) If $j = min\{i \mid a_i = b_i\}$ *then* $a_j = b_j = 0$.

Theorem 3.52. *[15, p. 100] Let θ_1 and θ_2 be discrete substitutions constant length n. Then $(\Omega_{\theta_1}, \sigma)$ and $(\Omega_{\theta_2}, \sigma)$ are topologically conjugate if and only if θ_1 and θ_2 have the same normal form.*

Our main results (in Chapter 5) will focus on classifying special type of tiling spaces which can be defined as a suspension of discrete substitution spaces and hence the results of Coven and Keane [15]will be extremely useful. However, topological conjugacy classification of a class of $\mathbb{Z}-$actions does not generally yield a topological classification of the spaces formed by their suspensions.

4. TILING SPACES

From the point of view of symbolic dynamics, this chapter will provide a basic introduction to the tilings of Euclidean space $\mathbb{R}^d$ by translations of a finite number of basic tile types called prototiles. The relation between tilings and dynamics will be clarified using a kind of dynamical system called a tiling dynamical system. Since tilings have geometric structure, tiling systems can be considered as a type of symbolic systems for which the group acting on them is continuous ($\mathbb{R}^d$) rather than discrete ($\mathbb{Z}^d$). To study these spaces we need a related compact metric space which is called the tiling space. The main method considered here for constructing tiling spaces is the tiling substitutions which are analogous to substitution systems in symbolic dynamics. We will focus on the one dimensional tiling spaces associated to a primitive aperiodic substitutions.

4.1 Tilings of $\mathbb{R}^d$

Definition 4.1. *A set $B \subseteq \mathbb{R}^d, d \geqslant 1$ is called a* tile *if it is compact and equal to the closure of its interior, that is, $B = cl(int(B))$*

Tiles in $\mathbb{R}$ are mainly closed intervals and in $\mathbb{R}^2$ are often polygons.

Definition 4.2. *[27] A* tiling $\mathcal{T}$ *of* $\mathbb{R}^d$ *is a collection of labelled tiles that pack and covers* $\mathbb{R}^d$ *in the sense that any two tiles have pairwise disjoint interiors and the union of all tiles is* $\mathbb{R}^d$*. If* $\mathcal{T}$ *is a tiling and* $x \in \mathbb{R}^d$ *then* $\mathcal{T} + x$ *is a shifting of* $\mathcal{T}$ *by a vector* x *and this shifting is said to be a* translation*. Two tiles* B_1, B_2 *are said to be* equivalent*, denoted* $B_1 \sim B_2$*, if one is a translation of the other and they have the same label. Equivalence classes representatives are said to be* prototiles*.*

Definition 4.3. *Let* $\mathcal{P}$ *be a finite set of inequivalent prototiles in* $\mathbb{R}^d$*. A* full tiling space*, denoted by* $X_{\mathcal{P}}$*, is the set of all tilings of* $\mathbb{R}^d$ *by translations of the elements of* $\mathcal{P}$*.*

Our central interest will be the action of $\mathbb{R}^d$ on $X_{\mathcal{P}}$ by translation:

Definition 4.4. *The* translation action *of* $\mathbb{R}^d$ *on* $X_{\mathcal{P}}$ *is the map* $T : \mathbb{R}^d \times X_{\mathcal{P}} \to X_{\mathcal{P}}$ *such that for each* $x \in \mathbb{R}^d$, $T_x(\mathcal{T}) = \{B + x \mid B \in \mathcal{T}\}$*.*

Patches can be defined as any subset of tiles from a tiling $\mathcal{T}$ [28]. The following definition define a special case.

Definition 4.5. *A* partial tiling Y *[2] is a collection of tiles in* $\mathbb{R}^d$ *with pairwise disjoint interiors such that the union of tiles in* Y *is connected. This union is said to be the* support *of* Y *and denoted by* $supp(Y)$*. Let* $\mathcal{P}$ *be a set of prototiles. A* $\mathcal{P}-$patch Y *[27] is a finite partial tiling of a tiling* $\mathcal{T} \in X_{\mathcal{P}}$*.*

It is clear that any tiling of $\mathbb{R}^d$ is a partial tiling with support $\mathbb{R}^d$.

The notion of equivalence of tiles can be extended to patches, and a set of equivalence classes representatives of patches is denoted by $\mathcal{P}^*$. The subset of patches of n tiles, called the $n-$patches, is denoted by $\mathcal{P}^{(n)} \subseteq \mathcal{P}^*$.

Definition 4.6. *[28] A tiling of $\mathbb{R}^d$ is said to be a* simple *tiling if:*

(1) There are only a finite number of prototiles, up to translation. In other words, there exists a finite collection of prototiles p_i such that each tile is a translation of one of the p_i.

(2) Each tile is a polytope. In one dimension, that means an interval. In 2 dimensions, it means a polygon. In three dimensions, it means a polyhedron.

(3) Tiles meet edge to edge. An edge of one tile cannot overlap with an edge of a neighboring tile.

We will consider only the simple tilings.

4.2 Tiling dynamical system

To study the dynamical properties of tilings we need to define the tiling dynamical system and hence we need first to know the topology of these tilings.

We will start by defining a tiling metric:
Let $\mathcal{T}, \mathcal{T}'$ be tilings of $\mathbb{R}^d$, then $\mathcal{T}, \mathcal{T}'$ are said to be $\epsilon-$close if they agree in a ball B_r of radius $r = 1/\epsilon$ around the origin up to translation by a vector x such that $\| x \| \leqslant \epsilon$. Let:

$$R(\mathcal{T}, \mathcal{T}') = sup\{r \in \mathbb{R} \mid \exists\, x, y \in \mathbb{R}^d \text{ with norms } \leqslant 1/2r \text{ such that} \mathcal{T}+x, \mathcal{T}'+y \text{ agree on } B_r\}.$$

Then:

$$d(\mathcal{T},\mathcal{T}') = min\{\sqrt{2}/2,\ 1/R(\mathcal{T},\mathcal{T}')\}.$$

Remark 4.7. *[28, p. 6] Let $\mathcal{T},\mathcal{T}'$ be tilings of $\mathbb{R}^d$, if $\mathcal{T}'$ is close to $\mathcal{T}$ then $\mathcal{T}'$ need not be a translation of $\mathcal{T}$.*

Lemma 4.8. *[27, Lemma 2.8] The tiling metric space $(X_{\mathcal{P}}, d)$ is complete.*

Theorem 4.9. *[27, Theorem 2.9] Suppose $X_{\mathcal{P}}$ is a simple tiling space. Then $(X_{\mathcal{P}}, d)$ is compact. Moreover, the action T of $\mathbb{R}^d$ by translation on $X_{\mathcal{P}}$ is continuous.*

Since we consider the translation action of $\mathbb{R}^d$, the orbit of a tiling $\mathcal{T}$ will be the set :

$$\mathcal{O}(\mathcal{T}) = \{\mathcal{T} + x \mid x \in \mathbb{R}^d\}.$$

Definition 4.10. *The set $\Omega_{\mathcal{T}} := cl(\mathcal{O}(\mathcal{T}))$ is said to be the* hull *of $\mathcal{T}$.*

Definition 4.11. *Let $X_{\mathcal{P}}$ be a full d-dimensional tiling space and let T denote the translation action of $\mathbb{R}^d$. A tiling space X is a closed T-invariant subset $X \subseteq X_{\mathcal{P}}$. The pair (X, T) a said to be a* tiling dynamical system.

It is clear that the hull of a tiling $\mathcal{T}$ is a tiling space. Such tiling spaces will play a key role in the rest of this thesis.

The tiling dynamical system can be considered as a new type of symbolic systems[27] where the prototiles of $\mathcal{P}$ represent the letters, the tilings represent the bisequences and the full tiling space $X_{\mathcal{P}}$ represents the full shift. The theory of tiling dynamical systems contains the theory of $\mathbb{Z}^d$ symbolic dynamics.

Definition 4.12. *Let $X_{\mathcal{P}}$ be a full tiling space and let $\mathcal{F} \subseteq \mathcal{P}^*$. Let $X_{\setminus \mathcal{F}} \subseteq X_{\mathcal{P}}$ be the set of all tilings $\mathcal{T} \in X_{\mathcal{P}}$ such that no patch Y in $\mathcal{T}$ is equivalent to any patch in $\mathcal{F}$. We call such a set $\mathcal{F}$ a set of* forbidden patches*.*

One can show that for any $\mathcal{F} \subseteq \mathcal{P}^*$, the set $X_{\setminus \mathcal{F}}$ is a tiling space (i.e., it is closed and $T-$invariant). Moreover, it is clear that every tiling space $X \subseteq X_{\mathcal{P}}$ is defined by a set $\mathcal{F}$ of forbidden patches. However, the set T is not unique.

Definition 4.13. *A tiling space $X \subseteq X_{\mathcal{P}}$ is said to be a* finite type *tiling space if there exists a finite $\mathcal{F} \in \mathcal{P}^*$ so that $X = X_{\setminus \mathcal{F}}$.*

Definition 4.14. *A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is said to be a* periodic tiling *if $\{g \in \mathbb{R}^d \mid T_g(\mathcal{T}) = \mathcal{T}\} \neq \{\mathbf{0}\}$. Otherwise, $\mathcal{T}$ is said to be* aperiodic*(or not periodic). A nonempty tiling space $X\,(\subseteq X_{\mathcal{P}})$ is said to be an* aperiodic tiling space *if it contains no periodic tilings (i.e., every $\mathcal{T} \in X$ is an aperiodic tiling).*

Remark 4.15. *[27, p. 88] It is not necessary for $X_{\mathcal{P}} \neq \phi$ to have a periodic tiling.*

We concern about the so called aperiodic substitution:

Definition 4.16. *A primitive substitution θ is said to be aperiodic (non-periodic in [5]) if at least one (equivalently, each) θ-periodic bi-infinite word is aperiodic* *under the shift map.*

In a tiling space, a tiling $\mathcal{T}$ is repetitive [27] if for any patch Y in $\mathcal{T}$ there is an $r > 0$ such that for any $g \in \mathbb{R}^d$ there is a translation $T_g(B)$ of B in $\mathcal{T}$ such that $supp(T_g(B)) \subseteq B_r + g$.

In other words, a copy of Y occurs "nearby" any given location g in $\mathcal{T}$. Since all periodic tilings are repetitive and repetitivity can be considered as a generalization of periodicity.

Definition 4.17. *Two repetitive tilings $\mathcal{T}_1, \mathcal{T}_2$ are said to be* locally isomorphic *if $cl\mathcal{O}(\mathcal{T}_1) = cl\mathcal{O}(\mathcal{T}_2)$*

Geometrically, two locally isomorphic tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ have exactly the same patches. Dynamically, local isomorphism means $\mathcal{T}_1$ and $\mathcal{T}_2$ belong to the same minimal tiling dynamical system.

In an aperiodic tiling space, there is always a properly repetitive tiling (i.e. repetitive and aperiodic).

Proposition 4.18. *[27, Theorem 5.8] Let X_θ be a substitution tiling space corresponding to a primitive tiling substitution θ. Then any $\mathcal{T} \in X_\theta$ is repetitive. Moreover, any $\mathcal{T}_1, \mathcal{T}_2 \in X_\theta$ are locally isomorphic. In particular, (X_θ, T) is minimal.*

Proposition 4.19. *[27, Theorem 6.1] If θ is a primitive tiling substitution then the corresponding tiling dynamical system (X_θ, T) is uniquely ergodic.*

4.3 Self–similar tilings

Kwapisz, in [22], studies abstract self-affine tiling actions which are class of minimal expansive actions of $\mathbb{R}^d$ on a compact space, they include the translation actions on the compact spaces associated to aperiodic repetitive tilings in $\mathbb{R}^d$ and show that in the self–similar case, the existence of a homeomorphism between tiling spaces implies topological conjugacy of the actions up to a linear rescaling.

Self–similarity means that the tiling is invariant under a substitution rule. A more general definition:

Definition 4.20. *Given a linear isomorphism $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ that is expanding (i.e., all its eigenvalues are greater than one in modulus), a tiling $\mathcal{T}$ is said to be $\Lambda-$*self–affine *if and only if there is a homeomorphism $\Phi : \Omega_{\mathcal{T}} \to \Omega_{\mathcal{T}}$ such that $\Phi(\mathcal{T}) = \mathcal{T}$ and:*

$$T_{\Lambda(g)} \circ \Phi = \Phi \circ T_g \textit{ for all } g \in \mathbb{R}^d$$

.

Furthermore, $\mathcal{T}$ is self–similar *if and only if it is $\Lambda-$self-affine with Λ that is conformal, i.e., $\Lambda = \lambda U$ where $\lambda > 1$ and U is an orthogonal transformation of $\mathbb{R}^d$.*

Self-affine tilings are necessarily aperiodic.

Remark 4.21. *We will consider the aperiodic primitive substitution θ which has at least one $\theta-$periodic tiling $\mathcal{T}$ which is not shift periodic and we will use such $\mathcal{T}$ to construct our main substitution tiling space as the hull of $\mathcal{T}$. to* *distinguish such spaces we will use the notation* Ω_θ *instead of* $\Omega_{\mathcal{T}}$*. That is:*

$$\Omega_\theta := cl\mathcal{O}(\mathcal{T})$$

Since $\mathcal{T}$ *is also repetitive then the system* (Ω_θ, σ) *is minimal.*

Theorem 4.22. *[22, Theorem 1.1] Suppose that* θ *and* $\tilde{\theta}$ *are repetitive tilings of finite local complexity that are* $\Lambda-$*self–similar and* $\tilde{\Lambda}-$*self-similar, respectively. Let* $T = (T_g)_{g\in\mathbb{R}^d}$ *and* $\tilde{T} = (\tilde{T}_g)_{g\in\mathbb{R}^d}$ *be the translation actions on the corresponding tiling spaces* Ω_θ *and* $\Omega_{\tilde{\theta}}$ *. If there is a homeomorphism* $h_0 : \Omega_\theta \to \Omega_{\tilde{\theta}}$*, then there is a linear isomorphism* $A : \mathbb{R}^d \to \mathbb{R}^d$ *and a homeomorphism* $h : \Omega_\theta \to \Omega_{\tilde{\theta}}$ *conjugating* $(T_g)_{g\in\mathbb{R}^d}$ *to the linear rescaling* $(\tilde{T}_{A(g)})_{g\in\mathbb{R}^d}$*, i.e.,* $h \circ T_g = \tilde{T}_{A(g)} \circ h$ $(\forall g \in \mathbb{R}^d)$*.*

Remark 4.23. *[22, p. 1747] Since the tiling spaces locally are products of a Cantor set by* $\mathbb{R}^d$*,* h_0 *must be an orbit equivalence (mapping orbits to orbits). Hence,* $\tilde{T}$ *is conjugated to a reparametrization (time change) of* T *. However, arbitrary reparametrizations of* T *are hardly ever conjugated to a linear rescaling of* T*. The assumption of self-similarity on both* T *and* $\tilde{T}$ *is therefore crucial.*

Remark 4.24. *[27, p. 89] In dimension 1, the tiling of* $\mathbb{R}$*, self–similar and self–affine are coincide.*

4.4 One dimensional substitution tiling spaces

In what follows, we shall only consider the dynamical systems Ω_θ arising from one-dimensional substitution tilings. these tiling spaces of $\mathbb{R}$ which are compact and connected and they are locally the product of a Cantor set with an arc and each of their arc components is dense[5].

Given a primitive substitution $\theta : \mathcal{A} \to \mathcal{A}^*$ with $card(\mathcal{A}) = d \geqslant 2$, let $w_L := (w_1, ..., w_d)$ be a positive left eigenvector for the PF-eigenvalue λ of the incidence matrix M_θ. The prototiles for θ are intervals such that $\mathcal{P} = \{P_i\} = \{[0, w_i], i = 1, \ldots, d\}$, consider P_i to be distinct from P_j for $i \neq j$ even if $w_i = w_j$. Let $\mathcal{T}$ be a tiling of $\mathbb{R}$ defined by $\mathcal{P}$ (i.e. $\mathcal{T} \in X_{\mathcal{P}}$) such that $\mathcal{T} = \{B_i\}_{i=-\infty}^{i=\infty}$ and $B_i \cap B_{i+1}$ is a singleton for each i. Assume that the indexing is such that $0 \in B_0 \setminus B_1$.
If $\theta(i) = i_1 i_2 \ldots i_{k(i)}$, then $\lambda_i = \sum_{j=1}^{k(i)} w_{i_j}$. Thus, $|\lambda B_i| = \sum_{j=1}^{k(i)} |B_{i_j}|$ where $B_j = P_{i_j} + \sum_{k=1}^{j-1} w_{i_k}$. This process is said to be inflation and substitution [6] and extends to a map Φ taking a tiling $\mathcal{T} = \{B_i\}_{i=-\infty}^{i=\infty}$ of $\mathbb{R}$ by prototiles $\mathcal{P}$ to a new tiling, $\Phi(\mathcal{T})$, of $\mathbb{R}$ by prototiles defined by inflating, substituting, and suitably translating each B_i.
Define $X_n = \vee_{i=1}^n S_i$ as a wedge of n oriented circles $S_1, ..., S_n$ with the circumference of $S_i = w_i$, and let $f_\theta : X_n \to X_n$ be the linear map, with expansion constant λ, that follows the pattern θ. That is, if $\theta(a_i) = a_{i_1} a_{i_2} ... a_{i_{k(i)}}$, then f_θ maps the circle S_i around the circles $S_{i_1} S_{i_2} ... S_{i_{k(i)}}$, in that order and preserving orientation. We call f_θ the map of the rose associated with θ. In the next chapter we will consider the case where there is only two circles, following the work of Coelho, Parry and Williams (Example 2.43).

Theorem 4.25. *[7, Theorem 2.1] Suppose that θ and θ' are primitive non-periodic substitutions with induced inflation and substitution homeomorphisms f_θ and $f_{\theta'}$ acting on the associated tiling spaces Ω_θ and $\Omega_{\theta'}$. Then Ω_θ and $\Omega_{\theta'}$ are homeomorphic if and only if there exist positive integers m and n such that f_θ^n and $f_{\theta'}^m$ are topologically conjugate.*

4.5 Asymptotic composants

Recall that, a composant of a point x in a topological space X is the union of the proper compact connected subsets of X containing x. In (one-dimensional) tiling spaces, composants and arc components are identical [5].

For a primitive and aperiodic substitution θ, the composants of the tiling space Ω_θ coincide with the orbits of the translation of $\mathbb{R}$ on Ω_θ.

Definition 4.26. *Two composants C and C' of Ω_θ are said to be* forward (backward) asymptotic *if there are orientation preserving parametrizations $\gamma : \mathbb{R} \to C : \gamma' : \mathbb{R} \to C'$ such that $d(\gamma(t), \gamma'(t)) \to 0$ as $t \to -\infty (t \to \infty)$.*

It is clear that, if C and C' are backward (forward) asymptotic composants of Ω_θ and $h : \Omega_\theta \to \Omega_\psi$ is an orientation preserving homeomorphism, then $h(C)$ and $h(C')$ are backward (forward) asymptotic composants of Ω_ψ. If h is orientation reversing, then $h(C)$ and $h(C')$ are forward (backward) asymptotic composants of Ω_ψ.

Definition 4.27. *Two substitutions $\theta_1 : \mathcal{A} \to \mathcal{A}^*$ and $\theta_2 : \mathcal{B} \to \mathcal{B}^*$ are* weakly equivalent, *denoted by $\theta_1 \sim_w \theta_2$, if there are sequences of positive integers $\{n_i\}, \{m_i\}$ and maps $f_i : \mathcal{A} \to \mathcal{B}^*$, $g_i : \mathcal{B} \to \mathcal{A}^*$, $i = 1, 2, 3, ...,$ such that the following infinite diagram:*

$$\begin{tikzcd}
 & {} \arrow[dl, "f_1"'] & & {} \arrow[ll, "\theta_1^{m_1}"'] \arrow[dl, "f_2"'] & & {} \arrow[ll, "\theta_1^{m_2}"'] \arrow[dl, "f_3"'] & & {} \arrow[ll, "\theta_1^{m_3}"'] \\
{} & & {} \arrow[ul, "g_1"'] \arrow[ll, "\theta_2^{n_1}"] & & {} \arrow[ul, "g_2"'] \arrow[ll, "\theta_2^{n_2}"] & & {} \arrow[ul, "g_3"'] \arrow[ll, "\theta_2^{n_3}"] & \cdots
\end{tikzcd}$$

is commutative.

Let θ be a primitive, aperiodic substitution. The tiling space Ω_θ has a finite (non-zero) number of asymptotic composants. Barge and Diamond [5] used these asymptotic composants to define a closely related substitution $\theta^*$and prove that for primitive, aperiodic substitutions θ_1 and θ_2, T_{θ_1} and T_{θ_2} are homeomorphic if and only if θ_1^* (or its reverse) and θ_2^* are weakly equivalent.

Proposition 4.28. *[5, Lemma 3.5] For any substitution θ, composants C and C' of Ω_θ are backward (forward) asymptotic if and only if given $\mathcal{T} \in C$ and $\mathcal{T}' \in C'$, there is $s \in \mathbb{R}$ such that $d(T_{t+s}(\mathcal{T}), T_t(\mathcal{T}')) \to 0$ as $t \to \infty\, (t \to -\infty)$.*

Definition 4.29. *[5] A substitution θ is said to be* proper *if it possess only one periodic, (hence fixed), bi-infinite word.*

Proposition 4.30. *[5, Lemma 2.1] The following are equivalent:*

(1) $\theta : \mathcal{A} \to \mathcal{A}^* \setminus \epsilon$ is proper;

(2) for some $k \in \mathbb{N}$ and $b, e \in \mathcal{A}$, $\theta^k(i) = b \dots e$ for each $i \in \mathcal{A}$;

(3) there are $b, e \in \mathcal{A}$ such that for all sufficiently large k and all $i \in A$, $\theta^k(i) = b \dots e$.

Proposition 4.31. *[5, Corollary 3.9] Let θ be a substitution, Ω_θ has at least two but only finitely many composants that are backward asymptotic to some other composant, and at least two but only finitely many composants forward asymptotic to some other composant. Each contains a periodic point (under inflation and substitution) which, if θ is proper, is of period at most $n^2 - n$.*

Remark 4.32. *[5, p. 1344] Anderson and Putnam in ([2]), prove that if the substitution θ forces its border, then Ω_θ is homeomorphic with $\varprojlim f_\theta$, where f_θ is the map of the rose associated with θ. If θ is proper, then θ forces its border. This cannot be used directly to distinguish Ω_{θ_1} from Ω_{θ_2}, since θ_1 and θ_2 can be proper with Ω_{θ_1} homeomorphic to Ω_{θ_2} but $\theta_1 \nsim_w \theta_2$. If θ is not proper, then $\varprojlim f_\theta$ is not homeomorphic with Ω_θ but is homeomorphic with a simple quotient of Ω_θ.*

Remark 4.33. *A substitution θ is said to have a prefix problem if for some $a \neq b, \theta(b)$ is a prefix of $\theta(a)$ (that is, for some $a \neq b, \theta(a) = \theta(b)w$ for some word w). A suffix problem is defined similarly. Since we focus on constant length substitutions hence, we have no prefix or suffix problem.*

4.6 Denjoy continua

Imagine splitting open a line of irrational slope on the torus, splitting less and less as you move out along the line in both directions. The closure, in the torus, of the resulting pair of lines is said to be a Denjoy continuum. We will follow the procedures of Barge and Williams [8] to construct such continuum. Recall that, a rigid transformation is a transformation that does not change the size or shape of a figure for instance, the rotations, reflections and translations. We will consider the (Rigid) rotation of the circle $\mathbb{R}/\mathbb{Z}$ by the irrational number α, that is, $r_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $r_\alpha(x) = x + \alpha$.
Let $d_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving homeomorphism which satisfies the following:

(1) the rotation number of d_α is α,

(2) there is a Cantor set $C_\alpha \subset \mathbb{R}/\mathbb{Z}$ on which d_α acts minimally,

(3) if u and v are any two components of $(\mathbb{R}/\mathbb{Z}) \setminus C_\alpha$ then there exist an integer n such that $d_\alpha^n(u) = v$.

Then d_α is said to be Denjoy homeomorphism.
There is a Cantor function $h_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ that semi-conjugates d_α with r_α. h_α is a monotone surjection that collapses the components of $(\mathbb{R}/\mathbb{Z}) \setminus C_\alpha$ (and so maps C_α onto $\mathbb{R}/\mathbb{Z}$) with $r_\alpha \circ h_\alpha = h_\alpha \circ d_\alpha$. That is the diagram:

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} \arrow[r, "h_\alpha"] \arrow[d, "d_\alpha"'] & \mathbb{R}/\mathbb{Z} \arrow[d, "r_\alpha"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "h_\alpha"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

is commutative.

The suspension of d_α:

$$susp(d_\alpha) = (\mathbb{R}/\mathbb{Z} \times [0,1])/((x,1) \sim (d_\alpha(x), 0))$$

is homeomorphic to $\mathbb{T}^2$ and h_α induces a semi-conjugacy $H_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ between the "Denjoy flow" (the natural flow) on $susp(d_\alpha)$ and the "irrational flow" (the natural flow) on the suspension of r_α:

$$susp(r_\alpha) = (\mathbb{R}/\mathbb{Z} \times [0,1])/((x,1) \sim (r_\alpha(x), 0))$$

which is also homeomorphic with $\mathbb{T}^2$. The Denjoy continuum, $\mathbb{D}_\alpha$ is the suspension of $d_\alpha |_{C_\alpha}$:

$$\mathbb{D}_\alpha = susp(d_\alpha |_{C_\alpha}) = (C_\alpha \times [0,1])/((x,1) \sim (d_\alpha(x), 0)) \subseteq susp(d_\alpha) \simeq \mathbb{T}^2.$$

Such continua also arise in the suspension of a minimal(Denjoy) homeomorphism of a Cantor set and occur as attractors. They are all locally the same (locally trivial fiber bundles over the circle with fiber a Cantor set) and have the same Čech cohomology. Robbert Fokkink [19] has proved that there are uncountably many distinct Denjoy continua, in Theorem 4.34, he shows that the Denjoy continua corresponding to irrational slopes α_1 and α_2 are homeomorphic if and only if there is a 2×2 unimodular integer matrix that maps lines of slope α_1 (in the plane) to lines of slope α_2 (α_1 equivalent to α_2).

Barge and Williams [8] reproved Fokkinks theorem by an approach that emphasizes the connection between the topology of a Denjoy continuum and the geometry of the continued fraction expansion of its slope.

Theorem 4.34. *(Classification of Denjoy continua).[19, Theorem 2.5] Let α_1 and α_2 be two irrationals with associated Denjoy continua $\mathbb{D}_{\alpha_1}$ and $\mathbb{D}_{\alpha_2}$. Then $\mathbb{D}_{\alpha_1}$ and $\mathbb{D}_{\alpha_2}$ are homeomorphic if and only if α_1 and α_2 are equivalent.*

Denjoy continuum is homeomorphic to the suspension of a Denjoy circle homeomorphism restricted to its minimal set.

Anderson and Putnam [2] have shown that all substitution tiling spaces occur as hyperbolic attractors of diffeomorphisms. From Barge and Diamond [5] We have that, every orientable hyperbolic one-dimensional attractor is either: homogeneous, and hence a solenoid; or non-homogeneous, and then a one-dimensional substitution tiling space (the inhomogeneity is represented by the asymptotic composants). One-dimensional substitution tiling spaces can be viewed as suspensions of substitution minimal homeomorphisms of Cantor sets.

Definition 4.35.

(1) Let Ω_{θ_1} and Ω_{θ_2} are one–dimensional tiling spaces, then their shift homeomorphisms are said to be orbit equivalent *if there is a homeomorphism $h : \Omega_{\theta_1} \to \Omega_{\theta_2}$ takes orbits into orbits.*

(2) Two maps are said to be flow equivalent *if there is a homeomorphism between their suspensions taking trajectories of one to trajectories of the other and preserves orientation*

Locating asymptotic composants in T_θ is easier if θ is proper.

One can rewrite any primitive aperiodic substitution as a proper substitution:

Proposition 4.36. *[5, Corollary 3.4] Let θ be a primitive aperiodic substitution, then there is a proper primitive aperiodic substitution ψ such that Ω_θ is homeomorphic to Ω_ψ under an orientation preserving homeomorphism.*

This rewriting allow us to move from an arbitrary substitution to a proper one without altering the tiling space .

5. TILING SPACES AND ATTRACTORS

Smale [29, p. 788] introduced a hyperbolic attractor of a smooth map on a $3-$manifold that is homeomorphic to a dyadic solenoid. Using a similar construction, one can construct similar attractors on the solid torus corresponding to multiplication by any given integer $n > 1$ on an $n-$adic solenoid. As pointed out by Barge and Diamond [5], a one–dimensional, orientable hyperbolic attractor that is not homeomorphic to such a solenoid is homeomorphic to a substitution tiling space over a finite alphabet, based on the fundamental results of Williams [31], [32]. Coelho, Parry and Williams [14] examine in detail an attractor of a diffeomorphism of solid surface of genus two that admits the structure of a substitution tiling space for a substitution of constant length.

The classification of one–dimensional solenoids has been established for some time. Another class of attractors that has been well studied are those that arise from Sturmian substitutions, or Denjoy homeomorphisms. These were classified in [19] with the aid of the maximal equicontinuous factor, which in this case are the linear flows on the torus. These were used to classify the Denjoy minimal sets with one pair of asymptotic orbits. These spaces do have overlap with substitutions dynamical systems and include suspensions of all Sturmian sequences. These flows are particular examples of flows which are almost one–to–one extensions of their maximal equicontinuous factors.

Here we shall consider attractors that are obtained by replacing one orbit of a solenoid with a pair of asymptotic orbits. However, unlike the case of the Denjoy continua, we show that for any given $p-$adic solenoid(p is prime), one obtains a family of attractors, which are not all homeomorphic but all of which correspond to replacing a single orbit of the suspension flow on the solenoid with a pair of asymptotic orbits.

More generally, in this Chapter we shall examine families of hyperbolic attractors for diffeomorphisms of the solid surface of genus two which admit the structure of a substitution tiling space for a substitution of constant length. These spaces are similar to solenoids in that they admit an almost one to one map onto a solenoid. For the simplest families the classification mirrors that of the solenoids as shown in Theorem (5.8). In the more general case the classification is more complex. The key elements in our classification are a detailed analysis of a factor map onto a solenoid, the structure of affine maps of solenoids and the rigidity of substitution tiling spaces determined by Barge, Swanson [7] and Kwapisz [22].

5.1 The construction of the main system

Recall from section 2.2, for a given $n \geq 2$, the n-adic integers are given by the set of formal series

$$Z(n) := \left\{ \sum_{i=0}^{\infty} z_i \, n^i \mid \text{ for all } i, \, z_i \in \{0, 1, \dots, n-1\} \right\}.$$

Under addition $Z(n)$ is a compact abelian group, which contains a natural copy of the integers in which 1 is identified with $1n^0 + \sum_{i=1}^{\infty} 0 \, n^i$ and -1 is identified with $\sum_{i=0}^{\infty} (n-1) \, n^i$. For any given n, $Z(n)$ is homeomorphic to the Cantor set in the topology induced by the product topology and any of its compatible metrics. The adding machine map is the translation map

$$\tau : Z(n) \to Z(n) \, ; \, \tau(z) = z + \mathbf{1}.$$

Definition 5.1. *Let $h : X \to X$ be a homeomorphism. The* suspension flow *is the flow that is induced by translation:*

$$\phi_h : susp(h) \times \mathbb{R} \to susp(h)\,;\ ((x, s), t) \mapsto (x, s + t)$$

which is continuously extended for all t.

Observe that for $n \in \mathbb{Z}$, the time n of the flow corresponds to h^n on $X \times \{0\}$.

We will consider the solenoid to be the suspension (mapping cylinder) of the adding machine map. That is, for a given $n \geq 2$, the n–solenoid is

$$\Sigma(n) := \mathrm{susp}(\tau),$$

the suspension of the adding machine map $\tau : Z(n) \to Z(n)$.

The space $\Sigma(n)$ supports an additive topological group structure in which it is both compact and connected (a continuum), with identity element $\mathbf{0} = \mathbf{0}\times\{0\}$. The suspension flow maps $(\mathbb{R}, +)$ continuously (but not bi-continuously) and isomorphically onto the orbit of $\mathbf{0}$, $\phi_\tau\left(\{\mathbf{0}\} \times \mathbb{R}\right)$.

The classification of the solenoids is well known [26],[1].

Theorem 5.2. *[26, p. 198] $\Sigma(m)$ is homeomorphic to $\Sigma(n)$ if and only if m and n have the same prime factors.*

Proof. For completeness, we provide the outline of a proof. By Pontryagin duality, $\Sigma(m)$ is homeomorphic (even topologically isomorphic) to $\Sigma(n)$ if the dual group of $\Sigma(m)$ is isomorphic to the dual group of $\Sigma(n)$. These dual groups are in turn isomorphic to the first Čech cohomology groups (with integer coefficients) of the respective spaces. Thus, these dual groups are isomorphic to $(\mathbb{Z}[1/m], +)$ and $(\mathbb{Z}[1/n], +)$ respectively. These groups are isomorphic precisely when m and n have the same prime factors [4]. Conversely, if $\Sigma(m)$ is homeomorphic to $\Sigma(n)$, then their Čech cohomology groups are isomorphic. $\square$

Recall that, a *substitution* on the finite alphabet $\mathcal{A} = \{a_0, \ldots, a_{n-1}\}$ is a map

$$\theta : \mathcal{A} \to \mathcal{A}^*,$$

where $\mathcal{A}^*$ is the set of finite, non–empty words in $\mathcal{A}$. The *length* of a word is the number of symbols in the word. A substitution is said to be of *constant length* when the length of each word $\theta(a_i)$, $0 \leq i < n$, is the same. A substitution is *primitive* when there exists some $k \in \mathbb{Z}^+$ such that each a_i, $0 \leq i < n$, occurs in each word $\theta^k(a_j)$, $0 \leq j < n$, where θ^k is understood to be applied recursively via concatenation.

We will consider θ to be a primitive substitution on the alphabet $\mathcal{A} = \{a_0, \ldots, a_{n-1}\}$. There is a standard associated dynamical system (Ω_θ, σ) which is a subshift of the two-sided shift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. (where $\mathcal{A}^{\mathbb{Z}}$ is homeomorphic to the Cantor set in the product topology induced by the discrete topology on $\mathcal{A}$.) such that:

$$\Omega_\theta := \{\, (w_i) \in \mathcal{A}^{\mathbb{Z}} \mid \text{ for all } k \in \mathbb{Z},\ n \in \mathbb{Z}^+, \text{ there are } \ell \in \mathbb{Z}^+ \text{ and } 0 \leq i < n \\ \text{such that } w_k \cdots w_{k+n} \text{ is a subword of } \theta^\ell(a_i) \,\}.$$

The substitution map θ is extended to Ω_θ by concatenation:

$$\ldots w_{-1} \cdot w_0 w_1 \ldots \;\mapsto\; \ldots\; \theta(w_{-1}) \cdot \theta(w_0)\theta(w_1) \ldots\;,$$

It is well known that for a primitive substitution θ the subshift (Ω_θ, σ) is minimal (every orbit is dense) and that Ω_θ is homeomorphic to the Cantor set [3].

Definition 5.3. *The* substitution tiling space $\Sigma(\theta)$ *of the primitive substitution* θ *of constant length is the suspension of* $\sigma : \Omega_\theta \to \Omega_\theta$.

In general, to obtain the substitution tiling space one needs to choose lengths of tiles corresponding to the entries of a (positive) left eigenvector associated to the PF–eigenvalue of the associated incidence matrix [5]. In the case of a constant length substitution, these lengths are constant for all tiles, and so the suspension as above yields the substitution tiling space, which then means that the tilings based on the periodic points of the substitution are self–similar with similarity scaling given by the length of the substitution. ([13] discusses the effect of tile lengths on the dynamics, but we will not need to consider that here). Thus, the rigidity theorem of Kwapisz (Theorem 4.22) applies to the spaces we consider and will be used in some of our results.

The classification of solenoids can alternatively be approached by examining the topological conjugacy of adding machines in arbitrarily small clopen sets as in [1], but the classification of the adding machines up to topological conjugacy is not itself sufficient for topological classification. Thus, while the classification of substitution minimal sets up to topological conjugacy has recently been significantly advanced [16], this is not of itself sufficient to classify the corresponding tiling spaces formed by their suspension.

Another approach to classifying spaces was followed in [19] where the maximal equicontinuous factor of Denjoy flows (linear flows on the torus) were used to classify the Denjoy minimal sets with one pair of asymptotic orbits. These spaces do have overlap with substitutions dynamical systems and include suspensions of all Sturmian sequences. These flows are particular examples of flows which are almost one to one extensions of their maximal equicontinuous factors.

Definition 5.4. *A minimal continuous action of the abelian group G on the compact metric space X is* almost automorphic *if it is an almost one–to–one extension of its maximal equicontinuous factor. Equivalently, the action of G admits a factor map π onto the equicontinuous action of G on a compact abelian group A so that $\pi^{-1}(y)$ is a single point for at least one $y \in A$.*

In general, proximal relation $\sim_p$ (Definition 2.22) is not an equivalence relation on X; however, in the case that the action of A on X is almost automorphic, $\sim_p$ is an equivlence relation and the induced action on the quotient space $X/\sim_p$ is the (a) maximal equicontinuous factor with the quotient map as the factor map. In what follows we will explore almost automorphic flows and the extent to which this factorisation can be used to classify spaces topologically.

In [14], the authors detail certain properties of a hyperbolic attractor that is homeomorphic to the substitution tiling space of the substitution

$$\theta : \{0,1\} \to \{0,1\}^*$$

$$\theta(0) = 001\,;\, \theta(1) = 011$$

(Example 2.43). This corresponds to a hyperbolic attractor of a diffeomorphism of a solid genus two surface, hereafter referred to as S.

5.2 Difference one substitutions

Just as Smale's original construction of the solenoid can be modified by adjusting the number of times n that the solid torus wraps around inside itself, we can realize similarly defined substitution tiling spaces as attractors of smooth maps $S \to S$ as done in the original [14] construction by varying the number of times that the handles wrap around each other.

Figure 5.1 illustrates the image of a smooth injective map $f : S \to S$ for which the substitution tiling space of the substitution

$$0 \mapsto 10\,;\ 1 \mapsto 00$$

is an attractor.

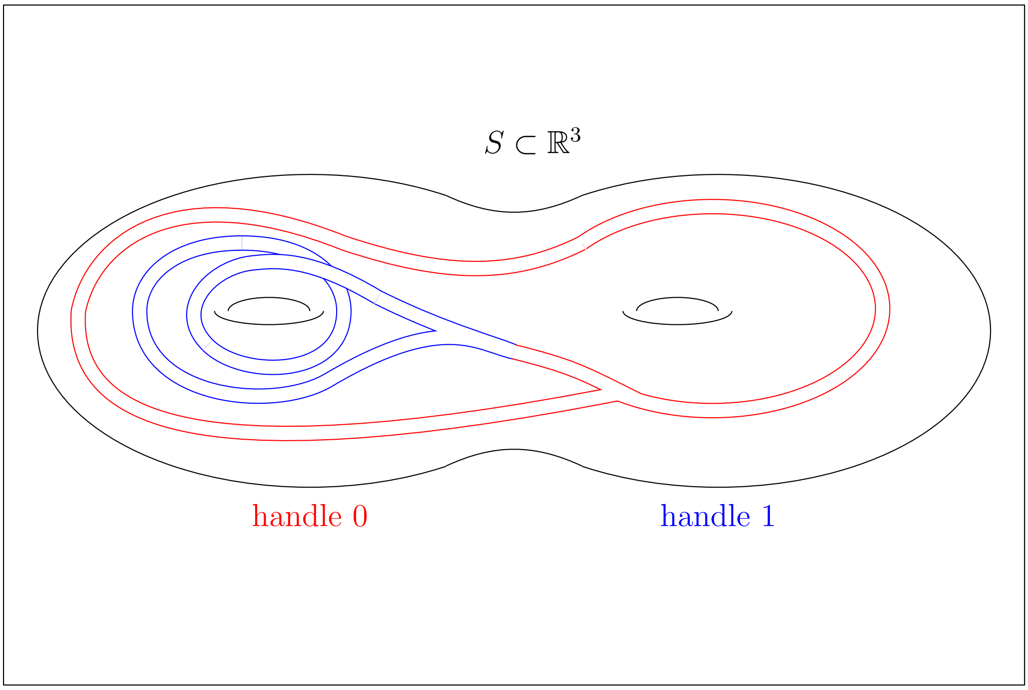


Fig. 5.1: Attractor is $\Sigma(\theta)$ for $\theta : 0 \mapsto 10; 1 \mapsto 00$

As they produce tiling spaces quite similar to the classical solenoids, we restrict ourselves to substitutions of constant length. Moreover, due to the extreme complexity of the general case, we shall restrict ourselves here to substitutions on the two letter alphabet $\{0, 1\}$ where the quite detailed results of [15] (section 3.5) will allow us to provide satisfying classification results.

Definition 5.5. *We call a substitution* $\theta : \{0,1\} \to \{0,1\}^*$ *of* difference one *when it is primitive, of constant length* $n \geq 2$ *and its difference set* J_1 *(definition 3.32) consists of precisely one element.*

In [15] there is an explicit description of a continuous surjective map

$$\pi : \Omega_\theta \to Z(n),\ x \mapsto \lim_{i\to\infty} \tau^{k_i}(\mathbf{0}), \tag{5.1}$$

where the sequence $\{k_i\}$ satisfies the condition that $\lim_{i\to\infty} \sigma^{k_i}(\omega) = x$ and ω is periodic under the application of θ extended to $\Sigma(\theta)$ by concatenation. This map is also developed in the more general constant length case in [24] and [17] for its properties as a factor map (i.e., it satisfies $\pi \circ \sigma = \tau \circ \pi,$) onto the maximal equicontinuous factor. If we define the *exceptional set* as

$$E := \{\, z \in Z(n) \,|\, \pi^{-1}(z) \text{ consists of more than one point} \,\},$$

then for a large class of substitutions of constant length (those exhibiting coincidence), the set E is of zero measure and its complement is a dense G_δ subset [24],[17]. In the case of two symbols as we consider here in proposition 3.48 there is a more detailed description of E and the π fibers of its points. In particular, it follows that for a difference one substitution we have:

(1) E consists of the τ–orbit $\{\, \tau^n(\zeta) \,|\, n \in \mathbb{Z} \,\}$ of the single point

$$\zeta := \sum_{i=0}^{\infty} z\, n^i,$$

where z is the unique element of J_1 and

(2) for $z \in E$, $\pi^{-1}(z)$ consists of exactly 2 points.

The map π then extends to a map $\mathrm{susp}(\pi)$ of the suspensions that conjugates the suspensions flows. To be specific, for $t \in [0,1)$

$$\mathrm{susp}(\pi)\,(x,t) = (\pi(x),t)\,.$$

In a similar way we consider the set $\mathrm{susp}(E)$, which in the difference one case consists of the entire ϕ_τ orbit of $(\zeta,0)$. By virtue of the construction, for all points $z \notin \mathrm{susp}(E)$, $\mathrm{susp}(\pi)^{-1}(z)$ consists of a single point and $\mathrm{susp}(\pi)^{-1}(E)$ consists of two ϕ_σ–orbits. In order to better understand the topology of $\Sigma(\theta)$ we will need to understand these orbits in detail.

Lemma 5.6. *$\pi^{-1}(E)$ consists of the σ-orbits of two points $x, \tilde{x} \in \Omega_\theta$ which approach each other asymptotically under application of both σ and σ^{-1}. Thus, $\mathrm{susp}(\pi)^{-1}(\mathrm{susp}(E))$ consists of the ϕ_σ–orbits of the two points $(x,0), (\tilde{x},0) \in \Sigma(\theta)$ that are asymptotic under the flow ϕ_σ in both directions:*

(1)

$$\mathrm{susp}(\pi)^{-1}(E) = \phi_\sigma(\{(x,0)\} \times \mathbb{R}) \cup \phi_\sigma(\{(\tilde{x},0)\} \times \mathbb{R})$$

(2)

$$lim_{t\to\pm\infty}\mathrm{d}\left(\phi_\sigma((x,0),t), \phi_\sigma((\tilde{x},0)),t)\right) = 0.$$

Proof. There are two cases:

1. $J_1 = \{z\}$ for $0 < z < i-1$ or
2. $J_1 = \{z\}$ for $z \in \{0, i-1\}$

In all cases, we let $\zeta(k)$ denote the integer $\sum_{j=0}^{k-1} z\, n^j$.

In the first case by Proposition 3.48, the two points $(x_i)_{i\in\mathbb{Z}}, (\tilde{x}_i)_{i\in\mathbb{Z}} \in \Omega_\theta$ in $\pi^{-1}(\zeta)$ satisfy:

$$x_{-\zeta(k)} \cdots x_{-\zeta(k)+n^k-1},\;\; \tilde{x}_{-\zeta(k)} \cdots \tilde{x}_{-\zeta(k)+n^k-1} \in \{\,\theta^k(0), \theta^k(1)\,\}.$$

That is, the n^k terms of the points $x, \tilde{x}$ starting from index $-\zeta(k)$ are given by the two words $\theta^k(0)$ and $\theta^k(1)$. As shown directly by induction, this means that in this difference one case, $x, \tilde{x}$ differ only in the term of index 0 and thus approach each other asymptotically under application of both σ and σ^{-1}. The lemma now follows in this first case.

In the second case, ζ is in the τ–orbit of $\mathbf{0}$ and is identified either with the integer 0 ($z = 0$) or -1 ($z = n - 1$), and E is the natural copy of $\mathbb{Z}$ within $Z(n)$. In this case by Proposition 3.48, the two points $(x_i)_{i\in\mathbb{Z}}, (\tilde{x}_i)_{i\in\mathbb{Z}} \in \Sigma(\theta)$ in $\pi^{-1}(\zeta)$ satisfy either ($z = 0$)

$$x_0 \cdots x_{n^k-1}\,,\ \ \tilde{x}_0 \cdots \tilde{x}_{n^k-1} \in \{\, \theta^k(0), \theta^k(1)\, \}$$

and

$$x_{-n^k} \cdots x_{-1},\ \ \tilde{x}_{-n^k} \cdots \tilde{x}_{-1} = \theta^k(a_{n-1}) = \theta^k(b_{n-1})$$

or ($z = n - 1$)

$$x_{-n^k} \cdots x_{-1},\ \ \tilde{x}_{-n^k} \cdots \tilde{x}_{-1} \in \{\, \theta^k(0), \theta^k(1)\, \}$$

and

$$x_0 \cdots x_{n^k-1},\ \ \tilde{x}_0 \cdots \tilde{x}_{n^k-1} = \theta^k(a_0) = \theta^k(b_0)$$

In the general setting the points $x, \tilde{x}$, which are fixed by θ^2, might not both lie in Ω_θ, but a basic combinatorial argument reveals that in the difference one case, they both lie in Ω_θ independent of whether $z = 0$ or $z = n - 1$.

Again, in this case, the points $x, \tilde{x}$ differ only in the term of a single index (either the term of index 0 or -1), and thus approach each other asymptotically under application of both σ and σ^{-1}. The lemma now follows in this case as well. $\square$

Remark 5.7. *By definition, a difference one substitution in normal form, see Definition 3.51, differs in the first place only, and so E consists of the orbit of the identity element as in the second case above.*

Theorem 5.8. *For any difference one substitutions θ and θ' of prime length p, $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic if and only if either the shift of Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ or the shift of Ω_θ is conjugate to the inverse of the shift on $\Omega_{\theta'}$.*

Proof. First assume that either the shift of Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ or the shift of Ω_θ is conjugate to the inverse of the shift on $\Omega_{\theta'}$. Generally, if the homeomorphism f is conjugate to either g or g^{-1}, then the suspension of f is homeomorphic to the suspension of g. If f conjugate to g then there is a homeomorphism from the suspension of f to the suspension of g which preserve the orientation of the flow while if f conjugate to g^{-1} then there is a homeomorphism from the suspension of f to the suspension of g which reverse the orientation of the flow. As $\Sigma(\theta)$ is the suspension of the shift of Ω_θ and $\Sigma(\theta')$ is the suspension of the shift of $\Omega_{\theta'}$, it follows directly that $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic.

Conversely, assume that $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic. As the topological class of $\Sigma(\theta)$ is the same for all topologically conjugate substitutions, without loss of generality, in what follows, θ and θ' are assumed to be difference one substitutions of prime length p in normal form. (This follows since each difference one substitution is conjugate to the uniquely determined such substitution in normal form by the results of [15].) By the rigidity theorem of Kwapisz 4.22, there is a homeomorphism $h : \Sigma(\theta) \to \Sigma(\theta')$ that conjugates the flow ϕ_θ with a linear rescaling by some $a \neq 0$ of the flow $\phi_{\theta'}$. Such a homeomorphism clearly preserves the proximal relation, and so h induces a continuous map $\overline{h}$ on the maximal equicontinuous factors that makes the following diagram commute:

$$\begin{tikzcd} \Sigma(\theta) \arrow[r, "h"] \arrow[d, "\mathrm{susp}(\pi)"'] & \Sigma(\theta') \arrow[d, "\mathrm{susp}(\pi')"] \\ \Sigma(p) \arrow[r, "\overline{h}"] & \Sigma(p) \end{tikzcd}$$

As the maps $\mathrm{susp}(\pi)$ and $\mathrm{susp}(\pi')$ factor the suspension flows on $\Sigma(\theta)$ and $\Sigma(\theta')$ onto the suspension flow on the solenoid $\Sigma(p)$, it follows that $\overline{h}$ commutes the action of the flow ϕ_τ on $\Sigma(p)$ with the linear rescaling by a of the flow ϕ_τ on $\Sigma(p)$. As previously discussed, the group structure on $\Sigma(p)$ is induced by the action of the flow ϕ_τ. As $\overline{h}$ corresponds to muliplication by a along an orbit, it follows that $\overline{h}$ is an affine map; that is, $\overline{h}$ can be represented as the composition of a translation and an endomorphism:

$$\overline{h}(x) = t + e(x),$$

where $e : \Sigma(p) \to \Sigma(p)$ is an endomorphism and $t \in \Sigma(p)$. By our assumption that θ and θ' are in normal form, the corresponding exceptional sets in $\Sigma(p)$ are formed by the orbit of the identity element as indicated in remark 5.7. As h must map the two asymptotic pair of orbits in $\Sigma(\theta)$ to those of $\Sigma(\theta')$ determined in Lemma 5.6, $\overline{h}$ must map the ϕ_τ orbit of the identity element of $\Sigma(p)$ to itself. As e preserves the identity element and the ϕ_τ orbit of the identity element is a subgroup of $\Sigma(p)$ (algebraically isomorphic to $(\mathbb{R}, +)$), we must have that t is in the ϕ_τ orbit of the identity, say $t = \phi_\tau(\mathbf{0}, T)$. However, this means that if we postcompose h with the homeomorphism given by the flow map $\phi_{\theta'}^{-T}$ we obtain a homeomorphism with the same properties as h making the diagram commute but with the corresponding translation by the identity element. Thus, without loss of generality we assume that for our given h the corresponding translation is by t being the identity element. In other words, $\overline{h} = e$ for some endomorphism e.

As both $\mathrm{susp}(\pi)$ and $\mathrm{susp}(\pi')$ are almost one–to–one, it follows from the commutative diagram that for almost all points x of $\Sigma(p)$, $(\overline{h})^{-1}(x)$ consists of a single point. Thus, e must in fact be an automorphism since all points have the same number of preimages under an affine map as determined by the kernel of the corresponding endomorphism. The automorphism group of $\Sigma(p)$ for prime p consists of all automorphisms $\alpha(k)$, where $\alpha(k)$ acts as multiplication by some k of the form $\pm p^n$ along the subgroup formed by the ϕ_τ orbit of the identity. In the case at hand, we have $k = a$. However, we also have by standard results that the substitution homeomorphism s' on $\Sigma(\theta')$ makes the following diagram commute:

$$\begin{tikzcd}
\Sigma(\theta') \arrow[r, "s'"] \arrow[d, "\mathrm{susp}(\pi')"'] & \Sigma(\theta') \arrow[d, "\mathrm{susp}(\pi')"] \\
\Sigma(p) \arrow[r, "\alpha(p)"] & \Sigma(p)
\end{tikzcd}$$

and s' conjugates the flow $\phi_{\theta'}$ with the linear rescaling by p of the flow $\phi_{\theta'}$. Also, as s' maps the $\mathrm{susp}(\pi')^{-1}(\mathbf{0})$ to itself. Thus, with $a = \pm p^n$ we have that the homeomorphism $H : \Sigma(\theta) \to \Sigma(\theta')$ given by $H(x) := (s')^{-n} \circ h(x)$ we have that H conjugates the flow ϕ_θ on $\Sigma(\theta)$ with either the flow $\phi_{\theta'}$ or the flow $-\phi_{\theta'}$, given by $-\phi_{\theta'}(x,t) = \phi_{\theta'}(x,-t)$ on $\Sigma(\theta')$. In general, for the suspension flow of a homeomorphism $R : X \to X$, the time one map corresponds to R on the section $X \times \{0\}$ and the time -1 map corresponds to R^{-1} on the section $X \times \{0\}$. Thus, we have that the shift on Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ or the shift of Ω_θ is conjugate to the inverse of the shift on $\Omega_{\theta'}$ according as $a = p^n$ or $a = -p^n$. □

While it seems quite possible that the above result might apply more generally to substitutions of lengths that are not prime, the above proof cannot be directly adapted as the corresponding solenoids have richer automorphism groups, allowing for multiplication by any factor of the length of the substitution.

Corollary 5.9. *For difference one substitutions θ and θ' of prime length p, $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic if and only if either θ and θ' have the same normal form or θ has the same normal form as the reverse of θ'.*

Proof. By Theorem 5.8, $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic if and only if either the shift on Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ or the shift of Ω_θ is conjugate to the inverse of the shift on $\Omega_{\theta'}$. By Theorem 3.52, Ω_θ is conjugate to the shift on $\Omega_{\theta'}$ if and only if they have the same normal form. The inverse shift on $\Omega_{\theta'}$ is given by the shift on $\Omega_{\theta'_r}$, where θ'_r is the reverse of θ', given by

$$\theta'_r(0) = a_{p-1} \cdots a_0 \text{ and } \theta'_r(1) = b_{p-1} \cdots b_0 \,,$$

given

$$\theta'(0) = a_0 \cdots a_{p-1} \text{ and } \theta'(1) = b_0 \cdots b_{p-1} \,.$$

The result then follows from another application of Theorem 3.52. □

Example 5.10. Consider the length 5 difference one substitutions θ and θ' given by:

$$\theta(0) = 01100 \text{ and } \theta(1) = 01110$$

and

$$\theta'(0) = 10001 \text{ and } \theta'(1) = 10011 \,.$$

Then θ and the reverse of θ' have the same normal form given by

$$0 \mapsto 00011 \text{ and } 1 \mapsto 10011 \,,$$

and so $\Sigma(\theta)$ and $\Sigma(\theta')$ are homeomorphic.

Example 5.11. Consider the length 5 difference one substitutions θ and θ' given by:

$$\theta(0) = 01100 \text{ and } \theta(1) = 01110$$

and

$$\theta'(0) = 10001 \text{ and } \theta'(1) = 00001\,.$$

Then θ has normal form as above, θ' is in normal form and the reverse of θ' has normal form given by

$$0 \mapsto 10111 \text{ and } 1 \mapsto 00111\,,$$

and so $\Sigma(\theta)$ and $\Sigma(\theta')$ are *not* homeomorphic.

As indicated by these examples, our results allow to algorithmically decide whether two given difference one substitution tiling spaces of the same prime length are homeomophic.

5.3 Difference d substitutions

A complete classification of the more general case is not readily found, but there are some quite general sufficient conditions for two of the tiling spaces to be homeomorphic as we detail in this section.

Definition 5.12. *We call a substitution $\theta : \{0,1\} \to \{0,1\}^*$ of* difference d *when it is primitive, of constant length $n \geq 2$ and its difference set J_1 consists* *of precisely d elements and $\theta(0)$ and $\theta(1)$ agree on the first symbol, the final symbol or on both.*

While much of the ingredients in the proof of Theorem 5.8 will apply to the case of difference d substitutions of prime length, the proof does not directly carry over as the translation element t could correspond a self homeomorphism of the tliling spaces that is not obtained from a time T map of the flow and might represent a sort of symmetry that would require some adjustment to the proof – assuming it could be generalised to this case.

BIBLIOGRAPHY

[1] J.M. Aarts and R.J. Fokkink, *The classification of solenoids.*, Proc. Amer. Math. Soc., 111 :1161-1163, 1991.

[2] J. E. Anderson and I. F. Putnam, *Topological invariants for substitution tilings and their associatedC^* algebras.*, Ergodic Theory and Dynamical Systems,18: 509-537, 1998.

[3] M. Baake and U. Grimm, *Aperiodic order.*, Cambridge University Press, 2013.

[4] R. Baer, *Abelian groups without elements of finite order.*, Duke Math. J., 3, no.1:68-122, 1937.

[5] M. Barge and B. Diamond, *A complete invariant for the topology of one-dimensional substitution tiling spaces.*, Ergodic Theory and Dynamical Systems, 21(5): 1333-1358, 2001.

[6] M. Barge and B. Diamond, *Cohomology in one-dimensional substitution tiling spaces.*, Proc. Amer. Math. Soc., 136(6): 2183-2191, 2008.

[7] M. Barge and R. Swanson, *Rigidity in one-dimensional tiling spaces.*, Topology and its applications, 154:3095-3099, 2007.

[8] M. Barge and R. Williams, *Classification of Denjoy continua.*, Topology and its applications, 106:77-89, 2000.

[9] L. Block and J. Keesling, *A Characterization of Adding Machine Maps.*, Topology and Its Applications 140, no. 2-3: 151-61, 2004.

[10] Broer, H., Takens, F., *Dynamical Systems and Chaos*, Springer, 2011.

[11] Brin, M., Stuck, G., *Introduction to Dynamical Systems.*, Cambridge University Press, 2002.

[12] A. Clark and S. Hurder, *Homogeneous matchbox manifolds.*, Trans. Amer. Math. Soc., vol. 365, 6: 31513191, 2013.

[13] A. Clark and L. Sadun, *When size matters: subshifts and their related tiling spaces.*, Ergodic Theory and Dynamical Systems , 23: 10431057, 2003.

[14] Z. Coelho, W. Parry and R. Williams, *A note on Livic's periodic point theorem.*, Topological dynamics and applications (Minneapolis, MN, 1995):223-230, Contemp. Math., 215, Amer. Math. Soc., Providence, RI , 1998.

[15] E.M. Coven and M.S. Keane, *The Structure of Substitution Minimal Sets.*, Trans. Amer. Math. Soc., 162:89 -102, 1971.

[16] E.M. Coven, F.M. Dekking and M.S. Keane, *Topological conjugacy of constant length substitution dynamical systems.*, Indagationes Mathematicae, 28, no.1: 91-107, 2017.

[17] F. M. Dekking, *The spectrum of dynamical systems arising from substitutions of constant length.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete , 41: 221-239, 1978.

[18] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[19] R. Fokkink, *The structure of trajectories.*, TU, Delft,Ph.D. Thesis, 1991.

[20] P. Kurka, *Topological and Symbolic Dynamics.*, Paris: Societe Mathematique De France, 2003.

[21] Kuratowski, K., *Topology*,Vol.I, Academic Press, London, 1966.

[22] J. KWAPISZ, *Rigidity and mapping class group for abstract tiling spaces.*, Ergodic Theory and Dynamical Systems, 31:1745-1783, 2011.

[23] Lee, J., *Introduction to Smooth Manifolds*, Springer, New York, 2003.

[24] John C. Martin , *Substitution Minimal Flows .*, American Journal of Mathematics, 93, No. 2 :503-526, 1971.

[25] Massey, William S., *A Basic Course in Algebraic Topology*, Springer-Verlag, New York, 1991.

[26] C. McCord, *Inverse limit sequences with covering maps.*, Trans. Amer. Math. Soc., 114:197209, 1965.

[27] E. A. Robinson, Jr., *Symbolic Dynamics and Tilings of* $\mathbb{R}^d$, Proceedings of Symposia in Applied Mathematics, vol. 60, 2004.

[28] L. Sadun, *Topology of Tiling Spaces.*, University Lecture Series, 46, Amer. Math. Soc., Providence, RI, (2008).

[29] S. Smale, *Differentiable dynamical systems.*, Bull. Amer. Math. Soc., 73:747-817, 1967.

[30] J. Vries, *Elements of Topological Dynamics.*, Springer Science+Business Media Dordrecht , 1993.

[31] R. F. Williams, *Expanding attractors.*, Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques., 43: 169203, 1974.

[32] R. F. Williams, *Classification of one dimensional attractors.*,Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), American Mathematical Society, Providence, RI :341361 1970.

[33] R. F. Williams, *One-dimensional non-wandering sets.*, Topology 6, 4:473-487, 1967.