Jordan-Lie Inner Ideals of Finite Dimensional Associative Algebras

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Abstract

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A subspace B of a Lie algebra L is said to be an inner ideal if $[B,[B,L]] \subseteq B$. Suppose that L is a Lie subalgebra of an associative algebra A. Then an inner ideal B of L is said to be Jordan-Lie if $B^2 = 0$.

In this thesis, we study Jordan-Lie inner ideals of finite dimensional associative algebras (with involution) and their corresponding Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic not 2 or 3.

Let A be a finite dimensional associative algebra over $\mathbb{F}$. Recall that A becomes a Lie algebra $A^{(-)}$ under the Lie bracket defined by $[x,y] = xy - yx$ for all $x, y \in A$. Put $A^{(0)} = A^{(-)}$ and $A^{(k)} = [A^{(k-1)}, A^{(k-1)}]$ for all $k \geq 1$. Let L be the Lie algebra $A^{(k)}$ ($k \geq 0$). In the first half of this thesis, we prove that every Jordan-Lie inner ideal of L admits Levi decomposition. We get full classification of Jordan-Lie inner ideals of L satisfying a certain minimality condition.

In the second half of this thesis, we study Jordan-Lie inner ideals of Lie subalgebras of finite dimensional associative algebras with involution. Let A be a finite dimensional associative algebra over $\mathbb{F}$ with involution $*$ and let $K^{(1)}$ be the derived Lie subalgebra of the Lie algebra K of the skew-symmetric elements of A with respect to $*$. We classify $*$-regular inner ideals of K and $K^{(1)}$ satisfying a certain minimality condition and show that every bar-minimal $*$-regular inner ideal of K or $K^{(1)}$ is of the form eKe^* for some idempotent e in A with $e^*e = 0$. Finally, we study Jordan-Lie inner ideals of $K^{(1)}$ in the case when A does not have "small" quotients and show that they admit $*$-invariant Levi decomposition.

I would like to dedicate this thesis to ....
My father "Mohammed Ali Shlaka" who passed away when I was four years old...
My grandfather "Saeed Shlaka" who supported me during my life and passed away when I finished my undergraduate study...
I owe all that I am to them.

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Contents

Chapter 1

Introduction

The ground ground field $\mathbb{F}$ is algebraically closed of characteristic $p \geq 0$. In this thesis we study Jordan-Lie inner ideals of finite dimensional associative algebras.

1.1 Overview

Let L be a Lie algebra. A subspace B of L is said to be an *inner ideal* of L if $[B,[B,L]] \subseteq B$. Note that every ideal is an inner ideal. On the other hand, there are inner ideals which are not even subalgebras. This makes them notoriously difficult to study. Inner ideals were first introduced by Benkart [13]. She showed that inner ideals and ad-nilpotent elements of Lie algebras are closely related [14]. Since certain restrictions on the ad-nilpotent elements yield an elementary criterion for distinguishing the non-classical from classical simple Lie algebras in positive characteristic, inner ideals play a fundamental role in classifying Lie algebras. It was shown in [28] that inner ideals play role similar to that of one-sided ideals in associative algebras and can be used to develop Artinian structure theory for Lie algebras.

Premet ([33] and [34]) proved that every finite dimensional simple Lie algebra over an algebraically closed field of characteristic not 2 or 3 must have nonzero extremal elements. Recall that an element x of a Lie algebra L is said to be *extremal* if $[x,[x,L]] \subseteq \mathbb{F}x$. Since one dimensional inner ideals of a Lie algebra L are spanned by extremal elements, finite dimensional simple Lie algebras over an algebraically closed field of characteristic not 2 or 3 must have one dimensional inner ideals. Moreover, it follows from [14], [32] and [17] that the classical Lie algebras over an algebraically closed field of characteristic greater than 5 can be characterized as nondegenerate finite dimensional simple Lie algebras which are generated by one dimensional inner ideals. Recall that a Lie algebra L is said to be *nondegenerate* if it has no non-zero absolute zero divisors (an element $x \in L$ is said to be an *absolute zero divisor* or *sandwich element* if $[x,[x,L]] = 0$).

Further motivation for studying inner ideals comes from [27], where Fernández López et al showed that if L is an arbitrary nondegenerate Lie algebra over a commutative ring Φ with 2 and 3 invertible, then every abelian inner ideal of finite length in L gives rise to a finite $\mathbb{Z}$-grading of L. Combining this with Zelmanov's classification [41] (see also [36] and [37]) of simple Lie algebras with finite $\mathbb{Z}$-gradings, we get that every nondegenerate simple Lie algebra over fields of characteristic 0 or $p > 4n+1$ (n is the largest integer with $L_n \neq 0$ in the grading) with a nonzero abelian inner ideal of finite length is isomorphic to either a (derived) Lie subalgebra of a simple associative ring (with involution) with a finite $\mathbb{Z}$-grading by taking the Lie commutator, or the Tits-Kantor-Koecher algebra of a Jordan algebra of a nondegenerate symmetric bilinear form, or an algebra of exceptional type (E_6, E_7, E_8, F_4, or G_2).

Inner ideals of classical Lie algebras were classified by Benkart [13] and completed by Benkart and Fernández López [15], using the fact that these algebras can be obtained as the derived Lie subalgebras of simple Artinian associative rings (with involution). Benkart's classification [13] of inner ideals of Lie algebras is similar to McCrimmon's one [31] of the derived Jordan subalgebras of simple associative rings (with involution). In a series of papers, Fernández López et al ([25], [26], [27], [28], [29] and [24]) show a strong connection between inner ideals of Lie algebras and inner ideals of Jordan systems (algebras and pairs, see [23]).

Benkart's classification was generalised by Fernández López, Garcia and Gómez Lozano [26] to the case of infinite dimensional finitary simple Lie algebras. Recall that an algebra is said to be *finitary* if it consists of finite-rank transformations of a vector space. Finitary simple Lie algebras over a field of characteristic 0 were classified by Baranov [2]. He proved that any infinite dimensional finitary central simple Lie algebra over a field of characteristic zero is isomorphic to either the finitary special linear Lie algebra, or the finitary unitary Lie algebra, or the *finitary symplectic* Lie algebra $\mathfrak{fsp}(V,\phi)$, or the *finitary orthogonal* Lie algebra $\mathfrak{fso}(V,\psi)$, where ϕ (resp. ψ) is a skew-symmetric (resp. symmetric) bilinear forms, defined on a vector space V over a field of characteristic 0. These results were further extended by Baranov and Strade [8] to the case of positive characteristic.

Inner ideals of another class of infinite dimensional Lie algebras were studied by Baranov and Rowley [6]. They proved that a simple locally finite Lie algebra over an algebraically closed field of characteristic 0 is diagonal (see [1] for definition) if and only if it has a non-zero proper inner ideal. Recall that an algebra is said to be *locally finite* if every finitely generated subalgebra is finite dimensional. The classification of diagonal simple locally finite Lie algebras was obtained in [1].

Benkart's and Benkart and Fernández López's results were further generalised by Fernández López, Brox and Gómez Lozano. They classified the inner ideals of the derived Lie subalgebras of centrally closed prime associative algebra over a field of characteristic $\neq 2,3$ [24] and of centrally closed prime ring with involution of characteristic $\neq 2,3,5$ [16].

In this thesis we use approach similar to Benkart's one to study inner ideals of the derived Lie subalgebras of finite dimensional associative algebras (with involution). These algebras generalise the class of simple Lie algebras of classical type and are closely related to the so-called root-graded Lie algebras [4]. They are also important in developing representation theory of non-semisimple Lie algebras (see [9]). As we do not require our algebras to be semisimple or semiprime we have a lot more inner ideals to take care of (as every ideal is automatically an inner ideal), so some reasonable restrictions are needed. We believe that such a restriction is the notion of a Jordan-Lie inner ideal introduced by Fernández López in [24]. We need some notation to state the definition.

Let A be a finite dimensional associative algebra over $\mathbb{F}$. Recall that A becomes a Lie algebra $A^{(-)}$ under the Lie bracket defined by $[x,y] = xy - yx$ for all $x,y \in A$. Put $A^{(0)} := A^{(-)}$ and $A^{(k)} = [A^{(k-1)}, A^{(k-1)}]$, $k \geq 1$. Suppose that A has an involution $*$. Recall that an involution is a linear transformation $*$ of an algebra A satisfying $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a,b \in A$. Note that we will work with involution of the first kind only, that is, $*$ is $\mathbb{F}$-linear. We denote by $\mathfrak{u}^*(A) := \{a \in A \mid a^* = -a\}$ the vector space of the skew symmetric elements of A. Recall that $K = \mathfrak{u}^*(A)$ is a Lie algebra with the Lie bracket defined by $[x,y] = xy - yx$ for all $x,y \in K$. We denote $\mathfrak{su}^*(A) := [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$. It is well known that the classical simple Lie algebras $\mathfrak{sl}_n$, $\mathfrak{sp}_n$ and $\mathfrak{so}_n$ can be defined as $\mathfrak{su}^*(A)$ for suitable involution simple associative algebras. Put $K^{(0)} := K$ and $K^{(k)} = [K^{(k-1)}, K^{(k-1)}]$ for all $k \geq 1$.

Let B be an inner ideal of $L = A^{(k)}$ or $K^{(k)}$ for some $k \geq 0$. We say that B is *Jordan-Lie* if $B^2 = 0$. Jordan-Lie inner ideals of $A^{(-)}$ were introduced in [24] by Fernández López. In some literature, see for example [16, Section 3], Jordan-Lie inner ideals of K are called *isotropic inner ideals*, as they correspond to isotropic subspaces of algebras with involution. The first motivation of studying Jordan-Lie inner ideals comes from [13, Theorem 5.1], where Benkart showed that if A is simple Artinian ring of characteristic not

2 or 3, then every inner ideal of the Lie algebra $[A,A]/Z(A)\cap[A,A]$ ($Z(A)$ is the centre of A) has square zero, that is, every inner ideal of such Lie algebra is Jordan-Lie.

Let V be a finite dimensional vector space over a division ring Δ of characteristic not 2 or 3 with involution and let $A=\mathrm{End}_\Delta V$. Suppose that A has involution and has dimension greater than 16 over its centre $Z(A)$. It follows from Benkart and Fernández López results [15, Theorem 6.1] that every proper inner ideal of $L=\mathfrak{su}^*(A)/\mathfrak{su}^*(A)\cap Z(A)$ is either Jordan-Lie or Clifford (in the later case, L is the finitary orthogonal Lie algebra $\mathfrak{fso}(V,\psi)$). Recall that an inner ideal B of $\mathfrak{fso}(V,\psi)$ is said to be *Clifford* if $B=[x,H^\perp]$, where H is a hyperbolic plane of V, x is a non-zero isotropic vector of H and $H^\perp=\{v\in V\mid \psi(v,H)=0\}$ [16]. Moreover, if Δ is a field, then their results are also true for inner ideals of the Lie algebra $\mathfrak{su}^*(A)$ [15, Theorem 6.3].

Further motivation comes from [24], where Fernández López showed that Jordan-Lie inner ideals are important in constructing the so-called standard inner ideals. Let A be an associative algebra over a ring of scalars Φ with $\frac{1}{2}\in\Phi$. An inner ideal of $A^{(-)}$ is called *standard* if it is of the form $B+\Omega$, where B is a Jordan-Lie inner ideal of A and Ω is a Φ-submodule of the centre $Z(A)$ of A. The usefulness of standard inner ideals comes from [24, Corollary 5.5], where it is proved that every abelian inner ideal of the (derived) Lie algebra of a non-unital centrally closed prime associative algebra over a field of characteristic not 2 or 3 is standard. Moreover, this is also true in the case when A is unital and every zero square element of A is Von Neumann regular.

In recent paper [16, Theorem 6.3] Brox, Fernández López and Gómez Lozano classify abelian inner ideals of the Lie algebra $\mathfrak{u}^*(A)$, where A is a centrally closed prime ring with involution of characteristic not 2, 3, or 5. They proved that every abelian inner ideal of $\mathfrak{u}^*(A)$ is either (i) isotropic (i.e. Jordan-Lie), or (ii) Clifford (of the form $[x,H^\perp]$), or (iii) standard (of the form $B\oplus\Omega$) or (iv) special, where A in (iii) and (iv) is unital with involution of the second kind and $*$ in (ii) is of orthogonal type. An inner ideal of $\mathfrak{u}^*(A)$ is said to be *special* if it is of the form $\{b+f(b)\mid b\in B\}$, where B is a Jordan-Lie (isotropic) inner ideal of $\mathfrak{u}^*(A)$ and $f:B\to\mathfrak{u}^*(Z(A))$ is a non-zero F-linear map with $[B,[B,\mathfrak{u}^*(A)]]\subseteq\ker f$ ($F=\{z\in Z(A)\mid z^*=z\}$ is a field of characteristic not 0) [16].

Further motivation of studying Jordan-Lie inner ideals comes from [6, Corollary 5.6], where Baranov and Rowley proved that if L is a locally finite infinite dimensional Lie algebra over an algebraically closed field of characteristic 0, then L is finitary if and only if L has a minimal regular inner ideal. A subspace B of $L=A^{(k)}$ (resp. $K^{(k)}$), $k\geq 0$, is said to be *regular* (resp. $*$*-regular*) *inner ideal* if $B^2=0$ and $BAB\subseteq B$ (resp. $\mathfrak{u}^*(BAB)\subseteq B$), see also Propositions 2.5.21 and 3.7.6 for alternative description of regular and $*$-regular inner ideals in terms of the orthogonal pairs of one-sided ideals of A.

Regular inner ideals were first defined in [6] and were recently used in [5] to classify zero product subsets of simple rings. Note that every regular inner ideal is Jordan-Lie (see Lemmas 2.5.15) and every $*$-regular inner ideal is Jordan-Lie (see Lemma 3.7.1). It was also proved in [5, 4.11] that all maximal abelian inner ideals of simple rings are regular. The regularity conditions $B^2 = 0$ and $BAB \subseteq B$ imply the original one ($[B,[B,L]] \subseteq B$) and are much easier to check, so it is an interesting question to describe the class of all finite dimensional algebras A such that all Jordan-Lie inner ideals of $A^{(k)}$ are regular. We believe that most algebras A are in this class. However, exceptions do exist, as we show in Example 2.5.17 that there is an algebra that contains a Jordan-Lie inner ideal which is not regular. In addition to point spaces (see Definition 2.1.6) are example of Jordan-Lie inner ideal which are not $*$-regular.

1.2 Outline of Thesis

Chapter 2 consists mainly of joint work with Alexander Baranov [7]. Recall that the ground field $\mathbb{F}$ is algebraically closed of characteristic $p \geq 0$. Let A be a finite dimensional associative algebra over $\mathbb{F}$ and let R be the radical of A. Let B be a Jordan-Lie inner ideal of $L = A^{(k)}$ $(k \geq 0)$, that is, B is an inner ideal with $B^2 = 0$. Denote by $\bar{B}$ the image of B in $\bar{L} = L/R \cap L$. Let X be an inner ideal of $\bar{L}$. We say that B is *X-minimal* (or simply, *bar-minimal*) if $\bar{B} = X$ and for every inner ideal B' of L with $\bar{B}' = X$ and $B' \subseteq B$ we have $B' = B$. Let e and f be idempotents in A. Then (e, f) is said to be an *idempotent pair in A*. An idempotent pair (e, f) in A is said to be *orthogonal* if $ef = fe = 0$ and *strict* if for each simple component S of $\bar{A} = A/R$, the projections of $\bar{e}$ and $\bar{f}$ on S are both either zero or non-zero. We are now ready to state the main results that will be proved in Chapter 2.

Theorem 1.2.1. *Let A be a finite dimensional associative algebra and let B be a Jordan-Lie inner ideal of $L = A^{(k)}$ $(k \geq 0)$. Suppose $p \neq 2, 3$. Then B is bar-minimal if and only if $B = eAf$ where (e, f) is a strict orthogonal idempotent pair in A.*

Corollary 1.2.2. *Let A be a finite dimensional associative algebra and let $L = A^{(k)}$ $(k \geq 0)$. Let B be a Jordan-Lie inner ideal of L. Suppose $p \neq 2, 3$ and B is bar-minimal. Then B is regular.*

Let B be an inner ideal of $L = A^{(k)}$ $(k \geq 0)$. Then we say that B *splits* in A if there is a Levi (i.e. maximal semisimple) subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$ (Definition 2.5.5).

Corollary 1.2.3. *Let A be a finite dimensional associative algebra and let $L = A^{(k)}$ $(k \geq 0)$. Let B be a Jordan-Lie inner ideal of L. Suppose $p \neq 2, 3$. Then B splits in A.*

In the proof of Theorem 1.2.1 we use the following result, which describes the poset of bar-minimal Jordan-Lie inner ideals of L and has an independent interest (see Section 2 for the definitions of the relations $\leq$, $\overset{\mathscr{L}\mathscr{R}}{\leq}$ and $\overset{\mathscr{L}\mathscr{R}}{\sim}$).

Theorem 1.2.4. *Let A be an Artinian ring or a finite dimensional associative algebra and let (e, f) and (e', f') be idempotent pairs in A. Suppose that (e, f) is strict. Then the following hold.*

(i) If $(e, f) \neq (0, 0)$ then $eAf \neq 0$.

(ii) $eAf \subseteq e'Af'$ if and only if $(e, f) \overset{\mathscr{L}\mathscr{R}}{\leq} (e', f')$.

(iii) Suppose that (e', f') is strict. Then $eAf = e'Af'$ if and only if $(e, f) \overset{\mathscr{L}\mathscr{R}}{\sim} (e', f')$.

(iv) Suppose that $eAf \subseteq e'Af'$. Then there exists a strict idempotent pair (e'', f'') in A such that $(e'', f'') \leq (e', f')$, $(e'', f'') \overset{\mathscr{L}\mathscr{R}}{\sim} (e, f)$ and $e''Af'' = eAf$.

Remark 1.2.5. It is well-known that every finite dimensional unital algebra is Artinian as a ring. In particular, semisimple finite dimensional algebras are Artinian. However, this is not true for non-unital algebras (e.g. for the one dimensional algebra over $\mathbb{Q}$ with zero multiplication). This is why we refer to both Artinian rings and finite dimensional algebras in the theorem above.

Chapter 3 contains some results in joint work with Alexander Baranov. In this chapter, we study Jordan-Lie inner ideals of the derived Lie subalgebras of finite dimensional associative algebras with involution. Let A be a finite dimensional associative algebra over $\mathbb{F}$ with involution $*$ (of the first kind) and let R be its radical. In [38] and [39] Taft proved that there is a $*$-invariant Levi subalgebra S of A. Recall that $K = \mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$ is a Lie algebra and $K^{(1)} = \mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$ is a subalgebra of $\mathfrak{u}^*(A)$. We denote by $\mathrm{sym}(A) := \{a \in A \mid a^* = a\}$ the vector space of symmetric elements of A. Note that $\mathrm{sym}(A)$ is a Jordan subalgebra of A (see [20]). Recall that a subspace B of $K^{(k)}$ ($k \geq 0$) is said to be a $*$-regular inner ideal if $B^2 = 0$ and $\mathfrak{u}^*(BAB) \subseteq B$. We are ready now to state some of our main results that will be proved in Chapter 3.

Theorem 1.2.6. *Let A be a finite dimensional associative algebra with involution and let B be a Jordan-Lie inner ideal of $K^{(k)}$ ($k = 0, 1$). Suppose that $p \neq 2$ and B is bar-minimal. Then B is $*$-regular if and only if $B = eKe^*$ for some idempotent e in A with $e^*e = 0$.*

Let L be a finite dimensional Lie algebra and let B be a subspace of L. Suppose that there is a quasi Levi (see Definition 2.1.4) decomposition $L = Q \oplus N$ of L such that $B = B_Q \oplus B_N$, where $B_Q = B \cap Q$ and $B_N = B \cap N$. Then we say that B *splits in* L and Q is a *B-splitting quasi Levi subalgebra* of L (Definition 2.5.4). We also say that an inner ideal B of $K^{(k)}$ ($k \geq 0$) *$*$-splits in* A if there is a $*$-invariant Levi (i.e. maximal semisimple) subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$ (Definition 3.6.7).

Corollary 1.2.7. *Let A be a finite dimensional associative algebra with involution. Suppose that $p \neq 2$. Then every $*$-regular inner ideal of $K^{(k)}$ ($k = 0, 1$) $*$-splits in A.*

The theorem and its corollary show that every bar-minimal $*$-regular inner ideal of $\mathfrak{su}^*(A)$ generated by idempotent e in A with $e^*e = 0$ and admits a Levi decomposition in A. We are going to show that all Jordan-Lie inner ideals (not just $*$-regular ones) admit a Levi decomposition in A under some natural restrictions on A (absence of "small" quotients). Such algebras are said to be admissible (see Definition 3.5.1). Our motivation to study this associative algebras comes from [10, Theorem 6.3], where Baranov and Zalesskii proved that if A is admissible over an algebraically closed field of characteristic 0, then the Lie algebra $\mathfrak{su}^*(A)$ is perfect, that is $[\mathfrak{su}^*(A), \mathfrak{su}^*(A)] = \mathfrak{su}^*(A)$. We believe that this is also true in the case when A is admissible over an algebraically closed field of characteristic not 2 or 3 (the proof is similar to that of characteristic 0 with some extra cases to be considered), so we may assume that $K^{(1)} = \mathfrak{su}^*(A)$ is perfect if A is admissible and $p \neq 2,3$. Now, we are ready to state our main results which will be proved in Chapter 3.

Theorem 1.2.8. *Let A be a finite dimensional associative algebra with involution and let B be a bar-minimal Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. Suppose that A is admissible and $p \neq 2,3$. Then B $*$-splits in A.*

Corollary 1.2.9. *Let A be a finite dimensional associative algebra with involution. Suppose that A is admissible and $p \neq 2,3$. Then the following holds.*
(i) Every Jordan-Lie inner ideal of $K^{(1)}$ $*$-splits in A.
(ii) Every Jordan-Lie inner ideal of $K^{(1)}$ splits in $K^{(1)}$.

1.3 Notations and Conventions

- A is a finite dimensional associative algebra (with involution in Chapter 3).
- $*$ is an involution of the first kind.
- S is a Levi subalgebra of A ($*$-invariant Levi in Chapter 3) .
- R is the radical of A.
- $\bar{A} = A/R$
- $A^{(-)}$ the Lie algebra of A with the Lie bracket defined by $[x,y] = xy - yx$ for all $x,y \in A$, where xy is the usual multiplication of A.
- $A^{(k)}$ ($k \geq 0$) is the derived Lie subalgebra of A, where $A^{(0)} = A^{(-)}$, $A^{(1)} = [A,A]$ and $A^{(k)} = [A^{(k-1)}, A^{(k-1)}]$ for all $k \geq 1$.
- $K = \mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$ the Lie algebra of A consisting of skew symmetric elements of A with respect to the involution $*$.
- $K^{(1)} = \mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$.

- $K^{(0)} = K$ and $K^{(k)} = [K^{(k-1)}, K^{(k-1)}]$ for all $k \geq 1$.
- $\mathscr{P}_1(A)$ the 1-perfect radical of A is the largest 1-perfect ideal of A (Definition 2.4.5).
- $\mathscr{P}_a(A)$ the admissible radical of A is the largest admissible ideal of A (Definition 3.5.5)
- L-perfect inner ideal is an inner ideal B of a Lie algebra L such that $[B,[B,L]] = B$.
- $\mathrm{core}_L(B)$ the core of the inner ideal B of a Lie algebra L (Definition 2.4.13).
- $\mathscr{M}_n$ the algebra of all $n \times n$-matrices over $\mathbb{F}$.
- $\mathfrak{gl}_n$ the general linear algebra over $\mathbb{F}$.
- $\mathfrak{sl}_n$ the special linear algebra over $\mathbb{F}$.
- $\mathfrak{sp}_{2n}$ the symplectic Lie algebra over $\mathbb{F}$:

$$\mathfrak{sp}_{2n} = \{\begin{pmatrix} X & X_1 \\ X_2 & -X^t \end{pmatrix} \mid X, X_1 \in \mathscr{M}_n \text{ with } X_i^t = X_i, \quad i = 1,2\}.$$

- $\mathfrak{so}_m$ ($m = 2n+1$ or $2n$) the orthogonal algebra:

$$\mathfrak{so}_{2n} = \{\begin{pmatrix} X & X_1 \\ X_2 & -X^t \end{pmatrix} \mid X, X_i \in \mathscr{M}_n \text{ with } X_i^t = -X_i, \quad i = 1,2\} \text{ and}$$

$$\mathfrak{so}_{2n+1} = \{\begin{pmatrix} \multicolumn{2}{c}{\mathfrak{so}_{2n}} & \begin{matrix} Y_1 \\ Y_2 \end{matrix} \\ -Y_2^t & -Y_1^t & 0 \end{pmatrix} \mid Y_1, Y_2 \in M_{n1}\}.$$

- τ_ε ($\varepsilon = \pm 1$ or simply $\pm$) is a canonical involution defined on $\mathscr{M}_m$ by $X \mapsto X^{\tau_\varepsilon} = J_\varepsilon^{-1} X^t J_\varepsilon$ where $J_\varepsilon = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}$ (I_n is the identity $n \times n$-matrix) when $m = 2n$ and $J_+ = diag(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, 1)$ when $m = 2n+1$; τ_+ is called *orthogonal* and τ_- is called *symplectic* (see 3.1.3). Moreover, if $*$ admits τ_+ (resp. τ_-) in $\mathscr{M}_m$, then we say that $*$ is *canonical of orthogonal (resp. symplectic) type* of A.

- $\mathrm{sym}^{\rho}_{\tau_\varepsilon}(\mathscr{M}_m)$ ($\rho = \pm 1$ or simply $\pm$) is the subspace of $\mathscr{M}_m$ defined by

$$\mathrm{sym}^{\rho}_{\tau_\varepsilon}(\mathscr{M}_{2n}) = \{\begin{pmatrix} X & Y_1 \\ Y_2 & \rho X^t \end{pmatrix} \mid X, Y_1, Y_2 \in \mathscr{M}_n, \quad Y_1^t = \rho\varepsilon Y_1, \quad Y_2^t = \rho\varepsilon Y_2\};$$

$$\mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n+1}) = \{\begin{pmatrix} \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n}) & \begin{matrix} Y_3 \\ Y_4 \end{matrix} \\ \begin{matrix} \rho Y_4^t & \rho Y_3^t \end{matrix} & \alpha \end{pmatrix} \mid Y_3, Y_4 \in \mathscr{M}_{n1}, \quad \alpha \in \mathbb{F}\},$$

  where $\alpha = 0$ if $\rho = -1$.

- $\mathrm{sym}^{\rho}(\mathscr{M}_m) = \{X \in \mathscr{M}_m \mid X^t = \rho X\}$.

Chapter 2

Jordan-Lie inner ideals of finite dimensional associative algebras

In this chapter we study Jordan-Lie inner ideals of Lie algebras come from finite dimensional associative algebras over algebraically closed fields of characteristic not 2 or 3. In particular, we will prove Theorem 1.2.1, Corollary 1.2.2, Corollary 1.2.3 and Theorem 1.2.4. We introduce and describe some special types of inner ideals such as cores of inner ideals, L-perfect inner ideals, bar-minimal inner ideals and regular inner ideals, which will be used to prove the main results. The relation between inner ideals and idempotent pairs will be discussed as well.

Outline of Chapter 2

(**Section** 2.1) We discuss some background results related to Lie algebras derived from associative algebras and Jordan-Lie inner ideals of such Lie algebras.

(**Section** 2.2) We prove Theorem 1.2.4, which is one of our main results that describes the poset of Jordan-Lie inner ideals generated by idempotents.

(**Section** 2.3) We study inner ideals of Lie algebras derived from semisimple associative algebras. We highlight some preliminary results related to inner ideals of such Lie algebras and state the most important ones.

(**Section** 2.4) We introduce and describe some classes of inner ideals and associative algebras: 1-perfect associative algebras, cores of inner ideals and L-perfect inner ideals.

(Section 2.5) We introduce and describe bar-minimal and regular inner ideals and their relation with L-perfect inner ideals. We define Levi decomposition of inner ideals. We prove that every bar-minimal regular inner ideal is generated by a pair of idempotents.

(Section 2.6) Using Theorem 1.2.4 (proved in section 2.2) and the notion of 1-perfect associative algebras with the properties of L-perfect inner ideals, we prove the main results of this chapter. In particular, we will prove that bar-minimal Jordan-Lie inner ideals are generated by idempotents (Theorem 1.2.1) and are regular (Corollary 1.2.2). As a corollary, we show that all Jordan-Lie inner ideals split in their associative algebras (Corollary 1.2.3).

2.1 Background Materials

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$.

Throughout this chapter, unless otherwise specified, A is a finite dimensional associative algebra over $\mathbb{F}$, $R = \operatorname{rad} A$ is the radical of A, S is a Levi (i.e. maximal semisimple) subalgebra of A, so $A = S \oplus R$; $L = A^{(k)}$ for some $k \geq 0$, $\operatorname{rad} L$ is the solvable radical of L and $N = R \cap L$ is the *nil-radical* of L. If V is a subspace of A, we denote by $\bar{V}$ its image in $\bar{A} = A/R$. In particular, $\bar{L} = (L+R)/R \cong L/N$. Since R is a nilpotent ideal of A the ideal $N = R \cap L$ of L is also nilpotent, so $N \subseteq \operatorname{rad} L$. It is easy to see that $N = \operatorname{rad} L$ if $p = 0$ and $k \geq 1$, so L/N is semisimple in that case. Recall that a Lie algebra M is called *perfect* if $[M,M] = M$.

Definition 2.1.1. Let Q be a Lie algebra. We say that Q is a *quasi (semi)simple* if Q is perfect and $Q/Z(Q)$ is (semi)simple.

It follows from Herstein [21, Theorem 4] that if A is simple ring of characteristic different from 2, then $A^{(1)} = [A,A]$ is a quasi simple Lie ring. Moreover, if A has an involution $*$ and of dimensional greater than 16 over its center, then $\mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$ is a quasi simple Lie algebra [21, Theorem 10]. Recall that $\mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$.

Furthermore, Martindale III and Meirs [30, Theorem 6.1] showed that if A is semiprime of characteristic not 2, then $\mathfrak{u}^*(A)/Z(\mathfrak{u}^*(A))$ is semiprime.

The following fact is a particular case of [21, Theorem 4].

Lemma 2.1.2. *Let $p \neq 2$, $n \geq 2$ and let $A = \mathscr{M}_n$. Then $[A,A] = \mathfrak{sl}_n$ is quasi simple. In particular, $A^{(\infty)} = A^{(1)}$.*

Note that the case of $p = 2$ is exceptional indeed as the algebra $\mathfrak{sl}_2$ is solvable in characteristic 2.

Proposition 2.1.3. *Suppose that A is semisimple and $p \neq 2$. Then $[A,A]$ is quasi semisimple. In particular, $A^{(\infty)} = A^{(1)}$.*

Proof. Since A is semisimple, $A = \bigoplus_{i \in I} S_i$ where the S_i are simple ideals of A. Since $\mathbb{F}$ is algebraically closed, $S_i \cong \mathscr{M}_{n_i}$ for some n_i. Note that $[S_i,S_i] = 0$ if $n_i = 1$ and $[S_i,S_i] \cong \mathfrak{sl}_{n_i}$ if $n_i \geq 2$. Now the result follows from Lemma 2.1.2.

□

Definition 2.1.4. Let M be a finite dimensional Lie algebra and let Q be a quasi semisimple subalgebra of M. We say that Q is a *quasi Levi subalgebra* of M if there is a solvable ideal P of M such that $M = Q \oplus P$. In that case we say that $M = Q \oplus P$ is a *quasi Levi decomposition* of M.

Recall that $N = R \cap L$ is the nil-radical of $L = A^{(k)}$.

Proposition 2.1.5. *Let S be a Levi subalgebra of A and let $L = [A,A]$ and $Q = [S,S]$. Suppose that $p \neq 2$. Then $N = [S,R] + [R,R]$, Q is a quasi Levi subalgebra of L and $L = Q \oplus N$ is a quasi Levi decomposition of L. Moreover, $N = [S,R]$ if $R^2 = 0$.*

Proof. We have $L = [A,A] = [S \oplus R, S \oplus R] = [S,S] + [S,R] + [R,R] = Q \oplus N$ where $Q = [S,S]$ is quasi semisimple by Proposition 2.1.3 and $[S,R] + [R,R] = L \cap R = N$ is the nil-radical of L, as required.

□

A subspace B of A is said to be a *Lie inner ideal* of A if B is an abelian inner ideal of $L = A^{(-)}$, that is $[B,[B,L]] \subseteq B$. A subspace B of A is said to be a *Jordan inner ideal* of A if B is an inner ideal of the Jordan algebra $A^{(+)}$ [24]. If $B^2 = 0$, then B is an inner ideal of the Jordan algebra $A^{(+)}$ if and only if it is an inner ideal of the Lie algebra $A^{(-)}$. Indeed, since $B^2 = 0$, one has

$$[b,[b',x]] = -bxb' - b'xb \tag{2.1.1}$$

for all $b, b' \in B$ and all $x \in A$. This justifies the following definition.

Definition 2.1.6. [24] An inner ideal B of $L = A^{(k)}$ is said to be *Jordan-Lie if* $B^2 = 0$.

It follows from Benkart's result [13, Theorem 5.1] that if A is a simple Artinian ring of characteristic not 2 or 3, then every inner ideal of $[A,A]/(Z(A) \cap [A,A])$ is Jordan-Lie. For $b, b' \in B$ and $x \in L$, we denote by $\{b,x,b'\}$ the Jordan triple product

$$\{b,x,b'\} := bxb' + b'xb.$$

The following lemma follows immediately from (2.1.1) and the definition.

Lemma 2.1.7. *Let $L = A^{(k)}$ for some $k \geq 0$ and let B be a subspace of L. Then B is a Jordan-Lie inner ideal of L if and only if $B^2 = 0$ and $\{b,x,b'\} \in B$ for all $b, b' \in B$ and $x \in L$.*

Recall that our algebra A is non-unital in general. Let $\hat{A} = A + \mathbb{F}1_{\hat{A}}$ be the algebra obtained from A by adding the external identity element. The following lemma shows that the Jordan-Lie inner ideals of $\hat{A}^{(k)}$ are exactly those of $A^{(k)}$ for all $k \geq 0$.

Lemma 2.1.8. *Let B be a subspace of A. Then B is a Jordan-Lie inner ideal of $A^{(k)}$ if and only if B is a Jordan-Lie inner ideal of $\hat{A}^{(k)}$ ($k \geq 0$).*

Proof. Note that $\hat{A}^{(k)} = A^{(k)}$ for all $k \geq 1$, so we only need to consider the case when $k = 0$, i.e. $A^{(k)} = A^{(-)}$. If B is a Jordan-Lie inner ideal of A then $[B,[B,\hat{A}]] = [B,[B,A+\mathbb{F}1_{\hat{A}}]] = [B,[B,A]] \subseteq B$, so B is a Jordan-Lie inner ideal of $\hat{A}$. Suppose now that B is a Jordan-Lie inner ideal of $\hat{A}$. Then $\tilde{B} = (B+A)/A$ is a Jordan-Lie inner ideal of $\hat{A}/A \cong \mathbb{F}$. Since $\tilde{B}^2 = 0$, we get that $\tilde{B} = 0$, so $B \subseteq A$. Therefore, B is a Jordan-Lie inner ideal of A.

□

Recall that idempotents e and f are said to be *orthogonal* if $ef = fe = 0$.

Lemma 2.1.9. *Let A be a ring and let $Z(A)$ be the center of A. Let e and f be idempotents in A such that $fe = 0$. Then*

(i) $eAf \cap Z(A) = 0$;

(ii) $B = eAf \cap A^{(k)}$ is a Jordan-Lie inner ideal of $A^{(k)}$ for all $k \geq 0$;

(iii) eAf is a Jordan-Lie inner ideal of $A^{(-)}$ and of $A^{(1)}$;

(iv) there exists an idempotent g in A such that g is orthogonal to e and $eAf = eAg$,

Proof. (i) Let $z \in eAf \cap Z(A)$. Then $z = eaf$ for some $a \in A$. Since $z \in Z(A)$, we have $0 = [e,z] = [e,eaf] = eaf = z$. Therefore, $eAf \cap Z(A) = 0$.

(ii) We have $B^2 \subseteq eAfeAf = 0$ and $[B,[B,A^{(k)}] \subseteq BA^{(k)}B \cap A^{(k)} \subseteq eAf \cap A^{(k)} = B$, as required.

(iii) This follows from (ii) as $eAf = [e,eAf] \subseteq [A,A]$.

(iv) Put $g = f - ef$. Then $g^2 = (f-ef)^2 = f^2 - eff = f - ef = g$, so g is an idempotent in A. Since $ge = (f-ef)e = 0$ and $eg = e(f-ef) = ef - ef = 0$, e and g are orthogonal. It remains to note that $eAg = eA(f-ef) \subseteq eAf$ and $eAf = eAf(f-ef) = eAfg \subseteq eAg$. Therefore, $eAf = eAg$, as required.

□

We note the following standard properties of inner ideals.

Lemma 2.1.10. *Let L be a Lie algebra and let B be an inner ideal of L.*

(i) If M is a subalgebra of L, then $B \cap M$ is an inner ideal of M.

(ii) If P is an ideal of L, then $B+P/P$ is an inner ideal of L/P.

2.2 Idempotent pairs

The aim of this section is to prove Theorem 1.2.4, which describes the poset of Jordan-

Lie inner ideals generated by idempotents. We start by recalling some well known relations on the sets of idempotents.

Definition 2.2.1. Let A be a ring and let e and e' be idempotents in A. Then

(1) e is said to be *left dominated by* e', written $e \overset{\mathscr{L}}{\leq} e'$, if $e'e = e$.

(2) e is said to be *right dominated by* e', written $e \overset{\mathscr{R}}{\leq} e'$, if $ee' = e$.

(3) e is said to be *dominated by* e', written $e \leq e'$, if e is a left and right dominated by e', that is, if $e \overset{\mathscr{L}}{\leq} e'$ and $e \overset{\mathscr{R}}{\leq} e'$, or equivalently, $ee' = e'e = e$.

(4) Two idempotents e and e' are called *left equivalent*, written $e \overset{\mathscr{L}}{\sim} e'$, if $e \overset{\mathscr{L}}{\leq} e'$ and $e' \overset{\mathscr{L}}{\leq} e$.

(5) Two idempotents e and e' are called *right equivalent*, written $e \overset{\mathscr{R}}{\sim} e'$, if $e \overset{\mathscr{R}}{\leq} e'$ and $e' \overset{\mathscr{R}}{\leq} e$.

Remark 2.2.2. (1) It is easy to see that $\overset{\mathscr{L}}{\leq}$ and $\overset{\mathscr{R}}{\leq}$ are *preorder* relations, $\leq$ is a *partial order* and $\overset{\mathscr{L}}{\sim}$ and $\overset{\mathscr{R}}{\sim}$ are *equivalences.* Note that if A is Artinian, then the set of all idempotents satisfies the descending chain condition with respect to the partial order $\leq$.

(2) If e and e' are idempotents in A, then it is easy to check that $e \leq e'$ if and only if $e' = e + e_1$ for some idempotent e_1 in A with $e_1 e = ee_1 = 0$.

The following lemma is well-known.

Lemma 2.2.3. *Let A be a ring and let e and e' be idempotents in A. Then*

(i) $e \overset{\mathscr{L}}{\leq} e'$ if and only if $eA \subseteq e'A$.

(ii) $e \overset{\mathscr{L}}{\sim} e'$ if and only if $eA = e'A$.

(iii) If $e \overset{\mathscr{L}}{\leq} e'$, then there is an idempotent e'' in A such that $e'' \leq e'$ and $e'' \overset{\mathscr{L}}{\sim} e$.

Proof. (i) Since $e \overset{\mathscr{L}}{\leq} e'$, we have $eA = e'eA \subseteq e'A$. On the other hand, if $eA \subseteq e'A$, then $e = ee \in e'A$, so $e'e = e$, as required.

(ii) This follows from (i).

(iii) Put $e'' = e'ee' = ee'$. Then $e''^2 = ee'ee' = eee' = ee' = e''$, so e'' is an idempotent. Since $e'e'' = e'(e'ee') = e'ee' = e''$ and $e''e' = (e'ee')e' = e'ee' = e''$, we have $e'' \leq e'$. It remains to note that $e''e = (ee')e = e(e'e) = ee = e$ and $ee'' = e(ee') = ee' = e''$, so $e \overset{\mathscr{L}}{\sim} e''$, as required.

□

We say that (e,f) is an *idempotent pair* in A if both e and f are idempotents in A. Moreover, (e,f) is *orthogonal* if $ef = fe = 0$.

Definition 2.2.4. Let A be a ring and let e, e', f and f' be idempotents in A. We say that

(1) (e,f) is *left-right dominated by* (e',f'), written $(e,f) \overset{\mathscr{L}\mathscr{R}}{\leq} (e',f')$, if $e \overset{\mathscr{L}}{\leq} e'$ and $f \overset{\mathscr{R}}{\leq} f'$.

(2) (e,f) is *dominated by* (e',f'), written $(e,f) \leq (e',f')$, if $e \leq e'$ and $f \leq f'$.

(3) (e,f) and (e',f') are *left-right equivalent*, written $(e,f) \overset{\mathscr{L}\mathscr{R}}{\sim} (e',f')$, if $(e,f) \overset{\mathscr{L}\mathscr{R}}{\leq} (e',f')$ and $(e',f') \overset{\mathscr{L}\mathscr{R}}{\leq} (e,f)$.

Using Remark 2.2.2, we get the following.

Remark 2.2.5. (1) The relation $\overset{\mathscr{L}\mathscr{R}}{\leq}$ is a preorder, $\leq$ is a partial order and $\overset{\mathscr{L}\mathscr{R}}{\sim}$ is an equivalence. If A is Artinian, then the set of all idempotent pairs satisfies the descending chain condition with respect to $\leq$.

(2) $(e,f) \leq (e',f')$ if and only if $e' = e + e_1$ and $f' = f + f_1$ for some idempotents e_1 and f_1 in A with e and e_1 (resp. f and f_1) orthogonal.

Lemma 2.2.6. *Let A be a ring. Let (e,f) and (e',f') be idempotent pairs in A with $(e,f) \overset{\mathscr{L}\mathscr{R}}{\leq} (e',f')$. Then there is an idempotent pair (e'',f'') in A such that $(e'',f'') \leq (e',f')$ and $(e'',f'') \overset{\mathscr{L}\mathscr{R}}{\sim} (e,f)$.*

Proof. This follows from Lemma 2.2.3(iii).

□

Proposition 2.2.7. *Let A be a simple ring and let e, e', f and f' be non-zero idempotents in A. Then we have the following.*

(i) $eAf \neq 0$.

(ii) $eAf \subseteq e'Af'$ if and only if $(e,f) \overset{\mathscr{L}\mathscr{R}}{\leq} (e',f')$.

(iii) $eAf = e'Af'$ *if and only if* $(e,f) \overset{\mathscr{LR}}{\sim} (e',f')$.

Proof. (i) Note that AeA is a two-sided ideal of A containing e. Since A is simple, $AeA = A$. Similarly, $AfA = A$. If $eAf = 0$ then $A^2 = AeAAfA = AeAfA = 0$, which is a contradiction.

(ii) Suppose first that $eAf \subseteq e'Af'$. Then $e'eaf = eaf$ for all $a \in A$, so $(e'e - e)af = 0$ for all $a \in A$. Hence, $e'e - e$ belongs to the left annihilator H of Af in A. Note that H is a two-sided ideal of A. Since A is simple, we have $H = A$ or 0. As $f \notin H$ (because $f(ff) = f \neq 0$), $H = 0$, so $e'e - e = 0$, or $e'e = e$. Hence, $e \overset{\mathscr{L}}{\leq} e'$. Similarly, we obtain $f \overset{\mathscr{R}}{\leq} f'$. Therefore, $(e,f) \overset{\mathscr{LR}}{\leq} (e',f')$. Suppose now that $(e,f) \overset{\mathscr{LR}}{\leq} (e',f')$. Then $e'e = e$ and $ff' = f$, so $eAf = e'eAff' \subseteq e'Af'$, as required.

(iii) This follows from (ii).

$\square$

Definition 2.2.8. (1) Let A be a semisimple Artinian ring and let $\{S_i \mid i \in I\}$ be the set of its simple components. Let e and f be non-zero idempotents in A and let e_i (resp. f_i) be the projection of e (resp. f) to S_i for each $i \in I$. Then the pair (e,f) is said to be *strict* if for each $i \in I$, e_i and f_i are both either non-zero or zero.

(2) Let A be an Artinian ring or a finite dimensional algebra and let R be its radical. Let e and f be non-zero idempotents in A. We say that (e,f) is *strict* if $(\bar{e},\bar{f})$ is strict in $\bar{A} = A/R$.

The following lemma follows directly from the definition and Proposition 2.2.7(i).

Lemma 2.2.9. *Let A be a semisimple Artinian ring and let (e,f) be a non-zero strict idempotent pair in A. Then $eAf \neq 0$.*

Now, we are ready to prove Theorem 1.2.4.

Proof of Theorem 1.2.4. Recall that (e,f) and (e',f') are idempotent pairs in A with (e,f) being strict.

(i) By Definition 2.2.8 (2), $(\bar{e},\bar{f})$ is a strict idempotent pair in $\bar{A}$, so by Proposition 2.2.9, $\bar{e}\bar{A}\bar{f} \neq 0$. Therefore, $eAf \neq 0$, as required.

(ii) We need to show that $eAf \subseteq e'Af'$ if and only if $(e,f) \overset{\mathscr{LR}}{\leq} (e',f')$. If $(e,f) \overset{\mathscr{LR}}{\leq} (e',f')$, then $eAf = e'eAff' \subseteq e'Af'$, as required.

Suppose now that $eAf \subseteq e'Af'$. We need only to check that $e \overset{\mathscr{L}}{\leq} e'$ (the proof for $f \overset{\mathscr{R}}{\leq} f'$ is similar). Assume to the contrary that $e'e \neq e$. Then $r = e'e - e \neq 0$. Fix minimal $n \geq 1$ such that $r \notin R^n$. By taking quotient of A by R^n we can assume that $R^n = 0$ and $r \in M$ where $M = R^{n-1}$ if $n > 1$ and $M = A$ (with A being semisimple) if $n = 1$. Since $MR \subseteq R^n = 0$, the right A-module M is actually an $\bar{A}$-module. Note that $re = (e'e-e)e = e'e - e = r$, so $r\bar{e} = r \neq 0$. Let $\{S_i \mid i \in I\}$ be the set of the simple components of $\bar{A}$ and let $\bar{e}_i$ be the projection of $\bar{e}$ to S_i. Since $r\bar{e} \neq 0$, there is $i \in I$ such that $r\bar{e}_i \neq 0$, so $r\bar{e}_iS_i$ is a non-zero unital right S_i-submodule of M. Moreover, it is isomorphic to a direct sum of copies of the natural S_i-module. Since $\bar{e}_i \neq 0$ and (e,f) is strict, $\bar{f}_i \neq 0$, so $r\bar{e}S_i\bar{f} = r\bar{e}_iS_i\bar{f}_i \neq 0$. In particular, there is $a \in A$ such that $r\bar{e}\bar{a}\bar{f} \neq 0$. As $r = e'e - e$, we have that $(e'e-e)\bar{e}\bar{a}\bar{f} \neq 0$, or equivalently, $e'x \neq x$ where $x = e\bar{e}\bar{a}\bar{f} = eaf$. On the other hand, $x \in eAf \subseteq e'Af'$, so $e'x = x$, a contradiction. Therefore, $e \overset{\mathscr{L}}{\leq} e'$, as required.

(iii) This follows from (ii).

(iv) This follows from (iii) and Lemma 2.2.6.

□

2.3 Jordan-Lie inner ideals of semisimple algebras

Recall that A is a finite dimensional associative algebra over $\mathbb{F}$ (unless otherwise stated). If A is simple then A can be identified with $\operatorname{End} V$ for some finite dimensional vector space V over $\mathbb{F}$. By fixing a basis of V we can represent the algebra $\operatorname{End} V$ in the matrix form $\mathscr{M}_n$, where $n = \dim V$. We say that $\mathscr{M}_n$ is a *matrix realization* of A. Recall that every idempotent of $\mathscr{M}_n$ is diagonalizable (as its minimal polynomial is a divisor of $t^2 - t$). Since orthogonal idempotents commute, we get the following.

Lemma 2.3.1. *Let (e,f) be an orthogonal idempotent pair in A. Suppose A is simple. Then there is a matrix realization of A such that e and f can be represented by the diagonal matrices $e = diag(1,\dots,1,0,\dots,0)$ and $f = diag(0,\dots,0,1,\dots,1)$ with $\operatorname{rk}(e)+\operatorname{rk}(f) \leq n$.*

Benkart proved that if A is a simple Artinian ring of characteristic not 2 or 3, then every inner ideal of $[A,A]/(Z(A)\cap[A,A])$ is induced by idempotents [13, Theorem 5.1]. We will need a slight modification of this result.

Theorem 2.3.2. *Let A be a simple Artinian ring of characteristic not 2 or 3. Let B be Jordan-Lie inner ideal of $[A,A]$. Then there exists orthogonal idempotent pair (e,f) in A such that $B = eAf$.*

Proof. Let Z be the center of A and let $\hat{B}$ be the image of B in $\hat{A} = [A,A]/(Z\cap[A,A])$. Then $\hat{B}$ is an inner ideal of $\hat{A}$ and by [13, Theorem 5.1], there are idempotents e and f in A with $fe = 0$ such that $\hat{B}$ is the image of eAf in $\hat{A}$. We wish to show that $B = eAf$. Let $b \in B$. Then $b = eaf + z$ for some $a \in A$ and $z \in Z$. As $B^2 = 0$ (because B is Jordan-Lie),

$$0 = b^2 = (eaf+z)(eaf+z) = e(2az)f + z^2.$$

Hence, by Lemma 2.1.9(i), we obtain $z^2 = e(-2az)f \in eAf \cap Z(A) = 0$. Therefore, $z = 0$ and $B \subseteq eAf$. Conversely, let $a \in A$. Then there is $z \in Z$ such that $eaf + z \in B$. As above, we obtain $z = 0$. Therefore, $eaf \in B$, so $B = eAf$. Since $fe = 0$, by Lemma 2.1.9(iv), there is an idempotent g in A such that g and e are orthogonal and $B = eAf = eAg$.

□

Lemma 2.3.3. *Let B be a Jordan-Lie inner ideal of $L = [A,A]$. Suppose A is simple and $p \neq 2,3$. Then there is a matrix realization $\mathscr{M}_n$ of A and integers $1 \le k < l \le n$ such that $B = span\{e_{st} \mid 1 \le s \le k < l \le t \le n\}$, where e_{st} are matrix units.*

Proof. This follows from Theorem 2.3.2 and Lemma 2.3.1.

□

Recall that every simple Artinian ring A is Von Neumann regular, i.e. for every $x \in A$ there is $y \in A$ such that $x = xyx$ [19].

Lemma 2.3.4. *Let A be a simple Artinian ring of characteristic not 2 or 3 and let B be a Jordan-Lie inner ideal of $A^{(1)}$. Then $B = [B,[B,A^{(1)}]]$.*

Proof. We need only to show that $B \subseteq [B,[B,A^{(1)}]]$. Let $b \in B$. By Theorem 2.3.2, $B = eAf$ for some orthogonal idempotents e and f in A, so $b = eaf$ for some $a \in A$. Since A is Von Neumann regular, $b = bxb$ for some $x \in A$. Hence, $eaf = b = bxb = (eaf)x(eaf)$. Put $y = fxe = [f, fxe] \in A^{(1)}$. Then $b = byb$, so $[b,[b,y]] = -2byb = -2b$. This implies $b \in [B,[B,A^{(1)}]$, as required.

$\square$

Let L be a finite dimensional semisimple Lie algebra and let $\{L_i \mid i \in I\}$ be the set of the simple components of L. If B is an inner ideal of L and the ground field is of characteristic $p \neq 2,3,5,7$ then $B = \bigoplus_{i \in I} B_i$, where $B_i = B \cap L_i$ (see [29, Proposition 2.3]). As the following lemma shows we need less restrictions on p if $L = [A,A]$ and B is Jordan-Lie.

Lemma 2.3.5. *Suppose A is semisimple and $p \neq 2,3$. Let $\{S_i \mid i \in I\}$ be the set of the simple components of A and let B be a Jordan-Lie inner ideal of $L = [A,A]$. Then $B = \bigoplus_{i \in I} B_i$, where $B_i = B \cap S_i$ is a Jordan-Lie inner ideal of $L_i = [S_i, S_i]$.*

Proof. Let $\psi_i : L \to L_i$, $\psi_i((x_1, \ldots, x_i, \ldots) = x_i$, be the natural projection. We need to show that $\psi_i(B) = B_i$. By Lemma 2.1.10, $\psi_i(B)$ is a Jordan-Lie inner ideal of L_i. Clearly, $B_i \subseteq \psi_i(B)$. On the other hand, by Lemma 2.3.4

$$\psi_i(B) = [\psi_i(B), [\psi_i(B), L_i]] \subseteq [B,[B,L_i]] \subseteq B \cap L_i \subseteq B_i$$

for all $i \in I$. Therefore, $B = \bigoplus_{i \in I} B_i$. Since $B \subseteq [A,A]$ we have $B_i \subseteq [S_i, S_i]$, as required.

$\square$

The following proposition first appeared in [35, Lemma 6.6] in the case $p = 0$.

Lemma 2.3.6. *Suppose A is semisimple and $p \neq 2,3$. Let B be a Jordan-Lie inner ideal of $L = [A,A]$. Then there exists a strict orthogonal idempotent pair (e, f) in A such that $B = eAf$.*

Proof. Let $\{S_i \mid i \in I\}$ be the set of the simple components of A. Using Theorem 2.3.2 and Lemma 2.3.5 we get that $B = \bigoplus_{i \in I} e_i S_i f_i$ for some orthogonal idempotent pairs (e_i, f_i) in S_i. Moreover, we can assume that $e_i = f_i = 0$ if $B_i = B \cap S_i = 0$. Put $e = \sum_{i \in I} e_i$ and $f =$

$\sum_{i\in I} f_i$. Then (e,f) is a strict orthogonal idempotent pair in A and $eAf = \bigoplus_{i\in I} e_iS_if_i = B$, as required.

$\square$

Lemma 2.3.7. *Suppose A is semisimple and $p \neq 2,3$. Let B be a Jordan-Lie inner ideal of $A^{(-)}$. Then B is a Jordan-Lie inner ideal of $[A,A]$.*

Proof. Let $b \in B$. Since A is Von Neumann regular, there is $x \in A$ such that $b = bxb$. As $b^2 = 0$,

$$b = bxb = b(xb) - (xb)b = [b,xb] \in [A,A].$$

Therefore, $B \subseteq [A,A]$, so B is a Jordan-Lie inner ideal of $[A,A]$.

$\square$

Lemmas 2.3.6 and 2.3.7 imply that all Jordan-Lie inner ideals of $A^{(-)}$ are generated by idempotents, which is essentially known, see for example [24, Theorem 6.1(2)]. We summarize description of Jordan-Lie inner ideals of $A^{(k)}$ in the following proposition.

Proposition 2.3.8. *Suppose A is semisimple, $p \neq 2,3$ and $k \geq 0$. Let B be a subspace of A. Then B is a Jordan-Lie inner ideal of $A^{(k)}$ if and only if $B = eAf$ where (e,f) is a strict orthogonal idempotent pair in A.*

Proof. By Proposition 2.1.3, $A^{(k)} = A^{(1)}$ if $k \geq 1$. The "only if" part now follows from Lemmas 2.3.6 ($k \geq 1$) and 2.3.7 ($k = 0$), and the "if" part follows from Lemma 2.1.9(iii).

$\square$

2.4 *L*-Perfect inner ideals

1-perfect associative algebras and their associated Lie algebras

Definition 2.4.1. The associative algebra A is said to be *Lie solvable* if the Lie algebra $A^{(-)}$ is solvable.

The following is well known.

Lemma 2.4.2. *Let $p \neq 2$. Then the following are equivalent.*

(i) A is Lie solvable.

(ii) There is a descending chain of ideals $A = A_0 \supset A_1 \supset \cdots \supset A_r = \{0\}$ of A such that $\dim A_i/A_{i+1} = 1$ *for* $0 \le i \le r-1$.

(iii) There is a descending chain of subalgebras $A = A_0 \supset A_1 \supset \cdots \supset A_r = \{0\}$ of A such that A_{i+1} is an ideal of A_i and $\dim A_i/A_{i+1} = 1$ *for* $0 \le i \le r-1$.

Proof. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are obvious (as $A^{(i)} \subseteq A_i$ for all i). To prove $(i) \Rightarrow (ii)$, suppose that A is Lie solvable. Let R be the radical of A and let $S = A/R$. Then S is a Lie solvable semisimple algebra, so by Lemma 2.1.2 and Proposition 2.1.3, $S \cong \mathbb{F}^m$ direct sum of m copies of $\mathbb{F}$ for some m. If $S = 0$, then $A = R$ is nilpotent, so such a chain exists. Suppose that $S \neq 0$. Since all simple components of S are 1-dimensional, all composition factors of the S-bimodule R/R^2 are one-dimensional, so there is a chain of ideals in A/R^2 with 1-dimensional quotients. The lemma now follows by induction on the degree of nilpotency of R.

□

Definition 2.4.3. An associative algebra is said to be 1-*perfect* if it has no ideals of codimension 1.

We note the following obvious properties of 1-perfect ideals.

Lemma 2.4.4. *(i) The sum of* 1*-perfect ideals is* 1*-perfect.*

(ii) If P is a 1*-perfect ideal of A and Q is a* 1*-perfect ideal of A/P then the full preimage of Q in A is a* 1*-perfect ideal of A.*

Lemma 2.4.4(i) implies that every algebra has the largest 1-perfect ideal.

Definition 2.4.5. The largest 1-perfect ideal $\mathscr{P}_1(A)$ of A is called the 1-*perfect radical* of A.

The following proposition shows that $\mathscr{P}_1(A)$ has radical-like properties indeed.

Proposition 2.4.6. *(i)* $\mathscr{P}_1(A)^2 = \mathscr{P}_1(A)$*;*

(ii) $\mathscr{P}_1(\mathscr{P}_1(A)) = \mathscr{P}_1(A)$*;*

(iii) $\mathscr{P}_1(A/\mathscr{P}_1(A)) = 0$*;*

(iv) If $p \neq 2$ *then* $\mathscr{P}_1(A)$ *is the smallest ideal of* A *such that* $A/\mathscr{P}_1(A)$ *is Lie solvable.*

Proof. (i) and (ii) are obvious; (iii) follows from Lemma 2.4.4(ii).

(iv) Let N be an ideal of A such that A/N is Lie solvable. Then it follows from Lemma 2.4.2 that $N \supseteq \mathscr{P}_1(A)$. It remains to prove that $A/\mathscr{P}_1(A)$ is Lie solvable. By Lemma 2.4.2, it is enough to construct a chain of subalgebras $\mathscr{P}_1(A) = A_r \subset A_{r-1} \subset \ldots \subset A_0 = A$ of A such that A_{i+1} is an ideal of A_i of codimension 1 for $0 \leq i \leq r-1$. Put $A_0 = A$ and suppose $A_k \subset \ldots \subset A_0 = A$ has been constructed. If A_k is not 1-perfect then we denote by A_{k+1} any ideal of A_k of codimension 1. If A_k is 1-perfect then by part (i), $A_k^s = A_k$ for all s so A_k is actually an ideal of A: $AA_k = AA_kA_k \ldots A_k \subseteq A_0A_1A_2 \ldots A_k \subseteq A_k$ (and similarly $A_kA \subseteq A_k$). This implies that $A_k = \mathscr{P}_1(A)$, as required.

□

Importance of 1-perfect algebras is shown by the following result from [4].

Theorem 2.4.7. *[4] If* A *is* 1*-perfect and* $p \neq 2$*, then* $[A,A]$ *is a perfect Lie algebra.*

Combining this result with Proposition 2.4.6(iv) we get the following.

Lemma 2.4.8. *Let* $p \neq 2$*. Then* $A^{(\infty)} = \mathscr{P}_1(A)^{(1)}$.

Proof. Since $A/\mathscr{P}_1(A)$ is Lie solvable, there is $n \geq 0$ such that $(A/\mathscr{P}_1(A))^{(n)} = 0$, so $A^{(n+1)} \subseteq \mathscr{P}_1(A)^{(1)}$. As $\mathscr{P}_1(A)$ is 1-perfect, by Theorem 2.4.7, $\mathscr{P}_1(A)^{(1)}$ is perfect, so $A^{(\infty)} = A^{(n+1)} = \mathscr{P}_1(A)^{(1)}$.

□

L-perfect inner ideals

Definition 2.4.9. Let L be a Lie algebra and let B be an inner ideal of L. We say that B is *L-perfect* if $B = [B,[B,L]]$.

It is known that every inner ideal of a semisimple Lie algebra L is L-perfect if $p \neq 2,3,5,7$, see for example [29, Proposition 2.3] (or [6, Lemmas 2.19 and 2.20] for characteristic zero). As the following lemma shows we need less restrictions on p if $L = [A,A]$ and B is Jordan-Lie.

Lemma 2.4.10. *Suppose A is semisimple, $k \geq 0$ and $p \neq 2,3$. Then every Jordan-Lie inner ideal of $L = A^{(k)}$ is L-perfect.*

Proof. Suppose first that $k \geq 1$. Then $A^{(k)} = A^{(1)}$ by Proposition 2.1.3. Therefore, this follows from Lemma 2.3.5 and Lemma 2.3.4.

Suppose now that $k = 0$. Let B be a Jordan-Lie inner ideal of $A^{(-)}$. Then by Lemma 2.3.7, B be is Jordan-Lie inner ideal of $A^{(1)}$, so B is $A^{(1)}$-perfect by above. This obviously implies that B is $A^{(-)}$-perfect.

□

Lemma 2.4.11. *Let L be a Lie algebra and let B be an inner ideal of L. If B is L-perfect, then B is an inner ideal of $L^{(k)}$ for all $k \geq 0$.*

Proof. Suppose $B \subseteq L^{(k)}$ for some $k \geq 0$. Then

$$B = [B,[B,L]] \subseteq [L^{(k)},[L^{(k)},L]] \subseteq [L^{(k)},L^{(k)}] = L^{(k+1)},$$

so the result follows by induction on k.

□

Lemma 2.4.12. *Let B be an L-perfect Jordan-Lie inner ideal of $L = A^{(k)}$ ($k \geq 0$). If $p \neq 2$ then $B \subseteq \mathscr{P}_1(A)$ and B is a Jordan-Lie inner ideal of $\mathscr{P}_1(A)^{(1)}$.*

Proof. Since B is L-perfect, by Lemma 2.4.11, $B \subseteq L^{(\infty)} = A^{(\infty)}$, so B is a Jordan-Lie inner ideal of $A^{(\infty)}$. It remains to note that $A^{(\infty)} = \mathscr{P}_1(A)^{(1)}$ by Lemma 2.4.8. □

The core of inner ideals

Let B be an inner ideal of L. Then $[B,[B,L]] \subseteq B$. It is well known that $[B,[B,L]]$ is an inner ideal of L (see for example [14, Lemma 1.1]). Put $B_0 = B$ and consider the following inner ideals of L:

$$B_n = [B_{n-1},[B_{n-1},L]] \subseteq B_{n-1} \quad \text{for} \quad n \geq 1. \tag{2.4.1}$$

Then $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$. As L is finite dimensional, this series terminates. This motivates the following definition.

Definition 2.4.13. Let L be a finite dimensional Lie algebra and let B be an inner ideal of L. Then there is an integer n such that $B_n = B_{n+1}$. We say that B_n is *the core of* B, denoted by $\mathrm{core}_L(B)$.

Lemma 2.4.14. *Let L be a finite dimensional Lie algebra and let B be an inner ideal of L. Then*

(i) $\mathrm{core}_L(B)$ *is L-perfect;*

(ii) B is L-perfect if and only if $B = \mathrm{core}_L(B)$;

(iii) $\mathrm{core}_L(B)$ *is an inner ideal of $L^{(k)}$ for all $k \geq 0$.*

Proof. (i) and (ii) follow from Definitions 2.4.9 and 2.4.13.

(iii) follows from (i) and Lemma 2.4.11. □

Remark 2.4.15. Let $k \geq 0$. If S is a Levi subalgebra of A, then $A = S \oplus R$, so $A^{(k)} = S^{(k)} \oplus N$, where $N = R \cap A^{(k)}$. Moreover, $\bar{A}^{(k)} = A^{(k)}/N = A^{(k)}/R \cap A^{(k)}$ is the image of $A^{(k)}$ in $\bar{A} = A/R$.

Lemma 2.4.16. *Let B be a Jordan-Lie inner ideal of $L = A^{(k)}$ ($k \geq 0$). If $p \neq 2,3$, then*

(i) $\bar{B} = \overline{\mathrm{core}_L(B)}$.

(ii) If $\mathrm{core}_L(B) = 0$*, then* $B \subset N$*.*

Proof. (i) Since $\bar{A}$ is semisimple and $\bar{B}$ is a Jordan-Lie inner ideal of $\bar{L} = \bar{A}^{(k)}$, by Lemma 2.4.10, $\bar{B}$ is $\bar{L}$-perfect. Hence, by Lemma 2.4.14, $\bar{B} = \mathrm{core}_{\bar{L}}(\bar{B}) = \overline{\mathrm{core}_L(B)}$.

(ii) This follows from (i).

□

2.5 Bar-minimal and regular inner ideals

Recall that $L = A^{(k)}$ for some $k \geq 0$, $N = R \cap L$, and $\bar{B}$ is the image of a subspace B of L in $\bar{L} = L + R/R \cong L/N$.

Bar-minimal inner ideals

Definition 2.5.1. Let $L = A^{(k)}$ and let X be an inner ideal of $\bar{L}$. Suppose that B is an inner ideal of L. We say that B is X*-minimal* (or simply, *bar-minimal*) if for every inner ideal B' of L with $\bar{B}' = X$ and $B' \subseteq B$ one has $B' = B$.

Lemma 2.5.2. *Let* $k \geq 0$ *and let* B *be a Jordan-Lie inner ideal of* $L = A^{(k)}$*. Suppose that* B *is bar-minimal and* $p \neq 2, 3$*. Then the following hold.*

(i) $B = \mathrm{core}_L B$*.*

(ii) B *is* L*-perfect.*

(iii) B *is a Jordan-Lie inner ideal of* $L^{(m)} = A^{(k+m)}$ *for all* $m \geq 0$*.*

Proof. (i) By definition of the core, $\mathrm{core}_L(B)$ is an inner ideal of L contained in B. By Lemma 2.4.16(i), $\overline{\mathrm{core}_L(B)} = \bar{B}$. Since B is bar-minimal, we have $B = \mathrm{core}_L B$.

(ii) This follows directly from (i) and Lemma 2.4.14(i).

(iii) This follows from (ii) and Lemma 2.4.11.

□

Recall that a Lie algebra L is said to be perfect if $L = [L, L]$. We will need the following result.

Lemma 2.5.3. *Let L be a perfect Lie algebra and let B be an L-perfect inner ideal of L. Suppose that $L = \bigoplus_{i\in I} L_i$, where each L_i is an ideal of L. Then $B = \bigoplus_{i\in I} B_i$, where $B_i = B \cap L_i$. Moreover, if $L = A^{(k)}$ ($k \geq 0$) and B is bar-minimal then B_i is a $\bar{B}_i$-minimal inner ideal of L_i, for all $i \in I$.*

Proof. Note that $[B,[B,L_i]] \subseteq B \cap L_i = B_i$, for all $i \in I$. Therefore,

$$B = [B,[B,L]] = \sum_{i\in I}[B,[B,L_i]] \subseteq \sum_{i\in I} B_i \subseteq B,$$

so $B = \sum_{i\in I} B_i$. As $B_i \cap B_j \subseteq L_i \cap L_j = 0$ for all $i \neq j$, $B = \bigoplus_{i\in I} B_i$. Clearly, if B is bar-minimal, then each B_i is $\bar{B}_i$-minimal.

□

Split inner ideals

Let L be a Lie algebra and let Q be a subalgebra of L. Recall that Q is said to be a quasi Levi subalgebra of L if Q is quasi semisimple and there is a solvable ideal P of L such that $L = Q \oplus P$.

Definition 2.5.4. Let L be a finite dimensional Lie algebra and let B be a subspace of L. Suppose that there is a quasi Levi decomposition $L = Q \oplus N$ of L such that $B = B_Q \oplus B_N$, where $B_Q = B \cap Q$ and $B_N = B \cap N$. Then we say that B *splits in* L and Q is a *B-splitting quasi Levi subalgebra* of L.

Definition 2.5.5. Let B be a subspace of A. Suppose that there is a Levi subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$. Then we say that B *splits* in A and S is a *B-splitting* Levi subalgebra of A.

Lemma 2.5.6. *Let $L = A^{(k)}$ ($k \geq 1$) and let B be a subspace of L. Suppose $p \neq 2$. If B splits in A, then B splits in L.*

Proof. Suppose that B splits in A. Then there is a B-splitting Levi subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$. Clearly, $Q = [S,S] = S^{(k)}$ is a quasi semisimple subalgebra of L, $N = L \cap R$ is a solvable ideal of L, and $L = Q \oplus N$ is a quasi Levi decomposition of L. It is easy to see that $B_S \subseteq Q$ and $B_R \subseteq N$, so B splits in L.

$\square$

Lemma 2.5.7. *Let B be an inner ideal of $L = A^{(k)}$ $(k \geq 0)$. Suppose $B = eAf$ for some orthogonal idempotents e and f of A. Then (i) B splits in A and (ii) if $k \geq 1$ then B splits in L.*

Proof. (i) Since e and f are orthogonal, By Wedderburn-Malcev theorem there is a Levi subalgebra S of A such that $e, f \in S$. Thus, $B = eAf = e(S \oplus R)f = eSf \oplus eRf$ as required.

(ii) This follows directly from (i) and Lemma 2.5.6.

$\square$

Proposition 2.5.8. *Let $C \subseteq B$ be subspaces of A such that $\bar{C} = \bar{B}$. If C splits in A, then B splits in A.*

Proof. Suppose C splits in A. Then there exists a Levi subalgebra S of A such that $C = C_S \oplus C_R$, where $C_S = C \cap S$ and $C_R = C \cap R$. Put $B_S = B \cap S$ and $B_R = B \cap R$. Then $C_S \subseteq B_S$, $C_R \subseteq B_R$ and $B_S + B_R \subseteq B$. Since $\bar{B} = \bar{C}$, we have

$$B_S \cong \bar{B}_S \subseteq \bar{B} = \bar{C} \cong C/C_R \cong C_S \subseteq B_S,$$

so $B_S \cong \bar{B} \cong B/B_R$. Since $B_S \cap B_R = 0$, we have $B = B_S \oplus B_R$ as required.

$\square$

Corollary 2.5.9. *Let $L = A^{(k)}$ $(k \geq 0)$ and let B be an inner ideal of L. Suppose that $p \neq 2,3$. If $\mathrm{core}_L(B)$ splits in A, then B splits in A.*

Proof. By Lemma 2.4.14, $\overline{\mathrm{core}_L(B)} = \bar{B}$. Since $\mathrm{core}_L(B) \subseteq B$ and $\mathrm{core}_L(B)$ splits, by Proposition 2.5.8, B splits.

$\square$

Definition 2.5.10. Let G be a subalgebra of A. We say that G is *large in A* if $\bar{G} = \bar{A}$ (equivalently, there is a Levi subalgebra S of A such that $S \subseteq G$; or equivalently, $G/\operatorname{rad} G$ is isomorphic to A/R).

Remark 2.5.11. Let G be a large subalgebra of A and let B be a subspace of $\mathscr{P}_1(G)$. Then $\mathrm{rad}(G) = G \cap R$ and $\mathrm{rad}(\mathscr{P}_1(G)) = \mathscr{P}_1(G) \cap \mathrm{rad}(G) = \mathscr{P}_1(G) \cap R$, so the image $\bar{B}$ of B in A/R is isomorphic to the images of B in $G/\mathrm{rad}(G)$ and $\mathscr{P}_1(G)/\mathrm{rad}(\mathscr{P}_1(G))$, respectively. Thus, we can use the same notation $\bar{B}$ for the images of B in all these quotient spaces.

Proposition 2.5.12. *Let B be a subspace of A. Let G be a large subalgebra of A and let C be a subspace of $\mathscr{P}_1(G)$. Suppose that $C \subseteq B$, $\bar{C} = \bar{B}$, and C splits in $\mathscr{P}_1(G)$. Then B splits in A.*

Proof. Put $R_1 = \mathrm{rad}\,\mathscr{P}_1(G)$. By Remark 2.5.11, $R_1 \subseteq \mathrm{rad}(G) \subseteq R$. Let S_1 be a C-splitting Levi subalgebra of $\mathscr{P}_1(G)$, so $C = C_{S_1} \oplus C_{R_1}$, where $C_{S_1} = C \cap S_1$ and $C_{R_1} = C \cap R_1$. Note that S_1 is a semisimple subalgebra of A, so by Wedderburn-Malcev Theorem there is a Levi subalgebra S of A such that $S_1 \subseteq S$. Since $S_1 \subseteq S$ and $R_1 \subseteq R$, C splits in A, so the result follows from Proposition 2.5.8.

□

Since A is large in A, we get the following corollary.

Corollary 2.5.13. *Let B be a subspace of A and let C be a subspace of $\mathscr{P}_1(A)$. Suppose that $C \subseteq B$, $\bar{C} = \bar{B}$, and C splits in $\mathscr{P}_1(A)$. Then B splits in A.*

Proposition 2.5.14. *Let B be a Jordan-Lie inner ideal of $L = A^{(k)}$ ($k \geq 0$). Let G be a large subalgebra of A and let $B' = B \cap G^{(k)}$. Suppose $p \neq 2,3$ and $\bar{B}' = \bar{B}$. Put $C = \mathrm{core}_{G^{(k)}}(B')$. Then C is a Jordan-Lie inner ideal of $\mathscr{P}_1(G)^{(1)}$ such that $C \subseteq B$ and $\bar{C} = \bar{B}$.*

Proof. Note that $B' = B \cap G^{(k)}$ is a Jordan-Lie inner ideal of $G^{(k)}$. By Lemmas 2.4.14(i) and 2.4.16(i), $C = \mathrm{core}_{G^{(k)}}(B')$ is a $G^{(k)}$-perfect Jordan-Lie inner ideal of $G^{(k)}$ with $C \subseteq B' \subseteq B$ and $\bar{C} = \bar{B}' = \bar{B}$. It remains to note that by Lemma 2.4.12, C is Jordan-Lie inner ideal of $\mathscr{P}_1(G)^{(1)}$.

□

Regular inner ideals

In this section we describe bar-minimal regular inner ideals of $A^{(k)}$ $(k \geq 0)$. We start with the following result which is a slight generalization of [6, Lemma 4.1].

Lemma 2.5.15. *Let $L = A^{(k)}$ for some $k \geq 0$ and let B be a subspace of L such that $B^2 = 0$. Then the following hold.*

(i) If $p \neq 2$ then B is a Jordan-Lie inner ideal of L if and only if $bLb \subseteq B$ for all $b \in B$.

(ii) $BAB \subseteq L \cap A^{(1)}$.

(iii) If $BAB \subseteq B$, then B is a Jordan-Lie inner ideal of L.

Proof. (i) This follows from Lemma 3.1.3 as

$$\{b,x,b'\} = bxb' + b'xb = (b+b')x(b+b') - bxb - b'xb'.$$

(ii) $bxb' = [b, xb'] \in [A^{(k)}, A] \subseteq A^{(k)} \cap A^{(1)} = L \cap A^{(1)}$ for all $b, b' \in B$ and $x \in L$.

(iii) This is obvious as $[B,[B,L]] \subseteq BAB$.

□

Definition 2.5.16. Let B be a subspace of $L = A^{(k)}$ $(k \geq 0)$. Then B is said to be a *regular* inner ideal of L (with respect to A) if $B^2 = 0$ and $BAB \subseteq B$.

Regular inner ideals were first defined in [6] (in characteristic zero) and were recently used in [5] to classify maximal zero product subsets of simple rings. Note that every regular inner ideal is Jordan-Lie (see Lemma 2.5.15). However, the converse is not true as the following example shows.

Example 2.5.17. Let $\mathfrak{n}_4(\mathbb{F}) \subset M_4(\mathbb{F})$ be the set of all strictly upper triangular 4×4 matrices. Let A be the direct sum of two nilpotent ideals T_4 and T_4' with both of them isomorphic to $\mathfrak{n}_4(\mathbb{F})$. Clearly, $A^4 = 0$. Let $\{e_{ij} \mid 1 \leq i < j \leq 4\}$ and $\{e'_{ij} \mid 1 \leq i < j \leq 4\}$ be the standard bases of T_4 and T_4', respectively, consisting of matrix units. Consider the following elements of A:

$$b_1 = e_{12} + e'_{34}, \quad b_2 = e_{34} + e'_{12}, \quad a = e_{23} + e'_{23}, \quad b = e_{14} + e'_{14}.$$

Let $A_1 = A^2 + span\{b_1, b_2, a\}$. Then A_1 is a subalgebra of A as $A_1^2 \subseteq A^2 \subset A_1$. Consider the subspace $B = span\{b_1, b_2, b\}$ of A_1. It is easy to check that $B^2 = 0$ and B is a Jordan-Lie inner ideal of A_1. Moreover, B is not regular as $b_1 a b_2 = e_{14} \notin B$.

Note that B is also a non-regular Jordan-Lie inner ideal of the unital algebra $\hat{A}_1 = A_1 + \mathbb{F}1_{\hat{A}}$, by Lemma 2.1.8.

Lemma 2.5.18. *Let A be any ring and let e and f be idempotents in A with $fe = 0$. Then the following hold.*

(i) If $eAf \subseteq A^{(k)}$ ($k \geq 0$), then eAf is a regular inner ideal of $A^{(k)}$.

(ii) eAf is a regular inner ideal of $A^{(-)}$ and $A^{(1)}$.

Proof. (i) By Lemma 2.1.9(ii), eAf is a Jordan-Lie inner ideal of $A^{(k)}$ ($k \geq 0$). It remains to note that $(eAf)A(eAf) \subseteq eAf$.

(ii) This follows from (i) and Lemma 2.1.9(iii).

□

The following result is proved in [6, Proposition 4.12] in the case $p = 0$.

Proposition 2.5.19. *Suppose A is semisimple, $p \neq 2, 3$ and $k \geq 0$. Then every Jordan-Lie inner ideal of $A^{(k)}$ is regular.*

Proof. This follows from Proposition 2.3.8 and Lemma 2.5.18(i).

□

We will need the following two results which were first proved in [6] in the case when $p = 0$. One can easily check that their proofs in [6] apply to any p.

Proposition 2.5.20. *[6, Proposition 4.8] Let A be an associative ring. Then*

(i) A is Von Neumann regular if and only if $\mathscr{R}\mathscr{L} = \mathscr{R} \cap \mathscr{L}$ for all left $\mathscr{L}$ and right $\mathscr{R}$ ideals in A.

(ii) every x in A with $x^2 = 0$ is Von Neumann regular if and only if $\mathscr{R}\mathscr{L} = \mathscr{R} \cap \mathscr{L}$ for all left $\mathscr{L}$ and right $\mathscr{R}$ ideals in A with $\mathscr{L}\mathscr{R} = 0$.

Proposition 2.5.21. *[6, Proposition 4.9] Let B be a subspace of $L = A^{(k)}$ $(k \geq 0)$. Then B is a regular inner ideal of L if and only if there exist left $\mathscr{L}$ and right $\mathscr{R}$ ideals of A such that $\mathscr{L}\mathscr{R} = 0$ and*

$$\mathscr{R}\mathscr{L} \subseteq B \subseteq \mathscr{R} \cap \mathscr{L}.$$

In particular, if A is Von Neumann regular then every regular inner ideal of L is of the form $B = \mathscr{R}\mathscr{L} = \mathscr{R} \cap \mathscr{L}$.

Let $\mathscr{L}$ be a left ideal of A and let X be a left ideal of $\bar{A}$. Then $\mathscr{L}$ is said to be X-minimal if $\bar{\mathscr{L}} = X$ and for every left ideal $\mathscr{L}'$ of A with $\mathscr{L}' \subseteq \mathscr{L}$ and $\bar{\mathscr{L}}' = X$ one has $\mathscr{L} = \mathscr{L}'$. We will need the following theorem from [11].

Theorem 2.5.22. *[11] Let A be a left Artinian associative ring and let $\mathscr{L}$ be a left ideal of A. If $\mathscr{L}$ is $\bar{\mathscr{L}}$-minimal, then $\mathscr{L} = Ae$ for some idempotent $e \in \mathscr{L}$.*

Theorem 2.5.23. *Let B be a bar-minimal Jordan-Lie inner ideal of $L = A^{(k)}$ $(k \geq 0)$. Then B is regular if and only if $B = eAf$ for some orthogonal idempotent pair (e, f) in A.*

Proof. Suppose first that $B = eAf$ for some orthogonal idempotent pair (e, f) in A. Then by Lemma 2.5.18, B is regular.

Suppose now that B is regular. Then by Proposition 2.5.21, there are left $\mathscr{L}$ and right $\mathscr{R}$ ideals of A such that $\mathscr{L}\mathscr{R} = 0$ and $\mathscr{R}\mathscr{L} \subseteq B \subseteq \mathscr{R} \cap \mathscr{L}$. Hence, $\overline{\mathscr{R}\mathscr{L}} = \bar{\mathscr{R}}\bar{\mathscr{L}} \subseteq \bar{B} \subseteq \bar{\mathscr{L}} \cap \bar{\mathscr{R}}$. Since $\bar{A}$ is Von Neumann regular (because it is semisimple), by Proposition 2.5.20, $\bar{\mathscr{R}}\bar{\mathscr{L}} = \bar{B}$. Let $\mathscr{L}' \subseteq \mathscr{L}$ (resp. $\mathscr{R}' \subseteq \mathscr{R}$) be an $\bar{\mathscr{L}}$-minimal left (resp. $\bar{\mathscr{R}}$-minimal right) ideal of A. Then by Theorem 2.5.22, $\mathscr{L}' = Af$ and $\mathscr{R}' = eA$ for some idempotents $e \in \mathscr{R}'$ and $f \in \mathscr{L}'$. Note that $fe \in \mathscr{L}'\mathscr{R}' \subseteq \mathscr{L}\mathscr{R} = 0$. Put $B' = \mathscr{R}'\mathscr{L}' \subseteq B$. Then $B' = eAAf = eAf$ (as $eAf = eeAf \subseteq eAAf \subseteq eAf$). Since $B'^2 = 0$, by Proposition 2.5.21, B' is a regular inner ideal of L. As $\overline{B'} = \overline{\mathscr{R}'}\,\overline{\mathscr{L}'} = \bar{\mathscr{R}}\bar{\mathscr{L}} = \bar{B}$ and B is bar-minimal, $B = B'$. Thus, $B = eAf$ for some idempotents e and f in A with $fe = 0$. Therefore, by Lemma 2.1.9(iv), $B = eAf = eAg$ for some idempotent g in A with $ge = eg = 0$.

□

2.6 Proof of the main results

The aim of this section is to prove that bar-minimal Jordan-Lie inner ideals are generated by idempotents (Theorem 1.2.1) and are regular (Corollary 1.2.2). As a corollary, we show that all Jordan-Lie inner ideals split (Corollary 1.2.3). Recall that S is a Levi subalgebra of A, $L = A^{(k)} = S^{(k)} \oplus N$, for some $k \geq 0$, $N = R \cap L$, and $\bar{B}$ is the image of B in $\bar{L} = L + R/R \cong L/N$.

First we consider the case when A is 1-perfect. Then $L = [A,A]$ is a perfect Lie algebra for $p \neq 2$ (see Proposition 2.4.7). The following theorem will be proved in steps.

Theorem 2.6.1. *Let $L = [A,A]$ and let B be a Jordan-Lie inner ideal of L. Suppose that $p \neq 2,3$, A is 1-perfect and B is bar-minimal. Then the following hold.*

(i) B splits in A.

(ii) $B = eAf$ for some strict orthogonal idempotent pair (e, f) in A.

(iii) B is regular.

First we will consider the case when $R^2 = 0$.

Theorem 2.6.2. *Let $L = [A,A]$ and let B be a Jordan-Lie inner ideal of L. Suppose that $p \neq 2,3$, A is 1-perfect, B is bar-minimal and $R^2 = 0$. Then B splits in A.*

Theorem 2.6.2 first appeared in Rowley's thesis [35] in the case when $p = 0$ and we use some of his ideas below. Unfortunately, his proof is incomplete and contains some inaccuracies. In particular, the proof of [35, Proposition 6.12] is incorrect. We will need the following lemma.

Lemma 2.6.3. *Let $L = [A,A]$ and $Q = [S,S]$. Suppose that $p \neq 2$, A/R is simple, $RA = 0$ and R is an irreducible left A-module. Then the following hold.*

(i) $N = R$.

(ii) Every Jordan-Lie inner ideal of Q is a Jordan-Lie inner ideal of L.

(iii) Let G be a large subalgebra of A and let B be a Jordan-Lie inner ideal of $[G,G]$. Then B is a Jordan-Lie inner ideal of L.

Proof. (i) Let $r \in R$. Since R is irreducible as S-module, $r = sr$ for some $s \in S$. As $RA = 0$, $r = sr = [s,r] \in [S,R] = N$ by Proposition 2.1.5, so $R = N$.

(ii) This follows from (iii) as $Q = [S,S]$ and S is a large subalgebra of A.

(iii) Since G is a large subalgebra of A, it contains a Levi subalgebra of A. Without loss of generality we can assume $S \subseteq G$. Let $x \in L$. Since $L = [A,A] \subseteq Q \oplus R$, $x = q + r$ for some $q \in Q$ and $r \in R$. As $RA = 0$, for all $b, b' \in B$ we have

$$\{b,x,b'\} = bxb' + b'xb = b(q+r)b' + b'(q+r)b = bqb' + b'qb = \{b,q,b'\} \in B,$$

i.e. B is an inner ideal of L, as required.

$\square$

Recall that A is a 1-perfect finite dimensional associative algebra, R is the radical of A with $R^2 = 0$ and S is a Levi subalgebra of A, so by Proposition 2.1.5, $L = [A,A]$ is a perfect Lie algebra, $Q = [S,S]$ is a quasi Levi subalgebra of L and $L = Q \oplus N$ is a quasi Levi decomposition of L, where $N = [S,R]$.

Proposition 2.6.4. *Theorem 2.6.2 holds if A/R is simple, $RA = 0$ and R is an irreducible left A-module. Moreover, $B \subseteq S'$ for some Levi subalgebra S' of A.*

Proof. By Lemma 2.6.3, R coincides with the nil-radical N of L. We identify $\bar{A}$ with S. Recall that B is bar-minimal. We are going to prove that there is a Levi subalgebra S' of A such that $B \subseteq S'$, so B splits in A. Since $S \cong A/R$ is simple, by Lemma 2.3.3, there is a matrix realization $\mathscr{M}_n$ of S and integers $1 \le k < l \le n$ such that $\bar{B}$ is the space spanned by $E = \{e_{st} \mid 1 \le s \le k < l \le t \le n\}$ where $\{e_{ij} \mid 1 \le i,j \le n\}$ is the standard basis of S consisting of matrix units. Since R is an irreducible left S-module, it can be identified with the natural n-dimensional left S-module V. Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of V. Fix $b_{st}^{(1)} \in B$ such that $\overline{b_{st}^{(1)}} = e_{st}$ for all s and t. Then $b_{st}^{(1)} = e_{st} + r_{st}$, where $r_{st} \in R$. Put

$$\Lambda_1 = \{b_{st}^{(1)} = e_{st} + r_{st} : 1 \le s \le k < l \le t \le n\} \subseteq B.$$

Since $e_{ts} \in L$, by Lemma 3.1.3, $b_{st}^{(2)} = b_{st}^{(1)} e_{ts} b_{st}^{(1)} \in B$. Let $r_{st} = \sum_{i=1}^n \alpha_i^{st} e_i$, where $\alpha_i^{st} \in \mathbb{F}$. Then

$$b_{st}^{(2)} = b_{st}^{(1)} e_{ts} b_{st}^{(1)} = (e_{st} + \sum_{i=1}^n \alpha_i^{st} e_i) e_{ts} (e_{st} + \sum_{i=1}^n \alpha_i^{st} e_i) = e_{ss}(e_{st} + \sum_{i=1}^n \alpha_i^{st} e_i) = e_{st} + \alpha_s^{st} e_s.$$

Hence, the set

$$\Lambda_2 = \{b_{st}^{(2)} = e_{st} + \alpha_s^{st} e_s : 1 \le s \le k < l \le t \le n\} \subseteq B.$$

Put $b_{1t}^{(3)} = b_{1t}^{(2)} = e_{1t} + \alpha_1^{1t} e_1$ and for $s > 1$ set $b_{st}^{(3)} = \{b_{st}^{(2)}, e_{t1}, b_{1t}^{(2)}\}$. Then by Lemma 3.1.3, $b_{st}^{(3)} \in B$. Since $RA = 0$, for $s > 1$ we have

$$\begin{aligned} b_{st}^{(3)} &= \{b_{st}^{(2)}, e_{t1}, b_{1t}^{(2)}\} = b_{st}^{(2)} e_{t1} b_{1t}^{(2)} + b_{1t}^{(2)} e_{t1} b_{st}^{(2)} \\ &= (e_{st} + \alpha_s^{st} e_s) e_{t1} b_{1t}^{(2)} + (e_{1t} + \alpha_1^{1t} e_t) e_{t1} b_{st}^{(2)} \\ &= e_{s1}(e_{1t} + \alpha_1^{1t} e_1) + e_{11}(e_{st} + \alpha_s^{st} e_s) \\ &= e_{st} + \alpha_1^{1t} e_s. \end{aligned}$$

Denote $\beta_t = \alpha_1^{1t}$ for all t. Then $b_{st}^{(3)} = e_{st} + \beta_t e_s \in B$ for all s and t. Thus

$$\Lambda_3 = \{b_{st}^{(3)} = e_{st} + \beta_t e_s : 1 \le s \le k < l \le t \le n\} \subseteq B.$$

Let $q = \sum_{j=l}^{n} \beta_j e_j \in R$. Then $q^2 \in R^2 = 0$. Define the special inner automorphism $\varphi : A \to A$ by $\varphi(a) = (1+q)a(1-q)$ for all $a \in A$. Since $RA = 0$, by applying φ to all $b_{st}^{(3)} \in \Lambda_3$ we obtain

$$\begin{aligned} \varphi(b_{st}^{(3)}) &= (1 + \sum_{j=l}^{n} \beta_j e_j)(e_{st} + \beta_t e_s)(1-q) \\ &= (e_{st} + \beta_t e_s)(1 - \sum_{j=l}^{n} \beta_j e_j) = e_{st} + \beta_t e_s - \beta_t e_s = e_{st} \in \varphi(B) \end{aligned}$$

Therefore,

$$E = \{e_{st} \mid 1 \le s \le k < l \le t \le n\} \subseteq \varphi(B) \cap S.$$

Note that $\varphi(r) = r$ for all $r \in R$. Hence, $\varphi(B) = \varphi(B)_S \oplus \varphi(B)_R$, where $\varphi(B)_S = \varphi(B) \cap S$ and $\varphi(B)_R = \varphi(B) \cap R$. By changing the Levi subalgebra S of A to $S' = \varphi^{-1}(S)$ we obtain $B = B_{S'} \oplus B_R$, where $B_{S'} = B \cap S'$ and $B_R = B \cap R$. Therefore, B splits in A.

It remains to show that $B \subseteq S'$. Let $P = [B_{S'}, [B_{S'}, S'^{(1)}]] \subseteq S'^{(1)}$. Then $P \subseteq [B, [B, A^{(1)}]] \subseteq B$, so $P \subseteq B \cap S'^{(1)}$. Since $\bar{A}$ is semisimple and $\bar{B}_{S'} = \bar{B}$, we get that

$$\bar{P} = [\bar{B}_{S'}, [\bar{B}_{S'}, \bar{S}'^{(1)}]] = [\bar{B}, [\bar{B}, \bar{A}^{(1)}]] = \bar{B}.$$

Note that $B' = B \cap S'^{(1)}$ is a Jordan-Lie inner ideal of $S'^{(1)}$. As $P \subseteq B'$, we have $\bar{B} = \bar{P} \subseteq \bar{B}'$, so $\bar{B}' = \bar{B}$. By Lemma 2.6.3, B' is a Jordan-Lie inner ideal of L. Since $\bar{B}' = \bar{B}$ and B is bar-minimal, we have $B = B' \subseteq S'$, as required. □

Proposition 2.6.5. *Theorem 2.6.2 holds if A/R is simple and $RA = 0$.*

Proof. Since A is 1-perfect, $SR = R$, so R as a left S-module is a direct sum of copies of the natural left S-module V. The proof is by induction on the length $\ell(R)$ of the left S-module R, the case $\ell(R) = 1$ being clear by Proposition 2.6.4. Suppose that $\ell(R) > 1$. Consider any maximal submodule T of R. Then $\ell(T) = \ell(R) - 1$ and T is an ideal of A. Let $\tilde{} : A \to A/T$ be the natural epimorphism of A onto $\tilde{A} = A/T$. Denote by $\tilde{R}$ and $\tilde{B}$ the images of R and B, respectively, in $\tilde{A}$. Since $\ell(\tilde{R}) = 1$, by Proposition 2.6.4, $\tilde{B}$ is contained in a Levi subalgebra of $\tilde{A}$. Therefore, $B \subseteq S_1 \oplus T$ for some Levi subalgebra S_1 of A. Put $G = S_1 \oplus T$. Then G is clearly 1-perfect (i.e. $G = \mathscr{P}_1(G)$), $\mathrm{rad}(G) = T$, $G = S_1 \oplus T$ is a Levi decomposition of G and $C = B \cap G^{(1)}$ is a Jordan-Lie inner ideal of $G^{(1)} = \mathscr{P}_1(G)^{(1)}$. Put $P = [B, [B, G^{(1)}]] \subseteq C$. Then

$$\bar{P} = [\bar{B}, [\bar{B}, \bar{G}^{(1)}]] = [\bar{B}, [\bar{B}, \bar{A}^{(1)}]] = \bar{B},$$

so $\bar{C} = \bar{B}$. Let C' be any $\bar{C}$-minimal inner ideal of $G^{(1)}$ contained in C. Since G is 1-perfect and $\ell(T) < \ell(R)$, by the inductive hypothesis, C' splits in G. Since $C' \subseteq C \subseteq B$ and $\bar{C'} = \bar{C} = \bar{B}$, by Proposition 2.5.12, B splits in A.

□

Proposition 2.6.6. *Theorem 2.6.2 holds if A/R is simple and $AR = 0$.*

Proof. The proof is similar to that of Proposition 2.6.5.

□

Proposition 2.6.7. *Theorem 2.6.2 holds if A/R is simple and R is isomorphic to the natural A/R-bimodule A/R with respect to the right and left multiplication.*

Proof. Recall that B is a Jordan-Lie inner ideal of $L = [A, A]$ such that B is bar-minimal. As in the proof of Proposition 2.6.4, we fix standard bases $\{e_{ij} \mid 1 \le i, j \le n\}$ and $\{f_{ij} \mid 1 \le i, j \le n\}$ of S and R, respectively, consisting of matrix units, such that the action of S on R corresponds to matrix multiplication and $\bar{B}$ is the space spanned by $E = \{e_{st} \mid 1 \le s \le k < l \le t \le n\} \subseteq S$. We identify $\bar{A}$ with S. We are going to prove that there is a Levi subalgebra S' of A such that $B = B_{S'} \oplus B_R$, where $B_{S'} = B \cap S'$ and $B_R = B \cap R$. Put

$$R_0 = span\{f_{st} \mid 1 \le s \le k < l \le t \le n\} \subseteq N.$$

CLAIM 1: $R_0 \subseteq B$. Fix any $b_{st} \in B$ such that $\bar{b}_{st} = e_{st}$. Then $b_{st} = e_{st} + r_{st}$, with $r_{st} \in N$. By Lemma 3.1.3, $b_{st} f_{ts} b_{st} \in B$. Since $R^2 = 0$, we have

$$b_{st} f_{ts} b_{st} = (e_{st} + r_{st}) f_{ts} (e_{st} + r_{st}) = f_{ss}(e_{st} + r_{st}) = f_{st}.$$

Therefore, $f_{st} \in B$ for all s and t as required.

CLAIM 2: For every $b_{st} = e_{st} + \sum_{i,j=1}^{n} \alpha_{ij}^{st} f_{ij} \in B$ we have

$$\theta(b_{st}) = e_{st} + \sum_{i>k} \alpha_{it}^{st} f_{it} + \sum_{j<l} \alpha_{sj}^{st} f_{sj} \in B.$$

Since $b_{st} \in B$, by Lemma 3.1.3, $b_{st} e_{ts} b_{st} \in B$. We have

$$\begin{aligned} b_{st} e_{ts} b_{st} &= (e_{st} + \sum_{i,j}^{n} \alpha_{ij}^{st} f_{ij}) e_{ts} b_{st} = (e_{ss} + \sum_{i}^{n} \alpha_{it}^{st} f_{is})(e_{st} + \sum_{i,j}^{n} \alpha_{ij}^{st} f_{ij}) \\ &= e_{st} + \sum_{i}^{n} \alpha_{it}^{st} f_{it} + \sum_{j}^{n} \alpha_{sj}^{st} f_{sj} = \theta(b_{st}) + \sum_{i=1}^{k} \alpha_{it}^{st} f_{it} + \sum_{j=l}^{n} \alpha_{sj}^{st} f_{sj}. \end{aligned}$$

Since $\sum_{i=1}^{k} \alpha_{it}^{st} f_{it} + \sum_{j=l}^{n} \alpha_{sj}^{st} f_{sj} \in R_0 \subseteq B$ and $b_{st} e_{ts} b_{st} \in B$, we have $\theta(b_{st}) \in B$ as required.

By claim 2, there are some $\alpha_{ij}^{st} \in \mathbb{F}$ such that

$$b_{st} = e_{st} + \sum_{i>k} \alpha_{it}^{st} f_{it} + \sum_{j<l} \alpha_{sj}^{st} f_{sj} \in B,$$

for all $1 \le s \le k < l \le t \le n$.

(1) Define the special inner automorphism $\varphi_1 : A \to A$ by $\varphi_1(a) = (1 + q_1) a (1 - q_1)$ for all $a \in A$, where

$$q_1 = \sum_{j<l} \alpha_{1j}^{1n} f_{nj} - \sum_{i>k} \alpha_{in}^{1n} f_{i1} \in R.$$

Put $B_1 = \varphi_1(B)$. Set $b_{st}^{(1)} = \varphi_1(b_{st})$ for all s and t. Then

$$\begin{aligned}
b_{1n}^{(1)} &= (1+q_1)b_{1n}(1-q_1)\\
&= (1+\sum_{j<l}\alpha_{1j}^{1n}f_{nj}-\sum_{i>k}\alpha_{in}^{1n}f_{i1})(e_{1n}+\sum_{i>k}\alpha_{in}^{1n}f_{in}+\sum_{j<l}\alpha_{1j}^{1n}f_{1j})(1-q_1)\\
&= (e_{1n}+\sum_{i>k}\alpha_{in}^{1n}f_{in}+\sum_{j<l}\alpha_{1j}^{1n}f_{1j}+\alpha_{11}^{1n}f_{nn}-\sum_{i>k}\alpha_{in}^{1n}f_{in})(1-q_1)\\
&= (e_{1n}+\sum_{j<l}\alpha_{1j}^{1n}f_{1j}+\alpha_{11}^{1n}f_{nn})(1-\sum_{j<l}\alpha_{1j}^{1n}f_{nj}+\sum_{i>k}\alpha_{in}^{1n}f_{i1})\\
&= e_{1n}+\sum_{j<l}\alpha_{1j}^{1n}f_{1j}+\alpha_{11}^{1n}f_{nn}-\sum_{j<l}\alpha_{1j}^{1n}f_{1j}+\alpha_{nn}^{1n}f_{11}\\
&= e_{1n}+\alpha_{11}^{1n}f_{nn}+\alpha_{nn}^{1n}f_{11}.
\end{aligned}$$

Since $(B_1)^2=0$, we have

$$\begin{aligned}
0 &= (b_{1n}^{(1)})^2=(e_{1n}+\alpha_{11}^{1n}f_{nn}+\alpha_{nn}^{1n}f_{11})(e_{1n}+\alpha_{11}^{1n}f_{nn}+\alpha_{nn}^{1n}f_{11})\\
&= \alpha_{11}^{1n}f_{1n}+\alpha_{nn}^{1n}f_{1n}=(\alpha_{11}^{1n}+\alpha_{nn}^{1n})f_{1n}.
\end{aligned}$$

Thus, $\alpha_{11}^{1n}=-\alpha_{nn}^{1n}$. Put $\alpha=\alpha_{11}^{1n}$. Then

$$b_{1n}^{(1)}=e_{1n}+\alpha f_{11}-\alpha f_{nn}\in B_1 \tag{2.6.1}$$

(2) Consider the special inner automorphism $\varphi_2: A\to A$ defined by $\varphi_2(a)=(1+\alpha f_{n1})a(1-\alpha f_{n1})$ for all $a\in A$. Put $B_2=\varphi_2(B_1)$. Then by applying φ_2 to (2.6.1), we obtain

$$\begin{aligned}
b_{1n}^{(2)} &= \varphi_2(b_{1n}^{(1)})=(1+\alpha f_{n1})(e_{1n}+\alpha f_{11}-\alpha f_{nn})(1-\alpha f_{n1})\\
&= (e_{1n}+\alpha f_{11}-\alpha f_{nn}+\alpha f_{nn})(1-\alpha f_{n1})=e_{1n}+\alpha f_{11}-\alpha f_{11}=e_{1n}\in B_2.
\end{aligned}$$

Put $b_{st}^{(2)}=\theta(\varphi_2(b_{st}^{(1)}))\in B_2$ for all s and t. Then $b_{st}^{(2)}=e_{st}+\sum_{i>k}\beta_{it}^{st}f_{it}+\sum_{j<l}\beta_{sj}^{st}f_{sj}$, where $\beta_{ij}^{st}\in\mathbb{F}$.

Put $b_{1n}^{(3)}=b_{1n}^{(2)}=e_{1n}$, $b_{st}^{(3)}=b_{st}^{(2)}$ for $t\neq n$ and $b_{sn}^{(3)}=\{b_{sn}^{(2)},e_{n1},e_{1n}\}$ for $s\neq 1$. Then by Lemma 3.1.3, $b_{sn}^{(3)}\in B_2$ for all s and t. Thus, for $s\neq 1$ we have

$$\begin{aligned}
b_{sn}^{(3)} &= \{b_{sn}^{(2)},e_{n1},e_{1n}\}=b_{sn}^{(2)}e_{n1}e_{1n}+e_{1n}e_{n1}b_{sn}^{(2)}=b_{sn}^{(2)}e_{nn}+e_{11}b_{sn}^{(2)}\\
&= (e_{sn}+\sum_{i>k}\beta_{in}^{sn}f_{in}+\sum_{j<l}\beta_{sj}^{sn}f_{sj})e_{nn}+e_{11}(e_{sn}+\sum_{i>k}\beta_{in}^{sn}f_{in}+\sum_{j<l}\beta_{sj}^{sn}f_{sj})\\
&= e_{sn}+\sum_{i>k}\beta_{in}^{sn}f_{in}\in B_2.
\end{aligned} \tag{2.6.2}$$

Note that $b_{1n}^{(3)}=e_{1n}$ is also of the shape (2.6.2) with all $\beta_{in}^{1n}=0$.

(3) Consider the special inner automorphism $\varphi_3 : A \to A$ defined by $\varphi_3(a) = (1+q_3)a(1-q_3)$ for $a \in A$, where

$$q_3 = -\sum_{i>k}\sum_{j=2}^{k} \beta_{in}^{jn} f_{ij}.$$

Put $B_3 = \varphi_3(B_2)$ and $b_{st}^{(4)} = \varphi_3(b_{st}^{(3)}) \in B_3$. By applying φ_3 to $b_{sn}^{(3)}$ in (2.6.2) (for all s), we obtain

$$\begin{aligned} b_{sn}^{(4)} &= \varphi_3(b_{sn}^{(3)}) = (1+q_3)b_{sn}^{(3)}(1-q_3) \\ &= (1-\sum_{i>k}\sum_{j=2}^{k}\beta_{in}^{jn} f_{ij})(e_{sn}+\sum_{i>k}\beta_{in}^{sn} f_{in})(1-q_3) \\ &= (e_{sn}+\sum_{i>k}\beta_{in}^{sn} f_{in}-\sum_{i>k}\beta_{in}^{sn} f_{in})(1+\sum_{i>k}\sum_{j=2}^{k}\beta_{in}^{jn} f_{ij}) \\ &= e_{sn}+\sum_{j=2}^{k}\beta_{nn}^{jn} f_{sj} \in B_3. \end{aligned} \tag{2.6.3}$$

Since $(B_3)^2 = 0$, for all $1 \le s, r \le k$ we have

$$0 = b_{sn}^{(4)} b_{rn}^{(4)} = (e_{sn}+\sum_{j=2}^{k}\beta_{nn}^{jn} f_{sj})(e_{rn}+\sum_{j=2}^{n}\beta_{nn}^{jn} f_{rj}) = \beta_{nn}^{rn} f_{sn}.$$

Thus, $\beta_{nn}^{rn} = 0$ for all $1 \le r \le k$. Substituting in (2.6.3) we obtain

$$b_{sn}^{(4)} = e_{sn} \in B_3 \text{ for all } 1 \le s \le k.$$

Put $b_{sn}^{(5)} = b_{sn}^{(4)} = e_{sn}$ and $b_{st}^{(5)} = \theta(b_{st}^{(4)}) \in B_3$ for $t \neq n$. Then for $t \neq n$ we have

$$b_{st}^{(5)} = e_{st} + \sum_{i>k}\gamma_{it}^{st} f_{it} + \sum_{j<l}\gamma_{sj}^{st} f_{sj} \text{ for some } \gamma_{ij}^{st} \in \mathbb{F}.$$

Put $b_{sn}^{(6)} = b_{sn}^{(5)} = e_{sn}$ and $b_{st}^{(6)} = \{e_{sn}, e_{n1}, b_{1t}^{(5)}\}$ for all $t \neq n$. Then by Lemma 3.1.3, $b_{st}^{(6)} \in B_3$. Thus, for $t \neq n$ we have

$$\begin{aligned} b_{st}^{(6)} &= \{e_{sn}, e_{n1}, b_{1t}^{(5)}\} = e_{sn}e_{n1}b_{1t}^{(5)} + b_{1t}^{(5)}e_{n1}e_{sn} = e_{s1}b_{st}^{(5)} + 0 \\ &= e_{s1}(e_{1t}+\sum_{i>k}\gamma_{it}^{1t} f_{it}+\sum_{j<l}\gamma_{1j}^{1t} f_{1j}) = e_{st}+\sum_{j<l}\gamma_{1j}^{1t} f_{sj}. \end{aligned} \tag{2.6.4}$$

Note that $b_{sn}^{(6)} = e_{sn}$ is also of the shape (2.6.4) with all $\gamma_{1i}^{1n} = 0$.

(4) We define the final special inner automorphism $\varphi_4 : A \to A$ by $\varphi_4(a) = (1+q_4)a(1-q_4)$ for $a \in A$, where

$$q_4 = \sum_{i=l}^{n-1} \sum_{j<l} \gamma_{1j}^{1i} f_{ij}.$$

Put $B_4 = \varphi_4(B_3)$ and $b_{st}^{(7)} = \varphi_4(b_{st}^{(6)}) \in B_4$ for all s and t. Then by applying φ_4 to $b_{st}^{(6)}$ in (2.6.4), we obtain (for all s and t)

$$\begin{aligned} b_{st}^{(7)} &= (\varphi_4(b_{st}^{(6)})) = (1+q_4)b_{st}^{(6)}(1-q_4) \\ &= (1+\sum_{i=l}^{n-1}\sum_{j<l}\gamma_{1j}^{1i}f_{ij})(e_{st}+\sum_{j<l}\gamma_{1j}^{1t}f_{sj})(1-q_4) \\ &= (e_{st}+\sum_{j<l}\gamma_{1j}^{1t}f_{sj}+\sum_{i=l}^{n-1}\gamma_{1s}^{1i}f_{it})(1-\sum_{i=l}^{n-1}\sum_{j<l}\gamma_{1j}^{1i}f_{ij}) \\ &= e_{st}-\sum_{j<l}\gamma_{1j}^{1t}f_{sj}+\sum_{i=l}^{n-1}\gamma_{1s}^{1i}f_{it}+\sum_{j<l}\gamma_{1j}^{1t}f_{sj} \\ &= e_{st}+\sum_{i=l}^{n-1}\gamma_{1s}^{1i}f_{it} \in B_4. \end{aligned} \tag{2.6.5}$$

Since $(B_4)^2 = 0$, we have (for all $1 \le s,r \le k < l \le t,q \le n$)

$$0 = b_{st}^{(7)}b_{rq}^{(7)} = (e_{st}+\sum_{i=l}^{n-1}\gamma_{1s}^{1i}f_{it})(e_{rq}+\sum_{i=l}^{n-1}\gamma_{1r}^{1i}f_{iq}) = \gamma_{1r}^{1t}f_{sq}.$$

Thus, $\gamma_{1r}^{1t} = 0$ for all $1 \le r \le k < l \le t \le n$. Substituting in (2.6.5) we obtain $b_{st}^{(7)} = e_{st}$ for $1 \le s \le k < l \le t \le n$. Thus,

$$E = \{e_{st} \mid 1 \le s \le k < l \le t \le n\} \subseteq B_4 \cap S.$$

Denote by φ the automorphism $\varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ of A and L. Then we have $E \subseteq \varphi(B) \cap S$. Note that $\varphi_i(R_0) = R_0$ for all $i = 1,2,3,4$ (because $R^2 = 0$). Hence, $\varphi(B) = \varphi(B)_S \oplus \varphi(B)_R$, where $\varphi(B)_S = \varphi(B) \cap S$ and $\varphi(B)_R = \varphi(B) \cap R = B_R$. Now, by changing the Levi subalgebra S to $S' = \varphi^{-1}(S)$ we obtain $B = B_S \oplus B_R$, where $B_{S'} = B \cap S'$ and $B_R = B \cap R$. □

Proposition 2.6.8. *Theorem 2.6.2 holds if $A/R \cong S_1 \oplus S_2$, where $S_1 \cong M_{n_1}(\mathbb{F})$, $S_2 \cong M_{n_2}(\mathbb{F})$ and $R \cong M_{n_1 n_2}(\mathbb{F})$ as an S_1-S_2-bimodule such that $RS_1 = S_2R = 0$.*

Proof. Recall that B is a Jordan-Lie inner ideal of $L = [A,A]$ such that B is bar-minimal.

We identify $\bar{A}$ with S. By Lemma 2.3.5, $\bar{B} = X_1 \oplus X_2$, where $X_i = \bar{B} \cap S_i$ are Jordan-Lie inner ideals of $S_i^{(1)}$. As in the proof of Proposition 2.6.5, we fix standard bases $\{e_{ij} \mid 1 \le i, j \le n_1\}$, $\{g_{ij} \mid 1 \le i, j \le n_2\}$ and $\{f_{ij} \mid 1 \le i \le n_1,\ 1 \le j \le n_2\}$ of S_1, S_2 and R, respectively, consisting of matrix units, such that the action of S_1 and of S_2 on R corresponds to matrix multiplication and $X_i = span\{E_i\}$, where

$$E_1 = \{e_{st} \mid 1 \le s \le k_1 < l_1 \le t \le n_1\} \subseteq S_1,$$

$$E_2 = \{g_{rq} \mid 1 \le r \le k_2 < l_2 \le q \le n_2\} \subseteq S_2.$$

Put $R_0 = span\{f_{sq} \mid 1 \le s \le k_1,\ l_2 \le q \le n_2\} \subseteq N$.

CLAIM 1: $R_0 \subseteq B$. Fix any $b_{st}, c_{rq} \in B$ such that $\bar{b}_{st} = e_{st}$ and $\bar{c}_{rq} = g_{rq}$. Then $b_{st} = e_{st} + r_{st}$ and $c_{rq} = g_{rq} + r'_{rq}$, with $r_{st}, r'_{rq} \in N$. By Lemma 3.1.3, $\{b_{st}, f_{tr}, c_{rq}\} \in B$. Since $R^2 = 0$ and $S_2R = RS_1 = 0$, we have

$$\{b_{st}, f_{tr}, c_{rq}\} = b_{st}f_{tr}c_{rq} + c_{rq}f_{tr}b_{st} = b_{st}f_{tr}c_{rq} + 0 = (e_{st} + r_{st})f_{tr}(g_{rq} + r'_{rq}) = f_{sq} \in B.$$

Therefore, $f_{sq} \in B$ for all $1 \le s \le k_1$ and $l_2 \le q \le n_2$ as required.

CLAIM 2: For every $b_{st} = e_{st} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \alpha_{ij}^{st} f_{ij} \in B$ we have

$$\theta(b_{st}) = e_{st} + \sum_{j<l_2} \alpha_{sj}^{st} f_{sj} \in B$$

Since $b_{st} \in B$, by Lemma 3.1.3, $b_{st}e_{ts}b_{st} \in B$. Since $RS_1 = 0$ and $R^2 = 0$, we have

$$\begin{aligned} b_{st}e_{ts}b_{st} &= (e_{st} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \alpha_{ij}^{st} f_{ij})e_{ts}b_{st} = e_{ss}(e_{st} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \alpha_{ij}^{st} f_{ij}) \\ &= e_{st} + \sum_{j=1}^{n_2} \alpha_{sj}^{st} f_{sj} = \theta(b_{st}) + \sum_{j=l_2}^{n_2} \alpha_{sj}^{st} f_{sj}. \end{aligned}$$

Since $\sum_{j=l_2}^{n_2} \alpha_{sj}^{st} f_{sj} \in R_0 \subseteq B$ and $b_{st}e_{ts}b_{st} \in B$, we have $\theta(b_{st}) \in B$ as required.

Put $A_2 = S_2 \oplus R$ and $L_2 = [A_2, A_2]$. Denote $B_2 = B \cap L_2$. By Lemma 2.1.10, B_2 is an inner ideal of L_2. Moreover, B_2 is a Jordan-Lie inner ideal as $(B_2)^2 = 0$. Note that $\bar{B}_2 = X_2$ (because B_2 contains the preimage of X_2 in B). By Lemma 2.5.3, B_2 is X_2-minimal. Thus, B_2 satisfies the conditions of Proposition 2.6.6. Hence, B_2 splits. Thus, there is a special inner automorphisms $\varphi_2 : A \to A$ such that $E_2 \subseteq \varphi_2(B_2) \subseteq \varphi_2(B)$. We will deal with the inner ideal $\varphi_2(B)$ of L. Note that $\overline{\varphi_2(B)} = \bar{B} = X$ and $E_2 \subseteq \varphi_2(B)$. Our aim is to modify $\varphi_2(B)$ in such a way that it contains both E_1 and E_2.

Put $b_{st}^{(1)} = \theta(\varphi_2(b_{st})) \in \varphi_2(B)$ for all $1 \le s \le k_1 < l_1 \le t \le n_1$. Then

$$b_{st}^{(1)} = e_{st} + \sum_{j<l_2} \alpha_{sj}^{st} f_{sj}$$

for all s and t. Put $b_{1t}^{(2)} = b_{1t}^{(1)} = e_{1t} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{1j}$ and for $s > 1$ set $b_{st}^{(2)} = \{b_{st}^{(1)}, e_{t1}, b_{1t}^{(1)}\}$. Then by Lemma 3.1.3, $b_{st}^{(2)} \in \varphi_2(B)$. Since $RS_1 = 0$, for $s > 1$ we have

$$\begin{aligned} b_{st}^{(2)} &= \{b_{st}^{(1)}, e_{t1}, b_{1t}^{(1)}\} = b_{st}^{(1)} e_{t1} b_{1t}^{(1)} + b_{1t}^{(1)} e_{t1} b_{st}^{(1)} \\ &= (e_{st} + \sum_{j<l_2} \alpha_{sj}^{st} f_{sj}) e_{t1} b_{1t}^{(1)} + b_{1t}^{(1)} e_{t1} (e_{st} + \sum_{j<l_2} \alpha_{sj}^{st} f_{sj}) \\ &= e_{s1}(e_{1t} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{1j}) + 0 = e_{st} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{sj}. \end{aligned}$$

Thus, for all s and t we have

$$b_{st}^{(2)} = e_{st} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{sj}. \tag{2.6.6}$$

Consider the special inner automorphism $\varphi : A \to A$ defined by $\varphi(a) = (1+q)a(1-q)$ for all $a \in A$, where

$$q = \sum_{i=l_1}^{n_1} \sum_{j<l_2} \alpha_{1j}^{1i} f_{ij}.$$

Since $RS_1 = 0$ and $R^2 = 0$, by applying φ to (2.6.6) we obtain

$$\begin{aligned} \varphi(b_{st}^{(2)}) &= (1+q) b_{st}^{(2)} (1-q) = b_{st}^{(2)}(1-q) \\ &= (e_{st} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{sj})(1 - \sum_{i=l_1}^{n_1} \sum_{j<l_2} \alpha_{1j}^{1i} f_{ij}) \\ &= e_{st} + \sum_{j<l_2} \alpha_{1j}^{1t} f_{sj} - \sum_{j<l_2} \alpha_{1j}^{1t} f_{sj} = e_{st} \in \varphi(\varphi_2(B)). \end{aligned}$$

Thus, $e_{st} \in \varphi(\varphi_2(B))$ for all $1 \le s \le k_1 < l_1 \le t \le n_1$. Now, by applying φ to $g_{rq} \in X_2 \subseteq \varphi_2(B)$ and using $S_2 R = 0$, we obtain

$$\begin{aligned} \varphi(g_{rq}) &= (1+q) g_{rq} (1-q) = (1+q) g_{rq} = (1 + \sum_{i=l_1}^{n_1} \sum_{j<l_2} \alpha_{1j}^{1i} f_{ij}) g_{rq} \\ &= g_{rq} + \sum_{i=l_1}^{n_1} \alpha_{1r}^{1i} f_{iq} \in \varphi(\varphi_2(B)). \end{aligned}$$

Since $(\varphi(\varphi_2(B)))^2 = 0$ and both e_{st} and $\varphi(g_{rq})$ are in $\varphi(\varphi_2(B))$, we have

$$0 = e_{st}\varphi(g_{rq}) = e_{st}(g_{rq} + \sum_{i=l_1}^{n_1} \alpha_{1r}^{1i} f_{iq}) = \alpha_{1r}^{1t} f_{sq}.$$

Hence, $\alpha_{1r}^{1t} = 0$ for all $1 \le r \le k_2$ and all $l_1 \le t \le n_1$. Thus, $\varphi(g_{rq}) = g_{rq} \in \varphi(\varphi_2(B))$ for all r and q. Therefore,

$$E_1 = \{e_{st} : 1 \le s \le k_1 < l_1 \le t \le n_1\} \subseteq \varphi(\varphi_2(B)) \cap S$$

and

$$E_2 = \{g_{rq} : 1 \le r \le k_2 < l_2 \le q \le n_2\} \subseteq \varphi(\varphi_2(B)) \cap S.$$

Put $E = E_1 \cup E_2 \subseteq \varphi(\varphi_2(B)) \cap S$. Since $R^2 = 0$, one can easily check that $\varphi(\varphi_2(R_0)) = R_0$. By changing the Levi subalgebra S to $S' = \varphi^{-1}(\varphi_2^{-1}(S))$ we prove that B splits in A.

□

We will need the following result.

Lemma 2.6.9. *Let S be a semisimple finite dimensional associative algebra and let $\{S_i \mid i \in I\}$ be the set of its simple components. Suppose that M is an S-bimodule. Then M is a direct sum of copies of U_{ij}, for $i, j \in I \cup \{0\}$, where U_{00} is the trivial 1-dimensional S-bimodule, U_{i0} is the natural left S_i-module with $U_{i0}S = 0$, U_{0j} is the natural right S-module with $SU_{0j} = 0$ and U_{ij} is the natural S_i-S_j-bimodule for $i, j > 0$.*

Proof. Let $\hat{S} = S + \mathbb{F}1_{\hat{S}}$, where $1_{\hat{S}}$ is the unity of $\hat{S}$. Then $\hat{S}$ is a unital algebra. Set $1_{\hat{S}}m = m1_{\hat{S}} = m$ for all $m \in M$. Then M is a unital $\hat{S}$-bimodule. Note that $\hat{S} = \bigoplus_{i \in I \cup \{0\}} S_i$, where $S_0 = \mathbb{F}(1_{\hat{S}} - 1_S)$ is a 1-dimensional simple component of $\hat{S}$. Thus, as a unital $\hat{S}$-bimodule M is a direct sum of copies of the natural S_i-S_j-bimodules U_{ij} such that $U_{ij} = S_i U S_j$, for all i and j. It remains to note that $U_{i0}S = 0$ and $SU_{0j} = 0$.

□

Now, we are ready to prove Theorem 2.6.2.

Proof of Theorem 2.6.2. Recall that A is 1-perfect with $R^2 = 0$, $p \neq 2, 3$ and B is a bar-minimal Jordan-Lie inner ideal of $L = [A, A]$. Let $\{S_i \mid i \in I\}$ be the set of the simple components of S. We identify $\bar{A}$ with S. By Lemma 2.6.9, the S-bimodule R is a direct sum of copies of the natural left S_i-module U_{i0}, the natural right S_j-module U_{0j} and the natural S_i-S_j- bimodule U_{ij} for all $i,j \in I$. Note that the S-bimodule R has no components isomorphic to the trivial 1-dimensional S-bimodule U_{00} as A is 1-perfect with $R^2 = 0$.

The proof is by induction on the length $\ell(R)$ of the S-bimodule R. If $\ell(R) = 1$, then $R = U_{ij}$ for some i and j. Note that $(i,j) \neq (0,0)$. Let $A_1 = (S_i + S_j) \oplus R$ and let A_2 be the complement of $S_i + S_j$ in S. Then A_1 and A_2 are 1-perfect. Note that $A_2A_1 = A_1A_2 = 0$ so both A_1 and A_2 are ideals of A with $A = A_1 \oplus A_2$. Hence $L = L_1 \oplus L_2$, where $L_i = [A_i, A_i]$ for $i = 1,2$. Since L satisfies the conditions of Lemma 2.5.3, we have $B = B_1 \oplus B_2$, where B_i is a $\bar{B}_i$-minimal Jordan-Lie inner ideal of L_i, $i = 1,2$. Since A_2 is semisimple, B_2 splits in A_2. Note that B_1 satisfies the conditions of one of the Propositions 2.6.5, 2.6.6, 2.6.7 and 2.6.8, so B_1 splits in A_1. Therefore, B splits in A.

Assume that $\ell(R) > 1$. Consider any maximal S-submodule T of R, so $\ell(T) = \ell(R) - 1$. Then T is an ideal of A. Let $\tilde{A} = A/T$. Denote by $\tilde{B}$ and $\tilde{R}$ the images of B and R in $\tilde{A}$. Since $\ell(\tilde{R}) = 1$, by the base of induction, $\tilde{B}$ splits, so there is a Levi subalgebra $S' \cong S$ of $\tilde{A}$ such that $\tilde{B} = \tilde{B}_{S'} \oplus \tilde{B}_R$, where $\tilde{B}_{S'} = \tilde{B} \cap S'$ and $\tilde{B}_R = \tilde{B} \cap \tilde{R}$. Let P be the full preimage of $\tilde{B}_{S'}$ in B. Then $\tilde{P} = \tilde{B}_{S'} \subseteq S'$, so P is a subspace of B with $\bar{P} = \bar{B}$. Let G be the full preimage of S' in A. Then G is clearly 1-perfect (i.e. $G = \mathscr{P}_1(G)$), $\mathrm{rad}(G) = T$, $G/T \cong S$ and $P \subseteq B \cap G$. Put $P_1 = [P, [P, S'^{(1)}]] \subseteq G^{(1)}$. Then $P_1 \subseteq [B, [B, A^{(1)}]] \subseteq B$, so $P_1 \subseteq B \cap G^{(1)}$. Note that $B' = B \cap G^{(1)}$ is a Jordan-Lie inner ideal of $G^{(1)}$ (because $G^{(1)}$ is a subalgebra of $A^{(1)}$). Since $\bar{P}_1 = [\bar{P}, [\bar{P}, \bar{S}'^{(1)}]] = [\bar{B}, [\bar{B}, \bar{A}^{(1)}]] = \bar{B}$, we get that $\bar{B} = \bar{P}_1 \subseteq \bar{B}' \subseteq \bar{B}$, so $\bar{B}' = \bar{B}$. Note that G is a large subalgebra of A (see Definition 2.5.10). Let $B'' \subseteq B'$ be a $\bar{B}'$-minimal Jordan-Lie inner ideal of $G^{(1)}$. As G is 1-perfect and $\ell(T) < \ell(R)$, by the inductive hypothesis, B'' splits in G. Since $B'' \subseteq B' \subseteq B$ and $\bar{B}'' = \bar{B}' = \bar{B}$, by Proposition 2.5.12, B splits in A.

□

The following result follows from Theorem 2.6.2 and Proposition 2.5.8.

Corollary 2.6.10. *Let $L = [A,A]$ and let B be a Jordan-Lie inner ideal of L. Suppose that $p \neq 2,3$, A is 1-perfect, and $R^2 = 0$. Then B splits in A.*

Now, we are ready to prove Theorem 2.6.1.

Proof of Theorem 2.6.1. (i) Recall that B is bar-minimal. Since $R = \operatorname{rad} A$ is nilpotent, there is an integer m such that $R^{m-1} \neq 0$ and $R^m = 0$. The proof is by induction on m. If $m = 2$, then by Theorem 2.6.2, B splits. Suppose that $m > 2$. Put $T = R^2 \neq 0$ and consider $\tilde{A} = A/T$. Let $\tilde{B}$ and $\tilde{R}$ be the images of B and R in $\tilde{A}$. Then we have $\tilde{R} = \operatorname{rad}\tilde{A}$, $\tilde{R}^2 = 0$ and $\tilde{A}$ satisfies the conditions of the Corollary 2.6.10. Hence, there is a Levi subalgebra S' of $\tilde{A}$ such that $\tilde{B} = \tilde{B}_{S'} \oplus \tilde{B}_R$, where $\tilde{B}_{S'} = \tilde{B} \cap S'$ and $\tilde{B}_R = \tilde{B} \cap \tilde{R}$. Let P be the full preimage of $\tilde{B}_{S'}$ in B. Then $\tilde{P} = \tilde{B}_{S'} \subseteq S'$, so P is a subspace of B with $\bar{P} = \bar{B}$. Let G be the full preimage of S' in A. Then G is a large subalgebra of A with $P \subseteq G \cap B$. Put $P_1 = [P,[P,S'^{(1)}]]$ and $B_1 = B \cap G^{(1)}$. Then $P_1 \subseteq [B,[B,A^{(1)}]] \subseteq B$ and $P_1 \subseteq [G,[G,G]] \subseteq G^{(1)}$, so $P_1 \subseteq B \cap G^{(1)} = B_1$. Since $\bar{P}_1 = [\bar{P},[\bar{P},\bar{S}'^{(1)}]] = [\bar{B},[\bar{B},\bar{A}^{(1)}]] = \bar{B}$, we get that $\bar{B} = \bar{P}_1 \subseteq \bar{B}_1 \subseteq \bar{B}$, so $\bar{B}_1 = \bar{B}$. As $G^{(1)}$ is a Lie subalgebra of $A^{(1)}$, $B_1 = B \cap G^{(1)}$ is a Jordan-Lie inner ideal of $G^{(1)}$. Put $B_2 = \operatorname{core}_{G^{(1)}}(B_1)$. Then by Proposition 2.5.14, B_2 is a Jordan-Lie inner ideal of $\mathscr{P}_1(G)^{(1)}$ such that $B_2 \subseteq B$ and $\bar{B}_2 = \bar{B}$. Let $B_3 \subseteq B_2$ be any $\bar{B}_2$-minimal inner ideal of $\mathscr{P}_1(G)^{(1)}$. Since $\mathscr{P}_1(G)$ is 1-perfect and $\operatorname{rad}(\mathscr{P}_1(G))^{m-1} \subseteq T^{m-1} = R^{2(m-1)} = 0$, by the inductive hypothesis, B_3 splits in $\mathscr{P}_1(G)$. Since $\bar{B}_3 = \bar{B}_2 = \bar{B}$, by Lemma 2.5.12, B splits in A.

(ii) We wish to show that $B = eAf$ for some strict orthogonal idempotent pair (e, f) in A. By (i), there is a B-splitting Levi subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$. Let $\{S_i \mid i \in I\}$ be the set of the simple components of S, so $S = \bigoplus_{i \in I} S_i$. We identify $\bar{A}$ with S. By Lemma 2.3.5, we have $\bar{B} = \bigoplus_{i \in I} X_i$, where $X_i = \bar{B} \cap S_i$ for all $i \in I$. Put $J = \{i \in I \mid X_i \neq 0\}$. By Lemma 2.3.3, for each $r \in J$ there is a matrix realization $M_{n_r}(\mathbb{F})$ of S_r and integers $1 \leq k_r < l_r \leq n_r$ such that X_r is spanned by the set

$$E_r = \{e^r_{st} \mid 1 \leq s \leq k_r < l_r \leq t \leq n_r\} \subseteq S_r$$

where $\{e^r_{ij} \mid 1 \leq i, j \leq n_r\}$ is a basis of S_r consisting of matrix units. Let $e = \sum_{r \in J} \sum_{i=1}^{k_r} e^r_{ii}$ and $f = \sum_{r \in J} \sum_{j=l_r}^{n_r} e^r_{jj}$. Then (e, f) is a strict orthogonal idempotent pair in A with $B_S = \bigoplus_{i \in J} X_i = eSf$. Note that eAf is a Jordan-Lie inner ideal of $[A,A]$ with $\overline{eAf} = eSf = \bar{B}$. We are going to show that $eRf \subseteq B_R$. This will imply $eAf = B$ as B is bar-minimal.

By Lemma 2.6.9, the S-bimodule R is a direct sum of copies of the natural left S_i-module U_{i0}, the natural right S_j-module U_{0j}, the natural S_i-S_j- bimodule U_{ij} and the trivial 1-dimensional S-bimodule U_{00} for all $i, j \in I$. Let M be any minimal S-submodule of R. It is enough to show that $eMf \subseteq B$. Fix $r, q \in I$ such that $M \cong U_{rq}$. We can assume that $r, q \in J$ (otherwise $eMf = \{0\} \subseteq B$). Let $\{f^{rq}_{ij} \mid 1 \leq i \leq n_r,\ 1 \leq j \leq n_q\}$ be the standard basis of M consisting of matrix units, such that the action of S_r-S_q on M corresponds to matrix multiplication. Note that

$$eMf = span\{f_{st}^{rq} \mid 1 \leq s \leq k_r,\ l_q \leq t \leq n_q\}.$$

We need to show that $f_{st}^{rq} \in B$ for all s and t. First, consider the case when $r = q$. Then $s \leq k_r < l_r \leq t$, so $s \neq t$. Since $e_{st}^{r} \in B$ and $f_{ts}^{rr} = [e_{tt}^{r}, f_{ts}^{rr}] \in L$, by Lemma 2.5.15, we have

$$e_{st}^{r} f_{ts}^{rr} e_{st}^{r} = f_{ss}^{rr} e_{st}^{r} = f_{st}^{rr} \in B,$$

as required. Assume now $r \neq q$. Fix any $e_{sj}^{r} \in E_r$ and $e_{it}^{q} \in E_q$. Since $e_{sj}^{r}, e_{it}^{q} \in B$ and $f_{ji}^{rq} = [e_{jj}^{r}, f_{ji}^{rq}] \in L$, using Lemma 3.1.3, we obtain

$$\{e_{sj}^{r}, f_{ji}^{rq}, e_{it}^{q}\} = e_{sj}^{r} f_{ji}^{rq} e_{it}^{q} + e_{it}^{q} f_{ji}^{rq} e_{sj}^{r} = f_{st}^{rq} + 0 \in B,$$

as required.

(iii) Since $B = eAf$, by Lemma 2.5.18, B is regular.

□

Corollary 2.6.11. *Let $L = [A,A]$ and let B be a Jordan-Lie inner ideal of L. Suppose that $p \neq 2,3$ and A is 1-perfect. Then B splits in A.*

Proof. Let $B' \subseteq B$ be a bar-minimal Jordan-Lie inner ideal of L. Then by Theorem 2.6.1(i), B' splits in A. Therefore, by Lemma 2.5.8, B splits in A.

□

Now we are ready to prove the main results of this paper.

Proof of Theorem 1.2.1. Suppose first that B is bar-minimal. We need to show that $B = eAf$ for some strict orthogonal idempotent pair (e,f) in A. By Lemma 2.5.2(ii), B is L-perfect, so by Lemma 2.4.12, $B \subseteq \mathscr{P}_1(A)$ and B is a Jordan-Lie inner ideal of $L_1 = \mathscr{P}_1(A)^{(1)}$. Let $C \subseteq B$ be a $\bar{B}$-minimal Jordan-Lie inner ideal of L_1. Since $\mathscr{P}_1(A)$ is 1-perfect, by Theorem 2.6.1, there exists a strict orthogonal idempotent pair (e,f) in $\mathscr{P}_1(A)$ such that $C = e\mathscr{P}_1(A)f$. Note that $\mathscr{P}_1(A)$ is a two-sided ideal of A, so

$$CAC = e\mathscr{P}_1(A)fAe\mathscr{P}_1(A)f \subseteq e\mathscr{P}_1(A)f = C$$

Hence, by Lemma 2.5.15(iii), C is an inner ideal of L with $C \subseteq B$ and $\bar{C} = \bar{B}$. Since B is bar-minimal, $C = B$. As $e, f \in \mathscr{P}_1(A)$, we have

$$e\mathscr{P}_1(A)f \subseteq eAf = eeAf \subseteq e\mathscr{P}_1(A)Af \subseteq e\mathscr{P}_1(A)f.$$

Therefore, $e\mathscr{P}_1(A)f = eAf$ and $B = C = eAf$ as required.

Suppose now that $B = eAf$, where (e, f) is a strict orthogonal idempotent pair in A. We need to show that B is bar-minimal. Let $C \subseteq B$ be a $\bar{B}$-minimal Jordan-Lie inner ideal of L. Then by the "if" part $C = e_1Af_1$ for some strict orthogonal idempotent pair (e_1, f_1) in A, so $e_1Af_1 \subseteq eAf$ and $\bar{e}_1\bar{A}\bar{f}_1 = \bar{B} = \bar{e}\bar{A}\bar{f}$. Then by Theorem 1.2.4(iv), there is a strict idempotent pair (e_2, f_2) in A such that $(e_2, f_2) \leq (e, f)$, that is, $ee_2 = e_2e = e_2$ and $f_2f = ff_2 = f_2$. Moreover, by Theorem 1.2.4(iv), $e_2Af_2 = e_1Af_1 = C$, so $\bar{e}_2\bar{A}\bar{f}_2 = \bar{B} = \bar{e}\bar{A}\bar{f}$. We are going to show that $e_2 = e$ (the proof of $f_2 = f$ is similar). Since (e, f) is strict, by Theorem 1.2.4(iii) , $\bar{e}_2 \overset{\mathscr{L}}{\sim} \bar{e}$, so $\bar{e} = \bar{e}_2\bar{e} = \overline{e_2e} = \bar{e}_2$. Hence, there is $r \in R$ such that $e_2 = e + r$. We have

$$e + r = e_2 = ee_2 = e(e + r) = e + er,$$

so $er = r$. Similarly, $re = r$. Since e_2 is an idempotent,

$$e + r = e_2 = e_2^2 = (e + r)^2 = e + 2r + r^2.$$

Therefore, $r^2 = -r$ and $r^{2^k} = -r$ for all $k \in \mathbb{N}$. As R is nilpotent, we get $r = 0$, so $e_2 = e$. Similarly, $f_2 = f$. Therefore, $B = eAf = e_2Af_2 = C$, as required.

□

Proof of Corollary 1.2.2. Since B is bar-minimal, by Theorem 1.2.1, there exists a strict orthogonal idempotent pair (e, f) in A such that $B = eAf$. Therefore, by Lemma 2.5.18, B is regular. □

Proof of Corollary 1.2.3. Let $C \subseteq B$ be a $\bar{B}$-minimal Jordan-Lie inner ideal of L. Then by Theorem 1.2.1, there exists a strict orthogonal idempotent pair (e, f) in A such that $C = eAf$, so by Lemma 2.5.7(i), C splits in A. Therefore, by Proposition 2.5.8, B splits in A. □

Chapter 3

Jordan-Lie Inner Ideals of Finite Dimensional Associative Algebras with Involution

In this chapter we study Jordan-Lie inner ideals of Lie algebras obtained from finite dimensional associative algebras with involution. We use the same approach as in Chapter 2. However, the case of algebras with involution is technically more difficult and more cases must be considered.

Outline of Chapter 3

(Section 3.1) We discuss some background results related to Lie algebras derived from associative algebras with involution and Jordan-Lie inner ideals of such Lie algebras.

(Section 3.2) We describe the relation between inner ideals and idempotents and recall some known results on inner ideals and point spaces.

(Section 3.3) We study inner ideals of Lie subalgebras of semisimple associative algebras with involution.

(Section 3.4) We describe the structure of the so-called $*$-indecomposable associative algebras and their corresponding Lie algebras. We provide some results that describe the derived Lie subalgebras of $*$-indecomposable associative algebras.

(Section 3.5) We study inner ideals of Lie subalgebras of admissible associative algebras.

(Section 3.6) We introduce and describe the structure of inner ideals that admit a $*$-invariant Levi decomposition.

(Section 3.7) We prove some of the main results stated in Chapter 1. In particular, we prove that every $*$-regular bar-minimal Jordan-Lie inner ideal is generated by $*$-orthogonal idempotent in the associative algebra (Theorem 1.2.6). As a consequence, we get Corollary 1.2.7 which shows that every $*$-regular inner ideal $*$-splits in the associative algebra.

(Section 3.8) We prove the remaining main results. In particular, we show that if the associative algebra A is admissible, then every bar-minimal Jordan-Lie inner ideal of the corresponding Lie algebra $\mathfrak{su}^*(A)$ $*$-splits in A (Theorem 1.2.8). As a corollary, we show that every Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$ $*$-splits in A and splits in the Lie algebra $\mathfrak{su}^*(A)$ as well (Corollary 1.2.9).

3.1 Background Materials

Throughout this chapter, unless otherwise specified, $\mathbb{F}$ is an algebraically closed field of characteristic $p \neq 2$, A is a finite dimensional associative algebra over $\mathbb{F}$ with involution $*$ (of the first kind), $R = \operatorname{rad} A$ is the radical of A, S is a $*$-invariant Levi (i.e. maximal semisimple) subalgebra of A (see [38] and [39] for the existence of this subalgebra), so $A = S \oplus R$. We denote by K the vector space $\mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$ of skew symmetric elements of A. Then K is a Lie algebra over $\mathbb{F}$. As $p \neq 2$, K can be represented in the form:

$$K = \mathfrak{u}^*(A) = \{a - a^* \mid a \in A\}. \tag{3.1.1}$$

Since both S and R are $*$-invariant, we have $K = \mathfrak{u}^*(S) \oplus \mathfrak{u}^*(R)$. Moreover, we denote $K^{(1)}$ the Lie algebra $\mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$. Note that the relation between $K^{(1)}$ and $A^{(1)} = [A, A]$ was highlighted in [12], where Baxter proved that if A is a simple ring with involution of dimension greater than 16 over its centre $Z(A)$ or $Z(A) = (0)$, then $A^{(1)} = [K, \operatorname{sym}(A)] + K^{(1)}$ and $K^{(1)} = [\operatorname{sym}(A), \operatorname{sym}(A)]$ (recall that $\operatorname{sym}(A) = \{a \in A \mid a^* = a\}$ is the vector space of the symmetric elements of A). Furthermore, we denote by $\operatorname{rad} K$ the *solvable radical* of K and $\mathfrak{u}^*(R) = \mathfrak{u}^*(A) \cap R$ the *nil-radical* of K. If V is a subspace of A, we denote by $\bar{V}$ its image in $\bar{A} = A/R$. In particular, $\bar{K} \cong K/\mathfrak{u}^*(R) \cong \mathfrak{u}^*(S)$. Since R is a nilpotent ideal of A, $\mathfrak{u}^*(R)$ is a nilpotent ideal of K, so $\mathfrak{u}^*(R) \subseteq \operatorname{rad} K$. It is easy to see that if $p = 0$, then $\mathfrak{u}^*(S)$ is semisimple, so $\mathfrak{u}^*(R) = \operatorname{rad} K$.

Since $*$ is $\mathbb{F}$-linear, by [16, 2.1], K is also a Jordan triple system with the product $\{x,y,z\} = xyz + zyx$ for all $x,y,z \in K$ and the quadratic operator $P_a(x) = axa$. Let B be a subspace of K. We say that B is a *Lie inner ideal* of K if B is an abelian inner ideal of K. Moreover, B is said to be a *Jordan inner ideal* of K if $\{B,K,B\} \subseteq B$ [16]. We denote

$$\{b,x,b'\} := bxb' + b'xb \text{ for all } b,b' \in B \text{ and } x \in K. \tag{3.1.2}$$

If B is a subspace of $K = \mathfrak{u}^*(A)$ such that $B^2 = 0$, then B is a Lie inner ideal of K if and only if it is a Jordan inner ideal of K. Indeed, since $B^2 = 0$, we have

$$[b,[b',x]] = -(bxb' + b'xb) = -\{b,x,b'\} \text{ for all } b,b' \in B \text{ and } x \in \mathfrak{u}^*(A).$$

This justifies the following definition. Recall that $K^{(0)} = K$ and $K^{(k)} = [K^{(k-1)}, K^{(k-1)}]$ for all $k \geq 1$.

Definition 3.1.1. [24] Let A be an associative algebra with involution. An inner ideal B of $K^{(k)}$ ($k \geq 0$) is said to be *Jordan-Lie if* $B^2 = 0$.

Remark 3.1.2. In some literature, see for example [16, Section 3], Jordan-Lie inner ideals of K are called *isotropic inner ideals,* as they correspond to isotropic subspaces of algebras with involution.

The following lemma follows immediately from the definition.

Lemma 3.1.3. *Let B be a subspace of $K^{(k)}$ ($k \geq 0$) with $B^2 = 0$. Then B is a Jordan-Lie inner ideal of $K^{(k)}$ if and only if $\{b,x,b'\} \in B$ for all $b,b' \in B$ and $x \in K^{(k)}$.*

Lemma 3.1.4. *Let e be an idempotent in A with $e^*e = 0$. Then*

(i) $eKe^* = \mathfrak{u}^*(eAe^*)$.

(ii) $eKe^* \cap Z(A) = 0$.

(iii) eKe^* is a Jordan-Lie inner ideal of both K and $K^{(1)}$.

Proof. By (3.1.1),

$$\mathfrak{u}^*(eAe^*) = \{eae^* - ea^*e^* \mid a \in A\} = \{e(a-a^*)e^* \mid a \in A\} = e\,\mathfrak{u}^*(A)e^* = eKe^*.$$

(ii) Let $z \in eKe^* \cap Z(A)$. Then $ez = z$ and $ze = 0$, so $0 = [e,z] = ez - ze = z$, as required.
(iii) By (i), $eKe^* \subseteq \mathfrak{u}^*(A) = K$. Let $x,a,y \in K$. Then

$$\{exe^*, a, eye^*\} = exe^*aeye^* + eye^*aexe^* = e(xe^*aey + ye^*aex)e^* \in eKe^*.$$

Since $e^*e = 0$, we have $(eKe^*)^2 = 0$, so by Lemma 3.1.3, eKe^* is a Jordan-Lie inner ideal of K. It remains to show that $eKe^* \subseteq K^{(1)} = \mathfrak{su}^*(A)$. Let $x \in eKe^*$. Then $ex = xe^* = x$ and $xe = e^*x = 0$, so

$$[e-e^*, x] = (e-e^*)x - x(e-e^*) = ex + xe^* = 2x.$$

Note that $e - e^* \in \mathfrak{u}^*(A) = K$. Since $p \neq 2$, by using (i), we get that $x = \frac{1}{2}[e-e^*, x] \in [K, eKe^*] \subseteq [K,K]$, as required.

□

Remark 3.1.5. The results of Lemma 3.1.4 are also true when A is an associative algebra with involution $*$ over a commutative ring Φ with $\frac{1}{2} \in \Phi$ and $*$ is Φ-linear.

We will need the following well known facts, see for example [6, Lemma 4.5].

Lemma 3.1.6. *Suppose that A contains an ideal D such that $A = D \oplus D^*$. Then*
(i) $\mathfrak{u}^*(A) = \{x - x^* \mid x \in D\}$.
(ii) Let φ be the projection of A on D. Then the restriction of φ to $\mathfrak{u}^*(A)$ is an isomorphism of the Lie algebras $\mathfrak{u}^*(A)$ and $D^{(-)}$. Moreover, if P is a $*$-invariant subalgebra of D, then $\varphi(\mathfrak{u}^*(P)) = \varphi(P)^{(-)}$.

Recall that A is a finite dimensional associative algebra with involution. Suppose that A is simple. Then A can be identified with $\operatorname{End} V$ for some finite dimensional vector space V over $\mathbb{F}$. By fixing a basis E of V we can represent the algebra $\operatorname{End} V$ in the matrix form $\mathscr{M}_m$ ($m = 2n$, or $2n+1$ for some positive integer n), where $m = \dim V$. We say that $\mathscr{M}_m$ is a *matrix realization* of A. Moreover, the basis E and the matrix realization of A are called *canonical* if $*$ in the chosen basis has the following form ($\varepsilon = \pm 1$, or simply $\varepsilon = \pm$):

$$X^* = X^{\tau_\varepsilon} = J_\varepsilon^{-1} X^t J_\varepsilon \text{ for all } X \in \mathscr{M}_m, \tag{3.1.3}$$

where $J_\varepsilon = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}$ (I_n is the identity $n \times n$-matrix) if $m = 2n$ and

$$J_+ = diag(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, 1) \text{ if } m = 2n+1.$$

Note that $J_\varepsilon^{-1} = \varepsilon J_\varepsilon$. Moreover, we say that τ_+ is *orthogonal and* τ_- is *symplectic*. If $*$ admits τ_+ (resp. τ_-) in $\mathscr{M}_m$, then we say that $*$ is a *canonical involution of orthogonal (resp. symplectic) type* of A.

The following proposition is well known, see for example [3, Proposition 2.3] and [10, Lemma 2.1].

Proposition 3.1.7. *Suppose that* $\dim V = m$ *(*$m = 2n+1$ *or* $2n$*). If* $*$ *is an involution of* $A = \operatorname{End} V$*, then* A *has a canonical matrix realization* $\mathscr{M}_m$*. In particular,* $\mathfrak{u}^*(A) \cong \mathfrak{so}_m$ *or* $\mathfrak{sp}_{2n}$ *and* V *is the natural* $\mathfrak{u}^*(A)$*-module.*

Let $\mathscr{M}_m$ be a canonical matrix realization of A. To find $\mathfrak{u}^*(A)$, we need to consider two cases. Suppose first that $m = 2n$. Then $J_\varepsilon^{-1} = \varepsilon J_-$ and $\mathscr{M}_{2n}$ can be represented in the form $\mathscr{M}_{2n} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathscr{M}_n\}$. Let $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_{2n}$. Then

$$\begin{aligned} Y^* &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\tau_\varepsilon} = \varepsilon J_\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t J_\varepsilon = \begin{pmatrix} 0 & \varepsilon I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon b^t & \varepsilon d^t \\ a^t & c^t \end{pmatrix} \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix} = \begin{pmatrix} d^t & \varepsilon b^t \\ \varepsilon c^t & a^t \end{pmatrix}. \end{aligned}$$

Hence, $Y - Y^* = \begin{pmatrix} a - d^t & b - \varepsilon b^t \\ c - \varepsilon c^t & -(a - d^t)^t \end{pmatrix}$. Since $\mathfrak{u}^*(A) = \{Y - Y^* \mid Y \in \mathscr{M}_{2n}\}$ (see

(3.1.1)), we get that

$$\mathfrak{u}^*(A) = \{ \begin{pmatrix} X & X_1 \\ X_2 & -X^t \end{pmatrix} \mid X, X_i \in \mathscr{M}_n \text{ with } X_i^t = X_i, \} = \mathfrak{sp}_{2n} \text{ (if } \varepsilon = -) \tag{3.1.4}$$

and

$$\mathfrak{u}^*(A) = \{ \begin{pmatrix} X & X_1 \\ X_2 & -X^t \end{pmatrix} \mid X, X_i \in \mathscr{M}_n \text{ with } X_i^t = -X_i, \} = \mathfrak{so}_{2n} \text{ (if } \varepsilon = +). \tag{3.1.5}$$

Suppose now that $m = 2n+1$. Then $* = \tau_+$, $\quad J_+ = diag(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, 1)$, $\quad J_+^{-1} = J_+$ and $\mathscr{M}_{2n+1}$ can be represented in the form

$$\mathscr{M}_{2n+1} = \{ \begin{pmatrix} \mathscr{M}_{2n} & & x \\ & & y \\ v & w & \alpha \end{pmatrix} \mid x, y \in \mathscr{M}_{n1}, \quad v, w \in \mathscr{M}_{1n}, \quad \alpha \in \mathbb{F}\}.$$

Let $\mathscr{Y} = \begin{pmatrix} Y & & x \\ & & y \\ v & w & \alpha \end{pmatrix} \in \mathscr{M}_{2n+1}$. Then as above, we have

$$\mathscr{Y}^* = \begin{pmatrix} Y & & x \\ & & y \\ v & w & \alpha \end{pmatrix}^* = J_+ \begin{pmatrix} Y^t & & v^t \\ & & w^t \\ x^t & y^t & \alpha \end{pmatrix} J_+ = \begin{pmatrix} Y^* & & w^t \\ & & v^t \\ y^t & x^t & \alpha \end{pmatrix}.$$

Thus, $\mathscr{Y} - \mathscr{Y}^* = \begin{pmatrix} Y - Y^* & & x - w^t \\ & & y - v^t \\ -(y - v^t)^t & -(x - w^t)^t & 0 \end{pmatrix} \in \mathfrak{u}^*(A)$. Therefore,

$$\mathfrak{u}^*(A) = \{ \begin{pmatrix} \mathfrak{so}_{2n} & & Y_1 \\ & & Y_2 \\ -Y_2^t & -Y_1^t & 0 \end{pmatrix} \mid Y_1, Y_2 \in M_{n1}\} = \mathfrak{so}_{2n+1}. \tag{3.1.6}$$

3.2 Inner ideals and idempotents

Throughout this section, unless otherwise specified, V is a finite dimensional vector space over $\mathbb{F}$ and $\psi : V\times V \to \mathbb{F}$ is a nondegenerate symmetric or skew symmetric bilinear form, that is, $\psi(v,w) = \varepsilon\psi(w,v)$ for all $v,w \in V$, where $\varepsilon = \pm$.

For every $x \in \operatorname{End} V$, define $*_\psi(x)$ by the following property

$$\psi(*_\psi(x)v,w) = \psi(v,xw) \quad \text{for all } v,w \in V.$$

Then the map $*_\psi : \operatorname{End} V \to \operatorname{End} V$ is an involution of the algebra $\operatorname{End} V$, called the *adjoint involution* with respect to ψ [3]. The following fact is well known (see [22, Chapter 1, Introduction]).

Proposition 3.2.1. *[22] The map $\psi \to *_\psi$ induces a one-to-one correspondence between the equivalence classes of nondegenerate symmetric and skew-symmetric bilinear forms on V modulo multiplication by a factor in $\mathbb{F}^\times$ and involutions (of the first kind) on* $\operatorname{End} V$.

For every $v,w \in V$, we denote by $w^*v \in \operatorname{End} V$ the linear operator on V defined by

$$w^*v(x) = \psi(x,w)v \quad \text{for all} \quad x \in V. \tag{3.2.1}$$

Lemma 3.2.2. *[15, 3.3] For the linear operator $w^*v \in \operatorname{End} V$, the following hold.*

(i) $(w^*v)^* = \varepsilon v^*w$.

(ii) Every $a \in \operatorname{End} V$ can be written in the form $a = \sum_{i=1}^n w_i^*v_i$, where both the v_i's and the w_i's are linearly independent.

(iii) $(w_1^*v_1)(w_2^*v_2) = w_2^*\psi(v_2,w_1)v_1$ for all $v_1,v_2,w_1,w_2 \in V$.

(iv) The operator defined by $[u,v] := u^*v - v^*u$ belongs to $\mathfrak{u}^*(\operatorname{End} V)$.

(v) $\mathfrak{u}^*(\operatorname{End} V) = [V,V]$.

Definition 3.2.3. Let V be a vector space over $\mathbb{F}$. An idempotent e in $\operatorname{End} V$ is said to be *isotropic* if it satisfies the following equivalent conditions (i) $e^*e = 0$, (ii) eV is a totally isotropic subspace.

Remark 3.2.4. [15, 3.6] To justify the definition we need to show that (i) holds if and only if (ii) holds. By Lemma 3.2.2(ii), $e = \sum_{i=1}^n w_i^*v_i$, where the w_i's and the v_i's are linearly independent vectors in V. Since $e^2 = e$, $\psi(v_i, w_j) = \delta_{ij}$ for all i and j. Moreover, $e^*e = 0$ if and only if $\psi(v_i, v_j) = 0$ for all i and j, or equivalently, eV is a totally isotropic subspace of V (because v_i form a basis of eV).

Definition 3.2.5. [15, Definition 5.6] Let L be a Lie algebra. A subspace P of L is said to be a *point space* if $[P,P] = 0$ and $\mathrm{ad}_x^2 L = \mathbb{F}x$ for every non-zero element $x \in P$.

In [18], Draper et al showed that the classical Lie algebras of types A_n, B_{n+1}, and D_{n+1} contain point spaces of dimension n. For instance, $\bigoplus_{i=1}^n \mathbb{F}e_{i,n+1}$ is a point space of $A_n = \mathfrak{sl}_{n+1}$. However, every nonzero point space of a classical Lie algebra of type C_n is one dimensional.

Proposition 3.2.6. *[15, 5.7] Let L be a Lie algebra. Then*

(i) Every point space of L is an abelian inner ideal of L

(ii) Any subspace of a point space of L is also a point space of L.

Definition 3.2.7. [15, 5.12] Let P be a point space of the orthogonal Lie algebra $\mathfrak{so}(V, \psi)$. If there exists a nonzero vector $u \in V$ in the image of every nonzero $a \in P$, then P is called a *Type* 1 *point space*. Point spaces which are not of Type 1 are called *Type* 2 *point spaces.*

Let W be a totally isotropic subspace of V of dimension greater than 1. Suppose that u is a nonzero vector in W. Then $P = [u,W]$ is a point space of $\mathfrak{so}(V, \psi)$. The following proposition is a particular case of the results proved in [15].

Proposition 3.2.8. *[15, Proposition 5.13] Every Type* 1 *point space P of the orthogonal algebra $\mathfrak{so}(V, \psi)$ is of the form $[u,W]$, where W is a totally isotropic subspace of V of dimension greater than* 1 *and u is a nonzero vector of W. Moreover, W is uniquely determined by P and if $\dim W > 2$, then $[u,W] = [v,W]$ implies that $v = \alpha u$ for some $\alpha \in \mathbb{F}$.*

The following result classified point spaces of the orthogonal Lie algebras $\mathfrak{so}(V, \psi)$. It is a particular case of the results proved in [15].

Theorem 3.2.9. *[15, Theorem 5.16] Let $A = \operatorname{End} V$ and let $K = \mathfrak{u}^*(A)$. Suppose that P is a point space of $\mathfrak{so}(V,\psi)$. Then either P is of Type 1 or $P = eKe^*$ for some isotropic idempotent e of rank 3 and P is a point space of Type 2.*

Lemma 3.2.10. *Suppose that A is simple and $*$ is canonical of orthogonal type. Let P be a Type 1 point space of $[K,K] = \mathfrak{su}^*(A)$. Then there is a canonical matrix realization $\mathscr{M}_n$ of A and $k \leq n$ such that $P = span\{e_{1,n+t} - e_{t,n+1} \mid 1 < t \leq k\}$, where $e_{i,n+j}$ are matrix units.*

Proof. We identify A with $\operatorname{End} V$ for some finite dimensional vector space V over $\mathbb{F}$. By Proposition 3.2.8, $P = [u,W]$ for some totally isotropic subspace W of V and a nonzero vector $u \in W$. Let $\hat{W}$ be a maximal totally isotropic subspace of V containing W. Put $u = w_1$. Let $\{w_1,\dots,w_n\}$ be a basis of $\hat{W}$ such that $\{w_1,\dots,w_k\}$ ($k \geq 2$) form a basis of W. Then by Lemma 3.2.2(iv),

$$P = [u,W] = [w_1,W] = span\{[w_1,w_i] \mid 1 \leq i \leq k\} = span\{w_1^* w_i - w_i^* w_1 \mid 1 \leq i \leq k\}.$$

Fix any basis $E = \{w_1,\dots,w_n,v_1,\dots v_n,v\}$ (v omitted if $m = 2n$) of V such that $\psi(v,v) = 1$, $\psi(w_i,v_j) = \delta_{ij}$ and $\psi(w_i,w_j) = \psi(v_i,v_j) = \psi(w_i,v) = \psi(v_i,v) = 0$ for all $1 \leq i,j \leq n$. Then by using (3.2.1), we get that

$$[w_1,w_i](w_j) = w_1^* w_i(w_j) - w_i^* w_1(w_j) = \psi(w_j,w_1)w_i - \psi(w_j,w_i)w_1 = 0,$$

$$[w_1,w_i](v) = w_1^* w_i(v) - w_i^* w_1(v) = \psi(v,w_1)w_i - \psi(v,w_i)w_1 = 0 \quad \text{and}$$

$$[w_1,w_i](v_j) = w_1^* w_i(v_j) - w_i^* w_1(v_j) = \psi(v_j,w_1)w_i - \psi(v_j,w_i)w_1 = \delta_{j1}w_i - \delta_{ji}w_1.$$

Hence, $[w_1,w_i] = e_{i,n+1} - e_{1,n+i}$ in terms of matrix units e_{ij} in the chosen basis. Note that this matrix realization of A is canonical. Moreover, the space P in this basis is spanned by $\{e_{1,n+t} - e_{t,n+1} \mid 1 < t \leq k \leq n\}$, as required.

□

Definition 3.2.11. Let A be a ring with involution and let e be an idempotent in A. We say that e is a $*$-*orthogonal idempotent* if $e^*e = ee^* = 0$.

Lemma 3.2.12. *Let e be an idempotent in A with $e^*e = 0$. Then there is a $*$-orthogonal idempotent g in A such that* *(i) $ge = e$ and $eg = g$.*

(ii) $eKe^* = gKg^*$.

Proof. (i) Put $g = e - \frac{1}{2}ee^*$. Then $g^* = e^* - \frac{1}{2}ee^*$. Since $e^*e = 0$, we get that $g^2 = g$,

$$g^*g = (e^* - \frac{1}{2}ee^*)(e - \frac{1}{2}ee^*) = 0 \quad \text{and}$$

$$gg^* = (e - \frac{1}{2}ee^*)(e^* - \frac{1}{2}ee^*) = ee^* - \frac{1}{2}ee^* - \frac{1}{2}ee^* = 0.$$

Therefore, g is a $*$-orthogonal idempotent in A. It remains to note that $ge = (e - \frac{1}{2}ee^*)e = e$ and $eg = e(e - \frac{1}{2}ee^*) = g$, as required.

(ii) By (i) there is a $*$-orthogonal idempotent g in A such that $eg = g$ and $ge = e$. Hence, $e^* = e^*g^*$ and $g^* = g^*e^*$. Since $eKe^*, gKg^* \subseteq \mathfrak{u}^*(A) = K$,

$$eKe^* = geKe^*g^* \subseteq gKg^* \quad \text{and} \quad gKg^* = egKg^*e^* \subseteq eKe^*,$$

so $eKe^* = gKg^*$, as required.

□

Remark 3.2.13. The results of Lemma 3.2.12 can be applied to an associative algebra A with involution $*$ over a commutative ring Φ with $\frac{1}{2} \in \Phi$ and $*$ is Φ-linear. Moreover, they can also be applied to a semisimple Artinian ring A with involution of characteristic not 2.

By using Lemma 3.2.12(ii) and Benkart and Fernández López results [15, Theorem 6.1,6.3], we get the following result

Theorem 3.2.14. *Let A be a simple Artinian ring of characteristic $\neq 2,3$ and let $K = \mathfrak{u}^*(A)$. Suppose that $\dim A > 16$ and B is a Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. Then B satisfies one of the following:*

(i) If $*$ is canonical of symplectic type, then $B = eKe^*$ for some $*$-orthogonal idempotent e in A.

(ii) If $*$ is canonical of orthogonal type, then either $B = eKe^*$ for some $*$-orthogonal idempotent e in A or B is a Type 1 point space of dimension greater than 1.

Let e be a $*$-orthogonal idempotent in $A = \operatorname{End} V$. Since $e^*e = 0$, by Remark 3.2.4, e is isotropic and eV is a totally isotropic subspace of V. Moreover, e^* is also isotropic as $(e^*)^*e^* = ee^* = 0$. Let W be a maximal totally isotropic subspace of V containing eV. Let $\{w_1,\dots,w_n\}$ be a basis of W such that $\{w_1,\dots,w_k\}$ $(k \geq 1)$ is a basis of eV. Let $v_1,\dots,v_n$ be linearly independent vectors in V such that $\psi(w_i,v_j) = \delta_{ij}$ and $\psi(v_i,v_j) = 0$. Put $E = \{w_1,\dots w_n, v_1,\dots,v_n\}$ and $U = span(E)$. If $U \neq V$, then there is $v \in V$ such that $\psi(v,U) = 0$ and $\psi(v,v) = 1$ and $V = U \oplus span\{v\}$ (Note that such v exists because $\mathbb{F}$ is algebraically closed). Let

$$E' = \begin{cases} E & \text{, if } U = V \\ E \cup \{v\} & \text{, if } U \neq V. \end{cases}$$

Then E' is a basis of V. Note that the matrix realization $[\psi]_{E'}$ of ψ is J_ε in (3.1.3), where $J_\varepsilon = \begin{pmatrix} 0 & Id_n \\ \varepsilon Id_n & 0 \end{pmatrix}$ in the case when $U = V$ and $J = diag(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, 1)$ in the case when $U \neq V$. Moreover, the matrix realization of e and e^* with respect to E' are of the form

$$e = diag(\underbrace{1,\dots,1}_{k},0,\dots,0) \text{ and } e^* = diag(\underbrace{0,\dots,0}_{n},\underbrace{1,\dots,1}_{k},0\dots,0)\ (k \leq n) \tag{3.2.2}$$

Hence, we get the following well known result.

Lemma 3.2.15. *Suppose that* $\dim V = m$ *($m = 2n$ or $2n+1$). Let e be a $*$-orthogonal idempotents in $A = \operatorname{End} V$. Then there is a canonical matrix realization $\mathscr{M}_m$ of A such that e and e^* are in the form (3.2.2).*

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $p \neq 2$ and A is a finite dimensional associative algebra over $\mathbb{F}$ with involution.

Lemma 3.2.16. *Suppose that A is simple of dimension greater than* 16 *and $p \neq 3$. Let B be a Jordan-Lie inner ideal of $[K,K] = \mathfrak{su}^*(A)$. Then there is a canonical matrix realization $\mathscr{M}_m$ ($m = 2n+1$ or $2n$) of A and $k \leq n$ such that B is one of the following: (e_{ij} are matrix units)*

(i) If $*$ is canonical of symplectic type, then $B = span\{e_{s,n+t} + e_{t,n+s} \mid 1 \leq s \leq t \leq k \leq n\}$.

(ii) If $*$ is canonical of orthogonal type, then either (a) $B = span\{e_{1,n+t} - e_{t,n+1} \mid 1 < t \leq k \leq n\}$, or (b) $B = span\{e_{s,n+t} - e_{t,n+s} \mid 1 \leq s < t \leq k \leq n\}$.

Proof. By Theorem 3.2.14, B is either a Type 1 point space or $B = eKe^*$ for some $*$-orthogonal idempotent e in A. If B is a Type 1 point space, then by Lemma 3.2.10, B can be written in the form (ii)(a). Suppose that $B = eKe^*$. Then by Lemma 3.2.15, there is a canonical matrix realization $\mathscr{M}_m$ ($m = 2n+1$ or $2n$) of A and integer $k \leq n$ such that $e = diag(\underbrace{1,\ldots,1}_{k},0\ldots0)$ and $e^* = (\underbrace{0,\ldots0}_{n},\underbrace{1,\ldots,1}_{k},0,\ldots,0)$. Since $\mathscr{M}_m$ is canonical, $K = \mathfrak{u}^*(A) = \mathfrak{sp}_{2n}, \mathfrak{so}_m$ is of the form (3.1.4), (3.1.5) or (3.1.6). Now, simple calculations show that the space $B = eKe^*$ has the required forms as in (i) or (ii)(b).

□

3.3 Jordan-Lie inner ideals of semisimple associative algebras

Recall that A is a finite dimensional associative algebra with involution. Suppose that A is semisimple and $\{S_i \mid i \in I\}$ is the set of its simple components. Clearly, $*$ permutes the simple components of S. Therefore, for each $i \in I$ there exists a unique $i^* \in I$ such that $S_i^* = S_{i^*}$. Since $(i^*)^* = i$, the set I can be expressed as a disjoint union $I_0 \cup I_1 \cup I_1^*$, where $I_0 = \{i \in I \mid i^* = i\}$ and $I_1^* = \{i^* \mid i \in I_1\}$.

Definition 3.3.1. [10] Let A be an associative algebra with involution $*$. Then A is said to be *involution simple* if $A^2 \neq 0$ and A has no non-trivial $*$-invariant ideal.

The following proposition is known, see for example [3, Proposition 2.1].

Proposition 3.3.2. *Let A be an involution simple associative algebra. Then A is either simple as algebra or has exactly two non-zero ideals S_1 and S_2 such that both of them are simple algebras, $S_1^* = S_2$ and $A = S_1 \oplus S_2$.*

Proposition 3.3.3. *Suppose that $A = S_1 \oplus S_2$ and $K = \mathfrak{u}^*(A)$, where S_1 is a simple ideal of A with $S_1^* = S_2$. Let B be a Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. If $p \neq 3$, then the following hold.*

(i) $B=(e+f^*)K(f+e^*)$ *for some orthogonal idempotents* e *and* f *in* S_1.

(ii) $B=gKg^*$ *for some* $*$*-orthogonal idempotent* g *in* A.

Proof. (i) Let $\varphi : A \to S$, $\varphi(s_1+s_2)=s_1$ for all $s_1 \in S_1$ and $s_2 \in S_2$, be the projection of A onto S_1. Then by Lemma 3.1.6, the restriction of φ to $\mathfrak{u}^*(A)$ is an isomorphism of the Lie algebras $\mathfrak{u}^*(A)$ and $S_1^{(-)}$, so $\varphi(\mathfrak{u}^*(A))={S_1}^{(-)}$. Hence,

$$\varphi(\mathfrak{su}^*(A))=\varphi([\mathfrak{u}^*(A),\mathfrak{u}^*(A)])=[\varphi(\mathfrak{u}^*(A)),\varphi(\mathfrak{u}^*(A))]=[S_1^{(-)},S_1^{(-)}]=S_1^{(1)}.$$

Note that the restriction of φ to $\mathfrak{u}^*(A)$ is defined by $\varphi(s-s^*)=s$ for all $s\in S_1$ as $\mathfrak{u}^*(A)=\{s-s^*\mid s\in S_1\}$ (see Lemma 3.1.6(i)). In particular, the map $\varphi^{-1}: S_1^{(-)}\to\mathfrak{u}^*(A)$ is given by $\varphi^{-1}(s)=s-s^*$ for all $s\in S_1$. Since $\varphi(B)$ is a Jordan-Lie inner ideal of $\varphi(\mathfrak{su}^*(A))=S_1^{(1)}$, by Theorem 2.3.2, $\varphi(B)=eS_1f$ for some strict orthogonal idempotent pair (e,f) in S_1. Thus,

$$\begin{aligned} B &= \varphi^{-1}(eS_1f)=\{esf-f^*s^*e^*\mid s\in S_1\}\\ &= \{(e-f^*)(s-s^*)(f-e^*)\mid s\in S_1\}=(e+f^*)K(f+e^*), \end{aligned}$$

as required.

(ii) This follows from (i) by putting $g=e+f^*\in A$.

□

Benkart [13, Theorem 5.5] and Benkart and Fernández López [15, Theorem 6.1, 6.3] classified Lie inner ideals of simple finite dimensional associative algebras with involution. The following theorem is a slight generalization of their results to the case of involution simple algebras.

Theorem 3.3.4. *Suppose that* A *is an involution simple such that each simple ideal of* A *is of dimension greater than* 16. *Let* B *be a Jordan-Lie inner ideal of* $K=\mathfrak{su}^*(A)$. *If* $p\neq 3$, *then* B *satisfies one of the following:*

(i) $B=eKe^*$ *for some* $*$*-orthogonal idempotent* e *of* A, *or*

(ii) B *is a Type* 1 *point space of dimension greater than* 1.

Proof. This from Theorem 3.2.14(ii) and Proposition 3.3.3(ii).

□

Let A be an involution simple finite dimensional associative algebra. By Proposition 3.3.2, we can identify A with either $\operatorname{End} V$ or $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ for some finite dimensional vector spaces V, V_1 and V_2 with $\dim V_1 = \dim V_2$. Put $\dim V_1 = \dim V_2 = m$. Then the algebra $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ can be represented in the matrix form $\mathscr{M}_m \oplus \mathscr{M}_m$ with respect to fixed bases in V_1 and V_2. We say that $\mathscr{M}_m \oplus \mathscr{M}_m$ is *the matrix realization* of $\operatorname{End} V_1 \oplus \operatorname{End} V_2$. Moreover, the matrix realization of the algebra $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ with involution $*$ is said to be *canonical* if $*$ in the chosen basis is of the form: $(X_1, X_2 \in \mathscr{M}_m)$

$$(X_1, X_2) \mapsto (X_2^t, X_1^t), \tag{3.3.1}$$

where t is the transpose. It is known that any finite dimensional involution simple associative algebra over an algebraically closed field of characteristic not 2 or 3 has a canonical matrix realization, see for example [3].

The following classical result describes the structure of involution simple algebras which are not simple as algebras:

Proposition 3.3.5. *[3, Proposition 2.5] Let V_i, $i = 1, 2$, be vector spaces of dimension m. Put $S_i = \operatorname{End}(V_i)$. Let $*$ be an involution of the algebra $S_1 \oplus S_2$ such that $S_1^* = S_2$. Then for every matrix realization of S_1 there is a matrix realization of S_2 such that the corresponding matrix realization of $S_1 \oplus S_2$ is canonical. In particular, $\mathfrak{u}^*(S_1 \oplus S_2) = \{(X, -X^t) \mid X \in \mathscr{M}_m\} \cong \mathfrak{gl}_m$, V_1 is the natural $\mathfrak{u}^*(S_1 \oplus S_2)$-module and V_2 is the module dual to V_1.*

We will need the following fact.

Lemma 3.3.6. *Suppose that $A = S_1 \oplus S_2$ where S_1 is a simple ideal of A with $S_1^* = S_2$. Let B be a Jordan-Lie inner ideal of $K^{(1)}$. If $p \neq 3$, then there is a matrix realization of A such that B is the space spanned by $\{(e_{st}, -e_{ts}) \mid 1 \leq s \leq k < l \leq t \leq n\}$, where $e_{ij} \in \mathscr{M}_m$ are matrix units.*

Proof. By Proposition 3.3.3 (i) and (ii), we have $B = (e + f^*)K(f + e^*)$, where e and f are orthogonal idempotents in A and $(e + f^*)$ is a $*$-orthogonal idempotent in A. Therefore, the result follows from Proposition 3.3.5 and Lemma 2.3.3.

□

Recall that $\{S_i \mid i \in I\}$ is the set of the simple components of S, where I can be expressed as a disjoint union $I_0 \cup I_1 \cup I_1^*$ with $I_0 = \{i \in I \mid i^* = i\}$ and $I_1^* = \{i^* \mid i \in I_1\}$. Let n_i be an integer such that $S_i \cong \mathscr{M}_{n_i}$. By Propositions 3.1.7 and 3.3.5, we have the following (see also [10]).

Lemma 3.3.7. *Suppose that A is semisimple and $\{S_i \mid i \in I\}$ is the set of its simple components. Then $\mathfrak{u}^*(S) = \bigoplus_{i \in I_0 \cup I_1} Q_i$, where*

$$Q_i = \begin{cases} \mathfrak{u}^*(S_i) \cong \mathfrak{so}_{n_i}, \mathfrak{sp}_{n_i}, & \text{if } i \in I_0 \\ \mathfrak{u}^*(S_i \oplus S_{i^*}) \cong \mathfrak{gl}_{n_i}, & \text{if } i \in I_1 \end{cases}$$

Recall Definition 2.1.1 that a Lie algebra Q is said to be a *quasi (semi)simple* if Q is perfect (i.e. $[Q,Q] = Q$) and $Q/Z(Q)$ is (semi)simple. The following result is a particular case of [21, Theorem 10] and [30, Theorem 6.1].

Proposition 3.3.8. *Suppose that A is semisimple. Then $K^{(1)} = \mathfrak{su}^*(A)$ is quasi semisimple. In particular $K^{(1)} = K^{(\infty)}$.*

Proof. By Lemma 3.3.7, $K = \bigoplus_{i \in I_0 \cup I_1} Q_i$, where Q_i is either 0 or $\mathbb{F}$ in the case when $n_i = 1$ or Q_i is isomorphic to one of the Lie algebras $\mathfrak{so}_{n_i}$, $\mathfrak{sp}_{n_i}$ and $\mathfrak{gl}_{n_i}$ in the case when $n_i \geq 2$. Therefore, $Q_i^{(1)} = [Q_i, Q_i]$ is quasi simple for all i. Therefore, $Q = \bigoplus_{i \in I_0 \cup I_1} [Q_i, Q_i]$ is a quasi semisimple Lie algebra.

□

Recall that an element x in A is said to be *Von Neumann regular* if there is an element $y \in A$ such that $x = xyx$ [19]. Moreover, A is said to be *Von Neumann regular* if every element of A is von Neumann regular. Recall that semisimple Artinian rings are von Neumann regular.

Lemma 3.3.9. *Let A be a semisimple Artinian ring with involution of characteristic not 2. Let $x \in \mathfrak{u}^*(A)$. The following hold.*

(i) $x = xyx$ *for some* $y \in \mathfrak{u}^*(A)$.

(ii) Suppose that $x^2 = 0$. Then

(a) $x \in \mathfrak{su}^*(A)$.

(b) *there is* $y' \in \mathfrak{su}^*(A)$ *such that* $x = xy'x$.

Proof. (i) We have $x^* = -x$. Since A is Von Neumann regular, $x = xax$ for some $a \in A$. Put $y = \frac{1}{2}(a - a^*) \in \mathfrak{u}^*(A)$. Then

$$xyx = \frac{1}{2}x(a-a^*)x = \frac{1}{2}(xax - xa^*x) = \frac{1}{2}(x - (xax)^*) = \frac{1}{2}(x - x^*) = \frac{1}{2}(2x) = x.$$

(ii) By (i), $x = xyx$ for some $y \in \mathfrak{u}^*(A)$.

(a) Since $x^2 = 0$, we get that $[x,[x,y] = -2xyx = -2x$, so

$$x = -\frac{1}{2}[x,[x,y]] \in [\mathfrak{u}^*(A), \mathfrak{su}^*(A)] \subseteq \mathfrak{su}^*(A).$$

(b) Let $e = xy$. Then $e^2 = xyxy = xy = e$ and $e^* = yx$, so e is an idempotent in A with $e^*e = yxxy = 0$. By Remark 3.2.13, there is a $*$-orthogonal idempotent g in A such that $ge = e$ and $eg = g$, so $e^*g^* = (ge)^* = e^*$. Put $y' = g^*yg \in \mathfrak{u}^*(A)$. Since $exe^* = (xy)x(yx) = xyx = x$,

$$xy'x = (exe^*)(g^*yg)(exe^*) = (exe^*g^*)y(gexe^*) = (exe^*)y(exe^*) = xyx = x.$$

It remains to note that $(y')^2 = (g^*yg)(g^*yg) = 0$, so by (a), $y' \in \mathfrak{su}^*(A)$, as required.

□

Proposition 3.3.10. *Let A be a semisimple Artinian ring with involution of characteristic not 2. Let B be a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$. Then $B = [B,[B,\mathfrak{su}^*(A)]]$.*

Proof. By definition, $[B,[B,\mathfrak{su}^*(A)]] \subseteq B$. Let $b \in \mathfrak{su}^*(A)$. Since $b^2 = 0$, by Lemma 3.3.9 (ii)(b), $b = byb$ for some $y \in \mathfrak{su}^*(A)$. Therefore, $b = byb = -\frac{1}{2}[b,[b,y]] \in [B,[B,\mathfrak{su}^*(A)]]$, as required.

□

Proposition 3.3.11. *Let A be a semisimple Artinian ring with involution of characteristic not 2. Let B be a Jordan-Lie inner ideal of $\mathfrak{u}^*(A)$. Then*

(i) B is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$.

(ii) $B = [B, [B, \mathfrak{su}^*(A)]$.
(iii) $B = [B, [B, \mathfrak{u}^*(A)]]$.

Proof. (i) By Lemma 3.3.9 (ii)(a), $B \subseteq \mathfrak{su}^*(A)$, so B is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$.

(ii) This follows from (i) and Proposition 3.3.10.

(iii) By definition, $[B, [B, \mathfrak{u}^*(A)]] \subseteq B$. On the other hand, by (ii), $B = [B, [B, \mathfrak{su}^*(A)]] \subseteq [B, [B, \mathfrak{u}^*(A)]]$

□

Recall that $p \neq 2$.

Lemma 3.3.12. *Suppose that A is a semisimple, $\{S_i \mid i \in I\}$ is the set of the involution simple components of A and $p \neq 3$. Let B be a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$. Then $B = \bigoplus_{i \in I} B_i$, where $B_i = B \cap \mathfrak{su}^*(S_i)$.*

Proof. Let $\psi_i : \mathfrak{su}^*(A) \to \mathfrak{su}^*(S_i)$, $\psi_i((x_1, \ldots, x_i, \ldots) = x_i$, be the projection of $\mathfrak{su}^*(A)$ onto $\mathfrak{su}^*(S_i)$. Then $\psi_i(B)$ is a Jordan-Lie inner ideal of $\psi_i(\mathfrak{su}^*(A)) = \mathfrak{su}^*(S_i)$ for each $i \in I$. We need to show that $\psi_i(B) = B_i$ for all $i \in I$. We have $B_i \subseteq \psi_i(B)$. Since S_i is semisimple with involution (because S_i is involution simple), by Proposition 3.3.10,

$$\psi_i(B) = [\psi_i(B), [\psi_i(B), \mathfrak{su}^*(S_i)]] \subseteq [B, [B, \mathfrak{su}^*(S_i)]] \subseteq B_i.$$

Thus, $\psi_i(B) = B_i$ for each $i \in I$. Therefore, $B = \bigoplus_{i \in I} B_i$.

□

Theorem 3.3.13. *Suppose that A is semisimple, $p \neq 3$ and every simple component of A is of dimension greater than 16. Let $K = \mathfrak{u}^*(A)$ and let B be a Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. Then $B = eKe^* \oplus C$, where e is a $*$-orthogonal idempotent in A and C is a direct sum of Type 1 point spaces of dimensions greater than 1.*

Proof. Let $\{S_i \mid i \in I\}$ be the set of the involution simple components of A. Using Theorem 3.3.4 and Lemma 3.3.12 we get that $B = \bigoplus_{i \in I} B_i$, where B_i is either a Type 1 point space or $B_i = e_i K_i e_i^*$ for some $*$-orthogonal idempotents e_i in A_i. Put $e = \sum_{j \in J} e_j$, where $J = \{i \in I \mid B_i = e_i K e_i^*\}$. Then e is a $*$-orthogonal and $eKe^* = \bigoplus_{j \in J} e_j K_j e_j^*$. Put $C = \bigoplus_{i \notin J} B_i$. Then C is a direct sum of Type 1 point spaces and $B = eKe^* \oplus C$, as required.

□

3.4 $*$-indecomposable associative algebras

We start with the following definition.

Definition 3.4.1. (1) Let A be an associative algebra with involution. We say that A is a $*$-*indecomposable* if A cannot be represented as a direct sum of two $*$-invariant ideals.

(2) Let M be a non-zero S-S-submodule of A. We say that M is $*$-irreducible if M does not contain proper non-zero $*$-invariant submodules.

Let S be a semisimple finite dimensional associative algebra and let $\{S_i \mid i \in I\}$ be the set of its simple components. Let $\hat{S} = S + \mathbb{F}1_{\hat{S}}$, where $1_{\hat{S}}$ is the identity of $\hat{S}$. Then $\hat{S}$ is a unital algebra. Let M be an S-bimodule. Set $1_{\hat{S}}m = m1_{\hat{S}} = m$ for all $m \in M$. Then M is a unital $\hat{S}$-bimodule. Put $\hat{I} = I \cup \{0\}$. Then $\hat{S} = \bigoplus_{i\in\hat{I}} S_i$, where $S_0 = \mathbb{F}(1_{\hat{S}} - 1_S)$ is a 1-dimensional simple component of $\hat{S}$. Thus, as a unital $\hat{S}$-bimodule, M is a direct sum of copies of the natural simple S_i- S_j-bimodules $U_{ij} \cong V_i \oplus V_j^*$, where V_i is the natural left S_i-module and V_{j^*} is the natural right S_j-module. Note that $U_{i0}S = SU_{0j} = 0$. Recall the following lemma from Chapter 2 (Lemma 2.6.9).

Lemma 3.4.2. *Let S be a semisimple finite dimensional associative algebra and let $\{S_i \mid i \in I\}$ be the set of its simple components. Suppose that M is an S-bimodule. Then M is a direct sum of copies of U_{ij}, for $i, j \in \hat{I} = I \cup \{0\}$, where U_{00} is the trivial 1-dimensional S-bimodule, U_{i0} is the natural left S_i-module with $U_{i0}S = 0$, U_{0j} is the natural right S-module with $SU_{0j} = 0$ and U_{ij} is the natural S_i- S_j-bimodule for $i, j \neq 0$.*

Let M be as in Lemma 3.4.2. Then $M \cong \bigoplus_{i,j\in\hat{I}} U_{ij} \otimes \Lambda(i, j)$, where the $\Lambda(i, j)$ are vector spaces over $\mathbb{F}$. Suppose now that S is an algebra with involution $*$. Then $*$ permutes the simple components of S, so for each $i \in I$, there is $i^* \in I$ such that $S_i^* = S_{i^*}$. Note that $S_0^* = S_0$. Put $1_{\hat{S}} = \sum_{i\in\hat{I}} 1_i$, where 1_i is the identity of S_i for all $i \in \hat{I}$. Then for each $x \in M$, we have $x = 1_{\hat{S}}x1_{\hat{S}} = \sum_{i,j\in\hat{I}} 1_i x 1_j$, so $M = \bigoplus_{i,j\in\hat{I}} 1_i M 1_j$. Put $M_{ij} = 1_i M 1_j$ for all i and j. Then obviously we have $M_{ij} = \bigoplus_{i,j\in\hat{I}} 1_i M 1_j = U_{ij} \otimes \Lambda(i, j)$ for all $i, j \in \hat{I}$.

Suppose now that S is a $*$-invariant Levi subalgebra of A and $M = R = \operatorname{rad} A$. Then by above we have

$$R = \bigoplus_{i,j\in\hat{I}} R_{ij} = \bigoplus_{i,j\in\hat{I}} U_{ij} \otimes \Lambda(i, j), \tag{3.4.1}$$

where $R_{ij} = 1_i R 1_j$. Since $R^* = R$, we have

$$R_{ij}^* = (1_i R 1_j)^* = 1_j^* R 1_i^* = 1_{j^*} R 1_{i^*} = R_{j^* i^*}. \tag{3.4.2}$$

Suppose that $A = S \oplus R$ is $*$-indecomposable, $R^2 = 0$ and R is $*$-irreducible as S-bimodule. Then R is isomorphic to either $U_{ij} \oplus U_{j^*i^*}$ (if $i \neq j^*$) or U_{ij} (if $j^* = i$) for some $i, j, i^*, j^* \in \hat{I}$ with $(i, j) \neq (0,0)$. Since A is $*$-indecomposable, I contains only i, j and their duals. Recall that $\hat{I} = \{0\} \cup I_0 \cup (I_1 \cup I_1^*)$. We have the following cases.

1. If $j = 0$ and $i \neq 0$:
   (a) $i \in I_0$. Then $S = S_i$ and $R \cong U_{i0} \oplus U_{0i}$.
   (b) $i \in I_1 \cup I_1^*$. Then $S = S_i \oplus S_{i^*}$ and $R \cong U_{i0} \oplus U_{0i^*}$.
2. $i, j \in I_0$:
   (a) $i \neq j$. Then $S = S_i \oplus S_j$ and $R \cong U_{ij} \oplus U_{ji}$.
   (b) $i = j$. Then $S = S_i$ and $R \cong U_{ii}$.
3. $i \in I_0$ and $j \in I_1 \cup I_1^*$. Then $S = S_i \oplus S_j \oplus S_{j^*}$ and $R \cong U_{ij} \oplus U_{j^*i}$.
4. $i, j \in I_1 \cup I_1^*$:
   (a) $i = j$. Then $S = S_i \oplus S_{i^*}$ and $R \cong U_{ii} + U_{i^*i^*}$.
   (b) $i = j^*$. Then $S = S_i \oplus S_{i^*}$ and $R \cong U_{ii^*}$.
   (c) $i \neq j, j^*$. Then $S = S_i \oplus S_j \oplus S_{i^*} \oplus S_{j^*}$ and $R \cong U_{ij} \oplus U_{j^*i^*}$.

Note that if U is an irreducible S-submodule of R isomorphic to U_{ij} (so $U \subseteq R_{ij}$), then U^* is an irreducible submodule of $R_{j^*i^*}$ (because $U^* \subseteq R_{ij}^* = R_{j^*i^*}$). We proved the following proposition.

Proposition 3.4.3. *Suppose that A is $*$-indecomposable, $R^2 = 0$ and R is $*$-irreducible as S-bimodule. Then A has one of the following decompositions.*

(i) *S is involution simple and U is a natural left S-module with $US = 0$:*

(a) $A = S \oplus U \oplus U^*$ *where S is simple.*

(b) $A = S_1 \oplus S_1^* \oplus U \oplus U^*$ *where S_1 is simple and $S = S_1 \oplus S_1^*$.*

(ii) *S is involution simple and U is an irreducible S-bimodule:*

(a) $A = S \oplus R$ *where S is simple and $R = U$.*

(b) $A = S_1 \oplus S_1^* \oplus U \oplus U^*$ *where S_1 is simple, $S = S_1 \oplus S_1^*$ and $R = U \oplus U^*$ with $US_1 = S_1^* U = 0$.*

(c) $A = S_1 \oplus S_1^* \oplus R$ *where S_1 is simple, $S = S_1 \oplus S_1^*$ and R is an irreducible S_1-S_1^*-bimodule with $RS_1 = S_1^* R = 0$.*

(iii) *$S = S' \oplus S''$ where S' and S'' are involution simple ideals of S and U is an irreducible S'-S''-bimodule with $US' = S''U = 0$.*

(a) $A = S' \oplus S'' \oplus U \oplus U^*$ *where S', S'' are simple.*

(b) $A = S' \oplus S_2 \oplus S_2^* \oplus U \oplus U^*$ *where S' is simple and S_2 is a simple ideal of S'' with $S'' = S_2 \oplus S_2^*$.*

(c) $A = S_1 \oplus S_1^* \oplus S_2 \oplus S_2^* \oplus U \oplus U^*$ *where S_1 and S_2 are simple ideals of S' and S'', respectively, with $S' = S_1 \oplus S_1^*$ and $S'' = S_2 \oplus S_2^*$.*

From Definition 3.4.1(2), Lemma 3.4.2 and Proposition 3.4.3, we have the following lemma

Lemma 3.4.4. *Let S be a semisimple finite dimensional associative algebra with involution and let $\{S_i \mid i \in I\}$ be the set of its simple components. Suppose that M is an S-bimodule. Then M is a direct sum of copies of $*$-irreducible S-S-bimodules, each of them is either irreducible S_i-S_{i^*}-bimodule or isomorphic to $U \oplus U^*$, where U is either a natural left S_i-module or an irreducible S_i-S_j-bimodule with $j \neq i^*$.*

The following results describe the structure of the Lie algebra $\mathfrak{u}^*(A)$ when A is $*$-indecomposable with $R^2 = 0$.

Lemma 3.4.5. *Suppose that S is simple and $R = U_1 \oplus U_1^*$, where U_1 is a natural left S-module with $U_1 A = 0$. Then $\mathfrak{u}^*(S) \cong \mathfrak{so}_m, \mathfrak{sp}_{2n}$ ($m = 2n$ or $2n+1$) and $\mathfrak{u}^*(R) = \{r - r^* \mid r \in U\} \cong U$.*

Proof. This follows from (3.1.1) and Proposition 3.1.7.

□

Lemma 3.4.6. *Suppose that S is simple and $R = U$, where U is an irreducible S-bimodule with $U^* = U$. Then there is a canonical matrix realization of A such that $\mathfrak{u}^*(S) = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_m$ and as a vector space $\mathfrak{u}^*(R)$ is one of the following ($\rho = \pm 1$ or simply $\pm$):*

$$\mathrm{sym}^{\rho}_{\tau_\varepsilon}(\mathscr{M}_{2n}) := \{ \begin{pmatrix} X & Y_1 \\ Y_2 & \rho X^t \end{pmatrix} \mid X, Y_1, Y_2 \in \mathscr{M}_n, \quad Y_1^t = \rho\varepsilon Y_1, \quad Y_2^t = \rho\varepsilon Y_2 \};$$

$$\mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n+1}) := \{ \begin{pmatrix} \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n}) & & \begin{matrix} Y_3 \\ Y_4 \end{matrix} \\ \rho Y_4^t & \rho Y_3^t & \alpha \end{pmatrix} \mid Y_3, Y_4 \in \mathscr{M}_{n1}, \quad \alpha \in \mathbb{F} \},$$

where $\alpha = 0$ if $\rho = -$.

Note that as vector spaces $\mathrm{sym}^{-}_{\tau_-}(\mathscr{M}_{2n}) = \mathfrak{sp}_{2n}$, $\mathrm{sym}^{-}_{\tau_+}(\mathscr{M}_{2n}) = \mathfrak{so}_{2n}$ and $\mathrm{sym}^{-}_{\tau_+}(\mathscr{M}_{2n+1}) = \mathfrak{so}_{2n+1}$ (see (3.1.4), (3.1.5) and (3.1.6), respectively)

Proof. Since S is simple with $S^* = S$, by (3.1.3) and Proposition 3.1.7, there is a canonical matrix realization $\mathscr{M}_m$ ($m = 2n+1$, or $2n$) of S such that $* = \tau_\varepsilon$ ($\varepsilon = \pm$) and $\mathfrak{u}^*(S) \cong \mathfrak{sp}_{2n}$ or $\mathfrak{so}_m$. Since U is an irreducible S-bimodule, as a vector space $R = U \cong \mathscr{M}_m$. By identifying R with $\mathscr{M}_m$, we can fix bases $\{e_{ij} \mid 1 \le i, j \le m\}$ and $\{f_{ij} \mid 1 \le i, j \le m\}$ of S and R, respectively, consisting of matrix units such that the action of S on R corresponds to matrix multiplication. As $U^* = U$, by (3.1.1), $f_{ij} - f_{ij}^* \in \mathfrak{u}^*(U)$ for each i and j, so we need to find f_{ij}^*.

Suppose first that $m = 2n$. Then $* = \tau_\varepsilon$ and $\mathfrak{u}^*(S) \cong \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, so by (3.1.4) and (3.1.5), we have $e_{st}^* = e_{n+t,n+s}$, $e_{s,n+t}^* = \varepsilon e_{t,n+s}$ and $e_{n+s,t}^* = \varepsilon e_{n+t,s}$ for all $1 \le s \le t \le n$.

Hence, $f_{11}^* = (e_{11}f_{11}e_{11})^* = e_{11}^*f_{11}^*e_{11}^* = e_{n+1,n+1}f_{11}^*e_{n+1,n+1} \in \mathbb{F}f_{n+1,n+1}$. Thus, there is a non-zero $\alpha \in \mathbb{F}$ such that $f_{11}^* = \alpha f_{n+1,n+1}$. Since $(f_{11}^*)^* = f_{11}$ and $*$ is $\mathbb{F}$-linear, we get that

$$\begin{aligned} f_{11} &= (f_{11}^*)^* = (\alpha f_{n+1,n+1})^* = (\alpha e_{n+1,1}f_{11}e_{1,n+1})^* = \alpha e_{1,n+1}^* f_{11}^* e_{n+1,1}^* \\ &= \alpha^2 e_{1,n+1}f_{n+1,n+1}e_{n+1,1} = \alpha^2 f_{11}, \end{aligned}$$

so $\alpha = \pm 1$. Put $\rho = -\alpha$. Then $f_{11}^* = -\rho f_{n+1,n+1}$ ($\rho = \pm 1$ or simply $\pm$). Now, for each $1 \le s \le t \le n$, we have

$$f_{st}^* = (e_{s1}f_{11}e_{1t})^* = e_{1t}^*f_{11}^*e_{s1}^* = e_{n+t,n+1}(-\rho f_{n+1,n+1})e_{n+1,n+s} = -\rho f_{n+t,n+s}; \quad (3.4.3)$$

$$f_{s,n+t}^* = (e_{s1}f_{11}e_{1,n+t})^* = e_{1,n+t}^*f_{11}^*e_{s1}^* = (\varepsilon e_{t,n+1})(-\rho f_{n+1,n+1})e_{n+1,n+s} = -\rho\varepsilon f_{t,n+s}; \quad (3.4.4)$$

$$f_{n+s,t}^* = (e_{n+s,1}f_{11}e_{1t})^* = e_{1t}^*f_{11}^*e_{n+s,1}^* = e_{n+t,n+1}(-\rho f_{n+1,n+1})(\varepsilon e_{n+1,s}) = -\rho\varepsilon f_{n+t,s}. \quad (3.4.5)$$

Hence, $f_{st} + \rho f_{n+t,n+s}$, $f_{s,n+t} + \rho\varepsilon f_{t,n+s}$ and $f_{n+s,t} + \rho\varepsilon f_{n+t,s}$ belong to $\mathfrak{u}^*(R)$ for all $1 \le s \le t \le n$, so

$$\mathfrak{u}^*(R) = \{ \begin{pmatrix} X & Y_1 \\ Y_2 & \rho X^t \end{pmatrix} \mid X, Y_i \in \mathscr{M}_n \quad \text{with} \quad Y_i^t = \rho\varepsilon Y_i \} = \mathrm{sym}_{\tau_\varepsilon}^{\rho}(\mathscr{M}_{2n}).$$

Suppose now that $m = 2n+1$. Then $* = \tau_+$ and $\mathfrak{u}^*(S) \cong \mathfrak{so}_{2n+1}$, so by (3.1.6), $e_{st}^* = e_{n+t,n+s}$, $e_{s,n+t}^* = e_{t,n+s}$, $e_{n+s,t}^* = e_{n+t,s}$, $e_{sm}^* = e_{m,n+s}$, $e_{n+s,m}^* = e_{ms}$ and $e_{mm}^* = e_{mm}$. By above, we have $f_{11}^* = -\rho f_{n+1,n+1}$, so by using the same technique as in (3.4.3), (3.4.4) and (3.4.5), we get that $f_{st}^* = -\rho f_{n+t,n+s}$, $f_{s,n+t}^* = -\rho f_{t,n+s}$ and $f_{n+s,t}^* = -\rho f_{n+t,s}$, respectively. It remains to find f_{sm}^*, $f_{n+s,m}^*$ and f_{mm}^*. We have

$$f_{sm}^* = (e_{s1}f_{11}e_{1m})^* = e_{1m}^*f_{11}^*e_{1s}^* = -\rho e_{m,n+1}f_{n+1,n+1}e_{n+1,n+s} = -\rho f_{m,n+s};$$

$$f_{n+s,m}^* = (e_{n+s,1}f_{11}e_{1m})^* = e_{1m}^*f_{11}^*e_{n+s,1}^* = -\rho e_{m,n+1}f_{n+1,n+1}e_{n+1,s} = -\rho f_{ms};$$

$$f_{mm}^* = (e_{m1}f_{11}e_{1m})^* = e_{1m}^*f_{11}^*e_{m1}^* = e_{m,n+1}f_{n+1,n+1}e_{n+1,m} = -\rho f_{mm}.$$

Hence, $f_{st} + \rho f_{n+t,n+s}$, $f_{s,n+t} + \rho f_{t,n+s}$, $f_{n+s,t} + \rho f_{n+t,s}$, $f_{sm} + \rho f_{m,n+s}$, $f_{n+s,m} + \rho f_{ms}$

and $f_{mm} + \rho f_{mm}$ belong to $\mathfrak{u}^*(R)$ for all $1 \le s \le t \le n$. Thus

$$\mathfrak{u}^*(R) = \{ \begin{pmatrix} \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n}) & & Y_3 \\ & & Y_4 \\ \rho Y_4^t & \rho Y_3^t & \alpha + \rho\alpha \end{pmatrix} \mid Y_3, Y_4 \in \mathscr{M}_{n1}, \quad \alpha \in \mathbb{F} \} = \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n+1}).$$

□

Lemma 3.4.7. *Suppose that $S = S_1 \oplus S_1^*$, where S_1 is simple, and R is an irreducible S_1-S_1^*-bimodule with $RS_1 = S_1^* R = 0$. Then $\mathfrak{u}^*(R)$ is isomorphic to*

$$\mathrm{sym}^{\rho}(\mathscr{M}_n) := \{X \in \mathscr{M}_n \mid X^t = \rho X\} \quad (\rho = \pm).$$

Proof. Since $S = S_1 \oplus S_1^*$, by Proposition 3.3.5, there is a canonical matrix realization $\mathscr{M}_n \oplus \mathscr{M}_n$ of S such that $\mathfrak{u}^*(S) = \{(X, -X^t) \mid \in X \in \mathscr{M}_n\}$. By identifying R with $\mathscr{M}_n$, we can fix a standard bases $\{e_{ij} \mid 1 \le i, j \le n\}$, $\{g_{ij} \mid 1 \le i, j \le n\}$ and $\{f_{ij} \mid 1 \le i, j \le n\}$ of S_1, S_1^* and R, respectively, consisting of matrix units such that the action of S_1 and of S_1^* on R correspond to the matrix multiplication and $e_{ij}^* = g_{ij}^t = g_{ji}$. Then $e_{ij} - g_{ji} \in \mathfrak{u}^*(S)$ for each i and j. Since $R^* = R$, by (3.1.1), $f_{ij} - f_{ij}^* \in \mathfrak{u}^*(U)$ for all i and j. We need to fine f_{ij}^*. We have $f_{11}^* = (e_{11} f_{11} g_{11})^* = g_{11}^* f_{11}^* e_{11}^* = e_{11} f_{11}^* g_{11} \in \mathbb{F} f_{11}$, so there is a non-zero $\alpha \in \mathbb{F}$ such that $f_{11}^* = \alpha f_{11}$. Since $f_{11} = (f_{11}^*)^* = (\alpha f_{11})^* = \alpha^2 f_{11}$, we get that $\alpha = \pm 1$. Thus $f_{11}^* = \rho f_{11}$ for some $\rho = \pm$. Therefore,

$$f_{ij}^* = (e_{i1} f_{11} g_{1j})^* = g_{1j}^* f_{11}^* e_{i1}^* = e_{j1}(\rho f_{11}) g_{1i} = \rho f_{ji} \quad \text{for all} \quad 1 \le i, j \le n.$$

Hence, $f_{ij} - \rho f_{ji} = f_{ij} - f_{ij}^* \in \mathfrak{u}^*(R)$. This implies $\mathfrak{u}^*(R) \cong \mathrm{sym}^{\rho}(\mathscr{M}_n)$, as required.

□

Lemma 3.4.8. *Suppose that $S = S_1 \oplus S_2$ and $R = U \oplus U^*$, where S_i is simple with $S_i^* = S_i$ for each $i = 1, 2$ and U is an irreducible S_1-S_2-bimodule with $S_2 U = U S_1 = 0$. Then $\mathfrak{u}^*(S_i) \cong \mathfrak{so}_{m_i}, \mathfrak{sp}_{2n_i}$ for each $i = 1, 2$ and $\mathfrak{u}^*(R) = \{r - r^* \mid r \in U \cong \mathscr{M}_{m_1 m_2}\}$.*

Proof. This follows from (3.1.1) and Proposition 3.1.7.

□

Lemma 3.4.9. *Suppose that $S = S_1 \oplus S_1^* \oplus S_2$ and $R = U \oplus U^*$, where S_1 and S_2 are simple, $S_2^* = S_2$ and U is an irreducible S_1-S_2-bimodule with $US_1 = S_2U = 0$. Then $\mathfrak{u}^*(S_1 \oplus S_1^*) = \{s - s^* \mid s \in S_1 \cong \mathscr{M}_{m_1}\}$, $\mathfrak{u}^*(S_2) \cong \mathfrak{so}_{m_2}, \mathfrak{sp}_{2n_2}$ and $\mathfrak{u}^*(R) = \{r - r^* \mid r \in U \cong \mathscr{M}_{m_1 m_2}\}$.*

Proof. This follows from from (3.1.1), Proposition 3.1.7 and Proposition 3.3.5. □

3.5 Admissible algebras

Recall that a Lie algebra L is called *perfect* if $[L, L] = L$. Similarly, we say that an associative algebra A is *perfect* if $AA = A$. If P is an ideal of A. Then we say that P is perfect if $PP = P$.

Definition 3.5.1. [10] Let A be a finite dimensional associative algebra with involution. Then A is said to be *admissible* if A is perfect and for each maximal $*$-invariant ideal M of A one of the following holds (below $d = \sqrt{\dim A/M}$ and $*_M$ denote the involution of A/M induced by $*$):

(1) A/M is not simple and $d' = d/\sqrt{2} \geq 4$; i.e., $\mathfrak{u}^*(A/M) \cong \mathfrak{gl}_{d'}$.

(2) A/M is simple, $d \geq 6$ and $*_M$ is symplectic; i.e., $\mathfrak{u}^*(A/M) \cong \mathfrak{sp}_d$.

(3) A/M is simple, $d \geq 7$ and $*_M$ is orthogonal; i.e., $\mathfrak{u}^*(A) \cong \mathfrak{so}_d$.

Let P be a subalgebra of A. We denote by $\ll P \gg_A$ *the ideal of A generated by P*. Recall that S is a $*$-invariant Levi subalgebra of A.

Lemma 3.5.2. *Let S' be an admissible $*$-invariant ideal of S. Then $\ll S' \gg_A$ is admissible.*

Proof. Put $A' = S' \oplus R$. Then A' is a $*$-invariant ideal of A and S' is a $*$-invariant Levi subalgebra of A'. Put $P = \ll S' \gg_{A'}$. Note that $P = S' \oplus S'R + RS' + RS'R = \ll S' \gg_A$. Thus we need to show that P is admissible. We have P is the smallest ideal of A' containing S'. Since P^2 is also an ideal of A' containing S' and $P^2 \subseteq P$, we have $P^2 = P$, so P is perfect. Note that $\ll S' \gg_P$ is an ideal of P containing the Levi subalgebra S' of P. Since P is perfect, $\ll S' \gg_P = P$, that is, P is generated by S' as an ideal. Denote by R' the radical of P, so $P = S' \oplus R'$. Let M be a maximal $*$-invariant ideal of P. If $M \supseteq S'$, then $M \supseteq P$ which is a contradiction, so M does not contain S'. Hence, $S'/M \cap S' \neq 0$. We claim that $R' \subseteq M$. Assume for the contrary that M does not contain R'. Then $M + R' \neq M$ is a $*$-invariant ideal of P containing M. Since M is maximal, $M + R' = P$, so

$$P/M = (M + R')/M \cong R'/(M \cap R') \neq 0,$$

that is, P/M is a non-zero nilpotent quotient of P, so P is not perfect, a contradiction. Therefore, $R' \subseteq M$, as required. Now, we have

$$P/M = (S' \oplus R'/M) = (S' + M)/M \cong S'/(M \cap S').$$

Since $S'/(M \cap S')$ is a $*$-semisimple, P/M is isomorphic to an involution simple component of S'. Therefore, P is admissible.

□

Recall that S is a $*$-invariant Levi (i.e. maximal semisimple) subalgebra of A.

Lemma 3.5.3. *Suppose that S is admissible. Then A is admissible if and only if $A = \ll S \gg_A$.*

Proof. If $A = \ll S \gg_A$, then by Lemma 3.5.2, A is admissible. Conversely, suppose that A is admissible. Then A/R is admissible. Since $S \subseteq \ll S \gg_A$ and $A = S \oplus R$, $A/ \ll S \gg_A$ is a nilpotent quotient of A. As A is perfect, $A/ \ll S \gg_A = 0$, so $A = \ll S \gg_A$.

□

We note the following properties of admissible ideals.

Lemma 3.5.4. *(i) The sum of admissible ideals is admissible.*

(ii) If P is an admissible ideal of A and Q is an admissible ideal of A/P then the full preimage of Q in A is an admissible ideal of A.

Lemma 3.5.4(i) implies that every algebra has the largest admissible ideal.

Definition 3.5.5. The largest admissible ideal $\mathscr{P}_a(A)$ of A is called the *admissible radical* of A.

Let $S_a = \mathscr{P}_a(S)$ be the largest admissible ideal of S. Then $S = S_a \oplus S'$, where S' is the complement of S_a in S.

Lemma 3.5.6. $\ll S_a \gg_A = \mathscr{P}_a(A)$.

Proof. By Lemma 3.5.2, $P = \ll S_a \gg_A$ is an admissible ideal of A, so $P \subseteq \mathscr{P}_a(A)$. Put $R' = \mathrm{rad}(\mathscr{P}_a(A))$. Then $\mathscr{P}_a(A) = S_a \oplus R'$. Since $S_a \subseteq \ll S_a \gg_A = P$, $\mathscr{P}_a(A)/P$ is nilpotent, but $\mathscr{P}_a(A)$ is admissible, so $\mathscr{P}_a(A)/P = 0$. Therefore, $\mathscr{P}_a(A) = P = \ll S_a \gg_A$, as required.

□

Lemma 3.5.7. *(i) A is admissible if and only if $A = \mathscr{P}_a(A)$.*

(ii) $\overline{\mathscr{P}_a(A)}$ is semisimple.

(iii) $\mathrm{rad}(\mathscr{P}_a(A)) = \mathscr{P}_a(A) \cap R$.

Proof. (i) This follows from Lemmas 3.5.3 and 3.5.6.

(ii) This is obvious as $\overline{\mathscr{P}_a(A)}$ is an ideal of $\bar{A}$.

(iii) This follows from (ii).

□

The following proposition shows that $\mathscr{P}_a(A)$ has radical-like properties indeed.

Proposition 3.5.8. *(i) $\mathscr{P}_a(A)^2 = \mathscr{P}_a(A)$;*

(ii) $\mathscr{P}_a(\mathscr{P}_a(A)) = \mathscr{P}_a(A)$;

(iii) $\mathscr{P}_a(A/\mathscr{P}_a(A)) = 0$.

Proof. (i) and (ii) are obvious; (iii) follows from Lemma 3.5.4(ii).

□

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $p \neq 2$. Importance of admissible algebras over $\mathbb{F}$ is shown by the following results from [10] for $p = 0$ and [4] for $p \neq 2$.

Theorem 3.5.9. *[4, 10] Let A be a finite dimensional associative algebra over $\mathbb{F}$ with involution $*$ and let $K = \mathfrak{u}^*(A)$. If A is admissible, then $K^{(1)} = \mathfrak{su}^*(A)$ is a perfect Lie algebra.*

Definition 3.5.10. Let G be a $*$-invariant subalgebra of A. We say that G is *$*$-large (or, simply large) in A* if $\bar{G} = \bar{A}$ (equivalently, there is a $*$-invariant Levi subalgebra S of A such that $S \subseteq G$; or equivalently, $G/\operatorname{rad} G$ is isomorphic to A/R).

Lemma 3.5.11. *Let G be a large subalgebra of A. Then* $\operatorname{rad}(G) = G \cap R$*.*

Proof. This follows from Definition 3.5.10.

□

Remark 3.5.12. Let G be a large subalgebra of A and let B be a subspace of $\mathscr{P}_a(G)$. Then by Lemma 3.5.11, $\operatorname{rad}(G) = G \cap R$, so

$$(B + \operatorname{rad}(G))/\operatorname{rad}(G) \cong B/(B \cap \operatorname{rad}(G) = B/B \cap R \cong B + R/R = \bar{B}.$$

Moreover, by 3.5.7(ii), $\operatorname{rad}(\mathscr{P}_a(G)) = \mathscr{P}_a(G) \cap \operatorname{rad}(G) = \mathscr{P}_a(G) \cap R$, so

$$(B + \operatorname{rad}(\mathscr{P}_a(G)))/\operatorname{rad}(\mathscr{P}_a(G)) \cong B/(B \cap \operatorname{rad}(\mathscr{P}_a(G)) = B/B \cap R \cong B + R/R = \bar{B}.$$

Therefore, we can use the same notation $\bar{B}$ for the image of B in A/R, $G/\operatorname{rad} G$ and $\mathscr{P}_a(G)/\operatorname{rad} \mathscr{P}_a(G)$.

Proposition 3.5.13. *Suppose that A is admissible and $R^2 = 0$. Let G be a large subalgebra of A. Then G is admissible.*

Proof. Let S' be a $*$-invariant Levi subalgebra of G. Put $T = \operatorname{rad} G$. Then $G = S' \oplus T$. Note that T is a $*$-invariant S'-submodule of R and S' is a Levi subalgebra of A, so $A = S' \oplus R$ and S' is an admissible subalgebra of A. Hence, by Lemma 3.5.3, $\ll S' \gg_G$ is admissible, so we need to show that $G = \ll S' \gg_G$. We have $\ll S' \gg_G \subseteq G$. It remains to show that $G \subseteq \ll S' \gg_G$. Since $R^2 = 0$, we have $A = A^2 = (S' \oplus R)^2 = S' \oplus S'R + RS'$, so $R = S'R + RS'$. Since R is a completely reducible $\hat{S}'$-bimodule, where $\hat{S}' = S' + 1_{\hat{A}}$ (see

(3.4.1)), we see that R contains no trivial $\hat{S}'$-bimodules, so $T = TS' + S'T$. Therefore, $G = S' \oplus T = S' \oplus (TS' + S'T) \subseteq \ll S' \gg_G$, as required.

□

3.6 Bar-minimal inner ideals

Recall Definition 2.4.9 that an inner ideal B of a Lie algebra L is said to *L-perfect* if $B = [B,[B,L]]$. It is well known that if $p \neq 2,3,5,7$, then Jordan-Lie inner ideals of semisimple Lie algebras are L-perfect, see for example [29, Proposition 2.3] (or [6, Lemmas 2.19 and 2.20] for $p = 0$). In Chapter 2, we introduced this notion and showed that if A is a semisimple finite dimensional associative algebra over an algebraically closed field of characteristic $p \neq 2,3$, then every Jordan-Lie inner ideal B of $L = A^{(k)}$ ($k \geq 0$) is L-perfect (see Lemma 2.4.10). The following result shows that this is also true when A has involution and B is a Jordan-Lie inner ideal of $K^{(k)}$ ($k = 0,1$). Recall that $p \neq 2$.

Proposition 3.6.1. *Suppose that A is semisimple. Then every Jordan-Lie inner ideal of $K^{(k)}$ ($k = 0,1$) is $K^{(k)}$-perfect.*

Proof. This follows directly from Proposition 3.3.11 (for $k = 0$) and Proposition 3.3.10 for ($k = 1$).

□

Let L be a finite dimensional Lie algebra and let B be an inner ideal of L. Put $B_0 = B$ and $B_n = [B_{n-1},[B_{n-1},L] \subseteq B_{n-1}$ for $n \geq 1$. Then B_n is an inner ideal of L for all $n \geq 0$. Recall Definition 2.4.13 that if there is $n \in \mathbb{N}$ such that $B_n = B_{n+1}$, then B_n is said to be the *core* of B, denoted by $\mathrm{core}_L(B)$. In Chapter 2, we introduced this notion and described some basic properties related to it. Recall the following lemma (see Lemma 2.4.14).

Lemma 3.6.2. *Let L be a finite dimensional Lie algebra and let B be an inner ideal of L. Then*

(i) $\mathrm{core}_L(B)$ *is L-perfect;*

(ii) B is L-perfect if and only if $B = \mathrm{core}_L(B)$*;*

(iii) $\mathrm{core}_L(B)$ *is an inner ideal of* $L^{(k)}$ *for all* $k \geq 0$.

We also proved that if A is a finite dimensional associative algebra over an algebraically closed field of characteristic $p \neq 2,3$ and B is a Jordan-Lie inner ideal of $A^{(k)}$ $(k \geq 1)$, then $\bar{B} = \overline{\mathrm{core}_L(B)}$ (see Lemma 2.4.16). The following results show that this is also true in the case when A has involution and B is a Jordan-Lie inner ideal of $\mathfrak{u}^*(A)$ and $\mathfrak{su}^*(A)$ as well.

Lemma 3.6.3. *Let B be a Jordan-Lie inner ideal of $K^{(k)}$ $(k = 0,1)$. If $p \neq 3$, then*
(i) $\bar{B} = \overline{\mathrm{core}_{K^{(k)}}(B)}$.
(ii) If $\mathrm{core}_{K^{(k)}}(B) = 0$, then $B \subseteq \mathfrak{u}^*(R)$.

Proof. Since $\bar{A}$ is semisimple with involution and $\bar{B}$ is a Jordan-Lie inner ideal of $\bar{K}^{(k)}$, by Proposition 3.6.1, $\bar{B}$ is $\bar{K}^{(k)}$-perfect, so by Lemma 3.6.2(ii), $\bar{B} = \mathrm{core}_{\bar{K}^{(k)}}(\bar{B}) = \overline{\mathrm{core}_{K^{(k)}}(B)}$.
(ii) This follows from (i).

□

Bar-minimal inner ideals

Let A is a finite dimensional associative algebra with involution and let $K = \mathfrak{u}^*(A)$. Let B be an inner ideal of $K^{(k)}$ $(k \geq 0)$. Suppose that X is an inner ideal of $\bar{K}^{(k)}$ and $\bar{B} = X$. We say that B is X*-minimal* (or simply *bar-minimal*) if for every inner ideal B' of $K^{(k)}$ with $\bar{B}' = X$ and $B' \subseteq B$, one has $B' = B$.

Lemma 3.6.4. *Let B be a Jordan-Lie inner ideal of $K^{(k)}$ $(k = 0,1)$. Suppose that B is bar-minimal and $p \neq 3$. Then the following hold:*
(i) $B = \mathrm{core}_{K^{(k)}}(B)$.
(ii) B is $K^{(k)}$-perfect.

Proof. (i) We have $\mathrm{core}_{K^{(k)}}(B)$ is a Jordan-Lie inner ideal of $K^{(k)}$ contained in B. By Lemma 3.6.3, $\overline{\mathrm{core}_{K^{(k)}}(B)} = \bar{B}$. Since B is $\bar{B}$-minimal, we have $B = \mathrm{core}_{K^{(k)}}(B)$.
(ii) This follows directly from (i) and Lemma 3.6.2(i).

□

Recall that a Lie algebra L is said to be perfect if $L = [L,L]$.

Lemma 3.6.5. *Let L be a perfect Lie algebra and let B be an L-perfect inner ideal of L. Suppose that $L = \bigoplus_{i\in I} L_i$, where each L_i is an ideal of L. Then $B = \bigoplus_{i\in I} B_i$, where $B_i = B\cap L_i$. Moreover, if $L = K^{(1)} = \mathfrak{su}^*(A)$ and B is bar-minimal then B_i is a $\bar{B}_i$-minimal inner ideal of L_i, for all $i \in I$.*

Proof. Since L is perfect and B is L-perfect, as in the proof of Lemma 2.5.3, $B = \bigoplus_{i\in I} B_i$, where $B_i = B\cap L_i$. Clearly, if $L = K^{(1)}$ and B is bar-minimal, then B_i is a $\bar{B}_i$-minimal Jordan-Lie inner ideal of L_i, for all $i \in I$.

□

Split inner ideals

Let L be a Lie algebra and let Q be subalgebra of L. Recall that Q is called a *quasi Levi subalgebra* of L if Q is quasi semisimple and there is a solvable ideal P of L such that $L = Q\oplus P$ (see Definitions 2.1.1 and 2.1.4 for quasi semisimple and quasi Levi, respectively). Recall the following definitions from Chapter 2.

Definition 3.6.6. (1) Let L be a finite dimensional Lie algebra and let B be a subspace of L. Suppose that there is a quasi Levi decomposition $L = Q\oplus N$ of L such that $B = B_Q \oplus B_N$, where $B_Q = B\cap Q$ and $B_N = B\cap N$. Then we say that B *splits in* L and Q is a *B-splitting quasi Levi subalgebra* of L (Definition 2.5.4).

(2) Let A be an associative algebra (not necessarily with involution) and let R be the radical of A. Let B be a subspace of A. Suppose that there is a Levi subalgebra S' of A such that $B = B_{S'} \oplus B_R$, where $B_{S'} = B\cap S$ and $B_R = B\cap R$. Then we say that B *splits* in A and S is a *B-splitting Levi subalgebra* of A (Definition 2.5.5).

Recall that A is a finite dimensional associative algebra with involution $*$, $p \neq 2$, S is a $*$-invariant Levi subalgebra of A and R is the radical of A.

Definition 3.6.7. Let B be a subspace of A. Suppose that there is a $*$-invariant Levi subalgebra S'' of A such that $B = B_{S''} \oplus B_R$, where $B_{S''} = B\cap S''$ and $B_R = B\cap R$. Then we say that B *$*$-splits in A* (or simply *splits in A*) and S'' is a *B-$*$-splitting Levi subalgebra* of A.

Lemma 3.6.8. *Let B be a subspace of $K^{(1)} = \mathfrak{su}^*(A)$. Suppose that A is admissible. If B $*$-splits in A, then B splits in $K^{(1)}$.*

Proof. Suppose that B $*$-splits in A. Then there is a $*$-invariant Levi subalgebra S of A such that $B = B_S \oplus B_R$, where $B_S = B \cap S$ and $B_R = B \cap R$. Since S is admissible (because A is admissible), $Q = \mathfrak{su}^*(S)$ is a quasi semisimple subalgebra of $K^{(1)}$. Note that $N = K^{(1)} \cap R$ is a solvable ideal of $K^{(1)}$, so $K^{(1)} = Q \oplus N$ is a quasi Levi decomposition of $K^{(1)}$. It is easy to see that $B_S \subseteq Q$ and $B_R \subseteq N$, so B splits in $K^{(1)}$.

□

We will need the following result due to Taft [40].

Theorem 3.6.9. *[40, Corollary 2] Let A be a finite dimensional associative algebra over a field Φ of characteristic not 2 and let R be the radical of A. Let G be a set of non-singular linear transformations of A, each element of which is either an automorphism or an anti-automorphism of the algebra A. Let P be a G-invariant separable subalgebra of A. Then P may be embedded in a G-invariant Levi subalgebra of A.*

As a special case of Theorem 3.6.9, we get the following corollary

Corollary 3.6.10. *Let S' be a $*$-invariant semisimple subalgebra of A. Then there is a $*$-invariant Levi subalgebra of A containing S'.*

Lemma 3.6.11. *Let e be a $*$-orthogonal idempotent in A with $e^*e = 0$. Then*

(i) eKe^* $*$-splits in A;

(ii) Suppose that A is admissible. Then eKe^* splits in $K^{(1)}$.

Proof. (i) Since e is a $*$-orthogonal. By Corollary 3.6.10, there is a $*$-invariant Levi subalgebra S of A such that $e, e^* \in S$. Since $\mathfrak{u}^*(A) = \mathfrak{u}^*(S) \oplus \mathfrak{u}^*(R)$,

$$eKe^* = e\,\mathfrak{u}^*(A)e^* = e(\mathfrak{u}^*(S) \oplus \mathfrak{u}^*(R))e^* = e\,\mathfrak{u}^*(S)e^* \oplus e\,\mathfrak{u}^*(R)e^* = B_S \oplus B_R,$$

where $B_S = e\,\mathfrak{u}^*(S)e^* = B \cap S$ and $B_R = e\,\mathfrak{u}^*(R)e^* = B \cap R$, as required.

(ii) This follows directly from (i) and Lemma 3.6.8.

□

Proposition 3.6.12. *Suppose that $A = D \oplus D^*$, where D is an ideal of A. Then every Jordan-Lie inner ideal of $K^{(1)}$ $*$-splits in A.*

Proof. Let B be a Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$ and let $\varphi : A \to D$ be the natural projection of A onto D. By Lemma 3.1.6, the restriction of φ is a Lie algebra isomorphism of $\mathfrak{su}^*(A)$ onto $D^{(1)}$. Since B is a Jordan-Lie inner ideal of $K^{(1)}$, $\varphi(B)$ is a Jordan-Lie inner ideal of $D^{(1)}$, so by Corollary 1.2.3, $\varphi(B)$ splits in D. Therefore, B $*$-splits in A, as required.

□

Proposition 3.6.13. *Let B be a subspace of A. Let G be a large subalgebra of A and let C be a subspace of $\mathscr{P}_a(G)$. Suppose that $C \subseteq B$, $\bar{C} = \bar{B}$, and C $*$-splits in $\mathscr{P}_a(G)$. Then B $*$-splits in A.*

Proof. Put $R_1 = \operatorname{rad} \mathscr{P}_a(G)$. By Remark 3.5.12, $R_1 \subseteq \operatorname{rad}(G) \subseteq R$. Let S_1 be a $*$-invariant C-splitting Levi subalgebra of $\mathscr{P}_a(G)$, so $C = C_{S_1} \oplus C_{R_1}$, where $C_{S_1} = C \cap S_1$ and $C_{R_1} = C \cap R_1$. Note that S_1 is a $*$-invariant semisimple subalgebra of G and so of A. By Corollary 3.6.10, there is a $*$-invariant Levi subalgebra S of A containing S_1. Put $B_S = B \cap S$ and $B_R = B \cap R$. Then $C_{S_1} \subseteq B_S$, $C_{R_1} \subseteq B_R$ and $B_S + R_R \subseteq B$. Since $\bar{B} = \bar{C}$,

$$B_S \cong \bar{B}_S \subseteq \bar{B} = \bar{C} \cong C/C_{R_1} \cong C_{S_1} \subseteq B_S,$$

so $B_S \cong \bar{B} \cong B/B_R$. Since $S \cap R = 0$, we get that $B = B_S \oplus B_R$, as required.

□

Corollary 3.6.14. *Let B be a subspace of A and let C be a subspace of $\mathscr{P}_a(A)$. Suppose that $C \subseteq B$, $\bar{C} = \bar{B}$, and C $*$-splits in $\mathscr{P}_a(A)$. Then B $*$-splits in A.*

The proof of the following proposition is similar to the proof of Proposition 3.6.13.

Proposition 3.6.15. *Let $C \subseteq B$ be subspaces of A such that $\bar{C} = \bar{B}$. If C $*$-splits in A, then B $*$-splits in A.*

Corollary 3.6.16. *Let B be an inner ideal of $K^{(k)}$ ($K = 0,1$). Suppose that $p \neq 3$. If* $\mathrm{core}_{K^{(k)}}(B)$ *$*$-splits in A, then B $*$-splits in A.*

Proof. By Lemma 3.6.3, $\overline{\mathrm{core}_{K^{(k)}}(B)} = \bar{B}$. Since $\mathrm{core}_{K^{(k)}}(B) \subseteq B$ and $\mathrm{core}_{K^{(k)}}(B)$ $*$-splits, by Proposition 3.6.15, B $*$-splits.

□

3.7 $*$-regular inner ideals

In this section we prove that if B is a bar-minimal Jordan-Lie inner ideal of $K^{(0)} = K = \mathfrak{u}^*(A)$ or $K^{(1)} = \mathfrak{su}^*(A)$, then B is $*$-regular (see definition below) if and only if $B = eKe^*$ for some $*$-orthogonal idempotent e in A. We start with the following result which is a slight generalization of [6, Lemma 4.2].

Lemma 3.7.1. *[6, Lemma 4.2] Let B be a subspace of $K^{(k)}$ ($k = 0,1$) such that $B^2 = 0$. Then the following hold.*

(i) $\mathfrak{u}^*(BAB) \subseteq K^{(k)}$.

(ii) $\mathfrak{u}^*(BAB) = BAB \cap K^{(k)}$.

(iii) If $\mathfrak{u}^*(BAB) \subseteq B$, *then B is a Jordan-Lie inner ideal of* $K^{(k)}$.

Proof. (i) This is clear for $k = 0$. Let $k = 1$. Clearly, BAB is $*$-invariant, so by (3.1.1), $\mathfrak{u}^*(BAB) = \{x - x^* \mid x \in BAB\}$. Let $b, b' \in \mathfrak{u}^*(BAB)$ and let $a \in A$. Since $B^2 = 0$,

$$\begin{aligned} bab' - (bab')^* &= bab' - b'a^*b = b(ab' + b'a^*) - (ab' + b'a^*)b \\ &= b(ab' - (ab')^*) - (ab' - (ab')^*)b \\ &= [b, ab' - (ab')^*] \in [\mathfrak{u}^*(A), \mathfrak{u}^*(A)] = K^{(1)}. \end{aligned}$$

Therefore, $\mathfrak{u}^*(BAB) \subseteq K^{(1)}$, as required.

(ii) This is obvious.

(iii) Let $b, b' \in \mathfrak{u}^*(BAB)$ and let $x \in K^{(k)}$. Then

$$\{b, x, b'\} = bxb' + b'xb = bxb' - (bxb')^* \in \mathfrak{u}^*(BAB) \subseteq B,$$

so B is a Jordan-Lie inner ideal of $K^{(k)}$.

□

Definition 3.7.2. Let B be a subspace of $K^{(k)}$ $(k \geq 0)$. Then B is said to be a $*$*-regular (or simply, regular) inner ideal* of $K^{(k)}$ (with respect to A) if $B^2 = 0$ and $\mathfrak{u}^*(BAB) \subseteq B$.

Note that every $*$-regular inner ideal is a Jordan-Lie inner ideal (see Lemma 3.7.1(iii)). However, there are Jordan-Lie inner ideals which are not $*$-regular. For example, point spaces are Jordan-Lie inner ideals but they are not $*$-regular. Regular inner ideals were first defined in [6] (in characteristic zero) and were recently used in [5] to classify maximal zero product subsets of simple rings.

Lemma 3.7.3. *Let e be an idempotent in A with $e^*e = 0$. Then eKe^* is a $*$-regular inner ideal of $K^{(k)}$ $(k = 0, 1)$.*

Proof. By Lemma 3.1.4(iii), eKe^* is a Jordan-Lie inner ideal of $K^{(k)}$ $(k = 0, 1)$. It remains to note that $\mathfrak{u}^*((eKe^*)A(eKe^*)) \subseteq \mathfrak{u}^*(eAe^*) = e\,\mathfrak{u}^*(A)e^* = eKe^*$.

□

Remark 3.7.4. The result of Lemma 3.7.3 is also true when A is an associative algebra with involution $*$ over a commutative ring Φ with $\frac{1}{2} \in \Phi$ and $*$ is Φ-linear.

We will need the following results due to Baranov and Rowley [6, Proposition 4.8].

Proposition 3.7.5. *Let A be an associative ring. Then*

(i) A is Von Neumann regular if and only if $\mathscr{R}\mathscr{L} = \mathscr{R} \cap \mathscr{L}$ for all left $\mathscr{L}$ and right $\mathscr{R}$ ideals of A.

(ii) every square zero element x in A is Von Neumann regular if and only if $\mathscr{R}\mathscr{L} = \mathscr{R} \cap \mathscr{L}$ for all left $\mathscr{L}$ and right $\mathscr{R}$ ideals of A with $\mathscr{L}\mathscr{R} = 0$.

The following proposition is known in the case $p = 0$ [6, Proposition 4.11].

Proposition 3.7.6. *Let B be a subspace of $K^{(k)}$ $(k \geq 0)$. Then B is a $*$-regular inner ideal of K if and only if there exists a left ideal $\mathscr{L}$ of A with $\mathscr{L}\mathscr{L}^* = 0$ such that*

$$\mathfrak{u}^*(\mathscr{L}^*\mathscr{L}) \subseteq B \subseteq \mathscr{L}^* \cap \mathscr{L} \cap K^{(k)}.$$

In particular, if A is Von Neumann regular then every regular inner ideal B of $K^{(k)}$ is of the form $B = \mathscr{L}^*\mathscr{L} = \mathscr{L}^* \cap \mathscr{L}$ for some left ideal $\mathscr{L}$ with $\mathscr{L}\mathscr{L}^* = 0$.

Proof. Suppose that B is $*$-regular. Put $\mathscr{L} = AB + B$. Then $\mathscr{L}$ is a left ideal of A. Note that $\mathscr{L}^* = BA + B$, so $\mathscr{L}\mathscr{L}^* = 0$. Since B is $*$-regular, $\mathfrak{u}^*(BAB) \subseteq B$, so

$$\mathfrak{u}^*(\mathscr{L}^*\mathscr{L}) \subseteq \mathfrak{u}^*(BAB) \subseteq B \subseteq \mathscr{L}^* \cap \mathscr{L} \cap K^{(k)},$$

as required. On the other hand, let $\mathscr{L}$ be a left ideal of A such that $\mathscr{L}\mathscr{L}^* = 0$ and $\mathfrak{u}^*(\mathscr{L}^*\mathscr{L}) \subseteq B \subseteq \mathscr{L}^* \cap \mathscr{L} \cap K$. Then $B^2 \subseteq \mathscr{L}\mathscr{L}^* = 0$. Moreover,

$$\mathfrak{u}^*(BAB) \subseteq \mathfrak{u}^*(\mathscr{L}^*A\mathscr{L}) \subseteq \mathfrak{u}^*(\mathscr{L}^*\mathscr{L}) \subseteq B.$$

Therefore, B is $*$-regular.

$\square$

Let $\mathscr{L}$ be a left ideal of A and let X be a left ideal of $\bar{A}$. Then $\mathscr{L}$ is said to be X-minimal if $\bar{\mathscr{L}} = X$ and for every left ideal $\mathscr{L}'$ of A with $\mathscr{L}' \subseteq \mathscr{L}$ and $\bar{\mathscr{L}}' = X$ one has $\mathscr{L} = \mathscr{L}'$. We will need the following theorem from [11].

Theorem 3.7.7. *[11] Let A be a left Artinian associative ring and let $\mathscr{L}$ be a left ideal of A. If $\mathscr{L}$ is $\bar{\mathscr{L}}$-minimal, then $\mathscr{L} = Ae$ for some idempotent $e \in \mathscr{L}$.*

Now, we are ready to prove Theorem 1.2.6 and Corollary 1.2.7

Proof of Theorem 1.2.6. If $B = eKe^*$ for some idempotent e in A with $e^*e = 0$, then by Lemma 3.7.3, B is $*$-regular. On the other hand, suppose that B is $*$-regular. Then by Proposition 3.7.6, there is a left $\mathscr{L}$ ideal of A such that $\mathscr{L}\mathscr{L}^* = 0$ and $\mathfrak{u}^*(\mathscr{L}^*\mathscr{L}) \subseteq B \subseteq \mathscr{L}^* \cap \mathscr{L} \cap K^{(k)}$, so $\mathfrak{u}^*(\overline{\mathscr{L}^*\mathscr{L}}) = \mathfrak{u}^*(\bar{\mathscr{L}}^*\bar{\mathscr{L}}) \subseteq \bar{B} \subseteq \bar{\mathscr{L}} \cap \bar{\mathscr{L}}^*$. Since $\bar{A}$ is Von Neumann regular (because it is semisimple), by Proposition 3.7.5, $\mathfrak{u}^*(\bar{\mathscr{L}}^*\bar{\mathscr{L}}) = \bar{B}$. Let $\mathscr{L}_1 \subseteq \mathscr{L}$ be an $\bar{\mathscr{L}}$-minimal left ideal of A. Then by Theorem 3.7.7, $\mathscr{L}_1 = Af$ for some idempotent $f \in \mathscr{L}_1$, so $\mathscr{L}_1^* = f^*A$. Put $e = f^* \in \mathscr{L}_1^*$. Then $e^* = (f^*)^* = f \in \mathscr{L}_1$ and $e^*e = ff^* \in \mathscr{L}_1\mathscr{L}_1^* \subseteq \mathscr{L}\mathscr{L}^* = 0$. We have $\mathscr{L}_1 = Ae^*$ and $\mathscr{L}_1^* = eA$. Put $B' = \mathfrak{u}^*(\mathscr{L}_1^*\mathscr{L}_1) \subseteq B$. Since

$eAe^* = eeAe^* \subseteq eAAe^*$, we have $eAAe^* = eAe^*$, so

$$B' = \mathfrak{u}^*(\mathscr{L}_1^*\mathscr{L}_1) = \mathfrak{u}^*(eAAe^*) = \mathfrak{u}^*(eAe^*) = e\,\mathfrak{u}^*(A)e^* = eKe^*.$$

As $e^*e = 0$, by Proposition 3.7.3, B' is a $*$-regular inner ideal of $K^{(k)}$. As B is $\bar{B}$-minimal, $B' \subseteq B$ and $\bar{B}' = \mathfrak{u}^*(\bar{\mathscr{L}}'\bar{\mathscr{L}}') = \mathfrak{u}^*(\bar{\mathscr{L}}^*\bar{\mathscr{L}}) = \bar{B}$, we get that $B = B' = eKe^*$, as required.

□

Proof of Corollary 1.2.7. Let B be a $*$-regular inner ideal of $K^{(k)}$ $(k = 0,1)$. Let $B' \subseteq B$ be a $\bar{B}$-minimal $*$-regular inner ideal of $K^{(k)}$. Then by Theorem 1.2.6, $B' = eKe^*$, where e is an idempotent in A with $e^*e = 0$. By Lemma 3.2.12, there is a $*$-orthogonal idempotent $g \in A$ such that $gKg^* = eKe^* = B'$, so by Lemma 3.6.11, B' $*$-splits in A. Therefore, by Proposition 3.6.15, B $*$-splits in A.

□

3.8 Proof of the main results

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $P \neq 2$, A is a finite dimensional associative algebra with involution $*$ (of the first kind), R is the radical of A and S is a $*$-invariant Levi subalgebra of A, so $A = S \oplus R$.

Throughout this section, unless otherwise specified, A is admissible, so by Theorem 3.5.9, $K^{(1)} = \mathfrak{su}^*(A)$ is a perfect Lie algebra, that is, $[K^{(1)}, K^{(1)}] = K^{(1)}$.

The aim of this section is to prove the following theorem in steps.

Theorem 3.8.1. *Let B be a Jordan-Lie inner ideal of $K^{(1)}$. Suppose that $p \neq 3$, A is admissible, B is $\bar{B}$-minimal and $R^2 = 0$. Then B $*$-splits in A.*

We will prove Theorem 3.8.1 by induction on the length of the S-bimodule R. The base of the induction (the case of $*$-irreducible R) will be settled using the following three propositions, which will be proved in steps.

Proposition 3.8.2. *Theorem 3.8.1 holds if A is $*$-indecomposable as in Proposition 3.4.3(i), that is, if A/R is involution simple and $R = U \oplus U^*$ where U is the natural left S-module with $US = 0$. Moreover, if A/R is simple, then $B \subseteq S'$ for some $*$-invariant Levi subalgebra S' of A.*

Proposition 3.8.3. *Theorem 3.8.1 holds if A is $*$-indecomposable as in Proposition 3.4.3(ii), that is, if A/R is involution simple and $R = U + U^*$ where U is an irreducible S-bimodule with respect to the left and right multiplication.*

Proposition 3.8.4. *Theorem 3.8.1 holds if A is $*$-indecomposable as in Proposition 3.4.3(iii), that is, if $A/R \cong S_1 \oplus S_2$, where S_1 and S_2 are involution simple associative algebras and $R = U \oplus U^*$, where U is an irreducible S_1-S_2-bimodule such that $US_1 = S_2U = 0$.*

We will need the following well-known result

Theorem 3.8.5 (Malcev). *Let A be a finite dimensional associative algebra and let R be the radical of A. Suppose that A/R is separable. If S_1 and S_2 are two subalgebras of A such that $A = S_i \oplus R$ ($i = 1,2$) then there exists $q \in R$ such that $S_1 = (1-q)^{-1}S_2(1-q)$.*

For each $q \in R$, we denote by φ_q *the special inner automorphism* of A defined by $\varphi_q(a) = (1-q)^{-1}a(1-q)$ for all $a \in A$. Since $R^n = 0$ for some $n \in \mathbb{N}$, we have $(1-q)^{-1} = 1 + q + \ldots + q^{n-1}$. Thus,

$$\varphi_q(a) = (1 + q + \ldots + q^{n-1})a(1-q). \tag{3.8.1}$$

Moreover, if $R^2 = 0$, then

$$\varphi_q(a) = (1+q)a(1-q) = a + qa - aq = a + [q,a]. \tag{3.8.2}$$

Lemma 3.8.6. *Suppose that $R^2 = 0$. Let φ_q be a special inner automorphism of A for some $q \in \mathfrak{u}^*(R)$. Then*

(i) $\varphi_q(S)$ is a $*$-invariant Levi subalgebra of A;

(ii) $\varphi_q(K) \subseteq K$.

Proof. (i) By Theorem 3.8.5, $\varphi_q(S) = (1-q)^{-1}S(1-q) = (1+q)S(1-q)$ is a Levi subalgebra of A. Let $s \in S$. Since $q \in \mathfrak{u}^*(R)$, by (3.8.2),

$$\varphi_q(s)^* = ((1+q)s(1-q))^* = (1-q^*)s^*(1+q^*) = (1+q)s^*(1-q) \in \varphi_q(S),$$

as required.

(ii) This follows from (3.8.2).

□

Recall that $\bar{A} = A/R$.

Lemma 3.8.7. *Let $\varphi_q : A \to A$ be a special inner automorphism of A. Then $\overline{\varphi_q(a)} = \bar{a}$ for all $a \in A$.*

Proof. By (3.8.1), $\varphi_q(a) = (1+q+q^2+\dots q^{n-1})a(1-q) \in a+aR+Ra+RaR$, so $\overline{\varphi_q(a)} = \bar{a}$, as required.

□

Suppose that $R = U \oplus U^*$, where U is an S-bimodule. Then the algebra $\tilde{A} = A/U^*$ is not $*$-invariant. The following lemma describes the relation between inner ideals of $\tilde{A}$ and A.

Lemma 3.8.8. *Let U be a subspace of R. Suppose that $R^2 = 0$ and $R = U \oplus U^*$. Let B be a subspace of $\mathfrak{u}^*(A)$ and let $\tilde{B}$ be the image of B in $\tilde{A} = A/U^*$. Suppose that $\tilde{B}$ splits in $\tilde{A}$, then B $*$-splits in A.*

Proof. Let X be a subspace of $\mathfrak{u}^*(S)$ with $\bar{X} = \bar{B}$. Since $\tilde{B}$ splits in A, there is a $\tilde{B}$-splitting Levi subalgebra S' of $\tilde{A}$ such that $\tilde{B} = \tilde{B}_{S'} \oplus \tilde{B}_{\tilde{R}}$, where $\tilde{B}_{S'} = \tilde{B} \cap S'$ and $B_{\tilde{R}} = \tilde{B} \cap \tilde{R}$. Note that $\bar{\tilde{B}}_{S'} = \bar{B} = \bar{X}$. By Theorem 3.8.5, there is $q' \in \tilde{R} \cong U$ and a special inner automorphism $\varphi_{q'}$ of $\tilde{A}$ such that $\tilde{S} = \varphi_{q'}(S')$. Since $\tilde{B}_{S'} \subseteq S'$, $\varphi_{q'}(\tilde{B}_{S'}) \subseteq \varphi_{q'}(S') = \tilde{S}$. Moreover, by Lemma 3.8.7, $\overline{\varphi_{q'}(\tilde{B}_{S'})} = \bar{\tilde{B}}_{S'} = \bar{B} = \bar{X}$. Note that $\tilde{X} \subseteq \tilde{S}$ (because $X \subseteq S$), so both $\varphi_{q'}(\tilde{B}_{S'})$ and $\tilde{X}$ have the same image $\bar{X}$ in $\bar{\tilde{A}} = \tilde{A}/\tilde{U}$. Since both of them are subspaces of $\tilde{S}$ and $\tilde{S} \cap \tilde{U} = 0$, they must be equal. Thus, $\tilde{X} = \varphi_{q'}(\tilde{B}_{S'}) \subseteq \varphi_{q'}(\tilde{B}) \cap \tilde{S}$. Fix any $q_1 \in U$ such that $\tilde{q}_1 = q'$. Put $q = q_1 - q_1^* \in \mathfrak{u}^*(R)$. Then $\tilde{q} = \tilde{q}_1 = q'$. Consider the special inner automorphism φ_q of A. Since $R^2 = 0$, by (3.8.2), $\varphi_q(r) = r$ for all $r \in R$, so U^* is a φ_q-invariant. Hence, φ_q induces a special inner automorphism $\tilde{\varphi}_q$ of $\tilde{A} = A/U^*$. As $\tilde{q} = q'$, we see that $\tilde{\varphi}_q = \varphi_{\tilde{q}} = \varphi_{q'}$, so $\tilde{X} \subseteq \tilde{\varphi}_q(\tilde{B})$. Hence $X \subseteq \varphi_q(B) + U^*$. We wish to show that $X \subseteq \varphi_q(B)$. Let $x \in X$. Then $x = b + u^*$ for some $b \in \varphi_q(B)$ and $u \in U$. By Lemma 3.8.6, $\varphi_q(B) \subseteq \mathfrak{u}^*(A)$, so $b^* = -b$. Since $x^* = -x$, we must have $(u^*)^* = -u^*$. This implies $u = -u^* \in U^* \cap U = 0$, so $u^* = 0$. Therefore, $x = b \in \varphi_q(B)$, as required.

□

Lemma 3.8.9. *Suppose that A is admissible, $R^2 = 0$, S is simple and $R = U \oplus U^*$, where U is a natural left S-module with $US = 0$. Then the following hold.*

(i) Every Jordan-Lie inner ideal of $\mathfrak{su}^*(S)$ is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$.

(ii) Let G be a large subalgebra of A and let B be a Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$. Then B is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$.

Proof. *(i)* This follows from (ii) as S is a large subalgebra of A.

(ii) Since G is a large subalgebra of A, it contains a $*$-invariant Levi subalgebra of A. Without loss of generality we can assume $S \subseteq G$. Let $b, b' \in B$. We need to show that $\{b, x, b'\} \in B$ for all $x \in \mathfrak{su}^*(A)$, this will imply, by Lemma 3.1.3, that B is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A)$, as required. Since $\mathfrak{su}^*(A) \subseteq \mathfrak{su}^*(S) \oplus \mathfrak{u}^*(R)$, $x = q + r$ for some $q \in \mathfrak{su}^*(S)$ and $r \in \mathfrak{u}^*(R)$. As $R^2 = 0$ and $ARA = SRS = SUS + SU^*S = 0$,

$$\{b, x, b'\} = \{b, q, b'\} + \{b, r, b'\} = \{b, q, b'\} + brb' + b'rb = \{b, q, b'\} \in B,$$

as required.

□

Proof of Proposition 3.8.2

The following lemma represents a special case of Proposition 3.8.2.

Lemma 3.8.10. *Theorem 3.8.1 holds if A/R is simple and $R = U \oplus U^*$, where U is an irreducible left S-module with $US = 0$. Moreover, $B \subseteq S'$ for some $*$-invariant Levi subalgebra S' of A.*

Proof. We identify $\bar{A} = A/R$ with S. Since S is simple, by Lemma 3.4.5, $\mathfrak{u}^*(S) \cong \mathfrak{so}_m, \mathfrak{sp}_{2n}$ ($m = 2n+1$ or $2n$) and $\mathfrak{u}^*(R) = \{r - r^* \mid r \in U\}$. Let $\tilde{B}$ be the image of B in $\tilde{A} = A/U^* \cong$

$A\oplus U$. Since $R=U\oplus U^*$, by Lemma 3.8.8, to show that B $*$-splits in A, it is enough to show that $\tilde{B}$ splits in $\tilde{A}$. To simplify notations, we will re-denote $\tilde{A}$, $\tilde{S}$, $\tilde{R}$ and $\tilde{B}$ by A, S, R and B, respectively. Thus, $R=U$ and $A/U\cong S$. We need to show that B splits in A. Let $\{e_1,e_2,\dots,e_m\}$ be the standard basis of U. Since $\bar{B}$ is a Jordan-Lie inner ideal of $\bar{A}=S$ and S is simple, by Lemma 3.2.16, there is a canonical matrix realization $\mathscr{M}_m$ of S and integer $1\le k\le n$ such that the action of S on R corresponds to matrix multiplication and $\bar{B}$ is the space spanned by $\mathscr{E}$, where $\mathscr{E}$ is one of the following ($\varepsilon=\pm$):

$$E=\{e_{1,n+t}-e_{t,n+1}\mid 1<t\le k\le n\}\subseteq \mathfrak{su}^*(S)=\mathfrak{so}_m$$

$$E^\varepsilon=\{e_{s,n+t}-\varepsilon e_{t,n+s}\mid 1\le s\le t\le k\le n\}\subseteq \mathfrak{su}^*(S)=\mathfrak{so}_m,\mathfrak{sp}_{2n},$$

where $\{e_{ij}\mid 1\le i,j\le m\}$ is a standard basis of S consisting of matrix units. Our aim is to find a special inner automorphism $\varphi:A\to A$ such that $\mathscr{E}\subseteq\varphi(B)$. Since $\mathscr{E}=E$, E^+, or E^-, we need to consider three cases.

Case (1): Suppose that $\mathscr{E}=E=\{e_{1,n+t}-e_{t,n+1}\mid 1<t\le k\}\subseteq\mathfrak{so}_m=\mathfrak{su}^*(S)$. We wish to show that there is a special inner automorphism $\varphi_q:A\to A$ such that $E\subseteq\varphi_q(B)$ for some $q\in U$. Without loss of generality we can assume $m=2n+1$ (the case $m=2n$ will follow immediately). Fix any subset $\{b_t\mid 1<t\le k\}\subseteq B$ such that $\bar{b}_t=e_{1,n+t}-e_{t,n+1}$ for all t. Then $b_t=e_{1,n+t}-e_{t,n+1}+\sum_{i=1}^m\alpha_i^te_i$, where $\alpha_i^t\in\mathbb{F}$. Put $b_t^{(1)}=b_t(e_{n+t,1}-e_{n+1,t})b_t\in B$ (by Lemma 3.1.3). Since $UA=0$,

$$\begin{aligned}
b_t^{(1)} &= b_t(e_{n+t,1}-e_{n+1,t})b_t=(e_{1,n+t}-e_{t,n+1}+\sum_{i=1}^m\alpha_i^te_i)(e_{n+t,1}-e_{n+1,t})b_t\\
&= (e_{11}+e_{tt})(e_{1,n+t}-e_{t,n+1}+\sum_{i=1}^m\alpha_i^te_i)=e_{1,n+t}-e_{t,n+1}+\alpha_1^te_1+\alpha_t^te_t\in B.
\end{aligned}$$

Put $b_k^{(2)}=b_k^{(1)}\in B$ and for $t<k$ set $b_t^{(2)}=\{b_k^{(1)},e_{n+k,1}-e_{n+1,k},b_t^{(1)}\}\in B$ (by Lemma 3.1.3). Since $UA=0$,

$$\begin{aligned}
b_t^{(2)} &= b_k^{(1)}(e_{n+k,1}-e_{n+1,k})b_t^{(1)}+b_t^{(1)}(e_{n+k,1}-e_{n+1,k})b_k^{(1)}\\
&= (e_{1,n+k}-e_{k,n+1}+\alpha_1^ke_1+\alpha_k^ke_k)(e_{n+k,1}-e_{n+1,k})b_t^{(1)}\\
&\quad +(e_{1,n+t}-e_{t,n+1}+\alpha_1^te_1+\alpha_t^te_t)(e_{n+k,1}-e_{n+1,k})b_k^{(1)}\\
&= (e_{11}+e_{kk})(e_{1,n+t}-e_{t,n+1}+\alpha_1^te_1+\alpha_t^te_t)\\
&\quad +e_{tk}(e_{1,n+k}-e_{k,n+1}+\alpha_1^ke_1+\alpha_k^ke_k)\\
&= e_{1,n+t}+\alpha_1^te_1-e_{t,n+1}+\alpha_k^ke_t=e_{1,n+t}-e_{t,n+1}+\alpha_1^te_1+\alpha_k^ke_t\in B.
\end{aligned}$$

We have $b_t^{(2)}=e_{1,n+t}-e_{t,n+1}+\alpha_1^te_1+\alpha_k^ke_t\in B$ for all t. Consider the inner automorphism $\varphi_q : A \to A$ with

$$q = \sum_{j=2}^{k} \alpha_1^j e_{n+j} - \alpha_k^k e_{n+1} \in U \subseteq R.$$

Since $UA = 0$,

$$\begin{aligned}
\varphi_q(b_t^{(2)}) &= (1+q)b_t^{(2)}(1-q) = b_t^{(2)}(1-q) \\
&= (e_{1,n+t} - e_{t,n+1} + \alpha_1^t e_1 + \alpha_k^k e_t)(1 - \sum_{j=2}^{k} \alpha_1^j e_{n+j} + \alpha_k^k e_{n+1}) \\
&= e_{1,n+t} - \alpha_1^t e_1 - e_{t,n+1} - \alpha_k^k e_t + \alpha_1^t e_1 + \alpha_k^k e_t \\
&= e_{1,n+t} - e_{t,n+1} \in \varphi_q(B),
\end{aligned}$$

so

$$E = \{e_{1,n+t} - e_{t,n+1} \mid 1 < t \leq k\} \subseteq \varphi_q(B) \cap S,$$

as required.

Case (2): Suppose that $\mathscr{E} = E^+ = span\{e_{s,n+t} - e_{t,n+s} \mid 1 \leq s < t \leq k\}$. We wish to show that there is a special inner automorphism $\varphi : A \to A$ such that $E^+ \subseteq \varphi(B)$. Fix any subset $\{b_{st} \mid 1 \leq s < t \leq k\} \subseteq B$ such that $\bar{b}_{st} = e_{s,n+t} - e_{t,n+s}$.

CLAIM 1: Suppose that $b_{st} = e_{s,n+t} - e_{n+s,t} + \sum_{i=1}^{m} \alpha_i^{st} e_i \in B$ ($1 \leq s < t \leq k$), where $\alpha_i^{st} \in \mathbb{F}$. Then

$$\theta(b_{st}) := e_{s,n+t} - e_{t,n+s} + \alpha_1^{1t} e_s + \alpha_k^{sk} e_t \in B \quad \text{for all} \quad 1 \leq s < t \leq k.$$

Put $c_{st} = b_{st}(e_{n+t,s} - e_{n+s,t})b_{st}$. Then by Lemma 3.1.3, $c_{st} \in B$. Since $UA = 0$,

$$\begin{aligned}
c_{st} &= (e_{s,n+t} - e_{t,n+s} + \sum_{i=1}^{m} \alpha_i^{st} e_i)(e_{n+t,s} - e_{n+s,t})b_{st} \\
&= (e_{ss} + e_{tt})(e_{s,n+t} - e_{t,n+s} + \sum_{i=1}^{m} \alpha_i^{st} e_i) = e_{s,n+t} - e_{t,n+s} + \alpha_s^{st} e_s + \alpha_t^{st} e_t \in B.
\end{aligned}$$

Put $\theta(b_{1k}) = c_{1k} \in B$. For all the remaining indices s and t set $\theta(b_{st}) = \{c_{sk}, e_{n+k,1} - e_{n+1,k}, c_{1t}\} \in B$ (by Lemma 3.1.3). Since $UA = 0$, (for all $1 < s < t < k$)

$$\begin{aligned}
\theta(b_{st}) &= c_{sk}(e_{n+k,1} - e_{n+1,k})c_{1t} + c_{1t}(e_{n+k,1} - e_{n+1,k})c_{sk} \\
&= (e_{s,n+k} - e_{k,n+s} + \alpha_s^{sk} e_s + \alpha_k^{sk} e_k)(e_{n+k,1} - e_{n+1,k})c_{1t} \\
&\quad + (e_{1,n+t} - e_{t,n+1} + \alpha_1^{1t} e_1 + \alpha_t^{1t} e_t)(e_{n+k,1} - e_{n+1,k})c_{sk} \\
&= e_{s1}(e_{1,n+t} - e_{t,n+1} + \alpha_1^{1t} e_1 + \alpha_t^{1t} e_t) + e_{tk}(e_{s,n+k} - e_{k,n+s} + \alpha_s^{sk} e_s + \alpha_k^{sk} e_k) \\
&= e_{s,n+t} + \alpha_1^{1t} e_s - e_{t,n+s} + \alpha_k^{sk} e_t = e_{s,n+t} - e_{t,n+s} + \alpha_1^{1t} e_s + \alpha_k^{sk} e_t \in B.
\end{aligned}$$

Calculations show that $\theta(b_{1t})$ and $\theta(b_{sk})$ is also of the shape above. Since $\theta(b_{1k}) = c_{1k}$ is also of the shape above, we get that $\theta(b_{st}) = e_{s,n+t} - e_{t,n+s} + \alpha_1^{1t} e_s + \alpha_k^{sk} e_t \in B$ for all $1 \le s < t \le k$, as required.

Recall that $\{b_{st} \mid 1 \le s < t \le k\} \subseteq B$ with $\bar{b}_{st} = e_{s,n+t} - e_{t,n+s}$. Then $b_{st} = e_{s,n+t} - e_{t,n+s} + \sum_{i=1}^{m} \beta_i^{st} e_i$ for some coefficients $\beta_i^{st} \in \mathbb{F}$ for all $1 \le s < t \le k$.

CLAIM 2: There exists a special inner automorphism φ' of A such that

$$e_{s,n+k} - e_{k,n+s} \in \varphi'(B) \quad \text{and} \quad e_{s,n+t} - e_{t,n+s} + \beta_k^{tk} e_s \in \varphi'(B) \quad \text{for all} \quad t < k.$$

Since $\bar{b}_{1t} = e_{1,n+t} - e_{t,n+1} = \bar{b}_t \in E \cap E^+$, by Case (1), there is a special inner automorphism $\varphi_q : A \to A$ such that $\varphi_q(b_{1t}) = e_{1,n+t} - e_{t,n+1} \in \varphi_q(B)$ for all t. By using Claim 1, we get that

$$\theta(\varphi_q(b_{st})) = e_{s,n+t} - e_{t,n+s} + \beta_1^{1t} e_s + \beta_k^{sk} e_t \in \varphi_q(B) \text{ for all } s > 1.$$

Put $b'_{1t} = \varphi_q(b_{1t}) = e_{1,n+t} - e_{t,n+1} \in \varphi_q(B)$ and for $s > 1$ set $b'_{st} = \{\theta(\varphi_q(b_{st})), e_{n+t,1} - e_{n+1,t}, b'_{1t}\} \in \varphi_q(B)$ (by Lemma 3.1.3). Since $UA = 0$,

$$\begin{aligned}
b'_{st} &= \theta(\varphi_q(b_{st}))(e_{n+t,1} - e_{n+1,t})b'_{1t} + b'_{1t}(e_{n+t,1} - e_{n+1,t})\theta(\varphi_q(b_{st})) \\
&= (e_{s,n+t} - e_{t,n+s} + \beta_1^{1t} e_s + \beta_k^{sk} e_t)(e_{n+t,1} - e_{n+1,t})b'_{1t} \\
&\quad + (e_{1,n+t} - e_{t,n+1})(e_{n+t,1} - e_{n+1,t})\theta(\varphi_q(b_{st})) \\
&= e_{s1}(e_{1,n+t} - e_{t,n+1}) + (e_{11} + e_{tt})(e_{s,n+t} - e_{t,n+s} + \beta_1^{1t} e_s + \beta_k^{sk} e_t) \\
&= e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in \varphi_q(B).
\end{aligned}$$

Consider the special inner automorphism $\varphi_{q'} : A \to A$, where

$$q' = -\sum_{i=2}^{k-1} \beta_k^{ik} e_{n+i} \in U.$$

Put $b''_{st} = \varphi_{q'}(b'_{st})$ for all s and t. As $UA = 0$,

$$\begin{aligned}
b''_{1k} &= \varphi_{q'}(b'_{1k}) = (1 + q')b'_{1k}(1 - q') = b'_{1k}(1 - q') \\
&= (e_{1,n+k} - e_{k,n+1})(1 + \sum_{i=2}^{k-1} \beta_k^{ik} e_{n+i}) = e_{1,n+k} - e_{k,n+1} \in \varphi_{q'}(\varphi_q(B))
\end{aligned}$$

and (for $s > 1$)

$$\begin{aligned} b''_{sk} &= \varphi_{q'}(b'_{sk}) = b'_{sk}(1-q') = (e_{s,n+k} - e_{k,n+s} + \beta_k^{sk} e_k)(1 + \sum_{i=2}^{k-1} \beta_k^{ik} e_{n+i}) \\ &= e_{s,n+k} - e_{k,n+s} - \beta_k^{sk} e_k + \beta_k^{sk} e_k = e_{s,n+k} - e_{k,n+s} \in \varphi_{q'}(\varphi_q(B)). \end{aligned}$$

Therefore,

$$b''_{sk} = e_{s,n+k} - e_{k,n+s} \in \varphi_{q'}(\varphi_q(B)) \text{ for all } s. \tag{3.8.3}$$

For all $t < k$, we have

$$\begin{aligned} b''_{1t} &= \varphi_{q'}(b'_{1t}) = b'_{1t}(1-q') = (e_{1,n+t} - e_{t,n+1})(1 + \sum_{i=2}^{k-1} \beta_k^{ik} e_{n+i}) \\ &= e_{1,n+t} + \beta_k^{tk} e_1 - e_{t,n+1} = e_{1,n+t} - e_{t,n+1} + \beta_k^{tk} e_1 \in \varphi_{q'}(\varphi_q(B)) \end{aligned}$$

and (for $1 < s < t < k$)

$$\begin{aligned} b''_{st} &= \varphi_{q'}(b'_{st}) = b'_{st}(1-q') = (e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t)(1 + \sum_{i=2}^{k-1} \beta_k^{ik} e_{n+i}) \\ &= e_{s,n+t} + \beta_k^{tk} e_s - e_{t,n+s} - \beta_k^{sk} e_t + \beta_k^{sk} e_t \\ &= e_{s,n+t} - e_{t,n+s} + \beta_k^{tk} e_s \in \varphi_{q'}(\varphi_q(B)), \end{aligned}$$

so

$$b''_{st} = e_{s,n+t} - e_{t,n+s} + \beta_k^{tk} e_s \in \varphi_{q'}(\varphi_q(B)) \text{ for all } 1 \le s < t < k. \tag{3.8.4}$$

Put $\varphi' = \varphi_{q'} \circ \varphi_q$. Then φ' is a special inner automorphism of A with $b''_{sk} = e_{s,n+k} - e_{k,n+s} \in \varphi'(B)$ and $b''_{st} = e_{s,n+t} - e_{t,n+s} + \beta_k^{tk} e_s \in \varphi'(B)$ (by (3.8.3) and (3.8.4), respectively), as required.

CLAIM 3: There is a special inner automorphism $\varphi_{q_1} : A \to A$ such that

$$b^{(1)}_{1t} = e_{1,n+t} - e_{t,n+1} \in \varphi_{q_1}(\varphi'(B)) = B_1 \quad \text{for all} \quad 1 < t \le k; \tag{3.8.5}$$

$$b^{(1)}_{st} = e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in B_1 \quad \text{for all} \quad 1 < s < t \le k. \tag{3.8.6}$$

By Claim 2, there is a special inner automorphism $\varphi' : A \to A$ such that $b''_{sk} = e_{s,n+k} - e_{k,n+s} \in \varphi'(B)$ and $b''_{st} = e_{s,n+t} - e_{t,n+s} + \beta_k^{tk} e_s \in \varphi'(B)$ for $t < k$. Consider the special inner automorphism φ_{q_1} of A, where

$$q_1 = \sum_{j=2}^{k-1} \beta_k^{jk} e_{n+j} \in U.$$

Put $B_1 = \varphi_{q_1}(\varphi'(B))$ and $b^{(1)}_{st} = \varphi_{q_1}(b''_{st}) \in B_1$. Since $UA = 0$,

$$\begin{aligned} b^{(1)}_{1k} &= \varphi_{q_1}(b''_{1k}) = (1+q)b''_{1k}(1-q_1) = b''_{1k}(1-q_1) \\ &= (e_{1,n+k} - e_{k,n+1})(1 - \sum_{j=2}^{k-1} \beta^{jk}_k e_{n+j}) = e_{1,n+k} - e_{k,n+1} \in B_1 \end{aligned}$$

and (for all $t < k$)

$$\begin{aligned} b^{(1)}_{1t} &= \varphi_{q_1}(b''_{1t}) = b''_{1t}(1-q_1) = (e_{1,n+t} - e_{t,n+1} + \beta^{tk}_k e_1)(1 - \sum_{j=2}^{k-1} \beta^{jk}_k e_{n+j}) \\ &= e_{1,n+t} - \beta^{tk}_k e_1 - e_{t,n+1} + \beta^{tk}_k e_1 = e_{1,n+t} - e_{t,n+1} \in B_1. \end{aligned}$$

Hence, $b^{(1)}_{1t} = e_{1,n+t} - e_{t,n+1} \in B_1$ for all $1 < t \leq k$, so (3.8.5) holds. It remains to show that that (3.8.6) holds. Applying φ_{q_1} to b''_{st} for all $s > 1$, we get that

$$\begin{aligned} b^{(1)}_{sk} &= \varphi_{q_1}(b''_{sk}) = b''_{sk}(1-q_1) = (e_{s,n+k} - e_{k,n+s})(1 - \sum_{j=2}^{k-1} \beta^{jk}_k e_{n+j}) \\ &= e_{s,n+k} - e_{k,n+s} + \beta^{sk}_k e_k \in B_1 \end{aligned}$$

and (for all $t < k$)

$$\begin{aligned} b^{(1)}_{st} &= \varphi_{q_1}(b''_{st}) = b_{st}(1-q_1) = (e_{s,n+t} - e_{t,n+s} + \beta^{tk}_k e_s)(1 - \sum_{j=2}^{k-1} \beta^{jk}_k e_{n+j}) \\ &= e_{s,n+t} - \beta^{tk}_k e_s - e_{t,n+s} + \beta^{sk}_k e_t + \beta^{tk}_k e_s = e_{s,n+t} - e_{t,n+s} + \beta^{sk}_k e_t \in B_1. \end{aligned}$$

Therefore, $b^{(1)}_{st} = e_{s,n+t} - e_{t,n+s} + \beta^{sk}_k e_t \in B$ for all $1 < s < t \leq k$, so (3.8.6) holds, as required.

CLAIM 4: There are $k-2$ inner automorphisms φ_{q_ι} ($\iota = 1, \dots, k-2$) on A such that

$$b^{(k-2)}_{\iota t} = e_{\iota,n+t} - e_{t,n+\iota} \in B_{k-2} \quad \text{for all} \quad \iota < t \leq k; \tag{3.8.7}$$

$$b^{(k-2)}_{st} = e_{s,n+t} - e_{t,n+s} + \beta^{sk}_k e_k \in B_{k-2} \quad \text{for all} \quad k-2 < s < t \leq k, \tag{3.8.8}$$

where $B_{k-2} = \varphi_{q_{k-2}}(\dots \varphi_{q_1}(\varphi'(B))\dots)$

We are going to prove Claim 4 by induction on ι. The base of the induction (when $\iota = 1$) being clear by Claim 3. Suppose that $\iota > 1$. Put $\kappa = k-2$. By the inductive hypothesis there are $\kappa - 1$ inner automorphisms φ_{q_r} ($r = 1, \dots, \kappa-1$) on A such that

$$b^{(\kappa-1)}_{rt} = e_{r,n+t} - e_{t,n+r} \in B_{\kappa-1} \quad \text{for all} \quad r < t \leq k \quad \text{and}$$

$$b_{st}^{(\kappa-1)} = e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in B_{\kappa-1} \quad \text{for all} \quad \kappa-1 < s < t \leq k.$$

Consider the inner automorphism $\varphi_{q_\kappa} : A \to A$, where

$$q_\kappa = -\beta_k^{\kappa k} e_{n+\kappa} \in U.$$

Put $B_\kappa = \varphi_{q_\kappa}(B_{\kappa-1})$ and $c_{st}^{(\kappa)} = \varphi_q(b_{st}^{(\kappa)}) \in B_\kappa$ for all s and t. Since $UA = 0$, for all $r = 1, \ldots, \kappa-1$, we have

$$\begin{aligned}
c_{r\kappa}^{(\kappa)} &= \varphi_{q_\kappa}(b_{r\kappa}^{(\kappa-1)}) = (1+q_\kappa) b_{r\kappa}^{(\kappa-1)} (1-q_\kappa) = b_{r\kappa}^{(\kappa-1)}(1-q_\kappa) \\
&= (e_{r,n+\kappa} - e_{\kappa,n+r})(1+\beta_k^{\kappa k} e_{n+\kappa}) = e_{r,n+\kappa} - e_{\kappa,n+r} + \beta_k^{\kappa k} e_r \in B_\kappa
\end{aligned}$$

and (for all $t \neq \kappa$)

$$\begin{aligned}
c_{rt}^{(\kappa)} &= \varphi_{q_\kappa}(b_{rt}^{(\kappa-1)}) = b_{rt}^{(\kappa-1)}(1-q_\kappa) = (e_{r,n+t} - e_{t,n+r})(1+\beta_k^{\kappa k} e_{n+\kappa}) \\
&= e_{r,n+t} - e_{t,n+r} \in B_\kappa. \qquad (3.8.9)
\end{aligned}$$

Note that if $s \geq \kappa$, then $t > \kappa$, so

$$\begin{aligned}
c_{\kappa t}^{(\kappa)} &= \varphi_{q_\kappa}(b_{\kappa t}^{(\kappa-1)}) = b_{\kappa t}^{(\kappa-1)}(1-q_\kappa) = (e_{\kappa,n+t} - e_{t,n+\kappa} + \beta_k^{\kappa k} e_t)(1+\beta_k^{\kappa k} e_{n+\kappa}) \\
&= e_{\kappa,n+t} - e_{t,n+\kappa} - \beta_k^{\kappa k} e_t + \beta_k^{\kappa k} e_t = e_{\kappa,n+t} - e_{t,n+\kappa} \in B_\kappa \qquad (3.8.10)
\end{aligned}$$

and (for $s > \kappa$)

$$\begin{aligned}
c_{st}^{(\kappa)} &= \varphi_{q_\kappa}(b_{st}^{(\kappa-1)}) = b_{st}^{(\kappa-1)}(1-q_\kappa) = (e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t)(1+\beta_k^{\kappa k} e_{n+\kappa}) \\
&= e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in B_\kappa. \qquad (3.8.11)
\end{aligned}$$

Put $b_{r\kappa}^{(\kappa)} = \{c_{r\kappa}^{(\kappa)}, e_{n+t,\kappa} - e_{n+\kappa,t}, c_{\kappa t}^{(\kappa)}\} \in B_\kappa$ (by Lemma 3.1.3) and $b_{st}^{(\kappa)} = c_{st}^{(\kappa)} \in B_\kappa$ for all of the remaining indices. Then by (3.8.11), $b_{st}^{(\kappa)} = c_{st}^{(\kappa)} = e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in B_\kappa$ for all $k-2 < s < t \leq k$, so (3.8.8) is proved. It remains to show that (3.8.7) holds. For all $r = 1, \ldots, \kappa-1$, we have

$$\begin{aligned}
b_{r\kappa}^{(\kappa)} &= c_{\kappa t}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t}) b_{r\kappa}^{(\kappa)} + b_{r\kappa}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t}) b_{\kappa t}^{(\kappa)} \\
&= (e_{\kappa,n+t} - e_{t,n+\kappa})(e_{n+t,\kappa} - e_{n+\kappa,t}) c_{r\kappa}^{(\kappa)} + c_{r\kappa}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t}) c_{\kappa t}^{(\kappa)} \\
&= (e_{\kappa\kappa} + e_{tt})(e_{r,n+\kappa} - e_{\kappa,n+r} + \beta_k^{\kappa k} e_r) + c_{r\kappa}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t}) c_{\kappa t}^{(\kappa)} \\
&= -e_{\kappa,n+r} + (e_{r,n+\kappa} - e_{\kappa,n+r} + \beta_k^{\kappa k} e_r)(e_{n+t,\kappa} - e_{n+\kappa,t}) b_{\kappa t}^{(\kappa)} \\
&= -e_{\kappa,n+r} - e_{rt}(e_{\kappa,n+t} - e_{t,n+\kappa}) = e_{r,n+\kappa} - e_{\kappa,n+r} \in B_\kappa.
\end{aligned}$$

Combining this result with (3.8.9), we get that $b_{rt}^{(\kappa)} = e_{r,n+t} - e_{t,n+r} \in B_\kappa$ for all $r < t \leq k$.

By (3.8.10), $b_{\kappa t}^{(\kappa)} = c_{\kappa t}^{(\kappa)} = e_{\kappa,n+t} - e_{t,n+\kappa} \in B_\kappa$ for all $t > \kappa$, so

$$b_{\iota t}^{(\kappa)} = e_{\iota,n+t} - e_{t,n+\iota} \in B_\kappa \quad \text{for all} \quad \iota < t \le k \quad \text{where} \quad \iota = 1,\ldots,\kappa.$$

This proves (3.8.7), as $\kappa = k-2$, as required.

Now, we are going to define the final inner automorphism in order to complete the proof. By Claim 4, there are $k-2$ inner automorphisms φ_{q_ι} $(\iota = 1,\ldots,k-2)$ on A such that

$$b_{\iota t}^{(k-2)} = e_{\iota,n+t} - e_{t,n+\iota} \in \varphi_{q_{k-2}}(\ldots\varphi_{q_1}(\varphi'(B))\ldots)) = B_{k-2} \text{ for all } \iota < t \le k$$

and (for $s > k-2$)

$$b_{st}^{(k-2)} = b_{k-1,k}^{(k-2)} = e_{s,n+t} - e_{t,n+s} + \beta_k^{sk} e_t \in B_{k-2}.$$

Put $\nu = k-1$. Consider the final inner automorphism $\varphi_{q_\nu} : A \to A$, where

$$q_\nu = -\beta_k^{\nu k} e_{n+\nu}.$$

Put $B_\nu = \varphi_{q_\nu}(B_{k-2})$ and $b_{st}^{(\nu)} = \varphi_{q_\nu}(b_{st}^{(k-2)})$ for all s and t. Since $UA = 0$, for all $\iota = 1,\ldots,k-2$, we have

$$\begin{aligned} b_{\iota\nu}^{(\nu)} &= \varphi_{q_\nu}(b_{\iota\nu}^{(k-2)}) = (1+q_\nu)b_{\iota\nu}^{(k-2)}(1-q_\nu) = b_{\iota\nu}^{(k-2)}(1-q_\nu) \\ &= (e_{\iota,n+\nu} - e_{\nu,n+\iota})(1+\beta_k^{\nu k} e_{n+\nu}) = e_{\iota,n+\nu} - e_{\nu,n+\iota} + \beta_k^{\nu k} e_\iota \in B_\nu \end{aligned}$$

and (for all $t \ne \nu$)

$$b_{\iota t}^{(\nu)} = b_{\iota t}^{(k-2)}(1-q_\nu) = (e_{\iota,n+t} - e_{t,n+\iota})(1+\beta_k^{\nu k} e_{n+\nu}) = e_{\iota,n+t} - e_{t,n+\iota} \in B_\nu. \quad (3.8.12)$$

For $s = \nu$, we have

$$\begin{aligned} b_{\nu k}^{(\nu)} &= b_{\nu k}^{(k-2)}(1-q_\nu) = (e_{\nu,n+k} - e_{k,n+\nu} + \beta_k^{\nu k} e_k)(1+\beta_k^{\nu k} e_{n+\nu}) \\ &= e_{\nu,n+k} - e_{k,n+\nu} - \beta_k^{\nu k} e_k + \beta_k^{\nu k} e_k = e_{\nu,n+k} - e_{k,n+\nu} \in B_\nu. \end{aligned} \quad (3.8.13)$$

Put $b_{\iota\nu}^{(k)} = \{b_{\iota\nu}^{(\nu)}, e_{n+k,\nu} - e_{n+\nu,k}, b_{\nu k}^{(\nu)}\} \in B_\nu$ (by Lemma 3.1.3) and $b_{st}^{(k)} = b_{st}^{(\nu)} \in B_\nu$ for all the remaining indices s and t. Since $UA = 0$,

$$\begin{aligned} b_{\iota\nu}^{(k)} &= b_{\iota\nu}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k})b_{\nu k}^{(\nu)} + b_{\nu k}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k})b_{\iota\nu}^{(\nu)} \\ &= (e_{\iota,n+\nu} - e_{\nu,n+\iota} + \beta_k^{\nu k} e_\iota)(e_{n+k,\nu} - e_{n+\nu,k})b_{\nu k}^{(\nu)} + b_{\nu k}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k})b_{\iota\nu}^{(\nu)} \end{aligned}$$

$$
\begin{aligned}
&= -e_{\iota k}(e_{\nu,n+k}-e_{k,n+\nu})+(e_{\nu,n+k}-e_{k,n+\nu})(e_{n+k,\nu}-e_{n+\nu,k})b_{\iota\nu}^{(\nu)}\\
&= e_{\iota,n+\nu}+(e_{\nu\nu}+e_{kk})(e_{\iota,n+\nu}-e_{\nu,n+\iota}+\beta_k^{\nu k}e_\iota)=e_{\iota,n+\nu}-e_{\nu,n+\iota}\in B_\nu.
\end{aligned}
$$

Combining this with (3.8.12), we get that $b_{\iota t}^{(k)}=e_{\iota,n+t}-e_{t,n+\iota}\in B_\nu$ for all t. By (3.8.13), $b_{\nu k}^{(k)}=b_{\nu k}^{(\nu)}=e_{\nu,n+k}-e_{k,n+\nu}\in B_\nu$. Recall that $\nu=k-1$. Therefore, $b_{st}^{(k)}=e_{s,n+t}-e_{t,n+s}\in B_\nu=B_{k-1}$ for all $1\le s<t\le k$. Put $\varphi=\varphi_{k-1}\circ\ldots\circ\varphi_{q_1}\circ\varphi'$. Then $\varphi:A\to A$ is a special inner automorphism with

$$E^+=\{e_{s,n+t}-e_{t,n+s}\mid 1\le s<t\le k\}\subseteq\varphi(B)\cap S,$$

as required.

Case (3): Suppose that $\mathscr{E}=E^-=\{e_{s,n+t}+e_{t,n+s}\mid 1\le s\le t\le k\}\subseteq\mathfrak{su}^*(S)=\mathfrak{sp}_{2n}$. As proved in Case (2), there is a special inner automorphism $\varphi:A\to A$ such that

$$\{h_{st}=e_{s,n+t}+e_{t,n+s}\mid 1\le s<t\le k\}\subseteq\varphi(B)\cap S.$$

Put $h_{kk}=h_{sk}e_{n+s,s}h_{sk}$ and $h_{ss}=h_{st}e_{n+t,t}h_{st}$ for all $1\le s<t<k$. Since $e_{n+i,i}\in K$ for all $1\le i\le k$, by Lemma 3.1.3,

$$h_{ss}=h_{st}e_{n+t,t}h_{st}=(e_{s,n+t}+e_{t,n+s})e_{n+t,t}(e_{s,n+t}+e_{t,n+s})=e_{s,n+s}\in\varphi(B);$$

$$h_{kk}=h_{sk}e_{n+s,s}h_{st}=(e_{s,n+k}+e_{k,n+s})e_{n+s,s}(e_{s,n+k}+e_{k,n+s})=e_{k,n+k}\in\varphi(B).$$

Hence, $e_{i,n+i}\in\varphi(B)$ for all $1\le i\le k$, so

$$E^-=\{e_{s,n+t}+e_{t,n+s}\mid 1\le s\le t\le k\}\subseteq\varphi(B)\cap S,$$

as required.

Now, by Case (1), Case (2) and Case (3), there is a special inner automorphism $\varphi:A\to A$ such that $\mathscr{E}\subseteq\varphi(\tilde B)\cap\tilde S$. Since $R^2=0$, $\varphi(r')=r'$ for all $r\in\tilde R$. Therefore, $\varphi(\tilde B)=\varphi(\tilde B)_{\tilde S}\oplus\varphi(\tilde B)_{\tilde R}$, where $\varphi(\tilde B)_{\tilde S}=\varphi(\tilde B)\cap\tilde S$ and $\varphi(\tilde B)_{\tilde R}=\varphi(\tilde B)_{\tilde R}\oplus\tilde R$. By changing the Levi subalgebra $\tilde S$ into $S'=\varphi^{-1}(\tilde S)$, we get that $\tilde B=\tilde B_{S'}\oplus\tilde B_{\tilde R}$, where $\tilde B_{S'}=\tilde B\cap S'$ and $\tilde B_{\tilde R}=\tilde B\cap\tilde R$. Hence, $\tilde B$ splits in $\tilde A=A/U^*$. Therefore, by Lemma 3.8.8, B $*$-splits in A.

It remains to show that $B\subseteq S'$ for some $*$-invariant Levi subalgebra S' of A. We have $B=B_{S'}\oplus B_R$, where $B_{S'}=B\cap S'$ for some $*$-invariant Levi subalgebra S' of A. Put $P=[B_{S'},[B_{S'},\mathfrak{su}^*(S)]]\subseteq\mathfrak{su}^*(S)\cap B$. Since $\bar A$ is semisimple, by Lemma 3.6.1,

$$\bar P=[\bar B_{S'},[\bar B_{S'},\mathfrak{su}^*(\bar S)]]=[\bar B,[\bar B,\bar K^{(1)}]]=\bar B.$$

Note that $B' = B \cap \mathfrak{su}^*(S)$ is a Jordan-Lie inner ideal of $\mathfrak{su}^*(S)$ with $\bar{B}' = \bar{B}$ (because $\bar{B}' \subseteq \bar{B}$ and $\bar{B} = \bar{P} \subseteq \bar{B}'$). By Lemma 3.8.9, B' is a Jordan-Lie inner ideal of $K^{(1)}$. Since $\bar{B}' = \bar{B}$ and B is bar-minimal, we must have $B = B' \subseteq S'$, as required.

□

Now, we are ready to prove Proposition 3.8.2.

Proof of Proposition 3.8.2. We identify A/R with S. Since S is involution simple, by Proposition 3.3.2, S is either simple, or $S = S_1 \oplus S_1^*$, where S_1 is a simple ideal of S. Suppose first that $S = S_1 \oplus S_1^*$. Recall that $R = U \oplus U^*$, where U is the natural left S-module. Let $D = S_1 \oplus U$. Then D is an ideal of A and $A = D \oplus D^*$, so by Proposition 3.6.12, B $*$-splits in A.

Suppose now that S is simple. Since $SU = U$, as a left S-module U is a direct sum of copies of the irreducible left S-module V. Since $R = U \oplus U^*$, R is completely reducible and can be written as a direct sum of copies of $*$-irreducible S-S-submodules $V \oplus V^*$. The proof is by induction on the length $\ell(R)$. The case $\ell(R) = 2$ being clear by Lemma 3.8.10. Suppose now that $\ell(R) > 2$. Consider any maximal $*$-invariant submodule T of R. Then T is an ideal of A with $\ell(T) < \ell(R)$. Let $\tilde{}: A \to A/T$ be the natural epimorphism of A onto $\tilde{A} = A/T$. Denote by $\tilde{B}$ and $\tilde{R}$ the images of B and R, respectively, in $\tilde{A}$. Since $\ell(\tilde{R}) = 2$, by Lemma 3.8.10, $\tilde{B}$ is contained in a $*$-invariant Levi subalgebra S' of $\tilde{A}$. Note that S' is also a $*$-invariant Levi subalgebra of A. Let G be the full preimage of S' in A. Then G is a large subalgebra of A and $\operatorname{rad} G = T$. Since A is admissible and $R^2 = 0$, by Proposition 3.5.13, G is admissible, so by Lemma 3.5.7(i), $G = \mathscr{P}_a(G)$. Fix any $*$-invariant Levi subalgebra S'' of G. Put $P = [B,[B,\mathfrak{su}^*(S'')]] \subseteq G$. Then $P \subseteq [B,[B,\mathfrak{su}^*(G)]] \subseteq B$ (because $\mathfrak{su}^*(G)$ is a subalgebra of $K^{(1)}$), so $P \subseteq B \cap \mathfrak{su}^*(G) = B'$. Moreover, $\bar{P} = [\bar{B},[\bar{B},\mathfrak{su}^*(\bar{G})]] = \bar{B}$. By Lemma 2.1.10, B' is a Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$. Since $\bar{B} = \bar{P} \subseteq \bar{B}'$ and $\bar{B}' \subseteq \bar{B}$, we get that $\bar{B}' = \bar{B}$. Let $B'' \subseteq B'$ be any $\bar{B}'$-minimal Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$. Since G is admissible, $\ell(T) < \ell(R)$, by the inductive hypothesis B'' $*$-splits in $G = \mathscr{P}_a(G)$. Since $B'' \subseteq B' \subseteq B$ with $\bar{B}'' = \bar{B}' = \bar{B}$, by Lemma 3.6.13, B $*$-splits in A, as required.

□

Proof of Proposition 3.8.3

Recall Lemma 3.4.6 that if R is an irreducible S-bimodule, then as a vector space

$\mathfrak{u}^*(R)$ is one of the following:

$$\mathrm{sym}^{\rho}_{\tau_\varepsilon}(\mathscr{M}_{2n}) = \{\begin{pmatrix} X & Y_1 \\ Y_2 & \rho X^t \end{pmatrix} \mid X, Y_1, Y_2 \in \mathscr{M}_n, \quad Y_1^t = \rho\varepsilon Y_1, \quad Y_2^t = \rho\varepsilon Y_2\};$$

$$\mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n+1}) = \{\begin{pmatrix} \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_{2n}) & & \begin{matrix} Y_3 \\ Y_4 \end{matrix} \\ -\rho Y_4^t & -\rho Y_3^t & \alpha \end{pmatrix} \mid Y_3, Y_4 \in \mathscr{M}_{n1}, \quad \alpha \in \mathbb{F}\},$$

where $\alpha = 0$ if $\rho = -1$.

Note that as vector spaces $\mathrm{sym}^{-}_{\tau_-}(\mathscr{M}_{2n}) = \mathfrak{sp}_{2n}$, $\mathrm{sym}^{-}_{\tau_+}(\mathscr{M}_{2n}) = \mathfrak{so}_{2n}$ and $\mathrm{sym}^{-}_{\tau_+}(\mathscr{M}_{2n+1}) = \mathfrak{so}_{2n+1}$ (see (3.1.4), (3.1.5) and (3.1.6), respectively).

The following lemma represents a special case of Proposition 3.8.3.

Lemma 3.8.11. *Theorem 3.8.1 holds if A/R is simple and R is an irreducible S-bimodule with respect to left and right multiplication.*

Proof. We identify $\bar{A} = A/R$ with S. Since S is simple, by Lemma 3.4.5, $\mathfrak{u}^*(S) \cong \mathfrak{so}_m, \mathfrak{sp}_{2n}$ ($m = 2n+1$ or $2n$) and $\mathfrak{u}^*(R) \cong \mathrm{sym}^{\rho}_{\tau_\varepsilon}(\mathscr{M}_m)$. Recall that $\bar{B}$ is a Jordan-Lie inner ideal of $\bar{A} = S$. As in the proof of Lemma 3.8.10, we fix standard bases of $\{e_{ij} \mid 1 \le i, j \le m\}$ and $\{f_{ij} \mid 1 \le i, j \le m\}$ of S and R, respectively, consisting of matrix units, such that the action of S on R corresponds to matrix multiplication and $\bar{B}$ is the space spanned by $\mathscr{E}$, where $\mathscr{E}$ is one of the following ($\varepsilon = \pm$):

$$E = \{e_{1,n+t} - e_{t,n+1} \mid 1 < t \le k \le n\} \subseteq \mathfrak{su}^*(S) = \mathfrak{so}_m;$$

$$E^{\varepsilon} = \{e_{s,n+t} - \varepsilon e_{t,n+s} \mid 1 \le s \le t \le k \le n\} \subseteq \mathfrak{su}^*(S) = \mathfrak{so}_m, \mathfrak{sp}_{2n}.$$

Our aim to find a special inner automorphism $\varphi_q : A \to A$ for some $q \in \mathfrak{u}^*(R)$ such that $\mathscr{E} \subseteq \varphi(B)$. We need to consider three cases:

Case (1): $\mathscr{E} = E = \{e_{1,n+t} - e_{t,n+1} \mid 1 < t \le k \le n\} \subseteq \mathfrak{su}^*(S) = \mathfrak{so}_m$. Without loss of generality, we can assume that $m = 2n+1$ (the case $m = 2n$ will follow immediately). Recall that $\mathfrak{u}^*(R) = \mathrm{sym}^{\rho}_{\tau_+}(\mathscr{M}_m)$. Fix any $\{b_t \mid 1 < t \le k\} \subseteq B$ such that $\bar{b}_t = e_{1,n+t} - e_{t,n+1}$ for all t. First, we claim that

$$R_0^{\rho} = span\{f_{1,n+t} + \rho f_{t,n+1} \mid 1 < t \le k\} \subseteq B \cap \mathfrak{u}^*(R);$$

$$R_1^+ = span\{f_{l,n+l} \mid 1 \le l \le k\} \subseteq B \cap \mathfrak{u}^*(R) \quad \text{if} \quad \rho = +.$$

We have $R_0^\rho, R_1^+ \subseteq \mathfrak{u}^*(R)$. Recall that $\bar{b}_t = e_{1,n+t} - e_{t,n+1}$ for all t. Then $b_t = e_{1,n+t} - e_{t,n+1} + r_t$ for some $r_t \in \mathfrak{u}^*(R)$. Since $R^2 = 0$, by Lemma 3.1.3,

$$\begin{aligned} b_t(f_{n+t,1} + \rho f_{n+1,t})b_t &= (e_{1,n+t} - e_{t,n+1} + r_t)(f_{n+t,1} + \rho f_{n+1,t})b_t \\ &= (f_{11} - \rho f_{tt})(e_{1,n+t} - e_{t,n+1} + r_t) = f_{1,n+t} + \rho f_{t,n+1} \in B. \end{aligned}$$

Hence, $R_0^\rho \subseteq B \cap \mathfrak{u}^*(R)$. If $\rho = +$, then $f_{n+i,i} \in \mathfrak{u}^*(R)$ for all $1 \le i \le n$, so by Lemma 3.1.3,

$$b_t(f_{n+t,t})b_t = (e_{1,n+t} - e_{t,n+1} + r_t)f_{n+t,t}(e_{1,n+t} - e_{t,n+1} + r_t) = -f_{1,n+1} \in B$$

and

$$b_t(f_{n+1,1})b_t = (e_{1,n+t} - e_{t,n+1} + r_t)f_{n+1,1}(e_{1,n+t} - e_{t,n+1} + r_t) = -f_{t,n+t} \in B.$$

Therefore, $R_1^+ \subseteq B \cap \mathfrak{u}^*(R)$, as required.

Next, for every $b_t = e_{1,n+t} - e_{t,n+1} + r_t \in B$ ($r_t \in R \cap K$), we claim that

$$\begin{aligned} \vartheta(b_t) := \; & e_{1,n+t} - e_{t,n+1} + \sum_{i>k} \alpha_{1i}^t(f_{1,n+i} + \rho f_{i,n+1}) + \sum_{j>1} \alpha_{tj}^t(f_{t,n+j} + \rho f_{j,n+t}) \\ & + \sum_{j=1}^n \beta_{tj}^t(f_{tj} + \rho f_{n+j,n+t}) + \sum_{j=1}^n \beta_{1j}^t(f_{1j} + \rho f_{n+j,n+1}) \\ & + \gamma_{1m}^t(f_{1m} + \rho f_{m,n+1}) + \gamma_{tm}^t(f_{tm} + \rho f_{m,n+t}) \in B, \end{aligned} \tag{3.8.14}$$

where $\alpha_{1i}^t, \alpha_{tj}^t, \beta_{tj}^t, \beta_{1j}^t, \gamma_{1m}^t, \gamma_{1m}^t \in \mathbb{F}$.

Put $c_t = b_t(e_{n+t,1} - e_{n+1,t})b_t \in B$ (by Lemma 3.1.3). Since $r_t \in \mathfrak{u}^*(R) = \mathrm{sym}_{\tau_+}^\rho(\mathscr{M}_m)$, r_t is of the form

$$\begin{aligned} r_t = & \sum_{1 \le i < j \le n} \eta_{ij}^t(f_{i,n+j} + \rho f_{j,n+i}) + \sum_{i,j=1}^n \beta_{ij}^t(f_{ij} + \rho f_{n+j,n+i}) \\ & + \sum_{1 \le i < j \le n} \sigma_{ij}^t(f_{n+i,j} + \rho f_{n+j,i}) + \sum_{i=1}^n \gamma_{im}^t(f_{im} + \rho f_{m,n+i}) \\ & + \sum_{i=1}^n \lambda_{im}^t(f_{n+i,m} + \rho f_{mi}) + \zeta^t(f_{mm} + \rho f_{mm}) \in \mathfrak{u}^*(R). \end{aligned}$$

Since $R^2 = 0$ and $\rho^2 = 1$,

$$\begin{aligned} c_t = \; & b_t(e_{n+t,1} - e_{n+1,t})b_t = (e_{1,n+t} - e_{t,n+1} + \sum_{1 \le i \le j \le n} \eta_{ij}^t(f_{i,n+j} + \rho f_{j,n+i}) \\ & + \sum_{i,j=1}^n \beta_{ij}^t(f_{ij} + \rho f_{n+j,n+i}) + \sum_{1 \le i \le j \le n} \sigma_{ij}^t(f_{n+i,j} + \rho f_{n+j,i}) + \sum_{i=1}^n \gamma_{im}^t(f_{im} \end{aligned}$$

$$
\begin{aligned}
& +\rho f_{m,n+i}) + \sum_{i=1}^{n} \lambda_{im}^{t}(f_{n+i,m} + \rho f_{mi}) + \zeta^{t}(f_{mm} + \rho f_{mm}))(e_{n+t,1} - e_{n+1,t})b_{t} \\
= \; & (e_{11} + e_{tt} + \sum_{i=1}^{n} \eta_{it}^{t} f_{i1} - \sum_{i=1}^{n} \eta_{i1}^{t} f_{it} + \rho \sum_{j=1}^{n} \eta_{tj}^{t} f_{j1} - \rho \sum_{j=1}^{n} \eta_{1j}^{t} f_{jt} \\
& +\rho \sum_{j=1}^{n} \beta_{tj}^{t} f_{n+j,1} - \rho \sum_{j=1}^{n} \beta_{1j}^{t} f_{n+j,t} + \rho \gamma_{tm}^{t} f_{m1} - \rho \gamma_{1m}^{t} f_{mt})(e_{1,n+t} - e_{t,n+1} \\
& + \sum_{1 \le i \le j \le n} \eta_{ij}^{t}(f_{i,n+j} + \rho f_{j,n+i}) + \sum_{i,j=1}^{n} \beta_{ij}^{t}(f_{ij} + \rho f_{n+j,n+i}) \\
& + \sum_{1 \le i \le j \le n} \sigma_{ij}^{t}(f_{n+i,j} + \rho f_{n+j,i}) + \sum_{i=1}^{n} \gamma_{im}^{t}(f_{im} + \rho f_{m,n+i}) \\
& + \sum_{i=1}^{n} \lambda_{im}^{t}(f_{n+i,m} + \rho f_{mi}) + \zeta^{t}(f_{mm} + \rho f_{mm})) \\
= \; & e_{1,n+t} + \sum_{j=1}^{n} \eta_{1j}^{t} f_{1,n+j} + \rho \sum_{i=1}^{n} \eta_{i1}^{t} f_{1,n+i} + \sum_{j=1}^{n} \beta_{1j}^{t} f_{1j} + \gamma_{1m}^{t} f_{1m} \\
& - e_{t,n+1} + \sum_{j=1}^{n} \eta_{tj}^{t} f_{t,n+j} + \rho \sum_{i=1}^{n} \eta_{it}^{t} f_{t,n+i} + \sum_{j=1}^{n} \beta_{tj}^{t} f_{tj} + \gamma_{tm}^{t} f_{tm} \\
& + \sum_{i=1}^{n} \eta_{it}^{t} f_{i,n+t} + \sum_{i=1}^{n} \eta_{i1}^{t} f_{i,n+1} + \rho \sum_{j=1}^{n} \eta_{tj}^{t} f_{j,n+t} + \rho \sum_{j=1}^{n} \eta_{1j}^{t} f_{j,n+1} \\
& + \rho \sum_{j=1}^{n} \beta_{tj}^{t} f_{n+j,,n+t} + \rho \sum_{j=1}^{n} \beta_{1j}^{t} f_{n+j,n+1} + \rho \gamma_{tm}^{t} f_{m,n+t} + \rho \gamma_{1m}^{t} f_{m,n+1} \\
= \; & e_{1,n+t} - e_{t,n+1} + \sum_{j=1}^{n} (\eta_{1j}^{t} + \rho \eta_{j1}^{t}) f_{1,n+j} + \sum_{j=1}^{n} (\rho \eta_{1j}^{t} + \eta_{j1}^{t}) f_{j,n+1} \\
& + \sum_{j=1}^{n} (\eta_{tj}^{t} + \rho \eta_{jt}^{t}) f_{t,n+j} + \sum_{j=1}^{n} (\rho \eta_{tj}^{t} + \eta_{jt}^{t}) f_{j,n+t} \\
& + \sum_{j=1}^{n} \beta_{1j}^{t}(f_{1j} + \rho f_{n+j,n+1}) + \sum_{j=1}^{n} \beta_{tj}^{t}(f_{tj} + \rho f_{n+j,,n+t}) \\
& + \gamma_{1m}^{t}(f_{1m} + \rho f_{m,n+1}) + \gamma_{tm}^{t}(f_{tm} + \rho f_{m,n+t}) \\
= \; & e_{1,n+t} - e_{t,n+1} + \sum_{j=1}^{n} (\eta_{1j}^{t} + \rho \eta_{j1}^{t})(f_{1,n+j} + \rho f_{j,n+1}) \\
& + \sum_{j=1}^{n} (\eta_{tj}^{t} + \rho \eta_{jt}^{t})(f_{t,n+j} + \rho f_{j,n+t}) + \sum_{j=1}^{n} \beta_{1j}^{t}(f_{1j} + \rho f_{n+j,n+1}) \\
& + \sum_{j=1}^{n} \beta_{tj}^{t}(f_{tj} + \rho f_{n+j,,n+t}) + \gamma_{1m}^{t}(f_{1m} + \rho f_{m,n+1}) + \gamma_{tm}^{t}(f_{tm} + \rho f_{m,n+t}) \in B.
\end{aligned}
$$

Put $\alpha^t_{1j} = \eta^t_{1j} + \rho\eta^t_{j1}$ and $\alpha^t_{it} = \eta^t_{ti} + \rho\eta^t_{it}$ for all $1 \le i, j \le n$. Then

$$\begin{aligned}
c_t &= e_{1,n+t} - e_{t,n+1} + \sum_{j=1}^{n} \alpha^t_{1j}(f_{1,n+j} + \rho f_{j,n+1}) + \sum_{i=1}^{n} \alpha^t_{it}(f_{t,n+i} + \rho f_{i,n+t}) \\
&\quad + \sum_{i=1}^{n} \beta^t_{ti}(f_{ti} + \rho f_{n+i,n+t}) + \sum_{j=1}^{n} \beta^t_{1j}(f_{1j} + \rho f_{n+j,n+1}) \\
&\quad + \gamma^t_{1m}(f_{1m} + \rho f_{m,n+1}) + \gamma^t_{tm}(f_{tm} + \rho f_{m,n+t}) \\
&= \vartheta(b) + \sum_{j=1}^{k} \alpha^t_{1j}(f_{1,n+j} + \rho f_{j,n+1}) + \alpha^t_{t1}(f_{t,n+1} + \rho f_{1,n+t}) \in B.
\end{aligned}$$

Since $\sum_{j=2}^{k} \alpha^t_{1j}(f_{1,n+j} + \rho f_{j,n+1}) + \alpha^t_{t1}(f_{t,n+1} + \rho f_{1,n+t}) \in R^\rho_0 \subseteq B$ and $\alpha^t_{11} f_{1,n+1} \in R^+_1 \subseteq B$ (if $\rho = +$), we get that $\vartheta(b) \in B$, as required.

Now, we are going to define special inner automorphisms in order to complete the proof. Recall that we fix $\{b_t \mid 1 < t \le k\} \subseteq B$ such that $\bar{b}_t = e_{1,n+t} - e_{t,n+1}$. Then $b_t = e_{1,n+t} - e_{t,n+1} + r_t$ ($r_t \in \mathfrak{u}^*(R)$). By (3.8.14), there are coefficients such that

$$\begin{aligned}
b_t &= e_{1,n+t} - e_{t,n+1} + \sum_{i>k} \alpha^t_{1i}(f_{1,n+i} + \rho f_{i,n+1}) + \sum_{i>1} \alpha^t_{ti}(f_{t,n+i} + \rho f_{i,n+t}) \\
&\quad + \sum_{j=1}^{n} \beta^t_{tj}(f_{tj} + \rho f_{n+j,n+t}) + \sum_{j=1}^{n} \beta^t_{1j}(f_{1j} + \rho f_{n+j,n+1}) \\
&\quad + \gamma^t_{1m}(f_{1m} + \rho f_{m,n+1}) + \gamma^t_{tm}(f_{tm} + \rho f_{m,n+t}) \in B.
\end{aligned}$$

Consider the special inner automorphism $\varphi_{\mathfrak{q}_1} : A \to A$, where

$$\begin{aligned}
\mathfrak{q}_1 &= -\sum_{i>1} \alpha^k_{ki}(f_{n+1,n+i} + \rho f_{i1}) + \sum_{i>k} \alpha^k_{1i}(f_{n+k,n+i} + \rho f_{ik}) \\
&\quad - \sum_{j=1}^{n} \beta^k_{kj}(f_{n+1,j} + \rho f_{n+j,1}) + \sum_{j=1}^{n} \beta^k_{1j}(f_{n+k,j} + \rho f_{n+j,k}) \\
&\quad - \gamma^k_{km}(f_{n+1,m} + \rho f_{m1}) + \gamma^k_{1m}(f_{n+k,m} + \rho f_{mk}) \in \mathfrak{u}^*(R).
\end{aligned}$$

Put $B_{\mathfrak{q}_1} = \varphi_{\mathfrak{q}_1}(B)$ and $b^{(1)}_t = \varphi_{\mathfrak{q}_1}(b_t) \in B_{\mathfrak{q}_1}$ for all t. Since $R^2 = 0$,

$$\begin{aligned}
b^{(1)}_k &= \varphi_{\mathfrak{q}_1}(b_k) = (1 + \mathfrak{q}_1) b_k (1 - \mathfrak{q}_1) \\
&= (1 - \sum_{i>1} \alpha^k_{ki}(f_{n+1,n+i} + \rho f_{i1}) + \sum_{i>k} \alpha^k_{1i}(f_{n+k,n+i} + \rho f_{ik}) \\
&\quad - \sum_{j=1}^{n} \beta^k_{kj}(f_{n+1,j} + \rho f_{n+j,1}) + \sum_{j=1}^{n} \beta^k_{1j}(f_{n+k,j} + \rho f_{n+j,k}) \\
&\quad - \gamma^k_{km}(f_{n+1,m} + \rho f_{m1}) + \gamma^k_{1m}(f_{n+k,m} + \rho f_{mk}))(e_{1,n+k} - e_{k,n+1} \\
&\quad + \sum_{i>k} \alpha^k_{1i}(f_{1,n+i} + \rho f_{i,n+1}) + \sum_{i>1} \alpha^k_{ki}(f_{k,n+i} + \rho f_{i,n+k})
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{n}\beta_{1j}^{k}(f_{1j}+\rho f_{n+j,n+1})+\sum_{j=1}^{n}\beta_{kj}^{k}(f_{kj}+\rho f_{n+j,n+k}) \\
& +\gamma_{1m}^{k}(f_{1m}+\rho f_{m,n+1})+\gamma_{km}^{k}(f_{km}+\rho f_{m,n+k}))(1-\mathfrak{q}_1) \\
= \;& (e_{1,n+k}-e_{k,n+1}+\sum_{i>k}\alpha_{1i}^{k}(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i>1}\alpha_{ki}^{k}(f_{k,n+i}+\rho f_{i,n+k}) \\
& +\sum_{j=1}^{n}\beta_{1j}^{k}(f_{1j}+\rho f_{n+j,n+1})+\sum_{j=1}^{n}\beta_{kj}^{k}(f_{kj}+\rho f_{n+j,n+k})+\gamma_{1m}^{k}(f_{1m} \\
& +\rho f_{m,n+1})+\gamma_{km}^{k}(f_{km}+\rho f_{m,n+k})-\rho\sum_{i>1}\alpha_{ki}^{k}f_{i,n+k}-\rho\sum_{i>k}\alpha_{1i}^{k}f_{i,n+1} \\
& -\beta_{k1}^{k}f_{n+1,n+k}+\beta_{kk}^{k}f_{n+1,n+1}-\rho\sum_{j=1}^{n}\beta_{kj}^{k}f_{n+j,n+k}+\beta_{11}^{k}f_{n+k,n+k} \\
& -\beta_{1k}^{k}f_{n+k,n+1}-\rho\sum_{j=1}^{n}\beta_{1j}^{k}f_{n+j,n+1}-\rho\gamma_{km}^{k}f_{m,n+k}-\rho\gamma_{1m}^{k}f_{m,n+1})(1-\mathfrak{q}_1) \\
= \;& (e_{1,n+k}-e_{k,n+1}+\sum_{i>k}\alpha_{1i}^{k}f_{1,n+i}+\sum_{i>1}\alpha_{ki}^{k}f_{k,n+i}+\sum_{j=1}^{n}\beta_{1j}^{k}f_{1j}+\sum_{j=1}^{n}\beta_{kj}^{k}f_{kj} \\
& +\gamma_{1m}^{k}f_{1m}+\gamma_{km}^{k}f_{km}-\beta_{k1}^{k}f_{n+1,n+k}+\beta_{kk}^{k}f_{n+1,n+1}+\beta_{11}^{k}f_{n+k,n+k} \\
& -\beta_{1k}^{k}f_{n+k,n+1})(1+\sum_{i>1}\alpha_{ki}^{k}(f_{n+1,n+i}+\rho f_{i1})-\sum_{i>k}\alpha_{1i}^{k}(f_{n+k,n+i}+\rho f_{ik}) \\
& +\sum_{j=1}^{n}\beta_{kj}^{k}(f_{n+1,j}+\rho f_{n+j,1})-\sum_{j=1}^{n}\beta_{1j}^{k}(f_{n+k,j}+\rho f_{n+j,k}) \\
& +\gamma_{km}^{k}(f_{n+1,m}+\rho f_{m1})-\gamma_{1m}^{k}(f_{n+k,m}+\rho f_{mk})) \\
= \;& e_{1,n+k}-\sum_{i>k}\alpha_{1i}^{k}f_{1,n+i}+\rho\beta_{kk}^{k}f_{11}-\sum_{j=1}^{n}\beta_{1j}^{k}f_{1j}-\rho\beta_{1k}^{k}f_{1k}-\gamma_{1m}^{k}f_{1m} \\
& -e_{k,n+1}-\sum_{i>1}\alpha_{ki}^{k}f_{k,n+i}-\sum_{j=1}^{n}\beta_{kj}^{k}f_{kj}-\rho\beta_{k1}^{k}f_{k1}+\rho\beta_{11}^{k}f_{kk}-\gamma_{km}^{k}f_{km} \\
& +\sum_{i>k}\alpha_{1i}^{k}f_{1,n+i}+\sum_{i>1}\alpha_{ki}^{k}f_{k,n+i}+\sum_{j=1}^{n}\beta_{1j}^{k}f_{1j}+\sum_{j=1}^{n}\beta_{kj}^{k}f_{kj}+\gamma_{1m}^{k}f_{1m} \\
& +\gamma_{km}^{k}f_{km}-\beta_{k1}^{k}f_{n+1,n+k}+\beta_{kk}^{k}f_{n+1,n+1}+\beta_{11}^{k}f_{n+k,n+k}-\beta_{1k}^{k}f_{n+k,n+1} \\
b_k^{(1)} = \;& e_{1,n+k}-e_{k,n+1}+\beta_{kk}^{k}(f_{n+1,n+1}+\rho f_{11})+\beta_{11}^{k}(f_{n+k,n+k}+\rho f_{kk}) \\
& -\beta_{k1}^{k}(f_{n+1,n+k}+\rho f_{k1})-\beta_{1k}^{k}(f_{n+k,n+1}+\rho f_{1k})\in\varphi_{\mathfrak{q}_1}(B)=B_{\mathfrak{q}_1}.
\end{aligned}
$$

Since $(B_{\mathfrak{q}_1})^2=0$,

$$
\begin{aligned}
0 = \;& (b_k^{(1)})^2=(e_{1,n+k}-e_{k,n+1}+\beta_{kk}^{k}(f_{n+1,n+1}+\rho f_{11})+\beta_{11}^{k}(f_{n+k,n+k}+\rho f_{kk}) \\
& -\beta_{k1}^{k}(f_{n+1,n+k}+\rho f_{k1})-\beta_{1k}^{k}(f_{n+k,n+1}+\rho f_{1k}))(e_{1,n+k}-e_{k,n+1} \\
& +\beta_{kk}^{k}(f_{n+1,n+1}+\rho f_{11})+\beta_{11}^{k}(f_{n+k,n+k}+\rho f_{kk})-\beta_{k1}^{k}(f_{n+1,n+k}+\rho f_{k1})
\end{aligned}
$$

$$\begin{aligned}
&\quad -\beta_{1k}^k(f_{n+k,n+1}+\rho f_{1k}))\\
&= \beta_{11}^k f_{1,n+k}-\beta_{1k}^k f_{1,n+1}-\beta_{kk}^k f_{k,n+1}+\beta_{k1}^k f_{k,n+k}+\beta_{kk}^k\rho f_{1,n+k}-\beta_{11}^k\rho f_{k,n+1}\\
&\quad -\beta_{k1}^k\rho f_{k,n+k}+\beta_{1k}^k\rho f_{1,n+1}\\
&= (\beta_{11}^k+\rho\beta_{kk}^k)f_{1,n+k}-(\beta_{kk}^k+\rho\beta_{11}^k)f_{k,n+1}-(\beta_{1k}^k-\rho\beta_{1k}^k)f_{1,n+1}\\
&\quad +(\beta_{k1}^k-\rho\beta_{k1}^k)f_{k,n+k}.
\end{aligned}$$

Therefore,

$$\beta_{11}^k+\rho\beta_{kk}^k=0,\quad \beta_{1k}^k-\rho\beta_{1k}^k=0,\quad \text{and}\quad \beta_{k1}^k-\rho\beta_{k1}^k=0 \tag{3.8.15}$$

We need to consider two cases. Suppose first that $\rho=-1$. Then $\beta_{1k}^k=\beta_{k1}^k=0$ and $\beta=\beta_{kk}^k=\beta_{11}^k$, so

$$b_k^{(1)}=e_{1,n+k}-e_{k,n+1}+\beta(f_{n+1,n+1}-f_{11})+\beta(f_{n+k,n+k}-f_{kk})\in B_{\mathfrak{q}_1}.$$

Consider the special inner automorphism $\varphi_{\mathfrak{q}_2^-}:A\to A$, where

$$\mathfrak{q}_2^-=\beta(f_{n+1,k}-f_{n+k,1})\in\mathfrak{u}^*(R).$$

Since $R^2=0$,

$$\begin{aligned}
\varphi_{\mathfrak{q}_2^-}(b_k^{(1)}) &= (1+\mathfrak{q}_2^-)b_k^{(1)}(1-\mathfrak{q}_2^-)=(1+\beta(f_{n+1,k}-f_{n+k,1}))(e_{1,n+k}-e_{k,n+1}\\
&\quad +\beta(f_{n+1,n+1}-f_{11})-\beta(f_{n+k,n+k}-f_{kk}))(1-\mathfrak{q}_2)\\
&= (e_{1,n+k}-e_{k,n+1}+\beta(f_{n+1,n+1}-f_{11})+\beta(f_{n+k,n+k}-f_{kk})-\\
&\quad \beta f_{n+1,n+1}-\beta f_{n+k,n+k}))(1-\mathfrak{q}_2)\\
&= (e_{1,n+k}-e_{k,n+1}-\beta f_{11}-\beta f_{kk})(1+\beta(f_{n+k,1}-f_{n+1,k}))\\
&= e_{1,n+k}+\beta f_{11}-e_{k,n+1}+\beta f_{kk}-\beta f_{11}-\beta f_{kk}\\
&= e_{1,n+k}-e_{k,n+1}\in\varphi_{\mathfrak{q}_2^-}(B_{\mathfrak{q}_1}).
\end{aligned}\tag{3.8.16}$$

Suppose now that $\rho=+1$. Then by (3.8.15), we have $\beta'=\beta_{11}^k=-\beta_{kk}^k$, so

$$\begin{aligned}
b_k^{(1)} &= e_{1,n+k}-e_{k,n+1}+\beta'(f_{11}+f_{n+1,n+1})-\beta'(f_{kk}+f_{n+k,n+1})\\
&\quad -\beta_{1k}^k(f_{1k}+f_{n+k,n+1})-\beta_{k1}^k(f_{k1}+f_{n+1,n+k})\in B_{\mathfrak{q}_1}.
\end{aligned}$$

Consider the special inner automorphism $\varphi_{\mathfrak{q}_2^+}:A\to A$, where

$$\mathfrak{q}_2^+=\beta'(f_{n+1,k}+f_{n+k,1})-\beta_{1k}^k f_{n+k,k}-\beta_{k1}^k f_{n+1,1}\in\mathfrak{u}^*(R).$$

Since $R^2=0$,

$$\begin{aligned}
\varphi_{\mathfrak{q}_2^+}(b_k^{(1)}) &= (1+\mathfrak{q}_2^+)b_k^{(1)}(1-\mathfrak{q}_2^+)\\
&= (1+\beta'(f_{n+1,k}+f_{n+k,1})-\beta_{1k}^k f_{n+k,k}+\beta_{k1}^k f_{n+1,1})(e_{1,n+k}-e_{k,n+1}\\
&\quad +\beta'(f_{11}+f_{n+1,n+1})-\beta'(f_{kk}+f_{n+k,n+k})-\beta_{1k}^k(f_{1k}+f_{n+k,n+1})\\
&\quad -\beta_{k1}^k(f_{k1}+f_{n+1,n+k}))(1-\mathfrak{q}_2^+)\\
&= (e_{1,n+k}-e_{k,n+1}+\beta'(f_{11}+f_{n+1,n+1})-\beta'(f_{kk}+f_{n+k,n+k})\\
&\quad -\beta_{1k}^k(f_{1k}+f_{n+k,n+1})-\beta_{k1}^k(f_{k1}+f_{n+1,n+k})-\beta' f_{n+1,n+1}\\
&\quad +\beta' f_{n+k,n+k}+\beta_{1k}^k f_{n+k,n+1}+\beta_{k1}^k f_{n+1,n+k})(1-\mathfrak{q}_2^+)\\
&= (e_{1,n+k}-e_{k,n+1}+\beta' f_{11}-\beta' f_{kk}-\beta_{1k}^k f_{1k}-\beta_{k1}^k f_{k1})(1\\
&\quad -\beta'(f_{n+1,k}+f_{n+k,1})+\beta_{1k}^k f_{n+k,k}-\beta_{k1}^k f_{n+1,1}))\\
&= e_{1,n+k}-\beta' f_{11}+\beta_{1k}^k f_{1k}-e_{k,n+1}+\beta' f_{kk}+\beta_{k1}^k f_{k1}+\beta' f_{11}\\
&\quad -\beta' f_{kk}-\beta_{1k}^k f_{1k}-\beta_{k1}^k f_{k1}\\
&= e_{1,n+k}-e_{k,n+1}\in\varphi_{\mathfrak{q}_2^+}(B_{\mathfrak{q}_1}).
\end{aligned}\tag{3.8.17}$$

Put $B_{\mathfrak{q}_2}=\varphi_{\mathfrak{q}_2^\rho}(B_{\mathfrak{q}_1})$ and $b_t^{(2)}=\varphi_{\mathfrak{q}_2^\rho}(b_t^{(1)})\in B_{\mathfrak{q}_2}$ for all t. Then by (3.8.16) and (3.8.17), we have

$$b_k^{(2)}=e_{1,n+k}-e_{k.n+1}\in\varphi_{\mathfrak{q}_2^\rho}(B_{\mathfrak{q}_1})=B_{\mathfrak{q}_2}.$$

By (3.8.14), there are coefficients such that (for all $t<k$)

$$\begin{aligned}
\vartheta(b_t^{(2)}) &= e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i>1}\zeta_{ti}^t(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad +\sum_{j=1}^n\eta_{tj}^t(f_{tj}+\rho f_{n+j,n+t})+\sum_{j=1}^n\eta_{1j}^t(f_{1j}+\rho f_{n+j,n+1})\\
&\quad +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\xi_{tm}^t(f_{tm}+\rho f_{m,n+t})\in B_{\mathfrak{q}_2}.
\end{aligned}$$

Put $b_k^{(3)}=b_k^{(2)}\in B_{\mathfrak{q}_2}$. For all $t<k$, set $b_t^{(3)}=\{b_k^{(2)},e_{n+k,1}-e_{n+1,k},\vartheta(b_t^{(2)})\}\in B_{\mathfrak{q}_2}$ (by Lemma 3.1.3). Then for all $t<k$,

$$\begin{aligned}
b_t^{(3)} &= b_k^{(2)}(e_{n+k,1}-e_{n+1,k})\vartheta(b_t^{(2)})+\vartheta(b_t^{(2)})(e_{n+k,1}-e_{n+1,k})b_k^{(2)}\\
&= (e_{1,n+k}-e_{k,n+1})(e_{n+k,1}-e_{n+1,k})\vartheta(b_t^{(2)})\\
&\quad +(e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i>1}\zeta_{ti}^t(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad +\sum_{j=1}^n\eta_{tj}^t(f_{tj}+\rho f_{n+j,n+t})+\sum_{j=1}^n\eta_{1j}^t(f_{1j}+\rho f_{n+j,n+1})\\
&\quad +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\xi_{tm}^t(f_{tm}+\rho f_{m,n+t}))(e_{n+k,1}-e_{n+1,k})b_k^{(2)}
\end{aligned}$$

$$\begin{aligned}
&= (e_{11}+e_{kk})(e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})\\
&\quad +\sum_{i>1}\zeta_{ti}^t(f_{t,n+i}+\rho f_{i,n+t})+\sum_{j=1}^n \eta_{tj}^t(f_{tj}+\rho f_{n+j,n+t})\\
&\quad +\sum_{j=1}^n \eta_{1j}^t(f_{1j}+\rho f_{n+j,n+1})+\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\xi_{tm}^t(f_{tm}+\rho f_{m,n+t}))\\
&\quad +(e_{tk}-\rho\sum_{i>k}\zeta_{1i}^t f_{ik}+\zeta_{tk}^t f_{t1}-\rho\sum_{j=1}^n \eta_{1j}^t f_{n+j,k}-\rho\xi_{1m}^t f_{mk})(e_{1,n+k}-e_{k,n+1})\\
&= e_{1,n+t}+\sum_{i>k}\zeta_{1i}^t f_{1,n+i}+\sum_{j=1}^n \eta_{1j}^t f_{1j}+\xi_{1m}^t f_{1m}+\rho\zeta_{tk}^t f_{k,n+t}-e_{t,n+1}\\
&\quad +\rho\sum_{i>k}\zeta_{1i}^t f_{i,n+1}+\zeta_{tk}^t f_{t,n+k}+\rho\sum_{j=1}^n \eta_{1j}^t f_{n+j,n+1}+\rho\xi_{1m}^t f_{m,n+1}\\
&= e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{j=1}^n \eta_{1j}^t(f_{1j}+\rho f_{n+j,n+1})\\
&\quad +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{tk}^t(f_{t,n+k}+\rho f_{k,n+t})\in B_{\mathfrak{q}_2}.
\end{aligned}$$

Since $(B_{\mathfrak{q}_2})^2=0$, for all $t<k$, we have

$$\begin{aligned}
0 &= b_k^{(3)}b_t^{(3)} = (e_{1,n+k}-e_{k,n+1})(e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})\\
&\quad +\sum_{j=1}^n \eta_{1j}^t(f_{1j}+\rho f_{n+j,n+1})+\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{tk}^t(f_{t,n+k}+\rho f_{k,n+t}))\\
&= \rho\eta_{1k}^t f_{1,n+1}-\rho\eta_{11}^t f_{k,n+1},
\end{aligned}$$

so $\eta_{1k}^t=\eta_{11}^t=0$. Therefore, for all $1<t<k$,

$$\begin{aligned}
b_t^{(3)} &= e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^t(f_{1i}+\rho f_{n+i,n+1})\\
&\quad +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{tk}^t(f_{t,n+k}+\rho f_{k,n+t})\in B_{\mathfrak{q}_2}.
\end{aligned}$$

Put $b_2^{(4)}=b_2^{(3)}\in B_{\mathfrak{q}_2}$ and $b_k^{(4)}=b_k^{(3)}\in B_{\mathfrak{q}_2}$. For all $2<t<k$ set $b_t^{(4)}=\{b_2^{(3)},e_{n+2,1}-e_{n+1,2},b_t^{(3)}\}\in B_{\mathfrak{q}_2}$ (by Lemma 3.1.3). Then

$$\begin{aligned}
b_t^{(4)} &= b_2^{(3)}(e_{n+2,1}-e_{n+1,2})b_t^{(3)}+b_t^{(3)}(e_{n+2,1}-e_{n+1,2})b_2^{(3)}\\
&= (e_{1,n+2}-e_{2,n+1}+\sum_{i>k}\zeta_{1i}^2(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^2(f_{1i}+\rho f_{n+i,n+1})\\
&\quad +\xi_{1m}^2(f_{1m}+\rho f_{m,n+1})+\zeta_{2k}^2(f_{2,n+k}+\rho f_{k,n+2}))(e_{n+2,1}-e_{n+1,2})b_t^{(3)}\\
&\quad +(e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^t(f_{1i}+\rho f_{n+i,n+1})
\end{aligned}$$

$$
\begin{aligned}
& +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{tk}^t(f_{t,n+k}+\rho f_{k,n+t}))(e_{n+2,1}-e_{n+1,2})b_2^{(3)} \\
= \; & (e_{11}+e_{22}-\rho\sum_{i>k}\zeta_{1i}^2 f_{i2}-\rho\sum_{i\neq 1,k}\eta_{1i}^2 f_{n+i,2}-\rho\xi_{1m}^2 f_{m2}+\rho\zeta_{2k}^2 f_{k1})(e_{1,n+t} \\
& -e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^t(f_{1i}+\rho f_{n+i,n+1}) \\
& +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{tk}^t(f_{t,n+k}+\rho f_{k,n+t})) \\
& +(e_{t2}-\rho\sum_{i>k}\zeta_{1i}^t f_{i2}-\rho\sum_{i\neq 1,k}\eta_{1i}^t f_{n+i,2}-\rho\xi_{1m}^t f_{m2})(e_{1,n+2}-e_{2,n+1} \\
& +\sum_{i>k}\zeta_{1i}^2(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^2(f_{1i}+\rho f_{n+i,n+1}) \\
& +\xi_{1m}^2(f_{1m}+\rho f_{m,n+1})+\zeta_{2k}^2(f_{2,n+k}+\rho f_{k,n+2})) \\
= \; & e_{1,n+t}+\sum_{i>k}\zeta_{1i}^t f_{1,n+i}+\sum_{i\neq 1,k}\eta_{1i}^t f_{1i}+\xi_{1m}^t f_{1m}+\rho\zeta_{2k}^2 f_{k,n+t}-e_{t,n+1} \\
& +\zeta_{2k}^2 f_{t,n+k}+\rho\sum_{i>k}\zeta_{1i}^t f_{i,n+1}+\rho\sum_{i\neq 1,k}\eta_{1i}^t f_{n+i,n+1}+\rho\xi_{1m}^t f_{m,n+1} \\
= \; & e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^t(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^t(f_{1i}+\rho f_{n+i,n+1}) \\
& +\xi_{1m}^t(f_{1m}+\rho f_{m,n+1})+\zeta_{2k}^2(f_{t,n+k}+\rho f_{k,n+t})\in B_{\mathfrak{q}_2}.
\end{aligned}
$$

Consider the special inner automorphism $\varphi_{\mathfrak{q}_3}: A\to A$, where

$$
\begin{aligned}
\mathfrak{q}_3 \; = \; & \sum_{j=2}^{k-1}\sum_{i>k}\zeta_{1i}^j(f_{n+j,n+i}+\rho f_{ij})+\sum_{j=2}^{k-1}\sum_{i\neq 1,k}\eta_{1i}^j(f_{n+j,i}+\rho f_{n+i,j}) \\
& +\sum_{j=2}^{k-1}\xi_{1m}^j(f_{n+j,m}+\rho f_{mj})-\zeta_{2k}^2(f_{n+1,n+k}+\rho f_{k1})\in\mathfrak{u}^*(R).
\end{aligned}
$$

Put $B_{\mathfrak{q}_3}=\varphi_{\mathfrak{q}_3}(B_{\mathfrak{q}_2})$ and $b_t^{(5)}=\varphi_{\mathfrak{q}_3}(b_k^{(4)})\in B_{\mathfrak{q}_3}$. Since $R^2=0$,

$$
\begin{aligned}
b_k^{(5)} \; = \; & \varphi_{\mathfrak{q}_3}(b_k^{(4)})=(1+\mathfrak{q}_3)b_k^{(4)}(1-\mathfrak{q}_3)=(1+\sum_{j=2}^{k-1}\sum_{i>k}\zeta_{1i}^j(f_{n+j,n+i}+\rho f_{ij}) \\
& +\sum_{j=2}^{k-1}\sum_{i\neq 1,k}\eta_{1i}^j(f_{n+j,i}+\rho f_{n+i,j})+\sum_{j=2}^{k-1}\xi_{1m}^j(f_{n+j,m}+\rho f_{mj}) \\
& -\zeta_{2k}^2(f_{n+1,n+k}+\rho f_{k1}))(e_{1,n+k}-e_{k,n+1})(1-\mathfrak{q}_3) \\
= \; & (e_{1,n+k}-e_{k,n+1}-\rho\zeta_{2k}^2 f_{k,n+k})(1-\sum_{j=2}^{k-1}\sum_{i>k}\zeta_{1i}^j(f_{n+j,n+i}+\rho f_{ij}) \\
& -\sum_{j=2}^{k-1}\sum_{i\neq 1,k}\eta_{1i}^j(f_{n+j,i}+\rho f_{n+i,j})-\sum_{i=2}^{k-1}\xi_{1m}^j(f_{n+j,m}+\rho f_{mj})
\end{aligned}
$$

$$
\begin{aligned}
&+\zeta_{2k}^{2}(f_{n+1,n+k}+\rho f_{k1}))\\
=\;& e_{1,n+k}-e_{k,n+1}-\zeta_{2k}^{2}f_{k,n+k}-\rho\zeta_{2k}^{2}f_{k,n+k}\\
=\;& e_{1,n+k}-e_{k,n+1}-(\zeta_{2k}^{2}+\rho\zeta_{2k}^{2})f_{k,n+k}\in B_{\mathfrak{q}_3}.
\end{aligned}
$$

If $\rho=-1$, then $b_k^{(5)}=e_{1,n+k}-e_{k,n+1}\in B_{\mathfrak{q}_3}$. Suppose that $\rho=+1$. Then $b_k^{(5)}=e_{1,n+k}-e_{k,n+1}-2\zeta_k^2 f_{k,n+k}\in B_{\mathfrak{q}_3}$. We have $f_{k,n+k}\in R_1^+\subseteq B$. Since $R^2=0$ and $\mathfrak{q}_i\in\mathfrak{u}^*(R)\subseteq R$ (for all $i=1,2,3$), we have $R_1^+=\varphi_{\mathfrak{q}_i}(R_1^+)$ for each i, so $R_1^+\subseteq B_{\mathfrak{q}_3}$. Hence, $2\zeta_k^2 f_{k,n+k}\in B_{\mathfrak{q}_3}$. Thus, $b_k^{(5)}+2\zeta_k^2 f_{k,n+k}=e_{1,n+k}-e_{k,n+1}\in B_{\mathfrak{q}_3}$. Therefore, for any choice of ρ, we get that

$$e_{1,n+k}-e_{k,n+1}\in B_{\mathfrak{q}_3}. \tag{3.8.18}$$

Put $b_k^{(6)}=e_{1,n+k}-e_{k,n+1}\in B_{\mathfrak{q}_3}$ and for all $t<k$ set $b_t^{(6)}=b_t^{(5)}=\varphi_{\mathfrak{q}_3}(b_t^{(4)})\in B_{\mathfrak{q}_3}$. Since $R^2=0$, for all $t<k$, we have

$$
\begin{aligned}
b_t^{(6)} =\;& b_t^{(5)}=\varphi_{\mathfrak{q}_3}(b_t^{(4)})=(1+\mathfrak{q}_3)b_t^{(4)}(1-\mathfrak{q}_3)\\
=\;& (1+\sum_{j=2}^{k-1}\sum_{i>k}\zeta_{1i}^{j}(f_{n+j,n+i}+\rho f_{ij})+\sum_{j=2}^{k-1}\sum_{i\neq 1,k}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j})\\
&+\sum_{j=2}^{k-1}\xi_{1m}^{j}(f_{n+j,m}+\rho f_{mj})-\zeta_{2k}^{2}(f_{n+1,n+k}+\rho f_{k1}))(e_{1,n+t}-e_{t,n+1}\\
&+\sum_{i>k}\zeta_{1i}^{t}(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^{t}(f_{1i}+\rho f_{n+i,n+1})\\
&+\xi_{1m}^{t}(f_{1m}+\rho f_{m,n+1})+\zeta_{2k}^{2}(f_{t,n+k}+\rho f_{k,n+t}))(1-\mathfrak{q}_3)\\
=\;& (e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^{t}(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i\neq 1,k}\eta_{1i}^{t}(f_{1i}+\rho f_{n+i,n+1})\\
&+\xi_{1m}^{t}(f_{1m}+\rho f_{m,n+1})+\zeta_{2k}^{2}(f_{t,n+k}+\rho f_{k,n+t})-\rho\sum_{i>k}\zeta_{1i}^{t}f_{i,n+1}\\
&-\sum_{j=2}^{k-1}\eta_{1t}^{j}f_{n+j,n+1}-\rho\sum_{i\neq 1,k}\eta_{1i}^{t}f_{n+i,n+1}-\rho\xi_{1m}^{t}f_{m,n+1}-\rho\zeta_{2k}^{2}f_{k,n+t})(1-\mathfrak{q}_3)\\
=\;& (e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\zeta_{1i}^{t}f_{1,n+i}+\sum_{i\neq 1,k}\eta_{1i}^{t}f_{1i}+\xi_{1m}^{t}f_{1m}+\zeta_{2k}^{2}f_{t,n+k}\\
&-\sum_{j=2}^{k-1}\eta_{1t}^{j}f_{n+j,n+1})(1-\sum_{j=2}^{k-1}\sum_{i>k}\zeta_{1i}^{j}(f_{n+j,n+i}+\rho f_{ij})\\
&-\sum_{j=2}^{k-1}\sum_{i\neq 1,k}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j})-\sum_{j=2}^{k-1}\xi_{1m}^{j}(f_{n+j,m}+\rho f_{mj})\\
&+\zeta_{2k}^{2}(f_{n+1,n+k}+\rho f_{k1}))
\end{aligned}
$$

$$\begin{aligned}
&= e_{1,n+t}-\sum_{i>k}\zeta_{1i}^{t}f_{1,n+i}-\sum_{i\neq 1,k}\eta_{1i}^{t}f_{1i}-\rho\sum_{j=2}^{k-1}\eta_{1t}^{j}f_{1j}-\xi_{1m}^{t}f_{1m}\\
&\quad -e_{t,n+1}-\zeta_{2k}^{2}f_{t,n+k}+\sum_{i>k}\zeta_{1i}^{t}f_{1,n+i}+\sum_{i\neq 1,k}\eta_{1i}^{t}f_{1i}+\xi_{1m}^{t}f_{1m}\\
&\quad +\zeta_{2k}^{2}f_{t,n+k}-\sum_{j=2}^{k-1}\eta_{1t}^{j}f_{n+j,n+1}\\
&= e_{1,n+t}-e_{t,n+1}-\sum_{j=2}^{k-1}\eta_{1t}^{j}(f_{n+j,n+1}+\rho f_{1j})\in\varphi_{\mathfrak{q}_3}(B_{\mathfrak{q}_2})=B_{\mathfrak{q}_3}.
\end{aligned}$$

Note that for all $1<t,q<k$, we have

$$\begin{aligned}
0 &= b_q^{(6)}b_t^{(6)}=(e_{1,n+q}-e_{q,n+1}-\sum_{j=2}^{k-1}\eta_{1q}^{j}(f_{n+j,n+1}+\rho f_{1j}))(e_{1,n+t}-e_{t,n+1}\\
&\quad -\sum_{j=2}^{k-1}\eta_{1t}^{j}(f_{n+j,n+1}+\rho f_{1j}))\\
&= -\eta_{1t}^{q}f_{1,n+1}+\rho\eta_{1q}^{t}f_{1,n+1}=-(\eta_{1t}^{q}-\rho\eta_{1q}^{t})f_{1,n+1},
\end{aligned}$$

so

$$\eta_{1t}^{i}-\rho\eta_{1i}^{t}=0\quad\text{for all}\quad 1<i<k. \tag{3.8.19}$$

Consider the final special inner automorphism $\varphi_{\mathfrak{q}_4}:A\to A$, where

$$\mathfrak{q}_4=-\frac{1}{2}\sum_{i,j=2}^{k-1}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j})\in\mathfrak{u}^*(R).$$

Since $R^2=0$, by applying $\varphi_{\mathfrak{q}_4}$ to $b_t^{(6)}\in B_{\mathfrak{q}_3}$ for all t, we get that

$$\begin{aligned}
\varphi_{\mathfrak{q}_4}(b_k^{(6)}) &= (1+\mathfrak{q}_4)b_k^{(6)}(1-\mathfrak{q}_4)\\
&= (1-\frac{1}{2}\sum_{i,j=2}^{k-1}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j}))(e_{1,n+k}-e_{k,n+1})(1-\mathfrak{q}_4)\\
&= (e_{1,n+k}-e_{k,n+1})(1+\frac{1}{2}\sum_{i,j=2}^{k-1}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j}))\\
&= e_{1,n+k}-e_{k,n+1}\in\varphi_{\mathfrak{q}_4}(B_{\mathfrak{q}_3})
\end{aligned}$$

and for $t<k$, by using (3.8.19), we get that

$$\begin{aligned}
\varphi_{\mathfrak{q}_4}(b_t^{(6)}) &= (1+\mathfrak{q}_4)b_t^{(6)}(1-\mathfrak{q}_4)=(1-\frac{1}{2}\sum_{i,j=2}^{k-1}\eta_{1i}^{j}(f_{n+j,i}+\rho f_{n+i,j}))(e_{1,n+t}\\
&\quad -e_{t,n+1}-\sum_{i=2}^{k-1}\eta_{1t}^{i}(f_{n+i,n+1}+\rho f_{1i})(1-\mathfrak{q}_4)
\end{aligned}$$

$$
\begin{aligned}
&= (e_{1,n+t} - e_{t,n+1} - \sum_{i=2}^{k-1} \eta_{1t}^{i}(f_{n+i,n+1} + \rho f_{1i}) + \frac{1}{2}\sum_{j=2}^{k-1} \eta_{1t}^{j} f_{n+j,n+1} \\
&\quad + \rho \frac{1}{2}\sum_{i=2}^{k-1} \eta_{1i}^{t} f_{n+i,n+1})(1-\mathfrak{q}_4) \\
&= (e_{1,n+t} - e_{t,n+1} - \rho \sum_{i=2}^{k-1} \eta_{1t}^{i} f_{1i} - \frac{1}{2}\sum_{i=2}^{k-1} \eta_{1t}^{i} f_{n+i,n+1} \\
&\quad + \rho \frac{1}{2}\sum_{i=2}^{k-1} \eta_{1i}^{t} f_{n+i,n+1})(1-\mathfrak{q}_4) \\
&= (e_{1,n+t} - e_{t,n+1} - \rho \sum_{i=2}^{k-1} \eta_{1t}^{i} f_{1i} - \frac{1}{2}\sum_{i=2}^{k-1} (\eta_{1t}^{i} - \rho \eta_{1i}^{t}) f_{n+i,n+1})(1-\mathfrak{q}_4) \\
&= (e_{1,n+t} - e_{t,n+1} - \rho \sum_{i=2}^{k-1} \eta_{1t}^{i} f_{1i} + 0)(1 + \frac{1}{2}\sum_{i,j=2}^{k-1} \eta_{1i}^{j}(f_{n+j,i} + \rho f_{n+i,j})) \\
&= e_{1,n+t} + \frac{1}{2}\sum_{i=2}^{k-1} \eta_{1i}^{t} f_{1i} + \rho \frac{1}{2}\sum_{j=2}^{k-1} \eta_{1t}^{j} f_{1j} - e_{t,n+1} - \rho \sum_{i=2}^{k-1} \eta_{1t}^{i} f_{1i} \\
&= e_{1,n+t} - e_{t,n+1} + \frac{1}{2}\sum_{i=2}^{k-1} \eta_{1i}^{t} f_{1i} - \rho \frac{1}{2}\sum_{j=2}^{k-1} \eta_{1t}^{i} f_{1i} \\
&= e_{1,n+t} - e_{t,n+1} + \frac{1}{2}\sum_{i=2}^{k-1} (\eta_{1i}^{t} - \rho \eta_{1t}^{i}) f_{1i} = e_{1,n+t} - e_{t,n+1} + 0 \\
&= e_{1,n+t} - e_{t,n+1} \in \varphi_{\mathfrak{q}_4}(B_{\mathfrak{q}_3}).
\end{aligned}
$$

Hence, $\varphi_{\mathfrak{q}_4}(b_t^{(6)}) = e_{1,n+t} - e_{t,n+1}$ for all t. Put $\varphi_{q'} = \varphi_{\mathfrak{q}_4} \circ \varphi_{\mathfrak{q}_3} \circ \varphi_{\mathfrak{q}_2} \circ \varphi_{\mathfrak{q}_1}$. Then $\varphi_{q'}$ is a special inner automorphism with $q' \in \mathfrak{u}^*(R)$ and

$$E = \{e_{1,n+t} - e_{t,n+1} \mid 1 < t \leq k\} \subseteq \varphi_{q'}(B) \cap S.$$

Therefore, if $\mathscr{E} = E$, then there is a special inner automorphism $\varphi_{q'} : A \to A$ such that $\mathscr{E} \subseteq \varphi_{q'}(B) \cap S$, as required.

Case (2): Suppose that $\mathscr{E} = E^+ = \{e_{s,n+t} - e_{t,n+s} \mid 1 \leq s < t \leq k\} \subseteq \mathfrak{su}^*(S) = \mathfrak{so}_m$ ($m = 2n+1$ or $2n$). We need to show that there is a special inner automorphism $\varphi_q : A \to A$ for some $q \in \mathfrak{u}^*(R)$ such that $\mathscr{E} \subseteq \varphi(B) \cap S$. Without loss of generality we can assume $m = 2n+1$ (the case $m = 2n$ will follow immediately). Fix any subset $\{b_{st} \mid 1 \leq s < t \leq k\} \subseteq B$ such that $\bar{b}_{st} = e_{s,n+t} - e_{t,n+s}$ for all s and t. Recall that $\mathfrak{u}^*(R) = \mathrm{sym}_{\tau_+}^{\rho}(\mathscr{M}_m)$. Put

$$R_2^{\rho} = span\{f_{i,n+j} + \rho f_{j,n+i} \mid 1 \leq i \leq j \leq k\} \subseteq \mathfrak{u}^*(R).$$

CLAIM 1: $R_2^\rho \subseteq B$. Since $\bar{b}_{st} = e_{s,n+t} - e_{t,n+s}$, we have $b_{st} = e_{s,n+t} - e_{t,n+s} + r_{st}$ for some $r_{st} \in \mathfrak{u}^*(R)$. Since $R^2 = 0$, by Lemma 3.1.3,

$$\begin{aligned} b_{st}(f_{n+t,s} + \rho f_{n+s,t})b_{st} &= (e_{s,n+t} - e_{t,n+s} + r_{st})(f_{n+t,s} + \rho f_{n+s,t})b_{st} \\ &= (f_{ss} - \rho f_{tt})(e_{s,n+t} - e_{t,n+s} + r_{st}) = f_{s,n+t} + \rho f_{t,n+s} \in B, \end{aligned}$$

Note that if $\rho = +1$, then $f_{n+i,i} \in \mathfrak{su}^*(A)$ for all $1 \le i \le n$, so

$$b_{st} f_{n+s,s} b_{st} = (e_{s,n+t} - e_{t,n+s} + r_{st}) f_{n+s,s}(e_{s,n+t} - e_{t,n+s} + r_{st}) = -f_{t,n+t} \in B$$

and

$$b_{st} f_{n+t,t} b_{st} = (e_{s,n+t} - e_{t,n+s} + r_{st}) f_{n+t,t}(e_{s,n+t} - e_{t,n+s} + r_{st}) = -f_{s,n+s} \in B,$$

as required.

CLAIM 2: For every $b_{st} = e_{s,n+t} - e_{t,n+s} + r_{st} \in B$ ($r_{st} \in R \cap K$), we claim that

$$\begin{aligned} \theta(b_{st}) := \; & e_{s,n+t} - e_{t,n+s} + \sum_{j>k} \alpha_{sj}^{st}(f_{s,n+j} + \rho f_{j,n+s}) + \sum_{i>k} \alpha_{ti}^{st}(f_{t,n+i} + \rho f_{i,n+t}) \\ & + \sum_{j=1}^{n} \beta_{sj}^{st}(f_{sj} + \rho f_{n+j,n+s}) + \sum_{i=1}^{n} \beta_{ti}^{st}(f_{ti} + \rho f_{n+i,n+t}) \\ & + \gamma_s^{st}(f_{sm} + \rho f_{m,n+s}) + \gamma_t^{st}(f_{tm} + \rho f_{m,n+t}) \in B. \end{aligned}$$

By Lemma 3.1.3, $c_{st} = b_{st}(e_{n+t,s} - e_{n+s,t})b_{st} \in B$. Since $r_{st} \in R \cap K^{(1)} \subseteq \mathfrak{u}^*(R)$, r_{st} is of the form

$$\begin{aligned} r_{st} = & \sum_{1 \le i \le j \le n} \eta_{ij}^{st}(f_{i,n+j} + \rho f_{j,n+i}) + \sum_{i,j=1}^{n} \beta_{ij}^{st}(f_{ij} + \rho f_{n+j,n+i}) \\ & + \sum_{1 \le i \le j \le n} \zeta_{ij}^{st}(f_{n+i,j} + \rho f_{n+j,i}) + \sum_{i=1}^{n} \gamma_i^{st}(f_{im} + \rho f_{m,n+i}) \\ & + \sum_{j=1}^{n} \delta_j^{st}(f_{n+j,m} + \rho f_{mi}) + \zeta^{st}(f_{mm} + \rho f_{mm}) \in \mathfrak{u}^*(R). \end{aligned} \tag{3.8.20}$$

As $R^2 = 0$, we have

$$\begin{aligned} c_{st} &= b_{st}(e_{n+t,s} - e_{n+s,t})b_{st} \\ &= (e_{s,n+t} - e_{t,n+s} + \sum_{1 \le i \le j \le n} \eta_{ij}^{st}(f_{i,n+j} + \rho f_{j,n+i}) + \sum_{i,j=1}^{n} \beta_{ij}^{st}(f_{ij} + \rho f_{n+j,n+i}) \\ &\quad + \sum_{1 \le i \le j \le n} \zeta_{ij}^{st}(f_{n+i,j} + \rho f_{n+j,i}) + \sum_{i=1}^{n} \gamma_i^{st}(f_{im} + \rho f_{m,n+i}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\delta_j^{st}(f_{n+j,m}+\rho f_{mi})+\zeta^{st}(f_{mm}+\rho f_{mm}))(e_{n+t,s}-e_{n+s,t})b_{st}\\
=\;&(e_{ss}+e_{tt}+\sum_{i=1}^{n}\eta_{it}^{st}f_{is}-\sum_{i=1}^{n}\eta_{is}^{st}f_{it}+\rho\sum_{j=1}^{n}\eta_{tj}^{st}f_{js}-\rho\sum_{j=1}^{n}\eta_{sj}^{st}f_{jt}\\
&+\rho\sum_{j=1}^{n}\beta_{tj}^{st}f_{n+j,s}-\rho\sum_{j=1}^{n}\beta_{sj}^{st}f_{n+j,t}+\rho\gamma_t^{st}f_{ms}-\rho\gamma_s^{st}f_{mt})(e_{s,n+t}-e_{t,n+s}\\
&+\sum_{1\le i\le j\le n}\eta_{ij}^{st}(f_{i,n+j}+\rho f_{j,n+i})+\sum_{i,j=1}^{n}\beta_{ij}^{st}(f_{ij}+\rho f_{n+j,n+i})\\
&+\sum_{1\le i\le j\le n}\zeta_{ij}^{st}(f_{n+i,j}+\rho f_{n+j,i})+\sum_{i=1}^{n}\gamma_i^{st}(f_{im}+\rho f_{m,n+i})\\
&+\sum_{j=1}^{n}\delta_j^{st}(f_{n+j,m}+\rho f_{mi})+\zeta^{st}(f_{mm}+\rho f_{mm}))\\
=\;&e_{s,n+t}+\sum_{j=1}^{n}\eta_{sj}^{st}f_{s,n+j}+\rho\sum_{i=1}^{n}\eta_{is}^{st}f_{s,n+i}+\sum_{j=1}^{n}\beta_{sj}^{st}f_{sj}+\gamma_s^{st}f_{sm}\\
&-e_{t,n+s}+\sum_{j=1}^{n}\eta_{tj}^{st}f_{t,n+j}+\rho\sum_{i=1}^{n}\eta_{it}^{st}f_{t,n+i}+\sum_{j=1}^{n}\beta_{tj}^{st}f_{tj}+\gamma_t^{st}f_{tm}\\
&+\sum_{i=1}^{n}\eta_{it}^{st}f_{i,n+t}+\sum_{i=1}^{n}\eta_{is}^{st}f_{i,n+s}+\rho\sum_{j=1}^{n}\eta_{tj}^{st}f_{j,n+t}+\rho\sum_{j=1}^{n}\eta_{sj}^{st}f_{j,n+s}\\
&+\rho\sum_{j=1}^{n}\beta_{tj}^{st}f_{n+j,n+t}+\rho\sum_{j=1}^{n}\beta_{sj}^{st}f_{n+j,n+s}+\rho\gamma_t^{st}f_{m,n+t}+\rho\gamma_s^{st}f_{m,n+s}\\
=\;&e_{s,n+t}-e_{t,n+s}+\sum_{j=1}^{n}\eta_{sj}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i=1}^{n}\eta_{is}^{st}(\rho f_{s,n+i}+f_{i,n+s})\\
&+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})\\
&+\sum_{j=1}^{n}\eta_{tj}^{st}(f_{t,n+j}+\rho f_{j,n+t})+\sum_{i=1}^{n}\eta_{it}^{st}(\rho f_{t,n+i}+f_{i,n+t})\\
&+\sum_{j=1}^{n}\beta_{tj}^{st}(f_{tj}+\rho f_{n+j,n+t})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\\
=\;&e_{s,n+t}-e_{t,n+s}+\sum_{j=1}^{n}(\eta_{sj}^{st}+\rho\eta_{js}^{st})(f_{s,n+j}+\rho f_{j,n+s})\\
&+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})\\
&+\sum_{i=1}^{n}(\eta_{ti}^{st}+\rho\eta_{it}^{st})(f_{t,n+i}+\rho f_{i,n+t})+\sum_{j=1}^{n}\beta_{tj}^{st}(f_{tj}+\rho f_{n+j,n+t})
\end{aligned}
$$

$$+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\in B$$

Put $\alpha_{sj}^{st}=\eta_{sj}^{st}+\rho\eta_{js}^{st}$ and $\alpha_{it}^{st}=\eta_{ti}^{st}+\rho\eta_{it}^{st}$ for all $1\le i,j\le n$. Then

$$\begin{aligned}
c_{st} &= e_{s,n+t}-e_{t,n+s}+\sum_{j=1}^{n}\alpha_{sj}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i=1}^{n}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})\\
&\quad+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\\
&= \theta(b_{st})+\sum_{j=1}^{k}\alpha_{sj}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i=1}^{k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\in B.
\end{aligned}$$

By Claim 1, $\sum_{j=1}^{k}\alpha_{sj}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i=1}^{k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\in R_2^{\rho}\subseteq B$, so $\theta(b_{st})\in B$, as required.

CLAIM 3: There is a special inner automorphism $\varphi_q:A\to A$ for some $q\in\mathfrak{u}^*(R)$ such that

$$b_{sk}'''=e_{s,n+k}-e_{k,n+s}\in\varphi_q(B)=B_q\quad\text{for all}\quad 1\le s<k \tag{3.8.21}$$

and (for all $1\le s<t<k$)

$$\begin{aligned}
b_{st}''' &= e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{tk}(f_{s,n+i}+\rho f_{i,n+s})\\
&\quad+\sum_{i>k}\beta_{ki}^{tk}(f_{si}+\rho f_{n+i,n+s})+\gamma_k^{tk}(f_{sm}+\rho f_{m,n+s})\in B_q.
\end{aligned}\tag{3.8.22}$$

Recall that $\bar b_{st}=e_{s,n+t}-e_{t,n+s}$. Since $\bar b_{1t}=e_{1,n+t}-e_{t,n+1}=\bar b_t\in E\cup E^+$, by Case (1), there is a special inner automorphism $\varphi_{q'}:A\to A$ for some $q'\in\mathfrak{u}^*(R)$ such that $\varphi_{q'}(b_{1t})=e_{1,n+t}-e_{t,n+1}\in\varphi_{q'}(B)\cap S$. Note that $\overline{\varphi_{q'}(b_{st})}=e_{s,n+t}-e_{t,n+s}$ for all s and t, so $\varphi_{q'}(b_{st})=e_{s,n+t}-e_{t,n+s}+r_{st}$ for some $r_{st}\in K^{(1)}\cap R$. As $r_{st}\in\mathfrak{u}^*(R)$, r_{st} can be written in the form (3.8.20), so by Claim 2, for all $s>1$, we have

$$\begin{aligned}
\theta(\varphi_{q'}(b_{st})) &= e_{s,n+t}-e_{t,n+s}+\sum_{j>k}\alpha_{si}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})\\
&\quad+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})+\gamma_t^{st}(f_{t,m}+\rho f_{m,n+t})\in\varphi_{q'}(B).
\end{aligned}$$

Put $b_{1t}'=\varphi_{q'}(b_{1t})=e_{1,n+t}-e_{t,n+1}\in\varphi_{q'}(B)$. For all $s>1$, set $b_{st}'=\{\theta(\varphi_{q'}(b_{st})),e_{n+t,1}-e_{n+1,t},b_{1t}'\}\in\varphi_{q'}(B)$ (by Lemma 3.1.3). Since $R^2=0$, for all $s>1$, we have

$$b_{st}' = \theta(\varphi_{q'}(b_{st}))(e_{n+t,1}-e_{n+1,t})b_{1t}'+b_{1t}'(e_{n+t,1}-e_{n+1,t})\theta(\varphi_{q'}(b_{st}))$$

$$
\begin{aligned}
&= (e_{s,n+t}-e_{t,n+s}+\sum_{j>k}\alpha_{si}^{st}(f_{s,n+j}+\rho f_{j,n+s})+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})\\
&\quad+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t}))(e_{n+t,1}-e_{n+1,t})b'_{1t}\\
&\quad+(e_{1,n+t}-e_{t,n+1})(e_{n+t,1}-e_{n+1,t})\theta(\varphi_{q'}(b_{st}))\\
&= (e_{s1}+\rho\sum_{i>k}\alpha_{ti}^{st}f_{i1}+\rho\sum_{i=1}^{n}\beta_{ti}^{st}f_{n+i,1}+\rho\gamma_t^{st}f_{m1})(e_{1,n+t}-e_{t,n+1})\\
&\quad+(e_{11}+e_{tt})(e_{s,n+t}-e_{t,n+s}+\sum_{j>k}\alpha_{si}^{st}(f_{s,n+j}+\rho f_{j,n+s})\\
&\quad+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})\\
&\quad+\sum_{j=1}^{n}\beta_{sj}^{st}(f_{sj}+\rho f_{n+j,n+s})+\gamma_s^{st}(f_{sm}+\rho f_{m,n+s})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t}))\\
&= e_{s,n+t}+\rho\sum_{i>k}\alpha_{ti}^{st}f_{i,n+t}+\rho\sum_{i=1}^{n}\beta_{ti}^{st}f_{n+i,n+t}+\rho\gamma_t^{st}f_{m,n+t}\\
&\quad-e_{t,n+s}+\sum_{i>k}\alpha_{ti}^{st}f_{t,n+i}+\sum_{i=1}^{n}\beta_{ti}^{st}f_{ti}+\gamma_t^{st}f_{tm}\\
&= e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})\\
&\quad+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\in\varphi_{q'}(B).
\end{aligned}
$$

Since $\varphi_{q'}(B)^2=0$, for all $1<q\le k$, we have

$$
\begin{aligned}
0 &= b'_{1q}b'_{st}=(e_{1,n+q}-e_{q,n+1})(e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad+\sum_{i=1}^{n}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\\
&= \rho\beta_{tq}^{st}f_{1,n+t}-\rho\beta_{t1}^{st}f_{q,n+t},
\end{aligned}
$$

so $\beta_{ti}^{st}=0$ all $1\le i\le k$. Therefore,

$$
\begin{aligned}
b'_{st} &= e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ti}^{st}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad+\sum_{i>k}\beta_{ti}^{st}(f_{ti}+\rho f_{n+i,n+t})+\gamma_t^{st}(f_{tm}+\rho f_{m,n+t})\in\varphi_{q'}(B).
\end{aligned}
$$

Put $b''_{1t}=b'_{1t}=e_{1,n+t}-e_{t,n+1}\in\varphi_{q'}(B)$ and for $s>1$ set $b''_{st}=\{b'_{sk},e_{n+k,s}-e_{n+s,k},b'_{st}\}\in$

$\varphi_{q'}(B)$ (by Lemma 3.1.3). Since $R^2 = 0$,

$$\begin{aligned}
b''_{st} &= b'_{sk}(e_{n+k,s} - e_{n+s,k})b'_{st} + b'_{st}(e_{n+k,s} - e_{n+s,k})b'_{sk} \\
&= (e_{s,n+k} - e_{k,n+s} + \sum_{i>k} \alpha^{sk}_{ki}(f_{k,n+i} + \rho f_{i,n+k}) + \sum_{i>k} \beta^{sk}_{ki}(f_{ki} + \rho f_{n+i,n+k}) \\
&\quad + \gamma^{sk}_{k}(f_{km} + \rho f_{m,n+k}))(e_{n+k,s} - e_{n+s,k})b'_{st} \\
&\quad + (e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha^{st}_{ti}(f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta^{st}_{ti}(f_{ti} + \rho f_{n+i,n+t}) \\
&\quad + \gamma^{st}_{t}(f_{tm} + \rho f_{m,n+t}))(e_{n+k,s} - e_{n+s,k})b'_{sk} \\
&= (e_{ss} + e_{kk} + \rho \sum_{i>k} \alpha^{sk}_{ki} f_{is} + \rho \sum_{i>k} \beta^{sk}_{ki} f_{n+i,s} + \rho \gamma^{sk}_{k} f_{ms})(e_{s,n+t} - e_{t,n+s} \\
&\quad + \sum_{i>k} \alpha^{st}_{ti}(f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta^{st}_{ti}(f_{ti} + \rho f_{n+i,n+t}) + \gamma^{st}_{t}(f_{tm} + \rho f_{m,n+t})) \\
&\quad + e_{tk}(e_{s,n+k} - e_{k,n+s} + \sum_{i>k} \alpha^{sk}_{ki}(f_{k,n+i} + \rho f_{i,n+k}) + \sum_{i>k} \beta^{sk}_{ki}(f_{ki} + \rho f_{n+i,n+k}) \\
&\quad + \gamma^{sk}_{k}(f_{km} + \rho f_{m,n+k})) \\
&= e_{s,n+t} + \rho \sum_{i>k} \alpha^{sk}_{ki} f_{i,n+t} + \rho \sum_{i>k} \beta^{sk}_{ki} f_{n+i,n+t} + \rho \gamma^{sk}_{k} f_{m,n+t} \\
&\quad - e_{t,n+s} + \sum_{i>k} \alpha^{sk}_{ki} f_{t,n+i} + \sum_{i>k} \beta^{sk}_{ki} f_{ti} + \gamma^{sk}_{k} f_{tm} \\
&= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha^{sk}_{ki}(f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta^{sk}_{ki}(f_{ti} + \rho f_{n+i,n+t}) \\
&\quad + \gamma^{sk}_{k}(f_{tm} + \rho f_{m,n+t}) \in \varphi_{q'}(B).
\end{aligned}$$

Recall that $b''_{1t} = e_{1,n+t} - e_{t,n+1} \in \varphi_{q'}(B)$.

Consider the special inner automorphism $\varphi_{q''} : A \to A$, where

$$\begin{aligned}
q'' &= -\sum_{j=2}^{k-1} \sum_{i>k} \alpha^{jk}_{ki}(f_{n+j,n+i} + \rho f_{ij}) - \sum_{j=2}^{k-1} \sum_{i>k} \beta^{jk}_{ki}(f_{n+j,i} + \rho f_{n+i,j}) \\
&\quad - \sum_{j=2}^{k-1} \gamma^{jk}_{k}(f_{n+j,m} + \rho f_{mj}) \in \mathfrak{u}^*(R).
\end{aligned}$$

Note that $\varphi_q = \varphi_{q''} \circ \varphi_{q'}$ is a special inner automorphism of A with $q \in \mathfrak{u}^*(R)$ (because $q', q'' \in \mathfrak{u}^*(R)$). Put $B_q = \varphi_q(B) = \varphi_{q''}(\varphi_{q'}(B))$ and $b'''_{st} = \varphi_{q''}(b''_{st}) \in \varphi_{q''}(\varphi_{q'}(B)) = B_q$ for all s and t. Since $R^2 = 0$,

$$\begin{aligned}
b'''_{1k} &= \varphi_{q''}(b''_{1k}) = (1 + q'')b''_{1k}(1 - q'') \\
&= (1 - \sum_{j=2}^{k-1} \sum_{i>k} \alpha^{jk}_{ki}(f_{n+j,n+i} + \rho f_{ij}) - \sum_{j=2}^{k-1} \sum_{i>k} \beta^{jk}_{ki}(f_{n+j,i} + \rho f_{n+i,j})
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{1,n+k}-e_{k,n+1})(1-q'') \\
= \; & (e_{1,n+k}-e_{k,n+1})(1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij}) \\
& +\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})+\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj})) \\
= \; & e_{1,n+k}-e_{k,n+1}\in B_q=\varphi_{q''}(\varphi_{q'}(B))
\end{aligned}
$$

and (for $s>1$)

$$
\begin{aligned}
b'''_{sk} \; = \; & \varphi_{q''}(b''_{sk})=(1+q'')b''_{sk}(1-q'') \\
= \; & (1-\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})-\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) \\
& -\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{s,n+k}-e_{k,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{k,n+i}+\rho f_{i,n+k}) \\
& +\sum_{i>k}\beta_{ki}^{sk}(f_{ki}+\rho f_{n+i,n+k})+\gamma_k^{sk}(f_{km}+\rho f_{m,n+k}))(1-q'') \\
= \; & (e_{s,n+k}-e_{k,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{k,n+i}+\rho f_{i,n+k})+\sum_{i>k}\beta_{ki}^{sk}(f_{ki}+\rho f_{n+i,n+k}) \\
& +\gamma_k^{sk}(f_{km}+\rho f_{m,n+k})-\rho\sum_{i>k}\alpha_{ki}^{sk}f_{i,n+k}-\rho\sum_{i>k}\beta_{ki}^{sk}f_{n+i,n+k} \\
& -\rho\gamma_k^{sk}f_{m,n+k})(1-q'') \\
= \; & (e_{s,n+k}-e_{k,n+s}+\sum_{i>k}\alpha_{ki}^{sk}f_{k,n+i}+\sum_{i>k}\beta_{ki}^{sk}f_{ki}+\gamma_k^{sk}f_{km})(1 \\
& +\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})+\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) \\
& +\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj})) \\
= \; & e_{s,n+k}-e_{k,n+s}-\sum_{i>k}\alpha_{ki}^{sk}f_{k,n+i}-\sum_{i>k}\beta_{ki}^{sk}f_{ki}-\gamma_k^{sk}f_{km}+\sum_{i>k}\alpha_{ki}^{sk}f_{k,n+i} \\
& +\sum_{i>k}\beta_{ki}^{sk}f_{ki}+\gamma_k^{sk}f_{km}=e_{s,n+k}-e_{k,n+s}\in B_q.
\end{aligned}
$$

Therefore, $b'''_{sk}=e_{s,n+k}-e_{k,n+s}\in B_q=\varphi_q(B)$ for all s, so (3.8.21), is proved. It remains to show that (3.8.22) holds. By applying $\varphi_{q''}$ to b''_{st}, for all $t<k$, we get that

$$
b'''_{1t} \; = \; \varphi_{q''}(b''_{1t})=(1+q'')b''_{1t}(1-q'')
$$

$$
\begin{aligned}
&= (1-\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})-\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})\\
&\quad -\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{1,n+t}-e_{t,n+1})(1-q'')\\
&= (e_{1,n+t}-e_{t,n+1}+\rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+1}+\rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+1}+\rho\gamma_k^{tk}f_{m,n+1})(1\\
&\quad +\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})+\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})\\
&\quad +\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+f_{mj}))\\
&= e_{1,n+t}+\sum_{i>k}\alpha_{ki}^{tk}f_{1,n+i}+\sum_{i>k}\beta_{ki}^{tk}f_{1i}+\gamma_k^{tk}f_{1m}-e_{t,n+1}+\rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+1}\\
&\quad +\rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+1}+\rho\gamma_k^{tk}f_{m,n+1}\\
&= e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\alpha_{ki}^{tk}(f_{1,n+i}+\rho f_{i,n+1})+\sum_{i>k}\beta_{ki}^{tk}(f_{1i}+\rho f_{n+i,n+1})\\
&\quad +\gamma_k^{tk}(f_{1m}+\rho f_{m,n+1})\in B_q
\end{aligned}
$$

and (for $s>1$)

$$
\begin{aligned}
b'''_{st} &= \varphi_{q''}(b''_{st})=(1+q'')b''_{st}(1-q'')\\
&= (1-\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})-\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})\\
&\quad -\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{t,n+i}+\rho f_{i,n+t})\\
&\quad +\sum_{i>k}\beta_{ki}^{sk}(f_{ti}+\rho f_{n+i,n+t})+\gamma_k^{sk}(f_{tm}+\rho f_{m,n+t}))(1-q'')\\
&= (e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{t,n+i}+\rho f_{i,n+t})+\sum_{i>k}\beta_{ki}^{sk}(f_{ti}+\rho f_{n+i,n+t})\\
&\quad +\gamma_k^{sk}(f_{tm}+\rho f_{m,n+t})-\rho\sum_{i>k}\alpha_{ki}^{sk}f_{i,n+t}+\rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+s}\\
&\quad -\rho\sum_{i>k}\beta_{ki}^{sk}f_{n+i,n+t}+\rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+s}-\rho\gamma_k^{sk}f_{m,n+t}\\
&\quad +\rho\gamma_k^{tk}f_{m,n+s})(1-q'')\\
&= (e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{sk}f_{t,n+i}+\sum_{i>k}\beta_{ki}^{sk}f_{ti}+\gamma_k^{sk}f_{tm}+\rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+s}\\
&\quad +\rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+s}+\rho\sum_{j=2}^{k-1}\gamma_k^{tk}f_{m,n+s})(1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})+\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))\\
=\;& e_{s,n+t}+\sum_{i>k}\alpha_{ki}^{tk}f_{s,n+i}+\sum_{i>k}\beta_{ki}^{tk}f_{si}+\gamma_k^{tk}f_{sm}-e_{t,n+s}-\sum_{i>k}\alpha_{ki}^{sk}f_{t,n+i}\\
&-\sum_{i>k}\beta_{ki}^{sk}f_{ti}-\gamma_k^{sk}f_{tm}+\sum_{i>k}\alpha_{ki}^{sk}f_{t,n+i}+\sum_{i>k}\beta_{ki}^{sk}f_{ti}+\gamma_k^{sk}f_{tm}\\
&+\rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+s}+\rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+s}+\rho\gamma_k^{tk}f_{m,n+s}\\
=\;& e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{tk}(f_{s,n+i}+\rho f_{i,n+s})+\sum_{i>k}\beta_{ki}^{tk}(f_{si}+\rho f_{n+i,n+s})\\
&+\gamma_k^{tk}(f_{sm}+\rho f_{m,n+s})\in\varphi_{q''}(\varphi_{q'}(B))=B_q.
\end{aligned}
$$

Therefore, (for all $1<s<t<k$)

$$
\begin{aligned}
b_{st}''' =\;& e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{tk}(f_{s,n+i}+\rho f_{i,n+s})+\sum_{i>k}\beta_{ki}^{tk}(f_{si}+\rho f_{n+i,n+s})\\
&+\gamma_k^{tk}(f_{sm}+\rho f_{m,n+s})\in B_q=\varphi_{q''}(\varphi_{q'}(B))=\varphi_q(B),
\end{aligned}
$$

so (3.8.22) is proved, as required.

Claim 4: There is a special inner automorphism $\varphi_{q_1}: A\to A$ such that

$$
b_{1t}^{(1)}=e_{1,n+t}-e_{t,n+1}\in\varphi_{q_1}(B_q)=B_{q_1} \tag{3.8.23}
$$

and (for $s>1$)

$$
\begin{aligned}
b_{st}^{(1)} =\;& e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{t,n+i}+\rho f_{i,n+t})\\
&+\sum_{i>k}\beta_{ki}^{sk}(f_{ti}+\rho f_{n+i,n+t})+\gamma_k^{sk}(f_{tm}+\rho f_{m,n+t})\in B_{q_1}.
\end{aligned} \tag{3.8.24}
$$

By Claim 3, there is a special inner automorphism $\varphi_q: A\to A$ for some $q\in\mathfrak{u}^*(R)$ such that $b_{sk}'''=e_{s,n+k}-e_{k,n+s}\in B_q$ and (for $t<k$)

$$
\begin{aligned}
b_{st}''' =\;& e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{tk}(f_{s,n+i}+\rho f_{i,n+s})\\
&+\sum_{i>k}\beta_{ki}^{tk}(f_{si}+\rho f_{n+i,n+s})+\gamma_k^{tk}(f_{sm}+\rho f_{m,n+s})\in B_q.
\end{aligned}
$$

Consider the special inner automorphism $\varphi_{q_1}: A\to A$, where

$$
\begin{aligned}
q_1 =\;& \sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij})+\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j})\\
&+\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj})\in\mathfrak{u}^*(R).
\end{aligned}
$$

Put $B_{q_1} = \varphi_{q_1}(B_q)$ and $b_{st}^{(1)} = \varphi_{q_1}(b'''_{st})$ for all s and t. Since $R^2 = 0$,

$$\begin{aligned}
b_{1k}^{(1)} &= \varphi_{q_1}(b'''_{1k}) = (1+q_1)b'''_{1k}(1-q_1) \\
&= (1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij}) + \sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad + \sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{1,n+k}-e_{k,n+1})(1-q_1) \\
&= (e_{1,n+k}-e_{k,n+1})(1-\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij}) \\
&\quad -\sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) + \sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj})) \\
&= e_{1,n+k}-e_{k,n+1} \in \varphi_{q_1}(B_q) = B_{q_1}
\end{aligned}$$

and (for all $t < k$)

$$\begin{aligned}
b_{1t}^{(1)} &= \varphi_{q_1}(b'''_{1t}) = (1+q_1)b'''_{1t}(1-q_1) \\
&= (1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij}) + \sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad + \sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj}))(e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\alpha_{ki}^{tk}(f_{1,n+i}+\rho f_{i,n+1}) \\
&\quad + \sum_{i>k}\beta_{ki}^{tk}(f_{1i}+\rho f_{n+i,n+1}) + \gamma_k^{tk}(f_{1m}+\rho f_{m,n+1}))(1-q_1) \\
&= (e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\alpha_{ki}^{tk}(f_{1,n+i}+\rho f_{i,n+1}) + \sum_{i>k}\beta_{ki}^{tk}(f_{1i}+\rho f_{n+i,n+1}) \\
&\quad +\gamma_k^{tk}(f_{1m}+\rho f_{m,n+1}) - \rho\sum_{i>k}\alpha_{ki}^{tk}f_{i,n+1} - \rho\sum_{i>k}\beta_{ki}^{tk}f_{n+i,n+1} \\
&\quad -\rho\gamma_k^{tk}f_{m,n+1})(1-q_1) \\
&= (e_{1,n+t}-e_{t,n+1}+\sum_{i>k}\alpha_{ki}^{tk}f_{1,n+i}+\sum_{i>k}\beta_{ki}^{tk}f_{1i}+\gamma_k^{tk}f_{1m})(1 \\
&\quad -\sum_{j=2}^{k-1}\sum_{i>k}\alpha_{ki}^{jk}(f_{n+j,n+i}+\rho f_{ij}) - \sum_{j=2}^{k-1}\sum_{i>k}\beta_{ki}^{jk}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad -\sum_{j=2}^{k-1}\gamma_k^{jk}(f_{n+j,m}+\rho f_{mj})) \\
&= e_{1,n+t}-\sum_{i>k}\alpha_{ki}^{tk}f_{1,n+i}-\sum_{i>k}\beta_{ki}^{tk}f_{1i}-\gamma_k^{tk}f_{1m}-e_{t,n+1} \\
&\quad +\sum_{i>k}\alpha_{ki}^{tk}f_{1,n+i}+\sum_{i>k}\beta_{ki}^{tk}f_{1i}+\gamma_k^{tk}f_{1m} = e_{1,n+t}-e_{t,n+1} \in B_{q_1}.
\end{aligned}$$

Hence, $b^{(1)}_{1t} = e_{1,n+t} - e_{t,n+1} \in B_{q_1}$ for all t, so (3.8.23) is proved. It remains to show that (3.8.24) holds. By applying φ_{q_1} to b'''_{st} for all $s > 1$, we get that

$$\begin{aligned}
b^{(1)}_{sk} &= \varphi_{q_1}(b'''_{sk}) = (1+q_1)b'''_{sk}(1-q_1) \\
&= (1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha^{jk}_{ki}(f_{n+j,n+i}+\rho f_{ij}) + \sum_{j=2}^{k-1}\sum_{i>k}\beta^{jk}_{ki}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad + \sum_{j=2}^{k-1}\gamma^{jk}_k(f_{n+j,m}+\rho f_{mj}))(e_{s,n+k}-e_{k,n+s})(1-q_1) \\
&= (e_{s,n+k}-e_{k,n+s}+\rho\sum_{i>k}\alpha^{sk}_{ki}f_{i,n+k}+\rho\sum_{i>k}\beta^{sk}_{ki}f_{n+i,n+k}+\rho\gamma^{sk}_k f_{m,n+k})(1 \\
&\quad -\sum_{j=2}^{k-1}\sum_{i>k}\alpha^{jk}_{ki}(f_{n+j,n+i}+\rho f_{ij}) - \sum_{j=2}^{k-1}\sum_{i>k}\beta^{jk}_{ki}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad -\sum_{j=2}^{k-1}\gamma^{jk}_k(f_{n+j,m}+\rho f_{mj})) \\
&= e_{s,n+k}-e_{k,n+s}+\sum_{i>k}\alpha^{sk}_{ki}f_{k,n+i}+\sum_{i>k}\beta^{sk}_{ki}f_{ki}+\gamma^{sk}_k f_{km}+\rho\sum_{i>k}\alpha^{sk}_{ki}f_{i,n+k} \\
&\quad +\rho\sum_{i>k}\beta^{sk}_{ki}f_{n+i,n+k}+\rho\gamma^{sk}_k f_{m,n+k} \\
&= e_{s,n+k}-e_{k,n+s}+\sum_{i>k}\alpha^{sk}_{ki}(f_{k,n+i}+\rho f_{i,n+k})+\sum_{i>k}\beta^{sk}_{ki}(f_{ki}+\rho f_{n+i,n+k}) \\
&\quad +\gamma^{sk}_k(f_{km}+\rho f_{m,n+k}) \in B_{q_1}
\end{aligned}$$

and (for all $t < k$)

$$\begin{aligned}
b^{(1)}_{st} &= \varphi_{q_1}(b'''_{st}) = (1+q_1)b'''_{st}(1-q_1) \\
&= (1+\sum_{j=2}^{k-1}\sum_{i>k}\alpha^{jk}_{ki}(f_{n+j,n+i}+\rho f_{ij}) + \sum_{j=2}^{k-1}\sum_{i>k}\beta^{jk}_{ki}(f_{n+j,i}+\rho f_{n+i,j}) \\
&\quad + \sum_{j=2}^{k-1}\gamma^{jk}_k(f_{n+j,m}+\rho f_{mj}))(e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha^{tk}_{ki}(f_{s,n+i}+\rho f_{i,n+s}) \\
&\quad +\sum_{i>k}\beta^{tk}_{ki}(f_{si}+\rho f_{n+i,n+s})+\gamma^{tk}_k(f_{sm}+\rho f_{m,n+s}))(1-q_1) \\
&= (e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha^{tk}_{ki}(f_{s,n+i}+\rho f_{i,n+s})+\sum_{i>k}\beta^{tk}_{ki}(f_{si}+\rho f_{n+i,n+s}) \\
&\quad +\gamma^{tk}_k(f_{sm}+\rho f_{m,n+s})+\rho\sum_{i>k}\alpha^{sk}_{ki}f_{i,n+t}-\rho\sum_{i>k}\alpha^{tk}_{ki}f_{i,n+s} \\
&\quad +\rho\sum_{i>k}\beta^{sk}_{ki}f_{n+i,n+t}-\rho\sum_{i>k}\beta^{tk}_{ki}f_{n+i,n+s}+\rho\gamma^{sk}_k f_{m,n+t} \\
&\quad -\rho\gamma^{tk}_k f_{m,n+s})(1-q_1)
\end{aligned}$$

$$\begin{aligned}
&= (e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{tk} f_{s,n+i} + \sum_{i>k} \beta_{ki}^{tk} f_{si} + \gamma_k^{tk} f_{sm} + \rho \sum_{i>k} \alpha_{ki}^{sk} f_{i,n+t} \\
&\quad + \rho \sum_{i>k} \beta_{ki}^{sk} f_{n+i,n+t} + \rho \gamma_k^{sk} f_{m,n+t})(1 - \sum_{j=2}^{k-1} \sum_{i>k} \alpha_{ki}^{jk} (f_{n+j,n+i} + \rho f_{ij}) \\
&\quad - \sum_{j=2}^{k-1} \sum_{i>k} \beta_{ki}^{jk} (f_{n+j,i} + \rho f_{n+i,j}) - \sum_{j=2}^{k-1} \gamma_k^{jk} (f_{n+j,m} + \rho f_{mj})) \\
&= e_{s,n+t} - \sum_{i>k} \alpha_{ki}^{tk} f_{s,n+i} - \sum_{i>k} \beta_{ki}^{tk} f_{si} - \gamma_k^{tk} f_{sm} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk} f_{t,n+i} \\
&\quad + \sum_{i>k} \beta_{ki}^{sk} f_{ti} + \gamma_k^{sk} f_{tm} + \sum_{i>k} \alpha_{ki}^{tk} f_{s,n+i} + \sum_{i>k} \beta_{ki}^{tk} f_{si} + \gamma_k^{tk} f_{sm} \\
&\quad + \rho \sum_{i>k} \alpha_{ki}^{sk} f_{i,n+t} + \rho \sum_{i>k} \beta_{ki}^{sk} f_{n+i,n+t} + \rho \gamma_k^{sk} f_{m,n+t} \\
&= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk} (f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta_{ki}^{sk} (f_{ti} + \rho f_{n+i,n+t}) \\
&\quad + \gamma_k^{sk} (f_{tm} + \rho f_{m,n+t}) \in B_{q_1}.
\end{aligned}$$

Therefore, (for all $1 < s < t \leq k$)

$$\begin{aligned}
b_{st}^{(1)} &= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk} (f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta_{ki}^{sk} (f_{ti} + \rho f_{n+i,n+t}) \\
&\quad + \gamma_k^{sk} (f_{tm} + \rho f_{m,n+t}) \in B_{q_1}
\end{aligned}$$

and (3.8.24) is proved, as required.

CLAIM 5: There are $k-2$ inner automorphisms φ_{q_ι} $(\iota = 1, \ldots, k-2)$ such that

$$b_{\iota t}^{(k-2)} = e_{\iota,n+t} - e_{t,n+\iota} \in \varphi_{q_{k-2}}(\ldots \varphi_{q_1}(B_q) \ldots)) = B_{q_{k-2}} \quad \text{for all} \quad \iota < t \leq k \tag{3.8.25}$$

and (for all $k-2 < s < t \leq k$)

$$\begin{aligned}
b_{st}^{(k-2)} &= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk} (f_{t,n+i} + \rho f_{i,n+t}) \\
&\quad + \sum_{i>k} \beta_{ki}^{sk} (f_{ti} + \rho f_{n+i,n+t}) + \gamma_k^{sk} (f_{tm} + \rho f_{m,n+t}) \in B_{q_{k-2}}.
\end{aligned} \tag{3.8.26}$$

We are going to prove Claim 5 by induction on ι. The base of the induction (when $\iota = 1$) being clear by Claim 4. Suppose that $\iota > 1$. Put $\kappa = k-2$. By the inductive hypothesis there are $\kappa - 1$ inner automorphisms φ_{q_r} $(r = 1, \ldots, \kappa - 1)$ on A such that

$$b_{rt}^{(\kappa-1)} = e_{r,n+t} - e_{t,n+r} \in \varphi_{q_{\kappa-1}}(\ldots \varphi_{q_1}(B_q) \ldots) = B_{q_{\kappa-1}} \quad \text{for all} \quad r < t \leq k$$

and (for $\kappa-1<s<t\le k$)

$$b_{st}^{(\kappa-1)} = e_{s,n+t}-e_{t,n+s}+\sum_{i>k}\alpha_{ki}^{sk}(f_{t,n+i}+\rho f_{i,n+t}) + \sum_{i>k}\beta_{ki}^{sk}(f_{ti}+\rho f_{n+i,n+t})+\gamma_k^{sk}(f_{tm}+\rho f_{m,n+t})\in B_{q_{\kappa-1}}. \tag{3.8.27}$$

Consider the special inner automorphism $\varphi_{q_\kappa}:A\to A$, where

$$q_\kappa=-\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i}+\rho f_{i\kappa})-\sum_{i>k}\beta_{ki}^{\kappa k}(f_{n+\kappa,i}+\rho f_{n+i,\kappa})-\gamma_k^{\kappa k}(f_{n+\kappa,m}+\rho f_{m\kappa})\in\mathfrak{u}^*(R).$$

Put $B_\kappa=\varphi_{q_\kappa}(B_{\kappa-1})$ and $c_{st}^{(\kappa)}=\varphi_{q_\kappa}(b_{st}^{(\kappa-1)})\in B_\kappa$ for all s and t. Recall that $1\le r\le\kappa-1$. Since $R^2=0$,

$$\begin{aligned}
c_{r\kappa}^{(\kappa)} &= \varphi_{q_\kappa}(b_{r\kappa}^{(\kappa-1)})=(1+q_\kappa)b_{r\kappa}^{(\kappa-1)}(1-q_\kappa)\\
&= (1-\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i}+\rho f_{i\kappa})-\sum_{i>k}\beta_{ki}^{\kappa k}(f_{n+\kappa,i}+\rho f_{n+i,\kappa})\\
&\quad -\gamma_k^{\kappa k}(f_{n+\kappa,m}+\rho f_{m\kappa}))(e_{r,n+\kappa}-e_{\kappa,n+r})(1-q_\kappa)\\
&= (e_{r,n+\kappa}-e_{\kappa,n+r}+\rho\sum_{i>k}\alpha_{ki}^{\kappa k}f_{i,n+r}+\rho\sum_{i>k}\beta_{ki}^{\kappa k}f_{n+i,n+r}\\
&\quad +\rho\gamma_k^{\kappa k}f_{m,n+r})(1+\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i}+\rho f_{i\kappa})\\
&\quad +\sum_{i>k}\beta_{ki}^{\kappa k}(f_{n+\kappa,i}+\rho f_{n+i,\kappa})+\gamma_k^{\kappa k}(f_{n+\kappa,m}+\rho f_{m\kappa}))\\
&= e_{r,n+\kappa}+\sum_{i>k}\alpha_{ki}^{\kappa k}f_{r,n+i}+\sum_{i>k}\beta_{ki}^{\kappa k}f_{ri}+\gamma_k^{\kappa k}f_{rm}\\
&\quad -e_{\kappa,n+r}+\rho\sum_{i>k}\alpha_{ki}^{\kappa k}f_{i,n+r}+\rho\sum_{i>k}\beta_{ki}^{\kappa k}f_{n+i,n+r}+\rho\gamma_k^{\kappa k}f_{m,n+r}\\
&= e_{r,n+\kappa}-e_{\kappa,n+r}+\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{r,n+i}+\rho f_{i,n+r})\\
&\quad +\sum_{i>k}\beta_{ki}^{\kappa k}(f_{ri}+\rho f_{n+i,n+r})+\gamma_k^{\kappa k}(f_{rm}+\rho f_{m,n+r})\in B_{q_\kappa}.
\end{aligned}$$

and (for all $\kappa\ne t$)

$$\begin{aligned}
c_{rt}^{(\kappa)} &= \varphi_{q_\kappa}(b_{rt}^{(\kappa-1)})=(1+q_\kappa)b_{rt}^{(\kappa-1)}(1-q_\kappa)\\
&= (1-\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i}+\rho f_{i\kappa})-\sum_{i>k}\beta_{ki}^{\kappa k}(f_{n+\kappa,i}+\rho f_{n+i,\kappa})\\
&\quad -\gamma_k^{\kappa k}(f_{n+\kappa,m}+\rho f_{m\kappa}))(e_{r,n+t}-e_{t,n+r})(1-q_\kappa)\\
&= (e_{r,n+t}-e_{t,n+r})(1+\sum_{i>k}\alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i}+\rho f_{i\kappa})\\
&\quad +\sum_{i>k}\beta_{ki}^{\kappa k}(f_{n+\kappa,i}+\rho f_{n+i,\kappa})-\gamma_k^{\kappa k}(f_{n+\kappa,m}+\rho f_{m\kappa}))
\end{aligned}$$

$$c_{rt}^{(\kappa)} = e_{r,n+t} - e_{t,n+r} \in B_{q_\kappa}. \tag{3.8.28}$$

Note that if $s = \kappa$, then $t > \kappa$, so by applying φ_{q_κ} to $b_{st}^{(\kappa-1)}$ in (3.8.27), we get that

$$\begin{aligned}
c_{\kappa t}^{(\kappa)} &= \varphi_{q_\kappa}(b_{\kappa t}^{(\kappa-1)}) = (1+q_\kappa)b_{\kappa t}^{(\kappa-1)}(1-q_\kappa) \\
&= (1 - \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i} + \rho f_{i\kappa}) - \sum_{i>k} \beta_{ki}^{\kappa k}(f_{n+\kappa,i} + \rho f_{n+i,\kappa}) \\
&\quad -\gamma_k^{\kappa k}(f_{n+\kappa,m} + \rho f_{m\kappa}))(e_{\kappa,n+t} - e_{t,n+\kappa} + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{t,n+i} + \rho f_{i,n+t}) \\
&\quad + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{ti} + \rho f_{n+i,n+t}) + \gamma_k^{\kappa k}(f_{tm} + \rho f_{m,n+t}))(1-q_\kappa) \\
&= (e_{\kappa,n+t} - e_{t,n+\kappa} + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{ti} + \rho f_{n+i,n+t}) \\
&\quad + \gamma_k^{\kappa k}(f_{tm} + \rho f_{m,n+t}) - \rho \sum_{i>k} \alpha_{ki}^{\kappa k} f_{i,n+t} - \rho \sum_{i>k} \beta_{ki}^{\kappa k} f_{n+i,n+t} \\
&\quad -\rho \gamma_k^{\kappa k} f_{m,n+t})(1-q_\kappa) \\
&= (e_{\kappa,n+t} - e_{t,n+\kappa} + \sum_{i>k} \alpha_{ki}^{\kappa k} f_{t,n+i} + \sum_{i>k} \beta_{ki}^{\kappa k} f_{ti} + \gamma_k^{\kappa k} f_{tm})(1 \\
&\quad + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i} + \rho f_{i\kappa}) + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{n+\kappa,i} + \rho f_{n+i,\kappa}) \\
&\quad + \gamma_k^{\kappa k}(f_{n+\kappa,m} + \rho f_{m\kappa})) \\
&= e_{\kappa,n+t} - e_{t,n+\kappa} - \sum_{i>k} \alpha_{ki}^{\kappa k} f_{t,n+i} - \sum_{i>k} \beta_{ki}^{\kappa k} f_{ti} - \gamma_k^{\kappa k} f_{tm} \\
&\quad + \sum_{i>k} \alpha_{ki}^{\kappa k} f_{t,n+i} + \sum_{i>k} \beta_{ki}^{\kappa k} f_{ti} + \gamma_k^{\kappa k} f_{tm} \\
&= e_{\kappa,n+t} - e_{t,n+\kappa} \in B_{q_\kappa}.
\end{aligned} \tag{3.8.29}$$

Recall that $r = 1, \ldots, \kappa-1$. Put $b_{r\kappa}^{(\kappa)} = \{c_{r\kappa}^{(\kappa)}, e_{n+t,\kappa} - e_{n+\kappa,t}, c_{\kappa t}^{(\kappa)}\} \in B_{q_\kappa}$ $(\kappa < t \leq k)$ and $b_{st}^{(\kappa)} = c_{st}^{(\kappa)}$ for all of the remaining indices s and t. Then

$$\begin{aligned}
b_{r\kappa}^{(\kappa)} &= c_{r\kappa}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t})c_{\kappa t}^{(\kappa)} + c_{\kappa t}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t})c_{r\kappa}^{(\kappa)} \\
&= (e_{r,n+\kappa} - e_{\kappa,n+r} + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{r,n+i} + \rho f_{i,n+r}) + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{ri} + \rho f_{n+i,n+r}) \\
&\quad + \gamma_k^{\kappa k}(f_{rm} + \rho f_{m,n+r}))(e_{n+t,\kappa} - e_{n+\kappa,t})c_{\kappa t}^{(\kappa)} + c_{\kappa t}^{(\kappa)}(e_{n+t,\kappa} - e_{n+\kappa,t})c_{r\kappa}^{(\kappa)} \\
&= -e_{rt}(e_{\kappa,n+t} - e_{t,n+\kappa}) + (e_{\kappa,n+t} - e_{t,n+\kappa})(e_{n+t,\kappa} - e_{n+\kappa,t})c_{r\kappa}^{(\kappa)} \\
&= e_{r,n+\kappa} + (e_{\kappa\kappa} + e_{tt})(e_{r,n+\kappa} - e_{\kappa,n+r} + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{r,n+i} + \rho f_{i,n+r}) \\
&\quad + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{ri} + \rho f_{n+i,n+r}) + \gamma_k^{\kappa k}(f_{rm} + \rho f_{m,n+r})) \\
&= e_{r,n+\kappa} - e_{\kappa,n+r} \in B_{q_\kappa}.
\end{aligned}$$

Combining this with (3.8.28), we get that $b_{rt}^{(\kappa)} = e_{r,n+t} - e_{t,n+r} \in B_\kappa$ for all $r < t \leq k$.

By (3.8.29), $b_{\kappa t}^{(\kappa)} = c_{\kappa t}^{(\kappa)} = e_{\kappa,n+t} - e_{t,n+\kappa} \in B_{q_\kappa}$ for all $\kappa < t \leq k$. Recall that $\kappa = k-2$. Therefore,

$$b_{\iota t}^{(k-2)} = e_{\iota,n+t} - e_{t,n+\iota} \in B_{q_\kappa} \text{ for all } \iota = 1,\ldots,\kappa, \quad \iota < t \leq k$$

and (3.8.25) is proved. It remains to show that (3.8.26) holds. Recall (3.8.27) that for all $\kappa - 1 < s < t \leq k$, we have

$$\begin{aligned} b_{st}^{(\kappa-1)} &= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk}(f_{t,n+i} + \rho f_{i,n+t}) \\ &\quad + \sum_{i>k} \beta_{ki}^{sk}(f_{ti} + \rho f_{n+i,n+t}) + \gamma_k^{sk}(f_{tm} + \rho f_{m,n+t}) \in B_{q_\kappa}. \end{aligned}$$

Note that if $s > \kappa$, then $t > \kappa$, so by applying φ_{q_κ} to $b_{st}^{(\kappa-1)}$ for all $s > \kappa$, we get that

$$\begin{aligned} b_{st}^{(\kappa)} &= c_{st}^{(\kappa)} = \varphi_{q_\kappa}(b_{st}^{(\kappa-1)}) = (1+q_\kappa)b_{st}^{(\kappa-1)}(1-q_\kappa) \\ &= (1 - \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i} + \rho f_{i\kappa}) - \sum_{i>k} \beta_{ki}^{\kappa k}(f_{n+\kappa,i} + \rho f_{n+i,\kappa}) \\ &\quad - \gamma_k^{\kappa k}(f_{n+\kappa,m} + \rho f_{m\kappa}))(e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk}(f_{t,n+i} + \rho f_{i,n+t}) \\ &\quad + \sum_{i>k} \beta_{ki}^{sk}(f_{ti} + \rho f_{n+i,n+t}) + \gamma_k^{sk}(f_{tm} + \rho f_{m,n+t}))(1-q_\kappa) \\ &= (e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk}(f_{t,n+i} + \rho f_{i,n+t}) + \sum_{i>k} \beta_{ki}^{sk}(f_{ti} + \rho f_{n+i,n+t}) \\ &\quad + \gamma_k^{sk}(f_{tm} + \rho f_{m,n+t}))(1 + \sum_{i>k} \alpha_{ki}^{\kappa k}(f_{n+\kappa,n+i} + \rho f_{i\kappa}) \\ &\quad + \sum_{i>k} \beta_{ki}^{\kappa k}(f_{n+\kappa,i} + \rho f_{n+i,\kappa}) + \gamma_k^{\kappa k}(f_{n+\kappa,m} + \rho f_{m\kappa})) \\ &= e_{s,n+t} - e_{t,n+s} + \sum_{i>k} \alpha_{ki}^{sk}(f_{t,n+i} + \rho f_{i,n+t}) \\ &\quad + \sum_{i>k} \beta_{ki}^{sk}(f_{ti} + \rho f_{n+i,n+t}) + \gamma_k^{sk}(f_{tm} + \rho f_{m,n+t}) \in B_{q_\kappa}. \end{aligned}$$

Therefore, (3.8.26) is proved, as required.

Now, we are going to define the final special inner automorphism in order to complete the proof. By Claim 5, there are $k-2$ inner automorphisms φ_{q_ι} ($\iota = 1,\ldots,k-2$) such that

$$b_{\iota t}^{(k-2)} = e_{\iota,n+t} - e_{t,n+\iota} \in \varphi_{q_{k-2}}(\ldots\varphi_{q_1}(B_q)\ldots)) = B_{q_{k-2}} \quad \text{for all} \quad \iota < t \leq k$$

and (for all $k-2<s<t\le k$)

$$\begin{aligned} b_{st}^{(k-2)} &= e_{s,n+t}-e_{t,n+s}+\sum_{j>k}\alpha_{kj}^{sk}(f_{t,n+j}+f_{j,n+t})\\ &\qquad +\sum_{j>k}\beta_{kj}^{sk}(f_{tj}+f_{n+j,n+t})+\gamma_k^{sk}(f_{tm}+f_{m,n+t})\in B_{q_{k-2}}. \end{aligned}$$

Put $\nu=k-1$. Consider the finial special inner automorphism $\varphi_{q_\nu}:A\to A$, where

$$\begin{aligned} q_\nu &= -\sum_{j>k}\alpha_{kj}^{\nu k}(f_{n+\nu,n+j}+\rho f_{j\nu})-\sum_{j>k}\beta_{kj}^{\nu k}(f_{n+\nu,j}+\rho f_{n+j,\nu})\\ &\quad -\gamma_k^{\nu k}(f_{n+\nu,m}+\rho f_{m\nu})\in\mathfrak{u}^*(R). \end{aligned}$$

Put $B_{q_\nu}=\varphi_{q_\nu}(B_{q_{k-2}})$ and $b_{st}^{(\nu)}=\varphi_{q_\nu}(b_{st}^{(k-2)})$ for all s and t. Then for all $1\le\iota\le k-2$, we have

$$\begin{aligned} b_{\iota\nu}^{(\nu)} &= \varphi_{q_\nu}(b_{\iota\nu}^{(k-2)})=(1+q_\nu)b_{\iota\nu}^{(k-2)}(1-q_\nu)\\ &= (1-\sum_{j>k}\alpha_{kj}^{\nu k}(f_{n+\nu,n+j}+\rho f_{j\nu})-\sum_{j>k}\beta_{kj}^{\nu k}(f_{n+\nu,j}+\rho f_{n+j,\nu})\\ &\quad -\gamma_k^{\nu k}(f_{n+\nu,m}+\rho f_{m\nu}))(e_{\iota,n+\nu}-e_{\nu,n+\iota})(1-q_\nu)\\ &= (e_{\iota,n+\nu}-e_{\nu,n+\iota}+\rho\sum_{j>k}\alpha_{kj}^{\nu k}f_{j,n+\iota}+\rho\sum_{j>k}\beta_{kj}^{\nu k}f_{n+j,n+\iota}\\ &\quad +\rho\gamma_k^{\nu k}f_{m,n+\iota})(1+\sum_{j>k}\alpha_{kj}^{\nu k}(f_{n+\nu,n+j}+\rho f_{j\nu})\\ &\quad +\sum_{j>k}\beta_{kj}^{\nu k}(f_{n+\nu,j}+\rho f_{n+j,\nu})+\gamma_k^{\nu k}(f_{n+\nu,m}+\rho f_{m\nu}))\\ &= e_{\iota,n+\nu}+\sum_{j>k}\alpha_{kj}^{\nu k}f_{\iota,n+j}+\sum_{j>k}\beta_{kj}^{\nu k}f_{\iota j}+\gamma_k^{\nu k}f_{\iota m}\\ &\quad -e_{\nu,n+\iota}+\rho\sum_{j>k}\alpha_{kj}^{\nu k}f_{j,n+\iota}+\rho\sum_{j>k}\beta_{kj}^{\nu k}f_{n+j,n+\iota}+\rho\gamma_k^{\nu k}f_{m,n+\iota}\\ &= e_{\iota,n+\nu}-e_{\nu,n+\iota}+\sum_{j>k}\alpha_{kj}^{\nu k}(f_{\iota,n+j}+\rho f_{j,n+\iota})\\ &\quad +\sum_{j>k}\beta_{kj}^{\nu k}(f_{\iota j}+\rho f_{n+j,n+\iota})+\gamma_k^{\nu k}(f_{\iota m}+\rho f_{m,n+\iota})\in B_{q_\nu} \end{aligned}$$

and (for all $\nu\ne t\le k$),

$$\begin{aligned} b_{\iota t}^{(\nu)} &= \varphi_{q_\nu}(b_{\iota t}^{(k-2)})=(1+q_\nu)b_{\iota t}^{(k-2)}(1-q_\nu)\\ &= (1-\sum_{j>k}\alpha_{kj}^{\nu k}(f_{n+\nu,n+j}+\rho f_{j\nu})-\sum_{j>k}\beta_{kj}^{\nu k}(f_{n+\nu,j}+\rho f_{n+j,\nu})\\ &\quad -\gamma_k^{\nu k}(f_{n+\nu,m}+\rho f_{m\nu}))(e_{\iota,n+t}-e_{t,n+\iota})(1-q_\nu)\\ &= (e_{\iota,n+t}-e_{t,n+\iota})(1+\sum_{j>k}\alpha_{kj}^{\nu k}(f_{n+\nu,n+j}+\rho f_{j\nu}) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{j>k} \beta_{kj}^{\nu k}(f_{n+\nu,j} + \rho f_{n+j,\nu}) + \gamma_k^{\nu k}(f_{n+\nu,m} + \rho f_{m\nu})) \\
= \;& e_{\iota,n+t} - e_{t,n+\iota} \in B_{q_\nu}. \qquad (3.8.30)
\end{aligned}
$$

Note that if $s > k-2$, then only option remaining for s is $s = k-1 = \nu$. In that case $t = k$, so by applying φ_{q_ν} to $b_{st}^{(k-2)}$ for all $s > k-2$, we get that

$$
\begin{aligned}
b_{\nu k}^{(\nu)} &= \varphi_{q_\nu}(b_{\nu t}^{(k-2)}) = (1+q_\nu) b_{\nu k}^{(k-2)}(1-q_\nu) \\
&= (1 - \sum_{j>k} \alpha_{kj}^{\nu k}(f_{n+\nu,n+j} + \rho f_{j\nu}) - \sum_{j>k} \beta_{kj}^{\nu k}(f_{n+\nu,j} + \rho f_{n+j,\nu}) \\
&\quad - \gamma_k^{\nu k}(f_{n+\nu,m} + \rho f_{m\nu}))(e_{\nu,n+k} - e_{k,n+\nu} + \sum_{j>k} \alpha_{kj}^{\nu k}(f_{k,n+j} + \rho f_{j,n+k}) \\
&\quad + \sum_{j>k} \beta_{kj}^{\nu k}(f_{kj} + \rho f_{n+j,n+k}) + \gamma_k^{\nu k}(f_{km} + \rho f_{m,n+k}))(1-q_\nu) \\
&= (e_{\nu,n+k} - e_{k,n+\nu} + \sum_{j>k} \alpha_{kj}^{\nu k}(f_{k,n+j} + \rho f_{j,n+k}) + \sum_{j>k} \beta_{kj}^{\nu k}(f_{kj} + \rho f_{n+j,n+k}) \\
&\quad + \gamma_k^{sk}(f_{km} + \rho f_{m,n+k}) - \rho \sum_{j>k} \alpha_{kj}^{\nu k} f_{j,n+k} - \rho \sum_{j>k} \beta_{kj}^{\nu k} f_{n+j,n+k} \\
&\quad - \rho \gamma_k^{sk} f_{m,n+k})(1-q_\nu) \\
&= (e_{\nu,n+k} - e_{k,n+\nu} + \sum_{j>k} \alpha_{kj}^{\nu k} f_{k,n+j} + \sum_{j>k} \beta_{kj}^{\nu k} f_{kj} + \gamma_k^{\nu k} f_{km})(1 \\
&\quad + \sum_{j>k} \alpha_{kj}^{\nu k}(f_{n+\nu,n+j} + \rho f_{j\nu}) + \sum_{j>k} \beta_{kj}^{\nu k}(f_{n+\nu,j} + \rho f_{n+j,\nu}) \\
&\quad + \gamma_k^{\nu k}(f_{n+\nu,m} + \rho f_{m\nu})) \\
&= (e_{\nu,n+k} - e_{k,n+\nu} - \sum_{j>k} \alpha_{kj}^{\nu k} f_{k,n+j} - \sum_{j>k} \beta_{kj}^{\nu k} f_{kj} - \gamma_k^{\nu k} f_{km} \\
&\quad + \sum_{j>k} \alpha_{kj}^{\nu k} f_{k,n+j} + \sum_{j>k} \beta_{kj}^{\nu k} f_{kj} + \gamma_k^{\nu k} f_{km} \\
&= e_{\nu,n+k} - e_{k,n+\nu} \in B_{q_\nu}. \qquad (3.8.31)
\end{aligned}
$$

Put $b_{\iota\nu}^{(k)} = \{b_{\iota\nu}^{(\nu)}, e_{n+k,\nu} - e_{n+\nu,k}, b_{\nu k}^{(\nu)}\} \in B_{q_\nu}$ (by Lemma 3.1.3) and $b_{st}^{(k)} = b_{st}^{(\nu)} \in B_{q_\nu}$ for of all the remaining s and t. Then

$$
\begin{aligned}
b_{\iota\nu}^{(k)} &= b_{\iota\nu}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k}) b_{\nu k}^{(\nu)} + b_{\nu k}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k}) b_{\iota\nu}^{(\nu)} \\
&= (e_{\iota,n+\nu} - e_{\nu,n+\iota} + \sum_{j>k} \alpha_{kj}^{\nu k}(f_{\iota,n+j} + \rho f_{j,n+\iota}) + \sum_{j>k} \beta_{kj}^{\nu k}(f_{\iota j} + \rho f_{n+j,n+\iota}) \\
&\quad + \gamma_k^{sk}(f_{\iota m} + \rho f_{m,n+\iota}))(e_{n+k,\nu} - e_{n+\nu,k}) b_{\nu k}^{(\nu)} + b_{\nu k}^{(\nu)}(e_{n+k,\nu} - e_{n+\nu,k}) b_{\iota\nu}^{(\nu)} \\
&= -e_{\iota k}(e_{\nu,n+k} - e_{k,n+\nu}) + (e_{\nu,n+k} - e_{k,n+\nu})(e_{n+k,\nu} - e_{n+\nu,k}) b_{\iota\nu}^{(\nu)} \\
&= e_{\iota,n+\nu} + (e_{\nu\nu} + e_{kk})(e_{\iota,n+\nu} - e_{\nu,n+\iota} + \sum_{j>k} \alpha_{kj}^{\nu k}(f_{\iota,n+j} + \rho f_{j,n+\iota})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j>k}\beta_{kj}^{\nu k}(f_{\iota j}+\rho f_{n+j,n+\iota})+\gamma_k^{\nu k}(f_{\iota m}+\rho f_{m,n+\iota}))\\
= \quad & e_{\iota,n+\nu}-e_{\nu,n+\iota}\in B_{q_\nu}.
\end{aligned}
$$

Combining this with (3.8.30), we get that $b_{\iota t}^{(k)}=e_{\iota,n+t}-e_{t,n+\iota}\in B_{q_\nu}$ for $\iota=1,\dots,k-2$ and all $\iota<t\leq k$. By (3.8.31), $b_{\nu k}^{(k)}=b_{\nu k}^{(\nu)}=e_{\nu,n+k}-e_{k,n+\nu}\in B_{q_\nu}$. Recall that $\nu=k-1$. Thus, $b_{st}^{(k)}=e_{s,n+t}-e_{t,n+s}\in B_{q_\nu}=B_{q_{k-1}}$ for all $1\leq s<t\leq k$. Put $\varphi=\varphi_{q_{k-1}}\circ\ldots\circ\varphi_{q_1}\circ\varphi_q$. Then φ is a special inner automorphism with

$$E^+=\{e_{s,n+t}-e_{t,n+s}\mid 1\leq s<t\leq k\}\subseteq\varphi(B)\cap S,$$

as required.

Case (3): Suppose that $\mathscr{E}=E^-=\{e_{s,n+t}+e_{t,n+s}\mid 1\leq s\leq t\leq k\}\subseteq\mathfrak{su}^*(S)=\mathfrak{sp}_{2n}$. As in the proof of Case (2) there is a special inner automorphism $\varphi_q:A\to A$ for some $q\in\mathfrak{u}^*(R)$ such that

$$\{b_{st}=e_{s,n+t}+e_{t,n+s}\mid 1\leq s<t\leq k\}\subseteq\varphi_q(B)\cap S.$$

It remains to show that $e_{i,n+i}\in\varphi_q(B)\cap S$ for all $1\leq i\leq k$. Put $b_{kk}=b_{sk}e_{n+s,s}b_{sk}$ and $b_{ss}=b_{st}e_{n+t,t}b_{st}$ for all $1\leq s,t<k$. Since $e_{n+i,i}\in K$ for all $1\leq i\leq k$, by Lemma 3.1.3,

$$b_{ss}=b_{st}e_{n+t,t}b_{st}=(e_{s,n+t}+e_{t,n+s})e_{n+t,t}(e_{s,n+t}+e_{t,n+s})=e_{s,n+s}\in\varphi_q(B);$$

$$b_{kk}=b_{sk}e_{n+s,s}b_{st}=(e_{s,n+k}+e_{k,n+s})e_{n+s,s}(e_{s,n+k}+e_{k,n+s})=e_{k,n+k}\in\varphi_q(B).$$

Hence, $e_{i,n+i}\in\varphi_q(B)$ for all $1\leq i\leq k$, so

$$E^-=\{e_{s,t+n}+e_{t,s+n}\mid 1\leq s\leq t\leq k\}\subseteq\varphi_q(B)\cap S,$$

as required.

Now, by Case (1), Case (2) and Case (3), there is a special inner automorphism $\varphi_q:A\to A$ such that $\mathscr{E}\subseteq\varphi_q(B)\cap S$. Since $R^2=0$, $\varphi_q(r)=r$ for all $r\in R$. Therefore, $\varphi_q(B)=\varphi_q(B)_S\oplus\varphi_q(B)_R$, where $\varphi_q(B)_S=\varphi_q(B)\cap S$ and $\varphi_q(B)_R=\varphi_q(B)\oplus R$. By changing the Levi subalgebra S into $S'=\varphi_q^{-1}(S)$, we get that $B=B_{S'}\oplus B_R$, where $B_{S'}=B\cap S'$ and $B_R=B\cap R$. Since $q\in\mathfrak{u}^*(R)$, by Lemma 3.8.6, S' is $*$-invariant, so B $*$-splits in A.

□

Now, we are ready to proof Proposition 3.8.3.

Proof of Proposition 3.8.3. We identify A/R with S. Since S is involution simple, by Proposition 3.3.2, S is either simple with involution, or $S = S_1 \oplus S_1^*$, where S_1 is a simple ideal of S. If S is simple with involution, then by Lemma 3.8.11, B $*$-splits in A.

Suppose that $S = S_1 \oplus S_1^*$. Then by Lemma 3.4.3 (ii), R is either $*$-invariant irreducible S_1-S_1^*-bimodule with $S_1^*R = RS_1 = 0$ or $R = U \oplus U^*$ for some irreducible S_1-S_1^*-bimodule U with $S_1^*U = US_1 = 0$, so we have two cases.

Suppose first that $R = U \oplus U^*$. Put $D = S_1 \oplus U$. Since $R^2 = 0$, D is an ideal of A and $A = D \oplus D^*$, so by Proposition 3.6.12, B $*$-splits in A.

Suppose now that R is a $*$-invariant irreducible S_1-S_1^*-bimodule with $RS = S^*R = 0$. Recall that $\bar{A} = A/R$ is identified with S. Since $\bar{B}$ is Jordan-Lie inner ideal of $\bar{K}^{(1)} = \mathfrak{su}^*(\bar{A}) = \mathfrak{su}^*(S_1 \oplus S_1^*)$, by Proposition 3.3.3, $\bar{B} = (\bar{e} + \bar{f}^*)K(\bar{f} + \bar{e}^*)$ for some orthogonal idempotents $\bar{e}$ and $\bar{f}$ of S_1. By using Lemma 3.3.6, we fix standard bases $\{e_{ij} \mid 1 \leq i,j \leq n\}$, $\{e'_{ij} \mid 1 \leq i,j \leq n\}$ and $\{f_{ij} \mid 1 \leq i,j \leq n\}$ of S_1, S_1^* and R consisting of matrix units such that $e_{ij}^* = e'_{ji}$, the action of S_1-S_1^* on R correspond the matrix multiplication and $\bar{B}$ is the space spanned by $\mathscr{E} = \{e_{st} - e'_{ts} \mid 1 \leq s \leq k < l \leq t \leq n\}$. Note that A satisfies the conditions of Lemma 3.4.7, so $\mathfrak{u}^*(R) = \{X \in \mathscr{M}_n \mid X^t = \rho X\}$ $(\rho = \pm 1)$, that is, $\{f_{ij} + \rho f_{ji} \mid 1 \leq i \leq j \leq n\}$ is a basis of $\mathfrak{u}^*(R)$. Fix any subset $\{b_{st} \mid 1 \leq s \leq k < l \leq t \leq n\} \subseteq B$ such that $\bar{b}_{st} = e_{st} - e'_{ts}$ for all s and t. We need to show that there is a special inner automorphism $\varphi_q : A \to A$, for some $q \in \mathfrak{u}^*(A)$, such that $e_{st} - e'_{ts} \in \varphi_q(B)$. This will imply that B $*$-splits in A, as required.

First, we claim that

$$R_0^{\rho} = span\{f_{sr} + \rho f_{rs} \mid 1 \leq s \leq r \leq k\} \subseteq \mathfrak{u}^*(R) \cap B. \tag{3.8.32}$$

We have $R_0 \subseteq \mathfrak{u}^*(R)$. As $\bar{b}_{st} = e_{st} - e'_{ts}$ for all s and t, we have $b_{st} = e_{st} - e'_{ts} + r_{st}$ for some $r_{st} \in \mathfrak{u}^*(R)$. Since $RS_1 = S_1^*R = R^2 = 0$, by Lemma 3.1.3,

$$\begin{aligned}
\{b_{st}, f_{tq} + \rho f_{qt}, b_{rq}\} &= b_{st}(f_{tq} + \rho f_{qt})b_{rq} + b_{rq}(f_{tq} + \rho f_{qt})b_{st} \\
&= (e_{st} - e'_{ts} + r_{st})(f_{tq} + \rho f_{qt})b_{rq} + b_{rq}(f_{tq} + \rho f_{qt})b_{st} \\
&= f_{sq}(e_{rq} - e'_{qr} + r_{rq}) + (e_{rq} - e'_{qr} + r_{rq})(f_{tq} + \rho f_{qt})b_{st} \\
&= -f_{sr} + \rho f_{rt}(e_{st} - e'_{ts} + r_{st}) \\
&= -(f_{sr} + \rho f_{rs}) \in B.
\end{aligned}$$

Moreover, if $\rho = +$, then $f_{tt} \in \mathfrak{su}^*(A)$ for all $l \leq t \leq n$, so by Lemma 3.1.3,

$$b_{st} f_{tt} b_{st} = b_{st} f_{tt} b_{st} = (e_{st} - e'_{ts} + r_{st}) f_{tt} b_{st} = f_{st}(e_{st} - e'_{ts} + r_{st}) = f_{ss} \in B.$$

Therefore, $R_0^\rho \subseteq \mathfrak{u}^*(R) \cap B$, as required.

Next, for every $b_{st} = e_{st} - e'_{ts} + r_{st} \in B$ $(r_{st} \in \mathfrak{u}^*(R))$, we claim that

$$\theta(b_{st}) := e_{st} - e'_{ts} + \sum_{j>k} \alpha_{sj}^{st}(f_{sj} + \rho f_{js}) \in B \quad \text{for some } \alpha_{sj}^{st} \in \mathbb{F}. \tag{3.8.33}$$

By Lemma 3.1.3, $c_{st} = b_{st}(e_{ts} - e'_{st})b_{st} \in B$. Since $r_{st} \in \mathfrak{u}^*(R)$, r_{st} is of the form $r_{st} = \sum_{i,j=1}^n \eta_{ij}^{st}(f_{ij} + \rho f_{ji})$, where $\eta_{ij}^{st} \in \mathbb{F}$. As $RS_1 = S_1^*R = R^2 = 0$ and $\rho^2 = 1$,

$$\begin{aligned}
c_{st} &= b_{st}(e_{ts} - e'_{st})b_{st} = (e_{st} - e'_{ts} + \sum_{i,j=1}^n \eta_{ij}^{st}(f_{ij} + \rho f_{ji}))(e_{ts} - e'_{st})b_{st} \\
&= (e_{ss} + e'_{tt} - \sum_{i=1}^n \eta_{is}^{st} f_{it} - \rho \sum_{j=1}^n \eta_{sj}^{st} f_{jt})(e_{st} - e'_{ts} + \sum_{i,j=1}^n \eta_{ij}^{st}(f_{ij} + \rho f_{ji})) \\
&= e_{st} + \sum_{j=1}^n \eta_{sj}^{st} f_{sj} + \rho \sum_{i=1}^n \eta_{is}^{st} f_{si} - e'_{ts} + \sum_{i=1}^n \eta_{is}^{st} f_{is} + \rho \sum_{i,j=1}^n \eta_{sj}^{st} f_{js} \\
&= e_{st} - e'_{ts} + \sum_{j=1}^n \eta_{sj}^{st}(f_{sj} + \rho f_{js}) + \sum_{j=1}^n \eta_{js}^{st}(\rho f_{sj} + f_{js}) \\
&= e_{st} - e'_{ts} + \sum_{j=1}^n \eta_{sj}^{st}(f_{sj} + \rho f_{js}) + \rho \sum_{j=1}^n \eta_{js}^{st}(f_{sj} + \rho f_{js}) \\
&= e_{st} - e'_{ts} + \sum_{j=1}^n (\eta_{sj}^{st} + \rho \eta_{js}^{st})(f_{sj} + \rho f_{js}) \in B.
\end{aligned}$$

Put $\alpha_{sj}^{st} = (\eta_{sj}^{st} + \rho \eta_{js}^{st})$ for all $1 \le j \le n$. Then

$$c_{st} = e_{st} - e'_{ts} + \sum_{j=1}^n \alpha_{sj}^{st}(f_{sj} + \rho f_{js}) = \theta(b_{st}) + \sum_{j=1}^k \alpha_{sj}^{st}(f_{sj} + \rho f_{js}) \in B.$$

By (3.8.32), $\sum_{j=1}^k \alpha_{sj}^{st}(f_{sj} + \rho f_{js}) \in R_0^\rho \subseteq B$ for all $1 \le s \le k$, so $\theta(b_{st}) \in B$, as required.

Now, by (3.8.33), there are coefficients such that

$$b_{st} = e_{st} - e'_{ts} + \sum_{j>k} \alpha_{sj}^{st}(f_{sj} + \rho f_{js}) \in B, \text{ for all } 1 \le s \le k.$$

Put $b_{1t}^{(1)} = b_{1t} \in B$ and for $s > 1$ set $b_{st}^{(1)} = \{b_{sn}, e_{k1} - e'_{1k}, b_{1t}\} \in B$ (by Lemma 3.1.3). Since $RS_1 = S_1^*R = R^2 = 0$,

$$\begin{aligned}
b_{st}^{(1)} &= b_{sn}(e_{n1} - e'_{1n})b_{1t} + b_{1t}(e_{n1} - e'_{1n})b_{sn} \\
&= (e_{sn} - e'_{ns} + \sum_{j>k} \alpha_{sj}^{sn}(f_{sj} + \rho f_{js}))(e_{n1} - e'_{1n})b_{1t} + b_{1t}(e_{n1} - e'_{1n})b_{sn}
\end{aligned}$$

$$
\begin{aligned}
&= e_{s1}(e_{1t}-e'_{t1}+\sum_{j>k}\alpha^{1t}_{1j}(f_{1j}+\rho f_{j1}))+b_{1t}(e_{n1}-e'_{1n})b_{sn}\\
&= e_{st}+\sum_{j>k}\alpha^{1t}_{1j}f_{sj}+(e_{1t}-e'_{t1}+\sum_{j>k}\alpha^{1t}_{1j}(f_{1j}+\rho f_{j1}))(e_{n1}-e'_{1n})b_{sn}\\
&= e_{st}+\sum_{j>k}\alpha^{1t}_{1j}f_{sj}+(e'_{tn}-\rho\sum_{j>k}\alpha^{1t}_{1j}f_{jn})(e_{sn}-e'_{ns}+\sum_{j>k}\alpha^{sn}_{sj}(f_{sj}+\rho f_{js}))\\
&= e_{st}+\sum_{j>k}\alpha^{1t}_{1j}f_{sj}-e'_{ts}+\rho\sum_{j>k}\alpha^{1t}_{1j}f_{js}=e_{st}-e'_{ts}+\sum_{j>k}\alpha^{1t}_{1j}(f_{sj}+\rho f_{js})\in B,
\end{aligned}
$$

so

$$
b^{(1)}_{st}=e_{st}-e'_{ts}+\sum_{j>k}\alpha^{1t}_{1j}(f_{sj}+\rho f_{js})\quad\text{for all}\quad 1\le s\le k<l\le t\le n.
$$

Consider the inner automorphism $\varphi_{q_1}:A\to A$, where

$$
q_1=\sum_{j>k}\alpha^{1n}_{1j}(f_{nj}+\rho f_{jn})\in\mathfrak{u}^*(R).
$$

Since $RS_1=S_1^*R=R^2=0$,

$$
\begin{aligned}
\varphi_{q_1}(b^{(1)}_{1n}) &= (1+q_1)b^{(1)}_{1n}(1-q_1)\\
&= (1+\sum_{j>k}\alpha^{1n}_{1j}(f_{nj}+\rho f_{jn}))(e_{1n}-e'_{n1}+\sum_{j>k}\alpha^{1n}_{1j}(f_{1j}+\rho f_{j1}))(1-q_1)\\
&= (e_{1n}-e'_{n1}+\sum_{j>k}\alpha^{1n}_{1j}(f_{1j}+\rho f_{j1})-\alpha^{1n}_{1n}f_{n1}-\rho\sum_{j>k}\alpha^{1n}_{1j}f_{j1})(1-q_1)\\
&= (e_{1n}-e'_{n1}+\sum_{j>k}\alpha^{1n}_{1j}f_{1j}-\alpha^{1n}_{1n}f_{n1})(1-\sum_{j>k}\alpha^{1n}_{1j}(f_{nj}+\rho f_{jn}))\\
&= e_{1n}-\sum_{j>k}\alpha^{1n}_{1j}f_{1j}-\rho\alpha^{1n}_{1n}f_{1n}-e'_{n1}+\sum_{j>k}\alpha^{1n}_{1j}f_{1j}-\alpha^{1n}_{1n}f_{n1}\\
&= e_{1n}-e'_{n1}-\alpha^{1n}_{1n}(f_{n1}+\rho f_{1n})\in\varphi_{q_1}(B).
\end{aligned}
$$

Since $\varphi_{q_1}(B)^2=0$,

$$
\begin{aligned}
0 &= \varphi_{q_1}(b^{(1)}_{1n})^2=(e_{1n}-e'_{n1}-\alpha^{1n}_{1n}(f_{n1}+\rho f_{1n}))(e_{1n}-e'_{n1}-\alpha^{1n}_{1n}(f_{n1}+\rho f_{1n}))\\
&= -\alpha^{1n}_{1n}f_{11}+\rho\alpha^{1n}_{1n}f_{11}=-(\alpha^{1n}_{1n}-\rho\alpha^{1n}_{1n})f_{11}.
\end{aligned}
$$

We have two cases. Suppose first that $\rho=-$. Then $\alpha^{1n}_{1n}=0$, so

$$
\varphi_{q_1}(b^{(1)}_{1n})=e_{1n}-e'_{n1}\in\varphi_{q_1}(B). \tag{3.8.34}
$$

Suppose now that $\rho=+$. Then

$$
\varphi_{q_1}(b^{(1)}_{1n})=e_{1n}-e'_{n1}-\alpha^{1n}_{1n}(f_{n1}+f_{1n})\in\varphi_{q_1}(B).
$$

Consider the special inner automorphism $\varphi_{q_2}: A \to A$, where

$$q_2 = -\alpha_{1n}^{1n} f_{nn} \in \mathfrak{u}^*(R).$$

Since $S_1^* R = RS_1 = R^2 = 0$, we get that

$$\begin{aligned}
\varphi_{q_2}(\varphi_{q_1}(b_{1n}^{(1)})) &= (1+q_2)\varphi_{q_2}(b_{1n}^{(1)})(1-q_2)\\
&= (1-\alpha_{1n}^{1n} f_{nn})(e_{1n} - e'_{n1} - \alpha_{1n}^{1n}(f_{n1} + f_{1n}))(1-q_2)\\
&= (e_{1n} - e'_{n1} - \alpha_{1n}^{1n}(f_{1n} + f_{n1}) + \alpha_{1n}^{1n} f_{n1})(1-q_2)\\
&= (e_{1n} - e'_{n1} - \alpha_{1n}^{1n} f_{1n})(1+\alpha_{1n}^{1n} f_{nn})\\
&= e_{1n} + \alpha_{1n}^{1n} f_{1n} - e'_{n1} - \alpha_{1n}^{1n} f_{1n}\\
&= e_{1n} - e'_{n1} \in \varphi_{q_2}(\varphi_{q_1}(B)). \qquad (3.8.35)
\end{aligned}$$

Put $\varphi_q = \varphi_{q_1}$ (if $\rho = -$) or $\varphi_q = \varphi_{q_2} \circ \varphi_{q_1}$ (if $p = +$). By (3.8.34) and (3.8.35), for any choice of ρ, we get that $e_{1n} - e'_{n1} \in \varphi_q(B)$. Note that $\varphi_q : A \to A$ is a special inner automorphism with $q \in \mathfrak{u}^*(R)$. Put $b_{st}^{(2)} = e_{1n} - e'_{n1} \in \varphi_q(B)$. For all of the remaining indices set $b_{st}^{(2)} = \theta(\varphi_q(b_{st}^{(1)})) \in \varphi_q(B)$. Then by (3.8.33), there are coefficients such that

$$b_{st}^{(2)} = e_{st} - e'_{ts} + \sum_{j>k} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}) \in \varphi_q(B).$$

Put $b_{1n}^{(3)} = b_{1n}^{(2)} = e_{1n} - e'_{n1} \in B_q$. For all $s > 1$, set $b_{sn}^{(3)} = \{b_{sn}^{(2)}, e_{n1} - e'_{1n}, b_{1n}^{(2)}\} \in \varphi_q(B)$ (by Lemma 3.1.3). Since $RS_1 = S_1^* R = R^2 = 0$, for all $s > 1$, we have

$$\begin{aligned}
b_{sn}^{(3)} &= b_{sn}^{(2)}(e_{n1} - e'_{1n})b_{1n}^{(2)} + b_{1n}^{(2)}(e_{n1} - e'_{1n})b_{sn}^{(2)}\\
&= (e_{sn} - e'_{ns} + \sum_{j>k} \beta_{1j}^{1n}(f_{sj} + \rho f_{js}))(e_{n1} - e'_{1n})b_{1n}^{(2)} + b_{1n}^{(2)}(e_{n1} - e'_{1n})b_{sn}^{(2)}\\
&\quad e_{s1}(e_{1n} - e'_{n1}) + (e_{1n} - e'_{n1})(e_{n1} - e'_{1n})b_{sn}^{(2)}))\\
&= e_{sn} + (e_{11} + e'_{nn})(e_{sn} - e'_{ns} + \sum_{j>k} \beta_{1j}^{1n}(f_{sj} + \rho f_{js}))\\
&= e_{sn} - e'_{ns} \in \varphi_q(B).
\end{aligned}$$

Since $b_{1n}^{(3)} = e_{1n} - e'_{n1} \in \varphi_q(B)$, we get that $b_{sn}^{(3)} = e_{sn} - e'_{ns} \in \varphi_q(B)$ for all $1 \le s \le k$.

Put $b_{sn}^{(4)} = e_{sn} + e'_{ns} \in \varphi_q(B)$ and for all $t < n$ set

$$b_{st}^{(4)} = b_{st}^{(2)} = e_{st} - e'_{ts} + \sum_{j>k} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}) \in \varphi_q(B).$$

Since $\varphi_q(B)^2 = 0$, for all $1 \le s, r \le k$, we have

$$\begin{aligned} 0 &= b_{sn}^{(4)} b_{rt}^{(4)} = (e_{sn} - e'_{ns})(e_{rt} - e'_{tr} + \sum_{j>k} \beta_{1j}^{1t}(f_{rj} + \rho f_{jr})) = \rho \beta_{1n}^{1t} f_{sr}; \\ 0 &= b_{st}^{(4)} b_{rq}^{(4)} = (e_{st} - e'_{ts} + \sum_{j>k} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}))(e_{rq} - e'_{qr} + \sum_{j>k} \beta_{1j}^{1q}(f_{rj} + \rho f_{jr})) \\ &= \rho \beta_{1t}^{1q} f_{sr} - \beta_{1q}^{1t} f_{sr} = -(\beta_{1q}^{1t} - \rho \beta_{1t}^{1q}) f_{sr}. \end{aligned}$$

Hence, for all t, we have

$$\beta_{1n}^{1t} = 0 \quad \text{and} \quad \beta_{1j}^{1t} - \rho \beta_{1t}^{1j} = 0 \quad \text{for all} \quad l \le j < n. \tag{3.8.36}$$

Thus,

$$b_{st}^{(4)} = e_{st} - e'_{ts} + \sum_{k<j<n} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}) \in \varphi(B).$$

Consider the special inner automorphism $\varphi_{q_3} : A \to A$, where

$$q_3 = \frac{1}{2} \sum_{k<i,j<n} \beta_{1j}^{1i}(f_{ij} + \rho f_{ji}) \in \mathfrak{u}^*(R).$$

Since $RS_1 = S_1^* R = R^2 = 0$, by using (3.8.36) that $\beta_{1t}^{1j} - \rho \beta_{1j}^{1t} = 0$ (for all $k < j \le n$), we get that

$$\begin{aligned} \varphi_{q_3}(b_{st}^{(4)}) &= (1+q_3) b_{st}^{(4)} (1-q_3) = (1 + \frac{1}{2} \sum_{k<i,j<n} \beta_{1j}^{1i}(f_{ij} + \rho f_{ji}))(e_{st} - e'_{ts} \\ &\quad + \sum_{k<j<n} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}))(1-q_3) \\ &= (e_{st} - e'_{ts} + \sum_{k<j<n} \beta_{1j}^{1t}(f_{sj} + \rho f_{js}) - \frac{1}{2} \sum_{k<i<n} \beta_{1t}^{1i} f_{is} \\ &\quad - \rho \frac{1}{2} \sum_{k<j<n} \beta_{1j}^{1t} f_{js})(1-q_3) \\ &= (e_{st} - e'_{ts} + \sum_{k<j<n} \beta_{1j}^{1t} f_{sj} + \frac{1}{2} \sum_{k<j<n} (\rho \beta_{1j}^{1t} - \beta_{1t}^{1j}) f_{js})(1-q_3) \\ &= (e_{st} - e'_{ts} + \sum_{k<j<n} \beta_{1j}^{1t} f_{sj} + 0)(1 - \frac{1}{2} \sum_{k<i,j<n} \beta_{1j}^{1i}(f_{ij} + \rho f_{ji})) \\ &= e_{st} - \frac{1}{2} \sum_{k<j<n} \beta_{1j}^{1t} f_{sj} - \rho \frac{1}{2} \sum_{k<i<n} \beta_{1t}^{1i} f_{si} - e'_{ts} + \sum_{k<j<n} \beta_{1j}^{1t} f_{sj} \end{aligned}$$

$$= \quad e_{st} - e'_{ts} + \frac{1}{2} \sum_{1<j<n} (\rho \beta_{1j}^{1t} - \beta_{1t}^{1j}) f_{sj} = e_{st} - e'_{ts} \in \varphi_{q_3}(\varphi_q(B)).$$

By applying φ_{q_3} to $b_{sn}^{(4)} = e_{sn} - e'_{ns} \in \varphi_q(B)$, we get that

$$\begin{aligned}
\varphi_{q_3}(b_{sn}^{(4)}) &= (1+q_3) b_{sn}^{(4)} (1-q_3) \\
&= (1 + \frac{1}{2} \sum_{k<i,j<n} \beta_{1j}^{1i} (f_{ij} + \rho f_{ji}))(e_{sn} - e'_{ns})(1-q_3) \\
&= (e_{sn} - e'_{ns})(1 - \frac{1}{2} \sum_{k<i,j<n} \beta_{1j}^{1i} (f_{ij} + \rho f_{ji})) \\
&= e_{sn} - e'_{ns} \in \varphi_{q_3}(\varphi_q(B)).
\end{aligned}$$

Thus,

$$\mathscr{E} = \{e_{st} - e'_{ts} \mid 1 \le s \le k < l \le t \le n\} \subseteq \varphi_{q_3}(\varphi_q(B)) \cap S.$$

Therefore, by changing the Levi subalgebra S into $S' = \varphi_q^{-1}(\varphi_{q_3}^{-1}(S))$, we get that B $*$-splits in A, as required.

$\square$

Proof of Proposition 3.8.4

We will need the following lemma which represents a special case of Proposition 3.8.4.

Lemma 3.8.12. *Theorem 3.8.1 holds if $A/R \cong S_1 \oplus S_2$, where S_1 and S_2 are both simple with involutions and $R = U \oplus U^*$, where U is an irreducible S_1-S_2-bimodule such that $US_1 = S_2U = 0$.*

Proof. We identify $\bar{A} = A/R$ with $S_1 \oplus S_2$. Since S_1 and S_2 are simple with involutions, by Lemma 3.4.8, for each $i = 1,2$ we have $\mathfrak{u}^*(S_i) \cong \mathfrak{so}_{m_i}, \mathfrak{sp}_{2n_i}$ ($m_i = 2n_i + 1$ or $2n_i$) and $\mathfrak{u}^*(R) = \{(r, -r^*) \mid r \in U\} \cong U \cong \mathscr{M}_{m_1 m_2}$. Recall that $\bar{B}$ is a Jordan-Lie inner ideal of $\bar{K}^{(1)} = \mathfrak{su}^*(S)$. As S is semisimple, by Lemma 3.3.12, $\bar{B} = X_1 \oplus X_2$, where $X_i = \bar{B} \cap \mathfrak{su}^*(S_i)$ is a Jordan-Lie inner ideal of $\mathfrak{su}^*(S_i)$ for each $i = 1,2$. As in the proof of Lemma 3.8.10, we fix standard bases $\{e_{ij} \mid 1 \le i,j \le m_1\}$, $\{g_{ij} \mid 1 \le i,j \le m_2\}$ and $\{f_{ij} \mid 1 \le i \le m_1, 1 \le j \le m_2\}$ of S_1, S_2 and U, respectively, consisting of matrix units, such that the action of S_1 and of S_2 on U corresponds to matrix multiplication and X_i is the space spanned by

$\mathscr{E}_i \subseteq S_i$, where $\mathscr{E}_i$ is one of the following:

$$E_1 = \{e_{1,n_1+t} - e_{t,n_1+1} \mid 1 < t \le k_1 \le n_1\} \subseteq \mathfrak{su}^*(S_1) = \mathfrak{so}_{m_1};$$

$$E_1^\varepsilon = \{e_{s,n_1+t} - \varepsilon e_{t,n_1+s} \mid 1 \le s \le t \le k_1 \le n_1\} \subseteq \mathfrak{su}^*(S_1) = \mathfrak{so}_{m_1}, \mathfrak{sp}_{2n_1};$$

$$E_2 = \{h_{1q} = g_{1,n_2+q} - g_{q,n_2+1} \mid 1 \le q \le k_2 \le n_2\} \subseteq \mathfrak{su}^*(S_2) = \mathfrak{so}_{m_2};$$

$$E_2^\varepsilon = \{h_{rq}^\varepsilon = g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \mid 1 \le r \le q \le k_2 \le n_2\} \subseteq \mathfrak{su}^*(S_2) = \mathfrak{so}_{m_2}, \mathfrak{sp}_{2n_n}.$$

Put $A_2 = S_2 \oplus R$ and $B_2 = B \cap \mathfrak{su}^*(A_2)$. Then B_2 is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A_2)$. Since B_2 contains the preimage of X_2 in B, we have $\bar{B}_2 = X_2$. We may assume that B_2 is X_2-minimal (if not, then it contains X_2-minimal Jordan-Lie inner ideal of $\mathfrak{su}^*(A_2)$). Thus, B_2 satisfies the conditions of Proposition 3.8.2, so there is a Levi subalgebra S_2' of A_2 such that $B_2 = B_{2_{S'}} \oplus B_{2_R}$, where $B_{2_{S'}} = B_2 \cap S_2'$. Note that $\bar{B}_{2_{S'}} = \bar{B}_2 = X_2$. By Theorem 3.8.5, there is $q \in \mathfrak{u}^*(R)$ and a special inner automorphism φ_q of A such that $S_2 = \varphi_q(S_2')$. Since $B_{2_{S'}} \subseteq S_2'$, $\varphi_q(B_{2_{S'}}) \subseteq \varphi_q(S_2') = S_2$. Moreover, by Lemma 3.8.7, $\overline{\varphi_q(B_{2_{S'}})} = \bar{B}_{2_{S'}} = \bar{B}_2 = X_2$. Recall that $X_2 \subseteq S_2$, so both $\varphi_q(B_{2_{S'}})$ and X_2 have the same image in $\bar{A}_2 = A_2/R$. Since both of them are subspaces of S_2, they must be equal. Thus, $X_2 = \varphi_q(B_{2_{S'}}) \subseteq \varphi_q(B_2) \cap S$, so $\mathscr{E}_2 \subseteq \varphi_q(B_2) \cap S_2 \subseteq \varphi_q(B) \cap S$. We will deal with the Jordan-Lie inner ideal $B_q = \varphi_q(B)$ of $K^{(1)}$. Our aim is find a special inner automorphism of A such that $B_q = \varphi_q(B)$ contains both $\mathscr{E}_1$ and $\mathscr{E}_2$. This will imply that B $*$-splits in A. Let $\tilde{B}$ be the image of B in $\tilde{A} = A/U^* \cong A \oplus U$. Since $R = U \oplus U^*$, by Lemma 3.8.8, to show that B $*$-splits in A, it is enough to show that $\tilde{B}$ splits in $\tilde{A}$, that is, there is a special inner automorphism of $\tilde{A}$ such that $\tilde{B}_q = \varphi_q(\tilde{B})$ contains both $\tilde{\mathscr{E}}_1$ and $\tilde{\mathscr{E}}_2$. To simplify notations, we will re-denote $\tilde{A}$, $\tilde{S}$, $\tilde{R}$, $\tilde{B}$, $\tilde{\mathscr{E}}_1$ and $\tilde{\mathscr{E}}_2$ by A, S, R, B, $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively. Thus, $R = U$ and $A/U \cong S$. We need to show that B splits in A. Recall that $\mathscr{E}_1$ is either $E_1 \subseteq \mathfrak{so}_{m_1}$, or $E_1^+ \subseteq \mathfrak{so}_{m_1}$, or $E_1^- \subseteq \mathfrak{sp}_{2n_1}$. Hence, to complete the proof we need to consider three cases.

Case (1): Suppose that $\mathscr{E}_1 = E_1 \subseteq \mathfrak{su}^*(S_1) = \mathfrak{so}_{m_1}$. We wish to show that there is a special inner automorphism $\varphi_{\mathrm{q}} : A \to A$ such that $\mathscr{E}_1, \mathscr{E}_2 \subseteq \varphi_{\mathrm{q}}(B_q)$. Without loss of generality we can assume $m = 2n+1$ (the case $m = 2n$ will follow immediately). Fix any subset $\{b_t \mid 1 < t \le k_1\} \subseteq B_q$ such that $\bar{b}_t = e_{1,n_1+t} - e_{t,n_1+1}$ for all t. Then $b_t = e_{1,n+t} - e_{t,n_1+t} + r_t$ for some $r_t \in U$. Suppose that $\mathscr{E}_2 = E_2^\varepsilon$. Since $R^2 = S_2U = 0$, by Lemma 3.1.3, for all $h_{1q}^\varepsilon = e_{1,n_2+q} - e_{q,n_2+1} \in E_2^\varepsilon$, we have

$$\begin{aligned}\{b_t, f_{n_1+t,q}, h_{1q}^\varepsilon\} &= b_t f_{n_1+t,q} h_{1q}^\varepsilon + 0 = (e_{1,n_1+t} - e_{t,n_1+1} + r_t) f_{n_1+t,q} h_{1q}^\varepsilon \\ &= f_{1q}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = -\varepsilon f_{1,n_2+1} \in B_q;\end{aligned}$$

$$\begin{aligned}\{b_t, f_{n_1+t,1}, h^{\varepsilon}_{1q}\} &= b_t f_{n_1+t,1} h^{\varepsilon}_{1q} + 0 = (e_{1,n_1+t} - e_{t,n_1+1} + r_t) f_{n_1+t,1} h^{\varepsilon}_{1q} \\ &= f_{11}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = f_{1,n_2+q} \in B_q;\end{aligned}$$

$$\begin{aligned}\{b_t, f_{n_1+1,q}, h^{\varepsilon}_{1q}\} &= b_t f_{n_1+1,q} h^{\varepsilon}_{1q} + 0 = (e_{1,n_1+t} - e_{t,n_1+1} + r_t) f_{n_1+1,q} h^{\varepsilon}_{1q} \\ &= -f_{tq}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = \varepsilon f_{t,n_2+1} \in B_q;\end{aligned}$$

$$\begin{aligned}\{b_t, f_{n_1+1,1}, h^{\varepsilon}_{1q}\} &= b_t f_{n_1+1,1} h^{\varepsilon}_{1q} + 0 = (e_{1,n_1+t} - e_{t,n_1+1} + r_t) f_{n_1+1,1} h^{\varepsilon}_{1q} \\ &= -f_{t1}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = -f_{t,n_2+q} \in B_q.\end{aligned}$$

Hence, $f_{i,n_2+j} \in B_q$ for all $1 \le i \le k_1$ and $1 \le j \le k_2$. Note that this is also true in case when $\mathscr{E}_2 = E_2$ (because $h^{+}_{1q} = g_{1,n_2+q} - g_{q,n_2+1} = h_q \in E_2^{+} \cap E_2$), so for any choice of $\mathscr{E}_2$ we have

$$R_0 = span\{f_{i,n_2+j} \mid 1 \le i \le k_1, \quad 1 \le j \le k_2\} \subseteq U \cap B_q. \tag{3.8.37}$$

Recall that $b_t = e_{1,n_1+t} - e_{t,n_1+1} + r_t \in B_q$ for all t. As $r_t \in U = \mathscr{M}_{m_1 m_2}$,

$$\begin{aligned} r_t &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\alpha^t_{ij} f_{ij} + \beta^t_{ij} f_{i,n_2+j} + \zeta^t_{ij} f_{n_1+i,j} + \eta^t_{ij} f_{n_1+i,n_2+j}) \\ &\quad + \sum_{i=1}^{n_1}(\gamma^t_i f_{im_2} + \xi^t_i f_{n_1+i,m_2}) + \sum_{j=1}^{n_2}(\lambda^t_j f_{m_1 j} + \mu^t_j f_{m_1,n_2+j}) + \delta^t f_{m_1 m_2} \in U.\end{aligned}$$

Put $c_t = b_t(e_{n_1+t,1} - e_{n_1+1,t})b_t \in B_q$ (by Lemma 3.1.3). Since $US_1 = R^2 = 0$,

$$\begin{aligned} c_t &= b_t(e_{n_1+t,1} - e_{n_1+1,t})b_t = (e_{1,n_1+t} - e_{t,n_1+1} + r_t)(e_{n_1+t,1} - e_{n_1+1,t})b_t \\ &= (e_{11} + e_{tt})(e_{1,n_1+t} - e_{t,n_1+1} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\alpha^t_{ij} f_{ij} + \beta^t_{ij} f_{i,n_2+j} + \zeta^t_{ij} f_{n_1+i,j} \\ &\quad + \eta^t_{ij} f_{n_1+i,n_2+j}) + \sum_{i=1}^{n_1}(\gamma^t_i f_{im_2} + \xi^t_i f_{n_1+i,m_2}) + \sum_{j=1}^{n_2}(\lambda^t_j f_{m_1 j} + \mu^t_j f_{m_1,n_2+j}) \\ &\quad + \delta^t f_{m_1 m_2}) \\ &= e_{1,n_1+t} + \sum_{j=1}^{n_2}\alpha^t_{1j} f_{1j} + \sum_{j=1}^{n_2}\beta^t_{1j} f_{1,n_2+j} + \gamma^t_1 f_{1m_2} - e_{t,n_1+1} + \sum_{j=1}^{n_2}\alpha^t_{tj} f_{tj} \\ &\quad + \sum_{j=1}^{n_2}\beta^t_{tj} f_{t,n_2+j} + \gamma^t_t f_{tm_2} \\ &= e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j=1}^{n_2}\alpha^t_{1j} f_{1j} + \sum_{j=1}^{n_2}\alpha^t_{tj} f_{tj} + \sum_{j>k_2}\beta^t_{1j} f_{1,n_2+j} + \sum_{j>k_2}\beta^t_{tj} f_{t,n_2+j} \\ &\quad + \gamma^t_1 f_{1m_2} + \gamma^t_t f_{tm_2} + \sum_{j=1}^{k_2}\beta^t_{1j} f_{1,n_2+j} + \sum_{j=1}^{k_2}\beta^t_{tj} f_{t,n_2+j}\end{aligned}$$

$$= \quad b'_t + \sum_{j=1}^{k_2} \beta^t_{1j} f_{1,n_2+j} + \sum_{j=1}^{k_2} \beta^t_{tj} f_{t,n_2+j} \in B_q,$$

Since $\sum_{j=1}^{k_2} \alpha^t_{1j} f_{1,n+j} + \sum_{j=1}^{k_2} \alpha^t_{t1} f_{t,n+1} \in R_0 \subseteq B_q$, we get that

$$\begin{aligned} b'_t &= e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j=1}^{n_2} \alpha^t_{1j} f_{1j} + \sum_{j=1}^{n_2} \alpha^t_{tj} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\ &\quad + \sum_{j>k_2} \beta^t_{tj} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^t_t f_{tm_2} \in B_q. \end{aligned}$$

Suppose that $\mathscr{E}_2 = E_2^\varepsilon$. Since $S_1 S_2 = 0$, $(B_q)^2 = 0$ and $h^\varepsilon_{1q} = g_{1,n_2+q} - \varepsilon g_{q,n_2+1} \in \mathscr{E}_2 \subseteq B_q$, we get that

$$\begin{aligned} 0 &= b'_t h^\varepsilon_{1q} = (e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j=1}^{n_2} \alpha^t_{1j} f_{1j} + \sum_{j=1}^{n_2} \alpha^t_{tj} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\ &\quad + \sum_{j>k_2} \beta^t_{tj} f_{t,n_2+j}) + \gamma^t_1 f_{1m_2} + \gamma^t_t f_{tm_2})(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) \\ &= \alpha^t_{11} f_{1,n_2+q} - \varepsilon \alpha^t_{1q} f_{1,n_2+1} + \alpha^t_{t1} f_{t,n_2+q} - \varepsilon \alpha^t_{tq} f_{t,n_2+1}, \end{aligned}$$

for all $1 < t \le k_1$ and $1 < q \le n_2$, so $\alpha^t_{1j} = \alpha^t_{tj} = 0$ for all t and all $1 \le j \le k_2$. Note that this is also true when $\mathscr{E}_2 = E_2$ (because $h_q = g_{1,n_2+q} - g_{q,n_2+1} = h^+_{1q} \in E_2^+ \cap E$). Therefore,

$$\begin{aligned} b'_t &= e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \alpha^t_{tj} f_{tj} \\ &\quad + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} + \sum_{j>k_2} \beta^t_{tj} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^t_t f_{tm_2} \in B_q. \end{aligned}$$

Put $b''_{k_1} = b'_{k_1} \in B_q$ and for $t < k_1$ set $b''_t = \{b''_{k_1}, e_{n_1+k_1,1} - e_{n_1+1,k_1}, b'_t\} \in B_q$ (by Lemma 3.1.3). Since $US_1 = R^2 = 0$, we have $b''_{k_1}(e_{n_1+k_1,1} - e_{n_1+1,k_1}) = e_{11} + e_{k_1k_1}$ and $b'_t(e_{n_1+k_1,1} - e_{n_1+1,k_1}) = e_{tk_1}$, so

$$\begin{aligned} b''_t &= b''_{k_1}(e_{n_1+k_1,1} - e_{n_1+1,k_1}) b'_t + b'_t (e_{n_1+k_1,1} - e_{n_1+1,k_1}) b''_{k_1} \\ &= (e_{11} + e_{k_1k_1}) b'_t + e_{tk_1} b''_{k_1} \\ &= (e_{11} + e_{k_1k_1})(e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \alpha^t_{tj} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\ &\quad + \sum_{j>k_2} \beta^t_{tj} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^t_t f_{tm_2}) + e_{tk_1}(e_{1,n_1+k_1} - e_{k_1,n_1+1} + \sum_{j>k_2} \alpha^{k_1}_{1j} f_{1j} \\ &\quad + \sum_{j>k_2} \alpha^{k_1}_{k_1j} f_{k_1j} + \sum_{j>k_2} \beta^{k_1}_{1j} f_{1,n_2+j} + \sum_{j>k_2} \beta^{k_1}_{k_1j} f_{k_1,n_2+j} + \gamma^{k_1}_1 f_{1m_2} + \gamma^{k_1}_{k_1} f_{k_1m_2}) \\ &= e_{1,n_1+t} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} + \gamma^t_1 f_{1m_2} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^{k_1}_{k_1j} f_{tj} \\ &\quad + \sum_{j>k_2} \beta^{k_1}_{k_1j} f_{t,n_2+j} + \gamma^{k_1}_{k_1} f_{tm_2} \end{aligned}$$

$$
\begin{aligned}
&= e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\
&\quad + \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^{k_1}_{k_1} f_{tm_2} \in B_q.
\end{aligned}
$$

Therefore, for all t, we have

$$
\begin{aligned}
b''_t &= e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\
&\quad + \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^{k_1}_{k_1} f_{tm_2} \in B_q.
\end{aligned}
$$

Consider the special inner automorphism $\varphi_{\mathfrak{q}} : A \to A$, where

$$
\begin{aligned}
\mathfrak{q} &= \sum_{i=2}^{k_1} \sum_{j>k_2} \alpha^i_{1j} f_{n_1+i,j} - \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{n_1+1,j} + \sum_{i=2}^{k_1} \sum_{j>k_2} \beta^i_{1j} f_{n_1+i,n_2+j} \\
&\quad - \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{n_1+1,n_2+j} + \sum_{i=2}^{k_1} \gamma^i_1 f_{n_1+i,m_2} - \gamma^{k_1}_{k_1} f_{n_1+1,m_2} \in U.
\end{aligned}
$$

Since $US_1 = R^2 = 0$, we have $\mathfrak{q} b''_t \in U(S_1 + R) = 0$, so

$$
\begin{aligned}
\varphi_{\mathfrak{q}}(b''_t) &= (1+\mathfrak{q}) b''_t (1-\mathfrak{q}) = b''_t (1-\mathfrak{q}) \\
&= (e_{1,n_1+t} - e_{t,n_1+1} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} + \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} \\
&\quad + \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^{k_1}_{k_1} f_{tm_2})(1 - \sum_{i=2}^{k_1} \sum_{j>k_2} \alpha^i_{1j} f_{n_1+i,j} \\
&\quad + \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{n_1+1,j} - \sum_{i=2}^{k_1} \sum_{j>k_2} \beta^i_{1j} f_{n_1+i,n_2+j} + \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{n_1+1,n_2+j} \\
&\quad - \sum_{i=2}^{k_1} \gamma^i_1 f_{n_1+i,m_2} + \gamma^{k_1}_{k_1} f_{n_1+1,m_2}) \\
&= e_{1,n_1+t} - \sum_{j>k_2} \alpha^t_{1j} f_{1j} - \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} - \gamma^t_1 f_{1m_2} - e_{t,n_1+1} \\
&\quad - \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{tj} - \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{t,n_2+j} - \gamma^{k_1}_{k_1} f_{tm_2} + \sum_{j>k_2} \alpha^t_{1j} f_{1j} \\
&\quad + \sum_{j>k_2} \alpha^{k_1}_{k_1 j} f_{tj} + \sum_{j>k_2} \beta^t_{1j} f_{1,n_2+j} + \sum_{j>k_2} \beta^{k_1}_{k_1 j} f_{t,n_2+j} + \gamma^t_1 f_{1m_2} + \gamma^{k_1}_{k_1} f_{tm_2} \\
&= e_{1,n_1+t} - e_{t,n_1+1} \in \varphi_{\mathfrak{q}}(B_q).
\end{aligned}
$$

Therefore,

$$
E_1 = \{ e_{1,n_1+t} - e_{t,n_1+1} \mid 1 < t \le k_1 \} \subseteq \varphi_{\mathfrak{q}}(B_q) \cap S_1 \subseteq \varphi_{\mathfrak{q}}(B_q) \cap S.
$$

Since $S_2U = R^2 = 0$, we have $h^{\varepsilon}_{rq}\mathfrak{q} \in S_2U = 0$, so by applying $\varphi_{\mathfrak{q}}$ to E^{ε}_2, we get that

$$\begin{aligned}
\varphi_{\mathfrak{q}}(h^{\varepsilon}_{rq}) &= (1+\mathfrak{q})h^{\varepsilon}_{rq}(1-\mathfrak{q}) = (1+\mathfrak{q})h^{\varepsilon}_{rq} = (1+\sum_{i=2}^{k_1}\sum_{j>k_2}\alpha^{i}_{1j}f_{n_1+i,j} \\
&\quad -\sum_{j>k_2}\alpha^{k_1}_{k_1j}f_{n_1+1,j}+\sum_{i=2}^{k_1}\sum_{j>k_2}\beta^{i}_{1j}f_{n_1+i,n_2+j}-\sum_{j>k_2}\beta^{k_1}_{k_1j}f_{n_1+1,n_2+j} \\
&\quad +\sum_{i=2}^{k_1}\gamma^{i}_{1}f_{n_1+i,m_2}-\gamma^{k_1}_{k_1}f_{n_1+1,m_2})(g_{r,n_2+q}-\varepsilon g_{q,n_2+r}) \\
&= g_{r,n_2+q}-\varepsilon g_{q,n_2+r} \in \varphi_{\mathfrak{q}}(B_q),
\end{aligned}$$

so

$$E^{\varepsilon}_2 = \{g_{r,n_2+q}-\varepsilon g_{q,n_2+r} \mid 1 \le r < q \le k_2\} \subseteq \varphi_{\mathfrak{q}}(B_q) \cap S_2.$$

Since $E_2 \subseteq E^{+}_2 \subseteq \varphi_{\mathfrak{q}}(B_q)$, we get that $\mathscr{E}_2 \subseteq \varphi_{\mathfrak{q}}(B_q)$. Therefore, if $\mathscr{E}_1 = E_1$, then $\varphi_{\mathfrak{q}}(B_q)$ contains both $\mathscr{E}_1$ and $\mathscr{E}_2$, as required.

Case (2): Suppose that $\mathscr{E}_1 = E^{+}_1 = \{e_{s,n+t}-e_{t,n+s} \mid 1 \le s < t \le k\} \subseteq \mathfrak{su}^*(S_1) = \mathfrak{so}_{m_1}$. We need to show that there is a special inner automorphism $\varphi : A \to A$ such that $\mathscr{E}_1, \mathscr{E}_2 \subseteq \varphi(B_q)$. Fix any subset $\{b_{st} \mid 1 \le s < t \le k_1\} \subseteq B_q$ such that $\bar{b}_{st} = e_{s,n_1+t}-e_{t,n_1+s}$.

CLAIM 1: R_0 in (3.8.37) is a subspace of B_q. Since $\bar{b}_{1t} = e_{1,n_1+t}-e_{t,n_1+1} = \bar{b}_t \in E_1 \cap E^{+}_1$, by using the same technique that were used to prove (3.8.37) in Case (1), one can easily show that $R_0 \subseteq B_q$, as required.

CLAIM 2: For every $b_{st} = e_{s,n_1+t}-e_{t,n_1+s}+r_{st}$ ($r_{st} \in U$), we have

$$\begin{aligned}
\theta(b_{st}) &:= e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{st}_{sj}f_{sj}+\sum_{j>k_2}\alpha^{st}_{tj}f_{tj}+\sum_{j>k_2}\beta^{st}_{sj}f_{s,n_2+j} \\
&\quad +\sum_{j>k_2}\beta^{st}_{tj}f_{t,n_2+j}+\gamma^{st}_{s}f_{sm_2}+\gamma^{st}_{t}f_{tm_2} \in B_q,
\end{aligned}$$

where $\alpha^{st}_{sj}, \alpha^{st}_{tj}, \beta^{st}_{sj}, \beta^{st}_{tj}, \gamma^{st}_{t} \in \mathbb{F}$.

Since $r_{st} \in U \cong \mathscr{M}_{m_1m_2}$,

$$\begin{aligned}
r_{st} &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\alpha^{st}_{ij}f_{ij}+\beta^{st}_{ij}f_{i,n_2+j}+\zeta^{st}_{ij}f_{n_1+i,j}+\eta^{st}_{ij}f_{n_1+i,n_2+j}) \\
&\quad +\sum_{i=1}^{n_1}(\gamma^{st}_{i}f_{im_2}+\xi^{st}_{i}f_{n_1+i,m_2})+\sum_{j=1}^{n_2}(\lambda^{st}_{j}f_{m_1j}+\mu^{st}_{j}f_{m_1,n_2+j})+\delta^{st}f_{m_1m_2})
\end{aligned}$$

for some coefficients in $\mathbb{F}$. By Lemma 3.1.3, $c_{st} = b_{st}(e_{n_1+t,s}-e_{n_1+s,t})b_{st} \in B_q$. Since

$US_1 = R^2 = 0,$

$$\begin{aligned}
c_{st} &= b_{st}(e_{n_1+t,s} - e_{n_1+s,t})b_{st} = (e_{s,n_1+t} - e_{t,n_1+s} + r_{st})(e_{n_1+t,s} - e_{n_1+s,t})b_{st} \\
&= (e_{ss} + e_{tt})(e_{s,n_1+t} - e_{t,n_1+s} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\alpha_{ij}^{st} f_{ij} + \beta_{ij}^{st} f_{i,n_2+j} + \zeta_{ij}^{st} f_{n_1+i,j} \\
&\quad + \eta_{ij}^{st} f_{n_1+i,n_2+j}) + \sum_{i=1}^{n_1}(\gamma_i^{st} f_{im_2} + \xi_i^{st} f_{n_1+i,m_2}) \\
&\quad + \sum_{j=1}^{n_2}(\lambda_j^{st} f_{m_1 j} + \mu_j^{st} f_{m_1,n_2+j}) + \delta^{st} f_{m_1 m_2}) \\
&= e_{s,n_1+t} + \sum_{j=1}^{n_2}\alpha_{sj}^{st} f_{sj} + \sum_{j=1}^{n_2}\beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2} - e_{t,n_1+s} + \sum_{j=1}^{n_2}\alpha_{tj}^{st} f_{tj} \\
&\quad + \sum_{j=1}^{n_2}\beta_{tj}^{st} f_{t,n_2+j} + \gamma_t^{st} f_{tm_2} \\
&= e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j=1}^{n_2}\alpha_{sj}^{st} f_{sj} + \sum_{j=1}^{n_2}\alpha_{tj}^{st} f_{tj} + \sum_{j>k_2}\beta_{sj}^{st} f_{s,n_2+j} \\
&\quad + \sum_{j>k_2}\beta_{tj}^{st} f_{t,n_2+j} + \gamma_s^{st} f_{sm_2} + \gamma_t^{st} f_{tm_2} + \sum_{j=1}^{k_2}\beta_{sj}^{st} f_{s,n_2+j} + \sum_{j=1}^{k_2}\beta_{tj}^{st} f_{t,n_2+j} \\
&= \theta(b_{st}) + \sum_{j=1}^{k_2}\beta_{sj}^{st} f_{s,n_2+j} + \sum_{j=1}^{k_2}\beta_{tj}^{st} f_{t,n_2+j}) \in B_q.
\end{aligned}$$

Since $\sum_{j=1}^{k_2}\beta_{sj}^{st} f_{s,n_2+j} + \sum_{j=1}^{k_2}\beta_{tj}^{st} f_{t,n_2+j} \in R_0 \subseteq B_q$ (by Claim 1), we get that

$$\begin{aligned}
\theta(b_{st}) &= e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j=1}^{n_2}\alpha_{sj}^{st} f_{sj} + \sum_{j=1}^{n_2}\alpha_{tj}^{st} f_{tj} + \sum_{j>k_2}\beta_{sj}^{st} f_{s,n_2+j} \\
&\quad + \sum_{j>k_2}\beta_{tj}^{st} f_{t,n_2+j} + \gamma_s^{st} f_{sm_2} + \gamma_t^{st} f_{tm_2} \in B_q.
\end{aligned}$$

Suppose that $\mathscr{E}_2 = E_2^{\varepsilon}$. Since $(B_q)^2 = 0$ and $S_1S_2 = 0$, for all $h_{1q}^{\varepsilon} \in E_2^{\varepsilon}$, we have

$$\begin{aligned}
0 &= \theta(b_{st})h_{1q}^{\varepsilon} = (e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j=1}^{n_2}\alpha_{sj}^{st} f_{sj} + \sum_{j=1}^{n_2}\alpha_{tj}^{st} f_{tj} + \sum_{j>k_2}\beta_{sj}^{st} f_{s,n_2+j} \\
&\quad + \sum_{j>k_2}\beta_{tj}^{st} f_{t,n_2+j} + \gamma_{sj}^{st} f_{sm_2} + \gamma_{tj}^{st} f_{tm_2})(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) \\
&= \alpha_{s1}^{st} f_{s,n_2+q} - \varepsilon\alpha_{sq}^{st} f_{s,n_2+1} + \alpha_{t1}^{st} f_{t,n_2+q} - \varepsilon\alpha_{tq}^{st} f_{t,n_2+1}
\end{aligned}$$

for all $1 < q \le k_2$, so $\alpha_{sj}^{st} = \alpha_{tj}^{st} = 0$ for all $1 \le j \le k_2$. Note that this is also true when $\mathscr{E}_2 = E_2$ (because $h_q = g_{1,n_2+q} - g_{q,n_2+1} = h_{1q}^{+} \in E^{+} \cap E$). Therefore, for any choice of

$\mathscr{E}_2$, we get that

$$\begin{aligned}\theta(b_{st}) &= e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{st}_{sj}f_{sj}+\sum_{j>k_2}\alpha^{st}_{tj}f_{tj}+\sum_{j>k_2}\beta^{st}_{sj}f_{s,n_2+j}\\ &\quad+\sum_{j>k_2}\beta^{st}_{tj}f_{t,n_2+j}+\gamma^{st}_{sj}f_{sm_2}+\gamma^{st}_{tj}f_{tm_2}\in B_q,\end{aligned}$$

as required.

CLAIM 3: There is a special inner automorphism $\varphi_{q'}$ of A such that

$$\mathscr{E}_2\subseteq\varphi_{q'}(B_q)=B_{q'}; \tag{3.8.38}$$

$$b'''_{sk_1}=e_{s,n_1+k_1}-e_{k_1,n_1+s}\in B_{q'}; \tag{3.8.39}$$

$$b'''_{st}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{tk_1}_{k_1j}f_{sj}+\sum_{j>k_2}\beta^{tk_1}_{k_1j}f_{s,n+j}+\gamma^{tk_1}_{k_1}f_{sm_2}\in B_{q'} \tag{3.8.40}$$

for all $1\le s<t<k_1$.

Recall that $\bar{b}_{st}=e_{s,n+t}-e_{t,n+s}$. Since $\bar{b}_{1t}=e_{1,n+t}-e_{t,n+1}=\bar{b}_t\in E\cap E^+$, by Case (1), there is a special inner automorphism $\varphi_{\mathrm{q}}:A\to A$ such that $\mathscr{E}_2\subseteq\varphi_{\mathrm{q}}(B_q)$ and $\varphi_{\mathrm{q}}(b_{1t})=e_{1,n_1+t}-e_{t,n_1+1}\in\varphi_{\mathrm{q}}(B_q)$ for all t. By applying φ_{q} to b_{st} (for all $s>1$) and using Claim 2, we get that

$$\begin{aligned}\theta(\varphi_{\mathrm{q}}(b_{st})) &= e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{st}_{sj}f_{sj}+\sum_{j>k_2}\alpha^{st}_{tj}f_{tj}+\sum_{j>k_2}\beta^{st}_{sj}f_{s,n_2+j}\\ &\quad+\sum_{j>k_2}\beta^{st}_{tj}f_{t,n_2+j}+\gamma^{st}_{sj}f_{sm_2}+\gamma^{st}_{tj}f_{tm_2}\in\varphi_{\mathrm{q}}(B_q).\end{aligned}$$

Put $b'_{1t}=\varphi_{\mathrm{q}}(b_{1t})=e_{1,n+t}-e_{t,n+1}\in\varphi_{\mathrm{q}}(B_q)$. For all $s>1$, set $b'_{st}=\{\theta(\varphi_{\mathrm{q}}(b_{st})),e_{n+t,1}-e_{n+1,t},b'_{1t}\}\in\varphi_{\mathrm{q}}(B_q)$ (by Lemma 3.1.3). Since $US_1=R^2=0$, we have $\theta(\varphi_{\mathrm{q}}(b_{st}))(e_{n_1+t,1}-e_{n_1+1,t})=e_{s1}$, so

$$\begin{aligned}b'_{st} &= \theta(\varphi_{\mathrm{q}}(b_{st}))(e_{n_1+t,1}-e_{n_1+1,t})b'_{1t}+b'_{1t}(e_{n_1+t,1}-e_{n_1+1,t})\theta(\varphi_{\mathrm{q}}(b_{st}))\\ &= e_{s1}(e_{1,n_1+t}-e_{t,n_1+1})+(e_{1,n_1+t}-e_{t,n_1+1})(e_{n_1+t,1}-e_{n_1+1,t})b_{st}\\ &= e_{s,n_1+t}+(e_{11}+e_{tt})(e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{st}_{sj}f_{sj}+\sum_{j>k_2}\alpha^{st}_{tj}f_{tj}\\ &\quad+\sum_{j>k_2}\beta^{st}_{sj}f_{s,n_2+j}+\sum_{j>k_2}\beta^{st}_{tj}f_{t,n_2+j}+\gamma^{st}_{sj}f_{sm_2}+\gamma^{st}_{tj}f_{tm_2})\end{aligned}$$

$$= e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{tj}^{st} f_{tj} + \sum_{j>k_2} \beta_{tj}^{st} f_{t,n_2+j} + \gamma_{tj}^{st} f_{tm_2} \in \varphi_{\mathfrak{q}}(B_q).$$

Put $b''_{1t} = b'_{1t} = e_{1,n_1+t} - e_{t,n_1+1} \in B_{\mathfrak{q}}$ and for $s>1$, set $b''_{st} = \{b'_{sk_1}, e_{n_1+k_1,s} - e_{n_1+s,k_1}, b'_{st}\} \in \varphi_{\mathfrak{q}}(B_q)$ (by Lemma 3.1.3). Since $US_1 = R^2 = 0$, we have $b'_{st}(e_{n_1+k_1,s} - e_{n_1+s,k_1}) = e_{tk_1}$ and $b_{sk_1}(e_{n_1+k_1,s} - e_{n_1+s,k_1}) = e_{ss} + e_{k_1k_1}$, so

$$\begin{aligned}
b''_{st} &= b_{sk_1}(e_{n_1+k_1,s} - e_{n_1+s,k_1})b'_{st} + b'_{st}(e_{n_1+k_1,s} - e_{n_1+s,k_1})b_{sk_1} \\
&= (e_{ss} + e_{k_1k_1})(e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{tj}^{st} f_{tj} + \sum_{j>k_2} \beta_{tj}^{st} f_{t,n_2+j} + \gamma_{tj}^{st} f_{tm_2}) \\
&\quad + e_{tk_1}(e_{s,n_1+k_1} - e_{k_1,n_1+s} + \sum_{j>k_2} \alpha_{k_1j}^{sk_1} f_{k_1j} + \sum_{j>k_2} \beta_{k_1j}^{sk_1} f_{k_1,n_2+j} + \gamma_{k_1j}^{sk_1} f_{k_1m_2}) \\
&= e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{k_1j}^{sk_1} f_{tj} + \sum_{j>k_2} \beta_{k_1j}^{sk_1} f_{t,n_2+j} + \gamma_{k_1j}^{sk_1} f_{tm_2} \in \varphi_{\mathfrak{q}}(B_q).
\end{aligned}$$

Consider the special inner automorphism $\varphi_{q'}: A \to A$, where

$$q' = -\sum_{i=2}^{k_1-1} \sum_{j>k_2} \alpha_{k_1j}^{ik_1} f_{n_1+i,j} - \sum_{i=2}^{k_1-1} \sum_{j>k_2} \beta_{k_1j}^{ik_1} f_{n_1+i,n_2+j} - \sum_{i=2}^{k_1-1} \gamma_{k_1}^{jk_1} f_{n_1+i,m_2} \in U.$$

Put $B_{q'} = \varphi_{q'}(\varphi_{\mathfrak{q}}(B_{\mathfrak{q}}))$ and $b'''_{st} = \varphi_{q'}(b''_{st}) \in B_{q'}$ for all s and t. Since $US_1 = R^2 = 0$, we have $(1+q')b'_{1k_1} = b'_{1k_1}$, so

$$\begin{aligned}
b'''_{1k_1} &= \varphi_{q'}(b''_{1k_1}) = (1+q')b''_{1k_1}(1-q') = b'_{1k_1}(1-q') \\
&= (e_{1,n_1+k_1} - e_{k_1,n_1+1})(1 + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \alpha_{k_1j}^{ik_1} f_{n_1+i,j} \\
&\quad + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \beta_{k_1j}^{ik_1} f_{n_1+i,n_2+j} + \sum_{i=2}^{k_1-1} \gamma_{k_1}^{jk_1} f_{n_1+i,m_2})) \\
&\quad = e_{1,n_1+k_1} - e_{k_1,n_1+1} \in B_{q'} = \varphi_{q'}(\varphi_{\mathfrak{q}}(B_q))
\end{aligned}$$

and (for all $s>1$)

$$\begin{aligned}
b'''_{sk_1} &= \varphi_{q'}(b''_{sk_1}) = b''_{sk_1}(1-q') \\
&= (e_{s,n_1+k_1} - e_{k_1,n_1+s} + \sum_{j>k_2} \alpha_{k_1j}^{sk_1} f_{k_1j} + \sum_{j>k_2} \beta_{k_1j}^{sk_1} f_{k_1,n_2+j} + \gamma_{k_1}^{sk_1} f_{k_1m_2})(1 \\
&\quad + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \alpha_{k_1j}^{ik_1} f_{n_1+i,j} + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \beta_{k_1j}^{ik_1} f_{n_1+i,n_2+j} + \sum_{i=2}^{k_1-1} \gamma_{k_1}^{jk_1} f_{n_1+i,m_2}) \\
&= e_{s,n_1+k_1} - e_{k_1,n_1+s} - \sum_{j>k_2} \alpha_{k_1j}^{sk_1} f_{k_1j} - \sum_{j>k_2} \beta_{k_1j}^{sk_1} f_{k_1,n_2+j} - \gamma_{k_1}^{sk_1} f_{k_1m_2}
\end{aligned}$$

$$+\sum_{j>k_2}\alpha_{k_1 j}^{sk_1} f_{k_1 j}+\sum_{j>k_2}\beta_{k_1 j}^{sk_1} f_{k_1,n_2+j}+\gamma_{k_1}^{sk_1} f_{k_1 m_2}=e_{s,n_1+k_1}-e_{k_1,n_1+s}\in B_{q'}.$$

Therefore,

$$b'''_{sk_1}=e_{s,n_1+k_1}-e_{k_1,n_1+s}\in B_{q'}=\varphi_{q'}(\varphi_{\mathsf{q}}(B_{q'}))\quad \text{for all}\quad s,$$

so (3.8.39) is proved. Next, we need to show that (3.8.40) holds. Applying $\varphi_{q'}$ to all b''_{st} for all $t<k_1$, we get that

$$\begin{aligned}
b'''_{1t} &= \varphi_{q'}(b''_{1t})=b''_{1t}(1-q')=(e_{1,n_1+t}-e_{t,n_1+1})(1+\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha_{k_1 j}^{ik_1} f_{n_1+i,j}\\
&\quad+\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta_{k_1 j}^{ik_1} f_{n_1+i,n_2+j}+\sum_{i=2}^{k_1-1}\gamma_{k_1}^{ik_1} f_{n_1+i,m_2})\\
&= e_{1,n_1+t}-e_{t,n_1+1}+\sum_{j>k_2}\alpha_{k_1 j}^{tk_1} f_{1j}+\sum_{j>k_2}\beta_{k_1 j}^{tk_1} f_{1,n_2+j}+\gamma_{k_1}^{tk_1} f_{1m_2}\in B_{q'}
\end{aligned}$$

and (for all $s>1$)

$$\begin{aligned}
b'''_{st} &= \varphi_{q'}(b''_{st})=b''_{st}(1-q')=(e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1 j}^{sk_1} f_{tj}\\
&\quad+\sum_{j>k_2}\beta_{k_1 j}^{sk_1} f_{t,n_2+j}+\gamma_{k_1 j}^{sk_1} f_{tm_2})(1+\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha_{k_1 j}^{ik_1} f_{n_1+i,j}\\
&\quad+\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta_{k_1 j}^{ik_1} f_{n_1+i,n_2+j}+\sum_{i=2}^{k_1-1}\gamma_{k_1}^{ik_1} f_{n_1+i,m_2})\\
&= e_{s,n_1+t}+\sum_{j>k_2}\alpha_{k_1 j}^{tk_1} f_{sj}+\sum_{j>k_2}\beta_{k_1 j}^{tk_1} f_{s,n_2+j}+\gamma_{k_1}^{tk_1} f_{sm_2}-e_{t,n_1+s}-\sum_{j>k_2}\alpha_{k_1 j}^{sk_1} f_{tj}\\
&\quad-\sum_{j>k_2}\beta_{k_1 j}^{sk_1} f_{t,n_2+j}-\gamma_{k_1}^{sk_1} f_{tm_2}+\sum_{j>k_2}\alpha_{k_1 j}^{sk_1} f_{tj}+\sum_{j>k_2}\beta_{k_1 j}^{sk_1} f_{t,n_2+j}+\gamma_{k_1 j}^{sk_1} f_{tm_2}\\
&= e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1 j}^{tk_1} f_{sj}+\sum_{j>k_2}\beta_{k_1 j}^{tk_1} f_{s,n_2+j}+\gamma_{k_1}^{tk_1} f_{sm_2}\in B_{q'}.
\end{aligned}$$

Therefore, for all s, we have

$$b'''_{st}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1 j}^{tk_1} f_{sj}+\sum_{j>k_2}\beta_{k_1 j}^{tk_1} f_{s,n_2+j}+\gamma_{k_1}^{tk_1} f_{sm_2}\in B_{q'},$$

so (3.8.40) is proved. It remains to show that $\mathscr{E}_2\subseteq B_{q'}$. We have $\mathscr{E}_2\subseteq\varphi_{\mathsf{q}}(B_q)$. Since $S_2U=0$, by applying $\varphi_{q'}$ to $h_{rq}^{\varepsilon}=g_{r,n_2+q}-\varepsilon g_{q,n_2+r}\in E_2^{\varepsilon}\subseteq S_2$ (for all $1\le r\le q\le k_2$), we get that

$$\begin{aligned}\varphi_{q'}(h^{\varepsilon}_{rq}) &= (1+q'')h^{\varepsilon}_{rq}(1-q'') = (1+q'')h^{\varepsilon}_{rq} = (1-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha^{ik_1}_{k_1j}f_{n_1+i,j}\\ &\quad -\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta^{ik_1}_{k_1j}f_{n_1+i,n_2+j}-\sum_{i=2}^{k_1-1}\gamma^{ik_1}_{k_1}f_{n_1+i,m_2})(g_{r,n_2+q}-g_{q,n_2+r})\\ &= g_{r,n_2+q}-g_{q,n_2+r}\in\varphi_{q'}(\varphi_{\mathfrak{q}}(B_q))=B_{q'}.\end{aligned}$$

Thus, $E^{\varepsilon}_2\subseteq B_{q'}$. Since $E_2\subseteq E^{\varepsilon}_2\subseteq B_{q'}$, we get that $\mathscr{E}_2\subseteq B_{q'}=\varphi_{q'}(\varphi_{\mathfrak{q}}(B_q))$, so (3.8.38) holds and Claim 3 is proved, as required.

CLAIM 4: There is a special inner automorphism $\varphi_{q_1}: A\to A$ such that

$$\mathscr{E}_2\subseteq\varphi_{q_1}(B_{q'})=B_{q_1}; \tag{3.8.41}$$

$$b^{(1)}_{1t}=e_{1,n_1+t}-e_{t,n_1+1}\in\varphi_{q_1}(B_{q'})=B_{q_1}\quad\text{for all}\quad 1<t\le k_1; \tag{3.8.42}$$

$$b^{(1)}_{st}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{tk_1}_{k_1j}f_{sj}+\sum_{j>k_2}\beta^{tk_1}_{k_1j}f_{s,n_2+j}+\gamma^{tk_1}_{k_1}f_{sm_2}\in B_{q_1} \tag{3.8.43}$$

for all $1<s<t\le k_1$.

By Claim 3, there is a special inner automorphism $\varphi_{q'}$ on A such that $\mathscr{E}_2\subseteq B_{q'}$,

$$b'''_{sk_1}=e_{s,n_1+k_1}-e_{k_1,n_1+s}\in B_{q'}\quad\text{and}$$

$$b'''_{st}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha^{sk_1}_{k_1j}f_{tj}+\sum_{j>k_2}\beta^{sk_1}_{k_1j}f_{t,n_2+j}+\gamma^{sk_1}_{k_1}f_{tm}\in B_{q'}.$$

Consider the special inner automorphism $\varphi_{q_1}: A\to A$, where

$$q_1=\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha^{ik_1}_{k_1j}f_{n_1+i,j}+\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta^{ik_1}_{k_1j}f_{n_1+i,n_2+j}+\sum_{i=2}^{k_1-1}\gamma^{ik_1}_{k_1}f_{n_1+i,m}\in U.$$

Put $B_{q_1}=\varphi_{q_1}(B_{q'})$ and $b^{(1)}_{st}=\varphi_{q_1}(b'''_{st})\in B_{q_1}$. Since $US_1=R^2=0$,

$$\begin{aligned}b^{(1)}_{1k_1} &= \varphi_{q_1}(b'''_{1k_1})=(1+q_1)b'''_{1k_1}(1-q_1)=b'''_{1k_1}(1-q_1)\\ &= (e_{1,n_1+k_1}-e_{k_1,n_1+1})(1-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha^{ik_1}_{k_1j}f_{n_1+i,j}-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta^{ik_1}_{k_1j}f_{n_1+i,n_2+j}\end{aligned}$$

$$-\sum_{i=2}^{k_1-1}\gamma_{k_1}^{jk_1}f_{n_1+i,m}) = e_{1,n_1+k_1}-e_{k_1,n_1+1}\in\varphi_{q_1}(B_{q'})=B_{q_1}$$

and (for all $1<t<k_1$)

$$\begin{aligned}
b_{1t}^{(1)} &= \varphi_{q_1}(b'''_{1t}) = b'''_{1t}(1-q_1)\\
&= (e_{1,n_1+t}-e_{t,n_1+1}+\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{1j}+\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{1,n+j}+\gamma_{k_1}^{tk_1}f_{1m_2})(1\\
&\quad -\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha_{k_1j}^{ik_1}f_{n_1+i,j}-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta_{k_1j}^{ik_1}f_{n_1+i,n_2+j}-\sum_{i=2}^{k_1-1}\gamma_{k_1}^{jk_1}f_{n_1+i,m})\\
&= e_{1,n_1+t}-\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{1j}-\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{1,n_2+j}-\gamma_{k_1}^{tk_1}f_{1m}-e_{t,n_1+1}+\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{1j}\\
&\quad +\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{1,n+j}+\gamma_{k_1}^{tk_1}f_{1m_2} = e_{1,n_1+t}-e_{t,n_1+1}\in B_{q_1}.
\end{aligned}$$

Hence,

$$b_{1t}^{(1)} = e_{1,n_1+t}-e_{t,n_1+1}\in\varphi_{q_1}(B_{q'})=B_{q_1}\quad\text{for all}\quad 1<t\le k_1,$$

so (3.8.42) is proved. Next, we need to show that (3.8.43) holds. Applying φ_{q_1} to b'''_{st} for all $s>1$, we get that

$$\begin{aligned}
b_{sk_1}^{(1)} &= \varphi_{q_1}(b'''_{sk_1}) = b'''_{sk_1}(1-q_1) = (e_{s,n_1+k_1}-e_{k_1,n_1+s})(1\\
&\quad -\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha_{k_1j}^{ik_1}f_{n_1+i,j}-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta_{k_1j}^{ik_1}f_{n_1+i,n_2+j}-\sum_{i=2}^{k_1-1}\gamma_{k_1}^{jk_1}f_{n_1+i,m})\\
&\quad e_{s,n_1+k_1}-e_{k_1,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{sk_1}f_{k_1j}+\sum_{j>k_2}\beta_{k_1j}^{sk_1}f_{k_1,n_2+j}+\gamma_{k_1}^{sk_1}f_{k_1m}\in B_{q_1}
\end{aligned}$$

and (for all $t<k_1$)

$$\begin{aligned}
b_{st}^{(1)} &= \varphi_{q_1}(b'''_{st}) = b'''_{st}(1-q_1)\\
&= (e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{sj}+\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{s,n+j}+\gamma_{k_1}^{tk_1}f_{sm_2})(1\\
&\quad -\sum_{i=2}^{k_1-1}\sum_{j>k_2}\alpha_{k_1j}^{ik_1}f_{n_1+i,j}-\sum_{i=2}^{k_1-1}\sum_{j>k_2}\beta_{k_1j}^{ik_1}f_{n_1+i,n_2+j}-\sum_{i=2}^{k_1-1}\gamma_{k_1}^{jk_1}f_{n_1+i,m})\\
&= e_{s,n_1+t}-\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{sj}-\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{s,n_2+j}-\gamma_{k_1}^{tk_1}f_{sm}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{sk_1}f_{tj}\\
&\quad +\sum_{j>k_2}\beta_{k_1j}^{sk_1}f_{t,n_2+j}+\gamma_{k_1}^{sk_1}f_{tm}+\sum_{j>k_2}\alpha_{k_1j}^{tk_1}f_{sj}+\sum_{j>k_2}\beta_{k_1j}^{tk_1}f_{s,n+j}+\gamma_{k_1}^{tk_1}f_{sm_2}\\
&= e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{sk_1}f_{tj}+\sum_{j>k_2}\beta_{k_1j}^{sk_1}f_{t,n_2+j}+\gamma_{k_1}^{sk_1}f_{tm}\in B_{q_1}
\end{aligned}$$

Therefore, for all t, we have

$$b_{st}^{(1)} = e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{k_1 j}^{sk_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{sk_1} f_{t,n_2+j} + \gamma_{k_1}^{sk_1} f_{tm} \in B_{q_1},$$

so (3.8.43) holds. It remains to show that $\mathscr{E}_2 \subseteq B_{q_1}$. We have $\mathscr{E}_2 \subseteq B_{q'}$. Since $S_2 U = 0$, by applying φ_{q_1} to $h_{rq}^{\varepsilon} = e_{r,n_2+q} - \varepsilon e_{q,n_2+r} \in E_2^{\varepsilon} \subseteq S_2$ for all $1 \le r \le q \le k_2$, we get that

$$\begin{aligned}
\varphi_{q_1}(h_{rq}^{\varepsilon}) &= (1+q_1)h_{rq}^{\varepsilon}(1-q_1) = (1+q_1)h_{rq}^{\varepsilon} = (1 + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \alpha_{k_1 j}^{ik_1} f_{n_1+i,j} \\
&\quad + \sum_{i=2}^{k_1-1} \sum_{j>k_2} \beta_{k_1 j}^{ik_1} f_{n_1+i,n_2+j} + \sum_{i=2}^{k_1-1} \gamma_{k_1}^{jk_1} f_{n_1+i,m})(g_{r,n_2+q} - \varepsilon g_{q,n_2+r}) \\
&= g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in \varphi_{q_1}(B_{q'}) = B_{q_1},
\end{aligned}$$

so $E_2^{\varepsilon} \subseteq B_{q_1}$. Since $\mathscr{E}_2 \subseteq B_{q_1}$ (because $E_2 \subseteq E_2^{+} \subseteq B_{q_1}$), we get that $\mathscr{E}_2 \subseteq B_{q_1}$, so (3.8.41) holds and Claim 4 is proved, as required.

CLAIM 5: There are $k_1 - 2$ inner automorphisms $\varphi_{q_\iota} : A \to A$ ($\iota = 1, \ldots, k_1 - 2$) such that

$$\mathscr{E}_2 \subseteq \varphi_{q_{k_1-2}}(\ldots \varphi_{q_2}(\varphi_{q_1}(B_{q'}))\ldots) = B_{q_{k_1-2}}; \tag{3.8.44}$$

$$b_{\iota t}^{(k_1-2)} = e_{\iota,n_1+t} - e_{t,n_1+\iota} \in B_{q_{k_1-2}} \quad \text{for all} \quad \iota < t \le k_1; \tag{3.8.45}$$

$$b_{st}^{(k_1-2)} = e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{k_1 j}^{sk_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{sk_1} f_{t,n_2+j} + \gamma_{k_1}^{sk_1} f_{tm} \in B_{q_{k_1-2}} \tag{3.8.46}$$

for all $k_1 - 2 < s < t \le k_1$.

We will prove Claim 5 by induction on ι. The base of the induction (when $\iota = 1$) being clear by Claim 4. Suppose that $\iota > 1$. Put $\kappa = k_1 - 2$. By the inductive hypothesis there are $\kappa - 1$ inner automorphisms $\varphi_{q_r} : A \to A$ ($r = 1, \ldots, \kappa - 1$) such that $\mathscr{E}_2 \subseteq \varphi_{q_{\kappa-1}}(\ldots \varphi_{q_1}(B_{q'})\ldots) = B_{q_{\kappa-1}}$,

$$b_{rt}^{(\kappa-1)} = e_{r,n_1+t} - e_{t,n_1+r} \in B_{q_{\kappa-1}} \quad \text{for all} \quad \iota < t \le k_1$$

and (for all $\kappa - 1 < s < t \le k_1$)

$$b_{st}^{(\kappa-1)} = e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{k_1 j}^{sk_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{sk_1} f_{t,n_2+j} + \gamma_{k_1}^{sk_1} f_{tm} \in B_{q_{\kappa-1}}.$$

Consider the special inner automorphism $\varphi_{q_\kappa} : A \to A$, where

$$q_\kappa = -\sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,j} - \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,n_2+j} - \gamma_{k_1}^{\kappa k_1} f_{n_1+\kappa,m_2} \in U.$$

Put $B_{q_\kappa} = \varphi_{q_\kappa}(B_{q_{\kappa-1}})$ and $c_{st}^{(\kappa)} = \varphi_{q_\kappa}(b_{st}^{(\kappa-1)}) \in B_{q_\kappa}$ for all s and t. Since $US_1 = R^2 = 0$,

$$\begin{aligned}
c_{r\kappa}^{(\kappa)} &= \varphi_{q_\kappa}(b_{r\kappa}^{(\kappa-1)}) = (1+q_\kappa)b_{r\kappa}^{(\kappa-1)}(1-q_\kappa) = b_{r\kappa}^{(\kappa-1)}(1-q_\kappa) \\
&= (e_{r,n_1+\kappa} - e_{\kappa,n_1+r})(1 + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,j} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,n_2+j} \\
&\quad + \gamma_{k_1}^{\kappa k_1} f_{n_1+\kappa,m_2}) \\
&= e_{r,n_1+\kappa} - e_{\kappa,n_1+r} + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{rj} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{r,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{rm_2} \in B_{q_\kappa}
\end{aligned}$$

and (for all $\kappa \neq t \leq k_1$)

$$\begin{aligned}
c_{rt}^{(\kappa)} &= \varphi_{q_\kappa}(b_{rt}^{(\kappa-1)}) = b_{rt}^{(\kappa-1)}(1-q_\kappa) = (e_{r,n_1+t} - e_{t,n_1+r})(1 \\
&\quad + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,j} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{n_1+\kappa,m_2}) \\
&= e_{r,n_1+t} - e_{t,n_1+r} \in B_{q_\kappa}. \qquad (3.8.47)
\end{aligned}$$

Note that if $s \geq \kappa$, then $t > \kappa$, so

$$\begin{aligned}
c_{\kappa t}^{(\kappa)} &= \varphi_{q_\kappa}(b_{\kappa t}^{(\kappa-1)}) = b_{\kappa t}^{(\kappa-1)}(1-q_\kappa) \\
&= (e_{\kappa,n_1+t} - e_{t,n_1+\kappa} + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{t,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{tm_2})(1 \\
&\quad + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,j} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{n_1+\kappa,m_2}) \\
&= e_{\kappa,n_1+t} - e_{t,n_1+\kappa} - \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{tj} - \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{t,n_2+j} - \gamma_{k_1}^{\kappa k_1} f_{tm_2} \\
&\quad + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{t,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{tm_2} \\
&= e_{\kappa,n_1+t} - e_{t,n_1+\kappa} \in B_{q_\kappa} \qquad (3.8.48)
\end{aligned}$$

and for $s > \kappa$, we have

$$\begin{aligned}
c_{st}^{(\kappa)} &= \varphi_{q_\kappa}(b_{st}^{(\kappa-1)}) = b_{st}^{(\kappa-1)}(1-q_\kappa) \\
&= (e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha_{k_1 j}^{sk_1} f_{tj} + \sum_{j>k_2} \beta_{k_1 j}^{sk_1} f_{t,n_2+j} + \gamma_{k_1}^{sk_1} f_{tm_2})(1 \\
&\quad + \sum_{j>k_2} \alpha_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,j} + \sum_{j>k_2} \beta_{k_1 j}^{\kappa k_1} f_{n_1+\kappa,n_2+j} + \gamma_{k_1}^{\kappa k_1} f_{n_1+\kappa,m_2})
\end{aligned}$$

$$= \; e_{s,n_1+t} - e_{t,n_1+s} + \sum_{j>k_2} \alpha^{sk_1}_{k_1 j} f_{tj} + \sum_{j>k_2} \beta^{sk_1}_{k_1 j} f_{t,n_2+j} + \gamma^{sk_1}_{k_1} f_{tm_2} \in B_{q_\kappa},$$

so (3.8.46) is proved (as $\kappa = k_1 - 2$). Next, we need to show that (3.8.45) holds. Let $b^{(\kappa)}_{r\kappa} = \{c^{(\kappa)}_{r\kappa}, e_{n_1+t,\kappa} - e_{n_1+\kappa,t}, c^{(\kappa)}_{\kappa t}\} \in B_{q_\kappa}$ and $b^{(\kappa)}_{st} = c^{(\kappa)}_{st} \in B_{q_\kappa}$ for all of the remaining indices s and t. Since $US_1 = R^2 = 0$,

$$\begin{aligned}
b^{(\kappa)}_{r\kappa} &= c^{(\kappa)}_{r\kappa}(e_{n_1+t,\kappa} - e_{n_1+\kappa,t})c^{(\kappa)}_{\kappa t} + c^{(\kappa)}_{\kappa t}(e_{n_1+t,\kappa} - e_{n_1+\kappa,t})c^{(\kappa)}_{r\kappa} \\
&= (e_{r,n_1+\kappa} - e_{\kappa,n_1+r} + \sum_{j>k_2} \alpha^{\kappa k_1}_{k_1 j} f_{rj} + \sum_{j>k_2} \beta^{\kappa k_1}_{k_1 j} f_{r,n_2+j} + \gamma^{\kappa k_1}_{k_1} f_{rm_2})(e_{n_1+t,\kappa} \\
&\quad - e_{n_1+\kappa,t})c^{(\kappa)}_{\kappa t} + (e_{\kappa,n_1+t} - e_{t,n_1+\kappa})(e_{n_1+t,\kappa} - e_{n_1+\kappa,t})c^{(\kappa)}_{r\kappa} \\
&= -e_{rt}(e_{\kappa,n_1+t} - e_{t,n_1+\kappa}) + (e_{\kappa\kappa} + e_{tt})(e_{r,n_1+\kappa} - e_{\kappa,n_1+r} + \sum_{j>k_2} \alpha^{\kappa k_1}_{k_1 j} f_{rj} \\
&\quad + \sum_{j>k_2} \beta^{\kappa k_1}_{k_1 j} f_{r,n_2+j} + \gamma^{\kappa k_1}_{k_1} f_{rm_2}) \\
&= e_{r,n_1+\kappa} - e_{\kappa,n_1+r} \in B_{q_\kappa},
\end{aligned}$$

Combining this with (3.8.47), we get that $b^{(\kappa)}_{rt} = e_{r,n_1+t} - e_{t,n_1+r} \in B_{q_\kappa}$ for all $r < t \le k_1$. By (3.8.48), $b^{(\kappa)}_{\kappa t} = c^{(\kappa)}_{\kappa t} = e_{\kappa,n_1+t} - e_{t,n_1+\kappa} \in B_{q_\kappa}$ for all $\kappa < t \le k_1$, so

$$b^{(\kappa)}_{\iota t} = e_{\iota,n_1+t} - e_{t,n_1+\iota} \in B_{q_\kappa} \quad \text{for all} \quad \iota = 1, \ldots, \kappa, \quad \iota < t \le k_1,$$

so (3.8.45) is proved. It remains to show that (3.8.44) holds, that is, $\mathscr{E}_2 \subseteq B_{q_\kappa}$. Recall that $\mathscr{E}_2 \subseteq B_{q_{\kappa-1}}$. Since $S_2U = R^2 = 0$, by applying φ_{q_κ} to $h^\varepsilon_{rq} = g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in E^\varepsilon_2$,

$$\begin{aligned}
\varphi_{q_\kappa}(h^\varepsilon_{rq}) &= (1+q_\kappa)h^\varepsilon_{rq}(1-q_\kappa) = (1+q_\kappa)h^\varepsilon_{rq} = (1 - \sum_{j>k_2} \alpha^{\kappa k_1}_{k_1 j} f_{n_1+\kappa,j} \\
&\quad - \sum_{j>k_2} \beta^{\kappa k_1}_{k_1 j} f_{n_1+\kappa,n_2+j} - \gamma^{\kappa k_1}_{k_1} f_{n_1+\kappa,m_2})(g_{r,n_2+q} - \varepsilon g_{q,n_2+r}) \\
&= g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in B_{q_\kappa},
\end{aligned}$$

so $E^\varepsilon_2 \subseteq \varphi_{q_\kappa}(B_{q_{\kappa-1}}) = B_{q_\kappa}$. Since $E_2 \subseteq E^+_2 \subseteq B_{q_\kappa}$, we get that $\mathscr{E}_2 \subseteq B_{q_\kappa} = B_{q_{k_1-2}}$, so (3.8.44) holds and Claim 5 is proved, as required.

Now, we are going to define the final inner automorphism in order to complete the proof. By Claim 5, there are $k_1 - 2$ inner automorphisms φ_{q_ι} ($\iota = 1, \ldots, k_1 - 2$) on A such that $\mathscr{E}_2 \subseteq \varphi_{q_{k_1-2}}(\ldots \varphi_{q_2}(\varphi_{q_1}(B_{q'}))\ldots) = B_{q_{k_1-2}}$,

$$b^{(k_1-2)}_{\iota t} = e_{\iota,n_1+t} - e_{t,n_1+\iota} \in B_{q_{k_1-2}} \quad \text{for all} \quad \iota = 1, \ldots, k_1 - 2, \quad \iota < t \le k_1$$

and (for all $k_1-2<s<t\leq k_1$)

$$b_{st}^{(k_1-2)}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{sk_1}f_{tj}+\sum_{j>k_2}\beta_{k_1j}^{sk_1}f_{t,n_2+j}+\gamma_{k_1}^{sk_1}f_{tm}\in B_{q_{k_1-2}}.$$

Put $\nu=k_1-1$. Consider the inner automorphism $\varphi_{q_\nu}:A\to A$, where

$$q_\nu=-\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{n_1+\nu,j}-\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{n_1+\nu,n_2+j}-\gamma_{k_1}^{\nu k_1}f_{n_1+\nu,m_2}\in U.$$

Put $B_{q_\nu}=\varphi_{q_\nu}(B_{q_{k_1-2}})$ and $b_{st}^{(\nu)}=\varphi_{q_\nu}(b_{st}^{(k_1-2)})\in B_{q_\nu}$. Since $US_1=R^2=0$, $(1\leq\iota\leq k_1-2)$

$$\begin{aligned}
b_{\iota\nu}^{(\nu)} &= \varphi_{q_\nu}(b_{\iota\nu}^{(k_1-2)})=(1+q_\nu)b_{\iota\nu}^{(k_1-2)}(1-q_\nu)=b_{\iota\nu}^{(k_1-2)}(1-q_\nu)\\
&= (e_{\iota,n_1+\nu}-e_{\nu,n_1+\iota})(1+\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{n_1+\nu,j}+\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{n_1+\nu,n_2+j}\\
&\quad +\gamma_{k_1}^{\nu k_1}f_{n_1+\nu,m_2})\\
&= e_{\iota,n_1+\nu}-e_{\nu,n_1+\iota}+\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{\iota j}+\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{\iota,n_2+j}+\gamma_{k_1}^{\nu k_1}f_{\iota m_2}\in B_{q_\nu}
\end{aligned}$$

and (for all $\nu\neq t\leq k_1$)

$$\begin{aligned}
b_{\iota t}^{(\nu)} &= \varphi_{q_\nu}(b_{\iota t}^{(k_1-2)})=b_{\iota t}^{(k_1-2)}(1-q_\nu)=(e_{\iota,n_1+t}-e_{t,n_1+\iota})(1+\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{n_1+\nu,j}\\
&\quad +\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{n_1+\nu,n_2+j}+\gamma_{k_1}^{\nu k_1}f_{n_1+\nu m_2})=e_{\iota,n_1+t}-e_{t,n_1+\iota}\in B_{q_\nu}. \qquad (3.8.49)
\end{aligned}$$

Recall that for all $k_1-2<s<t\leq k_1$, we have

$$b_{st}^{(k_1-2)}=e_{s,n_1+t}-e_{t,n_1+s}+\sum_{j>k_2}\alpha_{k_1j}^{sk_1}f_{tj}+\sum_{j>k_2}\beta_{k_1j}^{sk_1}f_{t,n_2+j}+\gamma_{k_1}^{sk_1}f_{tm}\in B_{q_{k_1-2}}.$$

If $s>k_1-2$, then the only option remaining to s is $s=k_1-1=\nu$. In that case $t=k_1$, so by applying φ_{q_ν} to $b_{st}^{(k_1-2)}$ for all $s>k_2-2$, we get that

$$\begin{aligned}
b_{\nu k_1}^{(\nu)} &= \varphi_{q_\nu}(b_{\nu k_1}^{(k_1-2)})=b_{\nu k_1}^{(k_1-2)}(1-q_\nu)\\
&= (e_{\nu,n_1+k_1}-e_{k_1,n_1+\nu}+\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{k_1j}+\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{k_1,n_2+j}+\gamma_{k_1}^{\nu k_1}f_{k_1m_2})(1\\
&\quad +\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{n_1+\nu,j}+\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{n_1+\nu,n_2+j}+\gamma_{k_1}^{\nu k_1}f_{n_1+\nu,m_2})\\
&= e_{\nu,n_1+k_1}-e_{k_1,n_1+\nu}-\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{k_1j}-\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{k_1,n_2+j}-\gamma_{k_1}^{\nu k_1}f_{k_1m_2}\\
&\quad +\sum_{j>k_2}\alpha_{k_1j}^{\nu k_1}f_{k_1j}+\sum_{j>k_2}\beta_{k_1j}^{\nu k_1}f_{k_1,n_2+j}+\gamma_{k_1}^{\nu k_1}f_{k_1m_2}
\end{aligned}$$

$$= \quad e_{\nu,n_1+k_1} - e_{k_1,n_1+\nu} \in B_{q_\nu}.$$

Recall that $\nu = k_1 - 1$, so we get that

$$b^{(k_1-1)}_{k_1-1,k_1} = e_{k_1-1,n_1+k_1} - e_{k_1,n_1+k_1-1} \in B_{q_{k_1-1}}. \tag{3.8.50}$$

Put $b^{(k_1)}_{\iota\nu} = \{b^{(\nu)}_{\iota\nu}, e_{n_1+k_1,\nu} - e_{n_1+\nu,k_1}, b^{(\nu)}_{\nu k_1}\} \in B_{q_\nu}$ and $b^{(k_1)}_{st} = b^{(\nu)}_{st} \in B_{q_\nu}$ for all of the remaining indices s and t. Since $US_1 = R^2 = 0$,

$$\begin{aligned} b^{(k_1)}_{\iota\nu} &= b^{(\nu)}_{\iota\nu}(e_{n_1+k_1,\nu} - e_{n_1+\nu,k_1})b^{(\nu)}_{\nu k_1} + b^{(\nu)}_{\nu k_1}(e_{n_1+k_1,\nu} - e_{n_1+\nu,k_1})b^{(\nu)}_{\iota\nu} \\ &= (e_{\iota,n_1+\nu} - e_{\nu,n_1+\iota} + \sum_{j>k_2} \alpha^{\nu k_1}_{k_1 j} f_{\iota j} + \sum_{j>k_2} \beta^{\nu k_1}_{k_1 j} f_{\iota,n_2+j} + \gamma^{\nu k_1}_{k_1} f_{\iota m_2})(e_{n_1+k_1,\nu} \\ &\quad - e_{n_1+\nu,k_1})b^{(\nu)}_{\nu k_1} + (e_{\nu,n_1+k_1} - e_{k_1,n_1+\nu})(e_{n_1+k_1,\nu} - e_{n_1+\nu,k_1})b^{(\nu)}_{\iota\nu} \\ &= -e_{\iota k_1}(e_{\nu,n_1+k_1} - e_{k_1,n_1+\nu}) + (e_{\nu\nu} + e_{k_1k_1})(e_{\iota,n_1+\nu} - e_{\nu,n_1+\iota} \\ &\quad + \sum_{j>k_2} \alpha^{\nu k_1}_{k_1 j} f_{\iota j} + \sum_{j>k_2} \beta^{\nu k_1}_{k_1 j} f_{\iota,n_2+j} + \gamma^{\nu k_1}_{k_1} f_{\iota m_2}) \\ &= e_{\iota,n_1+\nu} - e_{\nu,n_1+\iota} \in B_{q_\nu}. \end{aligned}$$

Combining this with (3.8.49), we get that $b^{(k_1)}_{\iota t} = e_{\iota,n_1+t} - e_{t,n_1+\iota} \in B_{q_\nu} = B_{q_{k_1-1}}$ for all $1 \le \iota \le k_1 - 2$ and all $\iota < t \le k_1$. By (3.8.50), $b^{(k_1)}_{k_1-1,k_1} = e_{k_1-1,n_1+k_1} - e_{k_1,n_1+k_1-1} \in B_{q_{k_1-1}}$, so

$$b^{(k_1)}_{st} = e_{s,n_1+t} - e_{t,n_1+s} \in B_{q_{k_1-1}} \quad \text{for all} \quad 1 \le s < t \le k_1. \tag{3.8.51}$$

Thus, $E_1^+ \subseteq B_{q_{k_1-1}}$. Finally, we need to show that $\mathscr{E}_2 \subseteq B_{q_{k_1-1}}$. Recall that $\mathscr{E}_2 \subseteq B_{q_{k_1-2}}$. Since $S_2U = R^2 = 0$, by applying φ_{q_ν} to $h^\varepsilon_{rq} = g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in E_2^\varepsilon$ for all $1 \le r \le q \le k_2$, we get that

$$\begin{aligned} \varphi_{q_\kappa}(h^\varepsilon_{rq}) &= (1+q_\kappa)h^\varepsilon_{rq}(1-q_\kappa) = (1+q_\kappa)h^\varepsilon_{rq} = (1 - \sum_{j>k_2} \alpha^{\nu k_1}_{k_1 j} f_{n_1+\nu,j} \\ &\quad - \sum_{j>k_2} \beta^{\nu k_1}_{k_1 j} f_{n_1+\nu,n_2+j} - \gamma^{\nu k_1}_{k_1} f_{n_1+\nu,m_2})(g_{r,n_2+q} - \varepsilon g_{q,n_2+r}) \\ &= g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in B_{q_\kappa}, \end{aligned}$$

so $E_2^\varepsilon \subseteq \varphi_{q_\nu}(B_{q_{k_1-2}}) = B_{q_\nu}$. Thus, $\mathscr{E}_2 \subseteq B_{q_\nu}$ (because $E_2 \subseteq E_2^+ \subseteq B_{q_\kappa}$). Put $\varphi = \varphi_{q_{k_1-1}} \circ \ldots \circ \varphi_{q_1} \circ \varphi_{q'} \circ \varphi_{\mathrm{q}}$. Then $\varphi : A \to A$ is a special inner automorphism with $E_1^+, \mathscr{E}_2 \subseteq \varphi(B_q)$. Therefore, if $\mathscr{E}_1 = E_1^+$, then $\mathscr{E}_1, \mathscr{E}_2 \subseteq \varphi(B_q) \cap S$, as required.

Case (3): Suppose that $\mathscr{E}_1 = E_1^- = \{e_{s,n_1+t} + e_{t,n_1+s} \mid 1 \le s \le t \le k_1 \le n_1\} \subseteq S_1$. As in the proof of Case (2), there is a special inner automorphism $\varphi : A \to A$ such that $\mathscr{E}_2 \subseteq \varphi(B)$ and

$$\{x_{st} = e_{s,n_1+t} + e_{t,n_1+s} \mid 1 \le s \le t < k_1 \le n_1\} \subseteq \varphi(B) \cap S_1.$$

Put $x_{ss} = x_{st}e_{n_1+t,t}x_{st}$ and $x_{tt} = x_{st}e_{n_1+t,t}x_{st}$. Then by Lemma 3.1.3,

$$x_{ss} = x_{st}e_{n_1+t,t}x_{st} = (e_{s,n_1+t} + e_{t,n_1+s})e_{n_1+t,t}(e_{s,n_1+t} + e_{t,n_1+s}) = e_{s,n_1+s} \in \varphi(B);$$

$$x_{tt} = x_{st}e_{n_1+s,s}x_{st} = (e_{s,n_1+t} + e_{t,n_1+s})e_{n_1+s,s}(e_{s,n_1+t} + e_{t,n_1+s}) = e_{t,n_1+t} \in \varphi(B),$$

so $x_{ii} = e_{i,n_1+i} \in \varphi(B)$ for all $1 \le i \le k_1$. Hence, $E_1^- \subseteq \varphi(B) \cap S_1$.Therefore, if $\mathscr{E}_1 = E_1^-$, then $\varphi(B')$ contains both $\mathscr{E}_1$ and $\mathscr{E}_2$, as required.

From Case (1), Case (2) and Case (3), there is a special inner automorphism $\varphi : \tilde{A} \to \tilde{A}$ such that $\mathscr{E}_1, \mathscr{E}_2 \subseteq \varphi(\tilde{B}) \cap \tilde{S}$. Since $R^2 = 0$, $\varphi(r) = r$ for all $r \in R$, so by changing the Levi subalgebra S into $S' = \varphi^{-1}(\tilde{S})$, we get that $\tilde{B}$ splits in $\tilde{A}$. Therefore, by Lemma 3.8.8, B $*$-splits in A.

$\square$

Now, we are ready to prove Proposition 3.8.4.

Proof of Proposition 3.8.4. We identify A/R with $S = S_1 \oplus S_2$. Since S_1 and S_2 are involution simple algebras, by Proposition 3.3.2, (for each $i = 1, 2$), S_i is either simple with involution or $S_i = Q_i \oplus Q_i^*$, where Q_i is simple ideal of S_i.

Suppose first that S_1 and S_2 are both simple. Then by Lemma 3.8.12, B $*$-splits in A.

Next, suppose that $S_i = Q_i \oplus Q_i^*$ for each $i = 1, 2$. Then A is a direct sum of two ideals $A = D \oplus D^*$, where $D = Q_1 \oplus Q_2 \oplus U$, so by Proposition 3.6.12, B $*$-splits in A.

Suppose now that S_2 is simple with involution and $S_1 = Q_1 \oplus Q_1^*$, where Q_1 is simple. We identify $\bar{A}$ with $S = S_1 \oplus S_2$. Recall that $R = U \oplus U^*$ where U is an irreducible S_1-S_2-bimodule with $S_2U = US_1 = 0$. Since $S_1 = Q_1 \oplus Q_1^*$, we have $Q_1U = 0$ or $Q_1^*U = 0$. We will consider the case when $Q_1^*U = 0$. By Lemma 3.4.9, $\mathfrak{u}^*(Q_1 \oplus Q_1^*) = \{s - s^* \mid s \in Q_1\} \cong \mathscr{M}_{m_1m_2}$, $\mathfrak{u}^*(Q_2) \cong \mathfrak{so}_{m_2}, \mathfrak{sp}_{2n_2}$ ($m_2 = 2n_2 + 1$ or $2n_2$) and $\mathfrak{u}^*(R) = \{(r, -r^*) \mid r \in U \cong \mathscr{M}_{m_1m_2}\}$. Since $\bar{A}$ is semisimple, by Lemma 3.3.12, $\bar{B} = X_1 \oplus X_2$, where $X_i = \bar{B} \cap \mathfrak{su}^*(S_i)$ is Jordan-Lie of $\mathfrak{su}^*(S_i)$ for each $i = 1, 2$. As in the proof of Lemma 3.8.10, we fix standard bases $\{e_{ij} \mid 1 \le i, j \le m_1\}$, $\{e'_{ij} \mid 1 \le i, j \le m_1\}$, $\{g_{ij} \mid 1 \le i, j \le m_2\}$ and $\{f_{ij} \mid 1 \le i \le m_1, \quad 1 \le j \le m_2\}$ of Q_1, Q_1^*, S_2 and U, respectively, consisting of matrix units, such that the action of S_1 and of S_2 on U corresponds to matrix multiplication, X_1 is the space spanned by

$$\mathscr{E}_1 = \{e_{st} - e'_{ts} \mid 1 \le s \le k_1 < l_1 \le t \le m_1\} \subseteq \mathfrak{su}^*(Q_1 + Q_1^*) = \mathfrak{su}^*(S_1)$$

and X_2 is the space spanned by $\mathscr{E}_2$, where $\mathscr{E}_2$ is one of the following.

$$E_2 = \{h_q = g_{1,n_2+q} - g_{q,n_2+1} \mid 1 < q \le k_2\} \subseteq \mathfrak{su}^*(S_2) = \mathfrak{so}_{m_2};$$

$$E_2^\varepsilon = \{h_{rq}^\varepsilon = g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \mid 1 \le r \le q \le k_2\} \subseteq \mathfrak{su}^*(S_2) = \mathfrak{so}_{m_2}, \mathfrak{sp}_{2n_2}.$$

Put $A_2 = S_2 \oplus R$. Denote $B_2 = B \cap \mathfrak{su}^*(A_2)$. Then B_2 is a Jordan-Lie inner ideal of $\mathfrak{su}^*(A_2)$ (because $(B_2)^2 = 0$). Note that $\bar{B}_2 = X_2$ (because B_2 contains the preimage of X_2 in B). We may assume that B_2 is X_2-minimal (if not, then it contains X_2-minimal Jordan-Lie inner ideal of $\mathfrak{su}^*(A_2)$). Thus, B_2 satisfies the conditions of Proposition 3.8.2, so there is a Levi subalgebra S_2' of A_2 such that $B_2 = B_{2_{S'}} \oplus B_{2_R}$, where $B_{2_{S'}} = B_2 \cap S_2'$. Note that $\bar{B}_{2_{S'}} = \bar{B}_2 = X_2$. By Theorem 3.8.5, there is $q \in \mathfrak{u}^*(R)$ and a special inner automorphism φ_q of A such that $S_2 = \varphi_q(S_2')$. Since $B_{2_{S'}} \subseteq S_2'$, $\varphi_q(B_{2_{S'}}) \subseteq \varphi_q(S_2') = S_2$. Moreover, by Lemma 3.8.7, $\overline{\varphi_q(B_{2_{S'}})} = \bar{B}_{2_{S'}} = \bar{B}_2 = X_2$. Recall that $X_2 \subseteq S_2$, so both $\varphi_q(B_{2_{S'}})$ and X_2 have the same image in $\bar{A}_2 = A_2/R$. Since both of them are subspaces of S_2, they must be equal. Thus, $X_2 = \varphi_q(B_{2_{S'}}) \subseteq \varphi_q(B_2) \cap S$, so $\mathscr{E}_2 \subseteq \varphi_q(B_2) \cap S_2 \subseteq \varphi_q(B) \cap S$. We will deal with the Jordan-Lie inner ideal $B_q = \varphi_q(B)$ of $K^{(1)}$. Our aim is modify $B_q = \varphi_q(B)$ in such a way that it contains both $\mathscr{E}_1$ and $\mathscr{E}_2$. This will imply that B $*$-splits in A.

Let $\tilde{B}$ be the image of B in $\tilde{A} = A/U^* \cong A \oplus U$. Since $R = U \oplus U^*$, by Lemma 3.8.8, to show that B $*$-splits in A, it is enough to show that $\tilde{B}$ splits in $\tilde{A}$, that is, there is a special inner automorphism of $\tilde{A}$ such that $\tilde{B}_q = \varphi_q(\tilde{B})$ contains both $\tilde{\mathscr{E}}_1$ and $\tilde{\mathscr{E}}_2$. To simplify notations, we will re-denote $\tilde{A}$, $\tilde{S}$, $\tilde{R}$, $\tilde{B}$, $\tilde{\mathscr{E}}_1$ and $\tilde{\mathscr{E}}_2$ by A, S, R, B, $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively. Thus, $R = U$ and $A/U \cong S$. We need to show that B splits in A. Fix any subset $\{b_{st} \mid 1 \le s \le k_1 < l_1 \le t \le m_1\} \subseteq B_q$ such that $\bar{b}_{st} = e_{st} - e'_{ts}$. Then $b_{st} = e_{st} - e'_{ts} + r_{st}$, where $r_{st} \in U$. Suppose that $\mathscr{E}_2 = E_2^\varepsilon$. By Lemma 3.1.3, $\{b_{st}, x, h_{1q}^\varepsilon\} \in B_q$ for all $x \in U$. Since $Q_1^* U = S_2 U = R^2 = 0$, we have $h_{rq}^\varepsilon U b_{st} \in S_2 U = 0$, so

$$\{b_{st}, f_{tq}, h_{1q}^\varepsilon\} = b_{st} f_{tq} h_{1q}^\varepsilon + 0 = (e_{st} - e'_{ts} + r_{st}) f_{tq}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = -\varepsilon f_{s,n_2+1} \in B_q$$

and

$$\{b_{st}, f_{t1}, h_{1q}^\varepsilon\} = b_{st} f_{t1} h_{1q}^\varepsilon + 0 = (e_{st} - e'_{ts} + r_{st}) f_{t1}(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) = f_{s,n_2+q} \in B_q.$$

Hence, $f_{s,n_2+j} \in B_q$ for all $1 \le j \le k_2$. Note that this is also true when $\mathscr{E}_2 = E_2$ (because $h_q = g_{1,n_2+q} - g_{q,n_2+1} = h_{1q}^+ \in E_2^+ \cap E_2$). Therefore, for any choice of $\mathscr{E}_2$, we have

$$R_0 = span\{f_{s,n_2+j} \mid 1 \le s \le k_1, \quad 1 \le j \le k_2\} \subseteq B_q \cap U. \tag{3.8.52}$$

Put $b_{st}^{(1)} = b_{st}(e_{ts} - e'_{st})b_{st} \in B_q$ (by Lemma 3.1.3). Since $r_{st} \in U = \mathscr{M}_{m_1 m_2}$, there are coefficients such that

$$r_{st} = \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \alpha_{ij}^{st} f_{ij} + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \beta_{ij}^{st} f_{i,n_2+j} + \sum_{i=1}^{m_1} \gamma_i^{st} f_{im_2} \in U.$$

Since $US_1 = Q_1^* U = R^2 = 0$, we get that

$$\begin{aligned}
b_{st}^{(1)} &= b_{st}(e_{ts} - e'_{st})b_{st} = (e_{st} - e'_{ts} + r_{st})(e_{ts} - e'_{st})b_{st} \\
&= (e_{ss} + e'_{tt})(e_{st} - e'_{ts} + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \alpha_{ij}^{st} f_{ij} + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \beta_{ij}^{st} f_{i,n_2+j} + \sum_{i=1}^{m_1} \gamma_i^{st} f_{im_2}) \\
&= e_{st} + \sum_{j=1}^{n_2} \alpha_{sj}^{st} f_{sj} + \sum_{j=1}^{n_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2} - e'_{ts} \\
&= e_{st} - e'_{ts} + \sum_{j=1}^{n_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2} + \sum_{j=1}^{k_2} \beta_{sj}^{st} f_{s,n_2+j} \\
&= b_{st}^{(2)} + \sum_{j=1}^{k_2} \beta_{sj}^{st} f_{s,n_2+j} \in B_q.
\end{aligned}$$

Since $\sum_{j=1}^{k_2} \beta_{sj}^{st} f_{s,n_2+j} \in R_0 \subseteq B_q$,

$$b_{st}^{(2)} = e_{st} - e'_{ts} + \sum_{j=1}^{n_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2} \in B_q.$$

Suppose that $\mathscr{E}_2 = E_2^{\varepsilon}$. Since $B_q^2 = 0$ and $(Q_1 + Q_1^*)S_2 = 0$, for all $h_{1q}^{\varepsilon} \in E_2^{\varepsilon}$ $(1 < q \leq k_2)$, we have

$$\begin{aligned}
0 &= b_{st}^{(2)} h_{1q}^{\varepsilon} \\
&= (e_{st} - e'_{ts} + \sum_{j=1}^{n_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2})(g_{1,n_2+q} - \varepsilon g_{q,n_2+1}) \\
&= \alpha_{s1}^{st} f_{s,n_2+q} - \varepsilon \alpha_{sq}^{st} f_{s,n_2+1},
\end{aligned}$$

so $\alpha_{s1}^{st} = \alpha_{sq}^{st} = 0$. Hence, $\alpha_{si}^{st} = 0$ for all $1 \leq i \leq k_2$. Note that this is also true when $\mathscr{E}_2 = E_2$ (because $h_q = g_{1,n_2+q} - g_{q,n_2+1} = h_{1q}^{+} \in E_2^{+} \cap E_2$). Therefore, for any choice of $\mathscr{E}_2$,

$$b_{st}^{(2)} = e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2} \in B_q.$$

Put $b_{1t}^{(3)} = b_{1t}^{(2)} \in B_q$ and for $s > 1$ set $b_{st}^{(3)} = \{b_{st}^{(2)}, e_{t1} - e'_{1t}, b_{1t}^{(2)}\} \in B_q$ (by Lemma 3.1.3). Since $UQ_1 = Q_1^* U = R^2 = 0$, for all $s > 1$, we have

$$\begin{aligned}
b_{st}^{(3)} &= b_{st}^{(2)}(e_{t1} - e'_{1t})b_{1t}^{(2)} + b_{1t}^{(2)}(e_{t1} - e'_{1t})b_{st}^{(2)} \\
&= (e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2})(e_{t1} - e'_{1t})b_{1t}^{(2)} \\
&\quad + (e_{1t} - e'_{t1} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{1j} + \sum_{j>k_2} \beta_{1j}^{1t} f_{1,n_2+j} + \gamma_1^{1t} f_{1m_2})(e_{t1} - e'_{1t})b_{st}^{(2)} \\
&= e_{s1}(e_{1t} - e'_{t1} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{1j} + \sum_{j>k_2} \beta_{1j}^{1t} f_{1,n_2+j} + \gamma_1^{1t} f_{1m_2}) \\
&\quad + (e_{11} + e'_{tt})(e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{sj}^{st} f_{sj} + \sum_{j>k_2} \beta_{sj}^{st} f_{s,n_2+j} + \gamma_s^{st} f_{sm_2}) \\
&= e_{st} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} + \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} + \gamma_{1j}^{1t} f_{sm_2} - e'_{ts} \\
&= e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} + \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} + \gamma_1^{1t} f_{sm_2} \in B_q.
\end{aligned}$$

Hence, for all s and t, we have

$$b_{st}^{(3)} = e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} + \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} + \gamma_1^{1t} f_{sm_2} \in B_q.$$

Consider the inner automorphism $\varphi_{q'} : A \to A$, where

$$q' = \sum_{i=l_1}^{n_1} \sum_{j>k_2} \alpha_{1j}^{1i} f_{ij} + \sum_{i=l_1}^{n_1} \sum_{j>k_2} \beta_{1j}^{1i} f_{i,n_2+j} + \sum_{i=l_1}^{n_1} \gamma_1^{1i} f_{im_2} \in \mathfrak{u}^*(R) = U.$$

Since $US_1 = Q_1^* U = R^2 = 0$, we have $q' b_{st}^{(3)} \in U(S_1 + U) = 0$, so

$$\begin{aligned}
\varphi_{q'}(b_{st}^{(3)}) &= (1 + q')b_{st}^{(3)}(1 - q') = b_{st}^{(3)}(1 - q') \\
&= (e_{st} - e'_{ts} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} + \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} + \gamma_{1j}^{1t} f_{sm_2})(1 \\
&\quad - \sum_{i=l_1}^{n_1} \sum_{j>k_2} \alpha_{1j}^{1i} f_{ij} - \sum_{i=l_1}^{n_1} \sum_{j>k_2} \beta_{1j}^{1i} f_{i,n_2+j} - \sum_{i=l_1}^{n_1} \gamma_{1j}^{1i} f_{im_2}) \\
&= e_{st} - \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} - \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} - \gamma_1^{1t} f_{sm_2} - e'_{ts} + \sum_{j>k_2} \alpha_{1j}^{1t} f_{sj} \\
&\quad + \sum_{j>k_2} \beta_{1j}^{1t} f_{s,n_2+j} + \gamma_1^{1t} f_{sm_2} = e_{st} - e'_{ts} \in \varphi_{q'}(B_q).
\end{aligned}$$

Therefore,

$$\mathscr{E}_1 = \{e_{st} - e'_{ts} \mid 1 \le s \le k_1 < l_1 \le t \le m_1\} \subseteq \varphi_{q'}(B_q) \cap S_1.$$

It remains to show that $\mathscr{E}_2 \subseteq \varphi_{q'}(B_q)$. Since $S_2 U = 0$, by applying $\varphi_{q'}$ to $h^{\varepsilon}_{rq} = E^{\varepsilon}_2$ for all $1 \le r \le q \le k_2$, we get that

$$\begin{aligned}
\varphi_{q'}(h^{\varepsilon}_{rq}) &= (1+q')h^{\varepsilon}_{rq}(1-q') = (1+q')h^{\varepsilon}_{rq} = (1+\sum_{i=l_1}^{n_1}\sum_{j>k_2}\alpha^{1i}_{1j}f_{ij} \\
&\quad + \sum_{i=l_1}^{n_1}\sum_{j>k_2}\beta^{1i}_{1j}f_{i,n_2+j} + \sum_{i=l_1}^{n_1}\gamma^{1i}_{1}f_{im_2})(g_{r,n_2+q} - \varepsilon g_{q,n_2+r}) \\
&= g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \in \varphi_{q'}(B_q),
\end{aligned}$$

so

$$E^{\varepsilon}_2 = \{g_{r,n_2+q} - \varepsilon g_{q,n_2+r} \mid 1 \le r \le q \le k_2\} \subseteq \varphi_{q'}(B_q) \cap S_2.$$

Since $E_2 \subseteq E^{+}_2 \subseteq \varphi_{q'}(B_q)$, we get that $\mathscr{E}_2 \subseteq \varphi_{q'}(B_q)$. Thus, $\varphi_{q'}(B_q)$ contains both $\mathscr{E}_1$ and $\mathscr{E}_2$. Hence, $\tilde{B}$ splits in $\tilde{A}$. Therefore, by Lemma 3.8.6, B $*$-splits in A.

$\square$

Proof of the main results

Let Q be an algebra and let M be a Q-bimodule. We denote by $\ell(M)$, the *length of the S-bimodule M*. If Q is an algebra with involution and M is a $*$-invariant Q-bimodule, then we denote by $\ell^*(M)$, the *$*$-length of the $*$-invariant Q-bimodule M*.

Now, we are ready to prove Theorem 3.8.1.

Proof of Theorem 3.8.1. Recall that A is admissible with $R^2 = 0$, $p \ne 2,3$ and B is a $\bar{B}$-minimal Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. Let $\{S_i \mid i \in I\}$ be the set of the simple components of S. We identify $\bar{A}$ with S. By Lemma 3.4.4, as an S-bimodule R is a direct sum of copies of a $*$-irreducible S-S-bimodules each of them is either irreducible S_i-S_{i^*}-bimodule or isomorphic to $U \oplus U^*$, where U is either a natural left S_i-module with $US_i = 0$, or an irreducible S_i-S_j-bimodule U_{ij} with $S_j U_{ij} = U_{ij} S_i = 0$. Note that the S-bimodule R has no components isomorphic to the trivial 1-dimensional S-bimodule U_{00} as A is admissible with $R^2 = 0$.

The proof is by induction on the $*$-length $\ell^*(R)$ of the $*$-invariant S-bimodule R. Suppose that $\ell^*(R) = 1$, i.e. R is $*$-irreducible. Let A_2 be the maximal ideal of S such that

$A_2R = RA_2 = 0$. Let S' be the complement of A_2 in S and let $A_1 = S' \oplus R$. Then A_1 and A_2 are both admissible ideals of A with $A_2A_1 = A_1A_2 = 0$ and $A = A_1 \oplus A_2$. Put $K_i = \mathfrak{u}^*(A_i)$ for all $i = 1,2$. Then $K^{(1)} = K_1^{(1)} \oplus K_2^{(1)}$. Since B is bar-minimal Jordan-Lie inner ideals of $K^{(1)}$ and $K^{(1)}$ is perfect, by Lemma 3.6.5, $B = B_1 \oplus B_2$, where B_i is a $\bar{B}_i$-minimal Jordan-Lie inner ideal of $K_i^{(1)}$ (for each $i = 1,2$). Since A_2 is semisimple, B_2 $*$-splits in A_2. It remains to show that B_1 $*$-splits in A_1. By Proposition 3.4.3, A_1 has one of the prescribed decompositions. Therefore, B_1 satisfies the conditions of one of the Propositions 3.8.2, 3.8.3 and 3.8.4, so B_1 $*$-splits in A_1. Thus, B $*$-splits in A.

Suppose now that $\ell^*(R) > 1$. Consider any maximal $*$-invariant S-submodule T of R, so $\ell^*(T) < \ell^*(R)$. Then T is an ideal of A. Let $\tilde{A} = A/T$. Denote by $\tilde{B}$ and $\tilde{R}$ the images of B and R in $\tilde{A}$. Since $\ell^*(\tilde{R}) = 1$, by the base of induction, $\tilde{B}$ $*$-splits, so there is a $*$-invariant Levi subalgebra S' of $\tilde{A}$ such that $\tilde{B} = \tilde{B}_{S'} \oplus \tilde{B}_R$, where $\tilde{B}_{S'} = \tilde{B} \cap S'$ and $\tilde{B}_R = \tilde{B} \cap \tilde{R}$. Let P be the full preimage of $\tilde{B}_{S'}$ in B. Then $\tilde{P} = \tilde{B}_{S'} \subseteq S'$, so $\bar{P} = \bar{B}$. Let G be the full preimage of S' in A. Then G is a large subalgebra of A containing P, so $P \subseteq G \cap B$. Put $P_1 = [P,[P,\mathfrak{su}^*(S')]] \subseteq \mathfrak{su}^*(G)$. Then $P_1 \subseteq [B,[B,\mathfrak{su}^*(A)]] \subseteq B$, so $P_1 \subseteq B \cap \mathfrak{su}^*(G)$. Note that $B_G = B \cap \mathfrak{su}^*(G)$ is a Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$ containing P_1. Since

$$\bar{P}_1 = [\bar{P},[\bar{P},\mathfrak{su}^*(\bar{S}')]] = [\bar{B},[\bar{B},\mathfrak{su}^*(\bar{A})]] = \bar{B},$$

we get that $\bar{B} = \bar{P}_1 \subseteq \bar{B}_G \subseteq \bar{B}$, so $\bar{B}_G = \bar{B}$. Since A is admissible and $R^2 = 0$, by Proposition 3.5.13(ii), G is admissible (i.e. $G = \mathscr{P}_a(G)$). Let $B'_G \subseteq B_G$ be a $\bar{B}_G$-minimal Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$. Since G is admissible, $T^2 \subseteq R^2 = 0$ and $\ell^*(T) < \ell^*(R)$, by the inductive hypothesis B'_G $*$-splits in $G = \mathscr{P}_a(G)$. Since $B'_G \subseteq B_G \subseteq B$ and $\bar{B}'_G = \bar{B}_G = \bar{B}$, by Lemma 3.6.13, B $*$-splits in A.

□

The following result follows from Theorem 3.8.1 and Proposition 3.6.15.

Corollary 3.8.13. *Let B be a Jordan-Lie inner ideal of $K = \mathfrak{su}^*(A)$. Suppose that $p \neq 2,3$, A is admissible and $R^2 = 0$. Then B $*$-splits in A.*

Now, we are ready to prove Theorem 1.2.8.

Proof of Theorem 1.2.8. Recall that A is admissible, $p \neq 2,3$ and B is a $\bar{B}$-minimal Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. We need to show that B $*$-splits in A. Since R is nilpotent, there is an integer m such that $R^{m-1} \neq 0$ and $R^m = 0$. The proof is by induction on m. If $m = 2$, then by Theorem 3.8.1, B $*$-splits, as required.

Suppose now that $m > 2$. Put $T = R^2 \neq 0$ and consider $\tilde{A} = A/T$. Let $\tilde{B}$ and $\tilde{R}$ be the images of B and R in $\tilde{A}$. Then $\tilde{R} = \mathrm{rad}\tilde{A}$, $\tilde{R}^2 = 0$ and $\tilde{A}$ satisfies the conditions of the Corollary 3.8.13. Hence, there is a $*$-invariant Levi subalgebra S_1 of $\tilde{A}$ such that $\tilde{B} = \tilde{B}_{S_1} \oplus \tilde{B}_R$, where $\tilde{B}_{S_1} = \tilde{B} \cap S_1$ and $\tilde{B}_R = \tilde{B} \cap \tilde{R}$. Let P be the full preimage of $\tilde{B}_{S_1}$ in B. Then $\tilde{P} = \tilde{B}_{S_1} \subseteq S_1$, so $\bar{P} = \bar{B}$. Let G be the full preimage of S_1 in A. Then G is a large subalgebra of A with $P \subseteq G \cap B$. Put $P_1 = [P,[P,\mathfrak{su}^*(S_1)]] \subseteq \mathfrak{su}^*(G)$. Then $P_1 \subseteq [B,[B,\mathfrak{su}^*(A)]] \subseteq B$, so $P_1 \subseteq B \cap \mathfrak{su}^*(G)$. Put $B_G = B \cap \mathfrak{su}^*(G)$. Then B_G is a Jordan-Lie inner ideal of $\mathfrak{su}^*(G)$ containing P_1. Note that

$$\bar{P}_1 = [\bar{P},[\bar{P},\mathfrak{su}^*(\bar{S}_1)]] = [\bar{B},[\bar{B},\mathfrak{su}^*(\bar{A})]] = \bar{B},$$

so $\bar{B}_G = \bar{B}$. Fix any $*$-invariant Levi subalgebra S_2 of G. Since S_2 is admissible (because $S_2 \cong S$), by Lemma 3.5.2, $\ll S_2 \gg_G$ is an admissible ideal of G, so by Lemma 3.5.6, $\ll S_2 \gg_G = \mathscr{P}_a(G)$. Put $P_2 = [B_G,[B_G,\mathfrak{su}^*(S_2)] \subseteq B_G$. Then $P_2 \subseteq \ll S_2 \gg_G = \mathscr{P}_a(G)$, so $P_2 \subseteq \mathscr{P}_a(G) \cap B_G$ with

$$\bar{P}_2 = [\bar{B}_G,[\bar{B}_G,\mathfrak{su}^*(\bar{S}_2)] = [\bar{B}_G,[\bar{B}_G,\mathfrak{su}^*(\bar{G})]] = \bar{B}_G = \bar{B}.$$

Put $P_G = [P_2,[P_2,\mathfrak{su}^*(S_2)]] \subseteq B_G$. Then $P_G \subseteq \mathfrak{su}^*(\mathscr{P}_a(G))$, so $P_G \subseteq B_G \cap \mathfrak{su}^*(\mathscr{P}_a(G))$. Note that $B'_G = B_G \cap \mathfrak{su}^*(\mathscr{P}_a(G))$ is a Jordan-Lie inner ideal of $\mathfrak{su}^*(\mathscr{P}_a(G))$ containing P_G. Since

$$\bar{P}_G = [\bar{P}_2,[\bar{P}_2,\mathfrak{su}^*(\bar{S}_2)]] = [\bar{B},[\bar{B},\mathfrak{su}^*(\bar{A})]] = \bar{B},$$

$\bar{B} = \bar{P}_G \subseteq \bar{B}'_G$, but $\bar{B}'_G \subseteq \bar{B}_G = \bar{B}$, so $\bar{B}'_G = \bar{B}$. Let $B''_G \subseteq B'_G$ be a $\bar{B}'_G$-minimal Jordan-Lie inner ideal of $\mathfrak{su}^*(\mathscr{P}_a(G))$. Since $\mathscr{P}_a(G)$ is admissible and $\mathrm{rad}(\mathscr{P}_a(G))^{m-1} \subseteq T^{m-1} = R^{2(m-1)} = 0$, by the inductive hypothesis, B''_G $*$-splits in $\mathscr{P}_a(G)$. As $B''_G \subseteq B'_G \subseteq B_G \subseteq B$ and $\bar{B}''_G = \bar{B}'_G = \bar{B}$, by Lemma 3.6.13, B $*$-splits in A, as required.

□

Now, we are ready to proof Corollary 1.2.9.

Prof of Corollary 1.2.9. (i) Let B be a Jordan-Lie inner ideal of $K^{(1)} = \mathfrak{su}^*(A)$. Let $C \subseteq B$ be a $\bar{B}$-minimal Jordan-Lie inner ideal of $K^{(1)}$. Then by Theorem 1.2.8, C $*$-splits in A, so by Proposition 3.6.15 B $*$-splits in A.

(ii) This follows from (i) and Lemma 3.6.8, B $*$-splits in $K^{(1)}$.

□

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