Positive μ modification as an anti-windup mechanism

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Abstract

This paper proposes a simple anti-windup mechanism for a model reference adaptive control scheme subject to saturation constraints. The anti-windup compensator has, in essence, the same structure as positive μ modification for the same class of systems. It is shown how this structure can, under certain circumstances, display characteristics similar to anti-windup schemes proposed for linear control systems. In particular, it is shown that if the (unknown) ideal control signal eventually lies within the control constraints, then the response of the adaptive control system will converge to that of the reference system - provided certain conditions are satisfied. The paper illustrates the challenge of designing anti-windup compensators for model-reference adaptive control systems.

Key words: adaptive control, anti-windup, saturation

1 Introduction

Model reference adaptive controllers (MRAC) are well known in the adaptive control community [10,2] and are an appealing way to design adaptive control systems. The central idea is to use the error between the state-vector of the model system and that of the real system to govern adaptation of the controller gains. The field has become reinvigorated recently thanks to reports of the efficacy of MRAC-type controllers on real systems - see for example [31,4,1,21,24] and references therein. Unfortunately, as with most control systems, MRAC systems are vulnerable to the effects on input saturation and, in fact, adaptive systems appear to be especially sensitive to saturation because the nonlinearity not only causes traditional wind-up effects, described in the books [9,7,32,25], but also corrupts the manner in which the controller parameters are updated [2] - delivering a "double whammy" to the adaptive controller.

Researchers have long been aware of the sensitivity of adaptive controllers to input saturation and many papers have appeared on the topic see [33,22,15,14,29,27]

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and references therein for more detail. Most of these papers propose approaches to reducing saturation problems in adaptive systems which do not follow a traditional anti-windup structure which has long been used in linear control systems. The advantage to the traditional anti-windup approach is the well-defined, two-part structure: the main (baseline) controller is solely responsible for stabilisation and performance when saturation does not occur; in the event of saturation an additional element (the anti-windup compensator) then becomes active to deliver improved performance and enhanced stability properties. The anti-windup compensator remains inactive unless saturation occurs and, once saturation is over, allows a "graceful recovery" of the un-saturated behaviour [18,26,30]. The design of the anti-windup compensator is *completely separate* from that of the baseline controller.

The anti-windup approach has not been pursued as much in the world of adaptive control because, as this paper illustrates, it can be difficult to demonstrate *bonafide* anti-windup-like properties for adaptive control systems. There are exceptions of course and we refer the reader to the work by [13] which reports an adaptive control scheme for indirect adaptive controllers, the so-called pseudo-hedging technique described in [12] and elsewhere, and the technique given in [17]. In addition, a recent paper reports the development of a so-called model-reference anti-windup (MRAW) scheme for MRAC controllers but the architecture of this scheme is complicated and the properties this scheme bestows upon the closed-loop is difficult to discern. In fact, for linear systems the development of a MRAW scheme requires one to have a reasonably good ([28]) model of the plant; in MRAC this model is assumed to be unknown so the generalisation of this to MRAC schemes is not straightforward.

This paper proposes an anti-windup scheme for MRAC schemes and, unlike many adaptive schemes addressing input saturation, attempts to keep the architecture of the scheme as simple as possible. The only extra dynamics introduced to the system is an additional controller gain to be adapted when input saturation occurs. In fact, one can see that the architecture of the anti-windup scheme is, in essence, the same as the "positive μ modification" proposed recently [19] and generalising earlier work [15]. However, using ideas from anti-windup compensation (in particular those introduced in [26] - see also [32,5]) it can be proved that the scheme, under certain circumstances, allows statements about the recovery of un-saturated behaviour to be made. Note in [19] it was proved that the error between the ideal model state and the plant state was *bounded* provided the reference and initial state satisfied certain bounds. Here, it will be proved that, under certain assumptions, the error between the ideal model state and the plant state will *converge* provided that, roughly speaking, the ideal control signal is within the control bounds in steady state. This property is much more aligned with the spirit of traditional anti-windup compensation.

1.1 Notation

Notation is standard throughout. A positive (negative) definite matrix M is denoted $M > 0 \ (< 0)$. The scalar saturation function $\mathrm{sat}_{\bar{u}}(.) : \mathbb{R} \mapsto [-\bar{u}, \bar{u}]$ is defined as

$$\mathrm{sat}_{\bar{u}}(u) = \mathrm{sign}(u) \min\{|u|, \bar{u}\} \quad \bar{u} > 0 \tag{1}$$

The scalar deadzone function $\mathrm{Dz}_{\bar{u}}(.) : \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$\mathrm{Dz}_{\bar{u}}(u) = \mathrm{sign}(u) \max\{0, |u| - \bar{u}\} \quad \bar{u} > 0 \tag{2}$$

The saturation and deadzone functions satisfy the identity

$$\mathrm{sat}_{\bar{u}}(u) + \mathrm{Dz}_{\bar{u}}(u) = u \tag{3}$$

Note that both functions are globally Lipschitz with a Lipschitz constant of unity, meaning that

$$\|\phi(x_1 + x_2) - \phi(x_1)\| \leq \|x_2\| \quad \forall x_1, x_2 \in \mathbb{R} \tag{4}$$

where $\phi(.)$ is either the saturation or the deadzone function.

A signal $x(t)$ is said to belong the Lebesgue space $\mathcal{L}_2$ if its $\mathcal{L}_2$ norm is finite, i.e.

$$\|x\|_2 := \left(\int_0^\infty \|x(t)\|^2 dt \right)^{\frac{1}{2}} < \infty \tag{5}$$

Similarly a signal is said to belong to the Lebesgue space $\mathcal{L}_\infty$ if its $\mathcal{L}_\infty$ norm is finite, i.e.

$$\|x\|_\infty := \sup_{t \geq 0} \max_i |x_i(t)| < \infty$$

2 Preliminary results

Several results will be used in the proof of the main results. Most of these are standard results in nonlinear control [16], but there are two particular results which are introduced below.

Lemma 1 *Consider the deadzone nonlinearity (2) and let $k(t)$ be some continuous scalar function such that $k(t) \in [0, \bar{k}]$ for all $t \geq 0$ and some $\bar{k} > 0$. Then we have*

$$\|\mathrm{Dz}_{\bar{u}}[k(t)u]\| \leq \|\mathrm{Dz}_{\bar{u}}[\bar{k}u]\| \quad \forall u \in \mathbb{R} \tag{6}$$

Proof: The proof proceeds on a case by case basis:

(i) $|k(t)u| < \bar{u}$. In this case

$$\|\mathrm{Dz}_{\bar{u}}[k(t)u]\| = 0$$

so if either $|\bar{k}u| < \bar{u}$ or $|\bar{k}u| \geq \bar{u}$, the inequality in the theorem follows.

(ii) $|k(t)u| \geq \bar{u}$. In this case

$$\|\mathrm{Dz}_{\bar{u}}[k(t)u]\| = \|k(t)u\| - \bar{u} \tag{7}$$
$$\leq \|\bar{k}u\| - \bar{u} \tag{8}$$
$$= \|\mathrm{Dz}_{\bar{u}}[\bar{k}u]\| \tag{9}$$

which is exactly the inequality in the lemma. □

The results in this paper will be developed for plants of the form below

$$\dot{x} = Ax + B\lambda \mathrm{sat}(u) \tag{10}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ and λ is a positive scalar. The following straightforward lemma will be used to derive the main results.

Lemma 2 *Consider the plant dynamics (10). If A is Hurwitz, then the state $x(t)$ is bounded for all $u(t) \in \mathbb{R}$.*

Proof: The proof is similar to the first part of the proof of Theorem 1 in [19]. Let P_A solve the Lyapunov equation

$$A'P_A + P_A A = -Q_A < 0$$

Let $V(x) = x'P_A x$ and differentiate it along the trajectories of (10). This yields

$$\dot{V}(x) = -x'Q_A x + 2x_a' P_A \lambda B \mathrm{sat}(u) \tag{11}$$
$$\leq -\lambda_{min}(Q_A)\|x\|^2 + 2\lambda\|x\|\|P_A B\|\bar{u} \tag{12}$$
$$\leq -\|x\|(\lambda_{min}(Q_A)\|x\| - 2\lambda\|P_A B\|\bar{u}) \tag{13}$$

It can then be proved that $x(t)$ will converge to a ball surrounding the origin by application of Lemma 5.1 of [16]: the state is ultimately bounded. □

In the next section, the concept of well-posedness is important. A feedback system is said to be well-posed if unique solutions exist to the feedback equations. Well-posedness issues often arise when the output equation of one feedback element has the implicit form $y = f(y, x)$. If there exists a unique y solving this implicit equation, well-posedness can often be inferred.

Remark 1: Even in the case that A, B and λ are perfectly known, the analysis of the system (10) is not trivial. There is much discussion in the literature on the conditions under which, for a given $u = Fx$, the system (10) is stable - see [20] for a good overview. □

3 Main results

Consider again the plant in equation (10). The following assumption is made throughout the paper. This assumption is restrictive but compatible with that made in [19].

Assumption 3 *A is Hurwitz but unknown; B is perfectly known; and λ is unknown but positive.*

The reference model is given by the dynamics

$$\dot{x}_r = A_m x_r + B_m r \tag{14}$$

In order for the reference model to be compatible with the plant the following assumption is made

Assumption 4 *There exist a matrix $K_x^* \in \mathbb{R}^{1\times m}$ and a scalar K_r^* such that*

$$A_m = A + B\lambda K_x^* \quad B_m = B\lambda K_R^* \tag{15}$$

Note that only the existence of K_x^* and K_r^* is required; they are typically not known *a priori*. Assumption 4 is restrictive and is well known to limit the applicability of MRAC to plants with a very particular structure. Again, it is compatible with that made in [19]. In the absence of saturation, that is if $\mathrm{sat}(u(t)) \equiv u(t)$ in equation (10), it is well known (see for example [10,19]) that the adaptive control law

$$\begin{cases} u &= \hat{K}_x' x + \hat{K}_r' r \\ \dot{\hat{K}}_x &= -\Gamma_x x(e'PB) \quad \Gamma_x > 0 \\ \dot{\hat{K}}_r &= -\Gamma_r r(e'PB) \quad \Gamma_r > 0 \end{cases} \tag{16}$$

ensures convergence of the error $e(t) := x(t) - x_r(t)$ and boundedness of the gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$ where $P > 0$ is the solution to the Lyapunov equation

$$A_m' P + P A_m = -Q < 0 \tag{17}$$

Unfortunately in the presence of input saturation, when the control law (16) is applied the plant (10), error convergence is not guaranteed. Instead, the following control law, which has a natural anti-windup structure, is proposed:

$$\begin{cases} u &= \hat{K}_x' x + \hat{K}_r' r - \mu(x, x_m)\mathrm{Dz}(u) & \\ \dot{\hat{K}}_x &= -\Gamma_x x(e'PB) & \Gamma_x > 0 \\ \dot{\hat{K}}_r &= -\Gamma_r r(e'PB) & \Gamma_r > 0 \\ \dot{\hat{K}}_u &= -\Gamma_u (1 + \mu(x, x_m))\mathrm{Dz}(u)(e'PB) & \Gamma_u > 0 \end{cases} \tag{18}$$

where the error now is formed between the plant state and a modified reference model: $e(t) = x(t) - x_m(t)$ where

$$\dot{x}_m = A_m x_m + B_m r - B(1 + \mu(x, x_m))\hat{K}'_u \mathrm{Dz}(u) \tag{19}$$

This control structure, initially, appears to be different to the scheme proposed in [19], but it is essentially the same with "anti-windup" notation. The attractive feature of the control system (18)-(19) when written in the above form is that a classic anti-windup structure may be observed: in the absence of saturation i.e. $\mathrm{Dz}(u) \equiv 0$, the original unsaturated control law (16) is recovered; also the "anti-windup" terms in the control law and the reference model are only activated once saturation occurs i.e. $\mathrm{Dz}(u) \neq 0$. The aim of the remainder of the section is to prove that the interconnection of the plant (10) with the controller (18)-(19) ensures the plant state, approaches the ideal model state (14) provided some additional conditions are satisfied. The function $\mu(.,.) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is a static anti-windup term, similar to that found in static linear anti-windup schemes. In [19], this term was chosen as a constant scalar $\mu(x, x_m) = \mu$ and, as demonstrated in [19], it can be tuned to bestow good performance upon the system. The extra adaptive term $\hat{K}'_u(t)$ in the reference model state equation is present to alter the evolution of the reference model during saturation; it gives information about saturation to the reference model and is crucial for proving convergence in the results given later. Figure 1 shows a schematic of the the proposed scheme.

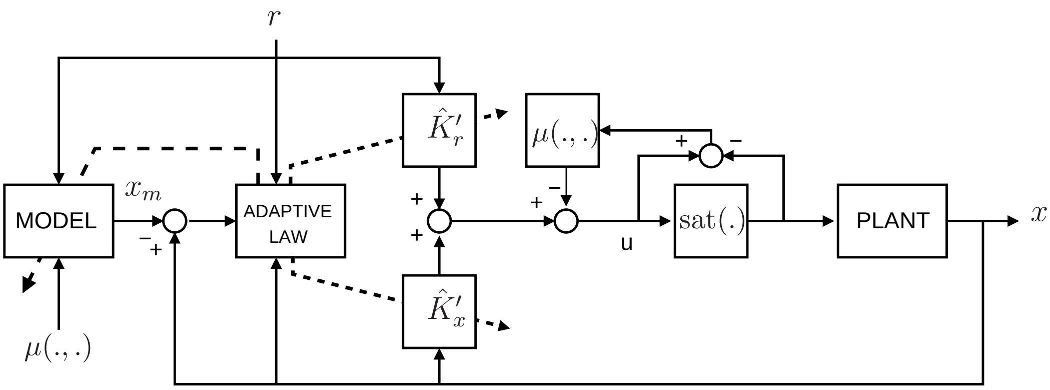

Fig. 1. Schematic of modified adaptive anti-windup scheme described by equation (18). The model is described by equation (19).

The first result is a stepping stone to the main result.

Proposition 5 *Consider the interconnection of the plant (10), the controller (18) and the dynamics (19) and let Assumptions 3 and 4 be satisfied. Assume further that* $\mu(x, x_m)$ *is such that the interconnection is well-posed,* $r \in \mathcal{L}_\infty$ *and that there exists a* $K_u^* = \lambda$*. Then the signals* $e = x - x_m$*,* $\hat{K}_x$*,* $\hat{K}_r$ *and* $\hat{K}_u$ *are all bounded and furthermore* $\lim_{t\to\infty} e(t) = 0$*.*

Proof: For simplicity in notation, the arguments of $\mu(x, x_m)$ are suppressed.

$$\dot{e} = Ax + B\lambda\mathrm{sat}(u) - A_m x_m - B_m r + B\hat{K}_u'(1+\mu)\mathrm{Dz}(u) \quad (20)$$
$$= Ax + B\lambda\mathrm{sat}(u) - A_m x_m - B_m r + BK_u^*(1+\mu)\mathrm{Dz}(u) + B(1+\mu)(\hat{K}_u' - K_u^*)\mathrm{Dz}(u) \quad (21)$$

Defining $\Delta K_u' := \hat{K}_u' - K_u^*$, then gives

$$\dot{e} = Ax + B\lambda\mathrm{sat}(u) - A_m x_m - B_m r + BK_u^*(1+\mu)\mathrm{Dz}(u) + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (22)$$
$$= Ax + B\lambda u - B\lambda\mathrm{Dz}(u) - A_m x_m - B_m r + BK_u^*(1+\mu)\mathrm{Dz}(u) + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (23)$$
$$= Ax + B\lambda u - A_m x_m - B_m r + B\lambda\mu\mathrm{Dz}(u) + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (24)$$

where the identity (3) has been used, along with the assumption that there exists a $K_u^* = \lambda$. From the expression for u in (18) it follows that

$$\dot{e} = Ax + B\lambda\hat{K}_x'x + B\lambda\hat{K}_r'r - A_m x_m - B_m r + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (25)$$

Adding and subtracting $B\lambda(K_x^*x + K_r^*r)$, then gives

$$\dot{e} = (A + B\lambda K_x^*)x + B\lambda K_r^*r + B\lambda\Delta K_x'x + B\lambda\Delta K_r'r - A_m x_m - B_m r + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (26)$$
$$= A_m e + B\lambda\Delta K_x'x + B\lambda\Delta K_r'r + B\Delta K_u'(1+\mu)\mathrm{Dz}(u) \quad (27)$$

where $\Delta K_x' := \hat{K}_x' - K_x^*$, $\Delta K_r' := \hat{K}_r' - K_r^*$ and Assumption 4 has been used. Next, choose the Lyapunov function

$$V = e'Pe + \lambda\Delta K_x'\Gamma_x^{-1}\Delta K_x + \lambda\Delta K_r'\Gamma_r\Delta Kr + \Delta K_u'\Gamma_u\Delta K_u \quad (28)$$

After using the adaptive updates from equation (18) and equation (27), the time derivative of the Lyapunov function (28) is given by

$$\dot{V} = e'(PA_m + A_m'P)e = -e'Qe \quad (29)$$

From this it follows that e, ΔK_x, ΔK_r and ΔK_u are bounded and then that $\hat{K}_x$, $\hat{K}_r$ and $\hat{K}_u$ are also bounded. To prove convergence of $e(t)$ Barbalat's Lemma needs to be applied and to do this $\dot{V}$ needs to be proven uniformly continuous. First note that the control signal is given by

$$u = \hat{K}_x'x + \hat{K}_r'r - \mu\mathrm{Dz}(u) \quad (30)$$

Writing $\mathrm{Dz}(u) = \sigma(u)u$ where $\sigma(.) : \mathbb{R} \mapsto [0, 1)$ then allows us to re-write this as

$$u = \kappa(u)[\hat{K}_x'x + \hat{K}_r'r] \quad (31)$$

where $\kappa(u) = (1+\sigma(u)\mu)^{-1}$. Because the system is assumed to be well-posed, $\kappa(u)$ exists and is unique for all $u \in \mathbb{R}$. Therefore, u will be bounded if x bounded, as $\hat{K}_x$ and $\hat{K}_r$ are bounded as proved above and r is bounded by assumption. However, by Assumption 3, A is Hurwitz, so from Lemma 2, x is bounded.

Therefore, referring to the equation (27), it follows that $\dot{e} \in \mathcal{L}_\infty$ since all terms on the right hand side are bounded. Next note that

$$\ddot{V} = -e' Q \dot{e}$$

which implies $\dot{V}$ is uniformly continuous and thus that $V(t)$ converges to zero by Barbalat's lemma. This then implies that $e(t)$ converges asymptotically since Q is positive definite. □

Remark 2: Convergence of the error depends on the system being well-posed and hence $\mu(x, x_m)$ being chosen such that $\kappa(u)$ exists and is unique. This will be the case if, for example μ is simply a scalar such that $\mu > -1$. □

Proposition 5 does not imply convergence to *ideal* behaviour: it simply says that the error between the plant state $x(t)$ will converge to the state of the model (19). Note that the model (19) is *not* the ideal model; it has been modified by the anti-windup term $B\hat{K}_u'(1 + \mu(x))\mathrm{Dz}(u)$. In fact we would like to prove that

$$\lim_{t\to\infty} e_m(t) = 0 \text{ where } e_m = x_m - x_r \tag{32}$$

If equation (32) holds, e_m converges to zero, which implies x_m converges to x_r, which is the state of the ideal model (14). Proposition 5 has already proved that that x converges to x_m, so equation (32) then implies that x will converge to x_r. The following result establishes conditions under which this convergence can occur. This is a natural generalisation of the "graceful return to linear behaviour" sought in anti-windup compensation for linear control systems [18].

Proposition 6 *Consider the interconnection of the plant (10) and the controller (18)-(19) and let Assumptions 3 and 4 be satisfied. Assume that $\mu(x, x_m)$ is such that the interconnection is well-posed and also that $\mu(x, x_m) \in [0, \bar{\mu}]$ for all $x, x_m \in \mathbb{R}^n$ and some $\bar{\mu} > 0$. Also assume $r \in \mathcal{L}_\infty$ and that there exists a $K_u^* = \lambda$. Then $\lim_{t\to\infty} e_m(t) = 0$ if the following conditions are also satisfied:*

(1) $\mathrm{Dz}(u^*) \in \mathcal{L}_2$
(2) $\Delta u \in \mathcal{L}_2$

where $u^* := K_x^* x(t) + K_r^* r(t)$ and $\Delta u := \Delta K_x(t)' x(t) + \Delta K_r(t)' r(t)$, and $\Delta K_x'(t)$ and $\Delta K_r(t)$ are as defined earlier.

Proof: From $e_m = x_m - x_r$ we have that

$$\dot{e}_m = A_m e - B(1+\mu)\hat{K}_u' \mathrm{Dz}(u) \tag{33}$$

Now, from equation (31) we have

$$u = \kappa(u)\left(\underbrace{K_x^* x + K_r^* r}_{=u^*} + \underbrace{\Delta K_x' x + \Delta K_r' r}_{=\Delta u}\right) \tag{34}$$

By the well-posedness assumption such a u exists and is unique. Furthermore, because $\mu(x, x_m) \geq 0$, then $\|\kappa(u)\| \leq 1$. Using this expression for u in (33), gives

$$\dot{e}_m = A_m e - B(1+\mu)\hat{K}_u \mathrm{Dz}[\kappa(u)(u^* + \Delta u)] \tag{35}$$

$$\begin{aligned} &= A_m e - B(1+\mu)\hat{K}_u(\mathrm{Dz}[\kappa(u)(u^* + \Delta u)] - \mathrm{Dz}[\kappa(u)u^*]) \\ &\quad - B(1+\mu)\hat{K}_u \mathrm{Dz}[\kappa(u)u^*] \end{aligned} \tag{36}$$

Next, forming a Lyapunov function $V_e(e_m) = e_m' P e_m$, we obtain

$$\begin{aligned} \dot{V}_e(e_m) &= e_m' Q e_m - 2e_m' PB\hat{K}_u(1+\mu)(\mathrm{Dz}[\kappa(u)(u^* + \Delta u)] - \mathrm{Dz}[\kappa(u)u^*]) \\ &\quad - 2e_m' PB\hat{K}_u(1+\mu)\mathrm{Dz}[\kappa(u)u^*] \end{aligned} \tag{37}$$

$$\leq -e_m' Q e_m + 2\|e_m\|\|PB\|\|\hat{K}_u\|(1+\bar{\mu})\|\Delta u\| + 2\|e_m\|\|PB\|\|\hat{K}_u\|(1+\bar{\mu})\|\mathrm{Dz}[\kappa(u)u^*]\| \tag{38}$$

where in the inequality we have used the Lipschitz property of the deadzone mentioned in Section 1.1. Proposition 5 implies that that $\hat{K}_u$ is bounded, viz $\|\hat{K}_u(t)\| \leq c_u$ for all $t \geq 0$ and some $c_u > 0$. Therefore we have

$$\dot{V}_e(e_m) \leq -e_m' Q e_m + 2\|e_m\|\|PB\|c_u(1+\bar{\mu})\|\Delta u\| + 2\|e_m\|\|PB\|c_u(1+\bar{\mu})\|\mathrm{Dz}(\kappa(u)u^*)\| \tag{39}$$

Next, because that $\|\kappa(u)\| \leq 1$, Lemma 1 then can be applied to obtain

$$\dot{V}_e(e_m) \leq -e_m' Q e_m + 2\|e_m\|\|PB\|c_u(1+\bar{\mu})\|\Delta u\| + 2\|e_m\|\|PB\|c_u(1+\bar{\mu})\|\mathrm{Dz}(u^*)\| \tag{40}$$

The Comparison Principle (section 5.4 in [16]) can now be applied to inequality (40) to prove convergence of e_m if (i) $\Delta u \in \mathcal{L}_2$, and (ii) $\mathrm{Dz}(u^*) \in \mathcal{L}_2$: exactly those conditions given in the proposition. □□

Remark 3: Proposition 6, roughly speaking, requires, u^* (the "ideal control signal") to eventually fall below the saturation limits i.e. $\lim_{t\to 0} |u^*(t)| < \bar{u}$, which is a similar condition to that assumed in the case of linear anti-windup schemes: the nominal linear control law should eventually lie within the saturation constraints. The additional assumption requires Δu also should decay to zero - this will be the case if ΔK_x and ΔK_r converge to zero. Note that the Lyapunov analysis of Proposition 5 only guarantees these gains are bounded, they do not necessarily converge; hence the extra assumptions in Proposition 6 □

The conditions in the above proposition are quite strong: they require the adaptive gains to converge to the ideal gains K_x^*, K_r^*, which is not only not guaranteed, but probably unlikely to be the case. Instead assume that the adaptive gains have steady state values

$$\lim_{t\to\infty} \hat{K}_x(t) := K_{x,ss} \tag{41}$$

$$\lim_{t\to\infty} \hat{K}_r(t) := K_{r,ss} \tag{42}$$

which then means that ΔK_x and ΔK_r will converge to steady state values (probably different from zero) which are defined as

$$\lim_{t\to\infty} \Delta K_x(t) := \Delta K_{x,ss} \tag{43}$$

$$\lim_{t\to\infty} \Delta K_r(t) := \Delta K_{r,ss} \tag{44}$$

These can be used to then define

$$\Delta u_{ss}(t) := \Delta\hat{K}_{x,ss}x(t) + \Delta\hat{K}_{r,ss}r(t) \tag{45}$$

and from there, the control law can be re-written as

$$u = \kappa(u)[u^* + \Delta u_{ss} + \Delta u - \Delta u_{ss}]$$

Then via a similar argument to that of the proof in Proposition 6, we can state the following result:

Proposition 7 *Consider the interconnection of the plant (10) and the controller (18)-(19) and let Assumptions 3 and 4 be satisfied. Assume that* $\mu(x, x_m)$ *is such that the interconnection is well-posed and also that* $\mu(x, x_m) \in [0, \bar{\mu}]$ *for all* $x, x_m \in \mathbb{R}^n$ *and some* $\bar{\mu} > 0$*. Also assume* $r \in \mathcal{L}_\infty$ *and that there exists a* $K_u^* = \lambda$*. Then* $\lim_{t\to\infty} e_m(t) = 0$ *if the following conditions are also satisfied:*

(1) $\mathrm{Dz}(u^* + \Delta u_{ss}) \in \mathcal{L}_2$
(2) $\Delta u - \Delta u_{ss} \in \mathcal{L}_2$

where u^* *and* Δu *are defined in Proposition 6 and* $\Delta u_{ss}(t)$ *is defined in equation (45).*

Proposition 7 is similar to Proposition 6 but the conditions are more practical. Condition (i) on the ideal control law now no longer requires $\lim_{t\to\infty} |u^*(t)| < \bar{u}$, but instead requires the ideal control law *plus some perturbation* to fall below the saturation limits i.e.

$$\lim_{t\to\infty} |u^*(t) + \Delta u_{ss}(t)| < \bar{u} \tag{46}$$

This may be seen as a stronger requirement than in Proposition 6, but it leads to a weaker second condition: now only $\Delta u - \Delta u_{ss}$ is required to converge in $\mathcal{L}_2$ which is, essentially, equivalent to requiring that $\hat{K}_x$ and $\hat{K}_r$ converge to *steady state values*, $\hat{K}_{x,ss}$ and $\hat{K}_{r,ss}$ respectively, *not* their *ideal* values K_x^* and K_r^*.

Remark 4: The results in this paper have been proved under Assumption 3 which required A to be Hurwitz, but assumes that it is otherwise *unknown*. For linear systems, it is well known (see [11,8] for example) that if A is not Hurwitz there is considerably more complexity in the design of an anti-windup compensator and estimating a subset of the domain of attraction becomes imperative. Note however, that estimating a region of attraction in the case that A is unknown is effectively impossible. Therefore, if A is known to not be Hurwitz, but is otherwise unknown, anti-windup design with *a priori* guarantees becomes extremely difficult. In this case, it would seem more sensible to use an estimate of a model and to design a robust anti-windup compensator to cope with the mismatch [28,6]. Note that some local conditions (size of initial state) have been given in [19] but they are extremely conservative and dependent on the (unknown) adaptive gains K_x^* and K_r^* - this is not a criticism of these results but a consequence of the difficulties of anti-windup when A is unknown and has unstable eigenvalues. □□

4 Simulation results

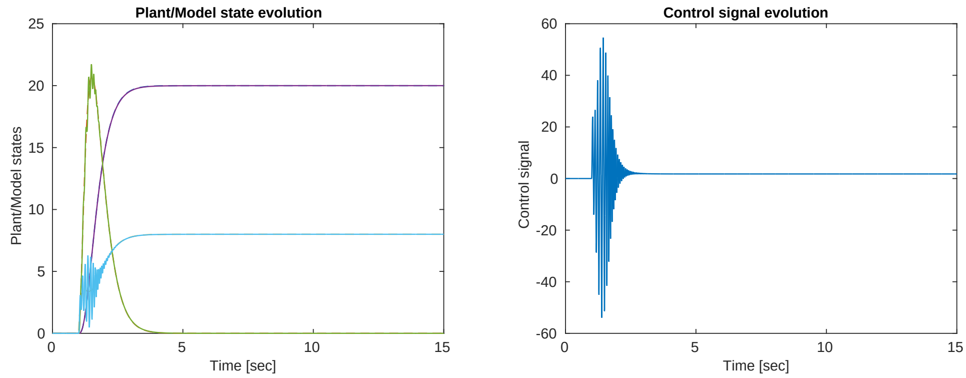

Fig. 2. Response of adaptive control system without input saturation: left, plant/model state evolution; right, control signal

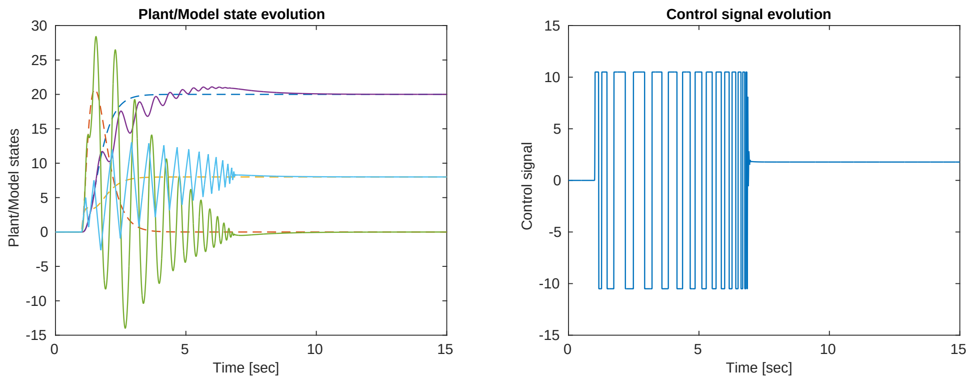

Fig. 3. Response of adaptive control system with input saturation: left, plant/model state evolution; right, control signal

We re-use the hydraulic actuator example from [23], but use an adaptive control law to control the system. The nominal adaptive control law was constructed according

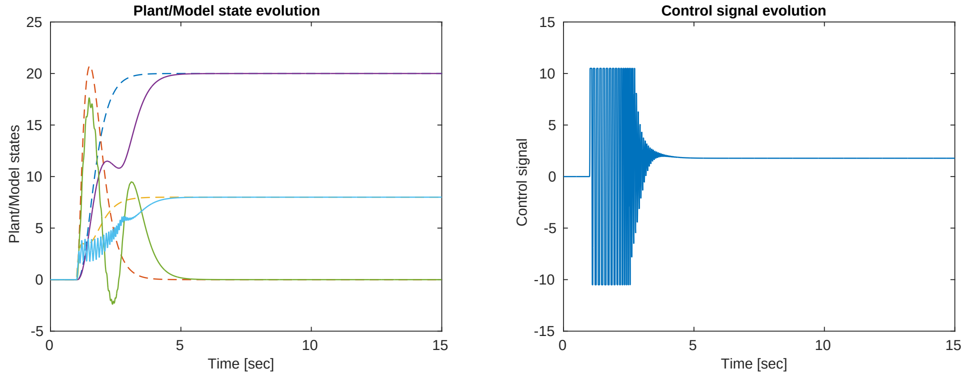

Fig. 4. Response of adaptive control system with input saturation and anti-windup: left, plant/model state evolution; right, control signal

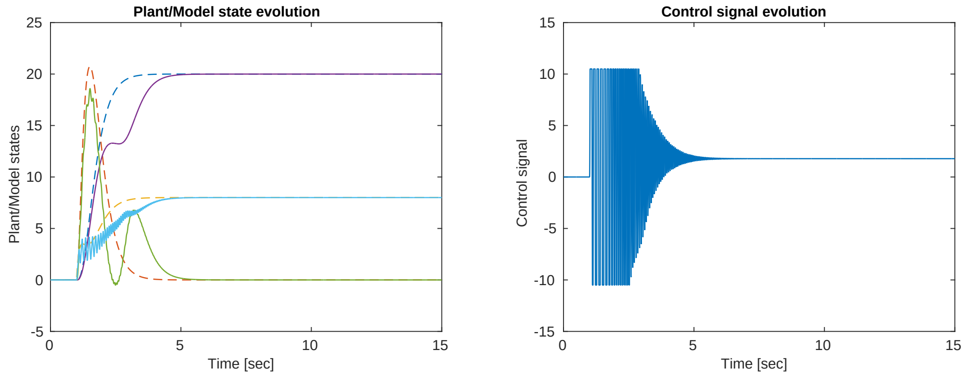

Fig. 5. Response of adaptive control system with input saturation and nonlinear anti-windup: left, plant/model state evolution; right, control signal

to equation (16) with $\Gamma_x = 2I_3$, $\Gamma_r = 2$ and $Q = I_3$ in equation (28). The nominal model was described by the dynamics in (14) where $A_m = A - BK$, $B_m = B$ and K was used to place the poles of the reference model at $s = -3, -4, -5$; this form of the reference model automatically satisfies Assumption 4. λ was set to 2 i.e. the control effectiveness is twice that which the controller expects. The nominal step response of the model is shown in Figure 2, where $r(t)$ is step with an amplitude of 20 cm. Note that the adaptive controller causes the system to behave well and after about a second all states have converged to the reference model states. However, note that the control signal is somewhat oscillatory and of high magnitude.

Figure 3 shows the behaviour of the system with input constraints of magnitude $\bar{u} = 10.5$ volts ([23]) are introduced. In this case, the adaptive controller takes a long time to converge to steady state behaviour and it does so in a highly oscillatory manner; clearly the saturation limits cause the performance of the adaptive controller to degrade.

Figure 4 shows the behaviour of the adaptive control scheme using the anti-windup modifications described by equations (18) and (19). Here $\Gamma_u = 2$ and $\mu(x, x_m)$ is simply chosen as a scalar $\mu = 2$ which satisfies the assumptions of Proposition 6. Note that in this case, convergence to steady state tracking is much improved with this occurring faster and without the oscillations observed when anti-windup is absent; clearly the adaptive controller with anti-windup performs better than without.

Consider again the adaptive control scheme with anti-windup modifications described by equations (18) and (19), with $\Gamma_u = 2$ as before, but with $\mu(x, x_m)$ chosen as a nonlinear function of the states x and x_m:

$$\mu(x, x_m) = 2 \min \left\{ 1 + \|x - x_m\|^2, 10 \right\} = 2 \min \left\{ 1 + \|e\|^2, 10 \right\} \qquad (47)$$

which satisfies the assumptions of Proposition 6, with $\mu(.) \leq \bar{\mu} = 20$. In this case we see from Figure 5, that there is more transient agreement between the actual and reference model states, although the corresponding control signal is much more active. It appears that nonlinear choices of $\mu(x, x_m)$ can lead to improved performance, in some sense, but care must be exercised in this choice.

As with any anti-windup-like approach to handling saturation, the design of the baseline controller is of paramount importance: the choice of adaptive parameters, Q, Γ_x and Γ_r, are central in determining the performance of the system. If Γ_x and Γ_r are chosen too small, and hence, adaptation is too slow, one may not anticipate good nominal performance and, therefore, performance under constraints cannot be expected to improve. Conversely, if adaptation is too fast, leading to good unconstrained performance, one might expect large control actions, leading to more saturation and therefore worse performance when control constraints are introduced. This will then lead to more saturation effects for the anti-windup compensator to attenuate, again implying poor performance. Indeed, for the example discussed here, too fast adaptation (i.e. Γ_x and Γ_r chosen large) leads to extensive periods of saturation and, consequently a slower return to desired behaviour.

5 Conclusion

This paper has shown how a simple anti-windup scheme can be developed for MRAC schemes. The anti-windup scheme has the structure of the positive-μ scheme introduced in [19]. For stable plants, it has been shown that in order for the scheme to exhibit bonafide anti-windup behaviour, the “ideal” (but unknown) control signal - plus a perturbation - must eventually lie within the saturation limits. This is a similar condition to that originally introduced in [26]. The results have been proved under the assumption that the plant A-matrix is unknown but Hurwitz; when this assumption is dropped, it is generally quite difficult to prove useful results about the performance of this anti-windup scheme due to the difficulty in estimating an accompanying region of attraction. The results have been developed for single-input systems, but the multi-input counterpart is a relatively straightforward extension.

The results in this paper have been proved, as in [19], under what might be described as *ideal* conditions. The effects of disturbances, measurement noise and unmodelled dynamics have not been accounted for; neither have the effects of so-called σ-modification or projection, which are typically used in practical adaptive systems [21]. Clearly, there is a need to understand the implications of these phenomena on the anti-windup scheme under consideration here. Moreover, it may be that anti-windup/positive μ modification is better understood in the context of other adaptive control techniques: the so-called Simple Adaptive Control approach (see for example [3]) appears particularly promising and similar anti-windup results may be anticipated for this scheme.

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