Twisted tensor products of n-groupoids and crossed complexes

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By

Heyam Khazaal Hassan Alkhayyat

Department of Mathematics
University of Leicester

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Statement

The accompanying thesis submitted for the degree of Ph.D. entitled 'Twisted tensor products of n-groupoids and crossed complexes' is based upon work conducted by the author in the department of Mathematics at the University of Leicester during the period between October 2015 and May 2019

All the work recorded in this thesis is original unless otherwise acknowledged in the text or by references. None of the work has been submitted for another degree in this or any other university.

Signed: .... Date: ....

In the name of God, the Gracious, the Merciful

They said, "Glory be to You! We have no knowledge except what You have taught us. It is you who are the Knowledgeable, the Wise."

This thesis is dedicated to my greatest achievement in this world, dad and mum. Dad, Mum you are the most wonderful persons in my life. Thank you for having faith in me when my own was lacking.

To

KHAZAAL HASSAN

Maleha Abdul-Ameer

My parents

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Abstract

For any 1-reduced simplicial set X, we define a crossed complex of groups $P^{\mathsf{Crs}}X$, which we define as a twisted tensor product of the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX. In fact, we prove that $P^{\mathsf{Crs}}X$ is contractible. Therefore $P^{\mathsf{Crs}}X$ is a crossed complex model for the path space of X. It is also an example of a crossed complex model of the total space of a fibration,

$$\Omega X \longrightarrow PX \longrightarrow X.$$

This generalises from chain complexes to crossed complexes the theorem proved by J. F. Adams, and P. J. Hilton in their paper [3]. Our definition of twisted tensor products of crossed complexes also defines a twisted tensor product of n-groupoids, for all n. This comes from the fact that there is an equivalence of categories (∞-groupoids $\longleftrightarrow$ crossed complexes) which was proved by R. Brown and P. J. Higgins in their paper [12].
We recall the classical Eilenberg-Zilber theorem for chain complexes, and its generalisation for crossed complexes, which show that the tensor product provides an algebraic model for the Cartesian product of the fibration

$$X \longrightarrow X \times Y \longrightarrow Y.$$

We also extend our theorems to 0-reduced simplicial sets X. In this case we generalise the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ from 1-reduced simplicial sets to the group-completed crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}X$ for 0-reduced simplicial sets and define a crossed complex of groupoids $P^{\mathsf{Crs}}X$, a twisted tensor product with the twisted boundary maps

$$\partial_n^P : P_n^{\mathsf{Crs}}X = (\hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X)_n \longrightarrow P_{n-1}^{\mathsf{Crs}} = (\hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X)_{n-1}, \qquad \partial^2 = 0.$$

We end by defining a contracting homotopy $\{\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X\}$ which shows that this crossed complex of groupoids is still a model for the path space on X.

Contents

1 Introduction

We are interested in the category of crossed complexes of groupoids and twisted tensor products of crossed complexes. The motivation for this thesis has come from two directions: firstly, from a wish to generalise J. F. Adams and P. J. Hilton's theorem for chain complexes [3], by constructing a crossed complex $P^{\mathsf{Crs}}X$ which is a model for the path space of X, as a twisted tensor product of the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX for 1-reduced simplicial set X. Secondly to define the general path crossed complex of groupoids $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ where $\hat{\Omega}^{\mathsf{Crs}}X$ is the group-completed crossed cobar construction for any 0-reduced simplicial set.

The definition of a crossed complex is motivated by the principal example: the fundamental crossed complex πX of a filtered space

$$X : X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X.$$

Here $\pi_1 X$ is the fundamental groupoid $\pi_1(X_1, X_0)$. For $n \geqslant 2$, $\pi_n X$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x_0)$ where $x_0 \in X_0$, together with the standard boundary operators $\partial : \pi_n(X_n, X_{n-1}) \to \pi_{n-1}(X_{n-1}, X_{n-2})$ and the actions of $\pi_1 X$ on $\pi_{n-1}(X_{n-1}, X_{n-2})$ [11]. The category of crossed complexes is a monoidal closed category which shares many properties of the category of chain complexes, but with some non-abelian features in dimensions one and two, it and may also be thought of as a reduced form of a simplicial groupoid [14], or as a strict ∞-groupoid [12].

A crossed complex of groupoids C is a sequence of groupoids C_n over a fixed object set C_0, which are C_0-indexed families of abelian groups for $n \geqslant 3$, equipped with C_1-actions and C_1-equivariant boundary maps ∂_n between them, which on the object sets will be the identity function, and $\partial_n^2 = 0$ for all n. Furthermore, $\partial_2 : C_2 \longrightarrow C_1$ is a crossed module of groupoids, and for $n \geqslant 3$, $\partial_2 C_2$ acts trivially on C_n.

The tensor product of two chain complexes A and B, is also a chain complex $C_n = (A \otimes B)_n$ such that :

$$C_n = \bigoplus_{p+q=n} A_p \otimes B_q$$

with the boundary homomorphism δ_n defined by:

$$\delta_n(a_p \otimes b_q) = (\delta_p a_p) \otimes b_q + (-1)^p a_p(\delta_q b_q)$$

which satisfies that $\delta^2 = 0$ [25]. The classical Eilenberg-Zilber theorem in its original form [33] gives a chain homotopy equivalence

$$C(X) \otimes C(Y) \simeq C(X \times Y)$$

where X, Y are simplicial sets, and $C(X)$ is the normalised free chain complex on the simplicial set X. This theorem was generalised to twisted products by E.H. Brown [4], and also generalised by A. Tonks for crossed complexes [31]. Our original aim was to combine the two generalisations to define a twisted Eilenberg-Zilber theorem for crossed complexes.

A. Tonks [31] gave a natural strong deformation retraction from the fundamental homotopy crossed complex of a product of simplicial sets $\pi(X \times Y)$ onto the tensor product of the corresponding crossed complexes $\pi X \otimes \pi Y$. For a fundamental crossed complex πX of a simplicial set X, A. Tonks had obtained a strong deformation retraction of $\pi(X \times Y)$ onto $\pi X \otimes \pi Y$ satisfying certain side conditions and interchange relations [31, 32].

Suppose first that X is a 1-reduced simplicial set. We introduce a free $\Omega^{\mathsf{Crs}}X$-module $P^{\mathsf{Crs}}X$ with basis $B = \{(\varnothing \otimes b_n), b_n \in \pi X\}$. $P^{\mathsf{Crs}}X$ is a twisted tensor product of the cobar construction $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX of a 1-reduced simplicial set X, with object set $P_1^{\mathsf{Crs}}X = P_0^{\mathsf{Crs}}X = \{(\varnothing \otimes *)\}$. It is twisted because we define a twisted boundary map $\partial_n^P : P_n^{\mathsf{Crs}}X \to P_{n-1}^{\mathsf{Crs}}X$ as

$$\partial_2^P(\varnothing \otimes b_2) = (s^{-1}b_2 \otimes *)$$

$$
\begin{aligned}
\partial_3^P(\varnothing \otimes b_3) &= (s^{-1}b_3 \otimes *) - (\varnothing \otimes d_3 b_3) - (\varnothing \otimes d_1 b_3) + (\varnothing \otimes d_2 b_3) \\
&\qquad + (\varnothing \otimes d_0 b_3) \\
\partial_n^P(\varnothing \otimes b_n) &= \sum_{i=1}^{n} (-1)^i (\varnothing \otimes d_i b_n) + \sum_{i=1}^{n} (s^{-1} b_{0\ldots i} \otimes b_{i\ldots n}), \quad n \geqslant 4 \text{ (Note 1.1, page 11).}
\end{aligned}
$$

which satisfy that $\partial_{n-1}^P \partial_n^P : P_n^{\mathsf{Crs}} X \to P_{n-2}^{\mathsf{Crs}} X$ is trivial. We prove that this crossed complex of groups $P^{\mathsf{Crs}}X$ is homotopy equivalent to the trivial crossed complex. It is therefore a crossed complex model for the path space of X, when X is 1-reduced simplicial set.

We extend our definition of the crossed complexes of groups $P^{\mathsf{Crs}}X$ to a crossed complexes of groupoids $P^{\mathsf{Crs}}X$ for a 0-reduced simplicial set X, but before we do this we generalise the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ to a crossed cobar construction of groupoids $\hat{\Omega}^{\mathsf{Crs}}X$ where X is a 0-reduced simplicial set. The group-completed crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}X$ is a free crossed chain algebra generated by the elements $s^{-1}a_{n+1}$ in dimension n for each non-degenerate $(n+1)$-simplex of X, together with extra generators $(s^{-1}a_1)^{-1}$ for each non-degenerate 1-simplex a_1 of X (Definition 5.4).

Now suppose X is only 0-reduced. The crossed complex of groupoids $P^{\mathsf{Crs}}X$ is a twisted tensor product of the crossed complex of groups πX, whose object set is $\{*\}$ and the crossed chain algebra $\hat{\Omega}^{\mathsf{Crs}}X$, whose object set will be defined in Definition 5.4. The crossed complex of groupoids $P^{\mathsf{Crs}}X$ will be a free crossed complex with the *same generators* as the ordinary, non-twisted, tensor product $\hat{\Omega}^{\mathsf{Crs}}X \otimes \pi X$. We write these generators as

$$x \otimes b \in P_{n+m}^{\mathsf{Crs}} X,$$

where

- x is a generator of degree $|x| = n$ in $\hat{\Omega}_n^{\mathsf{Crs}} X$, defined as:

$$x = \omega^{(0)} s^{-1} a_{n_1+1}^{(1)} \omega^{(1)} a_{n_2+1}^{(2)} \cdots \omega^{(r-1)} s^{-1} a_{n_r+1}^{(r)} \omega^{(r)}$$

where $r \geqslant 0$, each $\omega^{(i)} \in \hat{\Omega}_0^{\mathsf{Crs}} X$, $\omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2} \cdots (s^{-1}a_1^{(k)})^{\epsilon_k}$, each $a_{n_i+1}^{(i)}$ is a non-degenerate simplex in X_{n_i+1}, $n_i \geqslant 1$, and $\sum n_i = n$, $k \geqslant 0, a_1^{(i)} \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1$.

We know that $\hat{\Omega}_n^{\mathsf{Crs}} X$ is a (free) crossed chain algebra with the algebra structure defined by concatenation of words $x \otimes x' \mapsto xx'$.

- b is a generator of degree $|b| = m$ in πX, given by a non-degenerate m-simplex of X.

Before we define the twisted boundary maps ∂^P for $P^{\mathsf{Crs}} X$ we will give formulas for the boundary $\partial^{\otimes}$ for the ordinary, non-twisted, tensor product. This boundary map, in the context of chain complexes, would be $\partial^{\otimes} = \partial^{\hat{\Omega}} \otimes \mathrm{id} \pm \mathrm{id} \otimes \partial^{\pi}$. And then we define the twisted boundary map $\partial^P : P_n^{\mathsf{Crs}} X \to P_{n-1}^{\mathsf{Crs}} X$,which satisfies that $\partial_{n-1}^P \partial_n^P = 0$.

A crossed complex of groupoids is *pointed* if there is a specified object $* \in C_0$. If C is a pointed crossed complex of groupoids, then C is *contractible* to the basepoint $*$ if there is a family of functions $\eta_n : C_n \to C_{n+1}$ that define a *contracting homotopy*

$$h : * \simeq \mathrm{id}_C : \pi(\Delta[1]) \otimes C \to C$$

by

i. $h(0 \otimes c) = 0_*$ (or $*$ if $c \in C_0$),

ii. $h(1 \otimes c) = c$,

iii. $h(\sigma \otimes c) = \eta(c)$.

A family of functions $\eta_n : C_n \to C_{n+1}$, $(n \geqslant 0)$ defines a contracting homotopy via $h(\sigma \otimes c_n) = \eta_n(c_n)$ if and only if it satisfies

1. $\eta_0(c_0) \in C_1$ has source $*$ and target c_0,

2. $\eta_1(c_1) \in C_2$ has basepoint $*$ and boundary:

$$\partial_2 \eta_1(c_1) = -\eta_0(\mathrm{targ}(c_1)) + c_1 + \eta_0(\mathrm{src}(c_1)),$$

$$\begin{tikzcd}
* \arrow[rr, "\eta_0(x)"] \arrow[dr, "\eta_0(y)"'] & & x \arrow[dl, "c_1"] \\
& y &
\end{tikzcd}$$

Figure 1:

3. If $n \geqslant 2$ then, $\eta_n(c_n) \in C_{n+1}$ has basepoint $*$ and boundary:

$$\partial_{n+1}\eta_n(c_n) = c_n^{\eta_0(\mathfrak{p})} - \eta_{n-1}\partial_n(c_n),$$

4. For all $n \geqslant 1$,

$$\eta_n(c_n + c_n') = \eta_n(c_n) + \eta_n(c_n')$$

5. For all $n \geqslant 2$,

$$\eta_n(c_n^{c_1}) = \eta_n(c_n)$$

Important note we should point out.

Note 1.1. *We will use the symbols b_m which mean a simplex $b \in X_m$ of dimension m. While $b_{(m)}$ means the m^{th} vertex in the simplex b_m. We will also write, for example*

$$d_2 b_5 = b_{01345} \qquad d_1 b_1 = b_{(0)}$$

and

$$s_1 b_2 = b_{0112}$$

Structure of the thesis

In this thesis, we begin with recalling some background information on the category of simplicial sets and chain complexes [18, 27, 32],and [15] that will be used in the thesis, as well as reviewing the classical Eilenberg-Zilber theorem in its original form [33] which gives for simplicial sets X, Y a chain homotopy equivalence

$$C(X) \otimes C(Y) \simeq C(X \times Y)$$

where $C(X)$ is the normalised free chain complex on the simplicial set X. This theorem was generalised by A. Tonks in his paper [31] for crossed complexes, which shows that the tensor product provides an algebraic model for the Cartesian product and of trivial fibrations. We also recall E. H. Brown theorem [4] on chain equivalence of the chain complex of a total space of a twisted cartesian product of two simplicial sets, and a twisted tensor product of the corresponding chain complexes.

In Chapter 3, the definition of loop space and Adams' cobar construction is recalled [2], which is dual to the bar construction of Eilenberg and Mac Lane. We can think of the cobar construction as a chain complex analogous to the fibre space in the path loop fibration

$$\Omega X \to PX \to X$$

K. Hess and A. Tonks proved in their paper [19] that the Adams' cobar construction ΩCX of a 1-reduced simplicial set X, on the normalised chain complex is a strong deformation retract of the normalised chain on loop space CGX.

$$\begin{tikzcd} CGX \arrow[loop left, "\eta"] \arrow[r, shift right, "\psi"'] & \Omega CX \arrow[l, shift right, "\phi"'] \end{tikzcd}$$

They are obviously equivalent, as Ω and G are both models for the loop space.

We study also the generalised Adams cobar construction of a 0-reduced simplicial set which was defined by K. Hess and A. Tonks in their paper [19]. We end this chapter by introducing Baues' construction of the cobar construction $\Omega^{\mathsf{Crs}}X$ in the category of crossed complexes. If X is 1-reduced simplicial set, then the generators of the cobar construction have the form $\omega = s^{-1}x_1 \otimes s^{-1}x_2 \otimes \cdots \otimes s^{-1}x_n$, in dimension $\sum(|x_i| - 1)$, [14] we give some motivation for an intuitive definition of the twisted tensor product of pointed crossed complexes.

We begin Chapter 4 with defining a new crossed complex of groups $(P_n^{\mathsf{Crs}}X)$ in terms of a twisted tensor product of a free crossed chain algebra $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX for 1-reduced simplicial set X. It is a free $\Omega^{\mathsf{Crs}}X$-module with basis $B = (\varnothing \otimes b)$, and $b \in \pi X$. We explain the twisted boundary maps as:

$\partial_2^P(\varnothing \otimes b_2) = (s^{-1}b_2 \otimes *)$

$\partial_3^P(\varnothing \otimes b_3) = (s^{-1}b_3 \otimes *) - (\varnothing \otimes d_3b_3) - (\varnothing \otimes d_1b_3) + (\varnothing \otimes d_2b_3) + (\varnothing \otimes d_0b_3)$

$\partial_n^P(\varnothing \otimes b_n) = \sum_{i=1}^{n}(-1)^i(\varnothing \otimes d_ib_n) + \sum_{i=1}^{n}(s^{-1}b_{0\ldots i} \otimes b_{i\ldots n}), \qquad n \geqslant 4,$

then we prove that $(\partial^P)^2$ is trivial for all dimensions $n \geqslant 2$. For the general form of the generators $(\prod s^{-1}a_{n_i} \otimes b_m)$, we define a differential map ∂_n^P taking into account the order of terms and actions in dimensions one and two due to non-abelian features.

The main theorem in Chapter 4 is that we prove the crossed complex of groups $(P_n^{\mathsf{Crs}}X)$ is contractible by defining a contractible homotopy $\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$. First we recall the definition of the notion of contracting homotopy.

<u>Definition</u>:

Let C be a crossed complex with $C_0 = \{*\}$. A contracting homotopy is a homomorphism $h : \pi(\Delta[1]) \otimes C \to C$ that satisfies:

$$h(0 \otimes c) = 0_*,$$

$$h(1 \otimes c) = c.$$

Given a contracting homotopy we have $h : * \simeq id_C$, and so C is contractible because there is a homotopy equivalence:

$$h : * \simeq id_C \circlearrowright C \leftrightarrows \{*\}$$

From this contracting homotopy, we define the family of functions

$$\eta_n : C_n \to C_{n+1},\ (n \geqslant 1)$$

defined by

$$\eta_n(c) = h(\sigma \otimes c), \quad (c \in C_n)$$

where $(\sigma : 0 \to 1) \in (\mathbf{\Delta}[\mathbf{1}])$, conversely, given a family of functions η_n, we could define a contracting homotopy

$$h(0 \otimes c) = *, h(1 \otimes c) = c, h(\sigma \otimes c) = \eta(c)$$

In order for h to be well defined and commute with ∂^P, the family must satisfy the properties:

Proposition:

The family of functions $\eta_n : C_n \to C_{n+1}$ provides a contracting homotopy h, which is defined as $h(\sigma \otimes c_n) = \eta(c_n), \quad (n \geqslant 1)$ if η satisfies the properties that:

1. $\partial\eta(c_1) = c_1$,
2. $\partial\eta(c_n) = c_n - \eta\partial(c_n)$,
3. $\eta(c_n + c_n') = \eta(c_n) + \eta(c_n')$,
4. $\eta(c_n^{c_1}) = \eta(c_n)$.

and $\eta(*) = 0_*$.

Now we let $C = P^{\mathsf{Crs}}X$ and prove it is contractable by defining the functions η_n for all possible forms of the generating elements of $P^{\mathsf{Crs}}_n X$.

Definition:

Let $x = \prod s^{-1}a_{n_i}, \qquad a_{n_i} \in X_{n_i - 1}$ Define $\eta : P^{\mathsf{Crs}}_n X \to P^{\mathsf{Crs}}_{n+1} X$ as:

1. $\eta(\varnothing \otimes *) = 0_{(\varnothing \otimes *)}$,
2. $\eta(xs^{-1}a_r \otimes *) = (-1)^{|x|}(x \otimes a_r)$,
3. $\eta(x \otimes b_n) = 0_{(\varnothing \otimes *)}$.

At the end of the Chapter we present two examples to illustrate the definition of η_n.

Chapter 5, is concerned with extending our results in Chapter four on the crossed complex of groups $P^{\mathsf{Crs}}X$ from 1-reduced simplicial sets X to a crossed complexes of groupoids $P^{\mathsf{Crs}}X$ for 0-reduced simplicial sets. First, we need to generalise the definition of the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ to a group-completed crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}X$ whose objects form a free group whose generators correspond to the non-degenerate 1-simplices of X.

Definition:

For a 0-reduced simplicial set X, the group completed crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}X$ is a free crossed chain algebra generated by $s^{-1}a_{n+1}$ in dimension n for each non-degenerate (n)-simplex of X, together with extra generators $(s^{-1}a_1)^{-1}$ for each non-degenerate 1-simplex a_1 of X. The boundary of a generator $s^{-1}a_{n+1}$ is analogous to that of the cobar construction $\Omega^{\mathsf{Crs}}X$, in degree 0,

$$\hat{\Omega}^{\mathsf{Crs}}_0 X = \left\{ \omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2} \cdots (s^{-1}a_1^{(k)})^{\epsilon_k} \ : \ k \geqslant 0, a_1^{(i)} \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1 \right\}$$

the free group on $X_1 - s_0X_0$. The generators x of degree $|x| = n$ of the free crossed complex $\hat{\Omega}^{\mathsf{Crs}}X$ are given by words

$$x = \omega^{(0)} s^{-1} a^{(1)}_{n_1+1} \omega^{(1)} a^{(2)}_{n_2+1} \cdots \omega^{(r)} s^{-1} a^{(r)}_{n_r+1} \omega^{(r+1)},$$

where $r \geqslant 0$, each $\omega^{(i)} \in \hat{\Omega}^{\mathsf{Crs}}_0 X$, each $a^{(i)}_{n_i+1}$ is a non-degenerate simplex in X_{n_i+1}, $n_i \geqslant 1$, and $\sum n_i = n$. The source of $s^{-1}a_2$ is $s^{-1}a_{01} \cdot s^{-1}a_{12}$ and the target is $s^{-1}a_{02}$.

The basepoint $\mathfrak{p} = \beta(x)$ of x is the product of the basepoints of all of the terms in x.

Then, we define the crossed complex of groupoids $P^{\mathsf{Crs}}X$ as a kind of twisted tensor product of $\hat{\Omega}^{\mathsf{Crs}}X$ and πX:

Definition:

Let X be a 0-reduced simplicial set. The path crossed complex $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ is the twisted tensor product of the crossed complex of groups πX, and the free crossed complex of groupoids $\hat{\Omega}^{\mathsf{Crs}}X$. Its object set is

$$P^{\mathsf{Crs}}_0 X = (\hat{\Omega}^{\mathsf{Crs}}_0 X \otimes_\phi \pi_0 X) = \{(\omega \otimes *)\}$$

where

$$\omega = (s^{-1}a^{(1)}_1)^{\epsilon_1} (s^{-1}a^{(2)}_1)^{\epsilon_2} \cdots (s^{-1}a^{(k)}_1)^{\epsilon_k} \ : \ k \geqslant 0, a^{(i)}_1 \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1$$

and in Dimension 1 the generators are $\{(\omega \otimes b_1), (\omega s^{-1} a_2 \omega' \otimes *)\}$

$$(\omega \otimes b_1) : (\omega \otimes *) \to (\omega s^{-1} b_1 \otimes *)$$

and

$$(\omega s^{-1} a_2 \omega' \otimes *) : (\omega s^{-1} a_{01} s^{-1} a_{12} \omega' \otimes *) \to (\omega s^{-1} a_{02} \omega' \otimes *).$$

In dimension $n \geqslant 2$, the general form of a generator is:

$$(x \otimes y)$$

where

$$x = \omega^{(1)} s^{-1} a^{(1)}_{n_1+1} \omega^{(2)} a^{(2)}_{n_2+1} \omega^{(2)} \cdots \omega^{(k)} s^{-1} a^{(k+1)}_{n_k+1} \omega^{(k+1)}$$

where $k \geqslant 0$, and $\omega^{(i)} \in \hat{\Omega}_0^{\mathsf{Crs}} X$, each $a_{n_i+1}^{(i)}$ is a non-degenerate simplex in X_{n_i+1}, $n_i \geqslant 1$, and $\sum n_i = n$, $y_j \in \pi_j X$. We finish this Chapter by defining a twisted boundary map ∂_n^P for each $n \geqslant 1$ and prove $(\partial_n^P)^2 = 0$.

We finish this thesis with Chapter 6, in this Chapter we prove that the pointed crossed complex of groupoids $P^{\mathsf{Crs}} X$ is contractable to the basepoint. This comes from defining a homotopy $\eta_n : P_n^{\mathsf{Crs}} X \to P_{n+1}^{\mathsf{Crs}} X$.

A crossed complex of groupoids is *pointed* if there is a specified object $* \in C_0$. If C is a pointed crossed complex of groupoids, then C is *contractible* to the basepoint $*$ if there is a family of functions $\eta_n : C_n \to C_{n+1}$ that define a *contracting homotopy*

$$h : * \simeq \mathrm{id}_C : \pi(\Delta[1]) \otimes C \to C$$

by:

$$h(0 \otimes c) = 0_*, \qquad h(1 \otimes c) = c$$

$$h(\sigma \otimes c) = \eta(c)$$

The main proposition in this chapter show condition that $h : \pi(\Delta[1]) \otimes C \to C$ is a well defined homomorphism of crossed complexes of groupoids, and commutes with the boundary ∂, holds if and only if η satisfies the properties $(1 - 5)$ of Proposition:

<u>Proposition</u>:

A family of functions $\eta_n : C_n \to C_{n+1}$, $(n \geqslant 0)$ defines a contracting homotopy via $h(\sigma \otimes c_n) = \eta_n(c_n)$ if and only if it satisfies

1. $\eta_0(c_0) \in C_1$ has source $*$ and target c_0,

2. $\eta_1(c_1) \in C_2$ has basepoint $*$ and boundary:

$$\partial_2 \eta_1(c_1) = -\eta_0(\mathrm{targ}(c_1)) + c_1 + \eta_0(\mathrm{src}(c_1)),$$

$$\begin{tikzcd}
* \arrow[rr, "\eta_0(x)"] \arrow[dr, "\eta_0(y)"'] & & x \arrow[dl, "c_1"] \\
& y &
\end{tikzcd}$$

Figure 2:

3. If $n \geqslant 2$ then, $\eta_n(c_n) \in C_{n+1}$ has basepoint $*$ and boundary:

$$\partial_{n+1}\eta_n(c_n) = c_n^{\eta_0(\mathfrak{p})} - \eta_{n-1}\partial_n(c_n),$$

4. For all $n \geqslant 1$,

$$\eta_n(c_n + c'_n) = \eta_n(c_n) + \eta_n(c'_n)$$

5. For all $n \geqslant 2$,

$$\eta_n(c_n^{c_1}) = \eta_n(c_n)$$

<u>**Definition**</u>:

For every 0-reduced simplicial set X, and for $m \neq 0$ we define the contracting homotopy η_n on the general form generators $(x \otimes b_m)$ of $P_n^{\mathsf{Crs}}X$ as:

$$\eta_n(x \otimes b_m) = 0_{(\varnothing \otimes *)}.$$

While, for the generators when $m = 0$ we define η_n as:

<u>**Definition**</u>:

For a string of r one-simplices ω, define a homotopy $\eta : P_0^{\mathsf{Crs}} \to P_1^{\mathsf{Crs}}$ by:

1. $\eta_0(\varnothing \otimes *) = 0_{(\varnothing \otimes *)} \in P_1^{\mathsf{Crs}}X$,

2. $\eta_0(\omega \otimes *) : * \to s^{-1}a_1^{(1)} \otimes * \to s^{-1}a_1^{(1)}s^{-1}a_1^{(2)} \otimes * \to \cdots \to \omega \otimes *$,

   can be defined inductively by:

$$\eta_0(s^{-1}a_1^{(1)}s^{-1}a_1^{(2)}\dots s^{-1}a_1^{(r)}\otimes *) = (s^{-1}a_1^{(1)}s^{-1}a_1^{(2)}\dots s^{-1}a_1^{(r-1)}\otimes a^{(r)}) \\ + \eta_0(s^{-1}a_1^{(2)}\dots s^{-1}a_1^{(r-1)}\otimes *)$$

and for dimension 1 we define the homotopy $\eta_1 : P_1^{\mathsf{Crs}}X \to P_2^{\mathsf{Crs}}X$ as:

$$\eta_1(\omega s^{-1}a_2 \otimes *) = (\omega \otimes a_2)^{\eta_0(\omega\otimes *)}$$

and

$$\eta_1(xs^{-1}b_1 \otimes *) = \eta_1(x \otimes *) - (x \otimes b_1)^{\eta_0(\mathrm{src}(x\otimes *))}$$

finally for dimension $n \geqslant 2$, we make the definition:

Definition:

For dimension $n \geqslant 2$ we can define $\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$ as:

1. $\eta_2(\omega s^{-1}a_3 \otimes *) = (\omega \otimes a_3)^{\eta_0(\omega\otimes *)}$,
2. $\eta_2(xs^{-1}b_1 \otimes *) = \eta_2(x \otimes *) + (x \otimes b_1)^{\eta_0(\mathrm{src}(x\otimes *))}$,
3. $\eta_n(xs^{-1}a_r \otimes *) = (-1)^{|x|}(x \otimes a_r)^{\eta_0(\mathrm{src}(x\otimes *))}$
4. $\eta_n(xs^{-1}b_1 \otimes *) = \eta_n(x \otimes *) + (-1)^n(x \otimes b_1)^{\eta_0(\mathrm{src}(x\otimes *))}$.

We then prove theorem:

Theorem:

For $n \geqslant 0$, η_n satisfies the properties in Proposition. Therefore η is a contracting homotopy and, for any 0-reduced simplicial set, $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ is contractible : a model for the path space on X

2 Preliminaries

Introduction

In this chapter, we recall some preliminaries about the categorical definition of simplicial sets and homotopy. Furthermore, we introduce the classical Eilenberg-Zilber theorem and the generalised version of such for crossed complexes.
The structure of the chapter is as follows. In the first section, we recall some background information on simplicial sets, their structure and some of their properties. In the second section, we recall the definitions of chain complexes, and the Cartesian product and tensor products of chain complexes, in addition to, investigating how the classical Eilenberg-Zilber theorem for chain complexes was generalised to twisted products. In the third section, we recall from [31] the generalisation of the Eilenberg-Zilber theorem to crossed complexes, after we introduce the definitions of groupoids, crossed modules, crossed complexes and the equivalences between the categories, of crossed complexes and ∞-groupoids.

2.1 Simplicial Objects and Homotopy

We begin by recalling some standard definitions.

Definition 2.1. *[18, Page 4], [32, Page 18] Let* $\mathbf{\Delta}$ *be the ordinal number category whose objects are finite ordinal numbers* $[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$ *for* $n \geqslant 0$*(in other words,* $[n]$ *is a totally ordered set with* $n+1$ *elements). A morphism*

$$\alpha : [n] \to [m]$$

is an order-preserving set function, or alternatively a functor. Among all of the functors $[m] \to [n]$ *appearing in* $\mathbf{\Delta}$*, there are special ones, namely*

$$d^i : [n-1] \to [n] \quad 0 \leq i \leq n \qquad \textit{(cofaces)}$$

$$s^j : [n+1] \to [n] \quad 0 \le j \le n \qquad \textit{(codegeneracies)}$$

where by definition,

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \xrightarrow{I} j \to \cdots \to n).$$

d^i and s^j satisfy the following relations:

$$d^j d^i = d^i d^{j-1} \qquad \textit{if } i < j$$

$$s^j s^i = s^{i+1} s^j \qquad \textit{if } i \leqslant j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i \leqslant j \\ I & i = j \quad \textit{or} \quad i = j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$$

The maps d^i, s^j and these relations can be viewed as a set of generators and relations of $\mathbf{\Delta}$.

Proposition 2.2. *[15, Page 4] Every morphism $\alpha : [n] \to [m]$ can be uniquely decomposed as $\alpha = \delta\sigma$, where $\delta : [p] \to [m]$ is injective and $\sigma : [n] \to [p]$ is surjective. Moreover, if $d^i : [n-1] \to [n]$ is the injection which skips the value $i \in [n]$ and $s^j : [n+1] \to [n]$ is the surjection covering $j \in [n]$ twice, then $\delta = d^{i_r} \dots d^{i_1}$ and $\sigma = s^{j_s} \dots s^{j_1}$, where $m \geqslant i_r > \cdots > i_1 \geqslant 0$ and $0 \leqslant j_s < \cdots < j_1 < n$ and $m = n - s + r$. The decomposition is unique, with the i's in $[m]$ being the values not taken by α, and the j's being the elements of $[m]$ such that $\alpha(j) = \alpha(j+1)$.*

The relationship between the d^i and s^j in $\mathbf{\Delta}$ for $n \geqslant 2$ can be expressed by the diagram below [24, Page 3]:

$$
\begin{tikzcd}
{[n]} \arrow[r, shift left, "s^j"] \arrow[d, shift right, "s^i"'] & {[n-1]} \arrow[l, shift left, "d^j"] \arrow[d, shift left, "s^i"] \\
{[n-1]} \arrow[u, shift right, "d^i"'] \arrow[r, shift right, "s^{j-1}"'] & {[n-2]} \arrow[l, shift right, "d^{j-1}"'] \arrow[u, shift left, "d^i"]
\end{tikzcd}
$$

Figure 3:

Definition 2.3. *[18] A simplicial set is a contravariant functor $X : \mathbf{\Delta^{op}} \to \mathbf{Set}$, where* **set** *is the category of sets and $\mathbf{\Delta}$ is the simplex category. Denote $X([n]) = X_n, \quad n \geqslant 0$, the sets of n-simplices, together with maps*

$$d_i = X(d^i) : X_n \to X_{n-1} \qquad 0 \leqslant i \leqslant n \qquad \textit{(faces)}$$

and

$$s_j = X(s^j) : X_n \to X_{n+1} \qquad 0 \leqslant j \leqslant n \qquad \textit{(degeneracies)}$$

satisfying the simplicial identities dual to the cosimplicial identities given above.

The elements of X_0 are called the vertices of the simplicial set. A simplex x is degenerate if x is the image of some s_j.

Definition 2.4. ***Geometric realisation of any simplicial set*** *X [18, Page 7]*
The geometric realisation of any simplicial set X is a functor $|\ .\ | : \mathbf{S} \to \mathbf{Top}$ from the category $\mathbf{S}$ of simplicial sets to that $\mathbf{Top}$, of topological spaces, defined by

$$|X| = \bigsqcup_{n \geqslant 0} |\mathbf{\Delta[n]}| \times X_n \big/ \sim$$

where $|\mathbf{\Delta[n]}|$ is the realisation of the n-simplex given in the following example.

Example 2.5. *[18, Example (1.1)]*

There is a standard covariant functor

$$\mathbf{\Delta} \to \mathbf{Top}$$

$$\{0 \to 1 \to \cdots \to n\} \mapsto |\mathbf{\Delta[n]}|$$

where $|\mathbf{\Delta[n]}|$ *is the standard* n*-simplex in* ***Top*** *given by*

$$|\mathbf{\Delta[n]}| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, \qquad t_i \geqslant 0\}$$

Given

$$f : [n] \to [m],$$

the functor produces

$$([n] \xrightarrow{f} [m]) \mapsto (|\mathbf{\Delta[n]}| \xrightarrow{f_*} |\mathbf{\Delta[m]}|)$$

where f_* *is defined by*

$$f_*(t_0, \ldots, t_n) = f_*(t_0 v_0 + \cdots + t_n v_n)$$

$$= t_0 v_{f(0)} + t_1 v_{f(1)} + \cdots + t_n v_{f(n)}$$

$$= (\sum_{f(i)=0} t_i) v_0 + \cdots + (\sum_{f(i)=m} t_i) v_m$$

and we have used the notation

$$v_0 = (1, 0, \ldots, 0), v_1 = (0, 1, 0, \ldots, 0), \ldots, v_n = (0, 0, 0, \ldots, 0, 1).$$

This is the i^{th} *vertex of* $|\mathbf{\Delta[n]}|$, *as sent to the* $f(i)^{th}$ *vertex of* $|\mathbf{\Delta[m]}|$, *and the barycentric coordinates are mapped linearly.*

We see that the coface map d^i_* *sends* $|\mathbf{\Delta[n]}|$ *to* $|\mathbf{\Delta[n+1]}|$ *and that the codegeneracy map* s^j_* *sends* $|\mathbf{\Delta[n]}|$ *to* $|\mathbf{\Delta[n-1]}|$ *by collapsing together vertices* j *and* $j+1$.

A face of $[v_0, \ldots, v_n]$ *is defined as the simplex obtained by deleting one of the* v_i, *which we* *denote* $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$. *The union of all faces is the boundary of the simplex, and its complement is called the interior, or the open simplex.*

Definition 2.6. *[18, Page 6] As a simplicial complex, the* k^{th} *horn* $|\Lambda^k[n]|$ *on the* n*-simplex* $|\mathbf{\Delta[n]}|$ *is the sub-complex of* $|\mathbf{\Delta[n]}|$ *obtained by removing the interior of* $|\mathbf{\Delta[n]}|$ *and the interior of the face* $d_k\mathbf{\Delta[n]}$. *Let* $\Lambda^k[n]$ *refer to the associated simplicial set. This simplicial set consists of simplices* $[i_0, \ldots, i_m]$ *with* $0 \leqslant i_0 \leqslant, \ldots, \leqslant i_m \leqslant n$*, such that:*

(i) not all numbers $\{0, \ldots, n\}$ *are represented;*

(ii) we never have all numbers except k represented (this would be the missing the $(n-1)$*-face or degeneracy).*

That is

$$\Lambda^k[n] = \bigcup_{i \neq k} d^i_* \mathbf{\Delta[n-1]}$$

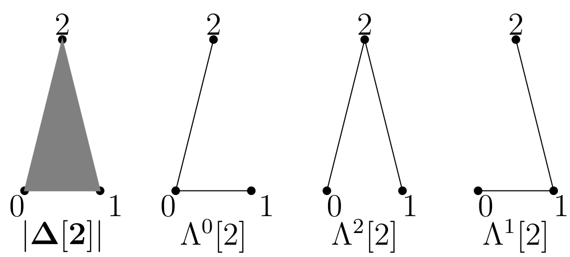


Figure 4: The three horns on $|\mathbf{\Delta[2]}|$

Definition 2.7. *[18, Page 10] The simplicial object* X *satisfies the extension condition, or Kan condition, if any morphism of simplicial sets* $\Lambda^k[n] \to X$ *can be extended to a simplicial morphism* $\mathbf{\Delta[n]} \to \mathbf{X}$. *Such an* X *is referred to as being fibrant. A map* $f : X \to Y$ *is also called a fibration if , when we have a horn in* X*, and a simplex in* Y *extending the image of the horn then we have a simplex in* X *extending the horn, as shown in the diagram below:*

$$
\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f \text{ is a fibration}"] \\
\mathbf{\Delta}[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

Figure 5:

Example 2.8. $\mathbf{\Delta}[\mathbf{0}]$ *does satisfy the Kan condition.*

2.2 The Classical Eilenberg-Zilber Theorem

The classical Eilenberg-Zilber theorem [16, 33] gives a strong deformation retraction of the chain complex of a Cartesian product of simplicial sets onto the corresponding tensor product of chain complexes.

Definition 2.9. *[15, Page 113] Let X and Y be simplicial sets, that is, X and Y are functors $\mathbf{\Delta^{op}} \to \mathbf{Set}$. The Cartesian product $X \times Y$ is the functor: $\mathbf{\Delta^{op}} \to \mathbf{Set}$ satisfying:*

1. $(X \times Y)_n = X_n \times Y_n = \{(x, y)|x \in X_n, y \in Y_n\}$,
2. *if* $(x, y) \in (X \times Y)_n$, *then* $d_i(x, y) = (d_i x, d_i y)$,
3. *if* $(x, y) \in (X \times Y)_n$, *then* $s_i(x, y) = (s_i x, s_i y)$.

Example 2.10. *[29, Page 45] We consider the two simplicial sets $X = \mathbf{\Delta}[2], Y = \mathbf{\Delta}[1]$, and their Cartesian product $X \times Y = \mathbf{\Delta}[2] \times \mathbf{\Delta}[1]$. Then $(\mathbf{\Delta}[1])_0$ is the set $\{0, 1\}$ of 0-simplices of $\mathbf{\Delta}[1]$, $(\mathbf{\Delta}[1])_1$ is the set $\{(00), (01), (11)\}$ of 1-simplices and so on. The Cartesian product of $(\mathbf{\Delta}[2])_1$ and $(\mathbf{\Delta}[1])_1$ will be*

$$
\begin{aligned}
(\mathbf{\Delta}[2])_1 \times (\mathbf{\Delta}[1])_1 \quad &= \quad (\mathbf{\Delta}[2] \times \mathbf{\Delta}[1])_1 = \\
&\big\{(00, 00), (01, 00), (02, 00), (11, 00), (12, 00), (22, 00),
\end{aligned}
$$

$$(00,01),(01,01),(02,01),(11,01),(12,01),(22,01),$$
$$(00,11),(01,11),(02,11),(11,11),(12,11),(22,11)\big\}.$$

Twelve of these are non-degenerate 1-simplices of $X \times Y$.

The Cartesian product $X \times Y$ contains three non-degenerate 3-simplices

$$(0012, 0111),\ (0112, 0011),\ (0122, 0001).$$

as shown in Figure 6.

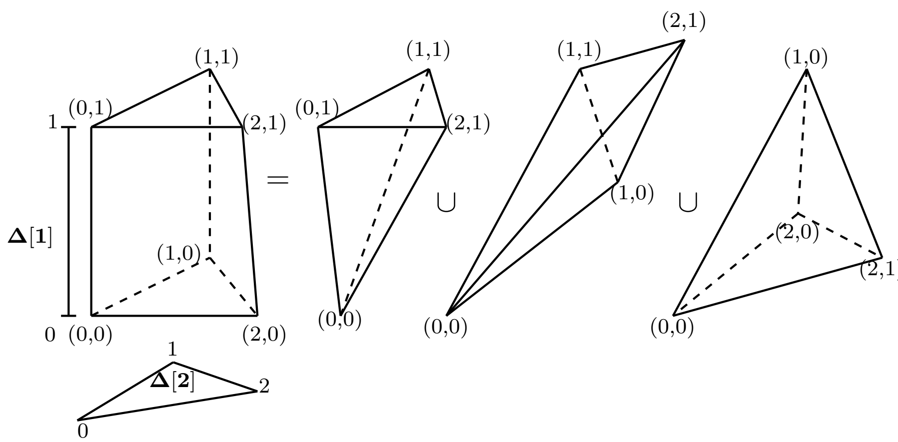


Figure 6: $\boldsymbol{\Delta}[\mathbf{1}] \times \boldsymbol{\Delta}[\mathbf{2}]$

2.2.1 Chain complexes

Definition 2.11. *[33, Definition(1.1.1)] A chain complex C is a sequence of abelian groups and homomorphisms $\delta : C_n \to C_{n-1}$ satisfying the condition that $\delta^2 : C_n \to C_{n-2}$ is zero. The kernel of δ_n is called the group of cycles of C_n, and denoted by Z_n. The image of δ_{n+1} is called the group of boundaries of C_n, denoted by B_n. From the rule that $\delta^2 = 0$, we have;*

$$0 \subset B_n \subset Z_n \subset C_n$$

Definition 2.12. *[33] For any chain complex (C_n, δ_n) the n^{th} Homology groups $\mathbb{H}_n$ are* *the quotient groups*

$$\mathbb{H}_n = Ker\ \delta_n \diagup Img\ \delta_{n+1}.$$

The elements of the Homology groups are cosets of $Img\ \delta_{n+1}$, called Homology classes.

Definition 2.13. *[29, 33] Let C and D be chain complexes such that*

$$C : \cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0,$$

$$D : \cdots \to D_n \xrightarrow{\delta_n} D_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} D_2 \xrightarrow{\delta_2} D_1 \xrightarrow{\delta_1} D_0.$$

Then, a chain complex morphism $f : C \to D$ is a sequence of morphisms of abelian groups $\{f_n\}$ where the $f_n : C_n \to D_n$ are compatible with the differentials, that is $f_{n-1}\delta_n = \delta_n f_n$ for every n.

Definition 2.14. *[29] Two chain complex morphisms $f, g : C_n \to D_n$ are homotopic if there exists some homotopy $H = \{h_n : C_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ satisfying*

$$g - f = \delta h + h\delta$$

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & C_{n+1} \arrow[r, "\delta"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] & C_n \arrow[r, "\delta"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, dashed, "h"'] & C_{n-1} \arrow[r, "\delta"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, dashed, "h"'] & \cdots \\
\cdots \arrow[r, "\delta"] & D_{n+1} \arrow[r, "\delta"] & D_n \arrow[r, "\delta"] & D_{n-1} \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

Figure 7:

Example 2.15. *[18, Page 5] Let X be any simplicial set. We can construct a chain complex $(C_n(X), \delta)$ as a sequence of a free abelian groups $\mathbb{Z}X_n$ on X_n, and homomorphisms*

$$\delta_n = \sum_i (-1)^i d_i \ : \mathbb{Z}X_n \longrightarrow \mathbb{Z}X_{n-1}$$

2.2.2 Tensor Products of Chain Complexes

Let A and B denote chain complexes. The tensor product $C = (A \otimes B)$ is also a chain complex such that :

$$C_n = \bigoplus_{p+q=n} (A_p \otimes B_q)$$

with the boundary homomorphism $\partial_n : C_n \to C_{n-1}$ defined as:

$$\partial_n(a_p \otimes b_q) = (\partial_p a_p) \otimes b_q + (-1)^p a_p \otimes (\partial_q b_q)$$

and this satisfies that $\partial^2 = 0$ [25].

Example 2.16. *In this example we will define the chain complexes for two simplices and then write the tensor product of these two chains.*
For $\mathbf{\Delta[1]}$*, the chain complex* $C(\mathbf{\Delta[1]})$ *is* $C_0 = \mathbb{Z}^2$*, which generated by two vertices* $\{0, 1\}$*,* $C_1 = \mathbb{Z}^3$ *which generated by three edges* $\{00, 01, 11\}$*,* $C_2 = \mathbb{Z}^4$ *which are generated by four triangles* $\{000, 001, 011, 111\}$*, and so on. Hence*

$$C\mathbf{\Delta[1]} : \cdots \to \mathbb{Z}^{\mathbf{4}} \to \mathbb{Z}^{\mathbf{3}} \to \mathbb{Z}^{\mathbf{2}}.$$

In the same manner for $\mathbf{\Delta[2]}$*, we have*

$$C(\mathbf{\Delta[2]}) : \cdots \to \mathbb{Z}^{\mathbf{10}} \to \mathbb{Z}^{\mathbf{6}} \to \mathbb{Z}^{\mathbf{3}}.$$

The normalized chain complex $C_N(\mathbf{\Delta[n]})$ *is for* $n \geqslant 0$ *the subchain complex of* $C_n(\mathbf{\Delta[n]})$ *generated by non-degenerate elements*

$$C_N\mathbf{\Delta[1]} : \cdots \to \mathbf{0} \to \mathbf{0} \to \mathbb{Z} \to \mathbb{Z}^{\mathbf{2}}$$

and

$$C_N\mathbf{\Delta[2]} : \cdots \to \mathbf{0} \to \mathbb{Z} \to \mathbb{Z}^{\mathbf{3}} \to \mathbb{Z}^{\mathbf{3}}$$

so

$$C_N(\mathbf{\Delta[1]}) \otimes \mathbf{C_N}(\mathbf{\Delta[2]}) : \cdots \to \mathbf{0} \to \mathbb{Z} \to \mathbb{Z}^{\mathbf{5}} \to \mathbb{Z}^{\mathbf{9}} \to \mathbb{Z}^{\mathbf{6}}$$

and

$$C_N(\mathbf{\Delta[1]} \times \mathbf{\Delta[2]}) : \cdots \to \mathbf{0} \to \mathbb{Z}^{\mathbf{3}} \to \mathbb{Z}^{\mathbf{8}} \to \mathbb{Z}^{\mathbf{12}} \to \mathbb{Z}^{\mathbf{6}}$$

Example 2.17.

$$C_n(\mathbf{\Delta}[1] \times \mathbf{\Delta}[1]) \underset{\varphi}{\overset{\phi}{\rightleftarrows}} \bigoplus_{p+q=n} C_p(\mathbf{\Delta}[1]) \otimes C_q(\mathbf{\Delta}[1])$$

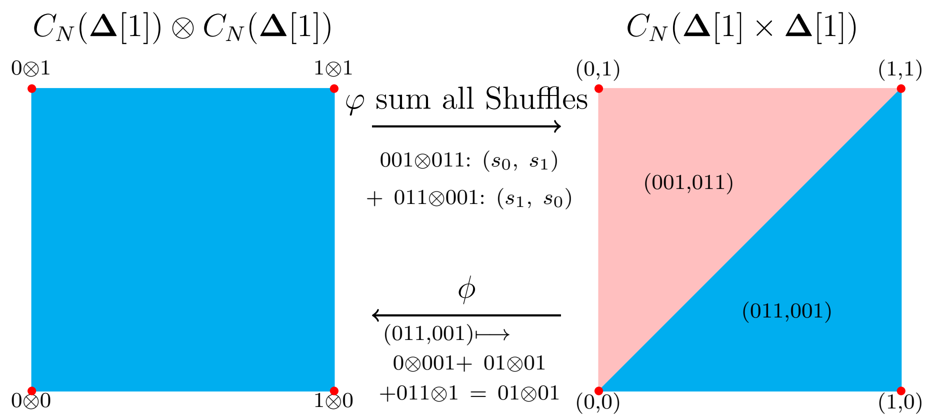


Figure 8:

Theorem 2.18. *(The classical Eilenberg- Zilber theorem) [16, 17]*
For any two simplicial sets X and Y there exists a strong deformation retract of chain complexes:

$$\begin{tikzcd} C(X \times Y) \arrow[loop left, "h"] \arrow[r, shift left, "\phi"] & C(X) \otimes C(Y) \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

where φ is the Eilenberg-Zilber map which sends generators of the tensor product of two chain complexes to a chain of products of two simplices as indexed by shuffles. This map *is a natural chain homotopy inverse of ϕ, where ϕ is the natural Alexander–Whitney map for the normalised free-chain complex on a simplicial set, that sends a generator (x, y) to $\sum d_{i+1} \dots d_n x \otimes d_0^i y$.*

$$\phi\varphi \simeq identity, \quad \varphi\phi \simeq identity.$$

For all vertices $v_i \in X$, $v'_i \in Y$,

$$\phi(v_i, v'_i) = v_i \otimes v'_i, \quad \varphi(v_i \otimes v'_i) = (v_i, v'_i).$$

2.3 The Twisted Eilenberg- Zilber Theorem

In 1958, E. H. Brown in his paper [4], generalised the classical Eilenberg-Zilber theorem to fibre spaces by using the twisted version. The generalisation is as follows: for every fibering $\rho : X \to B$ with fibre $A = \rho^{-1}(b_0)$, there is a twisted tensor product of the chains on the base space B and the chains of the fibre space A, with differential ∂_Φ, which is chain equivalent to the chain complex on X. The differential is

$$\partial_\Phi = \partial^I + \partial^{II}$$

where ∂^I, is the differential of the classical tensor product theorem and ∂^{II} is

$$\partial^{II} = (-1)^n (b_n \otimes a_m) \cap \Phi.$$

First, we recall from [26] a number of basic concepts necessary to understand E. H. Brown's generalisation.

First, let Λ be a commutative ring, with a unit 1, and let $\mathcal{A}$ be differential graded augmented Λ-module (DGA): a module graded by submodules $\mathcal{A}_s$, $s \geqslant 0$, with a homomorphism $\delta : \mathcal{A} \to \mathcal{A}$ (the differential) such that $\delta^2 = 0$, and an augmentation $\varepsilon : \mathcal{A} \to \Lambda$ which is Λ-linear epimorphism satisfying that $\varepsilon\delta = 0$ and $\varepsilon(\mathcal{A}_s) = 0$, for $s > 0$.

If $\mathcal{A}, \mathcal{B}$ are DGA Λ-modules, then $\mathcal{A} \otimes \mathcal{B}$ is the DGA Λ-module with the grading $(A \otimes B)_q = \bigoplus_{r+s=q}(A_r \otimes B_s)$, and the differential

$$\delta(a \otimes b) = (\delta a) \otimes b + (-1)^q a \otimes \delta b, \qquad a \in \mathcal{A}_q, b \in \mathcal{B}.$$

A DGA module $\mathcal{A}$ will be called a chain algebra if it has an associative product

$$\psi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

of degree zero. A DGA coalgebra is a DGA Λ-module $\mathcal{K}$ with a DGA associative (coproduct) homomorphism

$$\bigtriangledown : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}.$$

A DGA algebra $\mathcal{A}$ is connected if $\mathcal{A}_0 = \Lambda$ and it is n-reduced if $\mathcal{A}_i = 0,\ 1 \leqslant i \leqslant n$.

Definition 2.19. *[21, 26] Let $\mathcal{K}$ be a DGA coalgebra with differential ∂ with coproduct $\bigtriangledown : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$, let $\mathcal{L}, \mathcal{M}$ and $\mathcal{N}$ be Λ-modules and let $\mu : \mathcal{L} \times \mathcal{M} \to \mathcal{N}$ be a homomorphism. Let $C^n(\mathcal{K}, \mathcal{L}) = Hom(\mathcal{K}_n, \mathcal{L})$, $C^*(\mathcal{K}, \mathcal{L}) = \sum C^n(\mathcal{K}, \mathcal{L})$ and define $d : C^n(\mathcal{K}, \mathcal{L}) \to C^{n+1}(\mathcal{K}, \mathcal{L})$ by $dU = U\partial$. Let $U \in C^*(\mathcal{K}, \mathcal{L})$, $V \in C^*(\mathcal{K}, \mathcal{M})$ and $c \in \mathcal{K} \otimes M$.*
The cup product $U \smile V \in C^*(\mathcal{K}, \mathcal{N})$ is the composite

$$\mathcal{K} \xrightarrow{\bigtriangledown} \mathcal{K} \otimes \mathcal{K} \xrightarrow{U \otimes V} \mathcal{L} \otimes \mathcal{M} \xrightarrow{\mu} \mathcal{N}$$

and the cap product $c \frown U \in \mathcal{K} \otimes \mathcal{N}$ is the composite

$$\mathcal{K} \otimes \mathcal{M} \xrightarrow{\bigtriangledown \otimes I} \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{M} \xrightarrow{I \otimes U \otimes I} \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{M} \xrightarrow{I \otimes \mu} \mathcal{K} \otimes \mathcal{N}$$

Theorem 2.20. *[4] Let B be a pathwise connected space. For each fibering $\rho : X \to B$ with fibre $A = \rho^{-1}(\mathfrak{b}_\circ)$, there is*

- *a cochain $\Phi = \sum \Phi_q$*

- *a differential ∂_Φ on $C(B) \otimes_\Phi C(A)$ defined as:*

$$\partial_\Phi(b_n \otimes a_m) = (\partial b_n) \otimes a_m + (-1)^n (b_n \otimes \partial a_m + (b_n \otimes a_m) \frown \Phi)$$

- *a chain equivalence map $\varphi : C(B) \otimes_\Phi C(A) \to C(X)$*

Remark 2.21. *[4] The twisting cochain $\Phi = \sum \Phi_q$ used in E. H. Brown's theorem is a cochain which assigns to each q-chain of B a $(q-1)$-chain of the space of loops ΩB by twisting all loops $\alpha \in B$ based at b_0 to a loop α' in the space of loops ΩB, whose ending is at $x \in A$ with an initial point αx. Hence αx is a continuous action of ΩB on the fibre A and satisfies the identity:*

$$\partial \Phi_q = \Phi_{q-1}\partial - \sum_{i=1}^{q-1} (-1)^i \Phi_i \smile \Phi_{q-i}$$

such that $\partial_\Phi^2 = 0$.

The proof of theorem above, and indeed further details, can be found in [4].

Our first aim in this thesis was to try and generalise E. H. Brown's theorem from chain complexes to crossed complexes. The classical, non-twisted, Eilenberg–Zilber theorem was proved for crossed complexes by A. Tonks. We will end this chapter by presenting this result.

2.4 The Eilenberg–Zilber Theorem for crossed complexes

In this section we present Tonks' generalisation of the classical Eilenberg-Zilber theorem to a slightly non-abelian setting. In [31, 32], A. Tonks gave a natural strong deformation retraction from the fundamental homotopy crossed complex of a product of simplicial sets onto the tensor product of the corresponding crossed complexes.
We start this section by recalling some definitions.

2.4.1 Crossed modules and crossed complexes of groups

The notion of crossed complex, and of crossed module, is due to J.H.C. Whitehead, who called them group systems. They have been considered by many authors, especially H-J Baues.

Definition 2.22. *A crossed complex of groups C is a sequence of groups C_n, $n \geqslant 1$, and group homomorphisms ∂_n, called boundary maps,*

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$$

satisfying the following:

1. *$\partial_{n-1}\partial_n : C_n \to C_{n-2}$ is the trivial homomorphism for each $n \geqslant 3$*
2. *C_1 acts on each C_n for each $n \geqslant 2$ (and on itself by conjugation)*
3. *$\partial_n : C_n \to C_{n-1}$ preserves the group action for each $n \geqslant 2$*
4. *C_2 is not necessarily an abelian group, but if $a, b \in C_2$ then $a^{-1}ba = b^{\partial_2 a}$*
5. *For $n \geqslant 3$, $\partial_2 C_2$ acts trivially on C_n, and C_n is an abelian group.*

The map $\partial_2 : C_2 \to C_1$, satisfying (2,3,4) is called a crossed module of groups.

2.4.2 Actions of groupoids and crossed modules of groupoids

Groupoids are groups with many objects, or with many identities. Alternatively, they are categories in which every morphism is an isomorphism. They were first introduced by Brandt in 1926 [6]. We introduce some notation:

Definition 2.23. *A groupoid $\mathbb{G}$ consists of a set of objects $Ob(\mathbb{G}) = G_0$ and a set of morphisms or arrows $Arr(\mathbb{G}) = G_1$ together with*

1. *source and target maps* src, targ : $G_1 \to G_0$. *If an arrow* a *has source* x *and target* y *then we write* $a : x \to y$ *or* $x \xrightarrow{a} y$. *For* $x, y \in G_0$ *we write* $\mathbb{G}(x, y) = \{a : x \to y\}$, *the hom-set of all arrows from* x *to* y.

2. *a unit map* $\mathrm{id} : G_0 \to G_1$, *and we write* $\mathrm{id}(x) = \mathrm{id}_x : x \to x$

3. *a composition map* $\circ$ *which associates to every composable pair of arrows* $a : x \to y$ *and* $b : y \to z$ *the composite map* $b \circ a : x \to z$. *This composition is unital,* $\mathrm{id}_y \circ a = a \circ \mathrm{id}_x = a$, *and associative,* $(c \circ b) \circ a = c \circ (b \circ a) : x \to w$ *if* $c : z \to w$.

4. *an inverse map* $(-)^{-1} : G_1 \to G_1$ *such that if* $a : x \to y$ *then* $a^{-1} : y \to x$, $a^{-1} \circ a = \mathrm{id}_x$ *and* $a \circ a^{-1} = \mathrm{id}_y$.

A group is just a groupoid in which the object set is $\{*\}$. The definitions of group action and of crossed module of groups are extended to groupoids as follows.

Definition 2.24. *[31, 32] Suppose* $\mathbb{G}$ *and* $\mathbb{H}$ *are two groupoids over the same object set, and* $\mathbb{H}$ *is totally disconnected, that is,* $\mathbb{H}(x, y) = \varnothing$ *whenever* $x \neq y$. *An action of* $\mathbb{G}$ *on* $\mathbb{H}$ *is a collection of functions*

$$Arr(\mathbb{G}) \times Arr(\mathbb{H}) \xrightarrow{\alpha} Arr(\mathbb{H})$$

$$(g, h) \to h^g$$

where satisfies :

1. h^g *is defined if and only if* $\mathrm{src}(h) = \mathrm{targ}(g)$, *and then* $\mathrm{targ}(h^g) = \mathrm{src}(g)$,

2. $(h_2 \circ h_1)^g = h_2^g \circ h_1^g$ *for all* $h_1, h_2 : y \to y$ *in* $\mathbb{H}$ *and* $g : x \to y$ *in* $\mathbb{G}$.

3. $h^{g_2 \circ g_1} = (h^{g_2})^{g_1}$ *for all* $h : x \to x$ *in* $\mathbb{H}$ *and* $g_1 : z \to y, g_2 : y \to x$ *in* $\mathbb{G}$.

4. $h^{\mathrm{id}_y} = h$ *for all* $h : y \to y$ *in* $\mathbb{H}$.

5. $\mathrm{id}_y^g = id_x$ *for all* $g : x \to y$ *in* $\mathbb{G}$.

A group action is just a groupoid action in which the object set is $\{*\}$.

Definition 2.25. *[5] A crossed module of groupoids is a morphism of groupoids* $\partial : M \to P$ *over a fixed object set* O *together with an action* $(m, p) \mapsto m^p$ *of the groupoid* P *on the groupoid* M *satisfying the two axioms:*

1. $\partial(m^p) = p^{-1}(\partial m)p$

2. $m^{\partial n} = n^{-1}mn$

for all $m, n \in M$, $p \in P$.

Simple consequences of the axioms for a crossed module of groups $\partial : M \to P$ are:

- $\mathrm{Im}\,\partial$ is normal in P, because $\partial(m)p = p\partial(m^p)$.

- $\ker \partial$ is a central subgroup of M, because $mn = nm^{\partial n} = nm$ if $n \in \ker \partial$, and in particular $\ker \partial$ is an abelian group.

- $\ker \partial$ is acted on trivially by $\mathrm{Im}\,\partial$, because if $n \in \ker \partial$ then $n^{\partial m} = m^{-1}nm = n$.

- $\ker \partial$ inherits an action of $M/\,\mathrm{Im}\,\partial$.

- M is abelian if ∂ is the trivial homomorphism.

The cokernel $M/\,\mathrm{Im}\,\partial$ is usually called π_1 of the crossed module, and the kernel $\ker \partial$, which is a π_1-module, is usually called π_2 of the crossed module,

$$\pi_2 \to M \to P \to \pi_1.$$

All of these properties hold for crossed modules of groupoids, but they are slightly harder to state.

2.4.3 The equivalence of 2-groupoids and crossed modules

The material in this section comes from [9, 22, 28]. Crossed modules are algebraic models for connected homotopy 2-types, so are essentially the same thing as 2-groupoids.

($\Rightarrow$) Given a 2-groupoid structure $\mathbb{G} = (G_0, G_1, G_2)$, we define a crossed module $\lambda\mathbb{G}$ by assuming the object set $O = G_0$, and the set of arrows $P = G_1$, and define the source and target maps $s,\ t\ : P \to O \quad by \quad s_0,\ t_0 : G_1 \to G_0$ respectively.

Now let

$$M(x) = \{\alpha \in G_2\ |t_1\alpha = e_x \quad for \quad each \quad x \in G_0 = O\}$$

For each $\alpha \in M(x)$ we have $s_0(\alpha) = x$ since $s_0(\alpha) = s_0t_1(\alpha) = s_0e_x = x$. Thus we can characterise $M(x)$ as

$$M(x) = \{\alpha \in \mathbb{G} \mid s_0(\alpha) = t_0(\alpha) = x \ \ and \ \ t_1(\alpha) = e_x\}$$

Let M be the family $\{M(x)\}_{x\in O}$ and for $\alpha \in M(x)$, define $\partial(\alpha) = s_1(\alpha)$. Then $\partial(\alpha) \in P(x, x)$, and

$$\lambda\mathbb{G} = \ \left(M \xrightarrow{\partial} P \underset{t}{\overset{s}{\rightrightarrows}} O\right)$$

is a crossed module.

($\Leftarrow$) Now our aim is to show that $\mathbb{G}$ can be recovered from the crossed module (M, P, ∂). We have constructed, for any 2-groupoid $\mathbb{G}$, a crossed module $\lambda\mathbb{G}$, and this construction clearly gives a functor we now construct a functor in the. opposite direction

Proposition 2.26. *[22, Proposition (2.2)] Let (M, P, O) be a crossed module over groupoids. This induces a 2-groupoid $\mathbb{G}$ with $(G_1, G_0) = (P, O)$ and*

$$G_2 = P \rtimes M = \{(g, \alpha) \mid g \in G_1 \ and \ \alpha \in G(t(g))\ \}$$

The composition is given by $(g,\alpha)(g',\alpha') = (gg', \alpha^{g}\alpha')$, and the source and target maps are given by $s(g,\alpha) = g$ and $t(g,\alpha) = g\partial(\alpha)$. The source map s, and the target map t, and the inclusion map $P \hookrightarrow P \rtimes M, g \Rightarrow (g, 1)$, giving the identity map.

Theorem 2.27. *[22, Theorem (2.3)] [28] The functors*

$$\lambda : 2\text{-}\mathbf{Grpd} \to \mathbf{CrsMod}$$

and

$$\beta : \mathbf{CrsMod} \to 2\text{-}\mathbf{Grpd}$$

defined above are inverse equivalences.

Proof. See [22] □

2.4.4 Crossed complexes of groupoids

In this section, we will review a number of definitions and properties of crossed complexes of groupoids, including the fundamental crossed complex πX of a simplicial set X, and then we will introduce the Eilenberg–Zilber theorem for the fundamental crossed complex functor π, which was proved by A. Tonks in [31]. The concept of a crossed complex of groupoids was first introduced by Brown and Higgins, generalising the definition of crossed complex of groups to the case of a set of base points.

Definition 2.28. *A crossed complex of groupoids C is a sequence of groupoids C_n over a fixed object set C_0, which are totally disconnected groupoids for $n \geqslant 2$ and are C_0-indexed families of abelian groups for $n \geqslant 3$, equipped with C_1-actions and C_1-equivariant boundary maps between them*

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \underset{\text{targ}}{\overset{\text{src}}{\rightrightarrows}} C_0$$

The following axioms must be satisfied

- *each* $\partial_n : C_n \to C_{n-1}$ *is the identity function on the object sets, that is,*

$$\partial_n : C_n(x, x) \to C_{n-1}(x, x)$$

- *each* $\partial_{n-1}\partial_n : C_n \longrightarrow C_{n-2}$ *is trivial,*

- $\partial_2 : C_2 \longrightarrow C_1$ *is a crossed module of groupoids, and* $\partial_2 C_2$ *acts trivially on* C_n *if* $n \geqslant 3$.

We will usually write just $C_n(x)$ instead of $C_n(x, x)$ if $n \geqslant 2$, and we call $\beta(a) = \mathrm{src}(a)$ the basepoint $\mathfrak{p}$ of $a \in C_n$ for any $n \geqslant 1$.

A crossed complex of groups is just a crossed complex of groupoids C in which $C_0 = \{*\}$.

Remark 2.29. *Another simple consequence of the crossed complex axioms is that the image* $\partial_3 C_3$ *is central in* C_2, *because*

$$(\partial_3 c_3)^{-1}\, c_2\, \partial_3 c_3 \quad = \quad c_2^{\partial_2 \partial_3 c_3} \quad = \quad c_2.$$

Definition 2.30. *[13] A morphism of crossed complexes*

$$f : C \to D$$

is a family of morphisms of groupoids

$$f_n : C_n \to D_n \qquad\qquad n \geqslant 1$$

all inducing the same map of vertices $f_0 : C_0 \to D_0$ *and compatible with the boundary maps*

$$\partial_n^C : C_n \to C_{n-1}\ , \qquad \partial_n^D : D_n \to D_{n-1}$$

and compatible with the actions of C_1, D_1 *on* C_n, D_n.

The fundamental groupoid $\pi_1 C$ of a crossed complex C is the cokernel $C_1/\operatorname{Im}\partial_2$, that is, the quotient of the groupoid C_1 by $\partial_2(C_2)$. This groupoid acts on each C_n, for $n \geqslant 3$, and also on $\pi_2 C = \ker\partial_2$, because elements in the image of ∂_2 act trivially. This means that for each basepoint $x \in C_0$ we have a chain complex of $\pi_1 C$-modules

$$\cdots \to C_n(x) \to C_{n-1}(x) \to \cdots \to C_3(x) \to (\pi_2 C)(x).$$

Remark 2.31. *From now on we will use* additive notation *instead of multiplicative notation for the composition law in crossed complexes, even in C_1 and in C_2 which may be non-abelian. For example, the two crossed module axioms in Definition 2.25 will be written*

$$\partial(m^p) = -p + \partial m + p, \qquad m^{\partial n} = -n + m + n.$$

Remark 2.32. *[12] Theorem 2.27, may be extended to an equivalence of categories between crossed complexes and ∞-groupoids.*

2.4.5 Tensor product of crossed complexes

Brown and Higgins proved that the category of crossed complexes is equivalent to the category of strict (globular) ∞-groupoids, and also to the category of cubical ω-groupoids [12]. The category of cubical ω-groupoids has a tensor product with very good properties. It may be defined using the fact that the product of an r-dimensional cube with an s-dimensional cube is an $(r+s)$-dimensional cube.

Using the fact that the categories are equivalent, Brown and Higgins proved that the category of crossed complexes also has a tensor product. This tensor product includes non-abelian constructions related to the homotopy-addition lemma.

We will next give two explicit definitions of the tensor product of crossed complexes. The first one will be for the tensor product of crossed complexes of groups, and the more general one will be for the tensor product of crossed complexes of groupoids. These definitions can be found in [14], [20, P.2], [31, Definition (1.4)] and [11, Proposition (3-10)], for example.

Definition 2.33. *Let A, B be crossed complexes of groups. The tensor product $A \otimes B$ is the crossed complex of groups which has a presentation in terms of generators and relations as follows:*

Generators are given by symbols $a_r \otimes b_s$ in $(A \otimes B)_{r+s}$ for all elements $a_r \in A_r$ and $b_s \in B_s$, where $r, s \geqslant 0$ (and so $a_0 = *$ and $b_0 = *$). These are subject to the following equivariance, bilinearity and boundary relations,

$$a_r^{a_1} \otimes b_s = (a_r \otimes b_s)^{a_1 \otimes *} \qquad \text{for } r \geqslant 2, s \geqslant 0 \tag{1}$$

$$a_r \otimes b_s^{b_1} = (a_r \otimes b_s)^{* \otimes b_1} \qquad \text{for } s \geqslant 2, r \geqslant 0 \tag{2}$$

$$(a_r + a_r') \otimes * = a_r \otimes * \ + \ a_r' \otimes * \qquad \text{for } r \geqslant 1, \tag{3}$$

$$* \otimes (b_s + b_s') = * \otimes b_s \ + \ * \otimes b_s' \qquad \text{for } s \geqslant 1 \tag{4}$$

$$(a_1 + a_1') \otimes b_s = a_1' \otimes b_s \ + \ (a_1 \otimes b_s)^{a_1' \otimes *}, \qquad \text{for } s \geqslant 1 \tag{5}$$

$$a_r \otimes (b_1 + b_1') = (a_r \otimes b_1)^{* \otimes b_1'} \ + \ a_r \otimes b_1', \qquad \text{for } r \geqslant 1 \tag{6}$$

$$(a_r + a_r') \otimes b_s = a_r \otimes b_s \ + \ a_r' \otimes b_s \qquad \text{for } r \geqslant 2, s \geqslant 1 \tag{7}$$

$$a_r \otimes (b_s + b_s') = a_r \otimes b_s \ + \ a_r \otimes b_s' \qquad \text{for } s \geqslant 2, r \geqslant 1 \tag{8}$$

$$\partial_{1+1}(a_1 \otimes b_1) = - * \otimes b_1 \ - \ a_1 \otimes * \ + \ * \otimes b_1 \ + \ a_1 \otimes * \tag{9}$$

$$\partial_r(a_r \otimes *) = \partial_r a_r \otimes * \qquad \text{for } r \geqslant 2 \tag{10}$$

$$\partial_s(* \otimes b_s) = * \otimes \partial_s b_s \qquad \text{for } s \geqslant 2 \tag{11}$$

$$\partial_{r+1}(a_r \otimes b_1) = \partial_r a_r \otimes b_1 \ + \ (-1)^r \left(-a_r \otimes * + (a_r \otimes *)^{* \otimes b_1}\right) \qquad \text{for } r \geqslant 2 \tag{12}$$

$$\partial_{1+s}(a_1 \otimes b_s) = - * \otimes b_s \ + \ (* \otimes b_s)^{a_1 \otimes *} \quad - \quad a_1 \otimes \partial_s b_s \qquad \text{for } s \geqslant 2 \tag{13}$$

$$\partial_{r+s}(a_r \otimes b_s) = \partial_r a_r \otimes b_s \ + \ (-1)^r a_r \otimes \partial_s b_s \qquad \text{for } r, s \geqslant 2 \tag{14}$$

Definition 2.34. *Given crossed complexes of groupoids A and B, their tensor product $A \otimes B$ can be presented by generators $(a_r \otimes b_s) \in (A \otimes B)_{r+s}$ with source $(\mathrm{src}(a_r) \otimes \mathrm{src}(b_s))$ (and target $(\mathrm{targ}(a_r) \otimes \mathrm{targ}(b_s))$ if $r + s = 1$), subject to the following relations:*

1. *The equivariance relations*

$$a_r^{a_1} \otimes b_s = (a_r \otimes b_s)^{a_1 \otimes \operatorname{src}(b_s)} \qquad \text{for } r \geqslant 2$$

$$a_r \otimes b_s^{b_1} = (a_r \otimes b_s)^{\operatorname{src}(a_r) \otimes b_1} \qquad \text{for } s \geqslant 2$$

2. *The bilinearity relations*

$$(a_r + a_r') \otimes b_0 = a_r \otimes b_0 + a_r' \otimes b_0, \qquad \text{for } r \geqslant 1$$

$$a_0 \otimes (b_s + b_s') = a_0 \otimes b_s + a_0 \otimes b_s', \qquad \text{for } s \geqslant 1$$

$$(a_1 + a_1') \otimes b_s = a_1' \otimes b_s + (a_1 \otimes b_s)^{a_1' \otimes \operatorname{src}(b_s)}, \qquad \text{for } s \geqslant 1$$

$$a_r \otimes (b_1 + b_1') = (a_r \otimes b_1)^{\operatorname{src}(a_r) \otimes b_1'} + a_r \otimes b_1', \qquad \text{for } r \geqslant 1$$

$$(a_r + a_r') \otimes b_s = a_r \otimes b_s + a_r' \otimes b_s, \qquad \text{for } r \geqslant 2, s \geqslant 1$$

$$a_r \otimes (b_s + b_s') = a_r \otimes b_s + a_r \otimes b_s', \qquad \text{for } s \geqslant 2, r \geqslant 1$$

3. *The boundary relations*

$$\partial_r(a_r \otimes b_0) = \partial_r a_r \otimes b_0 \qquad \text{for } r \geqslant 2$$

$$\partial_s(a_0 \otimes b_s) = a_0 \otimes \partial_s b_s \qquad \text{for } s \geqslant 2$$

$$\partial_2(a_1 \otimes b_1) = -\operatorname{src} a_1 \otimes b_1 - a_1 \otimes \operatorname{targ} b_1 + \operatorname{targ} a_1 \otimes b_1 + a_1 \otimes \operatorname{src} b_1$$

$$\partial_{r+1}(a_r \otimes b_1) = \partial_r a_r \otimes b_1 \;+\; (-1)^r \left(-a_r \otimes \operatorname{src} b_1 + (a_r \otimes \operatorname{targ} b_1)^{\operatorname{src}(a_r) \otimes b_1}\right) \qquad \text{for } r \geqslant 2$$

$$\partial_{1+s}(a_1 \otimes b_s) = -\operatorname{src} a_1 \otimes b_s + (\operatorname{targ} a_1 \otimes b_s)^{a_1 \otimes \operatorname{src}(b_s)} \quad - \quad a_1 \otimes \partial_s b_s \qquad \text{for } s \geqslant 2$$

$$\partial_{r+s}(a_r \otimes b_s) = \partial_r a_r \otimes b_s + (-1)^r a_r \otimes \partial_s b_s \qquad \text{for } r, s \geqslant 2$$

2.4.6 Free crossed complexes

It is well known that any group can be defined via a presentation: first find a set of generators and specify the relations that hold between products of the generators and their inverses. In a similar way, any crossed complex (of groups or of groupoids) can be defined by a presentation. The generators of a crossed complex C will be:

- the object set C_0,
- generators for the groupoid C_1: a subset of $Arr(C_1)$ such that all arrows in C_1 can be expressed as composites of the arrows in this subset and their inverses,
- generators for the crossed C_1-module C_2: a subset of C_2 such that all elements in C_2 can be expressed as composites of elements a in this subset, and the elements a^{c_1} for any $c_1 \in C_1$, and their inverses,
- generators for the $\pi_1 C$-module C_n, for each $n \geqslant 3$.

To define the crossed complex C would then have to give all the relations that hold between expressions we can form using these generators. We would also have to specify the boundary relations by giving functions from the generators of C_n to expressions written using generators of C_{n-1}.

We have already seen examples: in the previous section we already gave definitions of the tensor product of crossed complexes using generators and relations.

The easiest crossed complexes that we use are the *free crossed complexes.* A crossed complex is free when it has a presentation with generators but no relations, except for axioms in the definition of a crossed complex and the formulas that define the boundary maps.

Example 2.35. *[31] Let X be a simplicial set with X_0 as its object set. We can construct a free crossed complex of groupoids $C = \pi X$, called the fundamental crossed complex of X, with the following presentation. The generators are elements $\overline{x} \in C_n$ for each non-degenerate n-simplex x of X, where the source and target of $\overline{x}_1$ are the objects $\overline{x}_{(0)}$ and $\overline{x}_{(1)}$*

respectively, and the boundary relations are

$$\partial^{\pi X}(\overline{x}) = \begin{cases} -\overline{d_1 x} + \overline{d_0 x} \ + \overline{d_2 x} & \overline{x} \in \ X_2, \\ \overline{d_2 x} \ + \overline{d_0 x}^{\overline{x}_{01}} \ - \overline{d_3 x} \ - \overline{d_1 x} & \overline{x} \in \ X_3, \\ \overline{d_0 x}^{\overline{x}_{01}} \ + \sum_{i=1}^{n} (-1)^i (\overline{d_i x}) & \overline{x} \in \ X_n, \quad n \geqslant 4 \end{cases}$$

Because the image of ∂_3 is central, we use any cyclic permutation of its terms, for example

$$\partial^{\pi X}(\overline{x}) \quad = \quad -\overline{d_2 x} + \partial^{\pi X}(\overline{x}) + \overline{d_2 x} \quad = \quad \overline{d_0 x}^{\overline{x}_{01}} - \overline{d_3 x} - \overline{d_1 x} + \overline{d_2 x}$$

if $\overline{x} \in X_3$.

The following result is very useful

Theorem 2.36. *If C and D are free crossed complexes then their tensor product $C \otimes D$ is still a free crossed complex.*

If we combine Example 2.35 with Definition 2.34 then we obtain the following

Example 2.37. *Let X and Y be two simplicial sets. Then the crossed complex of groupoids $C = \pi X \otimes \pi Y$ is the free crossed complex of groupoids with generators $\overline{x}_n \otimes \overline{y}_m$ in C_{n+m} for all non degenerate simplices $x \in X_n$ and $y_m \in Y_m$, with source $\overline{x_{(0)}} \otimes \overline{y_{(0)}}$ (and target $\overline{x_0} \otimes \overline{y_{(1)}}$ if $(n, m) = (0, 1)$ or $\overline{x_{(1)}} \otimes \overline{y_0}$ if $(n, m) = (1, 0)$). The boundary relations are:*

$$\partial_2^{\otimes}(\overline{x}_2 \otimes \overline{y}_0) = -(\overline{x}_{02} \otimes \overline{y}_0) + (\overline{x}_{12} \otimes \overline{y}_0) + (\overline{x}_{01} \otimes \overline{y}_0)$$

$$\partial_2^{\otimes}(\overline{x}_1 \otimes \overline{y}_1) = -(\overline{x}_{(0)} \otimes \overline{y}_1) - (\overline{x}_1 \otimes \overline{y}_{(1)}) + (\overline{x}_{(1)} \otimes \overline{y}_1) + (\overline{x}_1 \otimes \overline{y}_{(0)})$$

$$\partial_2^{\otimes}(\overline{x}_0 \otimes \overline{y}_2) = -(\overline{x}_0 \otimes \overline{y}_{02}) + (\overline{x}_0 \otimes \overline{y}_{12}) + (\overline{x}_0 \otimes \overline{y}_{01})$$

$$\partial_3^{\otimes}(\overline{x}_3 \otimes \overline{y}_0) = +(\overline{x}_{013} \otimes \overline{y}_0) + (\overline{x}_{123} \otimes \overline{y}_0)^{\overline{x}_{01} \otimes \overline{y}_{(0)}} - (\overline{x}_{012} \otimes \overline{y}_0) - (\overline{x}_{023} \otimes \overline{y}_0)$$

$$\partial_3^{\otimes}(\overline{x}_2 \otimes \overline{y}_1) = (\overline{x}_{01} \otimes \overline{y}_1) + (\overline{x}_{12} \otimes \overline{y}_1)^{(\overline{x}_{01} \otimes y_{(0)})} - (\overline{x}_2 \otimes \overline{y}_{(0)}) - (\overline{x}_{02} \otimes \overline{y}_1) + (\overline{x}_2 \otimes \overline{y}_{(1)})^{(\overline{x}_{(0)} \otimes y_1)}$$

$$\partial_3^{\otimes}(\overline{x}_1 \otimes \overline{y}_2) = (\overline{x}_{(1)} \otimes \overline{y}_2)^{\overline{x}_1 \otimes \overline{y}_{(0)}} - \overline{x}_1 \otimes \overline{y}_{01} - (\overline{x}_1 \otimes \overline{y}_{12})^{\overline{x}_{(0)} \otimes \overline{y}_{01}} - \overline{x}_{(0)} \otimes \overline{y}_2 + \overline{x}_1 \otimes \overline{y}_{02}$$

$$\partial_3^{\otimes}(\overline{x}_0 \otimes \overline{y}_3) = (\overline{x}_0 \otimes \overline{y}_{013}) + (\overline{x}_0 \otimes \overline{y}_{123})^{\overline{x}_{(0)} \otimes \overline{y}_{01}} - (\overline{x}_0 \otimes \overline{y}_{012}) - (\overline{x}_0 \otimes \overline{y}_{023})$$

$$\partial^{\otimes}_{n+m}(\overline{x}_n \otimes \overline{y}_m) = \begin{cases} (\overline{d_0 x_n} \otimes \overline{y}_m)^{\overline{x}_{01} \otimes \overline{y}_{(0)}} + \sum_{i=1}^{n} (-1)^i \overline{d_i x_n} \otimes \overline{y}_m \\ +(-1)^n (\overline{x}_n \otimes \overline{d_0 y_m})^{\overline{x}_{(0)} \otimes \overline{y}_{01}} + \sum_{j=1}^{m} (-1)^{n+j} \overline{x}_n \otimes \overline{d_j y_m} \end{cases}$$

The last relation is for $n+m \geqslant 4$, but if $n=0$ then the first line on the right hand side of this relation should be ignored, and if $m=0$ then the second line should be ignored. Because the other boundary relations, for $n+m \leq 3$, are not abelian expressions, we cannot say that they are special cases of the relation for $n+m \geqslant 4$. Their terms (including the signs and the actions) are the same, but the order is important.

Remark 2.38. *In general, if C and D are free crossed complexes, then it might be quite complicated to write down the boundary relations in the free crossed complex $C \otimes D$. First, we must use the boundary relations of Definition 2.34(3). For example, in the example we just calculated, we know that*

$$\begin{aligned} \partial_3(\overline{x}_1 \otimes \overline{y}_2) &= -\operatorname{src} \overline{x}_1 \otimes \overline{y}_2 + (\operatorname{targ} \overline{x}_1 \otimes \overline{y}_2)^{x_1 \otimes \operatorname{src} \overline{y}_2} \; - \; \overline{x}_1 \otimes \partial_2 \overline{y}_2 \\ &= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \; - \; \overline{x}_1 \otimes (-\overline{d_1 y_2} + \overline{d_0 y_2} \; + \overline{d_2 y_2}) \end{aligned}$$

Here we have used the boundary relation in the free crossed complex πY given in 2.35. The answer is still not in the form we need for the boundary relation of a free crossed complex: we have to use the bilinearity relations in Definition 2.34 to write $\partial_3(\overline{x}_1 \otimes \overline{y}_2)$ as a composite of generators, possibly with actions. For this example we can write

$$\begin{aligned} &\partial_3(\overline{x}_1 \otimes \overline{y}_2) = -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \; - \; \overline{x}_1 \otimes (-\overline{y}_{02} + \overline{y}_{12} \; + \overline{y}_{01}) \\ &= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \; - \; \big((\overline{x}_1 \otimes (-\overline{y}_{02}))^{\overline{x}_{(0)} \otimes (\overline{y}_{12} + \overline{y}_{01})} + \overline{x}_1 \otimes (\overline{y}_{12} \; + \overline{y}_{01})\big) \\ &= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \; - \; \big((\overline{x}_1 \otimes (-\overline{y}_{02}))^{\overline{x}_{(0)} \otimes (\overline{y}_{12} + \overline{y}_{01})} + \overline{x}_1 \otimes (\overline{y}_{12} \; + \overline{y}_{01})\big) \end{aligned}$$

The bilinearity relation also implies that

$$\overline{x}_1 \otimes (-\overline{y}_{02}) \;\; = \;\; -(\overline{x}_1 \otimes \overline{y}_{02})^{-x_{(0)} \otimes y_{02}}$$

and so

$$\partial_3(\overline{x}_1 \otimes \overline{y}_2) = -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}}$$

$$- \ \big((-(\overline{x}_1 \otimes \overline{y}_{02})^{-x_{(0)} \otimes y_{02}})^{\overline{x}_{(0)} \otimes (\overline{y}_{12} + \overline{y}_{01})} + \overline{x}_1 \otimes (\overline{y}_{12} + \overline{y}_{01}))\big)$$

$$= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \ - \ \big(-(\overline{x}_1 \otimes \overline{y}_{02})^{\partial_2(x_{(0)} \otimes \overline{y}_2)} + \overline{x}_1 \otimes (\overline{y}_{12} \ + \overline{y}_{01})\big)$$

$$= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \quad - \quad \overline{x}_1 \otimes (\overline{y}_{12} \ + \overline{y}_{01}) + (\overline{x}_1 \otimes \overline{y}_{02})^{\partial_2(x_{(0)} \otimes \overline{y}_2)}$$

$$= -\overline{x}_{(0)} \otimes \overline{y}_2 + (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \ - \ \overline{x}_1 \otimes (\overline{y}_{12} \ + \overline{y}_{01}) - x_{(0)} \otimes \overline{y}_2 + \overline{x}_1 \otimes \overline{y}_{02} + x_{(0)} \otimes \overline{y}_2$$

Since the image of ∂_3 is cyclic, this can be simplified:

$$\partial_3(\overline{x}_1 \otimes \overline{y}_2) = (\overline{x}_{(1)} \otimes \overline{y}_2)^{x_1 \otimes \overline{y}_{(0)}} \ - \ \overline{x}_1 \otimes (\overline{y}_{12} \ + \overline{y}_{01}) - x_{(0)} \otimes \overline{y}_2 + \overline{x}_1 \otimes \overline{y}_{02}$$

Now we can use bilinearity once more to obtain

$$\partial_3(\overline{x}_1 \otimes \overline{y}_2) = (\overline{x}_{(1)} \otimes \overline{y}_2)^{\overline{x}_1 \otimes \overline{y}_{(0)}} - \overline{x}_1 \otimes \overline{y}_{01} - (\overline{x}_1 \otimes \overline{y}_{12})^{\overline{x}_{(0)} \otimes \overline{y}_{01}} - \overline{x}_{(0)} \otimes \overline{y}_2 + \overline{x}_1 \otimes \overline{y}_{02}$$

2.4.7 Diagonal approximation and shuffles

In this section we will review the maps φ and ϕ defined by A. Tonks in his papers [31], and [32]. These crossed complex maps are analogues to the Eilenberg-Mac Lane map which sends generators of the tensor product to a sum of terms indexed by shuffles, and the Alexander-Whitney map for the normalised free-chain complex on a simplicial set, which sends a generator $(x, y) \longmapsto \sum_{i=0}^{n} x_{0\ldots i} \otimes y_{i\ldots n}$ respectively. $\varphi \ \phi$ is the identity, and there is a homotopy η between $\phi \ \varphi$ and the identity.

Remark 2.39. *[32, Proposition 2.2.6] For any simplicial set X. There is a crossed complex morphism, $\bigtriangledown$ of an approximation to the diagonal which acts on the generators $\overline{x} \in \pi_n X$ by,*

$$\bigtriangledown : \pi X \to \pi X \otimes \pi X$$

$$\bigtriangledown(*) = (* \otimes *)$$

$$\bigtriangledown(\overline{x}_{01}) = (\overline{x}_{(0)} \otimes \overline{x}_{01}) + (\overline{x}_{01} \otimes \overline{x}_{(1)})$$

$$\bigtriangledown(\overline{x}_{012}) = (\overline{x}_{01} \otimes \overline{x}_{12})^{(\overline{x}_{12} \otimes \overline{x}_{(2)})} + (\overline{x}_{012} \otimes \overline{x}_{(2)}) + (\overline{x}_{(0)} \otimes \overline{x}_{012})^{(\overline{x}_{02} \otimes \overline{x}_{(2)})}$$

$$\bigtriangledown(\overline{x}_{n}) = \textstyle\sum_{i=0}^{n} (\overline{x}_{0\ldots i} \otimes \overline{x}_{i\ldots n})^{(\overline{x}_{in} \otimes \overline{x}_{(n)})}$$

Proposition 2.40. *There are crossed complex homomorphisms*

$$\phi : \pi(X \times Y) \to \pi(X) \otimes \pi(Y)$$

natural in simplicial sets X, Y defined on generators by

$$\phi_n(x_{v_0\ldots v_n}, y_{v_0\ldots v_n}) = \sum_{i=0}^{n} \left(x_{v_0\ldots v_i} \otimes y_{v_i\ldots v_n}\right)^{(x_{v_0} \otimes y_{v_0 v_i})} \qquad for \quad n \geqslant 3$$

While in dimension 0 it is trivial and in dimension 1 and dimension 2 are defined as

$$\phi_1(x_{v_0v_1}, y_{v_0v_1}) = (x_{v_0v_1} \otimes ty_{v_0v_1}) + (sx_{v_0v_1} \otimes y_{v_0v_1})$$

and

$$\phi_2(x_{v_0v_1v_2}, y_{v_0v_1v_2}) = (x_{v_0v_1v_2} \otimes y_{v_2})^{(x_{v_0} \otimes y_{v_0v_2})} + (x_{v_0} \otimes y_{v_0v_1v_2}) + (x_{v_0v_1} \otimes y_{v_1v_2})^{(x_{v_0} \otimes y_{v_0v_1})}$$

These commute with the boundary map ∂ defined in Definition (2.34), and are associative. In the sense that the following diagram commutes.

$$\begin{tikzcd}
\pi(X \times Y \times Z) \arrow[r, "\phi"] \arrow[d, "\phi"'] & \pi(X \times Y) \otimes \pi Z \arrow[d, "\phi \otimes id"] \\
\pi X \otimes \pi(Y \times Z) \arrow[r, "id \otimes \phi"'] & \pi X \otimes \pi Y \otimes \pi Z
\end{tikzcd}$$

for simplicial sets X, Y, and Z.

For more detail the proof of the proposition above is in [31].

Example 2.41. *In this example we show that ϕ commutes with the boundary map $\partial_{\pi X}$ that is defined in Definition (2.34), for $n = 2$. We have $\partial\phi_2(x, y) = \phi_1\partial(x, y)$ with the cancellations which occur in the following diagram:*

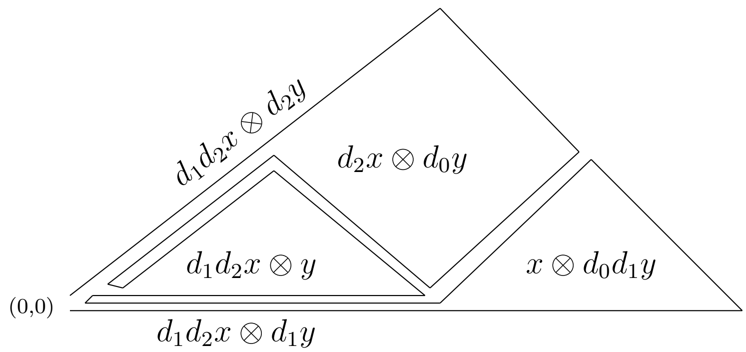


Figure 9: $(x(v_0v_1v_2), y(v'_0v'_1v'_2))$

Definition 2.42. *[32, Definition 2.2.7] A crossed differential graded algebra is a crossed complex $\mathbb{C}$, with a homomorphism $\mu : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ which make the diagram*

$$\begin{tikzcd}
\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C} \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & \mathbb{C}\otimes\mathbb{C} \arrow[d, "\mu"] \\
\mathbb{C}\otimes\mathbb{C} \arrow[r, "\mu"'] & \mathbb{C}
\end{tikzcd}$$

commute.

Dually one has that a crossed chain coalgebra πX is a crossed complex with a coassociative comultiplication forms a crossed chain coalgebra [20]:

$$\begin{tikzcd}
\pi X \arrow[rr, "\bigtriangledown"] \arrow[dr] & & \pi X \otimes \pi X \\
& \pi(X \times X) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Figure 10:

The homomorphism $\bigtriangledown : \pi X \to \pi X \otimes \pi X$ is termed the diagonal approximation map, as we have already said in Remark 2.39.

Proposition 2.43. *[31, Proposition 2.6], [32, Proposition 2.2.10] There are crossed complex homomorphisms*

$$\varphi : \pi X \otimes \pi Y \to \pi(X \times Y)$$

natural in simplicial sets X, Y, defined for all $(x, y) \in (X_p, Y_q)$ by

$$\varphi(x \otimes y) = \textstyle\sum_{(\sigma_0,\sigma_1)\in S_{p,q}} (-1)^{sg(\sigma)} (s_{\sigma_0} x, s_{\sigma_1} y) \qquad \text{where } (p, q) \neq (1, 1),$$

where $x \in X_1, y \in Y_1$, and $S_{p,q}$ denotes the set of (p, q)-shuffles we have,

$$\begin{aligned}\varphi(x \otimes y) &= -(s_0 x_{v_0v_1}, s_1 y_{v_0v_1}) + (s_1 x_{v_0v_1}, s_0 y_{v_0v_1}) \\ &= -(x_{v_0v_0v_1}, y_{v_0v_1v_1}) + (x_{v_0v_1v_1}, y_{v_0v_0v_1})\end{aligned}$$

which are associative in the sense that the following diagram commutes

$$\begin{tikzcd}
\pi X \otimes \pi Y \otimes \pi Z \arrow[r, "\varphi \otimes id"] \arrow[d, "id \otimes \varphi"'] & \pi(X \times Y) \otimes \pi Z \arrow[d, "\varphi"] \\
\pi X \otimes \pi(Y \times Z) \arrow[r, "\varphi"'] & \pi(X \times Y \times Z)
\end{tikzcd}$$

for simplicial sets X, Y, and Z.

Also the proof of this Proposition and for more detail can found in [31]

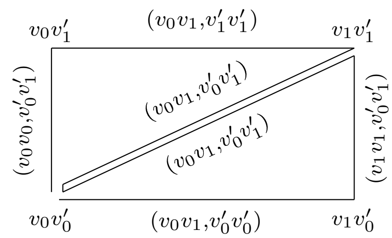
Example 2.44. *In this example we show that φ commutes with ∂ for dimension 2. Let $p = q = 1$*

$$\partial\varphi(x_{v_0v_1} \otimes y_{v_0v_1}) = -\partial(s_0x_{v_0v_1}, s_1y_{v_0v_1}) + \partial(s_1x_{v_0v_1}, s_0y_{v_0v_1})$$

$$= -(s_0x_{v_0}, y_{v_0v_1}) - (x_{v_0v_1}, s_0y_{v_1}) + (x_{v_0v_1}, y_{v_0v_1}) - (x_{v_0v_1}, y_{v_0v_1}) + (s_0x_{v_1}, y_{v_0v_1}) + (x_{v_0v_1}, s_0y_{v_0})$$

$$= -(x_{v_0v_0}, y_{v_0v_1}) - (x_{v_0v_1}, y_{v_1'v_1'}) + (x_{v_0v_1}, y_{v_0v_1}) - (x_{v_0v_1}, y_{v_0v_1}) + (x_{v_1v_1}, y_{v_0v_1}) + (x_{v_0v_1}, y_{v_0v_0})$$

which by the diagram:



The middle diagonal terms cancel, leaving $\varphi\partial(x(v_0v_1), y(v_0'v_1'))$.

Theorem 2.45. *[31, Theorem 3.1] There is a strong deformation retraction of crossed complexes*

$$\eta \circlearrowright \pi(X \times Y) \underset{\phi}{\overset{\varphi}{\leftrightarrows}} \pi(X) \otimes \pi(Y)$$

which is natural in X, Y, where η is a contracting homotopy $id \simeq \varphi\phi \;\; rel.(X_0 \times Y_0)$.

For simplicial sets X, and Y the composite

$$\pi(X \times Y) \xrightarrow{\phi} \pi X \otimes \pi Y \xrightarrow{\varphi} \pi(X \times Y)$$

is homotopic to the identity on $\pi(X \times Y)$ via a splitting homotopy. Thus $\pi X \otimes \pi Y$ is a strong deformation retract of $\pi(X \times Y)$.

The proof of this theorem could found in [32, P. 48].

3 The cobar construction

Introduction

The main of our aim in this chapter is to introduce the main theorem on chain algebras which was proved by J. F. Adams and P. J. Hilton in [3]. They showed, for a 1-reduced simplicial set X, that the twisted tensor product of chain loop algebra ΩCX and the chain complex CX is contractable. Moreover, we illustrate the theorem proved by K. Hess and A. Tonks in [19], on the loop group and the cobar construction for any 1-reduced simplicial set X.

The structure of the chapter is as follows. In section one, we introduce some preliminaries on path space and then show the theorem on chain algebra which was proved by J. F. Adams and P. J. Hilton. In section two we introduce the Adams cobar construction, which passes from one chain complex to another chain complex with different structure [2]. We also present the theorem that was proved by K. Hess and A. P. Tonks in [19], which shows that for any 1-reduced simplicial set X, Adams' cobar construction ΩX is a strong deformation retract on the chain on the Kan loop space CGX.

3.1 The cobar construction of Adams

Let CX be the normalised chain complex of 1-reduced simplicial set X. We have seen earlier that this is a differential graded coalgebra, using the Alexander–Whitney diagonal approximation,

$$CX \to CX \otimes CX$$

The classical cobar construction Ω is a functor that takes differential graded coalgebras to differential graded algebras. In algebraic topology, Adams in [1] introduced the cobar construction and proved that the differential graded algebra ΩCX is a model for the loop space on X. Recall that the loop space ΩX is defined as the space of all continuous maps $\gamma : S^1 \to X$. Two loops $\gamma, \gamma' : S^1 \to X$ may be composed. Therefore the chain complex $C(\Omega X)$ on the loop space has a multiplication operation. Adams defined the differential graded algebra ΩCX and proved that it is weakly equivalent to the differential graded algebra $C(\Omega X)$.

Definition 3.1 (Adams' cobar construction)**.** *Let Λ be a principal ideal domain of coefficients, and let C be a chain complex of Λ-modules which is 1-reduced: $C_0 = \Lambda$ and $C_1 = 0$. Suppose C has a comultiplication $\nabla : C \to C \otimes C$ given by an associative chain map such that, if $x \in C_r$, the components of $\nabla(x)$ in $C_0 \otimes C_r$ and in $C_r \otimes C_0$ are*

$$\begin{aligned} \nabla_{0,r}(x) &= 1 \otimes x \\ \nabla_{r,0}(x) &= x \otimes 1 \end{aligned}$$

respectively.

Adams defined the chain complex $\Omega(C)$ by

$$\Omega(C) = \Lambda + \sum_{r \geqslant 1} C^{\otimes r} \qquad (\textit{where } C^{\otimes r} = C \otimes C \otimes \cdots \otimes C, \quad r \textit{ times}).$$

If $x \in C_{n+1}$ then, in $\Omega(C)$, the element x has degree n and boundary

$$\partial_n^{\Omega}(x) = -d^C(x) + \sum_{2 \leq r \leq n-1} (-1)^r \nabla_{r,n+1-r}(x).$$

This is a (free) differential graded algebra with the multiplication induced by the maps

$$C^{\otimes r} \otimes C^{\otimes s} \;\cong\; C^{\otimes (r+s)} \;\subset\; \Omega C.$$

Definition 3.2. *For any simplicial set X, the normalised chain complex CX is a differential graded coalgebra and has a comultiplication*

$$\nabla : CX \to CX \otimes CX$$

given by the Alexander-Whitney diagonal approximation,

$$\nabla_{i,n-i}(x) = x_{0\ldots i} \otimes x_{i\ldots n}, \qquad \nabla_n(x) = \sum_{i=0}^{n} x_{0\ldots i} \otimes x_{i\ldots n}.$$

Theorem 3.3 (Adams)**.** *If X is a 1-reduced simplicial set, $X_0 = \{*\}$, $X_1 = \{s_0(*)\}$, then there is a homology equivalence between the cobar construction on the chains on X and the singular chain complex on the geometric realisation of the loop space on X,*

$$C(|\Omega X|) \sim \Omega(C(X)).$$

3.1.1 Kan's loop group and cobar construction

Kathryn Hess and Andrew Tonks showed in [19] that Adams' cobar construction is naturally a strong deformation retract of the normalised chains CGX on the Kan loop group GX.

Recall that the simplicial group GX is the *loop group* of a simplicial set X, and was first introduced by Kan. In each degree GX is a quotient of free groups

$$(GX)_n = F(X_{n+1})/F(s_0X_n) \cong F(X_{n+1} - s_0X_n)$$

In other words, it is the free group on the simplices that are not s_0-degenerate.

Let X be any simplicial set and $\mathcal{G}$ any simplicial group. A *twisting function* $\tau : X \to \mathcal{G}$ is a family of maps

$$\{\tau_m : X_m \to \mathcal{G}_{m-1}\}_{m \geqslant 1}$$

satisfying the following properties.

(i.) $d_0\tau(x) = -\tau(d_0x) + \tau(d_1x)$;

(ii.) $d_i\tau(x) = \tau(d_{i+1}x) \quad$ if $\; i \geqslant 1$;

(iii.) $s_i\tau(x) = \tau(s_{i+1}x), \quad$ if $\; i \geqslant 0$;

(v.) $\tau(s_0x) = e_m \quad$ if $\; x \in X_m$, the unit element of $\mathcal{G}_m$ being e_m.

In particular, a twisting function has degree -1 and is not a simplicial map.

Let $\tau : X \to GX$ be the *universal* twisting function from 0-reduced set X, to the simplicial group GX. The universal twisting function sends $x \in X_{n+1}$ to the image $\tau(x) = \overline{x}$ of the generator in $(GX)_n$.

As described in [19, page 1864], the shuffle map can be used to provide an algebra structure on the chains on the Kan loop group: the normalised chain complex CGX on the Kan loop group GX is a graded algebra with multiplication map

$$\mu : CGX \otimes CGX \to C(GX \times GX) \to CGX,$$

that is,

$$\mu(g_r \otimes g_s) = \sum_{\text{shuffles } \pi=(i,j)} (-1)^{sgn(i,j)} s_{i_s} \dots s_{i_1}(g_r) \cdot s_{j_r} \dots s_{j_1}(g_s) \qquad g_r \in G_r, \quad g_s \in G_s$$

.

Theorem 3.4. *[19] For any* 1*-reduced simplicial set X there is a strong deformation retract between Adams' cobar construction on the normalised chain complex ΩCX and the normalised chains on the Kan loop group CGX.*

$$\begin{tikzcd} CGX \arrow[loop left, "\eta"] \arrow[r, shift right, "\psi"'] & \Omega CX \arrow[l, shift right, "\phi"'] \end{tikzcd}$$

Here ϕ and ψ are homomorphisms of chain algebras and η is a chain homotopy from $\phi\psi$ to the identity map.

This strong deformation retract is actually Eilenberg-Zilber data in case of X is a simplicial suspension. More detail can be found in [19].

Proposition 3.5. *[19] For any simplicial map $\theta : GX \to GY$,(X, and Y are* 1*-reduced simplicial sets) there is a chain-level model ζ of θ, and then the diagram*

$$
\begin{tikzcd}
\Omega CX \arrow[r, "\zeta"] \arrow[d, "\phi"'] & \Omega CY \arrow[d, "\phi"] \\
CGX \arrow[r, "C\theta"'] & GCY
\end{tikzcd}
$$

Figure 11: $\zeta = \psi \circ C\theta \circ \phi : \Omega CX \to \Omega CY$

commutes up to chain homotopy.

The homomorphism ϕ in the theorem above was first described by Szczarba [30]: he gives the explicit formula for a twisting cochain λ_ϕ which is based on the twisting function $\tau : X \to GX$, but he did not prove that ϕ has a homotopy inverse that is also an algebra homomorphism.

3.1.2 The cobar construction of 0-reduced simplicial sets

In order to prove the previous theorem, Hess and Tonks needed to generalise the classical cobar construction of Adams from 1-reduced simplicial sets to 0-reduced simplicial sets. They introduced an extended cobar construction, that they denote $\hat{\Omega}$, and they defined ϕ and ψ for 0-reduced simplicial sets. They then proved the homotopy equivalence of CGX and $\hat{\Omega}CX$ using an acyclic-models argument.

Defining the Hess-Tonks cobar construction. Let R be a commutative ring with unit and let (C, ∂) be an R-free differential graded coalgebra with $C_0 = \mathrm{R}$. Consider first the ring Λ, in degree 0, given by the free associative R$-$algebra on the desuspension of C_1,

$$\Lambda = \sum_{r \geqslant 0} (s^{-1}C_1)^{\otimes r}.$$

Now let $B = \{x_j \ ; j \in J\}$ be a basis of C_1, so that Λ is the free algebra with generators $s^{-1}x_j$, and let K be the ring obtained from Λ by adjoining inverses λ_j of all elements of the form $(1 + s^{-1}x_j)$. The ring K is an algebra in degree 0 generated by $s^{-1}x_j$ and $\lambda_j = (1 + s^{-1}x_j)^{-1}$.

The *extended cobar construction* $\hat{\Omega}C$ of Hess and Tonks [19] is

$$\hat{\Omega}C = \sum_{r\geqslant 0, n\geqslant 2} \mathrm{K} \otimes (s^{-1}C_n \otimes \mathrm{K})^{\otimes r}.$$

In the case of 1-reduced chain complexes, $C_0 = 0$ and $K = 0$, so this is Adams' cobar construction.

The generators of in degree n of $\hat{\Omega}C$ therefore have the form

$$k = k_1 \otimes \cdots \otimes k_r, \quad n = \sum n_i$$

where either $k_i = s^{-1}c$ for some basis elements $c \in C_{n_i+1}$, or $n_i = 0$ and $k_i = \lambda_j$ for some $j \in J$. The unit $1 \in (\hat{\Omega}C)_0$ is the empty word. Since elements in degree zero do not have boundaries, the differential is the same as for the classical cobar construction: for all basis elements $c \in C_{n+1}, \quad n \geqslant 1$, the differential $\partial^{\hat{\Omega}}$ on $\hat{\Omega}$ is specified by

$$\partial_n^{\hat{\Omega}} s^{-1}c = -s^{-1}dc + (s^{-1} \otimes s^{-1})\nabla(c).$$

Definition 3.6. *Let X be a 0-reduced simplicial set and let $\tau : X \to G$ be any twisting function to a simplicial group. Then there is a canonical homomorphism of differential graded algebras*

$$\phi : \hat{\Omega}CX \to CG,$$

defined in positive degrees using the Szczarba operators Sz_i; see [30] and [19].

$$\begin{aligned}
\phi_0(\lambda_{x_1}) &= \tau x_1, \\
\phi_0(s^{-1}x_1) &= \tau(x_1)^{-1} - 1, \\
\phi_n(s^{-1}x_{n+1}) &= \sum_{i\in S_n} (-1)^{\Sigma i} Sz_i x, \quad n \geqslant 1
\end{aligned}$$

for any $x_{n+1} \in X_{n+1}$.

In the other direction,

Definition 3.7. *Let X be a 0-reduced simplicial set. The differential graded algebras map*

$$\psi : CGX \to \hat{\Omega}CX$$

from the chains on the loop group to the extended cobar construction of the 0-reduced simplicial set is determined as follows.

In degree 0, $\psi_0 : (CGX)_0 \to (\hat{\Omega}CX)_0$ is defined on the algebra generators by

$$\psi_0(\tau x) = \lambda_x, \qquad\qquad \psi_0(\tau x^{-1}) = 1 + s^{-1}x.$$

In degree 1, $\psi_1 : (CGX)_1 \to (\hat{\Omega}CX)_1$ is determined by

$$\psi_1(\tau x_1^{\alpha_1} \dots \tau x_r^{\alpha_r}) = \sum_{i=1}^{r} \psi_0 d_1(\tau x_1^{\alpha_1} \dots \tau x_{i-1}^{\alpha_{i-1}}) \otimes \psi_1(\tau x_i^{\alpha_i}) \otimes \psi_0 d_0(\tau x_{i+1}^{\alpha_{i+1}} \dots \tau x_r^{\alpha_r})$$

In degrees $\geqslant 2$, $\psi_n : (CGX)_n \to (\hat{\Omega}CX)_n$ is determined by

$$\psi_n(\tau x \cdot y) = \psi_n(y) - \sum_{i=0}^{n} x_{0\dots i+1} \otimes \psi_{n-1}(\tau d_1^i x . d_0^i y)$$

Hess and Tonks showed the following.

Proposition 3.8. *The map $\psi : CGX \to \hat{\Omega}CX$ is*

1. *well defined, that is, $\psi(\omega) = 0$ if ω is degenerate,*

2. *a chain map, i.e., for all $x \in X_{n+1}$ and $y \in (GX)_n$,*

$$\partial_n^{\hat{\Omega}} \psi_n(\tau x . y) = \psi_{n-1} \partial_n (\tau x . y),$$

3. *an algebra homomorphism.*

$$\psi_n(x.y) = \psi_r(x).\psi_s(y), \quad x \in (GX)_r, \quad y \in (GX)_s, \quad n = r + s,$$

4. a retraction of ϕ, that is, $\psi\phi$ is the identity.

Proof. See [19] $\square$

Theorem 3.9. *The cobar construction $\hat{\Omega}CX$ on the normalised chain complex of 0-reduced simplicial set X is naturally a strong deformation retraction of the normalised chains CGX on the Kan loop group GX.*

$$\begin{tikzcd} CGX \arrow[loop left, "\Phi"] \arrow[r, shift right, "\psi"'] & \hat{\Omega}CX \arrow[l, shift right, "\phi"'] \end{tikzcd}$$

Here ψ and ϕ are the Szczarba and the retraction maps respectively.

3.2 On the chain complex model of the path space

For any 0-reduced simplicial set X, there is a *simplicial fibration*

$$GX \to EX \to X$$

where EG may be identified with a certain *twisted cartesian product* of simplicial sets

$$EG = X \times_\tau GX$$

The simplicial set EG is contractible, and the simplicial fibration is a model for the path-loop fibration of spaces,

$$\Omega X \to PX \to X.$$

For any 1-reduced simplicial set X, the cobar construction on CX is an algebraic model for the loop space. A twisted tensor product of the chains CX and the chains on the loop space $C\Omega X$ should therefore be an algebraic model for the path space. That is, it should be contractible, since any path can be retracted to the constant path at the basepoint.

The following theorem was proved by J. F. Adams and P. J. Hilton in [3]

Theorem 3.10. *Let X be a 1-reduced simplicial set. The tensor product of the loop space ΩCX on the chains on a 1-reduced simplicial set X and the chain complex CX is contractable.*

Proof. Let $\mathrm{L} = CX$ be the free abelian group generated by elements $l_i \in C_iX$ with $l_0 = 1$ and augmented by $\alpha(1) = 1, \alpha(l_i) = 0, i \geqslant 1$. Let $\mathrm{K} = \Omega CX$ be the loop space on the chains on X generated by elements $k_i \in (\Omega CX)_i$. Define $C = \mathrm{L} \otimes \mathrm{K}$ as a tensor product of L, and K with the usual augmentation α. Next define a retraction $\eta : C_n \to C_{n+1}$ by:

$$\eta(1) = 0, \qquad \eta(k_i) = l_i, \qquad (\eta k_i)^2 = 0 \tag{15}$$

and for $x \in C_n,\ y \in \mathrm{K}_n$ define the homotopy η and a boundary map δ as:

$$\eta(xy) = \eta(x)y + (\alpha x)\eta(y), \tag{16}$$

$$\delta(xy) = (\delta x)y + (-1)^n x(\delta y), \tag{17}$$

The differential δ satisfies

$$\delta l_i = (1 - \eta\delta)\ k_i, \qquad\qquad l_i \in C_{n+1}, \quad k_i \in C_n \tag{18}$$

It is clear that η and δ are consistent with the two distributive laws and with the associative law of multiplication. □

Remark 3.11. *[3] The augmentation α is homotopic to the identity, i.,e, there is a homotopy η from the identity map $\mathbf{1}$ to the augmentation α such that*

$$(\delta\eta + \eta\delta)x = (1 - \alpha)x$$

for all $x \in C_n$.

Proof. Let $x \in C_n$, if $x = 1$, $(x \in C_0)$, this is trivial. If x is a generator of K_n, and $x = k$, then $(\delta\eta + \eta\delta)k = \delta\eta(k) + \eta\delta(k) = \delta(l) + \eta\delta(k)$ (from Theorem 3.10(1)), also from the same theorem in (4) we have $\delta l = (1-\eta\delta)(k)$, so $\delta(l) + \eta\delta(k) = (1-\eta\delta)(k) + \eta\delta(k) = k$.

If x is a generator of L_n, and $x = l$,

$(\delta\eta + \eta\delta)l = \delta\eta(l) + \eta\delta(l) = \delta\eta(\eta k) + \eta\delta(l) = 0 + \eta\delta(l)$ (by Theorem 3.10(1))

$$= \eta((1-\eta\delta)(k)) = \eta(k) - \eta^2(\delta(k)) = \eta(k) = l, \text{ (Theorem 3.10(1) and (4)).}$$

The prove above showed that if x on the generators of ΩCX and of CX, it satisfies $(\delta\eta + \eta\delta)x = (1-\alpha)x$.

Now if $x \in C_n,\ y \in \mathrm{K}_n$,

$(\delta\eta + \eta\delta)xy = \delta\eta(xy) + \eta\delta(xy) = \delta\big((\eta x)y + (\alpha x)(\eta y)\big) + \eta\big((\delta x)y + (-1)^n x(\delta y)\big)$ (Theorem 3.10(3))

$$= (\delta\eta x)y + (-1)^{n+1}(\eta x)(\delta y) + (\delta\alpha x)(\eta y) + (-1)^n(\alpha x)(\delta\eta y) + (\eta\delta x)y$$
$$+ (\delta\alpha x)(\eta y) + (-1)^n(\eta x)(\delta y) + (-1)^n(\alpha x)(\eta\delta y)$$
$$= (\delta\eta x)y + (\delta\alpha x)(\eta y) + (-1)^n(\alpha x)(\delta\eta y) + (\eta\delta x)y + (\delta\alpha x)(\eta y) + (-1)^n(\alpha x)(\eta\delta y)$$
$$= (\delta\eta x + \eta\delta x)y + (\delta\alpha x)(\eta y) + (-1)^n(\alpha x)(\delta\eta y + \eta\delta y) + (\delta\alpha x)(\eta y)$$

(the term $(-1)^{n+1}(\eta x)(\delta y)$ cancels with the term $(-1)^n(\eta x)(\delta y)$)

Now, if $n = 0$ then

$(\delta\eta + \eta\delta)(xy) = xy - (\alpha x)y + \alpha x(y - \alpha y) = xy - \alpha x\alpha y = (1-\alpha)xy$.

If $n > 0$,

$\alpha x = 0$ and $(\delta\eta + \eta\delta)(xy) = xy = (1-\alpha)xy$. □

Proposition 3.12. *δ is a differential on C.*

Proof. In case of the generators of K_n, it is clear from Theorem 3.10(3), so we need only verify the proposition on a generator of L_n. Let l be a generator of L_n satisfy that $\eta k = l$.

$\delta^2 l = \delta(1-\eta\delta)k = (\delta - \delta\eta\delta)k = (1-\delta\eta)\delta k$.

Now,

$(\delta\eta + \eta\delta)\delta k = (1-\alpha)\delta k$ (from Remark 3.11).

So

$\delta\eta\delta k = \delta k$ ($\delta^2 k = 0$, and $\alpha\delta k = 0$). Hence that implies $\delta^2 l = 0$. $\square$

4 A crossed complex of groups

In this chapter we will try to generalise the theorem of Adams and Hilton that we gave in the previous chapter.

We define a crossed complex of groups $P^{\mathsf{Crs}}X$, where X is a 1-reduced simplicial set, to try and obtain a crossed complex model for the path-loop fibration.

The cobar construction $\Omega^{\mathsf{Crs}}X$ for crossed complexes, for 1-reduced simplicial sets, was introduced by Baues and Tonks. We want to introduce a twisted tensor product of this crossed cobar construction $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX. It will have the same generators as the usual tensor product of crossed complexes of groups. The most important part of our construction will be to define the new twisted boundary maps $\partial^P : P_n^{\mathsf{Crs}}X \to P_{n-1}^{\mathsf{Crs}}X$.

To make our construction easy to define, we will introduce the idea of a free module over an algebra in the category of crossed complexes. Then our crossed complex of groups $P_n^{\mathsf{Crs}}X$ will be a free module over the crossed chain algebra $\Omega^{\mathsf{Crs}}X$. We will then only need to define the twisted boundary on the basis elements of the module.

The structure of the chapter is as follows. In first section, we begin with recalling the Baues–Tonks definition of the crossed cobar construction $\Omega^{\mathsf{Crs}}X$. We follow this by presenting the idea of free modules over crossed chain algebras, and then we can give our short definition of the path crossed complex $P^{\mathsf{Crs}}X$ as a module over $\Omega^{\mathsf{Crs}}X$. Next we expand this definition, and we calculate the boundary maps ∂^P on other generators of $P^{\mathsf{Crs}}X$. We will also prove that ∂^P is a differential, that is, we will prove its square is trivial.

In the second section, we define a contracting homotopy, which we can do by defining a family of maps $\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$ which raise the dimension by one and satisfy certain conditions. We then have $h : * \simeq id$, so $P^{\mathsf{Crs}}X$ is contractible. Therefore $P^{\mathsf{Crs}}X$ is a crossed complex model for the path space of X.

$$* \overset{h}{\simeq} \mathrm{Id}_P \circlearrowright P \underset{\pi}{\overset{i}{\rightleftarrows}} \{*\}$$

4.1 The crossed cobar construction

We begin this section by recalling the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ given by A. P. Tonks and H. J. Baues in [14]. In their paper, full details of the definition of $\Omega^{\mathsf{Crs}}X$ for a 1-reduced simplicial set X are given.

They define the interval object $\mathcal{I}$ in the category of crossed complexes to be given by the crossed complex $\mathcal{I} = \pi(\Delta[1])$. This has generators $\{0,1\} \in \mathcal{I}_0$ and $(\sigma : 0 \to 1) \in \mathcal{I}_1$. It has a map $\mu : \mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ given on the generators by $a \otimes b = 1$ for $a, b \in \{0, 1, \sigma\}$, except for $0 \otimes 0 = 0$ and $0 \otimes \sigma = \sigma \otimes 0 = \sigma$.

We write down some of the boundaries of tensor products $\mathcal{I}^{\otimes n}$ of copies of $\mathcal{I}$, which we will need later,

$$\begin{aligned}
\partial_2(\sigma \otimes \sigma) &= -(\mathrm{src}(\sigma) \otimes \sigma) - (\sigma \otimes \mathrm{targ}(\sigma)) + (\mathrm{targ}(\sigma) \otimes \sigma) + (\sigma \otimes \mathrm{src}(\sigma)) \\
&= -(0 \otimes \sigma) - (\sigma \otimes 1) + (1 \otimes \sigma) + (\sigma \otimes 0). \\
\partial_3(\sigma \otimes \sigma \otimes \sigma) &= -(\sigma \otimes \sigma \otimes \mathrm{targ}(\sigma)) - (\sigma \otimes \mathrm{src}(\sigma) \otimes \sigma)^{(1 \otimes \sigma \otimes 1)} - (\mathrm{targ}(\sigma) \otimes \sigma \otimes \sigma) \\
&\quad + (\sigma \otimes \sigma \otimes \mathrm{src}(\sigma))^{(1 \otimes 1 \otimes \sigma)} + (\sigma \otimes \mathrm{targ}(\sigma) \otimes \sigma) + (\mathrm{src}(\sigma) \otimes \sigma \otimes \sigma)^{(\sigma \otimes 1 \otimes 1)}
\end{aligned}$$

Definition 4.1. *[14,20] Let X be a 1-reduced simplicial set. The crossed cobar construction $\Omega^{\mathsf{Crs}}X$ is a free crossed chain algebra generated by the elements $s^{-1}a_{n+1}$ in degree n for each $(n+1)$-simplex of X and boundary map given by:*

$$\begin{aligned}
\partial_2^{\Omega} s^{-1}a_3 &= -s^{-1}a_{123} - s^{-1}a_{013} + s^{-1}a_{023} + s^{-1}a_{012} \\
\partial_3^{\Omega}(s^{-1}a_4) &= -s^{-1}a_{0123} - (s^{-1}a_{0134})^{\gamma_2} - s^{-1}a_{1234} + (s^{-1}a_{0124})^{\gamma_3}
\end{aligned}$$

$$+ (s^{-1}a_{012} \cdot s^{-1}a_{234}) + (s^{-1}a_{0234})^{\gamma_1}$$

and for dimension $n \geqslant 4$ *the differential defined by the formula*

$$\partial_n^{\Omega}(s^{-1}a_{n+1}) = \sum_{i=1}^{n}(-1)^{i+1}(s^{-1}d_i a_{n+1})^{\gamma_i} - \sum_{i=1}^{n}(-1)^{i+1}(s^{-1}a_{0\ldots i} \cdot s^{-1}a_{i\ldots n+1})$$

Here the actions are by the elements

$$\gamma_i = s^{-1}a_{i-1\ i\ i+1}$$

Remark 4.2. *The algebra structure of the cobar construction is given by the crossed complex homomorphism defined by concatenating the generators,*

$$\begin{aligned} \mu :\ \Omega^{\mathsf{Crs}}X \otimes \Omega^{\mathsf{Crs}}X \quad &\longrightarrow \quad \Omega^{\mathsf{Crs}}X \\ \mu(x \otimes x') \quad &= \quad xx' \end{aligned}$$

The generators of $\Omega^{\mathsf{Crs}}X$ *as a crossed complex are all of the words, or strings, of its generators as a crossed chain algebra. We can write a generator of degree* n *of the crossed complex* $\Omega^{\mathsf{Crs}}X$ *as*

$$x = s^{-1}a^{(1)}_{n_1} \cdots s^{-1}a^{(r)}_{n_r},$$

where $r \geqslant 0$*, each* $a^{(i)}_{n_i}$ *is a non-degenerate* $(n_i + 1)$*-simplex of* X *and* $n = \sum n_i$*.*

The boundary $\partial_n^{\Omega}x$ *of a general word* x *can be calculated using the boundary relations in the definition of the tensor product of crossed complexes.*

The crossed cobar construction is only defined here for a 1-reduced simplicial set, which has no non-degenerate 1-simplices. Therefore it is a crossed complex of groups, with basepoint given by the word $x = \varnothing$ of length $r = 0$,

$$\Omega_0^{\mathsf{Crs}}X = \{\varnothing\}$$

In dimension one we can see that $\Omega_1^{\mathsf{Crs}} X$ is the free group on $X_2 - \{s_0^2(*)\}$.

In dimension two, $\Omega_2^{\mathsf{Crs}} X$ is the free crossed module over $\Omega_1^{\mathsf{Crs}} X$, with two types of generators

$$s^{-1}a_3, \qquad\qquad s^{-1}a_2 s^{-1}a_2',$$

and boundary relations

$$\partial_2^{\Omega} s^{-1}a_3 = -s^{-1}a_{123} - s^{-1}a_{013} + s^{-1}a_{023} + s^{-1}a_{012} \tag{19}$$

$$\partial_2^{\Omega}(s^{-1}a_2 s^{-1}a_2') = -s^{-1}a_2' - s^{-1}a_2 + s^{-1}a_2' + s^{-1}a_2 \tag{20}$$

In dimension 3 we see that $\Omega_3^{\mathsf{Crs}} X$ is a free $\Omega_2^{\mathsf{Crs}} X$-module with four types of generators

$$s^{-1}a_4, \quad s^{-1}a_3 s^{-1}a_2, \quad s^{-1}a_2 s^{-1}a_3, \quad s^{-1}a_2 s^{-1}a_2' s^{-1}a_2''.$$

whose boundaries are given by

$$\begin{aligned}
\partial_3^{\Omega} s^{-1}a_4 &= -(s^{-1}a_{0134})^{(s^{-1}a_{123})} - (s^{-1}a_{1234}) + (s^{-1}a_{0124})^{(s^{-1}a_{234})} \\
&\quad + (s^{-1}a_{012}s^{-1}a_{234}) + (s^{-1}a_{0234})^{(s^{-1}a_{012})} - (s^{-1}a_{0123}), && (21)\\
\partial_3^{\Omega}(s^{-1}a_3 s^{-1}a_2) &= -(s^{-1}a_{123}s^{-1}a_2) + (s^{-1}a_3)^{(s^{-1}a_2)} + (s^{-1}a_{012}s^{-1}a_2) \\
&\quad + (s^{-1}a_{023}s^{-1}a_2)^{(s^{-1}a_{012})} - (s^{-1}a_3) - (s^{-1}a_{013}s^{-1}a_2)^{(s^{-1}a_{123})}. && (22)\\
\partial_3^{\Omega}(s^{-1}a_2 s^{-1}a_3) &= -(s^{-1}a_3) + (s^{-1}a_2 s^{-1}a_{013})^{(s^{-1}a_{123})} + (s^{-1}a_2 s^{-1}a_{123}) \\
&\quad + (s^{-1}a_3)^{(s^{-1}a_{012})} - (s^{-1}a_2 s^{-1}a_{012}) - (s^{-1}a_2 s^{-1}a_{023})^{(s^{-1}a_{012})} && (23)\\
\partial_3^{\Omega}(s^{-1}a_2 s^{-1}a_2' s^{-1}a_2'') &= -(s^{-1}a_2 s^{-1}a_2') - (s^{-1}a_2 s^{-1}a_2'')^{(s^{-1}a_2')} - (s^{-1}a_2' s^{-1}a_2'') \\
&\quad + (s^{-1}a_2 s^{-1}a_2')^{(s^{-1}a_2'')} + (s^{-1}a_2 s^{-1}a_2'') + (s^{-1}a_2' s^{-1}a_2'')^{(s^{-1}a_2)}, && (24)
\end{aligned}$$

4.2 Construction of the path crossed complex $(P^{\mathsf{Crs}} X, \partial^P)$

Definition 4.3. *Let A be an algebra in the category of crossed complexes, that is, a crossed complex A with an associative multiplication given by a homomorphism*

$$\mu : \ A \otimes A \to A.$$

Let M be a left A-module, that is, a crossed complex M with a homomorphism

$$\alpha : \ A \otimes M \to M$$

that respects the multiplication. We say that a subset B of M is a basis for the A-module M if the set

$$\{\alpha(a \otimes b); \quad a \in A, \ b \in B\}$$

forms a set of generators of the crossed complex M. The action of A on M is then given by multiplication in A,

$$\alpha(a \otimes \alpha(a' \otimes b)) = \alpha(a \ a' \otimes b).$$

Our main example of a module with a basis will be the path crossed complex $P^{\mathsf{Crs}}X$. We would like this to have the same generators as the usual non-twisted tensor product of $\Omega^{\mathsf{Crs}}X \otimes \pi X$,

$$x \otimes b_m \quad = \quad \left(\prod_{i=1}^{r} s^{-1} a^{(i)} \right) \otimes b_m$$

where $a^{(i)} \in X_{n_i+1}$ and $b_m \in X_m$. Therefore we can choose a basis

$$B = \{(\varnothing \otimes b) \mid b \text{ a non-degenerate element of } X\}.$$

The action of $\Omega^{\mathsf{Crs}}X$ on $B \subset P^{\mathsf{Crs}}X$ is given by

$$\alpha(x \otimes (\varnothing \otimes b_m)) = x \otimes b_m.$$

The elements $x \otimes b_m$ gives the set of generators that we want, and so we see that B is a basis.

Definition 4.4. *Consider the twisted tensor product $P^{\mathsf{Crs}}X = \Omega^{\mathsf{Crs}}X \otimes_\phi \pi X$ of the free crossed chain algebra $\Omega^{\mathsf{Crs}}X$ and the fundamental crossed complex πX, defined as the free $\Omega^{\mathsf{Crs}}X$-module with basis*

$$B = \{(\varnothing \otimes b_m) : b_m \text{ is a generator of } \pi X\},$$

whose boundaries are defined by the following formulas

$$\partial_2^P(\varnothing \otimes b_2) = (s^{-1}b_2 \otimes *)$$

$$\partial_3^P(\varnothing \otimes b_3) = (s^{-1}b_3 \otimes *) - (\varnothing \otimes d_3 b_3) - (\varnothing \otimes d_1 b_3) + (\varnothing \otimes d_2 b_3) + (\varnothing \otimes d_0 b_3)$$

$$\partial_n^P(\varnothing \otimes b_m) = \sum_{i=1}^{m}(-1)^i(\varnothing \otimes d_i b_n) + \sum_{i=1}^{m}(s^{-1}b_{0\ldots i} \otimes b_{i\ldots n}), \quad n \geqslant 4.$$

We call $P^{\mathsf{Crs}}X$, the path crossed complex of X. In the rest of this section we will make this definition more explicit. In the definition we have only given the definition of the boundary map on generators of $P^{\mathsf{Crs}}X$ of the form $(\varnothing \otimes b_m)$. In the next two theorems we use the fact that

$$\alpha : \Omega^{\mathsf{Crs}}X \otimes P^{\mathsf{Crs}}X \longrightarrow P^{\mathsf{Crs}}X, \qquad \alpha(x \otimes (x' \otimes b)) \;=\; (x \cdot x') \otimes b$$

is a homomorphism of crossed complexes. Therefore we can see that

$$\partial^P(x \otimes b_m) = \partial^P \alpha\big(x \otimes (\varnothing \otimes b_m)\big) = \alpha \partial^P(x \otimes (\varnothing \otimes b_m))$$

If $x = \varnothing$ this does not tell us anything new. In general, if $m, n \geqslant 3$, we know that the formula will have the form

$$\partial^P(x \otimes b_m) = \partial^\Omega x \otimes b_m + (-1)^{|x|}\alpha(x \otimes \partial^P(\varnothing \otimes b_m)).$$

In the case $m = 0$, $b_m = *$, we see that

$$\partial^P(x \otimes b_m) = \partial^\Omega(x) \otimes *$$

where the right hand side must be expanded using (1) and (3) from Definition 2.33 together with the formulas for ∂^Ω from the previous section. This is done in Theorem 4.5 below. Then in Theorem 4.6 we will give a general formula for

$$\partial^P(s^{-1}a_n \otimes b_m).$$

Theorem 4.5. *The boundary ∂_n^P of elements with generators of the forms*

$$p_n = (\prod s^{-1}a_{n_i+1} \otimes *), \qquad a_{n_i+1} \in X_{n_i+1}, \qquad n = \sum n_i$$

in $P_n^{\mathsf{Crs}}X$ for a 1-reduced simplicial set X are as follows.

1. *In dimension 2 we have two forms of generators and using (19) and (20) we find*

   I. $\partial_2^P(s^{-1}a_3 \otimes *) = -(s^{-1}a_{123} \otimes *) - (s^{-1}a_{013} \otimes *) + (s^{-1}a_{023} \otimes *) + (s^{-1}a_{012} \otimes *)$,

   II. $\partial_2^P(s^{-1}a_2 s^{-1}a_2' \otimes *) = -(s^{-1}a_2' \otimes *) - (s^{-1}a_2 \otimes *) + (s^{-1}a_2' \otimes *) + (s^{-1}a_2 \otimes *)$.

2. *In dimension 3 we use equations (21)–(24)*

   I. $$\begin{aligned}\partial_3^P(s^{-1}a_4 \otimes *) = &-(s^{-1}a_{0134} \otimes *)^{(s^{-1}a_{123}\otimes *)} - (s^{-1}a_{1234} \otimes *)\\ &+ (s^{-1}a_{0124} \otimes *)^{(s^{-1}a_{234}\otimes *)} + (s^{-1}a_{012}s^{-1}a_{234} \otimes *)\\ &+ (s^{-1}a_{0234} \otimes *)^{(s^{-1}a_{012}\otimes *)} - (s^{-1}a_{0123} \otimes *),\end{aligned}$$

   II. $$\begin{aligned}\partial_3^P(s^{-1}a_3 s^{-1}a_2 \otimes *) = &-(s^{-1}a_{123}s^{-1}a_2 \otimes *) + (s^{-1}a_3 \otimes *)^{(s^{-1}a_2\otimes *)}\\ &+ (s^{-1}a_{012}s^{-1}a_2 \otimes *) + (s^{-1}a_{023}s^{-1}a_2 \otimes *)^{(s^{-1}a_{012}\otimes *)}\\ &- (s^{-1}a_3 \otimes *) - (s^{-1}a_{013}s^{-1}a_2 \otimes *)^{(s^{-1}a_{123}\otimes *)},\end{aligned}$$

   III. $$\begin{aligned}\partial_3^P(s^{-1}a_2 s^{-1}a_3 \otimes *) = &-(s^{-1}a_3 \otimes *) + (s^{-1}a_2 s^{-1}a_{013} \otimes *)^{(s^{-1}a_{123}\otimes *)}\\ &+ (s^{-1}a_2 s^{-1}a_{123} \otimes *) + (s^{-1}a_3 \otimes *)^{(s^{-1}a_{012}\otimes *)}\\ &- (s^{-1}a_2 s^{-1}a_{012}' \otimes *) - (s^{-1}a_2 s^{-1}a_{023} \otimes *)^{(s^{-1}a_{012}\otimes *)}\end{aligned}$$

   IV. $$\begin{aligned}\partial_3^P(s^{-1}a_2 s^{-1}a_2' s^{-1}a_2'' \otimes *) = &-(s^{-1}a_2 s^{-1}a_2' \otimes *) - (s^{-1}a_2 s^{-1}a_2'' \otimes *)^{(s^{-1}a_2'\otimes *)}\\ &- (s^{-1}a_2' s^{-1}a_2'' \otimes *) + (s^{-1}a_2 s^{-1}a_2' \otimes *)^{(s^{-1}a_2''\otimes *)}\\ &+ (s^{-1}a_2 s^{-1}a_2'' \otimes *) + (s^{-1}a_2' s^{-1}a_2'' \otimes *)^{(s^{-1}a_2\otimes *)}.\end{aligned}$$

3. *For dimension $n \geqslant 4$ we can find ∂_n^P inductively,*

$$(\partial_n^P \prod s^{-1} a_{n_i+1}^{(i)} \otimes *) = (s^{-1} \partial_{n_i}^P a_{n_i+1}^{(1)} \prod a_{n_i+1}^{(i-1)} \otimes *)$$
$$+(-1)^{|a_{n_i+1}^{(1)}|}(s^{-1} a_{n_i+1}^{(1)} \partial_{n_i}^P \prod a_{n_i+1}^{(i-1)} \otimes *)$$

$$(s^{-1} \partial_{n_i}^P a_{n_i+1}^{(1)} \prod a_{n_i+1}^{(i-1)} \otimes *) = \textstyle\sum_{j=1}^{n_i} (-1)^{j+1} \big(s^{-1} d_j a_{n_i+1}^{(1)} \prod a_{n_i+1}^{(i-1)} \otimes *\big)^{(\gamma_j)^{(1)}}$$
$$- \textstyle\sum_{j=1}^{n_i} (-1)^{j+1} (s^{-1} a_{0\ldots j}^{(1)} s^{-1} a_{j\ldots n_i+1}^{(1)} \prod a_{n_i+1}^{(i-1)} \otimes *)$$
$$(\gamma_j)^{(1)} = (s^{-1} a_{j-1\ j\ j+1}^{(1)} \otimes *)$$

Proof. We will just prove **(1-I)**, and **(2-I)**, because the other cases will be similar but longer.

1. **(1-I)**

   $(s^{-1} a_3 \otimes *) = \alpha(s^{-1} a_3 \otimes (\varnothing \otimes *))$, so

   $$\begin{aligned}
   \partial_2^P(s^{-1} a_3 \otimes *) &= \partial_2^P \alpha(s^{-1} a_3 \otimes (\varnothing \otimes *)) = \alpha(\partial_2^\Omega(s^{-1} a_3 \otimes (\varnothing \otimes *))) \text{ by (19)} \\
   &= \alpha\big((-s^{-1} a_{123} - s^{-1} a_{013} + s^{-1} a_{023} + s^{-1} a_{012}) \otimes (\varnothing \otimes *)\big) \\
   &= \alpha\big(- (s^{-1} a_{123} \otimes (\varnothing \otimes *)) - (s^{-1} a_{013} \otimes (\varnothing \otimes *)) \\
   &\qquad + (s^{-1} a_{023} \otimes (\varnothing \otimes *)) + (s^{-1} a_{012} \otimes (\varnothing \otimes *))\big) \\
   &= -(s^{-1} a_{123} \otimes *) - (s^{-1} a_{013} \otimes *) + (s^{-1} a_{023} \otimes *) + (s^{-1} a_{012} \otimes *).
   \end{aligned}$$

2. **(2-I)**

   $(s^{-1} a_4 \otimes *) = \alpha(s^{-1} a_4 \otimes (\varnothing \otimes *))$, hence

   $$\begin{aligned}
   &\partial_3^P(s^{-1} a_4 \otimes *) = \partial_3^P \alpha(s^{-1} a_4 \otimes (\varnothing \otimes *)) = \alpha(\partial_3^\Omega(s^{-1} a_4 \otimes (\varnothing \otimes *))) \text{ by (21)} \\
   &\quad = \alpha\Big(\big(- s^{-1} a_{0123} - s^{-1} a_{0134}^{a_{123}} - s^{-1} a_{1234} + s^{-1} a_{0124}^{a_{234}} + (s^{-1} a_{012} \otimes s^{-1} a_{234}) \\
   &\qquad\qquad + s^{-1} a_{0234}^{a_{012}}\big) \otimes (\varnothing \otimes *)\Big) \\
   &\quad = \alpha\big(- (s^{-1} a_{0123} \otimes (\varnothing \otimes *)) - (s^{-1} a_{0134}^{a_{123}} \otimes (\varnothing \otimes *)) - (s^{-1} a_{1234} \otimes (\varnothing \otimes *))
   \end{aligned}$$

$$+\,(s^{-1}a^{a_{234}}_{0124}\otimes(\varnothing\otimes *))+(s^{-1}a_{012}\otimes(s^{-1}a_{234}\otimes *))+(s^{-1}a^{a_{012}}_{0234})\otimes(\varnothing\otimes *)\big)$$

$$=-(s^{-1}a_{0123}\otimes *)-(s^{-1}a^{a_{123}}_{0134}\otimes *)-(s^{-1}a_{1234}\otimes *)+(s^{-1}a^{a_{234}}_{0124}\otimes *)$$

$$+\,(s^{-1}a_{012}s^{-1}a_{234}\otimes *)+(s^{-1}a^{a_{012}}_{0234}\otimes *).$$

□

We give now a formula for ∂^P of general element $(s^{-1}a_{n+1}\otimes b_m)$.

Theorem 4.6. *Let X be a simplicial set with $X_0=X_1=\{*\}$. $P^{\mathsf{Crs}}X=\Omega^{\mathsf{Crs}}X\otimes_\phi \pi X$ is a path crossed complex with generators $(\varnothing\otimes b_m)$, with the differential defined on an element of form $(s^{-1}a_n\otimes b_m)_q,\quad q=m+n-1$, by the following formula:*

1. $\partial^P_3(s^{-1}a_2\otimes b_2)=-(s^{-1}a_2s^{-1}b_2\otimes *)-(\varnothing\otimes b_2)+(\varnothing\otimes b_2)^{(s^{-1}a_2\otimes *)}$,

2. $$\begin{aligned}\partial^P_q(s^{-1}a_n\otimes b_m)&=\textstyle\sum_{i=1}^m(-1)^{i+n-1}(s^{-1}a_n\otimes d_ib_m)\\ &+(-1)^{n-1}\textstyle\sum_{i=1}^m\big(s^{-1}a_ns^{-1}b_{0\dots i}\otimes b_{i\dots m}\big)\\ &+\textstyle\sum_{j=1}^{n-1}(-1)^{j+1}\big(s^{-1}d_ja_n\otimes b_m\big)^{\gamma_j}\\ &-\textstyle\sum_{j=1}^{n-1}(-1)^{j+1}(s^{-1}a_{0\dots j}s^{-1}a_{j\dots n}\otimes b_m)\end{aligned}$$

   where

   $$\gamma_j=(s^{-1}a_{j-1\;j\;j+1}\otimes *).$$

By induction, this specifies the differential on the whole of $P^{\mathsf{Crs}}X$.

Proof. We use the definition of the ordinary tensor product of crossed complexes that we introduced in Definition 2.33. Let us start with dimension $q=3=(1+2)$.

$$\partial^P_3(s^{-1}a_2\otimes b_2)=\partial^P_3\big(\alpha(s^{-1}a_2\otimes(\varnothing\otimes b_2))\big)=\alpha\partial^P_3(s^{-1}a_2\otimes(\varnothing\otimes b_2))$$

$$=-\alpha(\varnothing\otimes(\varnothing\otimes b_2))+\alpha(\varnothing\otimes(\varnothing\otimes b_2))^{(s^{-1}a_2\otimes *)}-\alpha(s^{-1}a_2\otimes\partial^P(\varnothing\otimes b_2))$$

$$=-\alpha(\varnothing\otimes(\varnothing\otimes b_2))+\alpha(\varnothing\otimes(\varnothing\otimes b_2))^{(s^{-1}a_2\otimes *)}-\alpha(s^{-1}a_2\otimes(s^{-1}b_2\otimes *))\text{Definition (4.4)}$$

$$=-(\varnothing\otimes b_2)+(\varnothing\otimes b_2)^{(s^{-1}a_2\otimes *)}-(s^{-1}a_2s^{-1}b_2\otimes *).$$

Now we need to prove the theorem when $q=1+m$

$$\partial^P_{1+m}(s^{-1}a_2\otimes b_m)=\partial^P_{1+m}\big(\alpha(s^{-1}a_2\otimes(\varnothing\otimes b_m))\big)=\alpha\partial^P_{1+m}(s^{-1}a_2\otimes(\varnothing\otimes b_m))$$

$$=-\alpha(\varnothing\otimes(\varnothing\otimes b_m))+\alpha(\varnothing\otimes(\varnothing\otimes b_m))^{(s^{-1}a_2\otimes *)}-\alpha(s^{-1}a_2\otimes\partial^P_m(\varnothing\otimes b_m))$$

$$=-\alpha(\varnothing\otimes(\varnothing\otimes b_m))+\alpha(\varnothing\otimes(\varnothing\otimes b_m))^{(s^{-1}a_2\otimes *)}$$

$$+\alpha\bigg(\textstyle\sum_{i=1}^m(-1)^{i+1}\big(s^{-1}a_2\otimes(\varnothing\otimes d_ib_m)\big)\bigg)-\alpha\bigg(\textstyle\sum_{i=1}^m(s^{-1}a_2\otimes(s^{-1}b_{0\dots i}\otimes b_{i\dots m}))\bigg)$$

$$=-(\varnothing\otimes b_m)+(\varnothing\otimes b_m)^{(s^{-1}a_2\otimes *)}+\textstyle\sum_{i=1}^m(-1)^{i+1}(s^{-1}a_2\otimes d_ib_m)$$

$$-\textstyle\sum_{i=1}^m(s^{-1}a_2s^{-1}b_{0\dots i}\otimes b_{i\dots m}).$$

$$\partial^P_{2+m}(s^{-1}a_3\otimes b_m)=\partial^P_{2+m}\big(\alpha(s^{-1}a_3\otimes(\varnothing\otimes b_m))\big)=\alpha\partial^P_{2+m}(s^{-1}a_3\otimes(\varnothing\otimes b_m))$$

$$=-\alpha(s^{-1}a_{123}\otimes(\varnothing\otimes b_m))-\alpha(s^{-1}a_{013}\otimes(\varnothing\otimes b_m))+\alpha(s^{-1}a_{023}\otimes(\varnothing\otimes b_m))+\alpha(s^{-1}a_{012}\otimes(\varnothing\otimes b_m))-\alpha(s^{-1}a_3\otimes\partial^P_m(\varnothing\otimes b_m))$$

$$=-\alpha(s^{-1}a_{123}\otimes(\varnothing\otimes b_m))-\alpha(s^{-1}a_{013}\otimes(\varnothing\otimes b_m))+\alpha(s^{-1}a_{023}\otimes(\varnothing\otimes b_m))+\alpha(s^{-1}a_{012}\otimes(\varnothing\otimes b_m))$$

$$+\alpha\bigg(\textstyle\sum_{i=1}^m(-1)^{i+2}\big(s^{-1}a_3\otimes(\varnothing\otimes d_ib_m)\big)\bigg)+(-1)^2\alpha\bigg(\textstyle\sum_{i=1}^m(s^{-1}a_3\otimes(s^{-1}b_{0\dots i}\otimes b_{i\dots m}))\bigg)$$

$$=-(s^{-1}a_{123}\otimes b_m)-(s^{-1}a_{013}\otimes b_m)+(s^{-1}a_{023}\otimes b_m)+(s^{-1}a_{012}\otimes b_m)+\textstyle\sum_{i=1}^m(-1)^i(s^{-1}a_3\otimes d_ib_m)$$

$$+\textstyle\sum_{i=1}^m(s^{-1}a_3s^{-1}b_{0\dots i}\otimes b_{i\dots m}).$$

And, for dimension $q\geqslant 4,\ n\geqslant 3$ we have

$$\partial^P_q(s^{-1}a_n\otimes b_m)=\partial^P_q\big(\alpha(s^{-1}a_n\otimes(\varnothing\otimes b_m))\big)=\alpha\partial^P_q(s^{-1}a_n\otimes(\varnothing\otimes b_m))$$

$$=\alpha(\partial_{n-1}s^{-1}a_n\otimes(\varnothing\otimes b_m))+(-1)^{n-1}\alpha(s^{-1}a_n\otimes\partial^P_m(\varnothing\otimes b_m))$$

$$=\alpha\big(\big(\textstyle\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}d_ia_n)^{\gamma_i}$$

$$-\textstyle\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}a_{0\dots i}s^{-1}a_{i\dots n})\big)\otimes(\varnothing\otimes b_m)\big)$$

$$+(-1)^{n-1}\alpha(s^{-1}a_n\otimes\big(\textstyle\sum_{i=1}^m(-1)^i(\varnothing\otimes d_ib_m)$$

$$+\textstyle\sum_{i=1}^n(s^{-1}b_{0\dots i}\otimes b_{i\dots m})\big)$$

$$=\alpha\bigg(\textstyle\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}d_ia_n)^{\gamma_i}\otimes(\varnothing\otimes b_m)\bigg)$$

$$-\alpha\Bigg(\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}a_{0\ldots i}s^{-1}a_{i\ldots n})\otimes(\varnothing\otimes b_m)\Bigg)$$
$$+(-1)^{n-1}\alpha\Bigg(s^{-1}a_n\otimes\sum_{i=1}^{m}(-1)^i(\varnothing\otimes d_ib_m)\Bigg)$$
$$+(-1)^{n-1}\alpha\Bigg(\sum_{i=1}^{m}(s^{-1}a_ns^{-1}b_{0\ldots i}\otimes(\varnothing\otimes b_{i\ldots m})\Bigg)$$
$$=\sum_{i=1}^{n-1}(-1)^{i+1}\alpha((s^{-1}d_ia_n)^{\gamma_i}\otimes(\varnothing\otimes b_m))$$
$$-\sum_{i=1}^{n-1}(-1)^{i+1}\alpha((s^{-1}a_{0\ldots i}s^{-1}a_{i\ldots n})\otimes(\varnothing\otimes b_m))$$
$$+(-1)^{n-1}\sum_{i=1}^{m}(-1)^i\alpha(s^{-1}a_n\otimes(\varnothing\otimes d_ib_m))$$
$$+(-1)^{n-1}(\sum_{i=1}^{m}\alpha(s^{-1}a_ns^{-1}b_{0\ldots i}\otimes(\varnothing\otimes b_{i\ldots m}))$$
$$=\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}d_ia_n\otimes b_m)^{\gamma_i}-\sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}a_{0\ldots i}s^{-1}a_{i\ldots n})\otimes b_m)$$
$$+\sum_{i=1}^{m}(-1)^{i+n-1}(s^{-1}a_n\otimes d_ib_m)$$
$$+(-1)^{n-1}\sum_{i=1}^{m}(s^{-1}a_ns^{-1}b_{0\ldots i}\otimes b_{i\ldots m})$$

□

Proposition 4.7. *The boundary map $\partial_n^P : P_n^{\mathsf{Crs}} \to P_{n-1}^{\mathsf{Crs}}$ which was defined in Definition 4.4 is a differential on the crossed complex group $P_n^{\mathsf{Crs}}X$.*

Proof. We will just prove that $\partial_{n-1}^P\partial_n^P(\varnothing\otimes b_n)=0$, for all $n\geqslant 3$.

We start with dimension $n=3$, and use Definition 4.4

$$\partial_2^P\partial_3^P(\varnothing\otimes b_3)=\partial_2^P\Bigg((s^{-1}b_3\otimes *)-(\varnothing\otimes d_3b_3)-(\varnothing\otimes d_1b_3)+(\varnothing\otimes d_2b_3)+(\varnothing\otimes d_0b_3)\Bigg)$$
$$=-(s^{-1}b_{123}\otimes *)\quad-(s^{-1}b_{013}\otimes *)\quad+(s^{-1}b_{023}\otimes *)\quad+(s^{-1}b_{012}\otimes *)$$
$$-(s^{-1}b_{012}\otimes *)\quad-(s^{-1}b_{023}\otimes *)\quad+(s^{-1}b_{013}\otimes *)\quad+(s^{-1}b_{123}\otimes *)$$
$$=0\qquad\text{(this also from Theorem 4.5).}$$

Now we need to show that $\partial_{n-1}^P\partial_n^P(\varnothing\otimes b_n)=0,\qquad n\geqslant 4.$

$$\partial^P_{n-1}\partial^P_n(\varnothing \otimes b_n) = \partial^P_{n-1}\Bigg(\textstyle\sum_{i=1}^n (-1)^i(\varnothing \otimes d_i b_n) + \sum_{i=1}^n (s^{-1}b_{0\ldots i} \otimes b_{i\ldots n})\Bigg) =$$

$$\sum_{i=1}^n (-1)^i \partial^P_{n-1}(\varnothing \otimes d_i b_n) + \sum_{i=1}^n \partial^P_{n-1}(s^{-1}b_{0\ldots i} \otimes b_{i\ldots n})$$

The terms of $\partial^P_n(\varnothing \otimes b_n)$ have the following form:

$$(-1)^i(\varnothing \otimes \widehat{b_n}), \qquad (1)$$

$$(s^{-1}b_{0\ldots i} \otimes b_{i\ldots n}), \qquad (2)$$

and the last element will be

$$(s^{-1}b_n \otimes *), \qquad (3)$$

where $\widehat{b_n}$ is the simplex b_n but after deleting the vertex i. When we take ∂^P_{n-1} for the terms (1) the elements which come out will be the same forms of elements in $(1), (2)$ and (3) but related to $\widehat{b_n}$ and $j = 0 \ldots n-1$. They are:

$$(-1)^{i+j}(\varnothing \otimes d_j\widehat{b_n}), \qquad (1-1)$$

$$(-1)^i(s^{-1}\widehat{b}_{0\ldots j} \otimes \widehat{b}_{j\ldots n}), \qquad (1-2)$$

and the last element will be

$$(-1)^i(s^{-1}\widehat{b}_n \otimes *), \qquad i = 1 \ldots n-1 \qquad (1-3)$$

all the terms in $(1-1)$ will cancel each other under the laws of simplices $(d_i d_j = d_{j-1}d_i)$.

The terms of

$$\partial^P_{n-1}(2) = \partial^P_{n-1}(s^{-1}b_{0\ldots i} \otimes b_{i\ldots n}) = \textstyle\sum_{j=1}^{n-i}(-1)^{j+i-1}(s^{-1}b_{0\ldots i} \otimes d_j b_{i\ldots n})$$

$$+ (-1)^{i-1}\textstyle\sum_{j=1}^{n-i}(s^{-1}b_{0\ldots i}s^{-1}b_{i\ldots j+i} \otimes b_{j+i\ldots n})$$

$$+ \textstyle\sum_{k=1}^{i}(-1)^{k+1}(s^{-1}d_k b_{0\ldots i} \otimes b_{i\ldots n})^{\gamma_k}$$

$$- \sum_{k=1}^{i}(-1)^{k+1}(s^{-1}d_k b_{0\ldots k}s^{-1}b_{k\ldots i}\otimes b_{i\ldots n})$$

have the following forms:

$$(-1)^{j+i-1}(s^{-1}b_{0\ldots i}\otimes d_j b_{i\ldots n}) \qquad (2-1)$$

$$(-1)^{i-1}(s^{-1}b_{0\ldots i}s^{-1}b_{i\ldots j+i}\otimes b_{j+i\ldots n}) \qquad (2-2)$$

$$(-1)^{k+1}(s^{-1}d_k b_{0\ldots i}\otimes b_{i\ldots n})^{\gamma_k} \qquad (2-3)$$

$$(-1)^{k+1}(s^{-1}d_k b_{0\ldots k}s^{-1}b_{k\ldots i}\otimes b_{i\ldots n}) \qquad (2-4)$$

If we take $\partial_{n-1}(3)$ we will use Theorem 4.5 which the formula is:

$$\partial_{n-1}(s^{-1}b_n\otimes *) = \sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}d_i b_n\otimes *)^{\gamma_i} - \sum_{i=1}^{n-1}(-1)^{i+1}(s^{-1}b_{0\ldots i}s^{-1}b_{i\ldots n}\otimes *)$$

which consists of two forms of elements,

$$(-1)^{i+1}(s^{-1}d_i b_n\otimes *)^{\gamma_i} \qquad (3-1)$$

and

$$(-1)^{i+2}(s^{-1}b_{0\ldots i}s^{-1}b_{i\ldots n}\otimes *) \qquad (3-2)$$

The elements in both terms $(1-3)$ and $(3-1)$ will cancel each other because of the fact that $P_2^{\mathsf{Crs}}X$ acts trivially on $P_n^{\mathsf{Crs}}X, n\geqslant 3$ so all elements on $(1-3)$ and $(3-1)$ have the same expression, but with opposite signs.

If $i=1$, the terms in $(2-1)$ will be $(-1)^j(\varnothing\otimes d_j b_{1\ldots n})$ which similar to the elements in $(1-2)$ where $j=1$ which are have the form $(-1)^i(\varnothing\otimes \widehat{b}_{1\ldots n})$, so all terms in $(1-2)$, and $(2-1)$ cancel each other in pairs.

The terms of $(2-2)$ and $(2-4)$ are equals but with opposite sign, so they cancel. In $(2-3)$ if $n-i=1$ the type of elements in this term will have the forms $(-1)^{k+1}(s^{-1}(d_k b_{0\ldots n-1}\otimes *)$ which are the same elements on $(3-2)$ in case of $i=2$ or $i=n-1$, since in this case the elements on $(3-2)$ will have the form $-(s^{-1}b_{1\ldots n}\otimes *)$ or $(-1)^{n+1}(s^{-1}b_{0\ldots n-1}\otimes *)$ also here we used the fact that $P_2^{\mathsf{Crs}}X$ acts trivially on $P_n^{\mathsf{Crs}}X$. otherwise the form of elements in $(3-2)$ will have the same form of elements in $(2-4)$ but with opposite sign, so they cancel in pairs. $\square$

4.3 Construction of the contracting homotopy

Recall that the interval object $\mathcal{I}$ in the category of crossed complexes is given by the fundamental crossed complex of the 1-simplex, $\mathcal{I} = \pi(\Delta[1])$. This has object set $\mathcal{I}_0 = \{0,1\}$ and just one generator $(\sigma : 0 \to 1) \in \mathcal{I}_1$.

Definition 4.8. *Two homomorphisms $f, g : C \to D$ are homotopic if there exists a homotopy $h : f \simeq g$ between f and g. That is, if there is a homomorphism*

$$h : \pi(\Delta[1]) \otimes C \to D$$

such that $hi_0 = f$ and $hi_1 = g$ [31].

$$\begin{tikzcd} C \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & \pi(\Delta[1]) \otimes C \arrow[r, "h"] & D \end{tikzcd}$$

Definition 4.9. *Let C be a crossed complex with $C_0 = \{*\}$. A contracting homotopy is a homotopy h between the constant homomorphism $0_* : C \to C$ and the identity function id_C. That is, it is a homomorphism*

$$h : \pi(\Delta[1]) \otimes C \to C$$

that satisfies:

i. $h(0 \otimes c) = 0_*$,

ii. $h(1 \otimes c) = c$,

We will also assume that, for $* \in C_0$, $h(\sigma \otimes *) = 0_* \in C_1$.

In other words, given a contracting homotopy we have $h : * \simeq id_C$. So C is contractible: there is a homotopy equivalence

$$\begin{tikzcd} C_n \arrow[loop left, "h : * \simeq id_C"] \arrow[r, shift right] & \{*\} \arrow[l, shift right] \end{tikzcd}$$

Given a contracting homotopy

$$h : \pi\Delta[1] \otimes C \to C$$

we consider the family of functions

$$\eta_n : C_n \to C_{n+1}, \ (n \geqslant 1)$$

defined by

$$\eta_n(c) = h(\sigma \otimes c), \quad (c \in C_n)$$

Conversely, given a family of functions η_n, we could define a contracting homotopy

$$h(0 \otimes c) = 0_*, \quad h(1 \otimes c) = c, \quad h(\sigma \otimes c) = \eta(c)$$

In order for h to be well defined and commute with $\partial : C_n \to C_{n-1}$, the family must satisfy some properties.

Proposition 4.10. *The family of functions* $\eta_n : C_n \to C_{n+1}$*, which is defined as* $h(\sigma \otimes c_n) = \eta(c_n), \quad (n \geqslant 1)$ *satisfies the properties that*

1. $\partial\eta(c_1) = c_1$,

2. $\partial\eta(c_n) = c_n - \eta\partial(c_n)$,

3. $\eta(c_n + c'_n) = \eta(c_n) + \eta(c'_n)$,

4. $\eta(c_n^{c_1}) = \eta(c_n)$.

and $\eta(*) = id_C$.

Remark 4.11. *The homotopy η which was defined in definition 4.9 satisfies the properties of Proposition 4.10* $(1-4)$ *if and only if h is well defined and commutes with ∂.*

Proof. $\Rightarrow$) Suppose that the contractable homomorphism map h, is well defined and commutes with ∂ then we need to prove that η satisfies the Properties $(1-4)$ of Proposition 4.10.

1. $\partial\eta(c_1) = \partial h(\sigma\otimes c_1) = h\partial(\sigma\otimes c_1) = -h(0\otimes c_1) - h(\sigma\otimes *) + h(1\otimes c_1) + h(\sigma\otimes *) = c_1$ (from Definition 2.33 and Definition 4.9),

2. $\partial\eta(c_n) = \partial h(\sigma\otimes c_n) = h\partial(\sigma\otimes c_n) = h(\partial_1\sigma\otimes c_n) - h(\sigma\otimes\partial c_n) = -h(\mathrm{src}(\sigma)\otimes c_n) + h((\mathrm{targ}(\sigma)\otimes c_n)^{(\sigma\otimes *)}) - h(\sigma\otimes\partial c_n) = h(0\otimes c_n) + h(1\otimes c_n)^{h(\sigma\otimes *)} - h(\sigma\otimes\partial c_n) = c_n - \eta\partial c_n$ (this is from Definition 4.9 (i) and (ii)),

3. by use of Definition 2.34 we have, $\eta(c_n + c'_n) = h(\sigma\otimes(c_n + c'_n)) = h\big((\sigma\otimes c_n)^{(\sigma_\otimes \mathrm{src}\, c'_n)} + (\sigma\otimes c'_n)\big) = h(\sigma\otimes c_n)^{(\sigma_\otimes c'_n)} + h(\sigma\otimes c'_n) = \eta(c_n)^{h(0\otimes c'_n)} + \eta(c'_n) = \eta(c_n) + \eta(c'_n)$,

4. because $\eta(c_n^{c_1}) = h(\sigma\otimes c_n^{c_1}) = h((\sigma\otimes c_n)^{(0\otimes c_1)}) = h(\sigma\otimes c_n)^{h(0\otimes c_1)} = \eta(c_n)^{0_*} = \eta(c_n)$.

$\Leftarrow$) Conversely, if we have $\eta : C_n \to C_{n+1}$ a family of functions that satisfies the properties $(1-4)$ of Proposition 4.10, then we need to prove that the contracting homotopy h given as $h(0\otimes c) = 0_*$, $h(\sigma\otimes c) = \eta(c)$ and $h(1\otimes c) = c$, is well defined and commutes with ∂.

1. $h(\sigma\otimes(c_n + c'_n)) = \eta(c_n + c'_n) = \eta(c_n) + \eta(c'_n) = h(\sigma\otimes c_n) + h(\sigma\otimes c'_n)$, (by Proposition 4.10 (3)).

2. $\partial h(\sigma \otimes c_n) = \partial \eta(c_n) = c_n - \eta\partial(c_n)$, (by Proposition 4.10)

$$= h(1 \otimes c_n) - h(\sigma \otimes \partial(c_n))$$

While

$h\partial(\sigma \otimes c_n) = h\big(- (0 \otimes c_n) + (1 \otimes c_n)^{(\sigma\otimes *)} - (\sigma \otimes \partial(c_n))\big) = -h(0 \otimes c_n) + h(1 \otimes c_n)^{h(\sigma\otimes *)} - h(\sigma \otimes \partial(c_n)) = h(1 \otimes c_n) - h(\sigma \otimes \partial(c_n))$.

$\square$

We want to define a family of functions $\eta : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$ which form a contracting homotopy.

Definition 4.12. *Let x be an element of $\Omega^{\mathsf{Crs}}X$ given by a word*

$$x = s^{-1}x_1 s^{-1}x_2 \dots s^{-1}x_k$$

where $x_i \in X_{n_i+1}$ and $\sum_{i=1}^{k} n_i = |x|$. Define $\eta : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$ as:

1. *$\eta(\varnothing \otimes *) = 0_{(\varnothing\otimes *)}$,*
2. *$\eta(xs^{-1}a_r \otimes *) = (-1)^{|x|}(x \otimes a_r)$,*
3. *$\eta(x \otimes b_n) = 0_{(\varnothing\otimes *)}$.*

Theorem 4.13. *The family of functions η in Definition 4.12 forms a contracting homotopy.*

Proof. The general form of the generators of $P^{\mathsf{Crs}}X$ of dimension n is

$$p_n = (xs^{-1}a_{q+1} \otimes b_r), \qquad |x| = m, \qquad m + q + r = n$$

1. Case $r = 0$, the generators of $P_n^{\mathsf{Crs}}X$ have the form

$$(xs^{-1}a_{q+1} \otimes *), \qquad |x| = m, \qquad m + q = n$$

i. In dimension 1 we have the only generator is $p_1 = (s^{-1}a_2 \otimes *)$, so by Definition 4.12 (2) we have

$$\eta(p_1) = (\varnothing \otimes a_2)$$

we need to show that $\partial_2\eta_1(p_1) = p_1$, so that the Proposition 4.10 holds.

$\partial_2\eta_1(p_1) = \partial_2(\varnothing \otimes a_2) = (s^{-1}a_2 \otimes *) = p_1$, (this from the definition of the boundary map ∂^P in Definition 4.4).

Hence for dimension 1 where $r = 0$, η satisfies Proposition 4.10.

ii. Assume $|x| \geqslant 3, \qquad q = 1, \qquad |x| + 1 = n,$

we need to show that $\partial_{n+1}\eta_n p_n = p_n - \eta_{n-1}\partial_n p_n$ where $p_n = (xs^{-1}a_2 \otimes *)$ so that the Proposition 4.10 holds.

From Definition 4.12 we have, $\eta_n(p_n) = \eta_n(xs^{-1}a_2 \otimes *) = (-1)^{|x|}(x \otimes a_2)$

and from Definition 4.6(2), the terms of $\partial_{n+1}(x \otimes a_2)$ will be

$$\partial_{n+1}(x \otimes a_2) = (-1)^{|x|}(xs^{-1}a_2 \otimes *) + (\partial^{\Omega}x \otimes a_2),$$

so the result of $\partial_{n+1}\eta_n(xs^{-1}a_2 \otimes *)$ is

$$\begin{aligned}\partial_{n+1}\eta_n(xs^{-1}a_2 \otimes *) &= (-1)^{|x|}\big((-1)^{|x|}(xs^{-1}a_2 \otimes *) + (\partial^{\Omega}x \otimes a_2)\big)\\ &= (xs^{-1}a_2 \otimes *) + (-1)^{|x|}(\partial^{\Omega}x \otimes a_2).\end{aligned}$$

Now;

first we will find $\partial_{|x|+1}(xs^{-1}a_2 \otimes *)$ by use Definition 2.33 the forms of the boundary of ordinary tensor product of crossed complexes,

$$\begin{aligned}\partial(xs^{-1}a_2 \otimes *) &= ((\partial^{\Omega}x)s^{-1}a_2 \otimes *) + (-1)^{|x|}\textstyle\sum_{i=1}^{1}(-1)^{i+1}(xs^{-1}d_i a_2 \otimes *)^{\gamma_i}\\ &\quad - (-1)^{|x|}\textstyle\sum_{i=1}^{1}(-1)^{i+1}(xs^{-1}a_{0\ldots i}s^{-1}a_{i\ldots 2} \otimes *)\\ &= ((\partial^{\Omega}x)s^{-1}a_2 \otimes *) + (-1)^{|x|+2}(x \otimes *)^{\gamma_i} - (-1)^{|x|+2}(x \otimes *)\\ &= ((\partial^{\Omega}x)s^{-1}a_2 \otimes *)\end{aligned}$$

that is because of Proposition 4.10(4) we can ignore the action.

$\eta\partial(xs^{-1}a_2 \otimes *) = (-1)^{|x|-1}(\partial^{\Omega}x \otimes a_2)$, (Definition 4.12).

And hence,

$$p_n - \eta\partial(p_n) = (xs^{-1}a_2 \otimes *) - (-1)^{|x|-1}(\partial^\Omega x \otimes a_2)$$
$$= (xs^{-1}a_2 \otimes *) + (-1)^{|x|}(\partial^\Omega x \otimes a_2).$$

again with dimension n where $|x| \geqslant 3, q = 1$, η satisfy Proposition 4.10.

iii. Assume $|x| \geqslant 2, \qquad q \geqslant 2, \qquad |x| + q = n$,

again we want to show that $\partial_{n+1}\eta_n p_n = p_n - \eta_{n-1}\partial_n p_n$ where $p_n = (xs^{-1}a_{q+1} \otimes *)$ from Definition 4.12(2) we have

$$\eta_n(p_n) = \eta_n(xs^{-1}a_{q+1} \otimes *) = (-1)^{|x|}(x \otimes a_{q+1}),$$
$$\partial_{n+1}\eta_n(xs^{-1}a_{q+1} \otimes *) = (-1)^{|x|}(\partial^\Omega x \otimes a_{q+1}) + (-1)^{|x|}\textstyle\sum_{i=1}^{q+1}(-1)^{i+|x|}(x \otimes d_i a_{q+1})$$
$$+ (-1)^{2|x|}\textstyle\sum_{i=1}^{q+1}(xs^{-1}a_{0\ldots i} \otimes a_{i\ldots q+1})$$
$$= (-1)^{|x|}(\partial^\Omega x \otimes a_{q+1}) + \textstyle\sum_{i=1}^{q+1}(-1)^{i}(x \otimes d_i a_{q+1}) + \sum_{i=1}^{q+1}(xs^{-1}a_{0\ldots i} \otimes a_{i\ldots q+1})$$

(this is by using the boundary laws of tensor products of crossed complexes Definition 2.33).

While,

$$\partial_n(p_n) = \partial_n(xs^{-1}a_{q+1} \otimes *) = ((\partial^\Omega x)s^{-1}a_{q+1} \otimes *) + (-1)^{|x|}(x\partial^\Omega a_{q+1} \otimes *)$$

(using Definition 2.33)

$$= ((\partial^\Omega x)s^{-1}a_{q+1} \otimes *) + (-1)^{|x|}\textstyle\sum_{i=1}^{q}(-1)^{i+1}(xs^{-1}d_i a_{q+1} \otimes *)$$
$$- (-1)^{|x|}\textstyle\sum_{i=1}^{q}(-1)^{i+1}(xs^{-1}a_{0\ldots i}s^{-1}a_{i\ldots q+1} \otimes *)$$

now from Proposition 4.10 (3) we have:

$$\eta_{n-1}\partial_n(xs^{-1}a_{q+1} \otimes *) = \eta_{n-1}((\partial^\Omega x)s^{-1}a_{q+1} \otimes *)$$
$$+ \eta_{n-1}\left((-1)^{|x|}\textstyle\sum_{i=1}^{q}(-1)^{i+1}(xs^{-1}d_i a_{q+1} \otimes *)\right)$$
$$- \eta_{n-1}\left((-1)^{|x|}\textstyle\sum_{i=1}^{q}(-1)^{i+1}(xs^{-1}a_{0\ldots i}s^{-1}a_{i\ldots q+1} \otimes *)\right)$$
$$= (-1)^{|x|-1}(\partial^\Omega x \otimes a_{q+1}) + (-1)^{2|x|}\textstyle\sum_{i=1}^{q}(-1)^{i+1}(x \otimes d_i a_{q+1})$$

$$- (-1)^{2|x|} \sum_{i=1}^{q} (-1)^{i+1} (xs^{-1}a_{0\ldots i} \otimes a_{i\ldots q+1})$$

Now

$$\begin{aligned} p_n - \eta_{n-1}\partial_n(p_n) &= (xs^{-1}a_{q+1} \otimes *) + \textstyle\sum_{i=1}^{q} (-1)^{i+1} (xs^{-1}a_{0\ldots i} \otimes a_{i\ldots q+1}) \\ &\quad - \textstyle\sum_{i=1}^{q} (-1)^{i+1} (x \otimes d_i a_{q+1}) - (-1)^{|x|-1} (\partial^{\Omega} x \otimes a_{q+1}) \\ &= (xs^{-1}a_{q+1} \otimes *) + \textstyle\sum_{i=1}^{q} (-1)^{i+1} (xs^{-1}a_{0\ldots i} \otimes a_{i\ldots q+1}) \\ &\quad + \textstyle\sum_{i=1}^{q} (-1)^{i} (x \otimes d_i a_{q+1}) + (-1)^{|x|} (\partial^{\Omega} x \otimes a_{q+1}) \end{aligned}$$

which satisfies property (2) of Proposition 4.10.

2. Case $r \neq 0$, we have two cases,

   i. If, $r = 2$, $\quad |x| = m$, $\quad m + q + 2 = n$,

      That is we have $p_n = (xs^{-1}a_{q+1} \otimes b_2)$ and we need to show that $\eta_{n-1}\partial_n p_n = p_n$ so that the Proposition 4.10, holds.

      $\partial_n(p_n) = \partial_n(xs^{-1}a_{q+1} \otimes b_2) = (\partial^{\Omega}(xs^{-1}a_{q+1}) \otimes b_2) + (-1)^n (xs^{-1}a_{q+1}s^{-1}b_2 \otimes *)$

      but from the Definition 4.12 (3) we have $\eta(\partial^{\Omega}(xs^{-1}a_{q+1}) \otimes b_2) = 0$, so, because of that we have

      $\eta\partial(xs^{-1}a_{q+1} \otimes b_2) = 0 + (-1)^n(-1)^n (xs^{-1}a_{q+1} \otimes b_2) = p_n,$

   ii. If, $r \geqslant 3$ and $n = q + |x| + r$

      Let $p_n = (xs^{-1}a_{q+1} \otimes b_r)$,

$$\begin{aligned} \partial(p_n) = \partial(xs^{-1}a_{q+1} \otimes b_r) &= ((\partial^{\Omega} x)s^{-1}a_{q+1} \otimes b_r) + (-1)^{|x|} (x\partial^{\Omega} a_{q+1} \otimes b_r) \\ &\quad + \textstyle\sum_{i=1}^{r} (-1)^{i+|x|+q} (xs^{-1}a_{q+1} \otimes d_i b_r) \\ &\quad + (-1)^{|x|+q} \textstyle\sum_{i=1}^{r-1} (xs^{-1}a_{q+1}s^{-1}b_{0\ldots i} \otimes b_{i\ldots r}) \\ &\quad + (-1)^{|x|+q} (xs^{-1}a_{q+1}s^{-1}b_r \otimes *) \end{aligned}$$

$$\begin{aligned} \eta_{n-1}\partial_n(xs^{-1}a_{q+1} \otimes b_r) &= 0 - 0 + 0 + (-1)^{2(|x|+q)} (xs^{-1}a_{q+1} \otimes b_r) \\ &= (xs^{-1}a_{q+1} \otimes b_r) = p_n, \end{aligned}$$

(Proposition 4.10 (3) and Definition 4.12 (3)).

$\square$

We will give two examples to help the reader understand the proof of the theorem above and furthermore to know how could calculate $\eta_n p_n$.

Example 4.14. *Here we introduce an example of Definition 4.12* (2)*.*
Let $p_3 = (s^{-1}a_3 s^{-1}a'_2 \otimes *)$,
we will use Proposition 4.10 (2) *to calculate* $\eta_3(s^{-1}a_3 s^{-1}a'_2 \otimes *)$.
First we find $\partial_3(s^{-1}a_3 s^{-1}a'_2 \otimes *)$ *by using Theorem 4.5* (II),

$$\partial_3(s^{-1}a_3 s^{-1}a'_2 \otimes *) = \sum_{j=1}^{2}(-1)^{j+1}(s^{-1}d_j a_3 s^{-1}a'_2 \otimes *)^{\gamma_j}$$
$$-\sum_{j=1}^{2}(-1)^{j+1}(s^{-1}a_{0\ldots j}s^{-1}a_{j\ldots 3}s^{-1}a'_2 \otimes *) + (-1)^2\sum_{i=1}^{1}(-1)^{i+1}(s^{-1}a_3 s^{-1}d_i a'_2 \otimes *)^{\gamma_i}$$
$$-(-1)^2\sum_{i=1}^{1}(-1)^{i+1}(s^{-1}a_3 s^{-1}a'_{0\ldots i}s^{-1}a'_{i\ldots 2} \otimes *)$$

$$= (s^{-1}a_{023}s^{-1}a'_2 \otimes *)^{(s^{-1}a_{012}\otimes *)} \quad -(s^{-1}a_{013}s^{-1}a'_2 \otimes *)^{(s^{-1}a_{123}\otimes *)} \quad -(s^{-1}a_{123}s^{-1}a'_2 \otimes *)$$
$$+(s^{-1}a_{012}s^{-1}a'_2 \otimes *) \quad +(s^{-1}a_3 \otimes *)^{(s^{-1}a'_{012}\otimes *)} \quad -(s^{-1}a_3 \otimes *)$$

The second step is find $\eta_2\partial_3 p_3$, *we will use Proposition 4.10* (3) *and* (4), *and Definition 4.12* (2)

$$\eta_2\partial_3(s^{-1}a_3 s^{-1}a'_2 \otimes *) = (-1)^1(s^{-1}a_{023}\otimes a'_2) \quad -(-1)^1(s^{-1}a_{013}\otimes a'_2) \quad -(-1)^1(s^{-1}a_{123}\otimes a'_2)$$
$$+(-1)^1(s^{-1}a_{012}\otimes a'_2) \quad +(\varnothing \otimes a_3) \quad -(\varnothing \otimes a_3)$$

Here we can ignore the action by using property (4) *of Proposition 4.10, and we get the result,*

$$= -1(s^{-1}a_{023}\otimes a'_2) \quad +(s^{-1}a_{013}\otimes a'_2) \quad +(s^{-1}a_{123}\otimes a'_2) \quad -(s^{-1}a_{012}\otimes a'_2)$$

now we need to find $(p_3 - \eta_2\partial_3 p_3)$

$$p_3 - \eta_2\partial_3(s^{-1}a_3s^{-1}a'_2 \otimes *) = (s^{-1}a_3s^{-1}a'_2 \otimes *) \quad + (s^{-1}a_{012} \otimes a'_2) \quad - (s^{-1}a_{123} \otimes a'_2)$$
$$- (s^{-1}a_{013} \otimes a'_2) \quad + (s^{-1}a_{023} \otimes a'_2)$$

so, the final step will be to calculate $(\partial_4\eta_3p_3)$ *by Proposition 4.10,*

$$\partial_4\eta_3(s^{-1}a_3s^{-1}a'_2 \otimes *) = p_3 - \eta_2\partial_3(s^{-1}a_3s^{-1}a'_2 \otimes *) = \partial_4(s^{-1}a_3 \otimes a'_2)$$

$\Leftrightarrow$

$$\eta_3(s^{-1}a_3s^{-1}a'_2 \otimes *) = (s^{-1}a_3 \otimes a'_2)$$

Example 4.15. *This example is related to the Definition 4.12 case* (3)*.*

Let $p_4 = (s^{-1}a_2 \otimes b_3)$ *we will calculate* $\eta_4(s^{-1}a_2 \otimes b_3)$ *by using the Proposition 4.10 property* (2)*.*

The calculation starts by finding $\eta_3\partial_4(s^{-1}a_2 \otimes b_3)$ *by calculating* $\partial_4(p_4)$*, by using Theorem 4.6*

$$\eta_3\partial_4(s^{-1}a_2 \otimes b_3) = \eta_3\bigg(\sum_{i=1}^{3}(-1)^{i+1}(s^{-1}a_2 \otimes d_ib_3) + (-1)^1 \sum_{i=1}^{3}(s^{-1}a_2s^{-1}b_{0\ldots i} \otimes b_{i\ldots 3})$$
$$+ \sum_{j=1}^{1}(-1)^{j+1}(s^{-1}d_ja_2 \otimes b_3)^{\gamma_j} - \sum_{j=1}^{1}(-1)^{j+1}(s^{-1}a_{0\ldots j}s^{-1}a_{j\ldots 2} \otimes b_3)\bigg)$$

$$= \eta_3\bigg((s^{-1}a_2 \otimes b_{023}) \quad -(s^{-1}a_2 \otimes b_{013}) \quad +(s^{-1}a_2 \otimes b_{012}) \quad -(s^{-1}a_2 \otimes b_{123})$$
$$- (s^{-1}a_2s^{-1}b_3 \otimes *) \quad + (\varnothing \otimes b_3)^{(s^{-1}a_{012} \otimes *)} \quad - (\varnothing \otimes b_3)\bigg)$$

From property (4) *of Proposition 4.10 this equals:*

$$= \eta_3(s^{-1}a_2 \otimes b_{023}) \quad - \eta_3(s^{-1}a_2 \otimes b_{013}) \quad + \eta_3(s^{-1}a_2 \otimes b_{012}) \quad - \eta_3(s^{-1}a_2 \otimes b_{123})$$
$$- \eta_3(s^{-1}a_2s^{-1}b_3 \otimes *) \quad + \eta_3(\varnothing \otimes b_3)^{(s^{-1}a_{012} \otimes *)} \quad - \eta_3(\varnothing \otimes b_3)$$

By Definition 4.12 (3) *is equal to*

$$= 0 - 0 + 0 - 0 - (-1)^1(s^{-1}a_2 \otimes b_3) + 0 - 0 = (s^{-1}a_2 \otimes b_3)$$

The second step to calculate $\eta_4(s^{-1}a_2 \otimes b_3)$ *will be:*

$$p_4 - \eta_3\partial_4(s^{-1}a_2 \otimes b_3) = (s^{-1}a_2 \otimes b_3) - (s^{-1}a_2 \otimes b_3) = 0$$

but from property (2) *of Proposition 4.10 we have:*

$\partial_5\eta_4(s^{-1}a_2 \otimes b_3) = p_4 - \eta_3\partial_4(s^{-1}a_2 \otimes b_3) = 0$

$\Leftrightarrow$

$\eta_4(s^{-1}a_2 \otimes b_3) = 0$

5 The general path crossed complex

Introduction

In the previous chapter we have defined a twisted tensor product $P^{\mathsf{Crs}}X = \Omega^{\mathsf{Crs}}(X) \otimes_{\phi} \pi X$ of crossed complexes for a 1-reduced simplicial set X. We have proved that this crossed complex is homotopy equivalent to the trivial crossed complex. It is therefore a crossed complex model for the path space of X.

In this chapter our objective is to extend all of our results to 0-reduced simplicial sets which are not necessarily 1-reduced. We obtain an extended crossed complex $(P^{\mathsf{Crs}}X, \partial^P)$.

Let X be a 0-reduced simplicial set, $X_0 = \{*\}$ and let πX be the fundamental crossed complex. This is a crossed complex of groups which has generators $b \in (\pi X)_n$ for each non-degenerate n-simplex b of X. The crossed complex $\hat{\Omega}^{\mathsf{Crs}}X$ is a crossed complex of groupoids. It is the free crossed chain algebra with graded algebra generators $s^{-1}a$ in degree n for each $(n+1)$-simplex $a \in X$. The structure of the chapter is as follows. In the first section, we generalise the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ to an extended 'group-completed' crossed cobar $\hat{\Omega}^{\mathsf{Crs}}X$ for any 0-reduced simplicial X and give its structure. In the second section, we consider a crossed complex that is simpler than the general path crossed complex $P^{\mathsf{Crs}}X$: it is the non-twisted tensor product of the crossed complex $\Omega^{\mathsf{Crs}}X$, and πX. For this non-twisted tensor product we know there is a boundary map $\partial^{\otimes}$. In the third section, we define the structure of the crossed complex of groupoids $P^{\mathsf{Crs}}X$, which is the twisted tensor product of the crossed complex of groups πX, and the free crossed complex of groupoids $\hat{\Omega}^{\mathsf{Crs}}X$. We define the boundary map ∂^P and prove it satisfies $\partial^P_{n-1}\partial^P_n = 0$.

5.1 The crossed cobar construction for 0-reduced simplicial sets

Let X be a 0-reduced simplicial set. We aim to introduce a crossed complex model for the path space PX, but before we do this we must introduce a crossed complex model for the loop space $\hat{\Omega}X$. That is, we must generalise the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ of Definition 4.1 from 1-reduced simplicial sets to 0-reduced simplicial sets. We know this is possible for chain complexes, by the work of Hess and Tonks [20], but for crossed complexes it will be a new construction.

For a 1-reduced simplicial set X, the crossed cobar construction $\Omega^{\mathsf{Crs}}X$ is a crossed complex of groups. If X is not 1-reduced (but only 0-reduced) then the crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}X$ is a crossed complex of groupoids. Since the cobar construction is a free algebra, the object set will be an infinite set, defined as a free monoid. The generators of this free monoid will be the non-degenerate 1-simplices of X.

We cannot see any obvious way to remove the condition that X is 0-reduced. If the simplicial set has more than one vertex, then there will be a loop space based at each vertex. These different loop spaces will be equivalent if X is connected, but they will be completely unrelated otherwise.

Definition 5.1. *Let X be a 0-reduced simplicial set. The crossed cobar $\Omega^{\mathsf{Crs}}X$ is a free crossed chain algebra generated by the elements $s^{-1}a_{n+1}$ in dimension n for each non-degenerate $(n+1)$-simplex of X. The basepoint of a generator $s^{-1}a_{n+1}$ in dimension $n \geqslant 1$ is*

$$\mathfrak{p} = \beta(s^{-1}a_{n+1}) = s^{-1}a_{01} \cdots s^{-1}a_{n\,n+1} \in \Omega_0^{\mathsf{Crs}}X$$

and the source and target of a generator $s^{-1}a_2$ in dimension 1 are

$$\operatorname{src}(s^{-1}a_2) = \beta(s^{-1}a_2) = s^{-1}a_{01}s^{-1}a_{12} \qquad \operatorname{targ}(s^{-1}a_2) = s^{-1}a_{02} \in \Omega_0^{\mathsf{Crs}}X.$$

The boundary map is given on the generators $s^{-1}a_{n+1}$, in dimension $n \geqslant 2$, by the following modification of the formulas in Definition 4.1:

$$\partial_2^{\Omega} s^{-1}a_3 = -s^{-1}a_{01} \cdot s^{-1}a_{123} - s^{-1}a_{013} + s^{-1}a_{023} + s^{-1}a_{012} \cdot s^{-1}a_{23}$$

$$\partial_3^\Omega s^{-1}a_4 = -s^{-1}a_{0123}\cdot s^{-1}a_{34} - s^{-1}a_{0134}{}^{s^{-1}a_{01}\cdot s^{-1}a_{123}\cdot s^{-1}a_{34}} - s^{-1}a_{01}\cdot s^{-1}a_{1234}$$
$$+ s^{-1}a_{0124}{}^{s^{-1}a_{01}\cdot s^{-1}a_{12}\cdot s^{-1}a_{234}} + s^{-1}a_{012}\cdot s^{-1}a_{234} + s^{-1}a_{0234}{}^{s^{-1}a_{012}\cdot s^{-1}a_{23}\cdot s^{-1}a_{34}}$$
$$\partial_n^\Omega s^{-1}a_{n+1} = \sum_{i=1}^{n}(-1)^{i+1}(s^{-1}d_i a_{n+1})^{\gamma_i} - \sum_{i=1}^{n}(-1)^{i+1}s^{-1}a_{0\ldots i}\cdot s^{-1}a_{i\ldots n+1}$$

where

$$\gamma_i = s^{-1}a_{01}\cdot s^{-1}a_{12}\cdots s^{-1}a_{i-2\ i-1}\cdot s^{-1}a_{i-1\ i\ i+1}\cdot s^{-1}a_{i+1\ i+2}\cdots s^{-1}a_{n\ n+1}$$

Proposition 5.2. *The boundary maps in Definition 5.1 are well-defined in the crossed complex of groupoids $\Omega^{\mathsf{Crs}}X$.*

Proof. Consider any generator $x = s^{-1}a_{n+1}$ in dimension $n \geqslant 2$. This has basepoint

$$\mathfrak{p} = \operatorname{src} s^{-1}a_{n+1} = s^{-1}a_{01}\cdots s^{-1}a_{n\,n+1} \in \Omega_0^{\mathsf{Crs}}X.$$

We must check that the terms in the expressions for $\partial_n^\Omega x$ in Definition 5.1 have the correct sources and targets to ensure they are composable in $\Omega_{n-1}^{\mathsf{Crs}}X$. We must also check that the composite $\partial_n^\Omega x \in \Omega_{n-1}^{\mathsf{Crs}}X$ has source and target equal to $\mathfrak{p}$ if $n=2$, and has basepoint equal to $\mathfrak{p}$ if $n \geqslant 3$.

$n=2$: We can write the expression

$$\partial_2^\Omega s^{-1}a_3 = -s^{-1}a_{01}s^{-1}a_{123} - s^{-1}a_{013} + s^{-1}a_{023} + s^{-1}a_{012}s^{-1}a_{23}$$

as a diagram:

$$\begin{tikzcd}
s^{-1}a_{02}\cdot s^{-1}a_{23} \arrow[d, "s^{-1}a_{023}"'] & & s^{-1}a_{01}\cdot s^{-1}a_{12}\cdot s^{-1}a_{23} = \mathfrak{p} \arrow[ll, "s^{-1}a_{012}\cdot s^{-1}a_{23}"'] \\
s^{-1}a_{03} \arrow[rr, "-s^{-1}a_{013}"'] & & s^{-1}a_{01}\cdot s^{-1}a_{13} \arrow[u, "-s^{-1}a_{01}\cdot s^{-1}a_{123}"']
\end{tikzcd}$$

In this diagram we have shown that the composite is defined and the result has source and target $\mathfrak{p}$.

$n = 3$: We could try to draw a diagram of the expression

$$\begin{aligned}\partial_3^\Omega(s^{-1}a_4) = &- s^{-1}a_{0123} \cdot s^{-1}a_{34} - s^{-1}a_{0134}{}^{s^{-1}a_{01}\cdot s^{-1}a_{123}\cdot s^{-1}a_{34}} - s^{-1}a_{01} \cdot s^{-1}a_{1234} \\ &+ s^{-1}a_{0124}{}^{s^{-1}a_{01}\cdot s^{-1}a_{12}\cdot s^{-1}a_{234}} + s^{-1}a_{012} \cdot s^{-1}a_{234} + s^{-1}a_{0234}{}^{s^{-1}a_{012}\cdot s^{-1}a_{23}\cdot s^{-1}a_{34}}\end{aligned}$$

but it would be a 3-dimensional cube. Instead, we will just check that the basepoints of all six terms are equal to $\mathfrak{p}$, so the composite is defined and also has basepoint $\mathfrak{p}$:

- $s^{-1}a_{0123}$ has basepoint $s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_{23}$.

  Therefore $s^{-1}a_{0123} \cdot s^{-1}a_{34}$ has basepoint $\mathfrak{p}$.

- The source of $s^{-1}a_{01} \cdot s^{-1}a_{123} \cdot s^{-1}a_{34}$ is $\mathfrak{p}$, and the target is $s^{-1}a_{01} \cdot s^{-1}a_{13} \cdot s^{-1}a_{34}$, which is the same as the basepoint of $s^{-1}a_{0134}$.

  Therefore $s^{-1}a_{0134}{}^{s^{-1}a_{01}\cdot s^{-1}a_{123}\cdot s^{-1}a_{34}}$ has basepoint $\mathfrak{p}$.

- $s^{-1}a_{1234}$ has basepoint $s^{-1}a_{12} \cdot s^{-1}a_{23} \cdot s^{-1}a_{34}$.

  Therefore $s^{-1}a_{01} \cdot s^{-1}a_{1234}$ has basepoint $\mathfrak{p}$.

- The source of $s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_{234}$ is $\mathfrak{p}$ and the target is $s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_{24}$, which is the same as the basepoint of $s^{-1}a_{0124}$.

  Therefore $s^{-1}a_{0124}{}^{s^{-1}a_{01}\cdot s^{-1}a_{12}\cdot s^{-1}a_{234}}$ has basepoint $\mathfrak{p}$.

- $s^{-1}a_{012} \cdot s^{-1}a_{234}$ has basepoint $\mathfrak{p}$.

- The source of $s^{-1}a_{012} \cdot s^{-1}a_{23} \cdot s^{-1}a_{34}$ is $\mathfrak{p}$ and the target is $s^{-1}a_{02} \cdot s^{-1}a_{23} \cdot s^{-1}a_{34}$ which is the same as the basepoint of $s^{-1}a_{0234}$.

  Therefore $s^{-1}a_{0234}{}^{s^{-1}a_{012}\cdot s^{-1}a_{23}\cdot s^{-1}a_{34}}$ has basepoint $\mathfrak{p}$.

$n \geqslant 4$: This is similar to the case $n = 3$, except now it is abelian too. We can see that half of the terms have the form $s^{-1}a_{0\ldots i} \cdot s^{-1}a_{i\ldots n+1}$, and these clearly have basepoint $\mathfrak{p}$. The other half of the terms have the form $s^{-1}d_i a_{n+1}{}^{\gamma_i}$ where the 1-dimensional element

γ_i has source $\mathfrak{p}$ and has target equal to the basepoint of the $(n-1)$-dimensional element $s^{-1}d_i a_{n+1}$. Therefore the composite of the terms in the boundary relation for $\partial_n^{\Omega} s^{-1} a_{n+1}$ exists in $(\Omega_{n-1}^{\mathsf{Crs}} X)(\mathfrak{p})$.

□

Example 5.3. *Let X be the 0-reduced simplicial set which is a model for S^1,*

$$X = S^1 = \Delta[1]/\partial\Delta[1]$$

which has one 0 *simplex* $*$, *one non-degenerate* 1*-simplex* σ, *and no non-degenerate simplices in dimensions* $n \geqslant 2$.

The crossed cobar construction is $\Omega^{\mathsf{Crs}} S^1$ *is the free crossed chain algebra generated by* $s^{-1}\sigma$. *Therefore* $\Omega^{\mathsf{Crs}} S^1$ *has object set given by the free monoid on one generator. In dimensions* $n \geqslant 1$ *it has only identity elements.*

$$\Omega^{\mathsf{Crs}} S^1 \cong \mathbb{N}.$$

The usual model for the loop space on S^1 is not the natural numbers $\mathbb{N}$, it is the integers $\mathbb{Z}$. We can introduce a new construction, which we call the *group-completed crossed cobar construction* $\hat{\Omega}^{\mathsf{Crs}}$, so that

$$\hat{\Omega}^{\mathsf{Crs}} S^1 \cong \mathbb{Z}.$$

If X is any 0-reduced simplicial set then the object set of $\hat{\Omega}^{\mathsf{Crs}} X$ will be a free group whose generators correspond to the non-degenerate 1-simplices of X. The group completed crossed cobar construction $\hat{\Omega}^{\mathsf{Crs}}$ is related to the extended cobar construction $\hat{\Omega}$ that we looked at for chain complexes in section 3.1.2.

Definition 5.4. *Let X be a 0-reduced simplicial set. The group-completed crossed cobar construction* $\hat{\Omega}^{\mathsf{Crs}} X$ *is a free crossed chain algebra generated by the elements* $s^{-1}a_{n+1}$ *in* *dimension n for each non-degenerate $(n+1)$-simplex of X, together with extra generators $(s^{-1}a_1)^{-1}$ for each non-degenerate 1-simplex a_1 of X. The source, target and boundary of a generator $s^{-1}a_{n+1}$ in dimension $n \geqslant 1$ is the same as in Definition 5.1.*

We have defined $\hat{\Omega}^{\mathsf{Crs}}X$ as a free crossed chain algebra. It is also free as a crossed complex of groupoids. As a crossed complex of groupoids, we know that the object set is

$$\left\{\omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2}\cdots(s^{-1}a_1^{(k)})^{\epsilon_k}\ :\ k \geqslant 0, a_1^{(i)} \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1\right\} \tag{25}$$

The generators x of degree $|x| = n$ of the free crossed complex $\hat{\Omega}^{\mathsf{Crs}}X$ are given by words

$$x = \omega^{(0)}s^{-1}a^{(1)}_{n_1+1}\omega^{(1)}a^{(2)}_{n_2+1}\cdots\omega^{(r-1)}s^{-1}a^{(r)}_{n_r+1}\omega^{(r)}, \tag{26}$$

where $r \geqslant 0$, each $\omega^{(i)} \in \hat{\Omega}^{\mathsf{Crs}}_0X$, each $a^{(i)}_{n_i+1}$ is a non-degenerate simplex in X_{n_i+1}, $n_i \geqslant 1$, and $\sum n_i = n$.

The basepoint $\mathfrak{p} = \beta(x)$ of x is the product of the basepoints of all of the terms in x. We point out that because there are inverses in degree zero, some cancellation might happen. For example,

$$\beta\left(s^{-1}a_3 \cdot (s^{-1}a_{23})^{(-1)} \cdot s^{-1}a_2'\right) = s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{01}'s^{-1}a_{12}'$$

Because $\hat{\Omega}^{\mathsf{Crs}}_nX$ is a (free) crossed chain algebra with the algebra structure

$$\hat{\Omega}^{\mathsf{Crs}}_nX \otimes \hat{\Omega}^{\mathsf{Crs}}_nX \to \hat{\Omega}^{\mathsf{Crs}}_nX$$

defined by concatenation of words

$$x \otimes x' \mapsto x \cdot x',$$

the boundary of an element x can be calculated from the relations in Definition 2.34 together with the boundary relations for the elements $s^{-1}a_{n_i+1}$ given in Definition 5.1.

If each $n_i \geqslant 3$ then the formula is long but easy. For example

$$
\begin{aligned}
&\partial_7^{\hat{\Omega}}(s^{-1}a_4 \cdot s^{-1}a_5') = \big(\partial_3 s^{-1}a_4\big) \cdot s^{-1}a_5' \ + \ (-1)^{|s^{-1}a_4|}\ s^{-1}a_4 \cdot \big(\partial_4 s^{-1}a_5'\big)\\
&= \Big(-s^{-1}a_{0123} \cdot s^{-1}a_{34} - s^{-1}{a_{0134}}^{s^{-1}a_{01}\cdot s^{-1}a_{123}\cdot s^{-1}a_{34}} - s^{-1}a_{01} \cdot s^{-1}a_{1234}\\
&\quad + s^{-1}{a_{0124}}^{s^{-1}a_{01}\cdot s^{-1}a_{12}\cdot s^{-1}a_{234}} + s^{-1}a_{012} \cdot s^{-1}a_{234} + s^{-1}{a_{0234}}^{s^{-1}a_{012}\cdot s^{-1}a_{23}\cdot s^{-1}a_{34}}\Big)\, s^{-1}a_5'\\
&\quad - s^{-1}a_4 \left(\sum_{j=1}^{4}(-1)^{j+1}(s^{-1}d_j a_5')^{\gamma_j'} - \sum_{j=1}^{4}(-1)^{j+1}s^{-1}a_{0\ldots j}' \cdot s^{-1}a_{j\ldots 5}'\right)\\
&= \sum_{i=1}^{3}(-1)^{i+1}\Big((s^{-1}d_i a_4 \cdot s^{-1}a_5')^{\gamma_i\cdot\mathfrak{p}'} - s^{-1}a_{0\ldots i}s^{-1}a_{i\ldots 4}s^{-1}a_5'\Big)\\
&\quad - \sum_{j=1}^{4}(-1)^{j+1}\Big((s^{-1}a_4 \cdot s^{-1}d_j a_5')^{\mathfrak{p}\cdot\gamma_j'} - s^{-1}a_4 s^{-1}a_{0\ldots j}' s^{-1}a_{j\ldots 5}'\Big)
\end{aligned}
$$

in the abelian group $\hat{\Omega}_6^{\mathsf{Crs}}X(\mathfrak{p}\cdot\mathfrak{p}')$, where $\mathfrak{p}$ and $\mathfrak{p}'$ are the basepoints of a_4 and a_5' respectively.

All the boundary formulas $\partial^{\hat{\Omega}}x$ can be calculated using the relations in Definitions 2.34 and 5.1. In low degrees the boundary formula will not be abelian so we must take more care. We write down the results in the following proposition

Proposition 5.5. *Consider a generator of the crossed complex of groupoids $\hat{\Omega}^{\mathsf{Crs}}X$,*

$$x = \omega^{(0)} \cdot \prod_{k=1}^{r} s^{-1}a_{n_k+1}^{(k)} \cdot \omega^{(k)}$$

with each $\omega^{(k)} \in \hat{\Omega}_0^{\mathsf{Crs}}X$ and each $a_{n_k+1}^{(k)} \in X_{n_k+1}$, as in (26).

If $n = |x| = \sum n_i \geqslant 4$ then the boundary $\partial_n^{\hat{\Omega}}x$ is given by

$$
\begin{aligned}
\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum_{\ell=1}^{k-1}n_\ell}\Bigg(&\Big(\omega^{(0)} \cdot \prod_{\ell=1}^{k-1} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot \omega^{(\ell)} \cdot s^{-1}d_i a_{n_k+1}^{(k)} \cdot \omega^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot \omega^{(\ell)}\Big)^{\gamma_i^{(k)}}\\
&- \quad \omega^{(0)} \cdot \prod_{\ell=1}^{k-1} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot \omega^{(\ell)} \cdot s^{-1}a_{0\ldots i}^{(k)} \cdot s^{-1}a_{i\ldots n_k+1}^{(k)} \cdot \omega^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot \omega^{(\ell)}\Bigg)
\end{aligned}
$$

Here the action is by

$$\gamma_i^{(k)} = \omega^{(0)} \cdot \prod_{\ell=1}^{k-1}\mathfrak{p}^{(\ell)} \cdot \omega^{(\ell)} \ \cdot\ \beta(s^{-1}a_{0\ldots i-1}^{(k)}) \cdot a_{i-1\,i\,i+1}^{(k)} \cdot \beta(s^{-1}a_{i+1\ldots n_k+1}^{(k)}) \cdot \prod_{\ell=k+1}^{r}\mathfrak{p}^{(\ell)} \cdot \omega^{(\ell)}$$

where $\mathfrak{p}^{(\ell)}$ *is the basepoint* $\beta(s^{-1}a^{(\ell)}_{n_\ell+1})$.

If $n \leq 3$ *then to save space we will not write the elements* ω *of degree 0:*

$$\begin{aligned}
&\partial_2^{\hat{\Omega}}(s^{-1}a_2 \cdot s^{-1}a'_2) \\
&= -(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_2) - (s^{-1}a_2 \cdot s^{-1}a'_{02}) + (s^{-1}a_{02} \cdot s^{-1}a'_2) + (s^{-1}a_2 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12}) \\
&\partial_3^{\hat{\Omega}}(s^{-1}a_2 \cdot s^{-1}a'_2 \cdot s^{-1}a''_2) \\
&= -(s^{-1}a_2 \cdot s^{-1}a'_2 \cdot s^{-1}a''_{01} \cdot s^{-1}a''_{12}) - (s^{-1}a_2 \cdot s^{-1}a'_{02} \cdot s^{-1}a''_2)^{(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_2 \cdot s^{-1}a''_{01} \cdot s^{-1}a''_{12})} \\
&\quad -(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_2 \cdot s^{-1}a''_2) + (s^{-1}a_2 \cdot s^{-1}a'_2 \cdot s^{-1}a''_{02})^{(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12} \cdot s^{-1}a''_2)} \\
&\quad +(s^{-1}a_2 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12} \cdot s^{-1}a''_2) + (s^{-1}a_{02} \cdot s^{-1}a'_2 \cdot s^{-1}a''_2)^{(s^{-1}a_2 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12} \cdot s^{-1}a''_{01} \cdot s^{-1}a''_{12})} \\
&\partial_3^{\hat{\Omega}}(s^{-1}a_3 \cdot s^{-1}a'_2) \\
&= -(s^{-1}a_{01} \cdot s^{-1}a_{123} \cdot s^{-1}a'_2) + (s^{-1}a_3 \cdot s^{-1}a'_{02})^{(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_{23} \cdot s^{-1}a'_2)} \\
&\quad +(s^{-1}a_{012} \cdot s^{-1}a_{23} \cdot s^{-1}a'_2) + (s^{-1}a_{023} \cdot s^{-1}a'_2)^{(s^{-1}a_{012} \cdot s^{-1}a_{23} \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12})} \\
&\quad -(s^{-1}a_3 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12}) - (s^{-1}a_{013} \cdot s^{-1}a'_2)^{(s^{-1}a_{01} \cdot s^{-1}a_{123} \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12})} \\
&\partial_3^{\hat{\Omega}}(s^{-1}a_2 s^{-1}a'_3) \\
&= -(s^{-1}a_2 \cdot s^{-1}a'_{023})^{(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_{012} \cdot s^{-1}a'_{23})} - (s^{-1}a_2 \cdot s^{-1}a'_{012} \cdot s^{-1}a'_{23}) \\
&\quad +(s^{-1}a_{02} \cdot s^{-1}a'_3)^{(s^{-1}a_2 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{12} \cdot s^{-1}a'_{23})} + (s^{-1}a_2 \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{123}) \\
&\quad +(s^{-1}a_2 \cdot s^{-1}a'_{013})^{(s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_{01} \cdot s^{-1}a'_{123})} - (s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a'_3)
\end{aligned}$$

5.2 The general path crossed complex: an example

In the previous section, in Example 5.3, we saw how to define the group-completed cobar construction $\widehat{\Omega}^{\mathsf{Crs}}S^1$ for the simplical model of the circle,

$$X = S^1 = \Delta[1]/\{0 \sim 1\}.$$

In this section we give an example of a crossed complex $P^{\mathsf{Crs}}S^1$ of groupoids which is

- contractible, and so it is a model for the path space on the circle, PS^1

- a kind of twisted tensor product of the fundamental crossed complex of S^1 and the group-completed cobar construction on S^1,

$$P^{\mathsf{Crs}}S^1 \;=\; \widehat{\Omega}^{\mathsf{Crs}}S^1 \otimes_\phi \pi S^1$$

It is a crossed complex of groupoids, so we first define the object set, then the groupoid structure. It is only 1-dimensional, so we will not need to define any crossed module or crossed complex structure. We have seen in the previous chapter how to define the twisted tensor product in higher dimensions. For the classical construction with chain complexes, the twisted boundary of the twisted tensor product is just

$$\partial^P(x \otimes b_n) \;=\; \partial^{\otimes}(x \otimes b_n) \;\pm\; \sum_{i=2}^{n-1} x\, s^{-1}b_{0\ldots i} \otimes b_{i\ldots n}.$$

The example we do now illustrates how to twist the tensor product in dimensions 0 and 1. Instead of twisting the boundary maps, we need to twist the source and target maps. We find that we just need to twist the target of an arrow in the groupoid, leaving the source as it was.

We know that $\pi(S^1)$ is a crossed complex of groups, which has a single basepoint $\pi(S^1) = \{*\}$. In dimension 1 it is the free group

$$\pi_1(S^1) \;=\; \langle b_1 \rangle \;\cong\; \mathbb{Z}.$$

All higher dimensional elements are the identity id_*.

We have seen in Example 5.3 that the object set of the group-completed crossed cobar construction $\widehat{\Omega}^{\mathsf{Crs}}S^1$ for S^1 is just the set

$$\widehat{\Omega}_0^{\mathsf{Crs}}S^1 \;=\; \{(s^{-1}b_1)^k \;:\; k \in \mathbb{Z}\} \;\cong\; \mathbb{Z},$$

and that all higher-dimensional elements in $\widehat{\Omega}^{\mathsf{Crs}}S^1$ are identities.

Definition 5.6. *We define the crossed complex of groupoids*

$$P^{\mathsf{Crs}}S^1 = \widehat{\Omega}^{\mathsf{Crs}}S^1 \otimes_\phi \pi S^1$$

as follows:

- *The object set is* $\{(s^{-1}b_1)^k \otimes * \ : \ k \in \mathbb{Z}\}$
- *The generators of the groupoid* $\widehat{\Omega}_1^{\mathsf{Crs}}S^1$ *are*

$$(s^{-1}b_1)^k \otimes b_1) : ((s^{-1}b_1)^k \otimes *) \longrightarrow ((s^{-1}b_1)^{k+1} \otimes *).$$

- *There are only identity elements in degree* $\geqslant 2$.

Another way of writing this is:

- the objects, in dimension $0+0$, are $\omega \otimes *$, where ω is an object of the group-completed cobar construction
- the arrows, in dimension $0+1$, are generated by $\omega \otimes b_1$, which has source $\omega \otimes *$ as usual, but has twisted target $\omega \cdot s^{-1}b_1 \otimes *$

The objects can be thought of as all integers k, and the generating arrows are arrows from $k \to k+1$.

A picture of the path crossed complex $P^{\mathsf{Crs}}S^1$ is:

$$\ldots \longrightarrow -k \longrightarrow \cdots \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow k \longrightarrow k+1 \longrightarrow \ldots$$

Theorem 5.7. *The following crossed complexes are isomorphic:*

$$P^{\mathsf{Crs}}S^1 \quad = \quad \widehat{\Omega}^{\mathsf{Crs}}S^1 \otimes_\phi \pi S^1 \quad \cong \quad \pi(\mathbb{R}) = \pi(\mathbb{Z} \times_\tau S^1)$$

5.3 The general path crossed complex: the definition

This is a crossed complex of groupoids, so we first define the object set, then the groupoid structure, and then the crossed module and crossed complex structure for each object.

Suppose that $X_0 = \{*\}$. The crossed complex $P^{\mathsf{Crs}}X$ will be an example of a twisted tensor product of:

- the crossed complex of groups πX, whose object set is $\{*\}$
- the crossed chain algebra $\hat{\Omega}^{\mathsf{Crs}}X$, whose object set $\hat{\Omega}^{\mathsf{Crs}}_0 X$ was defined in Definition 5.4, so is the free group on the desuspension of the non-degenerate 1-simplices of X. Its elements are thus words in the letters $s^{-1}a_1$, and $(s^{-1}a_1)^{-1}$ for $a_1 \in X_1 - \{s_0(*)\}$, with neutral element given by the empty word $\omega = \varnothing$.

We have already considered a simpler version of this construction in the previous chapter. In chapter 4, X was a 1-reduced simplicial set, and so $\Omega^{\mathsf{Crs}}_0 X = \{\varnothing\}$. The construction in this chapter will be more complicated but it will still be a *twisted tensor product.* The crossed complex of groupoids $P^{\mathsf{Crs}}X$ will be free crossed complex with the *same generators* as the ordinary, non-twisted, tensor product $\hat{\Omega}^{\mathsf{Crs}}X \otimes \pi X$. We write these generators as

$$x \otimes b \in P^{\mathsf{Crs}}_{n+m}X,$$

where

- x is a generator of degree $|x| = n$ in $\hat{\Omega}^{\mathsf{Crs}}_n X$, defined in (26).

  We know that $\hat{\Omega}^{\mathsf{Crs}}_n X$ is a (free) crossed chain algebra with the algebra structure defined by concatenation of words $x \otimes x' \mapsto xx'$.

- b is a generator of degree $|b| = m$ in πX, given by a non-degenerate m-simplex of X.

The boundary maps of $P^{\mathsf{Crs}}X$ will be more complicated than the boundary maps of the ordinary, non-twisted, tensor product.

5.3.1 The boundary of the non-twisted tensor product

Before we define the boundary maps for $P^{\mathsf{Crs}}X$ we will given now the explicit formulas for the ordinary, non-twisted, tensor product

$$\hat{\Omega}^{\mathsf{Crs}}X \otimes \pi X, \;\; \partial^{\otimes}$$

This boundary map, in the context of chain complexes, would be $\partial^{\otimes} = \partial^{\hat{\Omega}} \otimes \mathrm{id} \pm \mathrm{id} \otimes \partial^{\pi}$. The crossed complex formula for $\partial^{\otimes}$ will be similar, but with a more complicated (possibly non-abelian) formula if $n < 2$ or $m < 2$.

In chapter 4 we have seen that the twisted boundary maps have some extra terms with the form

$$(-1)^{|x|} \sum_{i=1}^{m} (x \cdot s^{-1} b_{0 \dots i} \otimes b_{i \dots m}).$$

In the following section we will modify the explicit formulas $\partial^{\otimes}$ to obtain a definition of ∂^{P}.

1. For the non-twisted tensor product, for $m = n = 1$, $\omega \in \hat{\Omega}^{\mathsf{Crs}}_0 X$ we have:

$$\begin{aligned} &\partial_2^{\otimes}(\omega s^{-1} a_2 \omega' \otimes b_1) \\ =& -(\omega s^{-1} a_{01} s^{-1} a_{12} \omega' \otimes b_1) - (\omega s^{-1} a_2 s^{-1} \omega' \otimes b_{(1)}) \\ &+ (\omega s^{-1} a_{02} \omega' \otimes b_1) + (\omega s^{-1} a_2 \omega' \otimes b_{(0)}) \end{aligned}$$

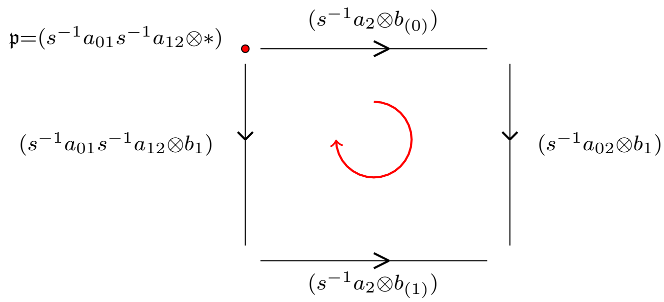


Figure 12: $\partial_2^{\otimes}(s^{-1}a_2 \otimes b_1)$

2. For $n \geqslant 2, m = 0$ we have:

•

$$\begin{aligned}
&\partial_2^{\otimes}(\omega s^{-1}a_3\omega' \otimes *)\\
=&-(\omega s^{-1}a_{01}s^{-1}a_{123}\omega' \otimes *) - (\omega s^{-1}a_{013}\omega' \otimes *)\\
&+(\omega s^{-1}a_{023}\omega' \otimes *) + (\omega s^{-1}a_{012}s^{-1}a_{23}\omega' \otimes *)
\end{aligned}$$

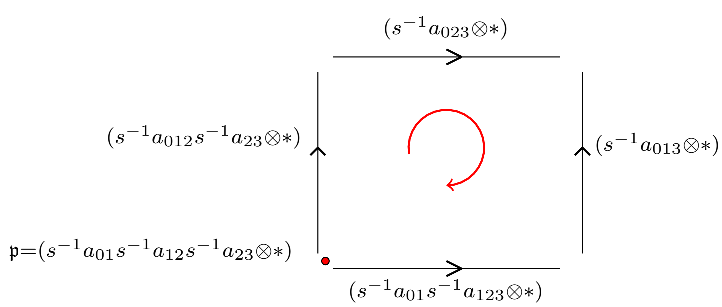


Figure 13: $\partial_2^{\otimes}(s^{-1}a_3 \otimes *)$

•

$$\partial_2^{\otimes}(\omega s^{-1}a_2 s^{-1}a_2'\omega' \otimes *)$$

$$
\begin{aligned}
= & -(\omega s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_2\omega'\otimes *) - (\omega s^{-1}a_2s^{-1}a'_{02}\omega'\otimes *) \\
& + (\omega s^{-1}a_{02}s^{-1}a'_2\omega'\otimes *) + (\omega s^{-1}a_2s^{-1}a'_{01}s^{-1}a'_{12}\omega'\otimes *)
\end{aligned}
$$

$$
\begin{tikzcd}
{} \arrow[r, "(s^{-1}a_2s^{-1}a'_{02}\otimes *)"] & {} \\
\mathfrak{p}=(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_{01}s^{-1}a'_{12}\otimes *) \arrow[u, "(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_2\otimes *)"] \arrow[r, "(s^{-1}a_2s^{-1}a'_{01}s^{-1}a'_{12}\otimes *)"'] & {} \arrow[u, "(s^{-1}a_{02}s^{-1}a'_2\otimes *)"']
\end{tikzcd}
$$

Figure 14: $\partial_2^{\otimes}(s^{-1}a_2s^{-1}a'_2\otimes *)$

- $$
\begin{aligned}
& \partial_3^{\otimes}(\omega s^{-1}a_4\omega'\otimes *) \\
= & -(\omega s^{-1}a_{0134}\omega'\otimes *)^{(s^{-1}a_{01}s^{-1}a_{123}s^{-1}a_{34}\otimes *)} - (\omega s^{-1}a_{01}s^{-1}a_{1234}\omega'\otimes *) \\
& + (\omega s^{-1}a_{0124}\omega'\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{234}\otimes *)} + (\omega s^{-1}a_{012}s^{-1}a_{234}\omega'\otimes *) \\
& + (\omega s^{-1}a_{0234}\omega'\otimes *)^{(s^{-1}a_{012}s^{-1}a_{23}s^{-1}a_{34}\otimes *)} - (\omega s^{-1}a_{0123}s^{-1}a_{34}\omega'\otimes *)
\end{aligned}
$$

- $$
\begin{aligned}
& \partial_3^{\otimes}(\omega s^{-1}a_3\omega's^{-1}a'_2\omega''\otimes *) \\
= & -(\omega s^{-1}a_{01}s^{-1}a_{123}\omega's^{-1}a'_2\omega''\otimes *) \\
& + (\omega s^{-1}a_3\omega's^{-1}a'_{02}\omega''\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{23}s^{-1}a'_2\otimes *)} \\
& + (\omega s^{-1}a_{012}s^{-1}a_{23}\omega's^{-1}a'_2\omega''\otimes *) \\
& + (\omega s^{-1}a_{023}\omega's^{-1}a'_2\omega''\otimes *)^{(s^{-1}a_{012}s^{-1}a_{23}s^{-1}a'_{01}s^{-1}a'_{12}\otimes *)} \\
& - (\omega s^{-1}a_3\omega's^{-1}a'_{01}s^{-1}a'_{12}\omega''\otimes *) \\
& - (\omega s^{-1}a_{013}\omega's^{-1}a'_2\omega''\otimes *)^{(s^{-1}a_{01}s^{-1}a_{123}s^{-1}a'_{01}s^{-1}a'_{12}\otimes *)}
\end{aligned}
$$

- $$\begin{aligned}&\partial_3^{\otimes}(\omega s^{-1}a_2\omega' s^{-1}a_3'\omega''\otimes *)\\
=&-(\omega s^{-1}a_{01}s^{-1}a_{12}\omega' s^{-1}a_3'\omega''\otimes *)\\
&+(\omega s^{-1}a_2\omega' s^{-1}a_{013}'\omega''\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{01}'s^{-1}a_{123}'\otimes *)}\\
&+(\omega s^{-1}a_2 s^{-1}a_{01}'\omega' s^{-1}a_{123}'\omega''\otimes *)\\
&+(\omega s^{-1}a_{02}\omega' s^{-1}a_3'\omega''\otimes *)^{(s^{-1}a_{012}s^{-1}a_{01}'s^{-1}a_{12}'s^{-1}a_{23}'\otimes *)}\\
&-(\omega s^{-1}a_2\omega' s^{-1}a_{012}'s^{-1}a_{23}'\omega''\otimes *)\\
&-(\omega s^{-1}a_2\omega' s^{-1}a_{023}'\omega''\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{012}'s^{-1}a_{23}'\otimes *)}\end{aligned}$$

- $$\begin{aligned}&\partial_3^{\otimes}(\omega^{(1)}s^{-1}a_2\omega^{(2)}s^{-1}a_2'\omega^{(3)}s^{-1}a_2''\omega^{(4)}\otimes *)\\
=&-(\omega^{(1)}s^{-1}a_2\omega^{(2)}s^{-1}a_2'\omega^{(3)}s^{-1}a_{01}''s^{-1}a_{12}''\omega^{(4)}\otimes *)\\
&-(\omega^{(1)}s^{-1}a_2\omega^{(2)}s^{-1}a_{02}'\omega^{(3)}s^{-1}a_2''\omega^{(4)}\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_2's^{-1}a_{01}''s^{-1}a_{12}''\otimes *)}\\
&-(\omega^{(1)}s^{-1}a_{01}s^{-1}a_{12}\omega^{(2)}s^{-1}a_2'\omega^{(3)}s^{-1}a_2''\omega^{(4)}\otimes *)\\
&+(\omega^{(1)}s^{-1}a_2\omega^{(2)}s^{-1}a_2'\omega^{(3)}s^{-1}a_{02}''\omega^{(4)}\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{01}'s^{-1}a_{12}'s^{-1}a_2''\otimes *)}\\
&+(\omega^{(1)}s^{-1}a_2\omega^{(2)}s^{-1}a_{01}'s^{-1}a_{12}'\omega^{(3)}s^{-1}a_2''\omega^{(4)}\otimes *)\\
&+(\omega^{(1)}s^{-1}a_{02}\omega^{(2)}s^{-1}a_2'\omega^{(3)}s^{-1}a_2''\omega^{(4)}\otimes *)^{(s^{-1}a_2s^{-1}a_{01}'s^{-1}a_{12}'s^{-1}a_{01}''s^{-1}a_{12}''\otimes *)}\end{aligned}$$

3. For $n \geqslant 0, m \geqslant 1$ we have:

- $$\begin{aligned}&\partial_3^{\otimes}(\omega s^{-1}a_3\omega'\otimes b_1)\\
=&-(\omega s^{-1}a_{01}s^{-1}a_{123}\omega'\otimes b_1)+(\omega s^{-1}a_3\omega'\otimes b_{(1)})^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{23}\otimes b_1)}\\
&+(\omega s^{-1}a_{012}s^{-1}a_{23}\omega'\otimes b_1)+(\omega s^{-1}a_{023}\omega'\otimes b_1)^{(s^{-1}a_{012}s^{-1}a_{23}\otimes *)}\end{aligned}$$

$$-\left(\omega s^{-1}a_3\omega'\otimes b_{(0)}\right)-\left(\omega s^{-1}a_{013}\omega'\otimes b_1\right)^{(s^{-1}a_{01}s^{-1}a_{123}\otimes *)}$$

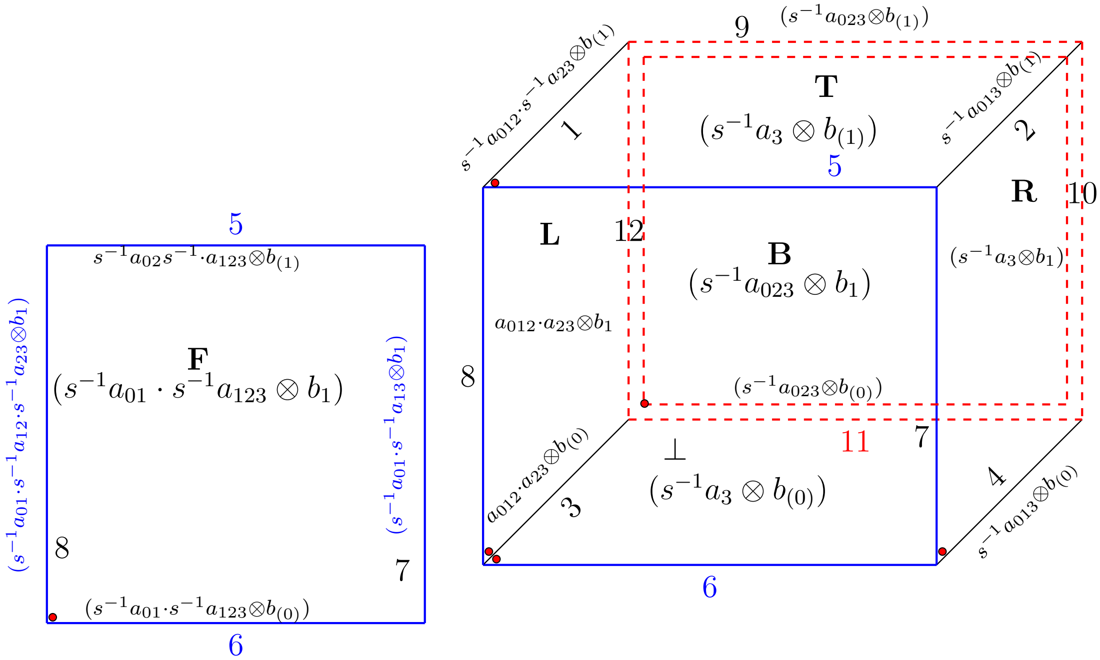


Figure 15: $\partial_2^{\otimes}(s^{-1}a_3\otimes b_1)$

- $$\begin{aligned}
&\partial_3^{\otimes}(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega''\otimes b_1)\\
&=+\left(\omega s^{-1}a_{02}\omega' s^{-1}a_2'\omega''\otimes b_1\right)^{(s^{-1}a_2s^{-1}a_{01}'s^{-1}a_{12}'\otimes b_{(0)})}-\left(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega''\otimes b_{(0)}\right)\\
&\quad-\left(\omega s^{-1}a_2\omega' s^{-1}a_{02}'\omega''\otimes b_1\right)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_2'\otimes b_{(0)})}-\left(\omega s^{-1}a_{01}s^{-1}a_{12}\omega' s^{-1}a_2'\omega''\otimes b_1\right)\\
&\quad+\left(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega''\otimes b_{(1)}\right)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{01}'s^{-1}a_{12}'\otimes b_1)}\\
&\quad+\left(\omega s^{-1}a_2\omega' s^{-1}a_{01}'s^{-1}a_{12}'\omega''\otimes b_1\right)
\end{aligned}$$

- $$\begin{aligned}
&\partial_2^{\otimes}(\omega\otimes b_2)\\
&=-(\omega\otimes b_{02})+(\omega\otimes b_{12})+(\omega\otimes b_{01})
\end{aligned}$$

$$\begin{tikzcd}[column sep=small]
(\omega\otimes b_{(0)}) \arrow[rr, "(\omega\otimes b_{01})"] \arrow[dr, "(\omega\otimes b_{02})"'] & \arrow[d, phantom, "\circlearrowright(\omega\otimes b_2)"] & (\omega\otimes b_{(1)}) \arrow[dl, "(\omega\otimes b_{12})"] \\
& (\omega\otimes b_{(2)}) &
\end{tikzcd}$$

Figure 16:

•

$$\begin{aligned}
&\partial_3^{\otimes}(\omega s^{-1}a_2\omega'\otimes b_2)\\
&= -(\omega s^{-1}a_2\omega'\otimes b_{01}) - (\omega s^{-1}a_2\omega'\otimes b_{(12)})^{(s^{-1}a_{01}s^{-1}a_{12}\otimes b_{01})}\\
&\quad -(\omega s^{-1}a_{01}s^{-1}a_{12}\omega'\otimes b_2)\\
&\quad +(\omega s^{-1}a_2\omega'\otimes b_{02}) + (\omega s^{-1}a_{02}\omega'\otimes b_2)^{(s^{-1}a_2\otimes b_{(0)})}
\end{aligned}$$

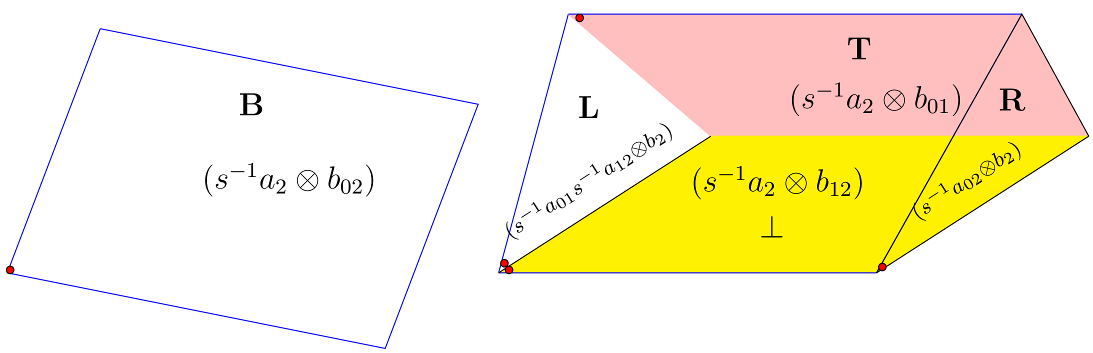


Figure 17:

•

$$\begin{aligned}
&\partial_3^{\otimes}(\omega\otimes b_3)\\
&= -(\omega\otimes b_{023}) + (\omega\otimes b_{013})\\
&\quad +(\omega\otimes b_{123})^{(\omega\otimes b_{01})} - (\omega\otimes b_{012})
\end{aligned}$$

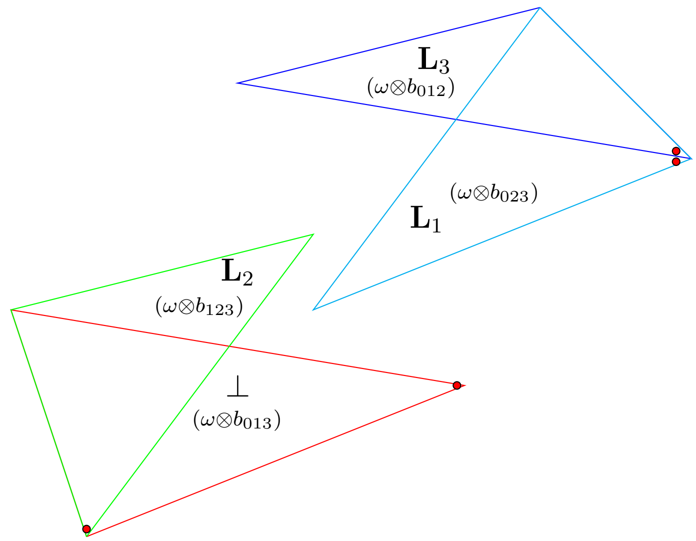


Figure 18:

$$
\begin{aligned}
&\partial^{\otimes}(x\otimes b)\\
&\qquad = (-1)^{|x|}(x\otimes d_0 b)^{\mathfrak{p}\otimes b_{01}} + \sum_{i=1}^{m}(-1)^{i+|x|}(x\otimes d_i b)\\
&+\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum_{\ell=1}^{k-1} n_\ell}\Bigg(\Big(\omega^{(0)}\cdot\prod_{\ell=1}^{k-1} s^{-1}a^{(\ell)}_{n_\ell+1}\cdot\omega^{(\ell)}\cdot s^{-1}d_i a^{(k)}_{n_k+1}\cdot\omega^{(k)}\cdot\prod_{\ell=k+1}^{r} s^{-1}a^{(\ell)}_{n_\ell+1}\cdot\omega^{(\ell)}\otimes b\Big)^{\gamma_i^{(k)}}\\
&\qquad\qquad - \quad \Big(\omega^{(0)}\cdot\prod_{\ell=1}^{k-1} s^{-1}a^{(\ell)}_{n_\ell+1}\cdot\omega^{(\ell)}\cdot s^{-1}a^{(k)}_{0\ldots i}\cdot s^{-1}a^{(k)}_{i\ldots n_k+1}\cdot\omega^{(k)}\cdot\prod_{\ell=k+1}^{r} s^{-1}a^{(\ell)}_{n_\ell+1}\cdot\omega^{(\ell)}\otimes b\Big)\Bigg)
\end{aligned}
$$

where the action is by

$$
\gamma_i^{(k)} = (\omega^{(0)}\cdot\prod_{\ell=1}^{k-1}\mathfrak{p}^{(\ell)}\cdot\omega^{(\ell)}\ \cdot\ \beta(s^{-1}a^{(k)}_{0\ldots i-1})\cdot a^{(k)}_{i-1\ i\ i+1}\cdot\beta(s^{-1}a^{(k)}_{i+1\ldots n_k+1})\cdot\prod_{\ell=k+1}^{r}\mathfrak{p}^{(\ell)}\cdot\omega^{(\ell)}\otimes *)
$$

where $\mathfrak{p}^{(\ell)}$ is the basepoint $\beta(s^{-1}a^{(\ell)}_{n_\ell+1})$.

5.3.2 The boundary of the twisted tensor product

In this section we will complete our construction of the crossed complex of groupoids $P^{\mathsf{Crs}}X$, which will be an example of a twisted tensor product of a (free) crossed chain algebra and a (free) crossed complex.

Definition 5.8. *Let X be a 0-reduced simplicial set. The* **path crossed complex** $P^{\mathsf{Crs}}X$ *is the twisted tensor product of the crossed complex of groups πX, and the free crossed complex of groupoids $\hat{\Omega}^{\mathsf{Crs}}X$. Its object set is*

$$P_0^{\mathsf{Crs}}X = (\hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X)_0 = \{(\omega \otimes *) \mid \omega \in \hat{\Omega}_0^{\mathsf{Crs}}X\}$$

where ω is any object of $\hat{\Omega}^{\mathsf{Crs}}X$,

$$\omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2}\cdots(s^{-1}a_1^{(k)})^{\epsilon_k}$$

for $k \geqslant 0$, $a_1^{(i)} \in X_1 - \{s_0(*)\}$, $\epsilon_i = \pm 1$.

In dimension 1 the generators are of form

$$(\omega \otimes b_1) : (\omega \otimes *) \to (\omega s^{-1}b_1 \otimes *)$$

which has twisted target, and

$$(\omega s^{-1}a_2\omega' \otimes *) : (\omega s^{-1}a_{01}s^{-1}a_{12}\omega' \otimes *) \to (\omega s^{-1}a_{02}\omega' \otimes *).$$

In any dimension $n+m \geqslant 1$, the general form of a generator is:

$$(x \otimes b) \in P_{n+m}^{\mathsf{Crs}}X$$

where b is a non-degenerate simplex in X_m and x is an n-dimensional generator of $\hat{\Omega}^{\mathsf{Crs}}X$,

$$x = \omega^{(0)}s^{-1}a_{n_1+1}^{(1)}\omega^{(1)}a_{n_2+1}^{(2)}\cdots\omega^{(k-1)}s^{-1}a_{n_k+1}^{(k)}\omega^{(k)}$$

with $\omega^{(i)} \in \hat{\Omega}_0^{\mathsf{Crs}}X$, $0 \leq i \leq k$, and each $a_{n_i+1}^{(i)}$ a non-degenerate simplex in X_{n_i+1}, $n_i \geqslant 1$, $\sum n_i = n$.

The boundary map $\partial^P_{n+m} : P^{\mathsf{Crs}}_{n+m}X \to P^{\mathsf{Crs}}_{n+m-1}X$ *is given for dimension* $n+m \geqslant 2$ *by the following formulas:*

<u>For $n+m=2$</u> we have four types of terms, but if $n=2$ and $m=0$ then the boundary is the same as the untwisted boundary, so we will not write them. The other two cases are

$$\partial^P_2(\omega \otimes b_2) = -(\omega \otimes b_{02}) + (\omega s^{-1}b_2 \otimes *) + (\omega s^{-1}b_{01} \otimes b_{12}) + (\omega \otimes b_{01})$$

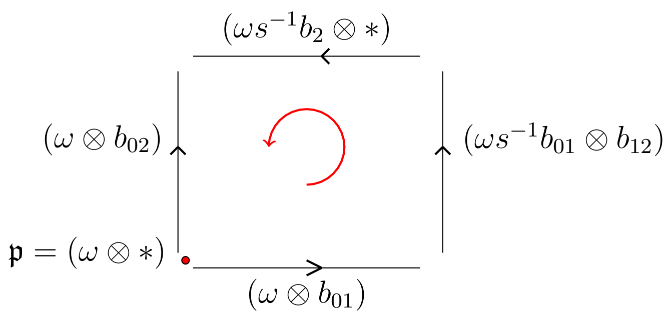


Figure 19: $\partial^P_2(\omega \otimes b_2)$

and

$$\partial^P_2(\omega s^{-1}a_2\omega' \otimes b_1) = -(\omega s^{-1}a_{01}s^{-1}a_{12}\omega' \otimes b_1) - (\omega s^{-1}a_2\omega' s^{-1}b_1 \otimes b_{(1)})$$

$$+(\omega s^{-1}a_{02}\omega' \otimes b_1) + (\omega s^{-1}a_2\omega' \otimes b_{(0)})$$

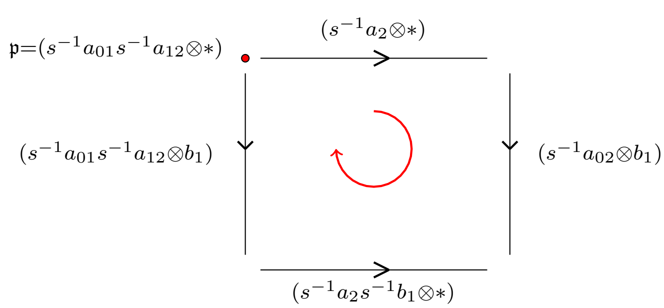


Figure 20: $\partial^P_2(s^{-1}a_2 \otimes b_1)$

<u>For $n+m=3$</u>, *the boundary map $\partial_3^P(x\otimes b_m)$ is in P_2^{Crs}, which is still non-abelian. There are eight types of generators, but when $m=0$ their boundaries are identical to the non-twisted version, so there are only four cases we need to define.*

$$\begin{aligned}\partial_3^P(\omega\otimes b_3) =&(\omega s^{-1}b_3\otimes b_{(3)})^{(s^{-1}b_{01}s^{-1}b_{12}\otimes b_{23})+(s^{-1}b_{01}\otimes b_{12})+(\omega\otimes b_{01})}\\ &+(\omega s^{-1}b_{012}\otimes b_{23})^{(s^{-1}b_{01}\otimes b_{12})+(\omega\otimes b_{01})}-(\omega\otimes b_{012})\\ &-(\omega\otimes b_{023})+(\omega\otimes b_{013})+(\omega s^{-1}b_{01}\otimes b_{123})^{(\omega\otimes b_{01})}\end{aligned}$$

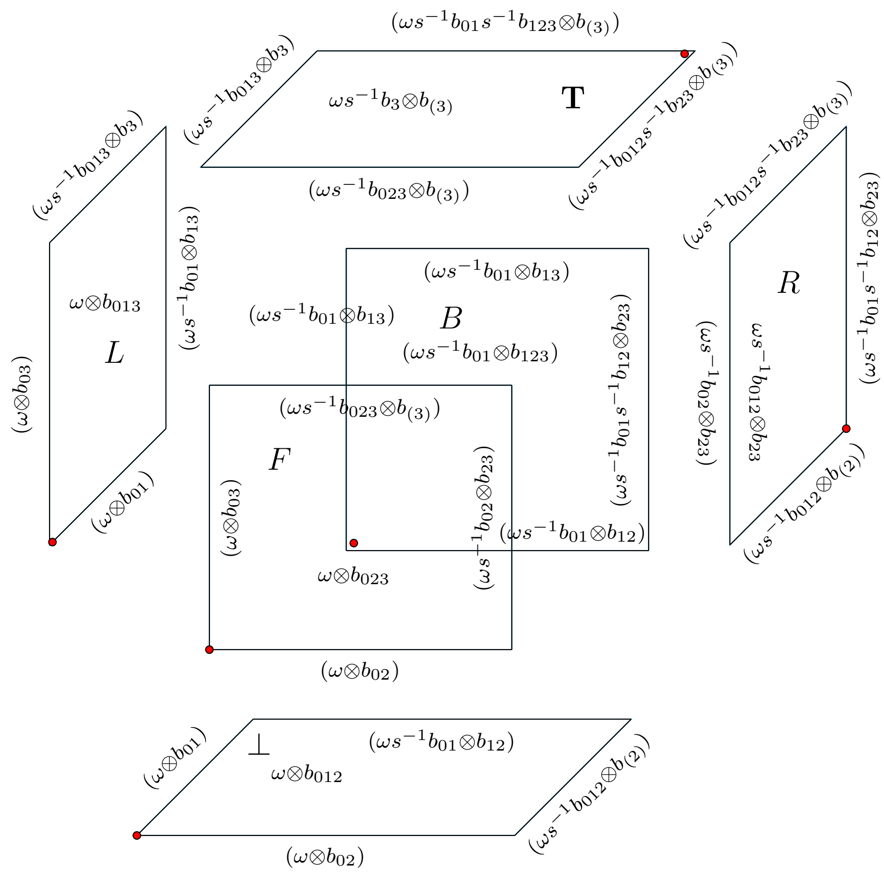


Figure 21: $\partial_3^P(\omega\otimes b_3)$

$$\partial_3^P(\omega s^{-1}a_2\omega'\otimes b_2) = -(\omega s^{-1}a_2\omega' s^{-1}b_{01}\otimes b_{12})^{(s^{-1}a_{01}s^{-1}a_{12}\otimes b_{01})}$$
$$-(\omega s^{-1}a_2\omega' s^{-1}b_2\otimes b_{(2)})^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01}\otimes b_{12})+(s^{-1}a_{01}s^{-1}a_{12}\otimes b_{01})}$$
$$-(\omega s^{-1}a_{01}s^{-1}a_{12}\omega'\otimes b_2)+(\omega s^{-1}a_2\omega'\otimes b_{02})$$
$$+(\omega s^{-1}a_{02}\omega'\otimes b_2)^{(s^{-1}a_2\otimes b_{(0)})}-(\omega s^{-1}a_2\omega'\otimes b_{01})$$

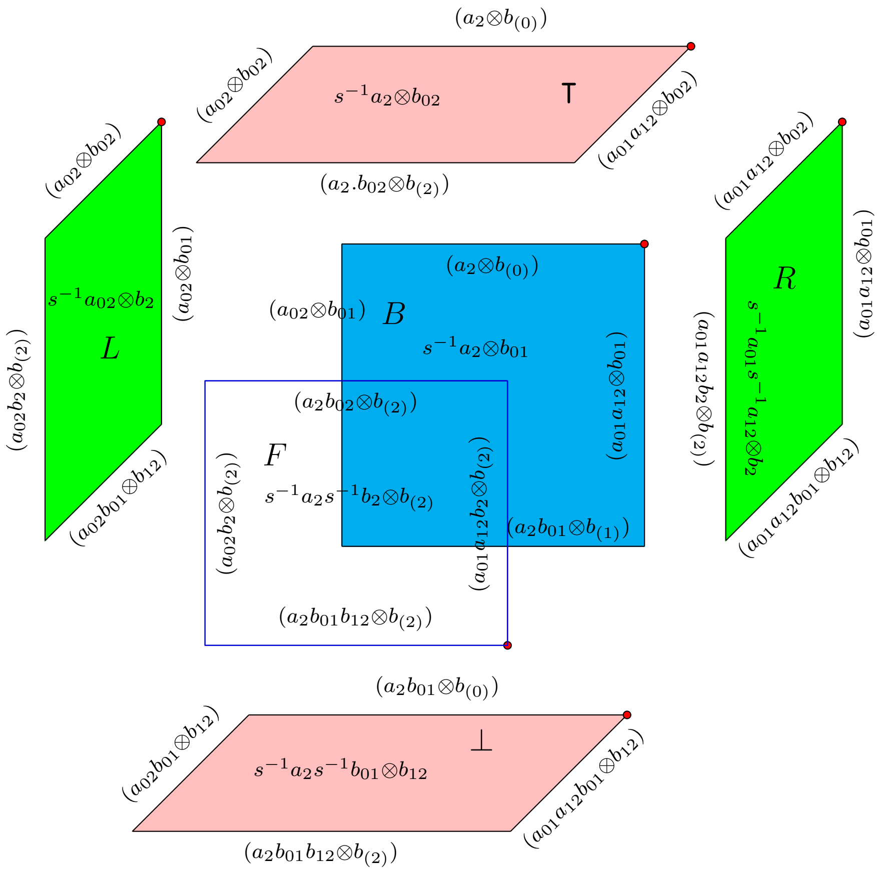


Figure 22: $\partial_3^P(\omega s^{-1}a_2\omega'\otimes b_2)$

$$
\begin{aligned}
\partial_3^P(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega''\otimes b_1) = &-(\omega s^{-1}a_2\omega' s^{-1}a_{02}'\omega''\otimes b_1)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_2'\otimes b_{(0)})}\\
&-(\omega s^{-1}a_{01}s^{-1}a_{12}\omega' s^{-1}a_2'\omega''\otimes b_1)\\
&+(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega'' s^{-1}b_1\otimes b_{(1)})^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{01}'s^{-1}a_{12}'\otimes b_1)}\\
&+(\omega s^{-1}a_2\omega' s^{-1}a_{01}'s^{-1}a_{12}'\omega''\otimes b_1)\\
&+(\omega s^{-1}a_{02}\omega' s^{-1}a_2'\omega''\otimes b_1)^{(s^{-1}a_{012}s^{-1}a_{01}'s^{-1}a_{12}'\otimes b_{(0)})}\\
&-(\omega s^{-1}a_2\omega' s^{-1}a_2'\omega''\otimes b_{(0)})
\end{aligned}
$$

$$
\begin{aligned}
\partial_3^P(\omega s^{-1}a_3\omega'\otimes b_1) = &-(\omega s^{-1}a_{01}s^{-1}a_{123}\omega'\otimes b_1)\\
&+(\omega s^{-1}a_3\omega'\otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_{23}\otimes b_1)}\\
&+(\omega s^{-1}a_{012}s^{-1}a_{23}\omega'\otimes b_1)\\
&+(\omega s^{-1}a_{023}\omega'\otimes b_1)^{(s^{-1}a_{012}s^{-1}a_{23}\otimes *)}\\
&-(\omega s^{-1}a_3\omega'\otimes *)\\
&-(\omega s^{-1}a_{013}\omega'\otimes b_1)^{(s^{-1}a_{01}s^{-1}a_{123}\otimes *)}
\end{aligned}
$$

For $m+n \geqslant 4$ *the formula looks somewhat long and complicated but it is still easy. Let us first rewrite the definitions of* x*,*

$$x = \omega^{(0)} s^{-1} a^{(1)}_{n_1+1} \omega^{(1)} a^{(2)}_{n_2+1} \cdots \omega^{(r-1)} s^{-1} a^{(r)}_{n_r+1} \omega^{(r)}, \quad \sum n_i = n, \omega^{(i)} \in \Omega_0^{\mathsf{Crs}} X$$

Then the definition of the boundary map will be:

$$\begin{aligned}
&\partial^P_q (x \otimes b_m) \\
=& \sum_{j=1}^{m} (-1)^{j+|x|} (x \otimes d_j b_m) + (-1)^{|x|} \sum_{j=1}^{m} \left(x s^{-1} b_{0\ldots j} \otimes b_{j\ldots m} \right)^{(\sum_j \Upsilon_j)} \\
&+ \sum_{k=1}^{r} \sum_{i=1}^{n_k} (-1)^{i+1+\sum_{\ell=1}^{k-1} n_\ell} \Bigg(\Big(\omega^{(0)} \cdot \prod_{\ell=1}^{k-1} s^{-1} a^{(\ell)}_{n_\ell+1} \cdot \omega^{(\ell)} \cdot s^{-1} d_i a^{(k)}_{n_k+1} \cdot \omega^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1} a^{(\ell)}_{n_\ell+1} \cdot \omega^{(\ell)} \otimes y_m \Big)^{\gamma_i^{(k)}} \\
&\quad - \quad \Big(\omega^{(0)} \cdot \prod_{\ell=1}^{k-1} s^{-1} a^{(\ell)}_{n_\ell+1} \cdot \omega^{(\ell)} \cdot s^{-1} a^{(k)}_{0\ldots i} \cdot s^{-1} a^{(k)}_{i\ldots n_k+1} \cdot \omega^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1} a^{(\ell)}_{n_\ell+1} \cdot \omega^{(\ell)} \otimes y_m \Big) \Bigg)
\end{aligned}$$

where the $\gamma_i^{(k)}$*-action is by*

$$\gamma_i^{(k)} = (\omega^{(0)} \cdot \prod_{\ell=1}^{k-1} \mathfrak{p}^{(\ell)} \cdot \omega^{(\ell)} \;\cdot\; \beta(s^{-1} a^{(k)}_{0\ldots i-1}) \cdot a^{(k)}_{i-1\, i\, i+1} \cdot \beta(s^{-1} a^{(k)}_{i+1\ldots n_k+1}) \cdot \prod_{\ell=k+1}^{r} \mathfrak{p}^{(\ell)} \cdot \omega^{(\ell)} \otimes *)$$

where $\mathfrak{p}^{(\ell)}$ *is the basepoint* $\beta(s^{-1} a^{(\ell)}_{n_\ell+1})$*,*

and where the Υ_j*-action is by*

$\Upsilon_j = ((\omega^{(0)} \cdot \prod_{\ell=1}^{k-1} \mathfrak{p}^{(\ell)} \cdot \omega^{(\ell)} s^{-1} b_{01} \ldots s^{-1} b_{j-2\ j-1} \otimes b_{j-1\ j})$

Conjecture 5.9. ∂^P_n *is a differential on* $P^{\mathsf{Crs}} X$*.*

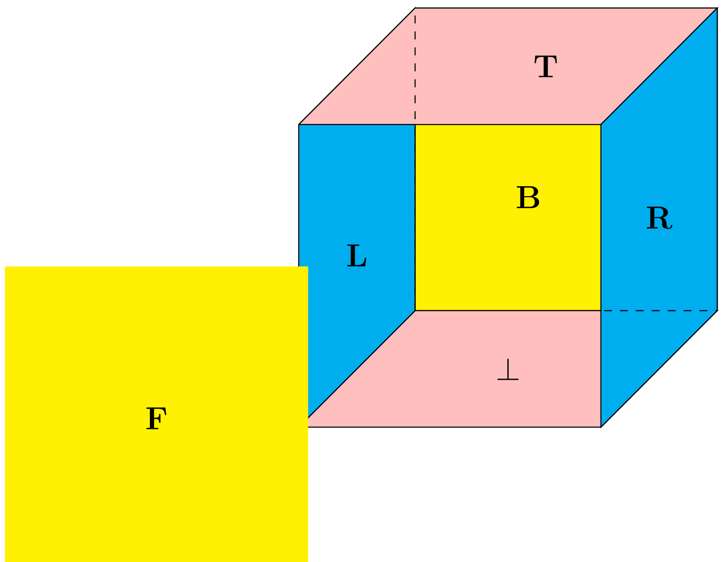
Proof. The only assertion to prove is $\partial^P_{q-1} \partial^P_q = 0$.

For dimension $q \leqslant 4$ the actions and order of terms are important, so we will divide the proof into two parts

A. in case of $q \leqslant 4$

Here we need to be a very careful in proof since the features are non-abelian. We could follow the instruction that we will do in example below for all cases of generators in dimensions $2, 3$ and 4.

Example 5.10. $\partial_3^{\mathsf{Crs}}(s^{-1}a_2 s^{-1}a'_2 \otimes b_1)\big) = (-R^{\gamma'_1} \ - \ F \ + \ T^{\Upsilon_1} \ + \ L \ + \ B^{\gamma_1} + \perp)$



$T = (s^{-1}a_2 s^{-1}a'_2 s^{-1}b_1 \otimes b_{(1)})^{\Upsilon_1} \qquad \Upsilon_1 = (s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_{01}s^{-1}a'_{12} \otimes b_1)$

$L = (s^{-1}a_2 s^{-1}a'_{01}s^{-1}a'_{12} \otimes b_1)$

$F = (s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_2 \otimes b_1)$

$\perp = (s^{-1}a_2 s^{-1}a'_2 \otimes b_{(0)})$

$R = (s^{-1}a_2 s^{-1}a'_{02} \otimes b_1)^{\gamma'_1} \qquad \gamma'_1 = (s^{-1}a_{01}s^{-1}a_{12}s^{-1}a'_2 \otimes b_{(0)})$

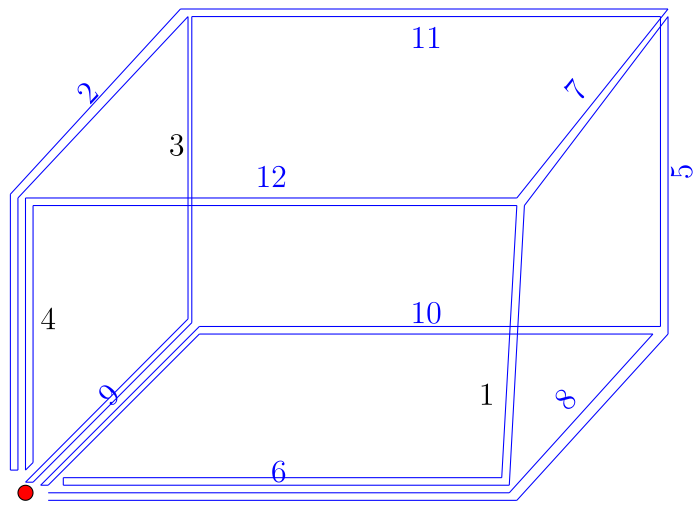
$B = (s^{-1}a_{02}s^{-1}a'_2 \otimes b_1)^{\gamma_1} \qquad \gamma_1 = (s^{-1}a_2 s^{-1}a'_{01}s^{-1}a'_{12} \otimes b_{(0)})$

We will label the edges by numbers to make the answer easier to handle

$\partial_2(-R^6 \ - \ F \ + \ T^4 \ + \ L \ + \ B^9 - \ \perp) =$

$-6 \ -8 \ -5 \ +7 \ +1 \ +6 \ -6 \ -1 \ +12 \ +4 \ -4 \ -12 \ -7 \ +11 \ +2 \ +4 \ -4 \ -$

$2 \; +3 \; +9 \; -9 \; -3 \; -11 \; +5 \; +10 \; +9 \; -9 \; -10 \; +8 \; +6$



Conjecture 5.11. *For all $p_4 = (x \otimes b), \partial_3^P \partial_4^P p_4$ is trivial.*

See the Appendix A for the way we hope that we could prove this conjecture in the future.

B: $q \geqslant 5$ in this case the proof may seem long and somewhat complicated due to the large number of symbols used. The twisted $\partial_q^P(x \otimes b_m)$ where x is the general generator element in $\hat{\Omega}^{\mathsf{Crs}}X$ will consist of four types of terms, and the square partial $(\partial^P)^2$ will consists from 16 part, and to make the proof easier for read and understand we will label each part by a number and each subparts which comes out from the square partial to sub numbers.

$\partial_q^P(x \otimes b_m) =$

$$\sum_{i=1}^{m} (-1)^{i+|x|}(x \otimes d_i b_m) \tag{27}$$

$$+(-1)^{|x|}\sum_{i=1}^{m}\big(x\cdot s^{-1}b_{0\ldots i}\otimes b_{i\ldots m}\big)^{\sum_i \Upsilon_i} \tag{28}$$

$$+\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1}n_\ell}\Big(\prod_{\ell=1}^{k-1}s^{-1}a^{(\ell)}_{n_\ell+1}\cdot s^{-1}d_i a^{(k)}_{n_k+1}\cdot\prod_{\ell=k+1}^{r}s^{-1}a^{(\ell)}_{n_\ell+1}\otimes b_m\Big)^{\gamma_i^{(k)}} \tag{29}$$

$$-\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1}n_\ell}\Big(\prod_{\ell=1}^{k-1}s^{-1}a^{(\ell)}_{n_\ell+1}\cdot s^{-1}a^{(k)}_{0\ldots i}\cdot s^{-1}a^{(k)}_{i\ldots n_k+1}\cdot\prod_{\ell=k+1}^{r}s^{-1}a^{(\ell)}_{n_\ell+1}\otimes b_m\Big) \tag{30}$$

we can see from above that the terms $(27),(29),(30)$ are non-twisting version whose we will symbolise them by d, and the twisting term 28, we will symbolise it by d'.
Now, the square partial $\partial^P_{q-1}\partial^P_q(x\otimes b_m)=\partial_{q-1}(27)+\partial_{q-1}(28)+\partial_{q-1}(29)+\partial_{q-1}(30)$ have 16 types of terms, where each part of parts $27,29,30$ consists of 3 non-twisting terms and one twisting, so from the square partial of the three parts $27,29,30$, we have 9 terms which are non- twisting we symbolise by $d\cdot d$, their second partial already equal zero and three terms which are twisting we symbolise by $d'\cdot d$.
The square partial of the twisting term 28 has three types of twisting terms $d\cdot d'$ and one type of term which is $d'\cdot d'$. So to prove $(\partial^P)^2=0$, we need to prove that $d\cdot d'+d'\cdot d+d'\cdot d'=0$, and to make the proof more readable let us give a label for these sub parts.

1. $d\cdot d'$ are the twisting terms coming out from square twisting partial of the non-twisting items of first partial.

   i. $\sum_{i=1}^{m}\sum_{j=1}^{m-1}(-1)^i\big(x\cdot s^{-1}\widehat{b}_{0\ldots r}\otimes\widehat{b}_{r\ldots m}\big)^{\sum_r\Upsilon_r},\quad \widehat{b}=b_{0\ldots\widehat{i}\ldots m}\qquad(27-2),$

   ii. $+(-1)^{|x|-1}\sum_{j=1}^{m}\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1}n_\ell}\big(\prod_{\ell=1}^{k-1}s^{-1}a^{(\ell)}_{n_\ell+1}\cdot s^{-1}\widehat{a}^{(k)}_{n_k+1}$
   $\cdot\prod_{\ell=k+1}^{r}s^{-1}a^{(\ell)}_{n_\ell+1}\cdot s^{-1}b_{0\ldots j}\otimes b_{j\ldots m}\big)^{\gamma_i^{(k)}+\sum_j\Upsilon_j},\quad \widehat{a}^{(k)}_{n_k+1}=a^{(k)}_{0\ldots\widehat{i}\ldots n_k+1}\qquad(29-2),$

iii. $+(-1)^{|x|}\sum_{j=1}^{m}\sum_{k=1}^{r}\sum_{i=1}^{n_k}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1} n_\ell}\big(\prod_{\ell=1}^{k-1} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot s^{-1}a_{0\ldots i}^{(k)}$
$\cdot\, s^{-1}a_{i\ldots n_k+1}^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot s^{-1}b_{0\ldots j} \otimes b_{j\ldots m}\big)^{\sum_j \Upsilon_j}$ $(30-2)$

2. $d' \cdot d$ are the twisting terms coming out from square non-twist partial of the twisting items of first partial. Here we also have three types of terms they are:

   i. $\sum_{j=1}^{m-i}\big(\sum_{i=1}^{m}(-1)^{j}\big(xs^{-1}b_{0\ldots i} \otimes d_j b_{i\ldots m}\big)\big)^{\sum_i \Upsilon_i}$ $(28-1)$,

   ii. $+\sum_{k=1}^{r+j-1}\sum_{i=1}^{n_k}\sum_{j=1}^{m}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1} n_\ell}\big(\prod_{\ell=1}^{k-1} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot s^{-1}d_i a_{n_k+1}^{(k)}$
   $\cdot \prod_{\ell=k+1}^{r} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot s^{-1}b_{0\ldots j} \otimes b_{j\ldots m}\big)^{\gamma_i^{(k)}+\sum_j \Upsilon_j}$, $(28-3)$,

   iii. $-\sum_{k=1}^{r+j-1}\sum_{i=1}^{n_k}\sum_{j=1}^{m}(-1)^{i+1+\sum\limits_{\ell=1}^{k-1} n_\ell}\big(\prod_{\ell=1}^{k-1} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot s^{-1}a_{0\ldots i}^{(k)}$
   $\cdot\, s^{-1}a_{i\ldots n_k+1}^{(k)} \cdot \prod_{\ell=k+1}^{r} s^{-1}a_{n_\ell+1}^{(\ell)} \cdot b_{0\ldots j} \otimes b_{j\ldots m}\big)^{\sum_j \Upsilon_j}$ $(28-4)$

3. and the final term will be $d' \cdot d'$ which is the twisted version of twisting term (28) it has the form
   $+\sum_{j=1}^{m-i}\sum_{i=1}^{m}\big(x \cdot s^{-1}b_{0\ldots i} \cdot s^{-1}b_{i+1\ldots j+i} \otimes b_{j+i\ldots m}\big)^{\sum_i \Upsilon_i+\sum_j \Upsilon_j}$ $(28-2)$

The elements which coming out of the terms $(28-2)$ will cancel in pairs with elements coming out of $(28-4)$ where $j \neq m$.

The other terms of $(28-4)$ will have the same expression of some terms coming out $(27-2)$ but with opposite signs, and the other elements which coming out the term $(27-2)$ will have the same form of the elements coming out $(28-1)$ but with opposite sign.

The elements in both terms $(29-2)$ and $(30-2)$ are similar to the elements in $(28-3)$, so all terms in $(29-2)$, $(30-2)$ and $(28-3)$ cancel each other.

□

6 Contracting homotopy

Introduction

In this Chapter, we define a contracting homotopy $\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$, which raises the dimension by one. We have $h : id \simeq *$, so $P^{\mathsf{Crs}}X$ is contractible.

6.1 The structure of the contracting homotopy

For convenience we will repeat the definitions from Section 4.3

Definition 6.1. *Two homomorphisms $f, g : C \to D$ are homotopic, if there exists a homomorphism*

$$h : \pi(\Delta[1]) \otimes C \to D$$

that satisfies $hi_0 = f$ and $hi_1 = g$.

$$\begin{tikzcd} C \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & \pi(\Delta[1]) \otimes C \arrow[r, "h"] & D \end{tikzcd}$$

Definition 6.2. *A crossed complex of groupoids is* pointed *if there is a specified object $* \in C_0$. If C is a pointed crossed complex of groupoids, then C is* contractible *to the basepoint $*$ if there is a family of functions $\eta_n : C_n \to C_{n+1}$ that define a* contracting homotopy

$$h : * \simeq \mathrm{id}_C : \pi(\Delta[1]) \otimes C \to C$$

so

i. $h(0 \otimes c) = 0_*$

ii. $h(1 \otimes c) = c$

and

iii. *for* $c \in C, \eta(c) = h(\sigma \otimes c)$

In other words, the family η_n defines a homomorphism h that provides a homotopy between the given trivial homomorphism $C \to \{*\} \to C$ and the identity homomorphism $\mathrm{id}_C : C \xrightarrow{=} C$. This gives a homotopy equivalence between the crossed complex C and the trivial crossed complex $\{*\}$.

$$* \overset{h}{\simeq} \mathrm{id}_C \circlearrowright C \rightleftarrows \{*\}$$

Proposition 6.3. *A family of functions* $\eta_n : C_n \to C_{n+1}$*,* $(n \geqslant 0)$ *defines a contracting homotopy via* $h(\sigma \otimes c_n) = \eta_n(c_n)$ *if and only if it satisfies*

1. $\eta_0(c_0) \in C_1$ *has source* $*$ *and target* c_0*,*
2. $\eta_1(c_1) \in C_2$ *has basepoint* $*$ *and boundary:*

$$\partial_2\eta_1(c_1) = -\eta_0(\mathrm{targ}(c_1)) + c_1 + \eta_0(\mathrm{src}(c_1)),$$

$$\begin{tikzcd}
* \arrow[rr, "\eta_0(x)"] \arrow[dr, "\eta_0(y)"'] & \arrow[d, phantom, "\circlearrowright"] & x \arrow[dl, "c_1"] \\
& y &
\end{tikzcd}$$

Figure 23:

3. *If* $n \geqslant 2$ *then,* $\eta_n(c_n) \in C_{n+1}$ *has basepoint* $*$ *and boundary:*

$$\partial_{n+1}\eta_n(c_n) = c_n^{\eta_0(\mathfrak{p})} - \eta_{n-1}\partial_n(c_n),$$

4. *For all* $n \geqslant 1$*,*

$$\eta_n(c_n + c_n') = \eta_n(c_n) + \eta_n(c_n')$$

5. *For all* $n \geqslant 2$*,*

$$\eta_n(c_n^{c_1}) = \eta_n(c_n)$$

Remark 6.4. *Using Definition 6.2 (i,ii,iii), if we are given η we can define h from η, or if we are given h then we can define η from h. The proposition means that the condition that $h : \pi(\Delta[1]) \otimes C \to C$ is a well defined homomorphism of crossed complexes of groupoids, and commutes with the boundary ∂, holds if and only if η satisfies the properties $(1-5)$ of Proposition 6.3.*

Proof. $\Rightarrow$) Let $h : \pi(\Delta[1]) \otimes C \to C$ be a homomorphism satisfies that $h(0 \otimes c) = 0_*$, $h(1 \otimes c) = c$, and $h(\sigma \otimes c) = \eta(c)$ which commutes with the boundary, ∂ and well defined. We want to prove $\eta : C_n \to C_{n+1}$ is a contracting homotopy.

1. Let $c \in C_n$, $\operatorname{src}(\eta(c)) = \operatorname{src}(h(\sigma \otimes c)) = h \operatorname{src}(\sigma \otimes c) = h(0 \otimes c) = 0_*$, and the $\operatorname{targ}(\eta(c)) = \operatorname{targ}(h(\sigma \otimes c)) = h \operatorname{targ}(\sigma \otimes c) = h(1 \otimes c) = c$. (Definition 6.2)

2. Let $c_1 : a \to b \in C_1$, $\partial\eta(c_1) = \partial h(\sigma \otimes c_1) = h\partial(\sigma \otimes c_1)$

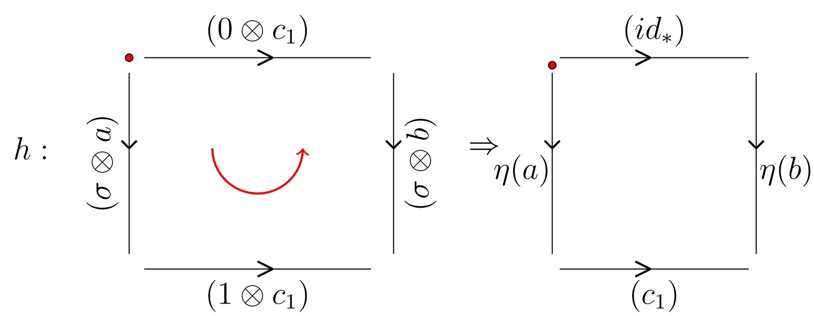


Figure 24: $\partial\eta(c)$ *in* dimension $1 = \ -\eta(b)\ + c_1\ +\ \eta(a)$

3. If $n = r$,

$$\partial\eta(c_r) = \partial h(\sigma \otimes c_r) = h\partial(\sigma \otimes c_r)$$

$= h\big(-(\operatorname{src}\ \sigma \otimes c_r) + (\operatorname{targ}\ \sigma \otimes c_r)^{(\sigma\otimes\mathfrak{p}_{c_r})} - (\sigma \otimes \partial_r c_r)\big)$ (by Definition 2.33 and thus the properties of ordinary tensor product of crossed complexes)

$= c_r^{\eta(\mathfrak{p}_{c_r})} - h(\sigma \otimes \partial_r c_r) = c_r^{\eta(\mathfrak{p}_{c_r})} - \eta\partial(c_r)$, ($\mathfrak{p}_{c_r}$ is the base point of c_r).

4. Finally we have

$$\eta(c_n + c_n') = h(\sigma \otimes (c_n + c_n')) = h\big((\sigma \otimes c_n)^{(\text{src } \sigma\otimes c_n')} + (\sigma \otimes c_n')\big)$$
$$= h(\sigma \otimes c_n)^{(\text{src } \sigma\otimes c_n')} + h(\sigma \otimes c_n') = \eta(c_n)^{h(0\otimes c_n')} + \eta(c_n') = \eta(c_n) + \eta(c_n'),$$

and

$$\eta(c_n^{c_n}) = h(\sigma \otimes c_n^{c_n}) = h((\sigma \otimes c_n)^{\text{src } (\sigma\otimes b)}) = \eta(c_n).$$

$\Leftarrow$) Let $\eta : C_n \to C_{n+1}$ be a family of functions satisfying 1 - 5 of Proposition 6.3 and define h by (i) (ii) and (iii) of Definition 6.2. To show that this gives a homomorphism, we need to show it is well defined and that it commutes with the boundary map, ∂. The first is given as follows

$$h(\sigma \otimes (c_n + c_n')) = \eta(c_n + c_n') = \eta(c_n) + \eta(c_n') = h(\sigma \otimes c_n) + h(\sigma \otimes c_n').$$

To see it commutes with the boundaries we note

$$\partial h(\sigma \otimes c_n) = \partial\eta(c_n) = c_n^{\eta(\mathfrak{p})} - \eta\partial(c_n)$$

whilst

$$h\partial(\sigma \otimes c_n) = h(-(0 \otimes c_n) + (1 \otimes c_n)^{(\sigma\otimes\mathfrak{p}_{c_r})} - (\sigma \otimes \partial c_n)) = c_n^{\eta(\mathfrak{p})} - h(\sigma \otimes \partial c_n) = c_n^{\eta(\mathfrak{p})} - \eta\partial c_n$$

follows by Definition 2.33 and Definition 6.2. $\square$

6.2 Contracting homotopy for $P^{\mathsf{Crs}}X$

In this section we define the contracting homotopy maps $\eta_n : P_n^{\mathsf{Crs}}X \to P_{n+1}^{\mathsf{Crs}}X$ which raise the dimension by one for a 0-reduced simplicial set X, for the group completed path complex $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ that we have introduced in Definition 5.8.

We will define the contracting homotopy inductively. We start by defining it in degree 0, and once we have defined η_0 we can define η_1, and so on. We can use the following definition for the partially-defined homotopies we will give:

Definition 6.5. *A k-contracting homotopy on a pointed crossed complex of groupoids C is a family of functions*

$$\{\eta_n : C_n \to C_{n+1} \,:\, n = 0, 1, \ldots k\}$$

which satisfy the conditions (1-5) of Proposition 6.3 for all elements c_n for $n \leq k$.

The group completed path complex $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ is a free crossed complex, with an infinite object set

$$P_0^{\mathsf{Crs}}X = \{(\omega \otimes *) \,:\, \omega \in \hat{\Omega}_0^{\mathsf{Crs}}X\}.$$

We know that

$$\omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2}\cdots(s^{-1}a_1^{(r)})^{\epsilon_r} \qquad (r \geqslant 0, a_1^{(i)} \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1)$$

is a word given by a string of non-degenerate 1-dimensional simplices of X and their 'formal' inverses. We want to think of the group completed path complex $P^{\mathsf{Crs}}X = \hat{\Omega}^{\mathsf{Crs}}X \otimes_\phi \pi X$ as a pointed crossed complex of groupoids: we specify a particular basepoint $\varnothing \otimes *$.

For dimension 1, the generators of the free groupoid $P_1^{\mathsf{Crs}}X$ are the elements

- $(\omega \otimes b_1)$ with source $(\omega \otimes *)$ and target $(\omega s^{-1}b_1 \otimes *)$
- $(\omega s^{-1}a_2\omega' \otimes *)$ with source $(\omega s^{-1}a_{01}s^{-1}a_{12}\omega' \otimes *)$ and target $(\omega s^{-1}a_{02}\omega' \otimes *)$.

The general form of a generator in higher degrees is $(x \otimes b_m) \in P_{n+m}^{\mathsf{Crs}}X$, where

$$x = \omega^{(0)}s^{-1}a_{n_1+1}^{(1)}\omega^{(1)}a_{n_2+1}^{(2)}\omega^{(2)}\cdots\omega^{(r-1)}s^{-1}a_{n_r+1}^{(r)}\omega^{(r)}$$

is a generator of $\hat{\Omega}^{\mathsf{Crs}}X$ in degree $n = |x| = \sum n_i$, and b is a generator of $\pi_m X$. The basepoint $\mathfrak{p}$ of this element is

$$\mathfrak{p} \;=\; \mathrm{src}(x \otimes b_m) \;=\; \mathrm{src}(x) \otimes * \;=\; \omega^{(0)}\mathfrak{p}^{(1)}\omega^{(1)}\cdots\omega^{(r-1)}\mathfrak{p}^{(r)}\omega^{(r)} \;\otimes\; *$$

where $\mathfrak{p}^{(i)} = \prod_{j=0}^{n_i} s^{-1}a_{j\;j+1}^{(i)}$.

Definition 6.6. *Consider a general object $\omega \otimes * \in P_0^{\mathsf{Crs}} X$ given by a string of r non-degenerate one-simplices and their inverses,*

$$\omega = (s^{-1}a_1^{(1)})^{\epsilon_1}(s^{-1}a_1^{(2)})^{\epsilon_2}\cdots(s^{-1}a_1^{(r)})^{\epsilon_r} \ : \ k \geqslant 0, a_1^{(i)} \in X_1 - \{s_0(*)\}, \epsilon_i = \pm 1.$$

We define a function $\eta_0 : P_0^{\mathsf{Crs}} \to P_1^{\mathsf{Crs}}$ in dimension 0 by:

1. *If $r = 0$ then $\eta_0(\varnothing \otimes *) = 0_{(\varnothing\otimes *)} \in P_1^{\mathsf{Crs}} X$,*
2. *If $r \geqslant 1$ and $\omega = \omega' \cdot (s^{-1}a_1^{(r)})^{\epsilon_r}$ where ω' has length $r-1$ then*

$$\eta_0(\omega \otimes *) : \varnothing \otimes * \longrightarrow \omega' \otimes * \longrightarrow \omega \otimes *$$

can be defined inductively by:

$$\eta_0(\omega \otimes *) = \eta_0(\omega' \cdot s^{-1}a_1^{(r)} \otimes *) = (\omega' \otimes a_1^{(r)}) + \eta_0(\omega' \otimes *) \qquad \text{if } \epsilon_r = +1$$

$$\eta_0(\omega \otimes *) = \eta_0(\omega' \cdot (s^{-1}a_1^{(r)})^{-1} \otimes *) = -(\omega \otimes a_1^{(r)}) + \eta_0(\omega' \otimes *) \qquad \text{if } \epsilon_r = -1$$

Remark 6.7. *To make it easier to read we have written out both of the two cases, for $\epsilon_r = \pm 1$, in Definition 6.6. This is redundant, as each of the two cases is really a consequence of the other one. If we are given the definition for $\epsilon = +1$, for example, we may rearrange it and write*

$$-(\omega' \otimes a_1^{(r)}) + \eta_0(\omega' \cdot s^{-1}a_1^{(r)} \otimes *) = \eta_0(\omega' \otimes *).$$

If $\omega' = \omega'' \cdot (s^{-1}a_1^{(r)})^{-1}$ this says

$$-(\omega' \otimes a_1^{(r)}) + \eta_0(\omega'' \otimes *) = \eta_0(\omega' \otimes *)$$

This is just the definition for $\epsilon = -1$.

Theorem 6.8. *The function $\eta_0 : P_0^{\mathsf{Crs}} \to P_1^{\mathsf{Crs}}$ in Definition 6.6 defines a 0-contracting homotopy.*

Proof. We need to prove η_0 is a well defined homomorphism that satisfies the properties of Proposition (6.3) for elements of degree 0. If $r = 0$, we can see that the source of $\eta_0(\varnothing \otimes *)$ is $(\varnothing \otimes *)$ and the target is $(\varnothing \otimes *)$ so it satisfies property (1) of Proposition 6.3. If $r \geqslant 1$, we see the source of $\eta(\omega \otimes *)$ is $(\varnothing \otimes *) = \mathrm{src}(\eta(\omega' \cdot (s^{-1}a_1^{(r)})^{\epsilon_r} \otimes *))$, and the target is $(\omega \otimes *)$ $(\epsilon_r = \pm 1)$, so it satisfies Proposition 6.3 (1). For the generators of the form $(\omega s^{-1}a_1^{(1)} \otimes *)$ where ω is a word of length n, we assume $\eta(\omega \otimes *)$ has source $(\varnothing \otimes *)$ and target $(\omega \otimes *)$, and since the source of $(\omega \otimes a_1^{(1)})$ is $(\omega \otimes *)$, and the target is $(\omega s^{-1}a_1^{(1)} \otimes *)$, we see $\eta_0(\omega s^{-1}a_1^{(1)} \otimes *)$ is well defined and hence η_0 satisfies Proposition 6.3 (1) by inductively. $\square$

Definition 6.9. *In dimension* 1 *we define a function* η_1 *on the generators of the free groupoid* $P_1^{\mathsf{Crs}}X$ *as follows.*

1. *For any generator* $(\omega \otimes b_1)$ *where* $\omega \in \hat{\Omega}_0^{\mathsf{Crs}}X$ *and* b_1 *is a non-degenerate 1-simplex of* X*, define*
$$\eta_1(\omega \otimes b_1) = 0_{(\varnothing \otimes *)} \tag{31}$$

2. *Consider a generator* $(\omega\, s^{-1}a_2\, \omega' \otimes *)$*, where* $\omega, \omega' \in \hat{\Omega}_0^{\mathsf{Crs}}X$ *and* a_2 *is a non-degenerate 2-simplex of* X*.*

   If $\omega' = \varnothing$ *then define*
$$\eta_1(\omega s^{-1}a_2 \otimes *) = (\omega \otimes a_2)^{\eta_0(\omega \otimes *)} \tag{32}$$

   If $\omega' = \omega'' \cdot s^{-1}a_1$ *then define inductively*
$$\eta_1(\omega s^{-1}a_2\omega' \otimes *) = \eta_1(\omega s^{-1}a_2\omega'' \otimes *) - (\omega s^{-1}a_2\omega'' \otimes a_1)^{\eta_0(\mathrm{src}(\omega s^{-1}a_2\omega'' \otimes *))}. \tag{33}$$

   If $\omega' = \omega'' \cdot (s^{-1}a_1)^{-1}$ *then define inductively*
$$\eta_1(\omega s^{-1}a_2\omega' \otimes *) = \eta_1(\omega s^{-1}a_2\omega'' \otimes *) + (\omega s^{-1}a_2\omega'' \otimes a_1)^{\eta_0(\mathrm{src}(\omega s^{-1}a_2\omega'' \otimes *))}. \tag{34}$$

Remark 6.10. *As in Remark 6.7, it is not necessary to give both of the last two definitions, because they imply each other. For example, we can rearrange the definition* (33) *to write it as*

$$\eta_1(\omega s^{-1}a_2\omega'' s^{-1}a_1 \otimes *) + (\omega s^{-1}a_2\omega'' \otimes a_1)^{\eta_0(\mathrm{src}(\omega s^{-1}a_2\omega''\otimes *))} = \eta_1(\omega s^{-1}a_2\omega'' \otimes *).$$

If we substitute $\omega'' = \omega'''(s^{-1}a_1)^{-1}$ *into this we get*

$$\eta_1(\omega s^{-1}a_2\omega''' \otimes *) + (\omega s^{-1}a_2\omega'' \otimes a_1)^{\eta_0(\mathrm{src}(\omega s^{-1}a_2\omega''\otimes *))} = \eta_1(\omega s^{-1}a_2\omega'''(s^{-1}a_1)^{-1} \otimes *).$$

This is the same as the definition (34)*.*

Example 6.11. *1. Let we have the element* $c_1 = (s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)$*, we use Proposition 6.3 to calculate* $\eta(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)$

$$\eta(\mathrm{src}(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)) = \eta(s^{-1}a_1^{(1)}s^{-1}a_{01}s^{-1}a_{12} \otimes *)$$
$$= (s^{-1}a_1^{(1)}s^{-1}a_{01} \otimes a_{12}) + (s^{-1}a_1^{(1)} \otimes a_{01}) + (\varnothing \otimes a_1^{(1)})$$
$$\eta(\mathrm{targ}(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)) = (s^{-1}a_1^{(1)} \otimes a_{02}) + (\varnothing \otimes a_1^{(1)})$$
$$\partial\eta(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *) = -\eta(\mathrm{targ}(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)) \quad + (s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)$$
$$+ \eta(\mathrm{src}(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)) \text{ (by Proposition 6.3 (2))},$$
$$= -(\varnothing \otimes a_1^{(1)}) \quad - (s^{-1}a_1^{(1)} \otimes a_{02}) \quad + (s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *)$$
$$+ (s^{-1}a_1^{(1)}s^{-1}a_{01} \otimes a_{12}) \quad + (s^{-1}a_1^{(1)} \otimes a_{01}) \quad + (\varnothing \otimes a_1^{(1)})$$

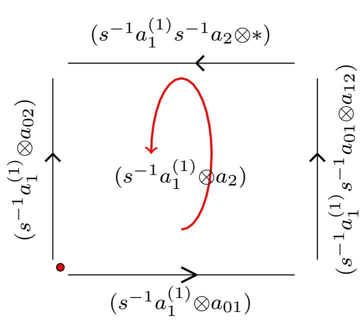


Figure 25:

$$= \partial(s^{-1}a_1^{(1)} \otimes a_2)^{(\varnothing \otimes a_1^{(1)})}$$

$$\Leftrightarrow$$

$$\eta(s^{-1}a_1^{(1)}s^{-1}a_2 \otimes *) = (s^{-1}a_1^{(1)} \otimes a_2)^{(\varnothing \otimes a_1^{(1)})}.$$

2. *let now calculate* $\eta(s^{-1}a_2s^{-1}a_1^{(1)}s^{-1}a_1^{(2)} \otimes *)$

$$\eta(\mathrm{src}(s^{-1}a_2s^{-1}a_1^{(1)}s^{-1}a_1^{(2)} \otimes *)) = (s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_1^{(1)} \otimes a_1^{(2)}) + (s^{-1}a_{01}s^{-1}a_{12} \otimes a_1^{(1)}) + (s^{-1}a_{01} \otimes a_{12}) + (\varnothing \otimes a_{01})$$

$$\eta(\mathrm{targ}(s^{-1}a_2s^{-1}a_1^{(1)}s^{-1}a_1^{(2)} \otimes *)) = (s^{-1}a_{02}s^{-1}a_1^{(1)} \otimes a_1^{(2)}) + (s^{-1}a_{02} \otimes a_1^{1}) + (\varnothing \otimes a_{02})$$

again from Proposition 6.3(2) ($\partial\eta(c_1) = -\eta(\mathrm{targ}\ \ c_1) + c_1 + \eta(\mathrm{src}\ \ c_1)$) *we have,*

$$\partial\eta(s^{-1}a_2s^{-1}a_1^1s^{-1}a_1^{(2)} \otimes *) = -(\varnothing \otimes a_{02}) \quad - (s^{-1}a_{02} \otimes a_1^1) \quad - (s^{-1}a_{02}s^{-1}a_1^{(1)} \otimes a_1^{(2)}) + (s^{-1}a_2s^{-1}a_1^{(1)}s^{-1}a_1^{(2)} \otimes *) \quad + (s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_1^{(1)} \otimes a_1^{(2)}) \quad + (s^{-1}a_{01}s^{-1}a_{12} \otimes a_1^{(1)}) \quad + (s^{-1}a_{01} \otimes a_{12}) \quad + (\varnothing \otimes a_{01})$$

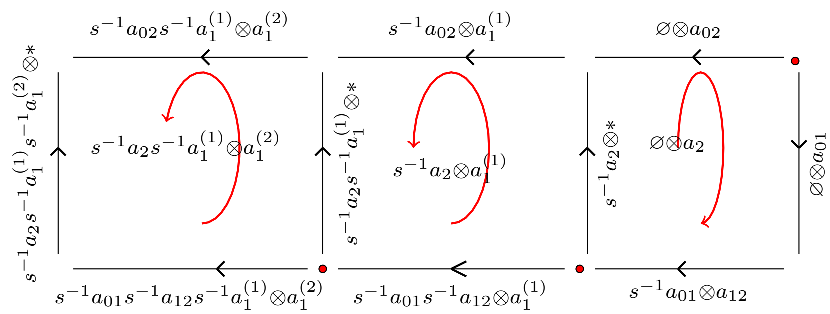


Figure 26:

$$= \partial\Big((\varnothing \otimes a_2) - \big(s^{-1}a_2 \otimes a_1^{(1)}\big)^{(s^{-1}a_{01}\otimes a_{12})+(\varnothing\otimes a_{01})} - \big(s^{-1}a_2s^{-1}a_1^{(1)} \otimes a_1^{(2)}\big)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_1^{(1)}\otimes a_1^{(2)})+(s^{-1}a_{01}\otimes a_{12})+(\varnothing\otimes a_{01})}\Big)$$

$$\Leftrightarrow$$

$$\eta(s^{-1}a_2s^{-1}a_1^1s^{-1}a_1^{(2)} \otimes *) = (\varnothing \otimes a_2) - \big(s^{-1}a_2 \otimes a_1^{(1)}\big)^{(s^{-1}a_{01}\otimes a_{12})+(\varnothing\otimes a_{01})} - \big(s^{-1}a_2s^{-1}a_1^{(1)} \otimes a_1^{(2)}\big)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}a_1^{(1)}\otimes a_1^{(2)})+(s^{-1}a_{01}\otimes a_{12})+(\varnothing\otimes a_{01})}.$$

Proposition 6.12. *The functions $\eta_0 : P_0^{\mathsf{Crs}}X \to P_1^{\mathsf{Crs}}X$ and $\eta_1 : P_1^{\mathsf{Crs}}X \to P_2^{\mathsf{Crs}}X$ in Definitions 6.6 and 6.9 define a 1-contracting homotopy on $P^{\mathsf{Crs}}X$.*

Proof. We need to show the function η_1 which we defined in Definition 6.9 satisfies part (2) of Proposition 6.3, that is, we need to show that

$$\partial_2\eta_1(c_1) = -\eta_0 \operatorname{targ}(c_1) + c_1 + \eta_0 \operatorname{src}(c_1).$$

There are three cases.

In the first case, suppose that $c_1 = \omega \otimes b_1$, for some $\omega \in \hat{\Omega}_0^{\mathsf{Crs}}X$ and some non-degenerate 1-simplex b_1 of X:

By Definition 6.9 (1) we have

$$\partial_2\eta_1(c_1) = 0_{(\varnothing\otimes *)}.$$

Also, from Definition 5.8, we know that c_1 has source $(\omega \otimes *)$, and target $(\omega s^{-1}b_1 \otimes *)$, and from Definition 6.6 (2) we have

$$\begin{aligned}
\eta_0(\operatorname{src}(c_1)) &= \eta_0(\omega \otimes *)\\
\eta_0(\operatorname{targ}(c_1)) &= (\omega \otimes b_1) + \eta_0(\omega \otimes *)\\
-\eta_0(\operatorname{targ}(c_1)) + c_1 + \eta_0(\operatorname{src}(c_1)) &= -\eta_0(\omega \otimes *) - (\omega \otimes b_1) + (\omega \otimes b_1) + \eta_0(\omega \otimes *)\\
&= 0_{\operatorname{src}(\eta_0(\omega\otimes *))}\\
&= 0_{(\varnothing\otimes *)}\\
&= \partial_2\eta_1(c_1).
\end{aligned}$$

In the second case, suppose $c_1 = (\omega s^{-1}a_2 \otimes *)$, for some non-degenerate 2-simplex a_2 of X:

From Definition 6.9 (2), we have $\eta_1(\omega s^{-1}a_2 \otimes *) = (\omega \otimes a_2)^{\eta_0(\omega\otimes *)}$

$$\partial_2(\eta_1(\omega s^{-1}a_2 \otimes *)) = \partial_2(\omega \otimes a_2)^{\eta_0(\omega\otimes *)} = -\eta_0(\omega \otimes *) + \partial(\omega \otimes a_2) + \eta_0(\omega \otimes *),$$

which by the Definition 5.8 equal to:

$$= -\eta_0(\omega \otimes *) - (\omega \otimes a_{02}) + (\omega s^{-1}a_2 \otimes *) + (\omega \otimes a_{01}) + (\omega s^{-1}a_{01} \otimes a_{12}) + \eta_0(\omega \otimes *)$$

While, by Definition 6.6 we have

$\eta_0(\mathrm{targ}(\omega s^{-1}a_2 \otimes *)) = \eta_0(\omega s^{-1}a_{02} \otimes *) = (\omega \otimes a_{02}) + \eta_0(\omega \otimes *)$,

and

$\eta_0(\mathrm{src}(\omega s^{-1}a_2 \otimes *)) = \eta_0(\omega s^{-1}a_{01}s^{-1}a_{12} \otimes *) = (\omega s^{-1}a_{01} \otimes a_{12}) + (\omega \otimes a_{01}) + \eta_0(\omega \otimes *)$

so we can see

$\partial_2(\eta_1(\omega s^{-1}a_2 \otimes *)) = -\eta_0(\mathrm{targ}(\omega s^{-1}a_2 \otimes *)) + (\omega s^{-1}a_2 \otimes *) + \eta_0(\mathrm{src}(\omega s^{-1}a_2 \otimes *))$ which satisfies the Proposition 6.3(2).

For the third case, suppose that $c_1 = (x \cdot s^{-1}a_1 \otimes *)$ for some generator x of $\hat{\Omega}^{\mathsf{Crs}}X$ in degree 1. Let us write $\mathfrak{p}$ for the source of x and $\mathfrak{q}$ for the target of x in $\hat{\Omega}^{\mathsf{Crs}}_0X$.

We assume, inductively, that condition (2) of Proposition 6.3 holds for the element $(x \otimes *)$,

$$\begin{aligned}
\partial_2\eta_1(x \otimes *) &= -\eta_0 \,\mathrm{targ}(x \otimes *) + (x \otimes *) + \eta_0 \,\mathrm{src}(x \otimes *) \\
&= -\eta_0(\mathfrak{q} \otimes *) + (x \otimes *) + \eta_0(\mathfrak{p} \otimes *).
\end{aligned}$$

From Definition 6.9, equation (33), we have

$$\begin{aligned}
\eta_1(c_1) &= \eta_1(x \otimes *) - (x \otimes a_1)^{\eta_0(\mathfrak{p}\otimes *)} \\
\partial_2(\eta_1(c_1)) &= \partial_2(\eta_1(x \otimes *)) - \partial_2((x \otimes a_1)^{\eta_0(\mathfrak{p}\otimes *)}) \\
&= -\eta_0(\mathfrak{q} \otimes *) + (x \otimes *) + \eta_0(\mathfrak{p} \otimes *) - (-\eta_0(\mathfrak{p} \otimes *) + \partial_2(x \otimes a_1) + \eta_0(\mathfrak{p} \otimes *))
\end{aligned}$$

by the inductive hypothesis and by Definition 2.25. Therefore

$$\partial_2(\eta_1(c_1)) = -\eta_0(\mathfrak{q} \otimes *) + (x \otimes *) - \partial_2(x \otimes a_1) + \eta_0(\mathfrak{p} \otimes *)$$

Now we need to use Definition of the boundary map, see Figure 20 in Definition 5.8:

$$\partial_2(x \otimes a_1) = -(\mathfrak{p} \otimes a_1) - c_1 + (\mathfrak{q} \otimes a_1) + (x \otimes *).$$

Therefore,

$$\begin{aligned}\partial_2(\eta_1(c_1)) &= -\eta_0(\mathfrak{q}\otimes *) - (\mathfrak{q}\otimes a_1) + (xs^{-1}a_1\otimes *) + (\mathfrak{p}\otimes a_1) + \eta_0(\mathfrak{p}\otimes *)\\ &= -\eta_0(\mathfrak{q}\cdot s^{-1}a_1\otimes *) + (xs^{-1}a_1\otimes *) + \eta_0(\mathfrak{p}\cdot s^{-1}a_1\otimes *)\end{aligned}$$

by Definition 6.6(2). But this says

$$\partial_2(\eta_1(c_1)) = -\eta_0\,\mathrm{targ}(c_1) + c_1 + \eta_0\,\mathrm{src}(c_1),$$

and we have finished the proof. □

Definition 6.13. *We define functions η_{n+m} on the generators $x\otimes b$ in degrees $n+m\geqslant 2$ of the free crossed complex $P^{\mathsf{Crs}}X$ as follows.*

1. *If b is given by a non-degenerate m-simplex where $m\geqslant 1$ then define*

$$\eta_{n+m}(x\otimes b) = 0_{(\varnothing\otimes *)} \tag{35}$$

2. *If $b=*$, the 0-simplex of X, then $x\neq\varnothing$ and we can suppose that it has the form*

$$x = x'\cdot s^{-1}a_{k+1}$$

   where a_{k+1} is a non-degenerate element of X_{k+1} and $|x'|+k=n=|x|$.

   If $k=0$ then define inductively

$$\eta_n(x\otimes *) = \eta_n(x'\otimes *) + (-1)^{n-k}(x'\otimes a_1)^{\eta_0(\beta(x'\otimes *))}. \tag{36}$$

   If $k\geqslant 1$ then define:

$$\eta_n(x\otimes *) = (-1)^{n-k}(x'\otimes a_{k+1})^{\eta_0(\beta(x'\otimes *))} \tag{37}$$

Remark 6.14. *We have not given the definition of* $\eta_n(x' \cdot (s^{-1}a_1)^{-1} \otimes *)$*. As in Remarks 6.7 and 6.10, the definition is implied by (and implies) the definition in equation* (36)*. If we let* $x' = x \cdot (s^{-1}a_1)^{-1}$ *so that* $x = x' \cdot s^{-1}a_1$ *then the definition in equation* (36) *says*

$$\eta_n(x \otimes *) = \eta_n(x \cdot (s^{-1}a_1)^{-1} \otimes *) + (-1)^n(x \cdot (s^{-1}a_1)^{-1} \otimes a_1).$$

Therefore by rearranging this equation we can give the definition of $\eta_n(x' \cdot (s^{-1}a_1)^{-1} \otimes *)$ *inductively as*

$$\eta_n(x \cdot (s^{-1}a_1)^{-1} \otimes *) = \eta_n(x \otimes *) - (-1)^n(x \cdot (s^{-1}a_1)^{-1} \otimes a_1). \tag{38}$$

Theorem 6.15. *The functions* η_{n+m} *which are given in Definition 6.13 define a contracting homotopy.*

Proof. Consider any element $c = x \otimes b$ where b is a non-degenerate m-simplex of X and x is a generator of degree n in $\hat{\Omega}^{\mathsf{Crs}}X$, as we have described in Definition 5.8.

We need to prove, for all $n + m \geqslant 2$, that $\eta(c)$ satisfies property (3) of Proposition 6.3,

$$\partial\eta(c) = c^{\eta_0(\beta(c))} - \eta\partial(c).$$

We will prove it by induction on the dimension of c.

Degree 2: To begin the induction, we will first consider an element c in degree 2. In this degree we must be careful because ∂c, $\eta\partial(c)$ and $\partial\eta(c)$ are non-abelian expressions.

There are three cases:

1. In the first case, suppose that $m > 0$. That is, b is not the basepoint of X.

2. In the second case, suppose $m = 0$. That is, $c = x \otimes *$. Suppose also that $x = x' \cdot s^{-1}a_1$, for some non-degenerate 1-simplex a_1 of X.

3. In the third case, suppose that $m = 0$, $c = x \otimes *$, where $x = x' \cdot s^{-1}a_{k+1}$ for some non-degenerate $(k+1)$-simplex a_{k+1} of X.

In the first case, Definition 6.13 says that $\eta(c)$ is trivial, so we need to prove that

$$0_{(\varnothing\otimes *)} = c^{\eta_0(\beta(c))} - \eta_1\partial_2(c)$$

where $c = x_0 \otimes b_2$ or $c = x_1 \otimes b_1$, so from definition 5.8, we have

$$\begin{aligned}
\partial_2 c &= -(x_0 \otimes b_{02}) + (x_0 s^{-1} b_2 \otimes *) + (x_0 s^{-1} b_{01} \otimes b_{12}) + (x_0 \otimes b_{01}) \\
\text{or } \partial_2 c &= -(\operatorname{src} x_1 \otimes b_1) - (x_1 s^{-1} b_1 \otimes *) + (\operatorname{targ} x_1 \otimes b_1) + (x_1 \otimes *)
\end{aligned}$$

Then, by Proposition 6.3(4), and Definition 6.9, we have

$$\begin{aligned}
\eta_1\partial_2 c &= -\eta_1(x_0 \otimes b_{02}) + \eta_1(x_0 s^{-1} b_2 \otimes *) + \eta_1(x_0 s^{-1} b_{01} \otimes b_{12}) + \eta_1(x_0 \otimes b_{01}) \\
&= 0_{(\varnothing\otimes *)} + (x_0 \otimes b_2)^{\eta_0(\operatorname{src}(x_0\otimes *))} + 0_{(\varnothing\otimes *)} + 0_{(\varnothing\otimes *)} \\
&= (x_0 \otimes b_2)^{\eta_0(\operatorname{src}(x_0\otimes *))} = c^{\eta_0(\beta(c))} \\
\text{or } \eta_1\partial_2 c &= -\eta_1(\operatorname{src} x_1 \otimes b_1) - \eta_1(x_1 s^{-1} b_1 \otimes *) + \eta_1(\operatorname{targ} x_1 \otimes b_1) + \eta_1(x_1 \otimes *) \\
&= 0_{(\varnothing\otimes *)} + (x_1 \otimes b_1)^{\eta_0(\operatorname{src}(x_1\otimes *))} - \eta_1(x_1 \otimes *) + 0_{(\varnothing\otimes *)} + \eta_1(x_1 \otimes *) \\
&= (x_1 \otimes b_1)^{\eta_0(\operatorname{src}(x_1\otimes *))} = c^{\eta_0(\beta(c))}
\end{aligned}$$

and so we always have $c^{\eta_0(\beta(c))} - \eta_1\partial_2(c)$, as we need.

In the second case, $m = 0$ and we can write

$$c = x \otimes * = x' \cdot s^{-1} a_1 \otimes *.$$

where a_1 is a non-degnerate element of X_1 and x' has degree 2.
Therefore by Equation (33)

$$\begin{aligned}
\eta_2 c &= \eta_2(x' \otimes *) + (x' \otimes a_1)^{\eta_0(\operatorname{src} x'\otimes *))} \\
\partial_3\eta_2 c &= \partial_3\eta_2(x' \otimes *) + \partial_3(x' \otimes a_1)^{\eta_0(\operatorname{src} x'\otimes *)}
\end{aligned}$$

In this case, we have two possibilities for x'. The first possibility is that $x' = x''s^{-1}a_3$, where x'', has degree zero. Then

$$\partial_2(x''s^{-1}a_3s^{-1}a_1 \otimes *) = -(x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{123} \cdot s^{-1}a_1 \otimes *)$$
$$- (x''s^{-1}a_{013}s^{-1}a_1 \otimes *) + (x'' \cdot s^{-1}a_{023} \cdot s^{-1}a_1 \otimes *)$$
$$+ (x'' \cdot s^{-1}a_{012} \cdot s^{-1}a_{23} \cdot s^{-1}a_1 \otimes *)$$

and so

$$\eta_1\partial_2(x''s^{-1}a_3s^{-1}a_1 \otimes *) =$$
$$+ (x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{123} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_3\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{01} \otimes a_{123})^{\eta_0(x''s^{-1}a_{01}\otimes *)} + (x'' \cdot s^{-1}a_{013} \otimes a_1)^{\eta_0\beta(x''\cdot s^{-1}a_{013}\otimes *)}$$
$$- (x'' \otimes a_{013})^{\eta_0(x''\otimes *)} + (x'' \otimes a_{023})^{\eta_0(x''\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{023} \otimes a_1)^{\eta_0\beta(x''\cdot s^{-1}a_{023}\otimes *)} + (x'' \otimes a_{012})^{\eta_0(x''\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{012} \otimes a_{23})^{(x''\cdot s^{-1}a_{012}\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{012} \cdot s^{-1}a_{23} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_3\otimes *)}$$

while

$$\eta_2(c) = \eta_2(x''s^{-1}a_3s^{-1}a_1 \otimes *) = (x'' \otimes a_3)^{\eta_0(x''\otimes *)} + (x''s^{-1}a_3 \otimes a_1)^{\eta_0\beta(x''s^{-1}a_3\otimes *)}$$
$$\partial_3\eta_2(c) = +(x''s^{-1}a_3s^{-1}a_1 \otimes *)^{\eta_0\beta(x''s^{-1}a_3\otimes *)} + (x'' \cdot s^{-1}a_{012} \cdot s^{-1}a_{23} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_3\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_{012} \otimes a_{23})^{\eta_0(x''\cdot s^{-1}a_{012}\otimes *)} - (x'' \otimes a_{012})^{\eta_0(x''\otimes *)} + (x''s^{-1}a_{023} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_{023}\otimes *)}$$
$$- (x'' \otimes a_{023})^{\eta_0(x''\otimes *)} + (x'' \otimes a_{013})^{\eta_0(x''\otimes *)} - (x''s^{-1}a_{013} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_{013}\otimes *)}$$
$$+ (x''s^{-1}a_{01} \otimes a_{123})^{\eta_0(x''s^{-1}a_{01}\otimes *)} - (x''s^{-1}a_{01}s^{-1}a_{123} \otimes a_1)^{\eta_0\beta(x''s^{-1}a_3\otimes *)}$$

The second possibility is that $x' = x'' \cdot s^{-1}a_2 \cdot s^{-1}a_2'$ and so

$$\partial_2(c) = \partial_2(x'' \cdot s^{-1}a_2 \cdot s^{-1}a_2' \cdot s^{-1}a_1 \otimes *) = -(x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_2' \cdot s^{-1}a_1 \otimes *)$$
$$- (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{02}' \cdot s^{-1}a_1 \otimes *) + (x'' \cdot s^{-1}a_{02} \cdot s^{-1}a_2' \cdot s^{-1}a_1 \otimes *)$$
$$+ (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{01}' \cdot s^{-1}a_{12}' \cdot s^{-1}a_1 \otimes *)$$
$$\eta_1\partial_2(c) = +(x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_2' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_2'\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes a_2')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{02}' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_{02}'\otimes *)} + (x'' \cdot s^{-1}a_2 \otimes a_{02}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$- (x'' \otimes a_2)^{\eta_0(x''\otimes *)} + (x'' \cdot s^{-1}a_{02} \otimes a_2')^{\eta_0(x''\cdot s^{-1}a_{02}\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{02} \cdot s^{-1}a_2' \otimes a_1)^{\eta_0(\mathrm{src}(x''\cdot s^{-1}a_{02}\cdot s^{-1}a_2'\otimes *))} + (x'' \otimes a_2)^{\eta_0(x''\otimes *)}$$
$$- (x'' \cdot s^{-1}a_2 \otimes a_{01}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)} - (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{01}' \otimes a_{12}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2 s^{-1}a_{01}'\otimes *)}$$
$$- (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{01}' \cdot s^{-1}a_{12}' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2 s^{-1}a_2'\otimes *))}$$

$$\eta_2(c) = \eta_2(x'' \cdot s^{-1}a_2 \cdot s^{-1}a_2' \cdot s^{-1}a_1 \otimes *) = -(x'' \cdot s^{-1}a_2 \otimes a_2')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_2' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_2'\otimes *)}$$
$$\partial_3(\eta_2(c)) = +(x'' \cdot s^{-1}a_2 \cdot s^{-1}a_2' \cdot s^{-1}a_1 \otimes *)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_2'\cdot s^{-1}a_1\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{01}' \cdot s^{-1}a_{12}' \otimes a_1)^{\eta_0\,\mathrm{src}((x''\cdot s^{-1}a_2\cdot s^{-1}a_2'\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{01}' \otimes a_{12}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_{01}'\otimes *)} + (x'' \cdot s^{-1}a_2 \otimes a_{01}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_{02} \cdot s^{-1}a_2' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_{02}\cdot s^{-1}a_2'\otimes *)+(x''\cdot s^{-1}a_2\otimes *)+\eta_0\,\mathrm{src}(x'' s^{-1}a_2\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{02} \otimes a_2')^{(x'' s^{-1}a_2\otimes *)+\eta_0\,\mathrm{src}((x''\cdot s^{-1}a_2\otimes *)} - (x'' \cdot s^{-1}a_2 \otimes a_{02}')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$- (x'' \cdot s^{-1}a_2 \cdot s^{-1}a_{02}' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_{02}'\otimes *)}$$
$$+ (x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes a_2')^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\otimes *)}$$
$$- (x'' \cdot s^{-1}a_{01} \cdot s^{-1}a_{12} \cdot s^{-1}a_2' \otimes a_1)^{\eta_0\,\mathrm{src}(x''\cdot s^{-1}a_2\cdot s^{-1}a_2'\otimes *)}$$

In the third case, $m = 0$ and we can write

$$c = x' s^{-1}a_{k+1} \otimes *$$

where a_{k+1} is non-degenerate element of X_{k+1} and $k = 1$ or 2 and x' has degree $2 - k$. If $k = 1$ then write $\mathfrak{p}$ and $\mathfrak{q}$ for the source and target of x' in $\hat{\Omega}_1^{\mathsf{Crs}} X$, so that

$$
\begin{aligned}
\partial_2 c =& \partial_2(x's^{-1}a_2 \otimes *) = -(\mathfrak{p}s^{-1}a_2 \otimes *) - (x's^{-1}a_{02} \otimes *) + (\mathfrak{q}s^{-1}a_2 \otimes *) + (x's^{-1}a_{01}s^{-1}a_{12} \otimes *) \\
\eta_1\partial_2 c =& -\eta_1(\mathfrak{p}s^{-1}a_2 \otimes *) - \eta_1(x's^{-1}a_{02} \otimes *) + \eta_1(\mathfrak{q}s^{-1}a_2 \otimes *) + \eta_1(x's^{-1}a_{01}s^{-1}a_{12} \otimes *) \\
&= -(\mathfrak{p} \otimes a_2)^{\eta_0(\mathfrak{p}\otimes *)} + (x' \otimes a_{02})^{\eta_0(\mathfrak{p}\otimes *)} - \eta_1(x' \otimes *) + (\mathfrak{q} \otimes a_2)^{\eta_0(\mathfrak{q}\otimes *)} + \eta_1(x' \otimes *) \\
&- (x' \otimes a_{01})^{\eta_0(\mathfrak{p}\otimes *)} - (x's^{-1}a_{01} \otimes a_{12})^{\eta_0(\mathfrak{p}s^{-1}a_{01}\otimes *)} \\
&= -(\mathfrak{p} \otimes a_2)^{\eta_0(\mathfrak{p}\otimes *)} + (x' \otimes a_{02})^{\eta_0(\mathfrak{p}\otimes *)} + (\mathfrak{q} \otimes a_2)^{\eta_0(\mathfrak{q}\otimes *)+\partial_2\eta_1(x'\otimes *)} \\
&- (x' \otimes a_{01})^{\eta_0(\mathfrak{p}\otimes *)} - (x's^{-1}a_{01} \otimes a_{12})^{\eta_0(\mathfrak{p}s^{-1}a_{01}\otimes *)} \\
&= -(\mathfrak{p} \otimes a_2)^{\eta_0(\mathfrak{p}\otimes *)} + (x' \otimes a_{02})^{\eta_0(\mathfrak{p}\otimes *)} + (\mathfrak{q} \otimes a_2)^{(x'\otimes *)+\eta_0(\mathfrak{p}\otimes *))} \\
&- (x' \otimes a_{01})^{\eta_0(\mathfrak{p}\otimes *)} - (x's^{-1}a_{01} \otimes a_{12})^{(x'\otimes a_{01})+\eta_0(\mathfrak{p}\otimes *)} \\
\eta_2 c =& \eta_2(x's^{-1}a_2 \otimes *) = -(x' \otimes a_2)^{\eta_0(\mathfrak{p}\otimes *))} \\
\partial_3\eta_2 c =& \Big((x's^{-1}a_2 \otimes *)^{(\mathfrak{p}s^{-1}a_{01}\otimes a_{12})+(\mathfrak{p}\otimes a_{01})} \\
&+ (x's^{-1}a_{01} \otimes a_{12})^{(\mathfrak{p}\otimes a_{01})} + (x' \otimes a_{01}) - (\mathfrak{q} \otimes a_2)^{(x'\otimes *)} - (x' \otimes a_{02}) + (\mathfrak{p} \otimes a_2) \Big)^{\eta_0(\mathfrak{p}\otimes *))} \\
&= c^{\eta_0\beta c} - \eta_1\partial_2 c
\end{aligned}
$$

If $k = 2$ then

$$
\begin{aligned}
\partial_2 c =& \partial_2(\omega s^{-1}a_3 \otimes *) \\
&= -(\omega s^{-1}a_{01}s^{-1}a_{123} \otimes *) - (\omega s^{-1}a_{013} \otimes *) + (\omega s^{-1}a_{023} \otimes *) + (\omega s^{-1}a_{012}s^{-1}a_{23} \otimes *) \\
\eta_1\partial_2 c =& \Big(-(\omega s^{-1}a_{01} \otimes a_{123})^{(\omega\otimes a_{01})} - (\omega \otimes a_{013}) + (\omega \otimes a_{023}) \\
&+ (\omega \otimes a_{012}) - (\omega s^{-1}a_{012} \otimes a_{23})^{(\omega s^{-1}a_{01}\otimes a_{12})+(\omega\otimes a_{01})} \Big)^{\eta_0(\omega\otimes *)} \\
\eta_2 c =& \eta_2(\omega s^{-1}a_3 \otimes *) = (\omega \otimes a_3)^{\eta_0(\omega\otimes *)} \\
\partial_3\eta_2 c =& \Big(+(\omega s^{-1}a_3 \otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}\otimes a_{23})+(s^{-1}a_{01}\otimes a_{12})+(\omega\otimes a_{01})}
\end{aligned}
$$

$$+ (\omega s^{-1}a_{012} \otimes a_{23})^{(s^{-1}a_{01}\otimes a_{12})+(\omega\otimes a_{01})} - (\omega \otimes a_{012})$$
$$\left. - (\omega \otimes a_{023}) + (\omega \otimes a_{013}) + (\omega s^{-1}a_{01} \otimes a_{123})^{(\omega\otimes a_{01})} \right)^{\eta_0(\omega\otimes *)}$$

We can see that $\partial_3 c_2 = c_2^{\eta_0 \beta c} - \eta_1 \partial_2 c_2$, which satisfies property (3) of Proposition 6.3, hence $\eta(c_2)$ define a contracting homotopy.

Degree $n + m \geqslant 3$: We now assume by induction that Property (3) of Proposition 6.3 holds for any element c of degree $< n + m$. We will now prove it for elements of degree $n + m$. Everything is abelian now.

As before, there are three cases:

1. In the first case, suppose that $m > 0$. That is, b is not the basepoint of X.

2. In the second case, suppose $m = 0$. That is, $c = x \otimes *$. Suppose also that $x = x' \cdot s^{-1}a_1$, for some non-degenerate 1-simplex a_1 of X.

3. In the third case, suppose that $m = 0$, $c = x \otimes *$, where $x = x' \cdot s^{-1}a_{k+1}$ for some non-degenerate $(k + 1)$-simplex a_{k+1} of X.

In the first case, $c = x_n \otimes b_m$, where b_m is a non-degenerate simplex of dimension $m \geqslant 1$ in X. Equation (35) in Definition 6.13 says that $\eta(c)$ is trivial, so we need to prove that

$$0_{(\varnothing\otimes *)} = c^{\eta_0(\beta(c))} - \eta_{n+m-1}\partial_{n+m}(c)$$

Suppose $m = 1$, so $c = x \otimes b_1$. Then the terms in the expression for

$$\partial_{n+1}(x \otimes b_1)$$

have the following form

$$(-1)^{n+1}(x \otimes *) \tag{39}$$

$$(-1)^n(x \cdot s^{-1}b_1 \otimes *)^\gamma \tag{40}$$

$$(y \otimes b_1) \tag{41}$$

where y is any term in the formula for $\partial^{\hat{\Omega}}(x)$. Because of Proposition 6.3(4, 5), we can ignore the action in the terms (40), and the terms in the expression for

$$\eta_n \partial_{n+1}(x \otimes b_1)$$

will be

$$(-1)^{n+1}\eta(x \otimes *) \tag{42}$$

$$(-1)^n \eta(x \cdot s^{-1}b_1 \otimes *) \tag{43}$$

$$\eta(y \otimes b_1) \tag{44}$$

But by our definition, the term (44) is trivial. Also we can expand (43) into two terms, by the inductive definition of η, and one of these terms cancels with (42). That is:

$$\begin{aligned}
\eta_n \partial_{n+1}(x \otimes b_1) &= (-1)^{|x|+1}\eta(x \otimes *) + (-1)^{|x|}\eta(x \cdot s^{-1}b_{01} \otimes *) \\
&= (-1)^{n+1}\eta(x \otimes *) + (-1)^n \eta(x \otimes *) + (x \otimes b_1)^{\eta_0(\beta(x \otimes *))} \\
&= (x \otimes b_1)^{\eta_0(\beta(x \otimes *))}
\end{aligned}$$

Hence, $\eta\partial(c) = c^{\eta_0 \beta c}$ as required.

Now suppose $c = x \otimes b_m$, where $m \geqslant 2$. We will show that

$$\eta_{n+m-1}\partial_{n+m} c = c^{\eta_0 \beta c}.$$

The terms of $\partial_{n+m}(x \otimes b_m)$ have one of the following forms

$$(-1)^{n+i} x \otimes d_i b_m \tag{45}$$

$$(-1)^n x \cdot s^{-1} b_{0 \dots i} \otimes b_{i \dots m} \tag{46}$$

$$y \otimes b_m \tag{47}$$

where y is any term in the formula for $\partial^{\hat{\Omega}}(x)$. The terms (45), (46), (47) might also have actions, but because of Proposition 6.3(4, 5) we can ignore them, and the terms of

$$\eta\partial_{n+m}(x \otimes b_m)$$

will be

$$(-1)^{n+i}\eta(x \otimes d_i b_m) \tag{48}$$

$$(-1)^n \eta(x \cdot s^{-1} b_{0\ldots i} \otimes b_{i\ldots m}) \tag{49}$$

$$\eta(y \otimes b_m) \tag{50}$$

But by our Definition 6.13, equation (35) all of these are trivial, except

$$(-1)^n \eta(x \cdot s^{-1} b_m \otimes *) \tag{51}$$

and by equation (36) we therefore have

$$\eta(\partial_{n+m} c) = (-1)^n \eta(x \cdot s^{-1} b_m \otimes *) = (-1)^{2n}(x \otimes b_m)^{\eta_0\beta(x\otimes *)} = (x \otimes b_m)^{\eta_0\beta(x\otimes *)} \tag{52}$$

$$= c^{\eta_0(\mathfrak{p})} \tag{53}$$

so that Proposition 6.3(3) holds.

In the second and third cases, we have

$$c = x \otimes * = x' \cdot s^{-1} a_{k+1} \otimes *$$

and we want to prove that

$$\partial_{n+1}\eta_n(c) = c^{\eta_0\beta(c)} - \eta_{n-1}\partial_n(c). \tag{54}$$

We will prove this by induction on the length of the word x.

In the second case, we have $k = 0$, and from Equation (36) we know that

$$\begin{aligned}\partial_{n+1}\eta_n(c) &= \partial_{n+1}\left(\eta_n(x' \otimes *) + (-1)^n (x' \otimes a_1)^{\eta_0\beta(x'\otimes *)}\right)\\ &= \partial_{n+1}\eta_n(x' \otimes *) + (-1)^n \partial_{n+1}(x' \otimes a_1)^{\eta_0\beta(x'\otimes *)}\\ &= (x' \otimes *)^{\eta_0\beta(x'\otimes *)} - \eta_{n-1}\partial_n(x' \otimes *) + (-1)^n \partial_{n+1}(x' \otimes a_1)^{\eta_0\beta(x'\otimes *)} \qquad \text{(A)}\end{aligned}$$

Here we have assumed (54) holds inductively, for $c = x' \otimes *$, since x' is a shorter word than x. We also know that

$$\begin{aligned}\eta_{n-1}\partial_n(c) &= \eta_{n-1}\partial_n(x' \cdot s^{-1}a_1 \otimes *)\\ &= \sum \eta_{n-1}(y \cdot s^{-1}a_1 \otimes *)\end{aligned}$$

where we take the sum over all terms y in the expression for $\partial_n^{\hat{\Omega}}(x')$, and we can ignore any actions. Therefore from Equation (36) we know that

$$\begin{aligned}\eta_{n-1}\partial_n(c) &= \sum \eta_{n-1}(y \otimes *) + \sum (-1)^{n-1}(y \otimes a_1)^{\eta_0\beta(y\otimes *)}\\ &= \eta_{n-1}\partial_n(x' \otimes *) + \sum (-1)^{n-1}(y \otimes a_1)^{\eta_0\beta(y\otimes *)} \qquad \text{(B)}\end{aligned}$$

If we combine (A) and (B) then we have

$$\begin{aligned}\partial_{n+1}\eta_n(c) + \eta_{n-1}\partial_n(c) &= (-1)^n \left(\partial_{n+1}(x' \otimes a_1) + (-1)^n (x' \otimes *) - \sum (y \otimes a_1)\right)^{\eta_0\beta(x'\otimes *)}\\ &= \left((x' \cdot s^{-1}a_1 \otimes *)^{(\beta x')\otimes a_1}\right)^{\eta_0\beta(x'\otimes *)}\\ &= (x \otimes *)^{\eta_0\beta(x\otimes *)} = c^{\eta_0\beta c}\end{aligned}$$

Finally, in the third case, $c = x' \cdot s^{-1}a_{k+1} \otimes *$ where $k \geqslant 1$. From our definition of the boundary of the cobar construction we can see that the terms in the expression for $\partial_n(x' \cdot s^{-1}a_{k+1} \otimes *)$ have one of the following forms

$$(y \cdot s^{-1}a_{k+1} \otimes *) \qquad (55)$$

$$(-1)^{|x|+i+1}(x' \cdot s^{-1}d_i a_{k+1} \otimes *) \qquad (56)$$

$$(-1)^{|x|+i+2}(x' \cdot s^{-1}a_{0\dots i} \cdot s^{-1}a_{i\dots k+1} \otimes *) \tag{57}$$

Here y denotes terms in the expression for $\partial^{\hat{\Omega}}(x')$, and $1 \leq i \leq k$. We do not write down the actions because they will disappear when we apply η. If $k \geqslant 2$ then by Equations (36) and (37)

$$\eta_{n-1}(y \cdot s^{-1}a_{k+1} \otimes *) = (-1)^{|y|}(y \otimes a_{k+1})^{\eta_0\beta(y\otimes *)} \tag{58}$$

$$\eta_{n-1}(x' \cdot s^{-1}d_i a_{k+1} \otimes *) = (-1)^{|x'|}(x' \otimes d_i a_{k+1})^{\eta_0\beta(x'\otimes *)} \tag{59}$$

$$\eta_{n-1}(x's^{-1}a_{0\dots i}s^{-1}a_{i\dots k+1} \otimes *) = (-1)^{|x'|+i-1}(x's^{-1}a_{0\dots i} \otimes a_{i\dots k+1})^{\eta_0\beta(x's^{-1}a_{0\dots i}\otimes *)} \tag{60}$$

$$\begin{aligned}\eta_{n-1}(x's^{-1}a_{0\dots k}s^{-1}a_{k\,k+1} \otimes *) &= \eta_{n-1}(x's^{-1}a_{0\dots k} \otimes *)\\ &\quad + (-1)^{n-1}(x's^{-1}a_{0\dots k} \otimes a_{k\,k+1})^{\eta_0\beta(x's^{-1}a_{0\dots k}\otimes *)}\\ &= (-1)^{|x'|}(x' \otimes a_{0\dots k})^{\eta_0\beta(x'\otimes *)}\\ &\quad + (-1)^{n-1}(x's^{-1}a_{0\dots k} \otimes a_{k\,k+1})^{\eta_0\beta(x's^{-1}a_{0\dots k}\otimes *)}\end{aligned} \tag{61}$$

where (60) is only for $1 \leq i < k$. If $k = 1$ then

$$\eta_{n-1}(y \cdot s^{-1}a_2 \otimes *) = (-1)^{n-2}(y \otimes a_2)^{\eta_0\beta(y\otimes *)} \tag{62}$$

$$\eta_{n-1}(x' \cdot s^{-1}d_1 a_2 \otimes *) = \eta_{n-1}(x' \otimes *) + (-1)^{n-1}(x' \otimes d_1 a_2)^{\eta_0\beta(x'\otimes *)} \tag{63}$$

$$\eta_{n-1}(x's^{-1}a_{01}s^{-1}a_{12} \otimes *) = \eta_{n-1}(x' \otimes *) + (-1)^{n-1}(x' \otimes a_{01})^{\eta_0\beta(x'\otimes *)} \tag{64}$$

$$+ (-1)^{n-1}(x's^{-1}a_{01} \otimes a_{12})^{\eta_0\beta(x's^{-1}a_{01}\otimes *)} \tag{65}$$

In the end we can see that the terms in the expression for $\eta_{n-1}\partial_n(x' \cdot s^{-1}a_{k+1} \otimes *)$ are exactly the same as the terms in the expression for

$$c^{\eta_0\beta(c)} - \partial_{n+1}\eta_n(c) = (x's^{-1}a_{k+1} \otimes *)^{\eta_0\beta(x's^{-1}a_{k+1}\otimes *)} - (-1)^{|x'|}\partial_{n+1}(x' \otimes a_{k+1})^{\eta_0\beta(x'\otimes *)}$$

□

References

[1] M. Arkowitz, *Introduction to homotopy theory*, Springer Science and Business Media, 2011.

[2] J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U. S. A. **42** (1956), 409- 412.

[3] J. F. Adams, P. J. Hilton, *On the chain algebra of a loop space*, Commentarii Mathematici Helvetici Journal 30.1 (1956), 305-330.

[4] E. H. Brown, *Twisted tensor products, I.*, Annals of Mathematics Vol.69, No.1, (1959), 223-246.

[5] A. L. Blakers, *Some relations between homology and homotopy groups*, Annals of Mathematics (1948), 428-461.

[6] R. Brown *From groups to groupoids*, Bull. London Math. Soc. 19(1987), 113-134.

[7] R. Brown, *Groupoid and crossed objects in algebraic topology*, Homology, Homotopy and Applications, Vol.1, No. 1, (1999), 1-78.

[8] R. Brown *Topology and Groupoids*, BookSurge LLC,North Carolina(2006), 113-134.

[9] R. Brown, B. Christopher Spencer, *Double groupoids and crossed modules*, Cahiers De Topologie Et Geometrie Differentielle, 17(1976), 343-362.

[10] R. Brown, P. J. Higgins, *On the algebra of cubes*, Journal of Pure and Applied Algebra 21.3 (1981), 233-260.

[11] R. Brown, P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes*, J. Pure and Appl. Algebra 47 (1987), 1-33.

[12] R. Brown, and P. J. Higgins, *The equivalence of ∞-groupoids and crossed complexes*, Cahiers De Topologie Et Geometrie Differentielle, 22.4 (1981): 371-386.

[13] R. Brown, M. Golasinski, *A model structure for the homotopy theory of crossed complexes*, Cahiers de topologie et géométrie différentielle catégoriques 30.1 (1989), 61-82.

[14] H. Baues, A. Tonks, *On the twisted cobar construction*, Max-Planck- Institut fur Mathematik, Gottfried-Claren-Strabe 26, 53225 Bonn, Germany, (1997), 121-229.

[15] E. B. Curtis, *Simplicial homotopy theory*, Advances in Mathematics 6.2 (1971), 107-209.

[16] S. Eilenberg, J. A. Zilber, *On products of complexes*, American Journal of Mathematics 75.1 (1953), 200-204.

[17] S. Eilenberg, S. Mac Lane, *On the groups H(*π,n)-II, Methods of computation, Ann. Math.60 (1954), 49-139.

[18] P. G. Goerss, J. F. Jardine, *Simplicial homotopy theory*, Springer Science and Business Media, 1999.

[19] K. Hess, A. Tonks, *The loop group and the cobar construction*, American mathematical society, V.138. No.5, May (2010), 1861-1876.

[20] K. Hess, A. Tonks, *Crossed complexes and twisted products*, private communication

[21] A. Hatcher *Algebraic Topology*, Cambridge University press, (2002).

[22] İ. İçen, *The equivalence of 2-groupoids and crossed modules*, Ccmmun. Fac. Sci. Univ. Ank. Series A1.Vol. 49. pp39-48(2000).

[23] J. Rubio, F. Sergeraert *Constructive homological algebra and applications*, arXiv preprint arXiv: (2012), 1208.3816.

[24] A. Joyal, Myles Tierney, *Notes on simplicial homotopy theory*, Preprint (2008).

[25] P. J. Hilton, S. Wylie *Homology theory, An introduction to algebraic topology*, Cambridge University press, CUP, 1960.

[26] T. Kadeishvili, S. Saneblidze, *A cubical model for a fibration*, Journal of Pure and Applied Algebra 196 (2005), 203-228.

[27] J. Faria Martins, *On the homotopy type and the fundamental crossed complex of the skeletal filtration of a CW-complex*, Homology, Homotopy and Applications, 9.1 (2007), 295-329.

[28] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed modules*, Homology, Homotopy and Applications, 9(2007), no.1, 75-106.

[29] F. Sergeraert, *Introduction to combinatorial homotopy theory*, Lecture Notes (2008).

[30] R. H. Szczarba, *The homology of twisted cartesian products*, Trans. Amer. Math.Soc.100 (1961), 197-216.

[31] A. P. Tonks, *On the Eilenberg–Zilber theorem for crossed complexes*, Journal of Pure and Applied Algebra 179.1 (2003), 199-220.

[32] A. P. Tonks, *Theory and applications of crossed complexes*, Ph.D.thesis, University of Wales(1993).

[33] C. A. Weibel *An Introduction to Homological Algebra*,Cambridge studies in Advanced Mathematics 38, Cambridge University press: (1994).

A Some data for the proof of the conjecture

We would like to try and prove Conjecture 5.11 by comparing the terms in the formulas for

$$\partial_3^{\otimes}\partial_4^{\otimes}(x \otimes b) \tag{66}$$

with the terms in the formulas for

$$\partial_3^{P}\partial_4^{P}(x \otimes b) \tag{67}$$

We know that the terms in (66) all cancel. We also know that the terms in (67) are quite similar to the terms in (66). We hope that this will give us enough insight to prove that the terms in (67) also all cancel. Unfortunately there are 48 terms (each 4-dimensional cube has 8 faces, and each of these cubes has 6 square faces) and we have not been able to prove they cancel yet.

So we collect below some of the data we have found so far. We think that we might need a good computer to check all of the possibilities and prove the conjecture.

The first two formulas are abelian,

$$\begin{aligned}
\partial_4^{\otimes}(s^{-1}a_2 \otimes b_3) &= (s^{-1}a_{02} \otimes b_3)^{(s^{-1}a_2 \otimes *)} - (s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes b_3) \\
&\quad + (s^{-1}a_2 \otimes b_{012}) - (s^{-1}a_2 \otimes b_{013}) + (s^{-1}a_2 \otimes b_{023}) - (s^{-1}a_2 \otimes b_{123})^{(s^{-1}a_{01}s^{-1}a_{12} \otimes b_{01})} \\
&= A^{\gamma} - B + C - D + E - F^{\Upsilon_1}
\end{aligned}$$

$$\begin{aligned}
\partial_4^{P}(s^{-1}a_2 \otimes b_3) &= A^{\gamma} - B + C - D + E - \widehat{F}^{\Upsilon_1} - \widehat{G}^{\Upsilon_2+\Upsilon_1} - \widehat{H}^{\Upsilon_3+\Upsilon_2+\Upsilon_1} \\
\widehat{F} &= (s^{-1}a_2 \cdot s^{-1}b_{01} \otimes b_{123}) \\
\widehat{G} &= (s^{-1}a_2 \cdot s^{-1}b_{012} \otimes b_{23})
\end{aligned}$$

$$\widehat{H} = (s^{-1}a_2 \cdot s^{-1}b_3 \otimes *)$$

The following formulas are not abelian but they are central. Their terms can be permuted cyclically, for example.

$$\begin{aligned}
\partial_3^{\otimes}(A^{\gamma}) &= -(s^{-1}a_{02} \otimes b_{023}) + (s^{-1}a_{02} \otimes b_{013}) + (s^{-1}a_{02} \otimes b_{123})^{(s^{-1}a_{02}\otimes b_{01})} - (s^{-1}a_{02} \otimes b_{012}) \\
&= -A_1 + A_2 + A_3^{\Upsilon_1(A)} - A_4 \\
\partial_3^{P}(A^{\gamma}) &= -A_1 + A_2 + \widehat{A}_3^{\Upsilon_1(A)} + \widehat{A}_5^{\Upsilon_3(A)+\Upsilon_2(A)+\Upsilon_1(A)} + \widehat{A}_6^{\Upsilon_2(A)+\Upsilon_1(A)} - A_4 \\
\widehat{A}_3 &= (s^{-1}a_{02}s^{-1}b_{01} \otimes b_{123}) \\
\widehat{A}_5 &= (s^{-1}a_{02}s^{-1}b_{3} \otimes *) \\
\widehat{A}_6 &= (s^{-1}a_{02}s^{-1}b_{012} \otimes b_{23})
\end{aligned}$$

$$\begin{aligned}
\partial_3^{\otimes}(B) &= +(s^{-1}a_{01}s^{-1}a_{12} \otimes b_{012}) - (s^{-1}a_{01}s^{-1}a_{12} \otimes b_{123})^{(s^{-1}a_{01}s^{-1}a_{12}\otimes b_{01})} \\
&\quad - (s^{-1}a_{01}s^{-1}a_{12} \otimes b_{013}) + (s^{-1}a_{01}s^{-1}a_{12} \otimes b_{023}) \\
&= B_1 - B_2^{\Upsilon_1(B)} - B_3 + B_4 \\
\partial_3^{P}(B) &= B_1 - \widehat{B}_2^{\Upsilon_1(B)=\Upsilon_1} - B_3 + B_4 + \widehat{B}_5^{\Upsilon_3(B)+\Upsilon_2(B)+\Upsilon_1(B)} + \widehat{B}_6^{\Upsilon_2(B)+\Upsilon_1(B)} \\
\widehat{B}_2 &= (s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01} \otimes b_{123}) \\
\widehat{B}_5 &= (s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{3} \otimes *) \\
\widehat{B}_6 &= (s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{012} \otimes b_{23})
\end{aligned}$$

$$\begin{aligned}
\partial_3^{\otimes}(C) &= -(s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes b_{012}) + (s^{-1}a_{2} \otimes b_{02}) \\
&\quad + (s^{-1}a_{02} \otimes b_{012})^{(s^{-1}a_{2}\otimes *)} - (s^{-1}a_{2} \otimes b_{01}) - (s^{-1}a_{2} \otimes b_{12})^{\Upsilon_1(C)=\Upsilon_1} \\
&= -C_1 + C_2 + C_3^{\gamma(C)} - C_4 - C_5^{\Upsilon_1(C)}
\end{aligned}$$

$$\partial_3^P(C) = -C_1 + C_2 + C_3^{\gamma(C)} - C_4 - \widehat{C}_5^{\Upsilon_1(C)} - \widehat{C}_6^{\Upsilon_2(C)+\Upsilon_1(C)}$$

$$\widehat{C}_5 = (s^{-1}a_2 \cdot s^{-1}b_{01} \otimes b_{12})$$

$$\widehat{C}_6 = (s^{-1}a_2 \cdot s^{-1}b_{012} \otimes *)$$

$$\partial_3^{\otimes}(D) = -(s^{-1}a_{02} \otimes b_{013})^{(s^{-1}a_2 \otimes *)} - (s^{-1}a_2 \otimes b_{03}) + (s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes b_{013})$$

$$+ (s^{-1}a_2 \otimes b_{13})^{(s^{-1}a_{01}s^{-1}a_{12} \otimes b_{01})} + (s^{-1}a_2 \otimes b_{01})$$

$$= -D_1^{\gamma(D)} - D_2 + D_3 + D_4^{\Upsilon_1(D)} + D_5$$

$$\partial_3^P(D) = -D_1^{\gamma(D)} - D_2 + D_3 + \widehat{D}_4^{\Upsilon_1(D)} + D_5 + \widehat{D}_6^{\Upsilon_2(D)+\Upsilon_1(D)}$$

$$\widehat{D}_4 = (s^{-1}a_2 s^{-1}b_{01} \otimes b_{13})$$

$$\widehat{D}_6 = (s^{-1}a_2 s^{-1}b_{013} \otimes *)$$

$$\partial_3^{\otimes}(E) = -(s^{-1}a_2 \otimes b_{02}) - (s^{-1}a_2 \otimes b_{23})^{(s^{-1}a_{01}s^{-1}a_{12} \otimes b_{02})} - (s^{-1}a_{01} \cdot s^{-1}a_{12} \otimes b_{023})$$

$$+ (s^{-1}a_2 \otimes b_{03}) + (s^{-1}a_{02} \otimes b_{023})^{(s^{-1}a_2 \otimes *)}$$

$$\partial_3^{\otimes}(E) = -E_1 - E_2^{\Upsilon_1(E)} - E_3 + E_4 + E_5^{\gamma(E)}$$

$$\partial_3^P(E) = -E_1 - \widehat{E}_2^{\Upsilon_1(E)} - E_3 + E_4 + E_5^{\gamma(E)} - \widehat{E}_6^{\Upsilon_2(E)+\Upsilon_1(E)}$$

$$\widehat{E}_2 = (s^{-1}a_2 s^{-1}b_{02} \otimes b_{23})$$

$$\widehat{E}_6 = (s^{-1}a_2 \cdot s^{-1}b_{023} \otimes *)$$

$$\partial_3^P(\widehat{F})^{\Upsilon_1} = (s^{-1}a_2 s^{-1}b_{01} \otimes b_{12}) - (s^{-1}a_{02}s^{-1}b_{01} \otimes b_{123})^{(s^{-1}a_{012} \otimes *)}$$

$$- (s^{-1}a_2 s^{-1}b_{01} \otimes b_{13}) + (s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01} \otimes b_{123})$$

$$+ (s^{-1}a_2 s^{-1}b_{01}s^{-1}b_{123} \otimes *)^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01}s^{-1}b_{12} \otimes b_{23}) + (s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01} \otimes b_{12})}$$

$$+ (s^{-1}a_2 s^{-1}b_{01} \otimes b_{23})^{(s^{-1}a_{01}s^{-1}a_{12}s^{-1}b_{01} \otimes b_{12})}$$

$$\partial_3^P(\widehat{F}) = F_1 - F_2^{\gamma(F)} - F_3 + F_4 + F_5^{\Upsilon_2(F)+\Upsilon_1(F)} + F_6^{\Upsilon_1(F)}$$

$$\begin{aligned}
\partial_3^P(\widehat{G})^{\Upsilon_2+\Upsilon_1} &= +(s^{-1}a_2 s^{-1}b_{012} \otimes *) - (s^{-1}a_{02} s^{-1}b_{012} \otimes b_{23})^{(s^{-1}a_2 s^{-1}b_{01} s^{-1}b_{12} \otimes *)} \\
&- (s^{-1}a_2 s^{-1}b_{01} s^{-1}b_{12} \otimes b_{23}) - (s^{-1}a_2 s^{-1}b_{012} s^{-1}b_{23} \otimes *)^{\Upsilon_3} \\
&+ (s^{-1}a_{01} s^{-1}a_{12} s^{-1}b_{012} \otimes b_{23}) + (s^{-1}a_2 s^{-1}b_{02} \otimes b_{23})^{(s^{-1}a_{01} s^{-1}a_{12} s^{-1}b_{012} \otimes *)}
\end{aligned}$$

$$\begin{aligned}
\partial_3^P(\widehat{H})^{\Upsilon_2+\Upsilon_2+\Upsilon_1} &= +(s^{-1}a_2 s^{-1}b_{023} \otimes *)^{-(s^{-1}a_{01} s^{-1}a_{12} s^{-1}b_{012} s^{-1}b_{23} \otimes *)} \\
&+ (s^{-1}a_2 s^{-1}b_{012} s^{-1}b_{23} \otimes *) - (s^{-1}a_{02} s^{-1}b_3 \otimes *)^{(s^{-1}a_{012} s^{-1}b_{01} s^{-1}b_{12} s^{-1}b_{23} \otimes *)} \\
&+ (s^{-1}a_{01} s^{-1}a_{12} s^{-1}b_3 \otimes *)
\end{aligned}$$