THE TRANSPORTATION OF TRIFUNCTORS IN THE TRICATEGORY OF BICATEGORIES

Thesis submitted for the degree of
Doctor of Philosophy

by

Andrew Smith
Department of Mathematics
University of Leicester

2019

Abstract

THE TRANSPORTATION OF TRIFUNCTORS IN THE TRICATEGORY OF BICATEGORIES

Andrew Smith

Tricategories, as the construction for the most general sort of weak 3-category being given by explicit coherence axioms, are a particularly important structure in the study of low-dimensional higher category theory. As such the correct notion of a morphism between tricategories, the Trifunctor, is also an important object of interest. Just as many constructions in mathematics can be realised by using functors between appropriate categories, these constructions can be generalised to the 3-dimensional level by using trifunctors between the appropriate tricategories. Of particular interest are trifunctors into the tricategory of bicategories.

Given a mathematical structure laid on top of a base object, it can be useful to transport that structure from the original object to a new object across a suitable sort of equivalence. The collection of trifunctors between two tricategories forms a tricategory of its own. So does the collection of functions from the objects of the source tricategory to the objects of the target tricategory, which form the object level of any trifunctor. Therefore in this case the appropriate notion of equivalence is that of biequivalence, and we would hope to be able to transport the structure of a trifunctor across a collection of biequivalences at the object level.

While the transport of structure at lower dimensions is achieved using monadic methods, at the general 3-dimensional level these haven't been developed. This thesis aims to provide a method for transporting the structure of a trifunctor into the tricategory of bicategories across object-indexed biequivalences. We do this by working directly from the definition of trifunctor: by constructing the data needed for the new trifunctor from the data of the original trifunctor and the biequivalences, and then proving that the axioms hold using diagram manipulation techniques.

Acknowledgements

I'll begin by offering my utmost gratitude to my supervisor Simona Paoli. She has been an insightful, steadfast and kind mentor throughout my years at Leicester, and without her support I could not have achieved as much as I have. Nor could I have made contact with so much of the academic community without her unfailing eye for new opportunities for me.
I would also like to thank the other members of staff of the Department of Mathematics, both academic and administrative, for making the Department the welcoming institution it is.
I thank the University of Leicester for the opportunity to hold a Graduate Teaching Assistant position, which financed my PhD studies and complemented my research with valuable teaching experience.

The companionship of my fellow PhD students has been a great comfort. Special mentions go to Scott Balchin and Dan Rust who, as the very first of Leicester's graduate students I met, gave me crucial support as I entered the Department and introduced me to academic life.
Liam Collard, Ollie Sutton and Drew Duffield were, together, the driving force behind the majority the social events that rounded out my time in the Mathematics Department: via the Criterion Quiz team and more besides. For this I am very grateful to them.
Finally, I should acknowledge my fellow graduate teaching assistant Kieran Fleming. We weathered our time here together stoically.

Last but very much not least, I would like to thank my parents for their unwavering belief and support.

Contents

Introduction

One useful tool in mathematics is the ability to take a given mathematical structure on a set and transport that structure across a bijection, with that bijection then becoming an isomorphism for the added structure as well. For instance, given a group G and a bijection $f\colon G \to S$, the set S becomes a group isomorphic to G using the operation $s * t = f(f^{-1}(s) \times f^{-1}(t))$. The bijection allows us to transport the group structure of G over to the set S.

The aim of this thesis is to take one particular type of structure - the structure of a trifunctor from any tricategory $\mathcal{T}$ to the tricategory of bicategories $\underline{Bicat}$ - and transport it across biequivalences in the tricategory of bicategories. Tricategories were first introduced by Gordon, Power and Street [GPS95] and they are the natural construction for a weak 3-dimensional categorical structure. As an example, the collection of all bicategories forms a tricategory in a natural way. As such, both tricategories and the morphisms between them - trifunctors - are important objects of study in the field of low-dimensional higher category theory.

There is considerable recent literature that makes use of tricategories (see Section 1.3). Although the results of this thesis do not immediately relate to these applications, the uses of tricategories demonstrate the need to continue developing the foundations of tricategorical theory. The technical complexity of working with the coherence cells and axioms of tricategories has slowed the research in this area. We believe the results of this thesis, obtained by working directly with those cells, are therefore a useful contribution to the literature. Just as many mathematical objects can be realised as functors, many objects of interest in 3-dimensional category theory can be realised as trifunctors. If we can transport the structure of a trifunctor then we can also transport these structures too. Moreover, the techniques developed to transport a trifunctor lead us to prove several properties concerning particular diagram manipulations (see Chapter 4) which are of independent interest to others engaged in research on tricategories.

We start by justifying the method used in this thesis to transport the trifunctor. The way this will be accomplished is by constructing the pasting diagrams given by the axioms of a trifunctor and then manipulating them step-by-step in order to show that the start and end diagrams (corresponding to each side of the axiom) are equal. At each step we need to ensure that the manipulation we make results in an equal diagram. Fortunately, there are many techniques we can use, based on the concepts of pseudonatural transformations and modifications between bicategories, that allow many different manipulations of the diagrams.

It is worthwhile to consider if more conceptual methods are available in the literature to attack the problem. In 1-dimensional category theory, the idea of transporting structure across an isomorphism can be realised by taking those structures to be algebras of a monad. Given any monad T on a category, the forgetful functor from the Eilenberg-Moore category of algebras is always an isofibration. This suggests that we could look at monadic ideas for transporting structure in higher-dimensional category theory. This is precisely how transport of structure is achieved at the 2-dimensional level: by the result of Kelly and Lack [KL04, Theorem 6.1], adjoint equivalences can be lifted to the 2-category of pseudoalgebras for a 2-monad. This gives us transport of structure at the 2-dimensional level.

Progress has been made in developing 3-dimensional monad theory, for example by Power [Pow07]. However, these developments turn out to be insufficient for the purposes of this thesis as they have focused only on the more specific case of monads on Gray-categories. Similarly, we currently lack a monadic description of trifunctors, but only have such a description for morphisms between Gray-categories [Buh14]. We see that a general method for transport of tricategorical structure is an open problem not covered by the existing frameworks.

We will therefore proceed with the more hands-on method, working directly from the definitions and the pasting diagrams. Diagram-manipulation methods have seen good use in the field: they were used to prove an important step towards the coherence theorem for tricategories [GPS95, Lemma 3.6] by showing that the structure of a tricategory could be transported across biequivalences of the hom-bicategories, and they also played a large role in the development of the theory of biequivalences in tricategories [Gur12].

Motivation

The motivation for considering the problem of transportation of trifunctors came from research into the alternate model of higher categories known as Tamsamani categories [Tam99]. Tamsamani categories are given by a recursive definition, with Tamsamani 1-categories being just categories and Tamsamani $(n+1)$-categories being simplicial objects in a 2-category of Tamsamani n-categories obeying certain conditions: the most important of which is that the Segal maps of the simplicial object are Tamsamani n-equivalences.

One question is how to compare Tamsamani 3-categories to tricategories. Cegarra and Heredia [CH14] have built a functor from tricategories to $\underline{Ta_3}$, but a functor from $\underline{Ta_3}$ to tricategories is unknown. Our result on transporting trifunctors could be used to attack this problem.

Tamsamani 3-categories $\underline{Ta_3}$ are in particular simplicial objects $\Delta^{op} \to \underline{Ta_2}$. After using the comparison between Tamsamani 2-categories and bicategories given by Lack and Paoli [LP08] these become simplicial objects $\Delta^{op} \to \underline{Bicat}$ such that the Segal maps $X_n \to X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$ are biequivalences of bicategories.

By interpreting the strict functors into the category of bicategories as trifunctors into the tricategory of bicategories, we could consider transporting these trifunctors across the biequivalences given by the Segal maps. This loses the strictness but makes it so that the image of each object n is exactly the n-fold pullback, which could simplify the process of understanding the Tamsamani 3-categories.

Outline

This thesis will proceed through the following chapters:

1. **Introduction to Higher Category Theory**: We start by exploring the ideas that led to the development of higher category theory. In particular, we will explore the reasons for studying weak higher categories and why strict n-categories are insufficient.
   We will also survey areas of mathematics where the results of low-dimensional higher category theory are utilised. Theoretical physics in particular has many uses for 3 dimensional weak categories. This provides potential uses of the results of this thesis.

2. **Bicategories**: As the first example of a weak higher category, one step above ordinary categories, bicategories are definitely noteworthy for the field of higher category theory as a whole. This chapter will survey the history of and main results on bicategories.
   These include the all-important coherence theorem, showing that the axioms of a bicategory are enough to show that all relevant diagrams of coherence cells commute. We will also illustrate how the results of ordinary category theory - thought of as taking place in the 2-category of categories - can be generalised to be internal to any bicategory. Many of these results, and especially those relating to adjunctions, are crucial to understanding operations on tricategories and the construction of the

tricategory of bicategories, <u>*Bicat*</u>, which is the target of the trifunctors we wish to study.

3. **Tricategories**: These 3-dimensional weak higher categories are the main object of study of this thesis. We will survey the history of tricategories and the main results on them to lay the groundwork.
   This will start with the definition of tricategories themselves, and the more specific Gray-categories. These provide the setting needed to formulate the coherence theorem: a crucial result in 3-dimensional category theory.
   From there, we survey the results on trifunctors and higher cells between tricategories. These trifunctors are the structures that this thesis aims to manipulate and transport, so we need to understand them thoroughly.

4. **Manipulating Tricategorical Pasting Diagrams**: The main results of this thesis are proved by means of manipulating pasting diagrams formed from tricategorical data, particularly in the tricategory of bicategories, in order to show that particular source and target diagrams arising from trifunctors are in fact equal. In this chapter we prove some original results that will help us in manipulating the diagrams.
   The first will use the coherence theorem for bicategories. Many of the cells used in the definitions result in coherence cells in the relevant bicategory. As such, the coherence theorem gives us a lot of flexibility when manipulating these cells in the diagram. We will mainly use these results to simplify the source and target diagrams of the axioms that need to be proved, allowing them to be verified.
   The results in this section are:

   - **Proposition 4.1.1**, which simplifies the definition of a trifunctor $F\colon \mathcal{T} \to \underline{Bicat}$.
   - **Proposition 4.1.2**, which simplifies the definition of a tritransformation $\theta\colon F \Rightarrow G\colon \mathcal{T} \to \underline{Bicat}$.
   - **Proposition 4.1.3**, which simplifies the definition of a biadjoint biequivalence between bicategories A and B.

   The other technique we will use is based on the definitions of pseudonatural transformation and modification. Many of the other cells in the diagrams being manipulated are components arising from pseudonatural transformations and modifications which have sources and targets which are composites of several arrows. Although the definitions of pseudonatural transformation and modification only specify that the components can be moved through the entire source to the entire target (or vice-versa) we will prove here that they can also be moved through any segment of their boundary. This opens up many more options for moving them through the diagrams.

The original results proved in this section are:

- **Proposition 4.2.1**, allowing us to move the 2-cell of a pseudonatural transformation through 2-cells even if those 2-cells don't cover the entire source or target.
- **Proposition 4.2.2**, allowing us to move a modification through some collection of pseudonaturality 2-cells even if those 2-cells don't add up to the pseudonaturality 2-cell of the source or target of the modification.

5. **Transporting a Trifunctor**: In this chapter the main result of the thesis will be proved. We will start by considering the original trifunctor $F\colon \mathcal{T} \to \underline{Bicat}$ and the object-indexed biequivalences: these will provide us with the components needed to construct the transported trifunctor.
We will then construct each piece of coherence data for the transported trifunctor $G\colon \mathcal{T} \to \underline{Bicat}$. At each level we'll start by identifying the source and target of each coherence cell of the transported trifunctor, demonstrate how each coherence cell can be constructed by pasting together cells coming from the original trifunctor and the biequivalences, and then prove that the results are suitably natural.
Once we have the 3-dimensional coherence cells of the transported trifunctor, we will prove that these actually form a trifunctor. We'll do this by considering the axioms for a trifunctor, constructing the diagrams that form the source and target of each axiom, and then manipulating the source diagram step-by-step to show that it is equal to the target axiom. In this way we prove that the axioms for a trifunctor hold for the newly-constructed transported trifunctor.

   In this chapter the original results proved are:

   - **Proposition 5.2.1**: we prove that the data constructed for the trifunctor G satisfies the first (simplified) axiom of a trifunctor.
   - **Proposition 5.2.2**: we prove that the data constructed for the trifunctor G satisfies the second (simplified) axiom of a trifunctor.
   - **Theorem 5.2.3**: since the data for G satisfies both trifunctor axioms, it is indeed a trifunctor $G\colon \mathcal{T} \to \underline{Bicat}$.

6. **Lifting the Biequivalences**: When considering transport of structure in ordinary categories - when we have a structure applied to a base object and an isomorphism

between that base object and some other - it is not only the case that we are able to transport the structure to the new object. We are also able to turn the isomorphism of the base objects into an isomorphism of the structure as well. Similarly with tricategorical structures; the object-indexed biequivalences should also become a tritransformation between the original and transported trifunctors. This is what we partially accomplish in this chapter: turning the family of biequivalences $S_A\colon FA \to GA$ into a tritransformation $S\colon F \Rightarrow G$.

As when constructing the transported trifunctor itself, we start by constructing the data that makes up $S\colon F \Rightarrow G$ from the data of the object-indexed biequivalences and the original trifunctor. We will work through the axioms of the tritransformations, constructing the diagrams for each of the tritransformation axioms and aiming to show that each pair of diagrams are equal by manipulating the pasting diagrams.

In this chapter we prove the original result:

- **Proposition 6.2.1**: we prove that the data constructed for the tritransformation S satisfies the first (simplified) axiom of a tritransformation.

We also make three conjectures:

- **Conjecture 6.3.1**: we conjecture that the data constructed for the tritransformation S satisfies the second (simplified) axiom of a tritransformation.
- **Conjecture 6.3.2**: we conjecture that the data constructed for the tritransformation S satisfies the third (simplified) axiom of a tritransformation.
- Conjecture 6.3.3: following immediately from the other two conjectures, S satisfies all three tritransformation axioms and is indeed a tritransformation $F \Rightarrow G$.

7. **Conclusions and Further Directions**: We conclude by summarising the main ideas and results of the thesis. After, we consider potential applications for the results of this thesis and directions for further work.

   One potential use relates to another model of weak higher category theory: Tamsamani categories [Tam99]. At the 3-dimensional level these Tamsamani categories can be viewed as functors into bicategories; our result can be applied in context of Tamsamani 3-categories and we envisage that this will lead to an explicit comparison between them and tricategories. There are also a few ways of potentially expanding the results of the thesis. One idea uses a Yoneda argument, allowing us to relate any trifunctor to one whose target is the tricategory of bicategories.

Chapter 1

Introduction to Higher Category Theory

In this chapter we will survey some of the literature on the background of higher category theory. We'll start with the philosophical ideas that led to the development of higher category theory as a field: it was needed to model many structures in homotopy theory, mathematical physics, and in category theory itself. Studying these ideas shows that the simplest type of higher categories - strict higher categories - aren't sufficient for what we want to use them for. This leads us to the much richer area of study given by weak higher category theory.

We'll then survey several approaches to weak higher categories. The explicit models given by bicategories and tricategories are central to this thesis and therefore covered in their own chapters, but there are also many other less direct models and it would be remiss not to be aware of them.

Finally, we will give an overview of the fields of mathematics where the ideas of higher category theory are used. We will particularly focus on applications of tricategories, as these are potential places which could benefit from the foundational results of this thesis.

1.1 Philosophy of Higher Category Theory

Just as category theory is the study of categories, which consist of objects and arrows between objects along with composition of arrows constrained by identity and associativity laws, higher category theory is the study of structures consisting of objects and arrows between objects and 2-cells between arrows (and so on: n-cells between $(n-1)$-cells) along with compositions in each direction. We must then understand what sorts of identity and associativity constraints are suitable. The simplest among higher categorical structures are those where the identity and associativity laws hold just as they do in a category at all levels (we call this complete strictness).

1.1.1 Strict n-categories

The simplest sorts of object in higher category theory are the strict n-categories.

Definition 1.1.1. A **Strict n-Category** $\mathcal{C}$ is defined recursively in the following way:
A strict 1-category is just a category.

Given the definition of a strict $(n-1)$-category, a strict n-category is a category enriched [Kel82] with respect to the Cartesian monoidal structure in strict $(n-1)$-categories. That is, it consists of:

- A collection of **Objects** $ob(\mathcal{C})$.
- For each pair of objects $A, B \in ob(\mathcal{C})$, a strict $(n-1)$-category denoted $\mathcal{C}(A, B)$. For each k between 0 and $(n-1)$, the k-cells of this $(n-1)$-category are the $(k+1)$-cells of $\mathcal{C}$. That is, the objects of this $(n-1)$-category are the **Morphisms** or **1-cells** of $\mathcal{C}$ (denoted $f\colon A \to B$), the 1-cells of this $(n-1)$-category are the 2-cells of $\mathcal{C}$ (denoted $\alpha\colon f \Rightarrow g$) and so on up to the top level where the $(n-1)$-cells of $\mathcal{C}(A, B)$ are the n-cells of $\mathcal{C}$ (similarly denoted with $\Rightarrow$).
- For each object $A \in ob(\mathcal{C})$ an object of $\mathcal{C}(A, A)$ labelled 1_A called the **Identity** of A.
- For each triple of objects $A, B, C \in ob(\mathcal{C})$ a functor

  $$\otimes\colon \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

  called **Composition**.

These are required to have the following two properties:

- The composition is **Associative**: i.e. for any four objects $A, B, C, D \in ob(\mathcal{C})$ the two functors

  $$- \otimes (- \otimes -)\colon \mathcal{C}(C, D) \times \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, D)$$

  and

  $$(- \otimes -) \otimes -\colon \mathcal{C}(C, D) \times \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, D)$$

  are equal.
- The **Identity Laws** hold: i.e. for any pair of objects $A, B \in ob(\mathcal{C})$ the functors

  $$1_B \otimes -\colon \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, B)$$

and

$$- \otimes 1_A \colon \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, B)$$

are both equal to the identity.

Unpacking this definition, we see that a strict n-category consists of cells in dimensions 0 all the way up to n with all compositions in all directions being unital and associative. One example has as its objects the collection of all categories, 1-cells given by functors, and 2-cells given by natural transformations. Composition of both functors and natural transformations is unital and associative, meaning that these do form a strict 2-category.

A related concept is the idea of a strict n-groupoid, which is the strict n-dimensional generalisation of the idea of groupoid.

Definition 1.1.2. A **Strict n-Groupoid** is a strict n-category in which every k-cell (for k running from 1 to n) $\alpha \colon f \Rightarrow g$ has a strict inverse. That is, there is a k-cell $\beta \colon g \Rightarrow f$ such that both $\alpha \circ \beta$ and $\beta \circ \alpha$ are equal to the identity for all composition operations defined on k-cells.

Having defined strict n-categories and strict n-groupoids, we will now explore why they are insufficient.

1.1.2 The Principle of Isomorphism

One of the key ideas behind the philosophy of category theory is the following.

Definition 1.1.3. The **Principle of Isomorphism** [Mak98] states that correct properties of objects in a fixed category should be invariant under isomorphism. In particular, it is philosophically incorrect to attempt to distinguish between isomorphic objects of a category.

As an example of the use of the principle of isomorphism, consider the similar **Principle of Equivalence**, which states that the correct notion of equivalence between categories is not isomorphism but instead the idea ordinarily denoted by equivalence. This follows from the principle of isomorphism in one of two different ways, depending on how you define equivalence.

- Consider an isomorphism to be a pair of functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$. Functors have a suitable concept of morphisms between them in the form of natural transformations so requiring the composites to be equal to the identity is too strict and breaks the principle of isomorphism. Instead, we should only require that both composites should be naturally isomorphic to the identity.

Therefore, our definition of equivalence should be a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ and a pair of natural isomorphisms $\eta\colon GF \Rightarrow 1_{\mathcal{C}}$ and $\varepsilon\colon 1_{\mathcal{D}} \Rightarrow FG$.

- Consider an isomorphism to be a functor $F\colon \mathcal{C} \to \mathcal{D}$ that is bijective on objects and bijective on morphisms. This notion is too strict and breaks the principle of isomorphism because we distinguish objects in the same isomorphism class if some are in the image and others are not. Instead, we should only be able to check whether or not each isomorphism class is hit by the functor.
  Thus, our definition of equivalence is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that is full and faithful and essentially surjective. (Note that such a functor is also 'essentially injective': since it is full and faithful it reflects isomorphisms and thus is injective on isomorphism classes).

Note that the definition of strict n-category given above breaches the principle of isomorphism. The definition requires that certain functors be equal but as we saw earlier the principle of isomorphism only allows us to define functors as being isomorphic, or even only equivalent at higher dimensions. It is this idea that motivates, in an abstract sense, the study of weakness in higher category theory: the axioms at lower levels should be satisfied only up to an isomorphism in the next level up.

1.1.3 Topological Motivations

Much of higher category theory was originally motivated by the study of algebraic topology, and in particular homotopy theory. Kan's notion of simplicial sets model topological spaces [GJ09] and this guides us to thinking of spaces as being built up from cells of different dimensions, just like higher categories are.

One tool for studying the homotopy theory of these simplicial sets is the Postnikov tower [Pos51]. The Postnikov tower of a space X is a tower of spaces X_n where each X_n has a vanishing homotopy group in dimensions higher than n. Such spaces are called **n-types**, and they are very amenable to being studied via n-categorical methods. Starting at $n = 1$, we have that 1-types are naturally modelled using groupoids via the following construction.

Definition 1.1.4. [GJ09] Given a topological space X, the **Fundamental Groupoid** of X is a category whose objects are the points of X and whose morphisms are homotopy classes of paths in X. The identity at a point is given by the homotopy class of the constant path at that point, composition is induced by concatenation of paths and each morphism is invertible with the inverse being given by reversing the direction of the path (these are respected by equivalence up to homotopy).

Despite only giving us an ordinary category, this definition implicitly requires a higher level of structure: the homotopies between the paths. We also note that the axioms of a groupoid are only satisfied up to a reparameterisation homotopy. This suggested the way forward to understanding the categorical behaviours of higher n-types. Grothendieck first intuited that n-types should be modelled by a suitable notion of n-groupoid [Gro]. This idea eventually developed into the **Homotopy Hypothesis**.

Definition 1.1.5. [BD95] The **Homotopy Hypothesis** states, in its strong form, that for a suitable notion of n-categories there should be an equivalence of $(n+1)$-categories where one part of it is given by the fundamental n-groupoid:

$$\Pi_n \colon \underline{n - Type} \longrightarrow \underline{n - Gpd}$$

and the other direction is given by the classifying space functor.

There is also a weaker, more easily satisfied, form of the Homotopy Hypothesis which only requires an equivalence between the homotopy categories, where the homotopy category of $\underline{n - Type}$ is taken by formally inverting the weak equivalences and the homotopy category of $\underline{n - Gpd}$ is taken by localising at suitably defined n-equivalences.

The Homotopy Hypothesis also demonstrates why strict n-categories are unsuitable. Although strict 2-groupoids can model 2-types [MS93], Simpson has shown that this fails immediately at the next level up: there is no strict 3-groupoid that can model the 3-type of the 2-sphere [Sim] because there is a non-trivial Whitehead product. If we are to satisfy the Homotopy Hypothesis and be able to study the Postnikov tower in higher dimensions, we'll need to move from strict n-groupoids to weak n-groupoids.

1.2 Models of Weak Higher Category Theory

Since strict higher categories aren't sufficient, we instead study the more general notion of weak higher categories. A weak n-category should be a categorical structure with cells in dimensions 0 to n with compositions in all directions, but the associativity and unit axioms hold only up to isomorphisms. Furthermore, these isomorphisms are required to be suitably compatible via coherence axioms. There are many possible models of weak n-categories.

The most explicit way to construct weak n-categories is to take a collection of objects and hom-$(n-1)$-categories (as when we constructed strict n-categories) and, at any level k below the top level of cells, replace any axiom in that level of cells by a $k+1$-cell that mediates the axiom. Doing this starting at $n = 2$ first gives you the definition of bicategory (developed by Benabou [Ben67]) and this lets you then develop the definition of a 3-dimensional weak category, the tricategory (first introduced by Gordon, Power and Street [GPS95]). These are the major objects of study of this thesis and will therefore be covered in more detail in the two following chapters, but there are two things we can note now.

First, both bicategories and tricategories satisfy the homotopy hypothesis (The proof for bicategories was given by Hardie, Kamps and Kieboom [HKK01] while the proof for tricategories was by Leroy [Ler94]); they are therefore suitable models of higher categories in 2 and 3 dimensions. Secondly, both bicategories and tricategories require their respective coherence theorems - stating that, despite only specific coherence cells being used in the definitions, all relevant diagrams of coherence cells give the same result - in order to work. The increase in complexity of both the coherence axioms and proof of the coherence theorem as we move from dimension 2 to dimension 3 indicates that the extension of this approach to higher dimensions is problematic. Indeed, although a potential definition in the same vein has been proposed by Trimble [Tri06], the coherence theorem for it has not yet been proved and without it we can't proceed further.

1.2.1 Combinatorial Models

In order to tackle the issues raised in the previous section, an alternate method for constructing weak higher categories is to set up combinatorial machineries that will encode the intuitions behind n-dimensional categories more indirectly. Although these models are generally less concrete - for example, by having composition maps that only arise indirectly, meaning that there may be many possible composites for any pair of composable morphisms - the combinatorics can handle all of the coherence issues automatically, rather than needing to encode them in explicit diagrams and then prove coherence theorems. This makes it possible to define these models in all dimensions.

Once a model has been shown to satisfy the homotopy hypothesis, it can be accepted as a suitable model of weak higher categories. A partial survey of these models is given by Leinster [Lei02]; the question of how to compare different models is largely an open problem. Some examples of these models are:

- Operadic models, most notably the Batanin model [Bat98]. These realise weak n-categories as algebras of a carefully constructed higher operad. For dimension 2, a sketch of the proof that Batanin 2-categories are equivalent to bicategories is given in [Bat98].

- Multi-simplicial models: these are the Tamsamani model [Tam99] and the similar Simpson model [Sim12]. These construct weak n-categories as simplicial objects in the category of weak $(n-1)$-categories, using the simplicial structure to control the coherences. The question of comparing Tamsamani categories to bicategories and tricategories is further along, with a complete comparison at the 2-dimensional level due to Lack and Paoli [LP08] and a method of taking a nerve of a tricategory to get a Tamsamani 3-category due to Cegarra and Heredia [CH14]. The other direction - taking a Tamsamani 3-category and constructing a tricategory - is a potential application of the results of this thesis.

- The opetopic model of Baez and Dolan [BD98].

- A more recent method by Paoli models weak n-categories using a subcategory of n-fold categories [Pao19] (i.e. the n-dimensional version of double-categories).

1.2.2 Infinite-Dimensional Category Theory

Although this thesis is focused on the finite-dimensional forms of higher category theory, it is worth being aware of the work done on structures with cells in all dimensions. These structures have important applications in algebraic geometry and mathematical physics.

The prototypical structure that was the motivation for studying infinite-dimensional category theory is the category of topological spaces, with the 1-cells being given by continuous maps and higher cells being given by higher homotopies. Homotopies are invertible up to homotopy so it is no surprise that the first infinite dimensional categories that were studied were **$(\infty, \mathbf{1})$-Categories**: those where the cells in dimensions 2 and above are all equivalences.

There are many models of $(\infty, 1)$-categories, of which the most well known are **Simplicially Enriched Categories** and **Quasi-Categories**. Quasi-categories were introduced by Boardman and Vogt [BV73] and the theory was developed by Joyal [Joy08] and Lurie [Lur09b].

The next step was to relax the requirement that certain cells be equivalences until higher dimensions are reached: this gives us the idea of **$(\infty, \mathbf{n})$-Categories**. A survey of the models of (∞, n)-categories is given by Bergner in [Ber11].

The final step to full generality is to consider infinite-dimensional categorical structures with cells in all dimensions but no requirements that the cells should be equivalences: **ω-Categories**. A model of ω-categories using complicial sets has been given by Verity [Ver08].

1.3 Applications of Tricategories

We conclude this chapter by surveying some existing applications of the theory of tricategories. Against the background of recent developments into less explicit models, one might ask if the classical notions of bicategories and tricategories are still relevant. The applications we will see show that, far from being only of historical significance, bicategories and tricategories are relevant and used in a variety of mathematical fields, often when the precise handling of the coherences is called for.

1.3.1 Applications in Logic

As one of the original developers of the definition of tricategory, Power was motivated by potential applications of 3-dimensional category theory to logic. He noted that attempting to generalise results about 2-monads up a dimension would require weakening 2-natural transformations to pseudonatural transformations [Pow95, Example 7.1] and would therefore need the weaker structure given by tricategories.

A more recent area where weak higher category theory is used in logic is in homotopy type theory (see, for example, the research of the Univalent Foundations Program [Pro13]): this is a merger of homotopy theory and logic via weak higher category theory that has applications to building proof assistants, a topic of considerable potential in theoretical computer science.

Gray categories have also been used in rewriting theory [FM18].

1.3.2 Applications in Mathematical Physics

Mathematical physics first needed the ideas of higher category theory to study higher cobordism categories [BD95]. This problem is very complex: despite considerable progress made by Lurie [Lur09a] the proof of the cobordism hypothesis is still partially open.

The main uses of specifically tricategorical theory in mathematical physics are when modelling 3-dimensional topological field theories. Examples include:

- Barrett, Meusburger and Schaumann [BMS12] study Gray-categories with duals, and their geometric properties, by means of a diagrammatic calculus. These Gray-categories help to simplify many of the calculations relating to TQFTs.

- Carqueville, Meusburger and Schaumann [CMS16] undertook a systematic study of defect TQFTs. They initiate this study by introducing symmetric monoidal functors on stratified and decorated bordisms, which can each be transformed into a tricategory with duals in a natural categorification of the idea of a pivotal bicategory.

1.3.3 Applications in Homotopy Theory

An important open problem in homotopy theory is the algebraic modelling of stable n-types. This is harder than modelling unstable n-types because of the notions of higher symmetry needed and the way these interact with the coherence data.

One tool that was introduced to help with this was the notion of symmetric monoidal bicategory [SP09], which can be viewed as a tricategory with one object and some extra structure. Once a coherence theorem was proved for symmetric monoidal bicategories [GO13] this tool was ready to be used. It led to the proof of the stable homotopy hypothesis in dimension 2 [GJO19].

1.3.4 Applications via Braided Monoidal Structures

Braided monoidal categories can be viewed as ordinary categories with some extra structure, but they can also be viewed as tricategories with a single object and 1-cell. As such, tricategorical results have applications in any area where braided monoidal categories have been used, such as representation theory [JS93] [JS95].

The situation where a braided monoidal structure has been taken one dimension higher - i.e. the braided monoidal bicategory - has also been studied [KV94] [BN96]. These have applications to 2-tangle invariants and 4-dimensional TQFTs. In [Gur11] the author proves a coherence theorem for braided monoidal bicategories and relates it to the coherence theorem for monoidal bicategories. They show how coherence for these structures can be interpreted topologically using up-to-homotopy operad actions and the algebraic classification of surface braids.

Other recent applications of braided structures have related them to quantum computation [Ver17].

Chapter 2

Bicategories

In this chapter we will survey the results on bicategories. These will be the objects of the tricategory that is the target of the trifunctor we want to transport. Furthermore, the techniques we will use to manipulate the pasting diagrams are based on the properties of bicategories and the cells between them. Therefore, it is crucial that we understand bicategories thoroughly before continuing.

2.1 Definitions

The idea behind the definition of bicategories is to take the definition of strict 2-category and, following the principle of isomorphism, replace every identity in 1-cells with a mediating isomorphism.

For this section, we will use $\otimes$ to mean composition in the direction of the 1-cells and $\circ$ to represent composition in the direction of the 2-cells.

Definition 2.1.1. [Ben67, Definition 1.1] A **Bicategory** $\mathcal{B}$ consists of:

- A collection of objects $ob(\mathcal{B})$.
- For each pair of objects $A, B \in ob(\mathcal{B})$, a **Hom-Category** $\mathcal{B}(A, B)$. The objects of these categories are the 1**-cells** of the bicategory and the morphisms of these categories are the 2**-cells** of the bicategory.
- For each triple of objects $A, B, C \in ob(\mathcal{B})$, a **Composition** functor $\otimes : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$.
- For each object $A \in ob(\mathcal{B})$, an **Identity** $id_A : A \to A$.

- For each four objects $A, B, C, D \in ob(\mathcal{B})$, an **Associator** given by a natural isomorphism

$$\begin{tikzcd} \mathcal{B}(C,D) \times \mathcal{B}(B,C) \times \mathcal{B}(A,B) \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & \mathcal{B}(B,D) \times \mathcal{B}(A,B) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "a"] \\ \mathcal{B}(C,D) \times \mathcal{B}(A,C) \arrow[r, "\otimes"'] & \mathcal{B}(A,D) \end{tikzcd}$$

- For each pair of objects $A, B \in ob(\mathcal{B})$ a **Left Unitor** given as a natural transformation

$$\begin{tikzcd} & \mathcal{B}(B,B) \times \mathcal{B}(A,B) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] & \\ 1 \times \mathcal{B}(A,B) \arrow[ur, "id_B\times 1"] \arrow[rr, "\cong"'] & {} & \mathcal{B}(A,B) \end{tikzcd}$$

- For each pair of objects $A, B \in ob(\mathcal{B})$ a **Right Unitor** given as a natural transformation

$$\begin{tikzcd} & \mathcal{B}(A,B) \times \mathcal{B}(A,A) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow r"] & \\ \mathcal{B}(A,B) \times 1 \arrow[ur, "1\times id_A"] \arrow[rr, "\cong"'] & {} & \mathcal{B}(A,B) \end{tikzcd}$$

satisfying the following axioms:

- The **Pentagon Identity**: for any four composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

the following diagram commutes:

$$\begin{tikzcd} & (k\otimes(h\otimes g))\otimes f \arrow[dr, "a"] & \\ ((k\otimes h)\otimes g)\otimes f \arrow[ur, "a\otimes 1"] \arrow[d, "a"'] & & k\otimes((h\otimes g)\otimes f) \arrow[d, "1\otimes a"] \\ (k\otimes h)\otimes(g\otimes f) \arrow[rr, "a"'] & & k\otimes(h\otimes(g\otimes f)) \end{tikzcd}$$

- The **Triangle Identity**: for any two 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

the following diagram commutes:

$$\begin{tikzcd}
(g \otimes id_B) \otimes f \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & g \otimes (id_B \otimes f) \arrow[dl, "1\otimes l"] \\
& g \otimes f &
\end{tikzcd}$$

Bicategories can be dualised in three different ways [Ben67, Section 3]. The first sends a bicategory $\mathcal{B}$ to $\mathcal{B}^{op}$ which has the same objects with the hom-categories being $\mathcal{B}^{op}(A,B) = \mathcal{B}(B,A)$. The second is $\mathcal{B}^{co}$ which has hom-categories $\mathcal{B}^{co}(A,B) = \mathcal{B}(A,B)^{op}$. Note that $\mathcal{B}^{co,op} = \mathcal{B}^{op,co}$; this gives the third dualisation.

It is also worth considering what the functors and higher transformations should be between bicategories. Once again, this involves taking the definitions for the strict versions and replacing all of the equalities of 1-cells with isomorphisms.

Definition 2.1.2. [Ben67, Definition 4.1] A **Pseudofunctor** or **Homomorphism** between bicategories $F : \mathcal{A} \to \mathcal{B}$ consists of:

- A function $F : ob(\mathcal{A}) \longrightarrow ob(\mathcal{B})$.
- For each pair of objects $A, B \in ob(\mathcal{A})$, a functor $F_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$.
- For each triple of objects $A, B, C \in ob(\mathcal{A})$, a **Compositor** given by a natural isomorphism

$$\begin{tikzcd}
\mathcal{A}(B,C) \times \mathcal{A}(A,B) \arrow[r, "F\times F"] \arrow[d, "\otimes"'] & \mathcal{B}(FB,FC) \times \mathcal{B}(FA,FB) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "\varphi"] \\
\mathcal{A}(A,C) \arrow[r, "F"'] & \mathcal{B}(FA,FC)
\end{tikzcd}$$

- For each object $A \in ob(\mathcal{A})$, a **Unitor** given by an invertible 2-cell $\sigma : id_{FA} \Rightarrow F(id_A)$.

such that:

- The compositor interacts properly with the associator: for each set of composable 1-cells in $\mathcal{A}$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

  the diagram

$$\begin{tikzcd}
(Fh \otimes Fg) \otimes Ff \arrow[r, "a"] \arrow[d, "\varphi\otimes 1"'] & Fh \otimes (Fg \otimes Ff) \arrow[d, "1\otimes\varphi"] \\
F(h \otimes g) \otimes Ff \arrow[d, "\varphi"'] & Fh \otimes F(g \otimes f) \arrow[d, "\varphi"] \\
F((h \otimes g) \otimes f) \arrow[r, "Fa"'] & F(h \otimes (g \otimes f))
\end{tikzcd}$$

commutes.

- The identity interacts well: for each 1-cell $f : A \to B$, the two diagrams

$$\begin{tikzcd}
id_{FB} \otimes F(f) \arrow[r, "\sigma\otimes 1"] \arrow[d, "l"'] & F(id_B) \otimes Ff \arrow[d, "\varphi"] \\
Ff & F(id_B \otimes f) \arrow[l, "Fl"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
Ff \otimes id_{FA} \arrow[r, "1\otimes\sigma"] \arrow[d, "r"'] & Ff \otimes F(id_A) \arrow[d, "\varphi"] \\
Ff & F(f \otimes id_A) \arrow[l, "Fr"]
\end{tikzcd}$$

commute.

A pseudofunctor where σ is an identity is called normal. A pseudofunctor where φ and σ are both identities, and thus $Fa = a$, $Fl = l$ and $Fr = r$, is called a **Strict Functor**.

Proposition 2.1.1. [Ben67, Theorem 4.3.1] Given two pseudofunctors

$$\mathcal{A} \xrightarrow{(F,\varphi,\sigma)} \mathcal{B} \xrightarrow{(G,\varphi',\sigma')} \mathcal{C}$$

they can be composed to produce a pseudofunctor $G \otimes F$ with the following components:

- The action on the objects and the homcategories is given by the composite of the respective actions.
- The new compositor is given by

$$\begin{tikzcd}
\mathcal{A}(B,C) \times \mathcal{A}(A,B) \arrow[r, "F\times F"] \arrow[d, "\otimes"'] & \mathcal{B}(FB,FC) \times \mathcal{B}(FA,FB) \arrow[r, "G\times G"] \arrow[d, "\otimes"] \arrow[dl, phantom, "\Leftarrow \varphi"] & \mathcal{C}(GFB,GFC) \times \mathcal{C}(GFA,GFB) \arrow[d, "\otimes"] \arrow[dl, phantom, "\Leftarrow \varphi'"] \\
\mathcal{A}(A,C) \arrow[r, "F"'] & \mathcal{B}(FA,FC) \arrow[r, "G"'] & \mathcal{C}(GFA,GFC)
\end{tikzcd}$$

- The new action on the units is given by the composite

$$id_{GFA} \xrightarrow{\sigma'} G(id_{FA}) \xrightarrow{G(\sigma)} GF(id_A)$$

  These satisfy the conditions because the original 2-cells did.

This composition is strictly unital and associative.

The next things to consider are the transformations between pseudofunctors.

Definition 2.1.3. [Lei98, Section 1.2] A **Pseudonatural Transformation** [Lei98, Section 1.2] between two pseudofunctors $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ consists of:

- For each object $A \in ob(\mathcal{A})$, a 1-cell $\alpha_A : FA \to GA$.
- For each pair of objects $A, B \in ob(\mathcal{A})$, a natural transformation

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{B}(GA,GB) \arrow[d, "-\otimes\alpha_A"] \arrow[dl, Rightarrow, "\alpha"] \\ \mathcal{B}(FA,FB) \arrow[r, "\alpha_B\otimes -"'] & \mathcal{B}(FA,GB) \end{tikzcd}$$

Satisfying the following conditions:

- The transformation respects identities: for each object $A \in ob(\mathcal{A})$, the following diagram commutes

$$\begin{tikzcd} id_{GA} \otimes \alpha_A \arrow[r, Rightarrow, "l"] \arrow[d, Rightarrow, "\sigma\otimes 1"'] & \alpha_A \arrow[r, Rightarrow, "r^{-1}"] & \alpha_A \otimes id_{FA} \arrow[d, Rightarrow, "1\otimes\sigma"] \\ G(id_A) \otimes \alpha_A \arrow[rr, Rightarrow, "\alpha_{id_a}"'] & & \alpha_A \otimes F(id_a) \end{tikzcd}$$

- The transformation respects composition: for each pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

the following diagram commutes

$$\begin{tikzcd}
(Gg \otimes Gf) \otimes \alpha_A \arrow[r, Rightarrow, "a"] \arrow[dddd, Rightarrow, "\varphi\otimes 1"'] & Gg \otimes (Gf \otimes \alpha_A) \arrow[r, Rightarrow, "1\otimes\alpha_f"] & Gg \otimes (\alpha_B \otimes Ff) \arrow[r, Rightarrow, "a^{-1}"] & (Gg \otimes \alpha_B) \otimes Ff \arrow[d, Rightarrow, "\alpha_g\otimes 1"] \\
& & & (\alpha_C \otimes Fg) \otimes Ff \arrow[d, Rightarrow, "a"] \\
& & & \alpha_C \otimes (Fg \otimes Ff) \arrow[d, Rightarrow, "1\otimes\varphi"] \\
& & & \\
G(g \otimes f) \otimes \alpha_A \arrow[rrr, Rightarrow, "\alpha_{g\otimes f}"'] & & & \alpha_C \otimes F(g \otimes f)
\end{tikzcd}$$

Once again, a pseudonatural transformation is called 'strict' if the components at all 1-cells are identities.

Pseudonatural transformations can be composed in both sensible directions. Given two pseudonatural transformations

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{B}
\end{tikzcd}$$

the components of $\alpha \circ \beta$ at each object are the composites of the respective components and the component at each 1-cell is given by

$$Hf \otimes (\beta_A \otimes \alpha_A) \xrightarrow{a^{-1}} (Hf \otimes \beta_A) \otimes \alpha_A \xrightarrow{\beta_f \otimes 1} (\beta_B \otimes Gf) \otimes \alpha_A \xrightarrow{a}$$
$$\beta_B \otimes (Gf \otimes \alpha_A) \xrightarrow{1\otimes\alpha_f} \beta_B \otimes (\alpha_B \otimes Ff) \xrightarrow{a^{-1}} (\beta_B \otimes \alpha_B) \otimes Ff$$

Similarly, it is possible to compose two pseudonatural transformations in the direction of the 1-cells, given transformations as in the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C}
\end{tikzcd}$$

However a decision has to be made, because although it is relatively easy to define the composite of a pseudonatural transformation with an identity on either side (i.e. $id_G \otimes \alpha$ and $\beta \otimes id_F$) these do not compose to give a consistent definition of the composite $\beta \otimes \alpha$: the interchange law fails. By convention, we define $\beta \otimes \alpha = (id_{G'} \otimes \alpha) \circ (\beta \otimes id_F)$ [Gur13, Prop. 5.1].

The final level of transformation is the modification. We do not need to weaken the definition because there are no cells of a higher level than the components of the modification so any isomorphism is an equality. However, we do need to adjust the definition given in the introduction to account for the fact that we are taking modifications between pseudonatural transformations.

Definition 2.1.4. [Lei98, Section 1.3] A **Modification** between two pseudonatural transformations $\Sigma : \alpha \Rightarrow \beta : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ is given by, for each object $A \in ob(\mathcal{A})$ a 2-cell $\Sigma_A : \alpha_A \Rightarrow \beta_A$ such that for all 1-cells $f : A \to B$ the diagram

$$\begin{tikzcd} Gf \otimes \alpha_A \arrow[r, Rightarrow, "1\otimes\Sigma_A"] \arrow[d, Rightarrow, "\alpha_f"'] & Gf \otimes \beta_A \arrow[d, Rightarrow, "\beta_f"] \\ \alpha_B \otimes Ff \arrow[r, Rightarrow, "\Sigma_B\otimes 1"'] & \beta_B \otimes Ff \end{tikzcd}$$

commutes.

Modifications can be composed in all three directions though again choices must be made.
As we shall see in the next chapter, the structure formed by bicategories, pseudofunctors, pseudonatural transformations and modifications is a tricategory.

One final useful definition is that of local properties.

Definition 2.1.5. [Ben67, Section 2.7]

- Let P be a property of categories. Then a bicategory is **Locally P** if all of its hom-categories are P.
- Let Q be a property of functors. Then a pseudofunctor is **Locally Q** if all of the functors on the hom-categories are Q.

2.1.1 Laxness

Our consideration of the principle of isomorphism led us to consider pseudofunctors (respectively pseudonatural transformations) as the natural morphisms between bicategories (respectively pseudofunctors). However, there are other possible classes of morphisms that are worth considering. These are the lax functors (resp. lax natural transformations), whose importance was first considered by Benabou [Ben67]. The definition of a lax functor (resp. lax natural transformation) is obtained from that of pseudofunctors (resp. pseudonatural transformations) by removing the requirement that the constraint cells be invertible.

Similarly, by reversing the direction of the constraint cells and their axioms, we get the concept of oplax functor (resp. oplax natural transformation).

One important example of an oplax natural transformation occurs in the form of an icon.

Definition 2.1.6. [Lac10b] An **Icon** (i.e. an Identity Component Oplax Natural transformation) between two pseudofunctors $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ consists of:

- The assertion that F and G agree on objects.
- For each pair of objects $A, B \in ob(\mathcal{A})$, a natural transformation

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{B}(FA,FB) \end{tikzcd}$$

which is

- Compatible with the identities: for each object $A \in ob(\mathcal{A})$ the diagram

$$\begin{tikzcd} & id_{FA} = id_{GA} \arrow[dl, Rightarrow, "\sigma_F"'] \arrow[dr, Rightarrow, "\sigma_G"] & \\ Fid_A \arrow[rr, Rightarrow, "\alpha_{id_A}"'] & & Gid_A \end{tikzcd}$$

  commutes.
- Compatible with the composition: for each pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

  the diagram

$$\begin{tikzcd} Fg \otimes Ff \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\alpha_f \otimes \alpha_g"'] & F(g \otimes f) \arrow[d, Rightarrow, "\alpha_{g\otimes f}"] \\ Gg \otimes Gf \arrow[r, Rightarrow, "\varphi"'] & G(g \otimes f) \end{tikzcd}$$

  commutes.

Examining this definition shows that this is equivalent to requiring an oplax transformation to have components that are the identity at each object, hence the name. Icons are useful because, unlike the difficulty we had in defining a horizontal composite for pseudonatural transformations, the structure formed by bicategories, pseudofunctors and icons is a strict 2-category Icon [Lac10b, Theorem 3.2].

2.2 2-Dimensional Category Theory

Many aspects of category theory can be developed internally in any bicategory. A survey of the development of this theory is the 2-Categories Companion by Lack [Lac10a]. 2-dimensional category theory is a rich subject so we will only present those key aspects that are relevant to the thesis: equivalences, adjunctions and monads. We will also explore the idea of limits within a bicategory.

2.2.1 Equivalences

Definition 2.2.1. [Lac10a, Exercise 2.2] Within a bicategory $\mathcal{B}$, an **Equivalence** between two objects is given by:

- Two 1-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

- Two invertible 2-cells $\eta : id_A \Rightarrow g \otimes f$ and $\varepsilon : f \otimes g \Rightarrow id_B$.

By the principle of isomorphism, this is the correct notion of equivalence within a bicategory. Thus, it is incorrect to attempt to distinguish equivalent objects.

2.2.2 Adjunctions

Definition 2.2.2. [Lac10a, Section 2.1] Within a bicategory $\mathcal{B}$, an **Adjunction** is given by:

- Two 1-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

- Two 2-cells $\eta : id_A \Rightarrow g \otimes f$ and $\varepsilon : f \otimes g \Rightarrow id_B$.

satisfying the usual triangle identities in the sense that the following diagrams commute:

$$\begin{tikzcd}
f \otimes id_A \arrow[r, Rightarrow, "1\otimes\eta"] \arrow[rrdd, Rightarrow, "r"'] & f \otimes (g \otimes f) \arrow[r, Rightarrow, "a^{-1}"] & (f \otimes g) \otimes f \arrow[d, Rightarrow, "\varepsilon\otimes 1"] \\
& & id_B \otimes f \arrow[d, Rightarrow, "l"] \\
& & f
\end{tikzcd}$$

$$\begin{tikzcd}
id_A \otimes g \arrow[d, Rightarrow, "\eta\otimes 1"'] \arrow[rrdd, Rightarrow, "l"] & & \\
(g \otimes f) \otimes g \arrow[d, Rightarrow, "a"'] & & \\
g \otimes (f \otimes g) \arrow[r, Rightarrow, "1\otimes\varepsilon"'] & g \otimes id_B \arrow[r, Rightarrow, "r"'] & g
\end{tikzcd}$$

If the two cells η and ε not only satisfy the triangle identities but are also invertible then this is an adjoint equivalence. Just as in Cat, given an equivalence it is possible to turn it into an adjoint equivalence by changing only one of the 2-cells.

The definition given above mirrors the definition of adjunctions in Cat given by unit and co-unit. The other definition of adjunctions in Cat, using the natural bijection between hom-sets, is generalised to the idea of mates under adjunctions. This theory is presented here in its most general form, as the use of mates is crucial when constructing tricategories and the cells between them.

Definition 2.2.3. [Lac10a] Let $(f, g, \eta, \varepsilon) : A \to B$ and $(h, k, \upsilon, \gamma) : C \to D$ be two adjunctions. Let $a : A \to C$ and $b : B \to D$ be two 1-cells. Then there is a bijective correspondence between 2-cells $\alpha : h \otimes a \Rightarrow b \otimes f$ and 2-cells $\beta : a \otimes g \Rightarrow k \otimes b$. This correspondence is called taking the **Mate** under the adjunctions. The correspondence is shown via the pasting diagrams

$$
\begin{tikzcd}
 & B \arrow[dl, "g"'] \arrow[dd, "id_B"] \\
A \arrow[r, phantom, "\overset{\varepsilon}{\Longrightarrow}"] \arrow[dr, "f"] \arrow[dd, "a"'] & {} \\
 & B \arrow[dd, "b"] \\
C \arrow[ur, phantom, "\overset{\alpha}{\Nearrow}"] \arrow[dr, "h"] \arrow[dd, "id_C"'] & \\
{} \arrow[r, phantom, "\overset{\upsilon}{\Longrightarrow}"] & D \arrow[dl, "k"] \\
C &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "id_A"'] & \\
{} \arrow[r, phantom, "\overset{\eta}{\Longrightarrow}"] & B \arrow[dl, "g"] \arrow[dd, "b"] \\
A \arrow[dr, phantom, "\overset{\beta}{\Searrow}"] \arrow[dd, "a"'] & \\
 & D \arrow[dl, "k"'] \arrow[dd, "id_D"] \\
C \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] \arrow[dr, "h"'] & {} \\
 & H
\end{tikzcd}
$$

along with suitable applications of the constraint cells. This correspondence is bijective because of the triangle identities.

The usual property of adjunctions in <u>*Cat*</u> comes from taking A and B to both be the terminal category with the identity adjunction between them: thus a and b pick out objects of the categories C and D respectively and the 2-cells (that is, natural transformations) are just morphisms from $ha \to b$ and $a \to kb$.

2.2.3 Monads

Definition 2.2.4. [Ben67, Section 5.4] A **Monad** in a bicategory is given by

- An object A.
- A 1-cell $t : A \to A$.
- A 2-cell $\eta : id_A \Rightarrow T$ called the unit.
- A 2-cell $\mu : T \otimes T \Rightarrow T$ called the multiplication.

satisfying the axioms

- The associativity law: the diagram

$$
\begin{tikzcd}
(T \otimes T) \otimes T \arrow[rr, Rightarrow, "a"] \arrow[d, Rightarrow, "\mu\otimes 1"'] & & T \otimes (T \otimes T) \arrow[d, Rightarrow, "1\otimes\mu"] \\
T \otimes T \arrow[dr, Rightarrow, "\mu"'] & & T \otimes T \arrow[dl, Rightarrow, "\mu"] \\
 & T &
\end{tikzcd}
$$

commutes.

- The unit laws: the diagrams

$$\begin{tikzcd}
T \otimes id_A \arrow[d, Rightarrow, "1\otimes\eta"'] \arrow[dr, Rightarrow, "r"] & & id_A \otimes T \arrow[dl, Rightarrow, "l"'] \arrow[d, Rightarrow, "\eta\otimes 1"] \\
T \otimes T \arrow[r, Rightarrow, "\mu"'] & T & T \otimes T \arrow[l, Rightarrow, "\mu"]
\end{tikzcd}$$

commute.

Note that a monad can also be defined as a lax functor from the terminal 2-category to the bicategory.

2.2.4 Pseudolimits

Just as we adjusted the definitions of functor and natural transformation earlier, we will also modify the definition of limit to account for the principle of isomorphism.

Definition 2.2.5.

- Given a diagram in a bicategory given by a pseudofunctor $D : \mathcal{J} \to \mathcal{B}$, a **Pseudocone** consists of
  - An object $C \in ob(\mathcal{B})$.
  - For each object $i \in ob(\mathcal{J})$, a 1-cell $\lambda_i : C \to Di$.
  - For each 1-cell $f : i \to j$ in $\mathcal{J}$, an invertible 2-cell $\Psi_f : Df \otimes \lambda_i \Rightarrow \lambda_j$.

  such that for every 2-cell $\alpha : f \Rightarrow g : i \to j$ of $\mathcal{J}$, the 2-cells Ψ_g and $\Psi_f \circ (D\alpha \otimes \lambda_i)$ are equal.

- A **Pseudolimit** is a pseudocone (L, μ, Φ) such that for any other pseudocone (C, λ, Ψ), there is a unique 1-cell $k : C \to L$ such that
  - For all $i \in ob(\mathcal{J})$, $\lambda_i = \mu_i \otimes k$.
  - For all 1-cells $f : i \to j$, $\Psi_f = \Phi_f \otimes k$.

A pseudolimit can also be realised more abstractly in terms of an isomorphism of hom-categories [Lac10a, Section 6.10].

One example of a pseudolimit is the pseudolimit of a single 1-cell, where $\mathcal{J}$ has two objects, a single non-trivial 1-cell going from one object to the other, and no non-trivial 2-cells. In normal category theory the limit of a single morphism is trivial, but the pseudolimit of a 1-cell can often be quite interesting.

In the 2-category $\underline{\mathrm{Cat}}$, the pseudolimit of a functor $F\mathcal{C} \to \mathcal{D}$ is given by a category $\mathcal{L}$ whose objects are triples (c, d, f) where $c \in ob(\mathcal{C})$, $d \in ob(\mathcal{D})$ and $f : Fc \to d$ is invertible. The morphisms of this category are given by pairs of morphisms on the objects such that all the data commutes. Then the cone has 1-cell components given by the obvious projections and a 2-cell component given by a natural transformation whose component at (c, d, f) is f.

2.3 The Coherence Theorem for Bicategories

As mentioned previously, the coherence theorem for bicategories states that every diagram constructed from instances of the constraint cells a, l, and r commutes. However, we cannot show this by the direct method of performing calculations on each individual diagram. Instead, the coherence theorem is proved by showing that every bicategory is biequivalent to a strict 2-category. We start this section by introducing the concepts needed to understand this proof.

2.3.1 Biequivalence

Biequivalences are the correct notion of equivalence between two bicategories. As with the definition of equivalence between categories, there are two equivalent ways of defining biequivalences.

Definition 2.3.1. [Gur12] A **Biequivalence** between two bicategories consists of either:

- A pair of pseudofunctors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ such that $G \otimes F$ is equivalent to the identity in the bicategory $\underline{Bicat}(\mathcal{A}, \mathcal{A})$ and $F \otimes G$ is equivalent to the identity in the bicategory $\underline{Bicat}(\mathcal{B}, \mathcal{B})$.

- A pseudofunctor $F\colon \mathcal{A} \to \mathcal{B}$ which is locally an equivalence and is biessentially surjective in the sense that every object $B \in ob(\mathcal{B})$ is equivalent to some object of the form FA for $A \in ob(\mathcal{A})$.

Given a suitably strong axiom of choice, these two definitions can be proved to be equivalent to each other. In Chapter 3, we will see how these are a specific case of biequivalence in any tricategory. The ability to take a pseudofunctor that is locally an equivalence and is biessentially surjective and complete it to a biequivalence pair (and then to a full biadjoint biequivalence [Gur12, Theorem 3.2]) is crucial for the usability of this thesis' result. The proof makes heavy use of the full structure of a biadjoint biequivalence and so is stated in the form where we begin with biadjoint biequivalences. Even so, given any application where biequivalences arise as biessentially surjective local equivalences, we can still apply our result just by first completing those biequivalences to biadjoint biequivalences.

2.3.2 The Yoneda Embedding

The next ingredient in the coherence theorem is a Yoneda Embedding.

Definition 2.3.2. [Lei98, Section 2.1] For any bicategory $\mathcal{B}$, the **Yoneda Embedding** $Y\colon \mathcal{B} \to \underline{Bicat}(\mathcal{B}^{op}, \underline{Cat})$ is a pseudofunctor that:

- sends an object $B \in ob(\mathcal{B})$ to the pseudofunctor $\mathcal{B}(-, B)\colon \mathcal{B}^{op} \to \underline{Cat}$.
- acts on a hom-category $\mathcal{B}(A, B)$ as the functor that sends a 1-cell f to

$$f \otimes -\colon \mathcal{B}(-, A) \to \mathcal{B}(-, B)$$

  and appropriately on 2-cells.

By a similar calculation to the Yoneda Lemma for categories, this is locally an equivalence.

2.3.3 Coherence Theorem

Theorem 2.3.1 (Coherence Theorem). [Lei98, Section 2.3] Every bicategory is biequivalent to a strict 2-category.

Proof. Consider the Yoneda embedding $Y\colon \mathcal{B} \to \underline{Bicat}(\mathcal{B}^{op}, \underline{Cat})$. It is locally an equivalence.
Now consider the image of the Yoneda embedding $im(Y)$. It is a strict 2-category because $\underline{Bicat}(\mathcal{B}^{op}, \underline{Cat})$ is. We can restrict the Yoneda embedding to $Y'\colon \mathcal{B} \to im(Y)$.
This restriction is locally an equivalence and is surjective on objects by construction. Therefore $\mathcal{B}$ is biequivalent to the strict 2-category $im(Y)$. □

This theorem ensures that every diagram in $\mathcal{B}$ constructed from instances of the constraint cells a, l, and r commutes because it does in the biequivalent 2-category.

2.4 2-Dimensional Monad Theory and Transport of Structure

The theory behind the transport of 2-dimensional structures, many of which arise from consideration of bicategories, is well established. The key paper for this is by Kelly and Lack [KL04].

This paper took the monadic approach to transport of structure: many applicable 2-dimensional structures can be realised as algebras of 2-monads on bicategories. Kelly and Lack studied these 2-monads using the tool of monoidal 2-categories; just as in 1-dimensional category theory, 2-monads can be viewed as monoids in the monoidal 2-category of endo-2-functors, and so their results about monoidal 2-categories also apply to monads.

As we are dealing with higher category theory here, there is a question of strictness versus weakness when considering 2-dimensional monads. Kelly and Lack's more general theorem [KL04, Theorem 6.1] allows transport of the structure of a pseudoalgebra of a 2-monad, and a strict algebra transported by this method can only be ensured to result in a pseudoalgebra. This may be preferred - the principle of isomorphism suggests that the correct notion of algebra of a 2-monad is a pseudoalgebra - but in the case that the strict algebras are preferred, the final result of Kelly and Lack also covers them. Given the extra condition that the 2-monad T is flexible (as defined by Kelly [Kel74]) the category of strict algebras of the flexible 2-monad is equivalent to the category of pseudoalgebras of an adjusted 2-monad T'. This means that the main result of Kelly and Lack also applies to the strict algebras of a flexible monad [KL04, Theorem 6.2].

Chapter 3

Tricategories

The next level of higher category theory is the tricategory. Tricategories were originally introduced by Gordon, Power and Street [GPS95] who also proved the coherence theorem by showing that every tricategory is triequivalent to a Gray-category, a somewhat stricter notion that is still not as strict as a totally strict 3-category.

A slightly different definition of tricategory is offered by Gurski [Gur07]. The key difference between the Gordon, Power and Street definition and the Gurski definition are that when Gordon, Power and Street require that certain constraint 2-cells be equivalences (in the sense of having some weak inverse) Gurski's definition specifies the weak inverse, and also the unit and co-unit needed to make them into an adjoint equivalence. Since any equivalence can be extended to an adjoint equivalence, these definitions are equivalent.

Throughout this chapter, adjoint equivalences in a bicategory given by the data $(f\colon A \to B, f^*\colon B \to A, \eta, \varepsilon)$ will be denoted by $\mathbf{f}\colon A \to B$, using the label of the first component and its source and target. The definitions in this chapter will be presented with all equivalences completed to adjoint equivalences, and can be converted to the original Gordon, Power and Street definition by taking those adjoint equivalences and considering only the primary component.

3.1 Introduction to Tricategories

3.1.1 Definitions

Definition 3.1.1. A **Tricategory** [GPS95][Defn. 2.2] [Gur07, Defn. 3.1.2] $\mathcal{T}$ consists of:

- A collection of **Objects** $ob(\mathcal{T})$.

- For each pair of objects $A, B \in ob(\mathcal{T})$, a **Hom-Bicategory** $\mathcal{T}(A, B)$. The objects of these bicategories are the 1-cells of $\mathcal{T}$, the 1-cells of these bicategories are the 2-cells of $\mathcal{T}$ and the 2-cells of these bicategories are the 3-cells of $\mathcal{T}$

- For each triple of objects $A, B \in ob(\mathcal{T})$, a **Composition Pseudofunctor** $\otimes\colon \mathcal{T}(B, C) \times \mathcal{T}(A, B) \to \mathcal{T}(A, C)$.

- For each object $A \in ob(\mathcal{T})$, an **Identity** given by a pseudofunctor from the terminal bicategory $1_A\colon 1 \to \mathcal{T}(A, A)$.

- For each four objects $A, B, C, D \in ob(\mathcal{T})$, an **Associator** given by an adjunction

$$\begin{tikzcd}
\mathcal{T}(C,D) \times \mathcal{T}(B,C) \times \mathcal{T}(A,B) \arrow[r, "\otimes\times1"] \arrow[d, "1\times\otimes"'] & \mathcal{T}(B,D) \times \mathcal{T}(A,B) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "\mathbf{a}"] \\
\mathcal{T}(C,D) \times \mathcal{T}(A,C) \arrow[r, "\otimes"'] & \mathcal{T}(A,D)
\end{tikzcd}$$

- For each pair of objects $A, B \in ob(\mathcal{T})$, a **Left Unitor** given as an adjunction

$$\begin{tikzcd}
& \mathcal{T}(B,B) \times \mathcal{T}(A,B) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathbf{l}"] & \\
1 \times \mathcal{T}(A,B) \arrow[ur, "1_B\times1"] \arrow[rr, "\cong"'] & {} & \mathcal{T}(A,B)
\end{tikzcd}$$

- For each pair of objects $A, B \in ob(\mathcal{T})$, a **Right Unitor** given as an adjunction

$$\begin{tikzcd}
& \mathcal{T}(A,B) \times \mathcal{T}(A,A) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathbf{r}"] & \\
\mathcal{T}(A,B) \times 1 \arrow[ur, "1\times1_A"] \arrow[rr, "\cong"'] & {} & \mathcal{T}(A,B)
\end{tikzcd}$$

- For every five objects $A, B, C, D, E \in ob(\mathcal{T})$, a **Pentagonator** π given as an invertible modification between the two pasting diagrams

$$\begin{tikzcd}
& \mathcal{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, "1\times\otimes\times1"] & & \mathcal{T}^3 \arrow[dr, "\otimes\times1"] & \\
\mathcal{T}^3 \arrow[dr, "1\times\otimes"'] \arrow[r, phantom, "\Swarrow 1\times a"] & & \mathcal{T}^3 \arrow[rr, "\otimes\times1"] \arrow[dl, "1\times\otimes"] \arrow[u, phantom, "\Swarrow a\times1"] \arrow[dr, phantom, "\Swarrow a"] & & \mathcal{T}^2 \arrow[dl, "\otimes"] \\
& \mathcal{T}^2 \arrow[rr, "\otimes"'] & & \mathcal{T} &
\end{tikzcd}
\overset{\pi}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, phantom, "="] & & \mathcal{T}^3 \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times1"] & \\
\mathcal{T}^3 \arrow[rr, "\otimes\times1"] \arrow[dr, "1\times\otimes"'] & & \mathcal{T}^2 \arrow[dr, "\otimes"] & \Leftarrow a & \mathcal{T}^2 \arrow[dl, "\otimes"] \\
& \mathcal{T}^2 \arrow[rr, "\otimes"'] \arrow[ur, phantom, "\Swarrow a"] & & \mathcal{T} &
\end{tikzcd}$$

where $\mathcal{T}^4$ is an abbreviation of $\mathcal{T}(D, E) \times \mathcal{T}(C, D) \times \mathcal{T}(B, C) \times \mathcal{T}(A, B)$ and similarly for the other abbreviations.

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, a **Middle Triangulator** μ given as an invertible modification between pasting diagrams

$$
\begin{tikzcd}
\mathcal{T}^2 \arrow[dr, "1\times 1_B\times 1"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & \arrow[d, phantom, "\Downarrow r^*\times 1"] & \\
\arrow[r, phantom, "\Leftarrow l\times 1"] & \mathcal{T}^3 \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"] \arrow[dr, phantom, "\Swarrow a"] & \mathcal{T}^2 \arrow[d, "\otimes"] \\
& \mathcal{T}^2 \arrow[r, "\otimes"'] & \mathcal{T}
\end{tikzcd}
\overset{\mu}{\Longrightarrow}
\begin{tikzcd}
\mathcal{T}^2 \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
* & \arrow[l, phantom, "\Leftarrow 1"] & \mathcal{T}^2 \arrow[d, "\otimes"] \\
& \mathcal{T}^2 \arrow[r, "\otimes"'] & \mathcal{T}
\end{tikzcd}
$$

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, a **Left Triangulator** λ given as an invertible modification between pasting diagrams

$$
\begin{tikzcd}
& \mathcal{T}^3 \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow l\times 1"] & \\
\mathcal{T}^2 \arrow[ur, "1_C\times 1\times 1"] \arrow[rr, "1"'] \arrow[d, "\otimes"'] & = & \mathcal{T}^2 \arrow[d, "\otimes"] \\
\mathcal{T} \arrow[rr, "1"'] & & \mathcal{T}
\end{tikzcd}
\overset{\lambda}{\Longrightarrow}
\begin{tikzcd}
& \mathcal{T}^3 \arrow[dr, "\otimes\times 1"] \arrow[d, "1\times\otimes"] & \\
\mathcal{T}^2 \arrow[ur, "1_C\times 1\times 1"] \arrow[d, "\otimes"'] \arrow[r, phantom, "="] & \mathcal{T}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] \arrow[r, phantom, "\Downarrow a"] & \mathcal{T}^2 \arrow[d, "\otimes"] \\
\mathcal{T} \arrow[ur, "1_C\times 1"] \arrow[rr, "1"'] & & \mathcal{T}
\end{tikzcd}
$$

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, a **Right Triangulator** ρ given as an invertible modification between pasting diagrams

$$
\begin{tikzcd}
\mathcal{T} \arrow[rr, "1"] & = & \mathcal{T} \\
\mathcal{T}^2 \arrow[u, "\otimes"] \arrow[rr, "1"] \arrow[dr, "1\times 1\times 1_A"'] & \arrow[d, phantom, "\Downarrow 1\times r^*"] & \mathcal{T}^2 \arrow[u, "\otimes"'] \\
& \mathcal{T}^3 \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
\overset{\rho}{\Longrightarrow}
\begin{tikzcd}
\mathcal{T} \arrow[rr, "1"] \arrow[dr, "1\times 1_A"] & \arrow[d, phantom, "\Downarrow r^*"] & \mathcal{T} \\
\mathcal{T}^2 \arrow[u, "\otimes"] \arrow[r, phantom, "="] \arrow[dr, "1\times 1\times 1_A"'] & \mathcal{T}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\Downarrow a"] & \mathcal{T}^2 \arrow[u, "\otimes"'] \\
& \mathcal{T}^3 \arrow[u, "\otimes\times 1"] \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
$$

satisfying the following three axioms. In the following diagrams we will replace $\otimes$ by concatenation for compactness. We should also note that, as the arrows in these diagrams come from the hom-bicategories, their composition is only weakly associative and unital. We therefore need to choose an association for them. By the coherence theorem for bicategories we get a unique way to transform our chosen association into any other, so the requirements that the following diagrams are equal still makes sense.

- For every five composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{j} E \xrightarrow{k} F$$

the two diagrams

$$\begin{tikzcd}
 & & & & (k(j(hg)))f \arrow[drr, "a"] & & & & \\
 & & (k((jh)g))f \arrow[urr, "(1a)1"] \arrow[drr, "a"] \arrow[rrrr, phantom, "\cong"] \arrow[ddr, phantom, "\Downarrow\pi"] & & & & k((j(hg))f) \arrow[drr, "1a"] \arrow[dd, phantom, "\Downarrow 1\pi"] & & \\
((k(jh))g)f \arrow[urr, "a1"] \arrow[drr, "a"] & & & & k(((jh)g)f) \arrow[urr, "1(a1)"] \arrow[drr, "1a"] & & & & k(j((hg)f)) \arrow[d, "1(1a)"] \\
(((kj)h)g)f \arrow[u, "(a1)1"] \arrow[drr, "a"] \arrow[rr, phantom, "\cong"] & & (k(jh))(gf) \arrow[rrrr, "a"] \arrow[drrrr, phantom, "\Downarrow\pi"] & & & & k((jh)(gf)) \arrow[rr, "1a"] & & k(j(h(gf))) \\
 & & ((kj)h)(gf) \arrow[u, "a1"] \arrow[rrrr, "a"] & & & & (kj)(h(gf)) \arrow[urr, "a"] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & & (k(j(hg)))f \arrow[drr, "a"] & & & & \\
 & & (k((jh)g))f \arrow[urr, "(1a)1"] \arrow[d, phantom, "\Downarrow\pi"] & & & & k((j(hg))f) \arrow[drr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & & \\
((k(jh))g)f \arrow[urr, "a1"] & & & & ((kj)(hj))f \arrow[uu, "a1"] \arrow[drr, "a"] \arrow[dd, phantom, "\Downarrow\pi"] & & & & k(j((hg)f)) \arrow[d, "1(1a)"] \\
(((kj)h)g)f \arrow[u, "(a1)1"] \arrow[urrrr, "a1"] \arrow[drr, "a"] & & & & & & (kj)((hg)f) \arrow[urr, "a"] \arrow[d, "1a"] \arrow[rr, phantom, "\cong"] & & k(j(h(gf))) \\
 & & ((kj)h)(gf) \arrow[rrrr, "a"] & & & & (kj)(h(gf)) \arrow[urr, "a"] & &
\end{tikzcd}$$

are equal, where the unmarked isomorphisms are the naturality isomorphisms of the associator.

- For every three composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the two diagrams

$$
\begin{tikzcd}
 & & & (h(1_Cg))f \arrow[d, dashed, "a"] \arrow[dr, "(1l)1"] & \\
 & (h1_C)g)f \arrow[urr, "a1"] \arrow[d, dashed, "a"] & \arrow[d, phantom, "\Downarrow\pi"] & h(1_Cg)f) \arrow[d, dashed, "1a"] \arrow[dr, dashed, "1(l1)"] \arrow[r, phantom, "\cong"] & (hg)f \arrow[d, "a"] \\
(hg)f \arrow[ur, "(r^*1)1"] \arrow[dr, "a"'] \arrow[r, phantom, "\cong"] & (h1_C)(gf) \arrow[rr, dashed, "a"] & \arrow[d, phantom, "\Rightarrow\mu"] & h(1_C(gf)) \arrow[r, dashed, "1l"] \arrow[ur, phantom, "\Leftarrow 1\lambda"] & h(gf) \\
 & h(gf) \arrow[u, "r^*1"] \arrow[urrr, "1"'] & & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & & (h(1_Cg))f \arrow[dr, "(1l)1"] & \\
 & (h1_C)g)f \arrow[ur, "a1"] & \arrow[d, phantom, "\Downarrow\mu 1"] & (hg)f \arrow[d, "a"] \\
(hg)f \arrow[ur, "(r^*1)1"] \arrow[urrr, "1"'] \arrow[dr, "a"'] & & \cong & h(gf) \\
 & h(gf) \arrow[urr, "1"'] & &
\end{tikzcd}
$$

are equal.

- For every three composable 1-cells

$$
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D
$$

the two diagrams

$$
\begin{tikzcd}
 & h((g1_B)f) \arrow[drr, "1a"] & & & \\
h(gf) \arrow[ur, "1(r^*1)"] \arrow[r, phantom, "\cong"] & (h(g1_B))f \arrow[u, dashed, "a"'] & \arrow[d, phantom, "\Downarrow\pi"] & h(g(1_Bf)) \arrow[dr, "1(1l)"] & \\
(hg)f \arrow[u, "a"] \arrow[ur, dashed, "(1r^*)1"] \arrow[r, dashed, "r^*1"] \arrow[drrr, "1"'] & ((hg)1_B)f \arrow[u, dashed, "a1"'] \arrow[ul, phantom, "\Rightarrow\rho 1"] \arrow[rr, dashed, "a"] & \arrow[d, phantom, "\Leftarrow\mu"] & (hg)(1_Bf) \arrow[u, dashed, "a"'] \arrow[d, dashed, "1l"] \arrow[r, phantom, "\cong"] & h(gf) \\
 & & & h(gf) \arrow[ur, "a"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & h((g1_B)f) \arrow[dr, "1a"] & & \\
h(gf) \arrow[ur, "1(r^*1)"] \arrow[rrrd, "1"'] & & h(g(1_Bf)) \arrow[l, phantom, "1\mu\Downarrow"] \arrow[dr, "1(1l)"] & \\
(hg)f \arrow[u, "a"] \arrow[drr, "1"'] & & \cong & h(gf) \\
 & & h(gf) \arrow[ur, "a"'] &
\end{tikzcd}
$$

are equal.

Much of the data given in the definition of a tricategory arises naturally by introducing invertible constraint 3-cells mediating axioms of bicategories. For example, the pentagonator replaces the pentagon identity that holds for bicategories. However, there are complications in this framework. Where only one triangle identity was needed in the definition of bicategories, three triangulator modifications are needed.

The first of the three axioms is known as the non-abelian 4-cocycle condition [GPS95, TA1], and is given by a version of a diagram known as K_5. However, the two axioms related to the unital conditions were first introduced by Gordon, Power and Street [GPS95] for their definition. Identifying these as the key axioms is crucial to the definition of a tricategory.

By the theory of mates under adjunctions, we can take the dual of a tricategory $\mathcal{T}^{op}$ [GPS95, Remark 2.4] which has the same objects, hom-bicategories $\mathcal{T}^{op}(A,B) = \mathcal{T}(B,A)$, the same identities and the adjoint equivalences being the opposites of those in $\mathcal{T}$. Then, the theory of mates under adjunctions in bicategories allows us to find suitable modifications and show that they obey the same axioms.

A tricategory is defined as being strict if each adjoint equivalence is the identity adjoint equivalence and modifications consist of the unique coherence isomorphisms in the hom-bicategories.

Next, we consider what the appropriate notion of morphism between tricategories should be.

Definition 3.1.2. A **Trifunctor** [GPS95, Defn. 3.2] [Gur07, Defn. 3.3.1] between two tricategories $F\colon \mathcal{S} \to \mathcal{T}$ consists of:

- A function $ob(\mathcal{S}) \to ob(\mathcal{T})$.
- For each pair of objects $A, B \in ob(\mathcal{S})$, a pseudofunctor $\mathcal{S}(A,B) \to \mathcal{T}(FA, FB)$.

- For each triple of objects $A, B, C \in ob(\mathcal{S})$, a **Compositor** given by an adjoint equivalence

$$
\begin{tikzcd}
\mathcal{S}(B,C) \times \mathcal{S}(A,B) \arrow[r, "F\times F"] \arrow[d, "\otimes"] & \mathcal{T}(FB,FC) \times \mathcal{T}(FA,FB) \arrow[d, "\otimes"] \arrow[dl, phantom, "\Leftarrow \chi"] \\
\mathcal{S}(A,C) \arrow[r, "F"'] & \mathcal{T}(FA,FC)
\end{tikzcd}
$$

- For each object $A \in ob(\mathcal{S})$, a **Unitor** given by an adjoint equivalence

$$
\begin{tikzcd}
1 \arrow[rr, "1_{FA}"] \arrow[dr, "1_A"'] & \Downarrow \iota & \mathcal{T}(FA,FA) \\
& \mathcal{S}(A,A) \arrow[ur, "F"'] &
\end{tikzcd}
$$

- For every four objects $A, B, C, D \in ob(\mathcal{S})$, an invertible modification between the two pasting diagrams

$$
\begin{tikzcd}
& \mathcal{S}^4 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & \arrow[d, phantom, "\Swarrow \chi\times 1"] & \mathcal{T}^3 \arrow[dr, "\otimes\times 1"] & \\
\mathcal{S}^2 \arrow[dr, "\otimes"'] & \Swarrow a & \mathcal{S}^2 \arrow[rr, "F\times F"] \arrow[dl, "\otimes"] & \Swarrow \chi & \mathcal{T}^2 \arrow[dl, "\otimes"] \\
& \mathcal{S} \arrow[rr, "F"'] & & \mathcal{T} &
\end{tikzcd}
\overset{\omega}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{S}^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] & \Swarrow 1\times\chi & \mathcal{T}^3 \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & \\
\mathcal{S}^2 \arrow[rr, "F\times F"'] \arrow[dr, "\otimes"'] & & \mathcal{T}^2 \arrow[dr, "\otimes"] & \Leftarrow a & \mathcal{T}^2 \arrow[dl, "\otimes"] \\
& \mathcal{S} \arrow[rr, "F"'] & \Swarrow \chi & \mathcal{T} &
\end{tikzcd}
$$

- For each pair of objects $A, B \in ob(\mathcal{S})$, an invertible modification between pasting diagrams

$$
\begin{tikzcd}
& \mathcal{T}^2 \arrow[dr, "\otimes"] & \\
\mathcal{T} \arrow[ur, "1_{FB}\times 1"] \arrow[r, phantom, "\overset{\iota\times 1}{\Rightarrow}"] & \mathcal{S}^2 \arrow[u, "F\times F"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\Searrow \chi"] \arrow[d, phantom, "\Downarrow l"] & \mathcal{T} \\
\mathcal{S} \arrow[u, "F"] \arrow[ur, "1_B\times 1"] \arrow[rr, "1"'] & & \mathcal{S} \arrow[u, "F"']
\end{tikzcd}
\overset{\gamma}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{T}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] & \\
\mathcal{T} \arrow[ur, "1_{FB}\times 1"] \arrow[rr, "1"'] & = & \mathcal{T} \\
\mathcal{S} \arrow[u, "F"] \arrow[rr, "1"'] & & \mathcal{S} \arrow[u, "F"']
\end{tikzcd}
$$

- For each pair of objects $A, B \in ob(\mathcal{S})$, an invertible modification between pasting diagrams

$$
\begin{tikzcd}
\mathcal{T} \arrow[rr, "1"] & & \mathcal{T} \\
\mathcal{S} \arrow[u, "F"] \arrow[rr, "1"] \arrow[dr, "1\times 1_A"'] & \Downarrow l & \mathcal{S} \arrow[u, "F"'] \\
& \mathcal{S}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
\overset{\delta}{\Rrightarrow}
\begin{tikzcd}
\mathcal{T} \arrow[rr, "1"] \arrow[dr, "1\times 1_{FA}"] & \Downarrow l & \mathcal{T} \\
\mathcal{S} \arrow[u, "F"] \arrow[dr, "1\times 1_A"'] \arrow[r, phantom, "\overset{1\times\iota}{\Searrow}"] & \mathcal{T}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{S} \arrow[u, "F"'] \\
& \mathcal{S}^2 \arrow[u, "F\times F"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

satisfying the two axioms

1. For every four composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

the two diagrams

$$\begin{tikzcd}
 & & & F(((kh)g)f) \arrow[d, phantom, "\cong"] \arrow[drr, "F(a1)"] & & & \\
 & F((kh)g)Ff \arrow[urr, "\chi"] \arrow[rr, "F(a1)"] & & F(k(hg))Ff \arrow[rr, "\chi"] \arrow[drr, phantom, "\Downarrow\omega"] & & F((k(hg))f) \arrow[dr, "Fa"] & \\
(F(kh)Fg)Ff \arrow[ur, "\chi 1"] & \arrow[r, phantom, "\Downarrow\omega 1"] & & (FkF(hg))Ff \arrow[u, "\chi 1"'] \arrow[dr, "a"] & & & F(k((hg)f)) \arrow[d, "F(1a)"] \\
((FkFh)Fg)Ff \arrow[u, "(\chi 1)1"] \arrow[r, "a1"] \arrow[dr, "a"'] & (Fk(FhFg))Ff \arrow[urr, "1(\chi 1)"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\pi"] & \arrow[r, phantom, "\cong"] & \arrow[d, phantom, "\Downarrow\omega 1"] & Fk(F(hg)Ff) \arrow[r, "1\chi"] & FkF((hg)f) \arrow[ur, "\chi"] \arrow[d, "1Fa"] \arrow[r, phantom, "\cong"] & F(k(h(gf))) \\
 & (FkFh)(FgFf) \arrow[dr, "a"'] & Fk((FhFg)Ff) \arrow[urr, "1(\chi 1)"] \arrow[d, "1a"] & & & FkF(h(gf)) \arrow[ur, "\chi"'] & \\
 & & Fk(Fh(FgFf)) \arrow[rr, "1(1\chi)"] & & Fk(FhF(gf)) \arrow[ur, "1\chi"] & &
\end{tikzcd}$$

and

$$
\begin{tikzcd}
 & & & F(((kh)g)f) \arrow[drr, "F(a1)"] \arrow[ddr, "Fa"] & & & \\
 & F((kh)g)Ff \arrow[urr, "\chi"] & {} \arrow[dr, phantom, "\Swarrow\omega"] & & & F((k(hg))f) \arrow[dr, "Fa"] \arrow[d, phantom, "\Downarrow F\pi"] & \\
(F(kh)Fg)Ff \arrow[ur, "\chi1"] \arrow[drr, "a"] & & & & F((kh)(gf)) \arrow[drr, "Fa"] & {} & F(k((hg)f)) \arrow[d, "F(1a)"] \\
((FkFh)Fg)Ff \arrow[u, "(\chi1)1"] \arrow[dr, "a"'] \arrow[r, phantom, "\cong"] & {} & F(kh)(FgFf) \arrow[r, "1\chi"] \arrow[d, phantom, "\cong"] & F(kh)F(gf) \arrow[ur, "\chi"] & {} \arrow[d, phantom, "\Downarrow\omega"] & & F(k(h(gf))) \\
 & (FkFh)(FgFf) \arrow[ur, "\chi1"] \arrow[rr, "1\chi"] \arrow[dr, "a"'] & {} \arrow[dr, phantom, "\cong"] & (FkFh)F(gf) \arrow[u, dashed, "\chi1"'] \arrow[dr, "a"] & {} & FkF(h(gf)) \arrow[ur, "\chi"'] & \\
 & & Fk(Fh(FgFf)) \arrow[rr, "1(1\chi)"'] & {} & Fk(FhF(gf)) \arrow[ur, "1\chi"'] & &
\end{tikzcd}
$$

are equal.

2. For every pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

the two diagrams

$$
\begin{tikzcd}[column sep=small]
 & & F((g1_B)f) \arrow[rrr, "Fa"] & \arrow[d, phantom, "\Rightarrow\omega"] & & F(g(1_Bf)) \arrow[ddr, "F(1l)"] & \\
 & & & {} & & FgF(1_Bf) \arrow[u, dashed, "\chi"] \arrow[dddr, "1Fl"] & \\
F(gf) \arrow[uurr, "F(r^*1)"] \arrow[r, phantom, "\cong"] & {} & F(g1_B)Ff \arrow[uu, "\chi"] \arrow[d, phantom, "\Rightarrow\delta 1"] & (FgF1_B)Ff \arrow[l, "\chi 1"'] \arrow[rr, "a"] \arrow[d, phantom, "\cong"] & & Fg(F1_BFf) \arrow[u, dashed, "1\chi"] \arrow[d, phantom, "\Rightarrow 1\gamma"] \arrow[r, phantom, "\cong"] & F(gf) \\
 & & (Fg1_{FB})Ff \arrow[ur, "(1\iota)1"] \arrow[r, "a"] & Fg(1_{FB}Ff) \arrow[urr, "1(\iota 1)"] \arrow[drrr, "1l"] \arrow[d, phantom, "\Downarrow\mu"] & & {} & \\
FgFf \arrow[uu, "\chi"] \arrow[uurr, "Fr^*1"] \arrow[urr, "r^*1"] \arrow[rrrrrr, "1"'] & & & {} & & & FgFf \arrow[uu, "\chi"']
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & F((g1_B)f) \arrow[r, "Fa"] & F(g(1_Bf)) \arrow[dr, "F(1l)"] \arrow[d, phantom, "\Downarrow F\mu"] & \\
F(gf) \arrow[ur, "F(r^*1)"] \arrow[rrr, "1"'] & & {} & F(gf) \\
 & & \arrow[u, phantom, "\cong"] & \\
FgFf \arrow[uu, "\chi"] \arrow[rrr, "1"'] & & & FgFf \arrow[uu, "\chi"']
\end{tikzcd}
$$

are equal.

The trifunctor is the structure that this thesis aims to transport. It is therefore crucial to understand this definition.

As with pseudofunctors, it is possible to define strictness, laxness and oplaxness for trifunctors. The definition for lax functor [GPS95, Defn. 3.1] replaces the pseudofunctors on the hom-bicategories with lax functors, replaces the adjoint equivalences with lax natural transformations and removes the requirement that the modifications be invertible.

A trifunctor is strict [Gur07, Defn. 3.3.3] if it is locally strict, χ and ι are identity adjoint equivalences, F maps the adjoint equivalences $\mathbf{a}$, $\mathbf{l}$ and $\mathbf{r}$ of the source tricategory to those of the target tricategory and the modifications are constructed from unique coherence cells.

We will also be considering how to lift the object-indexed biequivalences so that they become a biequivalence between the original and transported trifunctor. Therefore we also consider the morphisms between trifunctors, the tritransformations.

Definition 3.1.3. Given two trifunctors $F, G\colon \mathcal{S} \to \mathcal{T}$, a **Tritransformation** [GPS95, 3.3] [Gur13, Defn. 4.16] $\theta\colon F \Rightarrow G$ consists of:

- For each object A of $\mathcal{S}$ a 1-cell $\theta_A\colon FA \to GA$.
- For each pair of objects $A, B \in ob(\mathcal{S})$, an adjoint equivalence

$$\begin{tikzcd}
\mathcal{S}(A,B) \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{T}(FA,FB) \arrow[d, "\theta_B\otimes -"] \arrow[dl, phantom, "\swarrow\theta"] \\
\mathcal{T}(GA,GB) \arrow[r, "-\otimes\theta_A"'] & \mathcal{T}(FA,GB)
\end{tikzcd}$$

- For each triple of objects $A, B, C \in ob(\mathcal{S})$, an invertible modification Π that modulates how θ interacts with the compositors of the trifunctors. Its component at a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ is given by

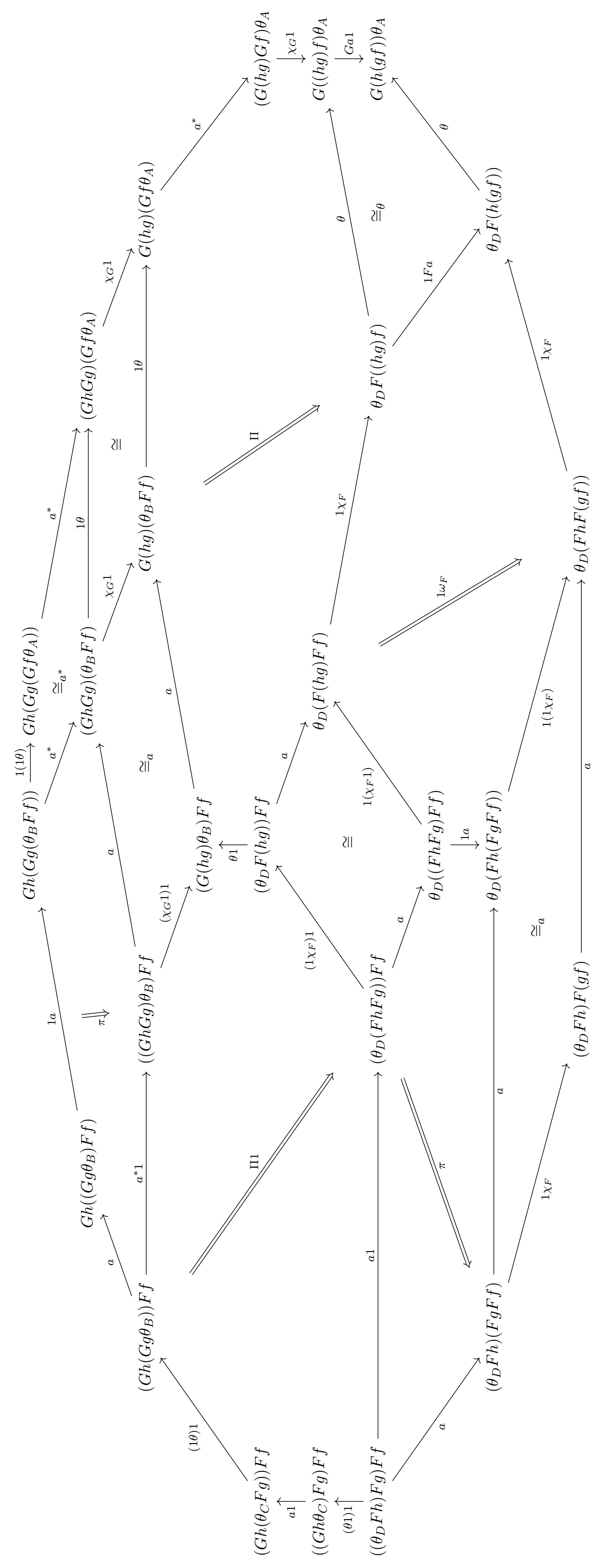
$$\begin{tikzcd}
& Gg(\theta_B Ff) \arrow[r, "1\theta"] & Gg(Gf\theta_A) \arrow[dr, "a^*"] & \\
(Gg\theta_B)Ff \arrow[ur, "a"] & & & (GgGf)\theta_A \arrow[d, "\chi_G 1"] \\
(\theta_C Fg)Ff \arrow[u, "\theta 1"] \arrow[dr, "a"'] & & & G(gf)\theta_A \\
& \theta_C(FgFf) \arrow[r, "1\chi_F"'] \arrow[uuu, phantom, "\Downarrow\Pi"] & \theta_C F(gf) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

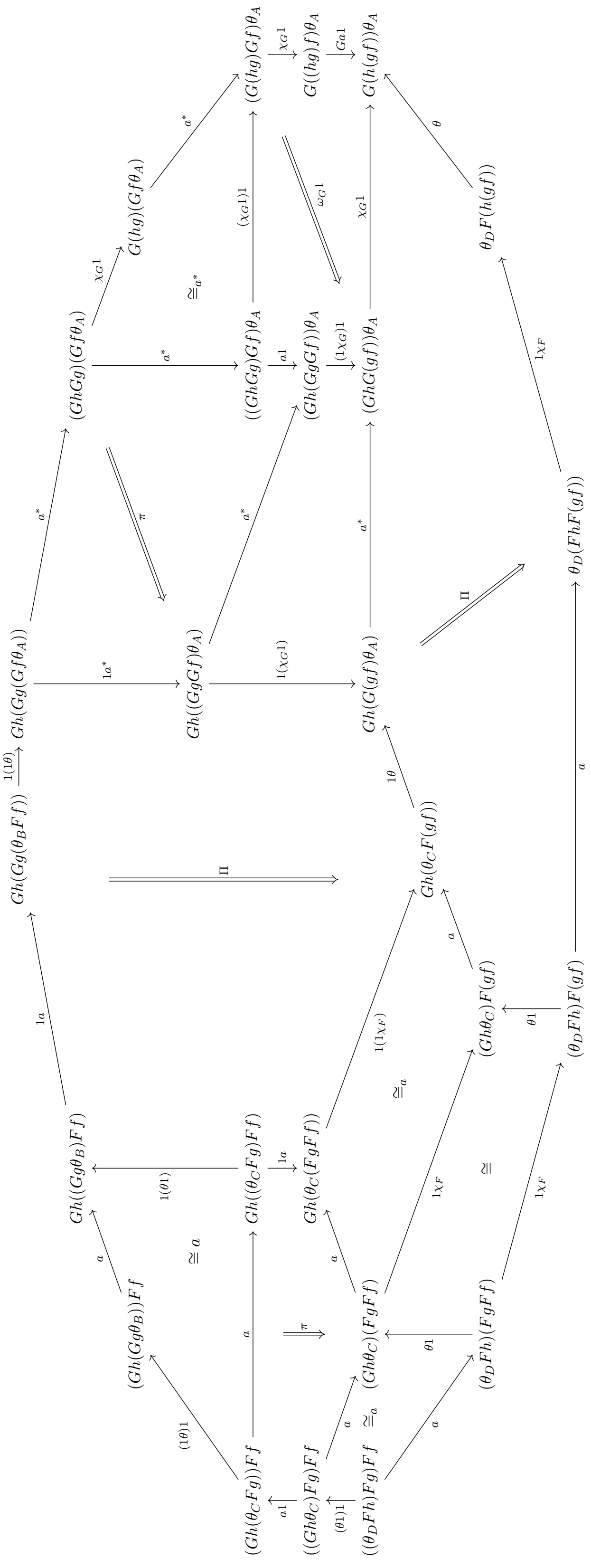
- For each object A of $\mathcal{S}$, an invertible modification M that modulates how θ interacts with the unitors.

$$\begin{tikzcd}
& 1 \arrow[dl, bend right, "1_A"'] \arrow[d, "1_{FA}"] \arrow[ddr, bend left, "\theta_A"] \arrow[dl, phantom, "\Leftarrow\iota_F"] \arrow[ddr, phantom, "\Leftarrow r^*"] & \\
\mathcal{S}(A,A) \arrow[r, "F"'] \arrow[d, "G"'] & \mathcal{T}(FA,FA) \arrow[dr, "\theta_A\otimes -"] \arrow[dl, phantom, "\swarrow\theta"] & \\
\mathcal{T}(GA,GA) \arrow[rr, "-\otimes\theta_A"'] & & \mathcal{T}(FA,GA)
\end{tikzcd}
\quad \overset{M}{\Rrightarrow} \quad
\begin{tikzcd}
& 1 \arrow[dl, bend right, "1_A"'] \arrow[ddl, bend left, "1_{GA}"] \arrow[ddr, bend left, "\theta_A"] \arrow[dl, phantom, "\Leftarrow\iota_G"] \arrow[ddr, phantom, "\swarrow l^*"] & \\
\mathcal{S}(A,A) \arrow[d, "G"'] & & \\
\mathcal{T}(GA,GA) \arrow[rr, "-\otimes\theta_A"'] & & \mathcal{T}(FA,GA)
\end{tikzcd}$$

These are required to obey the following three axioms.

1. For every triple of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ the following two diagrams are equal:


$Gh(Gg(\theta_B Ff))$
$Gh(Gg(Gf\theta_A))$
$Gh((Gg\theta_B)Ff)$
$(Gh(Gg\theta_B))Ff$
$((GhGg)\theta_B)Ff$
$(GhGg)(\theta_B Ff)$
$(GhGg)(Gf\theta_A)$
$G(hg)(\theta_B Ff)$
$G(hg)(Gf\theta_A)$
$(G(hg)Gf)\theta_A$
$G((hg)f)\theta_A$
$G(h(gf))\theta_A$
$(G(hg)\theta_B)Ff$
$(\theta_D F(hg))Ff$
$\theta_D(F(hg)Ff)$
$\theta_D F((hg)f)$
$\theta_D F(h(gf))$
$(Gh(\theta_C Fg))Ff$
$((Gh\theta_C)Fg)Ff$
$((\theta_D Fh)Fg)Ff$
$(\theta_D(FhFg))Ff$
$\theta_D((FhFg)Ff)$
$\theta_D(Fh(FgFf))$
$\theta_D(FhF(gf))$
$(\theta_D Fh)(FgFf)$
$(\theta_D Fh)F(gf)$

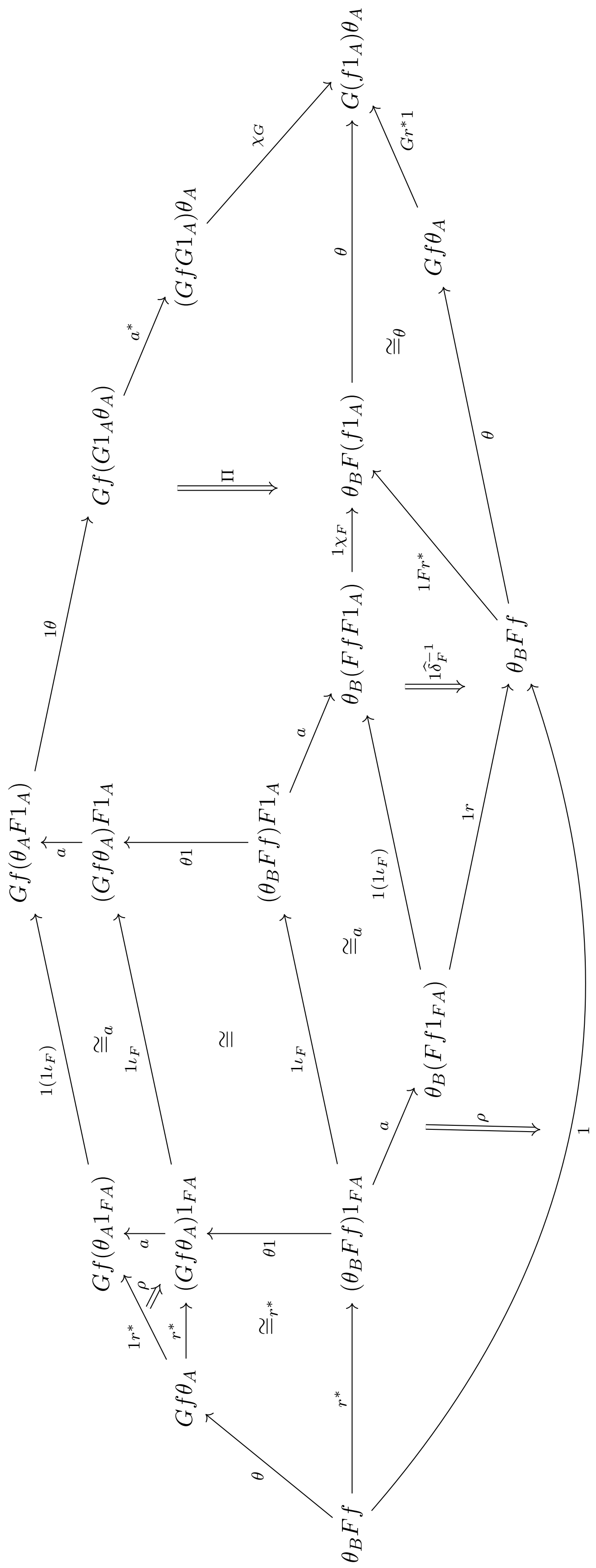


2. For every 1-cell $f\colon A \to B$, the following two diagrams are equal.

$$
\begin{tikzcd}
 & & & & G1_B(\theta_B F f) \arrow[dr, "1\theta"] & & & & \\
 & & & (G1_B\theta_B)Ff \arrow[ur, "a"] & & G1_B(Gf\theta_A) \arrow[dr, "a^*"] & & & \\
 & & (\theta_B F 1_B)Ff \arrow[ur, "\theta 1"] \arrow[rr, "a"] & & \theta_B(F1_B F f) \arrow[drr, "1\chi_F"] & & (G1_B G f)\theta_A \arrow[dr, "\chi_G 1"] & & \\
 & (\theta_B 1_{FB})Ff \arrow[ur, "(1\iota_F)1"] \arrow[rr, "a"] & & \theta_B(1_{FB}Ff) \arrow[ur, "1(\iota_F 1)"] \arrow[drrr, "1l"'] & & & \theta_B F(1_B f) \arrow[r, "\theta"] \arrow[d, "1Fl"] & G(1_B f)\theta_A \arrow[dr, "Gl1"] & \\
\theta_B F f \arrow[ur, "r^* 1"] \arrow[rrrrrr, "1"'] \arrow[drr, "l^* 1"'] & & & & & & \theta_B F f \arrow[rr, "\theta"] & & Gf\theta_A \\
 & & (1_{GB}\theta_B)Ff \arrow[rr, "a"'] & & 1_{GB}(\theta_B F f) \arrow[urr, "l"] \arrow[rr, "1\theta"'] & & 1_{GB}(Gf\theta_A) \arrow[urr, "l"'] & &
\arrow[from=1-5, to=3-5, phantom, "\Downarrow\Pi"]
\arrow[from=3-3, to=4-4, phantom, "\cong_a"]
\arrow[from=4-3, to=5-4, phantom, "\Rightarrow\mu"]
\arrow[from=3-5, to=4-6, phantom, "\Leftarrow 1\gamma"]
\arrow[from=4-7, to=5-8, phantom, "\cong_\theta"]
\arrow[from=5-7, to=6-7, phantom, "\cong_l"]
\arrow[from=5-4, to=6-4, phantom, "\Downarrow\lambda"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & G1_B(\theta_B Ff) \arrow[dr, "1\theta"] & & & & \\
 & & & (G1_B\theta_B)Ff \arrow[ur, "a"] & & G1_B(Gf\theta_A) \arrow[dr, "a^*"] & & & \\
 & & (\theta_B F1_B)Ff \arrow[ur, "\theta 1"] & & & & (G1_B Gf)\theta_A \arrow[dr, "\chi_G 1"] & & \\
 & (\theta_B 1_F B)Ff \arrow[ur, "(1\iota_F)1"] \arrow[drr, phantom, "\Rightarrow M1"] & & & \arrow[uul, phantom, "\cong_a"] \arrow[u, phantom, "\cong"] & \arrow[u, phantom, "\cong_{a^*}"] & \arrow[ur, phantom, "\Rightarrow \gamma 1"] & G(1_B f)\theta_A \arrow[dr, "Gl1"] & \\
\theta_B Ff \arrow[ur, "r^*1"] \arrow[drr, "l^*1"'] & & & & & & (1_G BGf)\theta_A \arrow[uu, "(\iota_G 1)1"'] \arrow[rr, "l1"] & \arrow[ur, phantom, "\Rightarrow \lambda"] & Gf\theta_A \\
 & & (1_G B\theta_B)Ff \arrow[uuuur, "(\iota_G 1)1"] \arrow[rr, "a"'] & & 1_G B(\theta_B Ff) \arrow[uuuuu, "\iota_G 1"] \arrow[rr, "1\theta"'] & & 1_G B(Gf\theta_A) \arrow[uuuul, bend left, "\iota_G 1"] \arrow[u, "a^*"] \arrow[urr, "l"'] & &
\end{tikzcd}
$$

3. For every 1-cell $f\colon A \to B$, the following two diagrams are equal.



$$
\begin{tikzcd}
 & & & Gf(\theta_A F1_A) \arrow[drr, "1\theta"] \arrow[dd, phantom, "\Downarrow M"] & & & & \\
 & & Gf(\theta_A 1_{FA}) \arrow[ur, "1(1\iota_F)"] & & & Gf(G1_A\theta_A) \arrow[dr, "a^*"] \arrow[dd, phantom, "\cong_{a^*}"] & & \\
 & Gf\theta_A \arrow[ur, "1r^*"] \arrow[rr, "1l^*"] \arrow[ddrrrr, "1"] & & Gf(1_{GA}\theta_A) \arrow[urr, "1(\iota_G 1)"'] \arrow[dr, "a^*"] \arrow[d, phantom, "\Downarrow \mu"] & & & (GfG1_A)\theta_A \arrow[ddr, "\chi_G"] \arrow[ddl, phantom, "\Downarrow \widehat{\delta}_G^{-1}1"] & \\
 & & & {} & (Gf1_{GA})\theta_A \arrow[uurr, "(1\iota_G)1"] \arrow[dr, "r1"] & & & \\
\theta_B Ff \arrow[uur, "\theta"] \arrow[ddrrr, bend right, "1"'] & & & & & & & G(f1_A)\theta_A \\
 & & & & & Gf\theta_A \arrow[urr, "Gr^*1"'] & & \\
 & & & \theta_B Ff \arrow[uurr, "\theta"'] \arrow[uul, phantom, "\cong"] & & & &
\end{tikzcd}
$$

(Note that the $\widehat{\delta}$ cells used in this axiom are mates (see Definition 2.2.3) of the modification δ defined as part of the data for a trifunctor under the adjunction

$r \dashv r^*$.)

The final key definition we will need for this thesis is that of a biadjoint biequivalence. A biequivalence in a tricategory is the generalisation of the notion of biequivalence between bicategories (see Section 2.3.1) and just as equivalences are the correct notion of equivalence between objects in a bicategory, the correct notion of equivalence between objects in a tricategory is that of biequivalence.

Also, just as every equivalence in a bicategory can be extended to an adjoint equivalence, every biequivalence 1-cell S (in the sense of having some 1-cell Ψ with $S \otimes \Psi$ and $\Psi \otimes S$ both equivalent to the identity) can be extended to a biadjoint biequivalence [Gur12, Theorem 4.5] as given by the following definition. This is incredibly useful, as the proofs of this thesis use the full structure of a biadjoint biequivalence, but we are still able to apply the transport of structure result even when only given a family of biequivalences just by choosing a way to complete them to full biadjoint biequivalences.

Definition 3.1.4. [Gur12, Definition 2.3] A **Biadjoint Biequivalence** between two objects A and B in a tricategory $\mathcal{T}$ consists of the following pieces of data:

- A pair of 1-cells $S\colon A \to B$ and $\Psi\colon B \to A$.
- 2-cells $\eta\colon 1_B \Rightarrow S \otimes \Psi$ and $\eta^*\colon S \otimes \Psi \Rightarrow 1_B$ forming an adjoint equivalence $\eta \dashv \eta^*$ in the hom-bicategory.
- Two cells $\varepsilon\colon \Psi \otimes S \Rightarrow 1_A$ and $\varepsilon^*\colon 1_A \Rightarrow \Psi \otimes S$ forming an adjoint equivalence $\varepsilon \dashv \varepsilon^*$.
- An invertible 3-cell

$$\begin{tikzcd}
\Psi \arrow[r, "r^*"] \arrow[dddrrr, bend right, "1"'] & \Psi \otimes 1_B \arrow[r, "1\otimes\eta"] & \Psi \otimes (S \otimes \Psi) \arrow[r, "a^*"] & (\Psi \otimes S) \otimes \Psi \arrow[d, "\varepsilon\otimes 1"] \\
 & & \arrow[dl, phantom, "\overset{\cong}{\Leftarrow}\,\Phi"] & 1_A \otimes \Psi \arrow[dd, "l"] \\
 & & & \\
 & & & \Psi
\end{tikzcd}$$

which modulates the triangle identity of an adjunction based around Ψ.

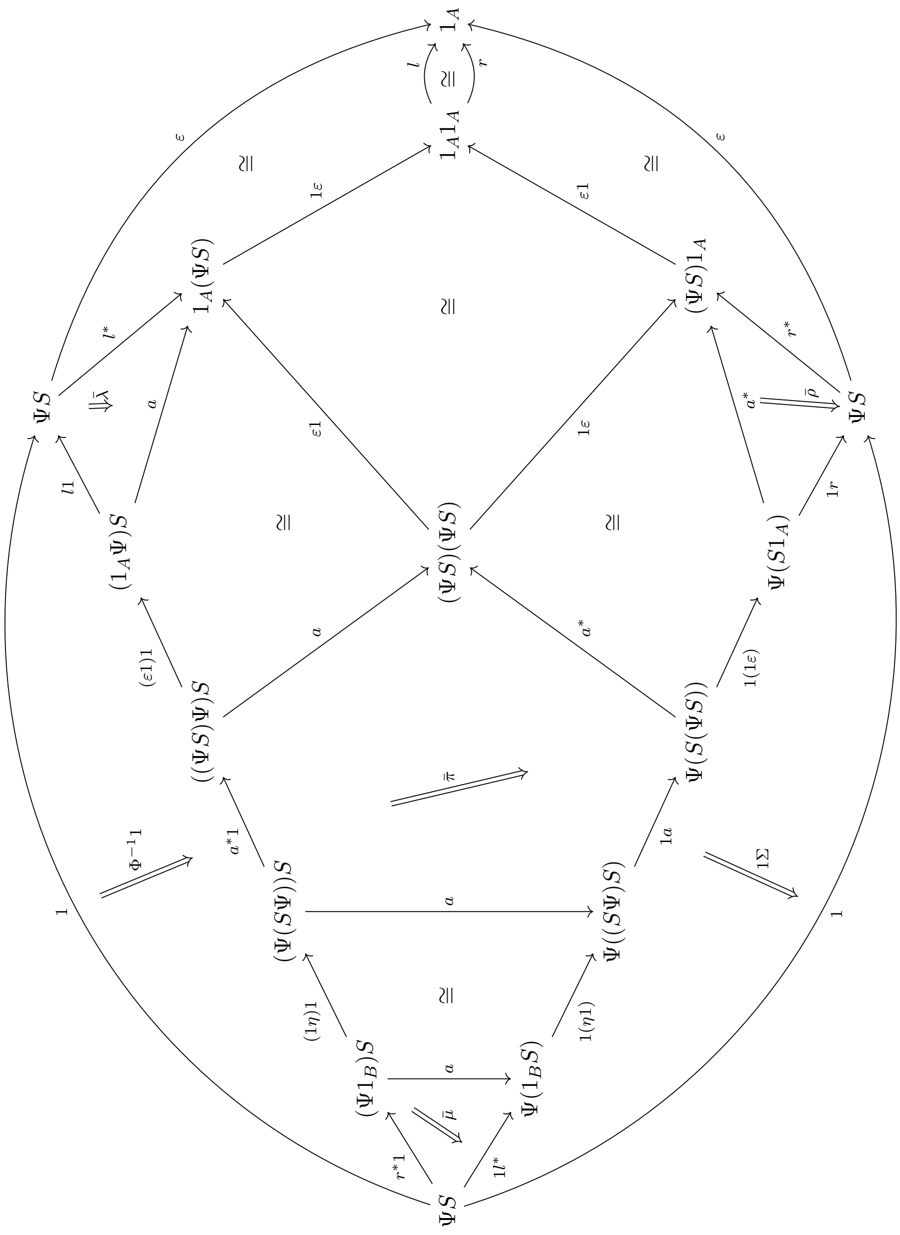
- An invertible 3-cell

$$\begin{tikzcd}
S \arrow[r, "l^*"] \arrow[dddrrr, bend right, "1"'] & 1_B \otimes S \arrow[r, "\eta\otimes 1"] & (S \otimes \Psi) \otimes S \arrow[r, "a"] & S \otimes (\Psi \otimes S) \arrow[d, "1\otimes\varepsilon"] \\
 & & \arrow[dl, phantom, "\overset{\cong}{\Leftarrow}\,\Sigma"] & S \otimes 1_A \arrow[dd, "r"] \\
 & & & \\
 & & & S
\end{tikzcd}$$

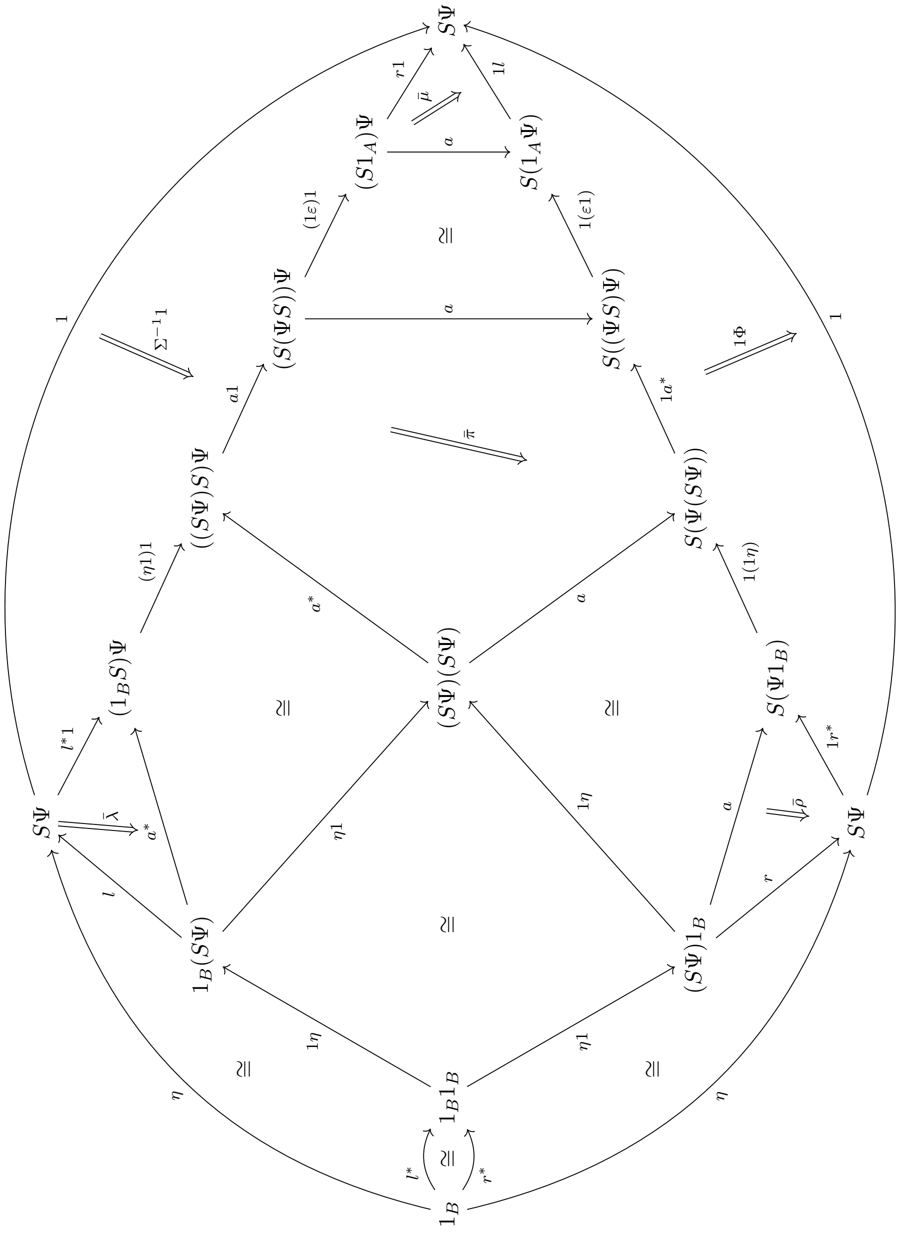
which modulates the triangle identity of an adjunction based around S.

These are required to obey the axioms that both of the following pasting diagrams - which are based around the graphs of all possible ways that a string of up to two instances of Ψ and up to two instances of S can be composed together, with instances of Σ and Φ attached along the appropriate edges - are equal to the identity:

1.



2.



3.1.2 Examples

The tricategory that is the main object of study of this thesis is the tricategory of bicategories, denoted, <u>*Bicat*</u>.

Definition 3.1.5. [Gur13, Section 5.1] The **Tricategory of Bicategories**, denoted <u>*Bicat*</u>, consists of:

- Objects labelled by bicategories
- Hom-bicategories given by the bicategories of pseudofunctors, pseudonatural transformations and modifications, <u>*Bicat*</u>$(\mathcal{A}, \mathcal{B})$.

- Horizontal composition of $\alpha\colon F \Rightarrow F'\colon \mathcal{A} \to \mathcal{B}$ and $\beta\colon G \Rightarrow G'\colon \mathcal{B} \to \mathcal{C}$ given by first calculating the whiskerings $G'\otimes\alpha$ and $\beta\otimes F$ and then defining $\beta\otimes\alpha = (G'\otimes\alpha)\circ(\beta\otimes F)$. The interchange isomorphism that makes this composition into a pseudofunctor is constructed from coherence cells and the pseudonaturality cell of the relevant pseudonatural transformation.

- The coherence 2-cells of this tricategory are given by adjoint equivalences with components that are the identity transformation at each collection of pseudofunctors and are given by coherence cells at each collection of pseudonatural transformations.

- All of the coherence 3-cells of <u>*Bicat*</u> are given by coherence cells in the target bicategory.

The tricategory of bicategories has particularly useful properties, and we will see how it aids our diagram manipulations in Chapter 4.

Other examples of tricategories include:

- A strict tricategory whose hom-bicategories are strict 2-categories is exactly a strict 3-category. Note that the definition given for 'strict tricategory' admits the possibility of strict tricategories whose hom-bicategories are not strict 2-categories and thus these strict tricategories are not necessarily strict 3-categories.

- Given a topological space X we can form its fundamental 3-groupoid whose objects are points of X, 1-cells are paths, 2-cells are homotopies between paths and 3-cells are equivalence classes of homotopies between 2-cells [Gur13, Section 5.2]. Once again, the homotopy hypothesis holds for tricategories.

3.2 Coherence Theorem for Tricategories

To properly interpret ideas about tricategories, we need a coherence theorem. By analogy to the coherence theorem for bicategories, we could hope that we could prove a coherence theorem by showing that every tricategory is triequivalent to some strict 3-category. Unfortunately, this last statement is false.

Proposition 3.2.1. [GPS95, Prop. 8.6 to Remark 8.8] Not every tricategory is equivalent to a strict 3-category.

Proof. Consider a tricategory with one object and one 1-cell. This amounts to a category (the category of 2-cells and 3-cells between that single arrow) with two monoidal structures (given by the compositions in the direction of 1-cells and 2-cells) related by an interchange isomorphism, which in turn is the same as a braided monoidal category [JS93, Prop. 5.3].

If this tricategory is triequivalent to a strict 3-category, then it is also triequivalent to its image under that equivalence, and thus to a strict braided monoidal category: i.e. one where the braiding is the identity. Since the identity is a symmetry on the monoidal structure, so must have been the original braiding.
Since there are braided monoidal categories which are not symmetric, these are not triequivalent to any strict 3-category. □

This failure of tricategories to be equivalent to strict 3-categories has consequences for homotopy theory. It is the reason why we cannot take the fact that trigroupoids model all 3-types and use that to claim that strict 3-groupoids model all 3-types. As seen earlier, an explicit counterexample was given by Simpson who showed that no realisation functor could possibly model the 3-type of the 2-sphere [Sim] due to the existence of a non-trivial Whitehead product. In other words, the homotopy hypothesis fails for strict 3-categories.

Since we cannot prove that all tricategories are equivalent to strict 3-categories, the coherence theorem is instead proved by showing that every tricategory is triequivalent to a member of some larger subclass of tricategories which, while not perfectly strict, have better properties than those of a general tricategory. Such a class of tricategories, which is as strict as possible while remaining triequivalence-dense, is called a **Semi-Strict** class of tricategories.

The class with which the coherence theorem was originally proved consists of tricategories called Gray-Categories.

3.2.1 Gray Categories

To start studying Gray-categories we first have to define a new tensor product on the 3-category $\underline{\text{2-Cat}}$. This tensor product, which we denote by $\otimes$, was first introduced by John Gray [Gra74, Theorem I.4.9] (albeit in a lax form rather than the now-more-commonly used pseudo form) and is thus called the Gray tensor product.

The Gray product has an explicit definition in terms of generators and relations [Gur13, Section 3.1] but applications of them usually use one of the Gray tensor product's two universal properties. The first gives the Gray tensor product a closed structure: for a 2-category $\mathcal{C}$, the functor $-\otimes\mathcal{C}$ is left adjoint to the functor $\underline{2-Cat}_{ps}(\mathcal{C},-)$ where the 2-category $\underline{2-Cat}_{ps}(\mathcal{C},\mathcal{D})$ consists of strict 2-functors, pseudonatural transformations and modifications. [BG17, Defn. 2.8]

The other universal property involves a specific class of pseudofunctors: Cubical functors. The property is that strict functors out of the Gray tensor product $F\colon \mathcal{C}_1\otimes\mathcal{C}_2 \to \mathcal{D}$

correspond naturally to cubical functors $\overline{F}\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$. Thus we will need to investigate the properties of these cubical functors.

Definition 3.2.1. A pseudofunctor $\prod \mathcal{C}_i \to \mathcal{D}$ is **Cubical** [Gur13, Defn. 3.1] if whenever we have 1-cells $f_1...f_n g_1...g_n$ with $g_i f_i$ composable in $\mathcal{C}_i$, satisfying the condition that whenever $i > j$, either f_i or g_j is an identity 1-cell, the compositor

$$\chi\colon F(g_1...g_n)F(f_1...f_n) \Rightarrow F(g_1 f_1 ... g_n f_n)$$

is an identity.

2-Categories and cubical functors form a multicategory [Gur13, Corollary 3.6]. The particular case where the product is of only two 2-categories is the most important for developing the theory of Gray-categories. These cubical functors can be given a characterisation in terms of the functors of each variable.

Proposition 3.2.2. [Gur13, Prop. 3.2] A cubical functor $F\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ uniquely determines and is uniquely determined by the following data:

- For each object $A \in ob(\mathcal{C}_1)$, a strict 2-functor $F_A\colon \mathcal{C}_2 \to \mathcal{D}$.
- For each object $B \in ob(\mathcal{C}_2)$, a strict 2-functor $F_B\colon \mathcal{C}_1 \to \mathcal{D}$.
- The assertion that for each pair of objects $A \in ob(\mathcal{C}_1)$ and $B \in ob(\mathcal{C}_2)$, $F_A(B) = F_B(A)$: i.e. both are equal to $F(A, B)$.
- For each pair of 1-cells $f_1\colon A \to A' \in \mathcal{C}_1$ and $f_2\colon B \to B' \in \mathcal{C}_2$, an invertible 2-cell

$$\begin{tikzcd} F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(f_1)"'] & F(A,B') \arrow[d, "F_{B'}(f_1)"] \arrow[dl, phantom, "\Swarrow \Sigma_{(f_1,f_2)}"] \\ F(A',B) \arrow[r, "F_{A'}(f_2)"'] & F(A',B') \end{tikzcd}$$

  which is the identity if either f_1 or f_2 is the identity.

These data satisfy the following three axioms for every diagram in $\mathcal{C}_1 \times \mathcal{C}_2$ of the form:

$$\begin{tikzcd} (A,B) \arrow[r, bend left, "(f_1,f_2)"] \arrow[r, bend right, "(g_1,g_2)"'] \arrow[r, phantom, "\Downarrow (\alpha_1,\alpha_2)"] & (A',B') \arrow[r, "(h_1,h_2)"] & (A'',B'') \end{tikzcd}$$

- The following two diagrams are equal:

$$\begin{tikzcd}
F(A,B) \arrow[r, bend left=50, "F_A(f_2)"] \arrow[r, "F_A(g_2)"'] \arrow[r, phantom, bend left=25, "\Downarrow F_A\alpha_2"] \arrow[d, "F_B(f_1)"] \arrow[d, bend right=60, "F_B(g_1)"'] \arrow[d, phantom, bend right=30, "\Leftarrow F_B\alpha_1"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B') \arrow[d, "F_{B'}(f_1)"] \\
F(A',B) \arrow[r, "F_{A'}(g_2)"'] & F(A',B')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(g_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B') \arrow[d, "F_{B'}(g_1)"'] \arrow[d, bend left=60, "F_{B'}(f_1)"] \arrow[d, phantom, bend left=30, "\Leftarrow F_{B'}\alpha_1"] \\
F(A',B) \arrow[r, "F_{A'}(f_2)"] \arrow[r, bend right=50, "F_{A'}(g_2)"'] \arrow[r, phantom, bend right=25, "\Downarrow F_{A'}\alpha_2"] & F(A',B')
\end{tikzcd}$$

- The following two diagrams are equal:

$$\begin{tikzcd}
F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(f_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B') \arrow[d, "F_{B'}(f_1)"] \\
F(A',B) \arrow[r, "F_{A'}(f_2)"] \arrow[d, "F_B(h_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A',B') \arrow[d, "F_{B'}(h_1)"] \\
F(A'',B) \arrow[r, "F_{A''}(f_2)"'] & F(A'',B')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(h_1f_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B') \arrow[d, "F_{B'}(h_1f_1)"] \\
F(A'',B) \arrow[r, "F_{A''}(f_2)"'] & F(A'',B')
\end{tikzcd}$$

- The following two diagrams are equal

$$\begin{tikzcd}
F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(f_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B') \arrow[r, "F_A(h_2)"] \arrow[d, "F_{B'}(f_1)"] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B'') \arrow[d, "F_{B''}(f_1)"] \\
F(A',B) \arrow[r, "F_{A'}(f_2)"'] & F(A',B') \arrow[r, "F_{A'}(h_2)"'] & F(A',B'')
\end{tikzcd}$$

$$\begin{tikzcd}
F(A,B) \arrow[r, "F_A(h_2f_2)"] \arrow[d, "F_B(f_1)"'] \arrow[dr, phantom, "\swarrow \Sigma"] & F(A,B'') \arrow[d, "F_{B''}(f_1)"] \\
F(A',B) \arrow[r, "F_{A'}(h_2f_2)"'] & F(A',B'')
\end{tikzcd}$$

The final useful property of cubical functors is the possibility of what Gordon, Power and Street termed **Nudging**.

Proposition 3.2.3. [GPS95, Section 4.5] Let $G\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ be a pseudofunctor with each functor $G(A,-)$ and $G(-,B)$ strict. Then G is isomorphic to a cubical functor $F\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ via an invertible icon.

One important use of this construct is to explain the symmetry of the Gray tensor product. If we reverse the direction of the condition in the definition of cubical functor, we can define opcubical functors. Opcubical functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ clearly correspond to cubical functors $\mathcal{C}_2 \times \mathcal{C}_1 \to \mathcal{D}$ and thus to strict functors $\mathcal{C}_2 \otimes \mathcal{C}_1 \to \mathcal{D}$. On the other hand, nudging provides a correspondence between opcubical functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ and cubical functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ and thus also strict functors $\mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{D}$. This gives a symmetry for the Gray tensor product.

We have not yet proved that the Gray tensor product does in fact satisfy the axioms we require for a monoidal category. The most elegant proof of this is given by Bourke and Gurski [BG17]. They prove that the Gray tensor product is a monoidal structure by giving it as a factorisation

$$\mathcal{C} * \mathcal{D} \longrightarrow \mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{C} \times \mathcal{D}$$

where $\mathcal{C} * \mathcal{D}$ is a tensor product - the so-called **Funny Tensor Product** - whose closed structure is given by an internal hom $\underline{2 - Cat}_f(\mathcal{C}, \mathcal{D})$ whose objects are strict 2-functors, whose 1-cells are plain transformations (collections of 1-cells $FA \to GA$ not required to satisfy any axioms) and whose 2-cells are modifications. The first part of the factorisation is bijective on objects and 1-cells and the second is locally full and faithful: these two classes of maps form an orthogonal factorisation system on $\underline{2 - Cat}$ which Bourke and Gurski use to prove that that the factor objects give a monoidal structure because both the source and target of the factorisation are both monoidal structures.

We are now able to define Gray-categories.

Definition 3.2.2. A **Gray-Category** is a category enriched in the monoidal structure given by the Gray tensor product on <u>2-Cat</u>. A Gray-category can be viewed as a tricategory whose hom-objects are strict 2-categories, whose association and identities are strict (because association and identities are always strict for enriched categories) but whose composition is a cubical functor.

It is in light of this that the motivation for cubical functors becomes clear. It is often the situation that the definition of the horizontal composites of a 2-cell by an identity ($\beta_1 \otimes id$ and $id \otimes \beta_2$) are obvious and strict. However, it is not obvious how to horizontally compose two 2-cells, so we have to make a choice to define $\beta_1 \otimes \beta_2 = (id \otimes \beta_2) \circ (\beta_1 \otimes id)$. Then $(\beta_1 \otimes \beta_2) \circ (\alpha_1 \otimes \alpha_2)$ is given by some association of

$$(id \otimes \beta_2) \circ (\beta_1 \otimes id) \circ (id \otimes \alpha_2) \circ (\alpha_1 \otimes id)$$

We can now see that the interchange law fails because the terms for β_1 and α_2 are the wrong way around. Therefore the interchange law works if either of these are the identity: exactly the condition of cubical functors.

Gordon, Power and Street proved the coherence theorem for tricategories by showing that every tricategory is triequivalent to a Gray-category. This is a long proof that takes up the bulk of their monograph and thus will not be presented here.

3.3 Low Dimensional Structures formed by Tricategories

Though tricategories and the transformations given above form a 4-dimensional categorical structure, it is also possible to construct lower dimensional structures whose objects are tricategories. This generalises the fact that bicategories, pseudofunctors and icons form a 2-dimensional structure (see Section 2.1.1), and so we should expect to see more icon-like behaviour.

3.3.1 A 2-Dimensional Structure

Definition 3.3.1. An **Ico-Icon** [GG09, Defn. 2] between two trifunctors or lax functors $\alpha\colon F \Rightarrow G\colon \mathcal{S} \to \mathcal{T}$ consists of:

- The assertion that F and G agree on objects.
- The assertion that F and G agree on 1-cells.
- For each pair of objects $A, B \in ob(\mathcal{S})$, an icon $\alpha_{AB}\colon F_{AB} \Rightarrow G_{A,B}\colon \mathcal{S}(A,B) \to \mathcal{T}(FA,FB) = \mathcal{T}(GA,GB)$.
- For each object $A \in ob(\mathcal{S})$, a 3-cell $M_A\colon \iota_A^F \Rightarrow \iota_A^G$.
- For each pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

  a 3-cell $\Pi_{gf}\colon \chi_{gf}^F \Rightarrow \chi_{gf}^G$.

such that

- For every pair of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\theta"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\eta"] & C \end{tikzcd}$$

  the following two diagrams are equal

$$
\begin{tikzcd}[column sep=small]
& FgFf \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[dr, "F\eta F\theta"] \arrow[d, phantom, "\Swarrow \alpha_\eta\alpha_\theta"] & & F(gf) \arrow[dr, "F(\eta\theta)"] \arrow[d, phantom, "\cong"] & \\
GgGf \arrow[dr, "G\eta G\theta"'] & {} & Fg'Ff' \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[dr, phantom, "\Swarrow \Pi"] & {} & F(g'f') \arrow[dl, equal] \\
& Gg'Gf' \arrow[rr, "\chi"'] & & G(g'f') &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& FgFf \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow \Pi"] & & F(gf) \arrow[dr, "F(\eta\theta)"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow \alpha_{\eta\theta}"] & \\
GgGf \arrow[rr, "\chi"'] \arrow[dr, "G\eta G\theta"'] & {} \arrow[d, phantom, "\cong"] & G(gf) \arrow[dr, "G(\eta\theta)"] & {} & F(g'f') \arrow[dl, equal] \\
& Gg'Gf' \arrow[rr, "\chi"'] & & G(g'f') &
\end{tikzcd}
$$

- For every 1-cell $f\colon A \to B$, the diagrams

$$
\begin{tikzcd}
& F1_BFf \arrow[r, "\chi"] & F(1_Bf) \arrow[dr, "Fl"] & \\
1_{FB}Ff \arrow[ur, "\iota 1"] \arrow[d, equal] \arrow[dr, "l"] & & & Ff \arrow[d, equal] \arrow[dl, equal] \\
1_{GB}Gf \arrow[r, phantom, "="] \arrow[dr, "l"'] & Ff \arrow[r, equal] \arrow[d, equal] \arrow[u, phantom, "\Downarrow\gamma"] & Ff \arrow[d, equal] \arrow[r, phantom, "="] & Gf \\
& Gf \arrow[r, equal] \arrow[ur, phantom, "="] & Gf \arrow[ur, equal] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& F1_BFf \arrow[r, "\chi"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Pi"] & F(1_Bf) \arrow[dr, "Fl"] \arrow[d, equal] & \\
1_{FB}Ff \arrow[ur, "\iota 1"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow M1"] & G1_BGf \arrow[r, "\chi"] & G(1_Bf) \arrow[dr, "Gl"] \arrow[r, phantom, "\Downarrow\alpha_l"] & Ff \arrow[d, equal] \\
1_{GB}Gf \arrow[ur, "\iota 1"] \arrow[dr, "l"'] & \arrow[r, phantom, "\Downarrow\gamma"] & {} & Gf \\
& Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}
$$

are equal.

- For every 1-cell $f\colon A \to B$, the diagrams

$$\begin{tikzcd}
 & FfF1_A \arrow[r, "\chi"] \arrow[dd, phantom, "\Downarrow\delta"] & F(f1_A) \arrow[dr, "Fr"] & \\
Ff1_{FA} \arrow[ur, "1\iota"] \arrow[drr, "r"] \arrow[d, equal] & & & Ff \arrow[d, equal] \arrow[dl, equal] \\
Gf1_{GA} \arrow[r, phantom, "="] \arrow[dr, "r"'] & Ff \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "="] & Ff \arrow[d, equal] \arrow[r, phantom, "="] & Gf \\
 & Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & FfF1_A \arrow[r, "\chi"] \arrow[d, equal] \arrow[dl, phantom, "\Downarrow 1M"] \arrow[dr, phantom, "\Downarrow\Pi"] & F(f1_A) \arrow[dr, "Fr"] \arrow[d, equal] & \\
Ff1_{FA} \arrow[ur, "1\iota"] \arrow[d, equal] & GfG1_A \arrow[r, dashed, "\chi"] \arrow[dd, phantom, "\Downarrow\delta"] & G(f1_A) \arrow[dr, "Gr"] \arrow[ur, phantom, "\Downarrow\alpha_r"] & Ff \arrow[d, equal] \\
Gf1_{GA} \arrow[ur, "1\iota"] \arrow[dr, "r"'] & & & Gf \\
 & Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}$$

are equal.

- For every triple of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the diagrams

$$\begin{tikzcd}
 & F(hg)Ff \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow\omega"] & F((hg)f) \arrow[dr, "Fa"] & \\
(FhFg)Ff \arrow[ur, "\chi 1"] \arrow[dr, "a"] \arrow[d, equal] & {} & & F(h(gf)) \arrow[d, equal] \\
(GhGg)Gf \arrow[r, phantom, "="] \arrow[dr, "a"'] & Fh(FgFf) \arrow[r, dashed, "1\chi"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1\Pi"] & FhF(gf) \arrow[ur, "\chi"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\Pi"] & G(h(gf)) \\
 & Gh(GgGf) \arrow[r, "1\chi"'] & GhG(gf) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & F(hg)Ff \arrow[r, "\chi"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Pi"] & F((hg)f) \arrow[d, equal] \arrow[dr, "Fa"] & \\
(FhFg)Ff \arrow[ur, "\chi 1"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\Pi 1"] & G(hg)Gf \arrow[r, dashed, "\chi"] \arrow[dr, phantom, "\Downarrow\omega"] & G((hg)f) \arrow[dr, "Ga"'] \arrow[r, phantom, "\Downarrow\alpha_a"] & F(h(gf)) \arrow[d, equal] \\
(GhGg)Gf \arrow[ur, "\chi 1"] \arrow[dr, "a"'] & & & G(h(gf)) \\
 & Gf \arrow[r, "1\chi"'] & Gf \arrow[ur, "\chi"'] &
\end{tikzcd}$$

are equal.

Comparing this definition with that of oplax natural transformations of tricategories, we see that these correspond to oplax natural transformations whose components at the objects and 1-cells are identities. However, the correct form of vertical composition of ico-icons is not that of oplax transformations because they now admit a strictly associative composition based on the strict composition of 3-cells in a tricategory.

We can form a bicategory, Ico-Icon, whose objects are tricategories, 1-cells are either trifunctors or lax functors and whose 2-cells are ico-icons. [GG09, Section 2].

3.3.2 A 3-Dimensional Structure

Definition 3.3.2. A **Pseudo-Icon** [GG09, Definitions 3 and 5] between trifunctors or lax functors $\alpha\colon F \Rightarrow G\colon \mathcal{S} \to \mathcal{T}$ consists of:

- The assertion that F and G agree on objects.
- For each pair of objects $A, B \in ob(\mathcal{S})$, a pseudonatural transformation $\alpha_{AB}\colon F_{AB} \Rightarrow G_{A,B}\colon \mathcal{S}(A,B) \to \mathcal{T}(FA, FB) = \mathcal{T}(GA, GB)$.
- For each object $A \in ob(\mathcal{S})$, an invertible 3-cell

$$\begin{tikzcd}
1_{FA} \arrow[r, "\iota^F_A"] \arrow[d, equal] \arrow[dr, phantom, "\swArrow M_A"] & F1_A \arrow[d, "\alpha 1_A"] \\
1_{GA} \arrow[r, "\iota^G_A"'] & G1_A
\end{tikzcd}$$

- For each triple of objects $A, B, C \in ob(\mathcal{S})$, an invertible modification whose components, at a pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are given by

$$\begin{tikzcd}
FgFf \arrow[r, "\chi"] \arrow[d, "\alpha_g\alpha_f"'] & F(gf) \arrow[d, "\alpha_{gf}"] \arrow[dl, phantom, "\swarrow\Pi_{gf}"] \\
GgGf \arrow[r, "\chi"'] & G(gf)
\end{tikzcd}$$

such that (in the following diagrams a bar above a 3-cell represents the composition of that 3-cell with an appropriate interchange isomorphism):

- For every 1-cell $f\colon A \to B$, the diagrams

$$\begin{tikzcd}
 & F1_BFf \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow\gamma"] & F(1_Bf) \arrow[dr, "Fl"] & \\
1_{FB}Ff \arrow[ur, "\iota 1"] \arrow[d, "1\alpha_f"'] \arrow[dr, "l"] & & & Ff \arrow[d, "\alpha_f"] \arrow[dl, equal] \\
1_{GB}Gf \arrow[r, phantom, "\cong"] \arrow[dr, "l"'] & Ff \arrow[r, equal] \arrow[d, "\alpha_f"] \arrow[dr, phantom, "="] & Ff \arrow[d, "\alpha_f"] \arrow[r, phantom, "="] & Gf \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & F1_BFf \arrow[r, "\chi"] \arrow[d, dashed, "\alpha_{1_B}\alpha_f"] \arrow[dr, phantom, "\Downarrow\Pi"] & F(1_Bf) \arrow[d, dashed, "\alpha_{1_Bf}"] \arrow[dr, "Fl"] & \\
1_{FB}Ff \arrow[ur, "\iota 1"] \arrow[d, "1\alpha_f"'] \arrow[r, phantom, "\Downarrow\overline{M1}"] & G1_BGf \arrow[r, dashed, "\chi"] \arrow[dr, phantom, "\Downarrow\gamma"] & G(1_Bf) \arrow[dr, "Gl"] \arrow[r, phantom, "\Downarrow\alpha_l"] & Ff \arrow[d, "\alpha_f"] \\
1_{GB}Gf \arrow[ur, "\iota 1"] \arrow[dr, "l"'] & & & Gf \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}$$

are equal.

- For every 1-cell $f\colon A \to B$, the diagrams

$$\begin{tikzcd}
 & FfF1_A \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow\delta"] & F(f1_A) \arrow[dr, "Fr"] & \\
Ff1_{FA} \arrow[ur, "1\iota"] \arrow[d, "\alpha_f 1"'] \arrow[dr, "r"] & & & Ff \arrow[d, "\alpha_f"] \arrow[dl, equal] \\
Gf1_{GA} \arrow[r, phantom, "\cong"] \arrow[dr, "r"'] & Ff \arrow[r, equal] \arrow[d, dashed, "\alpha_f"] \arrow[dr, phantom, "="] & Ff \arrow[d, dashed, "\alpha_f"] \arrow[r, phantom, "="] & Gf \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}$$

and

$$
\begin{tikzcd}
 & FfF1_A \arrow[r, "\chi"] \arrow[d, dashed, "\alpha_f\alpha_{1_A}"] \arrow[r, phantom, shift right=4, "\Downarrow\Pi"] & F(f1_A) \arrow[dr, "Fr"] \arrow[d, dashed, "\alpha_{f1_A}"] & \\
Ff1_{FA} \arrow[ur, "1\iota"] \arrow[d, "\alpha_f 1"'] \arrow[r, phantom, "\Downarrow\overline{1M}"] & GfG1_A \arrow[r, dashed, "\chi"] & G(f1_A) \arrow[dr, "Gr"] \arrow[r, phantom, "\Downarrow\alpha_r"] & Ff \arrow[d, "\alpha_f"] \\
Gf1_{GA} \arrow[ur, "1\iota"] \arrow[dr, "r"'] & & \Downarrow\delta & Gf \\
 & Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}
$$

are equal.

- For every triple of composable 1-cells

$$
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D
$$

the diagrams

$$
\begin{tikzcd}
 & F(hg)Ff \arrow[r, "\chi"] & F((hg)f) \arrow[dr, "Fa"] & \\
(FhFg)Ff \arrow[ur, "\chi 1"] \arrow[dr, "a"] \arrow[d, "(\alpha_h\alpha_g)\alpha_f"'] & & \Downarrow\omega & F(h(gf)) \arrow[d, "\alpha_{h(gf)}"] \\
(GhGg)Gf \arrow[r, phantom, "\cong"] \arrow[dr, "a"'] & Fh(FgFf) \arrow[r, dashed, "1\chi"] \arrow[d, dashed, "\alpha_h(\alpha_g\alpha_f)"] \arrow[r, phantom, shift right=4, "\Downarrow 1\Pi"] & FhF(gf) \arrow[ur, "\chi"] \arrow[d, dashed, "\alpha_h\alpha_{gf}"] \arrow[r, phantom, "\Downarrow\Pi"] & G(h(gf)) \\
 & Gh(GgGf) \arrow[r, "1\chi"'] & GhG(gf) \arrow[ur, "\chi"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & F(hg)Ff \arrow[r, "\chi"] \arrow[d, dashed, "\alpha_{hg}\alpha_f"] \arrow[r, phantom, shift right=4, "\Downarrow\Pi"] & F((hg)f) \arrow[dr, "Fa"] \arrow[d, dashed, "\alpha_{(hg)f}"] & \\
(FhFg)Ff \arrow[ur, "\chi 1"] \arrow[d, "(\alpha_h\alpha_g)\alpha_f"'] \arrow[r, phantom, "\Downarrow\overline{\Pi 1}"] & G(hg)Gf \arrow[r, dashed, "\chi"] & G((hg)f) \arrow[dr, "Ga"] \arrow[r, phantom, "\Downarrow\alpha_a"] & F(h(gf)) \arrow[d, "\alpha_{h(gf)}"] \\
(GhGg)Gf \arrow[ur, "\chi 1"] \arrow[dr, "a"'] & & \Downarrow\omega & G(h(gf)) \\
 & Gf \arrow[r, "1\chi"'] & Gf \arrow[ur, "\chi"'] &
\end{tikzcd}
$$

are equal.

Unlike ico-icons, pseudo-icons admit the possibility of transformations between them.

Definition 3.3.3. A **Modification between pseudo-icons** [GG09, Defn. 6] $\Gamma \colon \alpha \Rightarrow \beta \colon F \Rightarrow G \colon \mathcal{S} \to \mathcal{T}$ consists of a modification $\Gamma_{AB} \colon \alpha_{AB} \Rightarrow \beta_{AB}$ for each pair of objects $A, B \in ob(\mathcal{S})$ such that

- For every object $A \in ob(\mathcal{S})$ the following two diagrams are equal:

$$
\begin{tikzcd}
& 1_{FA} \arrow[rr, "\iota"] \arrow[dl, equal] \arrow[dr, equal] & & F1_A \arrow[dr, "\alpha_{1_A}"] \arrow[dl, phantom, "\Swarrow M"] & \\
1_{FA} \arrow[dr, equal] & & 1_{GA} \arrow[rr, "\iota"] \arrow[dl, equal] \arrow[dr, phantom, "="] & & G1_A \arrow[dl, equal] \\
& 1_{GA} \arrow[rr, "\iota"'] & & G1_A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1_{FA} \arrow[rr, "\iota"] \arrow[dl, equal] \arrow[dr, phantom, "="] & & F1_A \arrow[dr, "\alpha_{1_A}"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow \Gamma"] & \\
1_{FA} \arrow[rr, "\iota"] \arrow[dr, equal] & & F1_A \arrow[dr, "\beta_{1_A}"] \arrow[dl, phantom, "\Swarrow M"] & {} & G1_A \arrow[dl, equal] \\
& 1_{GA} \arrow[rr, "\iota"'] & & G1_A &
\end{tikzcd}
$$

- For every pair of composable 1-cells

$$
A \xrightarrow{f} B \xrightarrow{g} C
$$

the following two diagrams are equal:

$$
\begin{tikzcd}
& FgFf \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[dr, "\alpha_g \alpha_F"] & & F(gf) \arrow[dr, "\alpha_{(gf)}"] \arrow[dl, phantom, "\Swarrow \Pi"] & \\
FgFf \arrow[dr, "\beta_g \beta_f"'] \arrow[rr, phantom, "\Swarrow \Gamma\Gamma"] & & GgGf \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[dr, phantom, "="] & & G(gf) \arrow[dl, equal] \\
& GgGf \arrow[rr, "\chi"'] & & G(gf) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& FgFf \arrow[rr, "\chi"] \arrow[dl, equal] \arrow[dr, phantom, "="] & & F(gf) \arrow[dr, "\alpha_{gf}"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow \Gamma"] & \\
FgFf \arrow[rr, "\chi"] \arrow[dr, "\beta_g \beta_f"'] & & F(gf) \arrow[dr, "\beta_{gf}"] \arrow[dl, phantom, "\Swarrow \Pi"] & {} & G(gf) \arrow[dl, equal] \\
& GgGf \arrow[rr, "\chi"'] & & G(gf) &
\end{tikzcd}
$$

The structure formed by tricategories, trifunctors or lax functors, pseudo-icons and modifications is a tricategory [GG09, Theorem 7], which we will denote <u>Ps-Icon</u>

With the overview of the theory of tricategories complete, we are now ready to proceed to the original part of the thesis.

Chapter 4

Manipulating Tricategorical Pasting Diagrams

In this chapter we prove the first original results of the thesis. These results will provide us with techniques we can use to simplify and manipulate the pasting diagrams that make up the trifunctor axioms. Using these techniques will make the process of proving that one such diagram is equal to another - and thereby proving that the trifunctor axioms hold - significantly easier.

The first technique will use the coherence theorem for bicategories to allow us to ignore cells in the pasting diagram that are coherence cells in a particular bicategory. This is crucial for simplifying the source and target pasting diagrams and make them tractable.

The second technique considers pseudonatural transformations and modifications in a given bicategory whose sources and/or targets are composites. The definitions of pseudonatural transformation and modification imply that their component 2-cells can be moved through particular cells (For pseudonatural transformations, the images of 2-cells under a pseudofunctor. For modifications, 2-cell components of pseudonatural transformations.) attached along the entire source or target. The point of the second technique is that from this we can prove that any configuration of attaching such 2-cells to the pseudonatural transformation or modification works: both pseudonatural transformations and modifications can be moved through cells on any segment of the boundary.

Both of these techniques arise from the fact that the target tricategory is the tricategory of bicategories, which has particularly nice properties.

4.1 Simplifying Pasting Diagrams using the Coherence Theorem

The method described in this section is modelled on that used by Gurski to simplify the axioms for a biadjoint biequivalence in the tricategory of bicategories [Gur12, Theorem 3.2]. Using the method to simplify the definitions of trifunctor and tritransformation, and potentially any other tricategorical pasting diagram in the tricategory of bicategories, is original work.

To see why the coherence theorem for bicategories will allow us to simplify our pasting diagrams, note that taking components at a particular object of the bicategory will extract a coherence cell from many of the important cells we are considering in the pasting diagram. In particular, all of the coherence 2-cells of the tricategory of bicategories have components which are identity 1-cells in the bicategory, all of the coherence 3-cells have components which are the suitable coherence cells in the bicategory, and all of the pseudonaturality cells taken at a coherence 1-cell are coherence cells. Moving to the tricategory of bicategories also simplifies the interchange cells between two pseudonatural transformations, as the interchange cell becomes the pseudonaturality cell of the leading pseudonatural transformation.

We'll thus end up with a pasting diagram in a bicategory where many of the cells are coherence cells. Coherence cells in bicategories are natural, so in any given diagram they can be moved either towards the source or target. Then, once all the coherence cells are collected before or after all the other relevant cells, the coherence theorem ensures that any possible composition of these coherence cells gives the same result. This allows us to only consider the other remaining cells during the calculations.

As an example of this process in action, we can use it to simplify the definition of trifunctor into $\underline{Bicat}$ as follows:

Proposition 4.1.1. A **Trifunctor** from any tricategory $\mathcal{T}$ into the tricategory of bicategories $\underline{Bicat}$, $F\colon \mathcal{T} \to \underline{Bicat}$ consists of:

- A function $ob(\mathcal{T}) \to ob(\underline{Bicat})$.
- For each pair of objects $A, B \in ob(\mathcal{T})$, a pseudofunctor $\mathcal{T}(A, B) \to \underline{Bicat}(FA, FB)$.

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, a **Compositor** given by an adjoint equivalence

$$\begin{tikzcd}
\mathcal{T}(B,C) \times \mathcal{T}(A,B) \arrow[r, "F \times F"] \arrow[d, "\otimes"] & \underline{Bicat}(FB,FC) \times \underline{Bicat}(FA,FB) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "\chi"] \\
\mathcal{T}(A,C) \arrow[r, "F"'] & \underline{Bicat}(FA,FC)
\end{tikzcd}$$

- For each object $A \in ob(\mathcal{T})$, a **Unitor** given by an adjoint equivalence

$$\begin{tikzcd}
1 \arrow[rr, "id_{FA}"] \arrow[dr, "id_A"'] & {} \arrow[d, phantom, "\Downarrow\iota"] & \underline{Bicat}(FA,FA) \\
& \mathcal{T}(A,A) \arrow[ur, "F"'] &
\end{tikzcd}$$

- For every four objects $A, B, C, D \in ob(\mathcal{T})$, an invertible modification ω composed of 2-cells in the target bicategory which, by coherence, are determined exactly by 2-cells

$$\begin{tikzcd}
& F(hg)Ff(x) \arrow[rr, "\chi"] & {} \arrow[d, phantom, "\Downarrow\widetilde{\omega}"] & F((hg)f)(x) \arrow[dr, "Fa"] & \\
FhFgFf(x) \arrow[ur, "\chi 1"] \arrow[rr, "1\chi"'] & & FhF(gf)(x) \arrow[rr, "\chi"'] & & F(h(gf))(x)
\end{tikzcd}$$

(From here on, we will often take a 2-cell in a bicategory and refer to the unique cell determined from it by coherence by adding a tilde: e.g. α gives rise to the cell $\widetilde{\alpha}$.)

- For each pair of objects $A, B \in ob(\mathcal{Y})$, an invertible modification γ composed of 2-cells in the target bicategory which, by coherence, are determined exactly by 2-cells

$$\begin{tikzcd}
& F1_BFf(x) \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow\gamma", shift left=6] & F(1_Bf)(x) \arrow[dr, "Fl"] & \\
Ff(x) \arrow[ur, "\iota 1"] \arrow[rrr, "1"'] & & & Ff(x)
\end{tikzcd}$$

- For each pair of objects $A, B \in ob(\mathcal{T})$, an invertible modification δ composed of 2-cells in the target bicategory which, by coherence, are determined exactly by 2-cells

$$\begin{tikzcd}
Ff(x) \arrow[rrr, "Fr^*"] \arrow[dr, "1"'] & & & F(f1_A)(x) \\
& Ff(x) \arrow[r, "1\iota"'] \arrow[r, phantom, "\Downarrow\delta", shift left=6] & FfF1_A(x) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

satisfying two simpler axioms

- For every four composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

  the two diagrams

$$\begin{tikzcd}
 & F((kh)g)Ff(x) \arrow[r, "\chi"] \arrow[dr, "FaFf"] & F(((kh)g)f)(x) \arrow[dr, "F(af)"] \arrow[d, phantom, "\cong_\chi"] & & \\
 & & F(k(hg))Ff(x) \arrow[r, "\chi"] & F((k(hg))f)(x) \arrow[dr, "Fa"] \arrow[d, phantom, "\Downarrow\widetilde{\omega}"] & \\
F(kh)FgFf(x) \arrow[uur, "\chi Ff"] \arrow[r, phantom, "\Downarrow\widetilde{\omega}Ff"] & & & & F(k((hg)f))(x) \arrow[d, "F(ka)"] \\
 & & FkF(hg)Ff(x) \arrow[uu, "\chi Ff"] \arrow[r, "Fk\chi"] & FkF((hg)f)(x) \arrow[ur, "\chi"] \arrow[dr, "FkFa"] \arrow[r, phantom, "\cong_\chi"] \arrow[d, phantom, "\Downarrow Fk\widetilde{\omega}"] & F(k(h(gf)))(x) \\
 & FkFhFgFf(x) \arrow[uul, "\chi FgFf"] \arrow[ur, "Fk\chi_F Ff"] \arrow[dr, "FkFh\chi"] & & & FkF(h(gf))(x) \arrow[u, "\chi"] \\
 & & FkFhF(gf)(x) \arrow[urr, "Fk\chi"] & &
\end{tikzcd}$$

  and

$$\begin{tikzcd}
 & F((kh)g)Ff(x) \arrow[r, "\chi"] & F(((kh)g)f)(x) \arrow[dr, "F(af)"] \arrow[ddr, "Fa"] & & \\
 & & \arrow[d, phantom, "\Downarrow\widetilde{\omega}"] & F((k(hg))f)(x) \arrow[dr, "Fa"] \arrow[r, phantom, "\Swarrow F\pi"] & \\
F(kh)FgFf(x) \arrow[uur, "\chi Ff"] \arrow[drr, "F(kh)\chi"] & & & F((kh)(gf))(x) \arrow[dr, "Fa"] \arrow[d, phantom, "\Downarrow\widetilde{\omega}"] & F(k((hg)f))(x) \arrow[d, "F(ka)"] \\
 & \arrow[d, phantom, "\cong_\chi"] & F(kh)F(gf)(x) \arrow[ur, "\chi"] & & F(k(h(gf)))(x) \\
 & FkFhFgFf(x) \arrow[uul, "\chi FgFf"] \arrow[dr, "FkFh\chi"] & & & FkF(h(gf))(x) \arrow[u, "\chi"] \\
 & & FkFhF(gf)(x) \arrow[uu, "\chi F(gf)"] \arrow[urr, "Fk\chi"] & &
\end{tikzcd}$$

  are equal.

- For every pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

the two diagrams

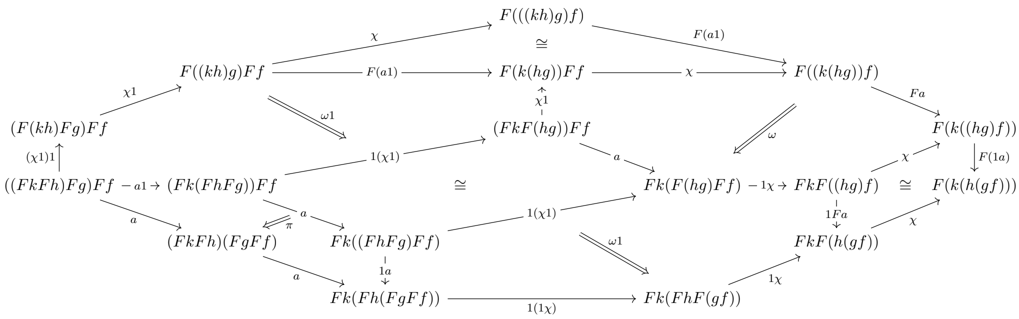
$$\begin{tikzcd}
 & F((g1_B)f)(x) \arrow[rr, "Fa"] & & F(g(1_Bf))(x) \arrow[dr, "F(gl)"] & \\
F(gf)(x) \arrow[ur, "F(r^*f)"] \arrow[r, phantom, "\cong_\chi"] & F(g1_B)Ff(x) \arrow[u, "\chi"] \arrow[rr, Rightarrow, "\widetilde{\omega}"] \arrow[dr, phantom, "\overset{\delta}{\Rightarrow}"] & & FgF(1_Bf)(x) \arrow[u, "\chi"] \arrow[dr, "FgFl"] \arrow[r, phantom, "\cong_\chi"] & F(gf)(x) \\
FgFf(x) \arrow[u, "\chi"] \arrow[ur, "Fr^*Ff"] \arrow[drr, "1"'] & & FgF1_BFf(x) \arrow[ul, "\chi Ff"'] \arrow[ur, "Fg\chi"] \arrow[r, phantom, "\overset{\gamma}{\Rightarrow}"] & & FgFf(x) \arrow[u, "\chi"'] \\
 & & FgFf(x) \arrow[u, "Fg\iota Ff"] \arrow[urr, "1"'] & &
\end{tikzcd}$$

and

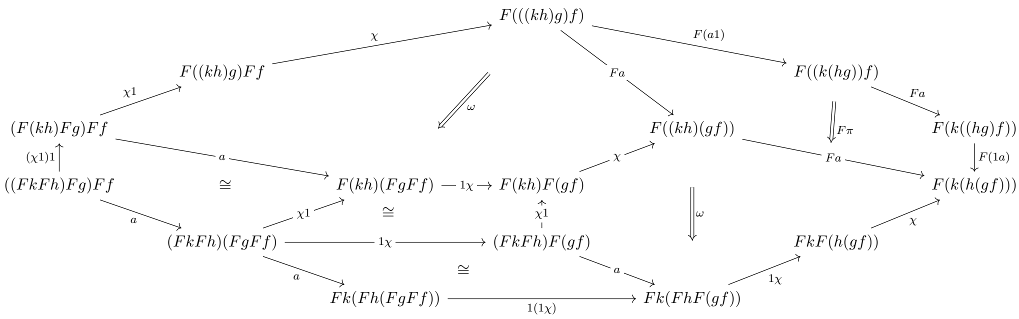
$$\begin{tikzcd}
 & F((g1_B)f)(x) \arrow[r, "Fa"] \arrow[d, phantom, "\Downarrow F\mu"] & F(g(1_Bf))(x) \arrow[dr, "F(gl)"] & \\
F(gf)(x) \arrow[ur, "F(r^*f)"] \arrow[rrr, "1"'] & & & F(gf)(x) \\
FgFf(x) \arrow[u, "\chi"] \arrow[rrr, "1"'] & & \cong & FgFf(x) \arrow[u, "\chi"']
\end{tikzcd}$$

are equal.

Proof. For the first trifunctor axiom, recall that the definition of a trifunctor between any tricategories (Definition 3.1.2) requires that the following two diagrams are equal:



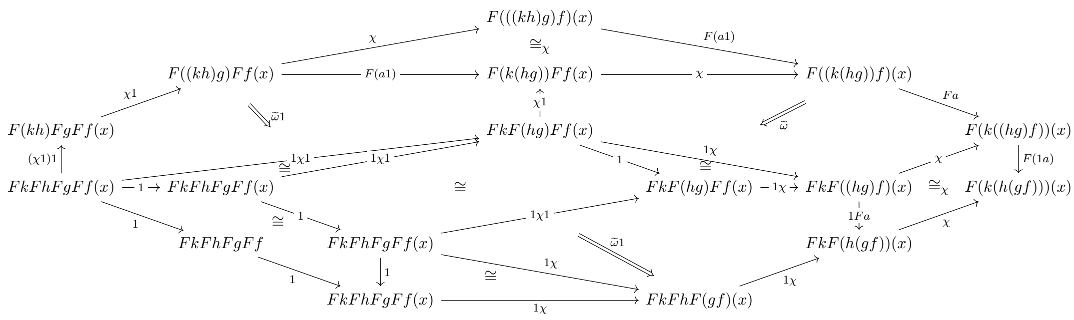
and

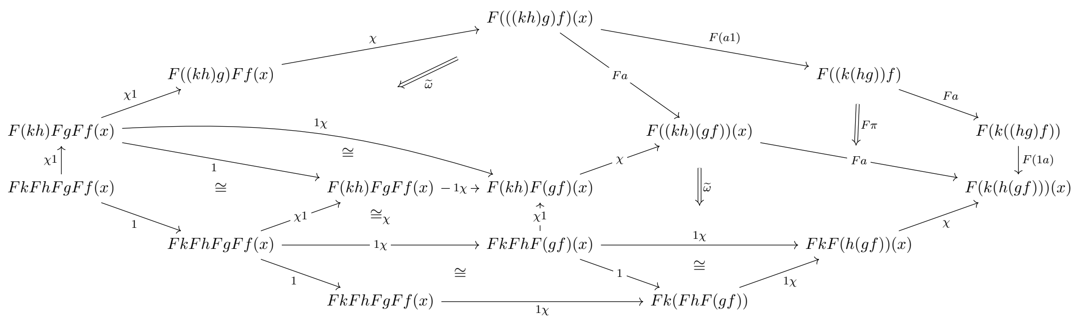


Interpret these diagrams where the target tricategory is the tricategory of bicategories, and then take the component at the object x in the bicategory $F(A)$. This has the following effects:

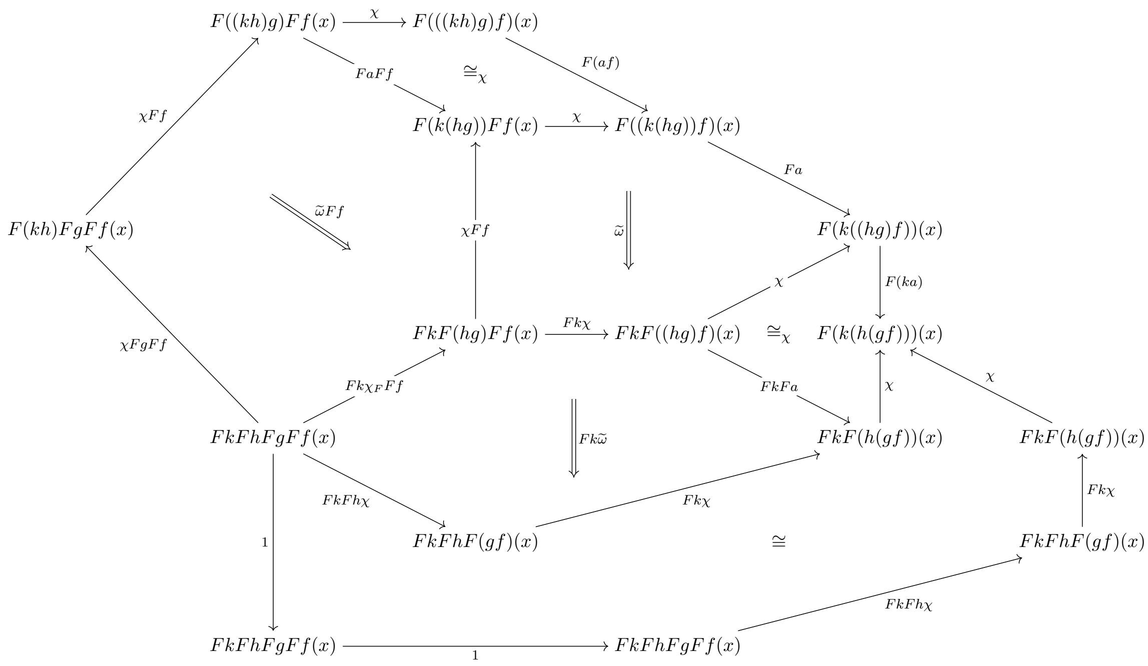
- Each instance of the associator becomes an identity, and all pseudonaturality squares for the associator become coherence cells of the bicategory $F(E)$.
- Each coherence 3-cell of the tricategory becomes a coherence cell of the bicategory.
- Any interchange isomorphisms become the pseudonaturality cell of the leading pseudonatural transformation.
- Any instance of the cell ω becomes an instance of the cell $\widetilde{\omega}$ plus a coherence cell.

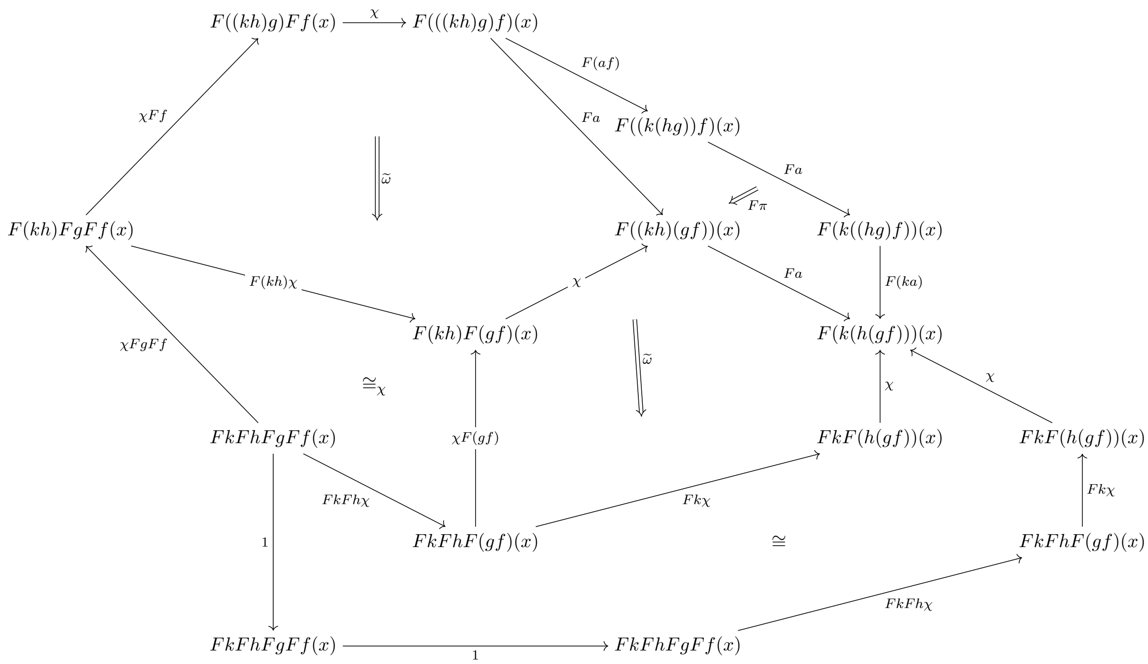
This results in the following two diagrams (where the unmarked $\cong$ cells are the coherence cells of a bicategory).





Now use the naturality of the coherence cells of the bicategory to move them through the other cells towards the target of these two diagrams. This results in the following two diagrams, as the coherence theorem means that we can amalgamate all the coherence cells into a single cell which will be the same no matter how we choose to do it.





These two diagrams are equal if and only if the original diagrams, from the axiom for a trifunctor, are equal. Since these two diagrams have the same coherence cell attached along the bottom, they are equal if and only if the two diagrams with that coherence cell removed (that is, the simplified diagrams for the axiom) are equal. Therefore the two diagrams from the original definition of a trifunctor are equal and the first axiom holds if and only if the simplified diagrams are equal.

The proof for the second axiom proceeds similarly.

□

When proving the main result in the next chapter, it is this definition of trifunctor we will use.

We can simplify the definition of a tritransformation in the tricategory $\underline{Bicat}$ as well. This will aid us in Chapter 6, when we partially prove that the object-indexed family of biequivalences can be lifted to be tritransformations between the original and the transported trifunctor.

Proposition 4.1.2. A tritransformation $\theta\colon F \Rightarrow G\colon \mathcal{T} \to \underline{Bicat}$ can be given as:

- For each object A of $\mathcal{T}$ a 1-cell $\theta_A\colon FA \to GA$.
- For each pair of objects $A, B \in ob(\mathcal{T})$, an adjoint equivalence

$$\begin{tikzcd} \mathcal{T}(A,B) \arrow[r, "F"] \arrow[d, "G"'] & \underline{Bicat}(FA,FB) \arrow[d, "\theta_B\otimes -"] \arrow[dl, Rightarrow, "\theta"] \\ \underline{Bicat}(GA,GB) \arrow[r, "-\otimes\theta_A"'] & \underline{Bicat}(FA,GB) \end{tikzcd}$$

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, an invertible modification Π whose component at a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{T}$ and at an object $x \in ob(FA)$ arises via coherence from a 2-cell

$$\begin{tikzcd} & Gg\theta_B Ff(x) \arrow[rr, "1\theta"] & & GgGf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\ \theta_C FgFf(x) \arrow[ur, "\theta 1"] \arrow[drr, "1\chi_F"'] & & \Downarrow\widetilde{\Pi} & & G(gf)\theta_A(x) \\ & & \theta_C F(gf)(x) \arrow[urr, "\theta"'] & & \end{tikzcd}$$

- For each object $A \in ob(\mathcal{T})$, an invertible modification M whose component at a given object $x \in ob(FA)$ arises via coherence from the 2-cell

$$\begin{tikzcd} & \theta_A F1_A(x) \arrow[dr, "\theta"] \arrow[d, phantom, "\Downarrow\widetilde{M}"] & \\ \theta_A(x) \arrow[ur, "1\iota_F"] \arrow[rr, "\iota_G 1"'] & {} & G1_A\theta_A(x) \end{tikzcd}$$

These cells are required to obey the following three simplified axioms.

1. For every triple of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ the following two diagrams are equal:

$$\begin{tikzcd}
 & & & GhGgGf\theta_A(x) \arrow[drr, "\chi_G 1"] \arrow[dd, phantom, "\cong_{\chi_G}"] & & & \\
 & GhGg\theta_B Ff(x) \arrow[urr, "1\theta"] \arrow[drr, "\chi_G 1"] \arrow[dd, phantom, "\Downarrow \widetilde{\Pi}1"] & & & & G(hg)Gf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\
Gh\theta_C FgFf(x) \arrow[ur, "1\theta 1"] & & & G(hg)\theta_B Ff(x) \arrow[urr, "1\theta"] & & & G((hg)f)\theta_A(x) \arrow[d, "Ga1"] \\
\theta_D FhFgFf(x) \arrow[u, "\theta 1"] \arrow[rr, "1\chi_F 1"] \arrow[drr, "1\chi_F"'] & & \theta_D F(hg)Ff(x) \arrow[ur, "\theta 1"] \arrow[rr, "1\chi_F"] \arrow[d, phantom, "\Downarrow 1\widetilde{\omega}_F"] & & \theta_D F((hg)f)(x) \arrow[urr, "\theta"] \arrow[d, "1Fa"] \arrow[u, phantom, "\Downarrow \widetilde{\Pi}"] \arrow[rr, phantom, "\cong_\theta"] & & G(h(gf))\theta_A(x) \\
 & & \theta_D FhF(gf)(x) \arrow[rr, "1\chi_F"'] & & \theta_D F(h(gf))(x) \arrow[urr, "\theta"'] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & & GhGgGf\theta_A(x) \arrow[ddr, "1\chi_G 1"] \arrow[drr, "\chi_G 1"] & & & \\
 & GhGg\theta_B Ff(x) \arrow[urr, "1\theta"] \arrow[dr, phantom, "\Downarrow 1\widetilde{\Pi}"] & & & & G(hg)Gf\theta_A(x) \arrow[dr, "\chi_G 1"] \arrow[dl, phantom, "\Downarrow \widetilde{\omega}_G 1"] & \\
Gh\theta_C FgFf(x) \arrow[ur, "1\theta 1"] \arrow[drr, "1\chi_F"] & & & & GhG(gf)\theta_A(x) \arrow[drr, "\chi_G 1"] \arrow[dd, phantom, "\Downarrow \widetilde{\Pi}"] & & G((hg)f)\theta_A(x) \arrow[d, "Ga1"] \\
\theta_D FhFgFf(x) \arrow[u, "\theta 1"] \arrow[drr, "1\chi_F"] \arrow[r, phantom, "\cong_\theta"] & & Gh\theta_C F(gf)(x) \arrow[urr, "1\theta"] & & & & G(h(gf))\theta_A(x) \\
 & & \theta_D FhF(gf)(x) \arrow[u, "\theta 1"] \arrow[rr, "1\chi_F"] & & \theta_D F(h(gf))(x) \arrow[urr, "\theta"] & &
\end{tikzcd}
$$

2. For every 1-cell $A \xrightarrow{f} B$ the following two diagrams are equal.

$$
\begin{tikzcd}
& & G1_B\theta_BFf(x) \arrow[rr, "1\theta"] \arrow[drr, phantom, "\Downarrow\widetilde{\Pi}"] & & G1_BGf\theta_A(x) \arrow[dr, "\chi_G1"] & \\
\theta_BF1_BFf(x) \arrow[urr, "\theta1"] \arrow[drrr, "1\chi_F"] \arrow[ddrr, phantom, "\Swarrow 1\gamma_F"] & & & & & G(1_Bf)\theta_A(x) \arrow[dd, "Gl1"] \\
& & & \theta_BF(1_Bf)(x) \arrow[urr, "\theta"] \arrow[ddl, "1Fl"] \arrow[dr, phantom, "\cong_\theta"] & & \\
\theta_BFf(x) \arrow[uu, "1\iota_F1"] \arrow[drr, "1"'] & & & & & Gf\theta_A(x) \\
& & \theta_BFf(x) \arrow[urrr, "\theta"] \arrow[rr, "\theta"'] \arrow[r, phantom, shift left=2, "\cong"] & & Gf\theta_A(x) \arrow[ur, "1"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & G1_B\theta_BFf(x) \arrow[r, "1\theta"] \arrow[dd, phantom, "\cong_{\iota_G}"] & G1_BGf\theta_A(x) \arrow[dr, "\chi_G1"] \arrow[ddr, phantom, "\Downarrow\gamma_G1"] & \\
\theta_BF1_BFf(x) \arrow[urr, "\theta1"] \arrow[r, phantom, "\Rrightarrow\widetilde{M}1"] & & & & G(1_Bf)\theta_A(x) \arrow[d, "Gl1"] \\
\theta_BFf(x) \arrow[u, "1\iota_F1"] \arrow[uurr, "\iota_G1"'] \arrow[drr, "1"'] \arrow[drrr, "\theta"] & & & & Gf\theta_A(x) \\
& & \theta_BFf(x) \arrow[r, "\theta"'] \arrow[u, phantom, "\cong"] & Gf\theta_A(x) \arrow[uuu, "\iota_G1"] \arrow[ur, "1"'] &
\end{tikzcd}
$$

3. For each 1-cell $A \xrightarrow{f} B$ the following two diagrams are equal.

$$
\begin{tikzcd}
& & & Gf\theta_AF1_A(x) \arrow[drr, "1\theta"] \arrow[ddr, phantom, "\Downarrow\widetilde{\Pi}"] & & & \\
& Gf\theta_A(x) \arrow[urr, "1\iota_F"] \arrow[d, phantom, "\cong_\theta"] & & & & GfG1_A\theta_A(x) \arrow[ddr, "\chi_G"] & \\
& & \theta_BFfF1_A(x) \arrow[uur, "\theta1"] \arrow[drr, "1\chi_F"] \arrow[dd, phantom, "\Downarrow 1\delta_F^{-1}"] & & & & \\
\theta_BFf(x) \arrow[uur, "\theta"] \arrow[urr, "1\iota_F"] & & & & \theta_BF(f1_A)(x) \arrow[rr, "\theta"] \arrow[d, phantom, "\cong_\theta"] & & G(f1_A)\theta_A(x) \\
& & \theta_BFf(x) \arrow[ull, "1"] \arrow[urr, "1Fr^*"] \arrow[rr, "\theta"'] & & Gf\theta_A(x) \arrow[urr, "Gr^*1"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & & Gf\theta_AF1_A(x) \arrow[drr, "1\theta"] \arrow[d, phantom, "\Downarrow 1\widetilde{M}"] & & & \\
& Gf\theta_A(x) \arrow[urr, "1\iota_F"] \arrow[rrrr, "1\iota_G1"'] & & {} & & GfG1_A\theta_A(x) \arrow[dr, "\chi_G"] \arrow[dl, phantom, "\delta_G^{-1}1"] & \\
\theta_BFf(x) \arrow[ur, "\theta"] \arrow[r, phantom, "\cong"] & & & & {} & & G(f1_A)\theta_A(x) \\
& & \theta_BFf(x) \arrow[ull, "1"] \arrow[rr, "\theta"'] & & Gf\theta_A(x) \arrow[uulll, "1"'] \arrow[urr, "Gr^*1"'] & &
\end{tikzcd}
$$

In this simplified axiom the adjunction from the right unitor becomes an identity and so the mate of the modification δ (as mentioned in definition 3.1.3 just becomes δ. As a result, the cells that appear in these diagrams are just δ^{-1} up to some coherence cell.

The proof of this proceeds similarly to Proposition 4.1.1.

Finally, we can also simplify the description of a biadjoint biequivalence between objects of *Bicat*.

Proposition 4.1.3. A **Biadjoint Biequivalence** between two bicategories $\mathcal{A}$ and $\mathcal{B}$ in the tricategory $\underline{Bicat}$ consists of the following pieces of data:

- A pair of pseudofunctors $S\colon \mathcal{A} \to \mathcal{B}$ and $\Psi\colon \mathcal{B} \to \mathcal{A}$.
- Pseudonatural transformations $\eta\colon 1_{\mathcal{B}} \Rightarrow S\Psi$ and $\eta^*\colon S\Psi \Rightarrow 1_{\mathcal{B}}$ forming an adjoint equivalence $\eta \dashv \eta^*$.
- Pseudonatural transformations $\varepsilon\colon \Psi S \Rightarrow 1_{\mathcal{A}}$ and $\varepsilon^*\colon 1_{\mathcal{A}} \Rightarrow \Psi S$ forming an adjoint equivalence $\varepsilon \dashv \varepsilon^*$.
- An invertible modification Φ whose components at each object x arise via coherence from 2-cells

$$\begin{tikzcd} \Psi(x) \arrow[r, "\Psi\eta"] \arrow[dr, bend right, "1"'] & \Psi S\Psi(x) \arrow[d, "\varepsilon\Psi"] \arrow[dl, phantom, "\overset{\cong}{\Swarrow}\widetilde{\boldsymbol{\Phi}}"] \\ & \Psi(x) \end{tikzcd}$$

- An invertible modification Σ whose components at each object x arise via coherence from 2-cells

$$\begin{tikzcd} S(x) \arrow[r, "\eta S"] \arrow[dr, bend right, "1"'] & S\Psi S(x) \arrow[d, "S\varepsilon"] \arrow[dl, phantom, "\overset{\cong}{\Swarrow}\widetilde{\boldsymbol{\Sigma}}"] \\ & S(x) \end{tikzcd}$$

These are required to obey the simplified axioms that both of the following pasting diagrams are equal to the identity:

1.

$$\begin{tikzcd} & \Psi S(x) \arrow[dr, "\varepsilon"] & \\ \Psi S(x) \arrow[ur, "1"] \arrow[r, "\Psi\eta S"] \arrow[dr, "1"'] \arrow[ur, phantom, "\widetilde{\Phi}^{-1}S"] \arrow[dr, phantom, "\Psi\widetilde{\Sigma}\Swarrow"] & \Psi S\Psi S(x) \arrow[u, "\varepsilon\Psi S"'] \arrow[d, "\Psi S\varepsilon"] \arrow[r, phantom, "\cong_{\varepsilon}"] & (x) \\ & \Psi S(x) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

2.

$$
\begin{tikzcd}
 & S\Psi(x) \arrow[d, "\eta S\Psi"] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3ex, "\widetilde{\Sigma}^{-1}\Psi"] & \\
(x) \arrow[ur, "\eta"] \arrow[dr, "\eta"'] \arrow[r, phantom, "\cong_\eta"] & S\Psi S\Psi(x) \arrow[r, "S\varepsilon\Psi"] & S\Psi(x) \\
 & \Psi S(x) \arrow[u, "S\Psi\eta"] \arrow[ur, "1"'] \arrow[ur, phantom, shift left=3ex, "S\widetilde{\Phi}\ \Rrightarrow"] &
\end{tikzcd}
$$

This simplification, in the specific case of biequivalences, was also noticed in the paper [Gur12, Theorem 3.2] which started by considering biequivalence in the tricategory *Bicat*. By expanding this technique and applying it to all tricategorical constructions we are able to get a great deal of use out of it. The pieces of data and axioms that govern constructions such as biequivalences, trifunctors and tritransformations will be far more easily wielded when it comes to using them in a diagram manipulation proof.

4.2 Moving Pseudonatural Transformations and Modifications

The other main advantage of working in the tricategory *Bicat* is that those cells that aren't coherence cells usually arise from pseudonatural transformations and modifications between bicategories. Our ability to move the pseudonaturality squares through other 2-cells and our ability to move the component cells of a modification through the pseudonaturality squares are key tools in manipulating the diagrams that we will be working with.

However, there is one subtlety, caused by the fact that many of the cells we are interested in have sources and targets that are composites. If our techniques are based only on directly using the definition of pseudonatural transformation, then we are only able to move such a cell through pseudonaturality squares if those squares are attached to the entire source (or entire target) of the cell. This limits our flexibility when attempting to manipulate this cell. In order to expand our options, we will show that we can, in fact, move a 2-cell through any contiguous arrangement of attached pseudonaturality squares.

Proposition 4.2.1. Let α be a pseudonatural transformation between the pseudofunctors $F, G\colon \mathcal{A} \to \mathcal{B}$, and $\beta\colon gf \Rightarrow kh$ be a 2-cell in the source bicategory $\mathcal{A}$ whose sources and targets are composites. Consider a pasting diagram where pseudonaturality squares of α are attached to some contiguous border of the image of β under one of the pseudofunctors (though not necessarily entirely along the source or target). For instance

$$\begin{tikzcd}
 & Fy \arrow[r, "\alpha"] \arrow[d, phantom, "\cong_\alpha"] & Gy \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow G\beta"] & \\
Fx \arrow[ur, "Ff"] \arrow[r, "\alpha"] \arrow[dr, "Fh"'] & Gx \arrow[ur, "Gf"] \arrow[dr, "Gh"] \arrow[d, phantom, "\cong_\alpha"] & & Gw \\
 & Fz \arrow[r, "\alpha"'] & Gz \arrow[ur, "Gk"'] &
\end{tikzcd}$$

Then we are able to move the instance of β through the pseudonaturality squares as the above pasting diagram is equal to

$$\begin{tikzcd}
 & Fy \arrow[r, "\alpha"] \arrow[dr, "Fg"] \arrow[d, phantom, "\Downarrow F\beta"] & Gy \arrow[dr, "Gg"] \arrow[d, phantom, "\cong_\alpha"] & \\
Fx \arrow[ur, "Ff"] \arrow[dr, "Fh"'] & & Fw \arrow[r, "\alpha"] \arrow[d, phantom, "\cong_\alpha"] & Gw \\
 & Fz \arrow[r, "\alpha"'] \arrow[ur, "Fk"] & Gz \arrow[ur, "Gk"'] &
\end{tikzcd}$$

Proof. We will need to construct additional pseudonaturality squares along the source of $G\beta$ in order to use the definition of pseudonaturality. Start by redrawing the first diagram so that the square attached to Gf is aligned upwards, and then add both the remaining pseudonaturality square along the source and also its inverse, keeping the diagram equal to the original.

$$\begin{tikzcd}
 & & Fy \arrow[d, "\alpha"] \arrow[rr, red, "\alpha"] \arrow[dr, red, "Fg"] & & \color{red}{Gy} \arrow[dl, red, "Gg"] \arrow[dl, phantom, "\color{red}{\cong_\alpha}"] \\
 & Fx \arrow[ur, "Ff"] \arrow[d, "\alpha"] \arrow[ddl, "Fh"'] \arrow[r, phantom, "\cong_\alpha"] & Gy \arrow[dr, "Gg"] \arrow[r, phantom, "\color{red}{\cong_\alpha}"] \arrow[d, phantom, "\Downarrow G\beta"] & \color{red}{Fw} \arrow[d, red, "\alpha"] & \\
 & Gx \arrow[ur, "Gf"] \arrow[dr, "Gh"] \arrow[dl, phantom, "\cong_\alpha"] & & Gw & \\
Fz \arrow[rr, "\alpha"'] & & Gz \arrow[ur, "Gk"'] & &
\end{tikzcd}$$

With a complete set of pseudonaturality squares along the source, we are able to use the definition of pseudonaturality to move β through them. This gives:

$$\begin{tikzcd}
 & & Fy \arrow[rr, "\alpha"] \arrow[dr, "Fg"] \arrow[dd, phantom, "\Downarrow F\beta"] & & Gy \arrow[ddl, "Gg"] \\
 & Fx \arrow[ur, "Ff"] \arrow[d, red, "\alpha"] \arrow[dr, "Fh"] \arrow[ddl, red, "Fh"'] & & Fw \arrow[d, "\alpha"] \arrow[ur, phantom, "\cong_\alpha"] & \\
 & \color{red}{Gx} \arrow[r, phantom, red, "\cong_\alpha"] \arrow[dr, red, "Gh"] & Fz \arrow[ur, "Fk"] \arrow[d, "\alpha"] \arrow[r, phantom, "\cong_\alpha"] & Gw & \\
\color{red}{Fz} \arrow[rr, red, "\alpha"'] \arrow[ur, phantom, red, "\cong_\alpha"] & & Gz \arrow[ur, "Gk"'] & &
\end{tikzcd}$$

Once the pseudonaturality squares in the bottom left have cancelled out, we are left with the diagram we were expecting and have successfully moved the instance of β through the original pair of pseudonaturality squares. □

Other possible variations of pseudonaturality squares attached to a 2-cell can be considered as corollaries of Proposition 4.2.1. For instance:

- In the situation where we start with the pseudonaturality cells attached closer to the target of the 1-cells, we have a diagram analogous to

$$\begin{tikzcd}
 & Fy \arrow[r, "\alpha"] \arrow[dr, "Fg"] \arrow[dd, phantom, "\Downarrow F\beta"] & Gy \arrow[dr, "Gg"] \arrow[d, phantom, "\cong_\alpha"] & \\
Fx \arrow[ur, "Ff"] \arrow[dr, "Fh"'] & & Fw \arrow[r, "\alpha"] \arrow[d, phantom, "\cong_\alpha"] & Gw \\
 & Fz \arrow[ur, "Fk"] \arrow[r, "\alpha"'] & Gz \arrow[ur, "Gk"'] &
\end{tikzcd}$$

  As the proposition shows that this diagram is equal to

$$\begin{tikzcd}
 & Fy \arrow[r, "\alpha"] \arrow[d, phantom, "\cong_\alpha"] & Gy \arrow[dr, "Gg"] \arrow[dd, phantom, "\Downarrow G\beta"] & \\
Fx \arrow[ur, "Ff"] \arrow[r, "\alpha"] \arrow[dr, "Fh"'] & Gx \arrow[ur, "Gf"'] \arrow[dr, "Gh"] \arrow[d, phantom, "\cong_\alpha"] & & Gw \\
 & Fz \arrow[r, "\alpha"'] & Gz \arrow[ur, "Gk"'] &
\end{tikzcd}$$

  we are able to move the 2-cell through the pseudonaturality cells if we choose.

- If we want to consider a situation where the pseudonaturality squares cover the entire source (or target) this is equivalent by coherence to a situation where the morphism

g (vis-a-vis k) in Proposition 4.2.1 is an identity so that that part of the construction has no effect.

- If we want to consider a situation where the pseudonaturality squares are only attached to the source (or only attached to the target) this is equivalent by coherence to a situation where the morphism h (vis-a-vis f) in Proposition 4.2.1 is an identity so that the pseudonaturality cells we see attached there in the proposition is just a coherence cell.

Similarly, we are able to prove a proposition that will allow us to move a modification cell through an arrangement of pseudonaturality squares even if they do not cover the entire source (or target).

Proposition 4.2.2. Let $M\colon \beta\alpha \Rrightarrow \delta\gamma$ be a modification whose source and target are composites of the pseudonatural transformations $\alpha\colon F \Rightarrow G$, $\beta\colon G \Rightarrow K$, $\gamma\colon F \Rightarrow H$ and $\delta\colon H \Rightarrow K$. Consider a pasting diagram where pseudonaturality squares are attached to some contiguous border of a component of M though not necessarily entirely along the source or target. For instance

$$\begin{tikzcd}
& Gx \arrow[r, "Gf"] \arrow[d, phantom, "\cong_\alpha"] & Gy \arrow[dr, "\beta_y"] \arrow[dd, phantom, "\Downarrow M_y"] & \\
Fx \arrow[ur, "\alpha_x"] \arrow[r, "Ff"] \arrow[dr, "\gamma_x"'] & Fy \arrow[ur, "\alpha_y"'] \arrow[dr, "\gamma_y"] \arrow[d, phantom, "\cong_\gamma"] & & Ky \\
& Hx \arrow[r, "Hf"'] & Hy \arrow[ur, "\delta_y"'] &
\end{tikzcd}$$

Then we are able to move the instance of β through the pseudonaturality squares as the above pasting diagram is equal to

$$\begin{tikzcd}
& Gx \arrow[r, "Gf"] \arrow[dr, "\beta_x"] \arrow[dd, phantom, "\Downarrow M_x"] & Gy \arrow[dr, "\beta_y"] \arrow[d, phantom, "\cong_\beta"] & \\
Fx \arrow[ur, "\alpha_x"] \arrow[dr, "\gamma_x"'] & & Kx \arrow[r, "Kf"] \arrow[d, phantom, "\cong_\delta"] & Ky \\
& Hx \arrow[r, "Hf"'] \arrow[ur, "\delta_x"] & Hy \arrow[ur, "\delta_y"'] &
\end{tikzcd}$$

Proof. We will need to construct additional pseudonaturality squares along the source of M in order to use the fact that it is a modification. Start by redrawing the first diagram so that the square attached to α_y is aligned upwards, and then add both the remaining pseudonaturality square along the source and also its inverse, keeping the diagram equal to the original.

$$\begin{tikzcd}
 & & Gx \arrow[rr, "Gf"] \arrow[d, "Gf"] \arrow[dr, "\beta_x"] & & Gy \arrow[ddl, "\beta_y"] \\
 & Fx \arrow[ur, "\alpha_x"] \arrow[d, "Ff"] \arrow[ddl, "\gamma_x"'] \arrow[r, phantom, "\cong_\alpha"] & Gy \arrow[dr, "\beta_y"] \arrow[dd, phantom, "\Downarrow M_y"] \arrow[r, phantom, "\cong_\beta"] & Kx \arrow[d, "Kf"] \arrow[ur, phantom, "\cong_\beta"] & \\
 & Fy \arrow[ur, "\alpha_y"] \arrow[dr, "\gamma_y"] \arrow[dl, phantom, "\cong_\gamma"] & & Ky & \\
Hx \arrow[rr, "Hf"'] & & Hy \arrow[ur, "\delta_y"'] & &
\end{tikzcd}$$

With a complete set of pseudonaturality squares along the source, we are able to use the definition of modification to move M through them. This gives:

$$\begin{tikzcd}
 & & Gx \arrow[rr, "Gf"] \arrow[dr, "\beta_x"] \arrow[dd, phantom, "\Downarrow M_x"] & & Gy \arrow[ddl, "\beta_y"] \\
 & Fx \arrow[ur, "\alpha_x"] \arrow[d, "Ff"] \arrow[dr, "\gamma_x"] \arrow[ddl, "\gamma_x"'] & & Kx \arrow[d, "Kf"] \arrow[ur, phantom, "\cong_\beta"] & \\
 & Fy \arrow[dr, "\gamma_y"] \arrow[r, phantom, "\cong_\gamma"] \arrow[dl, phantom, "\cong_\gamma"] & Hx \arrow[ur, "\delta_x"] \arrow[d, "Hf"] \arrow[r, phantom, "\cong_\delta"] & Ky & \\
Hx \arrow[rr, "Hf"'] & & Hy \arrow[ur, "\delta_y"'] & &
\end{tikzcd}$$

Once the pseudonaturality squares in the bottom left have cancelled out, we are left with the diagram we were expecting and have successfully moved the instance of M through the original pair of pseudonaturality squares. $\square$

As with the previous proposition, other potential arrangements can be considered as corollaries or special cases of this one.

Together, these two propositions provide plenty of tools for the diagram manipulations we will need to make when showing that the structure of a trifunctor can be transported. We are therefore ready to move on to proving the main result.

Chapter 5

Transporting a Trifunctor

With the techniques developed in the previous chapter, we are now ready to begin the transport of a trifunctor.

We start with a trifunctor $F : \mathcal{T} \Rightarrow \underline{Bicat}$ and a family of biequivalences between the bicategories FA and GA for each object $A \in ob(\mathcal{T})$ that we want to transport the trifunctor across. From these we will extract the cells that we are able to use to construct the transported trifunctor $G : \mathcal{T} \Rightarrow \underline{Bicat}$.

Once we have a catalogue of all available cells, we will begin constructing G. We will work through the (simplified) definition of a trifunctor step by step, identifying each coherence cell for G that we need to define and showing how they can be constructed from the cells given by F and the biequivalences. Each of these cells will form suitably natural structures: e.g. the 3-dimensional cells of G will be constructed from modifications in the definition of F and the biequivalences, as well as coherence cells, and so will also be modifications as required.

Once all of the cells of G have been constructed, we will prove that they do form a trifunctor by proving the two trifunctor axioms. We will do this by identifying the pasting diagrams that form the two sides of each axiom and then we will manipulate one into the other step-by-step. Since each manipulation will result in an equal diagram, this will prove that the two sides of the axiom are equal.

5.1 Setup: Available Data

In this section we will collect and overview the categorical data we have available for constructing the transported trifunctor. We start with the original trifunctor $F : \mathcal{T} \to \underline{Bicat}$ that we want to transport. As this is a trifunctor into the tricategory of bicategories, we can use the coherence theorem to give us the simplified definition (see Proposition 4.1.1), meaning that this trifunctor consists of:

- A function $ob(\mathcal{T}) \to ob(\underline{Bicat})$.
- For each pair of objects $A, B \in ob(\mathcal{T})$, a pseudofunctor $\mathcal{T}(A,B) \to \underline{Bicat}(FA, FB)$.
- For each triple of objects $A, B, C \in ob(\mathcal{T})$, an adjoint equivalence

$$\begin{tikzcd} \mathcal{T}(B,C) \times \mathcal{T}(A,B) \arrow[r, "F \times F"] \arrow[d, "\otimes"] & \underline{Bicat}(FB,FC) \times \underline{Bicat}(FA,FB) \arrow[d, "\otimes"] \arrow[dl, phantom, "\Leftarrow \boldsymbol{\chi}"] \\ \mathcal{T}(A,C) \arrow[r, "F"'] & \underline{Bicat}(FA,FC) \end{tikzcd}$$

- For each object $A \in ob(\mathcal{T})$, an adjoint equivalence

$$\begin{tikzcd} 1 \arrow[rr, "id_{FA}"] \arrow[dr, "id_A"'] & \Downarrow \boldsymbol{\iota} & \underline{Bicat}(FA,FA) \\ & \mathcal{T}(A,A) \arrow[ur, "F"'] & \end{tikzcd}$$

- For every string of three composable morphisms in $\mathcal{T}$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

  and object $x \in ob(FA)$, a 2-cell

$$\begin{tikzcd} & F(hg)Ff(x) \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow \widetilde{\omega}"] & F((hg)f)(x) \arrow[dr, "Fa"] & \\ FhFgFf(x) \arrow[ur, "\chi 1"] \arrow[r, "1\chi"'] & FhF(gf)(x) \arrow[rr, "\chi"'] & & F(h(gf))(x) \end{tikzcd}$$

- For each morphism $f : A \to B$ in $\mathcal{T}$ and object $x \in ob(FA)$, a 2-cell

$$\begin{tikzcd} & F1_BFf(x) \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow\gamma"] & F(1_Bf)(x) \arrow[dr, "Fl"] & \\ Ff(x) \arrow[ur, "\iota 1"] \arrow[rrr, "1"'] & & & Ff(x) \end{tikzcd}$$

- For each morphism $f : A \to B$ in $\mathcal{T}$ and object $x \in ob(FA)$, a 2-cell

$$\begin{tikzcd}
Ff(x) \arrow[rrr, "Fr^*"] \arrow[dr, "1"'] & & & F(f1_A)(x) \\
& Ff(x) \arrow[r, "1\iota"'] \arrow[u, phantom, "\Downarrow\delta" near end, shift left=8ex] & FfF1_A(x) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

satisfying the axioms

- **First Trifunctor Axiom**: For every four composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

the two diagrams

$$\begin{tikzcd}[column sep=small]
& F((kh)g)Ff(x) \arrow[r, "\chi"] \arrow[dr, "FaFf"] & F(((kh)g)f)(x) \arrow[dr, "F(af)"] \arrow[d, phantom, "\cong_\chi"] & & \\
& & F(k(hg))Ff(x) \arrow[r, "\chi"] & F((k(hg))f)(x) \arrow[dr, "Fa"] \arrow[dd, phantom, "\Downarrow\widetilde{\omega}"] & \\
F(kh)FgFf(x) \arrow[uur, "\chi Ff"] \arrow[rr, phantom, "\widetilde{\omega}Ff"] & & & & F(k((hg)f))(x) \arrow[d, "F(ka)"] \\
& & FkF(hg)Ff(x) \arrow[uu, "\chi Ff"] \arrow[r, "Fk\chi"] & FkF((hg)f)(x) \arrow[ur, "\chi"] \arrow[dr, "FkFa"] \arrow[r, phantom, "\cong_\chi"] \arrow[dl, phantom, "\Downarrow Fk\widetilde{\omega}"] & F(k(h(gf)))(x) \\
& FkFhFgFf(x) \arrow[uul, "\chi FgFf"] \arrow[ur, "Fk\chi_F Ff"] \arrow[dr, "FkFh\chi"'] & & & FkF(h(gf))(x) \arrow[u, "\chi"] \\
& & FkFhF(gf)(x) \arrow[urr, "Fk\chi"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
& F((kh)g)Ff(x) \arrow[r, "\chi"] & F(((kh)g)f)(x) \arrow[dr, "F(af)"] \arrow[ddr, "Fa"'] & & \\
& & \arrow[d, phantom, "\Downarrow \widetilde{\omega}"] & F((k(hg))f)(x) \arrow[dr, "Fa"] \arrow[d, phantom, "\Leftarrow F\pi"] & \\
F(kh)FgFf(x) \arrow[uur, "\chi Ff"] \arrow[drr, "F(kh)\chi"] & & {} & F((kh)(gf))(x) \arrow[dr, "Fa"] \arrow[d, phantom, "\Downarrow \widetilde{\omega}"] & F(k((hg)f))(x) \arrow[d, "F(ka)"] \\
& \arrow[r, phantom, "\cong_\chi"] & F(kh)F(gf)(x) \arrow[ur, "\chi"] & {} & F(k(h(gf)))(x) \\
& FkFhFgFf(x) \arrow[uul, "\chi FgFf"] \arrow[dr, "FkFh\chi"] & & & FkF(h(gf))(x) \arrow[u, "\chi"] \\
& & FkFhF(gf)(x) \arrow[uu, "\chi F(gf)"] \arrow[urr, "Fk\chi"] & &
\end{tikzcd}$$

are equal.

- **Second Trifunctor Axiom**: For every pair of composable 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

the two diagrams

$$\begin{tikzcd}
& F((g1_B)f)(x) \arrow[rr, "Fa"] & & F(g(1_Bf))(x) \arrow[dr, "F(gl)"] & \\
F(gf)(x) \arrow[ur, "F(r^*f)"] \arrow[r, phantom, "\cong_\chi"] & F(g1_B)Ff(x) \arrow[u, "\chi"] \arrow[rr, phantom, "\overset{\widetilde{\omega}}{\Longrightarrow}"] & & FgF(1_Bf)(x) \arrow[u, "\chi"] \arrow[dr, "FgFl"] \arrow[r, phantom, "\cong_\chi"] & F(gf)(x) \\
FgFf(x) \arrow[u, "\chi"] \arrow[ur, "Fr^*Ff"] \arrow[drr, "1"'] \arrow[r, phantom, "\overset{\delta}{\Rightarrow}"] & & FgF1_BFf(x) \arrow[ul, "\chi Ff"'] \arrow[ur, "Fg\chi"] \arrow[r, phantom, "\overset{\gamma}{\Rightarrow}"] & & FgFf(x) \arrow[u, "\chi"'] \\
& & FgFf(x) \arrow[u, "Fg\iota Ff"] \arrow[urr, "1"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& F((g1_B)f)(x) \arrow[rr, "Fa"] \arrow[rr, phantom, shift right=6ex, "\Downarrow F\mu"] & & F(g(1_Bf))(x) \arrow[dr, "F(gl)"] & \\
F(gf)(x) \arrow[ur, "F(r^*f)"] \arrow[rrrr, "1"'] & & & & F(gf)(x) \\
FgFf(x) \arrow[u, "\chi"] \arrow[rrrr, "1"'] \arrow[rrrru, phantom, "\cong"] & & & & FgFf(x) \arrow[u, "\chi"']
\end{tikzcd}$$

are equal.

We then need to identify where we are transporting the trifunctor to. We take a function $G : ob(\mathcal{T}) \to ob(\underline{Bicat})$ which will become the action on objects of our transported trifunctor.

Finally, we have the biequivalences along which we will transport the structure of a trifunctor. We need a biadjoint biequivalence for each object of $\mathcal{T}$ so that we can take the trifunctor from its original output at FA and translate it to what it should be at GA. We thus have a biadjoint biequivalence in $\underline{Bicat}$ for every object $A \in ob(\mathcal{T})$. Each of these biequivalences consists of (again: because we are working in $\underline{Bicat}$ we can use the simpler version given in Proposition 4.1.3):

- A pair of pseudofunctors $S_A : FA \to GA$ and $\Psi_A : GA \to FA$.
- Pseudonatural transformations $\eta_A : 1_{\mathcal{GA}} \Rightarrow S_A\Psi_A$ and $\eta^*_A : S_A\Psi_A \Rightarrow 1_{GA}$ forming an adjoint equivalence $\eta_A \dashv \eta^*_A$.
- Pseudonatural transformations $\varepsilon_A : \Psi_A S_A \Rightarrow 1_{FA}$ and $\varepsilon^*_A : 1_{FA} \Rightarrow \Psi_A S_A$ forming an adjoint equivalence $\varepsilon_A \dashv \varepsilon^*_A$.
- An invertible modification Φ_A whose components at each object x arise via coherence from 2-cells

$$\begin{tikzcd}
\Psi_A(x) \arrow[r, "\Psi_A\eta_A"] \arrow[dr, bend right, "1"'] & \Psi_A S_A \Psi_A(x) \arrow[d, "\varepsilon_A\Psi_A"] \arrow[dl, phantom, "\overset{\cong}{\Swarrow}\widetilde{\mathbf{\Phi_A}}"] \\
& \Psi_A(x)
\end{tikzcd}$$

- An invertible modification Σ_A whose components at each object x arise via coherence from 2-cells

$$\begin{tikzcd}
S_A(x) \arrow[r, "\eta_A S_A"] \arrow[dr, bend right, "1"'] & S_A\Psi_A S_A(x) \arrow[d, "S_A\varepsilon_A"] \arrow[dl, phantom, "\overset{\cong}{\Swarrow}\, \widetilde{\mathbf{\Sigma_A}}"] \\
 & S_A(x)
\end{tikzcd}$$

With the two simpler axioms that the following two diagrams

1.

$$\begin{tikzcd}
 & \Psi_A S_A(x) \arrow[dr, "\varepsilon_A"] & \\
\Psi_A S_A(x) \arrow[ur, "1"] \arrow[r, "\Psi_A\eta_A S_A"] \arrow[dr, "1"'] \arrow[ur, phantom, shift right=3ex, "\widetilde{\Phi}_A^{-1} S_A"] \arrow[dr, phantom, shift left=3ex, "\Psi_A\widetilde{\Sigma}_A \Swarrow"] & \Psi_A S_A \Psi_A S_A(x) \arrow[u, "\varepsilon_A \Psi_A S_A"'] \arrow[d, "\Psi_A S_A \varepsilon_A"] \arrow[r, phantom, "\cong_{\varepsilon_A}"] & (x) \\
 & \Psi_A S_A(x) \arrow[ur, "\varepsilon_A"'] &
\end{tikzcd}$$

2.

$$\begin{tikzcd}
 & S_A\Psi_A(x) \arrow[dr, "1"] \arrow[d, "\eta_A S_A \Psi_A"] & \\
(x) \arrow[ur, "\eta_A"] \arrow[dr, "\eta_A"'] \arrow[r, phantom, "\cong_{\eta_A}"] & S_A\Psi_A S_A\Psi_A(x) \arrow[r, "S_A\varepsilon_A\Psi_A"] \arrow[ur, phantom, shift left=3ex, "\widetilde{\Sigma}_A^{-1}\Psi_A"] \arrow[dr, phantom, shift right=3ex, "\Searrow\, S\widetilde{\Phi}_A"] & S_A\Psi_A(x) \\
 & \Psi_A S_A(x) \arrow[u, "S_A\Psi_A\eta_A"] \arrow[ur, "1"'] &
\end{tikzcd}$$

are both equal to the identity.

These are the data from which we will construct our transported trifunctor and these are the axioms which, along with the properties of pseudonatural transformations and modifications, will allow us to prove that our constructions do satisfy the properties of a trifunctor (and higher cells between trifunctors).

5.2 Construction of the Transported Trifunctor

Given the original trifunctor $F : \mathcal{T} \to \underline{Bicat}$ and the object-indexed biequivalences $FA \to GA$, we first need to construct the transported trifunctor $G : \mathcal{T} \to \underline{Bicat}$. We construct each of the components of G as follows.

5.2.1 Data with Dimension less than Three

- The action of G on objects has already been decided: it is given by the function $G : ob(\mathcal{T}) \to ob(\underline{Bicat})$.

- In order to define the action of G on the hom-bicategory $\mathcal{T}(A, B)$, we first use the part of the biequivalence Ψ_A in order to go from GA to FA, then use the action of the trifunctor F on the hom-bicategory getting us to FB, then compose with the other part of the biequivalence S_B in order to arrive at GB. This makes the action of G on the 1-cells $Gf = S_B F f \Psi_A$. However, we will need to choose an association in order to be able to state that the constructions of the higher cells are actually pseudonatural transformations and modifications. Both associations work; for this thesis we will arbitrarily pick that the action of G should be given by

$$\mathcal{T}(A, B) \xrightarrow{F} \underline{Bicat}(F(A), F(B)) \xrightarrow{-\otimes\Psi_A} \underline{Bicat}(G(A), F(B)) \xrightarrow{S_B\otimes -} \underline{Bicat}(G(A), G(B))$$

- When constructing the compositor χ_G, we note that the source of any given component of the compositor is going to be a 1-cell of the form $GgGf = S_C F g \Psi_B S_B F f \Psi_A$, where the two instances of F applied to a 1-cell are separated by instances of 1-cells from the biequivalence. Therefore, our strategy will be to use the appropriate 2-cell from the biequivalence to cancel those out, bringing the instances of F together so that the compositor of that trifunctor can be used. After taking care of the associations in order to bring the pairs of 1-cells together and apply the two cells, the compositor of G is given by:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{T}(B,C)\times\mathcal{T}(A,B) \arrow[r, "\otimes"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\Rightarrow \chi_F"] & \mathcal{T}(A,C) \arrow[d, "F"] \\
 & \underline{Bicat}(F(B),F(C))\times\underline{Bicat}(F(A),F(B)) \arrow[r, "\otimes"] \arrow[d, "1\times(-\otimes\Psi_A)"'] \arrow[dr, phantom, "\Rightarrow a^*"] & \underline{Bicat}(F(A),F(C)) \arrow[dd, "-\otimes\Psi_A"] \\
 & \underline{Bicat}(F(B),F(C))\times\underline{Bicat}(G(A),F(B)) \arrow[dl, "(-\otimes\Psi_B)\times 1"'] \arrow[d, bend right, "(-\otimes(\Psi_B\otimes S_B))\times 1"'] \arrow[d, bend left, "(-\otimes 1_{F(B)})\times 1"] \arrow[d, phantom, "\Rightarrow (-\otimes\varepsilon_B)\times 1"] \arrow[dr, "\otimes"] \arrow[dr, phantom, "\Rightarrow \otimes(r\times 1)"'] & \\
\underline{Bicat}(G(B),F(C))\times\underline{Bicat}(G(A),F(B)) \arrow[r, "(-\otimes S_B)\times 1"] \arrow[ur, phantom, "\Rightarrow a\times 1"] \arrow[dr, "1\times(S_B\otimes -)"'] & \underline{Bicat}(F(B),F(C))\times\underline{Bicat}(G(A),F(B)) \arrow[r, "\otimes"] & \underline{Bicat}(G(A),F(C)) \arrow[dd, "S_C\otimes -"] \\
 & \underline{Bicat}(G(B),F(C))\times\underline{Bicat}(G(A),G(B)) \arrow[u, phantom, "\Uparrow a^*"] \arrow[ur, "\otimes"'] \arrow[d, "(S_C\otimes -)\times 1"'] \arrow[dr, phantom, "\Rightarrow a"] & \\
 & \underline{Bicat}(G(B),G(C))\times\underline{Bicat}(G(A),G(B)) \arrow[r, "\otimes"'] & \underline{Bicat}(G(A),G(C))
\end{tikzcd}$$

This is a pseudonatural transformation because it is constructed from the pseudonatural transformation χ_F, coherence cells, and the 2-cell ε_B being applied to a different part of the composition (which is pseudonatural by the functoriality of composition).

We will note now that by the usual method of using the naturality of coherence cells and then the coherence theorem for bicategories, any modification starting or ending at χ_G has components that correspond exactly to a 2-cell starting or ending at

$$S_C F g \Psi_B S_B F f \Psi_A \xrightarrow{S_C F g \varepsilon_B F f \Psi_A} S_C F g F f \Psi_A \xrightarrow{S_C \chi_F \Psi_A} S_C F(gf)\Psi_A$$

- When defining the unitor ι_G, we note that we want the target of each component to be $G1_A = S_A F 1_A \Psi_A$. We will need to use an instance of the 2-cell η_A first in order to introduce those cells from the biequivalence, then use the unitor ι_F to introduce $F1_A$. This gives the unitor of G as:

$$\begin{tikzcd}
1 \arrow[rrr, "1_{G(A)}"] \arrow[d, "1_A"'] \arrow[dr, "1_{F(A)}"] \arrow[drr, "\Psi_A"'] \arrow[d, phantom, "\Leftarrow \iota_F"'] \arrow[drr, phantom, "\Downarrow l^*"'] \arrow[drrr, phantom, "\Rightarrow \eta_A"] & & & \underline{Bicat}(G(A),G(A)) \\
\mathcal{T}(A,A) \arrow[r, "F"'] & \underline{Bicat}(F(A),F(A)) \arrow[rr, "-\otimes\Psi_A"'] & & \underline{Bicat}(G(A),F(A)) \arrow[u, "S_A\otimes -"']
\end{tikzcd}$$

Again, we note that any modification starting or ending here has components that correspond uniquely to a 2-cell with source or target

$$1_{GA} \xrightarrow{\eta_A} S_A\Psi_A \xrightarrow{\iota_F} S_A F 1_A \Psi_A$$

That concludes the construction of all the data for G with dimension less than 3.

5.2.2 Three-Dimensional Data

We now need to construct the modifications for the trifunctor. As noted, we will be able to take the components of these modifications, simplify the sources and targets by removing coherence cells, and consider the unique cell the components correspond to under coherence. This will make the process of constructing the cells easier.

Each of the following constructions will give us a modification. This is because the cells are constructed out of the modifications coming from the trifunctor F, coherence cells and applications of 3-cells from the biequivalence to other parts of the composition (These give a modification by the functoriality of composition).

- In order to construct the modification ω_G, we will construct the corresponding cell $\widetilde{\omega}_G$ which is given by:

$$\begin{tikzcd}
\begin{array}{c} S_D F h \Psi_C S_C F g \\ \Psi_B S_B F f \Psi_A(x) \end{array} \arrow[r, "S_D F h \varepsilon_C F g \Psi_B S_B F f \Psi_A(x)"] \arrow[d, "S_D F h \Psi_C S_C F g \varepsilon_B F f \Psi_A(x)"'] \arrow[dr, phantom, "\cong_{\varepsilon_C}"] & \begin{array}{c} S_D F h F g \\ \Psi_B S_B F f \Psi_A(x) \end{array} \arrow[r, "S_D \chi_F \Psi_B S_B F f \Psi_A(x)"] \arrow[d, "S_D F h F g \varepsilon_B F f \Psi_A(x)"] \arrow[dr, phantom, "\cong_{\chi_F}"] & \begin{array}{c} S_D F(hg) \\ \Psi_B S_B F f \Psi_A(x) \end{array} \arrow[d, "S_D F(hg) \varepsilon_B F f \Psi_A(x)"] \\
\begin{array}{c} S_D F h \Psi_C S_C \\ F g F f \Psi_A(x) \end{array} \arrow[r, "S_D F h \varepsilon_C F g F f \Psi_A(x)"] \arrow[dd, "S_D F h \Psi_C S_C \chi_F \Psi_A(x)"'] \arrow[ddr, phantom, "\cong_{\varepsilon_C}"] & S_D F h F g F f \Psi_A(x) \arrow[r, "S_D \chi_F F f \Psi_A(x)"] \arrow[dd, "S_D F h \chi_F \Psi_A(x)"] \arrow[ddr, phantom, "\Swarrow S_D \widetilde{\omega}_{F, \Psi_A(x)}"] & S_D F(hg) F f \Psi_A(x) \arrow[d, "S_D \chi_F \Psi_A(x)"] \\
 & & S_D F((hg)f) \Psi_A(x) \arrow[d, "S_D F a \Psi_A(x)"] \\
\begin{array}{c} S_D F h \Psi_C S_C \\ F(gf) \Psi_A(x) \end{array} \arrow[r, "S_D F h \varepsilon_C F(gf) \Psi_A(x)"'] & S_D F h F(gf) \Psi_A(x) \arrow[r, "S_D \chi_F \Psi_A(x)"'] & S_D F(h(gf)) \Psi_A(x)
\end{tikzcd}$$

The strategy for constructing this cell was to notice that the final two 1-cells in the source and the final 1-cell in the target match those of $\widetilde{\omega}_F$ indicating that we should use a copy of that cell there. Once that $\widetilde{\omega}_F$ has been inserted, the remaining 1-cells match those of the source and target and need only be interchanged using pseudonaturality cells.

- We next construct the modification γ_G, again by considering the corresponding cell $\widetilde{\gamma}_G$. This cell is given by the diagram

$$
\begin{tikzcd}
\begin{array}{c} S_B F 1_B \Psi_B \\ S_B F f \Psi_A(x) \end{array} \arrow[rr, "\begin{array}{c} S_B F 1_B \varepsilon_B \\ F f \Psi_A(x) \end{array}"] \arrow[dr, phantom, "\cong_{\iota_F}"] & & \begin{array}{c} S_B F 1_B \\ F f \Psi_A(x) \end{array} \arrow[d, "S_B \chi_F \Psi_A(x)"] \\
\begin{array}{c} S_B \Psi_B S_B \\ F f \Psi_A(x) \end{array} \arrow[u, "\begin{array}{c} S_B \iota_F \Psi_B \\ S_B F f \Psi_A(x) \end{array}"] \arrow[r, "\begin{array}{c} S_B \varepsilon_B \\ F f \Psi_A(x) \end{array}"] \arrow[dr, phantom, "\Downarrow \widetilde{\Sigma}_B F f \Psi_A(x)"] & S_B F f \Psi_A(x) \arrow[ur, "\begin{array}{c} S_B \iota_F \\ F f \Psi_A(x) \end{array}"'] \arrow[dr, "1"] \arrow[d, phantom, "\cong"] & S_B F (1_B f) \Psi_A(x) \arrow[d, "S_B F l \Psi_A(x)"] \arrow[dl, phantom, "\Swarrow S_B \gamma_F \Psi_A(x)"] \\
S_B F f \Psi_A(x) \arrow[u, "\begin{array}{c} \eta_B S_B \\ F f \Psi_A(x) \end{array}"] \arrow[ur, "1"'] \arrow[rr, "1"'] & & S_B F f \Psi_A(x)
\end{tikzcd}
$$

  Here, the strategy was to first note that the final part of the source could have a copy of $\widetilde{\gamma}_F$ attached to them. Then, there are both an η_B cell and an ε_B cell present from the biequivalence: bringing them together using pseudonaturality allows them to be cancelled out by the 3-cell $\widetilde{\Sigma}_B$ from the biequivalence.

- We construct the final modification δ_G, and the corresponding cell $\widetilde{\delta}_G$, in a similar way to γ_G.

$$
\begin{tikzcd}[column sep=large, row sep=large]
 & \begin{matrix} S_B F f \Psi_A \\ S_A F 1_A \Psi_A(x) \end{matrix} \arrow[dr, "\substack{S_B F f \varepsilon_A \\ F 1_A \Psi_A(x)}"] \arrow[d, phantom, "\cong_{\varepsilon_A}"] & \\
\begin{matrix} S_B F f \Psi_A \\ S_A \Psi_A(x) \end{matrix} \arrow[ur, "\substack{S_B F f \Psi_A \\ S_A \iota_F \Psi_A(x)}"] \arrow[r, "\substack{S_B F f \\ \varepsilon_A \Psi_A(x)}"] & S_B F f \Psi_A(x) \arrow[r, "\substack{S_B F f \\ \iota_F \Psi_A(x)}"] & \begin{matrix} S_B F f \\ F 1_A \Psi_A(x) \end{matrix} \arrow[d, "S_B \chi_F \Psi_A(x)"] \\
S_B F f \Psi_A(x) \arrow[u, "\substack{S_B F f \\ \Psi_A \eta_A(x)}"] \arrow[ur, "1"'] \arrow[u, phantom, shift right=6ex, "\Leftarrow \substack{S_B F f \\ \widetilde{\Phi}_A^{-1}(x)}"] \arrow[rr, "S_B F r^* \Psi_A(x)"'] \arrow[urr, phantom, "\Rightarrow S_B \delta_F \Psi_A(x)"] & & S_B F(f 1_A) \Psi_A(x)
\end{tikzcd}
$$

5.3 First Trifunctor Axiom

Having constructed all the data for G, we now need to show that they form a trifunctor $\mathcal{T} \to \underline{Bicat}$. We do this by proving that both trifunctor axioms hold for the data given above.

Proposition 5.3.1. $G : \mathcal{T} \to \underline{Bicat}$ satisfies the first trifunctor axiom.

Proof. After substituting the data for G into the first trifunctor axiom (the version after it has been simplified using the coherence theorem for bicategories), we are asked to prove that the following source and target diagrams (Figures 5.1 and 5.2) are equal.

We will start with the first of these diagrams (the source of the axiom) and manipulate it to reach the second (the target), proving they are equal.

Looking at both diagrams (see Figures 5.3 and 5.4), we note that the cells around the upper-right corner in both of them are reminiscent of the cells in the first trifunctor axiom for $F : \mathcal{T} \to \underline{Bicat}$. The only differences are that these are composed with S_E and Ψ_A (as expected, given how G was constructed) and that the entire diagram for the axiom is not yet complete.

This suggests the strategy for proving that the two diagrams are equal: move the other instances of $\widetilde{\omega}_F$ in the first diagram towards the upper right using the diagram manipulation techniques provided by Propositions 4.2.1 and 4.2.2. This will let us complete and then use an instance of the first trifunctor axiom for F.

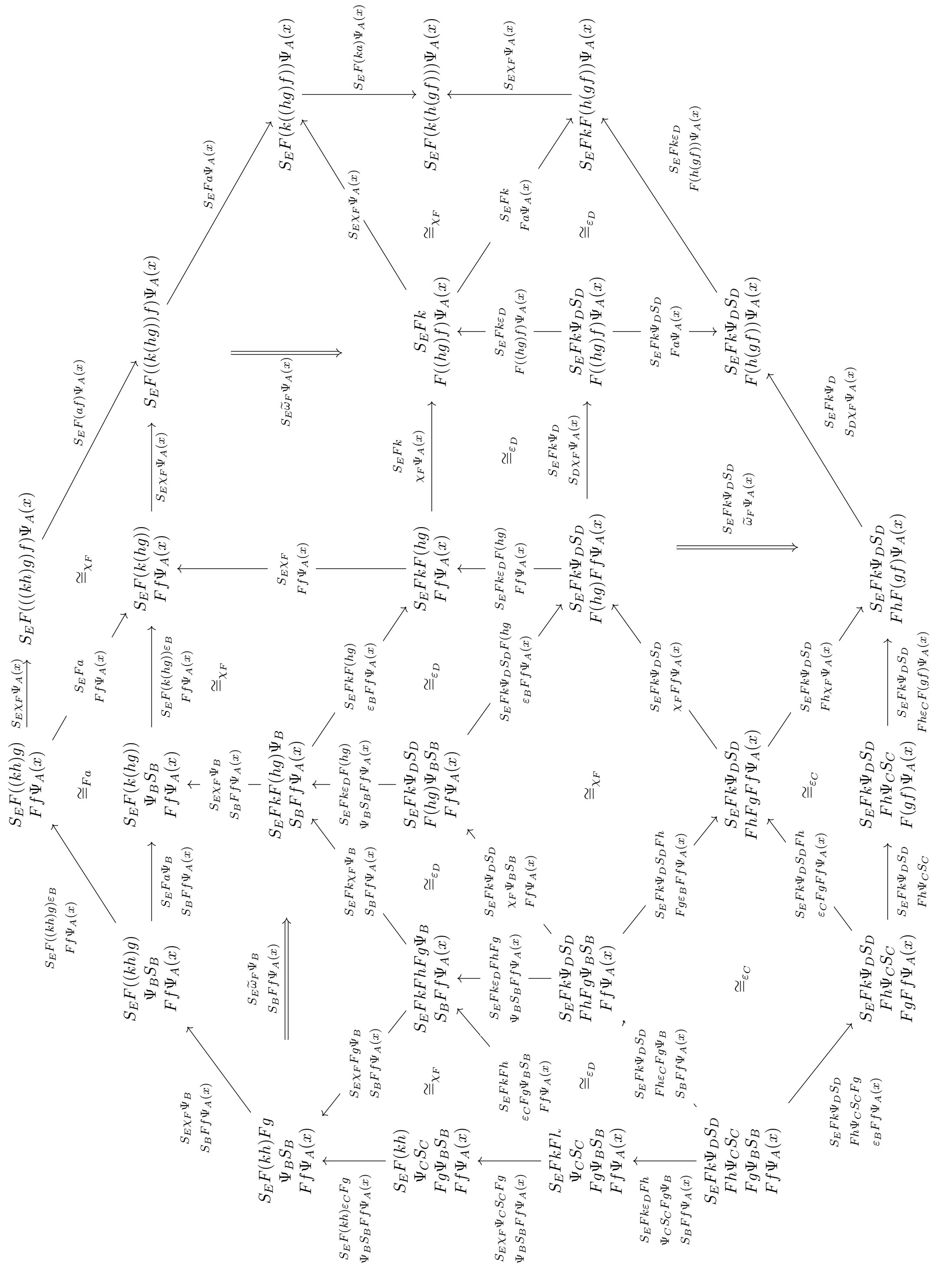


FIGURE 5.1: Trifunctor Axiom 1: Source

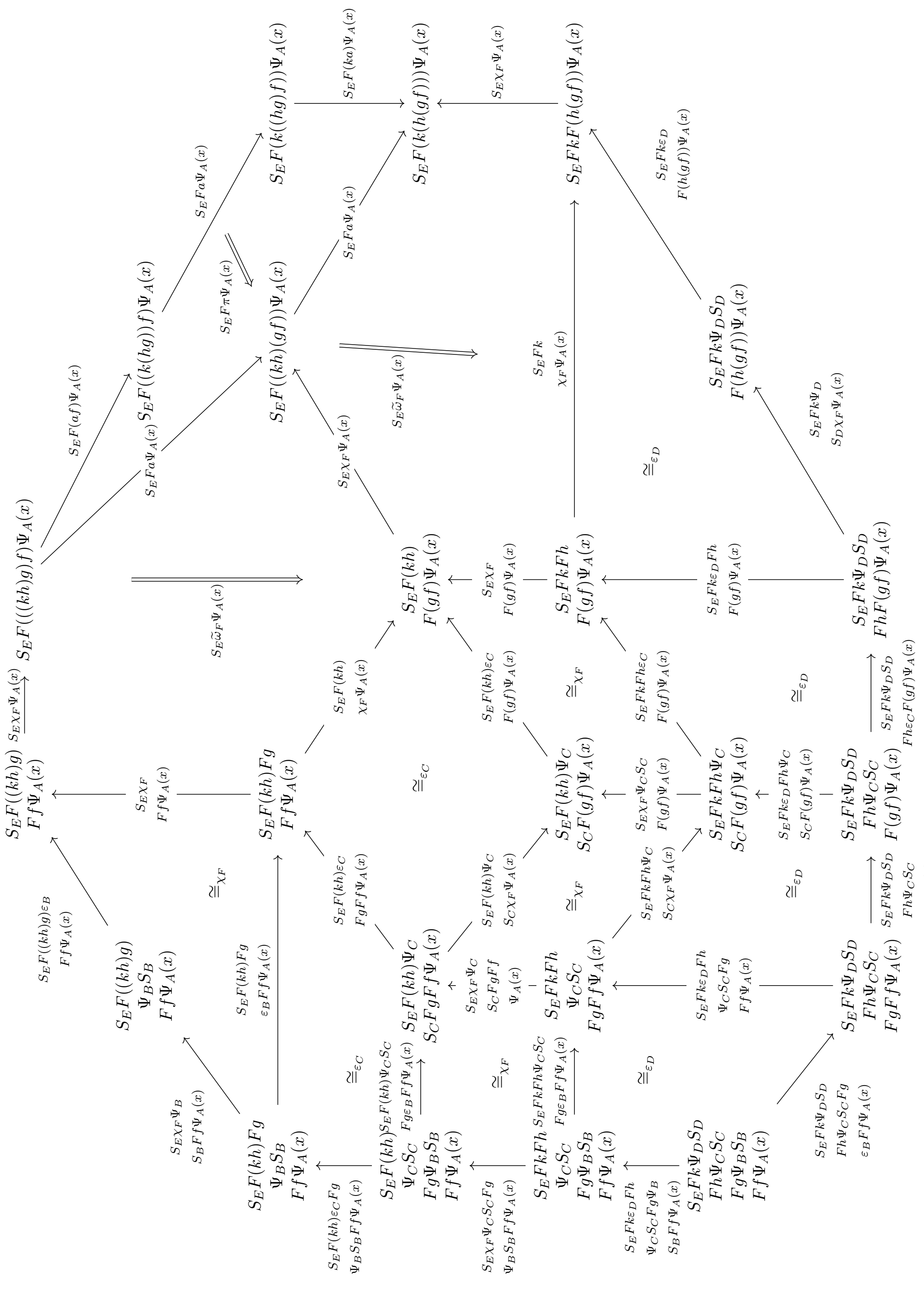


FIGURE 5.2: Trifunctor Axiom 1: Target

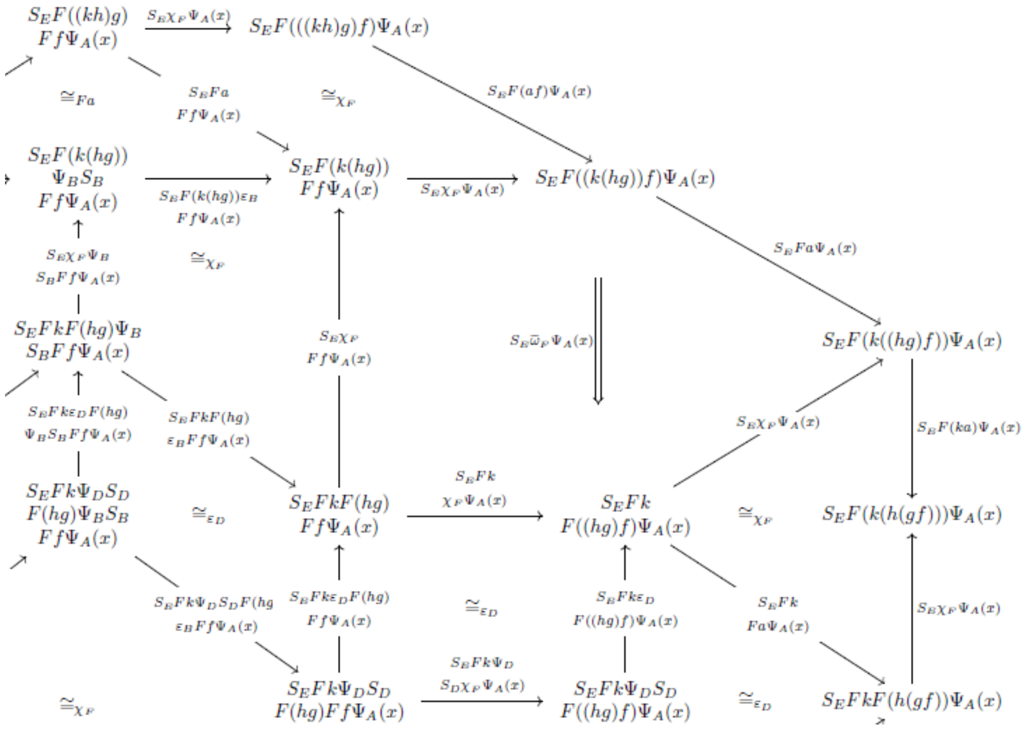


FIGURE 5.3: Part of the first trifunctor axiom for F in the source diagram

First, however, we are able to attach more of the relevant pseudonaturality cells to the other instances of $\widetilde{\omega}_F$, which will make it easier to move them. We do this by considering the following section (Figure 5.5) in between both of those $\widetilde{\omega}_F$ in the source.

By moving the χ_F cell through the ε_D cells using the pseudonaturality of ε_D, this is equal to the diagram in Figure 5.6.

This takes us from the source to the diagram labelled Trifunctor Axiom 1: Step 1 (Figure 5.7) where both instances of $\widetilde{\omega}_F$ not already in place in the top right corner have pseudonaturality cells attached on three of their edges.

We are now able to move both instances of $\widetilde{\omega}_F$ towards the upper right corner. First, we have the cell $S_E\widetilde{\omega}_F\Psi_BS_BFf\Psi_A(x)$ (marked in blue in Figure 5.8) which has the pseudonaturality cells for three of its pseudonatural transformations attached along its edges. Therefore, using the technique from Proposition 4.2.2, we are able to move that cell through the pseudonaturality cells.

We also have the cell $S_EFk\Psi_DS_D\widetilde{\omega}_F\Psi_A(x)$ (marked in red in Figure 5.9) which has pseudonaturality cells for ε_D attached all the way along its source. Therefore, by the pseudonaturality property of ε_D, we are able to move $S_EFk\Psi_DS_D\widetilde{\omega}_F\Psi_A(x)$ through them as well.

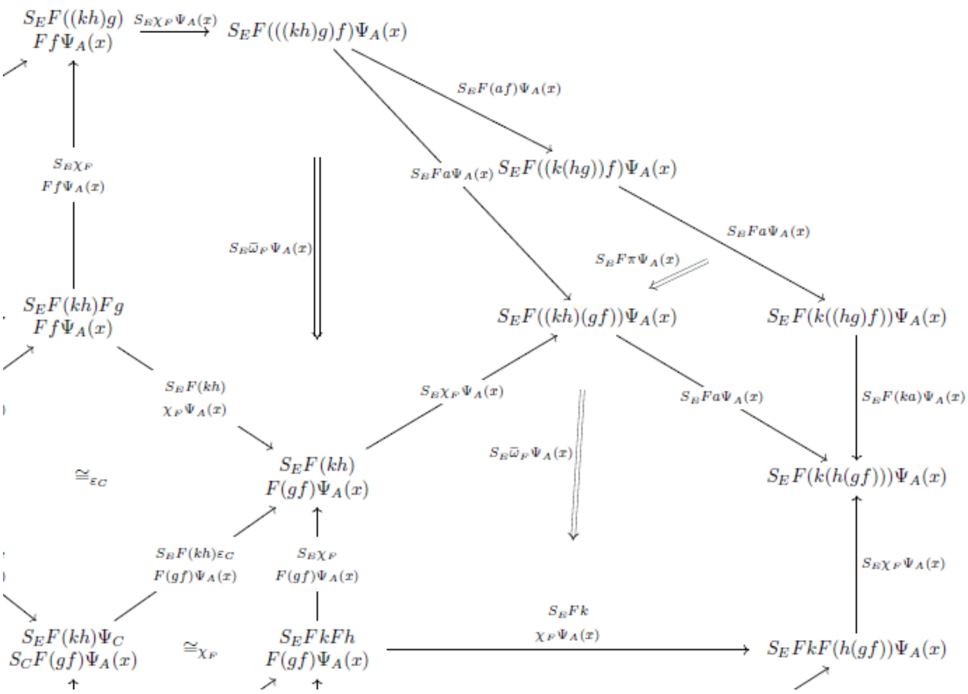


FIGURE 5.4: Part of the first trifunctor axiom for F in the target diagram

This takes us to the Step 2 diagram of Figure 5.10.

The cells in the upper-right corner now form an instance of the source of first trifunctor axiom for $F : \mathcal{T} \to \underline{Bicat}$ (taken at $\Psi_A(x)$, and then after the pseudofunctor S_E is applied). We are therefore able to use that axiom to turn this diagram into the diagram of Figure 5.11.

This differs from the target of the first trifunctor for G only in the bottom-left hand corner as shown in Figure 5.12.

Consider the two instances of the pseudonaturality cell of ε_C at the very bottom-left. They have the pseudonaturality cells for the pseudonatural transformation $\chi_F \circ \varepsilon_D$ pasted all the way along their source. Therefore we can pass the instances of ε_C through the pseudonaturality cells, arriving at the target (See Figure 5.13).

Step 4 is the target diagram we were aiming for. Since we were able to move from the source to the target diagram in a series of steps each of which was equal to the one before, we have proved that the source and target diagrams (as originally shown in Figures 5.1 and 5.2) are equal. Therefore G satisfies the first trifunctor axiom. □

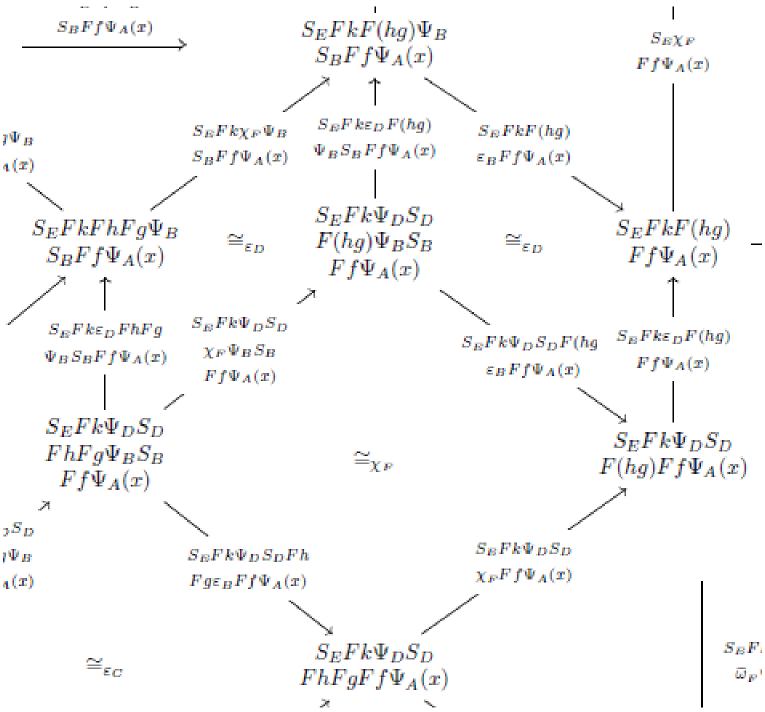


FIGURE 5.5: Three pseudonaturality cells in the centre of the source diagram

$$
\begin{tikzcd}
 & & \begin{array}{c} S_EFkF(hg)\Psi_B \\ S_BFf\Psi_A(x) \end{array} \arrow[drr, "\begin{array}{c} S_EFkF(hg) \\ \varepsilon_BFf\Psi_A(x) \end{array}"] & & \\
\begin{array}{c} S_EFkFhFg\Psi_B \\ S_BFf\Psi_A(x) \end{array} \arrow[urr, "\begin{array}{c} S_EFk\chi_F\Psi_B \\ S_BFf\Psi_A(x) \end{array}"] \arrow[drr, "\begin{array}{c} S_EFkFhFg \\ \varepsilon_BFf\Psi_A(x) \end{array}"'] & & \cong_{\chi_F} & & \begin{array}{c} S_EFkF(hg) \\ Ff\Psi_A(x) \end{array} \\
\begin{array}{c} S_EFk\Psi_DS_D \\ FhFg\Psi_BS_B \\ Ff\Psi_A(x) \end{array} \arrow[u, "\begin{array}{c} S_EFk\varepsilon_DFhFg \\ \Psi_BS_BFf\Psi_A(x) \end{array}"] \arrow[drr, "\begin{array}{c} S_EFk\Psi_DS_DFh \\ Fg\varepsilon_BFf\Psi_A(x) \end{array}"'] & \cong_{\varepsilon_D} & \begin{array}{c} S_EFkFhFg \\ Ff\Psi_A(x) \end{array} \arrow[urr, "\begin{array}{c} S_EFk\chi_F \\ Ff\Psi_A(x) \end{array}"'] & \cong_{\varepsilon_D} & \begin{array}{c} S_EFk\Psi_DS_D \\ F(hg)Ff\Psi_A(x) \end{array} \arrow[u, "\begin{array}{c} S_EFk\varepsilon_DF(hg) \\ Ff\Psi_A(x) \end{array}"'] \\
 & & \begin{array}{c} S_EFk\Psi_DS_D \\ FhFgFf\Psi_A(x) \end{array} \arrow[u, "\begin{array}{c} S_EFk\varepsilon_DFh \\ FgFf\Psi_A(x) \end{array}"] \arrow[urr, "\begin{array}{c} S_EFk\Psi_DS_D \\ \chi_FFf\Psi_A(x) \end{array}"'] & &
\end{tikzcd}
$$

FIGURE 5.6: The three pseudonaturality cells interchanged

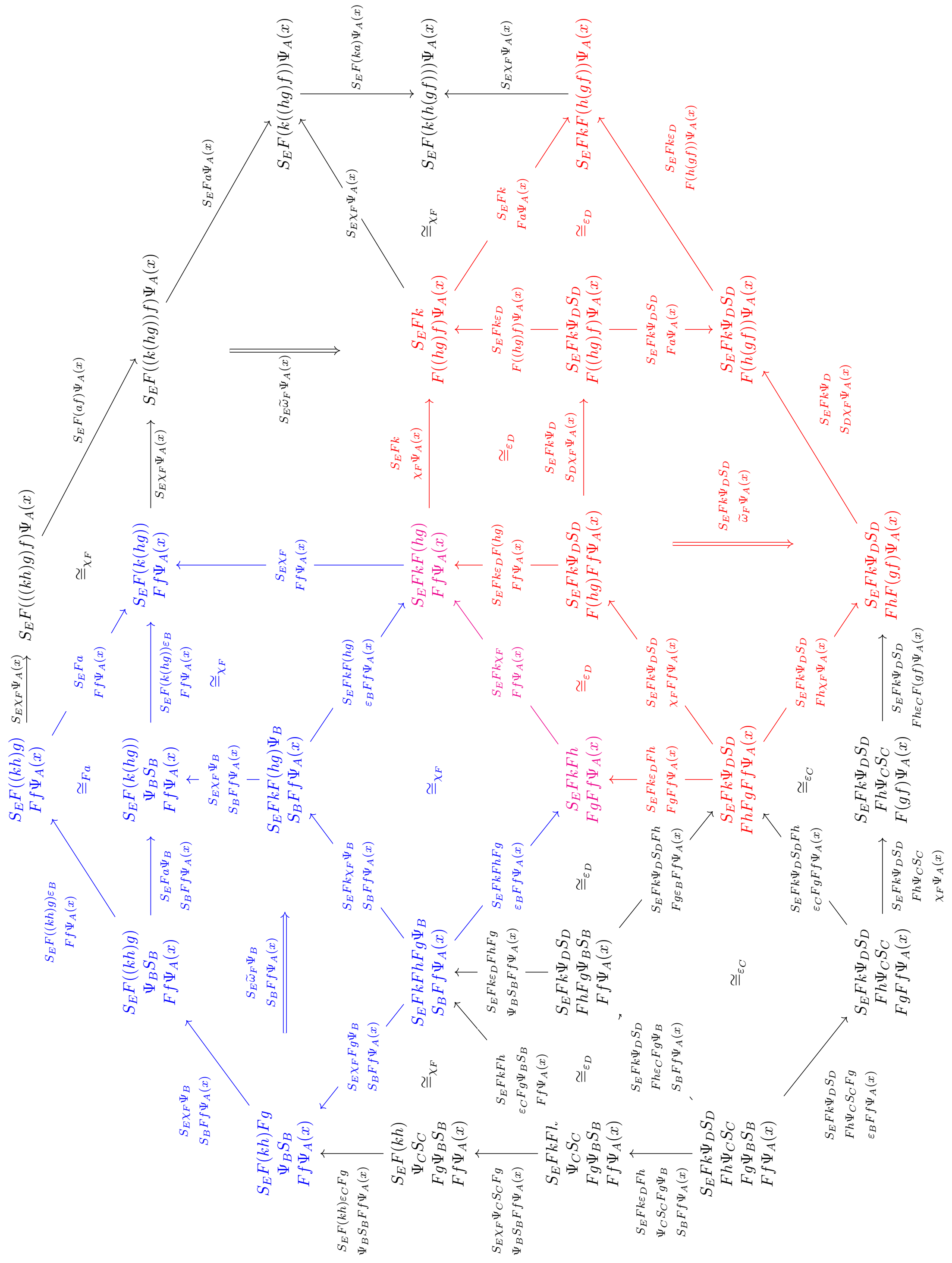


Figure 5.7: Trifunctor Axiom 1: Step 1

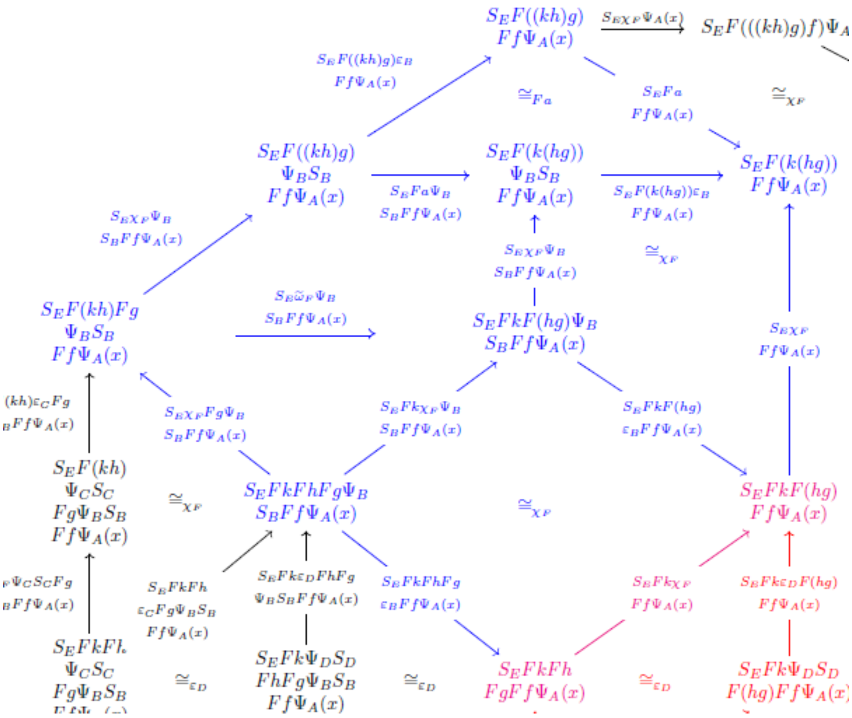


FIGURE 5.8: Moving the first $\widetilde{\omega}_F$ cell towards the top right

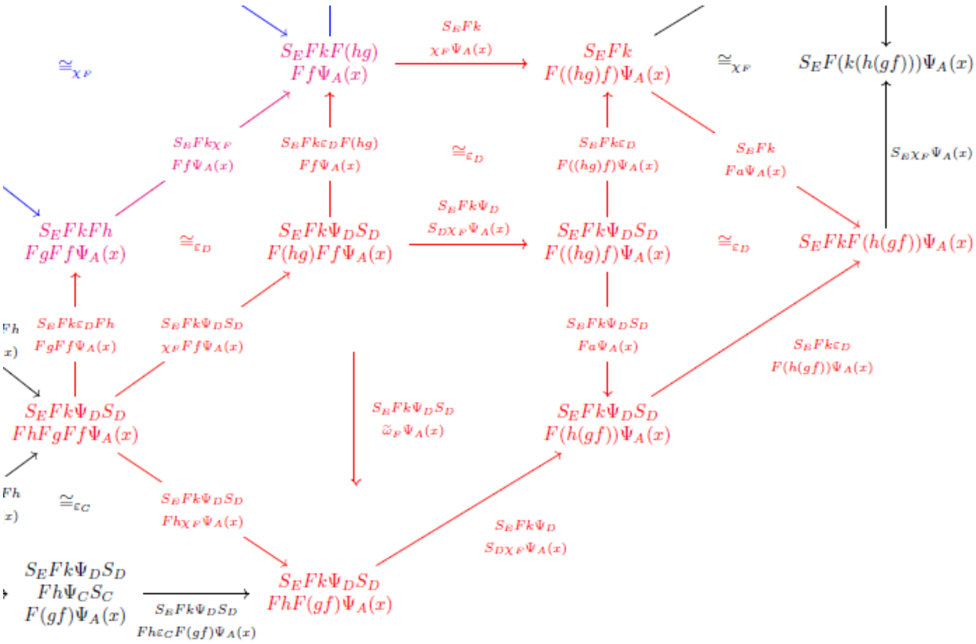


FIGURE 5.9: Moving the second $\widetilde{\omega}_F$ cell towards the top right

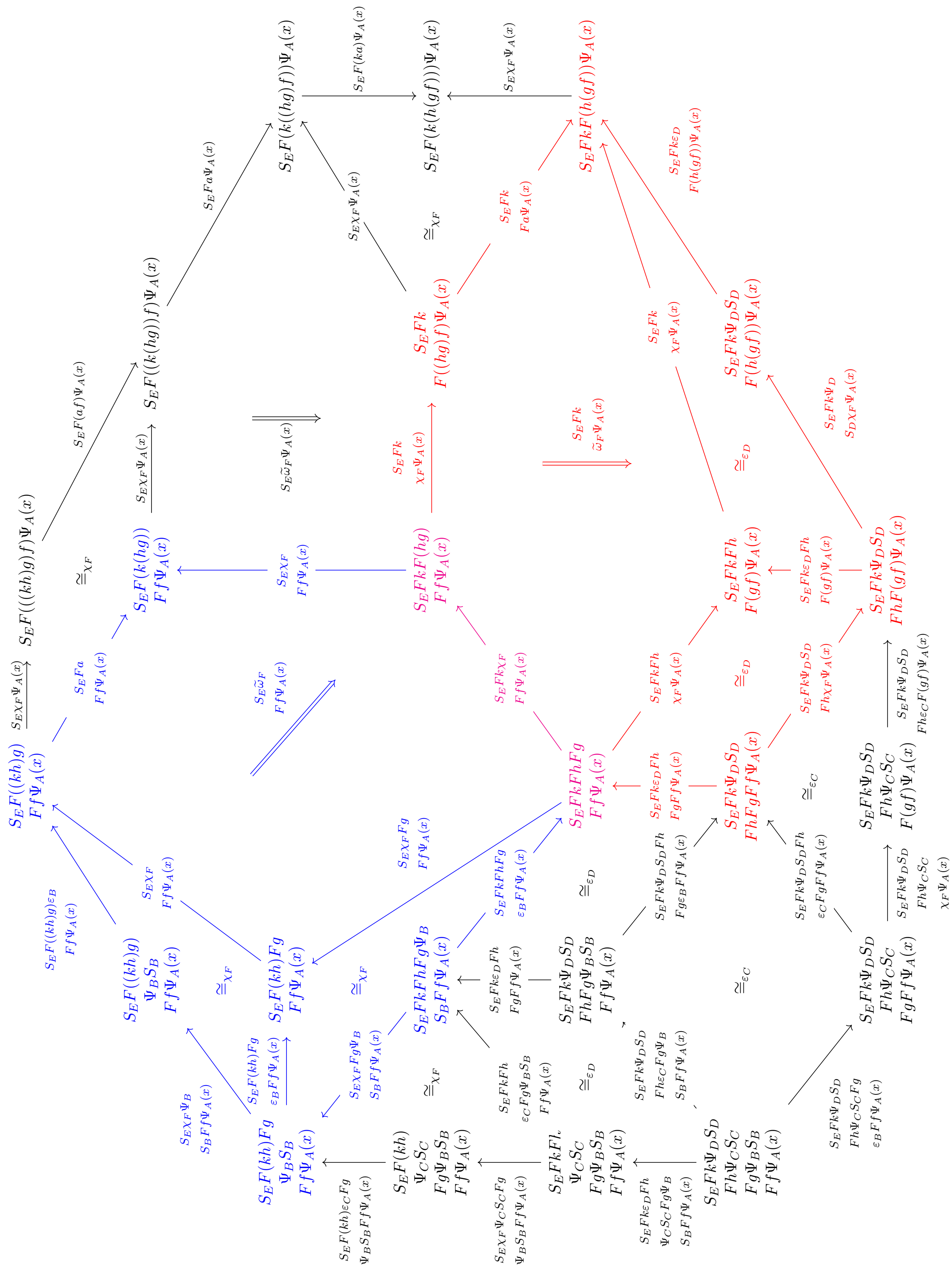


FIGURE 5.10: Trifunctor Axiom 1: Step 2

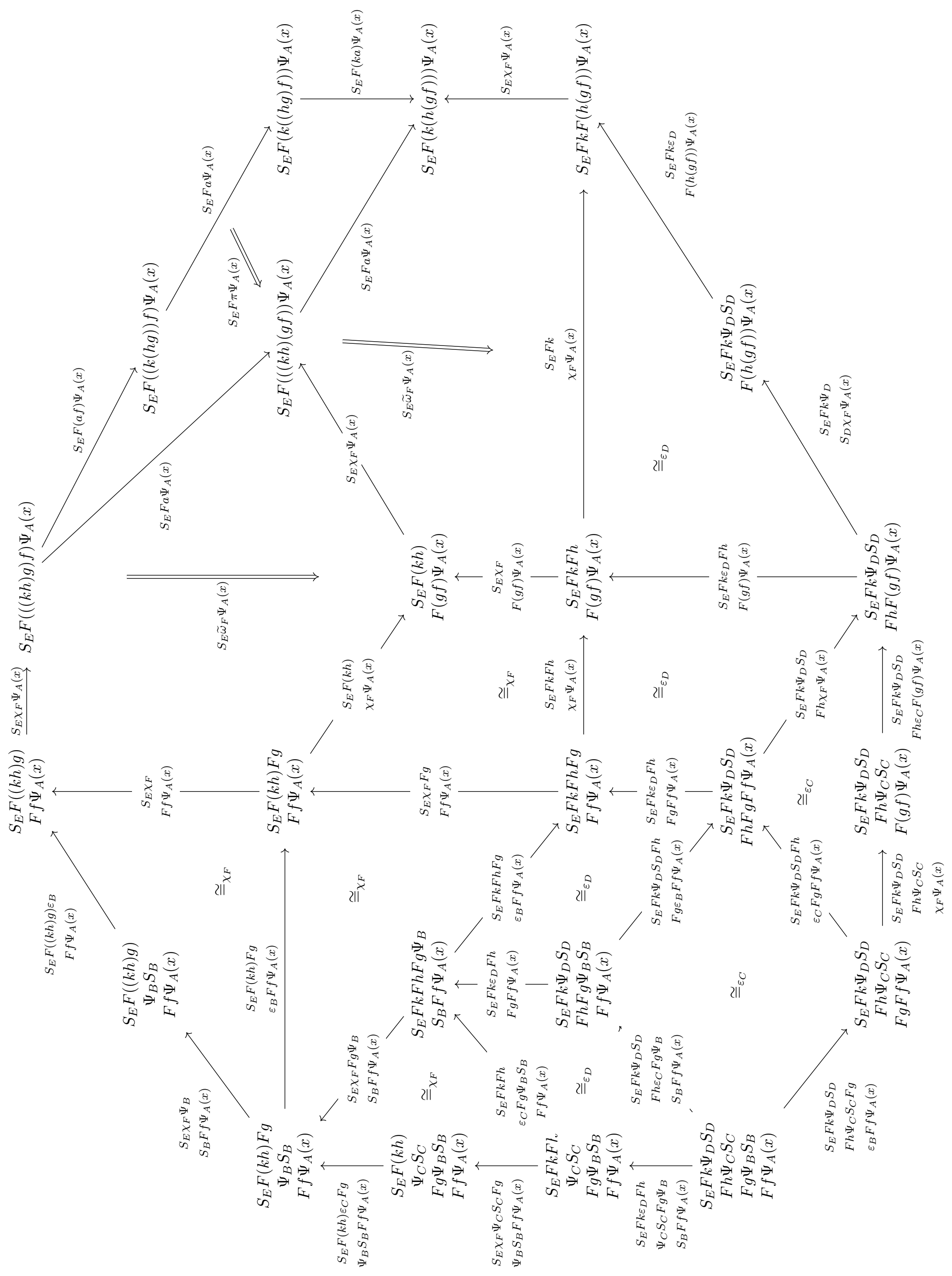


FIGURE 5.11: Trifunctor Axiom 1: Step 3

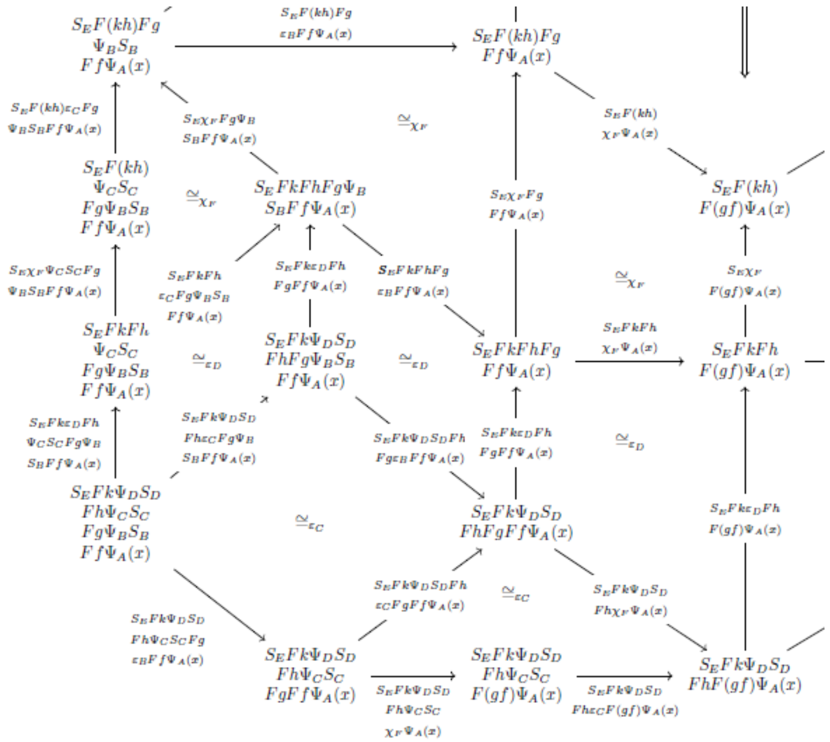


FIGURE 5.12: The bottom left of Step 3

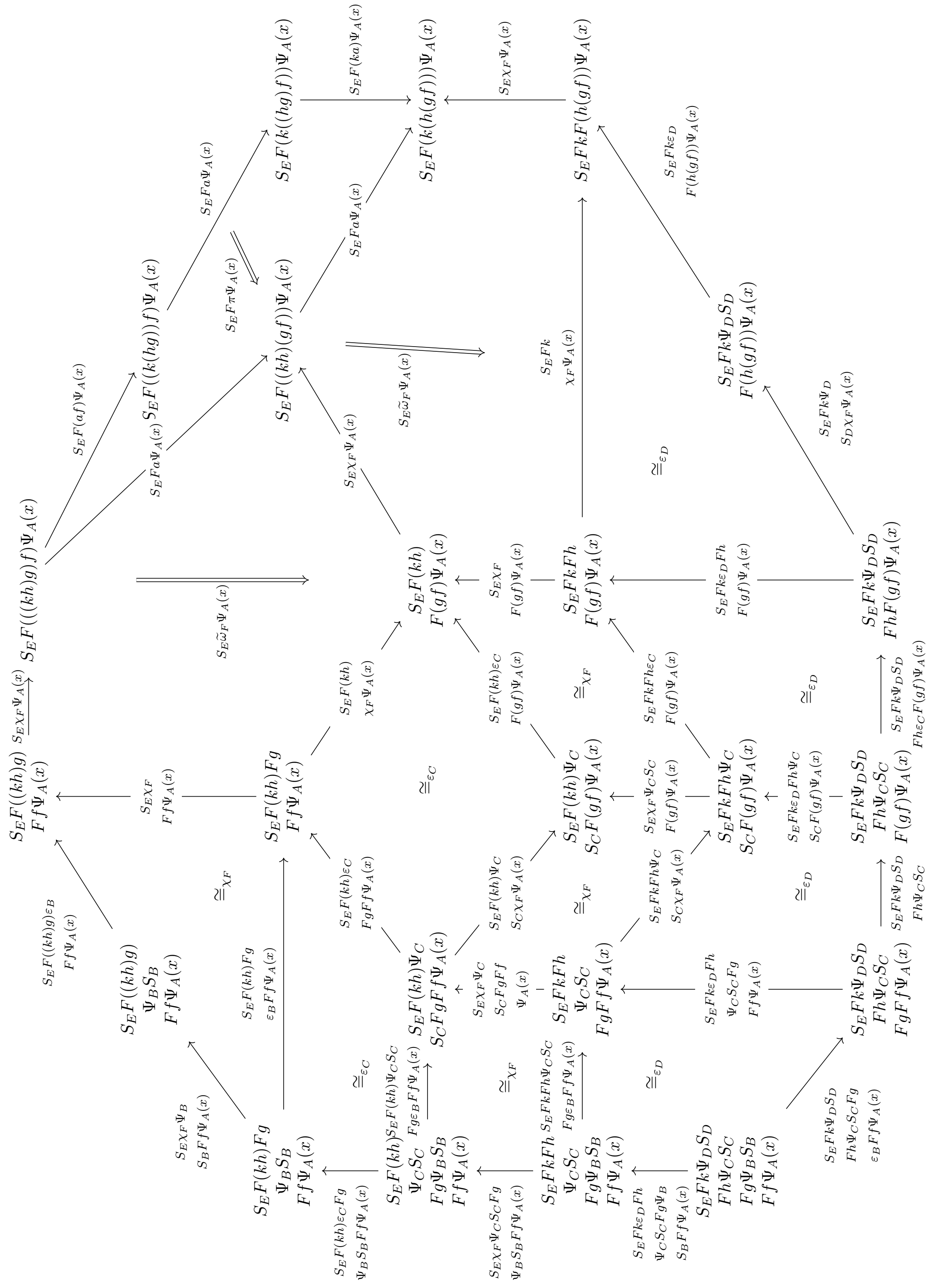


FIGURE 5.13: Trifunctor Axiom 1: Step 4 i.e. Target

5.4 Second Trifunctor Axiom

Proposition 5.4.1. $G : \mathcal{T} \to \underline{Bicat}$ satisfies the second trifunctor axiom.

Proof. After substituting the data for G into the second trifunctor axiom (the version after it has been simplified using the coherence theorem for bicategories), we are asked to prove that the two diagrams shown in the Figures 5.14 and 5.15 are equal.

Again, we will prove that these are equal by starting with the source of the second axiom and manipulating it until we reach the target diagram.

We notice two things when coming up with a strategy for doing so. First, that the cells along the top of each diagram (see Figures 5.16 and 5.17) resemble the second trifunctor axiom for $F : \mathcal{T} \to \underline{Bicat}$. (Albeit, as before, pre- and post-composed by the 1-cells of the biequivalence as expected for any diagram for G.)

Secondly, we note that the first diagram also contains cells ($\widetilde{\Phi}_B^{-1}$ and $\widetilde{\Sigma}_B$) coming from the biadjoint biequivalence between FB and GB that do not appear in the target diagram (See Figure 5.18).

Therefore, our strategy will be to first construct an instance of an axiom for a biadjoint biequivalence in order to remove $\widetilde{\Phi}_B^{-1}$ and $\widetilde{\Sigma}_B$. That will clear the way so that we construct the rest of the second trifunctor axiom for F along the top of the diagram.

To prepare this strategy, we first note that the $\cong_{\iota_F}$ cell in the lower centre of the source diagram has the pseudonaturality cells for ε_B attached along its entire source. Therefore, we are able to pass $\cong_{\iota_F}$ through them using the pseudonaturality of ε_B to reach the diagram of Figure 5.19.

The bottom centre diamond now consists exactly of a diagram of cells (see Figure 5.20) which the axioms for biadjoint biequivalences say is equal to the identity.

We are therefore able to remove those cells, and the obstruction caused by the cells $\widetilde{\Phi}_B^{-1}$ and $\widetilde{\Sigma}_B$, reaching the diagram of Figure 5.21.

We now attempt to complete the version of the second trifunctor axiom for F that we noticed along the top of the diagram. This is now only missing a cell coming from δ_F and a cell coming from γ_F, both of which can be found at the bottom of the diagram. The instance of δ_F has the pseudonaturality cells of three of the pseudonatural transformations it is modifying pasted along its edges (marked in blue in Figure 5.22) and so can be moved through them using the technique provided by Proposition 4.2.2.

Meanwhile the cell including γ_F has the pseudonaturality cells for ε_B attached along the entire source (marked in red in Figure 5.23) and so can be moved through them using the pseudonaturality of ε_B.

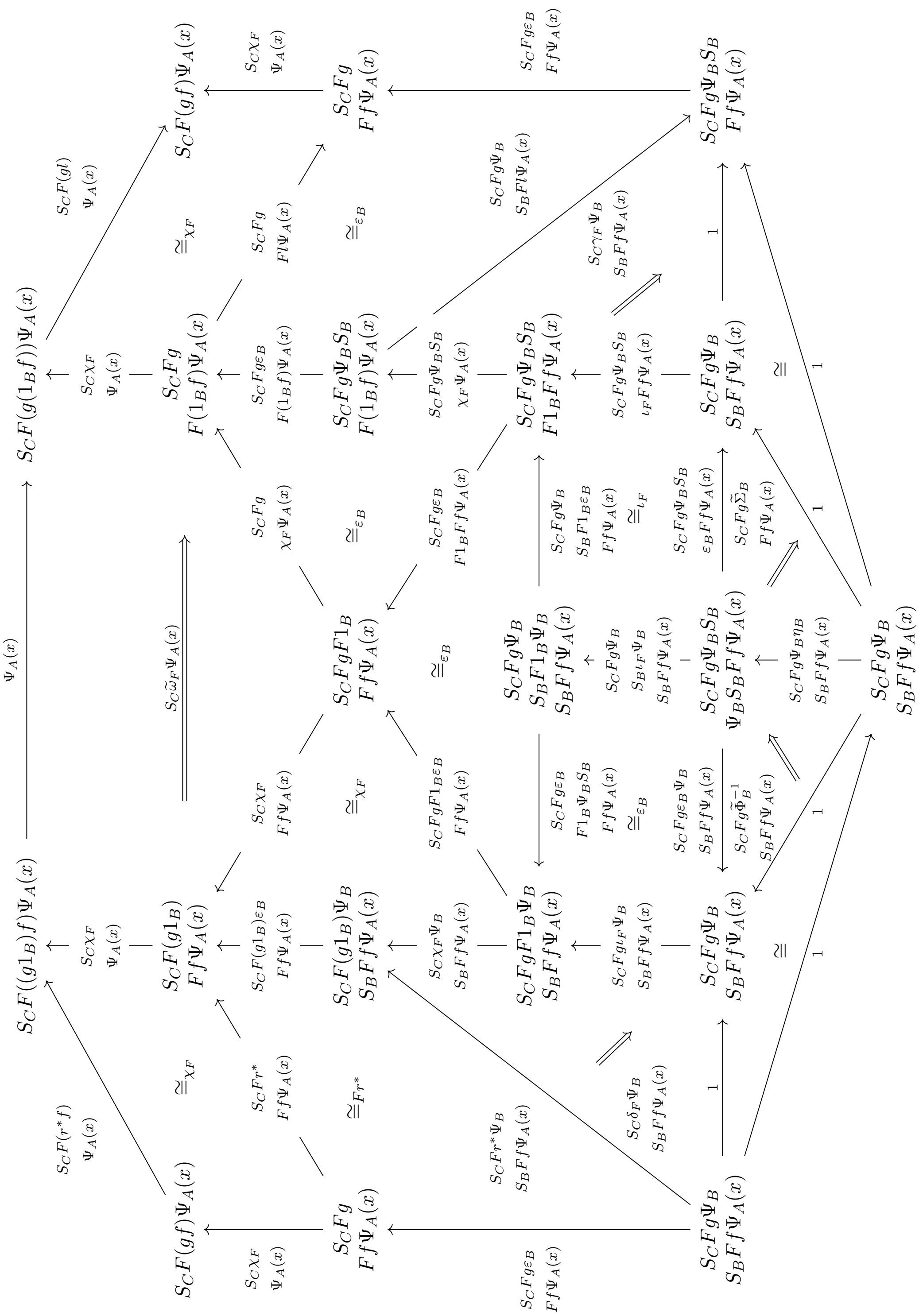


Figure 5.14: Trifunctor Axiom 2: Source

$$
\begin{tikzcd}
 & S_CF((g1_B)f)\Psi_A(x) \arrow[rr, "{S_CFa \atop \Psi_A(x)}"] \arrow[rr, phantom, shift right=6ex, "\Downarrow S_CF\mu\Psi_A(x)"] & & S_CF(g(1_Bf))\Psi_A(x) \arrow[dr, "{S_CF(gl) \atop \Psi_A(x)}"] & \\
S_CF(gf)\Psi_A(x) \arrow[ur, "{S_CF(r^*f) \atop \Psi_A(x)}"] \arrow[rrrr, "1"'] & & & & S_CF(gf)\Psi_A(x) \\
S_CFgFf\Psi_A(x) \arrow[u, "{S_C\chi_F \atop \Psi_A(x)}"] & & \cong & & S_CFgFf\Psi_A(x) \arrow[u, "{S_C\chi_F \atop \Psi_A(x)}"'] \\
{S_CFg\Psi_B \atop S_BFf\Psi_A(x)} \arrow[u, "{S_CFg\epsilon_B \atop Ff\Psi_A(x)}"] \arrow[rr, "1"'] & & {S_CFg\Psi_B \atop S_BFf\Psi_A(x)} \arrow[rr, "1"'] & & {S_CFg\Psi_B \atop S_BFf\Psi_A(x)} \arrow[u, "{S_CFg\epsilon_B \atop Ff\Psi_A(x)}"']
\end{tikzcd}
$$

FIGURE 5.15: Trifunctor Axiom 2: Target

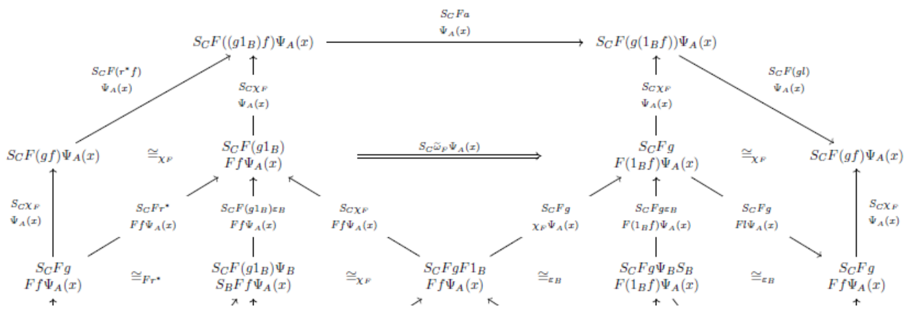


FIGURE 5.16: Part of the second trifunctor axiom for F in the source diagram.

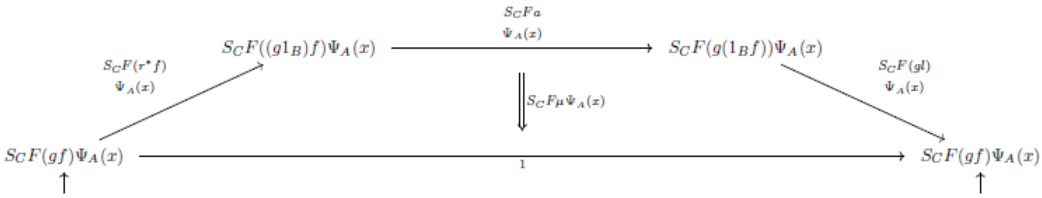


FIGURE 5.17: Part of the second trifunctor axiom for F in the target diagram.

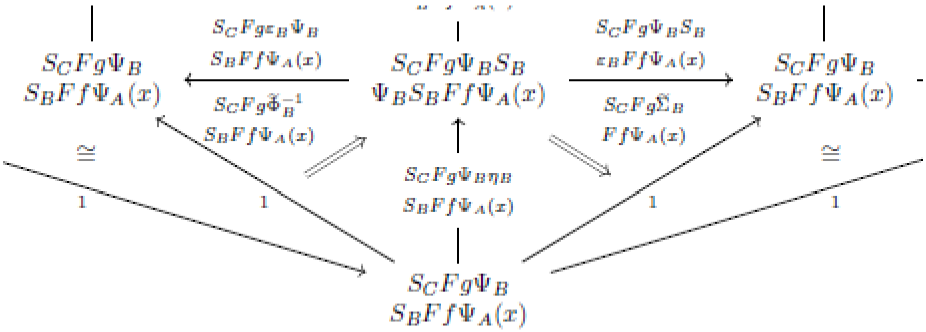


FIGURE 5.18: Cells from the Biadjoint Biequivalence in the source diagram

Doing both of these moves leads us to the diagram of Figure 5.24 (note that since the pseudonaturality cells on the other side are being taken at the 1-cell identity, their pseudonaturality 2-cells are identities as well).

The top of the Step 3 diagram (Figure 5.24) is now a complete instance of the second trifunctor axiom for $F : \mathcal{T} \to \underline{Bicat}$ (taken at $\Psi_A(x)$, and then after the pseudofunctor S_C is applied). Therefore we can replace it by the target of that axiom and get the Step 4 diagram of Figure 5.25.

This is exactly the target diagram we were trying to reach. Thus the source and target diagrams (Figures 5.14 and 5.15) are equal, and $G : \mathcal{T} \to \underline{Bicat}$ satisfies the second trifunctor axiom.

$\square$

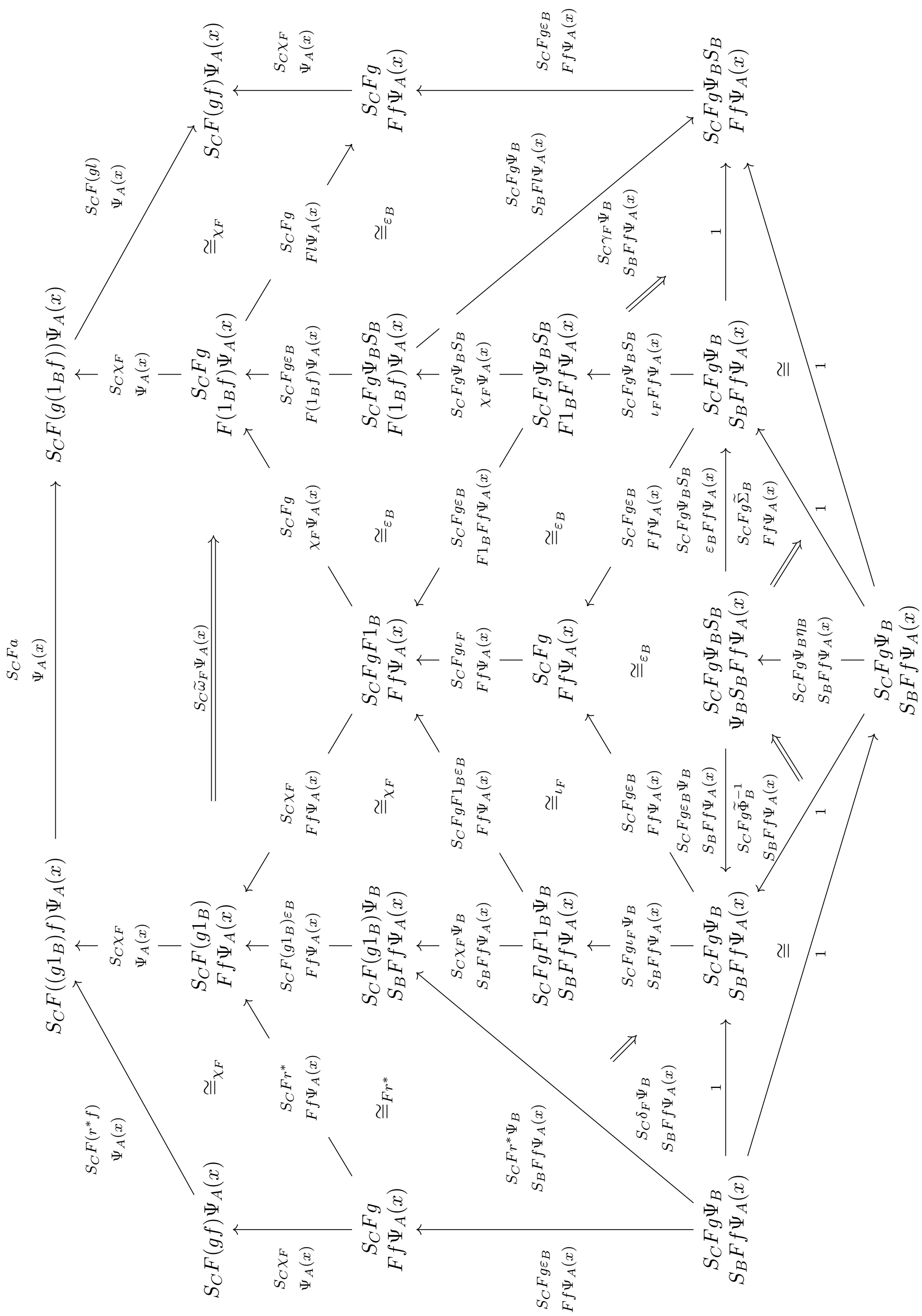


FIGURE 5.19: Trifunctor Axiom 2: Step 1

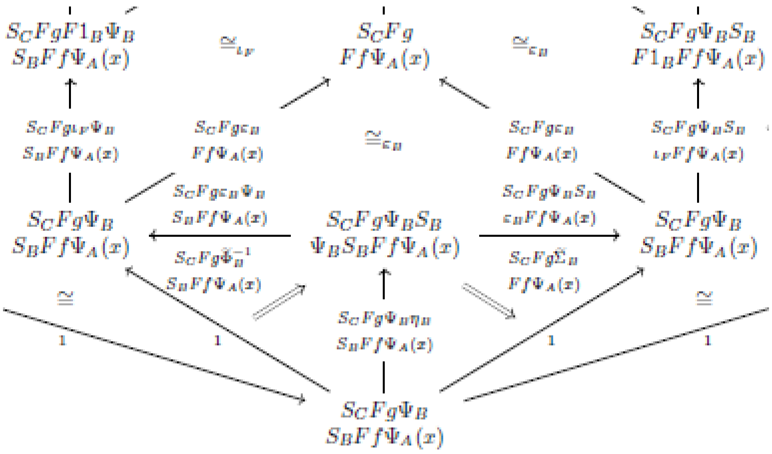


FIGURE 5.20: An instance of a biadjoint biequivalence axiom

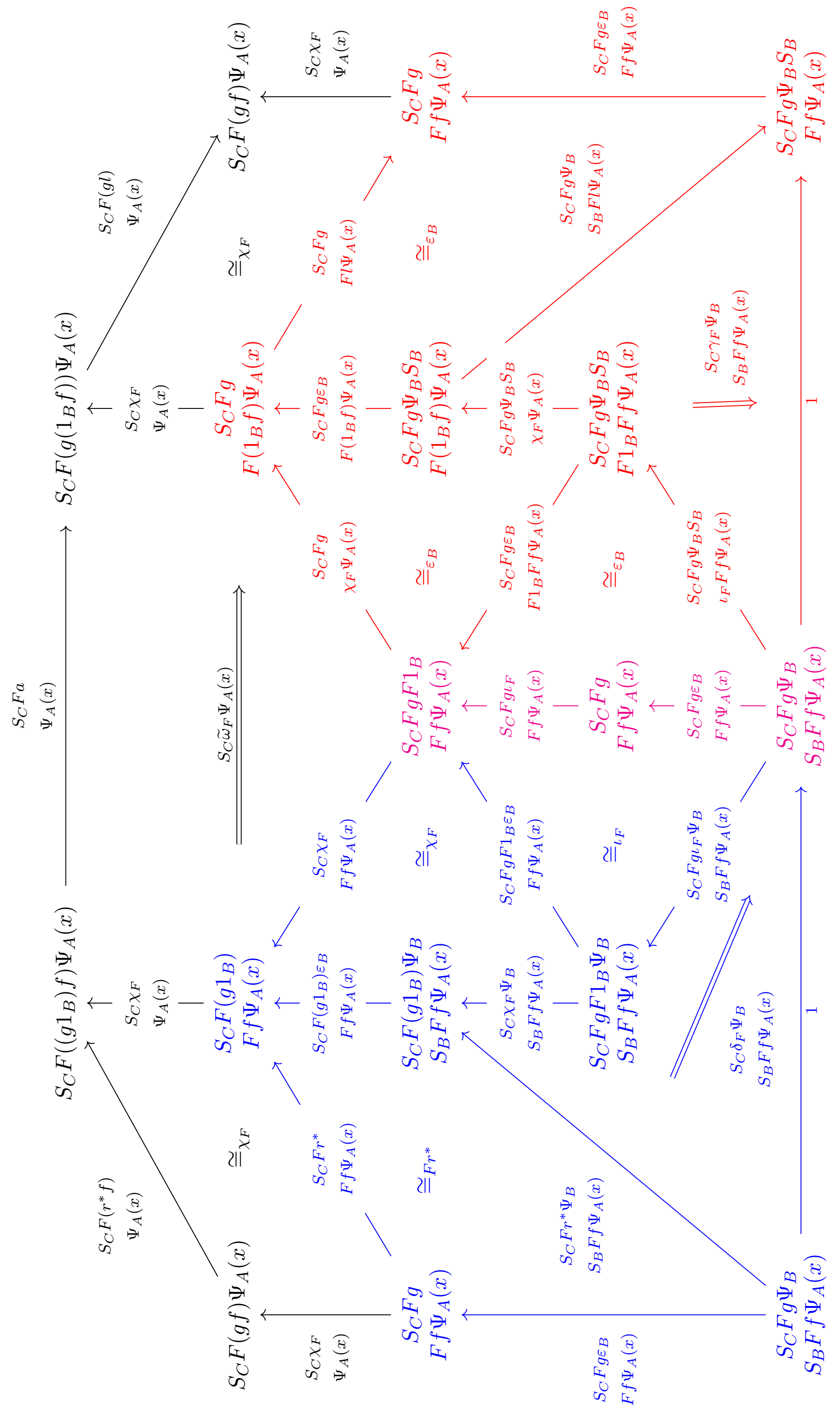


FIGURE 5.21: Trifunctor Axiom 2: Step 2

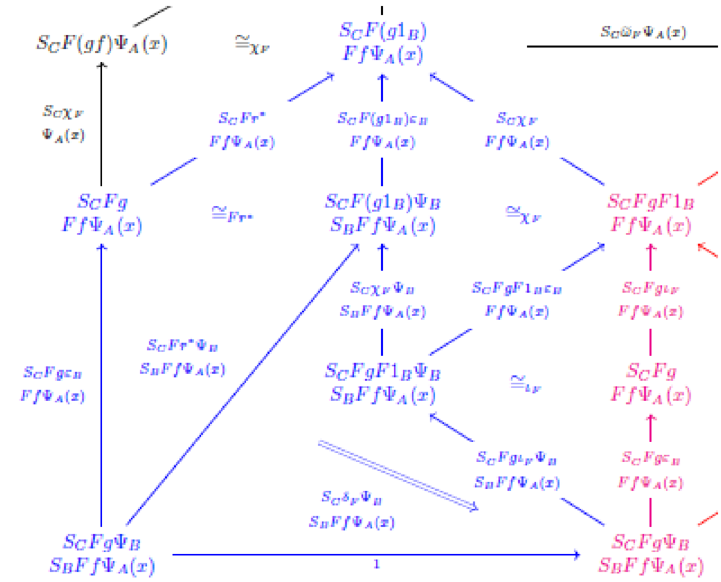


FIGURE 5.22: Moving δ_F upwards

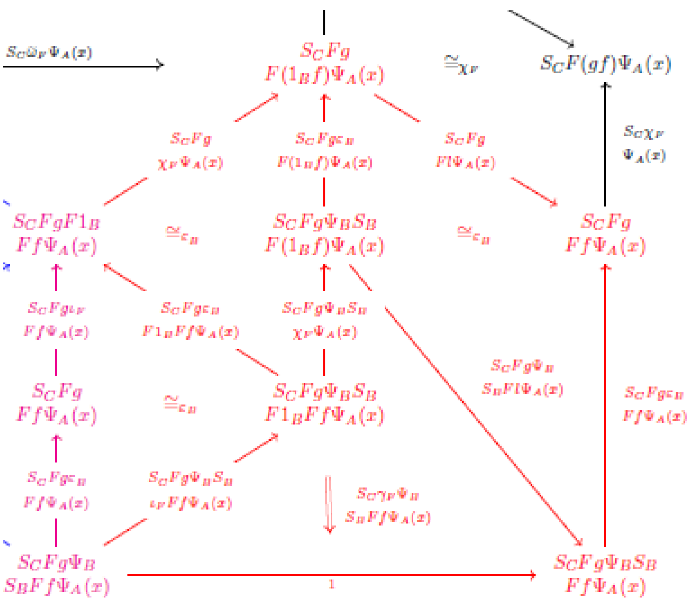


FIGURE 5.23: Moving γ_F upwards

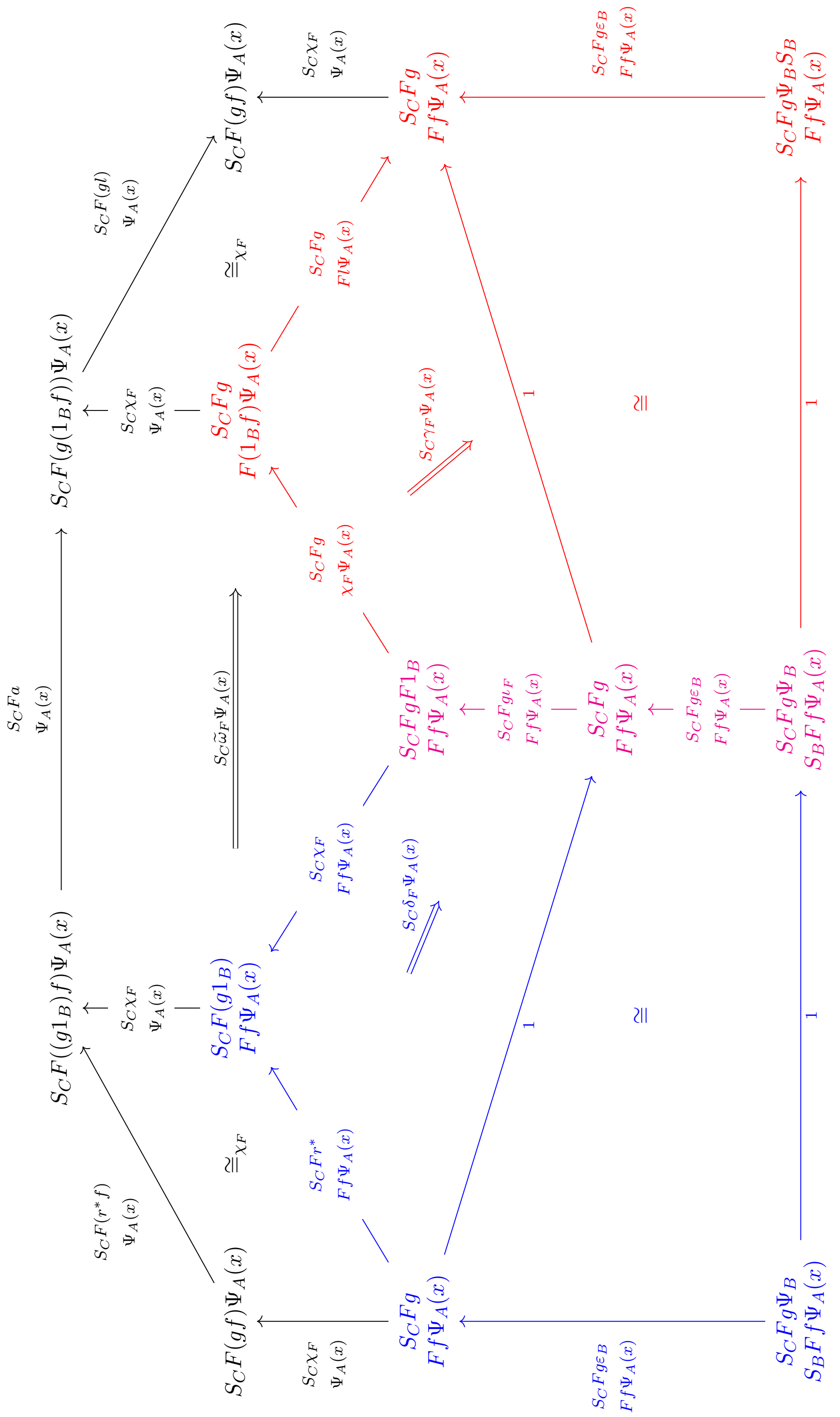


FIGURE 5.24: Trifunctor Axiom 2: Step 3

$$\begin{tikzcd}
 & S_CF((g1_B)f)\Psi_A(x) \arrow[rr, "\substack{S_CFa\\ \Psi_A(x)}"] \arrow[rr, phantom, shift right=6ex, "\Downarrow S_CF\mu\Psi_A(x)"] & & S_CF(g(1_Bf))\Psi_A(x) \arrow[dr, "\substack{S_CF(gl)\\ \Psi_A(x)}"] & \\
S_CF(gf)\Psi_A(x) \arrow[ur, "\substack{S_CF(r^*f)\\ \Psi_A(x)}"] \arrow[rrrr, "1"'] & & & & S_CF(gf)\Psi_A(x) \\
S_CFgFf\Psi_A(x) \arrow[u, "\substack{S_C\chi_F\\ \Psi_A(x)}"] & & \cong & & S_CFgFf\Psi_A(x) \arrow[u, "\substack{S_C\chi_F\\ \Psi_A(x)}"'] \\
\substack{S_CFg\Psi_B\\ S_BFf\Psi_A(x)} \arrow[u, "\substack{S_CFg\varepsilon_B\\ Ff\Psi_A(x)}"] \arrow[rr, "1"'] & & \substack{S_CFg\Psi_B\\ S_BFf\Psi_A(x)} \arrow[rr, "1"'] & & \substack{S_CFg\Psi_B\\ S_BFf\Psi_A(x)} \arrow[u, "\substack{S_CFg\varepsilon_B\\ Ff\Psi_A(x)}"']
\end{tikzcd}$$

FIGURE 5.25: Trifunctor Axiom 2: Step 4 i.e. Target

We have proved that the data given in our construction of $G : \mathcal{T} \to \underline{Bicat}$ satisfies both trifunctor axioms, and are therefore able to conclude:

Theorem 5.4.2. G as defined in Section 5.2 is a trifunctor between the tricategories $\mathcal{T}$ and $\underline{Bicat}$.

5.5 Examples

To aid our understanding of the above process we can examine particular examples to see what happens when we try transporting the structure of a trifunctor. One particularly simple example is when the source tricategory $\mathcal{T}$ is the category $\underline{\mathbf{1}}$ with two objects and one non-identity morphism between them (viewed as a tricategory where all of the higher cells are identities).

$$0 \xrightarrow{f} 1$$

Let us also suppose that the trifunctor $F : \underline{\mathbf{1}} \to \underline{Bicat}$ is strict: that is, it sends the identities of the two objects to the identities on the bicategories and it sends the identity 2-cells to identity pseudonatural transformations.
Then to set up the transport we need two biadjoint biequivalences given by $\Psi_0 \dashv S_0 : F0 \to G0$ and $\Psi_1 \dashv S_1 : F1 \to G1$ along with the related higher cells.
Then transporting the trifunctor across these biequivalences will let us see how this process interacts with identity 1-cells. The new trifunctor G sends the identity on 0 to $S_0F(1_0)\Psi_0 = S_0\Psi_0$. Similarly, it sends the identity on 1 to $S_1\Psi_1$. Since these are not equal to the identity, we see that G does not strictly preserves identities. Also, the morphism f is sent to $S_1F(f)\Psi_0$, which means that we would need to apply the (non-identity) compositor of G in order to compose it with the images of the identities.

We can also consider the case where the source tricategory is the category $\underline{\mathbf{2}}$ with three objects and two non-identity morphisms

$$0 \xrightarrow{f} 1 \xrightarrow{g} 2$$

Let us again take a strict trifunctor $F : \underline{\mathbf{2}} \to \underline{Bicat}$ and biadjoint biequivalences $\Psi_0 \dashv S_0 : F0 \to G0$, $\Psi_1 \dashv S_1 : F1 \to G1$ and $\Psi_2 \dashv S_2 : F2 \to G2$. This time, after transporting the trifunctor we will focus on the composition of the two non-identity morphisms. Even though the trifunctor we started with was strict, so that $F(gf) = FgFf$, this is not true for the transported trifunctor G. Because f is sent to $S_1Ff\Psi_0$ and g is sent to $S_2Fg\Psi_1$, we need to apply an instance of ε_1 in the middle to act as the compositor.

Both of these examples show that the transport of a strict functor need not be strict. This is to be expected, illustrating that the weak version of a concept is more natural.

One final example we shall consider is the following. For many concepts in higher category theory, the version for dimension $n-1$ can be realised in dimension n by taking all of the highest-level cells to be identities. In this case, that would mean finding a method for transporting a pseudofunctor from a bicategory into the 2-category of categories across an object-indexed collection of equivalences of categories. Although this 2-dimensional transport of structure can be accomplished using the more general methods of Kelly and Lack [KL04] we can also see how it arises as a special case of transporting trifunctors. The setup for this example is as follows:

- The source bicategory $\mathcal{B}$ can be realised as a tricategory whose 3-cells are all identities.
- Each category can be realised as a bicategory whose 2-cells are all identities. Then, because all the 2-cells are identities, pseudofunctors between these categories are just functors, pseudonatural transformations are just natural transformations and all modifications are identities. This gives a fully-faithful embedding of $\underline{Cat}$ in $\underline{Bicat}$.
- Next consider a trifunctor $F : \mathcal{B} \to \underline{Bicat}$ which lands entirely in the embedded version of $\underline{Cat}$. Since all the modifications involved are identities, in particular the modifications that make up the axioms of a trifunctor are identities: the axioms of a pseudofunctor are satisfied. Every pseudofunctor can be realised in this way.
- After identifying the objects GA for $A \in ob(\mathcal{B})$ to which the pseudofunctor will be transported, we then look at what happens to the biadjoint biequivalences between each FA and GA. This begins with pairs of functors $S_A : FA \to GA$ and $\Psi_A : GA \to FA$.
- We then get natural transformations $\eta_A : 1_{\mathcal{GA}} \Rightarrow S_A\Psi_A$ and $\varepsilon_A : \Psi_A S_A \Rightarrow 1_{FA}$. When we take these to be adjoint equivalences, the fact that all modifications are identities means that both all the η_As and the ε_As are actually invertible.
- The modifications Φ_A and Σ_A are identities, showing that the natural isomorphisms η_A and ε_A satisfy the triangle identities. Thus, the functors $S_A : FA \to GA$ and $\Psi_A : GA \to FA$ form an adjoint equivalence. Every setup of object-indexed adjoint equivalences can be interpreted in this way.

With this setup allowing us to interpret pseudofunctors as trifunctors, the method of transporting trifunctors gives us the following:

- The transported trifunctor acts on objects by sending A to GA as expected.
- The action on the hom-category $\mathcal{B}(A, B)$ is given by $S_B F(-)\Psi_A$. Although we had to choose an association when working with the tricategory of bicategories, we do not need that here.
- The constructions of the compositor χ_G and the unitor ι_G proceed as they did previously, though simplified because whenever the coherence cells of *Bicat* were used they are now identities. Since the components they are constructed out of - coherence cells, the natural transformations η_A and ε_A, and the compositor and unitor of F - are all strictly invertible, so are χ_G and ι_G.
- Finally, the 3-dimensional data ω_G, γ_G, and δ_G are constructed out of modifications. Since in this context all modifications are identities, this proves that ω_G, γ_G, and δ_G are all identities: that is, G satisfies the axioms of a pseudofunctor.

Thus our method of transporting the structure of a trifunctor generalises a method of transporting a pseudofunctor, as expected.

Chapter 6

Lifting the Biequivalences

Having transported the structure of a trifunctor across the family of biequivalences $S_A \dashv \Psi_A$, we now wish to give the biequivalences the structure of a tritransformation. This will have the effect of lifting them so that they also form a biequivalence between the original trifunctor $F : \mathcal{T} \to \underline{Bicat}$ and the newly constructed trifunctor $G : \mathcal{T} \to \underline{Bicat}$.

Recall that since we are working in the tricategory of bicategories, we are able to use the simplified definition of tritransformation given by Proposition 4.1.2. In other words, the tritransformations we intend to construct consist of:

- For each object A of $\mathcal{T}$ a 1-cell $\theta_A : FA \to GA$.
- For each pair of objects $A, B \in ob(\mathcal{T})$, an adjoint equivalence

$$\begin{tikzcd} \mathcal{T}(A,B) \arrow[r, "F"] \arrow[d, "G"'] & \underline{Bicat}(FA,FB) \arrow[d, "\theta_B \otimes -"] \arrow[dl, Rightarrow, "\theta"] \\ \underline{Bicat}(GA,GB) \arrow[r, "- \otimes \theta_A"'] & \underline{Bicat}(FA,GB) \end{tikzcd}$$

- For each triple of objects $A, B, C \in ob(\mathcal{T})$, an invertible modification Π whose component at a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{T}$ and at an object $x \in ob(FA)$ arises via coherence from a 2-cell

$$\begin{tikzcd} & Gg\theta_B Ff(x) \arrow[rr, "1\theta"] & & GgGf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\ \theta_C FgFf(x) \arrow[ur, "\theta 1"] \arrow[drr, "1\chi_F"'] & & {} \arrow[d, phantom, "\Downarrow \tilde{\Pi}"] & & G(gf)\theta_A(x) \\ & & \theta_C F(gf)(x) \arrow[urr, "\theta"'] & & \end{tikzcd}$$

- For each object $A \in ob(\mathcal{T})$, an invertible modification M whose component at a given object $x \in ob(FA)$ arises via coherence from the 2-cell

$$
\begin{tikzcd}
 & \theta_A F1_A(x) \arrow[dr, "\theta"] \arrow[d, phantom, "\Downarrow\widetilde{M}"] & \\
\theta_A(x) \arrow[ur, "1\iota_F"] \arrow[rr, "\iota_G 1"'] & {} & G1_A\theta_A(x)
\end{tikzcd}
$$

These cells are required to obey the following three simplified axioms.

1. **First Tritransformation Axiom**: For every triple of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ the following two diagrams are equal:

$$
\begin{tikzcd}
 & & & GhGgGf\theta_A(x) \arrow[drr, "\chi_G 1"] \arrow[d, phantom, "\cong_{\chi_G}"] & & & \\
 & GhGg\theta_B Ff(x) \arrow[urr, "1\theta"] \arrow[drr, "\chi_G 1"'] \arrow[d, phantom, "\Downarrow \widetilde{\Pi} 1"] & & {} & & G(hg)Gf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\
Gh\theta_C FgFf(x) \arrow[ur, "1\theta 1"] & {} & & G(hg)\theta_B Ff(x) \arrow[urr, "1\theta"'] & & & G((hg)f)\theta_A(x) \arrow[d, "Ga1"] \\
\theta_D FhFgFf(x) \arrow[u, "\theta 1"] \arrow[rr, "1\chi_F 1"] \arrow[drr, "1\chi_F"'] & & \theta_D F(hg)Ff(x) \arrow[ur, "\theta 1"] \arrow[rr, "1\chi_F"] \arrow[d, phantom, "\Downarrow 1\widetilde{\omega}_F"] & & \theta_D F((hg)f)(x) \arrow[urr, "\theta"] \arrow[d, "1Fa"] \arrow[u, phantom, "\Downarrow \widetilde{\Pi}"] \arrow[rr, phantom, "\cong_{\theta}"] & & G(h(gf))\theta_A(x) \\
 & & \theta_D FhF(gf)(x) \arrow[rr, "1\chi_F"'] & & \theta_D F(h(gf))(x) \arrow[urr, "\theta"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & GhGgGf\theta_A(x) \arrow[rr, "\chi_G 1"] \arrow[dr, "1\chi_G 1"] & & G(hg)Gf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\
 & GhGg\theta_B Ff(x) \arrow[urr, "1\theta"] & {} \arrow[d, phantom, "\Downarrow 1\widetilde{\Pi}"] & & GhG(gf)\theta_A(x) \arrow[drr, "\chi_G 1"] & {} \arrow[l, phantom, "\Leftarrow \widetilde{\omega}_G 1"] & G((hg)f)\theta_A(x) \arrow[d, "Ga1"] \\
Gh\theta_C FgFf(x) \arrow[ur, "1\theta 1"] \arrow[drr, "1\chi_F"] & {} \arrow[d, phantom, "\cong_\theta"] & Gh\theta_C F(gf)(x) \arrow[urr, "1\theta"] & & {} \arrow[d, phantom, "\Downarrow \widetilde{\Pi}"] & & G(h(gf))\theta_A(x) \\
\theta_D FhFgFf(x) \arrow[u, "\theta 1"] \arrow[drr, "1\chi_F"'] & & \theta_D FhF(gf)(x) \arrow[u, "\theta 1"] \arrow[rr, "1\chi_F"'] & & \theta_D F(h(gf))(x) \arrow[urr, "\theta"'] & &
\end{tikzcd}
$$

2. **Second Tritransformation Axiom**: For every 1-cell $A \xrightarrow{f} B$ the following two diagrams are equal:

$$
\begin{tikzcd}
 & G1_B\theta_BFf(x) \arrow[rrr, "1\theta"] & & & G1_BGf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\
\theta_BF1_BFf(x) \arrow[ur, "\theta 1"] \arrow[drr, "1\chi_F"] & & & \Downarrow\widetilde{\Pi} & & G(1_Bf)\theta_A(x) \arrow[dd, "Gl1"] \\
 & \Swarrow 1\gamma_F & \theta_BF(1_Bf)(x) \arrow[urrr, "\theta"] \arrow[ddl, "1Fl"] & & \cong_\theta & \\
\theta_BFf(x) \arrow[uu, "1\iota_F1"] \arrow[dr, "1"'] & & & & & Gf\theta_A(x) \\
 & \theta_BFf(x) \arrow[urrrr, "\theta"] \arrow[rrr, "\theta"'] & & \cong & Gf\theta_A(x) \arrow[ur, "1"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & G1_B\theta_BFf(x) \arrow[rr, "1\theta"] & & G1_BGf\theta_A(x) \arrow[dr, "\chi_G 1"] & \\
\theta_BF1_BFf(x) \arrow[ur, "\theta 1"] & \Rightarrow \widetilde{M}1 & & \Searrow \gamma_G 1 & G(1_Bf)\theta_A(x) \arrow[d, "Gl1"] \\
\theta_BFf(x) \arrow[u, "1\iota_F1"] \arrow[uur, "\iota_G 1"] \arrow[drrr, "\theta"] \arrow[dr, "1"'] & \cong_{\iota_G} & & & Gf\theta_A(x) \\
 & \cong & \theta_BFf(x) \arrow[r, "\theta"'] & Gf\theta_A(x) \arrow[uuu, "\iota_G 1"] \arrow[ur, "1"'] &
\end{tikzcd}
$$

3. **Third Tritransformation Axiom**: For each 1-cell $A \xrightarrow{f} B$ the following two diagrams are equal:

$$
\begin{tikzcd}
 & & & Gf\theta_AF1_A(x) \arrow[rr, "1\theta"] & & GfG1_A\theta_A(x) \arrow[ddr, "\chi_G"] & \\
 & Gf\theta_A(x) \arrow[urr, "1\iota_F"] & \cong_\theta & & \Downarrow\widetilde{\Pi}1 & & \\
 & & \theta_BFfF1_A(x) \arrow[uur, "\theta"] \arrow[drr, "1\chi_F"] & & & & \\
\theta_BFf(x) \arrow[uur, "\theta"] \arrow[urr, "1\iota_F"] & & \Downarrow 1\delta_F^{-1} & & \theta_BF(f1_A)(x) \arrow[rr, "\theta"] & & G(f1_A)\theta_A(x) \\
 & & \theta_BFf(x) \arrow[ull, "1"] \arrow[urr, "1Fr^*"] \arrow[rr, "\theta"'] & & Gf\theta_A(x) \arrow[urr, "Gr^*1"'] & \cong_\theta &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & Gf\theta_AF1_A(x) \arrow[dr, "1\theta"] & & \\
 & Gf\theta_A(x) \arrow[ur, "1\iota_F"] \arrow[rr, "1\iota_G 1"'] & \Downarrow 1\widetilde{M} & GfG1_A\theta_A(x) \arrow[dr, "\chi_G"] & \\
\theta_BFf(x) \arrow[ur, "\theta"] & \cong & & \Searrow \delta_G^{-1} & G(f1_A)\theta_A(x) \\
 & \theta_BFf(x) \arrow[ul, "1"] \arrow[rr, "\theta"'] & & Gf\theta_A(x) \arrow[uull, "1"] \arrow[ur, "Gr^*1"'] &
\end{tikzcd}
$$

This gives us the list of everything we need to construct in order to define the tritransformations, and the three axioms we will need to prove in order to confirm that they are tritransformations. As in the previous chapter, the tools we have to construct the cells of the tritransformation are the coherence cells of the trifunctor $F : \mathcal{T} \to \underline{Bicat}$ and the cells that form the family of biadjoint biequivalences $S_A \dashv \Psi_A$ (see section 5.1).

6.1 Constructing the Tritransformation

We will start by giving the biequivalences $S_A : FA \to GA$ the structure of a tritransformation $S : F \Rightarrow G$. This proceeds as follows:

- The 1-cell components of the tritransformation are the very 1-cells we are trying to give the structure of a tritransformation: $S_A : FA \to GA$.
- The adjoint equivalence part of the tritransformation needs to have the form

$$\begin{tikzcd}
\mathcal{T}(A,B) \arrow[r, "F"] \arrow[d, "F"'] & \underline{Bicat}(FA,FB) \arrow[ddd, "S_B\otimes -"] \\
\underline{Bicat}(FA,FB) \arrow[d, "-\otimes\Psi_A"'] & \\
\underline{Bicat}(GA,FB) \arrow[d, "S_B\otimes -"'] \arrow[ur, phantom, "\Swarrow \theta_S"] & \\
\underline{Bicat}(GA,GB) \arrow[r, "-\otimes S_A"'] & \underline{Bicat}(FA,GB)
\end{tikzcd}$$

We will define θ_S using the diagram

$$\begin{tikzcd}
\mathcal{T}(A,B) \arrow[r, "F"] & \underline{Bicat}(FA,FB) \arrow[rr, bend left=40, "-\otimes 1_{FA}"] \arrow[rr, bend right=40, "-\otimes(\Psi_A\otimes S_A)"'] \arrow[rrr, bend left=60, "S_B\otimes -"] \arrow[dr, "-\otimes\Psi_A"'] \arrow[rr, phantom, "\Downarrow -\otimes\varepsilon_A"] & & \underline{Bicat}(FA,FB) \arrow[r, "S_B\otimes -"] \arrow[u, phantom, "\Downarrow S_B\otimes r^*"] & \underline{Bicat}(FA,GB) \\
& & \underline{Bicat}(GA,FB) \arrow[ur, "-\otimes S_A"'] \arrow[r, "S_B\otimes -"'] \arrow[u, phantom, "\Downarrow a^*"] & \underline{Bicat}(GA,GB) \arrow[ur, "-\otimes S_A"'] \arrow[u, phantom, "\Downarrow a^*"] &
\end{tikzcd}$$

When considering the modifications including θ_S as part of the source or target, note that by coherence any 2-cell component of such a modification corresponds to one with

$$S_B Ff(x) \xrightarrow{S_B Ff\varepsilon_A^*} S_B Ff\Psi_A S_A(x)$$

instead.

- The components of the modification M_S at a pair of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ and object $x \in ob(FA)$ correspond to a 2-cell

$$
\begin{tikzcd}
 & S_C Fg\Psi_B S_B Ff\Psi_A S_A(x) \arrow[dr, "S_C Fg\varepsilon_B Ff\Psi_A S_A(x)"] \arrow[dd, phantom, "\Downarrow \widetilde{\Pi}_S"] & \\
S_C Fg\Psi_B S_B Ff(x) \arrow[ur, "S_C Fg\Psi_B S_B Ff\varepsilon^*_A(x)"] & & S_C FgFf\Psi_A S_A(x) \arrow[d, "S_C\chi_F \Psi_A S_A(x)"] \\
S_C FgFf(x) \arrow[u, "S_C Fg\varepsilon^*_B Ff(x)"] \arrow[dr, "S_C\chi_F(x)"'] & & S_C F(gf)\Psi_A S_A(x) \\
 & S_C F(gf)(x) \arrow[ur, "S_C F(gf)\varepsilon^*_A(x)"'] &
\end{tikzcd}
$$

We will define $\widetilde{\Pi}_S$ as

$$
\begin{tikzcd}
 & & S_C Fg\Psi_B S_B Ff\Psi_A S_A(x) \arrow[drr, "S_C Fg\varepsilon_B Ff\Psi_A S_A(x)"] \arrow[dr, phantom, "\Downarrow S_C Fg\beta^{\varepsilon}_B Ff\Psi_A S_A(x)"] & & \\
S_C Fg\Psi_B S_B Ff(x) \arrow[urr, "S_C Fg\Psi_B S_B Ff\varepsilon^*_A(x)"] \arrow[r, phantom, "\cong_{\varepsilon^*_B}"] & {} & S_C FgFf\Psi_A S_A(x) \arrow[u, "S_C Fg\varepsilon^*_B Ff\Psi_A S_A(x)"] \arrow[rr, "1"'] \arrow[drr, "S_C\chi_F \Psi_A S_A(x)"] & \cong & S_C FgFf\Psi_A S_A(x) \arrow[d, "S_C\chi_F \Psi_A S_A(x)"] \\
S_C FgFf(x) \arrow[u, "S_C Fg\varepsilon^*_B Ff(x)"] \arrow[urr, "S_C FgFf\varepsilon^*_A(x)"] \arrow[drr, "S_C\chi_F(x)"'] \arrow[rrrr, phantom, "\cong_{\chi_{F(g,f)}}"] & & & & S_C F(gf)\Psi_A S_A(x) \\
 & & S_C F(gf)(x) \arrow[urr, "S_C F(gf)\varepsilon^*_A(x)"'] & &
\end{tikzcd}
$$

- The components of the modification M_S at the object $x \in ob(FA)$ correspond to a 2-cell

$$\begin{tikzcd}
 & S_AF1_A(x) \arrow[dr, "S_AF1_A \varepsilon^*_A(x)"] \arrow[dd, phantom, "\Downarrow \widetilde{M}_S"] & \\
S_A(x) \arrow[ur, "S_A\iota_F(x)"] \arrow[dr, "\eta^*_A S_A(x)"'] & & S_AF1_A \Psi_A S_A(x) \\
 & S_A\Psi_A S_A(x) \arrow[ur, "S_A\iota_F \Psi_A S_A(x)"'] &
\end{tikzcd}$$

We will define $\widetilde{M}_S$ as

$$\begin{tikzcd}
 & S_AF1_A(x) \arrow[rr, "S_AF1_A\varepsilon^*_A(x)"] \arrow[dr, phantom, "\cong_{\iota_F}"] & & S_AF1_A\Psi_A S_A(x) \\
 & S_A(x) \arrow[u, "S_A\iota_F(x)"] \arrow[r, "S_A\varepsilon^*_A(x)"] \arrow[dr, phantom, "\Downarrow S_A\alpha_A^{\varepsilon(-1)}(x)"] & S_A\Psi_A S_A(x) \arrow[ur, "S_A\iota_F \Psi_A S_A(x)"] \arrow[r, phantom, "\cong"] & {} \\
S_A(x) \arrow[uur, "S_A\iota_F(x)"] \arrow[ur, "1"] \arrow[rr, "\eta_A S_A(x)"'] \arrow[u, phantom, "\cong"] \arrow[urr, phantom, "\Downarrow \widetilde{\Sigma}_A^{-1}(x)"] & & S_A\Psi_A S_A(x) \arrow[u, "1"'] \arrow[ul, "S_A\varepsilon_A(x)"] \arrow[uur, "S_A\iota_F \Psi_A S_A(x)"'] &
\end{tikzcd}$$

Now that the structure S has been entirely defined, we are ready to begin showing that it forms a tritransformation $F \Rightarrow G$.

6.2 First Tritransformation Axiom

Proposition 6.2.1. $S : F \Rightarrow G$ satisfies the first tritransformation axiom.

Proof. After substituting $\widetilde{\Pi}_S$ and $\widetilde{\omega}_G$ into the pasting diagrams for the simplified axiom, we see that proving this axiom holds is equivalent to showing that the following source (Figure 6.1) and target diagrams (Figure 6.2) are equal.

To arrive at a strategy for proving that these two diagrams are equal, we note that most of the cells are pseudonaturality cells for whichever pseudonatural transformation is appropriate. In each diagram there are only three that are not: a cell arising from β^{ε}_C (this

FIGURE 6.1: First Tritransformation Axiom for S: Source

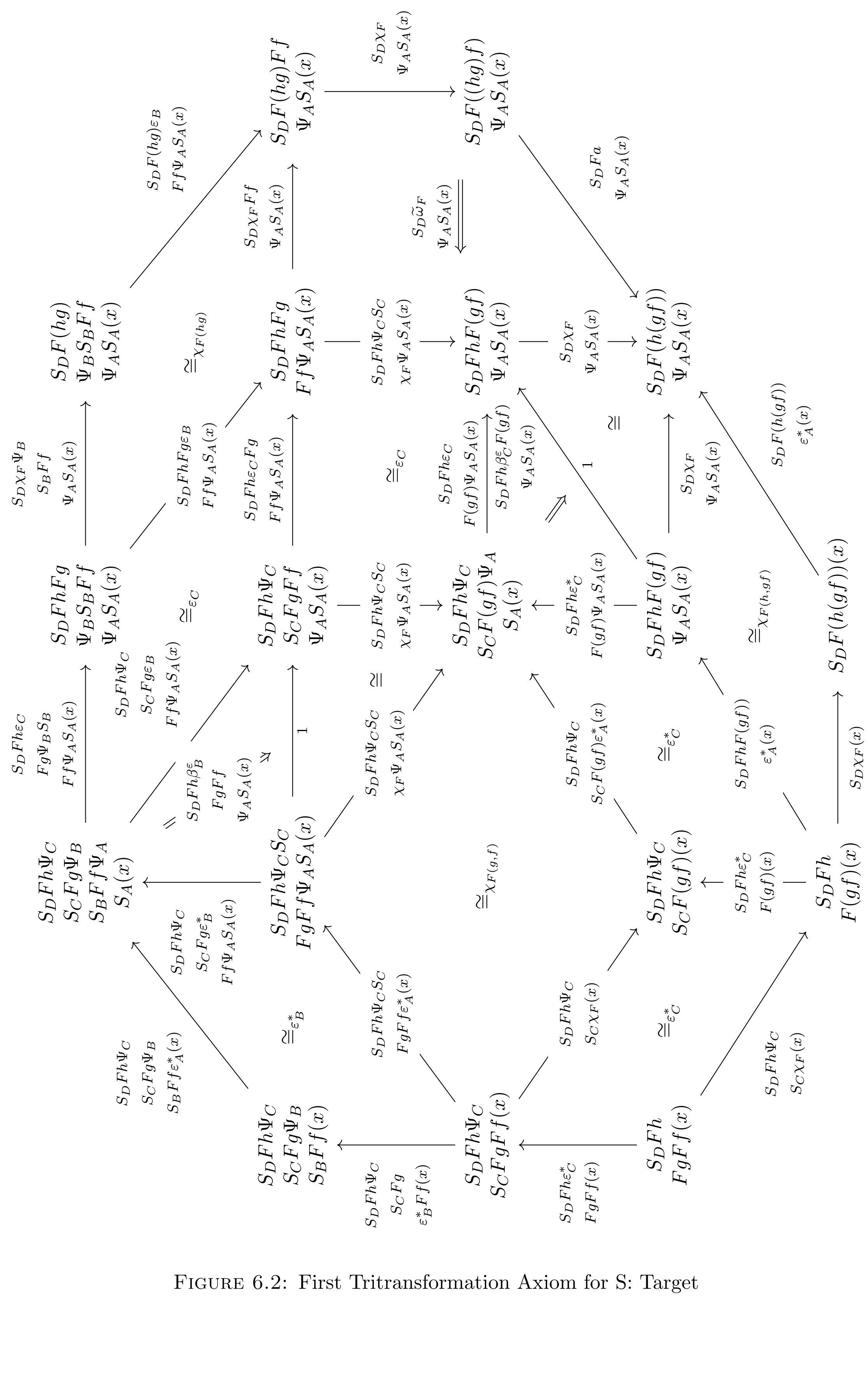


Figure 6.2: First Tritransformation Axiom for S: Target

is on the left side of the source and the lower middle of the target), a cell arising from β^{ε}_{B} (which is at the top right of the source and the top left of the target), and finally a cell arising from $\widetilde{\omega}_F$ (which is at the bottom of the source and on the right-hand side of the target. This suggest the strategy of moving β^{ε}_{C} down and right, β^{ε}_{B} left, and $\widetilde{\omega}_F$ up and right.

To start moving β^{ε}_{C} and β^{ε}_{B}, consider the following sections highlighted in the source diagram (Figure 6.3).

In the area around β^{ε}_{C}, highlighted in red (see Figure 6.4), the β^{ε}_{C} cell and the following coherence cell form a component of a modification from the three cells that make up the source to the target $S_D\chi_F\Psi_B S_B F f(x)$. The other two cells in the highlighted section are the pseudonaturality cells of the first part of the source. Therefore we are able to move β^{ε}_{C} and the coherence cell through those two cells.

A similar argument allows us to move β^{ε}_{B} through the section highlighted in blue. This takes us to the Step 1 diagram of Figure 6.5.

In the section highlighted in red in Step 1, we see that there is a path for the β^{ε}_{B} cell to move left, as both β^{ε}_{B} and the pseudonaturality cell of ε_B next to it have the pseudonaturality cells for $\chi_F(h, g)$ and ε_C attached along two of the three 1-cells in their source. Using the technique discussed Proposition 4.2.1, we can move them all the way left to arrive at the diagram Step 2 (see Figure 6.6).

The modification $\widetilde{\omega}_F$ now has the pseudonaturality cells of the transformations it is modifying attached along its entire source. We can move $\widetilde{\omega}_F$ up and right through them to arrive at the Step 3 diagram in Figure 6.7.

The final thing we need to do is move the β^{ε}_{C} cell right. We note two things here in order to come up with a strategy for that move. First, after Step 3 the β^{ε}_{C} cell has several bicategory coherence cells attached to it. These have the potential to make it very easy to move using the naturality of coherence cells, not least because the target of β^{ε}_{C} is an identity and so its pseudonaturality cell is a coherence cell. Secondly, we note that after our last move the $\widetilde{\omega}_F$ cell is still attached to the pseudonaturality cell of $\chi_F(h, gf)$. However, in the target diagram of this axiom we note that these two cells are not attached and that the β^{ε}_{C} cell comes in between them.

Therefore, in order to solve the problem and create room for β^{ε}_{C} to move into, we will introduce some coherence cells to make space between the $\widetilde{\omega}_F$ cell and the $\chi_F(h, gf)$ cell. This leads us to the following adjustment made to the Step 3 diagram (Figure 6.8).

There is now a large mass of coherence cells in the lower centre of the diagram (Figure 6.9).

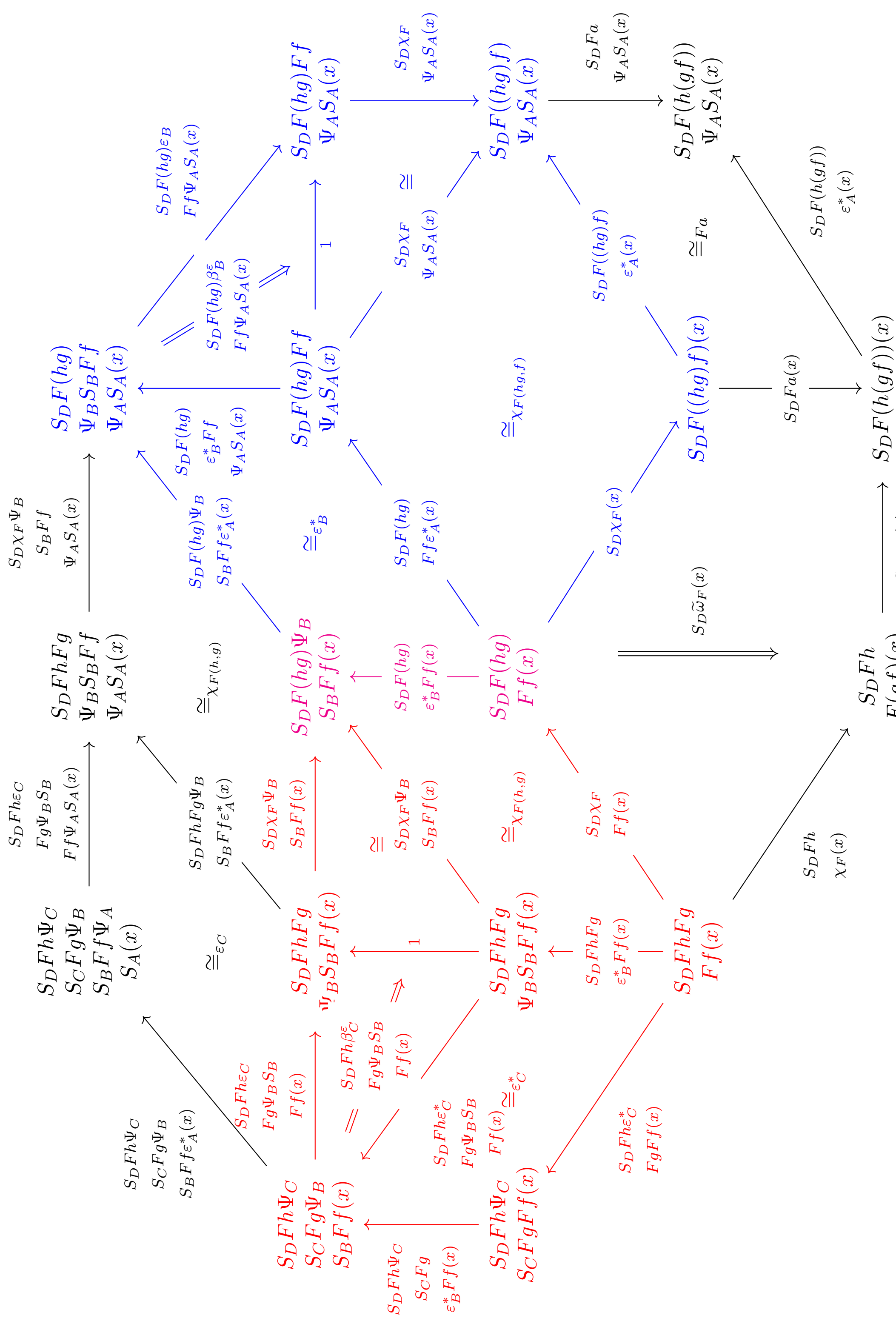


FIGURE 6.3: First Tritransformation Axiom for S: Source highlighted

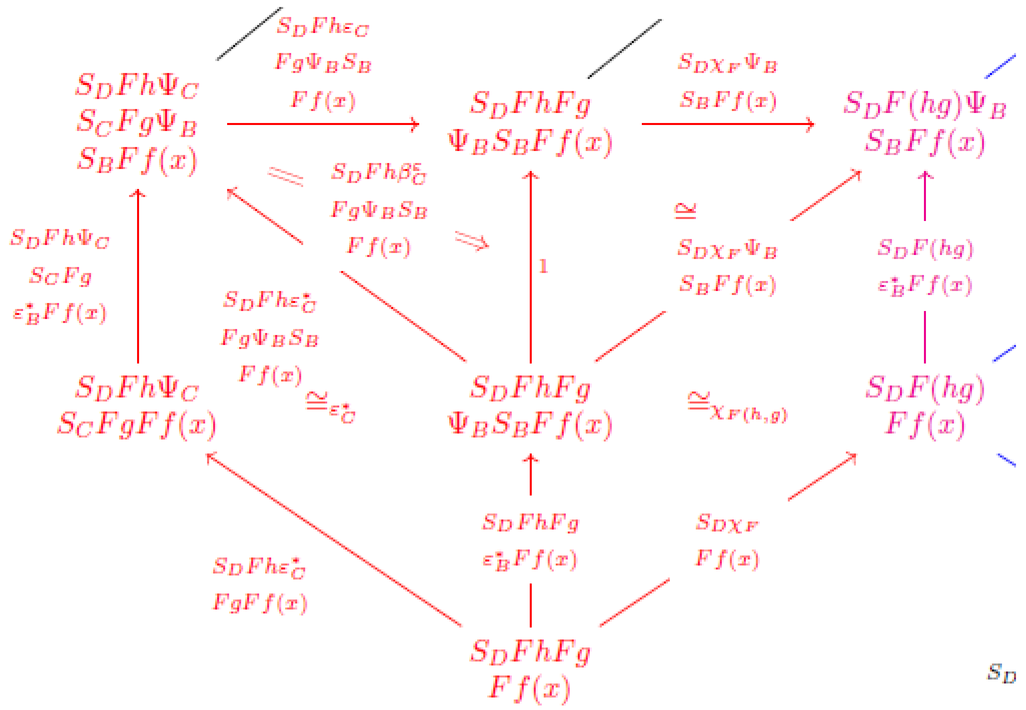


FIGURE 6.4: Moving β^ε_C through pseudonaturality cells

By the coherence theorem for bicategories, any diagram of coherence cells with the same source and target is equal, so we can replace these coherence cells with another pattern of coherence cells. We will choose the coherence cells so that $\beta\varepsilon_C$ has the pseudonaturality cells attached to its target. This gives us a second variation on the Step 3 diagram in Figure 6.10.

After these adjustments, we can move β^ε_C right by passing it through the coherence cells that are acting as the pseudonaturality cells of its target. This takes us to Step 4 (Figure 6.11).

We now clean up the coherence cells left on the left-hand side of the diagram. The target diagram does require that one coherence cell be left, attached to the target of the β^ε_B cell, but the rest can be removed. This adjustment gives us the diagram in Figure 6.12.

We now compare the diagram we have reached to the target diagram. The three key cells are in approximately the correct location - indeed, the $\widetilde{\omega}_F$ cell and the β^ε_C cells are in exactly the right location - but the β^ε_B cell isn't correctly placed yet. It's attached by the edge to the left-most boundary of Figure 6.12 (See Figure 6.13 for a close-up), but only its corner touches the boundary in the target diagram of Figure 6.2 (See Figure 6.14). The final steps of the proof will be used to correct this.

First, there are three pseudonaturality cells in the centre of diagram 4B (highlighted in red in Figure 6.15) that need to be adjusted in order to attach the right pseudonaturality

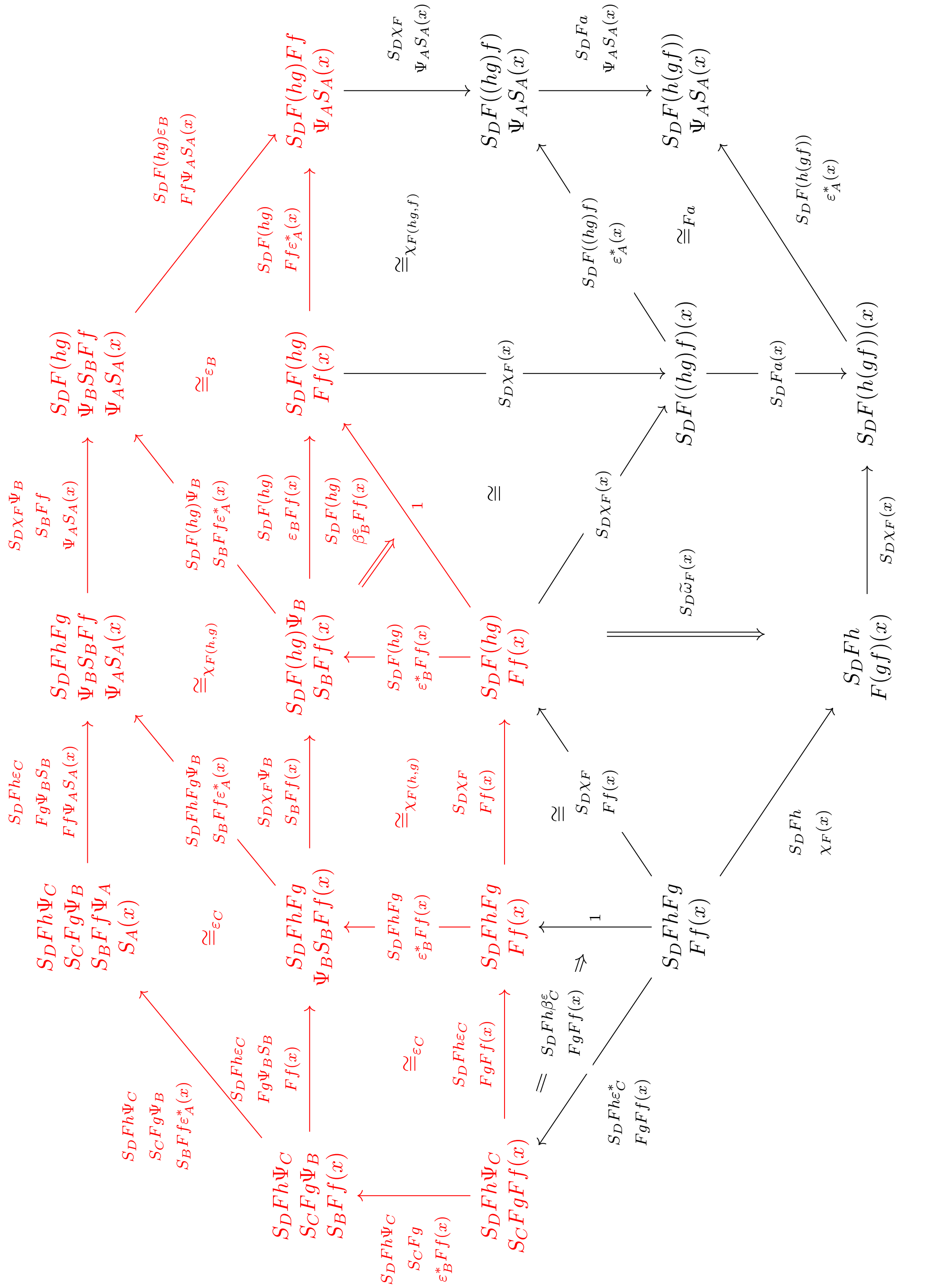


FIGURE 6.5: First Tritransformation Axiom for S: Step 1

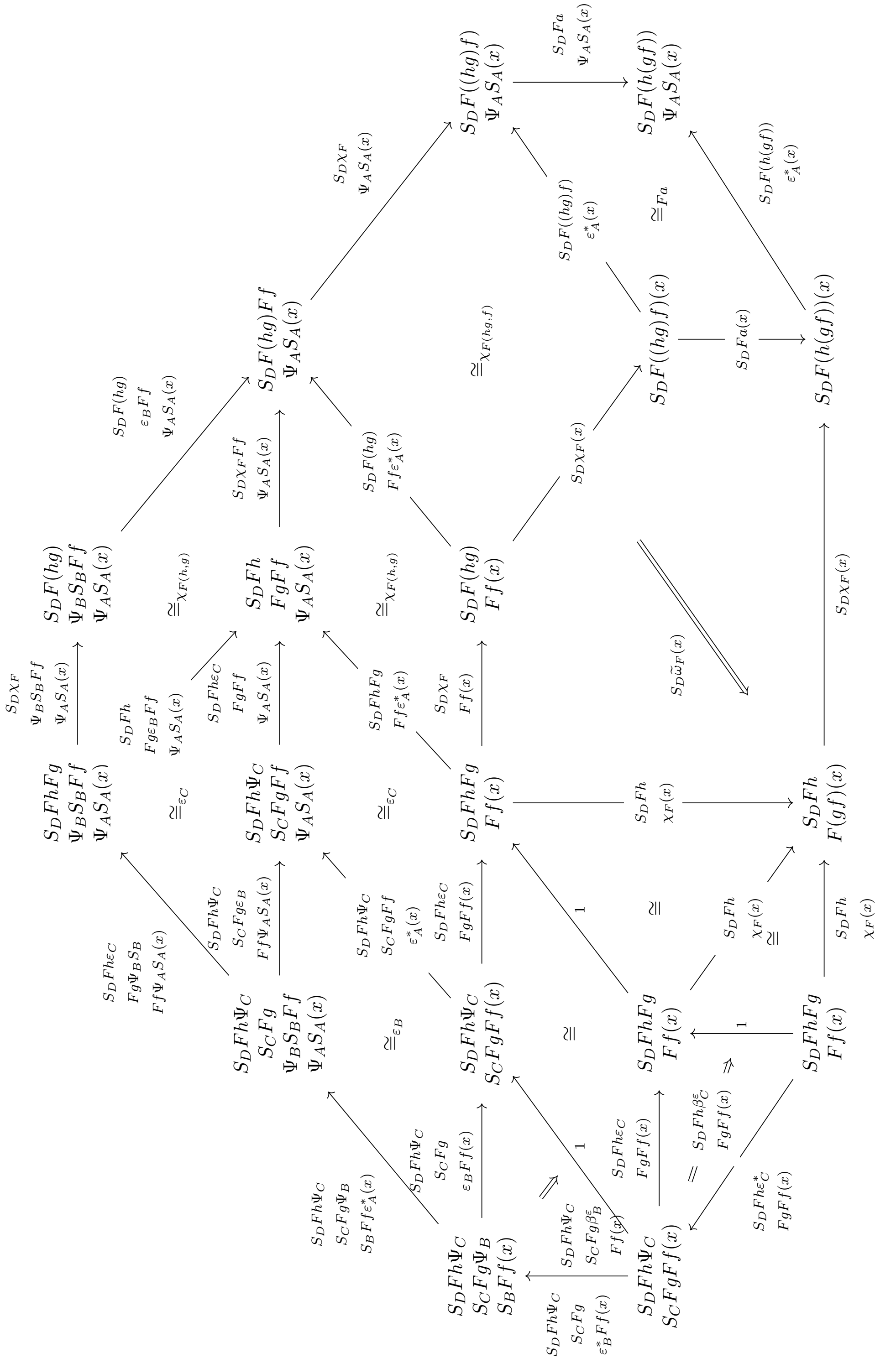


FIGURE 6.6: First Tritransformation Axiom for S: Step 2

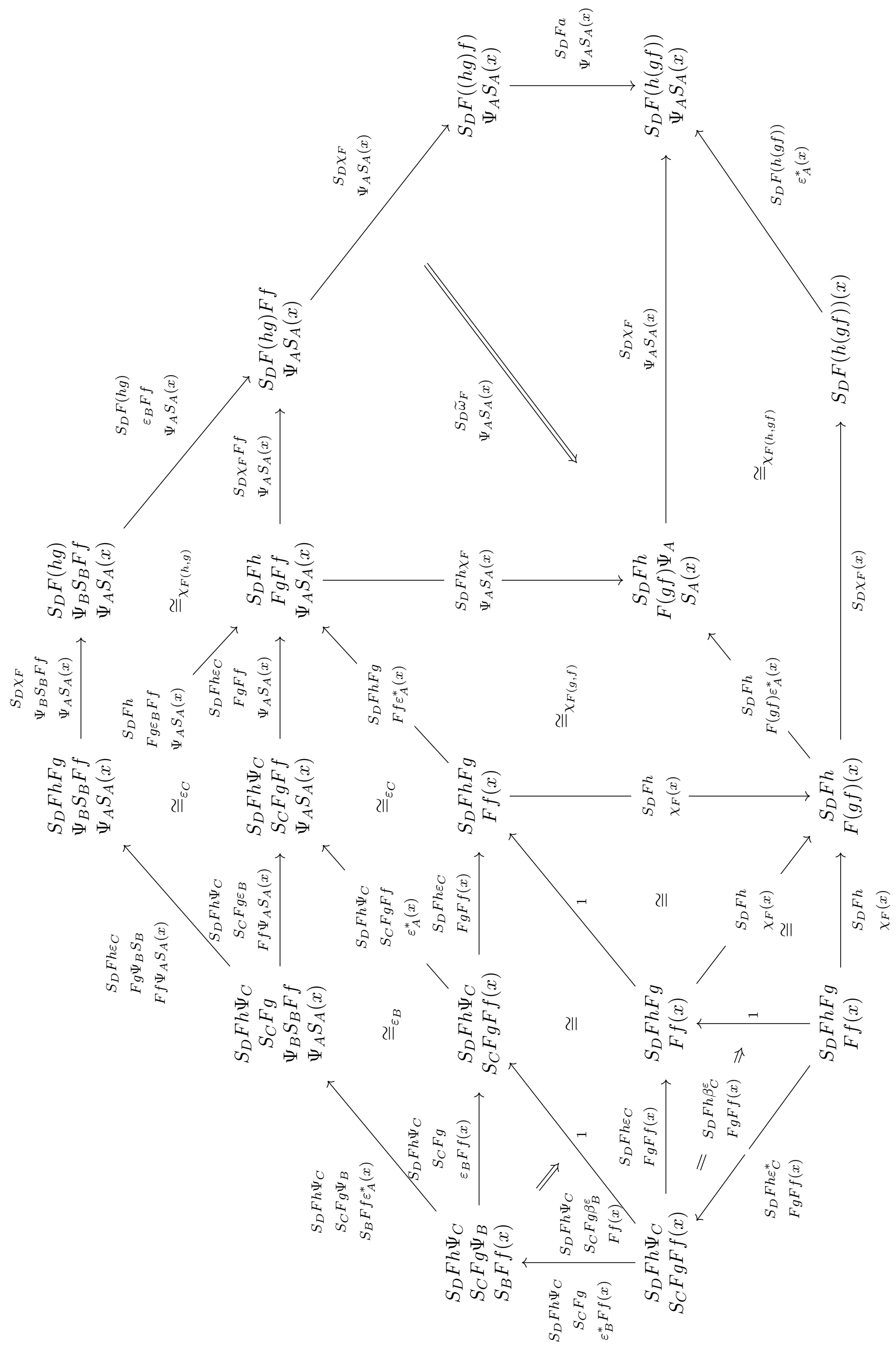


FIGURE 6.7: First Tritransformation Axiom for S: Step 3

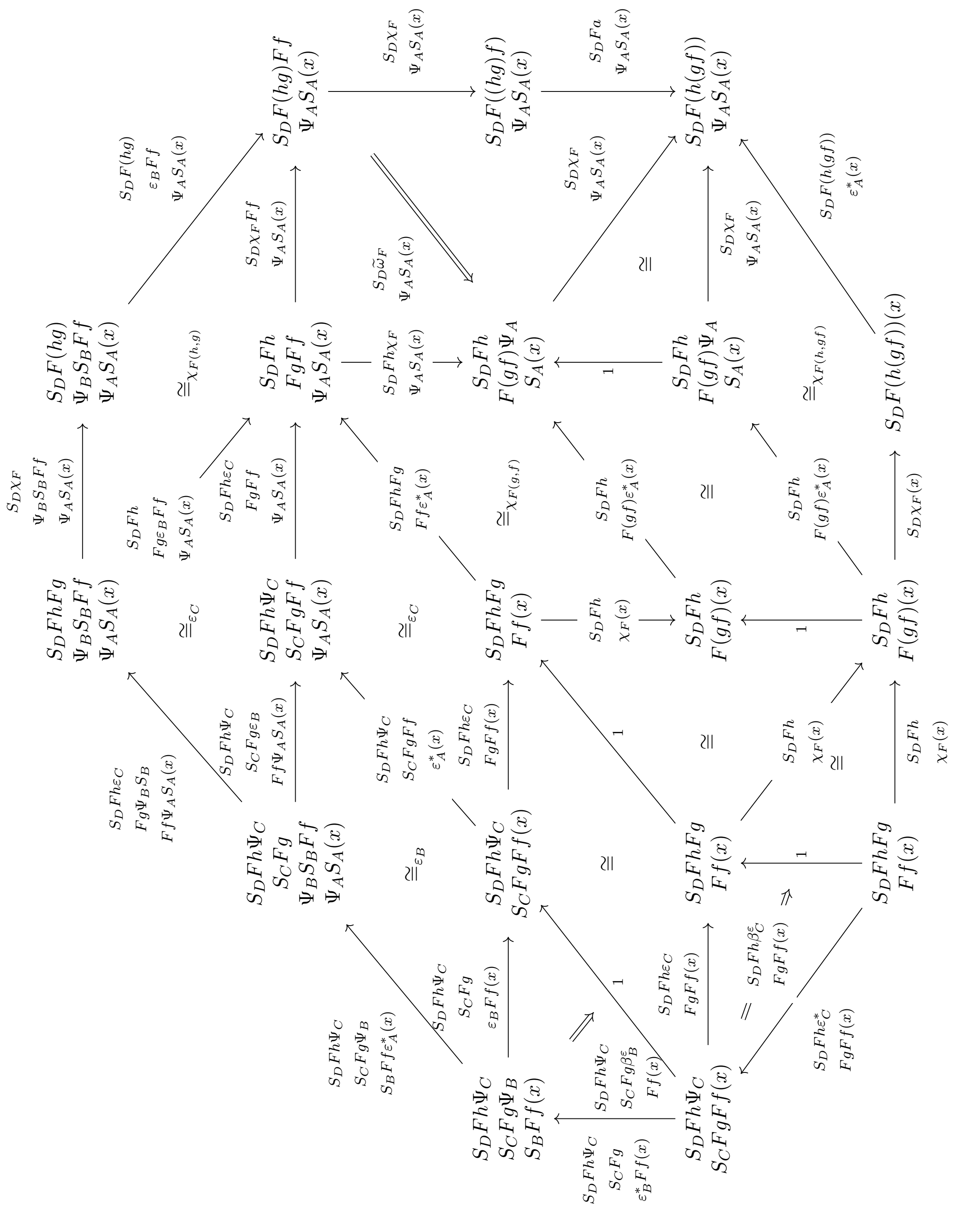


FIGURE 6.8: First Tritransformation Axiom for S: Step 3B

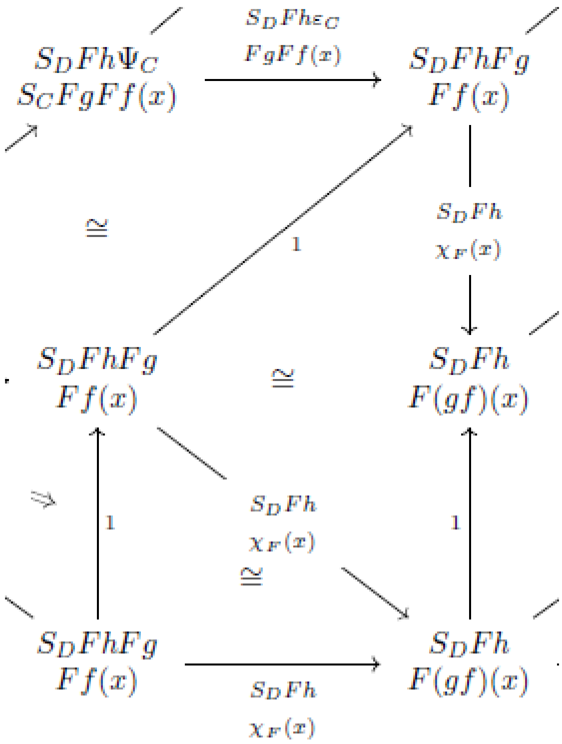


FIGURE 6.9: Coherence cells at the bottom of the Step 3 diagram

cells to the β_B^ε cell and neighbouring coherence cell. Moving the pseudonaturality cell of $\chi_F(g, f)$ through the ε_C pseudonaturality cell attaches it to the coherence cell neighbouring the β_B^ε cell and takes us to the diagram for Step 5 (Figure 6.16).

Before concluding, we note that our manipulations of this diagram have distorted the shape of the boundary of the diagram. We rectify this now: the diagram in Figure 6.17 is the same as Step 5, but with the boundary shape adjusted to be as it was in the source and target diagrams.

With this adjustment made to the layout, we can see the final step needed to move the β_B^ε cell to the correct location. As in the first step, β_B^ε and the coherence cell attached to it form a modification, and they have the appropriate pseudonaturality cells attached to the latter two 1-cells of the source. Therefore we can pass the modification through those pseudonaturality cells shown in Figure 6.18.

This move puts the β_B^ε cell in the correct location and we arrive (see Figure 6.19) at the target diagram as first shown in Figure 6.2. This series of steps shows that the source and target diagrams (Figures 6.1 and 6.2) are equal and therefore that S satisfies the first tritransformation axiom.

$\square$

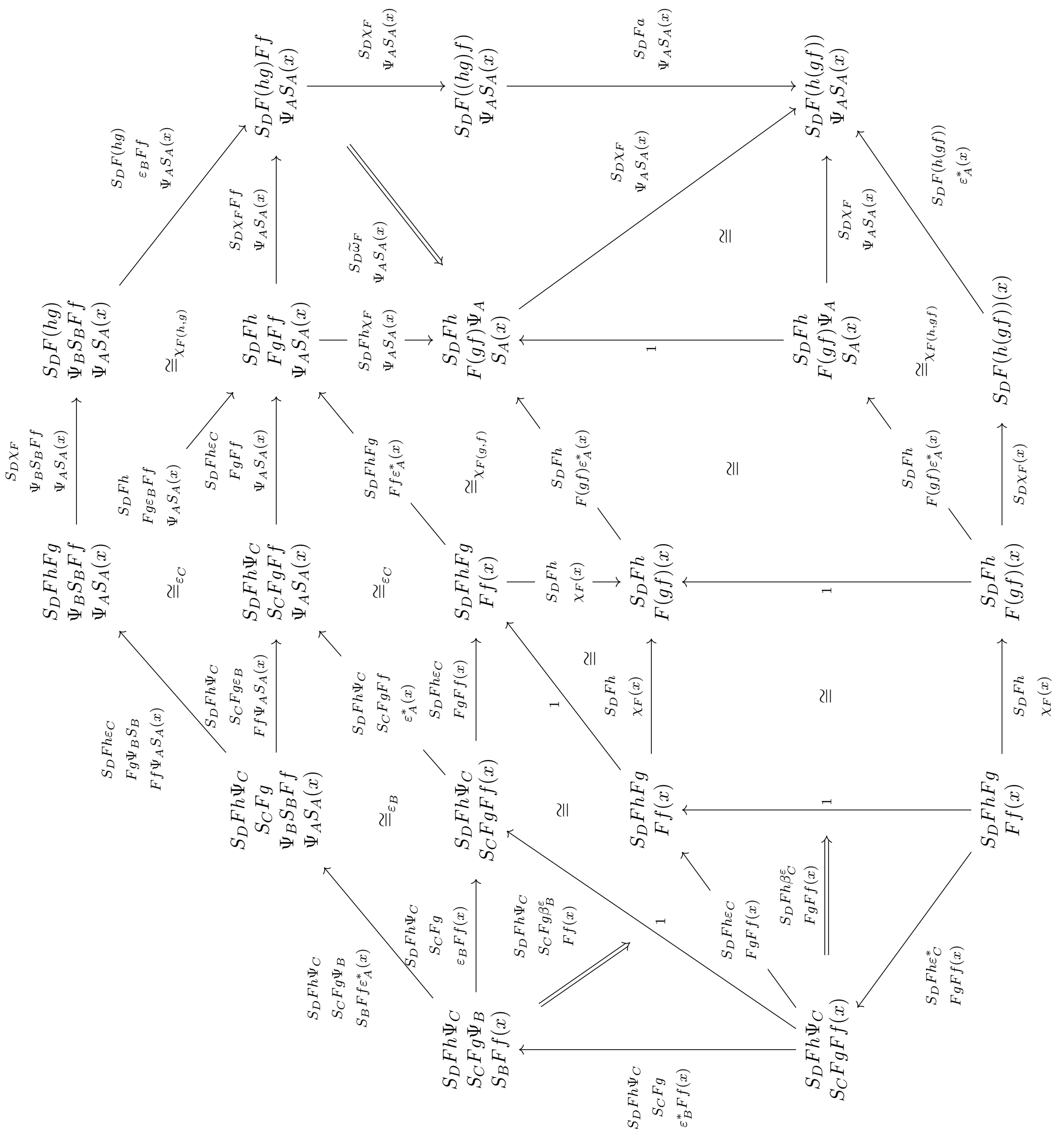


FIGURE 6.10: First Tritransformation Axiom for S: Step 3C

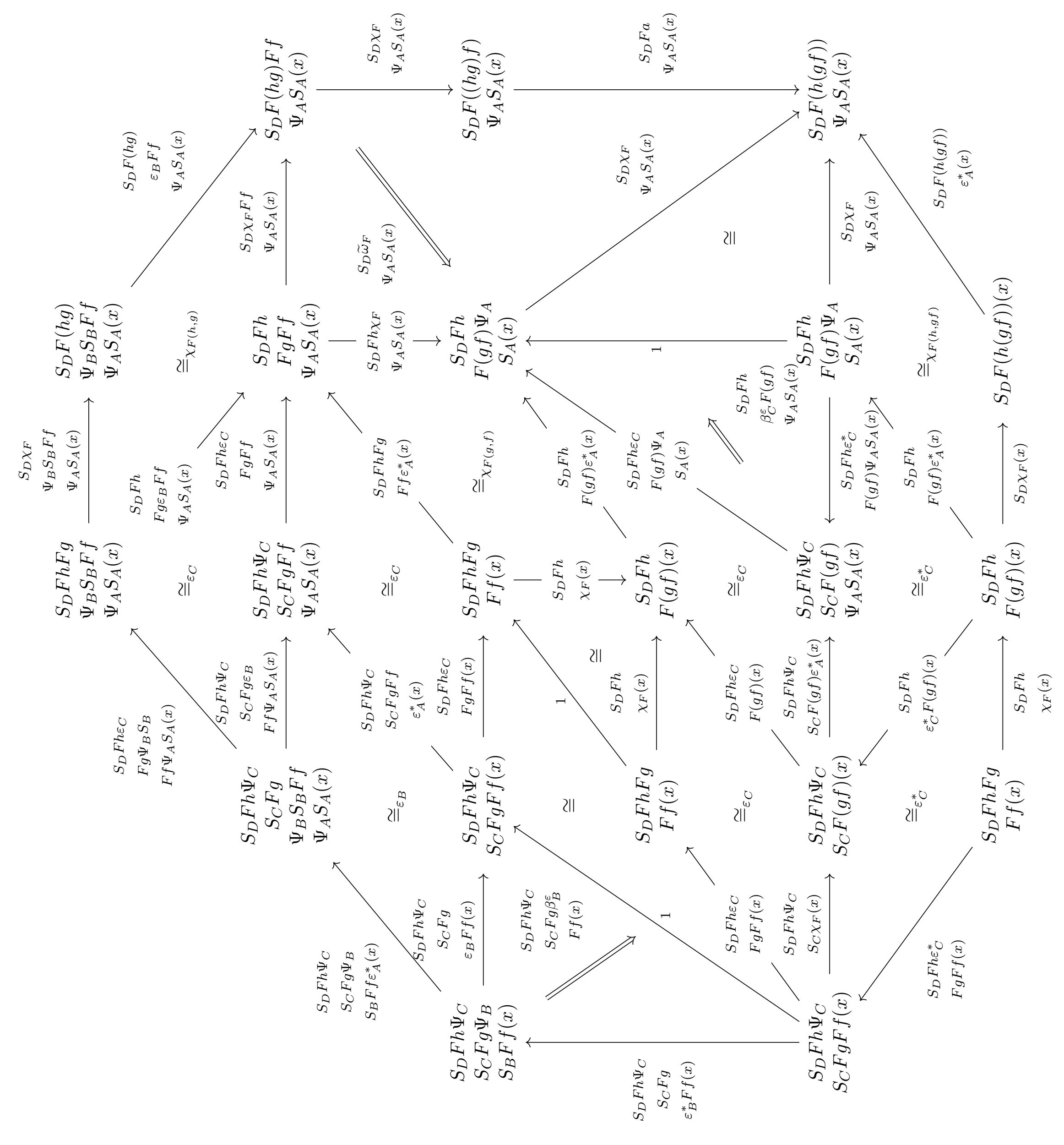


FIGURE 6.11: First Tritransformation Axiom for S: Step 4

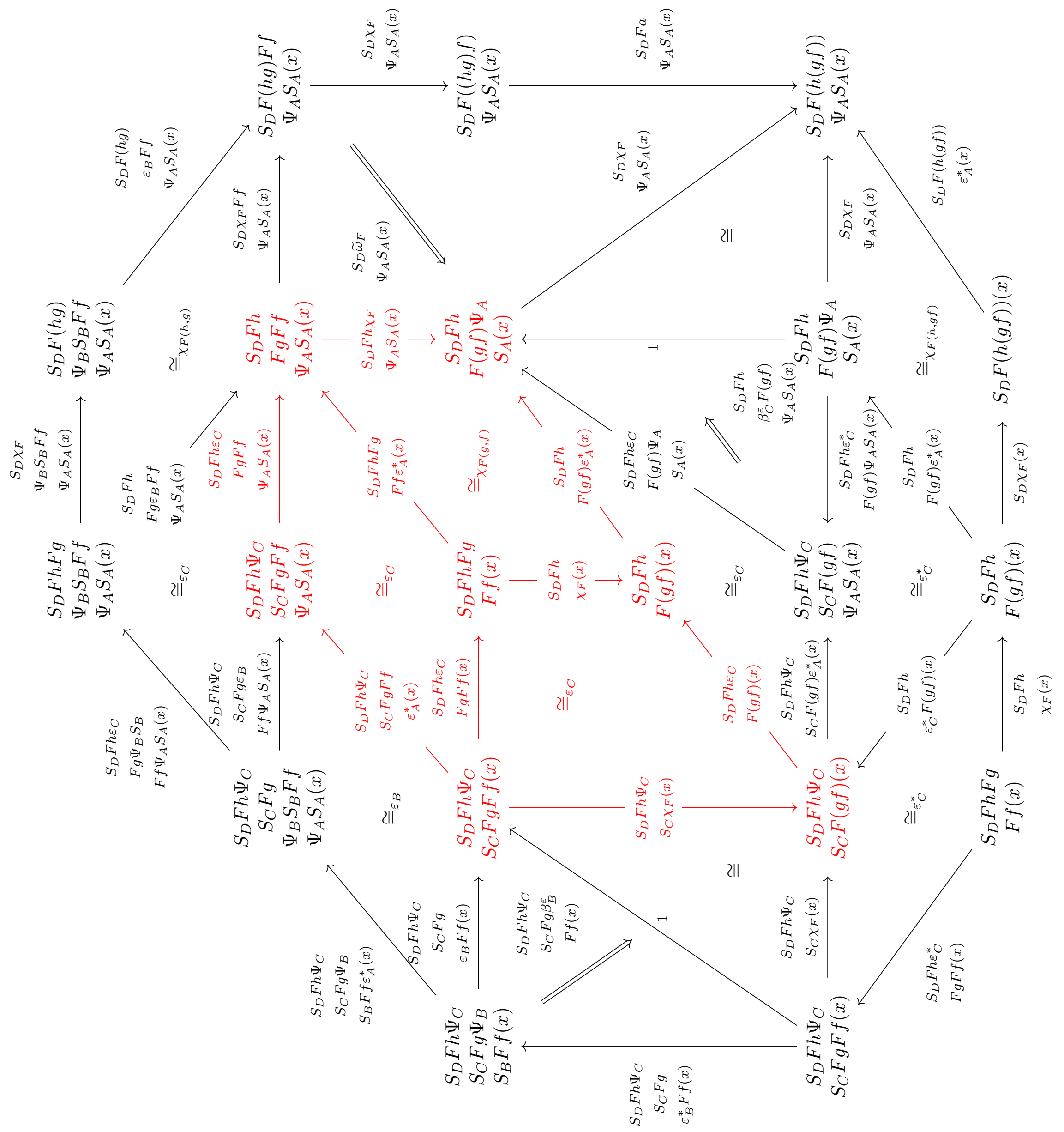


FIGURE 6.12: First Tritransformation Axiom for S: Step 4B

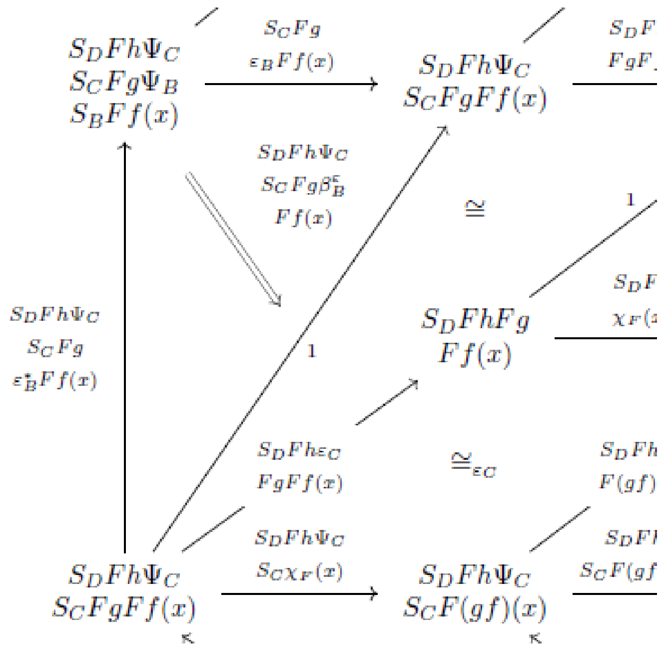


FIGURE 6.13: The β^{ε}_B cell is in the wrong place in Step 4B

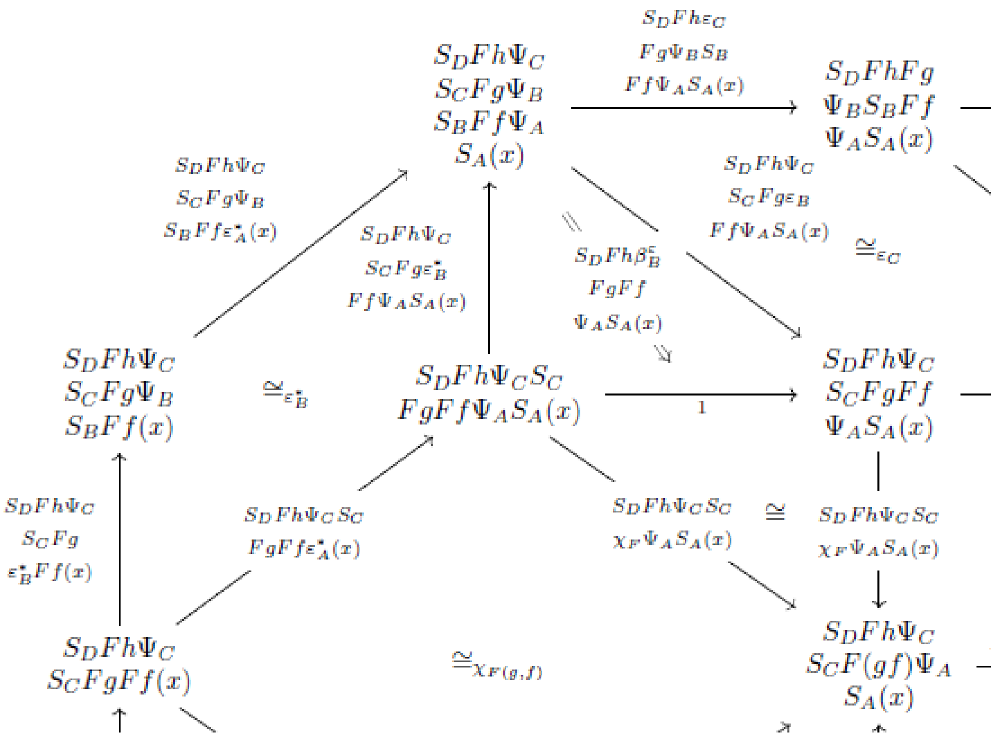


FIGURE 6.14: Where the β^{ε}_B cell should be

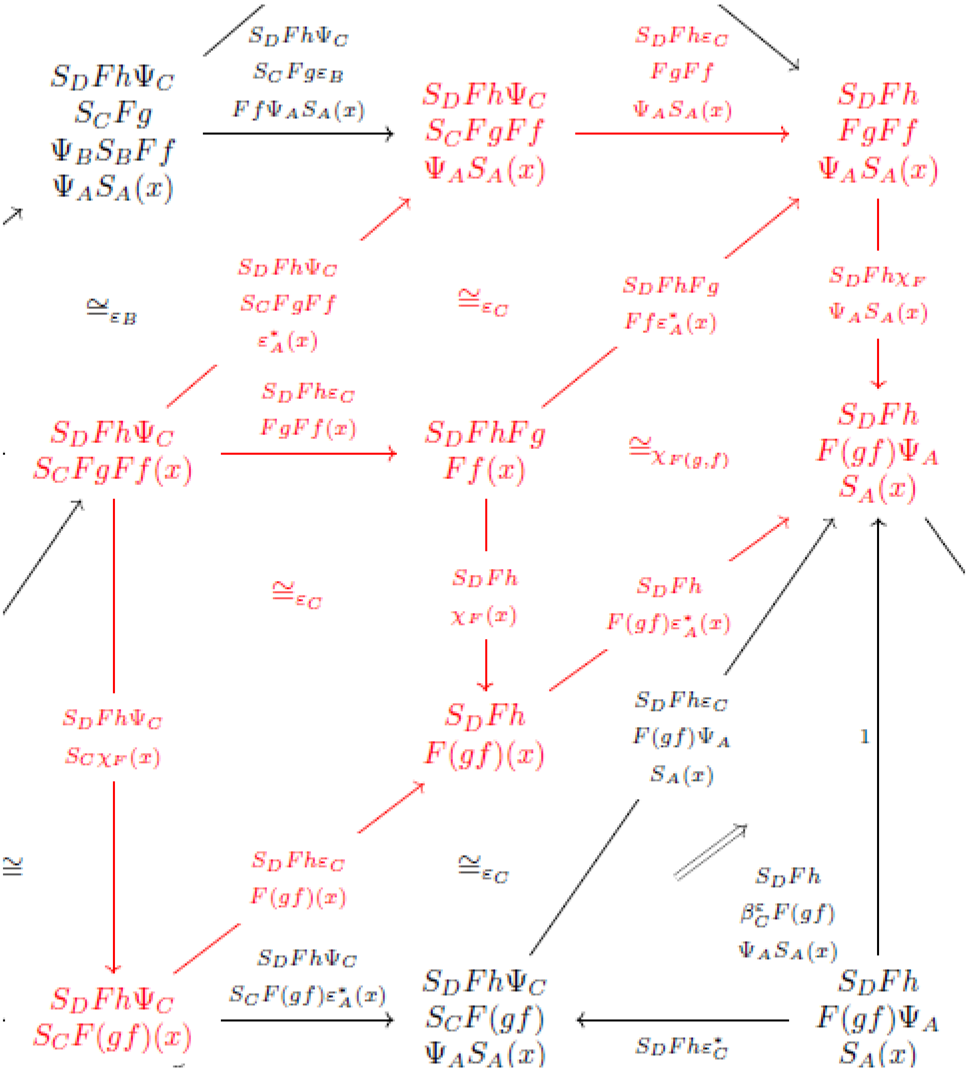


FIGURE 6.15: Attaching Pseudonaturality cells to β_B^ε

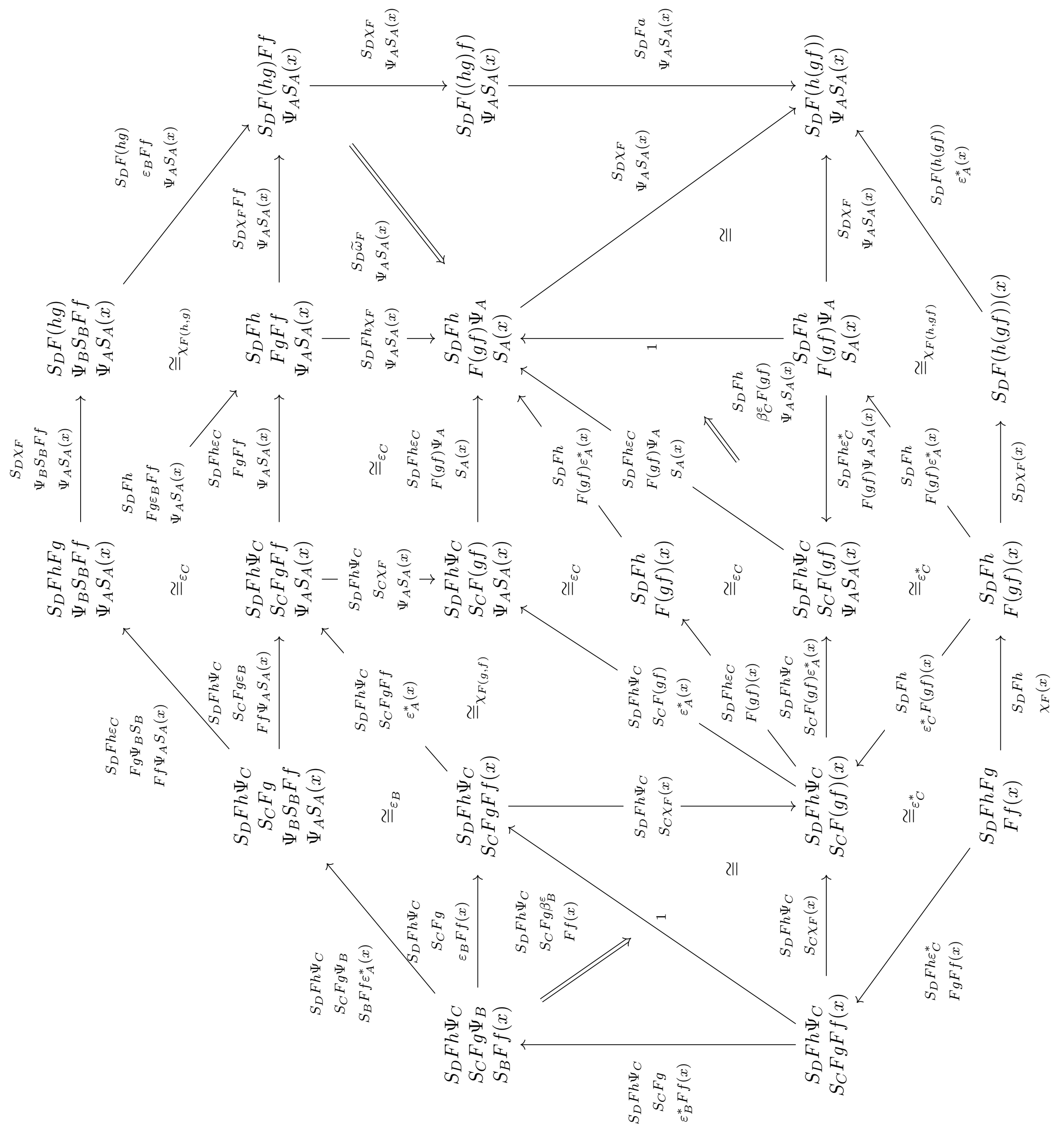


FIGURE 6.16: First Tritransformation Axiom for S: Step 5

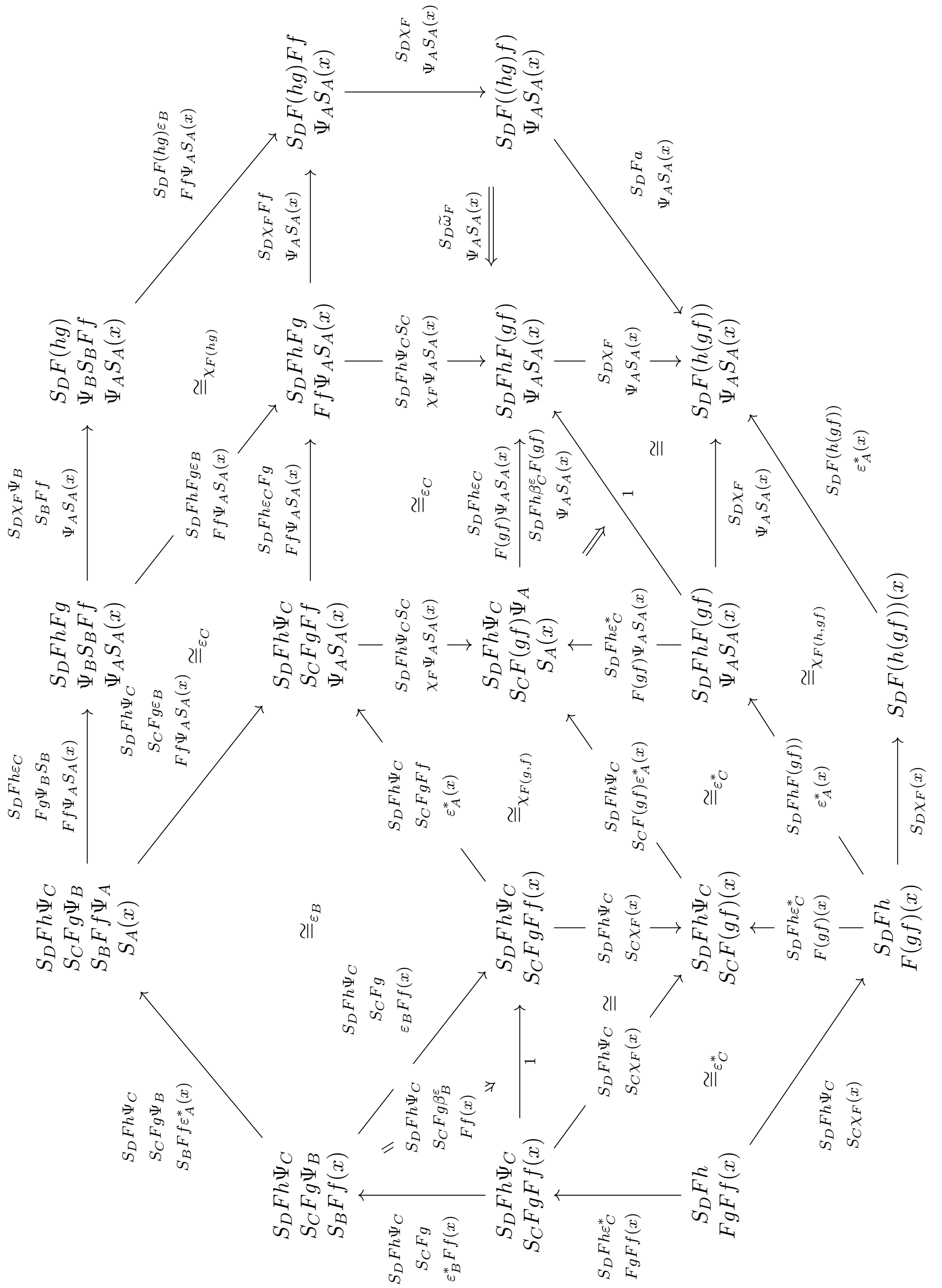


FIGURE 6.17: First Tritransformation Axiom for S: Step 5B

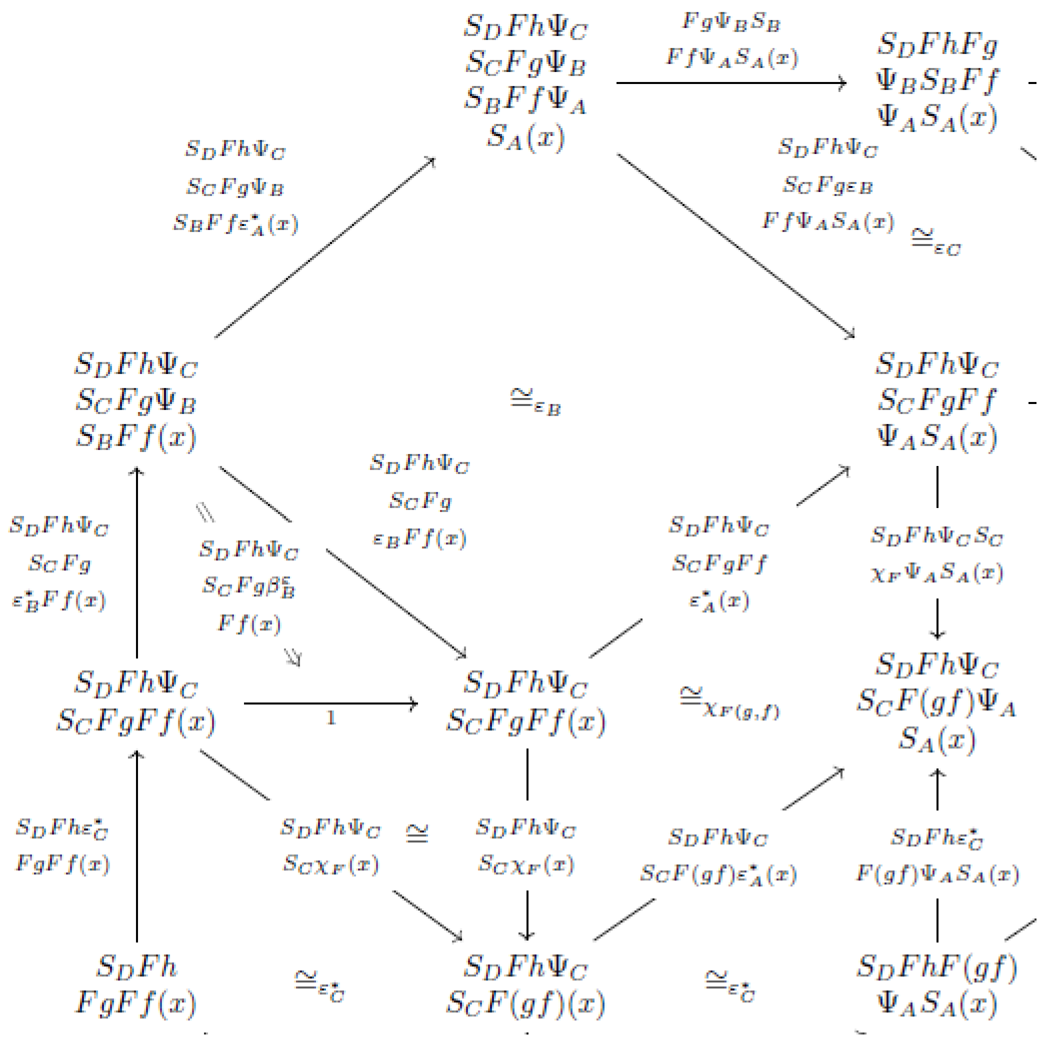


FIGURE 6.18: Moving β^ε_B to the correct location

$$
\begin{tikzcd}[column sep=small]
& \begin{matrix} S_DFh\Psi_C \\ S_CFg\Psi_B \\ S_BFf\Psi_A \\ S_A(x) \end{matrix} \arrow[r, "\begin{matrix} S_DFh\varepsilon_C \\ Fg\Psi_BS_B \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[dr, "\begin{matrix} S_DFh\Psi_C \\ S_CFg\varepsilon_B \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[d, phantom, shift left=8ex, "\begin{matrix} = \\ S_DFh\beta^{\varepsilon}_B \\ FgFf \\ \Psi_AS_A(x) \\ \Downarrow \end{matrix}"] & \begin{matrix} S_DFhFg \\ \Psi_BS_BFf \\ \Psi_AS_A(x) \end{matrix} \arrow[r, "\begin{matrix} S_D\chi_F\Psi_B \\ S_BFf \\ \Psi_AS_A(x) \end{matrix}"] \arrow[dr, "\begin{matrix} S_DFhFg\varepsilon_B \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[d, phantom, "\cong_{\varepsilon_C}"] & \begin{matrix} S_DF(hg) \\ \Psi_BS_BFf \\ \Psi_AS_A(x) \end{matrix} \arrow[dr, "\begin{matrix} S_DF(hg)\varepsilon_B \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[d, phantom, "\cong_{\chi_{F(hg)}}"] & \\
\begin{matrix} S_DFh\Psi_C \\ S_CFg\Psi_B \\ S_BFf(x) \end{matrix} \arrow[ur, "\begin{matrix} S_DFh\Psi_C \\ S_CFg\Psi_B \\ S_BFf\varepsilon^*_A(x) \end{matrix}"] \arrow[r, phantom, "\cong_{\varepsilon^*_B}"] & \begin{matrix} S_DFh\Psi_CS_C \\ FgFf\Psi_AS_A(x) \end{matrix} \arrow[u, "\begin{matrix} S_DFh\Psi_C \\ S_CFg\varepsilon^*_B \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[r, "1"'] \arrow[dr, "\begin{matrix} S_DFh\Psi_CS_C \\ \chi_F\Psi_AS_A(x) \end{matrix}"] & \begin{matrix} S_DFh\Psi_C \\ S_CFgFf \\ \Psi_AS_A(x) \end{matrix} \arrow[r, "\begin{matrix} S_DFh\varepsilon_CFg \\ Ff\Psi_AS_A(x) \end{matrix}"] \arrow[d, "\begin{matrix} S_DFh\Psi_CS_C \\ \chi_F\Psi_AS_A(x) \end{matrix}"] \arrow[d, phantom, shift right=6ex, "\cong"] \arrow[dr, phantom, "\cong_{\varepsilon_C}"] & \begin{matrix} S_DFhFg \\ Ff\Psi_AS_A(x) \end{matrix} \arrow[r, "\begin{matrix} S_D\chi_FFf \\ \Psi_AS_A(x) \end{matrix}"] \arrow[d, "\begin{matrix} S_DFh\Psi_CS_C \\ \chi_F\Psi_AS_A(x) \end{matrix}"] & \begin{matrix} S_DF(hg)Ff \\ \Psi_AS_A(x) \end{matrix} \arrow[d, "\begin{matrix} S_D\chi_F \\ \Psi_AS_A(x) \end{matrix}"] \\
\begin{matrix} S_DFh\Psi_C \\ S_CFgFf(x) \end{matrix} \arrow[u, "\begin{matrix} S_DFh\Psi_C \\ S_CFg \\ \varepsilon^*_BFf(x) \end{matrix}"] \arrow[ur, "\begin{matrix} S_DFh\Psi_CS_C \\ FgFf\varepsilon^*_A(x) \end{matrix}"] \arrow[dr, "\begin{matrix} S_DFh\Psi_C \\ S_C\chi_F(x) \end{matrix}"] \arrow[rr, phantom, "\cong_{\chi_{F(g,f)}}"] & & \begin{matrix} S_DFh\Psi_C \\ S_CF(gf)\Psi_A \\ S_A(x) \end{matrix} \arrow[r, "\begin{matrix} S_DFh\varepsilon_C \\ F(gf)\Psi_AS_A(x) \end{matrix}", "\begin{matrix} S_DFh\beta^{\varepsilon}_CF(gf) \\ \Psi_AS_A(x) \\ \Downarrow \end{matrix}"'] & \begin{matrix} S_DFhF(gf) \\ \Psi_AS_A(x) \end{matrix} \arrow[d, "\begin{matrix} S_D\chi_F \\ \Psi_AS_A(x) \end{matrix}"] & \begin{matrix} S_DF((hg)f) \\ \Psi_AS_A(x) \end{matrix} \arrow[l, Leftarrow, "\begin{matrix} S_D\widetilde{\omega}_F \\ \Psi_AS_A(x) \end{matrix}"'] \arrow[dl, "\begin{matrix} S_DFa \\ \Psi_AS_A(x) \end{matrix}"] \\
\begin{matrix} S_DFh \\ FgFf(x) \end{matrix} \arrow[u, "\begin{matrix} S_DFh\varepsilon^*_C \\ FgFf(x) \end{matrix}"] \arrow[drr, bend right=0, "\begin{matrix} S_DFh\Psi_C \\ S_C\chi_F(x) \end{matrix}"'] \arrow[r, phantom, "\cong_{\varepsilon^*_C}"] & \begin{matrix} S_DFh\Psi_C \\ S_CF(gf)(x) \end{matrix} \arrow[ur, "\begin{matrix} S_DFh\Psi_C \\ S_CF(gf)\varepsilon^*_A(x) \end{matrix}"] \arrow[r, phantom, "\cong_{\varepsilon^*_C}"] & \begin{matrix} S_DFhF(gf) \\ \Psi_AS_A(x) \end{matrix} \arrow[u, "\begin{matrix} S_DFh\varepsilon^*_C \\ F(gf)\Psi_AS_A(x) \end{matrix}"] \arrow[ur, "1"'] \arrow[r, "\begin{matrix} S_D\chi_F \\ \Psi_AS_A(x) \end{matrix}"'] \arrow[r, phantom, shift left=6ex, "\cong"] \arrow[dr, phantom, "\cong_{\chi_{F(h,gf)}}"] & \begin{matrix} S_DF(h(gf)) \\ \Psi_AS_A(x) \end{matrix} & \\
& \begin{matrix} S_DFh \\ F(gf)(x) \end{matrix} \arrow[u, "\begin{matrix} S_DFh\varepsilon^*_C \\ F(gf)(x) \end{matrix}"] \arrow[ur, "\begin{matrix} S_DFhF(gf)) \\ \varepsilon^*_A(x) \end{matrix}"] \arrow[r, "S_D\chi_F(x)"'] & S_DF(h(gf))(x) \arrow[ur, "\begin{matrix} S_DF(h(gf)) \\ \varepsilon^*_A(x) \end{matrix}"'] & &
\end{tikzcd}
$$

FIGURE 6.19: First Tritransformation Axiom for S: Step 6 i.e. Target

6.3 Conjectures for the other Axioms

In this section we conjecture that S as defined in section 6.1 also obeys the other two axioms of a tritransformation. We also explain the ideas behind a strategy for proving both axioms hold.

Conjecture 6.3.1. $S : F \Rightarrow G$ satisfies the second tritransformation axiom.

Idea of proof. After substituting $\widetilde{\Pi_S}$, $\widetilde{M_S}$ and $\widetilde{\gamma}_G$ into the diagrams for the second axiom, we see that proving it is equivalent to showing that the following source diagram (Figure 6.20) and target diagram (Figure 6.21) are equal (Note that these diagrams are in the opposite order to the way they were presented in Definition 3.1.3 and Proposition 4.1.2 in order to ensure that the more complicated diagram is the source).

As we come up with a strategy to turn the source diagram into the target diagram, there are three features of the diagram we have to note:

- The γ_F cell starts on the far right of the source, and needs to be moved to the bottom left of the target.
- The source diagram contains an instance of $\widetilde{\Sigma}_B$ and an instance of $\widetilde{\Sigma}_B^{-1}$ that the target doesn't. These will need to be cancelled with each other.
- The source diagram contains a cell that is an instance of $\alpha_B^{\varepsilon(-1)}$. This will need to be moved to the top right, and converted into an instance of β_B^{ε} by using the triangle identity for the adjoint equivalence $\varepsilon_B \dashv \varepsilon_B^*$. That is, because the triangle identity states that

$$\begin{tikzcd}
 & S_B\Psi_BS_B \arrow[rr, "1"] \arrow[dr, "\varepsilon_BS_B"] \arrow[d, phantom, "\Downarrow\beta_B^{\varepsilon}"] & & S_B\Psi_BS_B \\
S_B \arrow[ur, "S_B\varepsilon_B^*"] \arrow[rr, "1"'] & {} & S_B \arrow[ur, "S_B\varepsilon_B^*"'] \arrow[u, phantom, "\Downarrow\alpha_B^{\varepsilon}"] &
\end{tikzcd}$$

is equal to a coherence cell, the diagram

$$\begin{tikzcd}
 & & S_A \arrow[dr, "S_B\varepsilon_B^*"] \arrow[d, phantom, "\Downarrow\alpha_B^{\varepsilon(-1)}"] & \\
 & S_B\Psi_BS_B \arrow[ur, "\varepsilon_BS_B"] \arrow[rr, "1"] \arrow[dr, "\varepsilon_BS_B"] \arrow[d, phantom, "\Downarrow\beta_B^{\varepsilon}"] & {} & S_B\Psi_BS_B \\
S_B \arrow[ur, "S_B\varepsilon_B^*"] \arrow[rr, "1"'] & {} & S_B \arrow[ur, "S_B\varepsilon_B^*"'] \arrow[u, phantom, "\Downarrow\alpha_B^{\varepsilon}"] &
\end{tikzcd}$$

is equal to both β_B^{ε} and the combination of $\alpha_B^{\varepsilon(-1)}$ and a coherence cell. This should aid us in transforming one of these cells into the other.

Being able to deal with all three of these issues would allow us to manipulate the diagrams to show that they are equal and that S satisfies the second trifunctor axiom.

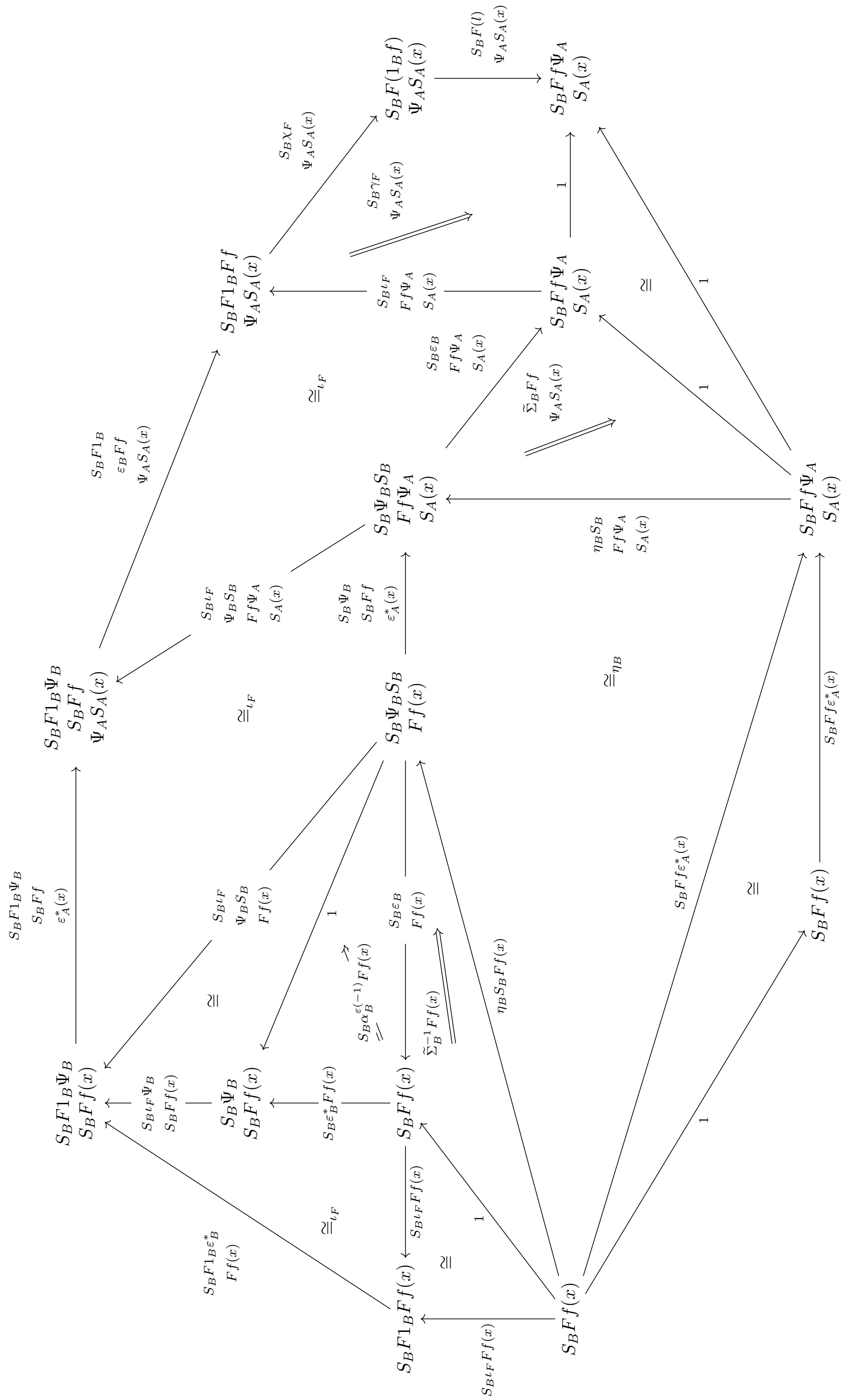


FIGURE 6.20: Second Tritransformation Axiom for S: Source

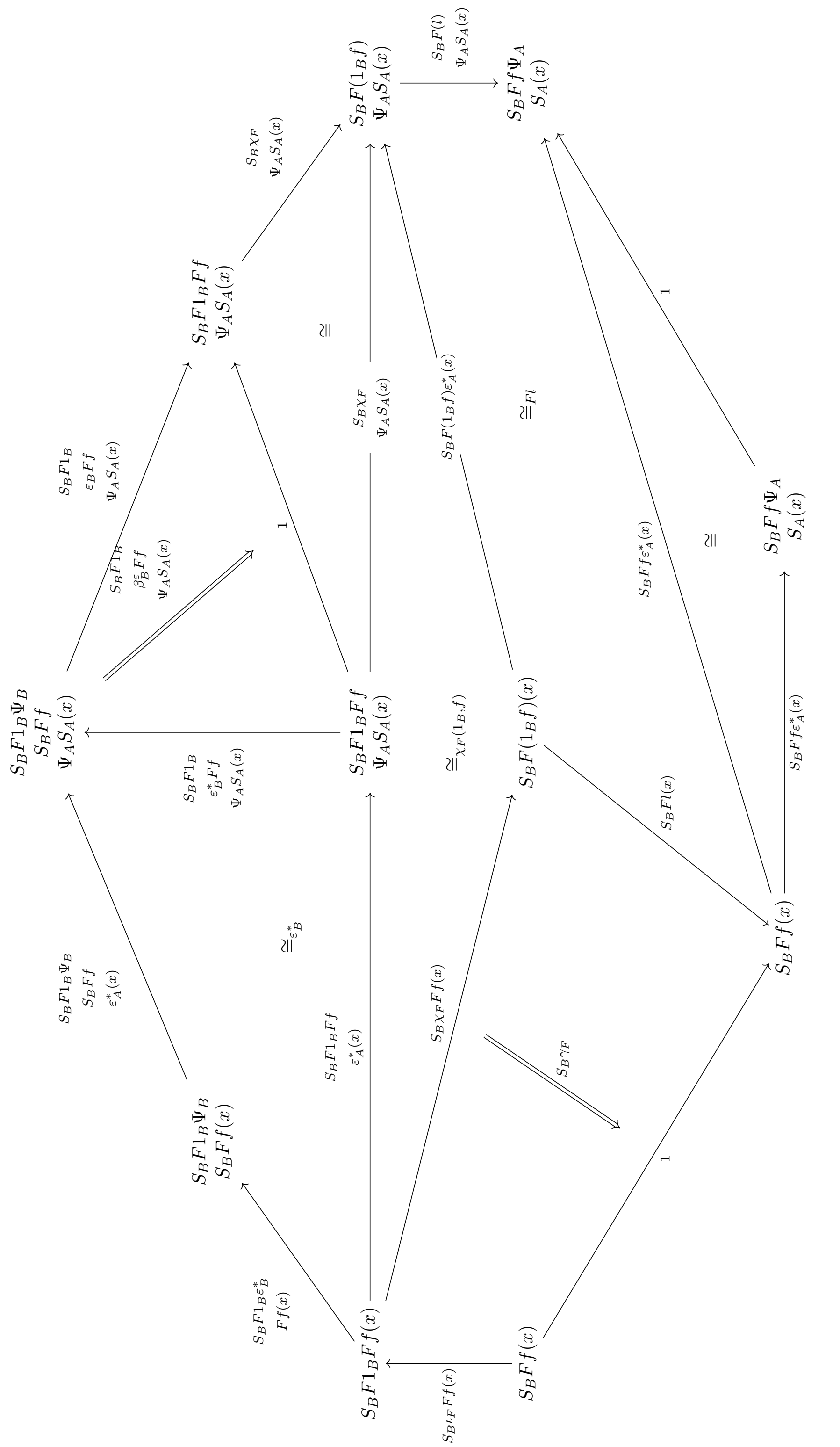


FIGURE 6.21: Second Tritransformation Axiom for S: Target

□

Conjecture 6.3.2. $S : F \Rrightarrow G$ satisfies the third tritransformation axiom.

Idea of proof. After substituting the defined values of Π_S, M_S and δ_G into the diagrams for the third axiom, we see that proving the third axiom is equivalent to proving that the following source diagram (Figure 6.22) and target diagram (Figure 6.23) are equal (as in the previous proof, they are presented here in the opposite order to the one in the definition, so that the more complicated diagram is the source).

As with the second axiom, there are three things to note when coming up with a strategy for proving that these two diagrams are equal:

- The δ_F^{-1} cell starts on the bottom right of the source and needs to be moved to the bottom left of the target.
- The source diagram contains an instance of $\widetilde{\Phi}_A$ and an instance of $\widetilde{\Sigma}_A^{-1}$ that the target doesn't. These will need to be removed by bringing them together and completing an instance of one of the biadjoint biequivalence axioms.
- The source diagram contains a cell that is an instance of $\alpha_A^{\varepsilon(-1)}$. This will need to be converted into an instance of β_A^{ε} using the triangle identity, just as in the previous proof.

If all three of these issues can be dealt with, then we can prove that the third axiom holds.

□

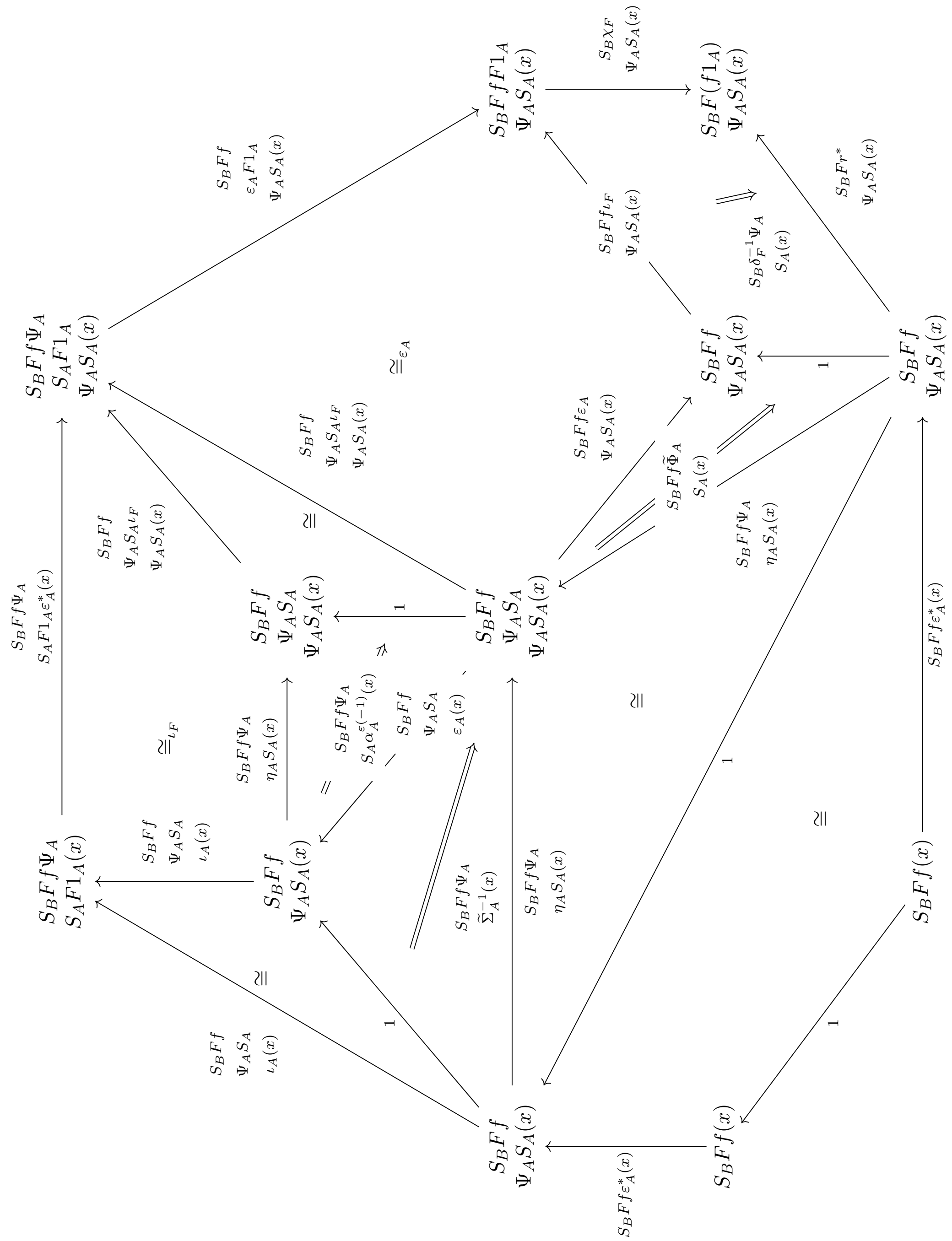


FIGURE 6.22: Third Tritransformation Axiom for S: Source

$$
\begin{tikzcd}
& S_BFf\Psi_A S_AF1_A\Psi_AS_A(x) \arrow[rr, "S_BFf\Psi_AS_AF1_A\varepsilon^*_A(x)"] \arrow[dr, phantom, "\cong_{\varepsilon^*_A}"] & & S_BFfF1_A\Psi_AS_A(x) \arrow[dr, "S_BFf\varepsilon_AF1_A\Psi_AS_A(x)"] \arrow[d, phantom, "\Downarrow S_BFf\beta^{\varepsilon}_BF1_A\Psi_AS_A(x)"] & \\
S_BFf\Psi_AS_A(x) \arrow[ur, "S_BFf\Psi_AS_A\iota_A(x)"] \arrow[dr, phantom, "\cong_{\varepsilon^*_A}"] & S_BFfF1_A(x) \arrow[u, "S_BFfF1_A\varepsilon^*_A(x)"'] \arrow[r, "S_BFfF1_A\varepsilon^*_A(x)"] \arrow[dr, "S_B\chi_F(x)"] & S_BFfF1_A\Psi_AS_A(x) \arrow[ur, "S_BFf\varepsilon^*_AF1_A\Psi_AS_A(x)"] \arrow[rr, "1"] \arrow[drr, "S_B\chi_F\Psi_AS_A(x)"] \arrow[d, phantom, "\cong_{\chi_F}"] & \cong & S_BFfF1_A\Psi_AS_A(x) \arrow[d, "S_B\chi_F\Psi_AS_A(x)"] \\
S_BFf(x) \arrow[u, "S_BFf\varepsilon^*_A(x)"] \arrow[ur, "S_BFf\iota_A(x)"] & & S_BF(f1_A)(x) \arrow[rr, "S_BF(f1_A)\varepsilon^*_A(x)"] & & S_BF(f1_A)\Psi_AS_A(x) \\
& S_BFf(x) \arrow[ul, "1"] \arrow[ur, "S_BFr^*(x)"'] \arrow[rr, "S_BFf\varepsilon^*_A(x)"] \arrow[u, phantom, "\Uparrow S_B\delta_F^{-1}"] & & S_BFf\Psi_AS_A(x) \arrow[ur, "S_BFr^*\Psi_AS_A(x)"'] \arrow[u, phantom, "\cong_{Fr^*}"] &
\end{tikzcd}
$$

FIGURE 6.23: Third Tritransformation Axiom for S: Target

If both of these conjectures could be proved, along with the first axiom which we have proved, we would reach the following conclusion:

Conjecture 6.3.3. S as defined in 6.1 is a tritransformation between the original trifunctor F and the trifunctor G defined in Section 5.2.

Chapter 7

Conclusions and Further Directions

In this thesis we have developed original techniques for working with pasting diagrams in the tricategory of bicategories. Using the coherence theorem for bicategories we were able to simplify the diagrams that made up axioms of key tricategorical structures such as trifunctors, tritransformations and biadjoint biequivalences. Without these new techniques the diagrams needed to prove the results of this thesis would have been intractable.
We also saw that the ability of pseudo-natural transformations and modifications to be passed through other cells applies to more than just the case specified by their definitions, where the cells they are being passed through cover the entire source (or target). This greatly expedited the task of manipulating the pasting diagrams; we saw many cases where a cell that needed to be moved had pseudo-naturality cells along part but not all of the source, or on part of the source and some part of the target.

We then demonstrated the method for transporting a trifunctor $F : \mathcal{T} \to \underline{Bicat}$ across a collection of biadjoint biequivalences $\Psi_A \vdash S_A : FA \to GA$. We noted that G should have an action on the hom-bicategories given by sending a cell f to $S_B F f \Psi_A$. Then we constructed the higher cells of G by noting what their sources and targets should be and filling them out with cells coming from the original trifunctor and the biequivalences. This typically involved noting a place in the boundary of the cell where the respective cell for F would fit, and then using pseudo-naturality cells to move the rest of the 1-cells into position around it.
Once all the data for G was constructed we were then able to prove that the axioms of a trifunctor held. These proofs started by substituting the data into the diagrams of the axioms. We then noted that, just as the data for G included places where instances of the data for F could fit, so also the axiom diagrams for G included patterns of cells along

the boundary where the axiom diagrams for F could fit. By moving the cells so as to complete the axiom for F, we were able to use that axiom as a step in showing that the axiom diagrams for G were equal.

With both axioms proved we have confirmed that the structure G that we defined is a trifunctor. Therefore, we have succeeded in transporting the structure of a trifunctor across the biequivalences.

Finally we constructed a reasonable candidate for the lifting of the family of object-indexed biequivalences $S_A : FA \to GA$ to a tritransformation $S : F \to G$, using similar methods to the construction of the transported trifunctor. We then proved that this structure satisfied the first tritransformation axiom, and conjectured that it also satisfies the other two. If these conjectures are true, and $S : F \to G$ is a tritransformation, then this shows that the constructed trifunctor isn't just arbitrary: it truly did arise as a result of the family of biequivalences.

A Potential Application

One potential application for this result is the problem that motivated it: the comparison of Tamsamani 3-categories to tricategories. We saw in the introduction how the setup needed for the result of this thesis arose: by using the result of Lack and Paoli [LP08] to turn a Tamsamani 3-category into a simplicial object in $\underline{Bicat}$, at which point we can interpret that simplicial object as a trifunctor and take the object-indexed biequivalences to be the Segal maps $X_n \to X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$.

Transporting across these biequivalences loses the strictness of the original Tamsamani 3-category: the new object is only a trifunctor from Δ^{op} to $\underline{Bicat}$ (called a pseudo-simplicial bicategory in [CH14]). However, we benefit as well, because now the image of each $[n] \in \Delta^{op}$ is the pullback $X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$, making the objects that much more meaningful. After all, if you interpret X_1 as the bicategory of all 1-cells then these pullbacks are the objects that allow composition, and are therefore the objects of interest when trying to define a tricategory corresponding to a particular Tamsamani 3-category.

We envisage using coherence theorem for trifunctors to take this transported trifunctor and simplify the proofs of the axioms of the tricategory we are trying to construct from the Tamsamani 3-category. If so, this would give us the other direction to the nerve on a tricategory given by Cegarra and Heredia [CH14].

Extending our Main Result

It would be interesting to obtain a method for transporting the structure of any trifunctor, not just those ending in $\underline{Bicat}$. Here is one possibility for achieving that.

Take any trifunctor $\mathcal{T} \to \mathcal{S}$ and biequivalences in $\mathcal{S}$ indexed by $ob(\mathcal{T})$. By a Yoneda argument, $\mathcal{S}$ is a full sub-tricategory of $\underline{Tricat}(\mathcal{S}^{op}, \underline{Bicat})$ and so we get a trifunctor $\mathcal{T} \to \underline{Tricat}(\mathcal{S}^{op}, \underline{Bicat})$ and biequivalences in $\underline{Tricat}(\mathcal{S}^{op}, \underline{Bicat})$ indexed by $ob(\mathcal{T})$. Finally, the trifunctor $\mathcal{T} \to \underline{Tricat}(\mathcal{S}^{op}, \underline{Bicat})$ corresponds to a trifunctor $\mathcal{T} \times \mathcal{S}^{op} \to \underline{Bicat}$.

This isn't exactly the setup needed to apply the main result as the object-indexed biequivalences in $\underline{Tricat}(\mathcal{S}^{op}, \underline{Bicat})$ end up placing some constraints on the biequivalences when curried over. Even so, the situation is close enough that it merits further study as a potential method for extending the result.

Bibliography

[Bat98] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics*, 136, 1998.

[BD95] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36, 1995.

[BD98] John Baez and James Dolan. Higher-dimensional algebra III: n-categories and the algebra of opetopes. *Advances in Mathematics.*, 135(2):145–206, 1998.

[Ben67] Jean Benabou. Introduction to Bicategories. In A. Dold and B. Eckmann, editors, *Reports of the Midwestern Category Seminar*, volume 47 of *Lecture Notes in Mathematics*, chapter 1, pages 1–77. Springer-Verlag, 1967.

[Ber11] Julie Bergner. Models for (∞, n)-categories and the cobordism hypothesis. *Proceedings of Symposia in Pure Mathematics*, 83, 2011.

[BG17] John Bourke and Nick Gurski. The Gray tensor product via factorisation. *Applied Categorical Structures*, 25(4):603–624, 2017.

[BMS12] John W. Barrett, Catherine Meusburger, and Gregor Schaumann. Gray categories with duals and their diagrams. 2012. ArXiv: 1211.0529.

[BN96] John Baez and M. Neuchl. Higher-dimensional algebra I: Braided monoidal 2-categories. *Advances in Mathematics.*, 121(2):196–244, 1996.

[Buh14] Lukas Buhne. Homomorphisms of Gray-categories as pseudo algebras. 2014. ArXiv: 1408.3481.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*, volume 347 of *Lecture Notes in Mathematics.* Springer, 1973.

[CH14] Antonio Cegarra and Benjamín Heredia. Comparing geometric realizations of tricategories. *Algebraic and Geometric Topology.*, 14(4):1997–2064, 2014.

[CMS16] Nils Carqueville, Catherine Meusburger, and Gregor Schaumann. 3-Dimensional defect tqfts and their tricategories. 2016. ArXiv: 1603.01171.

[FM18] Simon Forest and Samuel Mimram. Coherence of Gray categories via rewriting. In Helene Kirchner, editor, *3rd International Conference on Formal Structures for Computation and Deduction.*, LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., 2018.

[GG09] Richard Garner and Nick Gurski. The Low-Dimensional Structure formed by Tricategories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146, 2009.

[GJ09] Paul Goerss and John Jardine. *Simplicial Homotopy Theory.* Modern Birkhauser Classics. Birkhauser Basel, Basel, 2009.

[GJO19] Nick Gurski, Niles Johnson, and Angélica M. Osorno. The 2-dimensional stable homotopy hypothesis. *Journal of Pure and Applied Algebra*, 223(10):4348–4383, 2019.

[GO13] Nick Gurski and Angélica M. Osorno. Infinite loop spaces, and coherence for symmetric monoidal bicategories. *Advances in Mathematics.*, 246:1–32, 2013.

[GPS95] R. Gordon, A. J. Power, and Ross Street. *Coherence for Tricategories*, volume 117 of *Memoirs of the American Mathematical Society.* American Mathematical Society, 1995.

[Gra74] John Gray. *Formal Category Theory: Adjointness for 2-Categories*, volume 391 of *Lecture Notes in Mathematics.* Springer, 1974.

[Gro] Alexander Grothendieck. Pursuing Stacks. Available at http://groupoids.org.uk/pstacks.html.

[Gur07] Nick Gurski. *An Algebraic Theory of Tricategories.* PhD thesis, Yale University, 2007.

[Gur11] Nick Gurski. Loop spaces, and coherence for monoidal and braided monoidal bicategories. *Advances in Mathematics.*, 226(5):4225–4265, 2011.

[Gur12] Nick Gurski. Biequivalences in Tricategories. *Theory and Applications of Categories*, 26, 2012.

[Gur13] Nick Gurski. *Coherence in Three-Dimensional Category Theory*, volume 201 of *Cambridge Tracts in Mathematics.* Cambridge University Press, 2013.

[HKK01] K. A. Hardie, K. H. Kamps, and R. W. Kieboom. A homotopy bigroupoid of a topological space. *Applied Categorical Structures*, 9(3):311–327, 2001.

[Joy08] Andre Joyal. Notes on Quasicategories, 2008. Available here: http://www.math.uchicago.edu/ may/IMA/Joyal.pdf.

[JS93] A. Joyal and R. Street. Braided Tensor Categories. *Advances in Mathematics.*, 102(1):20–78, 1993.

[JS95] A. Joyal and R. Street. The category of representations of the general linear groups over a finite field. *Journal of Algebra.*, 176(3):908–946, 1995.

[Kel74] G. M. Kelly. *Doctrinal Adjunction.* Springer-Verlag, 1974.

[Kel82] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Notes.* Cambridge University Press, 1982.

[KL04] G.M. Kelly and Steve Lack. Monoidal Functors generated by Adjunctions, with Applications to Transport of Structure. *Fields Inst. Commun.*, 43, 2004.

[KV94] M. Kapranov and V. Voevodsky. Braided monoidal 2-categories and Manin-Schechtman higher braid groups. *Journal of Pure and Applied Algebra.*, 92(3):241–267, 1994.

[Lac10a] Steve Lack. A 2-Categories Companion. In John Baez and J. Peter May, editors, *Towards Higher Categories.* Springer New York, 2010.

[Lac10b] Steve Lack. Icons. *Applied Categorical Structures*, 18, 2010.

[Lei98] Tom Leinster. Basic Bicategories, 1998. ArXiv: 9810017.

[Lei02] Tom Leinster. A Survey of Definitions of n-Category. *Theory and Applications of Categories*, 10, 2002.

[Ler94] Olivier Leroy. Sur une notion de 3-catégorie adaptée à l'homotopie. *Univ. Montpellier II*, 1994.

[LP08] Steve Lack and Simona Paoli. 2-Nerves for bicategories. *K-Theory*, 38, 2008.

[Lur09a] J. Lurie. On the classification of topological field theories. *Current developments in mathematics, 2008*, pages 129–280, 2009.

[Lur09b] Jacob Lurie. *Higher Topos Theory.* Number 170 in Annals of Mathematical Studies. Princeton University Press, 2009.

[Mak98] Michael Makkai. Towards a Categorical Foundation of Mathematics. *Lecture Notes in Logic*, 11, 1998.

[MS93] I. Moerdijk and J. Svensson. Algebraic classification of equivariant homotopy 2-types. i. *J. Pure Appl. Algebra*, 89:187–216, 1993.

[Pao19] Simona Paoli. *Simplicial Methods for Higher Categories: Segal-type Models of Weak n-Categories*, volume 26 of *Algebra and Applications*. Springer, 2019.

[Pos51] Mikhail Postnikov. Determination of the homology groups of a space by means of the homotopy invariants. *Doklady Akad. Nauk SSSR*, 76, 1951.

[Pow95] A. J. Power. Why tricategories? *Information and computation.*, 120(2):251–262, 1995.

[Pow07] John Power. Three Dimensional Monad Theory. In Alexei Davydov, Michael Batanin, Michael Johnson, Stephen Lack, and Amnon Neemon, editors, *Categories in Algebra, Geometry and Mathematical Physics*, pages 405–426. American Mathematical Society, 2007.

[Pro13] The Univalent Foundations Program. *Homotopy type theory—univalent foundations of mathematics.* 2013. https://homotopytypetheory.org/book/.

[Sim] Carlos Simpson. Homotopy Types of Strict 3-Groupoids. ArXiv: 9810059.

[Sim12] Carlos Simpson. *Homotopy Theory of Higher Categories.* Number 19 in New Mathematical Monographs. Cambridge University Press, 2012.

[SP09] C. Schommer-Pries. *The classification of two-dimensional extended topological field theories.* PhD thesis, University of California, Berkeley, 2009.

[Tam99] Zouhair Tamsamani. Sur des notions de n-catégorie et n-groupoide non strictes via des ensembles multi-simpliciaux. *K-Theory*, 16, 1999.

[Tri06] Todd Trimble. Notes on Tetracategories. Available on John Baez's website here: http://math.ucr.edu/home/baez/trimble/tetracategories.html, 2006.

[Ver08] Dominic Verity. Weak Complicial Sets. *Advances in Mathematics*, 219, 2008.

[Ver17] Dominic Verdon. Coherence for braided and symmetric pseudomonoids. 2017. ArXiv: 1705.09354.