On The Design Of Output Feedback Controllers For LTI Systems Over Fading Channels

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***Abstract*—This paper considers linear time-invariant (LTI) control systems over fading channels in both continuous-time and discrete-time cases, and addresses the design of output feedback controllers that stabilize the closed-loop system in the mean square sense. It is shown that a sufficient and necessary condition for the existence of such controllers can be obtained by solving a convex optimization problem in the form of a semidefinite program (SDP). This condition is obtained by reformulating mean square stability as asymptotical stability of a suitable matrix comprising plant, controller and channel, and by introducing modified Hurwitz and Schur stability criteria.**

***Index Terms*—Output feedback control, Fading channels, Modified stability criterion, SDP.**

I. Introduction

Networked control systems have progressively become one of the most popular topics in recent years, see, e.g., [1]. In this area, stability and stablization under unreliable communication channels have been studied by numerous researchers. In particular, [2] has summarized the works on feedback control under data-rate constraints. [3] has investigated quantized feedback control. [4], [5] have studied the effect of time delays. [6], [7] have considered unreliable networks with packet losses and [8] have considered the signal-to-noise constraints.

In this paper, we focus on linear time-invariant (LTI) control systems in the presence of fading channels. As the use of wireless communication is becoming increasingly common, the fading channels are getting more attractive for the reason that they can be used to characterize several factors including signal attenuation, signal distortion, packet drop and noise disturbance, see, e.g., [9]–[12] and the references therein. In [9], it is shown how the stochastic variables responsible for the channel fading determine stability of the closed-loop system in the mean square sense. The result in [9] is extended to feedback stabilization of discrete-time multiple-input multiple-output (MIMO) plant over multiple fading channels by [11]. [10] considers the continuous-time case of feedback stabilization over stochastic multiplicative input channels.

It should be observed that the aforementioned works deal with the stabilization via state feedback, and for output feedback stabilization, only special cases are considered. Moreover, these works generally assume that the fading channels are uncorrelated. In wireless communication, when a non-orthogonal access scheme is adopted the fading experienced by different channels will be correlated, see [13] for more details. Moreover, in practice, the propagation channels between each pair of receiving and transmitting antennas are normally not statistically independent, which is characterized as spatial correlation. Our work aims at generalizing the results provided in [10], [11] by considering the design of output feedback controllers over multiple correlated fading channels. It is worth mentioning that such generalization is not trivial since the static (or fixed-order) output feedback design is a notoriously difficult non-convex optimization problem. Specifically, for both continuous-time and discrete-time systems, we consider the model in [11] where a MIMO plant is controlled in closed-loop by an output feedback controller over fading channels modeled as multiplicative white noise processes. The problem consists of establishing the existence of such a controller, in a desired semi-algebraic set, that makes the closed-loop system stable in the mean square sense. It is shown that a sufficient and necessary condition for the existence of such controllers can be obtained by solving a convex optimization problem in the form of a semidefinite program (SDP), and this allows us to test the given channels for stabilizability via checking the feasibility of the SDP. This condition is obtained as follows. First, mean square stability is reformulated as asymptotical stability of a suitable matrix comprising plant, controller and channel. Second, modified Hurwitz and Schur stability criteria are introduced to address the asymptotical stability in terms of the maximum of a polynomial over the positivity domain of a family of polynomials. Third, the Gram matrix method is exploited to convert the condition into establishing whether a polynomial is a sum of squares of polynomials (SOS) via a linear matrix inequality (LMI).

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II. Preliminaries

The notation used in the paper is as follows. The sets of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$. The notation $\mathcal{R}(\lambda)$ and $|\lambda|$ denote the real part and the magnitude of a complex number λ. The inverse and the transpose of a matrix A are denoted by A^{-1} and $A^{'}$. The notation $\det(A)$ denotes the determinant of a matrix A. The spectrum of a matrix A, i.e., the set of eigenvalues of A, is denoted by $\mathrm{spec}(A)$. The notation $\rho(A)$ denotes the spectral radius of A, i.e., $\rho(A) = \max_{\lambda\in\mathrm{spec}(A)} |\lambda|$. The vector obtained by stacking all the columns of a matrix A into one column vector is denoted by $\mathrm{vec}(A)$. The notation I denotes the identity matrix with the size specified by the context. For scalars $a_1, \dots, a_n$, the notation $\mathrm{diag}(a_1, \dots, a_n)$ denotes the diagonal matrix with its (i, i)-th entry equal to a_i. The degree of a polynomial $p(\cdot)$ is denoted by $\deg(p(\cdot))$. The symbol $\otimes$ denotes the Kronecker product. The operator $\mathcal{E}(\cdot)$ denotes the mathematical expectation. The acronym w.r.t. stands for "with respect to".

A. Problem Formulation

We consider the situation where a plant is controlled in closed-loop by an output feedback controller over fading channels.

The plant is described by

$$\left\{ \begin{array}{rcl} x^+ & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and x^+ denotes the derivative of the plant state in the continuous-time case and the plant state at the next time step in the discrete-time case.

The controller is initially supposed to be static for clarity of presentation[1], and is described by

$$v(t) = Ky(t) \tag{2}$$

where $v \in \mathbb{R}^m$ is the controller output and $K \in \mathbb{R}^{m\times p}$ has to be determined in the semi-algebraic set

$$\mathcal{K} = \left\{ K \in \mathbb{R}^{m\times p} : \ a_i(K) \geq 0, \ i = 1, \dots, n_a \right\} \tag{3}$$

where $a_i(K)$ are polynomials. It should be mentioned that semi-algebraic sets can be used to describe a wide range of sets, including the hyper-rectangles and hyper-spheres as special cases. Indeed, it is often required to restrict the admissible controllers to some desired bounded sets.

The fading channels are modeled in memoryless multiplicative form as

$$u(t) = \Xi(t)v(t) \tag{4}$$

where $\Xi(t) \in \mathbb{R}^{m\times m}$ represents the channel fading and has the diagonal structure

$$\Xi(t) = \mathrm{diag}(\xi_1(t), \xi_2(t), \dots, \xi_m(t)) \tag{5}$$

and $\xi_1(t), \dots, \xi_m(t)$ are assumed to be scalar-valued white noise processes with $\mu_i = \mathcal{E}(\xi_i(t)), \ \forall i = 1, \dots, m$. Let us define

$$\left\{ \begin{array}{l} \Pi = \mathrm{diag}(\mu_1, \dots, \mu_m) \text{ and } \Sigma = [\sigma_{ij}]_{i,j=1,\dots,m} \\ \sigma_{ij} = \mathcal{E}\left((\xi_i(t) - \mu_i)(\xi_j(t) - \mu_j)\right) \ \forall i, j = 1, \dots, m \end{array} \right. \tag{6}$$

and let $\sigma_i = \sqrt{\sigma_{ii}}$, $i = 1, \dots, m$.

The closed-loop system obtained by connecting the output of the controller (2) to the input of the unstable plant (1) over the fading channels (4) can be represented by

$$\left\{ \begin{array}{rcl} x^+ & = & A_{cl}(K,t)x(t) \\ A_{cl}(K,t) & = & A + B\Xi(t)KC. \end{array} \right. \tag{7}$$

Let us further define

$$X(t) = \mathcal{E}(x(t)x(t)'). \tag{8}$$

Definition 1: ([9]) The closed-loop system (7) is said to be stable in the mean square sense if $X(t)$ is well-defined for all $t \geq 0$ and

$$\lim_{t\to\infty} X(t) = 0 \ \forall x(0) \in \mathbb{R}^n. \tag{9}$$

The problem addressed in this paper is as follows.

Problem 1: For continuous-time and discrete-time cases, design an output feedback controller K in the set $\mathcal{K}$ such that the closed-loop system (7) is stable in the mean square sense.

[1] As it will be explained in Remark 3, the proposed methodology can be used also to design dynamic output feedback controllers.

B. Modified Routh-Hurwitz Table

The Routh-Hurwitz stability criterion provides a necessary and sufficient condition to determine whether all the roots of a univariate polynomial with real coefficients have negative real parts, see [14] for details. According to this criterion, the entries of the table are rational functions w.r.t. the coefficients of the polynomial. In order to transform the entries of the table into polynomial functions, let us introduce a modified Routh-Hurwitz table as follows.

Let us denote the characteristic polynomial of A as

$$v(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0 \tag{10}$$

where $s \in \mathbb{C}$ and $a_{n-1}, \dots, a_0 \in \mathbb{R}$ are the coefficients of $v(s)$. By multiplying each entry by their denominator, one obtains the modified Routh-Hurwitz table defined as

$$\begin{array}{cccc} 1 & a_{n-2} & a_{n-4} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{11}$$

where the number of rows is $n+1$ and the ij-th entry is

$$\begin{array}{rcl} a_{ij} & = & a_{i-1,1}a_{i-2,j+1} - a_{i-1,j+1}a_{i-2,1} \\ & & i = 3, \dots, n+1, j = 1, 2, \dots \end{array} \tag{12}$$

It is shown in [15] that all the roots of $v(s)$ have negative real parts if and only if the first column of the modified Routh-Hurwitz table contains positive entries only.

C. Modified Jury Table

Similar to the Routh-Hurwitz stability criterion, the Jury stability criterion provides a necessary and sufficient condition for establishing whether the roots of a univariate polynomial with real coefficients have magnitude smaller than 1 (see [16]). The entries of this table are rational functions w.r.t. the coefficients of the polynomial. Hereafter we introduce a modified Jury table where the entries are polynomial functions w.r.t. these coefficients.

Specifically, let us consider the characteristic polynomial $v(s)$ defined in (10). By removing the even rows and multiplying the odd rows by their denominator, one obtains the modified Jury table defined as

$$\begin{array}{ccccc} 1 & a_{n-1} & \cdots & a_1 & a_0 \\ 1 - a_0^2 & a_{n-1} - a_1a_0 & \cdots & a_1 - a_{n-1}a_0 & 0 \\ a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \\ a_{n+1,1} & 0 & & & \end{array} \tag{13}$$

where

$$\begin{array}{rcl} a_{ij} & = & a_{i-1,j}a_{i-1,1} - a_{i-1,n+4-i-j}a_{i-1,n+3-i} \\ & & i = 3, \dots, n+1, \ j = 1, 2 \dots \end{array} \tag{14}$$

It can be verified that the roots of $v(s)$ have magnitude smaller than 1 if and only if the first column of the modified Jury table contains positive entries only. It should be noted that the number of rows in the modified Jury Table does not have to be $n+1$ necessarily. For instance, whenever the spectrum of A includes zero, the number of rows will be reduced.

D. SOS Polynomials

Here we briefly introduce the class of SOS polynomials, see, e.g., [17] for details.

A polynomial $p(s) : \mathbb{R}^r \to \mathbb{R}$ is said to be SOS if and only if there exist polynomials $p_1(s), \ldots, p_k(s) : \mathbb{R}^r \to \mathbb{R}$ such that

$$p(s) = \sum_{i=1}^{k} p_i(s)^2. \tag{15}$$

SOS polynomials are nonnegative and one can establish whether a polynomial is SOS via an LMI feasibility test.

Indeed, let d be a nonnegative integer such that $2d \geq \deg(p(s))$. Then, $p(s)$ can be written according to the Gram matrix method, or square matrix representation (SMR), as

$$p(s) = b(s)^{'} (G + L(\alpha))\, b(s) \tag{16}$$

where $b(s) : \mathbb{R}^r \to \mathbb{R}^{\sigma(r,d)}$ is a vector containing all the monomials of degree less than or equal to d in s and

$$\sigma(r,d) = \frac{(r+d)!}{r!d!}, \tag{17}$$

$G \in \mathbb{R}^{\sigma(r,d)\times\sigma(r,d)}$ is a symmetric matrix satisfying

$$p(s) = b(s)^{'} G b(s), \tag{18}$$

$L(\alpha) : \mathbb{R}^{\omega(r,2d)} \to \mathbb{R}^{\sigma(r,d)\times\sigma(r,d)}$ is a linear parametrization of the linear set

$$\mathcal{L} = \{\tilde{L} = \tilde{L}' :\ b(s)^{'} \tilde{L} b(s) = 0\}, \tag{19}$$

and $\alpha \in \mathbb{R}^{\omega(r,2d)}$ is a free vector with

$$\omega(r,2d) = \frac{1}{2}\left(\sigma(r,d)(\sigma(r,d)+1)\right) - \sigma(r,2d). \tag{20}$$

It follows that $p(s)$ is SOS if and only if there exists α satisfying the LMI

$$G + L(\alpha) \geq 0. \tag{21}$$

III. Proposed Results

A. Stability Analysis

In this subsection, we consider the stability analysis of the closed-loop system (7).

First, let us define the vector

$$\mathrm{vec}(X(t)) = G v_s(X(t)) \tag{22}$$

where $X(t)$ is defined in (8), $G \in \mathbb{R}^{n^2 \times \frac{1}{2}n(n+1)}$ is a constant matrix, and $v_s(X(t))$ denotes the same vector with $\mathrm{vec}(X(t))$ where the entries of $X(t)$ that lie under its diagonal have been removed. We can observe that G is a full column rank matrix.

Let B_i denote the i-th column of B, and let $(KC)_i$ denote the i-th row of KC, i.e.,

$$B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix},\ KC = \begin{bmatrix} (KC)_1 \\ \vdots \\ (KC)_m \end{bmatrix}. \tag{23}$$

Theorem 1: In the continuous-time case, the closed-loop system (7) is stable in the mean square sense if and only if $\mathcal{R}(\lambda) < 0,\ \forall \lambda \in \mathrm{spec}(\Phi)$, where

$$\Phi = (G^{'}G)^{-1}G^{'}\Psi G \tag{24}$$

where

$$\begin{aligned}\Psi &= I \otimes (A + B\Pi KC) + (A + B\Pi KC) \otimes I \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij}(B_i(KC)_i) \otimes (B_j(KC)_j).\end{aligned} \tag{25}$$

In the discrete-time case, the closed-loop system (7) is stable in the mean square sense if and only if $\rho(\Phi) < 1$, where Φ is defined as (24) and

$$\begin{aligned}\Psi &= (A + B\Pi KC) \otimes (A + B\Pi KC) + \\ &\quad \sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij}(B_i(KC)_i) \otimes (B_j(KC)_j).\end{aligned} \tag{26}$$

Proof. In the continuous-time case, the closed-loop system (7) can be rewritten into the Itô form as follows:

$$dx(t) = (A + B\Pi KC)x(t)dt + \sum_{i=1}^{m} \sigma_i B_i (KC)_i x(t) d\omega_i(t)$$

where $d\omega_i(t) = \dfrac{\xi_i(t) - \mu_i}{\sigma_i} dt$. Thus, $\omega_i(t),\ i = 1, \ldots, m$ are standard scalar Wiener processes and $d\omega_i(t)\cdot d\omega_j(t) = \rho_{ij}dt = \dfrac{\sigma_{ij}}{\sigma_i\sigma_j}dt$. According to the Itô's formula, one has

$$\begin{aligned}\dot{X}(t) &= (A + B\Pi KC)X(t) + X(t)(A + B\Pi KC)^{'} + \\ &\quad \sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij}(B_i(KC)_i)X(t)(KC)_j^{'}B_j^{'}\end{aligned}$$

where $X(t)$ is defined in (8). Hence, it can be observed that

$$\mathrm{vec}(\dot{X}(t)) = \Psi \mathrm{vec}(X(t)),$$

where Ψ is defined in (25), which follows that $\lim_{t\to\infty} X(t) = 0$ if and only if $\mathcal{R}(\lambda) < 0,\ \forall \lambda \in \mathrm{spec}(\Psi)$.

Based on (22), it can be obtained that

$$G v_s(\dot{X}(t)) = \Psi G v_s(X(t)),$$

where G is a full column rank constant matrix. After pre-multiplying the above equation by $(G^{'}G)^{-1}G^{'}$, one has

$$v_s(\dot{X}(t)) = (G^{'}G)^{-1}G^{'}\Psi G v_s(X(t)).$$

Observe that $\lim_{t\to\infty} X(t) = 0$ is equivalent to $\lim_{t\to\infty} v_s(X(t)) = 0$, which follows that the closed-loop system (7) is stable in the mean square sense if and only if $\mathcal{R}(\lambda) < 0,\ \forall \lambda \in \mathrm{spec}(\Phi)$ where Φ is defined as (24)-(25).

In the discrete-time case, based on the Lemma 1 of [11], one has that

$$\mathrm{vec}(X(t+1)) = \Psi \mathrm{vec}(X(t)),$$

where Ψ is defined in (26). Thus, it is clear that $\lim_{t\to\infty} X(t) = 0$ if and only if $\rho(\Psi) < 1$. Similar to the continuous-time case, based on (22) it follows that

$$v_s(X(t+1)) = (G^{'}G)^{-1}G^{'}\Psi G v_s(X(t)).$$

Since $\lim_{t\to\infty} X(t) = 0$ is equivalent to $\lim_{t\to\infty} v_s(X(t)) = 0$, the closed-loop system (7) is stable in the mean square sense if and only if $\rho(\Phi) < 1$ where Φ is defined as (24) and (26). $\square$

Remark 1: The constant matrix G is introduced to reduce the size of matrix on which the modified stability criterion will be performed later from n^2 to $\frac{1}{2}n(n+1)$.

B. Controller Synthesis

Let us start with defining $k \in \mathbb{R}^{n_k}$ as

$$k = \ \mathrm{vec}(K), \tag{27}$$

and let $n_\Phi = \frac{1}{2}n(n+1)$ be the size of the square matrix Φ.

With K unknown, denote the characteristic polynomial of $\Phi(K)$ as

$$v(\lambda, k) = \det(\lambda I - \Phi(K)) = \lambda^{n_\Phi} + \sum_{j=0}^{n_\Phi - 1} c_j(k)\lambda^j \tag{28}$$

where $\lambda \in \mathbb{C}$ and $c_0(k), \ldots, c_{n_\Phi - 1}(k) \in \mathbb{R}$ are the coefficients of $v(\lambda, k)$.

Next, let us exploit the modified Routh-Hurwitz table and the modified Jury table introduced in Section II-B and Section II-C to derive an equivalent condition of Theorem 1. Let R and J be the modified Routh-Hurwitz table defined in (11)-(12) and the modified Jury table defined in (13)-(14) of the characteristic polynomial (28), respectively. Let us further define the general table

$$T = \begin{cases} R & \text{continuous-time case} \\ J & \text{discrete-time case.} \end{cases} \tag{29}$$

Hereafter we will analyze the positivity of the first column of the table T generally for both continuous-time and discrete-time cases. As defined in Section II-B and Section II-C, one can observe that all the entries of the modified table T are polynomials w.r.t k. Denote the number of entries in the first column of T with n_f and let $f_i(k) = T_{i1},\ i = 1, 2, \ldots, n_f$, which represents the first column of the table T. Let us further define the set

$$\hat{\mathcal{K}} = \left\{K \in \mathbb{R}^{m\times p} :\ f_i(k) - \varepsilon \geq 0,\ i = 1, \ldots, n_f - 1\right\} \tag{30}$$

where $\varepsilon > 0$ is introduced for considering positive values only of $f_i(k),\ i = 1, 2, \ldots, n_f - 1$. Observe that there is no need to go on with controller synthesis if any $f_i(k)$ is a negative constant number since in that case the plant is not stabilizable.

Lemma 1: Let us define

$$r \ = \sup_{K\in\mathcal{K}\cap\hat{\mathcal{K}}} f_{n_f}(k) \tag{31}$$

where $f_{n_f}(k)$ is the last entry in the first column of the table T. There exists a controller $K^* \in \mathcal{K}$ such that the closed-loop system (7) is stable in the mean square sense if and only if

$$r > 0 \tag{32}$$

for some $\varepsilon > 0$.

Proof. "⇒" Suppose there exists a controller $K^* \in \mathcal{K}$ such that the closed-loop system (7) is stable. This implies that there exists $\varepsilon > 0$ such that $K^* \in \hat{\mathcal{K}}$. Moreover, $f_{n_f}(k^*) > 0$, and hence $r > 0$.

"⇐" Suppose $r > 0$. This implies that there exist $K^* \in \mathcal{K}\cap\hat{\mathcal{K}}$ such that $f_{n_f}(k^*) > 0$, which implies that the first column of the table T contains only positive entries with $k = k^*$. Thus, there exists a controller $K^* \in \mathcal{K}$ such that the closed-loop system (7) is stable in the mean square sense. □

Let us observe that, if $r > 0$ for some $\varepsilon = \hat{\varepsilon} > 0$, then $r > 0$ for all $\varepsilon \in (0, \hat{\varepsilon}]$. This means that, in practice, ε can be simply chosen a priori as the smallest positive number allowed by the available computing platform.

Let us introduce the following assumption.

Assumption 1: The semi-algebraic set $\mathcal{K}$ is compact, and the polynomials $f_i(k),\ i = 1, \ldots, n_f - 1$ and $a_j(k),\ j = 1, \ldots, n_a$ have even degrees and their highest degree forms have no common zeros except 0.

Let us observe that Assumption 1 does not introduce strong restrictions. Indeed, it is reasonable to search for a controller in a compact set. Moreover, if the polynomials $f_i(k)$ and $a_j(k)$ have not even degrees, one can enforce this property without changing the problem by simply multiplying these polynomials by a linear function that is positive over $\mathcal{K}$. Lastly, the property that the highest degree forms of $f_i(k)$ and $a_j(k)$ have no common zeros except 0 is automatically satisfied for typical semi-algebraic set $\mathcal{K}$ such as hyper-ellipsoids, hyper-rectangles, etc.

For $\theta \in \mathbb{R}$, let us define the polynomial

$$g(k) = \theta - f_{n_f}(k) - \sum_{i=1}^{n_f - 1} \left(f_i(k) - \varepsilon\right)\gamma_i(k) - \sum_{j=1}^{n_a} a_j(k)\beta_j(k) \tag{33}$$

where $\gamma_i(k)$ and $\beta_i(k)$ are auxiliary polynomials.

Theorem 2: Suppose that Assumption 1 holds. Then, the condition (32) holds if and only if

$$\theta^* > 0 \tag{34}$$

where

$$\theta^* = \inf_{\theta,\gamma_i,\beta_j} \theta \quad \text{s.t.} \quad \begin{cases} g(k) & \text{is} & \text{SOS} \\ \gamma_i(k) & \text{is} & \text{SOS} \\ \beta_j(k) & \text{is} & \text{SOS} \\ \forall i & = & 1, \ldots, n_f - 1 \\ \forall j & = & 1, \ldots, n_a. \end{cases} \tag{35}$$

Proof. "⇒" Let us assume $r > 0$. Based on the definition of r in (31), one has

$$\theta^\# = r$$

where $\theta^\#$ is defined as

$$\theta^\# = \inf_{\theta}\ \theta \quad \text{s.t.} \quad \theta - f_{n_f}(k) > 0\ \forall K \in \mathcal{K}\cap\hat{\mathcal{K}}.$$

Since Assumption 1 holds, it follows from [18] that $\theta - f_{n_f}(k) > 0$ for all $K \in \mathcal{K}\cap\hat{\mathcal{K}}$ if and only if

$$\begin{cases} \exists s_0, \gamma_1, \ldots, \gamma_{n_f - 1}, \beta_1, \ldots, \beta_{n_a} \in \text{SOS} \\ \theta - f_{n_f}(k) = s_0 + \displaystyle\sum_{i=1}^{n_f - 1} \left(f_i(k) - \varepsilon\right)\gamma_i(k) + \sum_{j=1}^{n_a} a_j(k)\beta_j(k). \end{cases}$$

Hence, the condition (34)-(35) holds with $\theta^* = \theta^\#$.

"⇐" Let us assume that (34)-(35) hold, and suppose by contradiction that $r \leq 0$. It follows that any $\theta > r$ will satisfy that $\theta - f_{n_f}(k) > 0$ for all $K \in \mathcal{K}\cap\hat{\mathcal{K}}$. Thus, the optimal solution of θ^* in (35) should be less than or equal to 0, contradicting (34), which completes the proof. □

It should be mentioned that the condition for a polynomial depending linearly on some decision variables to be a SOS polynomial can be equivalently expressed via an LMI, see Section II-D for details. Thus, Theorem 2 states that one can establish whether $r > 0$ by solving the optimization problem (35), which is a SDP. In particular, this theorem provides a sufficient condition for any chosen degrees of the polynomials $\gamma_i(k)$ and $\beta_j(k)$. Moreover, this condition is also necessary when these degrees are large enough.

The following theorem explains how one can use the SDP (35) to determine a controller that solves Problem 1.

Theorem 3: Suppose that Assumption 1 holds. Let $\hat{\theta}$ be the solution of the SDP (35) with specified degrees of the polynomials $\gamma_i(k)$ and $\beta_j(k)$. Then,

$$\hat{\theta} = r \tag{36}$$

if and only if there exists K such that

$$\left\{ \begin{array}{ccc} \hat{\theta} - f_{n_f}(k) & = & 0 \\ g^*(k) & = & 0 \\ K & \in & \mathcal{K} \cap \hat{\mathcal{K}}. \end{array} \right. \tag{37}$$

Proof. "$\Rightarrow$" Suppose (36) holds. Then, the first and the third conditions in (37) hold with the maximizer K^* of (31). It follows that

$$g^*(k) = \hat{\theta} - f_{n_f}(k) - \sum_{i=1}^{n_f-1} \left(f_i(k) - \varepsilon\right) \gamma_i^*(k) - \sum_{j=1}^{n_a} a_j(k)\beta_j^*(k)$$

where $\gamma_i^*(k)$, $\beta_j^*(k)$ are the optimal solution in (35). Let $k^* = \mathrm{vec}(K^*)$, then one has

$$g^*(k^*) = -\sum_{i=1}^{n_f-1} \left(f_i(k^*) - \varepsilon\right) \gamma_i^*(k^*) - \sum_{j=1}^{n_a} a_j(k)\beta_j^*(k^*).$$

Let us observe that $K \in \mathcal{K} \cap \hat{\mathcal{K}}$ guarantees that

$$\left\{ \begin{array}{ll} f_i(k^*) - \varepsilon \geq 0 & i = 1, \ldots, n_f - 1 \\ a_j(k) \geq 0 & j = 1, \ldots, n_a \end{array} \right.$$

and that (35) ensures that $g^*(k)$, $\gamma_i^*(k)$ and $\beta_j^*(k)$ are SOS. Hence,

$$0 \leq g^*(k^*) \leq 0,$$

making the second line of (37) hold.

"$\Leftarrow$" Suppose that (37) holds for some K. As implied in Theorem 2, $\hat{\theta}$ is an upper bound of r. Hence, (37) states that the upper bound $\hat{\theta}$ can be achieved by some $K \in \mathcal{K} \cap \hat{\mathcal{K}}$. □

Searching for a controller K such that the condition (37) is satisfied can be done by, firstly, computing the set

$$\Omega = \{k \in \mathbb{R}^{n_k} : g^*(k) = 0\}. \tag{38}$$

Since $g^*(k)$ is a SOS polynomial, the set Ω can be found through computing vector of monomials in k belonging to the null space of a positive semidefinite Gram matrix of $g^*(k)$ (see [19]). After Ω has been found, one should check whether any K in the set Ω satisfies the other two constraints of (37).

Remark 2: The procedure for determining a controller that solves Problem 1 can be summarized as follows. First, we build the modified table T as described in Section II-B or Section II-C for (24). Then, we build and solve the SDP (35). Lastly, we check feasibility of (37) if $\hat{\theta} > 0$. In the case (37) is infeasible, one can increase the degrees of the polynomials $\gamma_i(k)$ and $\beta_j(k)$ and repeat.

Remark 3: Let us observe that the proposed methodology can be used not only to design static output feedback controllers as considered in (2), but also dynamic output feedback controllers. Indeed, this can be achieved as follows. First, one should replace (2) with

$$\left\{ \begin{array}{ccc} x_c^+ & = & A_c x_c(t) + B_c y(t) \\ v(t) & = & C_c x(t) + D_c y(t) \end{array} \right. \tag{39}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the controller state, and $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$ are matrices to be determined. Second, one should replace (27) with

$$k = \begin{pmatrix} \mathrm{vec}(A_c)' & \mathrm{vec}(B_c)' & \mathrm{vec}(C_c)' & \mathrm{vec}(D_c)' \end{pmatrix}'. \tag{40}$$

Lastly, one should replace A, B and (KC) in (25)-(26) with

$$\bar{A} = \begin{pmatrix} A & 0 \\ B_c C & A_c \end{pmatrix}, \; \bar{B} = \begin{pmatrix} B & 0 \end{pmatrix}', \; \bar{C} = \begin{pmatrix} D_c C & C_c \end{pmatrix}. \tag{41}$$

IV. Examples

The computations are done by Matlab with toolbox SeDuMi and SOSTOOLS. The set $\hat{\mathcal{K}}$ in (30) is defined with $\varepsilon = 10^{-3}$.

A. Example 1

Let us start by considering the continuous-time case with the plant (1) and the fading channel (6) as

$$A = \begin{pmatrix} 0.5 & -1 \\ 1 & 0.5 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Pi = \mu, \; \Sigma = 0.1.$$

The problem is to determine a controller (2) over a set $\mathcal{K}$ such that the closed-loop system (7) is stable in the mean square sense for some μ. Let us observe that the autonomous system (1) is unstable since we have $\mathrm{spec}(A) = \{0.5 \pm 1i\}$.

First, let us consider state feedback control, i.e., $C = I$ and $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, for different value of μ with $K \in \mathcal{K} = [-4, 4]^2$. The set $\mathcal{K}$ is described as in (3) by choosing $a_i = 16 - k_i^2, \; i = 1, 2$. Since Σ is fixed, it is clear that if the system is stabilizable with some μ, then it is also stabilizable with any $\bar{\mu} > \mu$ since a greater μ means a more reliable channel. Thus, based on bisection algorithm, we can obtain the minimum μ such that the state feedback controller $K \in \mathcal{K}$ can be successfully searched by our proposed method concluded in Remark 2. As a result, the minimum μ we obtained is $\mu^* = 0.448$, which coincides with the analytical result proposed in Theorem 3.1 in [10]. The solution of the controller K corresponding to such μ^* is $K^* = \begin{pmatrix} -1.154 & -3.558 \end{pmatrix}$. In fact, we have $\max_{\lambda \in \mathrm{spec}(\Phi(K^*))} \mathcal{R}(\lambda) = -0.0007 < 0$.

Next, let us elaborate our proposed method in detail with output feedback case. Assume $C = [1 \; 1]$ and $\mu = 0.6$, and that the controller is constrained into the set $\mathcal{K} = [-2, \; 2]$.

Then, the set $\mathcal{K}$ is described as in (3) by choosing $a_1 = 4 - K^2$. The first column of the table T in (29) has 3 entries, namely, $f_1(K), f_2(K), f_3(K)$, which are shown in Figure 1.

Next, we solve (35) finding the upper bound of r given by $\hat{\theta} = 0.278$ (the computation time is 0.36s). Meanwhile, we obtain the optimal $g^*(K)$ which is a SOS polynomial. According to Section II-D, we can easily find a positive semidefinite Gram matrix of $g^*(K)$. To get the set Ω defined in (38), we search for all K such that $b(K)$ defined in (16) belongs to the null space of the Gram matrix, leading to the solution $K^* = -1.395$ (see [19] for more details). In the end, we check the feasibility of the other two constraints in (37) with this K^* and find them feasible.

As shown in Figure 1, the found controller K^* is the maximizer of $f_3(K)$ in the set $K \in \mathcal{K} \cap \hat{\mathcal{K}}$ as we expected.

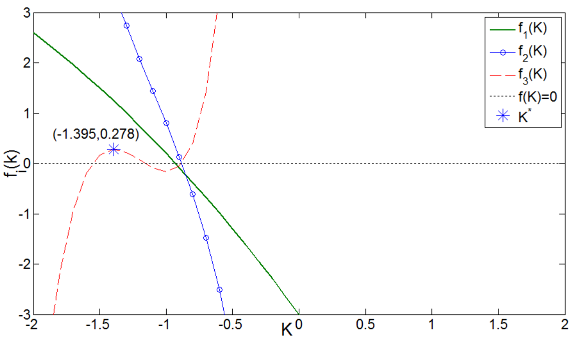

Fig. 1: $f_1(K), f_2(K), f_3(K)$ versus K

Thus, it can be concluded that the controller K^* can stabilize the system (7) in the mean square sense. In fact, we have $\max_{\lambda \in \text{spec}(\Phi(K^*))} \mathcal{R}(\lambda) = -0.029 < 0$.

B. Example 2

Next, we consider a discrete-time case with the plant (1) as

$$A = \begin{pmatrix} 1 & 0.3 \\ 1.1 & 0.5 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and the fading channels (6) as

$$\Pi = \text{diag}(\ 0.9 \quad 0.8\ ),\ \Sigma = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}.$$

The problem consists of searching for a controller (2) over the set $\mathcal{K} = [-1, 1]^2$ such that the closed-loop system (7) is stable in the mean square sense. First of all, let us observe that $\rho(A) = 1.376 > 1$, which implies that the plant is unstable.

Let us express the controller K as $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}'$ where $k_1, k_2 \in \mathbb{R}$. The set $\mathcal{K}$ is described as in (3) by choosing $a_i = 1 - k_i^2, i = 1, 2$. We compute the modified Jury table for $\Phi(K)$ in (24) based on (13)-(14). Then, we solve (35) finding the upper bound of r given by $\hat{\theta} = 0.791$ (the computational time is $7.125s$). In the end, by testing the feasibility of (37), we get the solution $K^* = \begin{pmatrix} -0.357 & -0.342 \end{pmatrix}'$.

Thus, we have that the controller K^* stabilizes the system (7) in the mean square sense. In fact, we have $\rho(\Phi(K^*)) = 0.648 < 1$. With the controller derived, Figure 2 shows that the trajectories of the closed-loop system converge to origin under different randomly generated initial conditions.

V. Conclusions

The paper has considered the design of stabilizing output feedback controllers for LTI systems over fading channels

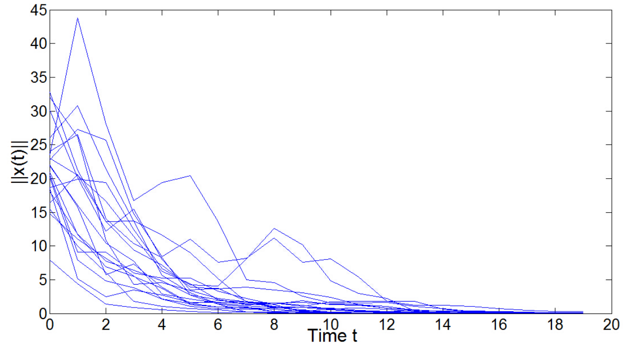

Fig. 2: Trajectory of $||x(t)||$ under different initial conditions

in both continuous-time and discrete-time cases. It has been shown that a sufficient and necessary condition for the existence of such controllers can be obtained by solving a convex optimization problem in the form of a SDP.

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