Stochastic Annuitization And Asset Allocation Under Uncertain Lifetime

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"Our greatest weakness lies in giving up. The most certain way to succeed is always to try just one more time."

Thomas Edison

Abstract

In this thesis, we investigate a pensioner's gains from access to annuities. We observe the optimal asset allocation and annuitization strategies for a pensioner whose retiring age is 65, with an individual pension wealth at retirement and with a guaranteed income from social security during the retirement period. We also observe with particular personal risk preferences towards risk, with a certain amount to buy more annuities after retirement and with certain limitations on pensioner's asset allocation and annuitization strategies. The pensioner's objective is to maximize the utility drawn from consumption during retirement with a Constant Relative Risk Aversion utility function.

We develop and solve two main models on stochastic volatility for the pensioner who receives an income after retirement from life annuities and investment performance which are under the Constant Elasticity of Variance model and Heston's Model. We start with the model proposed by Milevsky and Young in 2007 under the Geometric Brownian Motion model and address using the change variable technique. We extend the model under stochastic volatility and solve it using the combination of Legendre transform, dual theory and change variable technique.

By adopting the Legendre transform, dual theory and change variable approaches, the explicit solution for optimal investment, consumption and annuitization strategies is derived for the power utility. We use a numerical example to investigate the influence of life annuities and model parameters on the optimal strategy. The results show that the optimal strategy depends on model parameters and the presence of life annuities in the model affects the pensioner's decisions regarding optimal investment and annuity income level strategies for a period after retirement under the stochastic market price of risk. Besides, an annuity income plays a role in altering the consumption rates for all levels of risk aversion.

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Abbreviations

CARA	Constant Absolute Risk Aversion
CEV	Constant Elasticity of Variance
CRRA	Constant Relative Risk Aversion
CIR	Cox-Ingersoll-Ross
DC	Defined Contribution
FBV	Free Boundary Value
FOCs	First-order conditions
GBM	Geometric Brownian Motion
HARA	Hyperbolic Absolute Risk Aversion
HJB	Hamilton Jacobi Bellman
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation
SV	Stochastic Volatility

To my husband, parents & sister

Chapter 1

Introduction

1.1 Background

Pensions can sound a bit dull and gloomy, and it can be very alluring to put off accurately saving for them. Saving for your retirement is essential, but it is not always easy to know exactly how much a pensioner should be putting away, and this is why pension schemes can help. The pensioner saves a little wealth or income regularly during their working life or also known as the accumulation phase, and enjoys the income later after retirement (the decumulation phase). A pension scheme is a special kind of long-term savings plan.

As reported by Thurley [1], most people in the United Kingdom (UK) (75%) with a Defined Contribution (DC) pension scheme used a pension pot to buy an annuity at retirement. This happens because they took advantage of an annuity, as it provides guaranteed income for life and the pensioner no longer relies on the investment.

From April (2015), the UK government is allowing savers to take out their money from their pension pots when they reach the age of 55 and spend it as they see fit. Tax penalties will be reduced for those who withdraw their savings in a lump sum and this will be subject to marginal tax rates. These changes will empower individuals, allowing them to take control of their financial futures. Gerrard et al. [2] stated that pensioners had three principal degrees of freedom. First, they can decide what investment strategy to adopt in investing the fund. Second, they can choose how much of the fund to withdraw at any time between their retirement and the ultimate annuitization. Third, they can decide when to annuitize.

The UK government has concluded that the official retiring age for males and females will change to 65 starting November 2018 after a brief discussion among the experts and after taking into account the UK's economic status, as reported by the Department of Work and Pensions [3]. Before this, the state pension ages were different for males and females, fixed at 65 for males and 60 for females until 2020. According to Murthi et al. [4], individuals in the UK with an individual pension account needed to annuitize before the age of 75. However, Thurley [1] mentioned that starting April 2011, there would no longer be a specific age by which people have to annuitize effectively. Besides, the Office for National Statistic [5] reported males would live until the age of 79.2 and females until 82.9, and Mayhew et al. [6] stated gender is no longer a relevant consideration when pricing annuities.

In this thesis, we investigated optimal asset allocation, annuitization and consumption strategies for a pensioner whose retiring age is 65, with an individual pension wealth at retirement and with a guaranteed income from social security during the retirement period. Besides, a particular personal preference towards risk, a certain amount to buy more annuities after retirement and some limitations on pensioner's asset allocation and annuitization strategies have also been considered. The pensioner's objective is to maximize the utility drawn from consumption during retirement. This thesis is to develop the optimal asset allocation, optimal consumption and annuitization strategies for the pensioner who wishes to maximize the expected discounted utility drawn for future consumption.

We concentrated on adding life annuities (life insurance) to the model, where the pensioner can decide to buy more annuity after retirement when they need or choose to do so. We started developing the model with one similar to that of Milevsky and Young [7] under an open market structure where the pensioner can annuitize anything and anytime. Then we extended the model by considering the stochastic volatility market, with two different models of stochastic volatility, the Constant Elasticity of Variance (CEV) model and Heston's Stochastic Volatility (SV) model.

We recognized the analysis of optimization with life annuities, and that the stochastic volatility would be an interesting problem to investigate. Stochastic volatility refers to the volatility of asset prices which is not constant, as assumed in the Black-Scholes options pricing model. Stochastic volatility modelling attempts to correct this problem with Black-Scholes by allowing volatility to vary over time. Furthermore, it would be possible to identify the probable relationship between model parameters and optimization.

In this thesis, we do not consider gender annuitization. In order to setup a unisex annuity rate on retirement, we assume the retirement age 65 is made up of equals number of males and females. Thus, we used the force of mortality rates that reflects the whole population regardless of gender. Also, we assumed the pensioner has a 16-year life expectancy after retirement.

1.2 Literature review

1.2.1 Portfolio Optimization

The optimal investment-consumption was originally studied by Merton in [8] and [9] for the stochastic optimal control theory in portfolio selection problem over a fixed time horizon. Many authors have dealt with the issue of managing the pensioner's asset allocation on optimal investment-consumption at retirement. Shreve and Soner [10] considered the discounted problem of optimal consumption and investment in a risky asset, a risk-free asset and the proportional transaction costs under infinite-horizon. Zariphopoulou [11] discussed a portfolio problem where the investor could put their money into a stock and a money market account. Under an incomplete market, Miao and Wang [12] analyzed the implications of undiversifiable idiosyncratic risks using entrepreneurship as a motivating example. The development of the model becomes more attractive, as Guan and Liang [13] investigated an optimal reinsurance and investment problem for an insurer whose surplus process is approximated by a drifted Brownian motion.

The original model reported by Merton gave ways and ideas for researchers to expand and develop it to fit the pensioner's situation at the corresponding time as the study, especially, considering life insurance in the investor's portfolio. The pioneers in the insurance market were Yaari [14], followed by Richard [15], who introduced the concept of lifetime uncertainty, labor and insured wealth as further elements to be taken into account. Next, Pliska and Ye [16] studied random and unbounded lifetime for optimal life insurance and consumption for a wage earner. Then the research was extended by Huang and Milevsky [17] who included life insurance and pension annuity claims in one unified life-cycle model. Duarte et al. [18] extended it even further to discuss wage earner's problem on optimal consumption, investment, and insurance purchase decisions under the multidimensional Brownian motion. Meanwhile, Kwak et al. [19] studied the portfolio problem of a family combined with life insurance for a parent who receives deterministic labor income. Besides, Pirvu and Zhang [20] considered the issue of life insurance acquisition for a wage earner where the stock price is mean-reverting drift.

A particular asset universe composed of pension funds, the definition of a long-term horizon strategy and the stochastic elements are the main features of a retirement context portfolio allocation for the individual investor. See [7, 21, 2]. The most prominent life contingent claim is constant life annuity. This study focusses on life annuity.

An annuity is a contract that pays its beneficiary, the annuitant, a pre-specified amount for as long as he/she is alive, as mentioned by Finkelstein and Poterba [22]. An annuity contract will exchange the savings according to an annuity rate, which reflects how much income the pensioner will earn in a year. The rate offered often depends on several factors, including the pensioner's age, health, and where they live. According to Brown et al. [23], a financial contract between an annuitant (buyer) and an insurer (seller) that pays out the periodic income for as long as the annuitant is alive is an exchange for an initial premium called a constant life annuity.

The dynamic asset allocation with annuities in the restriction that financial wealth must be fully annuitized becomes huge in literature. Milevsky and Young [7] found the frequent repurchase of life annuities during retirement for gradual annuitization is optimal. Then, Horneff et al. [21] extended the work done by Milevsky and Young [7] by deriving the optimal consumption and saving strategies with constant life annuities, stock and bonds. The study on life annuities expands among the researchers. Gupta and Li [24] developed a framework that merges an annuity purchase decision with consumption-investment selections in retirement planning. Others studies on life annuities are [25, 26, 27]. We started with a similar model proposed by Milevsky and Young [7]; we solve it using different approaches and extends the model by considering the stochastic volatility market.

Portfolio optimization was first introduced by Markowitz [28] in a single period setting with a quadratic utility to maximize the trade-off between the expected return and variance. Then, Merton [8, 9] developed the portfolio-consumption model in continuous time. In the recent decade, there has been a growing development of the portfolio-consumption model to study optimal annuitization. Horneff et al. [29], compared different retirement payout approaches to show how people can optimize their retirement portfolios by simultaneously using investment-linked retirement rules along with life annuities.

On the other side, the vital component in portfolio optimization is the utility function. Epstein and Ji [30] formulated a model of utility for a continuous-time framework that captures the aversion to ambiguity about both volatility and drifts. Çanakoğlu and Özekici [31] and Kingston and Thorp [25] consider the portfolio selection problem in a stochastic market with Hyperbolic Absolute Risk Aversion (HARA) utility functions. Di Giacinto et al. [32] used a quadratic function as Gerrard et al. [2] known as the loss function for the pensioner's preferences.

Pratt [33] and Arrow [34] introduced the topic of the measure of risk aversion. Merton [8] considered special utility functions with algorithm power structures. Breuer and Gütler [35] investigated the performance of funds using different utility function. Meanwhile, Dokuchaev [36] considered a model with the expected utility of the terminal wealth with power and maximized the logarithmic utility functions in a discrete-time market serial correlation.

This study considered a Constant Relative Risk Aversion (CRRA) utility function since the optimal demand for life insurance is sensitive to the degree of risk aversion under CRRA. A lot of studies have been done considering CRRA utility function as the pensioner's risk preferences. Horneff et al. [37] studied the impact of risk aversion on the choice of distribution rule. Horneff et al. [26] defined a single non-durable consumption good, and Wang and Young [38] studied on commutable life annuities. Recently, Hulley et al. [39] analyzed the decumulation pattern of Australian Age Pensioners.

Another point that we need to consider when solving the portfolio optimization with annuitization is the mortality function. It is used to calculate the price of a life annuity and plays an essential role in the theory of life contingencies, especially when involving the limiting process. In actuarial science, the force of mortality represents the instantaneous rate of death at some age measured on an annualized basis. It is identical in concept to the failure rate, also called the hazard function by the reliability theory. Using age-dependent mortality, Kalemli-Ozcan and Weil [40] examined the role of declining mortality in explaining the rise of retirement in the twentieth century. Blake et al. [41] considered the probability the pensioner will survive until a certain age after retirement by comparing the purchases at the retirement age of the conventional life annuity with distribution programs involving differing exposures to equities during retirement. This study considers the constant force or mortality as discussed in Milevsky and Young [7], Wang and Young in [42] and [38] since we want to discuss a more straightforward situation, and our focus is on mathematical modelling.

1.2.2 Methods

Some methods have been widely used in literature to solve portfolio optimization. One famous technique is the martingale approach. It was introduced by Pliska [43] and Karatzas et al. [44] and Cox and Huang [45]. Xu and Shreve [46] and Karatzas et al. [47] combined the martingale approach and the duality method to solve utility maximization.

There were also researchers interested in the numerical approach, in particular, using the Markov Chain Approximation method. Initially, this method was used to directly solve the Partial Differential Equation (PDE). However, the direct solution is exceptionally complicated since Hamilton Jacobi Bellman's (HJB) equation from optimal control is highly non-linear. The pioneer for these approaches was Kushner [48]. Fitzpatrick and Fleming [49], Hindy et al. [50] and Munk [51] adapted this method.

Another leading approach to the solution of optimal decisions is a dynamic programming method. Dynamic programming gives easy access to the value function and the problem controls and portrays a significant role in solving stochastic control problems in finance. Merton [8] and [9] are among the pioneers who addressed the optimal decision using dynamic programming. Musumeci and Musumeci [52] and Papi and Sbaraglia [53] used classical dynamic programming for various levels and types of risk aversion and for the optimal asset-liability management model with transaction costs under discrete-time, respectively. Recently, Kraft and Steffensen [54] constructed the non-separable value function. Furthermore, Sørensen [55] used quasi-dynamic programming to provide the solution for the intertemporal investment problem.

In this thesis, we solve the optimization problem using stochastic dynamic programming. Vila and Zariphopoulou [56] were among the first who used it. Vila and Zariphopoulou [56] studied the intertemporal consumption and portfolio choice.

Later, Infanger [57] solved dynamic asset allocation problems, while Han and Hung [58] investigated the optimal asset allocation under stochastic inflation for DC pension plan.

Several techniques that can be considered and employed when solving portfolio optimization using dynamic programming. For example, Kingston and Thorp [25] and Chang and Rong [59] used the change variable technique. Also, Gao [60] used combinations of the variable change technique and power transformation. Another method that can be considered is the asymptotic method, as discussed by Noh and Kim [61], Gao [62] and Bayraktar et al. [63].

One of the most popular techniques is the Legendre transform-dual theory. This approach has been widely used in literature, for example, by Milevsky and Young [7] and Wang and Young [38] to solve the problem under the Geometric Brownian Motion (GBM) model. Xiao et al. [64], Gao [65, 62], and Bayraktar et al. [63] used it to solve the problem under stochastic volatility model and Chang and Chang [66] for the problem under the stochastic interest rate model.

This study uses the Legendre transform, duality theory, change variable technique (substitution method) and power transformation. An idea taken from Kingston and Thorp [25] was to solve our problem under GBM in Chapter 3 using change variable techniques. Then, ideas from Milevsky and Young [7], Wang and Young [38] and Gao [65] were used to solve the problem under stochastic volatility as in Chapters 4, 5 and 6 by means of the Legendre transform, dual theory, variable change technique and power transformation to derive the explicit solution optimal investment, consumption and annuity income level.

1.2.3 Stochastic Volatility

In the financial literature, the stochastic volatility (SV) model extended from the problem with constant volatility of risk asset price and recognized as an essential factor of stock dynamics. It can also demonstrate many well-known empirical features, such as volatility smile and volatility clustering. Therefore, many researchers have proposed various SV models such as mean-reverting as discussed by Hull and White [67] and Stein and Stein [68], CEV model by Cox and Ross [69] and Heston by Heston [70]. This study focused on two SV models, which are the CEV model and Heston's SV model.

Under the CEV model, Xiao et al. [64] studied the DC pension plan where the benefits were paid by an annuity. Gao [65] then used the variable change technique in the derivation of CEV option price to transform it into simple PDE and obtained an explicit solution for before and after retirement. Later Gao [60] derived the solution for CRRA and CARA utility functions with the correction factor. Gu et al. [71] gave a new explanation in literature when they studied the optimal excess-of-loss reinsurance and investment. Recently, Wu and Wu [72] considered the optimal proportional reinsurance and investment strategies for an insurance company.

Another well-known SV model is Heston's SV model and it has solved for various problems using different approaches. Kraft [73] presented a verification result for portfolio problems with stochastic volatility. Meanwhile, Li and Wu [74] provided a verification theorem without the usual Lipschitz assumptions. Besides, Li et al. [75] and Zhao et al. [76] studied the optimal-time consistent investment and optimal excess-of-loss, respectively, and both considered reinsurance strategies. Also, Chang and Rong [59] combined Heston's SV model with the stochastic interest rate to solve the investment and consumption problem. Recently, Chunxiang and Li [77] studied an optimal investment and excess-of-loss reinsurance problem with delay for an insurer under Heston.

1.3 Objectives and scope

This study aims to model the retirement phase for a representative pensioner who receives an income from an annuity purchased before the retirement and investment performance concerning the uncertain lifetime. The main focus of this study is on 'How annuitization affects the portfolio optimization in United Kingdom for the complete and incomplete market?'. The study focuses on a pensioner's income that allows the stochastic factor and the optimal control variables.

The aim can be divided further into subproblems. As a start, a model suitable for the UK pensioner needs to be defined. This model will then need to be extended to include pensioner's risk preferences and the decisions that represent a realistic financial market for the pensioner after retirement. Finally, methods that solve the problem mathematically need to be evaluated and/or constructed, and the take the necessary time for the derivation of the solution to find the result of the model. To solve the general research aim and each particular sub-problem, the following is needed:

(i) to define the model that captures the UK pensioners's characteristics, and calibrate the model with financial variables to ensure model validity.

(ii) to propose suitable solution methods and show the effect of the annuitization towards the portfolio optimization under stochastic volatility model for the complete and incomplete market.

(iii) to identify the effect of model variables towards the portfolio optimization problem under the stochastic volatility model when considering the life annuities.

1.4 Research significance

The study contributed to the fields of research of the Life-cycle model, especially in the niche field of portfolio optimization and annuitization among the pensioners in the UK, and also adaptive the dynamic programming in general. The main contribution of this study is to study the effect of annuitization towards portfolio optimization and extending the model into two stochastic volatility models, which are the Constant Elasticity of Variance (CEV) model and the Heston's model. Our main contribution can be divided into three sub-contributions as follows:

- First, we showed the effect of the annuitization towards the pensioner's decision. Here, we explained how the pensioner can optimize their income after retirement. We focused on pensioner's decision on optimal investment, consumption and annuity income level.
- Second, we have obtained the explicit solutions for the portfolio optimization under the CEV model and Heston's Model. We showed the effect of the annuitization towards the portfolio optimization under stochastic volatility model (this will be explained in Chapters 4 and 6 respectively).
- Lastly, this study discussed on 'How the model variables can affect optimization?'. The pensioner can decide how to optimize investment and consumption if they choose to invest in life annuities. The analysis showed how life annuities will affect the decision makes by the pensioner.

The primary motivation for this study is to explore the impacts of life annuities, and the stochastic problem on the optimal investment, consumption and annuity income level strategies for complete and incomplete market. The model is expected to be used in financial planning by an individual after retirement, where the model would give an idea for the pensioner to choose and decide their asset allocation.

1.5 Thesis structure

The rest of the thesis is arranged as follows. In Chapter 2, we gave a general overview of the portfolio optimization and dynamic programming, to convey the general idea on what we are going to discuss in the thesis. In Chapter 3, we started by solving the problem that is similar to the one studied by Milevsky and Young [7]; but, we solved it using the change variable technique alone or the substitution method. In Chapter 4, we extended the model by considering the stochastic volatility under the CEV model and applying the Legendre transform, duality theory and change variable technique to solve the problem. Next, Chapter 5 discussed the special case of the CEV model (when the elasticity factor of CEV model is zero) and applied a similar methodology as in Chapter 4, but using the different dual variable. Again, in Chapter 6 the original model was extended for different types of stochastic market price. Wherein this chapter, Heston's SV model was considered and solved using the same approaches as in Chapter 4. For the models in Chapters 3 until 6, we provided the numerical example and conclusions at the end of each chapter. For some relevant application of the proposed model, we discussed in Chapter 7. Finally, we gave a conclusion for our study in Chapter 8, together with the key findings and pointed out some possible ideas for further research.

Chapter 2

Stochastic optimization

2.1 Portfolio optimization

In this chapter, we first explained the general idea of optimization by explaining the assets invested in the financial market. There, the pensioner can spend his/her wealth on two assets, which are a riskless asset (bond) and a risky asset (stock).

2.1.1 The financial and pension annuity markets

The pensioner with future lifetime is described by the random variable τ and we assume that τ is an exponential random variable with parameter λ, also referred to as a force of mortality or hazard rate; $\mathbf{E}[\tau] = \frac{1}{\lambda}$. The pensioner can invest in a riskless asset whose price at time s, $S_{0,s}$, follows the process

$$\begin{aligned} \frac{dS_{0,s}}{S_{0,s}} &= r ds, \\ S_{0,0} &= S_0 > 0, \end{aligned} \tag{2.1}$$

for some fixed $r \geqslant 0$ and a risky asset $S_{1,s}$ at time s, whose price follows a geometric Brownian motion with drift μ and diffusion σ:

$$
\begin{aligned}
\frac{dS_{1,s}}{S_{1,s}} &= \mu ds + \sigma dB_s, \\
S_{1,0} &= S_1 > 0,
\end{aligned}
\tag{2.2}
$$

where $\mu > r$, $\sigma > 0$ and B is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_s\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1.1. *Following Zhang [78], we introduce, a pair (y_s, c_s) consisting of portfolio y_s and a consumption rate c_s which is said to be self-financing if the corresponding wealth process $W_s^{y,c}$, $s \in [0, \tau]$ satisfies*

$$
dW_s^{y,c} = y_s W_s^{y,c} \frac{dS_{1,s}}{S_{1,s}} + \big(1 - y_s\big) W_s^{y,c} \frac{dS_{0,s}}{S_{0,s}} - c_s ds. \tag{2.3}
$$

We derived the two-asset model from Definition 2.1.1. Let $W_s^{y,c}$ be the wealth at time s of the individual, y_s be the amount the pensioner invests in the risky asset at time s, and c_s be the consumption rate that the pensioner consumes at time s. Then, the wealth process for the pensioner who does not buy for annuities is as follows

$$
\begin{cases}
dW_s^{y,c} &= \big[rW_s^{y,c} + y_s\big(\mu - r\big) - c_s\big]ds + \sigma y_s dB_s; \\
W_0^{y,c} &= w > 0,
\end{cases}
\tag{2.4}
$$

where $w > 0$ is the fund wealth at retirement time $s = 0$ and B_s is standard Brownian motion. In literature, the wealth process as in (2.4) has been widely discussed (See [8, 79, 80, 2, 32]).

In this study, we consider he pensioner maximizes (over admissible set $\{c_s, y_s, A_s\}$) the expected utility of the discounted lifetime consumption in which A_s is the annuity purchasing process. A_s denotes the non-negative annuity income rate at time s after any annuity purchases at that time. The previous annuity purchases or a pre-existing annuity such as pension income could be the reason for the annuity income. This has also been discussed by Milevsky and Young [7] and Wang and Young [38]. By assuming that the pensioner has the existing annuity income at time s from the previous life annuities and that they will buy another annuity, a_s whenever he/she decides to do so, the annuitization wealth dynamics process is defined from the general wealth process in (2.4). Therefore, the annuitization wealth dynamics process is

$$\begin{cases} dW_s^{y,c} & = \left[rW_{s-}^{y,c} + y_s\big(\mu_1 - r\big) - c_s + A_{s-}\right]ds + \sigma_1 y_s dB_s - a_s dA_s; \\ W_{t-}^{y,c} & = w > 0, \end{cases} \tag{2.5}$$

where the pensioner can purchase an annuity at a price a_s per £ of annuity income at time s. Number one on the subscripts for μ and σ denotes the first model in this thesis. Following Milevsky and Young [7], the negative sign on the subscripts for wealth, W_{t-} and annuities, A_{t-} denotes the left-hand limit of those quantities before any annuity purchases.

Taking an idea from Wang and Young [42] and Ewald and Zhang [81], the time of death is considered as random, with

$$\mathbb{P}(\tau \in \big[s, s+ds\big] \mid \tau \geqslant s) = \lambda_s ds \tag{2.6}$$

where λ_s is the time dependent instantaneous mortality rate. Intuitively, the mortality rate λ_s describes the likelihood of the pensioner aged s dying in the interval $\big[s, s+ds\big]$ given he/she is still alive at time s. Here, we assumed that λ_s is a deterministic function of time.

The variable λ_s is also referred to force of mortality. Model with stochastic force of mortality have been considered in the actuarial literature, where it is typically assume that

$$\mathbb{P}(\tau > t \mid \tau > s, \mathcal{F}_s) = \mathbf{E}\Big[e^{-\int_s^t \lambda_s ds} \mid \mathcal{F}_s\Big]$$

where λ_s is a stochastic process. Then, we derived an equivalent form for the maximization utility. We considered the time of death as

$$\mathbb{P}(\tau > s) = e^{-\lambda s}, \tag{2.7}$$

where this also refers to the pensioner's likelihood of surviving until s with λ is the mortality rate. This means that the pensioner will survive until time s.

We assume that the random time τ is independent of any of the economic state variables. Then,

$$\mathbf{E}(\mathbf{1}_{\{s<\tau\}}) = \mathbb{P}(\tau > s).$$

The price of a life annuity, a_t, that pays £1 per year continuously until the individual dies is as given by Wang and Young in [42] and [38]

$$a_t = \int_t^{\infty} e^{-rs}(e^{-\lambda s})ds = \frac{1}{r+\lambda}e^{-(r+\lambda)t}, \tag{2.8}$$

where $\lambda > 0$ is the (constant) hazard rate that is used to price the annuities. We can say that he/she receives £1 per year continuously from a life annuity until he/she dies as the return for the price of a_t him/her pays to the life annuity.

2.1.2 Dynamic programming

Dynamic programming is a very convenient way of writing a broad set of dynamic problems in economic analysis. Most of the properties of this tool are now well established and understood. In this thesis, we are interested in the wealth process of the pensioner where they receive an annuity income from the existing annuity before retirement and will buy more after retirement when they decide to do so.

Before we discuss further, let us derive the Bellman equation generally for our problem. The principle of optimality is the fundamental concept in control theory, which usually appears in the form of a recurrence equation. Taking an idea from Chang [82], let $U(c)$ be the instantaneous utility function, which is of class C^2, and $(\alpha+\lambda)$ be the discounted rate where α is the interest rate. Under the wealth process denoted by equation (2.4), the pensioner looks for a consumption c_s^*, investment strategy y_s^* and annuity income A_s^*. First, we define the intertemporal optimization

$$\max_{c_s, y_s, A_s} \mathbf{E} \int_0^{\infty} e^{(\alpha+\lambda)s} U(c_s)ds \quad \text{subject to (2.4)}. \tag{2.9}$$

with $w_0 = w$ and $A_0 = A$ given. Therefore, the principle of optimality states that

$$V(w,A) = \max_{\{c_s,y_s,A_s\}\text{for}0\leqslant s\leqslant\Delta s} \mathbf{E}_{w,A}\Bigg\{\int_0^{\Delta s} e^{-(\alpha+\lambda)s}U(c_s)ds + \max_{\{c_s,y_s,A_s\}\text{for}\Delta s\leqslant s<\infty} \mathbf{E}_{\Delta s,w+\Delta w,A+\Delta A}\int_{\Delta s}^{\infty} e^{-(\alpha+\lambda)s}U(c_s)ds\Bigg\} \tag{2.10}$$

where $w+\Delta w = w(\Delta s)$ and $A+\Delta A = A(\Delta s)$. Using the intermediate value theorem, the first integral of (2.10) can be simplified as, with probability 1,

$$\int_0^{\Delta s} e^{-(\alpha+\lambda)s}U(c_s)ds = e^{-(\alpha+\lambda)\theta\Delta s}U(c_{\theta\Delta s})\Delta s \tag{2.11}$$

where $\theta = \theta(\omega)$ and $0 \leqslant \theta \leqslant 1$ such that $\theta\Delta s \in [0,\Delta s]$ and $c_{\theta\Delta s} \to c$ as $\Delta s \to 0$. Applying the change of variables $\epsilon = s-\Delta s$, the second integral of (2.10) becomes

$$\begin{aligned}
&\max_{\{c_s,y_s,A_s\}\text{for}\Delta s\leqslant s<\infty} \mathbf{E}_{\Delta s,w+\Delta w,A+\delta A}\int_{\Delta s}^{\infty} e^{-(\alpha+\lambda)s}U(c_s)ds \\
&\quad= \max_{\{c_{\epsilon+\Delta s},y_{\epsilon+\Delta s},A_{\epsilon+\Delta s}\}\text{for}0\leqslant\epsilon<\infty} \mathbf{E}_{\Delta s,w+\Delta w,A+\delta A}\int_0^{\infty} e^{-(\alpha+\lambda)(\epsilon+\Delta s)}U(c_{\epsilon+\Delta s})d\epsilon \\
&\quad= e^{-(\alpha+\lambda)\Delta s}\max_{\{c_\epsilon,y_\epsilon,A_\epsilon\}\text{for}0\leqslant\epsilon<\infty} \mathbf{E}_{\Delta s,w+\Delta w,A+\Delta A}\int_0^{\infty} e^{-(\alpha+\lambda)\epsilon}U(c_\epsilon)d\epsilon \\
&\quad= e^{-(\alpha+\lambda)\Delta s}V(w+\Delta w, A+\Delta A, s)
\end{aligned} \tag{2.12}$$

Relabeling $\{c_{\epsilon+\Delta s}, y_{\epsilon+\Delta s}, A_{\epsilon+\Delta s}\}$ as $\{c_\epsilon, y_\epsilon, A_\epsilon\}$ for the second equation of (2.12), and writing it as

$$0 = \max_{\{c_s,y_s,A_s\}\text{for}0\leqslant s\leqslant\Delta s} \mathbf{E}_{w,A}\Big\{e^{-(\alpha+\lambda)\theta\Delta s}U(c_{\theta\Delta s})\Delta s + e^{-(\alpha+\lambda)\Delta s}V(w+\Delta w, A+\Delta A) - V(w,A)\Big\} \tag{2.13}$$

for Δs, we have $e^{-(\alpha+\lambda)\Delta s} = 1-(\alpha+\lambda)\Delta s + o(\Delta s)$. Thus,

$$
\begin{aligned}
&e^{-(\alpha+\lambda)\Delta s}V(w+\Delta w, A+\Delta A) - V(w,A) \\
&= (1-(\alpha+\lambda)\Delta s)V(w+\Delta w, A+\Delta A) - V(w,A) + o(\Delta s) \\
&= [V(w+\Delta w, A+\Delta A, s) - V(w,A)] - (\alpha+\lambda)\Delta s V(w+\Delta w, A+\Delta A) + o(\Delta s)
\end{aligned} \tag{2.14}
$$

By Ito's lemma,

$$
V(w+\Delta w, A+\Delta A) - V(w,A) = V'(w)(\Delta w) + \frac{1}{2}V''(w)(\Delta w)^2 + o(\Delta s) \tag{2.15}
$$

The conditional expectation subject to our wealth process function (2.5), we have

$$
\begin{aligned}
&E_{w,A}[V(w+\Delta w, A+\Delta A) - V(w,A)] \\
&\quad = \left\{V'(w)[rw + y(\mu - r) - c + A] + \frac{1}{2}V''(w)\sigma^2 y^2\right\}\Delta s + o(\Delta s)
\end{aligned} \tag{2.16}
$$

Divide (2.13) by Δs, and then let $\Delta s \to 0$. Since, with probability 1, we have $w + \Delta w \to w, e^{-(\alpha+\lambda)\theta\Delta s} \to 1$ and $c_{\theta\Delta s} \to c$, as $\Delta s \to 0$, (2.13) is simplified as the following Bellman equation:

$$
0 = \max_{c,y,A}\left\{U(c) - (\alpha+\lambda)V + [rw + y(\mu - r) - c + A]V_w + \frac{1}{2}(\sigma y)^2 V_{ww}\right\} \tag{2.17}
$$

As the objective function depends on time and considering the utility function (3.1) in Bellman's equation in (2.17), it becomes as (2.18)

Proposition 2.1.1. *The value function V is a constrained viscosity solution of the Hamilton Jacobi Bellman (HJB) equation*

$$
\big(\alpha+\lambda\big)V = V_t + \max_{\{c_s, y_s, A_s\}}\left[\big\{rw + y\big(\mu - r\big) - c + A\big\}V_w + \frac{1}{2}\big(\sigma y\big)^2 V_{ww} + \frac{c^{1-\gamma}}{1-\gamma}\right]. \tag{2.18}
$$

Using dynamic programming, we can get the feedback form from the Bellman equation. However, the feedback form is not a closed-form solution. To solve the HJB equation and get the feedback form, we need to consider the First Order Condition (FOC) of the admissible variables. Therefore, we will use other approaches in dynamic programming to solve the feedback form until we get the closed-form solution or the explicit solution.

Chapter 3

Optimization using the change variable technique

3.1 Introduction

One of the most important financial decisions many people make is the choice of a portfolio of assets for their retirement. One of the most difficult decisions is on how to optimize their consumption and maximize the utility. Milevsky and Young [83], Milevsky et al. [84] and Milevsky and Young [7] have come up with a new explanation for the pensioner's reluctance to buy life annuities. This chapter studies the optimal decision on consumption and utility of a representative pensioner, who faces an uncertain lifetime with the effects of life annuities. This chapter focuses on the impact of life annuities encourages pensioner's consumption and utility. We followed the model from Milevsky and Young [7], where the annuity income variable is considered in the wealth process. This leads to a two-dimensional optimal control problem in a complete market. The optimal strategy depends on two state variables, wealth and the existing annuity income.

However, Milevsky and Young [7] only provided the numerical example for optimal annuitization without detailed explanations on how they solved the problem. Therefore, this chapter provided details on how to solve optimal annuitization using another approach of dynamic programming which is the change variable technique. Besides that, we provided a numerical example to show the impact of the model on the pensioner's decision. Unlike in the existing literature, we tested the effects of various model parameters such as the pensioner risk preference, investment volatility, initial wealth, existing amount of annuity income and time after retirement on the optimal consumption and utility.

This chapter's aim was to obtain an analytical solution by using the substitution method to solve the HJB equation and numerically calculate the impact of the model parameters on a pensioner's decision. This method was also discussed by Gerrard et al. [80] considering the income drawdown option and looking for optimal investment strategies to be adopted after retirement. Browne [85] focused on minimizing the risk of ruin by obtaining investment strategies. This method is used to simplify the HJB equation into Bernoulli ODE. In addition, the homogeneity property from CRRA utility is considered when reducing the dimension of the HJB equation. Kingston and Thorp [25] discussed the problem of maximizing the utility of consumption for an extension to the case HARA or Generalized Logarithmic Utility Model preferences via a substitution method, similar to the one in this chapter.

3.2 The model

In our case, we assumed that the pensioner maximizes (over admissible set $\{c_s, y_s, A_s\}$) the expected utility of the discounted lifetime consumption in which A_s is the annuity purchasing process. The wealth process as discussed in Chapter 2 in equation (2.5).

The idea that the pensioner received an income after retirement from the investment has been widely discussed in literature. Gerrard et al. [80] and Gerrard et al. [2] investigated the income drawdown option when allowing for the periodic fixed withdrawals from the fund. Meanwhile, Di Giacinto et al. [32] extended the model from Gerrard et al. [80] by adding no short-selling constraints on the control variable and a final capital requirement constraint on the state variable.

This is the difference with this study, where the pensioner is assuming he/she will receive an income after retirement from an annuitization process. This issue is a popular topic nowadays, for example, in Milevsky et al. [84], Milevsky and Young [7] and Wang and Young [86].

We used the change variable technique to solve the HJB equation until arriving at the closed-form solution. This method has been discussed by Chang [82], while

Gerrard et al. [80] used this method to solve optimal investment choices post-retirement and Browne [85] for optimal investment in minimizing the probability of ruin. Kingston and Thorp [25] settled for a real option to delay annuitization. This chapter solves the problem of optimization by taking an idea from Kingston and Thorp [25]. The direct substitution will turn the Bellman equation into a differential equation in the state variable alone since the first-order conditions enable us to express all control variables as functions of the state variables. Then, through this differential equation, the value function can be solved at least in principle.

Following Yaari [14], utility only comes from consumption since we considered a retirement without a bequest motive. Therefore, utility only comes from his/her consumption. The pensioner chooses to consume at a rate c_s at time s.

Definition 3.2.1. *The investment, consumption and annuity income* $\{y_s, c_s, A_s\}$ *are said to be admissible if*

(i) The control processes $\{y_s\}_{s\geqslant 0}$, $\{c_s\}_{s\geqslant 0}$ *and* $\{A_s\}_{s\geqslant 0}$ *are all adapted to filtration of* $\mathcal{F} = \{\mathcal{F}\}_{s\geqslant 0}$.

(ii) The controls $c_s \geqslant 0$ *and* $A_s \geqslant 0$ *are almost surely for all* $s > 0$.

(iii) $\int_0^s y_{s_1}^2 ds_1 < \infty$ *and* $\int_0^s c_{s_1} ds_1 < \infty$ *almost surely for all* $s \geqslant 0$.

(iv) The associated wealth and annuitization processes $W_s \geqslant 0$ *and* $A_s \geqslant 0$*, respectively, are almost surely for all* $s \geqslant 0$.

Denoted by $\mathcal{A}(w, A)$ *is the collection of all admissible strategies when the initial wealth and annuity income is* (w, A).

We assume that the individual is risk-averse and use CRRA pensioner preferences. The utility function for the individual is given by

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{3.1}$$

in which, $\gamma > 0$ and, $\gamma \neq 1$ with, γ is the (constant) relative risk aversion. [1] Davis and Norman [87] and Shreve and Soner [10] discussed the CRRA for the preferences

[1] Throughout the thesis, the Constant Relative Risk Aversion (CRRA) utility function will be used as pensioner preferences.

in the problem of consumption and investment in the presence of transaction costs. The homogeneity of CRRA allows a reduction of the dimension of the state space from two (one state w for wealth and one state A for the stochastic annuity rate) to one (for the ratio $z = \frac{w}{A}$ of wealth to income). Therefore, the HJB equation becomes a one-dimensional second-order ordinary differential equation, although it still can be degenerate.

The random time of an individual's death is given by τ. We assume that an individual seeks to maximize the expected utility of discounted consumption over admissible $\{y_s, c_s, A_s\}$ and discount his/her utility of consumption at the discount rate α. Therefore, following Ewald and Zhang [81], the maximization problem for an individual is obtain as follows, [2]

$$\begin{aligned} V(w, A, t) &= \max_{\{c_s, y_s, A_s\}} \mathbf{E_t}\left[\int_t^{\tau} e^{-\alpha(s-t)} U(c_s) ds\right] \\ &= \max_{\{c_s, y_s, A_s\}} \mathbf{E_t}\left[\int_t^{\infty} e^{-(\alpha+\lambda)(s-t)} U(c_s) ds\right]. \end{aligned} \tag{3.2}$$

where α is an individual discount rate and $(\alpha + \lambda)$ is the mortality-adjusted for the discount rate. Zhang [88] used the discount rate in the model of utility maximization with respect to consumption and labor supply but did not adjust for mortality. Besides that, Chang and Rong [59] and Chang and Chang [89] studied the maximization utility of consumption and also did not adjust for mortality. Recently, Ewald and Zhang [81] maximized the utility of consumption and labor supply under the assumption of the stochastic force of mortality. The discount rate used in the concept of the time the value of money-determining the present value of the future cash flows in the discounted cash flow analysis. It is more interesting for the investor's perspective. Note that U is a strictly increasing, concave function of consumption and $\mathbf{E_t}$ is the expectation conditional on $W_{t-} = w$ and $A_{t-} = A$. [3]

[2] Note that $\mathbf{E_t}(\mathbf{1}_{\{s<\tau\}}) = \mathbb{P}(\tau > s)$ and therefore $\mathbf{E_t}\Big(\int_t^{\tau} e^{-\alpha(s-t)} U(c_s) ds\Big) = \mathbf{E_t}\Big(\int_t^{\infty} e^{-\alpha(s-t)} U(c_s).\mathbf{1}_{\{s<\tau\}} ds\Big)$.

[3] This thesis will follow the same maximization problem as in (3.2), where most of the literature, they considered $(r + \lambda)$, whereas this thesis considers on $(\alpha + \lambda\ )$(mortality-adjusted of the discount rate).

Since the mortality discounting, the individual discounts future consumption at the riskless rate r, where it will increase the effective discount rate and it is separately corporate. The value function V is jointly concave in w, and A, as proved by Wang and Young [38]. By substituting the utility function in (3.1) into (3.2), the value function is

$$V(w,A,t) = \max_{\{c_s,y_s,A_s\}} \mathbf{E}\left[\int_t^{\infty} e^{-(\alpha+\lambda)(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds \Big| W_{t-}=w, A_{t-}=A\right]. \tag{3.3}$$

3.3 Solution of the model

Next, we derived the HJB equation based on the value function in (3.3). The derivation of the HJB equation uses the derivation discussed in Chang [82] as our reference. In this case, we considered the annuity income and the price of life annuity at time t.

3.3.1 The optimization

First, we started with the proposition of the HJB equation for our first model. We referred to the derivation of the general problem of the HJB equation, as discussed in Chapter 2.

Proposition 3.3.1. *The value function V is a constrained viscosity solution of the Hamilton Jacobi Bellman (HJB) equation*

$$\big(\alpha+\lambda\big)V = V_t + \max_{\{c,y,A\}}\left[\big\{rw+y\big(\mu_1-r\big)-c+A\big\}V_w+\frac{1}{2}\big(\sigma_1 y\big)^2 V_{ww}+\frac{c^{1-\gamma}}{1-\gamma}\right]. \tag{3.4}$$

We define, for any $(y,c)\in\mathbb{R}_+\times\mathbb{R}_+$, the functional operator $\mathcal{L}^{y,c}$ through its action on a test function $V\in\mathcal{C}^{2,1}(\mathbb{R}_+\times\mathbb{R}_+)$

$$\mathcal{L}^{y,c}V = -\big(\alpha+\lambda\big)V+V_t+\max_{\{c,y,A\}}\left[\big\{rw+y\big(\mu_1-r\big)-c+A\big\}V_w+\frac{1}{2}\big(\sigma_1 y\big)^2 V_{ww}+\frac{c^{1-\gamma}}{1-\gamma}\right]. \tag{3.5}$$

Theorem 3.3.1. *Suppose the function $v = v(w, A, t) \in \mathcal{C}^{2,1,1}(\mathbb{R}_+ \times \mathbb{R}_+ \times [t, \infty])$ is non-decreasing and concave with respect to w and non-decreasing with respect to A. Moreover, suppose v satisfies the following conditions on $\mathbb{R}_+ \times \mathbb{R}_+ \times [t, \infty]$:*

(i) $\mathcal{L}^{y,c}v \leqslant 0$ *for* $(y, c) \in \mathbb{R}_+ \times \mathbb{R}_+$.

(ii) $av_w - v_A \geqslant 0$.

Then, $v \geqslant V$ *on* $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. The function v satisfies the conditions of the theorem. We proved $v \geq V$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times [t, \infty]$ in two steps. First, we showed the theorem with two additional assumptions:

(i) v is bounded from below; that is $v \geq \underline{V} > -\infty$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times [t, \infty]$.

(ii) $v_w(0, A, t) < +\infty$ for all $A \geq 0$.

Next, we removed these assumptions and showed that the conclusion still holds.

Let $\tau_n^a \triangleq min\{t_1 \geq 0 : \int_0^{t_1} y_{t_1}^2 dt_1 \geq n\}$ and $\tau_n^b \triangleq min\{t_1 \geq 0 : A_{t_1} \geq n\}$. Define $\tau_n = n \wedge \tau_n^a \wedge \tau_n^b$, which is a stopping time with respect to the filtration $\mathcal{F}$; using the Ito's formula for semi-martingales, for any admissible strategy $\{y_t, c_t, A_t\}$,

$$\begin{aligned} e^{-(\alpha+\lambda)\tau_n} v(W_{\tau_n}, A_{\tau_n}, \tau_n) =& v(w, A, t) + \int_0^{\tau_n} e^{-(\alpha+\lambda)t} v_w(W_t, A_t, t)\sigma_1 y_t dB_t \\ &+ \int_0^{\tau_n} e^{-(\alpha+\lambda)t} \left[\mathcal{L}^{y_t, c_t} v(W_t, A_t, t) - \frac{c_t^{1-\gamma}}{1-\gamma} \right] dt \\ &+ \int_0^{\tau_n} e^{-(\alpha+\lambda)t} \big[v_A(W_t, A_t, t) - a v_w(W_t, A_t, t) \big] dA_t^{(c)} \\ &+ \sum_{0 \leq t \leq \tau_n} e^{-(\alpha+\lambda)t} \big[v(W_t, A_t, t) - v(W_{t-}, A_{t-}, t-) \big]. \end{aligned} \tag{3.6}$$

Here $A_t^{(c)}$ is the continuous part of A_t, that is $A_t^{(c)} = A_t - \sum_{0 \leq t_1 \leq t}(A_{t_1} - A_{t_1 -})$. Since v is non-decreasing and concave in w, $v_w^2(w, A, t) \leq v_w^2(0, A, t)$ for $w \leq 0$. Therefore,

$$\mathbf{E}^{w,A}\left[\int_0^{\tau_n} e^{-2(\alpha+\lambda)t} v_w^2(W_t, A_t, t)\sigma_1{}^2 y_t^2 dt\right] < \infty,$$

which implies

$$\mathbf{E}^{w,A}\left[\int_0^{\tau_n} e^{-(\alpha+\lambda)t} v_w(W_t, A_t, t)\sigma_1 y_t dB_t\right] = 0. \tag{3.7}$$

Here, $\mathbf{E}^{w,A}$ denotes conditional expectations given $W_{0-} = w$ and $A_{0-} = A$.

By taking expectations of (3.6) and using (3.7), the conditions in the statement of the theorem, and the additional assumptions at the beginning of the proof, we obtain

$$\begin{aligned}\mathbf{E}^{w,A}\left[e^{-(\alpha+\lambda)\tau_n}\underline{V}\right] &\leq \mathbf{E}^{w,A}\left[e^{-(\alpha+\lambda)\tau_n} v(W_{\tau_n}, A_{\tau_n}, \tau_n)\right] \\ &\leq v(w, A, t) - \mathbf{E}^{w,A}\left[\int_0^{\tau_n} e^{-(\alpha+\lambda)t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right].\end{aligned} \tag{3.8}$$

When deriving (3.8), we also used the fact that

$$\sum_{0\leq t\leq\tau_n} e^{-(\alpha+\lambda)t}\left[v(W_t, A_t, t) - v(W_{t-}, A_{t-}, t-)\right] \leq 0,$$

since condition *(ii)* in the Theorem 3.3.1 implies that v is non-increasing in the directions of jumps.

Since $\tau_n \nearrow \infty$ as $n \to \infty$, applying the monotonic convergence theorem to (3.8) yields

$$v(w, A, t) \geq \mathbf{E}^{w,A}\left[\int_0^{\infty} e^{-(\alpha+\lambda)t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right].$$

This implies that

$$
\begin{aligned}
v(w,A,t) &\geq \max_{\{y_t,c_t,A_t\}} \mathbf{E}^{w,A}\left[\int_0^\infty e^{-(\alpha+\lambda)t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \\
&= V(w,A,t),
\end{aligned}
$$

where we use the representation of V from (3.3).

Now, we showed that the conclusion holds when v is not bounded from below or when $v_w(0,A,t)$ is not finite. For a sequence $\psi_n \searrow 0$, define $v^{\psi_n}(w,A,t) \triangleq v(w+\psi_n, A+\psi_n, \psi_n)$. The function v^{ψ_n} is non-decreasing, twice-differentiable, and concave with respect to w and non-decreasing and differentiable with respect to A. Note that on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,\infty]$, $v^{\psi_n}(w,A,t)$ is bounded from below by $v(\psi_n,\psi_n,\psi_n)$ and that $v_w^{\psi_n}(0,A,t) = v_w(\psi_n, A+\psi_n, \psi_n) < +\infty$. Since $v_w^{\psi_n}(w,A,t) = v_w(w+\psi_n, A+\psi_n, \psi_n)$ and $v_A^{\psi_n}(w,A,t) = v_A(w+\psi_n, A+\psi_n, \psi_n)$, we have

$$
\begin{aligned}
0 \geq \mathcal{L}^{y,c}v(w+\psi_n, A+\psi_n, \psi_n) =& -(\alpha+\lambda)v^{\psi_n}(w,A,t) \\
&+ \Big[r(w+\psi_n) + (A+\psi_n)\Big]v_w^{\psi_n}(w,A,t) \\
&+ \left[(\mu_1 - r)yv_w^{\psi_n}(w,A,t) + \frac{1}{2}{\sigma_1}^2 y^2 v_{ww}^{\psi_n}(w,A,t)\right] \\
&+ \left[\frac{c^{1-\gamma}}{1-\gamma} - cv_w^{\psi_n}(w,A,t)\right] \\
=& \mathcal{L}^{y,c}v^{\psi_n}(w,A,t) + (\alpha+1)\psi_n v_w^{\psi_n}(w,A,t).
\end{aligned}
$$

Because $v_w^{\psi_n}(w,A,t) \geq 0$ and $\psi_n > 0$, it follows that $\mathcal{L}^{y,c}v^{\psi_n} \leq 0$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,\infty]$; that is, v^{ψ_n} satisfies condition (i) of this theorem.

We also have

$$
av_w^{\psi_n}(w,A,t) - v_A^{\psi_n}(w,A,t) = av_w(w+\psi_n, A+\psi_n, \psi_n) - v_A(w+\psi_n, A+\psi_n, \psi_n) \geq 0.
$$

Thus, v^{ψ_n} satisfies condition (ii) of this theorem. It follows that $v^{\psi_n} \geq V$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,\infty]$ for all n. Since $v(w,A,t)$ is continuous in both w and A, we conclude that $v(w,A,t) = \lim_{n\to\infty} v^{\psi_n}(w,A,t) \geq V(w,A,t)$. □

Proving Theorem 3.3.1 uses Wang and Young [38] as the guideline.

Next, it is optimal for the pensioner not to buy/purchase any annuities at the point (w, A, t). Following Ito's lemma, the value function that satisfies this situation is as in (3.4). Next, we assumed that the pensioner decides to move instantly from (w, A, t) to $(w - a\Delta A, A + \Delta A, t)$ as discussed by Milevsky and Young [7]. Then, we assumed that at the point (w, A, t) it is optimal to buy/purchase an annuity. Then, optimality implies

$$V(w, A, t) = V(w - a\Delta A, A + \Delta A, t). \tag{3.9}$$

By taking derivative for (3.9) for w and A, we have Milevsky et al. [84]

$$V_A(w, A, t) = aV_w(w, A, t). \tag{3.10}$$

By combining the HJB (3.4) in Proposition 3.3.1 and the boundary condition (3.10), the boundary value problem is defined as

$$\begin{cases} (\alpha + \lambda)V & = V_t + (rw + A)V_w + \max_y \left[\frac{1}{2}{\sigma_1}^2 y^2 V_{ww} + y(\mu_1 - r)V_w\right] \\ & \quad + \max_{c \geq 0}\left(- cV_w + \dfrac{c^{1-\gamma}}{1-\gamma}\right), 0 \leq \dfrac{w}{A} \leq \bar{z}; \\ V_A & = aV_w. \end{cases} \tag{3.11}$$

From (3.11), the first equation refers to the facts that it is optimal not to buy/purchase any annuities, meaning the pensioner continues investing, and A does not change, where it arises from the continuation region and $\bar{z} = \frac{w}{A}$. While the second equation comes from the boundary of the continuation region, it is optimal to buy/purchase more annuities.

3.3.2 The homogeneity of the value function

In this subsection, we used the homogeneity property of CRRA for dimension reduction to simplify the problem by specializing in transforming the HJB equation in (3.11) into the lower state variable, so that it will be easier to solve. The optimal strategy depends on the two-state variables, which are wealth w and existing annuity income A with respect to time t. The value function V is homogeneous of degree $(1-\gamma)$ in space (w, A, t). Taking advantage of the homogeneity of CRRA utility, we simplify our problem to a two-dimensional equivalent problem, where value function solves a non-linear differential equation. Following an idea from Fleming and Pang [90], we proved the following lemma.

Lemma 3.1. *$V(w, A, t)$ is homogeneous in w and A with an order $(1-\gamma)$.*

Proof. According to (2.5), for any $\kappa > 0$, we have

$$\begin{aligned} d\kappa W_t^{y,c} &= \kappa\big[rW_{t-}^{y,c} + y_t(\mu_1 - r) - c_t + A_{t-}\big]dt + \kappa\sigma_1 y_t dB_t - a_t d\kappa A_t \\ \kappa W_{0-}^{y,c} &= \kappa w \end{aligned}$$

Therefore,

$$V(kw, \kappa A, t) = \max_{y.,c.,A.} \mathbf{E}^{w,A} \int_0^\infty e^{-(\alpha+\lambda)t}\kappa^{1-\gamma}\frac{c_t^{1-\gamma}}{1-\gamma}dt = \kappa^{1-\gamma}V(w, A, t)$$

Thus, we have

$$V(w, A, t) = A^{1-\gamma}V(\frac{w}{A}, 1, t) = A^{1-\gamma}V(z, 1, t)$$

where $z = \frac{w}{A}$. That is, $V(w, A, t)$ is homogeneous in w and A.

□

From Lemma 3.1, we have

$$V(w, A, t) = A^{1-\gamma}\widetilde{V}(z, t) \tag{3.12}$$

also, for the utility function (3.1), it turns out that the value function V is homogeneous $(1-\gamma)$ with respect to wealth w and annuity income A. Defining $\widetilde{V}$ by $\widetilde{V}(z, t) = V(z, 1, t)$ with $V(w, A, t) = A^{1-\gamma}\widetilde{V}(\frac{w}{A}, t)$ and differentiate with respect to w , A and t,

$$\begin{aligned}
V_t &= A^{1-\gamma}\widetilde{V}_t, \\
V_w &= A^{-\gamma}\widetilde{V}_z, \\
V_{ww} &= A^{-\gamma-1}\widetilde{V_{zz}}, \\
V_A &= -A^{-\gamma}\frac{w}{A}\widetilde{V}_z + (1-\gamma)A^{-\gamma}\widetilde{V} \\
&= -A^{-\gamma}z\widetilde{V}_z + (1-\gamma)A^{-\gamma}\widetilde{V}.
\end{aligned} \tag{3.13}$$

Then, substituting (3.13) into the second equation of (3.11), we have

$$\big(z+a\big)\widetilde{V}_z - \big(1-\gamma\big)\widetilde{V} = 0. \tag{3.14}$$

Then, substituting (3.13) into the HJB equation (3.11),

$$\begin{aligned}
(\alpha+\lambda)\Big[A^{1-\gamma}\widetilde{V}\Big] =& A^{1-\gamma}\widetilde{V}_t + \max_{A>0}\Big[(rw+A)A^{-\gamma}\widetilde{V}_z\Big] \\
&+ \max_{y}\left[\frac{1}{2}\sigma_1^2 y^2\Big[A^{-\gamma-1}\widetilde{V_{zz}}\Big] + y(\mu_1 - r)\Big[A^{-\gamma}\widetilde{V}_z\Big]\right] \\
&+ \max_{c\geq 0}\left[-c\Big[A^{-\gamma}\widetilde{V_z}\Big] + \frac{c^{1-\gamma}}{1-\gamma}\right]
\end{aligned}$$

Next, we obtain the new boundary value problem due to homogeneity property,

$$\begin{cases}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z + \max\limits_{\bar{y}}\left[\frac{1}{2}{\sigma_1}^2\bar{y}^2\widetilde{V}_{zz} + \bar{y}(\mu_1 - r)\widetilde{V}_z\right] \\
\qquad\qquad + \max\limits_{\bar{c}\geq 0}\left(-c\widetilde{V}_z + \dfrac{\bar{c}^{1-\gamma}}{1-\gamma}\right), 0 \leq \dfrac{w}{A} \leq \bar{z} \\
\big(z+a\big)\widetilde{V}_z - \big(1-\gamma\big)\widetilde{V} = 0.
\end{cases} \tag{3.15}$$

where $\bar{c} = \frac{c}{A}$ and $\bar{y} = \frac{y}{A}$. Davis and Norman [87], Shreve and Soner [10], and Milevsky and Young [7] used the same transformation, while Milevsky et al. [84] used a slightly different transformation where they reduced the number of variables by defining the excess consumption that the individual requires.

3.3.3 The change variable technique

The change variable technique or is also known as a trial solution method. This method will turn the Bellman equation into a differential equation in the state variable alone since all control variables can be expressed as functions of state variables from the first-order conditions (FOCs). Therefore, we can solve the value function through this differential equation, at least in principle. The FOCs of HJB equation (3.15) for the optimal investment-consumption is defined as

a) Optimal investment strategy

$$\begin{aligned} \bar{y}^* {\sigma_1}^2 \widetilde{V}_{zz} + (\mu_1 - r)\widetilde{V}_z = 0 \\ \bar{y}^*_t = -\frac{(\mu_1 - r)}{{\sigma_1}^2}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}}. \end{aligned} \tag{3.16}$$

b) Optimal consumption

$$\begin{aligned} -\widetilde{V}_z + (\bar{c}^*)^{-\gamma} = 0 \\ \bar{c}^*_t = (\widetilde{V}_z)^{-\frac{1}{\gamma}}. \end{aligned} \tag{3.17}$$

Then, simplify it (3.15),

$$\begin{cases} (\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z - \frac{1}{2}\theta_1^2 \frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}} + \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}}, 0 \le z \le \bar{z} \\ (z+a)\widetilde{V}_z - (1-\gamma)\widetilde{V} = 0. \end{cases} \tag{3.18}$$

where $\theta_1 = \frac{\mu_1 - r}{\sigma_1}$, which also known as the Sharpe ratio. Meanwhile, the pensioner decision is an idea from proposition 6.2 by Milevsky and Young [7], where

Proposition 3.3.2. *For each value of $t \ge 0$, there exists a value of the wealth-to-income ratio $\bar{z}$ that solves*

$$(\bar{z}+a)\widetilde{V}_z - (1-\gamma)\widetilde{V} = 0 \tag{3.19}$$

where $\bar{z}$ satisfies the following arguments;

(i) when $z = \frac{w}{A} \geq \bar{z}$, *it is optimal to buy/purchase annuities*

$$\bar{z} = \frac{w - a\triangle A}{A + \triangle A} \tag{3.20}$$

which implies the optimal annuity income change

$$\triangle A = \frac{w - \bar{z}A}{\bar{z} + a} \tag{3.21}$$

this gives $\widetilde{V}(z,t) = \widetilde{V}(\bar{z},t)$.

(ii) when $z = \frac{w}{A} < \bar{z}$, *it is optimal not to buy/purchase any annuities. Then the value function* $\widetilde{V}(z,t)$,

$$(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z - \frac{1}{2}\theta_1^2\frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}} + \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}} \tag{3.22}$$

It follows that for each time point, the barrier $w = \bar{z}(t)A$ *is a ray emanating from the origin and lying in the first quadrant of* (w, A) *space.*

Then, we assume the solution of (3.22) is in the form of

$$\widetilde{V}(\bar{z},t) = P(t)\frac{\bar{z}^{1-\gamma}}{1-\gamma}. \tag{3.23}$$

The partial derivative of equation (3.23) is given by

$$\widetilde{V}_t = P'(t)\cdot\frac{\bar{z}^{1-\gamma}}{1-\gamma}, \tag{3.24}$$

$$\widetilde{V}_{\bar{z}} = P(t)\cdot\bar{z}^{-\gamma}, \tag{3.25}$$

$$\widetilde{V}_{\bar{z}\bar{z}} = -\gamma\cdot P(t)\cdot(\bar{z})^{-\gamma-1}. \tag{3.26}$$

Then, the optimal investment-consumption (3.16) and (3.17) respectively become,

a) Optimal investment strategy

$$\bar{y}_t^* = \bar{z}\frac{(\mu_1 - r)}{\gamma{\sigma_1}^2}. \tag{3.27}$$

b) Optimal consumption

$$\bar{c}_t^* = \bar{z}P(t)^{-\frac{1}{\gamma}}. \tag{3.28}$$

For (3.27), the optimal investment strategy is a constant dependent on $\bar{z}$, based on the ratio of wealth to annuity income as referred to by Merton [8] where the investment strategy is constant.

Next, we discuss the optimal utility when the individual is in the region to buy/purchase any annuities. Recalling that there exists $\bar{z}$ that solves (3.20), we have

$$\begin{aligned} \widetilde{V}(\bar{z}, t) &= \frac{(\bar{z} + a)}{(1-\gamma)}\widetilde{V}_z \\ &= \frac{(\bar{z} + a)}{(1-\gamma)}P(t) \cdot (\bar{z})^{-\gamma}. \end{aligned} \tag{3.29}$$

Then, substituting the Partial Differential Equation (PDE) in (3.24), (3.25) and (3.26) into HJB equation in (3.22) gives an ODE in $P(t)$

$$P'(t) + \gamma\Big[-pP(t) + \big(P(t)\big)^{\frac{\gamma-1}{\gamma}}\Big] = 0, \tag{3.30}$$

with

$$p = \frac{1-\gamma}{\gamma}\left(\frac{\alpha+\lambda}{1-\gamma} - r - \frac{1}{\bar{z}} - \frac{{\theta_1}^2}{2\gamma}\right). \tag{3.31}$$

The ODE equation in (3.30) is also known as Bernoulli ODE Kingston and Thorp [25] which is the simplest form obtained after solving the HJB equation using the change variable technique in dynamic programming. The verification of the Bernoulli ODE is followed from Kingston and Thorp [25] where it is showed that the solution to (3.30) is consistent to Milevsky and Young [7] as discussed in Appendix A, such that

$$\phi(t, \infty) = \big[P(t)\big]^{\frac{1}{\gamma}}.$$

Then,

$$
\begin{aligned}
P(t) &= \left[\phi(t,\infty)\right]^{\gamma} \\
&= \left[\int_t^{\infty} e^{-\left(\frac{(\alpha+\lambda)-\delta_1(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\bar{z}\gamma}\right)(s-t)} ds\right]^{\gamma}.
\end{aligned} \tag{3.32}
$$

This gave the final solution for optimization in (3.28). Therefore, the optimal consumption is

$$
\bar{c}_t^* = \bar{z}\left[\int_t^{\infty} e^{-\left(\frac{(\alpha+\lambda)-\delta_1(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\bar{z}\gamma}\right)(s-t)} ds\right]^{-1}. \tag{3.33}
$$

Note that the term in square brackets is defined as $\phi(t,\infty) \equiv \left[P(t)\right]^{\frac{1}{\gamma}}$.

The optimal utility depends on the annuitization strategy from Proposition 3.3.2 in the previous subsection. We considered when the individual who is in the region to buy/purchase any annuities and when they can optimize the utility.

From (3.29), we can get the optimal utility for a pensioner who is optimal not to buy/purchase annuities and who is optimal to buy/purchase annuities based on the optimal decision previously discussed . However, since the focus is on the optimal utility when the pensioner is optimal to buy/purchase annuities, by substituting (3.32) into optimization equations, the optimal value function is

$$
\widetilde{V}(\bar{z},t) = \left[\frac{(\bar{z}+a)}{(1-\gamma)}\right]\left[\int_t^{\infty} e^{-\left(\frac{(\alpha+\lambda)-\delta_1(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\bar{z}\gamma}\right)(s-t)} ds\right]^{\gamma} \bar{z}^{(-\gamma)}. \tag{3.34}
$$

3.4 Numerical example

This section provided a numerical example to illustrate the result of the model on optimal investment-consumption and optimal utility. The focus is on the impact of the level of wealth, risk aversion, amount of existing annuity and investment volatility towards the optimal investment-consumption and optimal utility.

Some assumptions on the parameter are needed, so that, our model is solvable and reliable. We use the values of the parameters that have been discussed and used in literature, for example by Gerrard et al. [80], Milevsky and Young [7] and

Milevsky and Posner [91]. Besides that, there are also parameters that we want to control. For example, the current annuity income A, risk aversion level of the pensioner γ, and the investment volatility σ_1.

TABLE 3.1:
Parameter Values for Chapter 3

r	$= 0.03$
μ_1	$= 0.08$
σ_1	$= 0.20$
α	$= 0.05$
τ	$= 16$

The mortality rate for exponential mortality is constant. As reported by the Department of Work and Pensions [3], the pension age for males and females is 65. According to the latest article from the Office for National Statistic [5], from 2015 to 2017, males in the UK had a life expectancy of 79.2 years at birth while females had a life expectancy of 82.9 years. As discussed before, this study does not focus on gender. Therefore, by taking the average life expectancy for both genders, the average life expectancy it is 81 years. Therefore, the expected remaining lifetime is $\frac{1}{\lambda} = 16$ years. This implies, $\lambda = \frac{1}{\tau}$. From the website of moneyfacts.co.uk [92] we see that the current annual annuity income rate for 2018 is 1.4%.

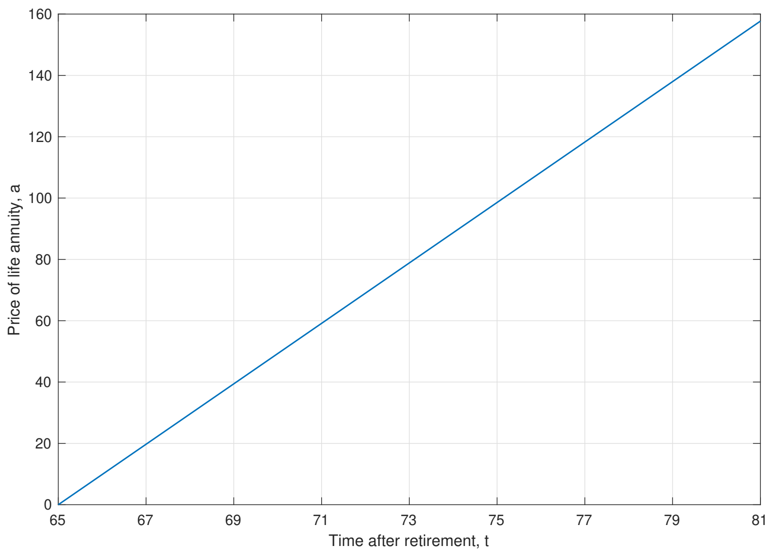

FIGURE 3.1: The Price of a life annuity for the pensioner at age 65 until 81 .

From the function of the price of a life annuity as in (2.8), we get found that the price of a life annuity is increase as the age increase as shown in Figure 3.1, with the mortality rate, $\lambda = \frac{1}{16}$ (which means that future life expectancy is 16 years) and the interest rate in the (annuity) market is $r = 0.03$, the price of £1 per year for life is £10.81. The higher the interest rate, the lower the present value of the life annuity. This explanation also can be referred to by Milevsky et al. [84].

The results are divided into three main sections. The first part of the results is on the utility maximization. Only a few been discussed in the literature since most of the researcher most interested in the optimal annuity time, and they even did not get the closed-form solution for optimal utility. The last two parts are on the optimal investment and consumption, where the result for optimization investment and consumption has widely discussed in the literature by using their own methodology and tested for different variables. This study focuses on the effects of the model parameters on the pensioner's decision, which is different from literature.

This section shows how the optimal investment, consumption and optimal utility will change/react with the changes in current wealth w, risk preferences γ, investment volatility σ_1 and age after retirement t. The result will help the pensioner to decide on annuitization and asset allocation for their pension pot.

3.4.1 Utility maximization

The first part of this section discusses the effects of model parameters on optimal utility. As mentioned before, this study focuses on the decisions made by the pensioner regarding when it is optimal to buy/purchase annuities during retirement. This gives the best decisions to the pensioner on how to optimize the utility.

Figure 3.2 shows that the pattern of the optimal utility with financial wealth just before retirement. Here, the optimal utility for optimal to buy/purchase annuities is concave smoothly. This concave graph shows a diminishing marginal utility of money and a justification for why people may exhibit risk aversion for the potentially significant losses with small probabilities. See [93] for further explanations.

The optimal utility is higher for the pensioner with $\gamma = 5$ (dislike risk) compared to the pensioner with $\gamma = 1.5$ (like risk). This explains why the pensioner with higher risk aversion will utilize more compared to the one with lower risk aversion since they do not want to take any risk on their investment.

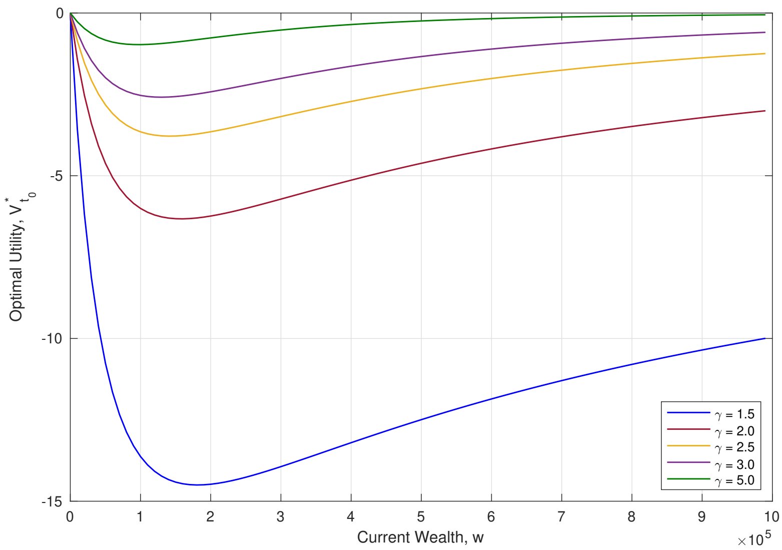

FIGURE 3.2: Optimal utility $V^*_{t_0}$ and pensioner's risk preferences γ in relation to the current wealth w. $t_0 = 65$, $A = £25,000$ and $\Delta A = 1.4$.

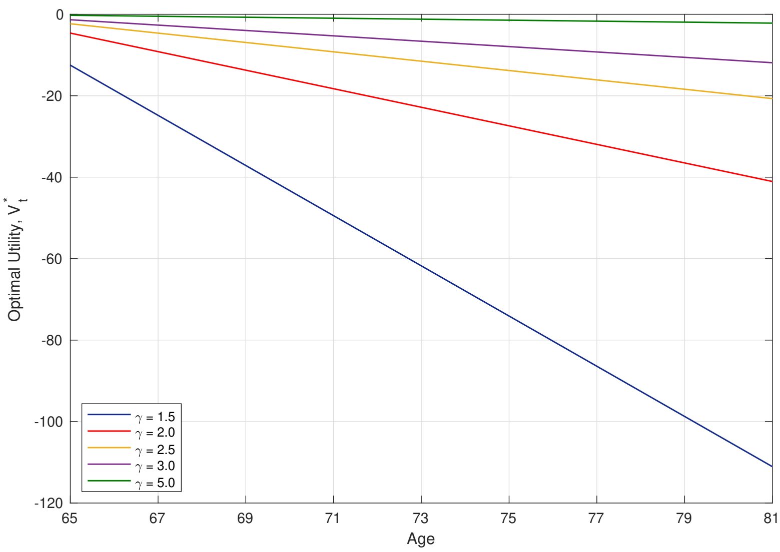

FIGURE 3.3: Optimal utility V^*_t and pensioner's risk preferences γ in relation to pensioner's age. $t_0 = 65$, $t = [0, 16]$, $\sigma_1 = 0.20$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

Figure 3.3 shows the effect of the pensioner's age after retirement on the optimal utility for risk aversion γ. From the figure, we can deduce that optimal utility increases with respect to the expected age after retirement. This shows that the pensioner becomes more satisfied with their life as they spent and allocated their pension wealth accordingly. Gupta and Li [24] analyzed this issue for 30 years and found that, at the early time of the first premium payment an annuity, the optimal utility increases until a certain period, and will later decrease.

3.4.2 Optimal investment strategy

This subsection discusses the relationship between the model parameters and the optimal investment strategy. First, the effects of the current wealth, w and risk aversion, γ on the optimal investment strategy are tested.

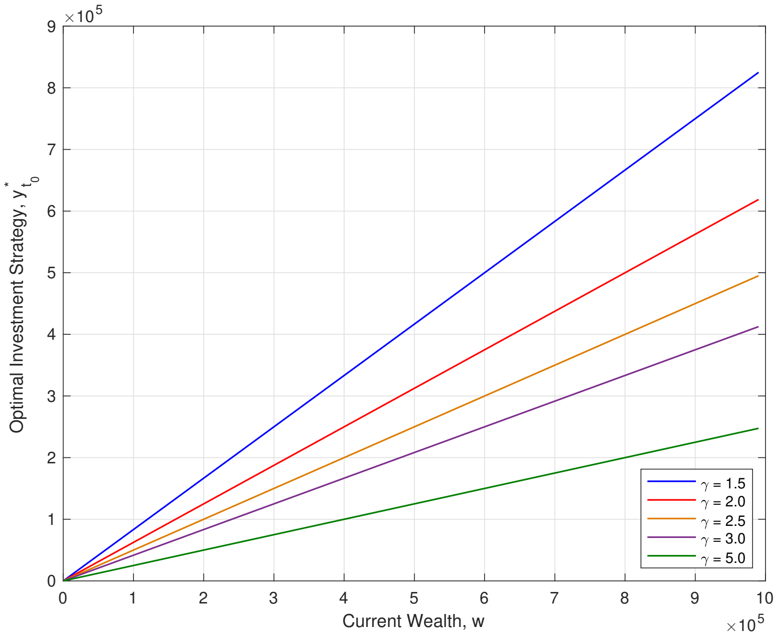

FIGURE 3.4: Optimal investment strategies y_0^* and risk aversion γ in relation to the current wealth for the pensioner at the retirement age. $t_0 = 65$, $A = £25,000$ and $\Delta A = 1.4$.

Figure 3.4 shows the relationship between the current wealth, w and pensioner's risk preferences, γ toward the optimal investment strategy, y_0^* at the age of retirement. As the current wealth increases, the optimal investment increases as well. Steffensen [94] mathematically proved the optimal demand for stocks increases in wealth for a fixed time point. The bigger the value of γ, the more risk-averse the pensioner. The pensioner with the lower risk aversion will invest more in risky assets compared to the one with higher risk aversion. This explains why the financial market is more aggressive among the less risk-averse pensioners.

Next, Figure 3.5 shows that the optimal investment strategy decreases with respect to investment volatility. This proves the higher the investment volatility, the riskier the security. Therefore, the pensioner will reduce their investment in risky assets even those with lower risk aversion.

Figure 3.6 clearly shows that an annuity income does not affect the optimal investment strategy and that it decreases with respect to risk aversion. Pirvu and

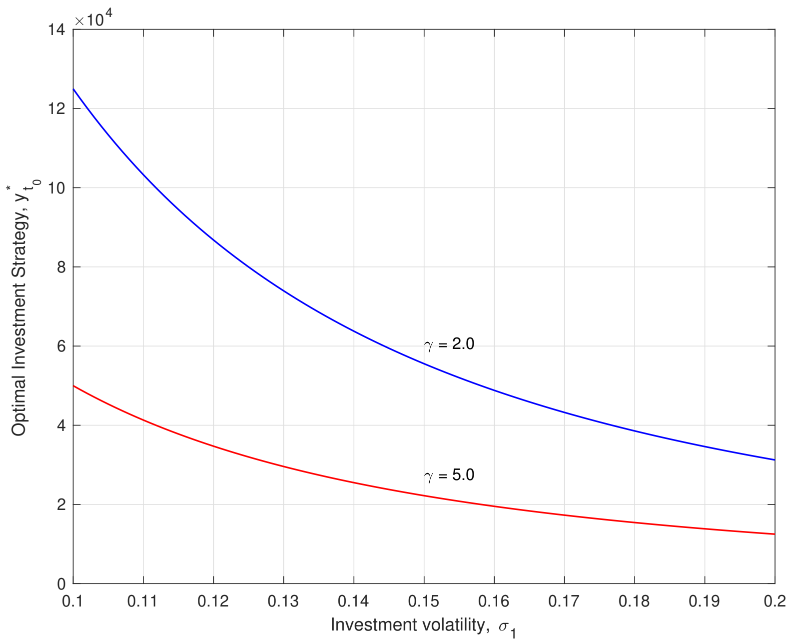

FIGURE 3.5: Optimal investment strategies y_0^* and pensioner's risk preferences γ in relation to investment volatility σ_1 for the pensioner at the retirement age. $t_0 = 65$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

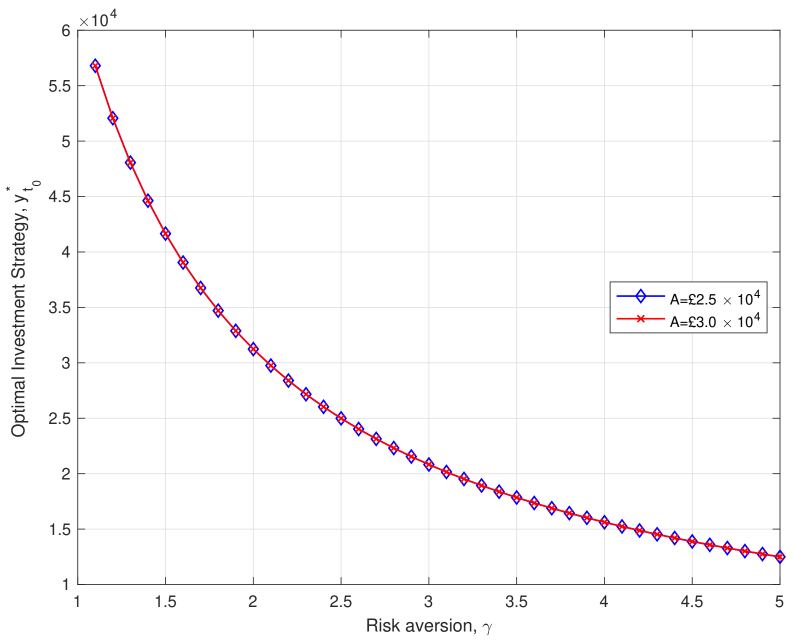

FIGURE 3.6: Optimal investment strategies y_0^* and annuity income level A in relation to risk aversion γ for the pensioner at retirement age. $t_0 = 65$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

Zhang [20] studied the complete market setting by looking the effect of the market price of risk on the model and found that a risk-averse wage earner invests less in the stock than a risk-seeking wage earner.

3.4.3 Optimal consumption

This subsection tests the model parameters on optimal consumption. For optimal consumption, besides testing the model with the current wealth w, investment volatility σ_1 and risk aversion γ, we also test it for the time after retirement and considering a 16-year period after retirement which means the individual will retire at 65 and be expected to live until age 81.

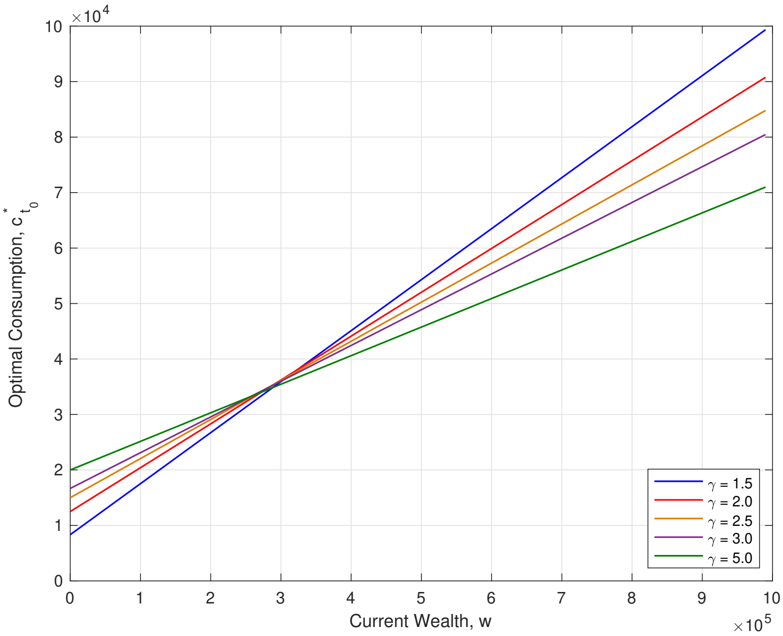

FIGURE 3.7: Optimal consumption $c^*_{t_0}$ and risk aversion γ in relation to the current wealth w at the retirement age. $t_0 = 65$, $A = £25,000$ and $\Delta A = 1.4$.

Figure 3.7 shows the impact of current wealth and risk aversion on optimal consumption at the retirement age (65 years old). From the figure, we can see the optimal consumption decreases with respect to current wealth. The intersection in the figure shows the change in the pensioner's decision on optimal consumption when they have a certain level of wealth in their pension pot to consume after retirement. When the current wealth is lower than the annuity income, the pensioner with higher risk aversion consumes more since they have extra wealth from the annuity income. However, if the current wealth is higher than the annuity income, the risk-averse pensioner will consume less since they do not want to risk not having enough wealth in the future.

Figure 3.8 shows that optimal consumption is almost stable as the investment volatility increases (even if it is slightly decreasing). The pensioner with higher risk aversion has a higher optimal consumption compared to lower risk aversion.

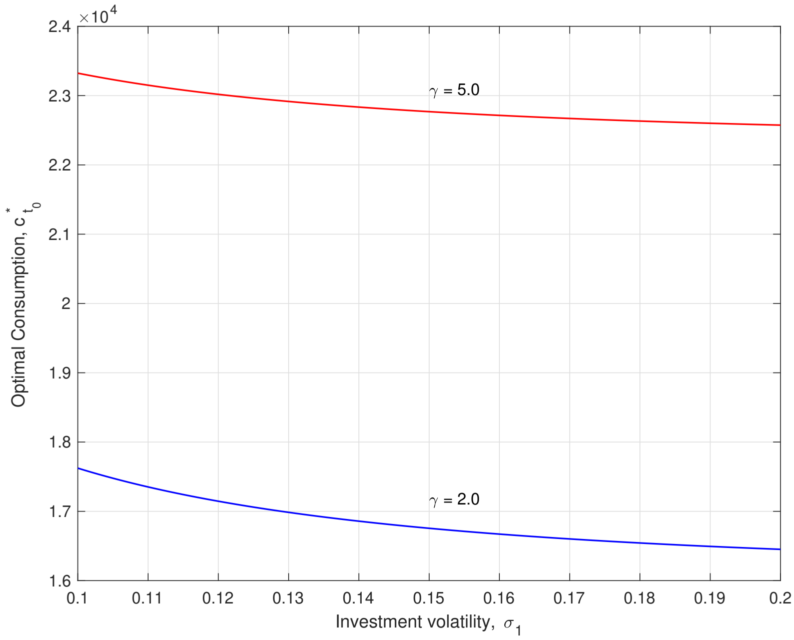

FIGURE 3.8: Optimal consumption $c^*_{t_0}$ and risk aversion γ in relation to investment volatility σ_1 at the retirement age. $t_0 = 65$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

This relates to how risk preferences affect the optimal investment strategy. As previously discussed, the higher risk-averse pensioner invests less in the risky assets, then they can consume more. Steffensen [94] concluded the consumption at time t is connected to investment at inverse relative risk aversion.

Besides that, an annuity income also affects the optimal consumption. The pensioner who decides to receive more annuity income can consume more. Logically, if you have more money/income, you will consume more. Figure 3.9 illustrates this.

Figure 3.10 illustrates the effect of the pensioner's age after retirement on the optimal consumption. The optimal consumption increases as the pensioner's age increases. Gupta and Li [24] got the same result when comparing the consumption levels under the optimal annuitization strategy and the self-annuitization strategy.

3.5 Conclusion

In this chapter, we considered an expected utility maximization problem with life annuities. The pensioner's optimal utility, investment strategies and consumption in a financial market are investigated with a riskless asset, a risky asset and life

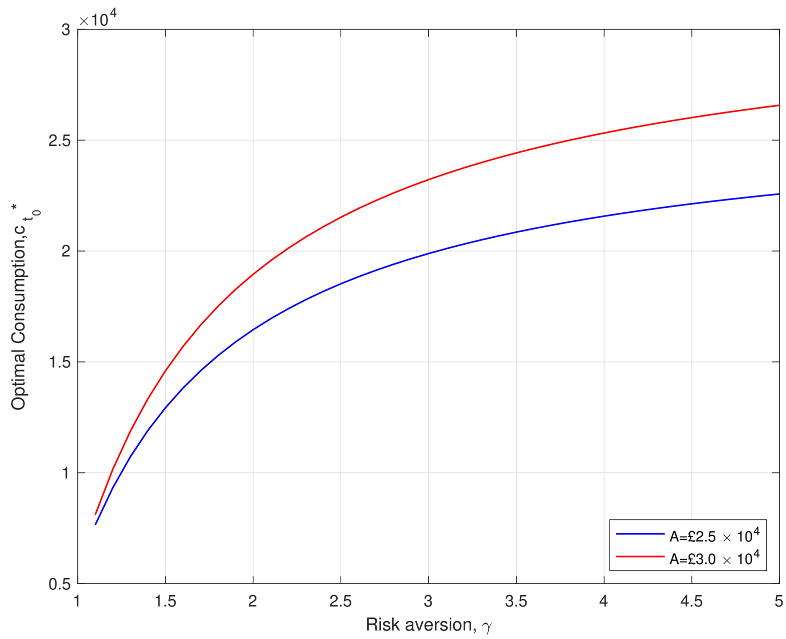

FIGURE 3.9: Optimal consumption $c^*_{t_0}$ and annuity income level A in relation to risk aversion γ at the retirement age. $t_0 = 65$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

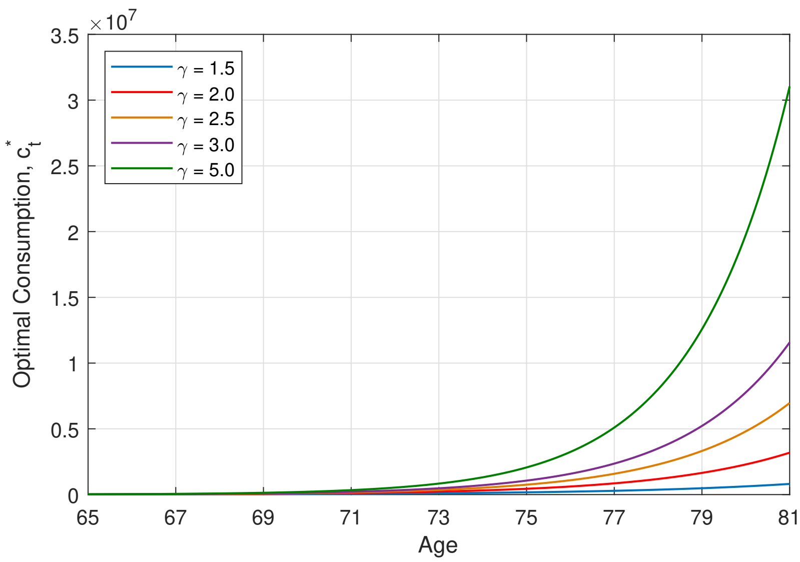

FIGURE 3.10: Optimal consumption c^*_t and pensioner's risk preferences γ in relation to the expected life after retirement. $t_0 = 65$, $t = [0, 16]$, $\sigma_1 = 0.20$, $A = £25,000$, $w_t = £50,000$ and $\Delta A = 1.4$.

annuities. The analysis provided a new explanation for the well-documented reluctance of retirees to purchase life annuities. In this chapter, we managed to solve the HJB equation and gets closed-form solution of portfolio optimization compared to Milevsky [7]. However, we were not able to find the closed-form solution for $\bar{z}$. Therefore, we extended our study by combining the techniques in dynamic programming as discussed in further chapters.

The numerical example showed the impact of the model on the pensioner's decision. Unlike in the existing literature, we analyzed the impact of various model parameters on the optimal investment strategy, consumption and utility.

We found that the individual's optimal utility, investment strategy and consumption depend on the current wealth, risk preferences, investment volatility and expected life after retirement. When the pensioner has a larger amount of current wealth, he/she will consume more. However, the existing annuity income will play a role in altering the optimal consumption rate for all levels of risk aversion. We also noticed that the investment volatility has a negative effect on the pensioner's decision on optimal utility, investment strategy and consumption.

Chapter 4

Optimal investment-consumption and annuitization under the Constant Elasticity of Variance (CEV) model

4.1 Introduction

This chapter extended the stochastic volatility model from Chapter 3. The chapter focused on the CEV model and studied the optimal investment strategy, consumption and annuity income level for after retirement periods when the pensioner receives an income after retirement from the annuity purchased before retirement and also from the investment performance. The CEV model with stochastic volatility is a natural extension of the Geometric Brownian Motion (GBM). Cox and Ross [69] originally proposed the CEV model as an alternative diffusion process for European option pricing. According to Black and Scholes [95], the advantage of the CEV model is that the volatility has a correlation with the risky asset price and the empirical bias exhibited can be explained by comparing it to the GBM model.

Many researchers have studied this model and usually it applied to analyze the options and asset pricing formula. Such examples are Beckers [96], Emanuel and MacBeth [97], Boyle [98], Basu and Samanta [99], Detemple and Tian[100] and Hsu et al. [101]. Lin and Li [102] and Wu and Wu [72] considered the optimal reinsurance-investment problem of an investor. Besides that, Xiao et al. [64] began to apply the CEV model to annuity contracts and solve it using Legendre transform and the duality theory. They were recently followed by Gao in [60] and [65].

In this chapter, we obtain the explicit solution by using the Legendre transform and duality theory to solve the Hamilton Jacobi Bellman (HJB) equation. We derive the explicit solution for the Constant Relative Risk Aversion (CRRA) utility function for after retirement period using Legendre transform, dual theory and variable change technique. This method is widely used to solve the HJB equation from non-linear PDE to linear PDE. Chang and Chang in [66] and [89] used this method to solve the problem on the Vasicek Model. We refer to the study done by Gao [60] who focused on CEV model to describe the stock price dynamics and Gao [65] who used this approach to solve the CEV-extended annuity contracts with a power (CRRA) and exponential (CARA) utility functions. The contribution of this chapter is discussed in Sabri [103].

The direction of this chapter is different from the literature since we considered the annuity income variable in the dynamics wealth process, and we focused the derivation on solving the purchasing boundary before solving the optimal strategy. Finally, in order to investigate the influence of the model parameters on the optimal investment, consumption and annuity income level, we analyzed using numerical examples.

4.2 The CEV model

The CEV spot price model is a one-dimensional diffusion model with the instantaneous volatility specified to be a power function of the underlying spot price, $\sigma = k_c S^{\beta}$.[1] As in the previous chapter, in the financial market, the pensioner can invest his/her wealth in two different assets, which are a riskless asset (bond) and a risky asset (stock) and also invest in life annuities. However, in this chapter,

[1] The alphabet on the subscripts for k is referred to k for CEV model.

the stock price process $S_{1,t}$ has independent increments with constant elasticity of variance.

4.2.1 The CEV process

The CEV model is a one-dimensional diffusion process that solves a stochastic differential equation

$$\begin{aligned} \frac{dS_{1,s}}{S_{1,s}} &= \mu_2 ds + k_c S_{1,s}^{\beta} dB_s \\ S_{1,0} &= S_1 > 0, \end{aligned} \tag{4.1}$$

where $\mu_2 > r$ is an expected instantaneous rate of return of the stock, $\{B_s, s > 0\}$ is the standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F} = \{\mathcal{F}_s\}$ is an augmented filtration generated by the Brownian motion. $k_c S_{1,s}^{\beta}$ is the instantaneous volatility with β is the elasticity parameter which satisfies the general condition $\beta < 0$ as discussed by Gao in [60], [65] and Gu et al. [104].

Remark 4.1. *If the elasticity parameter in (4.1), $\beta = 0$, then the CEV model is reduced to a GBM. If $\beta < 0$, the instantaneous volatility $k_c S_{1,t}^{\beta}$ increases as the stock price decreases and can generate a distribution with a fatter left tail. This special case of the CEV model will be discussed in the next chapter. If $\beta > 0$, the situation is reversed and unrealistic.*

Following Definition 2.1.1 and from (4.1), the CEV pension wealth process when the pensioner chooses to buy annuities and receive annuity income after retirement is defined as

$$\begin{cases} dW_s^{y,c} &= \left[rW_{s-}^{y,c} + y_s(\mu_2 - r) - c_s + A_{s-} \right] ds + y_s k_c S_{1,s}^{\beta} dB_s - a_s dA_s; \\ W_{t-} &= w > 0. \end{cases} \tag{4.2}$$

where the pensioner can purchase an annuity at a price a_s per £ of annuity income at the time, s. Following the ideas from Milevsky and Young [7] the negative sign on the subscripts for wealth and annuities denotes the left-hand limit of those quantities before annuity purchases.

Under the CEV wealth process denoted by (4.2), the investor looks for a strategy ($\{c_s^*, y_s^*, A_s^*\}$) to maximize the utility function:

$$\max_{\{c_s, y_s, A_s\}} \mathbf{E_t}\left[\int_t^\infty e^{-(\alpha+\lambda)s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] \tag{4.3}$$

4.3 Solution of the model

In this section, we derived the general framework to the optimization problem (4.3) by applying the maximum principle, dimension reduction, Legendre transforms and the dual theory.

4.3.1 The HJB equation

The value function is defined

$$V(w, A, S, t) = \max_{\{c_s, y_s, A_s\}} \mathbf{E}\left[\int_t^\infty e^{-(\alpha+\lambda)(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\middle| S_{1,t} = S, W_{t-} = w, A_{t-} = A\right]. \tag{4.4}$$

for $0 \leqslant s < \infty$. Applying Ito's lemma, the value function V satisfies the HJB equation as below.

$$\begin{aligned}(\alpha+\lambda)V = \max_{\{c,y,A\}} \Bigg[& V_t + (rw + (\mu_2 - r)y - c + A)V_w + \frac{1}{2}(yk_cS^\beta)^2 V_{ww} \\ & + yk_c^2 S^{2\beta+1}V_{wS} + \mu_2 S V_S + \frac{1}{2}(k_c S^{\beta+1})^2 V_{SS} + \frac{c^{1-\gamma}}{1-\gamma}\Bigg]\end{aligned} \tag{4.5}$$

Assume that at the point (w, A, S, t), it is optimal to instantaneously buy an annuity . In other words, assume that the pensioner moves instantly from (w, A, S, t) to $(w - a\Delta A, A + \Delta A, S, t)$ for some $\Delta A > 0$. Then, the optimality of the decision implies

$$V(w, A, S, t) = V(w - a\Delta A, A + \Delta A, S, t), \tag{4.6}$$

and by taking derivatives of (4.6) with respect to w and A, it yields

$$aV_w(w, A, S, t) = V_A(w, A, S, t). \tag{4.7}$$

Combining (4.5) and (4.7), the HJB equation associated with V given in (4.4),

$$\begin{cases} (\alpha+\lambda)V & = V_t + (rw + A)V_w + \max\limits_{y} \Big[(\mu_2 - r)yV_w + \frac{1}{2}(yk_cS^{\beta})^2 V_{ww} + yk_c^2 S^{2\beta+1} V_{wS} \Big] \\ & \quad + \max\limits_{c\geq 0} \Big(-cV_w + \frac{c^{1-\gamma}}{1-\gamma} \Big) + \mu_2 S V_S + \frac{1}{2}(k_c S^{\beta+1})^2 V_{SS}, 0 \leq \frac{w}{A} \leq \bar{z}, \\ V_A & = aV_w, \end{cases} \tag{4.8}$$

where V_t, V_w, V_S, V_A, V_{ww}, V_{SS} and V_{wS} denote the partial derivatives of first and second orders with respect to time, wealth, stock price and annuity income level.

4.3.2 Reducing the dimension of the maximization problem

In this subsection, we reduced the dimension of the free boundary value (FBV) problem in (4.8) by transforming the value function $V(w, A, S, t)$ into a function of three variables. Besides that, Milevsky and Robinson [105] studied when the probability of lifetime ruin is shown to depend only on the ratio of current wealth to desired consumption. Recently, Milevsky et al. [84] considered the excess consumption that the individual requires (the net consumption).

Then, define $\widetilde{V}$ by

$$\widetilde{V}(z, S, t) = V(z, 1, S, t) \tag{4.9}$$

So that $V(w, A, S, t) = \widetilde{V}(z, S, t)$, and taking the derivatives with respect to t, w , A and S

$$
\begin{aligned}
&V_t = \widetilde{V}_t, \quad V_w = \tfrac{1}{A}\widetilde{V}, \quad V_{ww} = \tfrac{1}{A^2}\widetilde{V}_{zz}, \quad V_A = -\tfrac{z}{A}\widetilde{V}_z, \quad V_{wS} = \tfrac{1}{A}\widetilde{V}_{zS}, \\
&V_S = \widetilde{V}_S, \quad V_{SS} = \widetilde{V}_{SS}.
\end{aligned}
$$

where $\bar{c} = \frac{c}{A}$ and $\bar{y} = \frac{y}{A}$, the same transformation as the previous chapter. The boundary condition in (4.8) becomes

$$
\begin{aligned}
-\frac{z}{A}\widetilde{V}_z &= a\frac{1}{A}\widetilde{V}_z, \\
(a-z)\widetilde{V}_z &= 0.
\end{aligned} \tag{4.10}
$$

Therefore, (4.8) becomes

$$
\begin{cases}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z + \max\limits_{\bar{y}}\left[(\mu_2 - r)\bar{y}\widetilde{V}_z + \dfrac{1}{2}(\bar{y}k_c S^{\beta})^2\widetilde{V}_{zz} + \bar{y}k_c^2 S^{2\beta+1}\widetilde{V}_{zS}\right] \\
\qquad + \max\limits_{\bar{c}\geq 0}\left[-\bar{c}\widetilde{V}_z + \dfrac{\bar{c}^{1-\gamma}}{1-\gamma}\right] + \mu_2 S\widetilde{V}_S + \dfrac{1}{2}(k_c S^{\beta+1})^2\widetilde{V}_{SS}, \\
(z+a)\widetilde{V}_z - (1-\gamma)\widetilde{V} = 0.
\end{cases} \tag{4.11}
$$

where $\bar{c} = \frac{c}{A}$ and $\bar{y} = \frac{y}{A}$, the same transformation as the previous chapter. We applied the first-order condition (FOC) of the HJB equation (4.11) to define the optimal strategies as below

a) Optimal investment strategy

$$
\begin{aligned}
\widetilde{V}_z(\mu_2 - r) + \bar{y}_t^* k_c^2 S^{2\beta}\widetilde{V}_{zz} + k_c^2 S^{2\beta+1}\widetilde{V}_{zS} &= 0 \\
\bar{y}_t^* k_c^2 S^{2\beta}\widetilde{V}_{zz} &= -(\mu_2 - r)\widetilde{V}_z - k_c^2 S^{2\beta+1}\widetilde{V}_{zS} \\
\bar{y}_t^* &= -\frac{(\mu_2 - r)\widetilde{V}_z + k_c^2 S^{2\beta+1}\widetilde{V}_{zS}}{k_c^2 S^{2\beta}\widetilde{V}_{zz}},
\end{aligned} \tag{4.12}
$$

b) Optimal consumption

$$\begin{aligned}
-\widetilde{V}_z + (1-\gamma)\frac{(\bar{c}_t^*)^{1-\gamma-1}}{1-\gamma} &= 0 \\
-\widetilde{V}_z + (\bar{c}_t^*)^{-\gamma} &= 0 \\
\bar{c}_t^* &= (\widetilde{V}_z)^{-\frac{1}{\gamma}}
\end{aligned} \quad . \tag{4.13}$$

For optimal annuity income level, we defined from the wealth-to-income ratio

$$\begin{aligned}
z_t &= \frac{w}{A_t^*} \\
A_t^* &= \frac{w}{z_t}.
\end{aligned} \tag{4.14}$$

Putting back (4.12) and (4.13) into the HJB equation (4.11), we obtained a PDE for the value function $\widetilde{V}$:

$$\begin{aligned}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z &+ \Bigg\{(\mu_2 - r)\left[-\frac{(\mu_2-r)}{k_c^2 S^{2\beta}}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - S\frac{\widetilde{V}_{zS}}{\widetilde{V}_{zz}}\right]\widetilde{V}_z \\
&+ \frac{1}{2}\left[-\frac{(\mu_2-r)}{k_c^2 S^{2\beta}}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - S\frac{\widetilde{V}_{zS}}{\widetilde{V}_{zz}}\right]^2 k_c^2 S^{2\beta}\widetilde{V}_{zz} \\
&+ \left[-\frac{(\mu_2-r)}{k_c^2 S^{2\beta}}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - S\frac{\widetilde{V}_{zS}}{\widetilde{V}_{zz}}\right] k_c^2 S^{2\beta+1}\widetilde{V}_{zS}\Bigg\} \\
&+ \left[-(\widetilde{V}_z)^{-\frac{1}{\gamma}}\widetilde{V}_z + \frac{\left((\widetilde{V}_z)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma}\right] + \mu_2 S\widetilde{V}_S + \frac{1}{2}(k_c S^{\beta+1})^2\widetilde{V}_{SS} \\
= \widetilde{V}_t + (rz+1)\widetilde{V}_z &+ \Bigg\{-\frac{1}{2}\frac{(\mu_2-r)^2}{k_c^2 S^{2\beta}}\frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}} - S(\mu_2-r)\frac{\widetilde{V}_z\widetilde{V}_{zS}}{\widetilde{V}_{zz}} - \frac{1}{2}k_c^2 S^{2\beta+2}\frac{\widetilde{V}_{zS}^2}{\widetilde{V}_{zz}}\Bigg\} \\
&+ \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}} + \mu_2 S\widetilde{V}_s + \frac{1}{2}(k_c S^{\beta+1})^2\widetilde{V}_{SS}
\end{aligned}$$

Now, we obtained a second-order nonlinear PDE for the value function $V(z,S,t)$ by simplifying the equation above.

$$
\begin{aligned}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z - \frac{1}{2}\frac{\left[(\mu_2 - r)\widetilde{V}_z + k_c^2 S^{2\beta+1}\widetilde{V}_{zS}\right]^2}{k_c^2 S^{2\beta}\widetilde{V}_{zz}} \\
+ \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}} + \mu_2 S\widetilde{V}_S + \frac{1}{2}(k_c S^{\beta+1})^2\widetilde{V}_{SS}
\end{aligned} \tag{4.15}
$$

4.3.3 The Legendre transform and dual theory

One of the approaches in dynamic programming is the Legendre Transform. Legendre transform is a method for explaining the content of the function by using a different independent variable, namely, the derivative of this function with respect to (one of) its argument(s). Following Xiao et al. [64], we gave a definition of the Legendre Transform

Definition 4.3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For $\rho > 0$, we write the Legendre Transform*

$$L(\rho) = \max_z \{f(z) - \rho z\}.$$

The function $L(\rho)$ is called the Legendre dual of the function $f(z)$.

If $f(z)$ is strictly convex, the maximum in the above equation will be attained at just one point, which is denoted by z_0. It arrives at the unique solution to the first-order condition,

$$\frac{df(z)}{dz} - \rho = 0.$$

Therefore,

$$L(\rho) = f(z_0) - \rho z_0.$$

In this study, we used the Legendre transform and duality approaches. The concept of duality means, for the systems which are said to exhibit duality, that there is the way to model the system in two ways. Each is independent of the other and both are valid. Based on the Definition 4.3.1 and the convexity of the value function $\widetilde{V}(z,S,t)$, the Legendre transform is defined as follows

$$\widehat{V}(\rho, S, t) = \max_{z>0} \left\{ \widetilde{V}(z, S, t) - \rho z \middle| 0 < z < \infty \right\}, \tag{4.16}$$

where $\rho > 0$ is the dual variable to z. The value of z where this optimum is attained is denoted by $g(\rho, S, t)$.

$$g(\rho, S, t) = \min_{z>0} \left\{ z \middle| \widetilde{V}(z, S, t) \geqslant \rho z + \widehat{V}(\rho, S, t) \right\} \tag{4.17}$$

According to Definition 4.3.1 and equation (4.16), we have

$$\rho = \widetilde{V}_z \tag{4.18}$$

The relationship between $\widehat{V}(\rho, S, t)$ and $g(\rho, S, t)$ can be determined by

$$g(\rho, S, t) = -\widehat{V}_\rho(\rho, S, t) \tag{4.19}$$

Remark 4.2. *We can choose either one of the functions $g(\rho, S, t)$ and $\widehat{V}(\rho, S, t)$ as the dual function of $\widetilde{V}(z, S, t)$. In this case, we chose $g(\rho, S, t)$ as the dual function for $\widetilde{V}(z, S, t)$.*

From (4.17), we have (4.18) and

$$\widehat{V}(\rho, S, t) = \widetilde{V}(g, S, t) - \rho g, \qquad g(\rho, S, t) = z_t. \tag{4.20}$$

From (4.20) and Definition 4.3.1, we gather z is the critical point. In this chapter, we needed to solve this critical point, or we assumed it as the purchasing boundary. Therefore, the purchasing boundary for this model is $g(\rho, S, t)$.

By differentiating (4.20) with respect to t, S and ρ. Xiao et al. [64] and Gao [60] also used the same transformation. The transformation rules for the derivatives of the value function $\widetilde{V}$ and the dual function $\widehat{V}$ can be given by

$$\begin{aligned} &\widetilde{V} = \widehat{V}, && \widetilde{V}_z = \rho, && \widetilde{V}_S = \widehat{V}, && \widehat{V}_\rho = -\tfrac{1}{\widetilde{V}_z(\rho,S,t)} = -g, \\ &\widetilde{V}_{zz} = -\tfrac{1}{\widehat{V}_{\rho\rho}}, && \widetilde{V}_{zS} = -\tfrac{\widehat{V}_{\rho S}}{\widehat{V}_{\rho\rho}}, && \widetilde{V}_{SS} = \widehat{V}_{SS} - \tfrac{\widehat{V}_{\rho S}^2}{\widehat{V}_{\rho\rho}}. \end{aligned} \tag{4.21}$$

Substituting (4.18), (4.20) and (4.21) into the HJB equation (4.15),

$$
\begin{aligned}
(\alpha+\lambda)[\widehat{V}+\rho g] &= \widehat{V}_t + (rg+1)\rho - \frac{\Big[(\mu_2-r)\rho + k_c^2 S^{2\beta+1}\Big(-\frac{\widehat{V}_{\rho S}}{\widehat{V}_{\rho\rho}}\Big)\Big]^2}{2k_c^2 S^{2\beta}\Big(-\frac{1}{\widehat{V}_{\rho\rho}}\Big)} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}} \\
&\quad + \mu_2 S\widehat{V}_S + \frac{1}{2}k_c^2 S^{2\beta+2}\Big(\widehat{V}_{SS} - \frac{\widehat{V}_{\rho S}^2}{\widehat{V}_{\rho\rho}}\Big) \\
&= \widehat{V}_t + (rg+1)\rho + \frac{1}{2}\frac{(\mu_2-r)^2\rho^2}{k^2 S^{2\beta}}\widehat{V}_{\rho\rho} - (\mu_2-r)\rho S\widehat{V}_{\rho S} + \frac{1}{2}k^2 S^{2\beta+2}\frac{\widehat{V}_{\rho S}^2}{\widehat{V}_{\rho\rho}} \\
&\quad + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}} + \mu_2 S\widehat{V}_S + \frac{1}{2}(k_c S^{\beta+1})^2\widehat{V}_{SS} - \frac{1}{2}k_c^2 S^{2\beta+2}\frac{\widehat{V}_{\rho S}^2}{\widehat{V}_{\rho\rho}} \\
&= \widehat{V}_t + (rg+1)\rho + \frac{1}{2}\frac{(\mu_2-r)^2\rho^2}{k_c^2 S^{2\beta}}\widehat{V}_{\rho\rho} - (\mu_2-r)\rho S\widehat{V}_{\rho S} + \mu_2 S\widehat{V}_S \\
&\quad + \frac{1}{2}k_c^2 S^{2\beta+2}\widehat{V}_{SS} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}
\end{aligned}
$$

Then, we reach at

$$
\begin{aligned}
0 &= \widehat{V}_t - (\alpha+\lambda)\widehat{V} - (\alpha+\lambda)\rho g + \frac{1}{2}\frac{(\mu_2-r)^2\rho^2}{k_c^2 S^{2\beta}}\widehat{V}_{\rho\rho} - (\mu_2-r)\rho S\widehat{V}_{\rho S} + \mu_2 S\widehat{V}_s \\
&\quad \frac{1}{2}k_c^2 S^{2\beta+2}\widehat{V}_{SS} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}
\end{aligned}
\tag{4.22}
$$

Now, differentiating (4.22) with respect to ρ and considering also (4.19)

$$
\begin{array}{lll}
\widehat{V}_{\rho t} = -g_t, & \widehat{V}_{\rho S} = -g_s, & \widehat{V}_{\rho\rho} = -g_\rho, \\
\widehat{V}_{\rho SS} = -g_{SS}, & \widehat{V}_{\rho\rho S} = -g_{\rho S}, & \widehat{V}_{\rho\rho\rho} = -g_{\rho\rho}.
\end{array}
\tag{4.23}
$$

substituting the derivatives in (4.22), we derived

$$
\begin{aligned}
0 = (-g_t) &- (\alpha\lambda)(-g) - [(\alpha+\lambda)\rho g_\rho + (\alpha+\lambda)g] + rg + rg_\rho\rho + 1 \\
&+ \left[\frac{1}{2}\frac{(\mu_2 - r)^2\rho^2}{k_c^2 S^{2\beta}}(-g_{\rho\rho}) + \frac{(\mu_2 - r)^2\rho}{k_c^2 S^{2\beta}}(-g_\rho)\right] \\
&- \Big[(\mu_2 - r)\rho S(-g_{\rho S}) + (\mu_2 - r)S(-g_S)\Big] + \mu_2 S(-g_S) + \frac{1}{2}k_c^2 S^{2\beta+2}(-g_{SS}) \\
&- \frac{1-\gamma}{\gamma}\Big(\frac{\gamma}{1-\gamma}\Big)\rho^{\frac{\gamma-1}{\gamma}-1}
\end{aligned}
$$

We simplified and arrived at

$$
\begin{aligned}
0 = g_t + rSg_S + \frac{1}{2}k_c^2 S^{2\beta+2} g_{SS} - rg - 1 + \left[\frac{(\mu_2 - r)^2}{k_c^2 S^{2\beta}} + (\alpha+\lambda) - r\right]\rho g_\rho \\
\frac{1}{2}\frac{(\mu_2 - r)^2\rho^2}{k_c^2 S^{2\beta}} g_{\rho\rho} - (\mu_2 - r)\rho S g_{\rho S} + \rho^{-\frac{1}{\gamma}}
\end{aligned}
\tag{4.24}
$$

The problem now is to solve (4.24) for g in order to replace and obtain the optimal investment, consumption and annuity income level.

4.3.4 The explicit solution of optimal strategies

In this subsection, we tried to find the explicit solutions of optimal investment, consumption and annuity income level for CRRA utility function via a variable change technique. This technique has also been used by Gao [60] to obtain explicit solutions for before and after retirement when benefits are paid under the form of annuities, which is quite similar to our problem with a different wealth process and objective function. Xiao et al. [64] used the same technique for the logarithmic utility function under the CEV model. Lin and Li [102] used it for the exponential utility function and recently, Chang and Chang [66] employed this technique for the power and logarithm utility function under the Vasicek model.

According to the CRRA utility function described by (3.1) in the previous chapter and considering the behavior of (4.24), we approximated the solution for (4.24) is given by

$$
g(\rho, S, t) = J(S,t)\rho^{-\frac{1}{\gamma}} + b(t). \tag{4.25}
$$

with the boundary conditions given by $b(\tau) = 0$ and $J(S,\tau) = 1$. Taking the derivatives of (4.25) with respect to t, S and ρ, we have

$$\begin{array}{lll} g_t = J_t\rho^{-\frac{1}{\gamma}} + b'(t), & g_S = J_S\rho^{-\frac{1}{\gamma}}, & g_\rho = -\frac{1}{\gamma}J\rho^{-\frac{1}{\gamma}-1}, \\ g_{SS} = J_{SS}\rho^{-\frac{1}{\gamma}}, & g_{\rho\rho} = \frac{1+\gamma}{\gamma^2}J\rho^{-\frac{1}{\gamma}-2}, & g_{\rho S} = -\frac{1}{\gamma}J_S\rho^{-\frac{1}{\gamma}-1} \end{array} \tag{4.26}$$

Putting (4.26) in (4.24) yields

$$\begin{aligned} 0 =& \Big[J_t\rho^{-\frac{1}{\gamma}} + b'(t)\Big] + rS\Big[J_S\rho^{-\frac{1}{\gamma}}\Big] + \frac{1}{2}k_c^2S^{2\beta+2}\Big[J_{SS}\rho^{-\frac{1}{\gamma}}\Big] - r\Big[J\rho^{-\frac{1}{\gamma}} + b(t)\Big] - 1 \\ &+ \left[\frac{(\mu_2 - r)^2}{k_c^2S^{2\beta}} + (\alpha+\lambda) - r\right]\rho\left[-\frac{1}{\gamma}J\rho^{-\frac{1}{\gamma}-1}\right] + \frac{1}{2}\frac{(\mu_2 r)^2\rho^2}{k_c^2S^{2\beta}}\left[\frac{1+\gamma}{\gamma}J\rho^{-\frac{1}{\gamma}-2}\right] \\ &- (\mu_2 - r)\rho S\left[-\frac{1}{\gamma}J_S\rho^{-\frac{1}{\gamma}-1}\right] + \rho^{-\frac{1}{\gamma}}. \end{aligned}$$

Simplifying it, we obtain

$$\begin{aligned} 0 = \rho^{-\frac{1}{\gamma}}&\left[J_t + \frac{\mu_2 - r(1-\gamma)}{\gamma}SJ_S + \frac{1}{2}k_c^2S^{2\beta+2}J_{SS} + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2S^{2\beta}\gamma^2}J\right. \\ &\left. + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}J + 1\right] + \Big[b'(t) - rb(t) - 1\Big]. \end{aligned} \tag{4.27}$$

We split (4.27) into two equations in order to eliminate the dependence of $\rho^{-\frac{1}{\gamma}}$. Therefore, we have

$$b'(t) - rb(t) - 1 = 0, \tag{4.28}$$

and

$$J_t + \frac{[\mu_2 - r(1-\gamma)]S}{\gamma}J_S + \frac{1}{2}k_c^2S^{2\beta+2}J_{SS} + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2S^{2\beta}\gamma^2}J + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}J + 1 = 0. \tag{4.29}$$

Taking into account the boundary condition at $b(\tau) = 0$, the solution for (4.28) is

$$b(t) = -\frac{1}{r}\Big[1 - e^{-r(\tau - t)}\Big]. \tag{4.30}$$

From (4.29), we noticed that the equation is in the form of S, $S^{2\beta+2}$, $S^{-2\beta}$ and also constant. It is difficult to solve the equation directly. First, we introduced the proposition to eliminate the constant and secondly, we used power transformation and the change variable technique to deal with S, $S^{2\beta+2}$ and $S^{-2\beta}$. For the first part, taking an idea from Chang and Chang in [66] and [89], we have

Proposition 4.3.1. *Assume that $J(S,t) = \int_t^\tau \widehat{J}(S,u)du + \widehat{J}(S,t)$ is a solution for (4.29), then $\widehat{J}(S,t)$ satisfies the following equation*

$$\begin{aligned} &\widehat{J}_t + \frac{[\mu_2 - r(1-\gamma)]S}{\gamma}\widehat{J}_S + \frac{1}{2}k_c^2 S^{2\beta+2}\widehat{J}_{SS} + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2 S^{2\beta}\gamma^2}\widehat{J} \\ &+ \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}\widehat{J} = 0, \qquad \widehat{J}(S,\tau) = 1. \end{aligned} \tag{4.31}$$

Proof. Introducing the following differential operator ∇ in any function $J(S,t)$;

$$\begin{aligned} \nabla J(S,t) = &\left[\frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2 S^{2\beta}\gamma^2} + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}\right]J + \left[\frac{[\mu_2 - r(1-\gamma)]S}{\gamma}\right]J_S \\ &+ \frac{1}{2}k_c^2 S^{2\beta+2}J_{SS}, \end{aligned}$$

We can write (4.29) as

$$\frac{\partial J(S,t)}{\partial t} + \nabla J(S,t) + 1 = 0. \tag{4.32}$$

According to $J(S.t) = \int_t^\tau \widehat{J}(S,u)du + \widehat{J}(S.t)$, we got

$$\begin{aligned} \frac{\partial J(S,t)}{\partial t} &= -\widehat{J}(S,t) + \frac{\partial \widehat{J}(S,t)}{\partial t} \\ &= \left(\int_t^\tau \frac{\partial \widehat{J}(S,u)}{\partial u}du - \widehat{J}(S,\tau)\right) + \frac{\partial \widehat{J}(S,t)}{\partial t} \end{aligned}$$

and

$$\nabla J(S,t) = \int_t^\tau \nabla \widehat{J}(S,u)du + \nabla \widehat{J}(S,t).$$

So, (4.32) can be rewritten as

$$\left(\int_t^\tau \frac{\partial \widehat{J}(S,u)}{\partial u}du - \widehat{J}(S,\tau)\right) + \frac{\partial \widehat{J}(S,t)}{\partial t} + \left(-\int_t^\tau \nabla \widehat{J}(S,u)du + \nabla \widehat{J}(S,t)\right) + 1 = 0,$$
$$\left[\int_t^\tau \left(\frac{\partial \widehat{J}(S,u)}{\partial u} + \nabla \widehat{J}(S,u)\right)du - \widehat{J}(S,\tau) + 1\right] + \left(\frac{\partial \widehat{J}(S,t)}{\partial t} + \nabla \widehat{J}(S,t)\right) = 0.$$

Finally, we obtain

$$\frac{\partial \widehat{J}(S,t)}{\partial t} + \nabla \widehat{J}(S,t) = 0, \qquad \widehat{J}(S,\tau) = 1.$$

Therefore, the proof is completed. □

Now, (4.29) becomes

$$J_t + \frac{[\mu_2 - r(1-\gamma)]S}{\gamma}J_S + \frac{1}{2}k_c^2 S^{2\beta+2}J_{SS} + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2 S^{2\beta}\gamma^2}J + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}J = 0. \tag{4.33}$$

Then, we proceed to the second part. Here, we use power transformation and variable change technique. This technique is used in the literature when dealing with the CEV model as Gao did in [60] and [65] when he used this to transform the non-linear equation into a linear in the portfolio selection.

Let

$$J(S,t) = H(l,t), \qquad \text{and} \qquad l = S^{-2\beta} \tag{4.34}$$

Differentiating (4.34) with respect to S and t, we have

$$J_t = H_t, \qquad J_S = -2\beta S^{-2\beta-1}H_l, \qquad J_{SS} = 4\beta^2 S^{-4\beta-2}H_{ll} + 2\beta(2\beta+1)S^{-2\beta-2}H_l.$$

Substituting into (4.33),

$$H_t + \frac{[\mu_2 r(1-\gamma)]}{\gamma}S\Big[-2\beta S^{-2\beta-1}H_l\Big] + \frac{1}{2}k_c^2 S^{2\beta+2}\Big[4\beta^2 S^{-4\beta-2}H_{ll} + 2\beta(2\beta+1)S^{-2\beta-2}H_l\Big] + \frac{(\mu_2-r)^2(1-\gamma)}{2k_c^2 S^{2\beta}\gamma 2}H + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}H = 0$$

Simplifying it, it yields

$$H_t + \beta\left[(2\beta+1)k_c^2 - \frac{2[\mu_2 - r(1-\gamma)]}{\gamma}l\right]H_l + 2\beta^2 k_c^2 l H_{ll} + \frac{(\mu_2-r)^2(1-\gamma)}{2k_c^2\gamma^2}lH + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}H = 0. \tag{4.35}$$

Next, we try to find the solution for (4.35), using the following proposition

Proposition 4.3.2. *Suppose that the solution of (4.35) is of the structure $H(l,t) = D(t)e^{E(t)l}$, with terminal conditions given by $D(\tau) = 1$ and $E(\tau) = 0$, then $D(t)$ and $E(t)$ are given by*

$$D(t) = e^{\left\{\eta_1\beta(2\beta+1)+\frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}(\tau-t)}\left[\frac{\eta_2-\eta_1}{\eta_2-\eta_1 e^{2\beta^2(\eta_1-\eta_2)(\tau-t)}}\right]^{\frac{2\beta+1}{2\beta}} \tag{4.36}$$

$$E(t) = k^{-2}Q(t) \qquad \textit{with} \qquad Q(t) = \frac{\eta_1 - \eta_1 e^{2\beta^2(\eta_1-\eta_2)(\tau-t)}}{1-\frac{\eta_1}{\eta_2}e^{2\beta^2(\eta_1-\eta_2)(\tau-t)}} \tag{4.37}$$

Proof. The solution of (4.35) is

$$H(l,t) = D(t)e^{E(t)l}, \qquad D(\tau) = 1, \qquad E(\tau) = 0 \tag{4.38}$$

Differentiate (4.38) with respect to v and t, we have

$$
\begin{aligned}
H_t &= D(t)E'(t)le^{E(t)l} + D'(t)e^{E(t)l} = HE'(t)l + H\frac{D'(t)}{D(t)}, \\
H_v &= D(t)E(t)e^{E(t)l} = HE(t), \\
H_{vv} &= D(t)E(t)e^{E(t)l} \cdot E(t) = HE^2(t).
\end{aligned}
$$

Substituting into (4.35), we have

$$
\begin{aligned}
&\left[HE'(t)l + D'(t)e^{E(t)l}\right] + \beta\left[(2\beta+1)k^2 - \frac{2[\mu_2 - r(1-\gamma)]}{\gamma}l\right]HE(t) + 2\beta^2k_c^2lHE^2(t) \\
&+ \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2\gamma^2}lH + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}H = 0.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
H\Bigg[&E'(t) + \frac{D'(t)}{D(t)} + \beta(2\beta+1)k_c^2E(t) - \frac{2\beta[\mu_2 - r(1-\gamma)]}{\gamma}lE(t) + 2\beta^2k_c^2lE^2(t) \\
&+ \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2\gamma^2}l + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma}\Bigg] = 0.
\end{aligned}
$$

Then, by eliminating the term H, we have as follows

$$
\begin{aligned}
&\frac{D'(t)}{D(t)} + \beta(2\beta+1)k_c^2E(t) + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma} \\
&\quad + l\left[E'(t) - \frac{2\beta[\mu_2 - r(1-\gamma)]}{\gamma}E(t) + 2\beta^2k_c^2E^2(t) + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2\gamma^2}\right] = 0.
\end{aligned}
\tag{4.39}
$$

Decomposing (4.39) in two conditions in order to eliminate the dependence in l and t,

$$E'(t) - \frac{2\beta[\mu_2 - r(1-\gamma)]}{\gamma}E(t) + 2\beta^2 k_c^2 E^2(t) + \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2\gamma^2} = 0, \qquad E(\tau) = 0, \tag{4.40}$$

$$\frac{D'(t)}{D(t)} + \beta(2\beta+1)k_c^2 E(t) + \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma} = 0, \qquad D(\tau) = 1. \tag{4.41}$$

For simplicity, let

$$h = -2\beta^2, \qquad i = \frac{2\beta[\mu_2 - r(1-\gamma)]}{\gamma}, \qquad j = -\frac{(\mu_2 - r)^2(1-\gamma)}{2\gamma^2}$$

Then (4.40) becomes

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{2\beta[\mu - r(1-\gamma)]}{\gamma}E(t) - 2\beta^2 k_c^2 E^2(t) - \frac{(\mu_2 - r)^2(1-\gamma)}{2k_c^2\gamma^2} \\ &= hk_c^2 E^2(t) + iE(t) + \frac{j}{k_c^2} \qquad \text{with} \qquad E(\tau) = 0. \end{aligned} \tag{4.42}$$

Integrating (4.42) both sides with respect to time t

$$\int \frac{dE(t)}{dt}dt = \int (hk_c^2 E^2(t) + iE(t) + \frac{j}{k_c^2})dt,$$

we have

$$\frac{1}{hk_c^2(m_1 - m_2)} \int \left(\frac{1}{E(t) - m_1} - \frac{1}{E(t) - m_2} \right) dE(t) = t + \mathcal{C} \tag{4.43}$$

where $\mathcal{C}$ is a constant and $m_{1,2}$ are the solutions of the quadratic equation

$$hk^2 m^2 + im + \frac{j}{k_c^2} = 0 \tag{4.44}$$

Now, we solve for $m_{1,2}$ using quadratic factorization

$$m_{1,2} = \frac{-i \pm \sqrt{i^2 - 4(hk_c^2)\left(\frac{j}{k^2}\right)}}{2hk_c^2}$$

Then, we arrived at

$$m_{1,2} = \frac{[\mu_2 - r(1-\gamma)] \pm \sqrt{\gamma\left[\mu_2^2 - r^2(1-\gamma)\right]}}{2\beta k_c^2 \gamma}$$

Therefore, m_1 and m_2 are given as

$$m_1 = \frac{[\mu_2 - r(1-\gamma)] + \sqrt{\gamma\left[\mu_2^2 - r^2(1-\gamma)\right]}}{2\beta k_c^2 \gamma} \tag{4.45}$$

and

$$m_2 = \frac{[\mu_2 - r(1-\gamma)] - \sqrt{\gamma\left[\mu_2^2 - r^2(1-\gamma)\right]}}{2\beta k_c^2 \gamma} \tag{4.46}$$

Notice that equations (4.42) and (4.43) are in the same form as (A.10) and (A.11) from Appendix A.2 in Gao [65] Then, we have the solution for (4.43), which is

$$E(t) = \frac{m_1 - m_1 e^{hk_c^2(m_1 - m_2)(t-\tau)}}{1 - \frac{m_1}{m_2} e^{hk_c^2(m_1 - m_2)(t-\tau)}} \tag{4.47}$$

Next, we defined $\eta_{1,2}$ from (4.45) and (4.46) by considering (4.47). We have,

$$\eta_1 = \frac{[\mu_2 - r(1-\gamma)] + \sqrt{\gamma\left[\mu_2^2 - r^2(1-\gamma)\right]}}{2\beta\gamma} \tag{4.48}$$

and

$$\eta_2 = \frac{[\mu_2 - r(1-\gamma)] - \sqrt{\gamma\left[\mu_2^2 - r^2(1-\gamma)\right]}}{2\beta\gamma} \tag{4.49}$$

Therefore, (4.47) becomes (4.37) as in the proposition.

Next, we solve for $D(t)$. Considering $Q(t)$ in (4.37), (4.41) can be rewritten as

$$\begin{aligned}\frac{D'(t)}{D(t)} &= -\beta(2\beta+1)k_c^2E(t) - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\\ &= -\beta(2\beta+1)Q(t) - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\end{aligned}$$

Therefore, we have

$$\frac{dD(t)}{D(t)} = \Bigg\{ -\beta(2\beta+1)Q(t) - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\Bigg\}dt \tag{4.50}$$

Next, by solving the integration of $Q(t)$ in (4.37), we reach

$$\int Q(t)dt = \eta_1 t + \frac{1}{2\beta^2}\ln(\eta_2 - \eta_1 e^{2\beta^2(\eta_1\eta_2)(\tau-t)}) + \mathcal{C} \tag{4.51}$$

where $\mathcal{C}$ is a constant. Finally, by integrating (4.50) with respect to t we get the solution for $D(t)$ as in the proposition.

□

Before we move to the optimal investment strategy, consumption and annuity income level solutions, we need to find ρ. From the boundary condition in (4.11), we solve for ρ by considering (4.21), (4.23) and (4.25)

$$\rho = \left[\frac{a-b(t)}{J(S,t)-\frac{1}{\gamma}J(S,t)}\right]^{-\gamma} \tag{4.52}$$

Then, we solved the optimal strategies. We used propositions so the solution of the optimal strategies is clearer and easier to define. First, the proposition for optimal investment strategy is defined as below:

Proposition 4.3.3. *The optimal investment strategy in the stock is given by*

$$y_t^* = A\Big[g(\rho,S,t)-b(t)\Big]\frac{\mu_2-r}{(k_cS^\beta)^2\gamma}K(t) \tag{4.53}$$

where

$$g(\rho, S, t) = J(S,t)\rho^{-\frac{1}{\gamma}} + b(t) \quad \textit{with} \quad J(S,t) = H(l,t)$$

$H(l,t)$ is defined in Proposition 4.3.2 . With also

$$b(t) = -\frac{1}{r}\Big[1 - e^{-r(\tau - t)}\Big].$$

and

$$K(t) = \left(1 - \frac{2\beta\gamma Q(t)}{\mu_2 - r}\right) \quad \textit{with} \quad Q(t) = \frac{\eta_1 - \eta_1 e^{2\beta^2(\eta_1 - \eta_2)(\tau - t)}}{1 - \frac{\eta_1}{\eta_2} e^{2\beta^2(\eta_1 - \eta_2)(\tau - t)}}$$

Proof. Recall the optimal investment strategy as previously defined in (4.12). Considering (4.18), (4.20), (4.21), (4.23), (4.25), (4.26), (4.34) and Proposition 4.3.2, we derive

$$\begin{aligned}
\bar{y}_t^* &= -\frac{(\mu_2 - r)\widetilde{V}_z + k_c^2 S^{2\beta+1}\widetilde{V}_{zS}}{k_c^2 S^{2\beta}\widetilde{V}_{zz}} \\
&= -\frac{(\mu_2 - r)\rho + k_c^2 S^{2\beta+1}\Big(- \frac{\widehat{V}_{\rho S}}{\widehat{V}_{\rho\rho}}\Big)}{k_c^2 S^{2\beta}\Big(- \frac{1}{\widehat{V}_{\rho\rho}}\Big)} \\
&= \frac{(\mu_2 - r)\rho\widehat{V}_{\rho\rho} - k_c^2 S^{2\beta+1}\widehat{V}_{\rho S}}{k_c^2 S^{2\beta}} \\
&= \frac{-(\mu_2 - r)\rho g_\rho + k_c^2 S^{2\beta+1} g_S}{k_c^2 S^{2\beta}} \\
&= \frac{\frac{(\mu_2 - r)}{\gamma} J\rho^{-\frac{1}{\gamma}} + k_c^2 S^{2\beta+1} J_S \rho^{-\frac{1}{\gamma}}}{k_c^2 S S^{2\beta}} \\
&= \frac{\frac{(\mu_2 - r)}{\gamma}[g - b(t)]}{k_c^2 S^{2\beta}} + \frac{k_c^2 S^{2\beta+1} J_S \Big[\frac{g - b(t)}{J}\Big]}{k_c^2 S^{2\beta}} \\
&= \frac{(\mu_2 - r)[g - b(t)]}{k_c^2 S^{2\beta}\gamma} - 2\beta\Big[g - b(t)\Big]\frac{H_v}{H}
\end{aligned}$$

Therefore, the optimal investment strategy is given by

$$\bar{y}_t^* = \Big[g - b(t)\Big]\frac{\mu_2 - r}{k_c^2 S^{2\beta\gamma}}\left(1 - \frac{2\beta\gamma Q(t)}{\mu_2 - r}\right)$$

As previously discussed, $\bar{y} = \frac{y}{A}$ and $g = g(\rho, S, t)$, then, we obtained the optimal investment strategy as in the proposition.

$\square$

Next, we define the proposition for optimal consumption.

Proposition 4.3.4. *The optimal consumption is given by*

$$c_t^* = A\frac{g(\rho, S, t) - b(t)}{D(t)e^{E(t)v}} \tag{4.54}$$

where

$$g(\rho, S, t) = J(S, t)\rho^{-\frac{1}{\gamma}} + b(t) \quad \textit{with} \quad J(S, t) = H(v, t)$$

$H(t)$ is defined in Proposition 4.3.2 . With also

$$b(t) = -\frac{1}{r}\Big[1 - e^{-r(\tau - t)}\Big],$$

$$D(t) = e^{\left\{\eta_1\beta(2\beta+1) + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}(\tau - t)}\left[\frac{\eta_2 - \eta_1}{\eta_2 - \eta_1 e^{2\beta^2(\eta_1 - \eta_2)(\tau - t)}}\right]^{\frac{2\beta+1}{2\beta}}$$

and

$$E(t) = k_c^{-2}Q(t) \qquad \textit{with} \qquad Q(t) = \frac{\eta_1 - \eta_1 e^{2\beta^2(\eta_1 - \eta_2)(\tau - t)}}{1 - \frac{\eta_1}{\eta_2}e^{2\beta^2(\eta_1 - \eta_2)(\tau - t)}}$$

Proof. Recall the optimal consumption as defined in (4.13). Considering (4.18), (4.25), (4.34) and Proposition 4.3.2, we derive

$$\begin{aligned}
\bar{c}_t^* &= \left(\widetilde{V}_z\right)^{-\frac{1}{\gamma}} \\
&= \rho^{-\frac{1}{\gamma}} \\
&= \frac{g - b(t)}{H}
\end{aligned}$$

Therefore, the optimal consumption is given by

$$\bar{c}_t^* = \frac{g - b(t)}{D(t)e^{E(t)v}}$$

Note that, $\bar{c} = \frac{c}{A}$ and $g = g(\rho, S, t)$. Finally, the optimal consumption is defined as in the proposition.

□

Finally, we define the proposition for optimal annuity income level.

Proposition 4.3.5. *The optimal annuity income level is given by*

$$A_t^* = \frac{w}{g(\rho, S, t)} \tag{4.55}$$

where

$$g(\rho, S, t) = J(S, t)\rho^{-\frac{1}{\gamma}} + b(t) \quad \textit{with} \quad J(S, t) = H(l, t)$$

$H(t)$ is defined in Proposition 4.3.2.

Proof. Suppose the optimal annuity income level is as in (4.14). As defined before in (4.20), the purchasing boundary z is defined as $z = g(\rho, S, t)$. Therefore, the optimal annuity income level is defined as in proposition. □

4.4 Numerical example

In this section, we analyzed how the optimal strategies change with the parameters of the model, and we compared the investment, consumption and annuity income level strategies with those from literature; specifically, our problem is when there are life annuities.

Throughout this section, the numerical example is presented to illustrate the results. The values of the parameters follow Gao [60] and Lin and Li [102]. In the example, unless stated otherwise, we used the following parameters values:

- $r = 0.03$; the interest rate for a riskless asset.
- $\mu_2 = 0.08$; the expected return of a risky asset.
- $\alpha = 0.05$; the discount rate.
- $t = [0, 16]$; the years after retirement until expected lifetime, where $t = 0$ is the year at retirement age which is 65 and $t = 16$ is when the pensioner is expected to live until 81. See [3, 5]
- $\tau = 16$; the expected lifetime after retirement.
- $\lambda = \frac{1}{\tau}$; the mortality rate based on the expected lifetime after retirement.
- $\sigma_2 = k_c S_{1,t}^{\beta}$; instantaneous volatility.
- $k_c = 16.16$; the constant parameter for the CEV model.
- $\beta = -1$; the constant elasticity parameter, Gao [65] stated $\beta < 0$, so that instantaneous volatility increases as the stock price decreases, and can generate a distribution with a fatter left tail.
- $S = 67$; the risky asset (stock) price.

As we discussed in Section 4.3, the Legendre transform, and the dual theory are used to solve the purchasing boundary $g(\rho, S, t)$. This is different from Chapter 3, where the purchasing boundary cannot be obtained. Since in this chapter we are able to obtain the purchasing boundary, we do not need the annuity income rate as in the previous chapter.

The price of a life annuity is the same as discussed in Section 3.4, where $a_t = 10.8108$. According to Milevsky et al. [84], the greater the interest rate, the lower the present value of the life annuity.

This section is divided into three main subsections. We focus the result on the optimal investment strategy, consumption and annuity income level. The CEV model on the optimal investment strategy and consumption has been widely discussed in the literature. For credits, we believe this study is the first to test the optimal annuity income level.

4.4.1 Optimal investment strategy

From Proposition 4.3.3, we apply the parameters as defined previously to compute the optimal investment strategy. We obtain the figures below to explain our model.

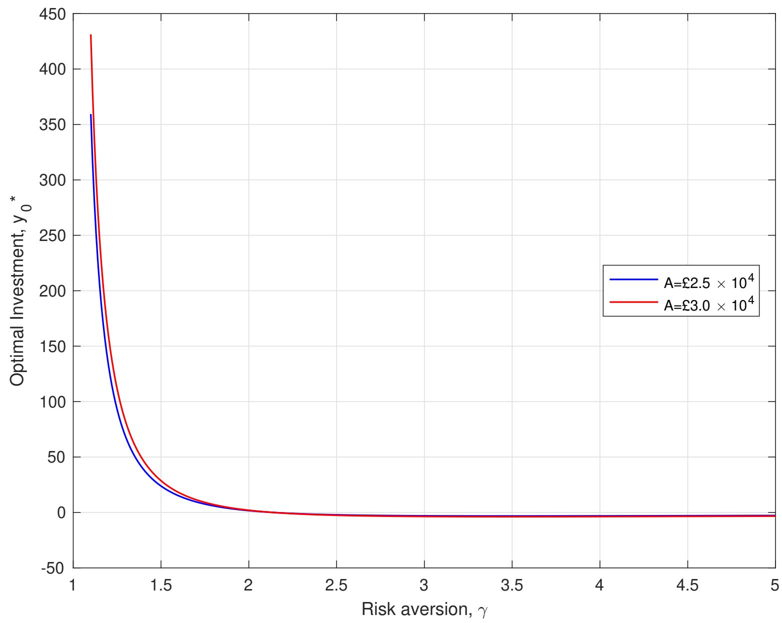

FIGURE 4.1: Sensitivities of risk aversion γ and optimal investment y^* with respect to annuity income level A at retirement age $t_0 = 65$.

Figure 4.1 illustrates the dynamic behavior of the optimal investment strategy y^* at age 65 (retirement age). According to this figure, the optimal investment strategy is decreasing with respect to pensioner's risk preferences γ. This implies the smaller the relative risk aversion, the more aggressive the investor is. Thus, the more he/she wishes to invest in the risky asset. Gao [60] and Pirvu and Zhang [20] got the same result, where the investor with higher risk aversion should invest less when considering the correction factor and mean-reverting returns, respectively.

At the same time, Li and Wu [74] obtained the same results when testing the stochastic volatility model for the one-dimensional Heston model. Then, Zheng et al. [106] investigated a robust optimal portfolio under a Cramer-Lundberg risk model for an ambiguity-averse-insurer and had the same result as this study. Most recently Yuan and Lai [107] studied the optimization with an exponential utility function and found that the larger the risk aversion, the more risk-averse the family and the less amount of wealth is invested in the stock.

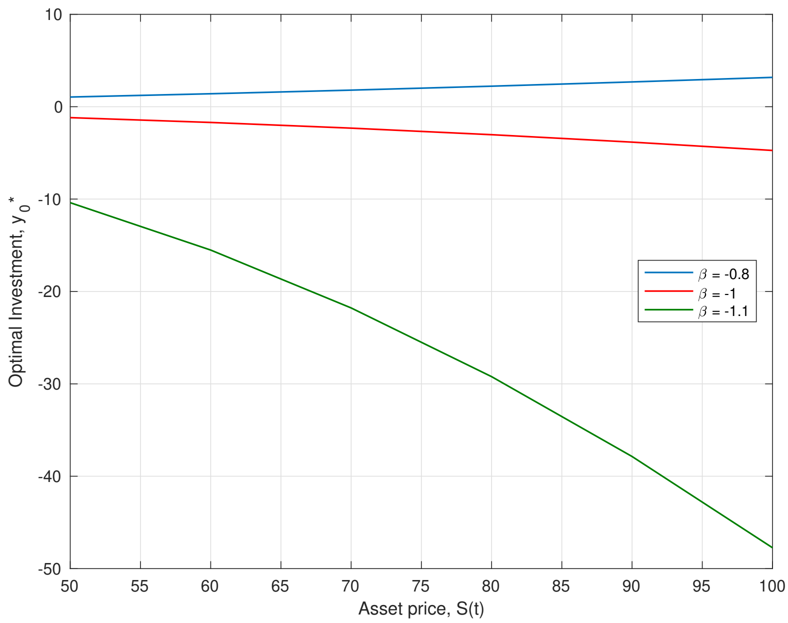

FIGURE 4.2: Sensitivities of asset price $S(t)$ and optimal investment y^* with respect to the elasticity factor of CEV model at retirement age $t_0 = 65$ when $\gamma = 2.5$ and $A = £25,000$.

Figure 4.2 represents the relationship between an asset price with the optimal investment strategy and the elasticity factor. This figure shows that as the asset price increases, the optimal investment strategy decreases. However, when the elasticity factor is approaching 0, the optimal investment increases. Chang et al. [108] showed the relationship when β closer to 0, and concluded that the amount invested in the stock under the CEV model is more than that under the GBM model. Meanwhile, Gu et al. [71] tested the relationship when $\beta > 0$. Both support our result, where the investor needs to reduce the amount invested in the stock as the asset price increases.

Figure 4.3 shows the optimal investment is decreasing in interest rate r as r is the risk-free interest rate. This explains that the larger the value of the interest rate, the higher the expected income of the risk-free asset. Hence, the investor wishes to invest less in the risky asset. Recently, Yuan and Lai [107] studied the household expenditure under the CEV model and concluded that the family reduces the amount invested in the stock in order to avoid the risk from investment. Besides that, Chang et al. [108] also showed the same findings when testing the relationship between the optimal investment strategy and the elasticity factor and the interest rate.

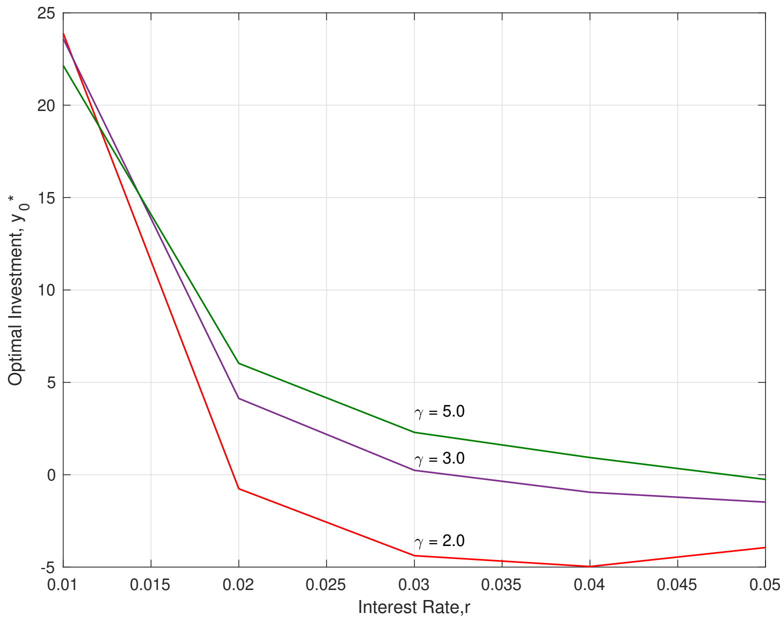

FIGURE 4.3: Sensitivities of interest rate r and optimal investment y^* with respect to risk aversion γ at retirement age $t_0 = 65$ when $\gamma = 2.5$ and $A =$ £$25,000$.

4.4.2 Optimal consumption

Now, we proceed to optimal consumption. We figure out the optimal consumption using Proposition 4.3.4.

Figure 4.4 illustrates the relationship between optimal consumption and annuity income level with risk aversion. The optimal consumption is decreasing with respect to pensioner's risk preferences. According to Gupta and Li [24], the higher the risk aversion preferences, the more stable the consumption is compared to less risk aversion preference for the discrete model. The same relationship between optimal consumption and risk preferences was investigated by Chang and Rong [59] and Zhang [88] for the one-dimensional Brownian motion of the Heston model and the presence of labor income, respectively.

Then, we can conclude that the pensioners who like risk will consume more compared to those pensioner who dislike risk since they believe in their investment performance. While, the pensioner who dislikes risk prefers to consume less, he/she will have enough wealth for a lifetime.

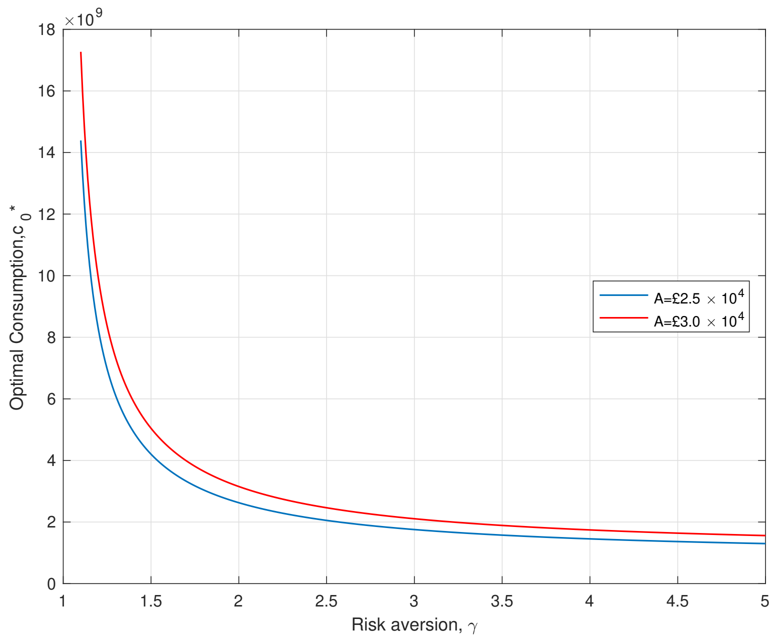

FIGURE 4.4: Sensitivities of risk aversion γ and optimal consumption c^* with respect to annuity income level A at retirement age $t_0 = 65$.

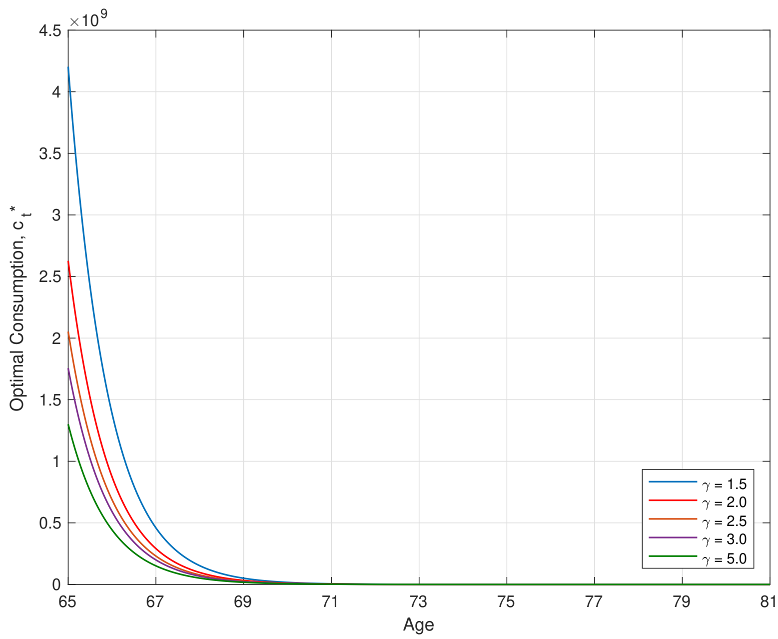

FIGURE 4.5: Sensitivities of post-retirement age and optimal consumption c^* with respect to risk aversion γ when $A = £25,000$.

Next, Figure 4.5 shows consumption for different years as a function of age. We observe that optimal consumption decreases with respect to the years after retirement. We observe the same approach as discussed by Ewald and Zhang [81], where we do not observe on a consumption hump at a given age and consideration being given to the life-cycle compared to the literature, for examples Feigenbaum [109], Kraft et al. [110] and Gourinchas and Parker [111] even though Banks et al. [112] stated that the reduction of consumption could not be explained by the life-cycle model.

Recently, Ewald and Zhang [81] discussed the consumption across all ages under historical changes in mortality. When Andreasson et al. [113] studied optimal consumption with a means-tested public pension at retirement, they proved that the effect on consumption decreases with age since the mortality risk increase as the expected sum future Age Pension decreases.

4.4.3 Optimal annuity income level

Finally, we calculate the optimal annuity income from Proposition 4.3.5. We present the result in the figures below. As we know, life annuities are risky assets as the annuity market value varies from day to day, yet Merton [114] stated that the income from an annuity is secure throughout the retiree's life.

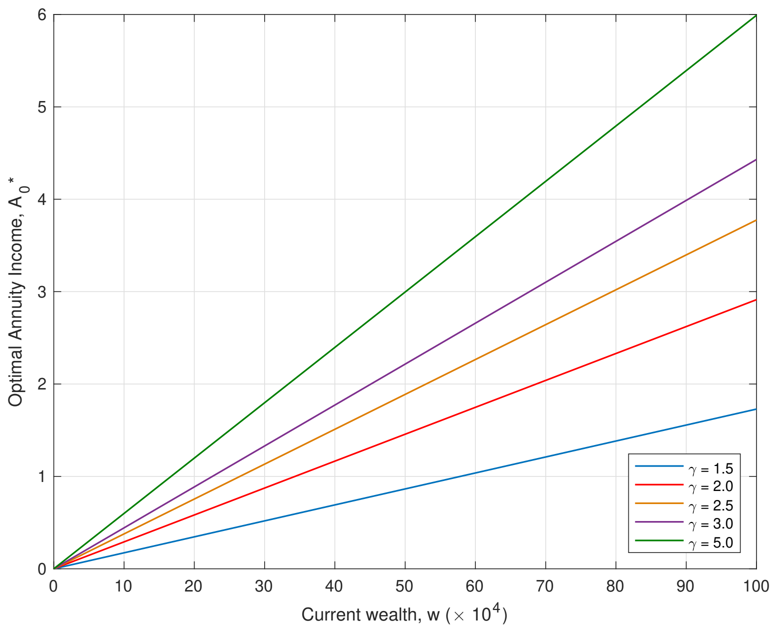

FIGURE 4.6: Sensitivities of the current wealth w and optimal annuity income level A^* with respect to risk aversion γ at retirement age $t_0 = 65$.

Figure 4.6 reveals the relationship between the pensioner's preferences and optimal annuity income to the current wealth. From the figure, we acknowledge that the annuity income increases as the wealth increases. This does not have a big impact on the findings. This is because, in order for the pensioner to have a better life after retirement, they need to have more income and will annuitize more whenever they want to do so. Gupta and Li [24] explained that individuals would be more aggressive and invest more in risky assets both for high and low-risk aversion with the existence of an annuity and the decisions may be influenced by the individual's life span, wealth and income status.

We are more interested in how the level of risk preferences impacts to the optimal annuity income. The figure shows that the higher risk aversion pensioner will optimize the annuity income more compared to the pensioner with lower risk aversion. This can why that the pensioner who dislikes risk (higher risk aversion) will annuitize more since annuity is the only way to secure the income after retirement and they do not consider investing in the risky assets. This is in contradiction with the pensioner who likes risk, and who will annuitize less. Gupta and Li [24] concluded that the low-risk aversion leads to allocating more wealth for the risky assets.

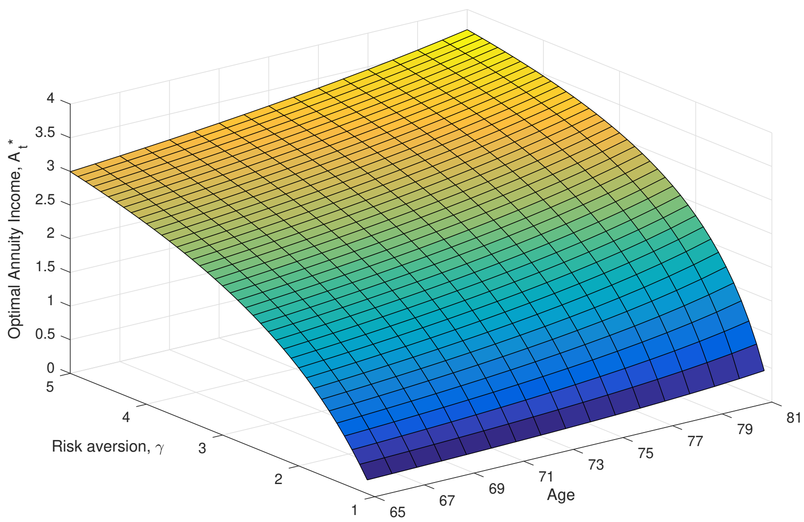

FIGURE 4.7: Sensitivities of the risk aversion γ and optimal annuity income level A^* with respect to post-retirement age.

Next, Figure 4.7 explains the effect of post-retirement age on optimal annuity income. From the figure, we find that the optimal annuity income level slightly increases with respect to age after retirement. Beshears et al. [115] did a survey on annuity payment and found that the total annual pension payments to age would increase since the lifestyle would become costly in the future. At the same time, they found that annuitization relative to a situation where annuitization is an "all or nothing" decision increased when allowing the individuals to annuitize a fraction of their wealth.

4.5 Conclusion

This chapter is concerned with optimal strategies on investment, consumption and annuity income level under the Constant Elasticity of Variance (CEV) model. By applying the dynamic programming principle, the Legendre transform, the duality theory, power transformation and the variable change technique, we obtain the closed-form solutions to the optimal investment, consumption and annuity income strategies when the risk preference of an investor is the Constant Relative Risk Aversion (CRRA) utility function. The numerical example is given to represent the conclusions and analyze the impact of the model parameter on the optimal investment, consumption and annuity income level strategies.

We found that the optimal investment, consumption and annuity income level strategies for CRRA utility function mainly depend on the post-retirement age and the pensioner's risk preferences. Interestingly, we found that the level of risk aversion has an inverse relationship with optimal annuity income level even though we already know that annuity is one of the risky assets.

Chapter 5

The special case of the Constant Elasticity of Variance (CEV) model

5.1 Introduction

This chapter discusses the particular case of the CEV Model mentioned in previous chapter. This happens when the stochastic volatility is assumed to be constant when $\beta = 0$ in (4.1) as mentioned by Gao [65]. This chapter aims its focus on the GBM model with the CRRA utility function to solve the optimal investment strategy, consumption, annuity income level and utility maximization at retirement. We discussed the optimization under the CEV model and its special case in a working paper by Sabri [103].

The study on the GBM is well known in literature and it was started by Merton in [8] and [9]. Then it expanded among the researchers in any area of study. For example, in annuitization, Milevsky and Robinson [105] considered the lifetime and eventual probability of ruin for an individual who wishes to consume a fixed periodic amount. Then, Milevsky and Young [7] examined the optimal annuitization, investment and consumption strategies for a pensioner facing a stochastic time of death. Besides that, Wang and Young [38] focused on how including commutable life annuities encourages annuitization. Also, recently, Di Giacinto et al. [32] deal with a constrained investment problem for a DC pension fund when the retiree is allowed to defer the purchase of the annuity.

This chapter follows the same approach as the previous chapter on the CEV model but considers the different dual variable. By considering a different dual variable, the maximization of utility can be determined compared to the previous chapter. We obtain the explicit solution for optimal investment strategy, consumption, and maximizing utility using the Legendre transform and the dual theory. We stayed true to the original intention of the study, where we use CRRA utility function and solve the discounted maximization utility of consumption. We refer to the study done by Milevsky and Young [7] which studied the optimization facing a stochastic time of death. Besides, Wang and Young [38] revealed the relationship between the commutability of life annuities and pensioners' willingness to annuitize.

The focus in this chapter is also on finding the solution for the purchasing boundary and from there, we are able to solve for the optimal investment strategy, consumption and utility maximization. Milevsky and Young [7] did not show clearly the solution to the purchasing boundary and only showed the numerical results for optimal annuitization while, Wang and Young [38] clearly showed the procedure to solve the purchasing boundary, but studied the effect of commutable life annuities on the optimal annuitization, consumption and investment. This is different from the focus of this chapter where we improvise the findings from Milevsky and Young [7] by showing clearly the solution of the purchasing boundary and the optimization by getting inspiration from Wang and Young [38]. Both of these references uses an interest rate in the maximization problem, while this chapter uses an adjusted discount rate. We used a numerical example to test the optimization model with different parameters.

5.2 The model

The model for this particular case of CEV model is GBM, same as discussed in Chapter 3 (3.2). From chapter 3, the wealth process for the pensioner who receives an income from an annuity purchasing and the investment performance under GBM is as discussed in Section 3.2 and equation (2.5) with σ_1 in 2.5 is become σ_3. The pensioner's objective function under $(\{c_t^*, y_t^*, A_t^*\})$ admissible strategy is

$$\max_{\{c_t, y_t, A_t\}} \mathbf{E}\left[\int_t^{\infty} e^{-(\alpha+\lambda)(s-t)} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right]$$

with $W_{t-} = w$ and $A_{t-} = A$.

5.3 Solution of the model

This subsection derived the solution to the optimization problem above, by applying the maximum principle, dimension reduction, the Legendre transform and the dual theory. We used the same HJB equation as in Proposition 2.1.1 since the model is now reduced to GBM and follows the same procedure as in Section 3.3.1.

In this chapter, there are two boundaries; considering the first boundary is the same as in (3.10), we define another boundary condition which is

$$V_{Aw}(w, A, t) = aV_{ww}(w, A, t) \tag{5.1}$$

and the combination of HJB equation and boundary condition (3.11) becomes

$$\begin{cases} (\alpha+\lambda)V & = V_t + (rw + A)V_w + \max\limits_{y}\left[\frac{1}{2}{\sigma_3}^2 y^2 V_{ww} + y(\mu_3 - r)V_w\right] \\ & \quad + \max\limits_{c \geq 0}\left(- cV_w + \frac{c^{1-\gamma}}{1-\gamma}\right), 0 \leq \frac{w}{A} \leq \bar{z}, \\ V_A & = aV_w, \\ V_{Aw} & = aV_{ww}. \end{cases} \tag{5.2}$$

From (5.2), the differential equation is the HJB equation through which we will determine the optimal investment, consumption and utility. The first boundary condition comes from the hypothesis that the pensioner buys annuity only when $\bar{z} = \frac{w}{A}$ for $(w, A) \in \mathbb{R}_1$. The last boundary condition is a smooth fit condition since the value $\bar{z}$ is chosen optimality.

5.3.1 Dimension reduction

The dimension of the HJB equation in (3.11) needs to be reduced so that it becomes easy to solve. From Lemma 3.1 in Chapter 3, we have

$$V(w, A, t) = A^{1-\gamma}\widetilde{V}(z, t)$$

as previously defined in Chapter 3. In this chapter, we also considered the partial derivatives of this equation since there are two boundary conditions. Therefore, by differentiating with respect to w, A and the partial derivatives, we have

$$\begin{aligned} V_w &= A^{-\gamma}\widetilde{V}_z, \\ V_A &= -A^{-\gamma}\frac{w}{A}\widetilde{V}_z + (1-\gamma)A^{-\gamma}\widetilde{V}, \\ &= -A^{-\gamma}z\widetilde{V}_z + (1-\gamma)A^{-\gamma}\widetilde{V}, \\ V_{ww} &= A^{-\gamma-1}\widetilde{V}_{zz}, \\ V_{Aw} &= -A^{-\gamma-1}z\widetilde{V}_{zz} - A^{-\gamma-1}\widetilde{V}_z + (1-\gamma)A^{-\gamma-1}\widetilde{V}_z. \end{aligned} \tag{5.3}$$

Then, we applied the dimension reduction in the boundary conditions in (5.2)

$$\begin{aligned} a(A^{-\gamma}\widetilde{V}_z) - \Big[- A^{-\gamma}z\widetilde{V}_z + (1-\gamma)A^{-\gamma}\widetilde{V}\Big] &= 0, \\ (z+a)\widetilde{V}_z - (1-\gamma)\widetilde{V} &= 0, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \Big[A^{-\gamma-1}\Big(- z\widetilde{V}_{zz} - \widetilde{V}_z + (1-\gamma)\widetilde{V}_z\Big)\Big] - A^{-\gamma-1}(a\widetilde{V}_{zz}) &= 0, \\ (z+a)\widetilde{V}_{zz} + \gamma\widetilde{V}_z &= 0. \end{aligned} \tag{5.5}$$

The differential equation in (5.2) is reduced to

$$
\begin{cases}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z + \max\limits_{\bar{y}}\left[\dfrac{1}{2}{\sigma_3}^2\bar{y}^2\widetilde{V}_{zz} + \bar{y}(\mu_3 - r)\widetilde{V}_z\right] \\
\qquad\qquad + \max\limits_{\bar{c}\geq 0}\left(- c\widetilde{V}_z + \dfrac{\bar{c}^{1-\gamma}}{1-\gamma}\right), 0 \leq \dfrac{w}{A} \leq \bar{z}, \\
\big(z+a\big)\widetilde{V}_z - \big(1-\gamma\big)\widetilde{V} = 0, \\
(z+a)\widetilde{V}_{zz} + \gamma\widetilde{V}_z = 0.
\end{cases} \tag{5.6}
$$

where $\bar{c} = \frac{c}{A}$ and $\bar{y} = \frac{y}{A}$. This is the same transformation as in the previous two chapters. From the differential HJB equation above, we solved FOCs for the optimal investment-consumption

i) Optimal investment strategy

$$
\begin{aligned}
\bar{y}^*{\sigma_3}^2\widetilde{V}_{zz} + (\mu_3 - r)\widetilde{V}_z = 0 \\
\bar{y}_t^* = -\frac{(\mu_3 - r)}{{\sigma_3}^2}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}}.
\end{aligned} \tag{5.7}
$$

ii) Optimal consumption

$$
\begin{aligned}
-\widetilde{V}_z + (\bar{c}^*)^{-\gamma} = 0 \\
\bar{c}_t^* = (\widetilde{V}_z)^{-\frac{1}{\gamma}}.
\end{aligned} \tag{5.8}
$$

Then, by substituting (5.7) and (5.8) into the differential equation in (5.6), we have

$$
\begin{aligned}
(\alpha+\lambda)\widetilde{V} =& \widetilde{V}_t + (rz+1)\widetilde{V}_z + \max_{\bar{y}^*}\left[\frac{1}{2}\sigma_3^2\left(- \frac{(\mu_3 - r)}{\sigma_3^2}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}}\right)^2\widetilde{V}_{zz}\right. \\
&\left. + \left(- \frac{(\mu_3 - r)}{\sigma_3^2}\frac{\widetilde{V}_z}{\widetilde{V}_{zz}}\right)(\mu_3 - r)\widetilde{V}_z\right] + \max_{\bar{c}^*\geqslant 0}\left[- \left(\widetilde{V}_z\right)^{-\frac{1}{\gamma}}\widetilde{V}_z + \frac{\left((\widetilde{V}_z)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma}\right], \\
=& \widetilde{V}_t + (rz+1)\widetilde{V}_z + \left(- \frac{1}{2}\frac{(\mu_3 - r)^2}{\sigma_3^2}\frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}}\right) + \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}}
\end{aligned}
$$

Then, we simplified it and considered the boundary conditions, and we arrived at

$$\begin{cases} (\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z - \frac{1}{2}\theta_3^2\frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}} + \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}}, 0 \le z \le \bar{z}, \\ (z+a)\widetilde{V}_z - (1-\gamma)\widetilde{V} = 0, \\ (z+a)\widetilde{V}_{zz} + \gamma\widetilde{V}_z = 0. \end{cases} \tag{5.9}$$

where $\theta_3 = \frac{\mu_3 - r}{\sigma_3}$, which also known as Sharpe ratio. Next, we followed Proposition 3.3.2 for the pensioner's decision.

5.3.2 Linearization using the Legendre transform

The Ordinary Differential Equation (ODE) in (5.9) is non-linear. We are searching for a concave solution of Free Boundary Problem (FBP); therefore, here we apply the Legendre transform to $\widetilde{V}$ to define the convex dual $\widehat{V}$.

We applied Definition 4.3.1 to define the Legendre Transform

$$\widehat{V}(\rho,t) = \max_{z>0}\left\{\widetilde{V}(z,t) - \rho z\right\}, 0 < z < \infty \tag{5.10}$$

Then, the critical value $\bar{z}$ solves the equation $\widetilde{V}_z(\bar{z}) - \rho = 0$; $\bar{z} = I(\rho)$, which I is the functional inverse of $\widetilde{V}_z$. In this chapter, we can see the dimension for the Legendre value function is two-dimensional compared to the CEV model, since in this chapter, the volatility is constant and the value function does not depend on the stochastic volatility.

By differentiating (5.10) with respect to ρ,

$$\begin{aligned} \widehat{V}_\rho(\rho,t) &= -\widetilde{V}_z^{-1}(\rho,t) = -\bar{z} \le 0 \\ \widehat{V}_{\rho\rho}(\rho,t) &= -\frac{1}{\widetilde{V}_{zz}(z)}\bigg|_{z=\widetilde{V}_z^{-1}(\rho,t)} \ge 0 \end{aligned} \tag{5.11}$$

Recover $\widetilde{V}$ from $\widetilde{V}(z,t) = \widehat{V}(\rho,t) + \rho z$. We also have a boundary condition at $z = 0$, where at this point, the pensioner has no wealth to invest in the risky asset. Then, we have

$$\rho_s = \widetilde{V}_z(0,t) \quad \text{and} \quad \rho_b = \widetilde{V}_z(\bar{z},t)$$

From (5.9) and considering (5.10), we get

$$(\alpha+\lambda)[\widehat{V}+\rho z] = \widehat{V}_t + rz\rho + \rho - \frac{1}{2}\theta_3^2 \frac{\rho^2}{\left(-\frac{1}{\widehat{V}_{\rho\rho}}\right)} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}$$

$$(\alpha+\lambda)\widehat{V} = \widehat{V}_t - (\alpha+\lambda)\rho z + rz\rho + \rho + \frac{1}{2}\theta_3^2\rho^2\widehat{V}_{\rho\rho} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}$$

Simplifying the equation above and (5.9) becomes

$$\begin{cases} (\alpha+\lambda)\widehat{V} = \widehat{V}_t + (\alpha+\lambda-r)\rho\widehat{V}_\rho + \rho + \frac{1}{2}\theta_3^2\rho^2\widehat{V}_{\rho\rho} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}; \\ \widehat{V}_\rho(\rho_s,t) = 0; \\ \widehat{V}_\rho(\rho_b,t) = -\bar{z}; \\ (1-\gamma)\widehat{V}(\rho_b,t) + \gamma\rho_b\widehat{V}_\rho(\rho_b,t) = a\rho_b; \\ \widehat{V}_\rho(\rho_b,t) + \gamma\rho_b\widehat{V}_{\rho\rho}(\rho_b,t) = a. \end{cases} \tag{5.12}$$

Since our work concerns the mathematical model corresponding to the annuitization strategy, we assumed the constant forces of mortality. We will lose the analytical tractability of the simple model if we consider age-dependent mortality. If we assume that the forces of mortality are constant for all $t \geq 0$, then we can obtain an explicit analytical solution of the value function $\widetilde{V}$ via the boundary-value problem given in (5.12). In this case, $\widetilde{V}$,$\widehat{V}$,ρ_s and ρ_b are independent of time, so (5.12) becomes the Ordinary Differential Equation (ODE),

$$\begin{cases} (\alpha+\lambda)\widehat{V} = (\alpha+\lambda-r)\rho\widehat{V}_\rho + \rho + q\rho^2\widehat{V}_{\rho\rho} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}; \\ \widehat{V}_\rho(\rho_s) = 0; \\ \widehat{V}_\rho(\rho_b) = -\bar{z}; \\ (1-\gamma)\widehat{V}(\rho_b) + \gamma\rho_b\widehat{V}_\rho(\rho_b) = a\rho_b; \\ \widehat{V}_\rho(\rho_b) + \gamma\rho_b\widehat{V}_{\rho\rho}(\rho_b) = a. \end{cases} \tag{5.13}$$

in which $q = \frac{1}{2}\left(\frac{\mu_3-r}{\sigma_3}\right)^2 = \frac{1}{2}\theta_3^2$. Refer to Remark 4.2. We can choose the dual function for $\widetilde{V}(z,t)$. Therefore, in this chapter, we chose the different dual function to compare to the previous chapter. The general solution of the ODE in (5.13) is given by

$$\widehat{V}(\rho) = D_1\rho^{B_1} + D_2\rho^{B_2} + \frac{\rho}{\alpha} + C\rho^{\frac{\gamma-1}{\gamma}} \tag{5.14}$$

with D_1 and D_2 constants to be determined by the boundary conditions. B_1, B_2 and C are given by

$$B_1 = \frac{1}{2m}\left[\Big(q - (\alpha+\lambda-r)\Big) + \sqrt{\Big(n - (\alpha+\lambda-r)\Big)^2 + 4q(\alpha+\lambda)}\right] \tag{5.15}$$

$$B_2 = \frac{1}{2m}\left[\Big(q - (\alpha+\lambda-r)\Big) - \sqrt{\Big(n - (\alpha+\lambda-r)\Big)^2 + 4q(\alpha+\lambda)}\right] \tag{5.16}$$

$$C = \frac{\gamma}{1-\gamma}\left[\alpha + \frac{(\alpha+\lambda-r)}{\gamma} - q\frac{1-\gamma}{\gamma^2}\right]^{-1} \tag{5.17}$$

Next, we solved it for D_1 and D_2 by expressing the boundary conditions in (5.13) and considered the general solution in (5.14). From $(1-\gamma)\widehat{V}(\rho_b)+\gamma\rho_b\widehat{V}_\rho(\rho_b) = a\rho_b$, we have

$$D_1\big[1+\gamma(B_1-1)\big]\rho_b^{B_1} + D_2\big[1+\gamma(B_2-1)\big]\rho_b^{B_2} + \frac{\rho_b}{\alpha} = \frac{\rho_b}{r+\lambda} \tag{5.18}$$

and from $\widehat{V}_\rho(\rho_b) + \gamma\rho_b\widehat{V}_{\rho\rho}(\rho_b) = a$,

$$D_1B_1\big[1+\gamma(B_1-1)\big]\rho_b^{B_1-1} + D_2B_2\big[1+\gamma(B_2-1)\big]\rho_b^{B_2-1} + \frac{1}{\alpha} = \frac{1}{r+\lambda} \tag{5.19}$$

Then, the boundary condition at ρ_s, from $\widehat{V}_\rho(\rho_s) = 0$, gives us

$$D_1B_1\rho_s^{B_1-1} + D_2B_2\rho_s^{B_2-1} + \frac{1}{\alpha} + \frac{\gamma-1}{\gamma}C\rho_s^{-\frac{1}{\gamma}} = 0 \tag{5.20}$$

and the boundary conditions at ρ_b, from $\widehat{V}_\rho(\rho_b) = -\bar{z}$,

$$D_1 B_1 \rho_b^{B_1-1} + D_2 B_2 \rho_b^{B_2-1} + \frac{1}{\alpha} + \frac{\gamma-1}{\gamma} C \rho_b^{-\frac{1}{\gamma}} = -\bar{z} \tag{5.21}$$

To solve for D_1 and D_2, from (5.18)

$$\begin{aligned} D_2\big[1+\gamma(B_2-1)\big]\rho_b^{B_2} &= \frac{\rho_b}{r+\lambda} - \frac{\rho_b}{\alpha} - D_1\big[1+\gamma(B_1-1)\big]\rho_b^{B_1} \\ D_2 &= \frac{-\rho_b\lambda}{\alpha(r+\lambda)\big[1+\gamma(B_2-1)\big]\rho_b^{B_2}} - D_1 \frac{\big[1+\gamma(B_1-1)\big]\rho_b^{B_1}}{\big[1+\gamma(B_2-1)\big]\rho_b^{B_2}} \end{aligned}$$

we substitute the equation above into (5.19)

$$D_1 B_1\big[1+\gamma(B_1-1)\big]\rho_b^{B_1-1} - \frac{B_2\lambda}{\alpha(r+\lambda)} - D_1 B_2\big[1+\gamma(B_1-1)\big]\rho_b^{B_1-1} + \frac{1}{\alpha} = \frac{1}{r+\lambda}$$

Therefore, we solved it for D_1

$$D_1 = -\frac{\lambda}{\alpha(r+\lambda)} \frac{1-B_2}{B_1-B_2} \frac{1}{\big[1+\gamma(B_1-1)\big]} \rho_b^{1-B_1} \tag{5.22}$$

Now, we are going to address D_2. Again from (5.18) we have

$$D_1 = \frac{-\rho_b\lambda}{\alpha(r+\lambda)\big[1+\gamma(B_1-1)\big]\rho_b^{B_1}} - D_2 \frac{\big[1+\gamma(B_2-1)\big]\rho_b^{B_2}}{\big[1+\gamma(B_1-1)\big]\rho_b^{B_1}}$$

Then, we substitute it into (5.19) and we get

$$D_2 B_1\big[1+\gamma(B_2-1)\big]\rho_b^{B_2-1} - D_2 B_2\big[1+\gamma(B_2-1)\big]\rho_b^{B_2-1} = \frac{\lambda}{\alpha(r+\lambda)} - \frac{\lambda B_1}{\alpha(r+\lambda)}$$

Therefore, we solved it for D_2

$$D_2 = -\frac{\lambda}{\alpha(r+\lambda)} \frac{B_1-1}{B_1-B_2} \frac{1}{\big[1+\gamma(B_2-1)\big]} \rho_b^{1-B_2} \tag{5.23}$$

Next, the substituted D_1 and D_2 go into (5.20) to solve for ρ_s

$$
\begin{aligned}
\frac{1-\gamma}{\gamma} C \rho_s^{-\frac{1}{\gamma}} =& -\frac{\lambda}{\alpha(r+\lambda)}\left[\frac{B_1(1-B_2)}{B_1-B_2}\frac{x^{B_1-1}}{\left[1+\gamma(B_1-1)\right]}+\frac{B_2(B_1-1)}{B_1-B_2}\frac{x^{B_2-1}}{\left[1+\gamma(B_2-1)\right]}\right] \\
&+\frac{1}{\alpha} \\
\rho_s =& \left[\frac{-\frac{\lambda}{\alpha(r+\lambda)}\left[\frac{B_1(1-B_2)}{B_1-B_2}\frac{x^{B_1-1}}{[1+\gamma(B_1-1)]}+\frac{B_2(B_1-1)}{B_1-B_2}\frac{x^{B_2-1}}{[1+\gamma(B_2-1)]}\right]+\frac{1}{\alpha}}{\frac{1-\gamma}{\gamma}C}\right]^{-\gamma}
\end{aligned} \tag{5.24}
$$

Expression (5.24) is in terms of x, therefore ρ_b is

$$
\rho_b = \frac{\rho_s}{x} \tag{5.25}
$$

Let us define the boundary conditions at ρ_s, as below

$$
\begin{aligned}
\widehat{V}_\rho(\rho_s) &= 0 \\
\widehat{V}_{\rho\rho}(\rho_s) &= 0
\end{aligned} \tag{5.26}
$$

From (5.26), we have

$$
D_1 B_1 \rho_s^{B_1-1} + D_2 B_2 \rho_s^{B_2-1} + \frac{1}{\alpha} + \frac{\gamma-1}{\gamma} C \rho_s^{-\frac{1}{\gamma}} = 0 \tag{5.27}
$$

and

$$
(B_1-1) D_1 B_1 \rho_s^{B_1-2} + (B_2-1) D_2 B_2 \rho_s^{B_2-2} + \frac{1-\gamma}{\gamma^2} C \rho_s^{\frac{-1-\gamma}{\gamma}} = 0 \tag{5.28}
$$

Next, substitute D_1 and D_2 into $\widehat{V}_\rho(\rho_s) + \gamma\rho_s \widehat{V}_{\rho\rho}(\rho_s) = 0$ by considering (5.27) and (5.28), we arrive at

$$
\frac{\lambda}{r+\lambda}\frac{B_1(1-B_2)}{B_1-B_2} x^{B_1-1} + \frac{\lambda}{r+\lambda}\frac{B_2(B_1-1)}{B_1-B_2} x^{B_2-1} = 1 \tag{5.29}
$$

The purchasing boundary $\bar{z}$ is obtained by substituting D_1 and D_2 in (5.22) and (5.23) respectively into (5.21),

$$
\begin{aligned}
\bar{z} =& \frac{\lambda}{\alpha(r+\lambda)}\left[\frac{B_1(1-B_2)[1+\gamma(B_2-1)]+B_2(B_1-1)[1+\gamma(B_1-1)]}{(B_1-B_2)[1+\gamma(B_1-1)][1+\gamma(B_2-1)]}\right] \\
&-\frac{1}{\alpha}-\frac{\gamma-1}{\gamma}C\rho_b^{-\frac{1}{\gamma}}
\end{aligned}
\tag{5.30}
$$

Next, we tried to find the explicit solutions for optimal investment, consumption, annuity income rate and utility and consider the purchasing boundary. First, we solved for optimal utility. To do this, from (5.10), we have

$$
\begin{aligned}
\widetilde{V}(\frac{w}{A}) &= \max_{z>0}\left[\widehat{V}(\rho)+\rho\frac{w}{A}\right] \\
&= \max_{z>0}\left[D_1\rho^{B_1}+D_2\rho^{B_2}+\frac{\rho}{\alpha}+C\rho^{\frac{\gamma-1}{\gamma}}+\rho\frac{w}{A}\right]
\end{aligned}
\tag{5.31}
$$

Then, we solved the critical point, ρ^* by taking the derivative of (5.31) with respect to ρ

$$
D_1B_1(\rho^*)^{B_1-1}+D_2B_2(\rho^*)^{B_2-1}+\frac{1}{\alpha}+\frac{\gamma-1}{\gamma}C(\rho^*)^{-\frac{1}{\gamma}}=-\frac{w}{A}
\tag{5.32}
$$

Then, we substituted the value of ρ^* into (5.31) to solve for optimal utility. The optimal investment strategy is solved from (5.7), considering (5.3)

$$
\begin{aligned}
y_t^* &= -\frac{\mu_3-r}{\sigma_3^2}A\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} \\
&= \frac{(\mu_3-r)}{\sigma_3^2}A\bar{z}
\end{aligned}
\tag{5.33}
$$

Next, from (5.8), (5.3) and (5.11) we solve for optimal consumption

$$
c_t^* = A\left(\frac{1}{\bar{z}}\right)^{-\frac{1}{\gamma}}
\tag{5.34}
$$

Lastly, the optimal annuity income level is solved considering the wealth-to-income ratio

$$\begin{aligned} \bar{z}_t &= \frac{w}{A_t^*} \\ A_t^* &= \frac{w}{\bar{z}_t} \end{aligned} \tag{5.35}$$

5.4 Numerical example

In this section, we provide the numerical example to illustrate the effect of model parameters on utility maximization, optimal investment, consumption and annuity income level strategies under the CRRA utility case.

The parameter values for this chapter are the same as those in Chapter 3. However, we have changed the number of subscripts on the parameter to represent the parameters for a different model - the list of parameters as in Table 5.1. The amount of a life annuity is calculated using (2.8) as discussed in Chapter 3.

TABLE 5.1:
Parameter Values for Chapter 5

r	$= 0.03$
μ_3	$= 0.08$
σ_3	$= 0.20$
α	$= 0.05$
τ	$= 16$

In this chapter, we obtain the purchasing boundary $\bar{z}$, which is different from that in the previous two chapters, as in Chapter 3 the purchasing boundary is not obtained while in Chapter 4, the purchasing boundary depends on the stock price $g(\rho, S, t)$. We have mentioned that in this chapter we use the different dual variables, thus we are able to solve for utility maximization, unlike in Chapter 4 when using the method in Chapter 4, we are not able to solve utility maximization.

This section is divided into four subsections. Firstly, we analyze the results on utility maximization using (5.31). Then, we proceed discussing optimal investment strategy applying (5.33) and adopting (5.34) to analyze the optimal consumption and lastly, (5.35) for optimal annuity income level. We noticed that the pattern of the graph and curve for each optimal strategy is mostly the same as in other models, and the only difference is the optimization value. As stated before, this chapter is a particular case of the CEV model, it is reduced to the GBM model. Therefore, the results are almost the same as the CEV model.

5.4.1 Utility maximization

We started the discussion of the model with utility maximization. By using (5.31), we applied the parameter as listed in Table 5.1 and tested using MATLAB to present the relationship between model parameters and optimization.

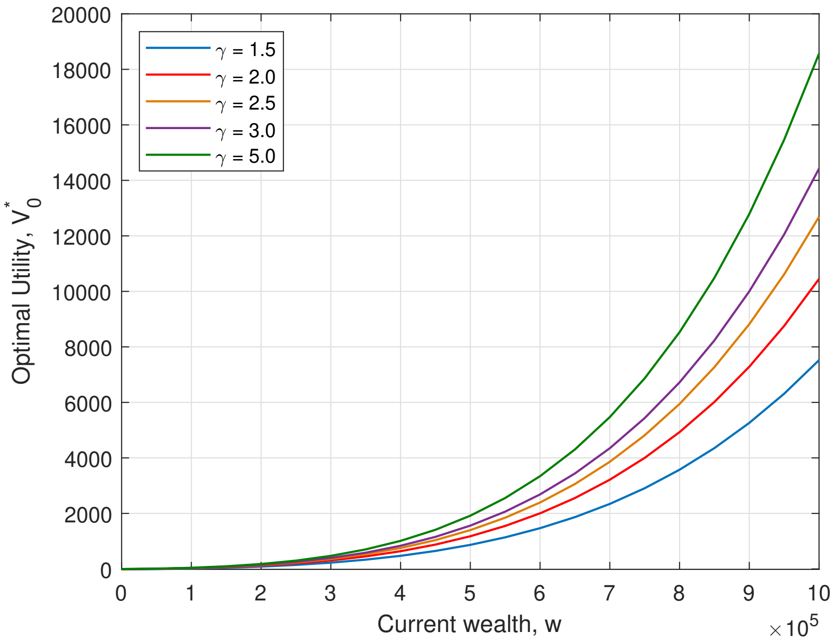

FIGURE 5.1: The impact of current wealth on the utility maximization at the retirement age$t_0 = 65$.

Figure 5.1 compares the maximized utility when the pensioner's risk preferences differ. We can see, as the current wealth increases, the optimal utility increases too. In order to maximize the utility, the pensioner needs to have enough wealth to continue living happily after retirement. The curve for optimal utility is convex. As shared by Singh in [116], the convex curve reflects a risk-loving individual, where the marginal utility of the individual's income increases as his/her money income increases.

From Figure 5.1 we also noticed that a pensioner's risk preferences are in an inverse relationship with the optimal utility. The risk-loving pensioner (low-risk aversion) optimizes lower compared to the pensioner who is not risk-loving (high-risk aversion). This happens when the pensioner with higher risk aversion can take advantage of the annuity income to achieve higher utility from consumption.

5.4.2 Optimal investment

We continue the model analysis by discussing the relationship between the model parameter and an optimal investment strategy by applying the model parameters in (5.33).

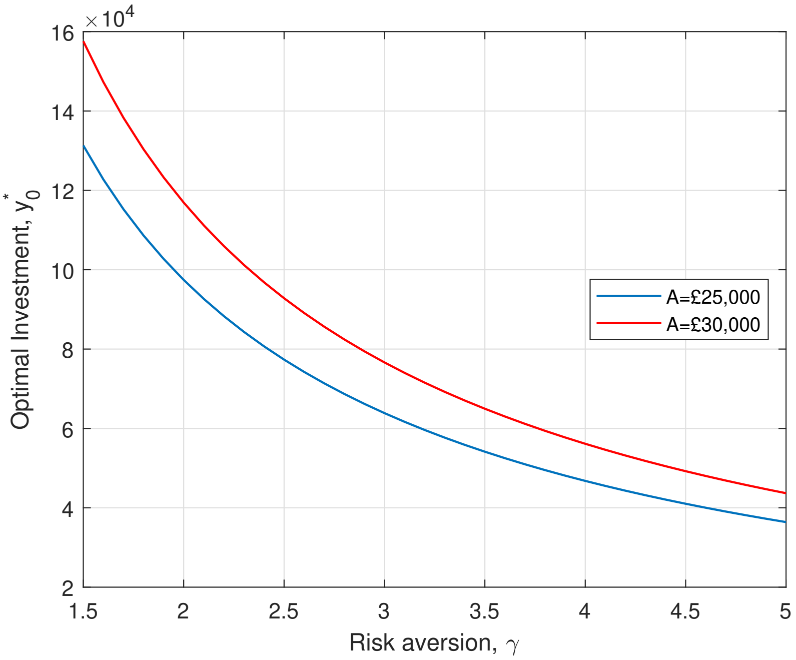

FIGURE 5.2: The impact of risk aversion on the optimal investment at the retirement age$t_0 = 65$.

Figure 5.2 illustrates the optimal investment for risk aversion comparing levels of annuity income. The figure clearly shows that an annuity income does not have an impact on the optimal investment strategy. The difference in optimal investment between the level of annuity income is small. The result is quite different from the previous two chapters, whereby solving the GBM model using the Legendre transform, the impact of annuity income towards the optimal investment is revealed.

5.4.3 Optimal consumption

We continue the discussion of the model with optimal consumption. Not surprisingly, the curve is almost the same as in the CEV model. However, the curve is contradicting the results for the GBM model using the substitution method.

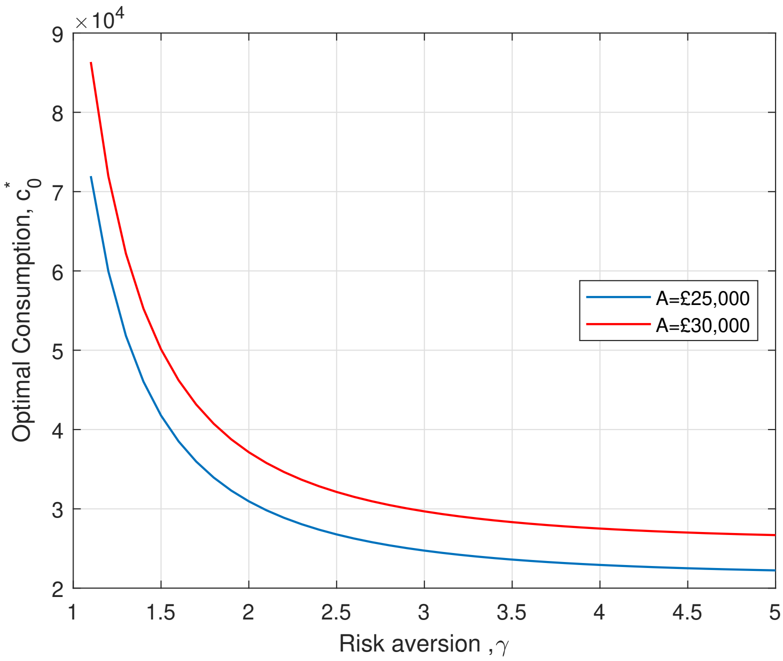

FIGURE 5.3: The impact of risk aversion on the optimal consumption at the retirement age$t_0 = 65$.

Figure 5.3 shows that optimal consumption decreases when risk aversion increases. This is the same result as in the CEV model in Chapter 4, yet, the result is vice versa to the GBM model in Chapter 3. This shows the non-monotonic behavior of consumption with risk preferences, as discussed by Wang and Young [38].

5.4.4 Optimal annuity income level

Finally, we analyze the impact of model parameters on the optimal annuity income level, applying the values of the parameters into (5.35).

Figure 5.4 is almost the same as Figure 4.6 in Chapter 4. According to Horneff et al. [29] when the annuitization decisions occur at retirement, the pensioner may prefer to combine (investment and life annuities) strategies. A pensioner with high risk-aversion is more attracted to those withdrawal plans or mixed strategies as shown in Figure 5.4, where a pensioner with higher risk aversion (who dislikes risk) believes in this mixed strategy because they know that they are guaranteed to receive income after retirement. This is consistent with other studies done by Horneff et al. [29], Brown et al. [117] and recently reported by Merton [114].

Next, we proceed to study the relationship between investment volatility and optimal annuity income with different levels of risk preferences. Figure 5.5 reports

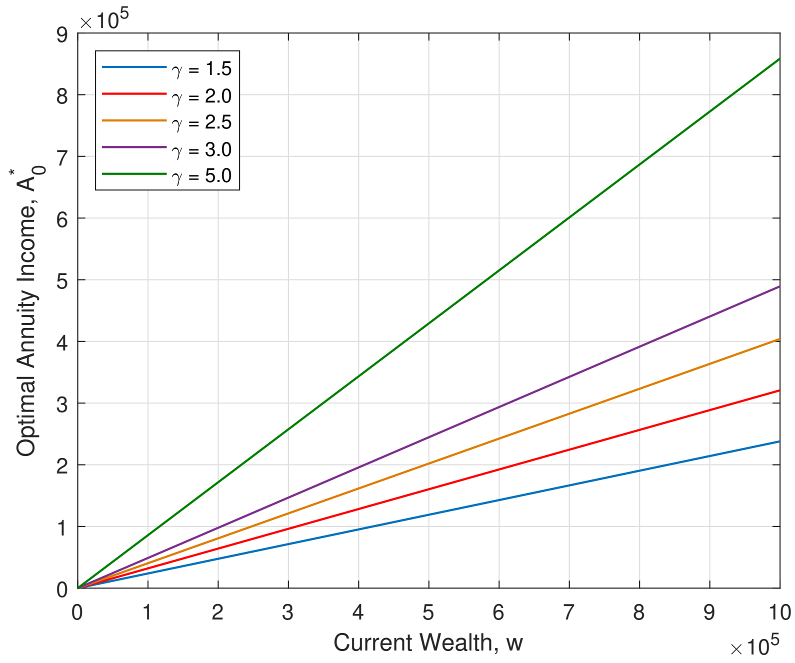

FIGURE 5.4: The impact of wealth status on the optimal annuity income level at the retirement age$t_0 = 65$.

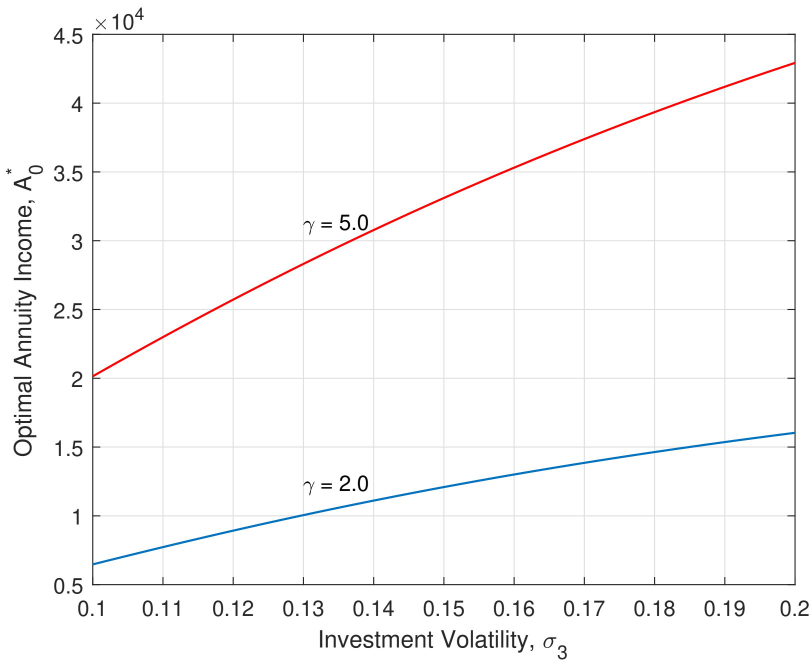

FIGURE 5.5: The impact of investment volatility on the optimal annuity income level at the retirement age$t_0 = 65$.

that as the investment volatility increases, the optimal annuity income level also increases. This is due to the fact that as the investment in the risky assets becomes unstable, the pensioner will choose to annuitize in life annuities.

5.5 Conclusion

In this chapter, we studied the particular case of the CEV model when the elasticity factor β in the CEV model is zero. We solved the GBM model with CRRA utility function for utility maximization, optimal investment, consumption and annuity income level strategies and obtained the closed-form solution. For this we used the Legendre transform and the duality theory to solve. We employed different dual variables compared to Chapter 4. We found the different dual variable choices affect how we derive the solution. For example, in this chapter, we were able to solve for maximization utility, but we could not obtain the optimal strategies for time-dependence.

The findings present a new summary for the well-documented question of the pensioner purchasing the life annuities. We found that an annuity income level affects the optimal investment and consumption. We also found that the decisions to annuitize depend on the individual's risk preferences. Besides that, the model parameters impact maximization utility, optimal investment, consumption and the annuity income level.

Chapter 6

Optimal investment-consumption and annuitization under Heston's Stochastic Volatility model

6.1 Introduction

The set-up of this chapter can support portfolio problems with stochastic volatility, where the state process drives the volatility of the stock. The chapter aimed to investigate the optimal investment strategy, consumption and annuity income under Heston's SV model when the pensioner receives the income after retirement from purchasing the annuity and the investment performance.

The Heston model is a mathematical model describing the evolution of the volatility of an underlying asset. It is a stochastic volatility model, the model that assumes the volatility of the asset is not constant, nor even deterministic, but follows a random process. The Heston model was initially proposed by Heston [70] for the price of a European call option on an asset. In 2005, Kraft [73] presented a verification result for portfolio problems with stochastic volatility. Meanwhile, Li and Wu [74] provided a verification theorem without the usual Lipschitz assumptions.

The study on Heston's SV model is popular among researchers who are interested in stochastic modelling. Forde et al. [118] used asymptotic formulae implied to the Heston model with saddle-point expansions. Foulon [119] solved the three-dimensional Heston-Hull-White model using Alternating Direction Implicit time discretization schemes. Kraft et al. [120] applied classical dynamic programming to the Heston model with recursive utility.

Recently, the Heston model has become popular in reinsurance problems. See [75, 77, 121]. Besides that, Li and Wu [74] and Chang and Rong [59] considered stochastic interest rates together with Heston's SV model on an investment and consumption problem, while Sun et al. [122] considered the stochastic inflation risk and salary income process and Zhang and Ge [123] solved optimal strategies for asset allocation. From literature, we noticed that none of the researchers studied life annuities under Heston's SV model. Therefore, as in previous chapters, this study assumed that the pensioner will invest in life annuities and risky assets. So, the income after retirement is from life annuities and investment performance. We also believe that we are the first to solve this problem using the Legendre transform and the dual theory. Wang et al. [124] proposed the same methodology under HARA utility and the financial market composed of a risky and a riskless asset. We also discussed this in a working paper, Sabri [125].

6.2 The Heston model

The Heston model is an extended version of the Black-Scholes SDE with the volatility that follows a so-called Cox-Ingersoll-Ross (CIR) process on the square volatility, Benhamou et al. [126]. This model takes the correlation between the two Brownian processes. The financial market is assumed to consist of one risk-free asset (bond) and one risky asset (stock). This study also considers the pensioner can invest in life annuities.

6.2.1 Heston's stochastic volatility model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where there are two correlated Brownian motions given with $\langle dB_t^1, dB_t^2 \rangle = \xi t$, $\xi \in [-1, 1]$. The price process $S_{0,t}$ of the riskless asset is according to the ordinary differential equation (ODE) as stated in Chapter 3 in (2.1). We refer the Heston's model that has been developed and studied by [73]. The price process for risky asset $S_{1,t}$ follows the Heston's SV model

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} &= (r + \mu_4 M_t)dt + \sqrt{M_t}dB_t^1, \\ S_{1,0} &= S_1 > 0, \\ dM_t &= k_H \big[\theta_H - M_t\big] dt + \sigma_4 \sqrt{M_t} dB_t^2, \\ M_0 &= M_1 > 0. \end{cases} \tag{6.1}$$

where μ_4, k_H, θ_H, and σ_4 are all positive constants. [1] $\{B_t^1\}$ and $\{B_t^2\}$ are two one-dimensional standard Brownian motions with $Cov(B_t^1, B_t^2) = \xi t$. M_t is the instantaneous variance, θ_H is the log variance, k_H is the rate at which M_t reverts to θ_H and σ_4 is the volatility of the volatility and determines the variance of M_t. Parameters k_H, θ_H, and σ_4 need to satisfy the Feller condition $2k_H\theta_H \geqslant \sigma_4^2$, to ensure that M_t is strictly positive. This can be referred to [75, 127, 123, 124].

Following Definition 2.1.1 and the Heston model (6.1), the Heston's wealth process when the pensioner buys no annuities is

$$\begin{cases} dW_t^{y,c} &= \big[rW_t^{y,c} + \mu_4 M_t y_t - c_t\big] dt + y_t \sqrt{M_t} dB_t^1; \\ W_0 &= w > 0. \end{cases} \tag{6.2}$$

From (6.2), the pensioner is assumed to receive an annuity income, A_t. The unrestricted life annuity assumed is the one where the pensioner can purchase more life annuities. Then, the Heston wealth process in (6.2) becomes

$$\begin{cases} dW_t^{y,c} &= \big[rW_{t-}^{y,c} + \mu_4 M_t y_t - c_t + A_{t-}\big] dt + y_t \sqrt{M_t} dB_t^1 - a_t dA_t; \\ W_{0-} &= w > 0, \end{cases} \tag{6.3}$$

where the pensioner can purchase an annuity at a price a_t per £ of an annuity income at time t. Based on an idea from Milevsky and Young [7], the negative sign on the subscripts for wealth and annuities denotes the left-hand limit of those quantities before any annuity purchases. Given τ as the random time of an individual's death. The pensioner is assumed to seek to maximize the expected utility of discounted consumption over admissible $\mathcal{A}(w, A, t) = \{y_t, c_t, A_t\}$ and

[1] The alphabet on subscripts for k and θ refers to k and θ for Heston's SV model and the number on subscripts for μ and σ refers to the number of the model in the thesis.

discount his/her utility consumption at the discount rate α. Therefore, as in Chapters 3 through 5, the maximized utility for an individual is as follows

$$V(w, A, M, t) = \max_{\{c_t, y_t, A_t\}} \mathbf{E}^{w,A,M}\left[\int_0^\infty e^{-(\alpha+\lambda)t}U(c_t)dt\right]. \tag{6.4}$$

with $U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$ and $\gamma \neq 1$. λ is a constant mortality force.

6.3 Solution of the model

Next, we derived the general framework for the optimization problem (6.4) by applying the maximum principle, dimension reduction, Legendre transforms and the dual theory.

6.3.1 The HJB equation

Suppose that at point (w, A, M, t), it is optimal not to purchase any annuities. Following Ito's Lemma, V in (6.4) satisfies the following HJB equation

$$\begin{aligned}(\alpha + \lambda)V = \max_{\{c_t, y_t, A_t\}} \Bigg\{ & V_t + \Big[rw + \mu_4 yM - c + A\Big]V_w + \frac{1}{2}y^2 MV_{ww} + k_H\Big[\theta_H - M\Big]V_M \\ & + \frac{1}{2}\sigma_4^2 MV_{MM} + \xi y\sigma_4 MV_{wM} + \frac{c^{1-\gamma}}{1-\gamma}\Bigg\}\end{aligned} \tag{6.5}$$

Next, assume that at the point (w, A, M, t), it is optimal to buy an annuity instantaneously. In other words, assume that the pensioner instantly moves from (w, A, M, t) to $(w - a\Delta A, A + \Delta A, M, t)$ for some $\Delta A > 0$. Therefore, the optimality of the decision implies

$$V(w, A, M, t) = V(w - a\Delta A, A + \Delta A, M, t) \tag{6.6}$$

where the derivatives are given by

$$aV_w(w, A, M, t) = V_A(w, A, M, t) \tag{6.7}$$

Then, by combining (6.5) and (6.7), the HJB equation associated with V given in (6.4), we have

$$\begin{cases} (\alpha+\lambda)V & = V_t + (rw + A)V_w + \max_y \left[\mu_4 yMV_w + \frac{1}{2}y^2 MV_{MM} + \xi y\sigma_4 MV_{wM} \right] \\ & \quad + \max_{c>0} \left[-cV_w + \frac{c^{1-\gamma}}{1-\gamma} \right] + k_H \left[\theta_H - M\right] V_M + \frac{1}{2}\sigma_4^2 MV_{MM}, 0 \leq \frac{w}{A} \leq \bar{z}; \\ V_A & = aV_w, \end{cases} \tag{6.8}$$

where V_t, V_w, V_M,V_{ww}, V_{MM} and V_{wM} denote the partial derivatives of first and second-order with respect to time, wealth and the instantaneous variance.

6.3.2 Reducing the dimension of the maximization problem

From (6.4) the value function V is in four variables dimension. Here, we reduced the dimension into a three-variable dimension, so it will be easier to apply the Legendre transform and solve the problem by transforming (w, A, M, t) to (z, M, t). Following the transformation method used by Milevsky et al. [84], the value function V is a function of the ratio $z = \frac{w}{A}$ and time t. Indeed, $V(w, A, M, t) = V(z, 1, M, t)$ and we defined $\widetilde{V}$ by

$$\widetilde{V}(z, M, t) = V(z, 1, M, t) \tag{6.9}$$

So that $V(w, A, M, t) = \widetilde{V}(z, M, t)$, and taking the derivatives with respect to t, w , A and M

$$\begin{aligned} &V_t = \widetilde{V}_t, \quad V_w = \tfrac{1}{A}\widetilde{V}, \quad V_{ww} = \tfrac{1}{A^2}\widetilde{V}_{zz}, \quad V_A = -\tfrac{z}{A}\widetilde{V}_z, \quad V_{wM} = \tfrac{1}{A}\widetilde{V}_{zM}, \\ &V_M = \widetilde{V}_M, \quad V_{MM} = \widetilde{V}_{MM}. \end{aligned}$$

Next, we substituted above derivatives into the HJB equation in (6.8)

$$\begin{aligned}(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z + \max_{\bar{y}}\left[\mu_4\bar{y}M\widetilde{V}_z + \frac{1}{2}\bar{y}^2 M\widetilde{V}_{zz} + \xi\bar{y}\sigma_4 M\widetilde{V}_{zM}\right] \\ + \max_{\bar{c}\geqslant 0}\left[-\bar{c}\widetilde{V}_z + \frac{\bar{c}^{1-\gamma}}{1-\gamma}\right] + k_H\left[\theta_H - M\right]\widetilde{V}_M + \frac{1}{2}\sigma_4^2 M\widetilde{V}_{MM}\end{aligned} \tag{6.10}$$

By applying the first-order conditions (FOCs), we are able to get the optimal strategies for optimal investment strategy and consumption.

i) Optimal investment strategy

$$\begin{aligned}\mu_4 M\widetilde{V}_z + \bar{y}_t^* M\widetilde{V}_{zz} + \xi\sigma_4 M\widetilde{V}_{zM} = 0, \\ \bar{y}_t^* = -\mu_4\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - \xi\sigma_4\frac{\widetilde{V}_{zM}}{\widetilde{V}_{zz}}\end{aligned} \tag{6.11}$$

ii) Optimal consumption

$$\begin{aligned}-\widetilde{V}_z + (1-\gamma)\frac{(\bar{c}_t^*)^{1-\gamma-1}}{1-\gamma} = 0, \\ \bar{c}_t^* = (\widetilde{V}_z)^{-\frac{1}{\gamma}}\end{aligned} \tag{6.12}$$

For the optimal annuity income level, we consider the wealth-to-income ratio as stated before.

$$\begin{aligned}z = \frac{w}{A_t^*}, \\ A_t^* = \frac{w}{z_t}\end{aligned} \tag{6.13}$$

Next, we substitute (6.11) and (6.12) into (6.10),

$$
\begin{aligned}
(\alpha+\lambda)\widetilde{V} &= \widetilde{V}_t + (rz+1)\widetilde{V}_z + \Bigg\{ \mu_4 M \left[-\mu_4 \frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - \xi\sigma_4 \frac{\widetilde{V}_{zM}}{\widetilde{V}_{zz}} \right] \widetilde{V}_z \\
&\quad + \frac{1}{2}\left[-\mu_4 \frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - \xi\sigma_4 \frac{\widetilde{V}_{zM}}{\widetilde{V}_{zz}} \right]^2 M\widetilde{V}_{zz} + \xi \left[-\mu_4 \frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - \xi\sigma_4 \frac{\widetilde{V}_{zM}}{\widetilde{V}_{zz}} \right] \sigma_4 M \widetilde{V}_{zM} \Bigg\} \\
&\quad + \left[-(\widetilde{V}_z)^{-\frac{1}{\gamma}}\widetilde{V}_z + \frac{\left((\widetilde{V}_z)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma} \right] + k_H\big[\theta_H - M\big]\widetilde{V}_M + \frac{1}{2}\sigma_4^2 M \widetilde{V}_{MM} \\
&= \widetilde{V}_t + (rz+1)\widetilde{V}_z + \left[-\frac{1}{2}\mu_4^2 M \frac{\widetilde{V}_z^2}{\widetilde{V}_{zz}} - \xi\mu_4\sigma_4 M \frac{\widetilde{V}_z\widetilde{V}_{zM}}{\widetilde{V}_{zz}} - \frac{1}{2}\xi^2\sigma_4^2 M \frac{\widetilde{V}_{zM}^2}{\widetilde{V}_{zz}} \right] \\
&\quad + \left[\frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}} \right] + k_H\big[\theta_H - M\big]\widetilde{V}_M + \frac{1}{2}\sigma_4^2 \widetilde{V}_{MM.}
\end{aligned}
$$

Simplify the above equation and obtain a second-order nonlinear PDE for value function $\widetilde{V}(z,M,t)$.

$$
\begin{aligned}
(\alpha+\lambda)\widetilde{V} = \widetilde{V}_t + (rz+1)\widetilde{V}_z - \frac{1}{2}\frac{\left[\mu_4\widetilde{V}_z + \xi\sigma_4\widetilde{V}_{zM}\right]^2}{\widetilde{V}_{zz}} + \frac{\gamma}{1-\gamma}\widetilde{V}_z^{\frac{\gamma-1}{\gamma}} \\
+ k_H\big[\theta_H - M\big]\widetilde{V}_M + \frac{1}{2}\sigma_4^2 M\widetilde{V}_{MM},
\end{aligned}
\tag{6.14}
$$

and the boundary condition in (6.8) becomes

$$
\begin{aligned}
-\frac{z}{A}\widetilde{V}_z &= a\frac{1}{A}\widetilde{V}_z, \\
(a-z)\widetilde{V}_z &= 0.
\end{aligned}
\tag{6.15}
$$

6.3.3 The Legendre transform and the dual theory

According to Definition 4.3.1, we define the Legendre transform for the Heston model case based on the convexity of the value function $\widetilde{V}(z,M,t)$.

$$
\widehat{V}(\rho,M,t) = \max_{z>0}\Big\{\widetilde{V}(z,M,t) - \rho z \Big| 0 < z < \infty\Big\}, \tag{6.16}
$$

where $\rho > 0$ is the dual variable to z. The value of z where this optimum is attained, denote by

$$g(\rho, M, t) = \min_{z>0} \left\{ z \middle| \widetilde{V}(z, M, t) \geqslant \rho z + \widehat{V}(\rho, M, t) \right\}. \tag{6.17}$$

From (6.16),

$$\rho = \widetilde{V}_z, \tag{6.18}$$

and the relationship between $\widehat{V}(\rho, M, t)$ and $g(\rho, M, t)$ can be determine by

$$g(\rho, M, t) = -\widehat{V}_\rho(\rho, M, t). \tag{6.19}$$

Remark 6.1. *We can choose either one of the functions $g(\rho, M, t)$ and $\widehat{V}(\rho, M, t)$ as the dual function of $\widetilde{V}(z, M, t)$. In this case, we choose $g(\rho, M, t)$ as the dual function of $\widetilde{V}(z, M, t)$ which is same as in CEV model (Chapter 4).*

Next, from (6.17) we have

$$\widehat{V}(\rho, M, t) = \widetilde{V}(g, M, t) - \rho g, \qquad g(\rho, M, t) = z. \tag{6.20}$$

Taking the derivatives of (6.20) with respect to t, z and M.

$$\begin{array}{llll} \widetilde{V}_t = \widehat{V}_t, & \widetilde{V}_z = \rho, & \widetilde{V}_M = \widehat{V}_M, & \widehat{V}_\rho = -\frac{1}{\widetilde{V}_z(\rho,M,t)} = -g, \\ \widetilde{V}_{zz} = -\frac{1}{\widehat{V}_{\rho\rho}}, & \widetilde{V}_{zM} = -\frac{\widehat{V}_{\rho M}}{\widehat{V}_{\rho\rho}}, & \widetilde{V}_{MM} = \widehat{V}_{MM} - \frac{\widehat{V}^2_{\rho M}}{\widehat{V}_{\rho\rho}}. & \end{array}$$

Substituting the above derivatives into HJB equation (6.14),

$$
\begin{aligned}
(\alpha+\lambda)[\widehat{V}+\rho g] &= \widehat{V}_t + (rg+1)\rho - \frac{1}{2}\frac{\Big[\mu_4\rho + \xi\sigma_4\Big(-\frac{\widehat{V}_{\rho M}}{\widehat{V}_{\rho\rho}}\Big)\Big]^2}{-\frac{1}{\widehat{V}_{\rho\rho}}} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}} \\
&\quad + k_H\big[\theta_H - M\big]\widehat{V}_M + \frac{1}{2}\sigma_4^2 M\left[\widehat{V}_{MM} - \frac{\widehat{V}_{\rho M}^2}{\widehat{V}_{\rho\rho}}\right] \\
&= \widehat{V}_t + (rg+1)\rho + \frac{1}{2}\mu_4^2\rho^2\widehat{V}_{\rho\rho} - \mu_4\rho\xi\sigma_4\widehat{V}_{\rho M} + k_H\big[\theta_H - M\big]\widehat{V}_M \\
&\quad + \frac{1}{2}\sigma_4^2 M\widehat{V}_{MM} + \frac{1}{2}\sigma_4^2 M(\xi^2-1)\frac{\widehat{V}_{\rho M}^2}{\widehat{V_{\rho\rho}}} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}
\end{aligned}
$$

Therefore, the equation becomes

$$
\begin{aligned}
0 &= \widehat{V}_t - (\alpha+\lambda)\widehat{V} - (\alpha+\lambda)\rho g + (rg+1)\rho + \frac{1}{2}\mu_4^2\rho^2\widehat{V}_{\rho\rho} - \mu_4\rho\xi\sigma_4\widehat{V}_{\rho M} \\
&\quad + k_H\big[\theta_H - M\big]\widehat{V}_M + \frac{1}{2}\sigma_4^2 M\widehat{V}_{MM} + \frac{1}{2}\sigma_4^2 M(\xi^2-1)\frac{\widehat{V}_{\rho M}^2}{\widehat{V}_{\rho\rho}} + \frac{\gamma}{1-\gamma}\rho^{\frac{\gamma-1}{\gamma}}
\end{aligned}
\tag{6.21}
$$

Next, considering (6.19) and differentiating (6.21) with respect to ρ,

$$
\begin{array}{lll}
\widehat{V}_{\rho t} = -g_t, & \widehat{V}_{\rho M} = -g_M, & \widehat{V}_{\rho\rho} = -g_\rho, \\
\widehat{V}_{\rho\rho M} = -g_{\rho M}, & \widehat{V}_{\rho MM} = -g_{MM}, & \widehat{V}_{\rho\rho\rho} = -g_{\rho\rho}.
\end{array}
\tag{6.22}
$$

Then, we have

$$
\begin{aligned}
0 &= -g_t + (\alpha+\lambda)g - (\alpha+\lambda)\rho g_\rho + rg + r\rho g_\rho + 1 - \frac{1}{2}\mu_4^2\rho^2 g_{\rho\rho} - \mu_4^2\rho g_\rho + \mu_4\rho\xi\sigma_4 g_{\rho M} \\
&\quad + \mu_4\xi\sigma_4 g_M - k_H\big[\theta_H - M\big]g_M - \frac{1}{2}\sigma_4^2 M g_{MM} + \frac{1}{2}\sigma_4^2 M(\xi^2-1)\frac{g_M^2 g_{\rho\rho} - 2g_\rho g_{\rho M} g_M}{g_\rho^2} \\
&\quad - \rho^{-\frac{1}{\gamma}}.
\end{aligned}
$$

Simplifying the equation above,

$$
\begin{aligned}
0 = g_t &+ \Big[(\alpha+\lambda) + \mu_4^2 - r\Big]\rho g_\rho - rg + \frac{1}{2}\mu_4^2\rho^2 g_{\rho\rho} - \mu_4\rho\xi\sigma_4 g_{\rho M} \\
&+ \Big[k_H\big[\theta_H - M\big] - \mu_4\xi\sigma_4\Big]g_M + \frac{1}{2}\sigma_4^2 M g_{MM} + \frac{1}{2}\sigma_4^2 M(1-\xi^2)\frac{g_M^2 g_{\rho\rho} - 2g_\rho g_{\rho M} g_M}{g_\rho^2} \\
&+ \rho^{-\frac{1}{\gamma}} - 1
\end{aligned}
\tag{6.23}
$$

The problem now is to solve (6.23) for g in order to replace and obtain the optimal investment, consumption and annuity income level.

6.3.4 The explicit solution for the optimal strategies

Through this subsection, we discussed the procedure to get the explicit solution of optimal investment, consumption and annuity income level for CRRA utility function via a variable change technique. We used the same technique to solve the CEV model in Chapter 4. We introduced the proposition for the explicit solution for optimal investment, consumption and annuity income level. Wang et al. [124] used the same technique for the Heston model but with a different approximate solution.

As discussed before, we used the CRRA utility function as described in (3.1). Therefore, the approximate solution, after considering the behavior of (6.23) is

$$
g(\rho, M, t) = J(M,t)\rho^{-\frac{1}{\gamma}} + b(t), \tag{6.24}
$$

with the boundary conditions stated by $b(\tau) = 0$ and $J(M,\tau) = 1$. Differentiating with respect to t, M and ρ gives,

$$
\begin{array}{lll}
g_t = J_t\rho^{-\frac{1}{\gamma}} + b'(t), & g_M = J_M\rho^{-\frac{1}{\gamma}}, & g_\rho = -\frac{1}{\gamma}J\rho^{-\frac{1}{\gamma}-1}, \\
g_{MM} = J_{MM}\rho^{-\frac{1}{\gamma}}, & g_{\rho\rho} = \frac{1+\gamma}{\gamma^2}J\rho^{-\frac{1}{\gamma}-2}, & g_{\rho M} = -\frac{1}{\gamma}J_M\rho^{-\frac{1}{\gamma}-1}.
\end{array}
$$

Next, we replace this into (6.23)

$$
\begin{aligned}
0 = {} & J_t\rho^{-\frac{1}{\gamma}} + b'(t) - \big[(\alpha+\lambda) + \mu_4^2 - r\big]\frac{1}{\gamma}J\rho^{-\frac{1}{\gamma}} - rJ\rho^{-\frac{1}{\gamma}} - rb(t) + \frac{1}{2}\mu_4^2\frac{1+\gamma}{\gamma^2}J\rho^{-\frac{1}{\gamma}} \\
& + \mu_4\xi\sigma_4\frac{1}{\gamma}J_M\rho^{-\frac{1}{\gamma}} + k_H\big[\theta_H - M\big]J_M\rho^{-\frac{1}{\gamma}} - \mu_4\xi\sigma_4 J_M\rho^{-\frac{1}{\gamma}} + \frac{1}{2}\sigma_4^2 M J_{MM}\rho^{-\frac{1}{\gamma}} \\
& + \frac{1}{2}\sigma_4^2 M(1-\xi^2)\left[\frac{J_M^2\rho^{-\frac{2}{\gamma}}\Big(\frac{1+\gamma}{\gamma^2}J\rho^{-\frac{1}{\gamma}-2}\Big) - 2\Big(\frac{1}{\gamma^2}JJ_M^2\rho^{-\frac{3}{\gamma}-2}\Big)}{\frac{1}{\gamma^2}J^2\rho^{-\frac{2}{\gamma}-2}}\right] + \rho^{-\frac{1}{\gamma}} - 1.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
0 = \rho^{-\frac{1}{\gamma}}\Bigg[& J_t + \left(\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right)J_M + \frac{1}{2}\sigma_4^2 M J_{MM} + \frac{\mu_4^2(1-\gamma)}{2\gamma^2}J \\
& + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}J + \frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)\frac{J_M^2}{J} + 1\Bigg] + \Big[b'(t) - rb(t) - 1\Big].
\end{aligned} \tag{6.25}
$$

Now, we can split (6.25) into two equations in order to remove the ρ term in the equation,

$$
b'(t) - rb(t) - 1 = 0, \tag{6.26}
$$

and

$$
\begin{aligned}
& J_t + \left[\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right]J_M + \frac{1}{2}\sigma_4^2 M J_{MM} + \frac{\mu_4^2(1-\gamma)}{2\gamma^2}J \\
& + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}J + \frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)\frac{J_M^2}{J} + 1 = 0.
\end{aligned} \tag{6.27}
$$

Taking into consideration the boundary condition at $b(\tau) = 0$, the solution for (6.26) is as follows

$$
b(t) = -\frac{1}{r}\Big[1 - e^{-r(\tau - t)}\Big]. \tag{6.28}
$$

According to (6.27), there is a constant in the equation. Hence, we wish to remove it. Then, the proposition is used to explain the procedure. The same procedure is used for the CEV model in Chapter 4 by taking the concept proposed by Chang and Chang in [66] and [89].

Proposition 6.3.1. *Consider that $J(M,t) = \int_t^\tau \widehat{J}(M,u)du + \widehat{J}(M,t)$ is a solution of (6.27), then $\widehat{J}(M,t)$ meet the following equation*

$$\widehat{J}_t + \left(\frac{\mu_4 \xi \sigma_4 (1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right)\widehat{J}_M + \frac{1}{2}\sigma_4^2 M \widehat{J}_{MM} + \frac{\mu_4^2(1-\gamma)}{2\gamma^2}\widehat{J}$$
$$+ \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\widehat{J} + \frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)\frac{\widehat{J}_M^2}{\widehat{J}} = 0, \quad \textit{with} \quad \widehat{J}(M,\tau) = 1. \tag{6.29}$$

Proof. Propose the following differential operator ∇ on any function $J(M,t)$,

$$\nabla J(M,t) = \left[\frac{\mu_4^2(1-\gamma)}{2\gamma^2} + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right] J + \left[\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right] J_M$$
$$+ \frac{1}{2}\sigma_4^2 M J_{MM} + \frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)\frac{J_M^2}{J}$$

Address (6.27) as

$$\frac{\partial J(M,t)}{\partial t} + \nabla J(M,t) + 1 = 0. \tag{6.30}$$

In accordance with $J(M,t) = \int_t^\tau \widehat{J}(M,u)du + \widehat{J}(M,t)$, we gain

$$\begin{aligned}\frac{\partial J(M,t)}{\partial t} &= -\widehat{J}(M,t) + \frac{\partial \widehat{J}(M,t)}{\partial t} \\ &= \left(\int_t^\tau \frac{\partial \widehat{J}(M,u)}{\partial u}du - \widehat{J}(M,t)\right) + \frac{\partial \widehat{J}(M,t)}{\partial t}\end{aligned}$$

and

$$\nabla J(M,t) = \int_t^{\tau} \nabla \widehat{J}(M,u)du + \nabla \widehat{J}(M,t).$$

Thus, (6.30) can be edited as

$$\left(\int_t^{\tau} \frac{\partial \widehat{J}(M,u)}{\partial u}du - \widehat{J}(M,\tau)\right) + \frac{\partial \widehat{J}(M,t)}{\partial t} + \left(\int_t^{\tau} \nabla \widehat{J}(M,u)du + \nabla \widehat{J}(M,t)\right) + 1 = 0,$$
$$\left[\int_t^{\tau} \left(\frac{\partial \widehat{J}(M,u)}{\partial u} + \nabla \widehat{J}(M,u)\right)du - \widehat{J}(M,\tau) + 1\right] + \left(\frac{\partial \widehat{J}(M,t)}{\partial t} + \nabla \widehat{J}(M,t)\right) = 0.$$

Overall, we achieve

$$\frac{\partial \widehat{J}(M,t)}{\partial t} + \nabla \widehat{J}(M,t) = 0,$$

where

$$\widehat{J}(M,\tau) = 1.$$

So, this completes the proof for the proposition.

□

From Proposition 6.3.1, (6.27) turns into

$$\begin{aligned} J_t &+ \left(\frac{\mu_4 \xi \sigma_4 (1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right) J_M + \frac{1}{2}\sigma_4^2 M J_{MM} + \frac{\mu_4^2(1-\gamma)}{2\gamma^2} J \\ &+ \frac{r(1-\gamma) - (\alpha+\lambda)}{\gamma} J + \frac{1}{2}\sigma_4^2 M (1-\xi^2)(\gamma - 1)\frac{J_M^2}{J} = 0. \end{aligned} \tag{6.31}$$

Next, we worked on finding the solution for (6.31), utilizing the following proposition

Proposition 6.3.2. *Expect that the solution for (6.31) is conjecture as $J(M,t) = D(t)e^{E(t)M}$, with terminal conditions given by $D(\tau) = 1$ and $E(\tau) = 0$, thus $D(t)$ and $E(t)$ are*

$$D(t) = e^{\left\{\psi_1\left(\frac{\mu_4\xi\sigma_4(1-\gamma)k_H}{\gamma}+k_H^2\theta_H\right)+\frac{\mu_4^2(1-\gamma)}{2\gamma^2}+\frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}(\tau-t)} \times \left[\frac{\psi_2-\psi_1}{\psi_2-\psi_1 e^{\left[\frac{1}{2}\sigma_4^2[1+(1-\xi^2)(\gamma-1)](\psi_1-\psi_2)(\tau-t)\right]}}\right]^{\frac{2k_H[\gamma k_H\theta_H+\mu_4\xi\sigma_4(1-\gamma)]}{\gamma\sigma_4^2[1+(1-\xi^2)(\gamma-1)]}} \tag{6.32}$$

$$E(t) = k_H Q(t) \quad \textit{with} \quad Q(t) = \frac{\psi_1-\psi_1 e^{\left\{\frac{1}{2}\sigma_4^2[1+(1-\xi^2)(\gamma-1)]\right\}(\psi_1-\psi_2)(\tau-t)}}{1-\frac{\psi_1}{\psi_2}e^{\left\{\frac{1}{2}\sigma_4^2[1+(1-\xi^2)(\psi_1-\psi_2)(\tau-t)]\right\}}} \tag{6.33}$$

Proof. The approximate solution for (6.31) is

$$J(M,t) = D(t)e^{E(t)M}, \qquad D(\tau)=1, \qquad E(\tau)=0. \tag{6.34}$$

Differentiating the approximate solution with respect to M and t, gives

$$\begin{aligned} J_t &= D(t)E'(t)Me^{E(t)M} + D'(t)e^{E(t)M} = JE'(t)M + J\frac{D'(t)}{D(t)}, \\ J_M &= D(t)E(t)e^{E(t)M} = JE(t), \\ J_{MM} &= D(t)E(t)e^{E(t)M}\cdot E(t) = JE^2(t) \end{aligned}$$

Adding the derivatives into (6.31) it becomes

$$\left[JE'(t)M + \frac{JD'(t)}{D(t)}\right] + \left[\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma} + k_H\big[\theta_H - M\big]\right]\big[JE(t)\big] + \frac{1}{2}\sigma_4^2 M\big[JE^2(t)\big] + \frac{\mu_4^2(1-\gamma)}{2\gamma^2}J + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}J + \frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)\frac{\big[JE(t)\big]^2}{J} = 0$$

We arrive at

$$J\Bigg[E'(t)M+\frac{D'(t)}{D(t)}+\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma}E(t)+k_H\theta_H E(t)-k_H ME(t)+\frac{1}{2}\sigma_4^2 ME^2(t)$$
$$+\frac{\mu_4^2(1-\gamma)}{2\gamma^2}+\frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}+\frac{1}{2}\sigma_4^2 M(1-\xi^2)(\gamma-1)E^2(t)\Bigg]=0.$$

Next, by erasing the term J, the equation becomes as follows

$$\frac{D'(t)}{D(t)}+\frac{\mu_4\xi\sigma_4(1-\gamma)}{\gamma}E(t)+k_H\theta_H E(t)+\frac{\mu_4^2(1-\gamma)}{2\gamma^2}+\frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}$$
$$+M\Big[E'(t)-k_H E(t)+\frac{1}{2}\sigma_4^2E^2(t)+\frac{1}{2}\sigma_4^2(1-\xi^2)(\gamma-1)E^2(t)\Big]=0. \tag{6.35}$$

We split (6.35) into two conditions in order to ignore the dependence in M and t.

$$E'(t)-kE(t)+\frac{1}{2}\sigma_4^2\big[1+(1-\xi^2)(\gamma-1)\big]E^2(t)=0,\qquad E(\tau)=0 \tag{6.36}$$

and

$$\frac{D'(t)}{D(t)}+\Bigg[\frac{\mu_4\xi\sigma_4(1-\gamma)k_H}{\gamma}+k_H^2\theta_H\Bigg]\frac{E(t)}{k_H}+\frac{\mu_4^2(1-\gamma)}{2\gamma^2}+\frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}=0,\qquad D(\tau)=1 \tag{6.37}$$

To make it easier to solve, let

$$h=-\frac{1}{2}\sigma_4^2\big[1+(1-\xi^2)(\gamma-1)\big],\qquad i=k,\qquad j=0.$$

Next, we solve the splitting one by one. We start with (6.36),

$$\begin{aligned}\frac{dE(t)}{dt}&=k_H E(t)-\frac{1}{2}\sigma_4^2\big[1+(1-\xi^2)(\gamma-1)\big]E^2(t),\\&=hE^2(t)+iE(t)\qquad\text{with}\qquad E(\tau)=0.\end{aligned} \tag{6.38}$$

Taking integration with respect to time t for both sides of (6.38),

$$\int \frac{dE(t)}{dt}dt = \int \big(hE^2(t) + iE(t)\big)dt,$$

we solve the integration and we have

$$\frac{1}{h(m_1 - m_2)} \int \left(\frac{1}{E(t) - m_1} - \frac{1}{E(t) - m_2} \right) dE(t) = t + \mathcal{C} \tag{6.39}$$

where $\mathcal{C}$ is a constant and $m_{1,2}$ are the solutions of the quadratic equation

$$hm^2 + im = 0. \tag{6.40}$$

Hence, using quadratic factorization, we have to solve for m_1 and m_2

$$m_1 = 0, \tag{6.41}$$

and

$$m_2 = \frac{i}{\frac{1}{2}\sigma_4^2\big[1 + (1 - \xi^2)(\gamma - 1)\big]}. \tag{6.42}$$

We found that equation (6.39) is in the same form as in (A.11) in Appendix A.2 in Gao [65]. Then, the solution for (6.39) is

$$E(t) = \frac{m_1 - m_1 e^{h(m_1 - m_2)(t-\tau)}}{1 - \frac{m_1}{m_2} e^{h(m_1 - m_2)(t-\tau)}}. \tag{6.43}$$

Next, we define ψ_1 and ψ_2 from m_1 and m_2 by considering (6.43),

$$\psi_1 = 0, \tag{6.44}$$

and

$$\psi_2 = \frac{1}{\frac{1}{2}\sigma_4^2\big[1 + (1 - \xi^2)(\gamma - 1)\big]} \tag{6.45}$$

Thus, in proposition statement (6.43) can be revised as (6.33). Next, we proceed solving for $D(t)$. Using $Q(t)$ in (6.33), (6.27) is rewritten as

$$\frac{D'(t)}{D(t)} = -\left[\frac{\mu_4\xi\sigma_4(1-\gamma)k_H}{\gamma} + k_H^2\theta_H\right]Q(t) - \frac{\mu_4^2(1-\gamma)}{2\gamma^2} - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}$$

and we have

$$\frac{dD(t)}{dt} = \left\{-\left[\frac{\mu_4\xi\sigma_4(1-\gamma)k_H}{\gamma} + k_H^2\theta_H\right]Q(t) - \frac{\mu_4^2(1-\gamma)}{2\gamma^2} - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}dt. \tag{6.46}$$

From $Q(t)$ in (6.33) and taking its integration, we get

$$\int Q(t) = \psi_1 t + \frac{2}{\sigma_4^2\left[1+(1-\xi^2)(\gamma-1)\right]}\ln\left(\psi_2 - \psi_1 e^{\left[\sigma_4^2[1+(1-\xi^2)(\gamma-1)](\psi_1-\psi_2)(\tau-t)\right]}\right) + \mathcal{C} \tag{6.47}$$

where $\mathcal{C}$ is a constant. Lastly, we integrate (6.46) with respect to time t and we solve for $D(t)$ as in (6.32) in the proposition statement.

□

From proving Proposition 6.3.2, we notice that one of the factors for (6.40), $m_1 = 0$ (6.41). In this situation, the solution for $D(t)$ and $E(t)$ becomes much easier to solve. Based on this case, it leads to $E(t) = 0$ for (6.33) and for $D(t)$, we have

$$\frac{dD(t)}{D(t)} = \left\{-\frac{\mu_4^2(1-\gamma)}{2\gamma^2} - \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}dt.$$

Solving the equation we arrive at

$$D(t) = e^{\left\{\frac{\mu_4^2(1-\gamma)}{2\gamma^2} + \frac{r(1-\gamma)-(\alpha+\lambda)}{\gamma}\right\}(\tau-t)}. \tag{6.48}$$

Before we solve for the optimal investment strategy, consumption and annuity income level, we need to solve for ρ. Using the boundary condition in (6.15) and considering (6.18), (6.20) and (6.24), we have

$$\rho = \left[\frac{a - b(t)}{J(S,t) - \frac{1}{\gamma} J(S,t)} \right]^{-\gamma} \tag{6.49}$$

Next, we focus on the solution for optimal strategies. The proposition is used during the derivation so that it clearer to refer to. First of all, the proposition for optimal investment strategy is defined as below:

Proposition 6.3.3. *The explicit solution for optimal investment strategy is defined as*

$$y_t^* = \frac{\mu_4}{\gamma} A[g(\rho, M, t) - b(t)] \tag{6.50}$$

where

$$g(\rho, M, t) = J(M,t)\rho^{-\frac{1}{\gamma}} + b(t), \qquad \textit{with} \qquad J(M,t) = D(t)e^{E(t)M}$$

while the solution for $J(M,t)$ is defined in Proposition 6.3.2 and

$$b(t) = -\frac{1}{r}\left[1 - e^{-r(\tau - t)}\right].$$

Proof. Call up the optimal investment strategy as defined in (6.11). Then, by considering (6.18), (6.20), (6.22), (6.24) and Proposition 6.3.2, we solve

$$
\begin{aligned}
\bar{y}_t^* &= -\frac{\widetilde{V}_z}{\widetilde{V}_{zz}} - \xi\sigma_4\frac{\widetilde{V}_{zM}}{\widetilde{V}_{zz}},\\
&= -\frac{\mu_4\rho}{\left(-\frac{1}{\widehat{V}_{\rho\rho}}\right)} - \xi\sigma_4\frac{\left(-\frac{\widehat{V}_{\rho M}}{\widehat{V}_{\rho\rho}}\right)}{\left(-\frac{1}{\widehat{V}_{\rho\rho}}\right)},\\
&= \mu_4\rho\widehat{V}_{\rho\rho} - \xi\sigma_4\widehat{V}_{\rho M},\\
&= -\mu_4\rho g_\rho + \xi\sigma_4 g_M,\\
&= -\mu_4\left[-\frac{1}{\gamma}J\rho^{-\frac{1}{\gamma}}\right] + \xi\sigma_4\left[J_M\rho^{-\frac{1}{\gamma}}\right],\\
&= \frac{\mu_4}{\gamma}\big[g-b(t)\big] + \xi\sigma_4 J_M\left[\frac{g-b(t)}{J}\right].
\end{aligned}
$$

Thus, we have

$$
\bar{y}_t^* = \big[g-b(t)\big]\frac{\mu_4}{\gamma}\big(1+\xi\sigma_4 E(t)\big) \tag{6.51}
$$

Since we already have solved that $E(t) = 0$, therefore

$$
\bar{y}_t^* = \frac{\mu_4}{\gamma}\big[g-b(t)\big]. \tag{6.52}
$$

As mentioned before, $\bar{y} = \frac{y}{A}$ and $g = g(\rho, M, t)$; we obtain the optimal investment strategy as in proposition statement.

□

From Proposition 6.3.3, we notice that the optimal investment strategy for our case does not depend on the $cov(B_s^1, B_s^2) = \xi t$. This makes our model reduce to complete market.

Secondly, the proposition for optimal consumption is introduced.

Proposition 6.3.4. *The explicit solution for optimal consumption for Heston's SV model is given by*

$$
c_t^* = A\left[\frac{\big[g(\rho, M, t) - b(t)\big]}{D(t)}\right] \tag{6.53}
$$

where

$$g(\rho, M, t) = J(M, t)\rho^{-\frac{1}{\gamma}} + b(t), \qquad \textit{with} \qquad J(M, t) = D(t)e^{E(t)M}$$

while the solutions for $J(M,t)$ and $D(t)$ are defined in Proposition 6.3.2 and

$$b(t) = -\frac{1}{r}\Big[1 - e^{-r(\tau - t)}\Big].$$

Proof. As defined in (6.12), we solve for optimal consumption by taking into account (6.18), (6.24) and Proposition 6.3.2. Therefore, we have

$$\begin{aligned}\bar{c}_t^* &= \big(\widetilde{V}_z\big)^{-\frac{1}{\gamma}} \\ &= \rho^{-\frac{1}{\gamma}} \\ &= \frac{\big[g - b(t)\big]}{J} \\ &= \frac{\big[g - b(t)\big]}{D(t)e^{E(t)M}}\end{aligned}$$

In Proposition 6.3.2, we showed that $E(t) = 0$, thus

$$\bar{c}_t^* = \frac{\big[g - b(t)\big]}{D(t)}$$

We also stated in Subsection 6.3.2, that $\bar{c} = \frac{c}{A}$ and in Proposition 6.3.2, $g = g(\rho, M, t)$. Therefore, we arrive at the final solution for the optimal consumption as in the proposition statement.

□

Finally, we defined the proposition for optimal annuity income level as below

Proposition 6.3.5. *The optimal annuity income level is given by*

$$A_t^* = \frac{w}{g(\rho, M, t)} \tag{6.54}$$

where

$$g(\rho, M, t) = J(M,t)\rho^{-\frac{1}{\gamma}} + b(t), \qquad \textit{with} \qquad J(M,t) = D(t)e^{E(t)M}$$

while the solutions for $J(M,t)$ and $D(t)$ are defined in Proposition 6.3.2.

Remark 6.2. *Proving Proposition 6.3.5, is as same as proving for Proposition 4.3.5.*

6.4 Numerical example

This section provided a numerical example to illustrate the effects of model parameters on the optimal strategies for investment and life annuities problem. We examined the impact of the key parameters on the optimal investment strategy, optimal consumption and optimal annuity income level for a pensioner after retirement.

The parameter values in this section are taken from literature such as Li et al. [75], Chang and Rong [59], Chunxiang and Li [77] and Zhang and Ge [123] that we have used as our guideline. The following parameters are fixed throughout this section:

- $r = 0.03$; the risk-free interest rate.
- $\mu_4 = 1.5$; the expected return of the risky asset.
- $\sigma_4 = 0.8$; the volatility of the volatility, or 'vol of vol', which determines the variance of $M(t)$.
- $\alpha = 0.05$; the discount rate.
- $k_H = 2$; the constant parameter for the Heston model.
- $\xi = 0.3$; the $Cov(B_t^1, B_t^2) = \xi t$, the correlation of two-dimensional Brownian motion.
- $\theta_H = 0.3$; long variance, long-run average price variance; as t tends to infinity, the expected value of $M(t)$ tends to θ_H.
- $M(0) = 0.36$; instantaneous variance, is a Cox-Ingersoll-Ross (CIR) process.

- $t = [0, 16]$; the years after retirement until expected lifetime, where $t = 0$ is the year at retirement age which is 65 and $t = 16$ as the pensioner is expected to live until 81. See [3, 5]
- $\tau = 16$; the expected lifetime after retirement.
- $\lambda = \frac{1}{\tau}$; the mortality rate based on the expected lifetime after retirement.

In this chapter, we were able to obtain the purchasing boundary $g(\rho, M, t)$. Hence, before we analyze the optimal strategies, we have to solve for the purchasing boundary by applying the parameters into (6.24) and Proposition 6.3.2 for the price of a life annuity as defined in (2.8) in Chapter 3.

The results are divided into three parts, which are optimal investment strategy, optimal consumption and optimal annuity income level. We havefound the Heston model has also become popular among the researchers who were interested in stochastic volatility. However, most studies only analyze the model for optimal investment and do not consider the presence of annuity income in the model. This is different from our model analysis, where we focused on three optimal strategies. Thus, we believe this result will fill the gap on the Heston model analysis in the literature .

6.4.1 Optimal investment strategy

We started the discussion with an optimal investment strategy. Here, Proposition 6.3.3 is used by inserting the values of the parameters.

Figure 6.1 plots the relationship between optimal investment and a pensioner's risk preferences for two different levels of annuity income. The graph shows the optimal investment decreases as the risk aversion increases. The higher the number of risk aversion is more dislike risk pensioner. This can explain why a risks-loving pensioner will invest more in stock assets compared to one who is more risk-averse. This happened since the individuals who dislike risk are not confident with the risky investment, and they prefer a more guaranteed and safety investment, for example, in life annuities.

The studies that came out with the same relationship were Li et al. [75], Chunxiang and Li [77] and Zhang et al. [121]. However, the way the model is tested is different in our study. Li et al. [75] solved for an optimal time-consistent strategy

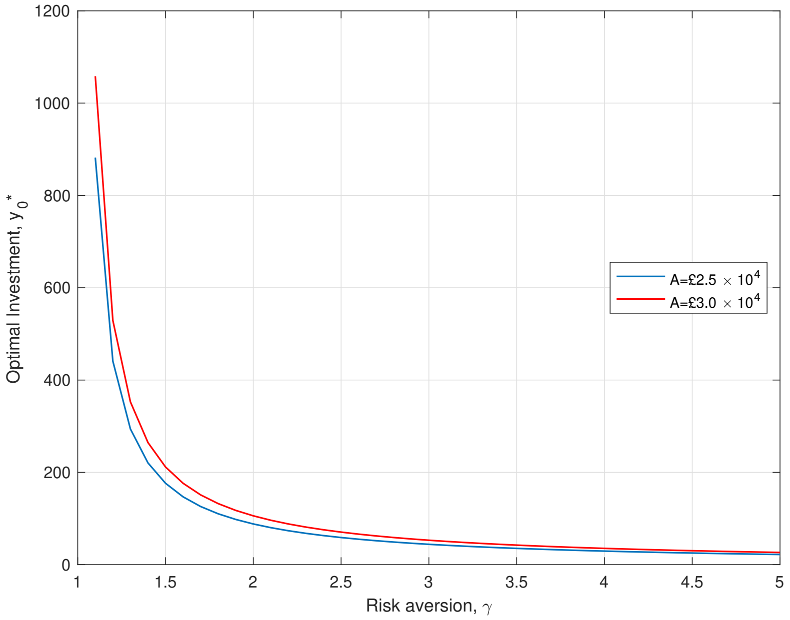

FIGURE 6.1: Effect on optimal investment of changes in the risk aversion for a different level of annuity income at retirement $t_0 = 65$.

for investment and reinsurance strategy. Then, Chunxiang and Li [77] considered the optimal investment and excess-of-loss reinsurance problem with a delay factor. In the most recent study, Zhang et al. [121] investigated the asset-liability management problem for an ordinary insurance system.

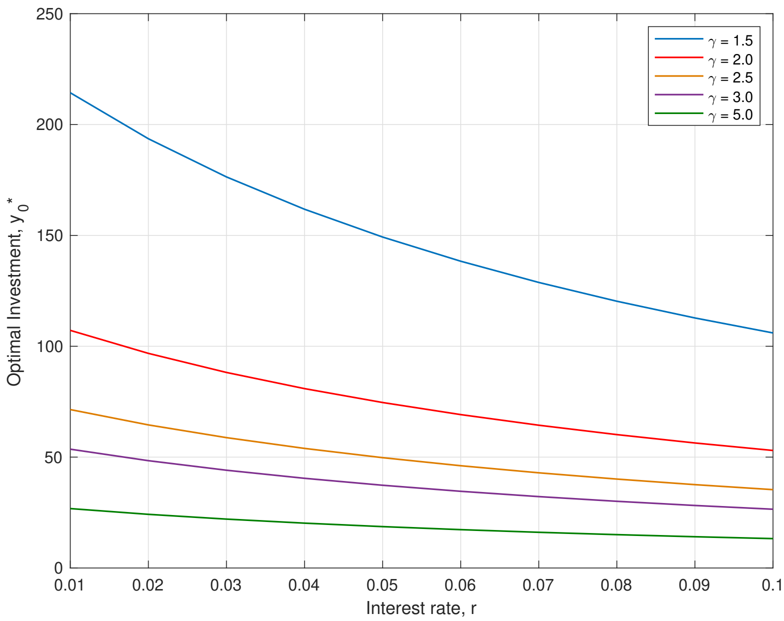

FIGURE 6.2: Effect on optimal investment of changes in the interest rate for a different level of risk aversion at retirement $t_0 = 65$.

Next, we investigated the impact of the risk-free interest rate on the optimal investment strategy as in Figure 6.2. Li et al. [75], Zhao et al. [76] and Zhang et al. [121] studied the relationship between optimal investment and the risk-free interest rate. For Li et al. [75] and Zhang et al. [121], we have discussed before how the model was tested. Zhao et al. [76] studied the optimal excess-of-loss reinsurance and investment problem for an insurer with a jump-diffusion risk model. These differ from our study. From the observation, the larger the risk-free interest rate, the smaller the amount of wealth invested in the risky asset. We can also say that no matter if the pensioner chooses to invest in life annuities or not, he/she will reduce their investment in the risky asset and continue to invest in the risk-free asset.

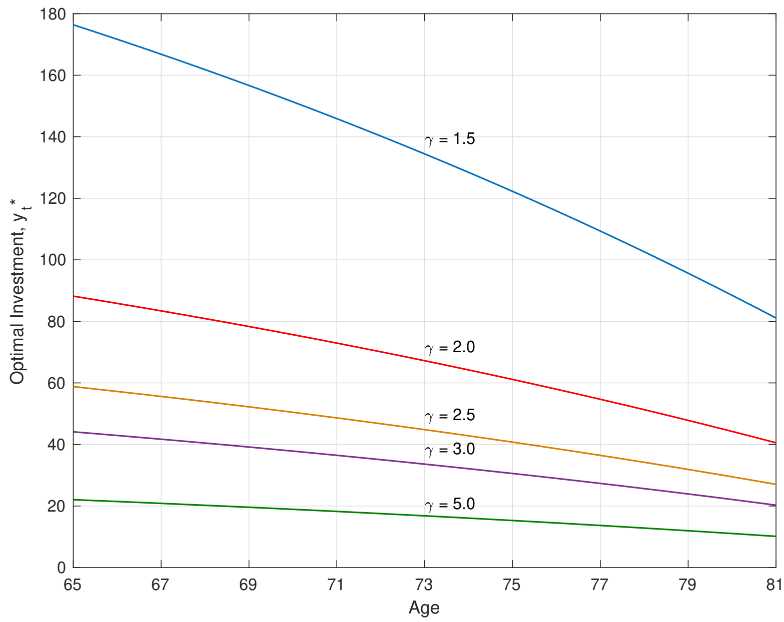

FIGURE 6.3: Effect on optimal investment of post-retirement age for a different level of risk aversion.

Finally, we tested the relationship of an optimal investment strategy with age-dependent. Figure 6.3 shows the pensioner will reduce the investment in the risky asset as their age increases in the decumulation phase or post-retirement. This result is different compared to the literature Li et al. [75], Zhao et al. [76], Chunxiang and Li [77] and Zhang et al. [121], since literature showed the investment would increase with respect to time t. This happens because the literature did not consider the existing annuity income after retirement. Blake et al. [128] assumed the pensioner was a rational life-cycle financial planner and has an Epstein-Zin utility function and allowed for an annuitization decision to be endogenously determined during the post-retirement (decumulation) phase.

Blake et al. [128] support the result for this study, as Blake et al. [128] found the bonds held at retirement are exchanged to life annuities during the post-retirement phase for optimal investment strategy and then gradually the remaining stocks is sold and more annuities are bought. Gupta and Li [24] stated the proportion invested in the risky asset decreases from a high value in the early period of an annuity income payment.

6.4.2 Optimal consumption

The discussions on the optimal consumption under the Heston model are popular as an optimal investment strategy. Some studies focusing on this are Chang and Rong [59] and Blake et al. [128].

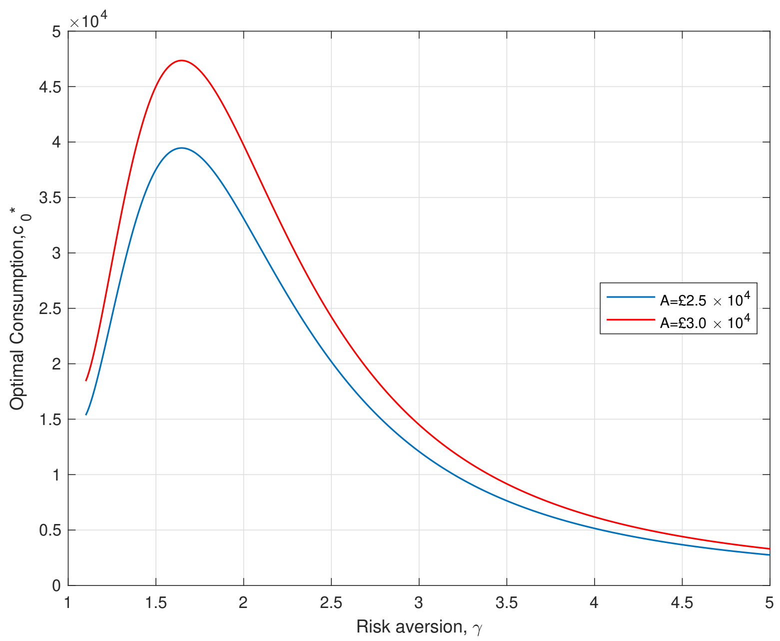

FIGURE 6.4: Effect on optimal consumption of risk aversion for a different level of annuity income.

Figure 6.4 shows the optimal consumption for a different level of annuity income changes when risk-aversion changes. From the graph, we notice the optimal consumption increases at lower risk aversion and will decreases as the risk aversion increases. This shows the consumption has a non-monotonic relationship with risk aversion. Wang and Young [38] showed the non-monotonic relationship between consumption and risk aversion for the GBM model in table form. According to Blake et al. [128], the consumption is more volatile as the risk aversion decreases (below baseline, $\gamma = 2.5$), as illustrated in Figure 6.4.

Besides, we also can say the more risk-averse the pensioner, the less they can consume at the age of retirement. This is because a more risk-averse pensioner does not want to risk not having enough money/wealth in the future to continue their happy life.

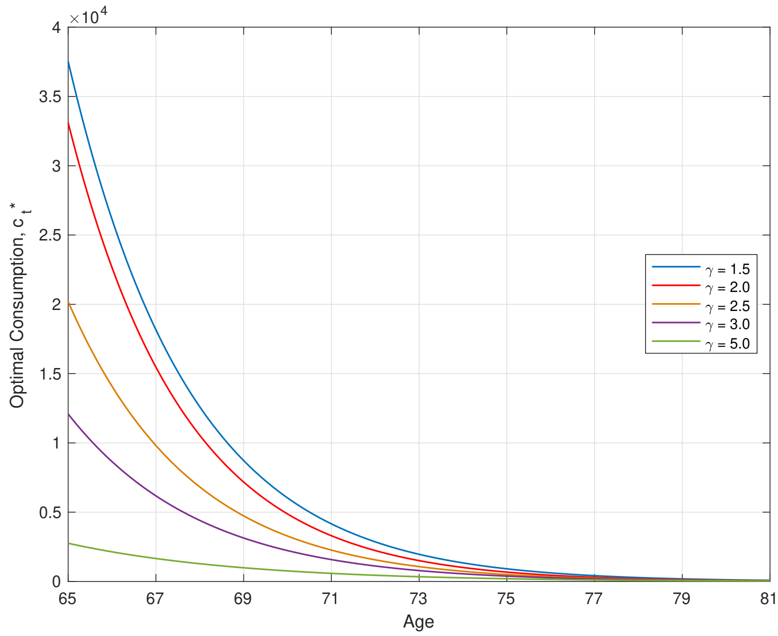

FIGURE 6.5: Effect on optimal consumption of post-retirement phase for a different level of risk aversion.

Then, we extended the discussion by considering the age after retirement. We wanted to see how it affected the pensioner's decisions on consumption. Figure 6.4 shows that the pensioner will consume less after retirement. This relationship is the same as the one observed on optimal consumption for the CEV model in Chapter 4.

6.4.3 Optimal annuity income level

The next discussion is on optimal annuity income level. To the best of our knowledge, we believe this discussion fills the gap in literature since we introduced the annuity income variable in the dynamic wealth process under the Heston model and analyze the optimal annuity income which has not been discussed yet.

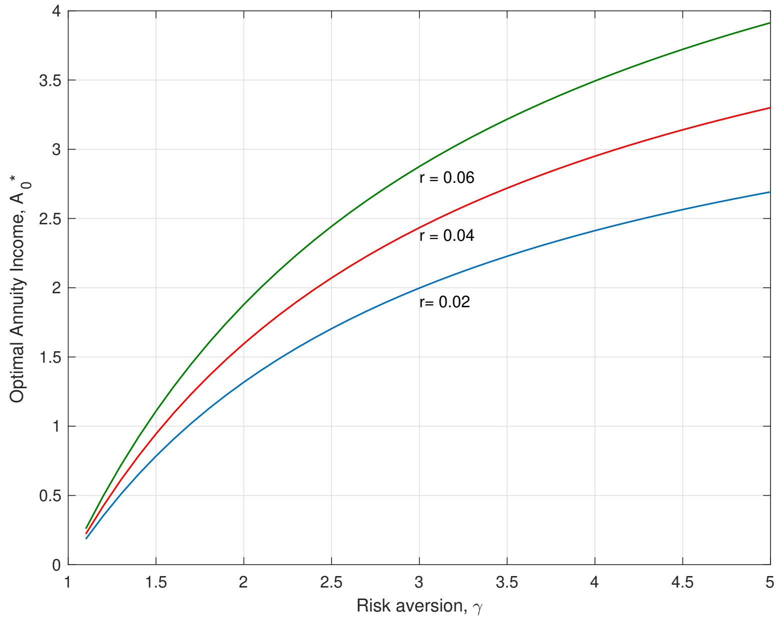

FIGURE 6.6: Effect on the optimal annuity income level of risk aversion for a different level of the interest rate.

First, we tested the relationship of an annuitization with a pensioner's risk preferences for a different level of the risk-free interest rate. Figure 6.6 shows the annuitization increases as the level of risk aversion increases. This means the pensioner who dislikes risks will annuitize more in life annuities since these are the only guaranteed income after retirement. We also notice the optimal annuity income for a higher interest rate is bigger. This happens because the pensioner will continue to invest in risk-free assets even if they have invested in life annuities and risky assets.

This relationship supports the result in Figure 6.2. When the pensioner decides to invest in life annuities and risk-free assets, he/she will also invest in a stock asset.

Lastly, we illustrated the relationship between optimal annuity income level and wealth status for age-dependence in Figure 6.7. The figure shows the optimal annuity income increases with respect to wealth status. This explains why pensioners are happy to invest more in life annuities if they are wealthy enough. Annuitization also increases after retirement. According to Blake et al. [128], the investor will purchase additional life annuities in the future to provide them with higher income, so after retirement they can survive off of the additional income after retirement.

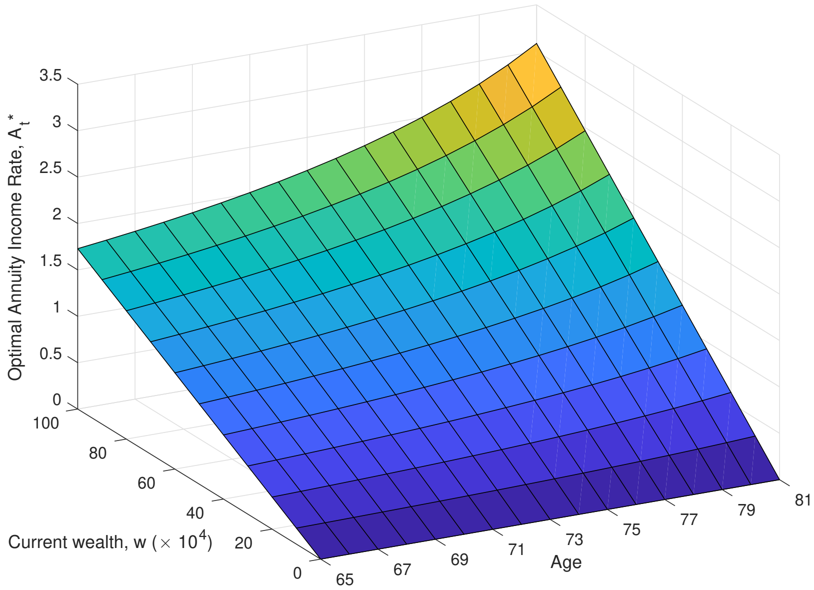

FIGURE 6.7: Effect on the optimal annuity income level of post-retirement and wealth for age-dependent.

6.5 Conclusion

This chapter examined the optimal strategies on investment, consumption and annuity income level for a pensioner who receives an income after retirement from the investment performance and life annuities under Heston's Stochastic Volatility model. The objective of this chapter was to maximize the expected utility of consumption under the CRRA utility function. We derived and solved the model using dynamic programming by applying the Legendre transform, the duality theory and the change variable technique. We believe this chapter is the first to study the Heston's SV model with the presence of the life annuity variable in the wealth process.

We found life annuities do not impact the investment strategy among the less risk averse pensioners compared to the more risk-averse ones. However, the presence of life annuities affects investment behavior after retirement. When the pensioner does not consider buying annuities before or after retirement (mostly discussed in literature), the investment increases as time increases. This is contradicting and does not happen when the pensioner decides to invest in life annuities. The pensioner will reduce their investment in risky assets no matter their risk aversion level.

Under the Heston model, we can see clearly the consumption has a non-monotonic relationship with the risk aversion level. The consumption becomes more volatile as the risk aversion become lower than the baseline $\gamma = 2.5$ as discussed in Blake et al. [128]. We also found pensioners will annuitize more after retirement since they want to have additional income after retirement. We believe the findings in this chapter provide a new explanation for the pensioner's decisions on optimization under the stochastic volatility market.

Chapter 7

Application of the Proposed Model

7.1 Introduction

In this chapter, we focused on the relevant applications of the proposed model. As we discussed throughout this thesis, we investigate the pensioner's gains from access to annuities.

7.2 Relevant Applications

Generally, our proposed model is applicable in pensioner's decision-making problems, to optimize the investment strategy, consumption and annuity after retirement, which requires a sufficient consideration before the final decision is made. This pensioner's objective is to maximize the utility drawn from consumption during retirement.

This study is on annuitization, where it is difficult to get a real market data of annuitization that can be used to simulate the model. It is well-known that the optimization model can be applied in pensioner's decision during and after retirement. Instead of that, we discussed possible applications that can be used in the proposed model.

Let assume that the pensioner wants to optimize his/her utility after retirement. At the first place, the pensioner needs to know their level of risk preference before they can put some investment in the existing market. The pensioner's risk tolerance (the degree of uncertainty the pensioners are willing to take on to achieve potentially greater rewards) is determined by a combination of factors, including the investment goals and experience, how much time they have to invest, pensioner's other financial resources and their "fear factor." Then, pensioner needs to know how much they have in their current pension pot and when they want to start the annuitization. By considering those situations, the pensioner can use the proposed model to predict their income after retirement. This model can help them to plan their income and consumption during and after retirement.

It is not limited to the pensioner side, but it is suitable to be implemented by the insurance company for the purpose of offering the best insurance plan for their clients.

Chapter 8

Conclusions

8.1 Findings

In this thesis, we modeled the retirement phase of the life cycle for UK pensioners. The model presented is realistic with respect to stochastic factors and control variables. It allows the pensioner to invest in life annuities, as this issue has become famous among researchers. The significant contribution of this study is the development of an investment-consumption model that allows the study of optimal investment behavior in the presence of a risky asset, life annuities and stochastic volatility market. This is accomplished by incorporating with continuous-time into the standard maximization model.

We have considered the investment allocation problem for a pensioner who invests their wealth in risky assets and life annuities in the decumulation phase. We referred to the model proposed by Milevsky and Young [7] and extended the proposed model by assuming the stochastic volatility variable. We also considered the pensioner's risk preferences in the model, where we assumed that the pensioner's preference is the CRRA utility function and we derived the model mathematically. We used dynamic programming or specifically stochastic dynamic programming to solve the model.

The main contribution of this study is developing the stochastic optimization model with life annuities under the stochastic volatility model. We considered two different volatility models, the CEV model and Heston's SV model. To the best of our knowledge, this will offer new explanations in literature. We divided our findings into four main criteria, wherein each criterion explains each optimal strategy. First, the key findings with respect to utility maximization are:

- The combination of Legendre transform, dual theory and change variable technique is a more reliable approach to solve optimization compared to the substitution method (change variable technique) alone (Chapter 3). Since the focus of the study is on asset allocation with the existing life annuities, it is better to solve it for purchasing boundary. In Chapter 3, the purchasing cannot be obtained using the change variable technique alone.
- The existence of life annuities in the model does not have much impact on utility maximization. According to the diminishing marginal utility of income and wealth, the individuals gain a smaller increase in satisfaction and happiness as the income increases. The utility is the satisfaction or happiness, and marginal utility is the increase in utility that results from an additional unit of consumption.

Then, the key findings with respect to investment strategy are:

- The existence of an annuity income does not much impact the optimal investment at the retirement, as the difference is small in every model we discussed. We can conclude that since investing in life annuities will provide a guaranteed income after retirement, the pensioner can continue to invest in the risky asset according to their risk preferences. The difference is seen in the GBM model when we solve it using the Legendre transform (Chapter 5).
- However, the optimal investment strategy is also age-dependent when allowing life annuities in the retirement process. As age-dependent, the optimal investment strategy decreases with the existence of annuity income in the wealth process as discussed in the numerical example in Chapter 6. This is different in literature, where when the pensioner does not consider life annuities, the optimal investment will increase over time after retirement. Steffensen [94] mathematically proved that , a) where the demand for the stock is decreasing for all investors. b) the initial optimal demand for stocks for a riskier investor is more significant than a less risky investment. c) the riskier investor will of course have the higher optimal demand for stocks than a less risky investor when holding at time t.

- For all the models, even it is a complete or incomplete market, the optimal investment strategy decreases with respect to the pensioner's risk preferences at retirement. According to Blake et al. [128], the lower risk aversion leads to higher post-retirement equity weighting but does not affect the age at which it is optimal to turn the remaining pension fund asset into annuities.

The key findings with respect to consumption are:

- Traditionally, consumption is a smooth, concave and monotone function of wealth for the risk-averse pensioner. However, we found that the consumption rate is not monotonic with the pensioner's risk preferences. As shown by the results of all the models discussed, the rate of consumption and risk aversion is not monotonic since there is no peak in the consumption rate. The rate of consumption is a monotonic function of wealth for a risk-averse individual, as illustrated in Figure 3.7 in Chapter 3. Wang and Young [38] obtained the same relationship under the GBM model with the existence of the surrender charge.
- We found that the existing annuity income plays a role in altering the optimal consumption rate for all levels of risk aversion. When an annuity income is higher than the wealth in the pension pot at retirement, the pensioner who dislikes risk will consume more since they know they have extra income from the life annuities purchased.

Lastly, the key findings with respect to annuity income level are:

- An annuity is a form of insurance. Thus, risk preferences are the natural factor to consider in evaluating an individual's willingness to annuitize. The pensioner who is not a risk lover will choose to continue with annuitization since the income from life annuities is more secure by looking at the historical view of the annuity rate, which is low and expected to remain so.
- The optimal annuity income is age-dependent. Referring to Figure 4.7 and Figure 6.7, the optimal annuity income increased after retirement. This proves the pensioner is happy to invest in life annuities and will purchase additional life annuities in the future.

8.2 Further study

The model presented has many benefits, but also some limitations which can give an idea for further research. For example, the utility function used in this study is CRRA, hence a more reliable utility function can be used, such as a recursive utility function. A recursive utility function can be built up from two components, which are a time aggregator that characterizes preferences in the absence of uncertainty and a risk aggregator that defines the certainty equivalent function that characterizes preferences over static gambles and is used to aggregate the risk associated with the future utility.

Besides that, the other model of stochastic volatility can be applied to test the optimization behavior under a different stochastic volatility model. Other stochastic volatility models that can be considered are the SABR model (Stochastic Alpha, Beta, Rho) and the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, both of which are another popular models for estimating stochastic volatility.

Finally, the model can be extended by taking care of the existence of the labor income or the pensioner's health status. Since both of these variables in the wealth process have become well-known, however, neither has been considered together with life annuities.

Appendix A

Appendix A : Verification of ODE

Equation (3.30) is a Bernoulli ODE of the form

$$P'(t) - \gamma p P(t) = -\gamma \big[P(t)\big]^{\frac{\gamma-1}{\gamma}}. \tag{A.1}$$

Then, we defined $\phi(t,\infty) = \big[P(t)\big]^{\frac{1}{\gamma}}$, and (A.1) can be rewrite as

$$\begin{aligned} d\phi + \big[-p\phi + 1\big]ds = 0 \\ \frac{d\phi}{dt} = p\phi(t,\infty) - 1. \end{aligned} \tag{A.2}$$

Then, we proposed the solution as

$$\phi(t,\infty) = \int_t^\infty e^{-\left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma} - \frac{(1-\gamma)}{\gamma\bar{z}}\right)(s-t)} ds. \tag{A.3}$$

From (A.3), we can write

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{d}{dt}\int_t^{\infty} e^{-\left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right)(s-t)}ds \\
&= -e^0 + \int_t^{\infty} \frac{\delta}{\delta t} e^{-\left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right)(s-t)}ds \\
&= -1 + \left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right) \times \int_0^{\infty} e^{-\left(\frac{(\alpha+\lambda-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right)(s-t)}ds.
\end{aligned} \tag{A.4}$$

Then, rearranging (A.4), we have

$$\begin{aligned}
\frac{d\phi}{dt} &= p\phi(t,\infty) - 1 \\
\frac{d\phi}{dt} &= \left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right)\left[\int_t^{\infty} e^{-\left(\frac{(\alpha+\lambda)-\delta(1-\gamma)}{\gamma}-\frac{(1-\gamma)}{\gamma\bar{z}}\right)(s-t)}ds\right] - 1,
\end{aligned} \tag{A.5}$$

which verifies equation (A.2).

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